Existence of Static Solutions to the Vlasov-Poisson and Einstein-Vlasov Systems as Fixed Points of a Mass-Preserving Algorithm

Master's thesis in Engineering mathematics and computational science

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Abstract

This thesis develops a new method for proving the existence of static, spherically symmetric solutions to the Einstein–Vlasov system by reformulating a mass-preserving algorithm as a fixed-point problem. Building on earlier work for the Vlasov–Poisson system, we first extend an existing fixed-point existence proof from isotropic to anisotropic solutions. We then establish the corresponding result for the Einstein–Vlasov system, using a simple isotropic ansatz.

Keywords: Einstein–Vlasov system, Vlasov–Poisson system, kinetic theory, general relativity, static solutions, steady state, collisionless matter, spherical symmetry

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1 Background

The Einstein–Vlasov system models collisionless matter in general relativity and is relevant in the study of galactic dynamics and cosmology. Of particular interest are the stationary solutions and their stability. The system also has a non-relativistic equivalent, the much simpler Vlasov-Poisson system, which models collisionless matter in classical gravity. Due to the great complexity of the Einstein equations it is common to introduce symmetry assumptions, which can reduce the system significantly [1].

The assumption of spherical symmetry in particular makes the system quite manageable analytically, and construction of static solutions reduces to an ODE [2]. When assumptions are weakened, however, such as in axial symmetry, things quickly become more complicated; the equations are large and the above-mentioned ODE method no longer applies [3]. In these cases iterative procedures take on greater importance. The so-called mass-preserving algorithm, which is the subject of this thesis, is an efficient method for constructing static solutions [4].

Since this algorithm is important in these cases it would be of interest to understand its properties analytically. The algorithm is also suspected to have a connection to stability, in that it appears unable to converge to unstable solutions, and there is some limited numerical evidence to support this [3]. Establishing criteria for non-linear stability is a central open problem regarding the Einstein-Vlasov system. Thus, if this connection can be established rigorously, the algorithm has the potential to be of very great significance.

In [4] Andréasson et al. managed to convert the mass-preserving algorithm into a fixed-point proof, proving the existence of spherically symmetric static solutions to the Vlasov-Poisson system. We extend this result to a wider class of solutions in the Vlasov-Poisson case, and successfully carry out the equivalent proof for the Einstein-Vlasov system.

2

Theory

2.1 The Vlasov Equation

The Vlasov equation, or collisionless Boltzmann equation, is a kinetic matter model describing the evolution of a distribution of collisionless particles. Unlike a fluid model, which describes the evolution of a field over physical space, e.g. density, kinetic models describe the evolution of the particle distribution in phase space. Since kinetic models retain full phase space information they can describe more nuanced behaviour than fluid models.

The equation was used by Vlasov to model plasma, but when coupled with a model of gravity it can also be used as a model of galactic dynamics, among other things. The equation takes the form

$$\frac{df}{dt} = 0,$$

or more explicitly

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{v}} f = 0,$$

where $\mathbf{x}$ is the position, $\mathbf{v}$ the velocity vector (here taken to be equal to momentum) and $\mathbf{F}$ the force. The physical interpretation is that the distribution $f(\mathbf{x}, \mathbf{v}, t)$ remains constant along trajectories $(\mathbf{x}(t), \mathbf{v}(t))$ in phase space. Setting $\partial_t f = 0$ yields the *static Vlasov equation*

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{v}} f = 0.$$

Coupling the Vlasov equation with the Poisson equation gives rise to the Vlasov-Poisson system, which models a collisionless self-gravitating system under Newtonian gravity. We may also couple it to the Einstein equations, yielding the Einstein-Vlasov system, describing the same kind of system but under general relativity. The Vlasov-Poisson system may be viewed as a Newtonian limit of the Einstein-Vlasov system.

2.2 The Gravitational Vlasov-Poisson System

The gravitational Vlasov-Poisson system takes the form

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} U \cdot \nabla_{\mathbf{v}} f = 0 \\ \Delta U = 4\pi\rho \\ \rho = \int f d\mathbf{v} \end{cases},$$

where U is the gravitational potential and ρ the spatial density. A solution of the form

$$f = \Phi(E, L), \tag{2.1}$$

where $E = \frac{1}{2}|\mathbf{v}|^2 + U$ is the particle energy and $L = |x \times v|^2$ the squared modulus of angular momentum, is guaranteed to be static since it depends only on conserved quantities. Further, a well-known result, known as Jeans' theorem, states that every spherically symmetric static solution must take the above form. If f depends only on E the solution is said to be *isotropic*, and is otherwise called *anisotropic*. A commonly used ansatz is the polytropic ansatz

$$\Phi(E, L) = (E_0 - E)_+^k (L - L_0)_+^l,$$

which includes cut-offs for energy and angular momentum. Here

$$(x)_+^k := \begin{cases} 0 \text{ if } x \leq 0, \\ x^k \text{ else.} \end{cases}$$

2.3 The Mass-Preserving Algorithm for Vlasov-Poisson

Given some initial density ρ of mass M we may compute the gravitational potential U according to

$$U(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}.$$

From this we may compute the particle energy E, and plugging it into our ansatz $f = \Phi(E, L)$ yields a distribution function that is static in the gravitational field of ρ. Integrating f yields a new density

$$\tilde{\rho}(\mathbf{x}) = \int f(\mathbf{x}, \mathbf{v}) d\mathbf{v}$$

which will not necessarily equal the original density ρ. Thus (f, U, ρ) will generally not constitute a static solution of the Vlasov-Poisson system. We may, however, normalize $\tilde{\rho}$ to again have mass M by letting

$$\rho_{new} = A(\tilde{\rho})\tilde{\rho},$$

where

$$A(\tilde{\rho}) = M \left(\int \tilde{\rho} \; dx \right)^{-1}.$$

It turns out that iterating the above procedure, reapplying it each time to ρ_{new}, results in convergence, provided the right ansatz and initial density are used. We call this the *mass-preserving algorithm*. Assuming the algorithm has converged, so that $\rho_{new} = \rho$, replacing our ansatz with $f = A(\tilde{\rho})\Phi(E, L)$ results in a static solution (f, U, ρ) of the Vlasov-Poisson system; the Poisson equation and static Vlasov equation are satisfied by construction, and with the new ansatz we have

$$\int f \; dv = A(\tilde{\rho})\tilde{\rho} =: \rho_{new} = \rho.$$

Now define $T : \rho \mapsto \rho_{new}$. If we can prove that T has a fixed point this proves the existence of a static solution of the form $f \propto \Phi(E, L)$. Thus using fixed-point techniques we may convert the algorithm into an existence proof. This is the main idea of [4], as well as the present thesis.

2.4 The Static Einstein-Vlasov System in Spherical Symmetry

In Schwarzschild coordinates the metric of a static, spherically symmetric spacetime takes the form

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2),$$

where $r \geq 0,\ \theta \in [0, \pi],\ \varphi \in [0, 2\pi]$. We enforce asymptotic flatness via boundary conditions

$$\lim_{r\to\infty} \lambda(r) = \lim_{r\to\infty} \mu(r) = 0,$$

and a regular centre by

$$\lambda(0) = 0. \tag{2.2}$$

The static Einstein-Vlasov system is given by

$$\begin{cases} \frac{w}{\varepsilon}\partial_r f - (\mu_r \varepsilon - \frac{L}{r^3\varepsilon})\partial_w f = 0 & (2.3) \\ e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho & (2.4) \\ e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p & (2.5) \end{cases}$$

where $\varepsilon = \varepsilon(r, w, L) = \sqrt{1 + w^2 + \frac{L}{r^2}}$ and the matter quantities are given by

$$\rho(r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \varepsilon(r, w, L) f(r, w, L) dL dw, \tag{2.6}$$

$$p(r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\varepsilon(r, w, L)} f(r, w, L) dL dw. \tag{2.7}$$

The variable L may again be thought of as the angular momentum squared and $w = \frac{x \cdot v}{r}$ as the momentum in the radial direction.

