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The Trinomial Asset Pricing Model

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Kandidatarbete inom civilingenjörsutbildningen vid Chalmers

Johan Björefeldt
Dick Hee
Edvin Malmgård
Vilhelm Niklasson
Tom Pettersson
Jakob Rados

Institutionen för matematiska vetenskaper
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Johan Björefeldt   Edvin Malmgård   Vilhelm Niklasson
Tom Pettersson   Jakob Rados

Kandidatarbete i matematik inom civilingenjörsprogrammet Teknisk matematik vid Chalmers

Dick Hee

Handledare: Simone Calogero
Examinator: Maria Roginskaya

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Populärvetenskaplig presentation

Har du någon gång blivit frustrerad av att priset på något som du planerade att köpa plötsligt steg precis innan du skulle köpa det? Kanske hade du länge planerat att åka på en semesterresa, men samma dag som du skulle köpa biljetter så höjde resebolaget priset med 3000 kr. Vad du skulle kunna haft nytta av i ett sådant läge är något som kallas för en option. En option är ett avtal mellan två parter som ger innehavaren rätten, men inte skyldigheten, att i framtiden köpa eller sälja en produkt (kallad underliggande tillgång) till ett förbestämt pris. För att få den här rättigheten betalar köparen av optionen en liten summa pengar till den som ställer ut optionen. En kritisk fråga är hur dessa optioner ska prissättas på ett rättvist sätt.

Det är i teorin möjligt att teckna optioner för många olika typer av produkter, även semesterresor. Vanligtvis används dock optioner på den finansiella marknaden och produkterna i fråga är aktier, det vill säga små ägarandelar i företag. För att kunna bestämma priset på en option måste antaganden göras kring hur värdet av den underliggande tillgången förändras. En modell för hur aktier kan modelleras beskrivs av den så kallade trinomialmodellen. Enligt denna modell så kan aktiepriset i varje tidsintervall, exempelvis varje dag eller varje timme, röra sig i en av tre möjliga riktningar med olika sannolikheter. Antingen rör sig priset upp eller ner med vissa specificerade procentsatser, eller så förblir priset oförändrat.

En av de vanligaste optionerna på den finansiella marknaden är den så kallade europeiska köpoptionen. Denna option ger ägaren möjligheten att i framtiden, vid den förbestämda lösendagen, köpa en aktie för ett visst lösenpris. Den avkastning (payoff) som optionen ger bestäms av skillnaden mellan aktiens pris på lösendagen och det förbestämda lösenpriset. Om aktiepriset är lägre än lösenpriset väljer givetvis optionsinnehavaren att inte nyttja optionen eftersom det är billigare att köpa aktien för dess verkliga pris och payoffen blir noll.

Ett rättvist pris för en option ska inte vara till varken säljarens eller köparens fördel. Den som till exempel säljer optionen ska alltså inte ha möjlighet att göra en garanterad vinst oavsett hur säljintäkterna investeras. Utifrån detta antagande går det att visa att det rättvisa priset för en option är den tidsjusterade (diskonterade) genomsnittliga payoffen givet vissa riskneutrala sannolikheter i trinomialmodellen. Eftersom det finns flera riskneutrala sannolikheter för aktiepriset i trinomialmodellen så kommer dock inte det rättvisa priset att vara unikt. Som en följd av detta är det inte alltid möjligt att replikera värdet av en option (även kallat hedga) i trinomialmodellen genom att endast investera i den underliggande aktien och en riskfri tillgång (räntepapper). Marknaden sägs således vara ofullständig.

Vid sidan om trinomialmodellen finns det många andra modeller för att prissätta optioner där den vanligaste är Black-Scholes-modellen. Denna modell introducerades 1973 och flera av personerna bakom modellen belönades med Sveriges Riksbanks pris i ekonomisk vetenskap till Alfred Nobels minne år 1997. Inom den finansiella marknaden anses detta vara det korrekta priset för optioner. En av fördelarna med deras modell är att priset kan uttryckas med hjälp av en matematisk formel som enkelt kan beräknas. En nackdel är dock att den inte fungerar till alla prissätta alla typer av optioner, vilket skapar ett behov av exempelvis trinomialmodellen som är mer generell men inte lika exakt. När tidsintervallen i trinomialmodellen sätts till väldigt små värden i kombination med vissa villkor på övriga parametrar i modellen närmar sig dock dess pris Black-Scholes-priset, förutsatt att det sistnämnda priset existerar.

En annan vanlig modell för optionsprissättning är binomialmodellen. Till skillnad från trinomialmodellen bygger denna modell på antagandet att priset för den underliggande tillgången endast kan röra sig i två olika riktningar i varje tidsintervall, antingen upp eller ner. En sådan begränsning i rörelsemönstret gör det enklare att implementera binomialmodellen än trinomialmodellen. Trinomialmodellen har dock fördelen att det krävs färre tidsintervall innan priset stabiliseras och närmar sig Black-Scholes-priset. Detta möjliggör kortare beräkningstider, vilket anses värdefullt.

Det är av speciellt intresse att använda trinomialmodellen för att prissätta optioner med mer komplexa payoff-strukturer än den som exempelvis beskrevs för den europeiska köpoptionen. Dessa så kallade exotiska optioner där payoffen exempelvis kan bero på den underliggande aktiens genomsnittliga värde under hela löptiden saknar ofta ett Black-Scholes-pris och det blir därför nödvändigt att tillämpa andra modeller. Trinomialmodellen kan för det mesta tillämpas och priset bestäms som tidigare beskrivits utifrån den diskonterade genomsnittliga payoffen. På grund av att beräkningarna ibland är väldigt tidskrävande är det dock inte alltid en bra lösning att använda trinomialmodellen för att prissätta sådana optioner.

Med kunskap om optioner och trinomialmodellen samt dess breda tillämpningsområden bör det stå klart att det finns många fördelar med att använda dem. Oavsett om det handlar om aktier eller semesterresor kan de implementeras med framgång.

Sammanfattning

Optioner har en stor betydelse på den finansiella marknaden. Under lång tid har matematiker arbetat med att ta fram det rättvisa priset för olika typer av optioner. I det här kandidatarbetet undersöker vi hur trinomialmodellen fungerar som prissättningsmetod. Vi ger en förklaring till modellen och därifrån härleder vi villkoren som krävs för att den ska kunna användas för att beräkna det rättvisa priset av europeiska optioner. Vi undersöker även hur modellen approximerar Black-Scholes-priset, samt applicerar trinomialmodellen för att prissätta sex olika typer av exotiska optioner.

I vår slutsats har vi kommit fram till att förutsatt vissa restriktioner konvergerar trinomialmodellen mot Black-Scholes-priset. Vi har även kommit fram till att trinomialmodellen är väldigt användbar för att beräkna det rättvisa priset för amerikanska optioner. Detta beror på att i jämförelse med den mindre avancerade binomialmodellen konvergerar trinomialmodellen snabbare till Black-Scholes-priset. Vid prissättning av exotiska optioner är slutsatsen att trinomialmodellen ofta kan vara användbar när optionens slutvärde inte beror på vägen av den underliggande tillgången.

Abstract

Options play an important part in financial markets. Throughout the years, several pricing theories have been developed to generate fair prices for options of different sorts. In this thesis we investigate the trinomial asset pricing model. After giving an explanation of its properties, we use the trinomial model to derive a fair price of standard European options. We study the trinomial model approximation of the Black-Scholes price and finally apply the trinomial model on six different exotic options.

We have found that, under certain conditions on the model parameters, the trinomial price converges to the Black-Scholes price. Furthermore, we have established that pricing American put options works well using the trinomial model. Regarding the investigated exotic options, we conclude that the trinomial model can often be suitable to use when pricing exotic options that are not path dependent. In relation to the less advanced binomial model, the trinomial model has the advantage of converging to the Black-Scholes price faster than the binomial model.

Preface

This thesis was produced at Chalmers University of Technology in Sweden, during the spring of 2016. The project group consisted of five students of the Industrial Engineering and Management programme, as well as one student of the Engineering Mathematics programme. The thesis has been written primarily in English, as we consider it to be the most fitting language for this type of project, however as it was done at a Swedish institution we have included abstract and popular science summary written in Swedish above, as well as an extended Swedish summary in Appendix D. All theorems included in this report have been produced by the project group, unless otherwise stated.

We would like to thank everyone involved in this project. We would like to especially thank our supervisor, Simone Calogero, who has been driving us forward and supporting us every step of the way. We were first introduced to option pricing concepts during his course Options and Mathematics; without him we would have never gotten the chance to work on this interesting project. Thank you!

While working on this thesis, the work has largely been done in group rather than on an individual basis. However, everyone involved has had particular responsibility for some specific sections, particularly during the writing process.

Johan Björefeldt has been in charge of the second chapter of this thesis, and has studied the compound option. Dick Hee has written about the barrier option and has been responsible for the fourth chapter. Edvin Malmgård has been in charge of the lookback option, and has also had the main responsibility for introduction, conclusion, and Appendix B. Vilhelm Niklasson has written the third chapter as well as the section on the cliquet option, he has also been in charge of assembling Appendix C. Tom Pettersson has investigated the Bermudan option, and has also written part of the first chapter. Jakob Rados has been in charge of Appendix A and Appendix D, he has also written part of the first chapter and has studied the Asian option.

A journal has been kept, detailing the efforts of each group member. It should be noted that while all group members have had particular responsibility for specific parts of this thesis, there have still been contributions by all members to all sections.

This project has been a challenge for us, the work has been complex at times and coordination has not always been easy. The fact that the background of the group members are so divided has been a challenge, but also an asset, as it means we all provide different perspectives. In the process of working we have learned a lot, and we are very happy we had the opportunity to work together on this project.

Contents

Introduction

One of the most central topics in financial mathematics is option pricing theory. In the early 1970s, Fisher Black and Myron Scholes derived a closed formula to calculate the price of European options on a non-dividend-paying stock [1]. Although they could not derive a corresponding formula for American options, the work by Black and Scholes laid a path for later research in financial mathematics, developing several pricing models and procedures for standard options. These models are, to this day, still popular and widely used. As one among them, the binomial model was the first lattice model introduced in 1979 by Cox, Ross, and Rubenstein [2]. This model quickly became one of the most applied methods for pricing options, because of its ability to price European and American derivatives by a simple algorithmic procedure.

Eventually the trinomial model was formulated, as an extension of the work of Cox, Ross, and Rubinstein. It was first introduced in 1986 by Phelim Boyle [3] with the advantages over the binomial model being that it is more flexible due to the extra degree of freedom and it possesses some important properties that the former model lacks. The trinomial model incorporates three possible values that an underlying asset can have in one time period, where the possible values will be greater than, the same as, or less than the current value. The main purpose of Boyle was to consider multidimensional markets in a discrete model.

The purpose of this thesis is to study the properties of the trinomial model, its convergence to Black-Scholes, and some applications of the model to an array of different exotic options. The initiative stems from a desire to provide the financial mathematics community with additional research on the topic. The thesis begins with an introduction to financial concepts and a short historical review on option pricing. Chapter two is devoted to formulating the trinomial model and interpreting it in a probabilistic sense. The chapter then shifts focus to illustrate how to correctly price self-financing portfolios in the context of the trinomial market, and under which conditions arbitrage opportunities may be prevented. Finally, the chapter ends with an investigation of whether the convergence to the Geometric Brownian Motion is faster for the trinomial model in comparison to the binomial model.

In chapter three we will derive the fair price of a European derivative by using the self-financing portfolio. The second part of the chapter will describe the important concept of hedging a derivative in the trinomial market model, and different ways to deal with the model incompleteness will also be presented.

The fourth chapter will focus on some important properties of the trinomial model and investigate whether the model converges to the Black-Scholes equation, and if so under which conditions. Convergence will be studied for both European as well as American options. Since there exists no closed formula for the theoretical Black-Scholes price of American options, we will use other means to verify convergence in this case. We will also study under what conditions on the relevant parameters this occurs and verify our findings by numerical experiments.

The fifth, and last chapter of the thesis is the most comprehensive one and will examine the applicability of the trinomial model on six exotic options. The options are Asian, barrier, Bermudan, cliquet, compound, and lookback options. They are classified as exotic because of their payoff structures, which are more complex than the standard vanilla European and American derivatives. The purpose of the chapter is to numerically test the trinomial model and compare the results with other methods.

Chapter 1

Background

In the beginning of this chapter some crucial financial terminology used throughout the thesis will be explained in more detail to provide a necessary foundation. Some basic concepts about options and their purpose on the financial market will also be introduced.

The latter part of the background consists of a historical review about options and option pricing.

1.1 Financial concepts and assets

The first part of this section consists of important financial concepts. Here we describe some crucial financial concepts that must be understood before introducing the trinomial model and pricing methods.

In the second part we introduce the financial assets that will be used throughout the thesis. Similar to financial concepts one has to understand the terminology of financial assets and their structure before it is possible to discuss pricing methods of options.

Financial concepts

Before introducing options there are a few concepts that must be understood, starting with the concept of financial asset. The term financial asset may be used to identify any object which can be bought and sold under a specific set of rules and whose value is derived from a contractual claim. Financial assets can be divided into two types, material assets such as gold, oil, coffee, and immaterial assets such as stocks. The value of the asset is the price determined by the buyer and the seller. The ask price is the minimum price at which the seller is willing to sell the asset. The bid price is the maximum price that the buyer is willing to pay for the asset. The price of the asset is set when the difference of these two values, called the bid-ask spread, becomes zero. The exchange of the asset takes place at this price.

There are two ways that assets can be exchanged or traded; in official markets or over the counter (OTC) [4]. In official markets all trades are regulated by a common legislation, while others are made over the counter, where all terms are agreed upon by the individual traders or institutions. The two most active regulated markets are the stock market and the options market. We define the market price of an asset as the price of that asset in the market where it is traded. From now on it will be referred to simply as the value of the asset or the price of the asset.

Now we need to consider different types of transactions of assets, other than just buying and selling; for instance short-selling. Short-selling an asset is in practice equivalent to selling an asset without owning it. In essence, this means that an investor borrows a number of shares of an asset from another investor and then immediately sells them in the market. The reason for this is either to speculate that the price of the asset will decrease or to hedge another position. In the future the investor must buy the asset to give it back to the lender and the investor will make a profit if the asset's current price is lower than the price when it was borrowed and sold. In general, the investor is said to have a short position on an asset if the investor will profit from a decrease of its value and a long position if the investor will profit from an increase of its value. We also have to remark that all transactions in that market are subject to transaction costs (e.g. broker's fees) and transaction delays (real markets are not instantaneous). In this thesis we will only consider the fair price of an option and hence we will assume assume that there are no transaction costs or delays.

Financial Assets

In this thesis we will consider three assets; stocks, assets in the money market, and options. A stock is a type of security that signifies ownership in a company and represents a claim on part of the company's assets and earnings. So holding shares of a stock in a specific company is equivalent to owning a fraction of the company. Stocks are traded on the stock market. For instance, over 300 company stocks are traded in the Stockholm stock exchange. The price per share at time $t > 0$ of a stock will be denoted by $S(t)$. Several companies choose to pay dividends to its shareholders. A dividend is a payout of a portion of a company's earnings to its shareholders. This means that a fraction of the asset price is deposited to the bank account of the shareholder. Dividends can also be payed as shares of stocks or other valuable property. Before the dividends is payed there is a day known as the Ex-dividend date, the investors that buy the stock after this date are not entitled to that year's dividends. In theory, the price of the stock will decrease with the same amount as the dividends the day after the Ex-dividend day. In this thesis we will however assume that there are no dividends paid due to simplicity, since dividends does not affect the generality of the pricing method.

A money market is a segment of the financial market in which investors can borrow or deposit money at a given interest rate which varies over time. A money market contains financial instruments with high liquidity and very short maturity times. Examples of short term loans traded on the money market are saving accounts, commercial papers, and treasury bills.The assets are typically considered risk-free. Holding a long position on a risk-free asset in the money market implies that the investor has lent money, while holding a short position implies that the investor borrowed money. The value of a risk-free asset at time t will be denoted by $B(t)$, $B(t_2) > B(t_1)$, for all $t_2 > t_1$. A risk-free asset is said to have a constant rate of interest $r \geq 0$ if $B(t) = B(0)e^{rt}$.

A financial derivative is defined as an asset whose value depends on the performance of another asset (or several different assets), which is called the underlying asset(s). In this thesis we consider options as our financial derivatives whose underlying asset generally is a single stock. A standard option is essentially a contract that gives to the buyer the right, but not the obligation, to buy or sell the underlying asset for a given fixed price at the time when the options expires, also known as the maturity T. This is also known as a vanilla option. There are two basic types of options, a call option and a put option, depending on whether the buyer has the right to buy or to sell the underlying asset. If the buyer has the right to buy the underlying asset then the option is called a call option and if the buyer has the right to sell the underlying asset the option is called a put option [4]. The transaction of buying and selling options involves two parties, a buyer or owner of the option and a writer of the option. If the buyer can only exercise the option at a given time $t = T > 0$ the option is called European, while if the buyer can exercise the option at any given time t in the interval $(0,T]$, the option is called American. We define the intrinsic value for a call option as

$$Y(t) = \max(S(t) - K,0) = (S(t) - K)_+ \tag{1.1}$$

and as

$$Y(t) = \max(K - S(t),0) = (K - S(t))_+ \tag{1.2}$$

for a put option, where K is the agreed upon strike price at maturity. $Y(t)$ is also the payoff of an American call or put option exercised at the time $t \leq T$, while $Y \equiv Y(T)$ is the payoff for the European options. Some useful terminology developed to describe the payoff over time are the following expressions. The option is said to be in the money if, for a call option, $S(t) > K$ (the opposite for a put option), to be at the money if $S(t) = K$ (the same holds for a put option), and to be out of the money if $S(t) < K$ (the opposite for a put option).

In conclusion, there are two main reasons for investors to invest in options; to hedge a position or to speculate [4]. The hedging strategy works as an insurance policy, options can be used to insure investments against a downturn. Hedging strategies are common in large companies with customers spread worldwide, hence creating revenue streams in different currencies and exposing the company to currency risks. Options give the investor the opportunity to restrict the downside of an investment while simultaneously generating a potential upside in a cost efficient way.

Speculations can be described as bets on movements of a underlying security. One advantage of options is that the investor can generate profit no matter which way the market is heading. The use of options in this manner is the reason why options have the reputation of being risky assets.

There exist other types of options than common European and American ones. One category of options is called exotic options, these are options that are more complex than standard European and American options. Unlike the standard options, an exotic option can have a path dependent payoff which means that the underlying asset's path affects the payoff of the option. Exotic options are generally traded over the counter. Exotic options can be used for several different purposes, for example to hedge another derivative, to speculate on the future movements of an asset or to reduce risk in investments [5].

1.2 Historical review

One could assume that options are another sophisticated financial instrument with roots no older than Wall Street itself. This assumption would be wrong as the concept traces back thousands of years, long before they began officially trading in 1973 [6]. The purpose of this section is to provide the reader with a brief summary of some historical events in the field of option pricing, with the research of Black and Scholes in 1973 as a point of chronological reference.

The beginning

The oldest accounted option trade is found in Aristotle's Politics [7]. In the eleventh chapter of the first book one can read about Thales of Miletus (624-547 BC), adopting the first option contracts in the olive business of ancient Greece. During winters Thales would predict a good crop on olives in the coming summer, and thus raised capital in order to pay deposits on all the oil-presses in Miletus and Chios. In doing so he secured an option to hire the oil-presses later on, and since there were no other buyers the price was set low. When summer came he eventually made a good fortune when he chose to exercise his option, renting out the now highly demanded oil-presses.

What Thales had made was, in essence, a purchase of a call option. Through his initial deposits he bought a right, without obligation, to hire all the oil-presses at a later point in time. Depending on the bountifulness of the harvest, he could choose to exercise or not to exercise his option in the same way modern options may or may not be exercised, depending on whether they end up in or out of the money.

Of course, with respect to pricing, no mathematical models were applied at this point in history. Logically so, since there was no need for it; Thales created his own monopoly and priced his oil-presses however he pleased. But in a historical point of view, as we know it, the financial instrument ”option” was born with Thales.

Pre Black-Scholes

Having concluded that the notion of options is thousands of years old, the story of pricing options is significantly younger [8]. Yet not so young as many believe. A widely spread assumption in modern finance is that option pricing theory started with Fischer Black and Myron Scholes through their revolutionary findings in 1973. While this year certainly constitutes one of the biggest milestones in the history of option pricing, the assumption of it originating back then is wrong. In fact, its true origin dates back to the very beginning of the twentieth century, with the work of the French mathematician Louis Bachelier.

In 1900, Bachelier published in his PhD thesis a closed formula for the pricing of standard European calls and puts [9]. For non-dividend-paying stocks, and for zero interest rates, he showed that the price of a European call should be

$$c(S,T) = SN\left(\frac{S-K}{\sigma\sqrt{T}}\right) - \sigma\sqrt{T}n\left(\frac{S-K}{\sigma\sqrt{T}}\right),$$

where S is the stock spot price, K the strike price, σ the volatility of the stock price (i.e. the instantaneous standard deviation of the stock price), T the time until maturity, $N(\cdot)$ the normal cumulative distribution function and $n(\cdot)$ the probability density function of the standard normal distribution. The formula was derived under the assumption that the stock prices follow an arithmetic Brownian motion. It also ignores discounting and assumes that stock prices can be negative. These assumptions are questionable to make, yet Bachelier had taken a first leap in the field of option pricing and gave the community a foundation to rely on for future research.

Approximately sixty years later, Case Sprenkle presented an adapted approach to the one of Bachelier [10]. He addressed the problematic assumption of existence of negative stock prices by assuming log-normal returns, as opposed to Bachelier who had used normal returns. Furthermore, Sprenkle also introduced the notion of risk averse investors. His closed formula became

$$c(S,T) = e^{\rho T} S N(d_1) - (1 - A) K N(d_2),$$

with

$$d_1 = \frac{1}{\sigma T}\Big[\log\frac{S}{K} + \big(\rho + \frac{1}{2}\sigma^2\big)\Big], d_2 = d_1 - \sigma T,$$

where ρ is the average rate of growth of the stock price and A is the degree of risk aversion. This formula does resemble the one by Black and Scholes that eventually would be derived, yet it did not receive much attention due to the number of involved parameters that required an estimation before computing the price according to this formula. For instance one has to calculate A and ρ, and Sprenkle did not include much information on how these should be properly computed in his article in 1961.

During the following decade, two mathematicians continued to improve this formula [8]. These were James Boness and Paul Samuelson. Boness improved the formula by allowing it to account for the time value of money through discounting, Samuelson modified the formula to allow the option to have a different level of risk from the stock. Their respective input to previous work yielded the formula

$$c(S,T) = S e^{(\rho-\alpha)T} N(d_1) - K e^{\alpha T} N(d_2),$$

where α was defined, by Samuelson, as the average rate of growth of the call's value, and d_1 and d_2 defined the same as by Sprenkle.

As we can see, substantial work was done prior to the year of 1973 when Myron Scholes and Fischer Black published their famous article "Option Pricing and Corporate Liabilities". It can even be concluded that the results of Bachelier, Sprenkle, Boness, and Samuelson resulted in something that very much resembles the Black-Scholes formula. Thus it should be said that the work of these predecessors of Black and Scholes had a greater part in the advancement of the field than one might think. Needless to say however, the discoveries of Black and Scholes were groundbreaking in the field of finance [8].

1973: The revolutionary year

In 1973, Fischer Black and Myron Scholes, with help from Robert Merton, introduced the world to the formula

$$c(S,t) = S N(d_1) - N(d_2) K e^{-r(T-t)},$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\Big[\log\frac{S}{K} + \big(r + \frac{\sigma^2}{2}\big)(T-t)\Big],$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The striking beauty with this formula is that it is completely independent of the risk aversion of the investor [1]. Apart from the standard properties of the stock, the Black-Scholes formula calculates the option price only depending on the volatility of the stock and the universal risk free rate. This makes the formula easy to use since the parameters are easy to obtain. Another immense advantage with the Black-Scholes formula, with regards to the work before 1973, is that it gives an explicit hedging strategy for the replication of the call, also only depending on the volatility and risk free rate apart from the standard properties of the stock.

As one could hereafter scientifically motivate theoretical prices of options, this led to a boom in options trading that legitimized sophisticated option markets. Institutions such as the Chicago Board Options Exchange (CBOE) were formed along with other option markets around the world. Scholes and Merton were later on awarded with a Nobel Prize in 1997 as a reward for these findings [8]. Two years prior to this event, Fischer Black had unfortunately deceased and thus was merely mentioned as a contributor [11].

Post Black-Scholes

Shortly after Black and Scholes presented their paper, reactions and alternative approaches to their methods piled up [3]. One of these was the model by Phelim Boyle, who in 1976 presented how Monte Carlo simulation provides an alternative method to obtain option valuation solutions through numerical experiments. Furthermore, discussion arose regarding some problematic assumptions raised in the Black-Scholes model [12]. A pair of these concerned the volatility and the risk free interest rate, which in the Black-Scholes model are assumed to be constant. Since it had been observed, especially after the market crash in 1987, that volatility is stochastic rather than constant this demanded further investigation. Thus there was a call for models taking this into account, and in 1993 Steven Heston derived a closed-form solution for the price of a European call option with stochastic volatility. The basic Heston model follows below:

$$dS(t) = \mu S(t)dt + \sqrt{\upsilon(t)}S(t)dW(t),$$

where $\upsilon(t)$ is the instantaneous variance and $W(t)$ is a Wiener process.

Another path of development post Black-Scholes was one of simplification [13]. Cox, Ross and Rubinstein concluded that the mathematical tools employed in the models of Black and Scholes were advanced, obscuring the underlying economics. Thus they presented a paper in 1979 containing their simplified option pricing formula in discrete time. The formula states

$$C = S\phi[a; n,p'] - Kr^{-n}\phi[a; n,p],$$

where ϕ is a binomial distributed function,

$$p = (r - d)/(u - d),$$

and

$$p' = (u/r)p,$$

and a is the smallest non-negative integer greater than $\log(K/Sd^n)/\log(u/d)$.

Based on these findings of Cox, Ross and Rubinstein, they continued in their thesis to propose a new valuation method of options [13]. This was the binomial model which models the prices of a risky asset $S(t)$ in the following way:

$$S(t) = \begin{cases} S(t-1)e^u & \text{with prob. } p_u \\ S(t-1)e^d & \text{with prob. } p_d \end{cases}, \; t \in \mathbb{I} = \{1,...,N\}.$$

Thus the price of the option is based on the value of the underlying asset. The purpose of this thesis is to investigate the trinomial model, the difference of which is that it also includes the event of the stock value remaining the same for the time step $t-1$ to t. Thus for the purpose of this section, we shall not go further into detail on the binomial model, as many mechanisms will be introduced later on.

Modern contributions

Recent discussions have involved questions regarding Brownian motions and how accurately they model the reality of the financial markets [14]. The problematic reality is that we may observe spikes or jumps in asset prices that Brownian motions effectively do not take into consideration sufficiently. As a consequence, modelling asset pricing with Lévy processes has recently become fashionable, as they capture these events more efficiently than classic Brownian motions. A Lévy process may be defined as a process $(L(t))_{0 \leq t \leq T}$ with the following conditions:

- $L_0 = 0$
- L has independent increments
- L has stationary increments
- L is stochastically continous, i.e.: for every $0 \leq t \leq T$ and $\epsilon > 0$: $\lim_{s \to t} P(|L(t) - L(s)| > \epsilon) = 0$

The Poisson process is an example of a Lévy process. Using Lévy processes we may model the asset price as

$$S(t) = S_0 e^{L(t)}, 0 \leq t \leq T,$$

where $L(t)$ is the Lévy process whose infinitely divisible distribution has been estimated from the data set available for the particular asset. For the purposes of this thesis, the depth of Lévy processes will not be assessed further, yet we can discuss the benefits of its usage in finance. These are that traders require models that effectively capture the behaviour of the implied volatility smiles in order to handle the risk of trades. Lévy processes provide tools to describe these observations and thus recent research has laid a portion of its focus on the applicability of these processes in finance.

Another recent development is the studies of non-linear partial differential equations to calculate option prices [15]. The idea is to make the pricing equations adaptable to certain situations, such as the market impact of the issuer. Market impact can be viewed as the feedback mechanism between the option hedging induced stock trading activity and the price mechanisms. Essentially it seeks to make the mathematical models more flexible to the reality of the financial world. Progress has been made in this field, with non-linear heat equations as well as non-linear Black-Scholes PDE's, and using these to incorporate market impact in the models. Aligning theory with reality, an ever so demanding challenge, is still a work in progress.

Chapter 2

Foundations of the trinomial model

Within this chapter we formulate the basic concepts of the trinomial model, and how it is used to calculate the future price of a risky asset. We also discuss variations in the approach to the trinomial model. Self-financing portfolios are studied, and we determine how to correctly price these portfolios in the context of the trinomial market. We also study the conditions that need to be applied to the market to prevent arbitrage opportunities.

In the second part of this chapter we look at the probabilistic interpretation of the model, formulating it in the context of probability theory. The Geometric Brownian Motion is discussed here, and we use the martingale condition to prove convergence of the trinomial price to the Geometric Brownian Motion.

2.1 Formulation of the trinomial model

The trinomial asset pricing model is governed by the following dynamics for the price $S(t)$ of a risky asset:

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{with prob. } p_u \\ S(t-1) & \text{with prob. } p_0 = 1 - p_u - p_d \\ S(t-1)e^{d} & \text{with prob. } p_d \end{cases}, \; t \in \mathbb{I} = \{1,...,N\}.$$

We assume that the risky asset is a stock. Here $u > 0$, $d < 0$, $p_u, p_d \in (0,1)$, $p_0 = 1 - p_u - p_d > 0$. Hence, the stock price at any given time step may rise, fall, or stay the same. We assume that $S_0 = S(0)$, i.e. the price of the asset at the present time $t = 0$ is known.

Clearly the number of possible prices at time t will escalate quickly as t increases. The figure below shows the trinomial tree for $N = 3$ (for three time steps), as we can see there are 10 possible values of $S(N)$ already.

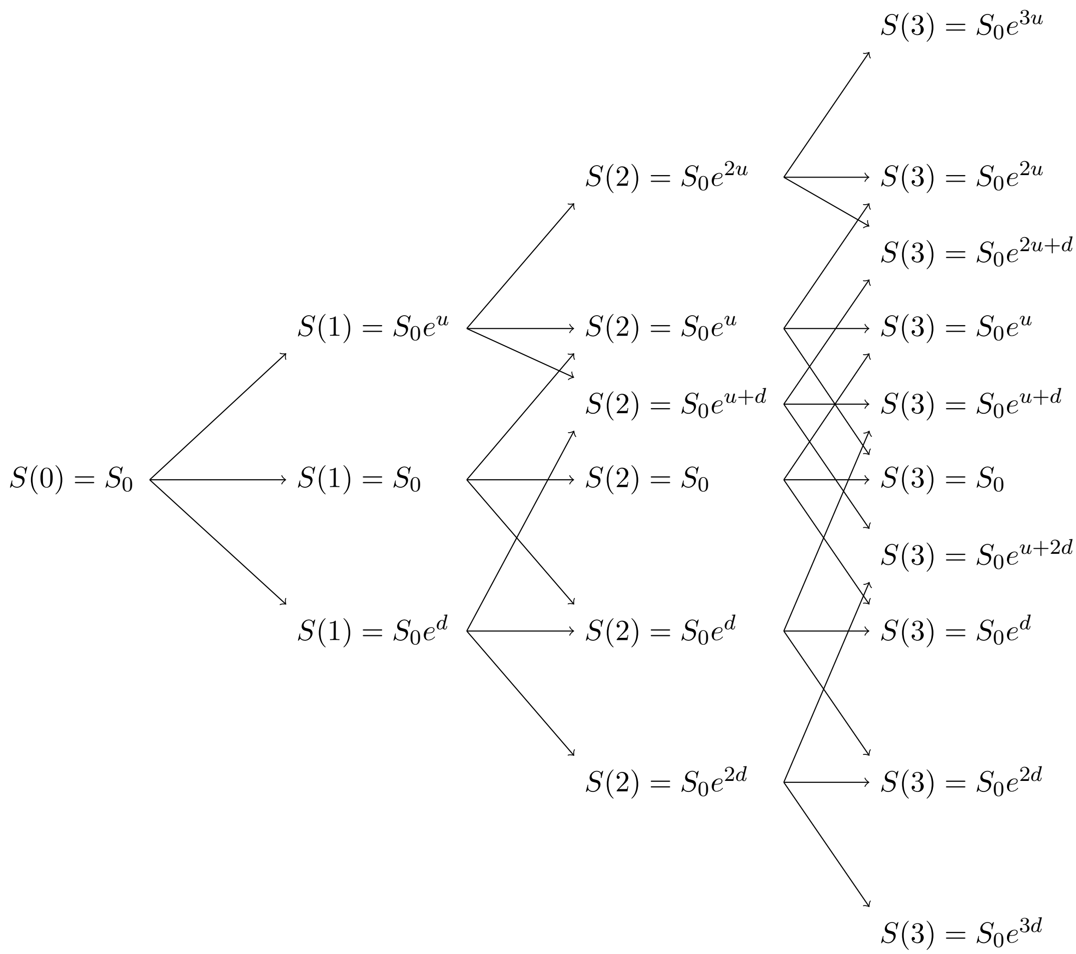

Figure 2.1: Evolution of the stock prices in the trinomial model when $u \neq d$.

Let us compute the number of possible prices at time t, denoted by ψ_t, analytically. We know that $S(t) = S_0 e^{N_u u + N_d d}$ where $N_u, N_d \in \{0,...,t\}$ and $N_u + N_d \leq t$, N_u and N_d clearly being the number of times the stock increases and decreases in value, respectively. By considering different ways of combining N_u and N_d we can see that

$$\psi_t = \sum_{N_u=0}^{t} \sum_{N_d=0}^{t-N_u} 1 = \sum_{N_u=0}^{t} (t - N_u + 1) = (t+1)t + t + 1 - \sum_{N_u=0}^{t} N_u$$

$$= (t+1)t + t + 1 - \frac{(t+1)t}{2} = \frac{(t+1)(t+2)}{2}.$$

In order to reduce the number of nodes in the trinomial tree, and simplify the model, we impose the recombination condition:

$$u = -d.$$

This will significantly limit the rate of expansion of the tree. From now on we will thus restrict the trinomial model to the form

$$S(t) = \begin{cases} S(t-1)e^u & \text{with prob. } p_u \\ S(t-1) & \text{with prob. } p_0 = 1 - p_u - p_d \\ S(t-1)e^{-u} & \text{with prob. } p_d \end{cases}, \ t \in \mathbb{I} = \{1,...,N\}.$$

The figure below shows the trinomial tree for $N = 3$, making use of the recombination condition; here we get 7 nodes at $t = 3$.

