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Majorana Fermions in Topological Quantum Matter

Master of Science Thesis in Fundamental Physics

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Gothenburg, Sweden 2012

THESIS FOR THE DEGREE OF
MASTER OF SCIENCE IN FUNDAMENTAL PHYSICS

Majorana Fermions in Topological Quantum Matter

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Chalmers Reproservice
Gothenburg, Sweden 2012

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Abstract

Two-dimensional topological insulators coupled to s-wave superconductors and external magnetic fields have been predicted to support Majorana edge states. The stability of these states against electron-electron interactions and spin-orbit interactions of Rashba and Dresselhaus type are studied using bosonization and renormalization group analysis.

To extend an earlier model, a local Umklapp interaction, corresponding to isolated impurities, is first studied. It is shown that the local Umklapp interaction is less relevant than the global Umklapp interaction, and that the Majorana states are stable against this disorder effect.

The model is then extended to also include Rashba and Dresselhaus spin-orbit interactions of constant interaction strength. It is shown that the Majorana states are stable also against the Rashba effect, but may be destabilized by the Dresselhaus effect. The Luttinger parameter is found to be a function of the Rashba interaction strength.

The thesis also contains an introduction to the theory of topological insulators, with focus on the field theoretical description and interaction processes.

Keywords: Topological insulators, Quantum spin Hall effect, Spin-orbit interactions, Majorana fermions

Acknowledgements

First of all, I must thank my supervisor Henrik Johannesson for giving me the chance to work on this project, for always managing to be so encouraging and inspiring, for his patient guidance and for his support and help when I applied for PhD positions. It has been a pleasure.

I would also like to thank Anders Ström and Erik Eriksson for the discussions, and their patience; Gabriele Ferretti, Mariana Malard and Anders Martinsson for helping me with some mathematical points; and, finally, Eran Sela for his willingness to answer questions and to allow me using the phase diagram graphics.

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List of notations

α	Lattice spacing	7
α_R	Rashba interaction strength	13
β_D	Dresselhaus interaction strength	14
Δ	Superconductor coupling strength	15
$\hat{K}$	Complex conjugation operator	7
$\mathbf{E}$	Electric field	13
$\mathbf{p}$	Momentum	13
μ	Chemical potential	9
$\psi_{L\downarrow}$	Left moving mode with spin down	9
$\psi_{R\uparrow}$	Right moving mode with spin up	9
$\rho_{L\downarrow}$	Density of left-movers	19
$\rho_{R\uparrow}$	Density of right-movers	19
σ^a	Pauli matrices, $a = x, y, z$	9
τ	Imaginary time	3
A	Magnetic vector potential	4
B	Magnetic field	15
K	Luttinger parameter	21
k_F	Fermi momentum	8
T	Time-reversal operator	7
v_F	Fermi velocity	9

Chapter 1

Introduction

Thanks to experimental and theoretical advances, topologically ordered states have become a focus of research in condensed matter physics. These states are characterized by topological invariants, unlike classical states, which are associated with broken symmetries. The quantum Hall states form the archetypal example of topological states, giving an exactly quantized resistance for a range of magnetic field strengths. Other examples of topological states are the more recently discovered topological superconductors and topological insulators, the latter being the topic of this thesis.

A two-dimensional (2+1 dimensions) topological insulator behaves like an ordinary, gapped insulator in the bulk, but its topologically protected edge states set it apart. In contrast, a three-dimensional (3+1) topological insulator is insulating in the bulk and has topologically protected surface states. This thesis will focus on the edge states of the two-dimensional topological insulators.

These edge states are helical, which means that the direction of motion depends on the spin. The topological protection enables dissipationless transport with effective spin currents. The topological insulators could thus be particularly useful in spintronics, but the dissipationless currents make them interesting in other electronic applications as well.

In 2008, it was shown that two-dimensional topological insulators coupled to conventional s-wave superconductors and external magnetic fields can support Majorana bound states along the edge [1, 2]. Such states correspond to Majorana fermions, which are hypothesized neutral fermions that are their own antiparticles [3]. Particle physicists have long tried to find evidence for these particles, but to no avail. There is, however, an increasing promise of discovering them as collective excitations in condensed matter systems [4]. In fact, signatures of Majorana fermions have recently been detected experimentally in quantum wires with strong spin-orbit interactions, when coupled to s-wave superconductors [5, 6, 7, 8]. Yet, it still remains an open question.

The Majorana states are interesting for fundamental reasons, but also because the topological protection makes them possible qubits [9]. Developing an understanding of how such states behave in materials is important in order to construct useful experiments. In particular, the stability of the Majorana edge states against interactions and disorder effects must be investigated.

Sela, Altland and Rosch [10] have shown that Majorana states in a two-dimensional topological insulator system can be protected in the case when

forward, dispersive and Umklapp scattering electron-electron interactions are present. In this thesis, we extend their model to also consider local Umklapp effects from isolated impurities and Rashba and Dresselhaus spin-orbit interactions. The model then describes a more realistic system, with disorder and spin-orbit effects likely to be present in experimental samples. We show that the Majorana edge states are stable also against such impurities and the Rashba spin-orbit interaction. The Dresselhaus spin-orbit interaction does, however, give rise to non-trivial effects and may destabilize the Majorana states.

The structure of the thesis is as follows. In chapter 2 the topological insulators are described in more detail. First the basic concepts of topology in physics are introduced by means of example, then we turn to the theory of topological insulators. Both topological band theory and the relevant field theory are described.

In chapter 3 we study the local Umklapp interaction using a Jordan-Wigner transformation as well as bosonization, followed by a renormalization group (RG) analysis. The results for the Rashba and Dresselhaus interactions are given in chapter 4. We conclude this thesis in chapter 5, with a summary of the results and their implications for future experimental work.

Chapter 2

Theory of topological insulators

In this chapter, topology is introduced in the context of condensed matter physics and relevant parts of topological insulator theory is described. For more extensive treatments of topology in physics, see references [11] and [12]. There are several good review articles introducing topological insulators, of which the author has found refs. [13, 14, 15] particularly helpful.

2.1 Topology and physics

The simplest example of topology in physics is probably the problem of a particle constrained to a ring threaded by a magnetic flux Φ , as described in e.g. ref. [11] and illustrated in fig. 2.1. The geometry requires periodicity with the period 2π . If the coordinate of the particle is measured in terms of an angular variable $\phi \in [0, 2\pi]$, then ϕ can be considered a field defined by the map

$$\phi : S^1 \rightarrow S^1, \quad (2.1)$$

$$\tau \mapsto \phi(\tau) \quad (2.2)$$

from the unit circle S^1 into another circle, where $\tau \in [0, \beta]$ is imaginary time. Such a mapping can be associated with a winding number W , counting the number of times $\phi(\tau)$ runs around the unit circle as τ goes from 0 to β , i.e.

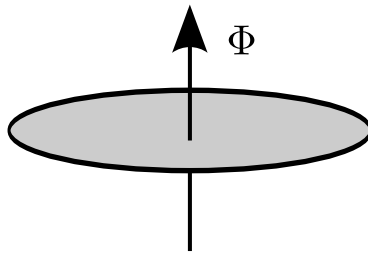


Figure 2.1 – Particle on a ring threaded by a magnetic flux Φ .

$2\pi W = \phi(\beta) - \phi(0)$. Obviously, $W \in \mathbb{Z}$ and cannot be changed by continuously deforming ϕ (a loop around the circle is a loop as long as the circle remains a circle).

The partition function is

$$\mathcal{Z} = \int_{\phi(\beta) - \phi(0) = 2\pi W} \mathcal{D}\phi e^{-\int d\tau (\frac{1}{2} \dot{\phi}^2 - iA\dot{\phi})}, \quad (2.3)$$

where $A = \frac{\Phi}{\Phi_0}$ is the vector potential of the magnetic field, and Φ_0 is the magnetic flux quantum. Applying the constraint $\phi(\beta) - \phi(0) = 2\pi W$ to the second term in the action yields

$$S_{top}[\phi] = iA \int_0^\beta d\dot{\phi} = iA(\phi(\beta) - \phi(0)) = i2\pi W A \quad (2.4)$$

and we can separate the partition function into different topological sectors,

$$\mathcal{Z} = \sum_W e^{2\pi i W A} \int_{\phi(\beta) - \phi(0) = 2\pi W} \mathcal{D}\phi e^{-\frac{1}{2} \int d\tau \dot{\phi}^2}. \quad (2.5)$$

S_{top} is a topological term, which cannot affect the equations of motion. It does, however, weight the contribution of different topological sectors in the functional integration.

In more formal terms (for an introduction, see references [11] or [12]), all fields with the same winding number are homotopic, meaning that they belong to an equivalence class of fields that can be continuously deformed into each other. Here, the homotopy group is $\pi_1(S^1) = \mathbb{Z}$. It is a topological invariant, encoded in the topological action (as it is $\sim W \in \mathbb{Z}$). Two theories are said to be topologically equivalent when the topological parts of their actions have the same homotopy group. From this property follows that the target spaces of the fields are homeomorphic, meaning that the spaces can be continuously deformed into each other.

The key message here is that such deformations will not change the topological properties of the system. In effect, continuous changes of the Hamiltonians will preserve certain properties.

2.2 Topological band theory

To get a first conceptual understanding of an topological insulator, let us consider the band theory of the system, rather than the underlying field theory description. Recall first the band theory of ordinary insulators, semiconductors and metals. The eigenvalues $E_n(k)$ form a structure of energy bands. Both insulators and semiconductors have a fully occupied (valence) band and an empty (conduction) band of higher energy. The size of the band gap (and hence the likelihood for an electron to jump to the upper band) determines whether the material is an insulator or a semiconductor. These two states are topologically equivalent, meaning that the Hamiltonian can be continuously deformed in such a way that an insulating state turns into a semiconducting one (or vice versa), without closing the band gap.

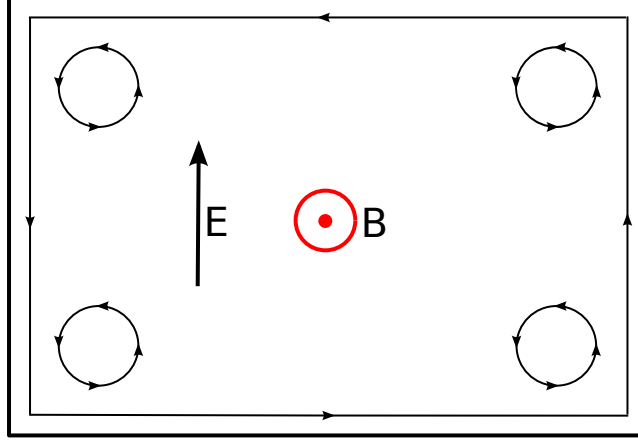


Figure 2.2 – The integer quantum Hall effect. Under a magnetic field B perpendicular to the plane and an electric field E across the system, the electrons in the bulk will be trapped in cyclotron orbits and electrons along the edge will form protected edge states.

2.2.1 The integer quantum Hall effect

The integer quantum Hall state provides a topologically non-trivial example of a gapped state. In short, consider a two-dimensional material subject to a strong, perpendicular magnetic field. The electrons on the surface will move in cyclotron orbits and become arranged in energy levels called Landau levels (LL). If N levels are filled, then there is an energy gap $\Delta E(\mathbf{k}) = E_{N+1}(\mathbf{k}) - E_N(\mathbf{k})$ up to the next level. We can form equivalence classes of Hamiltonians that can be smoothly varied without closing the gap, i.e. respecting the band structure. These classes are distinguished by their first Chern class [14]

$$n = \frac{1}{2\pi} \int d^2\mathbf{k} \mathcal{F}, \quad (2.6)$$

where $\mathcal{F}$ is the Berry's curvature $\mathcal{F} = \nabla \times \mathcal{A}$ and the Berry's connection $\mathcal{A} = i \sum_{m=1}^N \langle u_m | \nabla_{\mathbf{k}} | u_m \rangle$. The Chern class is always an integer. It corresponds to the Berry's phase picked up by a Bloch state $|u_m\rangle$ transported around the Brillouin zone, which can be compared with how the winding number corresponds to the “phase” picked up by the particle on the ring as it is transported around the ring. The analogy is strengthened by the fact that n is also a topological invariant.

When an electric field is applied across the system, the cyclotron orbits pick up a drift velocity, which yields a net current. The conductivity is quantized with

$$\sigma_{xy} = n \frac{e^2}{h}. \quad (2.7)$$

Since n is a topological invariant, the Hall conductivity is robust against smooth variations of the Hamiltonian [14]. This is what makes the quantization of the conductivity so precise, up to 1 ppb [16].

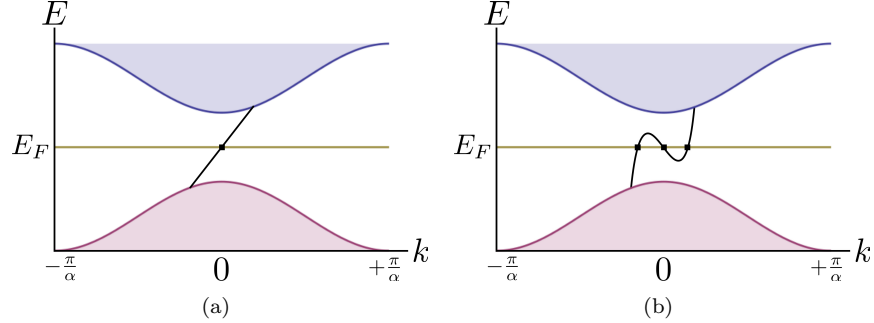


Figure 2.3 – If the gapless edge state of the quantum Hall system, between the bulk bands, originally crosses the Fermi energy once (a), the Hamiltonian can be smoothly varied such that the state crosses the Fermi energy any odd number of times (b). The number can, however, never become even, so the difference $N_R - N_L$ between the number of right and left moving modes is preserved.

Consider now an interface between a quantum Hall state with $n = 1$ and an ordinary insulator with $n = 0$. The Chern number then changes by $\Delta n = 1$ at the interface. In order to make the change in the topological invariant possible, the energy gap must disappear somewhere, giving rise to low energy electronic states at the interfaces. These states are chiral, meaning that they have a specific direction, defined by their dispersion relation and the direction of the electric field, as can be seen in fig. 2.2.