If we define the quasi-local mass

$$m(r) := \int_0^r 4\pi\rho(s)s^2 ds$$

then the field equation (2.4) along with the boundary condition (2.2) implies the following useful identity:

$$e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r}. \tag{2.8}$$

Further by (2.5)

$$\mu'(r) = e^{2\lambda(r)} \left(4\pi p(r) r + \frac{m(r)}{r^2} \right) \geq 0$$

and so μ is increasing. The compactness ratio $\frac{2m(r)}{r}$ appearing in (2.8) is of great importance; if $\frac{2m(r)}{r} = 1$ a trapped surface forms.

As with Vlasov-Poisson, solutions of the form (2.1), here using particle energy $E = e^{\mu}\varepsilon$, are guaranteed to be static. The analogue of Jeans' theorem however does not hold for Einstein-Vlasov, but it is still the norm to use ansätze of this form when studying static solutions.

2.5 The Mass-Preserving Algorithm for Einstein-Vlasov

For Einstein-Vlasov the mass-preserving algorithm proceeds analogously to the Vlasov-Poisson case, although there are now two matter quantities involved. We begin with some initial pair ρ, p, with ADM-mass

$$M := \lim_{r\to\infty} m(r) = \int_0^\infty 4\pi\rho(s)s^2 ds.$$

We compute λ according to (2.8) and μ according to

$$\mu(r) = -\lambda(r) - \int_r^\infty 4\pi(\rho + p)s\frac{1}{1-\frac{2m}{s}}ds.$$

This allows us to compute E and at this point we may plug our ansatz $f = \Phi(E, L)$ into (2.6) and (2.7), yielding new matter quantities $\tilde{\rho}, \tilde{p}$. We normalize according to

$$\rho_{new} = A(\tilde{\rho})\tilde{\rho},\ p_{new} = A(\tilde{\rho})\tilde{p},$$

where

$$A(\tilde{\rho}) = M\left(\int \tilde{\rho}\, dx\right)^{-1}.$$

Note that both matter quantities are scaled by the same amplitude $A(\tilde{\rho})$, rather than two separate ones. We may again iterate the above procedure, and doing so results in convergence, provided the right ansatz and initial matter quantities are used. This then constitutes the mass-preserving algorithm for Einstein-Vlasov.

As before, assuming the algorithm has converged to some (ρ, p) we replace our ansatz with $f = A(\tilde{\rho})\Phi(E, L)$. This results in a static solution $(f, \lambda, \mu, \rho, p)$; the field equations (2.4)-(2.5) are fulfilled by construction of λ, μ, and the ansatz both satisfies the static Vlasov equation (2.3) and ensures consistency of the matter quantities (2.6)-(2.7). Proving the map $T : (\rho, p) \mapsto (\rho_{new}, p_{new})$ has a fixed point thus proves the existence of a static solution of the form $f \propto \Phi(E, L)$, just like for Vlasov-Poisson.

2.6 Functional Analysis and Schauder's Theorem

The notion of a compact mapping is essential in functional analysis, as it ensures nice convergence properties in infinite-dimensional spaces. We recall the definition:

Definition 2.6.1. *A map between Banach spaces X, Y is called compact if it is continuous and maps bounded subsets of X to precompact subsets of Y.*

The following theorem is what we will use to prove our main results. It is referred to as Schauder's fixed-point theorem, or simply Schauder's theorem.

Theorem 2.6.2 (Schauder's fixed-point theorem)**.** *Let X be a real Banach space and $C \subseteq X$ a nonempty, closed, bounded and convex subset, and $F : C \to C$ a compact map. Then F has a fixed point in C.*

In order to prove compactness of our maps we will utilize the Rellich-Kondrachov theorem. The relevant part of the theorem may be formulated as follows:

Theorem 2.6.3 (Rellich-Kondrachov)**.** *Let $1 \leq p < n$ and consider an open, bounded, Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. Then the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in the space $L^q(\Omega)$, for all $1 \leq q < \frac{np}{n-p}$.*

We note that if $n = 3$ and $1 \leq p < 3$ this in particular implies that $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$.

3

Fixed Point Proof for Vlasov-Poisson with an Anisotropic Ansatz

In [4] the existence of a fixed point of T, and hence a static solution to the Vlasov-Poisson system, was shown in the spherically symmetric case using an isotropic ansatz

$$f = (E_0 - E)_+^k.$$

We extend this result to an anisotropic ansatz

$$f = (E_0 - E)_+^k L^l$$

i.e. the polytropic ansatz with $L_0 = 0$. For convenience we introduce cumulative mass functions

$$m(r) := \int_0^r 4\pi\rho(s)s^2 ds,\ m_{new}(r) := \int_0^r 4\pi\rho_{new}(s)s^2 ds.$$

Throughout this and other sections we will freely utilize abuse of notation encoding spherical symmetry, such as $\rho(\mathbf{x}) = \rho(r)$.

3.1 Choice of Domain and Initial Calculations

Let

$$E_0 = -1,\ M = 1,\ 3/2 < q < 3,$$
$$k > -1,\ l > \max\{k + 1/2,\ -k - 1/2\},$$
$$0 < \alpha < 1,\ \left(1 + (1-\alpha)\alpha^{k+l+1/2}\right)^{-\frac{1}{2l+3}} \leq r_1 < 1,$$

We will study T as an operator on the set

$$S := \{\rho \in L^q(B) \mid \rho \geq 0 \text{ is spherically symmetric},\ m(1) = 1,\ m(r_1) \geq \alpha\},$$

where B is the open ball centred at the origin with radius 1. We treat functions in S as being extended by 0 to all of $\mathbb{R}^3$.

In plain words S is the set of L^q densities supported on B with total mass $M = 1$ and with at most $1 - \alpha$ of that mass located outside of radius r_1. The set S is a closed, convex subset of the Banach space $L^q(B)$.