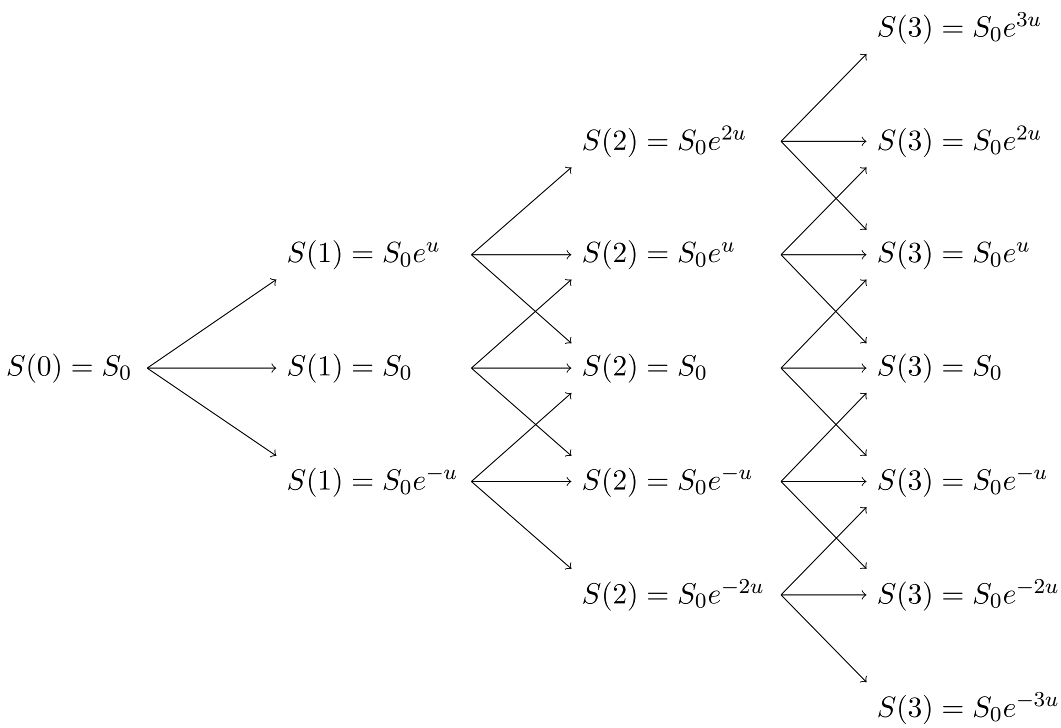

Figure 2.2: Evolution of stock prices in the trinomial model when $u = d$.

In this case the possible prices at time t are $S_0e^{u(N_u-N_d)}$ where $N_u, N_d \in \{0,...,t\}$, $N_u + N_d \leq t$, as we have seen before. Following the same logic as previously we arrive at

$$\psi_t = \sum_{i=-t}^{t} 1 = 2t + 1.$$

So for this case we obtain linear rate of growth for the trinomial tree, which (as we will see later) makes it a lot easier to manage.

2.1.1 Self-financing portfolios

We will denote by $B(t) = B(t-1)e^r$ the price at time t of a risk-free asset in the money market (i.e. a bond). We assume a constant interest rate $r \geq 0$ and that the initial price B_0 is known, so that

$$B(t) = B_0e^{rt}.$$

Definition 2.1.1. *Consider a portfolio* $\{h_S(t),h_B(t)\}_{t\in\mathbb{I}}$, *where* $h_S(t)$ *and* $h_B(t)$ *are the positions in the stock and the risk-free asset, respectively, at time* t. *Moreover,* $(h_S(t), h_B(t))$ *is the position in the interval* $(t-1, t]$. *This portfolio is called self-financing if and only if*

$$h_S(t)S(t-1) + h_B(t)B(t-1) = h_S(t-1)S(t-1) + h_B(t-1)B(t-1).$$

In addition, it is customary to choose $h_S(0) = h_S(1)$ and $h_B(0) = h_B(1)$. We can rewrite the definition of self-financing portfolios as

$$V(t) - V(t-1) = h_S(t)(S(t) - S(t-1)) + h_B(t)(B(t) - B(t-1)),$$

where $V(t)$ is the value of the self-financing portfolio at time t, which clearly shows that changes in the value of the portfolio are caused by changes in the value of the stock and the risk-free asset. It follows that changes in the portfolio position do not by themselves generate any value within the portfolio.

Next, we find ourselves in need to adjust the probabilities we use. The probabilities we have used thus far do not take into account the risk associated with investing in an asset, which is crucial when attempting to determine the asset price. Thus we will switch our focus to the risk-neutral probabilities q_{+1}, q_0, and q_{-1}. Let (q_{+1},q_0,q_{-1}) be a triple of real numbers defined by

$$q_{+1} + q_0 + q_{-1} = 1, \quad q_{+1}e^u + q_0 + q_{-1}e^{-u} = e^r. \tag{2.1}$$

The right equality comes from the fact that we need the current value of an asset to be its discounted value at a future point, therefore we impose this condition on the triple (q_{+1},q_0,q_{-1}). Choosing q_0 as the free variable we may represent the solution of the system (2.1) in the form

$$q_{+1} = \frac{e^r - e^{-u}}{e^u - e^{-u}} - q_0\frac{1 - e^{-u}}{e^u - e^{-u}}, \tag{2.2}$$

$$q_{-1} = \frac{e^u - e^r}{e^u - e^{-u}} - q_0\frac{e^u - 1}{e^u - e^{-u}}. \tag{2.3}$$

Later we will see that when the triple (q_{+1},q_0,q_{-1}) constitutes a probability, self-financing portfolios in the trinomial market are not arbitrage portfolios.

We will now study the valuation of self-financing portfolios, which is necessary in order to enable us to determine the fair price of a European derivative in the trinomial market. The following theorem posits a generalised valuation of the self-financing portfolio.

Theorem 2.1.1. *Let $V(N)$ be the value of a self-financing portfolio $\{h_S(t)h_B(t)\}_{t\in\mathbb{I}}$ at time N. Then the value at time $t < N$, for all $q_0 \in \mathbb{R}$, will be given by:*

$$V(t) = e^{-r(N-t)} \sum_{(x_{t+1}...x_N)\in\{+1,0,-1\}^{N-t}} q_{x_{t+1}}...q_{x_N} V(N,x) \tag{2.4}$$

In particular

$$V(0) = e^{-rN} \sum_{x\in\{+1,0,-1\}^N} (q_{+1})^{N_+(x)}(q_{-1})^{N_-(x)}(q_0)^{N_0(x)}V(N,x). \tag{2.5}$$

Moreover, letting

$V^{\pm1}(t+1) = V(t+1)$ *assuming* $x_{t+1} = \pm1$, *and* $V^0(t+1) = V(t+1)$ *assuming* $x_{t+1} = 0$,

then

$$V(t) = e^{-r}[q_{+1}V^{+1}(t+1) + q_0V^0(t+1) + q_{-1}V^{-1}(t+1)]. \tag{2.6}$$

Proof. **Part 1:** We begin by proving this theorem for the $t = N - 1$ case. The claim of the theorem then becomes as follows:

$$\begin{aligned} V(N-1) =& e^{-r}\Big[q_{-1}V(N,(x_1,...,x_{N-1}, -1)) \\ &+ q_{+1}V(N,(x_1,...,x_{N-1}, +1)) \\ &+ q_0V(N,(x_1,...,x_{N-1},0))\Big] \end{aligned} \tag{2.7}$$

In the right hand side of (2.7), we replace

$$V\Big(N,(x_1,...,x_{N-1},-1)\Big) = h_S(N)S(N-1)e^{-u} + h_B(N)B(N-1)e^r$$

$$V\Big(N,(x_1,...,x_{N-1},+1)\Big) = h_S(N)S(N-1)e^{u} + h_B(N)B(N-1)e^r$$

$$V\Big(N,(x_1,...,x_{N-1},0)\Big) = h_S(N)S(N-1) + h_B(N)B(N-1)e^r,$$

which all follow from the definition of portfolio value. So doing we obtain

$$\begin{aligned}\text{r.h.s (2.7)} &= e^{-r}q_{-1}h_S(N)S(N-1)e^{-u} + e^{-r}q_{-1}h_B(N)B(N-1)e^r \\ &+ e^{-r}q_{+1}h_S(N)S(N-1)e^{u} + e^{-r}q_{+1}h_B(N)B(N-1)e^r \\ &+ e^{-r}q_0h_S(N)S(N-1) + e^{-r}q_0h_B(N)B(N-1)e^r \\ &= e^{-r}h_S(N)S(N-1)(q_{-1}e^{-u} + q_{+1}e^u + q_0) \\ &+ (q_{-1} + q_{+1} + q_0)h_B(N)B(N-1) \\ &= h_S(N)S(N-1) + h_B(N)B(N-1) = V(N-1),\end{aligned} \tag{2.8}$$

by the definition of self-financing portfolio, which proves the claim for $t = N-1$.
Part 2: Now assume that the statement is true at time $t+1$, i.e.

$$V(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}...x_N)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}...q_{x_N}V(N,x) \tag{2.9}$$

Part 3: We now prove it at time t. Let

$$\begin{aligned} V^{+1}(t+1) &= h_S(t+1)S(t)e^u + h_B(t+1)B(t)e^r \\ V^{-1}(t+1) &= h_S(t+1)S(t)e^{-u} + h_B(t+1)B(t)e^r \\ V^{0}(t+1) &= h_S(t+1)S(t) + h_B(t+1)B(t)e^r \end{aligned} \tag{2.10}$$

Proceeding as we did in (2.8), using (2.10), we obtain

$$\begin{aligned}& e^{-r}q_{-1}V^{-1}(t+1) + e^{-r}q_{+1}V^{+1}(t+1) + e^{-r}q_0V^0(t+1) \\ &= h_S(t+1)S(t) + h_B(t+1)B(t) = V(t)\end{aligned}$$

Thus,

$$V(t) = e^{-r}\Big[q_{-1}V^{-1}(t+1) + q_{+1}V^{+1}(t+1) + q_0V^0(t+1)\Big] \tag{2.11}$$

Using our assumption in (2.9), we have:

$$V^{-1}(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}...x_n)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}...q_{x_N}V\Big(N,(x_1,...,x_t,-1,x_{t+2},...x_N)\Big)$$

$$V^{+1}(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}...x_n)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}...q_{x_N}V\Big(N,(x_1,...,x_t,+1,x_{t+2},...x_N)\Big)$$

$$V^{0}(t+1) = e^{-r(N-t-1)} \sum_{(x_{t+2}...x_n)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}...q_{x_N}V\Big(N,(x_1,...,x_t,0,x_{t+2},...x_N)\Big)$$

Inserting this into (2.11), we get:

$$V(t) = e^{-r(N-t)} \Bigg[q_{+1} \sum_{(x_{t+2}\dots x_n)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}\dots q_{x_N} V\Big(N,(x_1,\dots,x_t,-1,x_{t+2},\dots,x_N)\Big)$$

$$+ q_{-1} \sum_{(x_{t+2}\dots x_n)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}\dots q_{x_N} V\Big(N,(x_1,\dots,x_t,+1,x_{t+2},\dots,x_N)\Big)$$

$$+ q_0 \sum_{(x_{t+2}\dots x_n)\in\{+1,0,-1\}^{N-t-1}} q_{x_{t+2}}\dots q_{x_N} V\Big(N,(x_1,\dots,x_t,0,x_{t+2},\dots,x_N)\Big)\Bigg]$$

$$= e^{-r(N-t)} \sum_{(x_{t+2}\dots x_n)\in\{+1,0,-1\}^{N-t}} q_{x_{t+1}}\dots q_{x_N} V(N,x)$$

which completes the proof. □

Note that, in contrast to the binomial model, the value of self-financing portfolios at time t is not uniquely determined by the value at time N, due to the nature of the risk-neutral probabilities. In fact there exist infinitely many possible portfolio values at time t, which are parametrized by the arbitrary constant $q_0 \in \mathbb{R}$. It follows that the fair price of European options is also not uniquely defined in the trinomial model. We shall see later that the latter entails the existence of European derivatives which cannot be hedged. For this reason, the trinomial market model is said to be incomplete.

2.1.2 Arbitrage-free condition

Now we want to discuss the absence of arbitrage opportunities in the 1+1 dimensional trinomial market (i.e. a market consisting of one stock and one risk-free asset in the context of the trinomial model). We want to give conditions on r, u and q_0 so that the market is arbitrage free and at the same time we want (q_{+1},q_0,q_{-1}) to define a probability. The following two theorems accomplish this. We begin with the latter problem.

Theorem 2.1.2. *The following are equivalent:*

$$0 < q_0 < \frac{e^u - e^r}{e^u - 1} \tag{2.12}$$

$$q_{+1} > 0, \quad q_{-1} > 0, \quad q_{+1} + q_{-1} < 1 \tag{2.13}$$

Proof. Let us recall that $u > 0$, $r \geq 0$. We begin by showing that (2.12) implies (2.13). If (2.12) holds, then

$$q_{+1} = \frac{e^r - e^{-u}}{e^u - e^{-u}} - q_0 \frac{1 - e^{-u}}{e^u - e^{-u}} > \frac{e^r - e^{-u}}{e^u - e^{-u}} - \frac{e^u - e^r}{e^u - 1} \cdot \frac{1 - e^{-u}}{e^u - e^{-u}}$$

$$= \frac{(e^r - e^{-u})(e^u - 1) - (e^u - e^r)(1 - e^{-u})}{(e^u - 1)(e^u - e^{-u})}$$

$$= \frac{e^{r+u} - 1 - e^r + e^{-u} - e^u + e^r + 1 - e^{r-u}}{(e^u - 1)(e^u - e^{-u})}$$

$$= \frac{e^{r+u} + e^{-u} - e^u - e^{r-u}}{(e^u - 1)(e^u - e^{-u})}$$

$$= \frac{e^r(e^u - e^{-u}) - (e^u - e^{-u})}{(e^u - 1)(e^u - e^{-u})} = \frac{e^r - 1}{e^u - 1} \geq 0.$$

The last inequality holds since $r \geq 0$ and $u > 0$. Also, for q_{-1} we have

$$q_{-1} = \frac{e^u - e^r}{e^u - e^{-u}} - q_0 \frac{e^u - 1}{e^u - e^{-u}} > \frac{e^u - e^r}{e^u - e^{-u}} - \frac{e^u - e^r}{e^u - 1} \frac{e^u - 1}{e^u - e^{-u}},$$

$$\frac{(e^u - e^r)(e^u - 1) - (e^u - e^r)(e^u - 1)}{(e^u - 1)(e^u - e^{-u})} = 0.$$

Thus,

$$q_{-1} = \frac{e^u - e^r}{e^u - e^{-u}} - q_0 \frac{e^u - 1}{e^u - e^{-u}} > 0.$$

Finally since by definition $q_{+1} + q_0 + q_{-1} = 1$, it is easy to see that (2.12) implies that $q_{+1} + q_{-1} < 1$ when $q_0 > 0$.

Next we will show that (2.13) implies (2.12).

$$q_{+1} = \frac{e^r - e^{-u}}{e^u - e^{-u}} - q_0 \frac{1 - e^{-u}}{e^u - e^{-u}} > 0$$

$$\frac{e^r - e^{-u}}{e^u - e{-}u} > q_0 \frac{1 - e^{-u}}{e^u - e^{-u}}$$

$$\frac{(e^u - e^{-u})(e^r - e^{-u})}{(e^u - e^{-u})(1 - e^{-u})} > q_0$$

$$\frac{e^r - e^{-u}}{1 - e^{-u}} > q_0.$$

Thus,

$$\frac{e^u - e^r}{e^u - 1} > q_0.$$

Also, for q_{-1},

$$q_{-1} = \frac{e^u - e^r}{e^u - e^{-u}} - q_0 \frac{e^u - 1}{e^u - e^{-u}} > 0$$

$$\frac{e^u - e^r}{e^u - e^{-u}} > q_0 \frac{e^u - 1}{e^u - e{-}u}$$

$$\frac{(e^u - e^r)(e^u - e^{-u})}{(e^u - e^{-u})(e^u - 1)} > q_0$$

Thus,

$$\frac{e^u - e^r}{e^u - 1} > q_0.$$

In addition, since $q_{+1} + q_{-1} + q_0 = 1$ by definition, one can easily see that $q_0 > 0$, which completes the proof. $\square$

As we have seen here, the triple q_0, $q_{\pm 1}$ defines a probability if and only if

$$r < u, \quad 0 < q_0 < \frac{e^u - e^r}{e^u - 1}. \tag{2.14}$$

These are our conditions on r, u, and q_0. Whereas the second condition may not be very intuitive, the first one certainly is, as r determines the growth of value for the risk-free asset of our portfolio, and u the growth of the stock value. It would make little sense for a bond to grow more in value than a stock in a time period when both increase, thus the first condition comes quite naturally. With the following theorem we finally determine absence of arbitrage in our market.

Theorem 2.1.3. *Under condition* (2.14)*, the self-financing portfolios with value* (2.4) *are free of arbitrage.*

Proof. Let us first recall that a self-financing portfolio $\{h_S(t),h_B(t)\}_{t\in\mathbb{I}}$ is called an arbitrage if its value satisfies the following: $V(0)=0$, $V(N,x)\geq 0$ for all $x\in\{-1,0,1\}^N$, and there exists some $y\in\{-1,0,1\}^N$ such that $V(N,y)>0$.

We first prove the theorem for the 1-period case, for which

$$\begin{aligned} h_S(0) &= h_S(1) = h_S, \\ h_B(0) &= h_B(1) = h_B, \end{aligned}$$

i.e. the portfolio position in the 1-period model is constant over the interval [0,1]. The value of the portfolio at time $t=0$ is

$$V(0) = h_S S(0) + h_B B(0)$$

At time $t=1$ it is one of the following:

$$\begin{aligned} V(1) &= h_S S(0)e^u + h_B B(0)e^r \\ V(1) &= h_S S(0)e^{-u} + h_B B(0)e^r \\ V(1) &= h_S S(0) + h_B B(0)e^r \end{aligned}$$

For the portfolio to be an arbitrage, $V(0)=0$ must hold, so let us assume it does, i.e.

$$h_S S(0) + h_B B(0) = 0 \tag{2.15}$$

Also, $V(1)\geq 0$ must hold, i.e.

$$\begin{aligned} h_S S(0)e^u + h_B B(0)e^r &\geq 0 \\ h_S S(0)e^{-u} + h_B B(0)e^r &\geq 0 \\ h_S S(0) + h_B B(0)e^r &\geq 0 \end{aligned}$$

and also one of these inequalities must be strict. From (2.15) we have $h_S S(0) = -h_B B(0)$ and we get

$$\begin{aligned} h_S S(0)(e^u - e^r) &\geq 0 \\ h_S S(0)(e^{-u} - e^r) &\geq 0 \\ h_S S(0)(1 - e^r) &\geq 0 \end{aligned} \tag{2.16}$$

Since we must have at least one strict inequality, $h_S \neq 0$. We then have to analyze two cases. First, if $h_S > 0$, we obtain from (2.16):

$$\begin{aligned} e^u - e^r &\geq 0 \\ e^{-u} - e^r &\geq 0 \\ 1 - e^r &\geq 0 \end{aligned}$$

From the last inequality, we get $r \leq 0$, but since we know $r \geq 0$ we must have $r = 0$. Clearly, since we know $u > 0$, this means the second inequality cannot hold, and so there is a contradiction. Second, if $h_S < 0$, we obtain from (2.16):

$$\begin{aligned} e^{u} - e^{r} &\leq 0 \\ e^{-u} - e^{r} &\leq 0 \\ 1 - e^{r} &\leq 0 \end{aligned}$$

From the first inequality here, we get $r \geq u$, but since $r < u$ holds by our initial condition, there is a contradiction for this case also.

For the multiperiod model, we use that the value at time $t = 0$ of a self-financing portfolio satisfies

$$V(0) = e^{-rN} \sum_{x \in \{+1,0,-1\}^N} (q_{+1})^{N_+(x)} (q_{-1})^{N_-(x)} (q_0)^{N_0(x)} V(N,x),$$

as stated in (2.5). As $q_{\pm 1}, q_0$ are positive, $V(0) = 0$ can only hold if $V(N,x) \equiv 0$ along all paths. It follows that the portfolio cannot be an arbitrage, which completes the proof. □

Thus we have shown that this portfolio cannot be an arbitrage under these conditions. It follows that when our risk-neutral measure, the triple q_0, $q_{\pm 1}$, constitutes a probability it implies an absence of arbitrage opportunity in the market, and vice versa.

2.2 Probabilistic interpretation

When looking more closely at the trinomial model, it becomes relevant to develop a formulation for the model in the context of probability theory. An introduction to the basic concepts of probability theory can be found in Appendix B.

Defining the price of a stock (in the trinomial model context) as a stochastic process, as we do within this section, allows us to study the convergence of the stock price to the Geometric Brownian Motion. We will see that applying the martingale condition, which we derive herein also, aids us in proving this convergence.

2.2.1 Probabilistic formulation

We define

$$\Omega_N = \Big\{ \omega = (\gamma_1, \dots \gamma_N) : \gamma_i \in \{-1,0,1\}, i = 1,\dots,N \Big\}.$$

Given $p_{-1}, p_0, p_{+1} \in (0,1)$ such that $p_{-1} + p_0 + p_{+1} = 1$, and $\omega \in \Omega_N$, we define the finite probability space $(\Omega_N, \mathbb{P})$ where

$$\mathbb{P}(\omega) = p_{-1}^{N_{-1}(\omega)} p_0^{N_0(\omega)} p_{+1}^{N_{+1}(\omega)}$$

and where $N_{\pm 1}(\omega)$ is the number of ± 1 in the sample point ω and $N_0(\omega)$ the number of 0. The trinomial price of a stock as a stochastic process on the probability space $(\Omega_N, \mathbb{P})$ is then given by

$$S(t) = S_0 X(t), \text{ where } X(t) = \prod_{i=1}^{t} x_i \quad \forall t \in \{1,2,\dots\},$$

$$X(0) = 1, \text{ and for all } \omega = (\gamma_1,...,\gamma_N) \in \Omega,\ x_i(\omega) = \begin{cases} e^u \text{ if } \gamma_i = 1 \\ 1 \text{ if } \gamma_i = 0 \\ e^{-u} \text{ if } \gamma_i = -1 \end{cases}$$

This is our probabilistic formulation for the trinomial model.

2.2.2 Martingale condition

We have the expected value of $S(t)$

$$\mathbb{E}\big[S(t)\big] = S_0 \prod_{i=1}^{t} \mathbb{E}\big[x_i\big] = S_0 \prod_{i=1}^{t} (e^u p_{+1} + p_0 + e^{-u} p_{-1}) = S_0 (e^u p_{+1} + p_0 + e^{-u} p_{-1})^t$$

and the variance of $\log S(t)$

$$\mathbb{V}\mathrm{ar}\big[\log S(t)\big] = \mathbb{V}\mathrm{ar}\big[\log(S_0 \prod_{i=1}^{t} x_i)\big] = \mathbb{V}\mathrm{ar}\big[\log S_0\big] + \mathbb{V}\mathrm{ar}\big[\log \prod_{i=1}^{t} x_i\big]$$

$$= \sum_{i=1}^{t} \mathbb{V}\mathrm{ar}\big[\log x_i\big] = t\Big[\mathbb{E}\big[\log(x_i)^2\big] - \mathbb{E}\big[\log x_i\big]^2\Big]$$

$$= t\Big[u^2 p_{+1} + u^2 p_{-1}\Big] - t\Big[u^2 (p_{+1} - p_{-1})^2\Big] = tu^2\Big[p_{+1} + p_{-1} - (p_{+1} - p_{-1})^2\Big]$$

and the expected value of $S(t)$ given that $S(t-1)$ is known

$$\mathbb{E}\big[S(t)|S(t-1)\big] = \mathbb{E}\big[S_0 \prod_{i=1}^{t} x_i | S(t-1)\big]$$

$$= S_0 X(t-1) \mathbb{E}\big[x_i\big] = S_0 X(t-1)(e^u p_{+1} + p_0 + e^{-u} p_{-1}).$$

With this in mind, we can now state the martingale condition of the trinomial model in the form of the following theorem.

Theorem 2.2.1. *The discounted value of a stock, $e^{-rt}S(t)$, is a martingale if and only if $e^u p_{+1} + p_0 + e^{-u} p_{-1} = e^r$ holds.*

Proof. To prove that $e^{-rt}S(t)$ constitutes a martingale, we need to check that the following conditions hold

$$\mathbb{E}\big[e^{-rt}S(t)\big] < \infty, \tag{2.17}$$

$$\mathbb{E}\big[e^{-rt}S(t)|S(t-1)\big] = e^{-r(t-1)}S(t-1). \tag{2.18}$$

Let us begin by observing the first condition, (2.17)

$$\mathbb{E}\big[e^{-rt}S(t)\big] = S_0 e^{-rt} \prod_{i=1}^{t} \mathbb{E}\big[x_i\big] = S_0 e^{-rt} (e^u p_{+1} + p_0 + e^{-u} p_{-1})^t.$$

And then the second condition, (2.18)

$$\mathbb{E}\big[e^{-rt}S(t)|S(t-1)\big] = S_0 e^{-rt} X(t-1) \mathbb{E}\big[x_i\big] = S_0 e^{-rt} X(t-1)(e^u p_{+1} + p_0 + e^{-u} p_{-1}).$$

It is clear by this development of (2.18) that it holds if and only if

$$e^u p_{+1} + p_0 + e^{-u} p_{-1} = e^r, \tag{2.19}$$

and we can see that (2.17) also holds when we apply this condition, which completes the proof. □

Let us observe the expected value of the stock and the variance of the logarithm of the stock when (2.19) is satisfied. $\mathbb{E}\big[S(t)\big]$ and $\mathbb{V}\mathrm{ar}\big[\log S(t)\big]$ become

$$\mathbb{E}\big[S(t)\big] = S_0(e^u p_{+1} + p_0 + e^{-u} p_{-1})^t = S_0 e^{rt},$$

$$\mathbb{V}\mathrm{ar}\big[\log S(t)\big] = tu^2\Big[p_{+1} + p_{-1} - (p_{+1} - p{-1})^2\Big].$$

It follows that any of the triples (q_{+1},q_0,q_{-1}) defined as in (2.1) can be interpreted as a probability on the sample space Ω_N which makes the trinomial price a martingale. We denote by (Ω_N,$\mathbb{Q}$) this new probability space, i.e.

$$\mathbb{Q}(\omega) = q_{-1}^{N_{-1}(\omega)} q_0^{N_0(\omega)} q_{+1}^{N_{+1}(\omega)}.$$

The probability measure $\mathbb{Q}$ determined by (q_{+1},q_0,q_{-1}) is called a martingale probability measure, or risk-neutral measure. Note that there exists infinitely many risk-neutral measures, each determined by the arbitrary value of q_0.

2.2.3 Convergence to the Geometric Brownian Motion

The Geometric Brownian Motion (GBM) is a stochastic process

$$\tilde{S}(t) = S_0 e^{\alpha t + \sigma W(t)}$$

where $W(t)$ is a Brownian Motion, α is the instantaneous mean of log-return and σ is the instantaneous volatility of a stock with price $\tilde{S}(t)$. Given a partition $0 = t_0 < t_1 < ... < t_N = t$ of the interval $[0, t]$ with uniform size $t_{i+1} - t_i = h$, we temporarily adjust the trinomial model to

$$S(t_i) = \begin{cases} S(t_{i-1})e^u & \text{with prob. } p_u \\ S(t_{i-1}) & \text{with prob. } p_0 = 1 - p_u - p_d \\ S(t_{i-1})e^{-u} & \text{with prob. } p_d \end{cases}$$

We also define

$$X_i = \begin{cases} 1 & \text{with prob. } p_u \\ 0 & \text{with prob. } 1 - p_u - p_d \\ -1 & \text{with prob. } p_d \end{cases}$$

and

$$M_N = \sum_{i=1}^{N} X_i.$$

Using these definitions, we can express the stock price in the trinomial model as

$$S(t) = S(0)e^{uM_n}.$$

It is important for us to see that the trinomial model stock price converges to the Geometric Brownian Motion. If it does not, it will not be possible for the trinomial option price to converge to the Black-Scholes price. The following theorem proves the convergence of the trinomial stock price to the Geometric Brownian Motion.

Theorem 2.2.2. *$S(t) \to \tilde{S}(t)$ in distribution as $N \to \infty$.*

Proof. For $S(t) \to \tilde{S}(t)$ in distribution to hold, they must have the same expected value and the same variance. The expected value and variance of the logarithm of $\tilde{S}(t)$ divided by the initial stock price is:

$$\mathbb{E}\big[\log(\frac{\tilde{S}(t)}{S(0)})\big] = \alpha t$$

and

$$\mathbb{V}\mathrm{ar}\big[\log(\frac{\tilde{S}(t)}{S(0)})\big] = \sigma^2 t.$$

For the trinomial model, it holds that:

$$\mathbb{E}\big[\log(\frac{S(0)e^{uM_N}}{S(0)})\big] = \mathbb{E}\big[uM_N\big] = uN(p_u - p_d)$$

$$\mathbb{V}\mathrm{ar}\big[\log(\frac{S(0)e^{uM_N}}{S(0)})\big] = \mathbb{V}\mathrm{ar}\big[uM_N\big] = u^2 N(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2).$$

Setting these expected values and variances to be equal and using that $N = t/h$ gives us

$$\alpha t = \frac{ut(p_u - p_d)}{h} \Leftrightarrow \alpha = \frac{u(p_u - p_d)}{h} \tag{2.20}$$

and

$$\sigma^2 t = \frac{u^2 t(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2)}{h} \Leftrightarrow \sigma^2 = \frac{u^2(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2)}{h}. \tag{2.21}$$

Now we want to show that if we choose our parameters so that (2.20) and (2.21) are satisfied, then

$$uM_n \to \alpha t + \sigma W(t) \tag{2.22}$$

in distribution. This is equivalent to

$$\frac{uM_n - \alpha t}{\sqrt{t}\sigma} \to N(0,1) \tag{2.23}$$

in distribution. If we insert our values for α and σ given by (2.20) and (2.21) into (2.23), we obtain the following expression for what we want to prove:

$$\frac{uM_n - \frac{ut(p_u - p_d)}{h}}{\sqrt{t\frac{u^2(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2)}{h}}} \to N(0,1) \tag{2.24}$$

in distribution, which can be rewritten as

$$\frac{M_n - N(p_u - p_d)}{\sqrt{N(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2)h}} \to N(0,1) \tag{2.25}$$

in distribution. However, applying the central limit theorem (which can be found in Appendix B) to M_N gives us

$$\frac{\frac{\sum_{i=1}^{N} X_i}{N} - (p_u - p_d)}{\sqrt{\frac{p_u + p_d - p_u^2 + 2p_u p_d - p_d^2}{N}}} = \frac{M_n - N(p_u - p_d)}{\sqrt{N(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2)}} \to N(0,1)$$

in distribution as $N \to \infty$, which is exactly (2.25) but with the condition that $N \to \infty$, which completes the proof. $\square$

This result has significant implications for the relevance of the trinomial model. Not only does it mean that the model can be used to accurately approximate the Geometric Brownian Motion. It also follows from this that, under certain conditions, the trinomial model option price of European derivatives will converge to the Black-Scholes price, which we will further investigate later in the thesis. This is, in essence, what makes the trinomial model relevant both for academic study and application in practice. The relationship between the trinomial model and the Black-Scholes model will be examined further in later chapters of this thesis.

Chapter 3

Option pricing and hedging

Option pricing and hedging are two of the most important topics in option theory. The price of an option is the initial premium that the buyer pays to the seller in order to create a binding contract. How to find a fair price which does not trigger arbitrage is not trivial, and that is the first problem that we tackle within this chapter.

The second part of this chapter deals with the problem of hedging (or replicating) a derivative in the trinomial model. The incompleteness of our model makes hedging difficult and we will explore different ways to get around this problem.

3.1 Trinomial option pricing

The purpose of this section is to derive the fair price of a European derivative by using the trinomial model. We will be using the concept of a self-financing portfolio to do this and we shall explain why it is reasonable to say that our price is "fair".

We will begin by pricing derivatives where the payoff Y is a function of the value of the underlying asset at time of maturity, which means that $Y = g(S(N))$. The trinomial model can be used to price other types of derivatives as well, but to illustrate how the pricing approach works we will consider standard European derivatives.

3.1.1 Fair price for a European derivative

It is not obvious what "fair" price of a derivative means. The basic idea is that a fair price should favour neither the buyer nor the seller. In other words, no party should be able to make a guaranteed profit by buying or selling the derivative. If this was not the case, there would exist arbitrage opportunities in the market.

This interpretation of fair price makes it reasonable to associate the price of a derivative with the value of a self-financing hedging portfolio [16]. Assume that the seller invests the premium he gets from selling the derivative in the underlying asset and a risk-free bond. Moreover, suppose there is no cash flow in or out from this portfolio and that the value of the portfolio is exactly equal to the payoff of the derivative at the time of maturity. It may then appear natural to define the fair price of the derivative to be the same as the price of such a self-financing hedging portfolio.

Note that we have not proved that there exists a self-financing portfolio which satisfies the termination condition. However, this does not prevent us from defining the fair price to be equal to the value of such a portfolio [17]. We have previously shown that the value of a self-financing portfolio is not uniquely defined in the trinomial market model, thus also the price of a derivative will depend on the free parameter q_0.

3.1.2 Pricing a European derivative

Having justified that the fair price of a derivative should be equal to the value of a self-financing hedging portfolio, (2.5) suggests the following definition of fair price for European derivatives.

Definition 3.1.1. *The initial fair price* $\Pi_Y(0,q_0)$ *of a European derivative is defined by*

$$\Pi_Y(0,q_0) = e^{-rN} \sum_{x\in\{-1,0,1\}^N} (q_{-1})^{N_{-1}(x)}(q_0)^{N_0(x)}(q_{+1})^{N_{+1}(x)}Y(x). \tag{3.1}$$

In (3.1)*,* $Y(x)$ *denotes the payoff corresponding to the path* x *of the underlying asset;* N *is the number of steps until maturity;* $N_{-1}(x)$*,* $N_0(x)$ *and* $N_{+1}(x)$ *are, respectively, the number of steps that go downwards, horizontally, and upwards for the path* x*; and* r *is the risk-free interest rate.* q_{-1}*,* q_0*,* q_{+1} *are the risk-neutral probabilities introduced in Chapter 2, which implies that the market is arbitrage free.*

By the recurrence formula (2.6), the price of a European derivative at time $t \in \{0,1,\ldots,N-1\}$ satisfies

$$\Pi_Y(t,q_0) = e^{-r}[q_{-1}\Pi_Y^-(t+1,q_0) + q_0\Pi_Y^0(t+1,q_0) + q_{+1}\Pi_Y^+(t+1,q_0)], \tag{3.2}$$

where $\Pi_Y^-(t+1,q_0)$, $\Pi_Y^0(t+1,q_0)$, and $\Pi_Y^+(t+1,q_0)$ denote the one-step future prices depending on whether the price of underlying asset goes down, stays the same, or goes up respectively. It is possible to rewrite the initial price by using the multinomial theorem, and thus we have the following theorem for a European call option.