If there is a single such edge state, it will cross the Fermi energy exactly once. It is, however, possible to smoothly vary the Hamiltonian so that it crosses the Fermi energy any odd number of times, see fig. 2.3. The difference $N_R - N_L$ between the number of right and left moving modes is preserved under such a variation, as half of the added crossings must happen with positive group velocities, and half with negative group velocities. Furthermore, the smooth variation ensures that the new Hamiltonian is homeomorphic to the old one, which implies that the Chern number is conserved. This is encoded in the bulk-boundary correspondence [14]

$$N_R - N_L = \Delta n. \quad (2.8)$$

2.2.2 Topological insulators

As is clear from (2.7) and (2.8), the Chern number n and the Hall conductivity σ_{xy} are odd under time-reversal, so time-reversal symmetry must be broken, e.g. by a magnetic field, to allow the topologically protected integer quantum Hall edge states. Is it possible to find topologically non-trivial gapped states in a time-reversal symmetric system? Yes, the topological insulators, with $n = 0$, turn out to be examples of such states. These are sometimes called quantum spin Hall (QSH) states, as they utilize spin-orbit interactions. They were first proposed by Kane and Mele [17], building on work by Haldane [18].

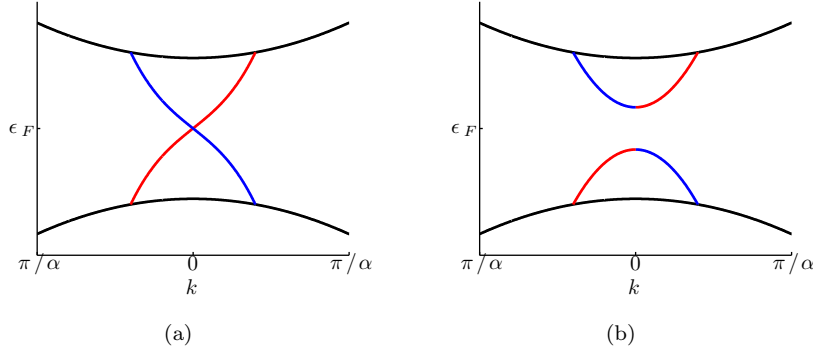


Figure 2.4 – The (a) allowed ungapped and (b) forbidden gapped configurations of a single Kramers’ pair for the edge states. Red and blue lines illustrate different spins, and the black bands are the bulk states. The gapped version is forbidden, as the degeneracy at $k = 0$ is removed.

One important result for time-reversal symmetric systems is Kramers’ theorem. It states that all eigenstates of an electron Hamiltonian commuting with the time-reversal operator T must have at least a twofold degeneracy [19]. To see this, recall that the anti-unitary time-reversal operator can be expressed as the product of a unitary operator U and complex conjugation $\hat{K}$, $T = U\hat{K}$, and that $T^2 = -1$ for spin $\frac{1}{2}$ particles. Now, assume that a nondegenerate state $|\xi\rangle$ existed. Then we would have $T|\xi\rangle = c|\xi\rangle$ for some constant c , and in effect $T^2|\xi\rangle = |c|^2|\xi\rangle$. This is not allowed as $|c|^2 \neq -1$, so by contradiction there must be another state $|\xi'\rangle$ with the same energy, which is related to $|\xi\rangle$ through time reversal [19]. Together the two states form a Kramers’ pair.

This theorem has interesting consequences for the edge states. To see this, let us study electronic states inside the bulk band gap. In particular, noting that the Γ and X points in the Brillouin zone ($k = 0$ and $k = \pm\frac{\pi}{\alpha}$, respectively, where α is the lattice spacing) transform onto themselves under time reversal, let us consider states at the Γ point. In the absence of spin-orbit interaction, these states trivially satisfy Kramers’ theorem, as states with different spins are automatically degenerate. In contrast, in the case with spin-orbit interactions, the energy levels of different spins are shifted. Since Kramers’ theorem requires the states to be degenerate, they must be pairwise connected at $k = 0$. Band gaps are thus only allowed when we have an even number of pairs of spin up and spin down states. Such pairs are known as Kramers’ pairs, as the different spins are related through time reversal [14]. The possible configurations for one and two Kramers’ pairs are shown in fig. 2.4 and 2.5, respectively.

This state of affairs can be described with a $\mathbb{Z}_2$ topological invariant ν [20]. When $\nu = 0$ there is an even number of Kramers’ pairs, and when $\nu = 1$ there is an odd number. A principle similar to the above bulk-boundary correspondence can now be formed for the number of Kramers’ pairs intersecting the Fermi energy N_K

$$N_K \mod 2 = \Delta\nu, \quad (2.9)$$

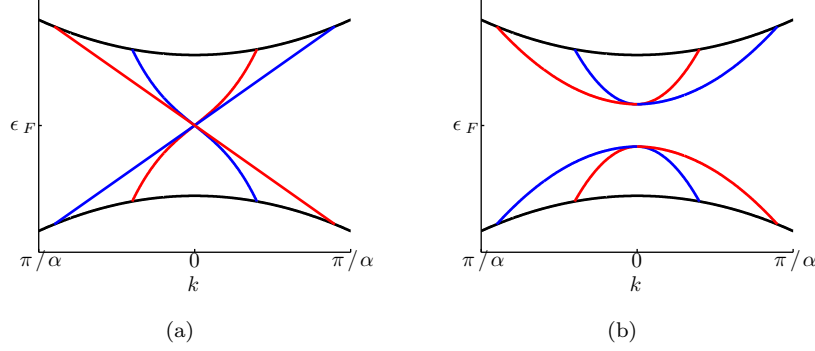


Figure 2.5 – (a) Ungapped and (b) gapped configurations of two Kramers’ pairs for the edge states. Both configurations are allowed. By means of smooth deformations of the Hamiltonian, the gap in (b) can be made as wide as that of the bulk insulator, i.e. it is a topologically uninteresting case. Red and blue lines illustrate different spins, and the black bands are the bulk states.

where $\Delta\nu$ is the change in ν across an interface between the two materials. Since the topological invariant must be changed at the interface, the gapless edge states for odd N_K ($\nu = 1$) are topologically protected against smooth variations of the Hamiltonian [14]. From now on, we will call the state with $\nu = 1$ a topological insulator and, for simplicity, restrict ourselves to the case with one Kramer’s pair ($N_K = 1$).

2.3 Field theory for topological insulators

It turns out that it is useful to study the electronic properties of these systems using quantum field theory, as it allows many powerful theoretical techniques and provides a natural way to describe excitations. The book by Altland and Simons [11] is a good introduction to the use of quantum field theory in condensed matter physics – the mathematics used is essentially the same as in particle physics, but the concepts it is used to represent and the overall philosophy are, at times, quite different. The book by Giamarchi [21] is a good reference for the use of QFT to describe one-dimensional (1+1) systems, such as the edge of a topological insulator.

In this language, the fermionic annihilation operator $\psi(x)$ written in its momentum components is

$$\psi_\sigma(x) = \frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} \psi_\sigma(k), \quad (2.10)$$

where $\sigma = \uparrow, \downarrow$ is a spin index. Since we are interested in low-energy excitations we can restrict the sum to momenta close to the Fermi momentum $\pm k_F$

$$\psi_\sigma(x) \approx \frac{1}{\sqrt{\Omega}} \left[\sum_{-\Lambda < k - k_F < \Lambda} e^{ikx} \psi_\sigma(k) + \sum_{-\Lambda < k + k_F < \Lambda} e^{ikx} \psi_\sigma(k) \right] \quad (2.11)$$

for some momentum cut-off Λ . We make the conventional choice of coordinate systems, with $k > 0$ (left term in the sum above) corresponding to right-movers and $k < 0$ (right term) corresponding to left-movers. Symbolically, this is written

$$\psi_\sigma(x) = \psi_{R\sigma}(x) + \psi_{L\sigma}(x). \quad (2.12)$$

The linearized low-energy free field theory for both spin species is [22]

$$H_0 = \int \frac{dk}{2\pi} \left[\psi_{R\uparrow}^\dagger v_F i \partial_x \psi_{R\uparrow} - \psi_{L\downarrow}^\dagger v_F i \partial_x \psi_{L\downarrow} + \psi_{R\downarrow}^\dagger v_F i \partial_x \psi_{R\downarrow} - \psi_{L\uparrow}^\dagger v_F i \partial_x \psi_{L\uparrow} \right], \quad (2.13)$$

where we immediately see that there is a Kramers' pair on each line (as spin flips and $i \rightarrow -i$ under time reversal [19]). Now, we said earlier that the topological insulator contains only one Kramers' pair. In this language, it is difficult to see how that would come about. The key is that the edge states are at the boundary of a two-dimensional (2+1) system, so we can in fact have a holographic system, with one Kramers' pair at one edge and the other at the opposite edge [22].

The spin up states move to the right at one edge, and to the left at the other (vice versa for spin down). We make the conventional choice of geometry with right-moving spin up electrons and left-moving spin down electrons at the upper edge, and the opposite at the lower edge. For our purposes, it is enough to only consider one of the edges, and we choose the upper one. This means that we have

$$\psi(x) = \psi_{R\uparrow}(x) + \psi_{L\downarrow}(x), \quad (2.14)$$

which has the spinor form

$$\Psi = \begin{pmatrix} \psi_{L\downarrow} \\ \psi_{R\uparrow} \end{pmatrix}. \quad (2.15)$$

This is known as a helical liquid, as the spin is determined by the direction of the particle¹. The free theory of the helical liquid has the Hamiltonian density

$$\begin{aligned} \mathcal{H}_0 &= \Psi^\dagger (v_F i \partial_x \sigma^z - \mu) \Psi \\ &= \psi_{L\downarrow}^\dagger (v_F i \partial_x - \mu) \psi_{L\downarrow} + \psi_{R\uparrow}^\dagger (-v_F i \partial_x - \mu) \psi_{R\uparrow}, \end{aligned} \quad (2.16)$$

where v_F is the Fermi velocity, σ^a , $a = x, y, z$ denotes the Pauli matrices and μ is the chemical potential. The next step is to consider interactions in the topological insulator edge states, and how they are restricted by time-reversal symmetry.

2.3.1 Allowed interactions

As discussed in the section 2.2.2, time-reversal symmetry is a fundamental property of the topological insulator state. Interactions and perturbations that break

¹The helical liquid theory can also be used to describe the states in quantum wires with strong spin-orbit interactions, when coupled to a s-wave superconductor [23], which is just the setup used in the recent experiments [5, 6, 7, 8].

time reversal, such as strong applied magnetic fields, will then act towards destroying the state. Interactions that respect time reversal invariance will thus be the most important ones in this context. (One can, of course, also consider weak perturbations of various kinds.) How will the interactions affect the excitations of the gapless edge states? Can they open up a gap?

We begin by studying time-reversal invariant interactions. Assume that a perturbation H' , which satisfies $[H', T] = 0$, is turned on at $t = 0$. If m right movers are excited at this time, can they be scattered back to m left movers through H' ? To model this, let the final state of m left movers, $|\Psi\rangle$, be related to the initial state of m right movers, $|\phi\rangle$, through time reversal, i.e. $|\Psi\rangle = T|\phi\rangle$. With $T = \hat{K}U$, where $\hat{K}$ is complex conjugation and U is a unitary operator, we then have [24]

$$\begin{aligned}
 \langle \Psi | H' | \phi \rangle &= \langle T\phi | H' | \phi \rangle &&= \langle \phi | H' | T\phi \rangle^* \\
 &= \langle H' \phi | T\phi \rangle^* &&= \langle \hat{K}H' \phi | \hat{K}T\phi \rangle \\
 &= \langle \hat{K}H' \phi | U^\dagger U | \hat{K}T\phi \rangle &&= \langle TH' \phi | T^2 \phi \rangle \\
 &= (-1)^m \langle TH' \phi | \phi \rangle &&= (-1)^m \langle H' T\phi | \phi \rangle \\
 &= (-1)^m \langle \Psi | H' | \phi \rangle.
 \end{aligned} \tag{2.17}$$

Clearly, the matrix element is zero for odd m . In the case of only one Kramers' pair, the only degenerate states are already connected by T . Then a single excitation cannot be scattered by a time reversal symmetric perturbation. In other words, one-particle backscattering, which tends to be the most relevant perturbation in normal metals is forbidden in topological insulators. This makes the edge states more protected – also from localization by disorder – and is related to their topological protection [17, 22].

However, two particle processes are not forbidden, even when there is only one Kramers' pair. They threaten to break the topological insulator state, and need to be studied further. Wu, Bernevig and Zhang [25] showed that the only allowed non-chiral interactions are the dispersive and Umklapp interactions, illustrated in figures 2.6 and 2.7, respectively². They can be written [10]

$$\mathcal{H}_{\text{dispersive}} = g_2 \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \psi_{R\uparrow}, \tag{2.18}$$

$$\mathcal{H}_{\text{um}} = g_u \psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow}^\dagger \psi_{R\uparrow} \partial_x \psi_{R\uparrow} + \text{h.c.}, \tag{2.19}$$

where h.c. denotes Hermitian conjugate. g_2 and g_u are the interaction strengths of the dispersive and Umklapp scattering processes, respectively.

Note that eq. (2.19) is a point-splitted version of the naive Umklapp term, $\psi_{L\downarrow}^\dagger \psi_{L\downarrow}^\dagger \psi_{R\uparrow} \psi_{R\uparrow} + \text{h.c.}$, which is forbidden by the exclusion principle. The point splitting can be considered a regularization of the theory. Another way to treat fields at arbitrarily close coordinates is through operator product expansions (OPE:s). We will return to aspects of the regularization later in the thesis.

It should also be noted that the Umklapp process only takes place during specific conditions. As is clear from the figure 2.7, two particles are moved from

²However, as we will come to see in chapter 4, the claim that these are the only two allowed interactions does not hold under all conditions, as one can also have inelastic backscattering when spin axial symmetry is broken.