For $\rho \in S$ the potential is given by

$$U_\rho(x) = -\int_B \frac{\rho(y)}{|x-y|}dy = -\frac{4\pi}{r}\int_0^r \rho(s)s^2 ds - 4\pi\int_r^1 \rho(s)s\ ds.$$

We see that U_ρ is increasing as a function of r and that $U_\rho(1) = -1$. We further have

$$\begin{aligned}
\tilde{\rho}(\mathbf{x}) &= \int_B (-1 - U_\rho(\mathbf{x}) - \frac{1}{2}|\mathbf{v}|^2)_+^k |\mathbf{x}\times\mathbf{v}|^{2l} d\mathbf{v} \\
&= 2\pi \int_0^\pi \int_0^\infty (-1 - U_\rho(\mathbf{x}) - \frac{1}{2}v^2)_+^k (|\mathbf{x}|v\sin(\varphi))^{2l} v^2 \sin(\varphi) dv\ d\varphi \\
&= 2\pi|\mathbf{x}|^{2l} \int_0^\pi \sin(\varphi)^{2l+1} d\varphi \int_0^\infty (-1 - U_\rho(\mathbf{x}) - \frac{1}{2}v^2)_+^k v^{2l+2} dv,
\end{aligned}$$

where we choose spherical coordinates (v, θ, φ) such that the pole aligns with $\mathbf{x}$, and the angle between $\mathbf{x}$ and $\mathbf{v}$ equals φ. Assuming that $U_\rho(\mathbf{x}) < -1$ we may make the substitution

$$u = \frac{-1 - U_\rho(\mathbf{x}) - \frac{1}{2}v^2}{-1 - U_\rho(\mathbf{x})},$$

yielding

$$\int_0^\infty (-1 - U_\rho(\mathbf{x}) - \frac{1}{2}v^2)_+^k v^{2l+2} dv = 2^{l+1/2}(-1 - U_\rho(\mathbf{x}))^{k+l+3/2} \int_0^1 u^k (1-u)^{l+1/2} du.$$

If $U_\rho(\mathbf{x}) \geq -1$ the original integral just equals 0. Thus we may write

$$\tilde{\rho}(\mathbf{x}) = c_{k,l}|\mathbf{x}|^{2l}(-1 - U_\rho(\mathbf{x}))_+^{k+l+3/2}, \tag{3.1}$$

where

$$c_{k,l} = 2^{l+3/2}\pi \int_0^\pi \sin(\varphi)^{2l+1} d\varphi \int_0^1 u^k (1-u)^{l+1/2} du. \tag{3.2}$$

The integral with respect to u is a beta integral and converges since our assumptions on k, l, q imply $k > -1$ and $l + 1/2 > -1$.

The function $\tilde{\rho}(r)$ is continuous, non-negative, spherically symmetric and vanishes outside of $\bar{B}$. If $U_\rho(0) = -1$ then $\tilde{\rho} \equiv 0$ and $A(\tilde{\rho}) = \infty$. This occurs precisely when ρ is a point mass placed at R, a distribution that may be approached from inside $L^q(B)$ by a sequence of step functions. Such behaviour prevents bounding of the range of T, making application of Schauder's theorem impossible.

In order to prevent this in the isotropic case, [4] introduces the constraint that ρ must be decreasing, allowing uniform bounding of U_ρ away from -1. The densities of anisotropic solutions are however not monotone, which is why we have instead introduced the tail-mass constraint $m(r_1) \geq \alpha$. Besides allowing for a more straightforward proof, this also yields sharp bounds.

3.2 Derivation of Uniform Bounds

We will need the following lemmas, proving uniform bounds on U_ρ and consequently on $A(\tilde{\rho})$.

Lemma 3.2.1. *Let $\rho \in S$. Then*

$$U_\rho(r) \leq \begin{cases} (\frac{1}{r_1} - 1)(1-\alpha) - \frac{1}{r_1}, & r \in [0, r_1], \\ (\frac{1}{r} - 1)(1-\alpha) - \frac{1}{r}, & r \in [r_1, 1]. \end{cases}$$

In particular, for $r \in [0, r_1]$,

$$-1 - U_\rho(r) \geq \delta := (\frac{1}{r_1} - 1)\alpha > 0.$$

Proof. Assume that $r \in [r_1, 1]$. Then

$$\begin{aligned} U_\rho(r) &= -\frac{4\pi}{r}\int_0^r \rho(s)s^2 ds - 4\pi \int_r^1 \rho(s) s ds \\ &= -\frac{4\pi}{r}\int_0^{r_1} \rho(s)s^2 ds - \frac{4\pi}{r}\int_{r_1}^r \rho(s)s^2 ds - 4\pi \int_r^1 \rho(s) s ds \\ &= -\frac{m(r_1)}{r} - 4\pi \int_{r_1}^1 f(s)\rho(s)s^2 ds \\ &\leq -\frac{m(r_1)}{r} - (1 - m(r_1)) \inf_{s \in [r_1, 1]} \{f(s)\} \\ &= -\frac{m(r_1)}{r} - (1 - m(r_1)) \\ &= (\frac{1}{r} - 1)(1 - m(r_1)) - \frac{1}{r} \\ &\leq (\frac{1}{r} - 1)(1 - \alpha) - \frac{1}{r}, \end{aligned}$$

where

$$f(s) = \begin{cases} \dfrac{1}{r}, & r_1 \leq s \leq r, \\ \dfrac{1}{s}, & r < s \leq 1. \end{cases}$$

Now assume $r \in [0, r_1]$. Since U_ρ is increasing we get

$$U_\rho(r) \leq U_\rho(r_1) \leq (\frac{1}{r_1} - 1)(1-\alpha) - \frac{1}{r_1}.$$

The bound $-1 - U_\rho \geq \delta$ follows directly. □

Remark 3.2.2. *This bound is in fact sharp, in the sense that it is a supremum, attained by placing a point mass $1-\alpha$ at $r = 1$ and a point mass α at $r = r_1$, which can be approached by a sequence of step functions inside S. For this reason we thought it might be of interest to include the bound over $(r_1, 1]$, even though we do not directly utilize it in our proof.*

Lemma 3.2.3. *Let $\rho \in S$. Then $U_\rho(r) \geq -\frac{1}{r}$.*

Proof. We have

$$\begin{aligned} U_\rho(r) &= -\frac{4\pi}{r}\int_0^r \rho(s)s^2 ds - 4\pi\int_r^1 \rho(s)s ds \\ &\geq -\frac{4\pi}{r}\int_0^r \rho(s)s^2 ds - 4\pi\int_r^1 \rho(s)s\frac{s}{r} ds \\ &= -\frac{4\pi}{r}\int_0^1 \rho(s)s^2 ds = -\frac{1}{r}. \end{aligned}$$

□

Lemma 3.2.4. *There exist constants $A_1, A_2 > 0$ such that*

$$A_1 \leq A(\tilde{\rho}) \leq A_2,\ \forall \rho \in S.$$

Proof. By Lemma 3.2.3 and the formula (3.1),

$$\begin{aligned} \int_B \tilde{\rho}(\mathbf{x})d\mathbf{x} &= \int_B c_{k,l}|\mathbf{x}|^{2l}(-1-U_\rho(\mathbf{x}))_+^{k+l+3/2}d\mathbf{x} \\ &= 4\pi c_{k,l}\int_0^1 r^{2l+2}(-1-U_\rho(r))_+^{k+l+3/2}dr \\ &\leq 4\pi c_{k,l}\int_0^1 r^{2l+2}\left(\frac{1}{r}-1\right)_+^{k+l+3/2}dr \\ &= 4\pi c_{k,l}\int_0^1 r^{2l+2}\left(\frac{1}{r}\right)^{k+l+3/2}dr \\ &= 4\pi c_{k,l}\int_0^1 r^{l-k+1/2}dr < \infty, \end{aligned}$$

where finiteness of the final integral follows from $k < l + 1/2$. This proves existence of A_1. Now by Lemma 3.2.1 we also have

$$\begin{aligned} \int_B \tilde{\rho}(\mathbf{x})d\mathbf{x} &= 4\pi c_{k,l}\int_0^1 r^{2l+2}(-1-U_\rho(r))_+^{k+l+3/2}dr \\ &\geq 4\pi c_{k,l}\delta^{k+l+3/2}\int_0^{r_1} r^{2l+2}dr > 0, \end{aligned}$$

where the final integral is also finite since $2l + 2 > -1$, and so no contradiction results. This proves the existence of A_2 and completes the proof. □