Theorem 3.1.1. *The initial price of a European call option with strike price* K *satisfies*

$$\Pi_Y(0,q_0) = e^{-rN} \sum_{m_1+m_2+m_3=N} \frac{N!}{m_1!m_2!m_3!}(q_{+1})^{m_1}(q_{-1})^{m_2}(q_0)^{m_3}(S_0e^{(m_1-m_2)u} - K)_+,$$

where $N_{+1} = m_1$*,* $N_{-1} = m_2$*, and* $N_0 = m_3$*.*

Proof. For a European call option we have the payoff function $g(S(N)) = (S(N) - K)_+$. Now we let $N_{+1}(x)$ and $N_{-1}(x)$ denote the number of times that the value of the stock goes up and down, respectively. Then

$$g(S(N)) = \left(S_0e^{\left(N_{+1}(x)-N_{-1}(x)\right)u} - K\right)_+ \tag{3.3}$$

Inserting (3.3) into (3.1), we obtain

$$\Pi_Y(0,q_0) = e^{-rN} \sum_{x\in\left\{-1,0,1\right\}^N} (q_{+1})^{N_{+1}(x)}(q_{-1})^{N_{-1}(x)}(q_0)^{N_0(x)} \left(S_0e^{\left((N_{+1}(x)-N_{-1}(x)\right)u} - K\right)_+,$$

which according to the multinomial theorem is equal to

$$e^{-rN} \sum_{N_{+1}(x)+N_0(x)+N_{-1}(x)=N} \frac{N!}{N_{+1}(x)!N_0(x)!N_{-1}(x)!} (q_{+1})^{N_{+1}(x)} (q_{-1})^{N_{-1}(x)} (q_0)^{N_0(x)} \left(S_0 e^{\left(N_{+1}(x)-N_{-1}(x)\right)u} - K \right)_+ .$$

And thus, changing notations to $N_{+1} = m_1$, $N_{-1} = m_2$, and $N_0 = m_3$, we obtain

$$\Pi_Y(0,q_0) = e^{-rN} \sum_{m_1+m_2+m_3=N} \frac{N!}{m_1!m_2!m_3!} (q_{+1})^{m_1} (q_{-1})^{m_2} (q_0)^{m_3} (S_0 e^{(m_1-m_2)u} - K)_+$$

which completes the proof. □

3.1.3 Price impact by the free parameter

As mentioned earlier in this chapter, the price of a derivative in the trinomial model will depend on the free parameter q_0. In order to see how this parameter impacts the initial price of a derivative, we have plotted the initial European call price as a function of q_0 by using (3.1), see Figure 3.1. Since we assume that the market is arbitrage free, we require q_0 to satisfy

$$0 < q_0 < \frac{e^u - e^r}{e^u - 1}.$$

We have tried four different numbers of steps N in the trinomial model (50, 100, 200 and 400) to see the effect of changing q_0 for different values of N. We have also chosen to vary K. The left plot in Figure 3.1 corresponds to $K = 8$ and the right plot corresponds to $K = 12$. The remaining three parameters, $S(0)$, u, and r, are kept constant and they are set to 10, 0.3, and 0.02, respectively. For the sake of comparison, we have also marked the corresponding binomial price with a cross in Figure 3.1. The plots were created in Matlab by using the function in Appendix C.2.

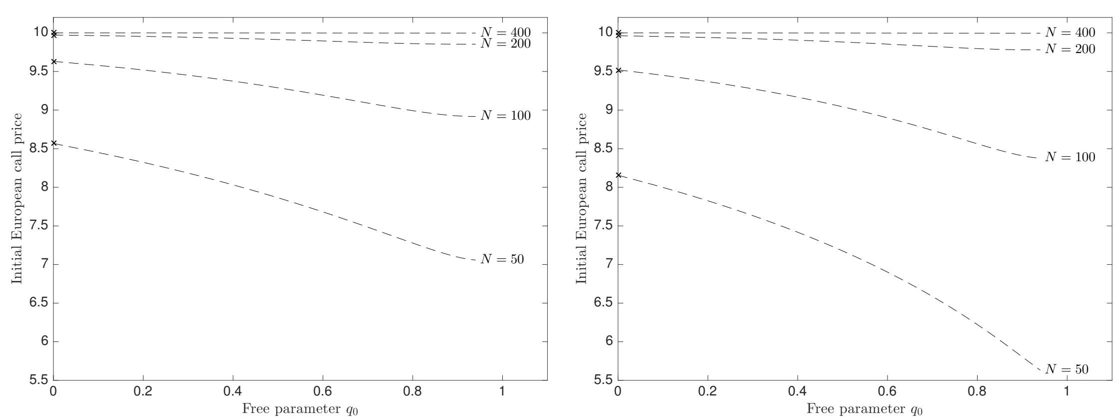

Figure 3.1: Initial European call prices for different values of q_0 and N, using the trinomial model. In the left figure, $K = 8$, and in the right figure, $K = 12$.

In Figure 3.1, we can see that the initial price for $q_0 = 0.0001$ calculated with the trinomial model is almost identical to the initial price calculated with the binomial model. This suggests that the trinomial price approaches the binomial price when q_0 approaches 0. This is reasonable since a very small value of q_0 basically means that the trinomial model behaves just like the binomial model in theory. Hence one could think of the binomial model as just a special case of the trinomial model, specifically when $q_0 = 0$.

As q_0 decreases, $q_{+1} + q_{-1}$ will increase (since $q_0 + q_{+1} + q_{-1} = 1$ and they are all positive) and Figure 3.1 shows that the initial European call price will then also slightly increase. Moreover, the price also seems to increase when we add more steps, i.e. increase N. Both of these behaviours can be explained in terms of volatility. The price of a European call option is an increasing function with respect to volatility [16]. One could argue that increasing q_{+1} and q_{-1}, or setting N to be a larger number, will actually result in a higher aggregated volatility. Higher values for q_{+1} and q_{-1} means that it becomes more likely that the price of the underlying asset will move far away from the initial price. Similarly, adding more steps makes it possible for the underlying asset to obtain more extreme values. Since all other parameters are kept constant the price will therefore increase, just as Figure 3.1 suggests.

3.1.4 Price boundaries for European derivatives

It is also interesting to note in Figure 3.1 that the initial price never seems to grow higher than S_0, no matter how much we increase the number of steps or decrease q_0. This demonstrates a very fundamental property of the price of a European call; that the derivative must always be cheaper than the underlying asset [18]. Otherwise the market would not be arbitrage free since one could take a short position in the derivative and a long position in the underlying asset and make a guaranteed profit.

By using similar arguments, it can be shown that the lower bound of a European call is its discounted intrinsic value [18]. For a European put, the upper bound is the discounted strike price and the lower bound is the discounted intrinsic value [18].

There is also a combined relationship between the price of a European put and call which is stated in the Put-Call Parity.

Theorem 3.1.2. ***(Put-Call Parity)*** *Let T be the termination date of a European call option and a European put option. If $t < T$ and $\tau = T - t$, then*

$$S(t) - c(t,S(t),K,T) = Ke^{-r\tau} - p(t,S(t),K,T),$$

where $c(t,S(t),K,T)$ denotes the price of the call and $p(t,S(t),K,T)$ denotes the price of the put at time t.

Proof. The proof is straightforward and the basic idea is to create a portfolio with alternating long and short positions in the stock, put, call, and bond. We refer to [19] for a complete proof of this theorem since it has little to do with the trinomial model.

3.2 Hedging

Having derived the fair price of a European derivative in the trinomial model, one of the two problems that were presented in the beginning of this chapter has been solved. The other problem has to do with hedging a derivative. The following sections describe the important concept of hedging a derivative in the trinomial market model and we present different ways to deal with the incompleteness of this model.

3.2.1 Hedging in complete and incomplete markets

Hedging is one of the most important concepts in modern finance. The idea behind hedging is to invest in one or several assets in such a way that they follow the price movements of another asset. Thus hedging is used to reduce the risk of a substantial loss, but in exchange for also missing potential profits.

There are different hedging strategies depending on what kind of asset one wants to hedge. Since derivatives depend on the performance of one (or several) underlying asset(s), derivatives can be effective when it comes to hedging an underlying asset. Conversely, a derivative can sometimes be hedged by investing in the underlying asset(s) as well as a risk-free bond.

A market is said to be complete if the arbitrage-free price of a derivative is uniquely defined. In such a market, the price will coincide with the value of a hedging (also commonly called replicating) strategy [20]. We have already shown that the standard trinomial model that we study is an example of an incomplete market since the price will depend on q_0 and hence not be uniquely defined. Another way to prove that this market is incomplete is to show that the standard European derivative is not replicable by only investing in the underlying asset and a risk-free bond. This is done in the following theorem.

Theorem 3.2.1. *It is not always possible to replicate a European derivative in the trinomial model.*

Proof. Consider the one step trinomial model, where $N = 1$, and a European derivative with payoff $g(S(N))$. Let $\{h_S, h_B\}$ be a constant portfolio where h_S denotes the number of shares of the underlying asset and h_B denotes the number of bonds in the portfolio. In order for this portfolio to replicate the derivative it must satisfy

$$h_S S_0 e^u + h_B B_0 e^r = g(S_0 e^u)$$

$$h_S S_0 + h_B B_0 e^r = g(S_0)$$

$$h_S S_0 e^{-u} + h_B B_0 e^r = g(S_0 e^{-u}).$$

If we let

$$A = \begin{pmatrix} S_0 e^u & B_0 e^r \\ S_0 & B_0 e^r \\ S_0 e^{-u} & B_0 e^r \end{pmatrix}, \qquad \mathbf{h} = \begin{pmatrix} h_S \\ h_B \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} g(S_0 e^u) \\ g(S_0) \\ g(S_0 e^{-u}) \end{pmatrix}, \tag{3.4}$$

the hedging condition can be expressed as the matrix equation $A\mathbf{h} = \mathbf{y}$. This system has more equations than unknown parameters and will only have a solution if $\mathbf{y}$ is in the column space of A. For the rest of the cases, $A\mathbf{h} = \mathbf{y}$ will not be solvable. Hence, in our trinomial model, it will not always be possible to hedge a European derivative by just investing in the underlying asset and a bond. This kind of market is therefore said to be incomplete. □

There are different ways to deal with this incompleteness of the trinomial model. One way is to try to find a portfolio which replicates the derivative as close as possible. Another way is to add another risky asset to our portfolio. A third way, which is often used in practice, is to fix q_0. We will take a closer look at the first two of these methods in the remaining part of this chapter. The third method will be examined in Chapter 4.

3.2.2 Least square hedging portfolio

Even though it is not always possible to replicate a European derivative in our model by only investing in the underlying asset and a bond, one might wonder if it is possible to find a hedging strategy that is close to being perfect. It is not obvious how different hedging strategies should be compared and what "close to being perfect" actually means. For example, do we accept a hedging strategy where our hedging portfolio might have a lower value than the asset that we are trying to hedge, or do we require our portfolio to have a value that is greater than or equal to this asset?

One way to find an approximation to a hedging portfolio is by using the least square method [21]. This portfolio will be the best approximation, in the least square sense, but it is not self-financing and it might be less valuable than the asset that we are trying to hedge.

Theorem 3.2.2. *Let A, $\mathbf{h}$ and $\mathbf{y}$ be specified as in expression* (3.4)*, then $A\mathbf{h} = \mathbf{y}$ will always admit a unique least square solution.*

Proof. The least square solution of $A\mathbf{h} = \mathbf{y}$ satisfies the equation

$$A^T A\mathbf{h} = A^T\mathbf{y}.$$

In our case we obtain

$$A^T A = \begin{pmatrix} S_0 e^u & S_0 & S_0 e^{-u} \\ B_0 e^r & B_0 e^r & B_0 e^r \end{pmatrix} \begin{pmatrix} S_0 e^u & B_0 e^r \\ S_0 & B_0 e^r \\ S_0 e^{-u} & B_0 e^r \end{pmatrix}$$

$$= \begin{pmatrix} S_0^2(e^{2u} + 1 + e^{-2u}) & B_0 e^r S_0(e^u + 1 + e^{-u}) \\ B_0 e^r S_0(e^u + 1 + e^{-u}) & 3B_0^2 e^{2r} \end{pmatrix}$$

and

$$A^T\mathbf{y} = \begin{pmatrix} S_0 e^u & S_0 & S_0 e^{-u} \\ B_0 e^r & B_0 e^r & B_0 e^r \end{pmatrix} \begin{pmatrix} g(S_0 e^u) \\ g(S_0) \\ g(S_0 e^{-u}) \end{pmatrix}$$

$$= \begin{pmatrix} S_0(e^u g(S_0 e^u) + g(S_0) + e^{-u} g(S_0 e^{-u})) \\ B_0 e^r(g(S_0 e^u) + g(S_0) + g(S_0 e^{-u})) \end{pmatrix}.$$

The set of least square solution is nonempty according to [22]. To show that $A^T A\mathbf{h} = A^T\mathbf{y}$ has a unique solution, we choose to study the determinant of $A^T A$. If the determinant is never equal to 0, then the rows of $A^T A$ are independent and the solution will be unique. The determinant of $A^T A$ is

$$|A^T A| = S_0^2 e^{2r} B_0^2 (3e^{2u} + 3 + 3e^{-2u} - e^{-2u} - 2e^{-u} - 2e^u - e^{2u} - 3)$$

$$= 2S_0^2 e^{2r} B_0^2 (e^{2u} + e^{-2u} - e^{-u} - e^u).$$

Since $2S_0^2 e^{2r} B_0^2$ is greater than 0, we only need to show that $e^{2u} + e^{-2u} - e^{-u} - e^u$ is never equal to 0 in order to prove that the same holds for $|A^T A|$. By rewriting $e^{2u} + e^{-2u} - e^{-u} - e^u$ we obtain

$$e^{2u} + e^{-2u} - e^{-u} - e^u = e^{-2u}(e^u - 1)^2(e^u + e^{2u} + 1).$$

The last expression is obviously greater than 0 when $u > 0$ and thus $|A^T A|$ is never equal to 0 and $A^T A\mathbf{h} = A^T\mathbf{y}$ has a unique solution, which completes the proof. □

The least square hedging portfolio for a multi-period model at time t is given by the least square solution of $A_t\mathbf{h_t} = \mathbf{y_t}$, where the subscript t indicates that the hedging condition must be adjusted to the current time instant. In order to demonstrate how well the least square hedging portfolio performs, we have chosen to compare the value of this portfolio at time of maturity to the actual payoff of a European call. Figure 3.2 shows such a plot for different values of $S(N)$ with parameters $u = 0.2$, $N = 10$, $S(0) = 10$, $B(0) = 10$, $r = 0.02$, $q_0 = 0.3$, and $K = 9$. The Matlab function that was used to generate this plot can be found in Appendix C.4.

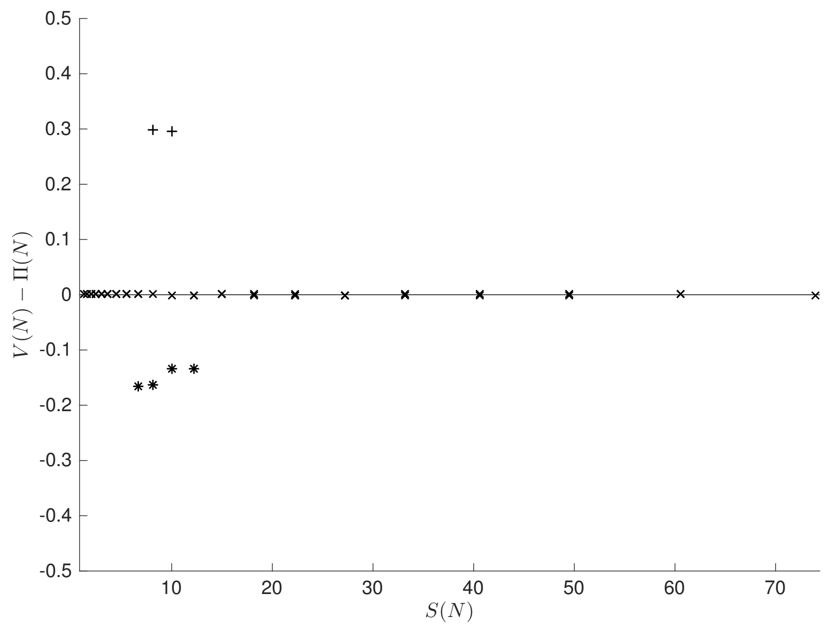

Figure 3.2: Performance of least square hedging portfolio at maturity.

The portfolio value is determined by $V(t) = h_S(t)S(t) + h_B(t)B(t)$. Since the least square hedging portfolio is not perfectly hedging the derivative, we may get different portfolio values in the same node depending on whether the last step to that node goes up, down or horizontally. This differs from the binomial case where the portfolio value in each node is uniquely defined. Since we are able to hedge the derivative perfectly in the binomial model, all paths to a specific node will result in the same value for the hedging portfolio.

In our example, we notice that the value of the least square hedging portfolio follows the actual payoff quite well. Two of the portfolios (3.5 percent) have a higher value than the payoff, four of the portfolios (7 percent) have a lower value, and 51 of the portfolios (89.5 percent) have a value that is identical to the payoff. However, the performance of the least square hedging portfolio depends on how we chose the parameters in the model. For example, it seems as though the least square hedging portfolio performs worse if we decrease the time until maturity. This makes sense as prices will then have less spread at time $N-1$, and thus it becomes more likely that the final step will determine whether the derivative expires in the money or not. If we can be sure at time $N-1$ that the derivative will expire in the money or out of the money, then we can replicate the derivative by having one share respectively zero shares of the underlying asset and a fixed number of bonds.

3.2.3 Hedging in the complete trinomial market model

We have seen that it is not possible to replicate a derivative in the trinomial model by only investing in the underlying asset and a risk-free bond. However, it turns out that adding a second risky asset to this model allows us to replicate a derivative perfectly. By doing so we get the completed trinomial market model. We will follow the structure of [17] when presenting this model. This means that we will assume that the risky assets are independent and we will also let go of our constraints regarding the change of states. We now consider a risky asset S^i which moves according to

$$S^i(t) = \begin{cases} S^i(t-1)e^{u_1} & \text{with prob. } p_{ui} \\ S^i(t-1)e^{m_1} & \text{with prob. } p_{mi} = 1 - p_{ui} - p_{di} \ , \\ S^i(t-1)e^{d_1} & \text{with prob. } p_{di} \end{cases} \quad t \in \mathbb{I} = \{1,...,N\}.$$

In order to determine a martingale measure Q for the completed trinomial market we proceed in the same way as in Chapter 2 and we get the equation system

$$\begin{cases} e^{u_1}q_{+1} + e^{m_1}q_0 + e^{d_1}q_{-1} = e^r \\ e^{u_2}q_{+1} + e^{m_2}q_0 + e^{d_2}q_{-1} = e^r \\ q_{+1} + q_0 + q_{-1} = 1. \end{cases} \tag{3.5}$$

Under the assumption of no arbitrage, the system has the solution

$$\begin{aligned} q_{+1} &= \frac{e^{m_1}(e^r - e^{d_2}) - e^{d_1}(e^r - e^{m_2}) - e^r(e^{m_2} - e^{d_2})}{e^{m_1}(e^{u_2} - e^{d_2}) - e^{u_1}(e^{m_2} - e^{d_2}) - e^{d_1}(e^{u_2} - e^{m_2})}, \\ q_0 &= \frac{e^{u_1}(e^{d_2} - e^r) - e^{d_1}(e^{u_2} - e^r) + e^r(e^{u_2} - e^{d_2})}{e^{m_1}(e^{u_2} - e^{d_2}) - e^{u_1}(e^{m_2} - e^{d_2}) - e^{d_1}(e^{u_2} - e^{m_2})}, \\ q_{-1} &= \frac{e^{u_1}(e^r - e^{m_2}) - e^{m_1}(e^r - e^{u_2}) - e^r(e^{u_2} - e^{m_2})}{e^{m_1}(e^{u_2} - e^{d_2}) - e^{u_1}(e^{m_2} - e^{d_2}) - e^{d_1}(e^{u_2} - e^{m_2})}. \end{aligned} \tag{3.6}$$

It can can be shown that q_{+1}, q_0 and q_{-1} given by (3.6) are positive numbers belonging to (0,1) if we choose the model parameters in a suitable way [17]. Thus q_{+1}, q_0, and q_{-1} constitute a probability measure Q under this assumption. This martingale measure is uniquely defined and the price of a derivative (or self-financing portfolio) will therefore be unique as well [17]. To construct a replicating hedging strategy $(\alpha_n^1,\alpha_n^2,\beta_n)$ where α_n^1, α_n^2 and β_n correspond, respectively, to the number of shares of the first risky asset, second risky asset, and risk-free bond, in the time interval $(n-1,n]$ where $0 < n < N$, we solve the system

$$\begin{cases} \alpha_n^1 e^{u_1} S_{n-1}^1 + \alpha_n^2 e^{u_2} S_{n-1}^2 + \beta_n e^{rn} = \Pi_n^u, \\ \alpha_n^1 e^{m_1} S_{n-1}^1 + \alpha_n^2 e^{m_2} S_{n-1}^2 + \beta_n e^{rn} = \Pi_n^m, \\ \alpha_n^1 e^{d_1} S_{n-1}^1 + \alpha_n^2 e^{d_2} S_{n-1}^2 + \beta_n e^{rn} = \Pi_n^d. \end{cases} \tag{3.7}$$

In this equation, S_{n-1}^1 and S_{n-1}^2 denote the prices of the risky assets at time $n-1$ and e^{rn} corresponds to the value of a risk-free bond at the same point of time. Moreover, Π_n^u, Π_n^m and Π_n^d denote the prices of the derivative at time n for different price movements in the last step. (3.7) has solution

$$\begin{aligned}
\alpha_n^1 &= \frac{e^{d_2}(\Pi_n^m - \Pi_n^u) + \Pi_n^u e^{m_2} - \Pi_n^m e^{u_2} + \Pi_n^d(-e^{m_2} + e^{u_2})}{S_{n-1}^1(e^{d_2}(e^{m_1} - e^{u_1}) + e^{m_2}e^{u_1} - e^{m_1}e^{u_2} + e^{d_1}(e^{u_2} - e^{m_2}))}, \\
\alpha_n^2 &= \frac{e^{d_1}(\Pi_n^m - \Pi_n^u) + \Pi_n^u e^{m_1} - \Pi_n^m e^{u_1} + \Pi_n^d(e^{u_1} - e^{m_1})}{S_{n-1}^2(-e^{m_2}e^{u_1} + e^{d_2}(e^{u_1} - e^{m_1}) + e^{d_1}(e^{m_2} - e^{u_2}) + e^{m_1}e^{u_2})}, \\
\beta_n &= \frac{e^{d_2}(\Pi_n^u e^{m_1} - \Pi_n^m e^{u_1}) + e^{d_1}(-\Pi_n^u e^{m_2} + \Pi_n^m e^{u_2}) + \Pi_n^d(e^{m_2}e^{u_1} - e^{m_1}e^{u_2})}{e^{rn}(e^{d_2}(e^{m_1} - e^{u_1}) + e^{m_2}e^{u_1} - e^{m_1}e^{u_2} + e^{d_1}(-e^{m_2} + e^{u_2}))}.
\end{aligned} \tag{3.8}$$

This shows that it is possible to find a replicating strategy in the completed trinomial market model. More generally, a market model with m number of states can only be complete if there exists at least $m-1$ risky assets [17].

Chapter 4

Approximation to Black-Scholes equation and pricing of vanilla options

We have yet to study some important properties of the trinomial model, for example its convergence. In this chapter we will study the convergence of the trinomial model for the European and American options. In reality there is no perfect model to price options theoretically. However, the possibly most widely used model among practitioners today is the Black-Scholes model. The model is not perfect as we will see in this chapter, but due to its wide use, we will check that the trinomial model converges to the Black-Scholes model for the European options. There is no theoretical Black-Scholes price for American put options. In order to check the convergence of the trinomial model for American put options we will need to use other means. We will also study for what parameters the model converges and verify this numerically in Matlab.

4.1 Convergence to Black-Scholes for European options

In this section we will investigate how the option prices obtained from the trinomial model converge to the Black-Scholes price. This will be investigated in two parts; first analytically, and later also numerically. We shall confirm that when the number of time steps approaches infinity, as well as some other conditions being fulfilled, the trinomial price converges to the Black-Scholes price.

4.1.1 Theoretical convergence

As previously seen the payoff of the regular European call and put are given by

$$\max(S(T) - K, 0)$$
$$\max(K - S(T), 0)$$

respectively, where K is the strike price, T is the time of maturity and $S(T)$ is the stock price at maturity. The Black-Scholes theoretical price V of the European call and put are obtained from the solutions of the Black-Scholes PDE [4]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV, \tag{4.2}$$

with the boundary conditions $V(T,S) = \max(S - K,0)$, $V(T,S) = \max(K - S,0)$ for the European call respectively put [1]. Hence to check the convergence of the trinomial prices for European options, we may investigate if the trinomial model converges to the Black-Scholes equation and under which conditions. The details are summarized in the following theorem.

Theorem 4.1.1. *Let p_u, p_d be the probabilities that the price of the underlying asset in the trinomial model goes up respectively down. Assume that $p_u = p_d = p$ where p is some constant. The trinomial model formulated as in Chapter 2 converges to the Black-Scholes model governed by an underlying asset with zero drift ($\alpha = 0$) if and only if*

$$u = \sigma\sqrt{\frac{h}{2p}},$$
$$q_0 = 1 - 2p, \quad 0 \leq p \leq 1/2.$$

Proof. As defined in Chapter 2, we assume the underlying asset to be governed by the following dynamics, given a uniform partition $0 = t_0 < t_1 \cdots < t_N = T$ on the interval $[0,T]$ with time differences $t_{i+1} - t_i = h$

$$S(t_i) = \begin{cases} S(t_{i-1})e^{u} & \text{with prob. } p_u \\ S(t_{i-1}) & \text{with prob. } 1 - p_u - p_d \\ S(t_{i-1})e^{-u} & \text{with prob. } p_d \end{cases}.$$

Where $i \in \mathbb{I} = \{1, \ldots, N\}$, p_u, p_d are the probabilities that the asset goes up respectively down and $1 - p_u - p_d$ is the probability that the stock value remains the same.

We let

$$X_i = \begin{cases} 1 & p_u \\ 0 & 1 - p_u - p_d \\ -1 & p_d \end{cases}$$

and

$$M_N = \sum_{i=1}^{N} X_i.$$

Using these definitions we can express the trinomial model stock price as

$$S(t_N) = S(0)e^{uM_N}. \tag{4.4}$$

In the Black-Scholes model, the underlying price is assumed to follow a Geometric Brownian Motion with mean rate of return μ and volatlity σ

$$S(t) = S(0)e^{\alpha t + \sigma W(t)}. \tag{4.5}$$

As shown in Theorem 2.2.2, the trinomial model stock price converges to the Geometric Brownian Motion if

$$\alpha = \frac{u(p_u - p_d)}{h},$$
$$\sigma^2 = \frac{u^2(p_u + p_d - p_u^2 + 2p_u p_d - p_d^2)}{h}.$$

For simplicity we set $p_u = p_d = p$ which gives $\alpha = 0$ and $\sigma^2 = \dfrac{u^2 2p}{h}$. Since we want σ to be a free parameter, we set

$$u = \sigma\sqrt{\frac{h}{2p}}.$$

Therefore for these values we have

$$S(0)e^{\sigma\sqrt{\frac{h}{2p}}M_N} \approx S(0)e^{\sigma W(t)}.$$

By the recurrence formula (2.11) for the trinomial model

$$V(t,S) = e^{-rh}[q_{+1}V(t+h,Se^u) + q_o V(t+h,S) + q_{-1}V(t+h,Se^{-u})]. \tag{4.6}$$

As we previously have formulated the trinomial model

$$q_{+1} = \frac{e^r - e^{-u}}{e^u - e^{-u}} - q_0 \frac{1-e^{-u}}{e^u - e^{-}u},$$

$$q_{-1} = \frac{e^u - e^r}{e^u - e^{-u}} - q_0 \frac{e^u - 1}{e^u - e^{-u}}.$$

We want to perform a Taylor expansion of $V(t+h,Se^u), V(t+h,S), V(t+h,Se^{-u})$ around t, S and Taylor expand $e^u = e^{\sigma\sqrt{\frac{h}{2p}}}$, $e^{-u} = e^{-\sigma\sqrt{\frac{h}{2p}}}$, and e^{rh} around zero with respect to h. We obtain

$$e^{rh} = 1 + rh + o(h), \tag{4.7}$$

$$e^{\sigma\sqrt{\frac{h}{2p}}} = 1 + \frac{\sigma^2 h}{4p} + o(h), \tag{4.8}$$

$$e^{-\sigma\sqrt{\frac{h}{2p}}} = 1 + \frac{\sigma^2 h}{4p} + o(h), \tag{4.9}$$

Taylor expansion of $V(t+h,Se^{\sigma\sqrt{\frac{h}{2p}}})$, also using (4.8), becomes

$$\begin{aligned} V(t+h,Se^{\sigma\sqrt{\frac{h}{2p}}}) =& V(t,S) + \frac{\partial V}{\partial t}(t,S)h + \frac{\partial V}{\partial S}(t,S)S(\sigma\sqrt{\frac{h}{2p}} + \frac{h\sigma^2}{4p}) \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(t,S)S^2\frac{h\sigma^2}{2p} + o(h). \end{aligned} \tag{4.10}$$

Taylor expansion of $V(t+h,Se^{-\sigma\sqrt{\frac{h}{2p}}})$, also using (4.9), becomes

$$\begin{aligned} V(t+h,Se^{-\sigma\sqrt{\frac{h}{2p}}}) =& V(t,S) + \frac{\partial V}{\partial t}(t,S)h + \frac{\partial V}{\partial S}(t,S)S(-\sigma\sqrt{\frac{h}{2p}} + \frac{h\sigma^2}{4p}) \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t,S)S^2\frac{h\sigma^2}{2p} + o(h). \end{aligned} \tag{4.11}$$

Finally the Taylor expansion of $V(t+h,S)$

$$V(t+h,S) = V(t,S) + \frac{\partial V}{\partial t}(t,S)h + o(h). \tag{4.12}$$

Since q_d, q_u are dependent of h we need take a close look at these as well. We use the taylor expansions of e^{rh}, $e^u = e^{\sigma\sqrt{\frac{h}{2p}}}$ and $e^{-u} = e^{-\sigma\sqrt{\frac{h}{2p}}}$ around zero with respect to h to express q_{+1} and q_{-1}. q_{+1} becomes

$$q_{+1} = \frac{e^{rh} - e^{-\sigma\sqrt{\frac{h}{2p}}}}{e^{\sigma\sqrt{\frac{h}{2p}}} - e^{-\sigma\sqrt{\frac{h}{2p}}}} - q_0 \frac{1 - e^{-\sigma\sqrt{\frac{h}{2p}}}}{e^{\sigma\sqrt{\frac{h}{2p}}} - e^{-\sigma\sqrt{\frac{h}{2p}}}}$$

$$= \frac{\sqrt{h}\sigma\frac{1}{\sqrt{2p}} + h(r - \frac{\sigma^2}{4p}) - q_0(\sqrt{h}\sigma\frac{1}{\sqrt{2p}} - \frac{h\sigma^2}{4p}) + o(h^{\frac{3}{2}})}{\sqrt{2}\sqrt{h}\sigma\sqrt{\frac{1}{p}} + o(h^{\frac{3}{2}})}$$

$$= \frac{1}{2} + \sqrt{\frac{hp}{2}}(\frac{r}{\sigma} - \frac{\sigma}{4p}) - q_0(\frac{1}{2} - \frac{\sqrt{h}\sigma}{4\sqrt{2p}}) + o(h),$$

and q_{-1} becomes

$$q_{-1} = \frac{e^{\sigma\sqrt{\frac{h}{2p}}} - e^{rh}}{e^{\sigma\sqrt{\frac{h}{2p}}} - e^{-\sigma\sqrt{\frac{h}{2p}}}} - q_0 \frac{e^{\sigma\sqrt{\frac{h}{2p}}} - 1}{e^{\sigma\sqrt{\frac{h}{2p}}} - e^{-\sigma\sqrt{\frac{h}{2p}}}}$$

$$= \frac{\sqrt{h}\sigma\frac{1}{\sqrt{2p}} + h(\frac{\sigma^2}{4p} - r) - q_0(\sqrt{h}\sigma\frac{1}{\sqrt{2p}} + \frac{h\sigma^2}{4p}) + o(h^{\frac{3}{2}})}{\sqrt{2}\sqrt{h}\sigma\sqrt{\frac{1}{p}} + o(h^{\frac{3}{2}})}$$

$$= \frac{1}{2} + \sqrt{\frac{hp}{2}}(\frac{\sigma}{4p} - \frac{r}{\sigma}) - q_0(\frac{1}{2} + \frac{\sqrt{h}\sigma}{4\sqrt{2p}}) + o(h).$$

By combining expressions for q_{+1} and q_{-1} we obtain

$$q_{+1} - q_{-1} = \sqrt{\frac{hp}{2}}(\frac{r}{\sigma} - \frac{\sigma}{4p} - \frac{\sigma}{4p} + \frac{r}{\sigma}) + 2q_0\frac{\sqrt{h}\sigma}{4\sqrt{2p}} + o(h)$$

$$= \sqrt{2hp}(\frac{r}{\sigma} - \frac{\sigma}{4p}) + q_0\frac{\sqrt{h}\sigma}{2\sqrt{2p}} + o(h),$$

$$q_{+1} - q_{-1} = 1 - q_0.$$

Inserting obtained Taylor expansions in (4.6) gives us

$$V(t,S) + V(t,S)rh + o(h) = V(t,S) + \frac{\partial V}{\partial t}(t,S)h$$

$$+\frac{\partial V}{\partial S}(t,S)S\Big[(q_{+1} - q_{-1})\sigma\sqrt{\frac{h}{2p}} + q_{+1} - q_{-1}\frac{\sigma^2 h}{4p}\Big]$$

$$+(1 - q_0)\frac{S^2 h\sigma^2}{4p}\frac{\partial^2 V}{\partial S^2}(t,S) + o(h),$$

which is equivalent to

$$V(t,S)rh = \frac{\partial V}{\partial t}(t,S)h + \frac{\partial V}{\partial S}(t,S)S\Big[h\sigma(\frac{r}{\sigma} - \frac{\sigma}{4p}) + q_0\frac{\sigma^2 h}{4p} + \frac{\sigma^2 h}{4p} - q_0\frac{\sigma^2 h}{4p}\Big]$$

$$+(1 - q_0)\frac{S^2 h\sigma^2}{4p}\frac{\partial^2 V}{\partial S^2}(t,S) + o(h),$$

which, in turn, is equivalent to

$$V(t,S)rh = \frac{\partial V}{\partial t}(t,S)h + \frac{\partial V}{\partial S}(t,S)Shr - (1-q_0)\frac{S^2 h\sigma^2}{4p}\frac{\partial^2 V}{\partial S^2}(t,S) + o(h).$$

Since $h > 0$ we can divide by h

$$V(t,S)r = \frac{\partial V}{\partial t}(t,S) + \frac{\partial V}{\partial S}(t,S)Sr - (1-q_0)\frac{S^2\sigma^2}{4p}\frac{\partial^2 V}{\partial S^2}(t,S) + \frac{o(h)}{h}.$$

By the definition of little o

$$o(x^\alpha) = b(x)x^\alpha,$$

where $b(x)$ is a bounded function which approaches zeros as $x \to 0$, hence $\frac{o(h)}{h} \to 0$ as $h \to 0$ and we obtain

$$V(t,S)r = \frac{\partial V}{\partial t}(t,S) + \frac{\partial V}{\partial S}(t,S)Sr - (1-q_0)\frac{S^2\sigma^2}{4p}\frac{\partial^2 V}{\partial S^2}(t,S).$$

In order to get the Black-Scholes equation, we need the condition

$$\frac{1-q_0}{2p} = 1 \Leftrightarrow q_0 = 1-2p.$$

If this condition is satisfied, we get the Black-Scholes equation

$$V(t,S)r = \frac{\partial V}{\partial t}(t,S) + \frac{\partial V}{\partial S}(t,S)Sr + \frac{\partial^2 V}{\partial S^2}(t,S)\frac{S^2\sigma^2}{2},$$

which completes the proof. □

We have now shown that the option prices obtained from the trinomial model converges to the Black-Scholes model when $h \to 0$. As mentioned earlier the European option prices are obtained by solving the Black-Scholes PDE with boundary conditions (payoff) $V(T,S) = (S-K)_+$, $V(T,S) = (K-S)_+$ for call respectively put in (4.2). Hence given the payoff $V(T,S(T)) = (S(T)-K)_+$, $V(T,S(T)) = (K-S(T))_+$ in the recurrence formulas (2.11), the trinomial price will converge to the Black-Scholes price of the European call respectively put. Analyzing the obtained result of the theorem we see that we require $q_0 = 1-2p$ which makes sense since for $p = 0.5$ we get $q_0 = 0$ and obtain the binomial model. It can be shown that it is numerically most efficient to set $p = 1/6$ [4].