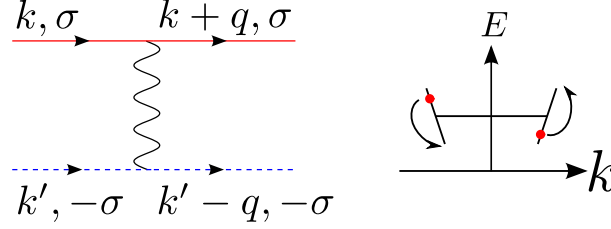


Figure 2.6 – Dispersive scattering, or the g_2 scattering channel. Two particles traveling in opposite directions (having opposite spins) interact and continue in their respective directions.

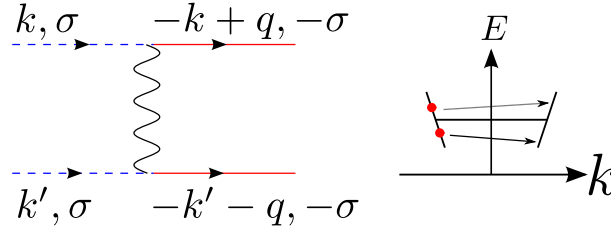


Figure 2.7 – Umklapp interaction. Two particles that initially travel in one direction (have the same spin) interact with each other and the lattice. After the interaction the two particles travel in the opposite direction (have the opposite spin) and there has been a transfer of $4k_F$ to or from the lattice.

the vicinity of one Fermi point ($k = k_F$) to the other one. Each one is changed by a momentum of $2k_F$, so a total momentum of $4k_F$ must be transferred to or from the lattice during the process. By just considering the naive form of the Umklapp term, $\psi_{L\downarrow}^\dagger \psi_{L\downarrow}^\dagger \psi_{R\uparrow} \psi_{R\uparrow} + \text{h.c.}$, and inserting the expansions of the fields from (2.12) and (2.11), we see that there will be a factor $e^{i4k_F x}$ in the Hamiltonian (2.19), which oscillates rapidly during the integration $H_{um} = \int dx e^{i4k_F x} \mathcal{H}_{um}$ [21, 25]. The term will tend to zero unless the system is at half-filling³ or $k_F = 0$. In the later parts of this thesis we will mainly focus on a model for zero momentum Majorana modes, so the Umklapp interaction will indeed be present.

Even though there are only two allowed non-chiral interactions, they do allow for a greater variation and richness in physics than can first be expected. The Umklapp term is used to describe the dynamics within the system and the interaction with localized impurities. One can also consider more general cases of disorder, such as annealed and quenched disorder. In those cases the coupling constant g_u in (2.19) is replaced by Gaussian variables $g_u(x, t)$ and $\varphi(x, t)$, as in $g_u(x, t) e^{i\varphi(x, t)}$. In the case of quenched disorder there is no time dependence – the disorder is quenched, or frozen in time [25].

In addition, there is the chiral forward scattering. It is illustrated in fig. 2.8

³In general, for the lattice model we have $k_F = v\pi/\alpha$ and $x = j\alpha$, where v is the filling fraction and $j \in \mathbb{Z}$. For $v = 1/2$, $\exp[i4k_F x] \equiv 1$, and the rapid phase fluctuation is suppressed [21].

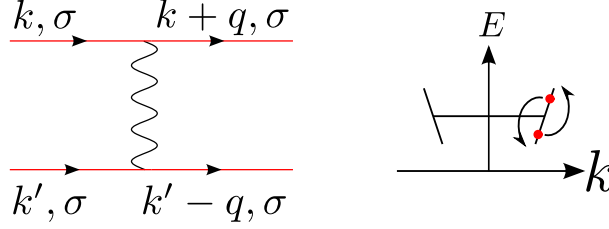


Figure 2.8 – Forward scattering, or the g_4 scattering channel. Two particles that travel in the same direction (have the same spin) interact and continue in the same direction.

and can be written [10]

$$\mathcal{H}_{\text{forward}} = \frac{g_4}{2} \left[\left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right)^2 + \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} \right)^2 \right], \quad (2.20)$$

where g_4 is the interaction strength of the forward scattering process.

In the limit of small momentum transfer, starting from the general formula for electron-electron interactions one can show that $g_4 = 2g_2$ [21, 26]. The dispersive and forward scattering terms (2.18) and (2.20) can then be combined into

$$\mathcal{H}_{\text{fw}} = g_2 \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \psi_{R\uparrow} + \frac{g_4}{2} \left[\left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right)^2 + \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} \right)^2 \right] \quad (2.21)$$

$$= g_2 \left(\Psi^\dagger \Psi \right)^2. \quad (2.22)$$

In appendix A we show explicitly that these interactions satisfy time reversal invariance.

In addition, one can consider spin-orbit interactions in the helical liquid. Couplings to other systems are by definition not inside the system, and can be treated as perturbations (as long as one stays within the regime of validity of perturbation theory, of course).

2.3.2 Spin-orbit interactions

The general idea of spin-orbit interaction is that a particle traveling through electromagnetic fields will interact with these through its magnetic moment (i.e. spin). In the case of the Hydrogen atom, the electron travels through the Coulomb potential from the atomic nucleus, and the resulting interaction gives rise to a splitting of its spectrum (the “fine structure”). Such atomic spin-orbit interactions are also the source of the topological insulator state, but the effects are much larger than in the case of the Hydrogen atom due to the heavy elements involved in these materials.

The quantum spin Hall state was first predicted to occur in graphene [17], being formed from a mirror symmetric atomic spin-orbit interaction. Unfortunately, graphene never really did work as a topological insulator, as its spin-orbit gap is very small (sub-Kelvin) [27, 28, 29], so another route was taken. However, it was recently shown that by depositing adatoms on graphene sheets, one can

reach several orders of magnitude larger gaps ($\sim 100 - 1000$ K) than in pure graphene [30, 31], so graphene could yet turn out to become a useful topological insulator.

Instead, the breakthrough came when Bernevig, Hughes and Zhang [32] predicted the atomic p-orbitals in HgTe quantum wells to give rise to the QSH state. The first topological insulator was soon realized in just such a system [22, 33], and such heterostructures remain the main way to achieve topological insulators in 2+1 dimensions [30, 34]. Recently, there have been experimental reports that also the InAs/GaSb quantum wells suggested by Liu et al. [35] can host the QSH state [36, 37]. In addition, the InAs/GaSb wells are reported to form good surfaces to superconductors [36, 38], thus being interesting candidates for the systems hosting Majorana fermions discussed in chapter 1.

In the quantum wells, other spin-orbit interactions such as the Rashba and Dresselhaus effects may also be considerable [39, 40]. For the HgTe quantum wells, the dominant spin-orbit interaction is the Rashba effect. InAs/GaSb quantum wells may, on the other hand, have comparable Rashba and Dresselhaus interaction strengths [40].

These spin-orbit effects are caused by inversion asymmetry of the crystal and the doping of the semiconductor materials, quite unlike the atomic spin-orbit interaction of the Hydrogen atom. A quantum well, in contrast, also has a microscopic potential which depends on the design of the heterostructure and on externally applied gate voltages. It is thus possible to tune this effect in experiments. In addition there may exist a momentum-dependent magnetic field $\mathbf{b}(\mathbf{p})$ intrinsic to the semiconductor structure, caused by bulk effects [41]. In order to be allowed by time reversal, $\mathbf{b}(\mathbf{p})$ must be odd in $\mathbf{p}$. (As has already been noted, an external magnetic field $\mathbf{B}$ would break time reversal symmetry.)

For a particle with spin vector $\boldsymbol{\sigma}$ moving through the electric field $\mathbf{E}$ and the $\mathbf{b}$ field, a general spin-orbit term can be written (with $\hbar = c = m_e = 1$) [19, 42]

$$H_{SO} = C_1 (\mathbf{E} \times \mathbf{p}) \cdot \boldsymbol{\sigma} + C_2 \boldsymbol{\sigma} \cdot \mathbf{b}(\mathbf{p}) \quad (2.23)$$

where $\mathbf{p}$ is the momentum of the electron and $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is the “vector” of the three Pauli matrices. When we have a constant electric field, $\mathbf{E} = |\mathbf{E}|\hat{z}$, the first term of (2.23) is known as Rashba spin-orbit interaction [39]. Restricting ourselves to the one-dimensional system of the edge of the topological insulator, it is quite simply $H_{Rashba} = -i|\mathbf{E}|\sigma^y\partial_x$. In field theory language for a helical liquid, the Rashba Hamiltonian takes the form [42, 43]

$$\mathcal{H}_R = \alpha_R(x) \Psi^\dagger (-i\partial_x \sigma^y) \Psi \quad (2.24)$$

or, in the component fields,

$$\mathcal{H}_R = \alpha_R(x) \left(\psi_{L\downarrow}^\dagger \partial_x \psi_{R\uparrow} - \psi_{R\uparrow}^\dagger \partial_x \psi_{L\downarrow} \right), \quad (2.25)$$

where $\alpha_R(x)$ is the spatially dependent Rashba interaction strength. This interaction is explicitly shown to be time reversal invariant in appendix A. It does, however, break parity symmetry as the electric field is odd under parity.

The second term of (2.23) is dubbed Dresselhaus spin-orbit interaction. Using a tight-binding model one can determine the form of $\mathbf{b}(\mathbf{p})$. In 2+1 dimensions, the dominating term in the Hamiltonian depends on $k_x\sigma^x - k_y\sigma^y$ [42, 43].

For the helical liquid we write

$$\mathcal{H}_D = \beta_D(x) \Psi^\dagger (-i\partial_x \sigma^x) \Psi \quad (2.26)$$

or

$$\mathcal{H}_D = -i\beta_D(x) \left(\psi_{R\uparrow}^\dagger \partial_x \psi_{L\downarrow} + \psi_{L\downarrow}^\dagger \partial_x \psi_{R\uparrow} \right), \quad (2.27)$$

where $\beta_D(x)$ is the spatially dependent Dresselhaus interaction strength. This interaction is shown to be time reversal invariant in appendix A.

Chapter 3

Majorana fermions and local Umklapp interactions

3.1 Introduction

We will here explore a model considered by Sela, Altland and Rosch (SAR) in ref. [10]. We will attempt to extend it by treating additional interactions, with inspiration from Wu, Bernevig and Zhang [25]. In particular, a local Umklapp interaction is treated in this chapter and spin-orbit interactions are treated in the next chapter.

The SAR model aims to describe a two-dimensional (2+1) topological insulator in the vicinity of an s-wave superconductor and an external magnetic field B along the edge. It thus consists of a free helical liquid, as in (2.16) and an additional coupling term in the Hamiltonian density, which can be written

$$\delta\mathcal{H} = B\psi_{L\downarrow}^\dagger\psi_{R\uparrow} + \Delta\psi_{L\downarrow}\psi_{R\uparrow} + \text{h.c.}, \quad (3.1)$$

where Δ is the strength of the superconductor coupling. Furthermore, we include forward and Umklapp scattering terms, i.e. (2.21) and (2.19). The full SAR Hamiltonian is thus

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{fw}} + \mathcal{H}_{\text{um}} + \delta\mathcal{H}. \quad (3.2)$$

Following SAR [10], we consider using their method of mapping a field Hamiltonian to an XYZ spin chain for an extended model, now also including a local Umklapp interaction. Our results are given below, in section 3.2. As we shall see, this approach is instructive, but also difficult to exploit for deriving useful results. To get further we then use the more general approach of bosonizing the field Hamiltonian and analyzing it by means of renormalization group methods. This is done in section 3.3.

3.2 Mapping the field theory model to an XYZ spin chain

3.2.1 The SAR model

Sela, Altland and Rosch [10] have shown that the Hamiltonian (3.2) can be mapped (up to a constant) onto the XYZ spin-chain model, with both staggered

and non-staggered magnetic fields, $H_{XYZ} = \sum_j \mathcal{H}_j$, where

$$\mathcal{H}_j = \sum_{a=x,y,z} J_a S_j^a S_{j+1}^a - \left[\mu + B (-1)^j \right] S_j^z. \quad (3.3)$$

The coupling constants $J_x = v_F + \Delta$, $J_y = v_F - \Delta$ and $J_z = \frac{g_2}{4} = \frac{g_4}{2} = g_u$ are determined using a Jordan-Wigner transformation¹

$$S_j^z = a_j^\dagger a_j - \frac{1}{2} \quad (3.4)$$

$$S_j^\pm = a_j^\dagger (-1)^j e^{i\pi \sum_{l=1}^{j-1} a_l^\dagger a_l} \quad (3.5)$$

and expanding the lattice fermions a_j in terms of the left and right movers,

$$a_j \sim e^{i\frac{\pi}{2}x} \psi_{R\uparrow}(x) + e^{-i\frac{\pi}{2}x} \psi_{L\downarrow}(x). \quad (3.6)$$

Since this mapping is exact, up to a constant, Sela, Altland and Rosch [10] were able to explore the phase diagram using known results from spin chain theory. Among other things they showed that the model has a Majorana quantum phase transition², i.e. a phase transition at $T = 0$ which supports Majorana modes around a critical point [44]. Since the phase diagram is most readily explained after introducing bosonization, we will save the details for section 3.3.5.

The question now is, can we use this method to extend the model in order to also describe other interactions and impurity effects? We will consider the case of a local Umklapp term, existing in only a single bond of the chain. This corresponds to a localized impurity, affecting electron-electron dynamics around the site.

3.2.2 Local Umklapp interaction

Following Wu, Bernevig and Zhang [25] we write the Umklapp term at half-filling in a point-splitting form, using the lattice constant α as the point-splitting. This gives a regularized theory, using the lattice constant as a UV cut-off. The Umklapp term takes the form

$$H_{um} = -g_u \int dx \psi_{R\uparrow}^\dagger(x) \psi_{R\uparrow}^\dagger(x + \alpha) \psi_{L\downarrow}(x + \alpha) \psi_{L\downarrow}(x) + \text{h.c.} \quad (3.7)$$

While this looks distinctly different from the earlier form (2.19), the two forms are shown to be approximately equivalent using a first-order Taylor expansion [21]

$$\psi_{R\uparrow}^\dagger(x + \alpha) \approx \psi_{R\uparrow}^\dagger(x) + \alpha \partial_x \psi_{R\uparrow}^\dagger, \quad (3.8)$$

¹This relation between the interaction strengths g_2 , g_4 and g_u , is introduced by the lattice and unimportant for the structure of the phase diagram [10]. It thus does not need to correspond to the relation $g_4 = 2g_2$ used in eq. (2.22).