Remark 3.2.5. *The lower bound on $A(\tilde{\rho})$ away from 0 is not directly utilized in our proof, but we nevertheless include it, as it is interesting in its own right.*

3.3 The Domain S is Invariant Under T

First, we show that $\rho_{new} \in L^q(B)$. By (3.1) and Lemma 3.2.3 we have

$$\begin{aligned} \|\rho_{new}\|_{L^q(B)}^q &= A(\tilde{\rho})^q c_{k,l}^q \int_B |\mathbf{x}|^{2lq}(-1-U_\rho(\mathbf{x}))^{q(k+l+3/2)}d\mathbf{x} \\ &\leq A_2^q c_{k,l}^q\int_0^1 r^{2lq+2}\left(\frac{1}{r}-1\right)^{q(k+l+3/2)}dr \\ &= A_2^q c_{k,l}^q\int_0^1 r^{q(l-k-3/2)+2}\left(1-r\right)^{q(k+l+3/2)}dr. \end{aligned} \tag{3.3}$$

This is a beta integral which converges since our assumptions on k, l, q imply

$$q(l-k-3/2)+2>-1,\ q(k+l+3/2)>-1$$

Clearly, ρ_{new} is non-negative, spherically symmetric and has mass 1. It remains to verify that $m_{new}(r_1) \geq \alpha$, where m_{new} is the quasi-local mass induced by ρ_{new}. We have

$$m_{new}(r_1) = \frac{\int_0^{r_1} \tilde{\rho}(s)s^2 ds}{\int_0^1 \tilde{\rho}(s)s^2 ds} = \frac{1}{1+\frac{\int_{r_1}^1 \tilde{\rho}(s)s^2 ds}{\int_0^{r_1} \tilde{\rho}(s)s^2 ds}}.$$

Now by Lemma 3.2.1 we have

$$\begin{aligned}\int_0^{r_1} \tilde{\rho}(s)s^2 ds &\geq c_{k,l}\delta^{k+l+3/2}\int_0^{r_1} s^{2l+2} ds\\ &= c_{k,l}\alpha^{k+l+3/2}\left(\frac{1}{r_1}-1\right)^{k+l+3/2}\frac{r_1^{2l+3}}{2l+3},\end{aligned}$$

and from Lemma 3.2.3

$$\begin{aligned}\int_{r_1}^1 \tilde{\rho}(s)s^2 ds &\leq c_{k,l}\int_{r_1}^1 s^{2l+2}(\frac{1}{s}-1)^{k+l+3/2} ds\\ &\leq c_{k,l}(\frac{1}{r_1}-1)^{k+l+3/2}\int_{r_1}^1 s^{2l+2} ds\\ &= c_{k,l}(\frac{1}{r_1}-1)^{k+l+3/2}\frac{1-r_1^{2l+3}}{2l+3}.\end{aligned}$$

Thus

$$m_{new}(r_1) \geq \frac{1}{1+\frac{1}{\alpha^{k+l+3/2}}\left(\frac{1}{r_1^{2l+3}}-1\right)} \geq \alpha,$$

where the final inequality follows from

$$r_1 \geq \left(1+(1-\alpha)\alpha^{k+l+1/2}\right)^{-\frac{1}{2l+3}}.$$

Thus $\rho_{new} \in S$ and $T(S) \subseteq S$, and we may consider T as an operator $T : S \to S$.

3.4 Compactness of T

We will decompose T into a number of different mappings T_i, and denote by S_i the range of T_i over its domain.

We first note that, by Hölder,

$$\begin{aligned}
\|U_\rho\|^q_{L^q(B)} &= \int_0^1 4\pi \left| \frac{4\pi}{r} \int_0^r \rho(s)s^2 ds + 4\pi \int_r^1 \rho(s)s\ ds \right|^q r^2 dr \\
&\leq \int_0^1 4\pi \left| \int_0^1 4\pi\rho(s)s^2 ds \right|^q r^{2-q} dr \\
&\leq \int_0^1 4\pi \int_0^1 (4\pi)^q |\rho(s)|^q s^{2q} ds\ r dr \\
&\leq \int_0^1 (4\pi)^q \int_0^1 4\pi |\rho(s)|^q s^2 ds\ r^{2q-1} dr \\
&= \|\rho\|^q_{L^q(B)} \int_0^1 (4\pi)^q r^{2q-1} dr.
\end{aligned}$$

Further,

$$\|\Delta U_\rho\|_{L^q(B)} = \|4\pi\rho\|_{L^q(B)} = 4\pi\|\rho\|_{L^q(B)}.$$

Thus the linear mapping $S \ni \rho \mapsto U_\rho \in W^{1,q}(B)$ is bounded, and hence also continuous. Since by Rellich-Kondrachov $W^{1,q}(B)$ is compactly embedded in $L^q(B)$ we conclude that $T_U : S \ni \rho \mapsto U_\rho \in L^q(B)$ is a compact mapping.

Now let

$$\phi(\xi) = (-1-\xi)^{k+l+3/2},\ \xi \in \left[-\frac{1}{r}, -1\right].$$

Since $k + l + 1/2 > 0$ we have

$$|\phi(\xi)'| = (k+l+3/2)(-1-\xi)^{k+l+1/2} \leq (k+l+3/2)\left(\frac{1}{r} - 1\right)^{k+l+1/2}.$$

Assume that $U_n \to U$ in S_U. Recalling that $-\frac{1}{r} \leq U_\rho \leq -1,\ \forall U_\rho \in S_U$, the mean value theorem then yields

$$\begin{aligned}
\|\tilde\rho_n - \tilde\rho\|^q_{L^q(B)} &= c_{k,l} \int_0^1 4\pi r^{2lq} \left|(-1-U_{\rho_n}(r))^{k+l+3/2} - (-1-U_\rho(r))^{k+l+3/2}\right|^q r^2 dr \\
&\leq c_{k,l}(k+l+3/2)^q \int_0^1 4\pi r^{2lq} \left(\frac{1}{r} - 1\right)^{q(k+l+1/2)} |U_{\rho_n} - U_\rho|^q r^2 dr \\
&\leq c_{k,l}(k+l+3/2)^q \int_0^1 4\pi\, |U_{\rho_n} - U_\rho|^q r^2 r^{q(l-k-1/2)} dr \\
&\leq c_{k,l}(k+l+3/2)^q \|U_{\rho_n} - U_\rho\|^q_{L^q(B)} \to 0,
\end{aligned}$$

where the last inequality follows since $q(l-k-1/2) > 0$. Hence the mapping $T_{\tilde\rho} : S_U \ni U_\rho \mapsto \tilde\rho \in L^q(B)$ is continuous. The functional $S_{\tilde\rho} \ni \tilde\rho \mapsto \int_B \tilde\rho\ d\mathbf{x} \in \mathbb{R}$ is continuous by linearity and boundedness:

$$\left|\int_B \tilde\rho\ d\mathbf{x}\right| = \|\tilde\rho\|_{L^1(B)} \leq \left(\frac{4\pi R^3}{3}\right)^{1-\frac{1}{q}} \|\tilde\rho\|_{L^q(B)}.$$