4.1.2 Numerical study of convergence

To numerically study the convergence rate to the Black-Scholes price, we create a table of errors (the absolute difference between the trinomial price and the Black-Scholes price) for different values of probabilities p and number of steps N. The results of these experiments can be seen in the table below. Note that the binomial case is a special case of the trinomial model, with $p = 1/2$ (the second column from the right). Also note that here we are using values $T = 10/252$, $S_0 = 10$, $K = 10$, $r = 0.01$, $\sigma = 0.2$. The Black-Scholes price was computed with the inbuilt Matlab function *blsprice*, our code to compute the trinomial price for European options is found in Appendix C.5.

Table 4.1: Error of the trinomial pricing model for different values of N and p.

N	p = 0.1	p = 0.2	p = 0.3	p = 0.4	p = 0.5	B-S price
10	0.0087420	0.0030177	0.0012903	0.0004725	0.0039171	0.1608920
20	0.0041316	0.0014988	0.0006540	0.0002368	0.0019732	0.1608920
30	0.0027148	0.0009971	0.0004379	0.0001605	0.0013185	0.1608920
40	0.0020226	0.0007471	0.0003291	0.0001214	0.0009900	0.1608920
50	0.0016119	0.0005973	0.0002636	0.0000976	0.0007925	0.1608920
60	0.0013399	0.0004976	0.0002199	0.0000816	0.0006607	0.1608920
70	0.0011465	0.0004264	0.0001886	0.0000701	0.0005665	0.1608920
80	0.0010018	0.0003730	0.0001651	0.0000614	0.0004958	0.1608920
90	0.0008896	0.0003315	0.0001468	0.0000547	0.0004408	0.1608920
100	0.0008000	0.0002983	0.0001321	0.0000493	0.0003968	0.1608920

We can see that in this case the error size decreases faster with the trinomial model for $p = 0.2$, $p = 0.3$, and $p = 0.4$, than for the binomial model ($p = 0.5$). The same is not true for $p = 0.1$, though.

In Figure 4.1 we study the error for different values of p and a set $N = 20$. The error for the binomial model ($p = \frac{1}{2}$) is marked with a dashed line. Clearly we can see that the error for the trinomial model is less than that of the binomial model for all $p \in (0.17, 0.5)$ approximately. For $p < 0.17$ the trinomial model seems to have a higher error than that of the binomial model.

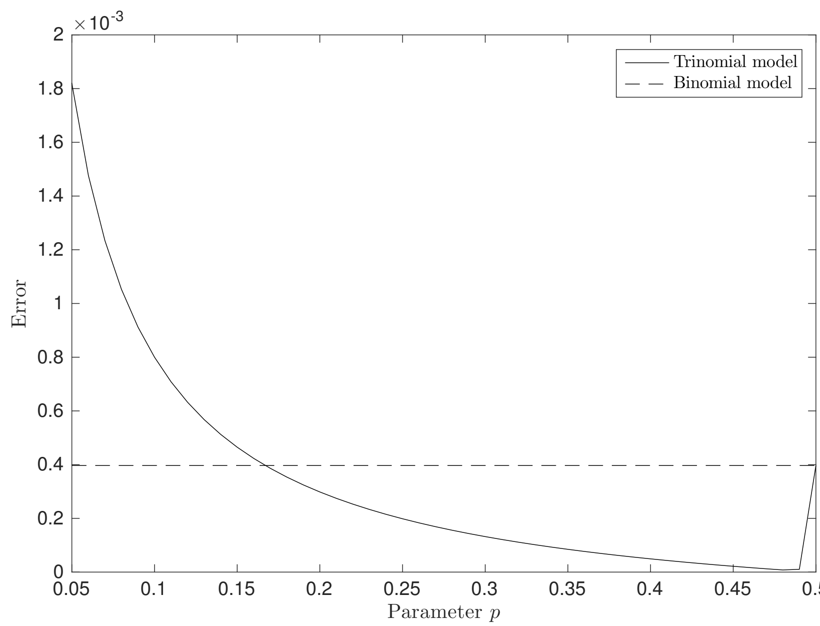

Figure 4.1: Error for the trinomial model as a function of p.

For Figure 4.1 above we use the values $T = 10/252$, $S_0 = 10$, $K = 10$, $r = 0.01$ $\sigma = 0.2$, and $N = 100$.

4.2 Convergence of the trinomial model price for American options

Here we will study the convergence of the trinomial model for American options. Since there is no theoretical Black-Scholes price we will require other means to verify the convergence.

4.2.1 Convergence to binomial model price

Since there is no theoretical Black-Scholes price for the American options, it is not possible to show the convergence of the trinomial pricing of American put options using the Black-Scholes formulas directly. In order to price the American derivatives we need to first define the fair price of the American options, we will follow the definitions in [16]. As stated in the previous chapter when defining the fair price of European options, it is natural to associate a fair price with the value of self-financing hedging portfolios [16]. By Theorem 1.1 in [16] we know that it is never optimal to exercise American call options prior to expiration, hence the fair price of American calls is identical to the fair price of European calls. This is however not true for American puts. Hence we only need to verify the trinomial price of American options for American puts, for which Theorem 1.1 [16] does not hold. Since it may be profitable to exercise an American put prior to expiration, we need to redefine hedging portfolios for American options.

The portfolio process $\{h_S(t),h_B(t)\}_{t\in\mathbb{I}}$ is said to be hedging an American derivative with intrinsic value $Y(t)$ if

$$V(N) = Y(N),\ \ V(t) \geq Y(t),\ \ t = 0,\ldots,N-1. \tag{4.13}$$

We can now define the fair price $\hat{\Pi}_Y$ of a American option with payoff Y. Since at time of maturity $V(N) = Y(N)$, we simply define the fair price at the expiration date as $\hat{\Pi}_Y(N) = Y(N)$. Since it might be more profitable to exercise the American put prior to expiration we define the fair price at $t = N-1$ as $\hat{\Pi}_Y(N-1) = \max(Y(N-1), \Pi_Y(N-1))$. Using our recurrence formula of the fair price for European options (3.2) suggests the following definition of fair price for American options.

Definition 4.2.1. *The fair price $\hat{\Pi}_Y(t)$ of a standard American derivative with payoff $Y(t) = g(S(t))$ at time $t \in \{0,\ldots,N\}$ is defined by the recurrence formula*

$$\hat{\Pi}_Y(t) = \begin{cases} Y(N) & t = N \\ \max(Y(t),e^{-r}[q_{-1}\hat{\Pi}_Y^{-}(t+1,q_0) + q_0\hat{\Pi}_Y^{0}(t+1,q_0) + q_{+1}\hat{\Pi}_Y^{+}(t+1,q_0)]) & t \in \mathbb{I}_{N-1} \end{cases}$$

where $\mathbb{I}_{N-1} = \{0,\ldots,N-1\}$.

A Matlab code implementing this in the trinomial model to price American put options can be found in Appendix C.7.

Since there is a corresponding fair price for the binomial model, we may check that the trinomial price of an American put converges to the same price as the binomial model price of the same American put. For this experiment we use the values $T = \frac{10}{252}$, $S_0 = 10$, $K = 10$, $r = 0.01$, $\sigma = 0.5$. To get the binomial model price we have simply set $p = 1/2$ in the trinomial model, code in Appendix C.7. We clearly see that the trinomial model (solid line) with $p = 0.4$ converges significantly faster than the binomial model (dashed line) in Figure 4.2 below.

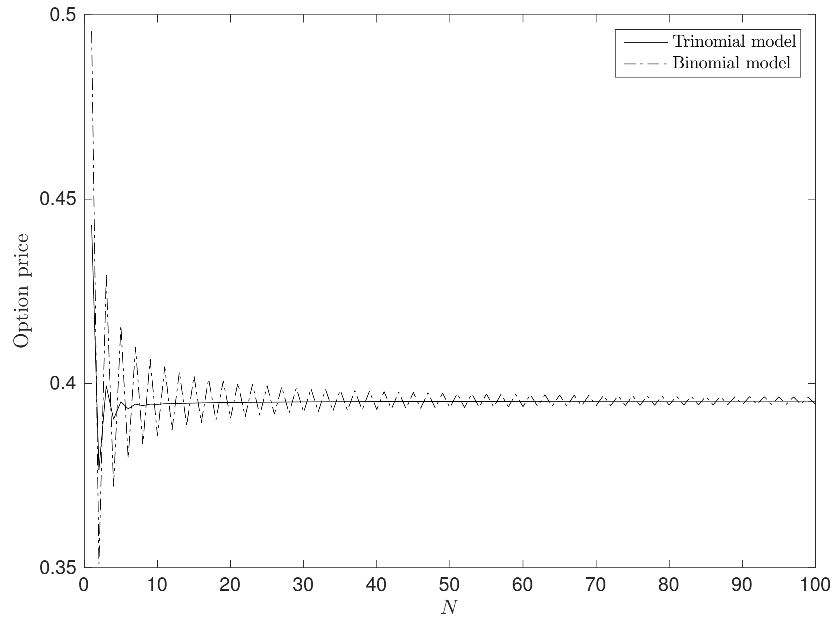

Figure 4.2: Trinomial model (solid line) and binomial model (dashed line) convergence of American put price as $N \to \infty$.

4.2.2 Convergence to the American perpetual put

Another way to validate the trinomial price of the American put is through the American perpetual put. Since there exists a closed Black-Scholes formula for the American perpetual put we can check that the trinomial price of American puts converges to the Black-Scholes price for American perpetual puts for $T \to \infty$, with N large. The price for a perpetual American put is

$$P = \frac{K}{1-h_2}\Big(\frac{h_2-1}{h_2}\frac{S}{K}\Big)^{h_2},$$

where

$$h_2 = \frac{1}{2} - \frac{r-q}{\sigma^2} - \sqrt{\Big(\frac{r-q}{\sigma^2} - \frac{1}{2}\Big)^2 + \frac{2r}{\sigma^2}}.$$

Here K is the strike, r is the interest rate, σ is the volatility of the underlying asset, and q is the yield of the underlying asset. For our experiment, we set the dividend $q = 0$. As can be seen Figure 4.3 the trinomial price clearly converges to the Black-Scholes price of a perpetual American put option. In the experiment which produced this figure, we set $S_0 = 10$, $K = 10$, $r = 0.01$, $\sigma = 0.3$, and $p = 0.4$. The Black-Scholes price of this option was computed to be 5.6018 (visualized by the dashed line).

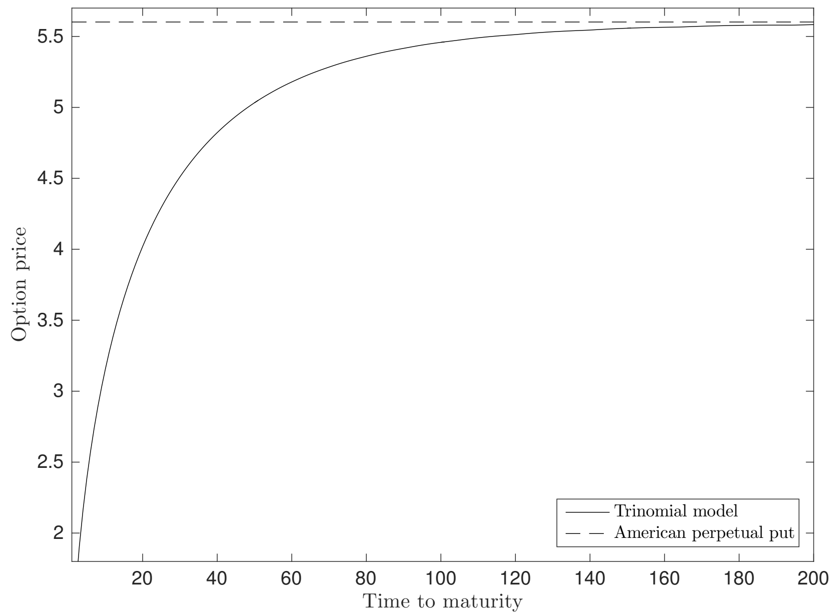

Figure 4.3: Trinomial model (solid line) and price of perpetual put (dashed line) convergence as $T \to \infty$. Matlab code for the trinomial price is found in Appendix C.7

From this analysis we see that the trinomial price of American puts approaches the perpetual American put as the time of maturity tends to infinity. This suggests that our trinomial model prices American puts correctly. To investigate this further we, also compare optimal exercise boundary of the American put computed by the trinomial model with the optimal exercise boundary of the American perpetual put.

Optimal exercise boundary of the American put

Unlike European options, American options can be exercised at any time before expiration, the optimal exercise curve tells us when it is optimal to exercise the American option. Plotting the pairs $(t,S(t))$ for each optimal exercise time to maturity t gives the optimal exercise boundary. When the price of the underlying asset is above this boundary, the option price satisfies the Black-Scholes PDE and it is not optimal to exercise the American put. In an analogous manner, when the value of the underlying asset is below this boundary, the option price is identical to the intrinsic value and it is optimal to exercise the American put. To date, there exists no closed form solution for the optimal exercise boundary for a general American put, and thus calculations must be carried out numerically [23].

Since the study will be done numerically we will get the optimal exercise boundary at discrete time points t_i. To find a point $(t_i,S(t_i))$ we look for the greatest price of the underlying asset which makes the option price identical to the intrinsic value at time t_i. In other words, $S(t_i)$ will lay on the dividing point between the price function of an American put and the intrinsic value $y = (K - S(t_i))_+$, see Figure 4.4.

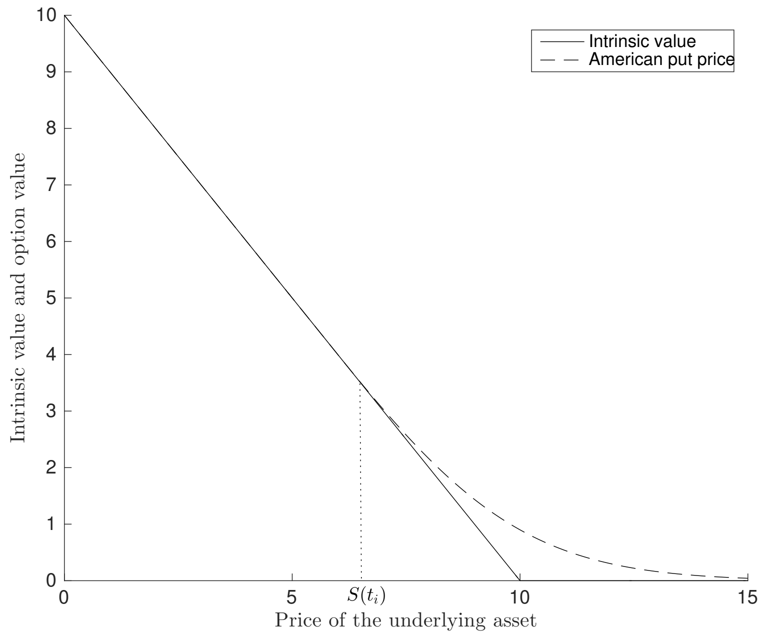

Figure 4.4: Visualization of $S(t_i)$.

By using Matlab, it is possible to find an approximation for $(t_i,S(t_i))$, and hence also an approximate optimal exercise boundary. We choose to fix $N = 200$, $r = 0.01$ and $\sigma = 0.3$, whereas we test three different values for K and we also let the time until maturity vary between 0 and 60. Figure 4.5 shows $(t_i,S(t_i))$ for $K = 5, 10, 15$ and $t_{i+1} - t_i = 0.1$. In Appendix C.7, the Matlab code for the trinomial price can be found.

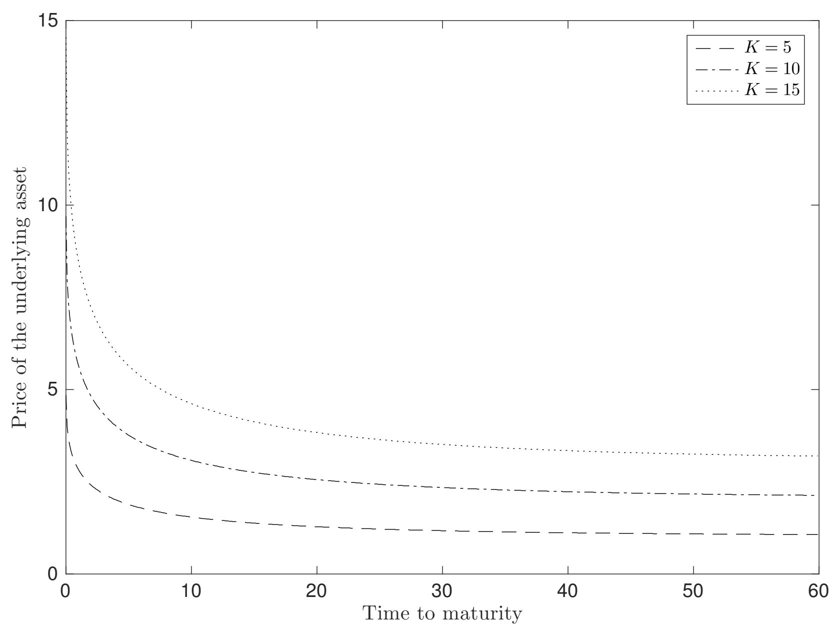

Figure 4.5: Optimal exercise boundary for three different values of K.

In Figure 4.5, we see that the optimal exercise boundary appears to be almost constant (or increase slowly) when there is a long time until maturity. For time instances close to maturity, the slope will increase drastically and the boundary will approach K. This behaviour is reasonable since a shorter time until maturity makes the American put less valuable if all other parameters are kept constant [24]. One could think of it as shifting the dashed line corresponding to the price of the American put downwards in Figure 4.4, and hence $S(t_i)$ will increase. Close to maturity, it makes sense that $S(t_i)$ is almost identical to K because the price will always be the same or very close to the intrinsic value.

Optimal exercise boundary of the American perpetual put

Since a perpetual American put has infinite maturity, the optimal exercise boundary will be constant. According to [25], this constant value B_∞ will satisfy

$$B_\infty = \frac{2rK}{2r + \sigma^2}. \tag{4.14}$$

Figure 4.6 shows the optimal exercise boundary of the put from the previous exercise with $K = 10$ together with the corresponding boundary of a perpetual American put. For sake of comparison, we have included the optimal exercise boundary for $N = 50$, $N = 200$ and $N = 800$ whereas $t_{i+1} - t_i = 0.1$ for all i. The Matlab code for the trinomial price of the American put can be found in Appendix C.7.

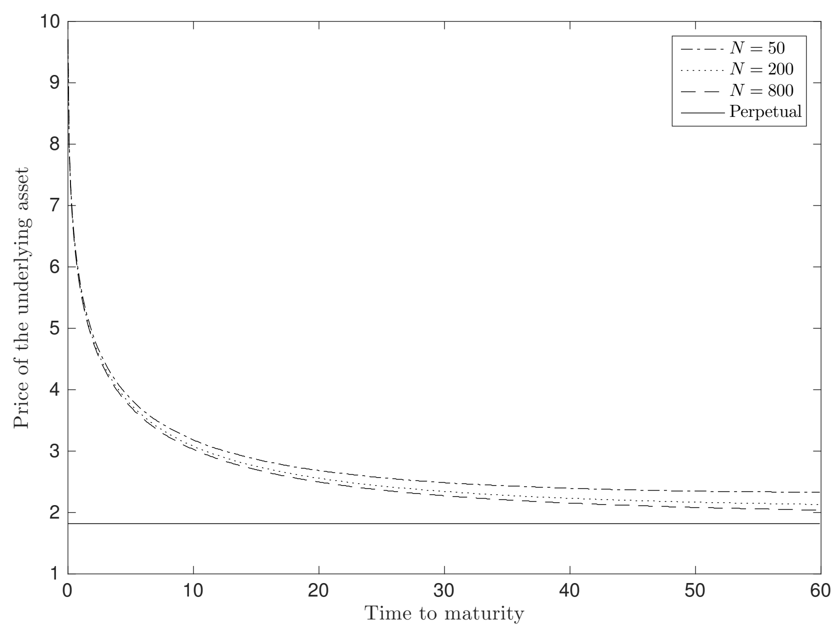

Figure 4.6: Optimal exercise boundary of an American put with $K = 10$ for different values of N compared to the boundary of a perpetual American put.

Figure 4.6 shows that the optimal exercise boundary of our American put is very close to the boundary of a perpetual American put when N is large. Hence the price of an American put should be very close to the price of a perpetual American put when the time until maturity is sufficiently large. This result is in line with our previous findings, that the trinomial price of American puts converges to the fair price of American puts.

To conclude we have found that the trinomial price of an American put converges to the same price as the binomial price of an American put and that the trinomial price of an American put converges to the corresponding American perpetual put as time of maturity $T \to \infty$. Finally we have also shown that the optimal exercise boundary of an trinomial model priced American put converges to the optimal exercise boundary of the corresponding American perpetual put.

4.3 Historical and implied volatility

In the Black-Scholes model the underlying asset return $S(t) - S(0)$ is assumed to be lognormally distributed, i.e.

$$\log\left(\frac{S(t)}{S_0}\right) \in N\left((\mu - \frac{\sigma^2}{2})t,\ \sigma^2 t\right). \tag{4.15}$$

The volatility of the stock is σ, the standard deviation and the return given by the asset over one year ($t = 1$) [4]. Intuitively it is a measure of the uncertainty of the return given by the asset. In the Black-Scholes model the risk-free-rate and volatility σ are the only parameters which can not be directly observed from the market [24], hence these parameters need to be approximated. Historical volatility is a method to approximate the unknown volatility σ. As the name suggests, historical volatility uses historical values of the asset $S(t)$ to approximate the volatility. Implied volatility gives an approximate value of the volatility as well, however, this is not used to price options using the Black-Scholes model. The implied volatility is obtained from the market and can be seen as what the market believes the volatility should be.

4.3.1 Historical volatility

Historical volatility is obtained by calculating the standard deviation of log-returns based on historical data of the asset. To calculate the historical volatility of a stock on the interval $[t_0,t]$ where t_0 is some time prior to t which possibly is the present time. Let $t_0 < t_1 < \cdots < t_n = t$ be the uniform partition of $[t_0,t]$ with constant time differences, $t_{i+1} - t_i = \Delta t$ for $i \in \{0,\ldots,n-1\}$. Let

$$R_i = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right) \quad \text{for} \quad i = 1,\ldots,n. \tag{4.16}$$

The corrected standard deviation of R_i is given by

$$\sqrt{\mathbb{V}\mathrm{ar}[R]} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}. \tag{4.17}$$

Since the stock return is lognormally distributed

$$\log\left(\frac{S(t)}{S_0}\right) \in N\left((\mu - \frac{\sigma^2}{2})t,\ \sigma^2 t\right), \tag{4.18}$$

we have that

$$\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2} \approx \sigma\sqrt{h}. \tag{4.19}$$

Let $T = t - t_0$, we define the T-historical volatility as

$$\hat{\sigma}_T(t) = \frac{1}{\sqrt{\Delta t}}\sqrt{\frac{1}{n-1}\sum_{i}^{n} = 1(R_i - \bar{R})^2}, \tag{4.20}$$

where

$$\bar{R} = \frac{1}{n}\sum_{i=1}^{n}R_i. \tag{4.21}$$

4.3.2 Implied volatility

The implied volatility is determined by looking at the actual market price. Consider a European call on a non-dividend-paying stock with market price 1.875, $S_0 = 21$, $K = 20$, $r = 0.1$ and $T = 0.25$. The implied volatility is the σ_{imp} such that the Black-Scholes price $c = 1.875$ with $S_0 = 21$, $K = 20$, $r = 0.1$.

This argument may be extended to the trinomial model under the assumption that it converges and that the trinomial price is one to one with respect to σ. Consider the same European call as above, market price 1.875, $S_0 = 21$, $K = 20$, $r = 0.1$ and $T = 0.25$. The implied volatitily can be approximated by the σ^*_{imp} such that the trinomial model converges to 1.875 with $S_0 = 21$, $K = 20$, $r = 0.1$.

In order for the implied volatility to be uniquely defined, the trinomial price c with respect to σ needs to be one-to-one. We will verify this nummerically. . In Figure 4.7, we have plotted the trinomial price as a function of the volatility. This graph suggests that the trinomial price is an increasing function with respect to this variable. The Matlab code for the trinomial price of European options is found in Appendix C.5.

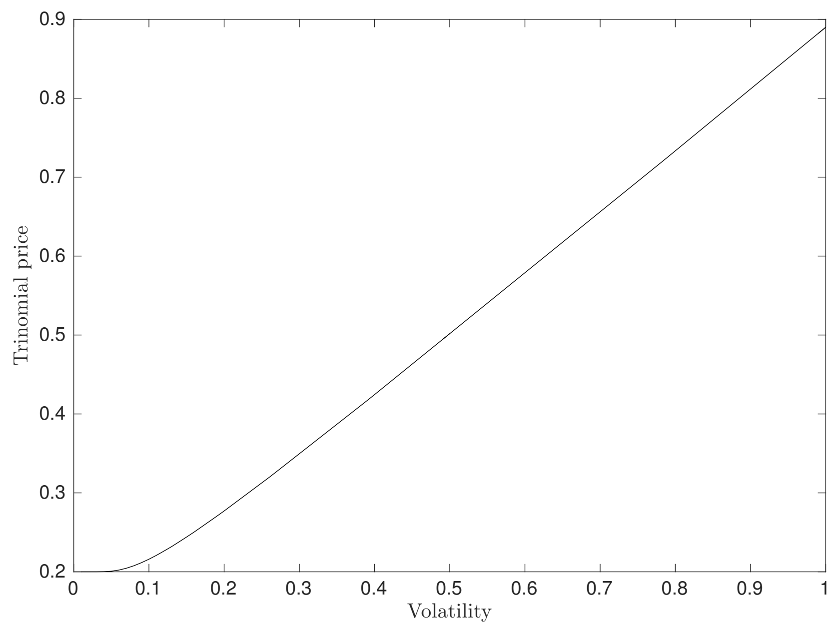

Figure 4.7: The trinomial price of an European call as a function of σ with $S(0) = 10$, $K = 9.8$ and $T = 10/252$.

A higher volatility means that it is more likely that the stock price will change drastically before time of maturity. Since the payoff has a lower limit (zero) but no upper limit, one has more to gain from a high volatility than to lose from it. Thus it makes sense that the call option is an increasing function of the volatility.

Since the call price appears to be a one-to-one map with respect to volatility, it is possible to calculate the implied volatility for different call prices. There is no closed formula to calculate the implied volatility, but by using numerical methods it is possible to get an approximation.

4.3.3 Volatility smile

When plotting the implied volatility as a function of the strike price K for options, a convex curve often arises, this pattern is called a volatility smile. In the Black-Scholes formulas and the trinomial model it is assumed that the underlying asset is lognormally distributed. This is however many times an incorrect assumption.

An example is the foreign currency options, which is options with currencies as underlying assets. Two requirements for the underlying asset to be lognormally distributed are that the volatility of the asset is constant and that the underlying asset is continuous. Neither of these assumptions are true for the exchange rates, the volatility changes and the exchange rates often jump discontinuously (often due to actions of central banks). This results in a volatility smile, when plotting the implied volatility against the strike price since the volatility is not constant[4].

After the stock market crash in October 1987, volatility smile patterns started to show up for equity options (options on assets that signify ownership in a corporation, e.g. stocks). Prior to the stock market crash, the implied volatility of options were much less dependent on the strike price. A possible reason for appearance of volatility smiles after the crash is leverage [4]. If a company's equity decreases in value, the company's leverage increases which makes the equity more risky and so the volatility increases. In the reverse case when the company's equity increases in value, the company's leverage decreases which makes the equity less risky and the volatility decreases. This results in a volatility smile when plotting the implied volatility against the strike price. Compared to the volatility smile of foreign exchange options, the volatility smile of equity options are often slightly skewed hence often called volatility skew.

We can illustrate this limitation of the Black-Scholes model and hence also the trinomial model by plotting the difference between the Black-Scholes price and market price as a function of K. Yet since r is an unknown variable, we need to have an understanding of the implications when r changes. By observing the formula

$$V(t,s) = e^{-rh}[q_{+1}V(t+h,se^{u},q_0) + q_oV(t+h,s,q_0) + q_{-1}V(t+h,se^{-u},q_0)],$$

we can see that the factor e^{-rh} will essentially be equal to one for r sufficiently small since h is present and very small. Furthermore, this conclusion can be drawn through numerical arguments. In Table 4.2 below we observe values of the trinomial price and Black-Scholes price for different values of K at $r = 0$ and $r = 0.05$. One can easily see that the results do not change significantly when computing the prices for such a large value of r, the implications of a changed r may then be dismissed.

Table 4.2: A comparison between the strike price (K), market price, trinomial model price (TMP) and Black-Scholes price (B-S price) of a European call option with current stock price 97.8, $N = 100$, $p = 0.3$, $T = 12/252$, $\sigma = 0.392$, $r = 0$ and $r = 0.05$. The Matlab code for the trinomial price of a European call is found in Appendix C.5.

K	Market price	TMP r=0	B-S price r=0	TMP r=0.05	B-S price r=0.05
104	0.2600	1.1962	1.1937	1.2524	1.2500
103	0.3700	1.4364	1.4343	1.5008	1.4987
102	0.5400	1.7125	1.7115	1.7856	1.7846
101	0.7600	2.0275	2.0283	2.1098	2.1106
100	1.0600	2.3850	2.3874	2.4768	2.4793
99	1.4600	2.7908	2.7910	2.8925	2.8927
97	2.4600	3.7406	3.7382	3.8622	3.8597
96	3.1500	4.2865	4.2834	4.4177	4.4146
95	3.8500	4.8797	4.8764	5.0202	5.0170
94	4.6000	5.5195	5.5164	5.6689	5.6657
93	5.4000	6.2045	6.2017	6.3620	6.3592
92	6.3000	6.9326	6.9302	7.0974	7.0950

The market prices we have used are for European call options on Apple stocks, these were obtained from Yahoo Finance 17 February 2016 with maturity date 4 March 2016. To obtain the corresponding prices using the trinomial model and Black-Scholes price we have used the 20-days historical volatility of the Apple stock.

Having concluded that fixing $r = 0$ implies no significant errors, we may now plot the difference between the trinomial price and market price as a function of K and fixing $r = 0$, without significant impact on our results. See Figure 4.8 below.

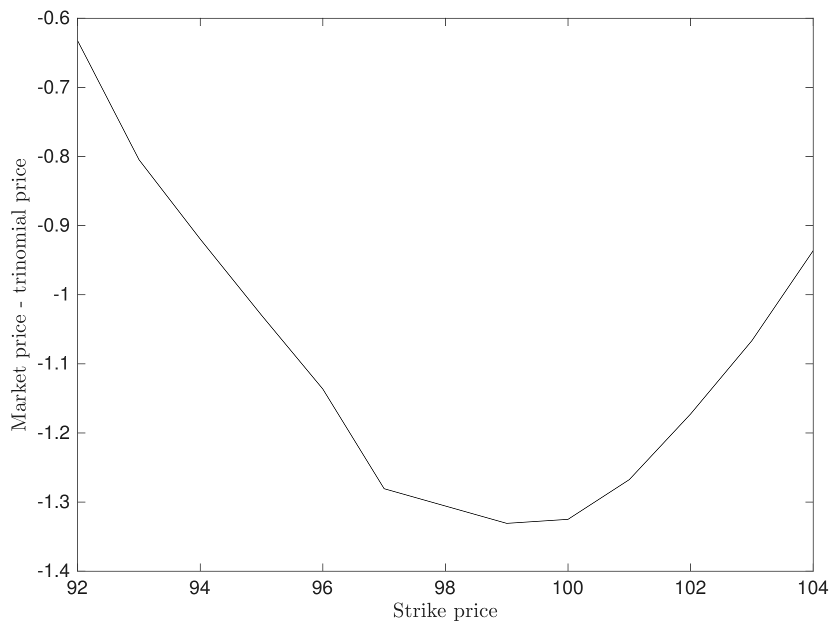

Figure 4.8: Difference between the market price and the trinomial price as a function of K. The Matlab code for the price of a European call is found in Appendix C.5

We see that the difference between the trinomial model prices and market prices increases when K is such that the derivative is either deeply in the money or out of the money, which indicates that the model may be defective under these conditions. We further investigate this by graphing the implied volatility as a function of the strike price K. To calculate the implied volatility we use Newton's method, the Matlab function which computes the implied volatility be found in Appendix C.6. Applying this function to the market prices used the previously and plotting the implied volatility as a function of the strike price yields Figure 4.9 below.

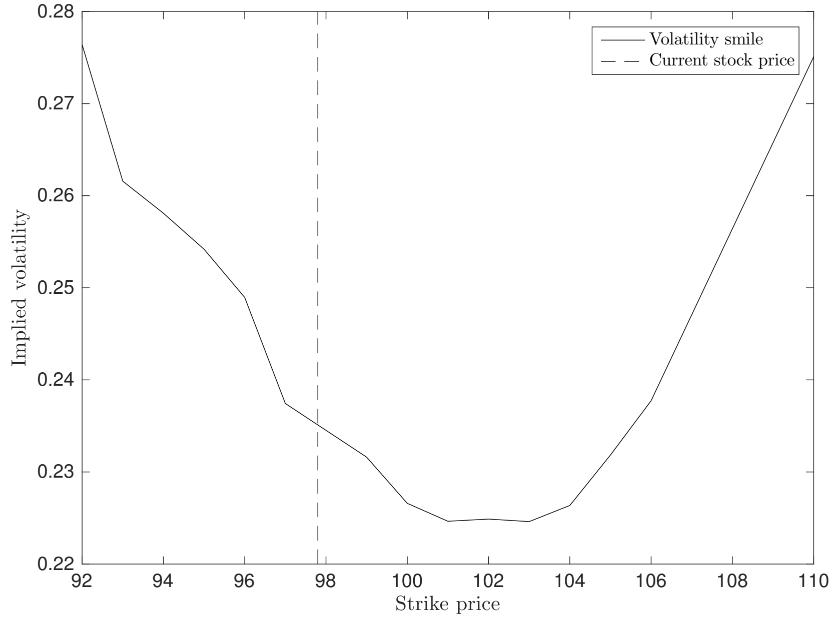

Figure 4.9: Volatility smile for Apple stock February 17, 2016.

Figure 4.9 shows the implied volatility for Apple options with the same expiration date but different strike prices K. It tells us that, for a given time to expiration, options whose strike price differs from the current price of the underlying asset (dashed line) has higher implied volatility. We can clearly see that the volatility smile we discussed in the beginning of this section arises. The volatility smile implies that deep out of the money and deep in the money options are priced with a higher or lower price in the market compared to the theoretical prices calculated by the Black-Scholes and trinomial models. The volatility smile also tells us that demand is greater for options that are in the money or out of the money.

The Black-Scholes model and trinomial model suggests that the volatility should be constant and independent of any other parameter, but as we can see in Figure 4.9, the implied volatility for actual market prices of the Apple options differ depending on K. When the volatility of a call differs from the volatility of a put with the same strike and expiration, it indicates the market's bias towards a call or a put. As a general rule, the lowest point of the volatility smile tends to correspond to the at the money price, but as we can see in Figure 4.9 the smile is right biased witch indicates that it is the upside calls relative to the at the money price. The market tends to behave in this way because institutions write a lot of upside calls for large long positions in the underlying as investors want to increase returns of their investment. Thus the market adjusts this by shifting the lowest point in the smile to the right to compensate for these institutional writers and sellers. If we plot the put and call in the same graph we should see that the lowest points are at the same price because of the potential arbitrage opportunities if they are not. This can actually occur in the opposite way when the shift is to the left, this implies that many companies sell puts against their heavily promoted stock during long run bull markets.