²In general, a quantum phase transition is achieved by varying a physical parameter at $T = 0$, resulting in a changed ground state [44]. One example is the destruction of the superconducting state by a critical magnetic field.

which yields

$$\begin{aligned}\psi_{R\uparrow}^\dagger(x) \partial_x \psi_{R\uparrow}^\dagger(x) &\approx \frac{1}{\alpha} \left[\psi_{R\uparrow}^\dagger(x) \psi_{R\uparrow}^\dagger(x+\alpha) - \psi_{R\uparrow}^\dagger(x) \psi_{R\uparrow}^\dagger(x) \right] \\ &= \frac{1}{\alpha} \psi_{R\uparrow}^\dagger(x) \psi_{R\uparrow}^\dagger(x+\alpha).\end{aligned}\quad (3.9)$$

By using this expression and the corresponding one with $\psi_{R\uparrow}^\dagger \rightarrow \psi_{L\downarrow}$, and including the $1/\alpha$ factors in the coupling constant, we do indeed see that (2.19) and (3.7) capture the same physics. It should be noted that the $1/\alpha$ factor is needed to balance the dimensions, as the differential operator also has units $1/\text{length}$, but that α is constant and does not, unlike the operators, scale under renormalizations or conformal transformations.

A local Umklapp interaction is acquired by inserting a delta function $\delta(x)$ in the integral (3.7). Call the new coupling constant g_L . The Hamiltonian density we want to try to map onto a spin chain then becomes

$$\mathcal{H}_{um}^{local} = g_L \psi_{R\uparrow}^\dagger(0) \psi_{R\uparrow}^\dagger(\alpha) \psi_{L\downarrow}(\alpha) \psi_{L\downarrow}(0) + \text{h.c.} \quad (3.10)$$

In order to map this onto a spin chain, we now need to invert the lattice fermion expansion (3.6) in order to express the Ψ fields in terms of lattice fermions. This is less straight-forward for a fixed position $x_0 = 0$ than when we integrate over all x , but can be done by approximating the fields as slow, $\psi_{R\uparrow}(x) \approx \psi_{R\uparrow}(x+\alpha)$. Such an approximation essentially means that we have an effective field theory, which describes the physics at length scales larger than the lattice spacing³. We have

$$a_j \sim e^{i\frac{\pi}{2}x} \psi_{R\uparrow}(x) + e^{-i\frac{\pi}{2}x} \psi_{L\downarrow}(x), \quad (3.11)$$

$$a_{j+1} \sim e^{i\frac{\pi}{2}(x+\alpha)} \psi_{R\uparrow}(x+\alpha) + e^{-i\frac{\pi}{2}(x+\alpha)} \psi_{L\downarrow}(x+\alpha). \quad (3.12)$$

Using units in which $\alpha = 1$, we write

$$a_{j+1} \approx ie^{i\frac{\pi}{2}x} \psi_{R\uparrow}(x) - ie^{-i\frac{\pi}{2}x} \psi_{L\downarrow}(x). \quad (3.13)$$

Combining (3.11) and (3.13) we get

$$\psi_{R\uparrow}(x) = \frac{e^{-i\frac{\pi}{2}x}}{2} (a_j - ia_{j+1}), \quad (3.14)$$

$$\psi_{L\downarrow}(x) = \frac{e^{i\frac{\pi}{2}x}}{2} (a_j + ia_{j+1}). \quad (3.15)$$

Using the above expressions and the inverses of the Jordan-Wigner transformations in (3.4) and (3.5) we are now ready to translate the local Umklapp Hamiltonian (3.10) into the spin operator language. After a straight-forward but rather tedious calculation one finds

$$\mathcal{H}_{um}^{local} = \frac{g_L}{8} \left[(S_0^+ S_2^- + S_2^+ S_0^-) \left(\frac{1}{2} + S_1^z \right) - S_1^z (S_0^z + S_2^z + 1) + S_0^z S_2^z \right]. \quad (3.16)$$

³If the local Umklapp has a finite interaction width and cannot be approximately described by the delta function, it is possible to change the slow field approximation and carry out a similar analysis to the one described here.

We find that $\mathcal{H}_{um}^{local}$ contains next-nearest neighbour couplings and three-site interactions, apart from the more familiar types of terms. It is also easy to see that site $j = 1$ is in some sense special, as can be expected from a fixed spin impurity. Now, the question is, can such a model be solved?

As it turns out, it is not clear how to approach this model analytically. Models with isotropic next-nearest neighbour interaction, such as the Majumdar-Ghosh and J_1 - J_2 models, have been studied in some detail [45, 46], but local, anisotropic next-nearest neighbour terms do not seem to be covered in the literature. The three-point interaction does not make things simpler either.

As an attempt to remove the latter difficulty, we form a mean field theory by letting $S_1^z \rightarrow \gamma = \langle S_1^z \rangle$. This approximation gives us

$$\tilde{\mathcal{H}}_{um}^{local} = \frac{gL}{8} \left[\left(S_0^+ S_2^- + S_2^+ S_0^- \right) \left(\frac{1}{2} + \gamma \right) - \gamma (S_0^z + S_2^z) + S_0^z S_2^z \right]. \quad (3.17)$$

Again, models of this kind do not seem to have been studied in the literature. The most promising way to study them may be through numerical studies. One of the questions involved with such a study is how to treat the other components of the spin on the first site? Instead of delving into the numerical tricks and complexities, one may try a “ghost site approach”. The idea of the ghost site approach is to put the impurity at position $j = \frac{1}{2}$ in the lattice, either between two lattice points or next to the one-dimensional edge. For details on this approach, see appendix B. Unfortunately, the resulting theory appears at least as complicated as the original one, making further progress difficult. In the following, we will instead use a more traditional field theory approach.

3.2.3 Conclusions

In general it seems to be possible to map field theories in 1+1 dimensions onto spin chain Hamiltonians, using the method of an inverse Jordan-Wigner transformation. It is formally possible to do this also for sites outside the lattice, although the basis for such a procedure awaits a rigorous proof. However, as above, one runs the risk of finding a rather unwieldy spin chain Hamiltonian, unless one does have the luck of Sela, Altland and Rosch [10] to find an already well-studied model. In our case with the local Umklapp scattering, this method does not seem practicable.

One can, of course, choose to bosonize the spin chain model and apply renormalization group analysis methods. It does, however, seem more straightforward to instead bosonize the original field theory model and apply the RG analysis to that theory. This is what is done in the later parts of this chapter.

3.3 Analysis of the field model

3.3.1 Bosonization

In $1 + 1$ and fewer dimensions, it is possible to transform fermion fields Ψ to boson fields ϕ , and vice versa. We have already seen an example of this equivalence when we mapped the field model onto lattice spins. The equivalence is more generally, and perhaps more elegantly, illustrated in the language of conformal field theory (CFT) [47], in which we can write the fermion fields as

vertex operators. This procedure, called bosonization, generally gives a simpler theory, as several fermion interactions can be expressed as free boson fields. For an introduction to bosonization, please refer to S  n  chal [48], Giamarchi [21] or Gogolin et al. [49].

The left and right moving fields and densities can be expressed [48, 49]

$$\Psi_{L\downarrow} = \frac{1}{\sqrt{2\pi\alpha}} \eta_{L\downarrow} e^{-i\sqrt{4\pi}\phi_{L\downarrow}(x)}, \quad (3.18)$$

$$\Psi_{R\uparrow} = \frac{1}{\sqrt{2\pi\alpha}} \eta_{R\uparrow} e^{i\sqrt{4\pi}\phi_{R\uparrow}(x)}, \quad (3.19)$$

$$\rho_{L\downarrow} = \frac{1}{\sqrt{\pi}} \partial_x \phi_{L\downarrow}, \quad (3.20)$$

$$\rho_{R\uparrow} = \frac{1}{\sqrt{\pi}} \partial_x \phi_{R\uparrow}. \quad (3.21)$$

In the following calculations, we will set $\alpha = 1$.

The η_L and η_R are Klein factors. They are included to account for the anticommuting nature of fermion fields and obey the Clifford algebra⁴. For general (not necessarily chiral) spin indices $\mu, \nu \in \{\uparrow, \downarrow\}$ the algebra has the familiar form [48]

$$\{\eta_{L\mu}, \eta_{L\nu}\} = \{\eta_{R\mu}, \eta_{R\nu}\} = 2\delta_{\mu\nu}, \quad (3.22)$$

$$\{\eta_{L\mu}, \eta_{R\nu}\} = 0. \quad (3.23)$$

We will find it convenient to introduce the fields⁵

$$\phi = \phi_{L\downarrow} + \phi_{R\uparrow}, \quad (3.24)$$

$$\theta = \phi_{R\uparrow} - \phi_{L\downarrow}. \quad (3.25)$$

Let us now bosonize our full model, including the local Umklapp interaction, i.e. bosonize eqs. (3.2) and (3.10). The free Hamiltonian (2.16) bosonizes to [48]

$$\begin{aligned} \mathcal{H}_0 &= \psi_{L\downarrow}^\dagger (v_F i \partial_x - \mu) \psi_{L\downarrow} + \psi_{R\uparrow}^\dagger (-v_F i \partial_x - \mu) \psi_{R\uparrow} \\ &= v_F \left[(\partial_x \phi_{R\uparrow})^2 + (\partial_x \phi_{L\downarrow})^2 \right] - \frac{\mu}{\sqrt{\pi}} \partial_x \phi. \end{aligned} \quad (3.26)$$

For the non-standard terms we have slightly more to do:

$$\begin{aligned} \mathcal{H}_{\text{fw}} &= g_2 \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \psi_{R\uparrow} + \frac{g_4}{2} \left[\left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right)^2 + \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} \right)^2 \right] \\ &= g_2 \rho_{L\downarrow} \rho_{R\uparrow} + \frac{g_4}{2} \left(\rho_{L\downarrow}^2 + \rho_{R\uparrow}^2 \right) \\ &= \frac{g_2}{\pi} \partial_x \phi_{L\downarrow} \partial_x \phi_{R\uparrow} + \frac{g_4}{2\pi} \left[(\partial_x \phi_{L\downarrow})^2 + (\partial_x \phi_{R\uparrow})^2 \right], \end{aligned} \quad (3.27)$$

⁴Technically, this procedure means that we have assumed that the system is in the thermodynamic limit. The Klein factors are, in the general case, really operators that change the fermion number. This has notable effect in systems of finite size L , but in the thermodynamic limit ($L \rightarrow \infty$) a changed fermion number means that k_F shifts by order $\frac{1}{L} \rightarrow 0$, which can be neglected. One can then use the η 's [26, 50].

⁵ ϕ corresponds to the total field along the edge, and θ is its dual. This basis is sometimes known as the helical edge basis [51], as it generalizes nicely and allows the separate treatment of two opposite edges of the topological insulator. In that case one would use $\phi_1 = \phi_{L\downarrow} + \phi_{R\uparrow}$ and $\phi_2 = \phi_{L\uparrow} + \phi_{R\downarrow}$ etc. [52].

$$\begin{aligned}
& B \left(\psi_{L\downarrow}^\dagger \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow} \right) \\
&= \frac{B}{2\pi} \left(\eta_{L\downarrow} \eta_{R\uparrow} e^{i\sqrt{4\pi}(\phi_{L\downarrow} + \phi_{R\uparrow})} + \eta_{R\uparrow} \eta_{L\downarrow} e^{-i\sqrt{4\pi}(\phi_{R\uparrow} + \phi_{L\downarrow})} \right) \\
&= \frac{B}{2\pi} \eta_{L\downarrow} \eta_{R\uparrow} \left(e^{i\sqrt{4\pi}\phi} - e^{-i\sqrt{4\pi}\phi} \right), \tag{3.28}
\end{aligned}$$

$$\begin{aligned}
& \Delta \left(\psi_{L\downarrow} \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger \right) \\
&= \frac{\Delta}{2\pi} \left(\eta_{L\downarrow} \eta_{R\uparrow} e^{i\sqrt{4\pi}(\phi_{R\uparrow} - \phi_{L\downarrow})} + \eta_{R\uparrow} \eta_{L\downarrow} e^{-i\sqrt{4\pi}(\phi_{R\uparrow} - \phi_{L\downarrow})} \right) \\
&= \frac{\Delta}{2\pi} \eta_{L\downarrow} \eta_{R\uparrow} \left(e^{i\sqrt{4\pi}\theta} - e^{-i\sqrt{4\pi}\theta} \right), \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{um} &= -g_u \psi_{R\uparrow}^\dagger(x) \psi_{R\uparrow}^\dagger(x + \alpha) \psi_{L\downarrow}(x + \alpha) \psi_{L\downarrow}(x) + \text{h.c.} \\
&= -g_u \left[\left(\psi_{L\downarrow}^\dagger \psi_{R\uparrow} \right)^2 + \left(\psi_{R\uparrow}^\dagger \psi_{L\downarrow} \right)^2 \right] \\
&= -\frac{g_u}{(2\pi)^2} \left[\eta_{L\downarrow} \eta_{R\uparrow} \eta_{L\downarrow} \eta_{R\uparrow} e^{i\sqrt{4\pi}(2\phi_{L\downarrow} + 2\phi_{R\uparrow})} \right. \\
&\quad \left. + \eta_{R\uparrow} \eta_{L\downarrow} \eta_{R\uparrow} \eta_{L\downarrow} e^{-i\sqrt{4\pi}(2\phi_{L\downarrow} + 2\phi_{R\uparrow})} \right]. \tag{3.30}
\end{aligned}$$

The local Umklapp term will behave similarly to the global Umklapp term above, except for the inclusion of a delta function. Since the delta function is not related to any field, it will not be affected by the bosonization.