Finally, assume that $\tilde{\rho}_n \to \tilde{\rho}$ in $S_{\tilde{\rho}}$. Then

$$\begin{aligned}
\left\| \frac{M\tilde{\rho}_n}{\int_B \tilde{\rho}_n \, d\mathbf{x}} - \frac{M\tilde{\rho}}{\int_B \tilde{\rho} \, d\mathbf{x}} \right\|_{L^q(B)} &= M \left\| \frac{\int_B \tilde{\rho} \, d\mathbf{x}(\tilde{\rho}_n - \tilde{\rho}) - (\int_B \tilde{\rho}_n \, d\mathbf{x} - \int_B \tilde{\rho} \, d\mathbf{x})\tilde{\rho}}{\int_B \tilde{\rho}_n \, d\mathbf{x} \int_B \tilde{\rho} \, d\mathbf{x}} \right\|_{L^q(B)} \\
&\leq M \frac{\int_B \tilde{\rho} \, d\mathbf{x} \|\tilde{\rho}_n - \tilde{\rho}\|_{L^q(B)} + |\int_B \tilde{\rho}_n \, d\mathbf{x} - \int_B \tilde{\rho} \, d\mathbf{x}| \|\tilde{\rho}\|_{L^q(B)}}{\int_B \tilde{\rho}_n \, d\mathbf{x} \int_B \tilde{\rho} \, d\mathbf{x}} \\
&= A(\tilde{\rho}_n) \frac{\int_B \tilde{\rho} \, d\mathbf{x} \|\tilde{\rho}_n - \tilde{\rho}\|_{L^q(B)} + |\int_B \tilde{\rho}_n \, d\mathbf{x} - \int_B \tilde{\rho} \, d\mathbf{x}| \|\tilde{\rho}\|_{L^q(B)}}{\int_B \tilde{\rho} \, d\mathbf{x}} \\
&\leq A_2 \frac{\int_B \tilde{\rho} \, d\mathbf{x} \|\tilde{\rho}_n - \tilde{\rho}\|_{L^q(B)} + |\int_B \tilde{\rho}_n \, d\mathbf{x} - \int_B \tilde{\rho} \, d\mathbf{x}| \|\tilde{\rho}\|_{L^q(B)}}{\int_B \tilde{\rho} \, d\mathbf{x}} \\
&\to 0,
\end{aligned}$$

where we used continuity of the above-mentioned functional to get the final limit. Thus $S_{\tilde{\rho}} \ni \tilde{\rho} \mapsto \rho_{new} \in L^q(B)$ is continuous, and $T : S \to S$ is compact as a whole.

3.5 Application of Schauder's Theorem

We begin by noting that the range $T(S)$ is bounded, by the uniform bound on $\|\rho_{new}\|^q_{L^q(B)}$ yielded in (3.3). With this stated we are ready to apply Schauder's theorem.

Theorem 3.5.1. *The operator T has a fixed point in S. In particular the Vlasov-Poisson system has a spherically symmetric steady state of the form $f \propto (E_0 - E)^k_+ L^l$.*

Proof. By boundedness of $T(S)$ we may define $r^* := \sup\{\|T\rho\|_{L^q(B)} : \rho \in S\}$. The set $S_{r^*} := \{\rho \in S : \|\rho\|_{L^q(B)} \leq r^*\}$ is closed, bounded and convex, and $T : S_{r^*} \to S_{r^*}$ is a compact operator. Hence T has a fixed point in S_{r^*} by Schauder's theorem. $\square$

4

Fixed Point Proof for Einstein-Vlasov with an Isotropic Ansatz

In this chapter, we prove the existence of a fixed point in the Einstein-Vlasov case. This is done using the convenient isotropic ansatz

$$f = (E_0 - E)_+^0. \tag{4.1}$$

The overall structure of the proof is very similar to the one for Vlasov-Poisson.

4.1 Choice of Domain and Initial Calculations

Let

$$\frac{1}{\sqrt{3}} < E_0 < 1,\ M = 1,\ R = \frac{2}{1 - E_0^2},\ 3/2 < q < 3$$
$$0 < \alpha < 1,\ R\alpha^{1/3} < r_1 < R,\ K \gg 0$$

and define the set

$$S := \{(\rho, p) \in L^q(B) \times L^q(B) :\ \rho, p \geq 0 \text{ are spherically symmetric,}$$
$$2m(r) + m(r)^{1/3} \leq r\ ,\ m(R) = 1,\ m(r_1) \geq \alpha,\ p \leq \frac{1}{3}\rho,\ \rho \leq K\},$$

where B is the open ball of radius R centred at the origin. The set S is a nonempty, closed subset of $L^q(B) \times L^q(B)$. The constraint $2m(r) + m(r)^{1/3} \leq r$ is convex since $\rho \mapsto m$ is linear and the function $\xi \mapsto 2\xi + \xi^{1/3}$ is increasing. The other constraints are all affine or linear. Hence S is also convex.

Remark 4.1.1. *Requiring $(\rho, p) \in L^p(B) \times L^p(B)$ may seem strange when $\rho \leq K$ ensures $(\rho, p) \in L^\infty(B) \times L^\infty(B)$. The L^q space is however needed for its topology, which we will use when considering questions like continuity and compactness.*

Plugging our ansatz (4.1) into (2.6)-(2.7) yields the formulas

$$\tilde{\rho}(r) = 4\pi \int_1^{E_0 e^{-\mu(r)}} x^2\sqrt{x^2 - 1}dx,$$
$$\tilde{p}(r) = \frac{4\pi}{3} \int_1^{E_0 e^{-\mu(r)}} (x^2 - 1)^{3/2}dx.$$

The matter quantities $\tilde{\rho}, \tilde{p}$ are decreasing and we have the useful identity $\tilde{\rho}(r) = F(E_0 e^{-\mu(r)})$ where F is the increasing function

$$F(x) = \frac{\pi}{2}\left(\sqrt{x^2-1}(2x^3 - x) - \ln\left(\sqrt{x^2-1} + x\right)\right).$$

A major difference from the Vlasov-Poisson case is that we in addition to bounding the tail-mass also have to introduce a bound excluding densities that are too compact. The reason is twofold:

First, we need to prevent the compactness ratio $\frac{2m(r)}{r}$ from approaching 1, as this introduces singularities. Second, it turns out to be necessary for producing a uniform lower bound on $\mu(0)$. The obvious linear compactness bound $\frac{2m(r)}{r} \leq \beta < 1$ however, does not suffice to produce the latter. This is why we have introduced a bound of the form

$$2m(r) + m(r)^{\gamma} \leq r,$$

where $0 < \gamma < 1$, which makes it straight-forward to derive a lower bound on $\mu(0)$, and by extension $A(\tilde{\rho})$. The bound implies in particular the linear bound $\frac{2m(r)}{r} \leq \frac{2}{3}$, which it approaches as $\gamma \to 1$. This linear bound is in fact physically meaningful; $\frac{2m(r)}{r} = \frac{2}{3}$ precisely corresponds to the formation of a photon sphere. The choice $\gamma = 1/3$ is based on numerical considerations and analytical convenience.

4.2 Derivation of Uniform Bounds

In this section we prove bounds on μ and $A(\tilde{\rho})$. First we use the tail-mass bound to produce an upper bound on μ.