Chapter 5

Exotic options

Exotic options are options that are more complex than standard European and American put and call options. Unlike these standard options, which are commonly referred to as vanilla options, an exotic option can have a payoff that depends on the path of the underlying asset and not just the price at maturity. Other features that can make these options more advanced are, for example, different constraints on the payoff or prespecified exercise dates prior to maturity. Exotic options can be used for several different purposes, for example to hedge another derivative or to speculate on the future movements of an asset. There is generally no exchange that handles the trade of exotic options, so they have to be traded over the counter.

Since the trinomial model has proved to result in faster convergence rates than the binomial model when pricing vanilla options, it seems reasonable that the trinomial model could be an efficient method to price exotic options. We will examine if this is true for Asian options, cliquet options, compound call and put options, lookback options, Bermudan options, and barrier options, which are all classified as exotic options. We will also discuss other pricing methods and describe what these options are being used for.

5.1 Asian options

Asian options were invented in the late 1980s, initially for trading crude oil with the purpose of reducing the risk of market manipulation at the maturity day, and to decrease the effect of volatility [26].

Introduction

An Asian option is a path-dependent exotic option. Hence pricing an Asian option is more difficult than pricing a standard European or American option. The payoff of an Asian option depends on the average price of the underlying asset during a certain time period, hence it is also sometimes known as an average option. The reason this option attracts investors is that it often costs less than a European option while simultaneously being less sensitive to volatility. These two properties are dependent; if the volatility of the underlying asset is high then the price of a European or American option on that asset will be high. Another advantage with the Asian option is that it is not as sensitive to extreme market conditions that might prevail on the expiration day. For the rest of this section we will assume that the underlying asset is a stock. We will also only investigate pricing methods using an arithmetic average, since there are no main differences in the payoff [27].

Mathematical representation

As mentioned in the introduction the payoff of an Asian option depends on the average price of the underlying asset. Let T be the maturity of the Asian option, N be the number of time steps until maturity, K be the strike price, and A the average of all stock prices until maturity. Then the payoff is calculated as

$$Y = \max(A - K{,}0),$$

for a call option, and

$$Y = \max(K - A{,}0),$$

for a put option, where the average A can be calculated either as an arithmetic or a geometric average. The arithmetic average is expressed as

$$A_n = \frac{1}{N}\Big(S(t_1) + S(t_2) + ... + S(t_n)\Big),$$

and the geometric average as

$$A_n = \left(\prod_{t=1}^{n} S(t)\right)^{1/n}.$$

Pricing methods

In the following section we will describe two different ways to price Asian options; by use of Monte Carlo simulations and the trinomial model. Since a closed formula to price Asian options with arithmetic average does not exist it has to be done numerically.

Monte Carlo simulation

One popular way to price Asian options is to use Monte Carlo simulations. This technique is very useful when pricing different path dependent options [28]. The technique involves first generating a large number of samples of the underlying asset, in this case through a Geometric Brownian Motion. These samples are then used to generate a statistic price of the option. Since each simulation generates a path of the underlying asset it is then possible to use the discrete time approach and to calculate the arithmetic average for each sample. After this one can apply the law of large numbers which states that with a high enough number of samples the mean of all sampled averages will converge to the statistic average of the underlying asset's path.

It is known that the solution for the price of a non dividend paying stock following the stochastic process of a Geometric Brownian Motion can be expressed as

$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right),$$

where W_t is a Brownian motion

$$W_t \sim N(0{,}t),$$

and

$$\log(\frac{S_t}{S_0}) \sim N\left((r - \frac{1}{2}\sigma^2)t{,}t\sigma^2\right).$$

The stock price is then obtained by setting

$$S_t(i) = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z(i)\right),$$

where $Z(i) \sim N(0,1)$, $i = 1,2,3,...,n$, and independent [28][29]. Now we can use the law of large numbers which states that

$$M = \frac{1}{n}\sum_{i=1}^{n} S_t(i) \to E[S_t], \text{ as } n \to \infty.$$

We also have to evaluate the variance of the estimator. The variance of M is

$$\mathbb{V}\mathrm{ar}(M) = \mathbb{V}\mathrm{ar}\left(\frac{1}{n}\sum_{i=1}^{n} S_t(i)\right) = \frac{1}{n^2}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{n} S_t(i)\right) = \frac{\mathbb{V}\mathrm{ar}(S_t)}{n}$$

and as n goes to infinity the variance of M will go to zero. Hence this fulfills two required characteristics of an estimator; convergence to the right mean, and a decreasing variance as n increases. This concludes the Monte Carlo pricing method.

Trinomial method

As discussed in the introduction, the Asian option is path dependent. Every path will generate an average A. To generate paths we use

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{with prob. } p_u \\ S(t-1) & \text{with prob. } p_0 = 1 - p_u - p_d \\ S(t-1)e^{-u} & \text{with prob. } p_d \end{cases}, \; t \in \mathbb{I} = \{1,...,N\}.$$

To calculate the initial price for Asian options we first have to calculate the average for each path. Then we have to calculate the expected value of all payoffs and discount them using e^{-rT}. Hence the initial price for an Asian call option can be expressed as

$$C = \mathbb{E}[e^{-rT}(A - K)^{+}].$$

This is in practice done by generating payoffs and using a recursive formula with the risk neutral probabilities and the discounting factor e^{-rT}.

Numerical results

One major issue with Monte Carlo simulations is that they do not provide an exact result, instead we need to simulate several times and then take the expected value of all simulations. To observe the accuracy of the model we calculate the variance of all those simulations. The trinomial model also has an issue; when the number of time steps increases the number of possible paths increases rapidly, making the computational time very demanding. In the table below we try to decide the optimal number of time steps with regards to computational time and price for the trinomial model. We use S_0=10, K=8, r=0.01, T=0.062, σ=0.2, and p=0.25.

Table 5.1: The computational time for different number of steps in the trinomial model using the function in Appendix C.10.

N	Computational time	Initial price
10	1.72 s	2.0022
11	5.10 s	2.0021
12	15.19 s	2.0021
13	45.47 s	2.0021
14	136.02 s	2.0021

We also need to calculate the computational times for the Monte Carlo method. Here we focus on the number of replicates instead of the number of steps

Table 5.2: The computational time for different number of replicates in the Monte Carlo simulation using the function in Appendix C.9.

Replicates	Computational time	Initial price
1000	2.11 s	2.0034
2500	5.28 s	2.0042
5000	10.40 s	2.0018
7500	15.60 s	2.0022
10000	20.76 s	2.0018
15000	31.02 s	2.0015

The calculations were made on a computer with 16 GB 2.8 GHz of RAM and a 2.8 GHz dual-core Intel Core i5 processor. From this we can conclude that N=12 seems like a good number of time steps with regards to both the computational time and the obtained price.

Next we need to compare the prices from the trinomial model with prices from the Monte Carlo simulations. Since the purpose of the Asian option is to reduce the effects of volatility, and hence also the price of the option, it is also interesting to compare the results with the Black-Scholes price of a European option. As an example we will study a call option with S_0=10, r=0.01, K=8, T=0.062, N=12, and p=0.25. For the Monte Carlo simulation we use 10000 replicates with the purpose of obtaining a good result without increasing the computational time excessively. We do this 100 times and take the mean of those results to obtain an estimate for the expected value. We also compute the variance of those results to judge the accuracy of the model. In Table 5.3, T-price denotes the price for the trinomial model, MC-price denotes the price for the Monte Carlo simulation, $\mathbb{V}\mathrm{ar}$ denotes the variance for the Monte Carlo simulation, and B-S price denotes the Black-Scholes price for a European call option.

Table 5.3: Prices for the trinomial model, Monte Carlo simulation, and the Black-Scholes model for a European call option.

σ	T-price	MC-price	Var	B-S price
0.1	2.0021	2.0015	$2.3 \cdot 10^{-6}$	2.0050
0.2	2.0021	2.0020	$6.5 \cdot 10^{-6}$	2.0050
0.3	2.0021	2.0016	$1.4 \cdot 10^{-5}$	2.0052
0.4	2.0021	2.0000	$3.2 \cdot 10^{-5}$	2.0088
0.5	2.0022	2.0033	$3.2 \cdot 10^{-5}$	2.0209
0.6	2.0033	2.0022	$6.7 \cdot 10^{-5}$	2.0442

Discussion

As seen in the tables above the Monte Carlo simulation tends to generate different prices, and without further investigation it is difficult to say conclusively how many replicates the model needs to converge to a fixed value. It is apparent that with higher volatility the variance increases, therefore variance reduction techniques likely need to be implemented. Our findings about the trinomial model show that it can be used to price Asian options, however since there does not exist a closed formula for computing Asian options with arithmetic average it is difficult to estimate the accuracy. Instead we recommend that other pricing methods are investigated to provide an accurate result.

5.2 Cliquet options

The following sections describe how a cliquet option works, what it is being used for, and how it can be priced by the trinomial model. A comparison regarding rate of price convergence is also made with the binomial model. Finally, we will discuss whether the trinomial model is a suitable pricing method based on our findings.

Introduction

The cliquet option is an exotic option that was introduced in the beginning of the twentyfirst century as a response to investors' demand for safer financial products [30]. The payoff of a cliquet option depends on the returns of the underlying asset between given reset dates prior to maturity. These returns might be locally floored and capped, and at maturity the sum of these modified local returns might also be globally floored and capped. All of these floors and caps are prespecified in the option contract. The payoff of the cliquet option is determined by the final truncation [30]. By imposing upper and lower boundaries on the payoff, the downside risk is reduced in exchange for less upside potential. This makes cliquet options very appealing to many different types of investors, such as pension funds and retail investors [31].

Having briefly described how a cliquet option works, it should be clear that it is path dependent. This means that the payoff depends not only on the final value of the underlying asset, but also on its path. The price of a cliquet option can only be calculated by numerical methods [31]. Several different ways to calculate the price have been suggested. Most literature on the topic focuses on techniques based on partial differential equations [32]. Because of the nature of the option, methods based on the binomial model can easily lead to computational time problems. Different ways to make such algorithms more efficient have been suggested, e.g. a technique based on singular points [32]. A common way to price cliquet options is also by means of Monte Carlo simulation.

Mathematical representation

As mentioned earlier, the payoff of a cliquet option depends on the returns of the underlying asset between given reset dates. Let T be the maturity of a cliquet option and denote by $t_1,...,t_m$ m number of consecutive reset dates in the interval $(0,T]$ where $t_m = T$. The return of the underlying asset S in the interval $[t_{i-1},t_i)$ is

$$R_i = \frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})}, \quad i = 1,...,m, \quad S(t_0) = S(0).$$

Moreover, if the cliquet option has a local floor F_l and a local cap C_l, the truncated local return $\bar{R}_i$ becomes

$$\bar{R}_i = \min(C_l, \max(R_i, F_l)) \quad i = 1,...,m.$$

Of course F_l must be smaller than C_l in order for this equation to be interesting. At time of maturity, the cliquet option has a final payoff Y of

$$Y = B \cdot \min\left(C_g, \max\left(\sum_{i=1}^{m} \bar{R}_i, F_g\right)\right) \tag{5.1}$$

when a global floor F_g and global cap C_g are imposed [30]. B is the notional amount which for sake of simplicity is set to one, which is in line with other literature [30][32]. The payoff described by (5.1) corresponds to the most general cliquet option, which is actually not very common on the market. More often, there exists no global cap.

Pricing methods

In the following sections, we describe a number of different ways to price cliquet options. We focus on a trinomial model approach but will also mention a few other techniques.

PDE methods and Closed-form price formula

The most popular way to price cliquet options has been by partial differential equation techniques. To solve such an equation numerically, the finite difference method can be used [33]. Since this thesis focuses on the trinomial asset pricing model, we will not describe this method in detail and we refer to other literature (e.g. [30]) for a more detailed presentation on this topic.

There exists no closed-form price formula for a general cliquet option, but recent research has found a way to express the initial price in a semi-closed form [31][34]. This solution involves the characteristic function of the periodical returns which cannot be computed exactly. Thus, numerical approximations are necessary even in this case. For certain types of cliquet options however, a closed form solution exists [35].

Tree methods

When pricing cliquet options by tree methods, we will assume that the reset dates are equally distributed and that the local floors and caps are kept constant. This is quite common in reality and it makes it easier to compare our results with other authors' work [32]. We will derive a theoretical valuation of the cliquet option in the trinomial model framework, but since the binomial model is just a special case of the trinomial, all arguments will hold for that model as well.

The first thing to notice when pricing cliquet options by tree methods is that the presence of local floors and caps will drastically reduce the number of paths that need to be investigated. Let N denote the number of time steps between two consecutive reset dates and assume $u = -d$ in the trinomial model. If returns are going to be in the interval (F_l,C_l), the number of ups N_u and downs N_d in each period must satisfy

$$\frac{S(t_{i-1})e^{N_u u - N_d u} - S(t_{i-1})}{S(t_{i-1})} < C_l \Leftrightarrow N_u - N_d < \left\lceil \frac{\log(C_l + 1)}{u} \right\rceil = \alpha, \tag{5.2}$$

and

$$\frac{S(t_{i-1})e^{N_u u - N_d u} - S(t_{i-1})}{S(t_{i-1})} > F_l \Leftrightarrow N_u - N_d > \left\lfloor \frac{\log(F_l + 1)}{u} \right\rfloor = \beta. \tag{5.3}$$

Let $j = \alpha - \beta$ and denote by P_j and P_0 the probabilities that (5.2) respectively (5.3) do not hold. Moreover, let P_i, $i \in \{1, \ldots, j-1\}$, be the probability that $N_u - N_d = \beta + i$ and define $\Omega = \{(N_u, N_d) : N_u - N_d = \beta + i\}$. Then P_j, P_0 and P_i can be calculated as

$$P_j = \sum_{N_u=\alpha}^{N} \sum_{N_d=0}^{(N_u-\alpha, N-N_u)_-} \binom{N}{N_u}\binom{N-N_u}{N_d} p_u^{N_u} p_d^{N_d} p_0^{N-N_u-N_d},$$

$$P_0 = \sum_{N_d=(0,-\beta)_+}^{N} \sum_{N_u=0}^{(N-N_d, N_d+\beta)_-} \binom{N}{N_d}\binom{N-N_d}{N_u} p_u^{N_u} p_d^{N_d} p_0^{N-N_u-N_d},$$

$$P_i = \sum_{\Omega} \binom{N}{N_u}\binom{N-N_u}{N_d} p_u^{N_u} p_d^{N_d} p_0^{N-N_u-N_d}, \qquad i = 1, \ldots, j-1,$$

where p_u, p_d and p_0 are defined as usual. The possible truncated returns $\bar{R}'_j, \bar{R}'_0$ $\bar{R}'_i$ associated with these probabilities are

$$\bar{R}'_j = C_l, \quad \bar{R}'_0 = F_l, \quad \text{and} \quad \bar{R}'_i = e^{N_u u - N_d u} - 1.$$

Now, the exact trinomial price V of a cliquet option at time $t = 0$ with m number of reset dates is

$$V = e^{-rT} \sum_{(x_1,\ldots,x_m)\in\{0,\ldots,j\}^m} Q_{x_1} \cdots Q_{x_m} \min\left(C_g, \max\left(\sum_{i=1}^{m} \bar{R}'_{x_i}, F_g\right)\right). \tag{5.4}$$

In this formula, Q_{x_i} corresponds to P_{x_i} under a risk neutral probability (q_{+1}, q_0, q_{-1}).

We have assumed that u (and thus also d) are kept constant within and between all reset dates. Since cliquet options tend to have a relatively long maturity, usually several years, this might not be very realistic. However, it would not be any more complicated to allow for different volatilities in different reset periods in the trinomial model. For the sake of simplicity, we have assumed that the volatility is not changing. Moreover, we have assumed that the underlying asset pays no dividends. It is possible to incorporate dividends in the trinomial model, but it would make calculations more complex. However, we should still be able to make fair comparisons with other pricing models under the same assumption.

Monte Carlo simulation

Another way to price cliquet options is by generating a large number of paths (e.g. by assuming that the stock price follows a Geometric Brownian Motion) and then calculating the average discounted payoff. This method, which is a type of Monte Carlo simulation, requires a large number of runs to be accurate. It is critical to apply risk neutral probabilities when using this method, otherwise the results will not be accurate.

Numerical results

In order to evaluate how well the trinomial model performs, it has been compared to both the binomial model and the Monte Carlo method. We consider a cliquet option with parameters $F_l = 0$, $C_l = 0.08$, $F_g = 0.16$, $C_g = \infty$, $T = 5$, $m = 5$, $\sigma = 0.2$, and $r = 0.03$. This is a standard cliquet contract studied in several articles [32]. In the trinomial model we set $p_u = p_d = 1/6$, which has proved to give stable results [4].

Figure 5.1 shows the theoretical price of this cliquet option when the number of steps N in each reset period varies from 10 to 200. The trinomial price corresponds to the solid line and the binomial price corresponds to the dashed line. As a reference, a horizontal line at $V = 0.174$ has been included, which corresponds to the average value obtained from Monte Carlo simulations of the price with 10^6 runs [32]. $V = 0.174$ is also the price for $N = 500$ in the trinomial model, which means that this value should be accurate. All calculations were carried out in Matlab and the code can be found in Appendix C.11 and Appendix C.12.

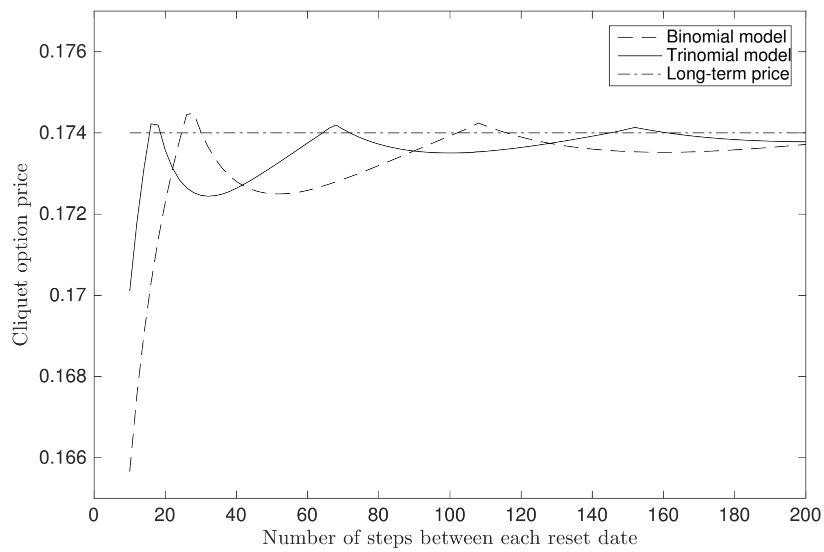

Figure 5.1: The trinomial model price (solid line) and the binomial model price (dashed line) of a cliquet option for different number of steps between each reset date. A horizontal line corresponding to the long-term convergence price is also included.

The two graphs in Figure 5.1 look quite similar, but the curve that represents the trinomial price appears to be left shifted in relation to the curve that corresponds to the binomial price. Moreover, this left shifting seems to increase if one considers the distances between the peeks. This suggests that the trinomial price converges faster than the binomial price.

It should be noted though that the binomial model allows for shorter computational time. The computational time is also heavily dependent on the volatility and the number of steps. Table 5.4 shows the trinomial price (TP) and the binomial price (BP) of the same cliquet option as before but with $\sigma = 0.1$, $\sigma = 0.2$ and $\sigma = 0.5$, and for $N = 100$, $N = 200$ and $N = 300$. The computational time in Matlab when using recursive functions to calculate the price is also included in parentheses under the price. The calculations were made on a laptop computer with 8 GB 1600 MHz of RAM and a 2.4 GHz dual-core Intel Core i5 processor.

Table 5.4: Prices of cliquet options and computational times in relation to σ and N.

N	**BP** $\sigma = 0.1$	**TP** $\sigma = 0.1$	**BP** $\sigma = 0.2$	**TP** $\sigma = 0.2$	**BP** $\sigma = 0.5$	**TP** $\sigma = 0.5$
$N = 100$	0.17344 (0.7200 s)	0.17323 (106.0 s)	0.17393 (0.1116 s)	0.17364 (12.46 s)	0.16376 (0.0647 s)	0.16465 (1.118 s)
$N = 200$	0.17338 (3.865 s)	0.17343 (1365 s)	0.17372 (0.2719 s)	0.17393 (115.4 s)	0.16494 (0.1230 s)	0.16463 (12.13 s)
$N = 300$	0.17345 (7.910 s)	0.17346 (3841 s)	0.17381 (0.7487 s)	0.17395 (209.6 s)	0.16460 (0.1275 s)	0.16468 (24.40 s)

It makes sense that the binomial model results in shorter computational time since the asset can only move in two directions, and hence fewer paths need to be examined than for the trinomial model. σ will also have an impact on the computational time since it affects u, and thereby also the restraints on $N_u - N_d$ according to (5.2) and (5.3). However, Gaudenzi and Zanette, who have made similar calculations for the binomial model (they did not study the trinomial model), arrived at shorter computational times [32]. It is possible that our code could be improved further or that it would be more suitable to use another programming software.

Another interesting thing to note in Table 5.4 is that a higher volatility does not necessarily imply a higher price for the cliquet option. In other words, Vega (the derivative of the option value with respect to the volatility) is not always positive and also Gamma (the second derivative of the option price with respect to the value of the underlying asset) will change sign [36]. The logic behind this is not obvious, and further investigations are required to make a conclusion regarding the inflexion point.

Discussion

Our findings show that the trinomial model can be used to price cliquet options. However, this method is very time consuming (especially for small volatilities) and it requires a large number of steps to be accurate. Even though the trinomial model results in faster price convergence than the binomial model, we consider the latter to be a better choice when pricing this type of option as it results in shorter processing times. It is possible to compensate for a slower rate of convergence by adding more steps in the binomial model without losing advantage with respect to computational time.

5.3 Compound options

Compound options are options for which the underlying asset is also an option. This means that a compound option has two separate strike prices and maturity times; for the compound option itself and for the underlying option. For the purposes of the following section, we will refer to the compound option as the first option, with strike price K_1 and maturity T_1, and refer to its underlying option as the second option, with strike price K_2, maturity T_2, and underlying stock with price $S(t)$ at time t, $T_1 \leq t \leq T_2$.

Introduction

The concept of a compound option may appear confusing at first glance, but its function is fairly simple. At the exercise date of the first option, one must decide whether to exercise it, depending of course on the strike price K_1 and the price of the second option, which in turn will depend on the underlying stock price $S(T_1)$. If the first option is exercised, the investor obtains a further option. Obviously the first option will be exercised only if the price of the second option at time T_1 exceeds the strike price K_1 [4].

Compound options are commonly used for currency or fixed income markets where there exists some insecurity regarding the second option's risk protection [37]. Specifically if the buyer is unsure about the need for hedging in a certain period. This can be useful for companies bidding for a foreign contract when the outcome of the bid is uncertain.

For example, let us say a Swedish construction company is bidding for a contract in the US worth $30 million. The outcome of the bid will not be known for three months, and the company would only be paid after six months. So for the next six months this company has tremendous exposure to changes in the price of the US dollar to the Swedish krona (assuming the company wins the contract), and so there is a clear need to hedge this risk. The company may not want to buy a standard option as the premium is high and the outcome of the bid is undecided. In this scenario, a good compromise is to buy a call on a dollar-put option with $K_1 = 3$ months and $K_2 = 6$ months. The premium will be significantly lower, and it will also limit the exposure of the company in case the contract is lost.

While the compound option has found many areas of application within business and finance, there are certain disadvantages to this derivative. Clearly compound options offer more versatility and hedging opportunities than simply buying the underlying option. However, according to [37], in case that both the first and second options are exercised, the total premium paid will generally be more than that of simply buying the second option in the first place.

Mathematical representation

Representing the final price (i.e. the payoff) for a compound option is fairly straightforward. Clearly it will depend on the types of the first and second option. We might divide compound options into four types:

- A call on a call option (CoC), with payoff $\max\{0, C_{und}(S(T_1), K_2, T_2 - T_1) - K_1\}$
- A call on a put option (CoP), with payoff $\max\{0, P_{und}(S(T_1), K_2, T_2 - T_1) - K_1\}$
- A put on a call option (PoC), with payoff $\max\{0, K_1 - C_{und}(S(T_1), K_2, T_2 - T_1)\}$
- A put on a put option (PoP), with payoff $\max\{0, K_1 - P_{und}(S(T_1), K_2, T_2 - T_1)\}$

Here the underlying call and put options have the values C_{und} and P_{und}, respectively.

Pricing methods

As compound options (under certain assumptions) has a closed form pricing formula the question of what the definitive price should be becomes very clear. However there are still other ways of approximating this value quite well. Here we will look specifically at approximation of the theoretical value using the trinomial model. However, we will first take a brief look at the closed form pricing formula.

Theoretical valuation

Let us consider the theoretical, closed form valuation of the European CoC option, as derived by Robert Geske in 1979 [38]. We assume here that the underlying asset follows a Geometric Brownian Motion, and that volatility remains constant. By risk-neutral valuation, the current value of this compound option is the discounted expected value of its payoff:

$$C = e^{-r(T_1-t)}\mathbb{E}\big[\max\{0, C_{und}(S(T_1),K_2,T_2-T_1)-K_1)\}\big]. \tag{5.5}$$

Here, C_{und} is given by the Black-Scholes formula:

$$C_{und}(S(T_1),K_2,T_2-T_1) = S(T_1)e^{-q(T_2-T_1)}N(d_1) - K_2e^{-r(T_2-T_1)}N(d_2)$$

where

$$d_2 = \frac{\log\big(\frac{S(T_1)}{K_2}\big) + \big(r-q-\frac{\sigma^2}{2}\big)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}, d_1 = d_2 - \sigma\sqrt{T_2-T_1},$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. Also q is the dividend yield (for the purposes of this thesis, we will set $q = 0$), r is the interest rate, and σ is the volatility of the underlying asset, as we have seen before.

Geske evaluates the payoff of the CoC option, using partial differential equation techniques, to develop his argument. For a complete view of this evaluation, we refer to [38]. Eventually Geske arrives at the following formula for the price of the CoC compound option:

$$C = Se^{-q(T_2-t)}N_2(D_2^\star,D_2;\rho) - K_2e^{-r(T_2-t)}N_2(D_1^\star,D_1;\rho) - K_1e^{-r(T_1-t)}N(D_1^\star).$$

Here $N_2(\cdot,\cdot;\rho)$ is the bivariate standard normal CDF with correlation coefficient ρ, $\rho = \sqrt{(T_1-t)/(T_2-t)}$, $D_1 = D_2 - \sigma\sqrt{T_2-t}$, and $D_1^\star = D_2^\star - \sigma\sqrt{T_1-t}$ where

$$D_2 = \frac{\log\big(\frac{S}{K_2}\big) + \big(r-q+\frac{\sigma^2}{2}\big)(T_2-t)}{\sigma\sqrt{T_2-t}} \text{ and } D_2^\star = \frac{\log\big(\frac{S}{S^\star}\big) + \big(r-q+\frac{\sigma^2}{2}\big)(T_1-t)}{\sigma\sqrt{T_1-t}}.$$

By evaluating the payoffs of the other three types of compound options in a similar way, Geske obtains their values also. It is important to keep in mind that this model to compute the prices for the different types of compound options assumes constant volatility, which means it will tend to underestimate the price. The Matlab code for this valuation method (for the CoC option) can be found in Appendix C.14.

Trinomial approximation

To calculate the price of compound options using the trinomial model we use a very similar technique to what has been shown previously. Once we have calculated the payoff we can simply use our regressive algorithm to arrive at today's price (at time $t = 0$). However here we have to think one step further, as there is an underlying option we need to consider for the payoff. First we calculate the stock price tree up to time T_1, and then we need the prices of the second option for each value of $S(T_1)$. To calculate these we treat T_1 as if it were $t = 0$, and generate stock price trees up to time T_2 for each value of $S(T_1)$. Using these stock price trees we can then calculate the different prices of the second option at time T_1 as we have done before.

We shall see in the following section that this way of approximating the theoretical price of the CoC option appears to be quite accurate. The Matlab code for this approach can be found in Appendix C.13.

Numerical results

In the figure below we show the trinomial model approximation of the CoC option price as we increase the value of N. Here we have used the values $S(0) = 500$, $K_1 = 300$, $K_2 = 150$, $T_1 = 5/12$, $T_2 = 30/12$, $r = 0.05$, $\sigma = 0.3$, and $p = 0.3$. We also show the theoretical price, which of course is not affected by changes in the value of N. As can be seen in the figure below, the trinomial price clearly converges to the theoretical price as $N \to \infty$.

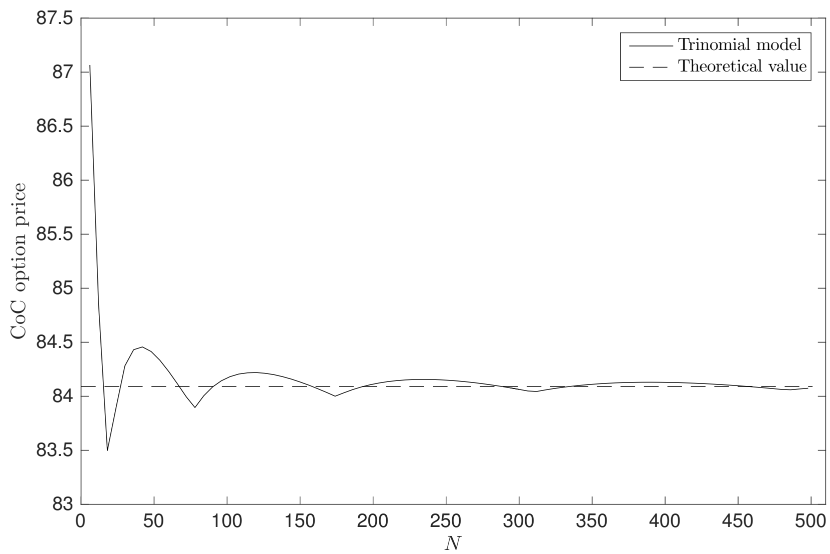

Figure 5.2: Trinomial model price (solid line) and the theoretical price (dashed line) of a CoC option as $N \to \infty$.

Interesting to note are the "wave-like" motions of the trinomial price in this figure. These are almost certainly caused by the way we handle the relation between the maturity times and N within our code. As there are no separately specified numbers of steps for the first and second options, we are forced to assign part of the total number of steps N to each of them. This involves some rounding of these numbers, which appears to be causing this behaviour of the trinomial price. Furthermore, we can see that the error of the trinomial price already becomes quite small (below 0.2) for approximately $N = 100$. For small N the approximation becomes quite unreliable, as is to be expected.

To be able to correctly judge the applicability of the trinomial approximation for CoC compound options, it becomes necessary to take computational time into account. In the table below we have compiled prices and computational times, for the theoretical pricing as well as the trinomial model pricing, for different values of N and σ. We use here the same variables as used in the figure above. The calculations were made on a desktop computer with 6 GB of RAM and a 2.5 GHz quad-core Intel Core Q8300 processor.

Table 5.5: Theoretical prices (TP) and trinomial model price approximations (TMP) of CoC options and computational times in relation to σ and N.

N	TP $\sigma = 0.1$	TMP $\sigma = 0.1$	TP $\sigma = 0.2$	TMP $\sigma = 0.2$	TP $\sigma = 0.5$	TMP $\sigma = 0.5$
$N = 100$	73.8754 (0.262 s)	73.9978 (3.848 s)	76.9026 (0.016 s)	77.1617 (2.209 s)	104.4289 (0.016 s)	105.3220 (2.010 s)
$N = 200$	73.8754 (0.014 s)	73.8103 (9.122 s)	76.9026 (0.012 s)	76.7931 (10.624 s)	104.4289 (0.014 s)	104.0508 (9.450 s)
$N = 300$	73.8754 (0.010 s)	73.8738 (20.141 s)	76.9026 (0.013 s)	76.8915 (20.301 s)	104.4289 (0.132 s)	104.5412 (20.807 s)

Clearly the computational time increases drastically for the trinomial price as we increase the value of N. This is quite intuitive considering the need to expand the trinomial tree accordingly which is quite consuming in terms of processing power. It is unclear from these experiments what effect changes in volatility has on computational time. It is also difficult to draw any conclusions regarding the changes in computational time for the theoretical price as we change N and σ. What is clear, however, is that the theoretical price is vastly more efficient than the trinomial model approximation, especially as we increase N to obtain more accurate results.

Discussion

It is clear that the trinomial model may be used to correctly approximate the theoretical value of a compound option, for example the CoC option as we have shown above. However this seems to be a suboptimal way of valuating the option, compared to simply using the theoretical valuation. The trinomial valuation is a lot more expensive in terms of processing time, even for lower values of N. Using our Matlab code (as seen in Appendix C.13 and C.14) the time to evaluate the CoC option price using the trinomial approach was significantly longer than Geske's closed form valuation model.

While approximating the theoretical price of the compound option may be an interesting experiment and theoretical exercise, it seems this approach is not to be recommended when attempting to price this type of option in practice.

5.4 Lookback options

A lookback option is an exotic option with a path dependency that allows the investor to take a look back at the historical price of the underlying asset. The payoff for a lookback call or put depends on the minimum or maximum stock price reached during the life time of the option [39]. It is possible to exercise the option based on the underlying asset's optimal value at any specific point. It is possible to break down the lookback option in two types; fixed strike and floating strike lookback options. Within the following section we only discuss European lookback options [39].

Introduction

Compared to standard European options, the strike price of a lookback option with floating strike is determined at maturity [39]. Lookback options with floating strikes are not really options, as they always will be exercised by the holder of the option.

This type of option is optimal when reducing uncertainties associated with the timing of the market. Compared to other standard European options the lookback option is not exercised at the market price of the underlying asset. A lookback option is instead exercised at the most optimal point in time. In the case of a call option, the option holder can take a look back over all the historical prices of the underlying asset and choose to exercise it at the one point where the underlying asset was priced at its highest over the option's life time. The opposite holds for a put option, which can be exercised at the lowest historical price of the underlying asset. The option settles the selected past market price against the floating strike [40].

As for the standard European options, the strike price is fixed at purchase [39]. The difference between standard European options and lookback options with a fixed strike is that, for the latter, the payoff is the maximum difference between the optimal underlying asset price and the fixed strike. The lookback option is actually exercised at the optimal price out of all prices reached during the life time of the option. For a call option, the payoff is fixed at the lowest price during the life time of the option. For a put option, it is fixed at the highest [40].

The main purpose of the lookback option is to help the investor with the a major issue involved in market timing [41]. This common issue is the difficulty involved when assessing when to enter and exit a position. Because of the way lookback options work, the issue of market timing becomes less important as profits are effectively guaranteed to be maximized. The chances of a contract expiring worthless are also much lower compared to other types of options. But for these reasons lookback options are generally more expensive compared to similar options, so the advantages are due to higher costs. Lookback options are not exchange traded products that are easily accessible on the various exchanges around the world, they are only bought and sold over the counter. The lookback option is often appealing to investors but they can be expensive and are also considered to be highly speculative.

As an example, suppose that a stock is currently trading at \$100, strike determined at expiration date. The stock falls to \$80 during the term of the contract and increases to \$115 by the expiration date. At the day of expiration the buyer of the lookback option will receive a cash settlement of \$115-\$80. The floating strike is set at the underlying asset's lowest price during the life of the option. So the lowest price is compared to the price of the stock at maturity [41].

Mathematical representation

The payoff of a lookback option depends on the path of the underlying asset during the life of the option. Let

$$M_t = \max S_u, u \in [0,t],$$

$$m_t = \min S_u, u \in [0,t],$$

where M_t is the maximum price and m_t the minimum price of the asset during the period $0 \leq u \leq t$. The representation of the four types of payoff functions for lookback options are given by:

- A call option (LC) with floating strike and payoff $LC_{float} = S_T - S_{\min}$
- A put option (LP) with floating strike and payoff $LP_{float} = S_{\max} - S_T$

- A call option (LC) with fixed strike K and payoff $LC_{fix} = \max\{S_{\max} - K, 0\}$
- A put option (LP) with fixed strike K and payoff $LP_{fix} = \max\{K - S_{\min}, 0\}$

where LC_{float} and LP_{float} are the payoff functions for a lookback call and put with floating strike, and LP_{fix} and LP_{fix} are payoff functions for lookback calls and puts with fixed strike.