We must now choose a Klein basis which simultaneously diagonalizes all Klein factors. The two Klein factors we have to consider are $C = \eta_{L\downarrow} \eta_{R\uparrow} \eta_{L\downarrow} \eta_{R\uparrow}$ and $D = \eta_{L\downarrow} \eta_{R\uparrow}$. We use the same representation as S  n  chal [48], using Pauli sigma matrices

$$\eta_{L\downarrow} = 1 \otimes \sigma^y, \tag{3.31}$$

$$\eta_{R\uparrow} = \sigma^x \otimes \sigma^x. \tag{3.32}$$

and find, in matrix notation, that

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.33}$$

$$D = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & +i \\ -i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix}. \tag{3.34}$$

The two matrices commute, so they can be simultaneously diagonalized. The eigenvectors of D are simple linear combinations of the eigenvectors of C , and

can be used to form an unitary transformation U . We find

$$C' = U^{-1}CU = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (3.35)$$

$$D' = U^{-1}DU = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \quad (3.36)$$

and we can pick eigenvalues -1 and $-i$ from the upper left corner, forgetting about the rest of the Klein space. In effect, this is a gauge choice. We can now rewrite the terms in eqs. (3.28), (3.29) and (3.30):

$$\begin{aligned} B \left(\psi_{L\downarrow}^\dagger \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow} \right) &= \frac{B}{2\pi} \eta_{L\downarrow} \eta_{R\uparrow} \left(e^{i\sqrt{4\pi}\phi} - e^{-i\sqrt{4\pi}\phi} \right) \\ &= \frac{B}{\pi} \sin \left(\sqrt{4\pi}\phi \right), \end{aligned} \quad (3.37)$$

$$\begin{aligned} \Delta \left(\psi_{L\downarrow} \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger \right) &= \frac{\Delta}{2\pi} \eta_{L\downarrow} \eta_{R\uparrow} \left(e^{i\sqrt{4\pi}\theta} - e^{-i\sqrt{4\pi}\theta} \right) \\ &= \frac{\Delta}{\pi} \sin \left(\sqrt{4\pi}\theta \right), \end{aligned} \quad (3.38)$$

$$\begin{aligned} \mathcal{H}_{um} &= -\frac{g_u}{(2\pi)^2} \eta_{L\downarrow} \eta_{R\uparrow} \eta_{L\downarrow} \eta_{R\uparrow} \left(e^{2i\sqrt{4\pi}(\phi_{L\downarrow} + \phi_{R\uparrow})} + e^{-2i\sqrt{4\pi}(\phi_{L\downarrow} + \phi_{R\uparrow})} \right) \\ &= \frac{g_u}{2\pi^2} \cos \left(\sqrt{16\pi}\phi \right). \end{aligned} \quad (3.39)$$

To get the same bosonized Hamiltonian as Sela, Altland and Rosch [10] we rescale the fields $\sqrt{4\pi}\phi \rightarrow \sqrt{4\pi}\phi + \frac{\pi}{2}$ and $\sqrt{4\pi}\theta \rightarrow \sqrt{4\pi}\theta - \frac{\pi}{2}$. Introducing the renormalized velocity v and the Luttinger parameter K

$$v = v_F + \frac{g_4}{2\pi} + \mathcal{O} \left(g_2^2, g_4^2, g_u^2, g_L^2 \right), \quad (3.40)$$

$$K = 1 - \frac{g_2}{2\pi v_F} + \mathcal{O} \left(g_2^2, g_4^2, g_u^2, g_L^2 \right), \quad (3.41)$$

we can now write the full bosonized Hamiltonian as

$$\begin{aligned} \mathcal{H} &= \frac{v}{2} \left(\frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right) - \frac{\mu}{\sqrt{\pi}} \partial_x \phi \\ &\quad + \frac{B}{\pi} \cos \left(\sqrt{4\pi}\phi \right) - \frac{\Delta}{\pi} \cos \left(\sqrt{4\pi}\theta \right) \\ &\quad - \frac{g_u}{2\pi^2} \cos \left(\sqrt{16\pi}\phi \right) - \frac{g_L}{2\pi^2} \delta(x) \cos \left(\sqrt{16\pi}\phi(x) \right). \end{aligned} \quad (3.42)$$

The parametrizations of v and K in (3.40) and (3.41) hold for small values of the coupling constants, but the theory (3.42) can be shown to hold for general interaction strengths [21, 53]. We will thus be able to explore the properties of the theory for generic values of the Luttinger parameter K , but will not attempt to find explicit parametrizations for larger values of the coupling constants.

3.3.2 Lagrangian formulation

The RG analysis uses the action, so we first need to rewrite the Hamiltonian density (3.42) as a Lagrangian density. As a first step, note that the quadratic part of the bosonized Hamiltonian density, $\mathcal{H}'_0 = \mathcal{H}_0 + \mathcal{H}_{fw}$, can be written in an explicitly canonical form by rescaling the fields

$$\phi \rightarrow \phi' = \frac{\phi}{\sqrt{K}}, \quad (3.43)$$

$$\theta \rightarrow \theta' = \sqrt{K}\theta, \quad (3.44)$$

which yields

$$\mathcal{H}'_0 = \frac{v}{2} \left[(\partial_x \phi')^2 + (\partial_x \theta')^2 \right]. \quad (3.45)$$

Since ϕ' and θ' are canonically conjugate fields, we can use Hamilton's equations to write⁶

$$\partial_x \theta' = -\frac{1}{v} \partial_t \phi', \quad (3.46)$$

$$\partial_x \phi' = -\frac{1}{v} \partial_t \theta'. \quad (3.47)$$

These relations make it possible to write the Lagrangian density $\mathcal{L}_0$ purely in the ϕ' or θ' fields. The kinetic Lagrangian is given by the Legendre transformation $\mathcal{L}_0 = \Pi_{\phi'} \partial_t \phi' - \mathcal{H}'_0$, or, using the θ' fields by $\mathcal{L}_0 = \Pi_{\theta'} \partial_t \theta' - \mathcal{H}'_0$. All interactions terms satisfy $\mathcal{L}_{int} = -\mathcal{H}_{int}$. We can then write $\mathcal{L}_0$ as

$$\mathcal{L}_0 = \frac{1}{2} \left[v^{-1} (\partial_t \phi')^2 - v (\partial_x \phi')^2 \right]. \quad (3.48)$$

Finally, since it is convenient to construct the partition function from the Euclidean action, we now rewrite this Lagrangian in imaginary time, $\tau = it$. This gives us

$$\mathcal{L}_0 = -\frac{1}{4} \left[v^{-1} (\partial_\tau \phi')^2 + v (\partial_x \phi')^2 \right]. \quad (3.49)$$

3.3.3 Partition function

The action $S = \int dt \int dx \mathcal{L}$ also has the Euclidean form $S_E = - \int d\tau \int dx \mathcal{L}$, which is then used to form the partition function

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\theta e^{-S_E[\phi, \theta]}. \quad (3.50)$$

We have $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$, where $\mathcal{L}_0$ is given by (3.49) and $\mathcal{L}_{int}$ denote the interaction terms. Let us begin by considering the local Umklapp term, $\mathcal{L}_{um}^{local}$,

⁶To see this, identify one term of (3.45) as the x -derivative of the field and the other term as its conjugate momentum Π . Then, by Hamilton's equations for a field φ , we have $\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial \Pi} = v\Pi$. Hence, the time derivative of one field is related to the x -derivative of the other.

on its own in the commensurate case, with the other interactions inactive, i.e. let $\mu = \Delta = B = g_u = 0$. We have

$$\mathcal{Z} = \int \mathcal{D}\phi' \exp \left[\int d\tau dx \left(-\frac{1}{4} v^{-1} (\partial_\tau \phi')^2 - \frac{1}{4} (\partial_x \phi')^2 \right) - \int d\tau dx \mathcal{L}_{um}^{local} [\phi'] \right], \quad (3.51)$$

where $\mathcal{L}_{um} = -\mathcal{H}_{um}$, with $\mathcal{H}_{um}$ being the last term of (3.42).

Since the local Umklapp interaction acts only at one point in space, at $x = 0$, we want to express the partition function in a local form, using space-independent fields, $\tilde{\phi}'$. To do this, we integrate over the fields at all positions except $x = 0$, closely following ref. [54]. $\mathcal{Z}$ can then be written

$$\mathcal{Z} \propto \int \mathcal{D}\phi' \mathcal{D}\tilde{\phi}' e^{-S_E[\phi']} \delta(\tilde{\phi}'(\tau) - \phi'(\tau, 0)). \quad (3.52)$$

We use the relation

$$\delta(\tilde{\phi}'(\tau) - \phi'(\tau, 0)) = \frac{1}{2\pi} \int dk_{\phi'}(\tau) \exp \left[ik_{\phi'}(\tau) (\tilde{\phi}' - \phi'(\tau, 0)) \right] \quad (3.53)$$

to write

$$\mathcal{Z} \propto \int \mathcal{D}\phi' \mathcal{D}\tilde{\phi}' \mathcal{D}k_{\phi'} \exp \left[-S_E[\phi'] + i \int d\tau k_{\phi'}(\tau) (\tilde{\phi}' - \phi'(\tau, 0)) \right]. \quad (3.54)$$

The exponential contains integrals of three types

$$I_1 = \int d\tau dx \left(v^{-1} (\partial_\tau \phi')^2 + v (\partial_x \phi')^2 \right), \quad (3.55)$$

$$I_2 = i \int d\tau k_{\phi'}(\tau) \tilde{\phi}'(\tau), \quad (3.56)$$

$$I_3 = -i \int d\tau k_{\phi'}(\tau) \phi'(\tau, 0). \quad (3.57)$$

These can be rewritten using Fourier sums, yielding [54]

$$\begin{aligned} I_1 &= \int d\tau dx \frac{1}{(\beta L)^2} \\ &\times \left(v^{-1} \sum_{q, \omega_n} (-i\omega_n \phi'_{q, \omega_n}) e^{i(qx - \omega_n \tau)} \sum_{q', \omega'_n} i\omega'_n \phi'^{\star}_{q', \omega'_n} e^{-i(q'x - \omega'_n \tau)} \right. \\ &\quad \left. + v \sum_{q, \omega_n} i q \phi'_{q, \omega_n} e^{i(qx - \omega_n \tau)} \sum_{q', \omega'_n} (-i q' \phi'^{\star}_{q', \omega'_n}) e^{-i(q'x - \omega'_n \tau)} \right) \\ &= \int d\tau dx \frac{1}{(\beta L)^2} \sum_{q, q', \omega_n, \omega'_n} \left(v^{-1} \omega_n \omega'_n \phi'_{q, \omega_n} \phi'^{\star}_{q', \omega'_n} e^{i(q-q')x} e^{i(\omega'_n - \omega_n)\tau} \right. \\ &\quad \left. + v q q' \phi'_{q, \omega_n} \phi'^{\star}_{q', \omega'_n} e^{i(q-q')x} e^{i(\omega'_n - \omega_n)\tau} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\beta L} \sum_{q, q', \omega_n, \omega'_n} \left(v q q' + v^{-1} \omega_n \omega'_n \right) \phi'_{q, \omega_n} \phi'^{\star}_{q', \omega'_n} \delta_{q, q'} \delta_{\omega_n, \omega'_n} \\
&= \frac{1}{\beta L} \sum_{q, \omega_n} \left(v q^2 + v^{-1} \omega_n^2 \right) |\phi'|^2,
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
I_2 &= i \int d\tau \beta^{-1} \sum_{\omega_n, \omega'_n} k(\omega_n) \tilde{\phi}'(\omega'_n) e^{i(-\omega_n - \omega'_n)\tau} \\
&= \frac{i}{\beta} \sum_{\omega_n} k(\omega_n) \tilde{\phi}'(-\omega_n),
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
I_3 &= -i \int d\tau \beta^{-1} \sum_{\omega_n} k(\omega_n) e^{-i\omega_n \tau} \frac{1}{\beta L} \sum_{q, \omega'_n} \phi'_{q, \omega'_n} e^{i(qx - \omega'_n \tau)} \\
&= -i \int d\tau \frac{1}{\beta^2 L} \sum_{q, \omega_n, \omega'_n} k(\omega_n) \phi'_{q, \omega'_n} e^{-i(\omega_n + \omega'_n)\tau} e^{iqx} \\
&= -\frac{i}{\beta L} \sum_{q, \omega_n} k(-\omega_n) \phi'_{q, \omega_n}.
\end{aligned} \tag{3.60}$$

Transforming the ϕ' fields back to the ϕ fields using (3.43) and inserting the above integrals in (3.54), the partition function can be written

$$\begin{aligned}
\mathcal{Z} &\propto \mathcal{D}\phi \mathcal{D}\tilde{\phi} \mathcal{D}k_\phi \exp \left[-\frac{1}{4\beta L} \sum_{q, \omega_n} \left[\left(\frac{1}{vK} \omega_n^2 + \frac{v}{K} q^2 \right) |\phi|^2 \right. \right. \\
&\quad \left. \left. - 4ik_\phi(-\omega_n) \phi(q, \omega_n) \right] + \frac{i}{\beta} \sum_{\omega_n} k_\phi(\omega_n) \tilde{\phi}(-\omega_n) \right. \\
&\quad \left. - \int d\tau \mathcal{L}_{um}^{local} [\tilde{\phi}] \right].
\end{aligned} \tag{3.61}$$

$\mathcal{Z}$ is Gaussian in ϕ , so we can perform the integration over ϕ . Note that we are looking for singular behaviour of the partition function, rather than its exact values, so constant prefactors can be neglected. We find

$$\begin{aligned}
\mathcal{Z} &\propto \int \mathcal{D}\tilde{\phi} \mathcal{D}k_\phi \exp \left[-\frac{1}{\beta L} \sum_{q, \omega_n} k_\phi(-\omega_n) \left(\frac{1}{vK} \omega_n^2 + \frac{v}{K} q^2 \right)^{-1} k_\phi(\omega_n) \right. \\
&\quad \left. + \frac{i}{\beta} \sum_{\omega_n} k_\phi(\omega_n) \tilde{\phi}(-\omega_n) - \int d\tau \mathcal{L}_{um}^{local} [\tilde{\phi}] \right].
\end{aligned} \tag{3.62}$$

Take the q -sum to the continuum limit and perform the resulting integral

$$\begin{aligned}
I_4 &= \frac{1}{\beta L} \sum_{q, \omega_n} \left(\frac{1}{vK} \omega_n^2 + \frac{v}{K} q^2 \right)^{-1} \\
&\rightarrow \frac{1}{\beta} \sum_{\omega_n} \int \frac{dq}{2\pi} \left(\frac{1}{vK} \omega_n^2 + \frac{v}{K} q^2 \right)^{-1} = \frac{1}{\beta} \sum_{\omega_n} \frac{1}{2|\omega_n|}.
\end{aligned} \tag{3.63}$$

Inserting this in (3.62) and then performing the Gaussian integral over k_ϕ gives us

$$\begin{aligned}
\mathcal{Z} &\propto \int \mathcal{D}\tilde{\phi} \mathcal{D}k_\phi \exp \left[-\frac{1}{\beta} \sum_{\omega_n} k_\phi(-\omega_n) k_\phi(\omega_n) \frac{K}{2|\omega_n|} \right. \\
&\quad \left. + \frac{i}{\beta} \sum_{\omega_n} k_\phi(\omega_n) \tilde{\phi}(-\omega_n) - \int d\tau \mathcal{L}_{um}^{local} [\tilde{\phi}] \right] \\
&= \int \mathcal{D}\tilde{\phi} \exp \left[-\frac{1}{\beta} \sum_{\omega_n} \frac{|\omega_n|}{2K} |\tilde{\phi}|^2 - \int d\tau \mathcal{L}_{um}^{local} [\tilde{\phi}] \right] \\
&= \int \mathcal{D}\tilde{\phi} \exp \left[-\int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} |\omega| \frac{|\tilde{\phi}(\omega)|^2}{2K} - \int d\tau \mathcal{L}_{um}^{local} [\tilde{\phi}] \right], \tag{3.64}
\end{aligned}$$

where, in the last step, we have taken the ω_n -sum to its continuum limit. The entire partition function is now written in terms of the $\tilde{\phi}$ field, which lives only at $x = 0$, i.e. the position of the impurity. In effect, we now have an entirely local partition function and are ready to start the RG analysis.