Lemma 4.2.1. *For every $\rho, p \in S$ we have*

$$\mu(r) \leq \ln(E_0) - \frac{1}{2}\ln\left(\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}\right), \ \forall r \in [0, r_1].$$

Proof. We have

$$\begin{aligned}
\mu(r_1) &= -\frac{1}{2}\ln\left(\frac{1}{1 - \frac{2m(r_1)}{r_1}}\right) - \int_{r_1}^{R} \frac{4\pi(\rho + p)s}{1 - \frac{2m(s)}{s}} ds \\
&= -\frac{1}{2}\ln\left(\frac{r_1}{r_1 - 2m(r_1)}\right) - \int_{r_1}^{R} \frac{4\pi(\rho + p)s^2}{s - 2m(s)} ds.
\end{aligned}$$

But

$$\begin{aligned}
\int_{r_1}^{R} \frac{4\pi(\rho + p)s^2}{s - 2m(s)} ds &\geq \int_{r_1}^{R} \frac{4\pi\rho(s)s^2}{s - 2m(s)} ds \geq \int_{r_1}^{R} \frac{4\pi\rho(s)s^2}{R - 2m(s)} ds \\
&= \int_{m(r_1)}^{1} \frac{1}{R - 2u} du = \frac{1}{2}\ln\left(\frac{R - 2m(r_1)}{R - 2}\right),
\end{aligned}$$

where we used $m'(r) = 4\pi\rho(r)r^2$ in the substitution step. Thus

$$\begin{aligned}
\mu(r_1) &\leq -\frac{1}{2}\ln\left(\frac{r_1}{r_1 - 2m(r_1)}\right) - \frac{1}{2}\ln\left(\frac{R-2m(r_1)}{R-2}\right)\\
&= -\frac{1}{2}\ln\left(\frac{r_1}{r_1-2m(r_1)}\frac{R-2m(r_1)}{R}\frac{R}{R-2}\right)\\
&= \ln(E_0) - \frac{1}{2}\ln\left(\frac{Rr_1 - 2m(r_1)r_1}{Rr_1 - 2m(r_1)R}\right)\\
&\leq \ln(E_0) - \frac{1}{2}\ln\left(\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}\right),
\end{aligned}$$

since

$$\frac{Rr_1 - 2m(r_1)r_1}{Rr_1 - 2m(r_1)R} = 1 + \frac{R - r_1}{\frac{Rr_1}{2m(r_1)} - R} \geq 1 + \frac{R - r_1}{\frac{Rr_1}{2\alpha} - R} = \frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}.$$

Because μ is increasing $\mu(r) \leq \mu(r_1)$ for $r \in [0, r_1]$ which completes the proof. □

In the following lemma we see how a lower bound on μ follows almost trivially from the compactness bound $2m(r) + m(r)^{1/3} \leq r$

Lemma 4.2.2. *Let $\rho, p \in S$. Then $\mu(0) \geq -2$.*

Proof. We have $\mu(0) = -\int_0^R \frac{4\pi(\rho+p)s^2}{s-2m(s)}ds \geq -\frac{4}{3}\int_0^R \frac{4\pi\rho(s)s^2}{m(s)^{\frac{1}{3}}}ds = -\frac{4}{3}\int_0^1 \frac{1}{u^{\frac{1}{3}}}du = -2.$ □

Finally, using the previous two lemmas, we may derive upper and lower bounds on $A(\tilde{\rho})$.

Lemma 4.2.3. *There are constants $0 < A_1, A_2$ such that $A_1 \leq A(\tilde{\rho}) \leq A_2$ for all $\rho, p \in S$*

Proof. By Lemma 4.2.2, and that $\tilde{\rho}$ is decreasing,

$$\int_0^R 4\pi\tilde{\rho}(s)s^2 ds \leq \frac{4\pi R^3}{3}\tilde{\rho}(0) \leq \frac{4\pi R^3}{3}F(E_0 e^2) < \infty.$$

This proves the existence of A_1. By Lemma 4.2.1,

$$\int_0^R 4\pi\tilde{\rho}(s)s^2 ds \geq \int_0^{r_1} 4\pi\tilde{\rho}(s)s^2 ds \geq \frac{4\pi r_1^3}{3}\tilde{\rho}(r_1) \geq \frac{4\pi r_1^3}{3}F\left(\sqrt{\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}}\right) > 0,$$

since $\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R} > 1$, which proves the existence of A_2. □

Remark 4.2.4. *Again, we include the lower bound on $A(\tilde{\rho})$ even if we do not directly utilize it for our proof.*

4.3 The Domain S is Invariant Under T

Since $\tilde{\rho}$ is decreasing and K is large Lemma 4.2.2 together with Lemma 4.2.1 imply that $\rho_{new} \leq K$. We also have that $m_{new}(R) = 1$ by construction. The bound $r_1 \geq R\alpha^{1/3}$ together with $\tilde{\rho}$ being decreasing yields

$$m_{new}(r_1) = \frac{\int_0^{r_1} \tilde{\rho}(s)s^2 ds}{\int_0^R \tilde{\rho}(s)s^2 ds} = \frac{1}{1 + \frac{\int_{r_1}^R \tilde{\rho}(s)s^2 ds}{\int_0^{r_1} \tilde{\rho}(s)s^2 ds}} \geq \frac{1}{1 + \frac{\int_{r_1}^R \tilde{\rho}(r_1)s^2 ds}{\int_0^{r_1} \tilde{\rho}(r_1)s^2 ds}} = \frac{r_1^3}{R^3} \geq \alpha,$$

and $p_{new} \leq \frac{1}{3}\rho_{new}$ since

$$\tilde{p} = \frac{4\pi}{3}\int_1^{E_0 e^{-\mu(r)}} (x^2-1)\sqrt{x^2-1}dx \leq \frac{4\pi}{3}\int_1^{E_0 e^{-\mu(r)}} x^2\sqrt{x^2-1}dx = \frac{1}{3}\tilde{\rho}.$$

From this we also get that $(\rho_{new}, p_{new}) \in L^q(B) \times L^q(B)$ since $p_{new} \leq \rho_{new} \leq K$. Finally we need to prove that $2m_{new}(r) + m_{new}(r)^{\frac{1}{3}} \leq r$. For $r \geq 3$ the inequality holds automatically since $m_{new} \leq 1$. It remains to show that it holds for $r < 3$. We have

$$m_{new}(r) = \frac{\int_0^r \tilde{\rho}(s)s^2 ds}{\int_0^R \tilde{\rho}(s)s^2 ds} \leq \frac{\int_0^r \tilde{\rho}(s)s^2 ds}{\int_0^{r_1} \tilde{\rho}(s)s^2 ds} \leq \frac{\tilde{\rho}(0)\int_0^r s^2 ds}{\tilde{\rho}(r_1)\int_0^{r_1} s^2 ds} = \frac{\tilde{\rho}(0)}{\tilde{\rho}(r_1)r_1^3} r^3 =: Cr^3.$$

Assume that $C \leq \frac{1}{27}$. Then

$$2m_{new}(r) + m_{new}(r)^{\frac{1}{3}} \leq 2Cr^3 + (Cr^3)^{\frac{1}{3}} \leq r$$

for $r \leq 3$. Thus it suffices to show that $\frac{\tilde{\rho}(0)}{\tilde{\rho}(r_1)r_1^3} \leq \frac{1}{27}$. But since F is increasing we get, by Lemma 4.2.1 and Lemma 4.2.2,

$$\frac{\tilde{\rho}(0)}{\tilde{\rho}(r_1)r_1^3} = \frac{F(E_0 e^{-\mu(0)})}{r_1^3 F(E_0 e^{-\mu(r_1)})} \leq \frac{F(E_0 e^2)}{r_1^3 F\left(\sqrt{\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}}\right)}.$$