Pricing methods

Here we will present the arbitrage-free price of lookback options with floating strikes using the Black-Scholes model. We assume here that the underlying asset follows a Geometric Brownian Motion, and that volatility remains constant [42]. The pricing method for a lookback option with a floating strike is a bit more complicated than for standard European options. Assume a risk-free rate $r \geq 0$ and a constant volatility $\sigma > 0$ for the underlying asset. The time to maturity is $T > 0$ and the option is priced at time t where $t < T$ [39]. The Black Scholes price for vanilla option is not dependent on the number of steps N, but the Black Scholes price for a lookback call option is dependent due to that the maximum (M) and minimum (m) price are dependent on N. This makes the calculations more difficult.

The arbitrage-free price at time t for a lookback call option with a floating strike is given by [39][41]

$$LC_{float} = S\Phi(a_i(S,m)) - me^{-r(T-t)}\Phi(a_2(S,m)) - \frac{S\sigma^2}{2r}(\Phi(-a_1(S,m)) \\ -e^{-r(T-t)}(\frac{m}{S})^{\frac{2r}{\sigma^2}}\Phi(-a_3(S,m))),$$

and similarly, the arbitrage-free price of the lookback put option with floating strikes is given by [39][41]

$$LP_{float} = -S\Phi(-a_i(S,M)) + Me^{-r(T-t)}\Phi(-a_2(S,M)) + \frac{S\sigma^2}{2r}(\Phi(a_1(S,M)) \\ -e^{-r(T-t)}(\frac{M}{S})^{\frac{2r}{\sigma^2}}\Phi(a_3(S,M))),$$

where, $S = S_t, M = M_t, m = m_t$ and

$$a_1(S,H) = \frac{\log\frac{S}{H} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$a_2(S,H) = \frac{\log\frac{S}{H} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = a_1(S,H) - \sigma\sqrt{T-t},$$
$$a_3(S,H) = \frac{\log\frac{S}{H} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = a_1(S,H) - \frac{2r\sqrt{T-t}}{\sigma},$$

and where $\Phi(\cdot)$ is the standard normal cumulative distribution function:

$$\Phi(a) = \frac{1}{\sqrt{2\phi}}\int_{-\infty}^{a} e^{-\frac{-x^2}{2}}dx.$$

Numerical results

In order to evaluate how well the trinomial model performs, it has been compared to both the theroetical price and Monte Carlo method. We consider a lookback with T= 1/4,σ= 0.3, p=0.3, r=0.05 S0=100. This is a standard lookback contract studied in several articles [39][42]. The calculations were made on a computer with 16 GB 2.8 GHz of RAM and a 2.8 GHz dual-core Intel Core i5 processor.

Table 5.6: Lookback prices and computational times in relation to N.

N	Trinomial model price	Black-Scholes price
$N = 10$	15.0455 (0.3629 s)	17.7709
$N = 13$	14.6798 (9.5517 s)	17.6767
$N = 16$	14.4180 (260.2545 s)	17.5976

The Matlab code for the trinomial model can be found in Appendix C.15 and Appendix C.16. The Black Scholes prices has been calculated with the modified formula for the Black Scholes price of a lookback option with a floating strike.

Discussion

As seen in Table 5.6 calculating the price of a lookback put option using the trinomial model is not an efficient method. The trinomial valuation is a lot more inefficient in terms of processing time, even for very low values of N. When $N = 16$ the time for calculations reaches 260 seconds and it becomes obvious that larger calculations will take a very long time. As can be seen in the table above, the error at such low values of N seems to be very large which indicates that the trinomial model for this type of calculations is inefficient. It is not obvious to see that the trinomial price converges to the theoretical price.

5.5 Bermudan options

The basic principle of Bermudan options is that they can be exercised at a finite set of times prior to the expiration date of the option [43]. This section will discuss the usage and implications of these principles, primarily with regards to theoretical pricing.

Introduction

As the reference to the Bermudan islands implies, Bermudan options can be viewed as an intermediary between standard European and American options. [43] While the American option can be exercised at any time point between present time and the expiration date T, and the European can only be exercised at time T, the Bermudan option offers exercise opportunities prior to T but only at predetermined dates.

A Bermudan option can apply to a variety of underlying assets, however, this option type tends to be most frequently used with foreign exchange and interest rate contracts [44]. These are mostly referred to as swaptions, options on interest rate swaps. If one wishes to invest in swaptions, the Bermudan characteristics of the derivative may be desirable. This is because the holder has the opportunity to exercise the derivative earlier than planned if deemed necessary, instead of having to wait until maturity T. Of course, a Bermudan swaption is logically priced higher than a European one, as a consequence of these added perks, but the amount of risk mitigated may be considered to outweigh the increased price.

In order to illustrate the nature of Bermudan options, assume a trinomial price tree with expiration date $T = 3$. Let the derivative be a Bermudan option with the possibility to exercise it prior to expiration at time $t = 1$ and, of course, at the normal expiration date $T = 3$. Let f be the expected pay-off at each node under the risk neutral probability measure, discounted with the risk free interest rate r. For each node, the respective f will constitute the option price since the fair price f of European, as well as American, options is a deterministic function of $S(t)$.

We can illustrate these functions in a trinomial tree. In the tree below, for simplicity, we have assumed that $u = -d$. The bolded functions serve to illustrate where the derivative is European and where it becomes American.

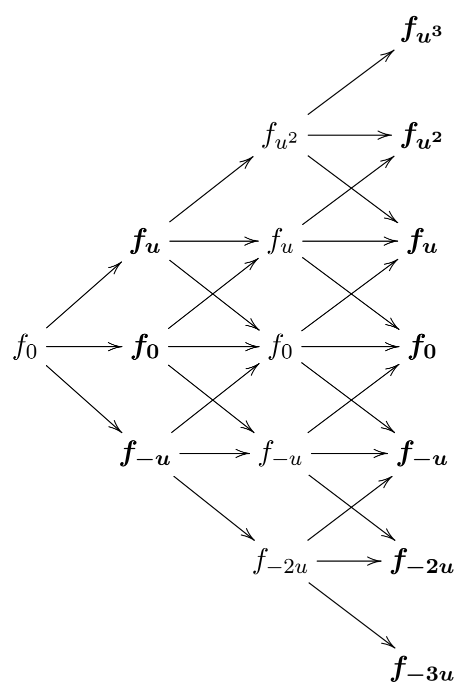

Figure 5.3: Example of trinomial tree for Bermudan options. f denotes the expected payoff at the given node under the risk neutral probability measure, discounted with the risk free interest rate r. When f is bold the payoff is American, else it is European.

This means that the Bermudan option switches between the characteristics of American and European options at every time step until maturity. Since we can conclude that sometimes the Bermudan option will act American and sometimes European, intuition would lead us to believe that the theoretical valuation of a Bermudan option should always lie between these respective values. Another notion of intuition with regards to this is that the Bermudan is obviously less flexible than the American yet more so than the European one. Flexibility, in this regard, corresponds to the amount of possible exercise dates. In conclusion, however, we may to this point believe that

$$\Pi_{European} \leq \Pi_{Bermudan} \leq \Pi_{American}$$

where Π is the price of the derivatives. We will later investigate this numerically and see that it in fact holds.

Mathematical representation

A Bermudan option is a pair (U,R) where U is the payoff function for the option and $R \subset [0,T]$ is the region of permitted exercise dates [43]. This implies that the Bermudan option is a hybrid with respect to t, and thus must be treated differently depending on whether we, at a given time, are permitted to exercise the option or not. When we are, we may treat the option as an American one with regards to valuation. This also implies that the time point t_i is one of the time steps $t_i \in [0,T]$ in which we can exercise the option. When the time t is not a permitted date to exercise the Bermudan option, we treat it as a European option with regards to valuation [43].

Since Bermudan options are essentially a mix of European and American characteristics, we briefly remind ourselves about their payoff functions. At the expiration date, N is the same for standard American and European derivatives, i.e.

$$Y(x) = g(S(N) - K) = \left(S_0 e^{\left(N_{+1}(x) - N_{-1}(x)\right)u} - K\right)_+$$

in the case of a call and

$$Y(x) = g(K - S(N)) = \left(K - S_0 e^{\left(N_{+1}(x) - N_{-1}(x)\right)u}\right)_+$$

in the case of a put [16], where the standard denotations apply. This, in turn, allows us to be general with respect to the maturity date T in the case of Bermudan options. Regardless of its region R of permitted exercise dates, it will always behave as the European and American derivatives at maturity.

Pricing method

We shall now describe the pricing procedure when adopting the trinomial model. In research today, the binomial model is also frequently used [45]. These models are popular as the option is not path dependent and thus can be solved analytically using these tools.

Proceeding backwards from maturity is when the pricing starts to differ from the European and American derivatives. In the case of the Bermudan option we continuously must identify the nature of the previous time step t. If $t \in R$, then we calculate the price as if it were American, i.e

$$\Pi_Y(t,q_0) = \max\left(Y(x,t), e^{-r}\left[q_{+1}\Pi_Y^{+1}(t+1,q_0) + q_0\Pi_Y^{0}(t+1,q_0) + q_{-1}\Pi_Y^{-1}(t+1,q_0)\right]\right),$$

where $Y(x,t)$ is the payoff at time t. Furthermore, we calculate they price according to

$$\Pi_Y(t,q_0) = e^{-r}\left[q_{+1}\Pi_Y^{+1}(t+1,q_0) + q_0\Pi_Y^{0}(t+1,q_0) + q_{-1}\Pi_Y^{-1}(t+1,q_0)\right]$$

for none-permitted time steps to exercise. Here we shall assume that q_0 is fixed by

$$q_0 = 1 - 2p$$

by imposing convergence to Black-Scholes. Repeating these steps until time $t = 0$ throughout the trinomial tree, one eventually yields the initial price of the Bermudan option.

Numerical results

As one may conclude, the pricing of Bermudan options is similar to pricing of standard European and American ones, with the extra requirement of identifying what sort of derivative the option becomes at each time step. Thus, for sufficiently small trinomial trees, the pricing can of course be solved analytically. It does, however, become computationally complex rather quickly which is why the relations between European, Bermudan, and American options will be demonstrated through numerical calculations. Below, in Figure 5.4 we have calculated the initial prices of the corresponding call derivatives.

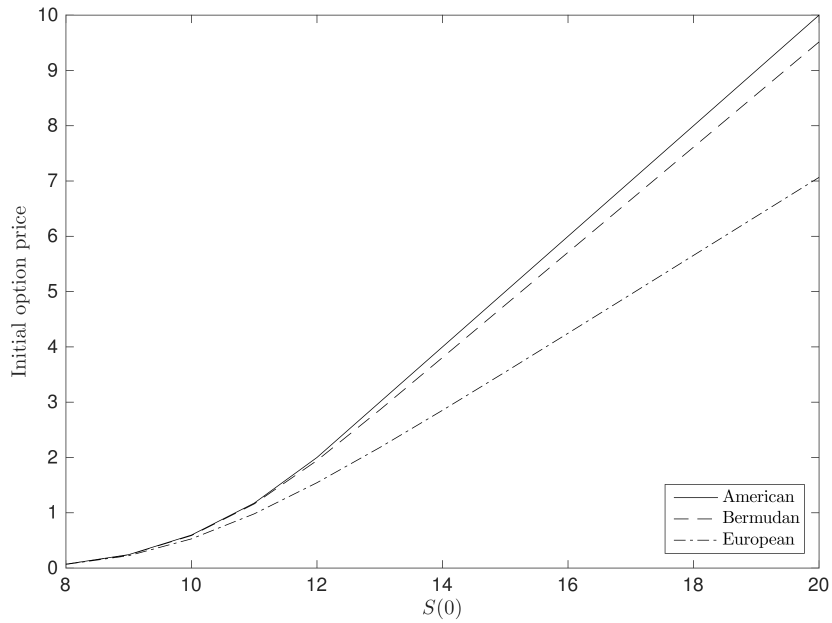

Figure 5.4: Price differences between derivatives for increasing initial stock value.

The calculations were based on a 36-period model with six possible exercise dates for the Bermudan option. The Matlab code for this experiment can be found in Appendix C.17. We can see that the relations between the three derivatives (American, European, and Bermudan) satisfy the following condition

$$\Pi_{European} \leq \Pi_{Bermudan} \leq \Pi_{American}$$

which is intuitively clear. For each time step that the Bermudan option is permitted to be exercised prior to expiration it will either be equal to, or larger, than the price of the European option. At the same time it will always be less or equal to the American one, depending on whether it behaves like a European or American option at the given time. Of course, as was mentioned previously, one may also argue for this since the Bermudan option is less flexible than the American option yet more so than the European one with regards to possibility to exercise. Thus it is logical that it is priced between these derivatives.

Furthermore, we shall make a comparison between the binomial and trinomial models. Similarly to what has been done before, we have computed the initial prices of Bermudan options for increasing periods N. In Figure 5.5 below follows the results from the binomial, as well as the trinomial model pricing.

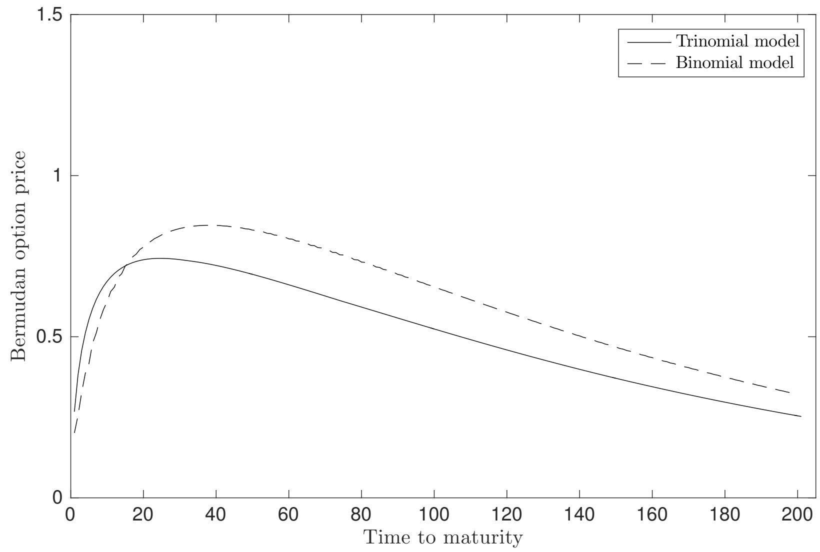

Figure 5.5: Comparison between binomial and trinomial price for Bermudan option.

In the experiment we have used a Bermudan option with strike $K = 10$, $S_0 = 10$ and six possible exercise dates in a 150 period model. The Matlab code used in this experiment can be found in Appendix C.17 and C.18. We note that the trinomial model generates a higher price than the binomial model, but only while N remains sufficiently small. As N increases, the binomial model eventually starts generating a higher price after which both prices seem to linearly decrease with a constant distance between them. Through numerous computations with different values of q_0, we have found that this difference between the models converges to zero as q_0 approaches zero. This is logical since q_0 approaching zero essentially means that the trinomial model becomes binomial. Furthermore, the continuous decrease in price for both models can be explained by the fact that while N increases, the amount of exercise dates stays the same, thus the derivative resembles a European option to an increasing extent. However, since there is no closed formula to calculate the Bermudan option price, it is difficult to say which model best converges to the correct theoretical price.

Discussion

We have seen that the price of the Bermudan option lies in between European and American option prices. In terms of methods to price the derivative, the applicability of the trinomial model on Bermudan options is clear. It is appropriate to use as the general uncertainty in the region of permitted exercise dates R makes it impossible to derive a closed formula for the theoretical option price.

Furthermore, since the binomial model is often employed when pricing Bermudan options, we have made a comparison between this model and the trinomial model. The findings show that the latter initially generates a higher price than the former, but as N increases the dynamics change and the binomial model eventually generates the higher price. As mentioned before, in the absence of a closed formula for the theoretical price, one can not easily determine which model calculates the more accurate price. However, since the trinomial model offers higher flexibility with regards to possible outcomes of the stock value, it seems logical that this model continuously would capture the more reliable result out of the two.

5.6 Barrier options

Barrier options are dependent on a pre-specified option. The payoff of the barrier option is identical to the option which it is dependent of if the barrier option is active. If the barrier option is inactive, the payoff is zero. Whether the barrier option is active or not is determined by the type of barrier option. In this section we will study the application of the trinomial model to price the barrier option on the standard European options.

Introduction

Barrier options are options whose payoff existence depend on whether the price of the underlying asset has reached a predefined asset price, or barrier. We will call a barrier option active if the payoff exist, else we will call it inactive. There are mainly two types of barrier options; knock-out options and knock-in options. For knock-out options to be active, the price of the underlying asset must not reach a certain barrier to get "knocked out". Knock-in options are the corresponding opposite, for a knock-in option to be active the price of the underlying asset must reach a certain barrier to get "knocked in". If the barrier option is active, its payoff is identical to the payoff of which the barrier option is on, otherwise the payoff is zero. An example of a knock-in barrier option is the up and in barrier option on a European call. Initially the option is inactive and has zero payoff; in order for the option to become active, the underlying stock must reach a predefined barrier. If the barrier option becomes active it stays active and its payoff is identical to the payoff of the European call [4].

The barrier options on the European options can be regarded as a more precise version of the European option in question. When using barrier options more accurate predictions are required since a part of the underlying asset domain is cut off, which also means that they are riskier. Due to the increased risk, barrier options are often cheaper than the corresponding unbounded option. There are several reasons to use barrier options over regular European calls or puts, these are some of the reasons [24].

The payoff for the barrier option may match beliefs about future behaviour of the market more closely. Using barrier options, one can cut off stock prices one finds unlikely and only pay for the scenarios one thinks are probable. For example, if one believes there is a high chance for a stock to increase to a price more than 105% of the current price in one year, but also believes that if the stock price ever decreases to 95%, then it will decline further and not come back. Then one can buy a down and out call with barrier at 95% of the current stock price and strike at 105% of the current stock price. In this way we only pay for scenarios we think are profitable enough.

Barrier options are also naturally cheaper than the corresponding standard options since the barrier options are exposed to more risk. Consider a European knock-out call, if one is able to predict a barrier level such that the option does not get knocked out, one will get the benefits of the regular call counterpart but for a lower premium.

Mathematical representation

Common barrier options are the down-and-out, down-and-in, up-and-out, up-and-in barrier options on regular European calls or puts. We shall study these more closely in this section.

Let B denote the barrier, K the strike-price of the European call or put. Furthermore, let $S(t)$ be the stock price at time t and let $m_s = \inf\{S(t),t \in (0,T)\}$ $M_s = \sup\{S(t),t \in (0,T)\}$, the payoff Y for the different types of barrier options on the european calls and puts is then given by:

- Up-and-out (UO)
  Active while asset spot price $< B$

$$Y_{UOcall} = \begin{cases} (S(T) - K)_+ & M_s < B \\ 0 & M_s \geq B \end{cases} \tag{5.7a}$$

$$Y_{UOput} = \begin{cases} (K - S(T))_+ & M_s < B \\ 0 & M_s \geq B \end{cases} \tag{5.7b}$$

- Down-and-out (DO)
  Active while asset spot price $> B$

$$Y_{DOcall} = \begin{cases} (S(T) - K)_+ & m_s > B \\ 0 & m_s \leq B \end{cases} \tag{5.8a}$$

$$Y_{DOput} = \begin{cases} (K - S(T))_+ & m_s > B \\ 0 & m_s \leq B \end{cases} \tag{5.8b}$$

- Up-and-in (UI)
  Inactive while asset spot price $< B$

$$Y_{UIcall} = \begin{cases} 0 & M_s \leq B \\ (S(T) - K)_+ & M_s > B \end{cases} \tag{5.9a}$$

$$Y_{UIput} = \begin{cases} 0 & M_s < B \\ (K - S(T))_+ & M_s \geq B \end{cases} \tag{5.9b}$$

- Down-and-in (DI)
  Inactive while asset spot price $> B$

$$Y_{DIcall} = \begin{cases} 0 & m_s > B \\ (S(T) - K)_+ & m_s \leq B \end{cases} \tag{5.10a}$$

$$Y_{DIput} = \begin{cases} 0 & m_s > B \\ (K - S(T))_+ & m_s \leq B \end{cases} \tag{5.10b}$$

In the knock-out options the barrier B must be defined such that the option is initally active and vice-versa for the knock-in options. It is important to note that when the option gets actived or inactivated, it stays that way.

Pricing methods

Due to the form of barrier options, the trinomial model is easily applied to price barrier options and there exists a closed formula for the theoretical price of barrier options on the European calls and puts.

Theoretical price

Theorem 5.6.1. *The Black-Scholes price of the down and out barrier option of a European call is given by*

$$c_{DO}(t,S) = \begin{cases} c_v(t,S) - \left(\frac{S}{B}\right)^{2\alpha} c_v(t,B^2/S), & K > B \\ c_v(t,S) + (B-K)c_d(t,S) - & \\ \quad \left(\dfrac{S}{B}\right)^{2\alpha} (c_v(t,B^2/S) + (B-K)c_d(t,B^2/S)), & K < B \end{cases}$$

where c_v is the corresponding standard European call, B is the barrier, K is the strike price and $\alpha = -\frac{r}{\sigma^2} + \frac{1}{2}$. c_d is the standard digital call, with barrier at B paying 1\$ if the underlying asset reaches the barrier.

Proof. We will follow the procedures done in [46]. If the asset price S is above the barrier, the pay-off behaves just like an regular European call option which of course satisfies the Black-Scholes PDE

$$\frac{\partial c_{DO}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c_{DO}}{\partial S^2} + rS\frac{\partial c_{DO}}{\partial S} = rc_{DO}, \tag{5.11}$$

for $B < S < \infty$ and $c_{DO}(S,T) = \max(S-K)^+, B < S < \infty$. If the asset price reaches the barrier B, the option ceases to exist and becomes worthless, we can translate this into a boundary condition on the PDE, $c_{DO}(B,t) = 0$. Hence the barrier option is essentially a regular option with the extra constraint $c_{DO}(B,t) = 0$.

The Black-Scholes formula can, after several change of variables, be transformed to the familiar heat-equation. With

$$S = Be^x, \tag{5.12a}$$

$$t = T - \tau/\frac{1}{2}\sigma^2, \tag{5.12b}$$

$$\alpha = -\frac{r}{\sigma^2} + \frac{1}{2}, \tag{5.12c}$$

$$\beta = -\frac{r^2}{\sigma^4} - \frac{1}{4} - \frac{r}{\sigma^2}, \tag{5.12d}$$

$$c_{DO} = Be^{\alpha x + \beta\tau}u(x,\tau). \tag{5.12e}$$

Our problem becomes reduced to

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < \tau, \quad 0 < x < \infty \\ u(x,0) = U(x) = \max(e^{x(1-\alpha)} - \dfrac{K}{B}e^{-\alpha x},0), \quad x > 0. \\ u(0,\tau) = 0 \end{cases} \tag{5.13}$$

In other words, we have the heat equation for an infinite long bar. $u(x,0)$ is the inital heat distribution over the bar and $u(0,\tau) = 0$ means that the temperature at the end of the bar ($x = 0$) is always zero. Since the heat equation is invariant under reflection in x, i.e. if $u(x,\tau)$ is a solution then so is $u(-x,\tau)$, we know that there exists a solution for $x < 0$. Hence we can solve our problem (5.13) by the method of images. We can replace the $u(0,\tau) = 0$ condition by a reflection which causes the heat to cancel each other at

$x = 0$ and solve our problem for all x, instead of just $x > 0$, that is to say we get the problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial x^2}, \ \ 0 < \tau, \ \ 0 < x < \infty \\ u(x,0) = \begin{cases} U(x), \ \ x > 0 \\ -U(-x), \ \ x < 0. \end{cases} \end{cases} \tag{5.14}$$

We can solve this problem by considering a standard European call with the same expiry and strike price but without barrier. We denote its value $c_v(S(t),K)$ and let $U_v(x,\tau)$ be the corresponding solution for the heat-equation.

When $S < K$ $c_v(S(T),K) = 0$. Thus, since $S = Be^x$, when $x < \log\left(\frac{K}{B}\right)$ then $S < K$ and $U_v(x,0) = 0$. If we assume that the strike price $K > B$, then $\log\left(\frac{K}{B}\right) > 0$. Thus, if we set $U(x) = 0$ for $x < 0$, $U(x)$ becomes defined $\forall x$ and $U(x) = U_v(x)$. We can now write

$$u(x,0) = U_v(x,\tau) - U_v(-x,\tau) \ \ \forall x.$$

Thus,

$$u(x,\tau) = U_v(x,\tau) - U_v(-x,\tau), \tag{5.15}$$

which precisely is the solution of our problem. Since the newly obtained problem satisfies the same conditions with the same PDE, by the uniqueness of solutions for the heat-equation the problems must be equivalent. Since

$$c_v(S,t,K) = c_v(Be^x,t(\tau),K) = He^{\alpha x + \beta\tau} U_v(x,\tau), \tag{5.16}$$

we obtain

$$U_v(x,\tau) = e^{-\alpha x - \beta\tau} c_v(Be^x,t(\tau),K)/B, \tag{5.17a}$$

$$U_v(-x,\tau) = e^{\alpha x - \beta\tau} c_v(Be^{-x},t(\tau),K)/B. \tag{5.17b}$$

If we now substitute back (5.15) into (5.12a) using (5.16), (5.17a,b) we obtain

$$c_{DO}(S(t)) = c_v(S(t),K) - \left(\frac{S}{B}\right)^{2\alpha} c_v(B^2/S(t),K), \tag{5.18}$$

which holds when the strike price is above the barrier $K > B$.

If the barrier is above the strike, $B > K$, then the reflected solution $U(-x)$ does no longer vanish above the barrier at expiry since the regular call does not vanish below the barrier. To fix this we let the pay-off for the vanilla call be zero for $S < B$. The new payoff of this modified vanilla call becomes

$$c_v(S(t),B) + (B - K)c_d(S(t),B).$$

Where $c_d(S(t),B)$ is a standard digital call, with barrier at B paying 1\$, i.e. if the stock reaches the barrier, we get 1\$, else nothing.

Hence, after reflection, we get

$$\begin{aligned} c_{DO}(S(t),B) = & c_v(S(t),B) + (B-K)c_d(S(t),B) - \\ & \left(\frac{S(t)}{B}\right)^{2\alpha} (c_v(B^2/S(t),B) + (B-K)c_d(B^2/S(t),B)), \end{aligned}$$

which is valid when $B > K$, which completes the proof. □

Since the value of the regular call equals the sum of a down-and-in call and down-and-out with the same strike price that is, $c = c_{DO} + c_{DI}$, we can easily obtain the value of the corresponding down-and-in call. The price of the up-and-in, up-and-out calls can be obtained in a similar fashion. We also note that digital options, europeans calls and puts have all closed forms hence there exists a closed form for the barrier options on European calls and puts [4].

Pricing using the trinomial model

Consider the the down-and-out barrier option on a European call with barrier $B = 97.5$ on a stock with the properties $T = 30/365$, $S_0 = 100$, $K = 95$, $r = 0.1$, $\sigma = 0.2$, and $B = 97.5$. To price this barrier option using the trinomial model we simply add the condiditon that if the stock ever reaches the barrier, the value of the option becomes equal to zero. This translates in to the boundary condition $c(B,t) = 0,\ t \geq 0$ and we get the following modified tree. Since the option behaves as an European call while active we simply apply our recurrence formula on a modified trinomial tree similar to example in Figure 5.6.

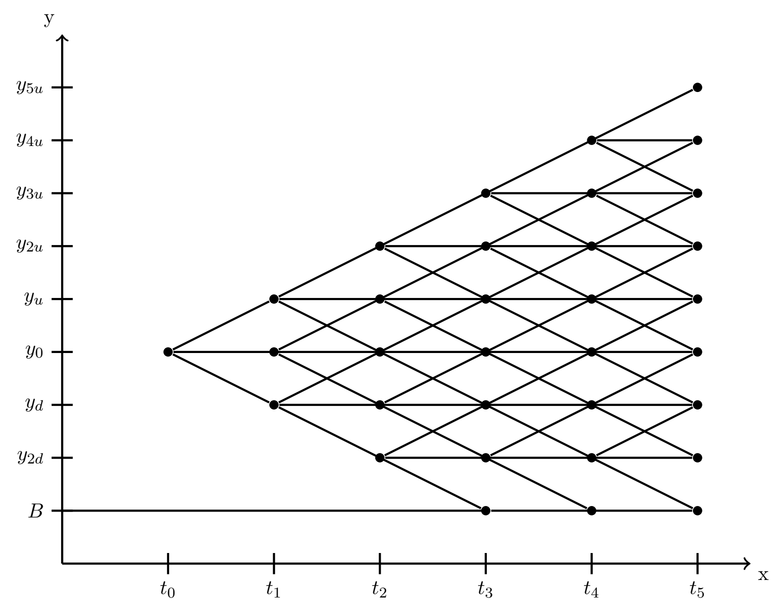

Figure 5.6: An example of trinomial tree for the European down-out-call with five time steps.

The payoff $Y_{DOcall}(T)$ is identical to the European call if the stock has not reached the barrier, that is

$$Y_{DOcall}(T) = \begin{cases} (S(T) - K)_+ & S(T) > B \\ 0 & S(T) \leq B. \end{cases} \tag{5.20}$$

Ideally we want to make one level of the possible stock prices to coincide with the barrier to easily determine whether the barrier option has been deactivated or not. That is, we want $B = S_0 e^{-ux_b} = S_0 e^{-\sigma\sqrt{\frac{h}{2p}}x_b}$ where x_b is an integer. Since h,p,σ already are determined, we choose $x_b = floor\left(\log(\frac{S_0}{B})/u\right)$ in Matlab. $floor(\cdot)$ gives the closest integer, less than or equal to the argument, hence we get an lower approximation of x_b, which means that the barrier is shifted upwards slightly compared to the actual barrier.

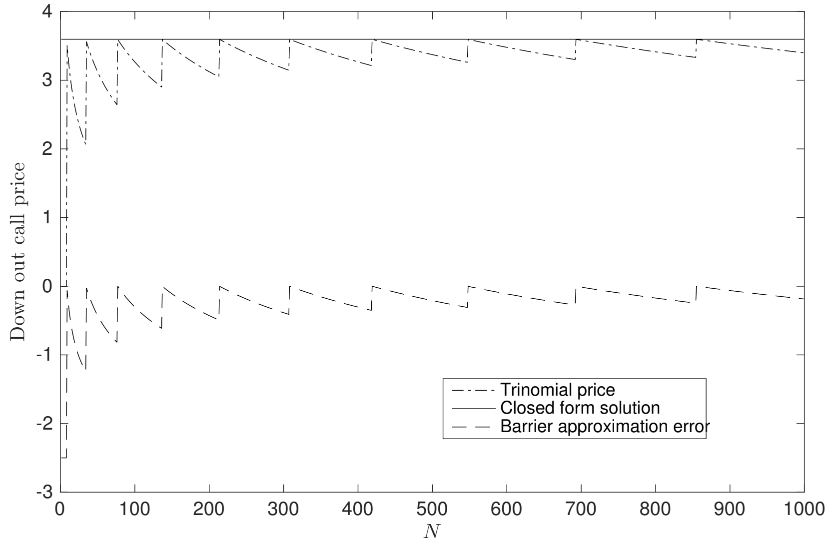

Figure 5.7: The trinomial price and Black-Scholes price of the European down-and-out call for increasing number of steps N, and $p = 0.3$. The barrier approximation error, $B - S_0 e^{-ux_b}$ are outlined as well, where x_b is approximated with $floor(\cdot)$.

In Figure 5.7 we have used the parameters $T = 30/365$, $r = 0.05$, $S_0 = 100$, $K = 95$, $B = 97.5, \sigma = 0.2$. The Matlab code in Appendix C.19 is identical to the code in Figure 5.7, except that we have used $floor(\cdot)$ to approximate x_b instead of $round(\cdot)$. We can see that the spikes in the trinomial price align almost perfectly with the barrier approximation errors, which suggest that the spike fluctuations are directly linked to the barrier approximation errors. We observe that in the case of the down-out-call, when the barrier is high and close to the current price S_0 the option becomes cheaper since less of the possible paths for the underlying asset are allowed which makes the option riskier and cheaper. When the barrier is far from the current underlying asset price the possible paths are many and the option becomes less risky hence more expensive compared to the previous case. This explains why the trinomial price moves with the barrier errors; when the barrier errors increase, the trinomial model prices the barrier option for a barrier that is higher than the actual barrier which makes the trinomial price cheaper than the actual price.

Since x_b always is given a lower approximation the barrier $S_0 e^{-ux_b}$ is always approximated to arrive earlier than the actual barrier. Hence as the error increases, the option becomes cheaper than the actual price and we get perfect barrier approximations at the top spikes of the trinomial price as we can see in Figure 5.7. With this in mind we notice that is more effective approximate x_b using $round(\cdot)$ instead of $floor(\cdot)$. $round(\cdot)$ approximates the argument to the closest integer, hence the absolute value of the error in x_b will be in [0,0.5) using $round(\cdot)$ and [0,1) using $floor(\cdot)$.

We will now investigate how the barrier B affects the trinomial price convergence. To get a quick view of how the trinomial price behaves for different barriers B we have outlined the trinomial price and Black-Scholes price of the down out call for different barriers, B, in Figure 5.8 and Figure 5.9 using $round(\cdot)$ respectively $floor(\cdot)$. In Figure 5.8, 5.9 we have used the parameters $T = 30/365,\ \ r = 0.05,\ \ S_0 = 100,\ \ K = 70,\ \ \sigma = 0.2,\ \ p = 0.3$ and barriers $B = 85, 88, 91, 94, 97$.

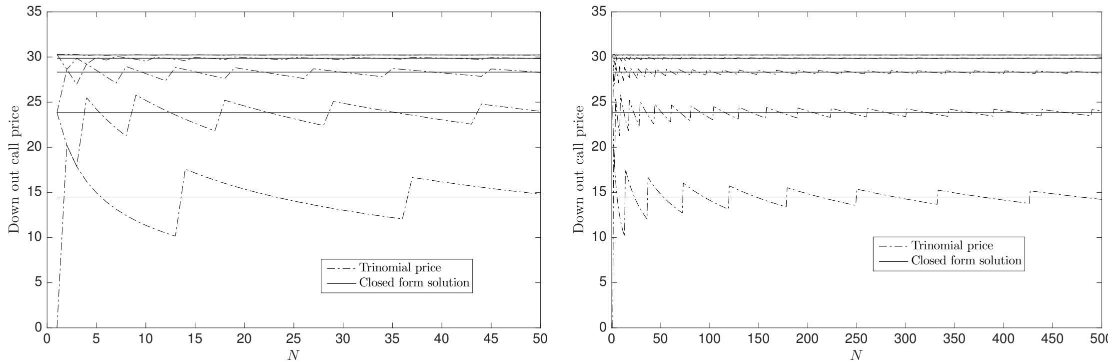

Figure 5.8: Trinomial price with x_b approximated by $round(\cdot)$ and Black-Scholes price of the European down-out-call. Since the down-and-out call price increases as B decreases, the most expensive down-out-call has $B = 85$ and the least expensive option has $B = 97$.