3.3.4 Renormalization group analysis

The renormalization group analysis is based on the idea to consider the systems at different scales. The theory is said to be renormalizable if, under a scaling transformation, the structure of the theory is preserved, but the coupling constants are changed. By exploring how the coupling constants change under the transformation one can quickly reach a conclusion of whether a specific interaction is relevant at the desired scale. In the context of condensed matter physics, we are primarily interested in long-range phenomena. We therefore use the RG apparatus to uncover the low-energy behaviour of a system. This is achieved by decomposing the field into a slow field (with long wave-lengths) and a fast field (with shorter wavelengths), and then integrating the fast fields away, thereby producing an effective model for the slow field.

Introductions to the RG procedure can be found in references [11], [21] and [49]. Unfortunately, these books use different notations and approaches. Here we will use a momentum shell renormalization using path integrals, such as the one described in ref. [11]. The current work has also been largely influenced by the techniques used in ref. [54], where an explicit calculation is given for the interaction from a point contact connecting two opposite edges of a topological insulator.

From now on, in order to simplify the notation, we will drop the tilde on the local field, writing ϕ instead of $\tilde{\phi}$. We decompose the field into fast and slow parts, i.e. $\phi(\tau) = \phi_f(\tau) + \phi_s(\tau)$, with

$$\phi_s(\tau) = \sum_{|\omega_n| < \frac{\Lambda}{b}} e^{-i\omega_n \tau} \phi_{\omega_n} = \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} e^{-i\omega \tau} \phi(\omega), \tag{3.65}$$

$$\phi_f(\tau) = \sum_{\frac{\Lambda}{b} < |\omega_n| < \Lambda} e^{-i\omega_n \tau} \phi_{\omega_n} = \int_{\frac{\Lambda}{b} < |\omega_n| < \Lambda} \frac{d\omega}{2\pi} e^{-i\omega \tau} \phi(\omega). \tag{3.66}$$

The full action is read off from (3.64)

$$S[\phi] = \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} |\omega| \frac{|\phi(\omega)|^2}{2K} + \int d\tau \mathcal{L}_{um}^{local}[\phi], \quad (3.67)$$

where

$$S_{um}^{local} = \int d\tau \frac{g_L}{2\pi^2} \cos(\sqrt{16\pi}\phi). \quad (3.68)$$

The first term of (3.67) separates into one part for the slow fields, $S_s[\phi_s]$, and one for the fast fields, $S_f[\phi_f]$, so the action can be written

$$S[\phi] = S_s[\phi_s] + S_f[\phi_f] + S_{um}^{local}[\phi_s, \phi_f]. \quad (3.69)$$

As mentioned above, we want to integrate away the fast modes ϕ_f to define an effective action S_{eff} for the slow modes ϕ_s . We write

$$e^{-S_{eff}[\phi_s]} = e^{-S_s[\phi_s]} \langle e^{-S_{um}^{local}[\phi_s, \phi_f]} \rangle_f, \quad (3.70)$$

where $\langle A \rangle_f \equiv \int \mathcal{D}\phi_f e^{-S_f[\phi_f]} A$. We can assume that g_L is small enough to justify the first-order cumulant approximation (see app. C)

$$\langle e^{-S_{um}^{local}} \rangle_f \approx e^{-\langle S_{um}^{local} \rangle_f}, \quad (3.71)$$

which leads to

$$e^{-S_{eff}} \approx e^{-S_s[\phi_s]} e^{-\langle S_{um}^{local}[\phi_s, \phi_f] \rangle_f}. \quad (3.72)$$

The average of S_{um}^{local} is

$$\begin{aligned} & \langle S_{um}^{local}[\phi_s, \phi_f] \rangle_f \\ &= \frac{g_L}{2\pi^2} \int \mathcal{D}\phi_f e^{-S_f[\phi_f]} \int d\tau \cos[\sqrt{16\pi}(\phi_s(\tau) + \phi_f(\tau))] \\ &= \frac{g_L}{(2\pi)^2} \int \mathcal{D}\phi_f e^{-S_f[\phi_f]} \int d\tau \left(e^{i\sqrt{16\pi}\phi_s} e^{i\sqrt{16\pi}\phi_f} + \text{h.c.} \right) \\ &= \frac{g_L}{(2\pi)^2} \int \mathcal{D}\phi_f e^{-S_f[\phi_f]} \int d\tau \left(e^{i\sqrt{16\pi}\phi_s} e^{i\sqrt{16\pi} \int_f \frac{d\omega}{2\pi} e^{i\omega\tau} \phi} + \text{h.c.} \right), \end{aligned} \quad (3.73)$$

where we have used the notation $\int_f \equiv \int_{\frac{\Lambda}{b} < |\omega| < \Lambda}$. After an inverse Hubbard-Stratonovich transformation (see app. D for details) we find

$$\begin{aligned} & \langle S_{um}^{local}[\phi_s, \phi_f] \rangle_f = -\frac{g_L}{(2\pi)^2} \int d\tau \left(e^{i\sqrt{16\pi}\phi_s} \int \mathcal{D}\phi_f \right. \\ & \quad \times \exp \left[\int_f \frac{d\omega}{2\pi} \left(i\sqrt{16\pi} e^{i\omega\tau} \phi_f(\omega) - \frac{|\omega|}{2K} |\phi_f|^2 \right) \right] + \text{h.c.} \Big) \\ &= -\frac{g_L}{(2\pi)^2} \int d\tau \left(e^{i\sqrt{16\pi}\phi_s} \exp \left[-\int_f \frac{d\omega}{2\pi} \frac{16\pi}{2} \frac{K}{|\omega|} \right] + \text{h.c.} \right) \\ &= -\frac{g_L}{2\pi^2} \int d\tau \cos(\sqrt{16\pi}\phi_s) e^{-\int_f d\omega \frac{4K}{|\omega|}}. \end{aligned} \quad (3.74)$$

Using $\int_f \frac{d\omega}{|\omega|} = \int_{\frac{\Lambda}{b} < |\omega| < \Lambda} \frac{d\omega}{|\omega|} = \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d\omega}{\omega} = \ln b$ and (3.68) we can finally write

$$\begin{aligned} \langle S_{um}^{local} [\phi_s, \phi_f] \rangle_f &= \frac{g_L}{2\pi^2} \int d\tau \cos(\sqrt{16\pi}\phi_s) e^{-4K \ln b} \\ &= \frac{g_L}{2\pi^2} \int d\tau \cos(\sqrt{16\pi}\phi_s) b^{-4K} \\ &= b^{-4K} S_{um}^{local} [\phi_s]. \end{aligned} \quad (3.75)$$

In effect, the effective action S_{eff} can be written

$$e^{-S_{eff}} \approx \exp \left(-S_s [\phi_s] - b^{-4K} S_{um}^{local} [\phi_s] \right), \quad (3.76)$$

i.e. entirely expressed in the slow fields, with a rescaled coupling constant in the interaction term, where we have seen $g_L \rightarrow b^{-4K} g_L$ as $\Lambda \rightarrow \frac{\Lambda}{b}$. The next step is now to rescale ω and τ so that the fields range over the same energy/time scale as in the unrenormalized action, in order to be able to compare the actions and find the scaling equation. To do this, we note that the change $\Lambda \rightarrow \bar{\Lambda} = \frac{\Lambda}{b}$ is countered by setting $\omega \rightarrow \bar{\omega} = b\omega$ and, in effect, $\tau \rightarrow \bar{\tau} = \frac{\tau}{b}$. The fields will then be rescaled as well. We choose $\bar{\phi}(\bar{\tau}) = \phi_s(\tau)$, which by (3.65) implies $\bar{\phi}(\bar{\omega}) = \frac{\phi(\omega)}{b}$. Applying these rescalings we write the effective action as

$$\begin{aligned} S_{eff} [\phi_s] &= S_s [\phi_s] + b^{-4K} S_{um}^{local} [\phi_s] \\ &= \int_{-\Lambda/b}^{\Lambda/b} \frac{d\omega}{2\pi} |\omega| \frac{|\phi_s(\omega)|^2}{2K} + b^{-4K} \int d\tau \frac{g_L}{2\pi^2} \cos(\sqrt{16\pi}\phi(\tau)) \\ &= \int_{-\Lambda}^{\Lambda} \frac{d\bar{\omega}}{2\pi b} |\bar{\omega}| \frac{b^2 |\bar{\phi}(\bar{\omega})|^2}{2K} + b^{1-4K} \int d\bar{\tau} \frac{g_L}{2\pi^2} \cos(\sqrt{16\pi}\bar{\phi}(\bar{\tau})) \\ &= \int_{-\Lambda}^{\Lambda} \frac{d\bar{\omega}}{2\pi} |\bar{\omega}| \frac{|\bar{\phi}(\bar{\omega})|^2}{2K} + b^{1-4K} \int d\bar{\tau} \frac{g_L}{2\pi^2} \cos(\sqrt{16\pi}\bar{\phi}(\bar{\tau})). \end{aligned} \quad (3.77)$$

The full rescaling of the coupling constant is thus $g_L \rightarrow \bar{g}_L = g_L b^{1-4K}$. After logarithmic differentiation and denoting $l = \ln b$, this equation becomes the scaling equation

$$\frac{d\bar{g}_L(l)}{dl} = \bar{g}_L(l) (1 - 4K) \quad (3.78)$$

subject to the initial condition $\bar{g}_L(l=0) = g_L$. We see that $\bar{g}_L$ changes with the strength of other electron-electron interactions. Specifically, it tends to zero for $K > \frac{1}{4}$ and to infinity for $K < \frac{1}{4}$ and that the renormalization group eigenvalue is $x_L = 1 - 4K$. We say that the local Umklapp interaction is relevant if $x_L > 0$ and irrelevant if $x_L < 0$. The case $x_L = 0$ is known as marginal, and its behaviour depends on the sign of the coupling constant.

One can also express this classification using the conformal dimension of the perturbation. To do this, we must first apply the transformation (3.43) to the $\bar{\phi}$ field in (3.77), in order to have the canonical form of the original action. We then have

$$\begin{aligned} S_{eff} [\bar{\phi}'] &= \int_{-\Lambda}^{\Lambda} \frac{d\bar{\omega}}{2\pi} |\bar{\omega}| \frac{|\bar{\phi}'(\bar{\omega})|^2}{2} \\ &\quad + b^{1-4K} \int d\bar{\tau} \frac{g_L}{2\pi^2} \cos(\sqrt{16\pi K} \bar{\phi}'(\bar{\tau})). \end{aligned} \quad (3.79)$$

The cosine can be written as a vertex operator, $e^{i\beta\Xi}$, where β is a constant and Ξ is a field. If Ξ is spinless, like the combined ϕ field in (3.24), then the vertex operator has zero conformal spin and conformal dimension $d = \frac{\beta^2}{4\pi}$. If Ξ is chiral, e.g. $\Xi = \phi_R$, then the conformal spin is generally non-zero and the conformal dimension $d = \frac{\beta^2}{8\pi}$ [49].

In the case of our local Umklapp interaction, we have from (3.79) that $d = 4K$. If D is the number of spatial dimensions + the number of time dimensions, a perturbation is relevant when $d < D$ and irrelevant when $d > D$. The case $d = D$ is the marginal one [49]. This knowledge will allow us to classify operators more quickly in the remaining parts of the thesis.

3.3.5 Effects on Majorana fermions and the phase diagram

First, let us consider our Hamiltonian (3.2) in the simplifying case $\mu = g_2 = g_4 = g_{um} = 0$ (i.e. no interactions and zero chemical potential). We write our Dirac fermion fields in terms of two Majorana fields

$$\psi_{L\downarrow}(x) = \frac{1}{\sqrt{2}} [i\chi_1(x) + \chi_2(x)], \quad (3.80)$$

$$\psi_{R\uparrow}(x) = \frac{1}{\sqrt{2}} [\bar{\chi}_1(x) + i\bar{\chi}_2(x)]. \quad (3.81)$$

The Hamiltonian (3.2) can then be written

$$\mathcal{H}_0 + \delta\mathcal{H} = \sum_{j=1}^2 \left(\chi_j i \frac{v_F}{2} \partial_x \chi_j - \bar{\chi}_j i \frac{v_F}{2} \partial_x \bar{\chi}_j + im_j \chi_j \bar{\chi}_j \right), \quad (3.82)$$

where the last term is a Majorana mass term [55], with $m_1 = \Delta - B$ and $m_2 = \Delta + B$ (Δ is taken to be positive). At $|B| = \Delta$, one mode becomes massless. That is a sign of a Majorana quantum phase transition⁷ in a transversal Ising model [10, 56]. One can also easily see that the spin chain (3.3) gets the form of a transversal Ising spin chain model with an S_j^z term [44] at $v_F = \Delta$, zero chemical potential and no interactions, so the spin chain and Majorana representations do match.