Since $R\alpha^{1/3} \leq r_1 \leq R$ we may write $r_1 = cR$ with $\alpha^{1/3} \leq c \leq 1$. Thus

$$\begin{aligned}
\sqrt{\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}} &= \sqrt{1 + \frac{1}{R}\frac{2\alpha(1-c)}{c - \frac{2\alpha}{R}}} \\
&\geq \sqrt{1 + \frac{1}{R}\frac{2\alpha(1-c)}{c}} \\
&= \sqrt{1 + \frac{2\alpha\left(\frac{1}{c} - 1\right)}{R}} \\
&=: \sqrt{1 + \frac{\Gamma}{R}}.
\end{aligned}$$

From $\ln\left(\sqrt{x^2-1} + x\right) \leq \sqrt{x^2-1}$ we have, for $x \geq 1$,

$$\begin{aligned}
F(x) &\geq \frac{\pi}{2}\sqrt{x^2-1}(2x^3 - x - 1) \\
&\geq \frac{\pi}{2}\sqrt{x^2-1}(x^2-1) \\
&= \frac{\pi}{2}(x^2-1)^{3/2}.
\end{aligned}$$

Since F is increasing we thus get

$$
\begin{aligned}
r_1^3 F\left(\sqrt{\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}}\right) &\geq c^3 R^3 F\left(\sqrt{1 + \frac{\Gamma}{R}}\right) \\
&\geq c^3 R^3 \frac{\pi}{2} \left(\frac{\Gamma}{R}\right)^{3/2} \\
&\longrightarrow \infty \text{ as } R \to \infty.
\end{aligned}
$$

But $R \to \infty$ as $E_0 \to 1^-$. Thus $C \to 0$ as E_0 approaches 1, and $T(S) \subseteq S$ for E_0 sufficiently close to 1.

Remark 4.3.1. *Numerically, we begin finding combinations r_1, α such that*

$$
\frac{F(E_0 e^2)}{r_1^3 F\left(\sqrt{\frac{Rr_1 - 2\alpha r_1}{Rr_1 - 2\alpha R}}\right)} \leq \frac{1}{27}
$$

at around $E_0 = 0.9999$ and above. This results in very large R-values and the corresponding solutions we find using the mass-preserving algorithm are extremely non-relativistic.

4.4 Continuity of T

Lemma 4.4.1. *Let $(\rho, p) \in S$. Then $\lambda \in L^q(B)$ and the mapping $S \ni (\rho, p) \mapsto \lambda \in L^q(B)$ is continuous.*

Proof. The bound $2m/r \leq 2/3$ implies that λ is bounded and thus $2m/r \in L^q(B)$ and $\lambda \in L^q(B)$. The function

$$
\phi(\xi) = -\frac{1}{2} \ln(1 - \xi), \ \xi \in [0, 2/3]
$$

is Lipschitz by the mean value theorem. Since $\lambda = \phi(2m/r)$ it thus suffices to prove that $S \ni (\rho, p) \mapsto 2m/r \in L^q(B)$ is continuous. Assume that $(\rho_n, p_n) \to (\rho, p)$ in S. Then, by Hölder,

$$
\begin{aligned}
\left\| \frac{2m_n}{r} - \frac{2m}{r} \right\|_{L^q(B)}^q &= \int_0^R 4\pi \left| \frac{2m_n(r)}{r} - \frac{2m(r)}{r} \right|^q r^2 dr \\
&= 2^q \int_0^R 4\pi \left| \int_0^r 4\pi(\rho_n(s) - \rho(s)) s^2 ds \right|^q r^{2-q} dr \\
&\leq 2^q \int_0^R 4\pi r^{q-1} \int_0^r (4\pi)^q |\rho_n(s) - \rho(s)|^q s^{2q} ds \ r^{2-q} dr \\
&\leq 2^q \int_0^R (4\pi)^q \int_0^r 4\pi |\rho_n(s) - \rho(s)|^q s^2 ds \ r^{2q-1} dr \\
&\leq 2^q \int_0^R (4\pi)^q \int_0^R 4\pi |\rho_n(s) - \rho(s)|^q s^2 ds \ r^{2q-1} dr \\
&= \|\rho_n - \rho\|_{L^q(B)}^q 2^q \int_0^R (4\pi)^q r^{2q-1} dr \to 0,
\end{aligned}
$$

completing the proof. □

Lemma 4.4.2. *Let* $(\rho, p) \in S$. *Then*

$$\int_r^R \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} ds \in L^q(B)$$

and the mapping

$$S \ni (\rho, p) \mapsto \int_r^R \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} ds \in L^q(B)$$

is continuous.

Proof. The mapping $S \ni (\rho, p) \mapsto 4\pi(\rho+p)s \in L^q(B)$ is continuous since it is linear and bounded. The bound $\frac{1}{1-2m/s} \leq 3$ implies that $\frac{1}{1-2m/s} \in L^q(B)$. By the mean value theorem

$$\phi(\xi) = \frac{1}{1-\xi},\ \xi \in [0, 2/3]$$

is Lipschitz. Thus, we may argue that $S \ni (\rho, p) \mapsto \frac{1}{1-2m/s} \in L^q(B)$ is continuous, analogously to the previous lemma. We conclude that the mapping

$$T_1 : S \ni (\rho, p) \mapsto \left(4\pi(\rho+p)s, \frac{1}{1-2m/s}\right) \in L^q(B) \times L^q(B)$$

is continuous. The mapping

$$T_2 : S_1 \ni \left(4\pi(\rho+p)s, \frac{1}{1-2m/s}\right) \mapsto \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \in L^q(B)$$

is bilinear. Since $\|\frac{1}{1-2m/r}\|^q_{L^q(B)} \geq \frac{4\pi R^3}{3}$ we have

$$\begin{aligned}
\left\| \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \right\|^q_{L^q(B)} &\leq 3^q \left\| 4\pi(\rho+p)s \right\|^q_{L^q(B)} \\
&\leq 3^q \frac{3}{4\pi R^3} \left\| \frac{1}{1-2m/r} \right\|^q_{L^q(B)} \left\| 4\pi(\rho+p)s \right\|^q_{L^q(B)}.
\end{aligned}$$

Thus T_2 is bounded, and hence also continuous. Finally, the mapping

$$T_3 : S_2 \ni \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \mapsto \int_r^R \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} ds \in L^q(B)$$

is linear, and by Hölder we have

$$\begin{aligned}
\left\| \int_r^R \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} ds \right\|_{L^q(B)}^q &= \int_0^R 4\pi \left| \int_r^R \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} ds \right|^q r^2 dr \\
&\leq \int_0^R 4\pi (R-r)^{q-1} \int_r^R \left| \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \right|^q ds\ r^2 dr \\
&= \int_0^R (R-r)^{q-1} \int_r^R 4\pi \left| \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \right|^q r^2 ds\ dr \\
&\leq \int_0^R (R-r)^{q-1} \int_r^R 4\pi \left| \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \right|^q s^2 ds\ dr \\
&\leq \int_0^R (R-r)^{q-1} \int_0^R 4\pi \left| \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \right|^q s^2 ds\ dr \\
&= \left\| \frac{4\pi(\rho+p)s}{1-\frac{2m}{s}} \right\|_{L^q(B)}^q \int_0^R (R-r)^{q-1} dr.
\end{aligned}$$