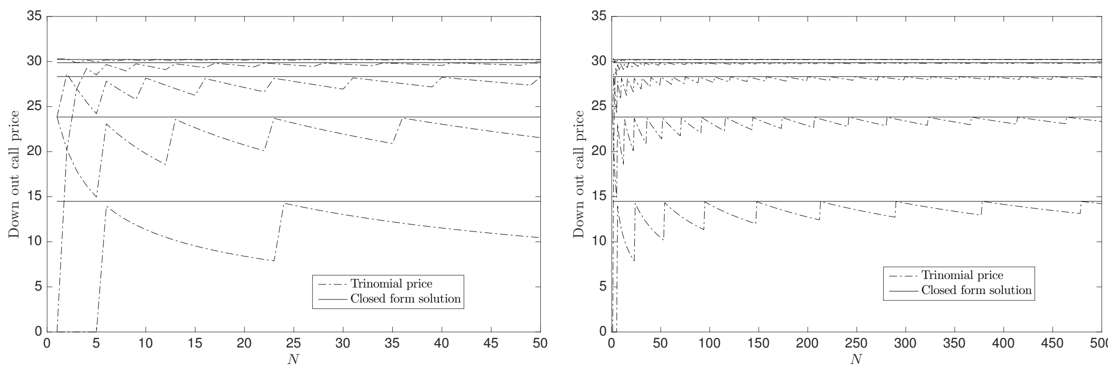

Figure 5.9: Trinomial price with x_b approximated by $floor(\cdot)$ and Black-Scholes price of the European down-out-call. Since the down-and-out call price increases as B decreases, the most expensive down-out-call has $B = 85$ and the least expensive option has $B = 97$.

In Figure 5.8 we see that the trinomial price converges almost instantly when B is far from the current price S_0 and that the barrier approximation error has a low impact on the price. As discussed earlier we know that when using $floor(\cdot)$ the barrier approximation is perfect at the top of the spikes. Looking at Figure 5.9 we see that the down out call converges almost instantly when B is close to S_0 as well, but here the impact of the barrier approximation errors are much bigger and in order to get the barrier approximation errors small we require many iterations.

In Table 5.7 we have compared the computational time of the trinomial price with the computational time of the Black-Scholes price. Matlab code for the Black-Scholes price of the down and out call can be found in Appendix C.20. It can be shown that it is nummerically most effective to use $p = 1/6$ in the trinomial model [4], hence we will use $p = 1/6$.

Table 5.7: Computational times of the trinomial model and Black-Scholes price of the European down-out-call with $T = 30/365,\ r = 0.05,\ S_0 = 100,\ K = 70,\ \sigma = 0.2,\ p = 1/6$ for different number of iterations N and barrier values B.

N	B=85	B=88	B=91	B=94	B=97
10	41.5μs (54.99%)	34.3μs (56.98%)	31.6μs (58.89%)	32.7μs (58.11%)	31.5μs (60.54%)
20	44.5μs (51.77%)	47.2μs (40.83%)	43.1μs (43.93%)	42.9μs (45.02%)	41.3μs (48.38%)
30	64.3μs (30.22%)	67.4μs (15.48%)	60.0μs (21.89%)	58.9μs (24.60%)	57.7μs (27.83%)
40	90.0μs (2.40%)	91.1μs (-14.15%)	85.0μs (-10.70%)	82.1μs (-5.11%)	78.4μs (1.84%)
50	128.2μs (-39.07%)	118.5μs (-48.53%)	116.1μs (-51.08%)	114.9μs (-47.14%)	105.5μs (-31.97%)

In Table 5.7 we have in parentheses, the computational time difference between the trinomial and closed formula price as percentage of the computation time of the closed formula price. If the difference percentage is negative the closed formula price computation time is shorter than the trinomial price computation time. We also display the time to compute the trinomial price in microseconds ($10^{-6}s$). The computations were done on a computer with a 2.3GHz Intel Core i7-4712MQ processor and 8 GB 1600MHz RAM. The computational times were obtained from the mean of 1000 runs and the variance of the computation times were at most of order $0.1\mu s$.

If we were to calculate the trinomial price error directly from the given trinomial price for a specific N, we would most likely get a lower approximation of the trinomial price error due to the fluctuations of the trinomial price, that is we might get a very good value even if the trinomial price fluctuates violently around the observed N. Since we are looking at such small values of N it is futile to look at the variance of the for example last ten points of the trinomial price with respect to N, and especially for the first values of N, $N \in \{1, \dots 10\}$ since the trinomial price might not have converged yet. Furthermore, since the length of the spikes varies with N, we do not either know how many steps back in N we need to take in account in order to get whole spikes to compute the variance or standard deviation of.

However since the trinomial model converges fast as seen in Figure 5.8, we have decided to only look at the trinomial model error induced by the barrier approximation errors. Since there exists a closed formula for the down and out call we can calculate the maximum error in the trinomial price from the barrier errors, under the assumption that the trinomial price has converged. The error in the trinomial price from the barrier approximation error will be given by:

$$c_{DO}(B) - c_{DO}(S_0 e^{-x_b u})$$

where we can choose the worst barrier approximation, i.e. $x_b = \log(S_0/B)/u + 0.5$.

Table 5.8: Here we have calculated the theoretical trinomial price percentage error due to maximum barrier approximation errors for different values of N and B with $T = 30/365,\ \ r = 0.05,\ \ \sigma = 0.2,\ \ S_0 = 100,\ \ K = 70,\ \ p = 1/6$.

N	B=85	B=88	B=91	B=94	B=97
10	0.127%	0.796%	3.409%	11.302%	37.214%
20	0.101%	0.621%	2.607%	8.474%	27.376%
30	0.087%	0.530%	2.204%	7.100%	22.738%
40	0.077%	0.471%	1.949%	6.244%	19.890%
50	0.071%	0.429%	1.768%	5.643%	17.912%

As we saw in Table 5.7, for the parameters used in Table 5.8 the trinomial price is computed faster than the theoretical price for $N = 30$ and below. In Table 5.8 we see that, when the barrier approaches $B = 91$ the approximation errors are rather small for $N = 30$ and it might be a good idea to price the barrier option using the trinomial model. When we increase B, it gets closer to S_0 and the barrier approximation errors have a greater impact on the trinomial price. To reduce the barrier approximation error we require large number of iterations as we can see in Figure 5.9 and it might be better to price the barrier option using the closed formula instead. By using an adaptive mesh it is possible to make a more precise barrier approximation and the trinomial model might be an efficient pricing method even when the barrier is close to the current asset price S_0 [4].

Discussion

Due to the form of the barrier options on European calls and puts, the trinomial model is easily applied to price the european barrier options, and due to the similarities there also exist a theoretical price. The trinomial price converges fast and is computed fast with small errors when the barrier B is far from the current stock price S_0. The barrier options carry a higher risk to the holder compared to the corresponding options, which implies both advantages and disadvantages. With barrier options one is able to make profit from more precise predictions of the market with the price of higher risk as this implies.

Conclusion

The purpose of this thesis was to investigate the trinomial asset pricing model through three main topics. First, by examining its properties. In doing so, we derived the fair price of a European derivative by using self-financing hedging portfolios, and we also examined the incompleteness of the model. The second topic of the investigation was a study of the conditions under which the trinomial price converges to the Black-Scholes price. Finally, we applied the model to six different exotic options, in order to determine whether the model is well suited for pricing them.

Through numerical studies of the convergence to Black-Scholes, we have found a clear advantage of the trinomial model compared to the binomial one. This follows by observing Figure 4.1, for $p \in (0.17, 0.5)$. In this approximated interval, the difference of error greatly favored the trinomial model, which means that the trinomial price converges faster to the Black-Scholes price than the binomial price. Furthermore, we have used different techniques for American options to show that the trinomial model, once again, is the fastest converging model. Thus we draw the consolidated conclusion that, compared to the binomial model, the trinomial model is more accurate for smaller N, i.e. it requires a lower number of steps and thus a shorter computational time to approximate the Black-Scholes price with the same precision.

It is theoretically possible for the trinomial model to price both non path-dependent as well as path-dependent exotic options. Yet for the path-dependent ones, we have found that the computations consume large amounts of time as a large number of steps is often required to provide accurate results. Even if it is true that the convergence of the trinomial model is faster compared to the binomial model when pricing these types of options, we consider the latter to be a better choice. This conclusion stems from the fact that it requires shorter processing time while providing sufficiently good results. Needless to say, however, we recommend other pricing methods to be investigated for the path-dependent exotic options, due to the computer power required to provide accurate results.

By using the least square method, we have found that it is possible to construct a portfolio which follows the actual payoff of the derivative quite well. We conclude that the method appears to generate accurate results when hedging financial assets, but that a call for future research on the topic is still justified.

For further research, we recommend this to be directed towards the area of the general trinomial model. In essence, one should exclude the assumptions $u = -d$, and investigate the implications on the model. As for the field of exotic options, we believe it is justified to initiate a rigorous study that compares a wide array of different pricing methods. Preferably one that would include Lévy processes and PDEs.

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Appendix A

Glossary

Arbitrage: A transaction that involves no risk of negative cash flow at any given time, while simultaneously offering a probability of positive cash flow in at least one time state t.

Asset: Tangible or intangible property that can be converted into cash.

Bond: A debt security where the issuer holds the owner's debt and hence is obliged to pay interest.

Derivative: A contract whose value is derived from an underlying asset.

Exercise: Putting into effect the rights specified in a contract.

Exotic option: Financial derivatives with a more complex structure than standard European or American options.

Fair price: The price of a self-financing hedging portfolio. The objective is that neither the investor or the owner shall be guaranteed to make a profit, nor a loss.

Geometric Brownian Motion: A stochastic process where the logarithm follows a Brownian Motion with a drift in continuous time.

Hedge: Investing in derivatives with the ambition to offset potential gains or losses that can be encountered from other investments or risks.

Intrinsic value: The difference between the stock price and the strike price, which always is positive or zero.

In, at, or out of the money: If the intrinsic value is greater than 0 the option is said to be in the money, if the stock price equals the strike price the option is said to be at the money, and if the intrinsic value is 0 and the option is not at the money it is out of the money,

Maturity: The time T until the contract expires.

Payoff: The profit from selling an asset.

Portfolio: A collection of positions on derivatives, held by an investor or an institution.

Risk-free: A asset is considered to be risk-free if i can guarantee some kind of future return.

Security: A fungible financial asset representing the value of, for instance, a stock, a bond, or an option.

Self-financing portfolio: A portfolio that does not require the investor to add cash, nor allows withdrawal upon creation of the portfolio. Hence the purchase of an asset must be financed by selling another one.

Volatility: The standard deviation of returns over time.

Appendix B

Introduction to probability

The purpose of this chapter is to give the reader knowledge of some basic concepts in probability theory.

Finite probability space

Let Ω be a set containing a finite number of elements $\omega_1,\omega_2,...\omega_M$ where ω_i is a selected outcome of an experiment (called a sample point or an atomic event). We will call this set a sample space. Moreover, let $p = (p_1,p_2,...,p_M)$ be M real numbers such that

$$0 < p_i < 1, \text{for all } i = 1,...,M, \quad \text{and} \quad \sum_{i=1}^{M} p_i = 1.$$

It is now possible to associate a probability to each of the sample points in Ω by defining p_i as the probability of the event $\{\omega_i\}$, i.e.

$$\mathbb{P}(\{\omega_i\}) = p_i, \quad i = 1,...,M.$$

Any generic event (combination of atomic events) can be written as the disjoint union of atomic events, e.g.

$$\{\omega_1,\omega_3,\omega_5\} = \{\omega_1\} \cup \{\omega_3\} \cup \{\omega_5\},$$

and the probability of such an event is equal to the sum of the individual atomic events. We also set

$$\mathbb{P}(\emptyset) = 0.$$

The pair (Ω,p) defined like this is called a finite probability space.

Random variable

A random variable $X(\omega)$ on Ω is a function that assigns a real number to each sample point ω_i of a probability space Ω. If $X(\omega_i) = c$, for all $i = 1,...,M$ then X is said to be non-random, or deterministic.

The image of a random variable X is the set defined as

$$\text{Im}(X) = \{x \in \mathbb{R} \text{ such that } X(\omega) = x, \text{ for some } \omega \in \Omega\}. \tag{B.1}$$

We also denote

$$\{X = a\} = \{\omega \in \Omega : X(\omega) = a\}. \tag{B.2}$$

And the probability that $X = a$ can be computed as

$$\mathbb{P}(X = a) = \sum_{i:X(\omega_i)=a} p_i. \tag{B.3}$$

The last two equations can easily be generalized to situations where X belongs to any open subset I of $\mathbb{R}$.

A random variable is said to be discrete if its range contains a finite or countably infinite number of points. Analogously, a random variable is said to be continuous if its range contains an interval (either finite of infinite) of real numbers. A random variable can also be both discrete and continuous, then it is called a mixed random variable.

Distribution functions

The cumulative distribution function of a random variable X is the function $F_X(x)$ defined by

$$F_X(x) = \mathbb{P}(X \leq x), \qquad -\infty < x < \infty.$$

It is easy to see that $F_X(x)$ is a non-decreasing function. The probability mass function $p_X(x)$ of a discrete random variable X is defined by

$$p_X(x) = \mathbb{P}(X = x). \tag{B.4}$$

If X is a continuous random variable, then the probability density function of X is defined by

$$f_X(x) = \frac{dF_X(x)}{dx}, \qquad -\infty < x < \infty, \tag{B.5}$$

provided the derivative exists.

Expectation and variance

In probability theory, the expected value of a random variable is the long-run average value obtained when repeating an experiment. The expected value of a random variable X, denoted μ_X or $\mathbb{E}(X)$, is defined by

$$\mu_X = \mathbb{E}(X) = \begin{cases} \sum_k x_k p_X(x_k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f_X(x) dx & \text{if } X \text{ is continuous.} \end{cases} \tag{B.6}$$

The variance measures how far a set of numbers are spread out when repeating an experiment. The variance of a random variable X, denoted σ_X^2 or $\mathrm{Var}(X)$, is defined by

$$\sigma_X^2 = \mathbb{V}\mathrm{ar}(X) = \begin{cases} \sum_k (x_k - \mu_X)^2 p_X(x_k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx & \text{if } X \text{ is continuous.} \end{cases} \tag{B.7}$$

By expanding this equation we can obtain the relationship

$$\sigma_X^2 = \mathbb{V}\mathrm{ar}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2, \tag{B.8}$$

which is commonly used to calculate the variance.

Independence and Correlation

Two events are said to be independent if one of the events does not affect the probability of the other one. Two events A and B are independent if and only if

$$\mathbb{P}(A,B) = \mathbb{P}(A)\mathbb{P}(B).$$

Similarly, two random variables $X_1, X_2 : \Omega \to \mathbb{R}$ are said to independent if the events $\{X_1 \in I_1\}, \{X_2 \in I_2\}$ are independent events, for all sets $I_1,I_2 \in \mathbb{R}$ i.e. the realization of one event does not affect the probability function of the other. So, two random variables X_1 and X_2 are independent if and only if

$$\mathbb{P}(X_1 \in I_1, X_2 \in I_2) = \mathbb{P}(X_1 \in I_1)\mathbb{P}(X_2 \in I_2).$$

The covariance of two random variables $X, Y : \Omega \to \mathbb{R}$ is defined by

$$\mathbb{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

If $\mathbb{Cov}(X,Y) > 0$, the variables X,Y tend to move in the same direction, i.e. if X increases then Y increases also. If $\mathbb{Cov}(X,Y) < 0$, the variables X,Y tend to move in opposite directions, i.e. if X increases then Y decreases. If $\mathbb{Cov}(X,Y) = 0$, the two variables X,Y are said to be uncorrelated, however that does not in general imply that they are independent.

Correlation is as covariance a measure of how much two random variables change together but as opposed to covariance, $\mathbb{Corr}(X,Y) \in [-1,1]$. The correlation of two random variables $X, Y : \Omega \to \mathbb{R}$ with non-zero $\mathbb{Var}[X]$ and $\mathbb{Var}[Y]$ is defined as

$$\mathbb{Corr}(X,Y) = \frac{\mathbb{Cov}(X,Y)}{\sqrt{\mathbb{Var}[X]\mathbb{Var}[Y]}}.$$

Stochastic processes and Martingales

Before defining Martingales, a crucial concept to finance, we need to define a stochastic process. A stochastic process is a family of random variables who are defined on a given probability space. The stochastic process is indexed by t, where t varies over the set $[0,T]$. Hence, $X(t) : \Omega \to \mathbb{R}, t \in [0,T]$ is a stochastic process. We then denote a stochastic process by $X(t)_{t\in[0,T]}$ and the value of the stochastic process on the sample $\omega \in \Omega$ is denoted by $X(t,\omega)$. From this it is possible for each fixed $\omega \in \Omega$ to obtain a curve $t \to X(t,\omega)$, called a path of the stochastic process.

A process which has the same paths for all samples is a non-random function of time, called a deterministic process. For a stochastic process to be discrete, t needs to run over a discrete set $\{t_1, t_2, ...\} \subset [0,T]$. For a discrete stochastic process on a finite probability space to be a martingale the following must hold:

$$\mathbb{E}[X_{i+1}|X_1,X_2,...,X_i] = X_i, \text{ for all i } \geq 1.$$

Central limit theorem

The central limit theorem (CLT) states that the distribution of the sum of a large number of independent, identically distributed variables under certain conditions, and each with finite variance and well-defined expected value, will be approximately normally distributed regardless of the underlying distribution. The central limit theorem has a number of variants but the random variables must be identically distributed in the theorem's most common form. Let

$$Z_n = \frac{X_n - n\mu}{\sqrt{n}\sigma} = \frac{M_n - \mu}{\sigma/\sqrt{n}},$$

where Z_n is standard normal with $\mathbb{E}(Z_n) = 0$ and $\mathbb{V}\mathrm{ar}(Z_n) = 0$. X_n is the sample sum, n is the sample size, μ is the mean and σ is the standard deviation. The central limit theorem states that the distribution of Z_n converges to the standard normal distribution as $n \to \infty$. The standard normal distribution has probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad z \in \mathbb{R}.$$

Appendix C

Matlab code

C.1 Stock prices in the trinomial model

```
% StockPrices creates a (2N+1)x(N+1) matrix with the prices of
% a stock calculated by the trinomial model. Each column
% corresponds to a time instant. The initial price can be
% found in position (N+1,1). If the current price is found in
% position (i,j), then (i+m,j+1), m=-1,0 or 1, corresponds to
% the subsequent stock price depending on if the price moves up,
% stays the same or moves down, respectively. u is the price change
% when the stock price goes up, N is the number of steps, and
% S0 is the initial stock price.

function S=StockPrices(u,N,S0)

S=zeros(2*N+1,N+1);
S(N+1,1)=S0;

for i=2:(N+1)
    S(:,i)=S(:,i-1);
    S(N+2-i,i)=S(N+3-i,i-1)*exp(u);
    S(N+i,i)=S(N+i-1,i-1)*exp(-u);
end

end
```

C.2 European option prices in the trinomial model

```
% OptionPrices creates a matrix with the prices of an option
% calculated by the trinomial model. Each column corresponds
% to a time instant. The initial price can be found in position
% (N+1,1). If the current price is found in postion (i,j), then
% (i+m,j+1), m=-1,0 or 1, corresponds to the subsequent option
% price depending on if the price moves up, stays the same or
% moves down, respectively. S is the trinomial tree prices of
% the underlying stock computed with the function StockPrices,
% g is the payoff function, u is the price change when the stock
% price goes up, r>=0 is the risk-free interest rate, and q0 is
% the free parameter in the trinomial model.

function P=OptionPrices(S,g,u,r,q0)

% Check input arguments
if (r<0) || (q0<0) || (q0>(exp(u)-exp(r))/(exp(u)-1))
    display('Error: invalid input parameters');
    P=0;
    return
end

M=size(S,1);
N=size(S,2);
P=zeros(M,N); % Option prices

syms x;
f=sym(g);

qu=(exp(r)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));
P(:,N)=subs(f,x,S(:,N)); % Calculate final price, i.e. payoff, of option

% Recurrence formula to calculate the option prices
for j=N-1:-1:1
    for i=(N-j+1):(M-(N-j))
        P(i,j)=exp(-r)*(qu*P(i-1,j+1)+q0*P(i,j+1)+qd*P(i+1,j+1));
    end
end

end
```

C.3 Least square hedging portfolio

```
% LSqHedgingPortfolio calculates the least square hedging
% portfolio. hs and hb are the number of stocks and bonds
% in this portfolio in the time interval (t-1,t], respectively.
% hs and hb are presented as two matrices of the same sizes as
% S and P, and they are interpreted in the same way as these.
% S is the trinomial tree prices of the underlying stock computed
% with the function StockPrices, P is the corresponding option
% prices computed with the function OptionPrices, B0 is the
% initial price of a bond, u is the price change when the stock
% price goes up, and r>=0 is the risk-free interest rate

function [hs,hb]=LSqHedgingPortfolio(S,P,B0,u,r)
M=size(S,1);
N=size(S,2);

% Number of stocks
hs=zeros(M,N-1);
% Number of bonds
hb=zeros(M,N-1);

% Least square method to calculate hs and hb
for j=1:N-1
    for i=(M+1)/2-(j-1):(M+1)/2+(j-1)
        A=[S(i-1,j+1) B0*exp(r*j)
           S(i,j+1) B0*exp(r*j)
           S(i+1,j+1) B0*exp(r*j)];
        y=[P(i-1,j+1);P(i,j+1);P(i+1,j+1)];
        h=(A.'*A)\(A.'*y);
        hs(i,j)=h(1);
        hb(i,j)=h(2);
    end
end

% Remove empty rows in hs matrix
hs=hs(2:M-1,:);
% Remove empty rows in hb matrix
hb=hb(2:M-1,:);

end
```

C.4 Value of least square hedging portfolio at maturity

```
% FinalHPValue computes the final value of a least square
% hedging portfolio. S is the trinomial tree prices of
% the underlying stock computed with the function StockPrices,
% B0 is the initial price of a bond, hs is the number of shares
% invested in the stock, hb is the number of shares invested in
% the bond, u is the price change when the stock price goes up,
% r>=0 is the risk-free interest rate, and Mi is the row index
% for the final postion in the trinomail tree.

% hs and hb are computed with the function LSqHedgingPortfolio.

function V=FinalHPValue(S,B0,hs,hb,u,r,Mi)
M=size(S,1);
N=size(S,2);
S0=S((M+1)/2,1);
V=[];

if N>2
    if Mi==1 % Postiton for higest value of S(N)
        V=hs(1,N-1)*S0*exp((N-Mi)*u)+hb(1,N-1)*B0*exp(r*(N-1));
    elseif Mi==M % Position for lowest value of S(N)
        V=hs(M-2,N-1)*S0*exp((N-Mi)*u)+hb(M-2,N-1)*B0*exp(r*(N-1));
    elseif Mi==2 % Position for second higest value of S(N)
        V(1)=hs(1,N-1)*S0*exp((N-Mi)*u)+hb(1,N-1)*B0*exp(r*(N-1));
        V(2)=hs(2,N-1)*S0*exp((N-Mi)*u)+hb(2,N-1)*B0*exp(r*(N-1));
    elseif Mi==(M-1) % Position for second lowest value of S(N)
        V(1)=hs(M-2,N-1)*S0*exp((N-Mi)*u)+hb(M-2,N-1)*B0*exp(r*(N-1));
        V(2)=hs(M-3,N-1)*S0*exp((N-Mi)*u)+hb(M-3,N-1)*B0*exp(r*(N-1));
    else % Other positions
        V(1)=hs(Mi-2,N-1)*S0*exp((N-Mi)*u)+hb(Mi-2,N-1)*B0*exp(r*(N-1));
        V(2)=hs(Mi-1,N-1)*S0*exp((N-Mi)*u)+hb(Mi-1,N-1)*B0*exp(r*(N-1));
        V(3)=hs(Mi,N-1)*S0*exp((N-Mi)*u)+hb(Mi,N-1)*B0*exp(r*(N-1));
    end
elseif N==2 % N=2 is an exception
    V=hs(1)*S0*exp((N-Mi)*u)+hb(1)*B0*exp(r*(N-1));
else
    display('Error: Nmax must be greater than 1')
    return;
end

end
```

C.5 Time adjusted European option prices in the trinomial model

```
% OptionPrices_h creates a matrix with the prices of an option
% calculated by the trinomial model. This function works just
% like the function OptionPrices, but OptionPrices_h takes
% the time parameter h=T/N into account. This makes it possible
% to price real-world options.

function P=OptionPrices_h(S,g,r,p,h,u)

% Check input arguments
if (r<0) || (p<0)
    display('Error: invalid input parameters');
    P=0;
    return
end

M=size(S,1);
N=size(S,2);
P=zeros(M,N);
q0=1-2*p;

syms x;
f=sym(g);

qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

% Calculate final price, i.e. payoff, of option
P(:,N)=subs(f,x,S(:,N));

% Recurrence formula to calculate option prices
for j=N-1:-1:1
    for i=(N-j+1):(M-(N-j))
        P(i,j)=exp(-r*h)*(qu*P(i-1,j+1)+q0*P(i,j+1)+qd*P(i+1,j+1));
    end
end

end
```

C.6 Implied volatility

```
% ImpliedVolatility computes the implied volatility of
% a European call option by using Newton's method. S0 is the
% price of the underlying stock, K is the strike price, r>=0
% is the risk-free interest rate, T is the time to maturity,
% histVol is the historical volatility, and marketPrice is the
% market price of the option.

function impVol=ImpliedVolatility(S0,K,r,T,histVol,marketPrice)

% B-S price using historic volatility
[BS_call, BS_put]=blsprice(S0,K,r,T,histVol);
% Difference between B-S price and market price
diff=abs(BS_call-marketPrice);
sigma_prev=histVol;

% Newton's method
while diff>0.005
    % d2 in B-S formula for call
    d2=(log(S0/K)+(r-(sigma_prev^2)/2)*T)/(sigma_prev*sqrt(T));
    % d1 in B-S formula for call
    d1=d2+sigma_prev*sqrt(T);

    % Derivative of d1 w.r.t. sigma
    derivative_d1=(-log(S0/K)-T)/(sigma_prev^2*sqrt(T))+sqrt(T)/2;
    % Derivative of d2 w.r.t. sigma
    derivative_d2=(-log(S0/K)-T)/(sigma_prev^2*sqrt(T))-sqrt(T)/2;

    % Derivative of fi (cum. std normal dist.) w.r.t. sigma for d1
    derivative_fi_d1=exp(-(d1^2)/2)/(sqrt(2*pi))*derivative_d1;
    % Derivative of fi (cum. std normal dist.) w.r.t. sigma for d2
    derivative_fi_d2=exp(-(d2^2)/2)/(sqrt(2*pi))*derivative_d2;

    % B-S price
    BS_call=S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2);
    % New volatility
    sigma_new=sigma_prev-(BS_call-marketPrice)/(S0*derivative_fi_d1-
            K*exp(-r*T)*derivative_fi_d2);

    d2=(log(S0/K)+(r-(sigma_new^2)/2)*T)/(sigma_new*sqrt(T));
    d1=d2+sigma_new*sqrt(T);
    BS_call=S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2);
    diff=abs(BS_call-marketPrice);
    sigma_prev=sigma_new;
end
impVol=sigma_prev;

end
```

C.7 American put option prices in the trinomial model

```
% AmericanPut computes the price of an American put option
% by using the trinomial model. S is the trinomial tree
% prices of the underlying stock computed with the function
% StockPrices, r>=0 is the risk-free interest rate, K is the
% strike price, N is the number of steps, p is the probability
% that the stock price goes up, h is the length of each time step,
% and u is price change when the stock price goes up.

function A=AmericanPut(S,K,r,N,p,h,u)
A=zeros(2*N+1,N+1);
A(:,N+1)=max(K-S(:,N+1),0);

q0=1-2*p;
qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

for j=N:-1:1
    for i=N+1-(j-1):N+1+(j-1)
        A(i,j)=max(max(K-S(i,j),0),exp(-r*h)*(qu*A(i-1,j+1)+
                        q0*A(i,j+1)+qd*A(i+1,j+1)));
    end
end
```

C.8 Optimal exercise boundary

```
% OptExeBound calculates the optimal exercise boundary
% of an American put option for different times to maturity.
% This is being done by slowly increasing the price of the
% underlying stock until the intrinsic value becomes less than
% the price of the option. sigma is the volatility of the
% underlying stock, K is the strike prie, r>=0 is the risk-free
% interest rate, N is the number of steps in the trinomial model,
% p is the probability that the stock price goes up, and T is
% the maximum time to maturity.

function [t,St]=OptExeBound(sigma,K,r,N,p,T)
t=[];
St=[];
tDelta=0.1;

for ti=0.001:tDelta:T
    Sti=0.0000001;
    h=ti/N;
    u=sigma*sqrt(h/(2*p));
    S=StockPrices(u,N,Sti);
    A=AmericanPut(S,K,r,N,p,h,u);
    while max(K-Sti,0)==A(N+1,1)
        Sti=Sti+1;
        S=StockPrices(u,N,Sti);
        A=AmericanPut(S,K,r,N,p,h,u);
    end
    Sti=Sti-1;
    S=StockPrices(u,N,Sti);
    A=AmericanPut(S,K,r,N,p,h,u);
    while max(K-Sti,0)==A(N+1,1)
        Sti=Sti+0.1;
        S=StockPrices(u,N,Sti);
        A=AmericanPut(S,K,r,N,p,h,u);
    end
    Sti=Sti-0.1;
    S=StockPrices(u,N,Sti);
    A=AmericanPut(S,K,r,N,p,h,u);
    while max(K-Sti,0)==A(N+1,1)
        Sti=Sti+0.01;
        S=StockPrices(u,N,Sti);
        A=AmericanPut(S,K,r,N,p,h,u);
    end
    t=[t ti];
    St=[St Sti];
end

end
```

C.9 Price of an Asian call option using Monte Carlo simulation

```
% AsianCall calculates the price of an Asian call option by using
% Monte Carlo simulation. T is the time to maturity, Nsteps is the
% number of steps, Nreps is the number of times a path and a price
% is calculated, S0 is the initial stock price, K is the strike price,
% r>=0 is the risk-free interest rate, and sigma is the volatility
% of the underlying stock.

% The stock path is calculated as
% S_t = S_0*exp(r-0.5*sigma^2+sigma*sqrt(t)*Z)
% where Z is N(0,1)

function AC=AsianCall(S0,K,r,T,sigma,N,reps)
dt=T/N;
R=exp(-r*T);
S = zeros(reps,N);
S(:,1) = S0;
drift = (r-0.5*sigma^2)*dt;
for n=1:reps
    for t=2:N
        dW = randn(1)*sqrt(dt);
        S(n,t) = S(n,t-1)*exp(drift+sigma*dW);
    end
    Average(n) = mean(S(n,:));
end
Payoff= max(Average-K,0);
% The arithmetic mean of all payoffs discounted with the factor R
AC=R*mean(Payoff);

end
```

C.10 Price of an Asian call option using the trinomial model

```
% RecursiveAsian calculates the trinomial price for
% Asian call option. The function is recursive and calculates
% the undiscounted price for a Asian call option.
%
% Q is a vector containing the risk-neutral probabilities qu,q0,and qd
% M is a vector containing u,0,-u
% V calculate the payoff for each potential path in the trinomial tree
%
% qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
% qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

function [V P_tot]=RecursiveAsian(V,N,K,n,P_tot,P,Q,M,allS)

allS_prev=allS;
S_prev=allS(end);
P_prev=P;

for i=1:3
    allS=[allS_prev S_prev*exp(M(i))];
    P=P_prev*Q(i);
    if n==N
        P_tot=P_tot+P;
        % Calculates the payoff for each path
        V=V+P*max(mean(allS(2:length(allS)))- K,0);
    else % Increases n until n=N
        [V P_tot]=RecursiveAsian(V,N,K,n+1,P_tot,P,Q,M,allS);
    end
end
```

C.11 Cliquet option prices in the trinomial model (main)

```
% CliquetPriceTrinomial calculates the trinomial cliquet option price.
% This function calls "RecursionCliquetTrinomial", which is recursive.
% m is the number of reset periods and N is the number of
% steps in each period.
% qu, qd, q0 and u are specified so that the price converges.
% Floc, Cloc, Fglob and Cglob are local and global caps and floors.
% Alpha and Beta are restraints on the number of ups and downs
% in each reset period.
% P_j and P_0 are probabilities associated with Alpha and Beta.

function [price]=CliquetPriceTrinomial(Floc,Cloc,Fglob,Cglob,T,m,N,sigma,r,p)

q0=1-2*p;
h=T/(N*m);
u=sigma*sqrt(h/(2*p));

% Calculate qu and qd (risk neutral meassure)
qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

% Alpha and P_j as defined in text
alpha=ceil(log(Cloc+1)/u);
P_j=0;
for Nu=alpha:N
    for Nd=0:min(Nu-alpha,N-Nu)
        P_j=P_j+nchoosek(N,Nu)*nchoosek(N-Nu,Nd)*qu^(Nu)*qd^(Nd)*q0^(N-Nu-Nd);
    end
end

% Beta and P_0 as defined in text
beta=floor(log(Floc+1)/u);
P_0=0;
for Nd=max(0,-beta):N
    for Nu=0:min(N-Nd,Nd+beta)
        P_0=P_0+nchoosek(N,Nd)*nchoosek(N-Nd,Nu)*qu^(Nu)*qd^(Nd)*q0^(N-Nu-Nd);
    end
end

j=alpha-beta;

P=zeros(1,m);
Z=zeros(1,m);
Q=0;
```

```
% Put all the constants in  a vector
constants=[u qu qd q0 Floc Cloc Fglob Cglob m N alpha beta j P_j P_0];

% Start the recursive algorithm
[price]=RecursionCliquetTrinomial(Z,P,1,Q,constants);
price=price*exp(-r*T);
end
```

C.12 Cliquet option prices in the trinomial model (recursive)

```
% RecursionCliquetTrinomial calculates the non-discounted trinomial
% price of a cliquet option. This function is recursive and it is called
% by the function CliquetPriceTrinomial.
% Local returns and final payoffs are being calculated as well as their
% associated probabilities.

function [Q]=RecursionCliquetTrinomial(Z,P,i,Q,constants)

% Define constants
u=constants(1); qu=constants(2); qd=constants(3); q0=constants(4);
Floc=constants(5); Cloc=constants(6);
Fglob=constants(7); Cglob=constants(8);
m=constants(9); N=constants(10);
alpha=constants(11); beta=constants(12); j=constants(13);
P_j=constants(14); P_0=constants(15);

% Treat all the cases where alpha is reached
P(i)=P_j;
Z(i)=Cloc;
if i==m % If the current reset date is the last one (i.e. maturity)
    P_final=prod(P);
    Z_final=sum(Z);
    Q=Q+P_final*max(Fglob,min(Cglob,Z_final));
elseif sum(Z)<=(Fglob-(N-i)*Cloc) % No possibility to go higher than Fglob
    P_final=prod(P(1:i));
    Q=Q+P_final*Fglob;
else
    Q=RecursionCliquetTrinomial(Z,P,i+1,Q,constants);
end

% Treat all the cases where beta is reached
P(i)=P_0;
Z(i)=Floc;
if i==m % If the current observation is the last one (i.e. maturity)
    P_final=prod(P);
    Z_final=sum(Z);
    Q=Q+P_final*max(Fglob,min(Cglob,Z_final));
elseif sum(Z)<=(Fglob-(N-i)*Cloc) % No possibility to go higher than Fglob
    P_final=prod(P(1:i));
    Q=Q+P_final*Fglob;
else
    Q=RecursionCliquetTrinomial(Z,P,i+1,Q,constants);
end

% Treat rest of the cases
for diff=beta+1:(beta+j-1)
    P_temp=0;
    for Nu=max(0,diff):min(N,N/2+floor(diff/2))
        Nd=Nu-diff;
        P_temp=P_temp+nchoosek(N,Nu)*nchoosek(N-Nu,Nd)*qu^Nu*qd^Nd*q0^(N-Nu-Nd);
    end
    P(i)=P_temp;
    R=exp(Nu*u-Nd*u)-1;
```

```
    Z(i)=max(Floc,min(R,Cloc));
    if i==m % If the current observation is the last one (i.e. maturity)
        P_final=prod(P);
        Z_final=sum(Z);
        Q=Q+P_final*max(Fglob,min(Cglob,Z_final));
    elseif sum(Z)<=(Fglob-(N-i)*Cloc) % No possibility to go higher than Fglob
        P_final=prod(P(1:i));
        Q=Q+P_final*Fglob;
    else
        Q=RecursionCliquetTrinomial(Z,P,i+1,Q,constants);
    end
end

end
```