By tuning B or Δ in real space, we can localize the massless Majorana mode around some position x_0 , for which $m_1(x_0) = 0$. One can easily see that this Majorana zero mode is stable against weak interactions around the QPT. The term describing interaction between two Majorana fermions can be written $\mathcal{H}_{int} \sim \chi_1 \bar{\chi}_1 \chi_2 \bar{\chi}_2$ [55]. By treating the gapped (massive) component as a c-number, $\chi_2 \bar{\chi}_2 \rightarrow \langle \chi_2 \bar{\chi}_2 \rangle$ one gets a mean-field approximation which only redefines m_1 and shifts the transition point [10].

When one considers generic interaction strengths and chemical potentials, the phase diagram quickly becomes more complex as well as multi-dimensional. The full diagram is given in fig. 3.1. For details about this diagram and derivations, please see [10]. Let us here focus on certain qualitative features.

As one example, at zero chemical potential and $B = \Delta = 0$, (3.3) becomes an XXZ spin chain model, with no qualitative differences between different J_z

⁷Recall the discussion about quantum phase transitions on page 16.

values, i.e. different interaction strengths. In fig. 3.1, this XXZ model corresponds to the horizontal red line in the upper figure. The $|B| = \Delta$ transition of the Ising model connects to this line at $J_z = 0$, and also extends onto non-zero interaction strengths, and forms “transition sheets”. This is expected to continue until the global Umklapp term becomes relevant at $K = \frac{1}{2}$ [10] (the global Umklapp term has conformal dimension $d = 4K$ as the local one, but lives in both one spatial dimension and time, i.e. $D = 2$ instead of $D = 1$). Classically, such a cosine term will tend to pin the field into one of the minima⁸, effectively forming a gap for $K < \frac{1}{2}$ [21].

At this point, we can form another XXZ model (an “XYX model”), perpendicular to the old one. The new XXZ model corresponds to the diagonal red line in the upper part of fig. 3.1. This XXZ model supports similar transition surfaces, specified by $\Delta = J_z - v_F$, which will merge with the old ones [10].

Similarly, we can expect something to happen when the local Umklapp term becomes relevant at $K = \frac{1}{4}$. However, since the two terms are structurally similar, we can conclude that the local Umklapp term will only reinforce the gapping effect of the global one. The one qualitative difference is that it may change the slope of the second transition surface for $K < \frac{1}{4}$, shifting the transition points somewhat. This is a minor effect, of much less qualitative importance than the actual transition. In effect, for $\mu = 0$, the local Umklapp interaction does not yield any new physics.

At $\mu \neq 0$ the picture gets even more involved, but the result is eventually the same. The main difference to the phase diagram is that it becomes harder to open up gaps. Specifically, to create an excitation gap in the helical liquid (or e.g. a quantum wire subject to Rashba interaction) with finite μ , a finite threshold magnetic field is required. The first XXZ line at $B = \Delta = 0$ opens up and becomes a surface in the $\Delta = 0$ plane, as can be seen in the lower part of fig. 3.1. The width of the surface decreases with increasing interactions, as lower threshold fields are required [10].

The other XXZ line also changes character, as it becomes a three-dimensional tubular structure that also merges with the surface in the $\Delta = 0$ plane. Inside the tube $K < \frac{1}{2}$, but it remains an ungapped Luttinger liquid even though we previously noted that the global Umklapp term (for the zero modes) becomes relevant at $K = \frac{1}{2}$. The fact that $\mu \neq 0$ means that we have a different type of quantum phase transition (commensurate-incommensurate, C-IC, instead of Kosterlitz-Thouless, KT), hence this change. The exterior of the tube has $K = \frac{1}{2}$ everywhere except for a singular line bordering the strongly interacting phase. This line has $K = \frac{1}{4}$. In effect, the global Umklapp term becomes relevant at this line, and continues to be relevant for stronger interactions. The Majorana transition surfaces touch this line [10, 57].

The local Umklapp term becomes relevant at the same value of the Luttinger parameter K as the global Umklapp term. It will thus reinforce the gap caused by the global Umklapp interaction, but not provide any qualitative differences.

⁸The field pinning also drives a magnetization in the plane, along an axis determined by the sign of g_{um} . For $g_{um} > 0$, the magnetization is along the x axis with order parameter $N_x = \Psi^\dagger \sigma^x \Psi \sim \cos \sqrt{4\pi}\phi$, while the magnetization along the y axis, $N_y = \Psi^\dagger \sigma^y \Psi \sim \sin \sqrt{4\pi}\phi \rightarrow 0$ for strong interactions [25]. In effect, the interacting system distinguishes between the x and y axes, whereas the non-interacting system does not.

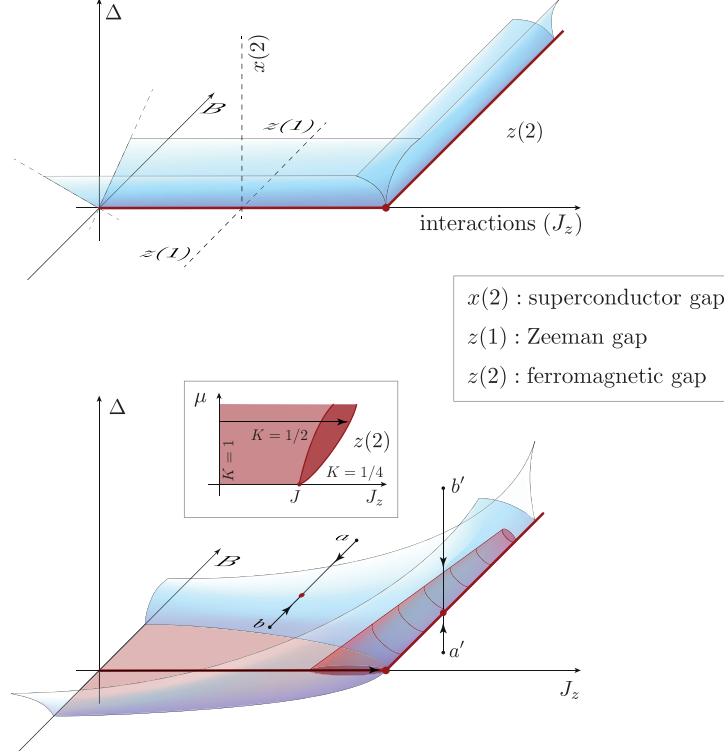


Figure 3.1 – Phase diagram for the SAR model [10], used with the authors’ permission. The top figure is the phase diagram of the XYZ model at $\mu = 0$. At the thick red lines, the model is reduced to XXZ spin chains. The blue surfaces are Ising transition surfaces, with Majorana modes. The phase diagram supports a few Néel ordered (antiferromagnetic) phases, denoted $a(n)$ where $a = x, y, z$ denotes the magnetization axis and n the degeneracy. These phases are associated with gaps, dominated by different perturbations. The bottom figure shows the case for $\mu \neq 0$, where the XXZ lines have expanded into surfaces and a tubular structure. The inset figure is the phase diagram of the XXZ model at $\Delta = B = 0$.

3.3.6 Conclusions

For general field theories and interaction terms in 1+1 dimensions, the method of bosonization seems more useful than that of mapping the theory to a spin chain model. The calculations certainly are not trivial, but the procedure of bosonization and RG analysis is well-known and general. In contrast, to have the spin chain method working well one needs to have the luck of stumbling on a recognizable theory.

We also conclude that the local Umklapp is less relevant than the global one, and that it does not have any noteworthy effect on the phase diagram. In effect, the helical liquid and the Majorana states are stable also in the presence of localized impurities.

Chapter 4

Spin-orbit interactions

In this chapter, we will investigate the effects of Rashba and Dresselhaus spin-orbit interactions. While these interactions do give rise to new terms in the Hamiltonian, we show that the Rashba spin-orbit interactions do not affect the Majorana transitions of the SAR model [10] or the stability of the Majorana fermions. The Dresselhaus spin-orbit interaction, however, may destabilize the Majorana fermions and needs further study.

4.1 Rashba spin-orbit interaction

Väyrynen and Ojanen [58] recently presented a neat way to diagonalize $\mathcal{H}_0 + \mathcal{H}_R$, from (2.16) and (2.25), for constant Rashba interaction strength, $\alpha_R(x) = \alpha_R$. Let $\mu = 0$ and

$$\Psi \rightarrow \Psi' = e^{-i\sigma^x \theta_\alpha/2} \Psi \quad (4.1)$$

be a rotation of the original spinor, choosing the angle θ_α such that $\cos \theta_\alpha = v_F/v_\alpha$ and $\sin \theta_\alpha = \alpha_R/v_\alpha$. This is simply a polar coordinate system with radius $v_\alpha = \sqrt{v_F^2 + \alpha_R^2}$ and angle θ_α , used to parametrize the two-dimensional parameter space spanned by v_F and α_R . The new fields are linear combinations of the old fields, and we denote them by $\Psi' = (\psi_-, \psi_+)^T$. In this parametrization the Hamiltonian takes the form

$$\mathcal{H}_0 + \mathcal{H}_R = v_\alpha \int dx \Psi'^\dagger \left(-i\partial_x \sigma^z \right) \Psi', \quad (4.2)$$

which has just the form of $\mathcal{H}_0$ in eq. (2.16). The prime on the z is just there to remind us that the rotation has taken us into a new coordinate system, (x', y', z') . So $\mathcal{H}_0 + \mathcal{H}_R$ becomes diagonal, but how do the interactions transform under this Rashba rotation? Actually, all terms except for the Umklapp interaction transform surprisingly simply. To see this, denote the rotation matrix by

$$s = e^{-i\sigma^x \theta_\alpha/2}. \quad (4.3)$$

For the combined forward and dispersive scattering terms from eq. (2.3.1) we have, for $g_4 = 2g_2$ ¹

$$\begin{aligned}\mathcal{H}_{fw} &= \left(\Psi^\dagger \Psi\right)^2 \\ &= \left(\Psi'^\dagger s s^{-1} \Psi'\right)^2 \\ &= \left(\Psi'^\dagger \Psi'\right)^2.\end{aligned}\tag{4.4}$$

The structure of the term is not changed, and hence the conformal dimension is unchanged. This term will then have the same effect as before.

When $\mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_{fw}$ is bosonized, we get a theory that looks as before, but the Fermi velocity v_F is replaced by v_α . This has the effect that parameters v and K in equations (3.40) and (3.41) get the new values

$$v' \approx v_\alpha + \frac{g_2}{2\pi},\tag{4.5}$$

$$K' \approx 1 - \frac{g_2}{2\pi v_\alpha}.\tag{4.6}$$

K' can also be written

$$K' \approx 1 - \frac{g_2}{2\pi} \frac{\sin \theta_\alpha}{\alpha_R},\tag{4.7}$$

i.e. the new Luttinger parameter K' depends explicitly both on the strength of the electron-electron interaction and on the Rashba interaction strength. The latter parameter is tunable in experiments, so the Rashba spin-orbit interaction allows for a direct control of the Luttinger parameter.

To continue with the other interactions, we first write $\delta\mathcal{H} = \Delta\mathcal{H}_\Delta + B\mathcal{H}_B$ and use (3.1) to implicitly define $\mathcal{H}_\Delta$ and $\mathcal{H}_B$. Under a Rashba rotation, we have for the perturbation from the magnetic field

$$\begin{aligned}\mathcal{H}_B &= \Psi^\dagger \sigma^x \Psi \\ &= \Psi'^\dagger s \sigma^x s^{-1} \Psi' \\ &= \Psi'^\dagger \sigma^{x'} \Psi'\end{aligned}\tag{4.8}$$

and for the perturbation from the superconductor

$$\begin{aligned}\mathcal{H}_\Delta &= \psi_{L\downarrow} \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger \\ &= \left(\cos \frac{\theta_\alpha}{2} \psi_- + i \sin \frac{\theta_\alpha}{2} \psi_+\right) \left(\cos \frac{\theta_\alpha}{2} \psi_+ + i \sin \frac{\theta_\alpha}{2} \psi_-\right) \\ &\quad + \left(\cos \frac{\theta_\alpha}{2} \psi_+^\dagger - i \sin \frac{\theta_\alpha}{2} \psi_-^\dagger\right) \left(\cos \frac{\theta_\alpha}{2} \psi_-^\dagger - i \sin \frac{\theta_\alpha}{2} \psi_+^\dagger\right),\end{aligned}\tag{4.9}$$

where we have used that [19]

$$e^{-i\sigma^x \theta_\alpha/2} = \cos \frac{\theta_\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin \frac{\theta_\alpha}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\tag{4.10}$$

¹As noted on page 12, we do indeed have $g_4 = 2g_2$ for low energies [21].