Thus T_3 is bounded, and hence also continuous, completing the proof. □

From Lemmas 4.4.1 and 4.4.2 we conclude that $T_\mu : S \ni (\rho, p) \mapsto \mu \in L^q(B)$ is continuous. Further, letting

$$\phi(\xi) = \int_1^{E_0 e^{-\xi}} x^2 \sqrt{x^2-1} dx,\ \xi \in [-2, \ln E_0]\ ,$$

we have that $\tilde{\rho} = \phi(\mu)$ and by the mean value theorem ϕ is Lipschitz. The analogous statement holds for $\tilde{p}$. Since by Lemmas 4.2.2 and 4.2.1 $-2 \leq \mu \leq \ln E_0$ it follows that $T_{(\tilde{\rho},\tilde{p})} : S_\mu \ni \mu \mapsto (\tilde{\rho}, \tilde{p}) \in L^q(B) \times L^q(B)$, is continuous. The proof that

$$S_{(\tilde{\rho},\tilde{p})} \ni (\tilde{\rho}, \tilde{p}) \mapsto (\rho_{new}, p_{new}) \in S$$

is continuous is the same as for the analogous statement in section 3.4. From this we conclude that $T : S \to S$ is continuous.

4.5 The Operator T is Compact

If we can prove a uniform upper bound on $\|\mu\|_{W^{1,q}(B)}$ over S then $\mu \in W^{1,q}(B)$ and $S \ni (\rho, p) \mapsto \mu \in W^{1,q}(B)$ maps subsets of S to bounded subsets of $W^{1,q}(B)$. Since by Rellich-Kondrachov $W^{1,q}(B)$ is compactly embedded in $L^q(B)$ this would imply that $T_\mu : S \ni (\rho, p) \mapsto \mu \in L^q(B)$ maps bounded subsets of S to precompact sets in $L^q(B)$. Together with the continuity results in the previous section this would yield that T is compact as a whole.

Since μ is increasing and $\mu(r) < 0$ we get

$$\|\mu\|_{L^q(B)} = \left(4\pi \int_0^R |\mu(s)|^q s^2 ds \right)^{1/q} \leq |\mu(0)| \left(4\pi R^3\right)^{1/q} \leq 2 \left(4\pi R^3\right)^{1/q}.$$

For μ', we have the following identity:

$$\mu'(r) = \frac{4\pi p(r) r^2}{r - 2m(r)} - \frac{m(r)}{r(r - 2m(r))}.$$

Since $p \leq \frac{1}{3}\rho$ and $\rho \leq K$ we get

$$\begin{aligned}
\left\| \frac{4\pi p(r) r^2}{r - 2m(r)} \right\|_{L^q(B)} &= \left(4\pi \int_0^R \left| \frac{4\pi p(s) s^2}{s - 2m(s)} \right|^q s^2 ds \right)^{1/q} \\
&\leq \left(4\pi R^2 \int_0^R \left(\frac{4\pi \rho(s) s^2}{3m(s)^{\frac{1}{3}}} \right)^q ds \right)^{1/q} \\
&\leq \left(4\pi R^2 \int_0^R \frac{4\pi \rho(s) s^2}{3^q m(s)^{\frac{q}{3}}} (4\pi K R^2)^{q-1} ds \right)^{1/q} \\
&= \left(4\pi R^2 \int_0^1 \frac{1}{3^q u^{\frac{q}{3}}} (4\pi K R^2)^{q-1} du \right)^{1/q} \\
&= \frac{4\pi}{3} R^2 K^{1-1/q} \left(\frac{3}{3-q} \right)^{1/q}.
\end{aligned}$$

Further, by $m(r) \leq (r - 2m(r))^3$,

$$\begin{aligned}
\left\| \frac{m(r)}{r(r - 2m(r))} \right\|_{L^q(B)} &= \left(4\pi \int_0^R \left| \frac{m(s)}{s(s - 2m(s))} \right|^q s^2 ds \right)^{1/q} \\
&\leq \left(4\pi R^2 \int_0^R \left| \frac{(s - 2m(s))^2}{s} \right|^q ds \right)^{1/q} \\
&\leq \left(4\pi R^2 \int_0^R \left| \frac{s^2}{s} \right|^q ds \right)^{1/q} \\
&= \left(\frac{4\pi R^{q+3}}{q+1} \right)^{1/q}.
\end{aligned}$$

Thus, by the triangle inequality,

$$\begin{aligned}
\|\mu\|_{W^{1,q}(B)} &= \|\mu\|_{L^q(B)} + \|\mu'\|_{L^q(B)} \\
&\leq 2\left(4\pi R^3\right)^{1/q} + \frac{4\pi}{3} R^2 K^{1-1/q} \left(\frac{3}{3-q} \right)^{1/q} + \left(\frac{4\pi R^{q+3}}{q+1} \right)^{1/q},
\end{aligned}$$

and $T : S \to S$ is compact.

4.6 Application of Schauder's Theorem

We will need the following final lemma:

Lemma 4.6.1. *The range $T(S) \subseteq L^q(B) \times L^q(B)$ is bounded.*

Proof. Thanks to Lemma 4.2.3 it suffices to bound $\|\tilde{\rho}\|_{L^q(B)}$ and $\|\tilde{p}\|_{L^q(B)}$. But

$$\|\tilde{p}\|^q_{L^q(B)} \leq \|\tilde{\rho}\|^q_{L^q(B)} \leq \frac{4\pi R^3}{3} K^q < \infty,$$

proving the lemma. □

At this point we are ready to apply Schauder's fixed-point theorem.

Theorem 4.6.2. *There exists E_0^* such that for all $E_0 \in [E_0^*, 1)$ the operator $T : S \to S$ has a fixed point, and in particular the Einstein-Vlasov system has a spherically symmetric steady state of the form $f \propto (E_0 - E)^0_+$.*

Proof. By Lemma 4.6.1 we may define $r^* = \sup\{\|T(\rho, p)\|_{L^q(B)\times L^q(B)} : (\rho, p) \in S\}$. The set $S_{r^*} = \{(\rho, p) \in S : \|(\rho, p)\|_{L^q(B)\times L^q(B)} \leq r^*\}$ is a nonempty, closed, bounded and convex subset of $L^q(B) \times L^q(B)$. Taking E_0^* close to one we have $T(S_{r^*}) \subseteq S_{r^*}$, by the results in section 4.3. Further $T : S_{r^*} \to S_{r^*}$ is compact. Thus by Schauder's theorem T has a fixed point in S_{r^*}. □

5

Conclusion and Future Research

With our successful proof for the Einstein-Vlasov system we have shown that the study of the mass-preserving algorithm in this case is analytically tractable. It might be possible to extend the proof to a wider class of solutions, either by establishing tighter bounds, or by making better assumptions on the initial matter quantities.

Since the suspected connection to stability relates to convergence it would be interesting to see whether one could establish that the operator T is a contraction. There is numerical evidence that this is the case, given the right ansatz, and so an analytical proof might be achievable.

Most useful of all would likely be an extensive numerical investigation of the connection between the algorithm and stability. This could establish whether the connection really exists, and if so with respect to what class of perturbations it yields stability/instability.

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