C.13 Trinomial model approximation of compound CoC option price

```
% EUCompound_Tri  returns the full price tree for CoC option
% Current price (at t=0) will be at position P(round(T1*N/T2)+1,1)
% Necessary to round off number of steps to fit time until expiration T1
% and T2
%
% Stock price tree calculated first until time T1
% Then calculating underlying option prices in different nodes at T1
% Obtain payoff for CoC in each node at time T1
% Regressive algorithm to obtain prices of CoC at each time t<T1

function P = EUCompound_Tri(S0, T1, T2, N, K1, K2, p, r, sigma)

% Checking input arguments
if (r<0) || (T1<0) || (T2<0) || (K1<0) || (K2<0)
    display('Error: invalid input parameters');
    P=0;
    return
end

h=T2/N;
u=sigma*sqrt(h/(2*p));
% Number of steps for compound and underlying option
N1=round(T1*N/T2);
N2=round(N-N1);

% Choosing payoff functions
g1=['max(0, x-' num2str(K1) ')'];
g2=['max(0, x-' num2str(K2) ')'];

S=StockPrices(u, N1, S0);
M=size(S,1);
P=zeros(M,N1+1); % Compound option prices
Und_prices=zeros(M,1);

% Calculating underlying asset prices at time T1
for i=1:M
    S_temp=StockPrices(u,N2,S(i,N1+1));
    Und_temp=OptionPrices_h(S_temp, g2, r, p, h, u);
    Und_prices(i,1)=Und_temp(N2+1,1);
end

% Calculate final prices, i.e. payoffs, of compound option
syms x;
f = sym(g1);
P(:,N1+1)=subs(f,x,Und_prices(:,1));

q0 = 1 - 2*p;
qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

% Recurrence formula to calculate option prices
for j=N1:-1:1
    for i=(N1-j+2):(M-(N1-j+1))
```

```
        P(i,j)=exp(-r*h)*(qu*P(i-1,j+1)+q0*P(i,j+1)+qd*P(i+1,j+1));
    end
end

end
```

C.14 Compound CoC option price using Geske's model

```
% EUCompound_BS returns the current price (at time t=0) of the European CoC option
%
% Computing each input parameter in turn
% Critical value of S calculated by using CP=undPayoff as seen below

function P=EUCompound_BS(S0, r, sigma, T1, T2, K1, K2)

% Checking input arguments
if (r<0) || (T1<0) || (T2<0) || (K1<0) || (K2<0)
    display('Error: invalid input parameters');
    P=0;
    return
end

% Calculating critical value of S
maxiter=5000;
tol=1e-6;
Sstar=fzero(@undPayoff,[0.000000001 100*S0],optimset('MaxIter', maxiter,
    'TolFun', tol),K1,K2,r,T2-T1,sigma);

D2star=(log(S0/Sstar)+(r+0.5*sigma^2)*T1)/(sigma*sqrt(T1));
D1star=D2star-sigma*sqrt(T1);
D2=(log(S0/K2)+(r+0.5*sigma^2)*T2)/(sigma*sqrt(T2));
D1=D2-sigma*sqrt(T2);
rho=sqrt(T1/T2);

% Calculating price as calculated by Geske
P=S0*mvncdf([D2star D2],[0 0],[1 rho; rho 1])-K2*exp(-r*T2)*mvncdf([D1star D1],
    [0 0],[1 rho; rho 1])-K1*exp(-r*T1)*normcdf(D1star);

end

function CP=undPayoff(Sint,K1,K2,r,T,sigma)
[callprice,putprice]=blsprice(Sint,K2,r,T,sigma,0);
% Select appropriate function
CP=callprice-K1;
end
```

C.15 European lookback put option with floating strike using trinomial model

```
% LookbackFloatPut calculates the trinomial price of a European lookback put option
% with floating strike. LookbackFloatPut returns the initial price of the option.
% The output is compared to the Black Scholes price.
% LookbackFloatPut generates every potential path and creates a payoffs every single
% path. LookbackFloatPut is dependent on the likelihood of every single path.
% This function is recursive.
% Q is a vector containing the risk-neutral probabilities qu,q0,and qd
% qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
% qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));
% M is a vector containing u,0,-u
% V calculate the payoff for each potential path in the trinomial tree

S_pre=S_n; % initially equal S0
P_pre=P;
for i=1:3
    S_n=S_pre*exp(M(i));
    P=P_pre*Q(i);
    if S_n>S_max % checking if S(t)> Smax if so => S(t)=Smax
        S_max=S_n;
    end
    if n==N % Calculating when n=N
       ProbTot=ProbTot+P; % Total probability sum(ProbTot)=1 when finished
       V=V+P*(S_max-S_n);
    else % recursive
        [V ProbTot]=LookbackFloatPut(S_n,T,N,V,ProbTot,Q,M,n+1,P,S_max);
    end
end

end
```

C.16 European lookback put option with fixed strike using trinomial model

```
% LookbackFixedPut calculates the trinomial price of a European lookback put
% option with fixed strike. The function returns the initial price of the option.
% LookbackFixedPut generates every potential path and creates a payoff for every
% path. The function is dependent on the likelihood of every single path.
% LookbackFixedPut is recursive.
% Q is a vector containing the risk-neutral probabilities qu,q0,and qd
% M is a vector containing u,0,-u
% V calculate the payoff for each potential path in the trinomial tree

S_pre=S_n; % initially equal S0
P_pre=P;
for i=1:3
    S_n=S_pre*exp(M(i)); % S(t)=S(t-1)*exp(M(i))
    P=P_pre*Q(i); % probability
    if S_n>S_max  % checking if S(t)> Smax if so => S(t)=Smax
        S_max=S_n;
    end
    if n==N % Calculating when n=N
       ProbTot=ProbTot+P; % Total probability sum(ProbTot)=1 when finished
       V=V+P*max(S_max-K,0);
       count=count+1;
       % skips lines that cant find any new maximum anyway
    elseif S_n*exp(M(1)*(N-n))<S_max
        ProbTot=ProbTot+P;
        V=V+P*max(S_max-K,0);
        count=count+1;
    elseif S_n*exp(M(1)*(N-n))<K
        ProbTot=ProbTot+P;
        count=count+1;
    else % recursive
        [V ProbTot count]=LookbackFixedPut(S_n,T,N,V,ProbTot,Q,M,n+1,P,S_max,K,
        count);
    end
end

end
```

C.17 Bermudan option prices in the trinomial model

```
% BermOptionPrices computes Bermudan prices using the Trinomial model.
% There are two possible returns to the function, depending on whether we
% wish to compare the Bermudan price with American/European derivatives or
% if we wish to compare the Trinomial Bermudan price with the Binomial
% Bermudan price.
% The function begins with creating several matrices, one for the payoff
% tree and one each for the European-, American-, and Bermudan derivative.
% All price trees take their bases in the payoff tree, then recursively
% their respective prices based on the properties of the derivatives.

function  P=BermOptionPrices(S,u,r,h,p,ex_dates,K,put_True)
q0 = 1-2*p;
% Check input arguments
if (r<0) || (q0<0) || (q0>(exp(u)-exp(r))/(exp(u)-1))
    display('Error: invalid input parameters');
    P=0;
    return
end

M=size(S,1);
N=size(S,2);

% Price trees for different derivatives
PTree=zeros(M,N); % Pay-off tree used for american tree later
PEuro=zeros(M,N); % European tree used later
PAmer=zeros(M,N); % American tree used later
PBerm=zeros(M,N); % Bermudan tree used later

exercise_True = zeros(1,N);

AmStep = N / ex_dates;
% Identifying steps at which we treat the Berm option as American
for i=AmStep:AmStep:N
    exercise_True(i) = 1;
end

qu=(exp(r*h)-exp(-u))/
(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));

qd=(exp(u)-exp(r*h))/
(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

% Calculate entire pay-off tree with respect to S
if put_True == 1
 for j=N:-1:1
    for i=(N-j+1):(M-(N-j))
            PTree(i,j) = max(0,K-S(i,j));
    end
 end

else
    for j=N:-1:1
        for i=(N-j+1):(M-(N-j))
            PTree(i,j) = max(0,S(i,j)-K);
        end
```

```
    end
end

% Recurrence formula to calculate the European option prices
PEuro(:,N) = PTree(:,N);
for j=N-1:-1:1
    for i=(N-j+1):(M-(N-j))
        PEuro(i,j)=exp(-r)*(qu*PEuro(i-1,j+1)
        +q0*PEuro(i,j+1)+qd*PEuro(i+1,j+1));
    end
end
% Recurrence formula to calculate the American option prices

PAmer(:,N) = PTree(:,N);
for j=N-1:-1:1
    for i=(N-j+1):(M-(N-j))
       PAmer(i,j) = max(PTree(i,j),exp(-r)*
       (PAmer(i-1,j+1)*qu+q0*PAmer(i,j+1) + qd*PAmer(i+1,j+1)));
    end
end

% Recurrence formula to calculate the Bermudan option prices

PBerm(:,N) = PTree(:,N);
for j=N-1:-1:1
    for i=(N-j+1):(M-(N-j))
        if exercise_True(j) == 1
           PBerm(i,j) = max(PTree(i,j),exp(-r)*
           (PBerm(i-1, j+1)*qu+q0*PBerm(i,j+1)+qd*PBerm(i+1,j+1)));
        else
           PBerm(i,j) = exp(-r)*(PBerm(i-1, j+1)*
           qu+q0*PBerm(i,j+1)+qd*PBerm(i+1,j+1));
    end
end
% Initial prices
P = [PEuro(N,1) PBerm(N,1) PAmer(N,1)];
% Use this for executing the comparison between derivatives
P = PBerm(N,1)
% Use this for executing the comparison with the binomial model
end
```

C.18 Bermudan option prices in the binomial model

```
% BinomBerm computes Bermudan prices using the binomial model. We begin by
% creating a payoff tree that forms a basis for the price tree. The
% function creates different payoff trees depending on whether we have a
% put or a call option. Then we use the payoff at maturity to recursively
% compute the prices backwards.

function  P=BinomBerm(S,u,r,h,ex_dates,K,put_True)
M=size(S,1);
N=size(S,2);

exercise_True = zeros(1,N);

PTree=zeros(M,N); % Pay-off tree used for american tree later
PBerm=zeros(M,N); % Bermudan tree used later

qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u));

AmStep = N / ex_dates;
% Identifying steps at which we treat the Berm option as American
for i=AmStep:AmStep:N
    exercise_True(i) = 1;
end
% Here a payoff tree is created, Binomial style
if put_True == 1
 for j=N:-1:1
    for i=1:(M-(N-j))
            PTree(i,j) = max(0,K-S(i,j));
    end
 end

else
    for j=N:-1:1
        for i=1:(M-(N-j))
            PTree(i,j) = max(0,S(i,j)-K);
        end
    end
end
PBerm(:,N) = PTree(:,N);
% Here we calculate the Bermudan prices by using the exercise_true variable,
% Identifying where the derivative becomes American and European
for j=N-1:-1:1
    for i=1:(M-(N-j))
        if exercise_True(j) == 1
            PBerm(i,j) = max(PTree(i,j),exp(-r)
            *(PBerm(i+1,j+1)*qu+qd*PBerm(i+1,j+1)));
        else
            PBerm(i,j) = exp(-r)*(PBerm(i,j+1)
            *qu+qd*PBerm(i+1,j+1));
        end
    end
end
P = PBerm(1,1);
end
```

C.19 Trinomial price of down and out barrier option on European call

```
% BarrierOptionDOCTMP(T,S0,K,r,sigma,p,N,B)
% Computes the trinomial price of barrier option on european down and out   call
% S0 current stock price
% T is the time to maturity, T=1 is one year
% K is the strike price of the underlying European call
% r>0 is the intrest rate of the bond
% Sigma is the volatility of the underlying stock
% B is the barrier
% p is the probability that the stock goes up or down, 0<p<=1/2 atleast
% N is the number of iterations to compute

function P=BarrierOptionDOCTMP(T,S0,K,r,sigma,p,N,B)

%N=#Partitions the time interval is divided into
h = T/N;

u=sigma*sqrt(h/(2*p));

%xb= number of steps down to reach barrier
xb=round(log(S0/B)/u);

%We only need to calculate the stock prices at maturity

%The stock Nu steps up from the middle S0 is given by
%exp(log(S0)+Nu*u)

%The stock Nd steps down from the middle S0 is given by
%exp(log(S0)-Nd*u)

SEnd=zeros(2*N+1,1);
for i=1:N
SEnd(i)=exp(log(S0)+u*(N-i+1)); %Starts from the start
SEnd(end+1-i)=exp(log(S0)-u*(N-i+1)); %Starts from the bottom
end
SEnd(N+1)=S0;

g=@(x)max(x-K,0);  %Payoff of european Call

P=zeros(2*N+1,N+1); % Option prices

q0 = 1 - 2*p;
qu=(exp(r*h)-exp(-u))/(exp(u)-exp(-u))-q0*(1-exp(-u))/(exp(u)-exp(-u));
qd=(exp(u)-exp(r*h))/(exp(u)-exp(-u))-q0*(exp(u)-1)/(exp(u)-exp(-u));

P(1:N+xb,end)=g(SEnd(1:N+xb)); % Payoff at maturity
%Barrier arrives a N+1+xb, if we hit the barrier the values is instantly
%zero


% Recurrence formula to calculate option prices
for j=N:-1:1
    for i=N+2-j:N+j;
        if i<N+1+xb %If i>=N+1+xb we have reached the barrier -> P=0
```

```
            P(i,j)=exp(-r*h)*(qu*P(i-1,j+1)+q0*P(i,j+1)+qd*P(i+1,j+1));
        end
    end
end
end
```

C.20 Black-Scholes price of down and out barrier option on European call

```
% BSDownOutCall(S0,T,K,r,sigma,q,B)
% Computes the theoretical Black-Scholes price of barrier options
% on a European down and out call
% S0 current stock price
% T is the time to maturity, T=1 is one year
% K is the strikeprice of the underlying European call
% r>0 is the intrest rate of the bond
% Sigma is the volatility of the underlying stock
% q is dividend
% B is the barrier
function Cdo=BSDownOutCall(S0,T,K,r,sigma,q,B)

Lb=(r-q+(sigma^2)/2)/(sigma^2);
x1=log(S0/B)/(sigma*sqrt(T))+Lb*sigma*sqrt(T);
y1=log(B/S0)/(sigma*sqrt(T))+Lb*sigma*sqrt(T);

Cdo=S0*normcdf(x1)*exp(-q*T)-K*exp(-r*T)*normcdf(x1-sigma*sqrt(T))...
    -S0*exp(-q*T)*(B/S0)^(2*Lb)*normcdf(y1)+K*exp(-r*T)*...
    (B/S0)^(2*Lb-2)*normcdf(y1-sigma*sqrt(T));
end
```

Appendix D

Utökad svensk sammanfattning

Introduktion

Trinomialmodellen härstammar ifrån binomialmodellen, och utvecklades av Phelim Boyle 1986. Fördelarna med trinomialmodellen kontra binomialmodellen var att den ansågs vara mer flexibel samt innehöll några viktiga egenskaper som binomialmodellen saknade.

Syftet med detta kandidatarbete är att studera egenskaperna hos trinomialmodellen, hur den konvergerar mot Black-Scholes-priset, samt applicera den på olika typer av exotiska optioner med målsättning att hitta lämpliga användningsområden för modellen.

Detta kommer att ske genom att först härleda grunderna kring trinomialmodellen, sedan studera prissättning samt hedning. Vi kommer även att härleda villkoren för att trinomialmodellen ska konvergera till Black-Scholes-priset. Avslutningsvis kommer modellen att appliceras på olika exotiska optioner.

Bakgrund

Innan grunderna kring trinomialmodellen introduceras behöver generella finansiella koncept och tillgångar förtydligas för att skapa en nödvändig förståelse genom arbetet.

Finansiella koncept

Det första konceptet vi introducerar är finansiella tillgångar. En finansiell tillgång definieras som ett objekt vilket kan köpas och säljas under specifika regler samt för ett pris som kan härledas från ett avtal. Finansiella tillgångar kan delas in i två generella grupper, materiella och immateriella. Exempel på materiella tillgångar är olja, guld och kaffe medan en immateriell tillgång exempelvis kan vara en aktie.

Det finns två sätt att utbyta tillgångar; på officiella reglerade marknader samt genom handel över disk. På officiella marknader lyder samtliga transaktioner under samma regelverk, men då handel sker över disk avtalas reglerna mellan köparen och säljaren. Vi väljer utifrån detta att definiera priset för en tillgång som det pris den handlas för på en reglerad marknad (marknadspris).

Finansiella tillgångar

I det här arbetet kommer vi att arbeta med tre typer av finansiella tillgångar; aktier, optioner och räntepapper vilka betecknas som riskfria tillgångar. En aktie representerar en del av ett företag och värderas efter företagets förmåga att generera framtida kassaflöden. Räntepapper används för att göra köp- och säljtransaktioner även vid utebliven likviditet hos investeraren.

Den finansiella tillgången vi kommer att fokusera på är optioner. En option är en finansiell tillgång vars värde beror på en underliggande tillgång, i vårt fall en aktie. En standardoption är ett finansiellt derivat som ger köparen rättigheten men inte skyldigheten att i framtiden köpa eller sälja den underliggande tillgången till ett förbestämt pris. Det finns två typer av standardoptioner, köpoptioner samt säljoptioner. En köpoption ger investeraren möjligheten att köpa den underliggande tillgången, och en säljoption ger investeraren möjligheten att sälja den underliggande tillgången.

Grunderna kring trinomialmodellen

I detta kapitel formulerar vi de grundläggande koncepten kring trinomialmodellen samt hur den kan användas för att beräkna prisen av en riskfyllt tillgång så som en option. Vi kommer även att titta på självfinansierande portföljer samt vilka villkor som måste vara uppfyllda för att marknaden ska vara arbitragefri.

Formulering av trinomialmodellen

Trinomialmodellen modellerar aktiepriset $S(t)$ vid tiden t på följande vis:

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{med sannolikheten } p_u \\ S(t-1) & \text{med sannolikheten } p_0 = 1 - p_u - p_d \\ S(t-1)e^{d} & \text{med sannolikheten } p_d \end{cases}, \; t \in \mathbb{I} = \{1,...,N\}. \quad \text{(D.1)}$$

Här är $u > 0$, $d < 0$, $p_u, p_d \in (0,1)$, $p_0 = 1 - p_u - p_d > 0$. Aktiepriset kan öka, minska eller vara oförändrat i varje tidssteg. Vi antar att $S_0 = S(0)$, vilket betyder att priset vid tiden $t = 0$ är känt.

Det går enkelt att se att antalet möjliga vägar ökar snabbt vilket gör modellen svår att hantera. Vi väljer därför att göra förenklingen att $u = -d$. Detta minskar antalet noder i modellen. Således kommer vi nu enbart att ta hänsyn till den begränsade modellen på formen

$$S(t) = \begin{cases} S(t-1)e^{u} & \text{med sannolikheten } p_u \\ S(t-1) & \text{med sannolikheten } p_0 = 1 - p_u - p_d \\ S(t-1)e^{-u} & \text{med sannolikheten } p_d \end{cases}, \; t \in \mathbb{I} = \{1,...,N\}. \quad \text{(D.2)}$$

Nu kommer antalet noder i aktieträdet vid tiden t beskrivas av $\psi_t = 2t + 1$ istället för $\psi_t = \frac{(t+1)(t+2)}{2}$, vilket gör den lättare att hantera.

Självfinansierande portfölj och en arbitragefri marknad

En självfinansierande portfölj beskrivs som en portfölj med en position i en aktie och en position i en riskfri tillgång. Portföljens värde kan inte påverkas genom att förändra positionerna utan enbart av värdeförändringarna på aktien samt den riskfria tillgången.

För att beräkna värdet av en självfinansierande portfölj introduceras riskneutrala sannolikheter i trinomialmodellen. De riskneutrala sannolikheterna q_{+1}, q_0 och q_{-1} definieras enligt

$$q_{+1} + q_0 + q_{-1} = 1, \quad q_{+1}e^u + q_0 + q_{-1}e^{-u} = e^r. \tag{D.3}$$

Den högra olikheten kommer ifrån det faktum att det nuvarande värdet måste vara det framtida diskonterade värdet. Värdet av en självfinansierande portfölj beräknas som det diskonterade genomsnittliga slutvärdet för portföljen givet dessa riskneutrala sannolikheter. Ett fullständigt bevis kring självfinansierande portföljer går att hitta i det fullständiga kandidatarbetet.

Nu behöver vi introducera begreppet arbitrage till de riskneutrala sannolikheterna. För att marknaden ska vara arbitragefri måste villkoren för r,u och q_0 hittas samtidigt som (q_{+1},q_0,q_{-1}) definieras som sannolikheter. Efter utredning blir slutsatsen att marknaden är arbitragefri om och endast om

$$r < u, \quad 0 < q_0 < \frac{e^u - e^r}{e^u - 1}. \tag{D.4}$$

Optionsprissättning och hedging av optioner

Prissättningen och hedging är två centrala begrepp när det kommer till optionsteori. Tillvägagångssättet för att hitta det rättvisa priset är inte helt trivialt. Det rättvisa priset är det pris där varken köparen eller utställaren av optioner är garanterad en vinst, vilket betyder att det inte finns möjlighet till arbitrage.

Att hedga innebär att replikerar den underliggande tillgången. Ofullständigheten av vår modell gör detta svårt och vi måste undersöka alternativa sätt för att lyckas med detta.

Definitionen av det rättvisa priset för en europeisk option

Definitionen av ett rättvist pris är att priset inte ska vara till favör varken för säljaren eller för köparen. Med andra ord betyder det att varken köparen eller säljaren ska ha möjlighet att göra en garanterad vinst. Om detta skulle vara fallet så skulle det innebära att det finns arbitragemöjligheter i marknaden.

Denna tolkning gör det möjligt att associera det rättvisa priset med värdet på en självfinansierande hedgande portfölj. Antag att säljaren investerar sin premium från försäljningen av optionen i den underliggande tillgången samt i en riskfri tillgång, att det inte finns något kassaflöde varken in eller ut ur portföljen samt att värdet på portföljen är lika med payoffen från optionen på lösendagen. På så sätt är det möjligt att definiera det rättvisa priset som priset för en sådan självfinansierande portfölj.

Genom att ha definierat det rättvisa priset på detta sätt kan vi med hjälp av de riskneutrala sannolikheterna samt en rekursiv formel härleda det initiala priset för en europeisk option. En fullständig utläggning av detta går att hitta i kandidatarbetet.

Hedging

Efter att ha tagit fram det rättvisa priset för en europeisk option i trinomialmodellen finns det ytterligare ett avsnitt som måste diskuteras, hedging. Hedging är en av de viktigaste delarna i modern finansteori. Idén bakom konceptet är att investera i en eller flera olika tillgångar så att de följer prisrörelserna hos en annan tillgång. På så sätt är det möjligt att reducera risken för en position, dock på bekostnad av eventuella vinster.

Hedging i en ofullständig marknad

En marknad sägs vara fullständig om de arbitragefria priset för ett derivat är entydigt. På en sådan marknad sammanfaller priset för derivatet med värdet på den hedgande portföljen. Eftersom trinomialmodellen beror av den fria parametern q_0 är inte priset entydigt och därför är marknaden ofullständig. Ett annat sätt att påvisa detta är att visa att det generellt sett inte går att replikera genom att enbart investera i den underliggande tillgången samt en riskfri tillgång.

Även fast det inte går att exakt replikera en europeisk option med hjälp av trinomialmodellen finns det lösningar som kringgår problemet med en ofullständig marknad. Det ena är att använda sig av minstakvadratmetoden för att skapa en lösning som är så nära den verkliga lösningen som möjligt. Det andra är att lägga till ytterligare en riskfylld tillgång i portföljen.

Black-Scholes-priset och trinomialmodellen

En av de viktigaste egenskaperna hos trinomialmodellen är att dess pris konvergerar mot Black-Scholes-priset. Black-Scholes-priset är det mest vedertagna priset för prissättning av optioner. Således, för att kunna säga att trinomialmodellen ger ett rättvist pris och fungerar som prissättningsmetod för europeiska optioner måste den generera samma pris som Black-Scholes. Black-Scholes-priset får man genom att lösa den partiala differentialekvationen Black och Scholes tog fram. Således, om vi kan härleda den ekvationen från trinomialmodellen vet vi att vi kommer få det rättvisa priset.

Genom att definiera trinomialmodellen enligt tidigare, skriva den som en Geometrisk Brownsk Rörelse, använda den rekursiva formeln med de riskneutrala sannolikheterna, samt genom att applicera taylorutveckling får vi slutligen att trinomialmodellen uppfyller Black-Scholes PDE om och endast om $u = \sigma\sqrt{\frac{h}{2p}}$ och $q_0 = 1 - 2p$ där p är en konstant som ligger mellan 0 och 0.5. Det fullständiga beviset finns i kandidatarbetet.

Slutsatsen av detta är att trinomialmodellen fungerar som prissättningsmetod för att prissätta europeiska optioner men eftersom det redan finns en matematisk formel för detta är det inte det optimala sättet att prissätta europeiska optioner. Amerikanska optioner har däremot ingen exakt matematisk formel för det rättvisa priset, men vi har lyckats visa numeriskt att trinomialmodellen fungerar bra för att prissätta amerikanska optioner.

Exotiska optioner

Exotiska optioner är optioner som är mer komplicerade än vanliga europeiska och amerikanska optioner. Till skillnad från europeiska och amerikanska optioner kan exotiska optioner ha en avkastning som beror på vägen den underliggande tillgången tar fram till lösendagen. Användningsområden för exotiska optioner varierar och kan vara allt från möjligheten till att hedga ett annat derivat till att minska hur volatiliteten hos den underliggande finansiella tillgången påverkar avkastningen.

Eftersom trinomialmodellen har visat sig konvergera snabbare än binomialmodellen när det kommer till europeiska och amerikanska optioner är det intressant att undersöka hur väl den fungerar som prissättningsmetod när det kommer till exotiska optioner. Vi kommer att undersöka hur väl modellen fungerar för att prissätta asiatiska optioner, cliquetoptioner, compoundoptioner, lookbackoptioner, bermudiska optioner samt barriäroptioner.

Asiatiska optioner

En asiatisk option beror på medelvärdet av vägen som den underliggande tillgången tar fram till lösendagen. Detta gör den svår att prissätta och det finns inget exakt pris för en asiatisk option. Syftet med optionen är att minska effekterna av volatilitet hos den underliggande tillgången. Genom att avkastningsfunktionen beror på medelvärdet av samtliga värden fram till lösendagen elimineras risken för extremvärden på lösendagen, vilket annars kunde ha gjort avkastningen till noll. Genom att känsligheten för volatilitet minskas blir priset för en asiatisk option också lägre än priset för en amerikansk eller europeisk option.

För att få ett perspektiv på hur effektiv trinomialmodellen är jämförs den mot Monte Carlo-simuleringar. I de numeriska resultaten ser vi att varken trinomialmodellen eller Monte Carlo-simuleringar kan klassas som speciellt bra för att prissätta asiatiska optioner. Variansen i Monte Carlo-simuleringarna är för hög för att modellen ska kunna generera ett exakt resultat med ett rimligt antal simuleringar. Trinomialmodellen genererar exakta resultat som vid anblick ser väldigt bra ut, men tiden det tar att prissätta optioner gör att det inte är ett optimalt alternativ vid prissättning. Eftersom det inte finns något exakt rättvist pris när det kommer till asiatiska optioner är det också svårt att bedöma hur bra modellen är. Vi rekommenderar därför att andra prissättningsmetoder måste undersökas och jämföras med våra resultat för att med säkerhet kunna säga hur bra trinomialmodellen är.

Cliquetoptioner

Cliquetoptioner introducerades i början av tvåtusentalet som ett svar på investerares behov av säkrare finansiella produkter. Payoffen från en cliquetoption beror på avkastningen genererat av den underliggande tillgången mellan olika datum för omstart innan lösendagen. Vid dessa datum för omstart nollställs lösenpriset till det nuvarande aktiepriset och en eventuell payoff betalas ut, således kan en cliquetoption ses som en följd av optioner där lösenpriset är det initiala värdet på aktien.

Från denna korta beskrivning av cliquetoptionen är det tydligt att dess payoff beror på vägen som den underliggande tillgången tar. Priset för en cliquetoption kan enbart beräknas numeriskt och därför är det extra intressant att undersöka hur väl trinomialmodellen kan användas för att prissätta den.

De undersökningar som har genomförts i kandidatarbetet påvisar att trinomialmodellen kan användas för att prissätta cliquetoptioner. Det ska dock sägas att denna metod är väldigt tidskrävande (framför allt för låga volatiliteter) samt behöver ett stort antal steg för att vara exakt. Även om trinomialmodellen konvergerar snabbare än binomialmodellen måste binomialmodellen anses vara den bättre av de två, på grund av kortare beräkningstider. Trots att binomialmodellen konvergerar långsammare går det att kompensera för det genom att lägga till fler steg, detta utan att förlora fördelen med en snabbare beräkningshastighet.

Compoundoptioner

Compoundoptioner är optioner vars underliggande tillgång också är en option. Detta betyder att compoundoptioner har två separata lösendatum och två separata förbestämda lösenpriser. Strukturen är uppbyggd så att investeraren vid det första lösendatumet måste besluta om den vill använda sin rätt till att utnyttja den första optionen. Compoundoptioner används primärt i valuta- och fasta räntemarknader när investeraren vill skydda sig mot en möjlig risk den andra optionen avser. Detta kan exempelvis vara framtida projekt i en annan valuta där det inte går att säga med säkerhet att projektet kommer att genomföras.

När det gäller prissättning av compoundoptioner härledde Robert Geske en exakt formel under 1979. Trinomialmodellen kommer således att jämföras mot detta pris. För att beräkna priset med trinomialmodellen beräknar vi först priset av den andra optionen vid den första lösendagen sedan beräknar vi det initiala priset av dessa optioner vid tiden $t = 0$.

Slutsatsen av de numeriska beräkningarna är att trinomialmodellen konvergerar mot det rättvisa priset, men att det känns suboptimalt att använda en prissättningsmodell när det existerar ett exakt värde. Trinomialmodellen fungerar således mycket bra för att sätta rätt pris men det är inte den bästa lösningen.

Lookbackoptioner

En lookbackoption är en exotisk option som är beroende av vägen den underliggande tillgången tar. Den gör det möjligt för investeraren att se tillbaka på de historiska priserna för den underliggande tillgången, därav namnet. Det primära syftet med lookbackoptioner är att hjälpa investeraren med marknadstiming.

Det finns två typer av lookbackoptioner; de med ett bestämt lösenpris och de med ett rörligt lösenpris. För en lookbackoption med ett bestämt lösenpris ges payoffen av den maximala skillnaden mellan lösenpriset och den underliggande tillgången. Payoffen för en lookbackoption med rörligt lösenpris beror på de maximala eller minimala värden den underliggande tillgången har fram till lösendagen.

Det går att härleda Black-Scholes-priset för en lookbackoption med ett rörligt lösenpris och vi jämför därför trinomialmodellen mot detta pris. Slutsatsen är att trinomialmodellen inte är en effektiv prissättningsmetod när det kommer till lookbackoptioner med rörligt lösenpris. Det är både väldigt tidskrävande, även för ett lågt antal steg samt att det inte är uppenbart att priset som ges av trinomialmodellen konvergerar mot Black-Scholes-priset.

Bermudiska optioner

En bermudisk option är en option som ger investeraren möjligheten att köpa eller sälja den underliggande tillgången till det förbestämda priset vid ett bestämt antal tillfällen fram till lösendagen. Det är således en kombination av en amerikansk och en europeisk option. Detta medför i sin tur att priset för en bermudisk option ligger mellan (eller är lika med) en europeisk och en amerikansk option.

Karaktären av optionen gör att prissättningsmetoden blir en kombination av den rekursiva formel som används för amerikanska och europeiska optioner. I våra numeriska resultat kommer vi fram till att trinomialmodellen anses vara lämplig för att prissätta bermudiska optioner. Eftersom de dagar då det är möjligt att köpa eller sälja den underliggande tillgången innan lösendagen varierar från kontrakt till kontrakt medför detta att det inte går att ta fram någon exakt formel för att prissätta bermudiska optioner. För att verifiera våra resultat har vi därför jämfört dem med binomialmodellen som ofta används för att prissätta bermudiska optioner. Vi finner att trinomialmodellen är fördelaktig mot binomialmodellen av samma anledningar som vid prissättningen av europeiska och amerikanska optioner, på grund av konvergeringshastighet och flexibilitet.

Barriäroptioner

Barriäroptioner är en variant av andra optioner där en barriär har introducerats. Payoffen för en barriäroption är identisk med payoffen för den förbestämda optionen om barriären är aktiv, om inte är den lika med noll.

Det finns generellt sett två typer av barriäroptioner; knock-out- och knock-in-optioner. För att en knock-out-option ska bli aktiv måste den underliggande tillgången inte ha nått en fördefinierad barriär. Knock-in-optioner är motsatsen, för dessa måste den underliggande tillgången ha nått den fördefinierade barriären för att optionen ska vara aktiv.

När det gäller prissättning av barriäroptioner har europeiska barriäroptioner undersökts. Eftersom en europeisk barriäroption är en variant av en vanlig europeisk option med bivillkoret att om den underliggande tillgången når eller undviker en viss barriär finns det många likheter med prissättningen för hur europeiska optioner prissätts. Det är möjligt att härleda ett analytiskt Black-Scholes-pris för barriäroptioner och det fungerar även bra att använda trinomialmodellen för att prissätta dem. När barriären är långt ifrån det initiala priset är trinomialmodellen både väldigt exakt och kan även vara snabbare än beräkningstiden för den analytiska formen av optionspriset. Därför anses trinomialmodellen vara en bra prissättningsmetod för barriäroptioner när barriären befinner sig på avstånd från det initiala priset på den underliggande tillgången.

Slutsats

Genom numerisk analys av konvergeringshastigheten för trinomialmodellen mot Black-Scholes-priset är slutsatsen att trinomialmodellen konvergerar snabbare än binomialmodellen. Vi har även använt oss av tekniker som påvisar att trinomialmodellen är effektivare än binomialmodellen när det kommer till att prissätta amerikanska optioner.

Numeriska studier av minstakvadratmetoden har visat att det är möjligt att konstruera en portfölj som följer den riktiga payoffen för ett derivat relativt väl. Vi kan se att denna metod verkar ge bra resultat när det kommer till att hedga derivat, men vidare forskning inom området är att rekommendera.

Det är även teoretiskt möjligt för trinomialmodellen att prissätta olika typer av exotiska optioner både när payoffen beror av vägen den underliggande tillgången tar och när den inte gör det. Det visar sig dock att den är väldigt tidskrävande, framför allt när det kommer till exotiska optioner som beror av den underliggande tillgångens väg. Även om trinomialmodellen konvergerar snabbare än binomialmodellen är binomialmodellen att föredra. Detta beror på tillräckligt bra resultat samt en kortare beräkningstid. Vi rekommenderar dock att ytterligare studier genomförs på området med fokus på andra prissättningsmetoder samt beräkningstider.