By expanding the parantheses and using the exclusion principle, we find

$$\begin{aligned}
\mathcal{H}_\Delta &= \cos^2 \frac{\theta_\alpha}{2} \psi_- \psi_+ + i \sin \frac{\theta_\alpha}{2} \cos \frac{\theta_\alpha}{2} (\psi_- \psi_- + \psi_+ \psi_+) - \sin^2 \frac{\theta_\alpha}{2} \psi_+ \psi_- \\
&\quad + \cos^2 \frac{\theta_\alpha}{2} \psi_+^\dagger \psi_-^\dagger - i \sin \frac{\theta_\alpha}{2} \cos \frac{\theta_\alpha}{2} (\psi_+^\dagger \psi_+^\dagger + \psi_-^\dagger \psi_-^\dagger) - \sin^2 \frac{\theta_\alpha}{2} \psi_-^\dagger \psi_+^\dagger \\
&= \cos^2 \frac{\theta_\alpha}{2} (\psi_- \psi_+ + \psi_+^\dagger \psi_-^\dagger) - \sin^2 \frac{\theta_\alpha}{2} (\psi_+ \psi_- + \psi_-^\dagger \psi_+^\dagger) \\
&= \psi_- \psi_+ + \psi_+^\dagger \psi_-^\dagger.
\end{aligned} \tag{4.11}$$

The forms of these perturbations and their conformal dimensions are thus also preserved. However, terms of the Umklapp type (both global and local) transform less trivially. It is enough to treat only the global Umklapp, as the local one yields the same result multiplied by a delta function $\delta(x)$. For the global Umklapp of (2.19) we find

$$\begin{aligned}
\mathcal{H}_{um} &= g_u \left(\psi_{R\uparrow}^\dagger \partial_x \psi_{R\uparrow}^\dagger \psi_{L\downarrow} \partial_x \psi_{L\downarrow} + \psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow}^\dagger \psi_{R\uparrow} \partial_x \psi_{R\uparrow} \right) \\
&= g_u \left[\left(\cos^4 \frac{\theta_\alpha}{2} + \sin^4 \frac{\theta_\alpha}{2} \right) (\psi_+^\dagger \partial_x \psi_+^\dagger \psi_- \partial_x \psi_- + \psi_-^\dagger \partial_x \psi_-^\dagger \psi_+ \partial_x \psi_+) \right. \\
&\quad + i \sin \frac{\theta_\alpha}{2} \cos^3 \frac{\theta_\alpha}{2} (\psi_+^\dagger \partial_x \psi_+^\dagger \psi_- \partial_x \psi_+ + \psi_-^\dagger \partial_x \psi_-^\dagger \psi_+ \partial_x \psi_- \\
&\quad - \psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_- - \psi_-^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_+ + \psi_-^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+ \\
&\quad + \psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+ - \psi_-^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_+ - \psi_+^\dagger \partial_x \psi_-^\dagger \psi_+ \partial_x \psi_+) \\
&\quad + i \sin^3 \frac{\theta_\alpha}{2} \cos \frac{\theta_\alpha}{2} (\psi_+^\dagger \partial_x \psi_-^\dagger \psi_+ \partial_x \psi_+ + \psi_-^\dagger \partial_x \psi_+^\dagger \psi_- \partial_x \psi_- \\
&\quad - \psi_-^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+ + \psi_-^\dagger \partial_x \psi_-^\dagger \psi_+ \partial_x \psi_- + \psi_-^\dagger \partial_x \psi_+^\dagger \psi_- \partial_x \psi_- \\
&\quad + \psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_- - \psi_+^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_- - \psi_+^\dagger \partial_x \psi_+^\dagger \psi_- \partial_x \psi_+) \\
&\quad + \frac{1}{2} \sin^2 \theta_\alpha (\psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+ + \psi_+^\dagger \partial_x \psi_-^\dagger \psi_+ \partial_x \psi_- \\
&\quad + \psi_-^\dagger \partial_x \psi_+^\dagger \psi_- \partial_x \psi_+ + \psi_-^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_- \\
&\quad \left. - \psi_+^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_+ - \psi_-^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_-) \right].
\end{aligned} \tag{4.12}$$

This expression is not only long, it is also quite interesting. (Fortunately, it is also Hermitian and time-reversal symmetric.) In particular, it contains terms of three types. First, the terms with coefficient $\cos^4 \frac{\theta_\alpha}{2} + \sin^4 \frac{\theta_\alpha}{2}$ look like ordinary Umklapp terms (in the new basis). Indeed, the whole expression reduces to these terms when the rotation angle θ_α is zero or a multiple of π . There are also terms with even numbers of right and left movers, with coefficient $\frac{1}{2} \sin^2 \theta_\alpha$ and terms with odd numbers of right and left movers, with coefficients $i \sin \frac{\theta_\alpha}{2} \cos^3 \frac{\theta_\alpha}{2}$ or $i \sin^3 \frac{\theta_\alpha}{2} \cos \frac{\theta_\alpha}{2}$.

The terms with coefficient $\frac{1}{2} \sin^2 \theta_\alpha$ all have conformal dimension $d = 2 + 2K$. They will thus be irrelevant for all positive K (repulsive interactions). We show this using the method of operator product expansions (OPE:s) in conformal field theory (CFT) [47]. In terms of complex coordinates $z = v\tau - ix$ and $\bar{z} = v\tau + ix$,

a vertex operator of some field Ξ is

$$V_a(z, \bar{z}) =: e^{ia\Xi(z, \bar{z})} :, \quad (4.13)$$

where the colons denote normal ordering. The vertex operators satisfy the following two OPE:s [47]

$$\partial\Xi(z) V_a(w, \bar{w}) \sim \frac{V_a(w, \bar{w})}{z - w}, \quad (4.14)$$

$$V_a(z, \bar{z}) V_{-a}(w, \bar{w}) \sim |z - w|^{-2a^2/g\pi} + \dots, \quad (4.15)$$

where $g = 4$ if Ξ has zero conformal spin and $g = 8$ for a chiral field Ξ ².

Consider e.g. the term $\psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+$. We bosonize it using analogues of the bosonization relations (3.18) and (3.19). Note that, in general, the prefactors in (3.18) and (3.19) can be dropped if we normal order the vertex operators [47]. Applying the transformation (3.43) and using the OPE (4.15), we find

$$\begin{aligned} \psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+ &\sim: e^{i\sqrt{4\pi}\phi_+} :: \partial_x e^{-i\sqrt{4\pi}\phi_-} :: e^{i\sqrt{4\pi}\phi_-} :: \partial_x e^{-i\sqrt{4\pi}\phi_+} : \\ &\sim: e^{i\sqrt{4\pi}\phi_+} :: \partial_x \phi_+ e^{-i\sqrt{4\pi}\phi_+} :: \partial_x \phi_- e^{-i\sqrt{4\pi}\phi_-} :: e^{i\sqrt{4\pi}\phi_-} : \\ &\sim: e^{i\sqrt{4\pi K}\phi'_+} :: \partial_x \phi'_+ e^{-i\sqrt{4\pi K}\phi'_+} :: \partial_x \phi'_- e^{-i\sqrt{4\pi K}\phi'_-} :: e^{i\sqrt{4\pi K}\phi'_-} : \\ &\sim \epsilon^{-2\frac{4\pi K}{8\pi}} : \partial_x \phi_+ :: \partial_x \phi_- : \epsilon^{-2\frac{4\pi K}{8\pi}} \\ &= \epsilon^{-2K} : \partial_x \phi_+ :: \partial_x \phi_- : \end{aligned} \quad (4.16)$$

where ϵ is the point splitting distance, which has dimension length. This term has conformal dimension $d = 2 + 2K$, as each of the differential operators has dimension one. The same holds for the other terms with the same coefficients.

As an aside, we can here note the importance of the bosonization procedure and the transformation (3.43). In a purely fermionic language we would not see that the conformal dimension depends on the Luttinger parameter K . If one would rewrite

$$\psi_+^\dagger \partial_x \psi_-^\dagger \psi_- \partial_x \psi_+ = \psi_+^\dagger \partial_x \psi_+ \partial_x (\psi_-^\dagger \psi_-) - \psi_+^\dagger \partial_x \psi_+ \psi_-^\dagger \partial_x \psi_-, \quad (4.17)$$

all factors would seem to have the form of kinetic energies ($\psi \partial_x \psi$) or the derivative of an electron density ($\partial_x (\psi^\dagger \psi)$), both with the conformal dimension 2. The total dimension of the term would then be 4, which is the case of the theory without interactions ($K = 1$). Indeed, the interacting fermionic theory is always compared to the free theory, whereas we in the bosonization scheme compare the interaction terms to a continuum of “free” theories defined by K .

Now we turn to the terms with odd numbers of right and left movers. We first note what they represent: two particles interact and one of them change direction. While the full process is elastic, with a momentum transfer of $2k_F$ to or from the lattice, the backscattering part is inelastic. We will call the process inelastic one-particle backscattering. This process should be active when there is disorder, or one of the participating states is at $k = 0$, such that rapid phase oscillations are suppressed [59].

²C.f. the discussion on page 28.

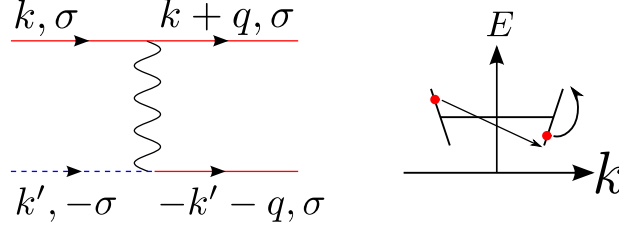


Figure 4.1 – Inelastic backscattering. Two incoming particles traveling in opposite directions (having opposite spins) interact, and one of the particles change direction (spin).

The difference to the restrictions of interactions we considered in section 2.3.1 (and to what was considered in ref. [25]) is that the electric field of the Rashba interaction breaks axial spin symmetry. If that symmetry is broken, the S^z component is no longer conserved, and terms of this type may be generated by means of an $SU(2)$ transformation, such as the Rasha rotation s above. Schmidt *et al.* [59] recently found the same process using a momentum-dependent $SU(2)$ transformation. The process was in fact noted already by Kane and Mele [17], but seems to have been mostly neglected (or forgotten) since then [60].

The terms for this process have conformal dimension $d = K + 2$ [60, 61]. The easiest way to see this is to take e.g. the term $\psi_+^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_-$ and remove the derivative on the ψ_- , as it is not needed for regularization. We then have

$$\begin{aligned}
 \psi_+^\dagger \partial_x \psi_+^\dagger \psi_+ \partial_x \psi_- &= \psi_+^\dagger \psi_- \partial_x \psi_+^\dagger \psi_+ \\
 &= \psi_+^\dagger \psi_- \partial_x (\psi_+^\dagger \psi_+) + \psi_+^\dagger \psi_- \psi_+^\dagger \partial_x \psi_+ \\
 &= \psi_+^\dagger \psi_- \partial_x \rho_+ \\
 &\sim e^{i\sqrt{4\pi}\phi_+} e^{i\sqrt{4\pi}\phi_-} \partial_{xx}^2 \phi_+ \\
 &\sim e^{i\sqrt{4\pi K}\phi'_+} e^{i\sqrt{4\pi K}\phi'_-} \partial_{xx}^2 \phi'_+,
 \end{aligned} \tag{4.18}$$

where $\rho_+ = \psi_+^\dagger \psi_+$ is the density of the $+$ electrons, and we have applied the transformation (3.43). The term has indeed the conformal dimension $d = K + 2$, as the vertex operator $e^{i\sqrt{4\pi K}\phi'_+}$ has conformal dimension $K/2$ and $\partial_{xx}^2 \phi_+$ has dimension 2. The same result holds for the other terms of this type.

These terms are then irrelevant for all positive K (repulsive interactions). Similarly to the terms with coefficient $\frac{1}{2} \sin^2 \theta$ then, they will not affect the relevant parts of the phase diagram or the Majorana transitions at all.

4.2 Dresselhaus spin-orbit interaction

If we include also the Dresselhaus spin-orbit interaction from (2.26) with constant interaction strength β_D , we get the three-dimensional parameter space spanned by v_F , α_R and β_D . The method used before for the Rashba spin-orbit interaction here generalizes to a spherical coordinate system, with the new radius $v_\beta = \sqrt{v_F^2 + \alpha_R^2 + \beta_D^2}$ and the additional angle θ_β , chosen such that $\cos \theta_\beta = \beta_D/v_\beta$ and $\sin \theta_\beta = v_\alpha/v_\beta$.

To diagonalize $\mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_D$ (with $\mu = 0$), we can use the aggregate of the Rashba rotation (with rotation matrix s), followed by a rotation with matrix

$$t = e^{-i\sigma^{y'} \theta_\beta / a}, \quad (4.19)$$

where

$$a = \frac{2\theta_\beta}{\theta_\beta - \frac{\pi}{2}}. \quad (4.20)$$

Again, the prime denotes that we are in the Rashba-rotated coordinate system, (x', y', z') . The new spinor is $\tilde{\Psi}' = t\Psi' = ts\Psi$, and we denote its components by $\tilde{\Psi}' = (\tilde{\psi}_-, \tilde{\psi}_+)^T$. It is clear that the forward and dispersive interactions of eq. (2.22) retain their structure as

$$\begin{aligned} \mathcal{H}_{fw} &= g_2 \Psi^\dagger \Psi = g_2 \tilde{\Psi}'^\dagger t s s^{-1} t^{-1} \tilde{\Psi}' \\ &= g_2 \tilde{\Psi}'^\dagger \tilde{\Psi}'. \end{aligned} \quad (4.21)$$

This implies yet another set of values for the renormalized velocity and the Luttinger parameter. The new values are

$$v'' \approx v_\beta + \frac{g_2}{2\pi}, \quad (4.22)$$

$$K'' \approx 1 - \frac{g_2}{2\pi v_\beta} = 1 - \frac{g_2}{2\pi \alpha_R} \sin \theta_\alpha \sin \theta_\beta. \quad (4.23)$$

In the Rashba case, the Umklapp interaction gave rise to 32 terms in total, which could be combined into 24 terms. In this case, the additional rotation gives an unmanageable “worst case” number of $32 \cdot 16 = 512$ terms. However, the combined rotation is just an $SU(2)$ transformation, which does not give rise to any other kinds of terms than those already included [59]. The perturbation from the superconductor also does not yield any new effects, as it is form-invariant,

$$\mathcal{H}_\Delta = \psi_{L\downarrow} \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger = \tilde{\psi}_- \tilde{\psi}_+ + \tilde{\psi}_+^\dagger \tilde{\psi}_-^\dagger. \quad (4.24)$$

The magnetic field, however, rotates also into the z'' direction:

$$\begin{aligned} \mathcal{H}_B &= \Psi^\dagger \sigma^x \Psi = \tilde{\Psi}'^\dagger t s \sigma^x s^{-1} t^{-1} \tilde{\Psi}' = \tilde{\Psi}'^\dagger t \sigma^x t^{-1} \tilde{\Psi}' \\ &= \tilde{\Psi}'^\dagger \begin{pmatrix} \cos \theta_\beta & \sin \theta_\beta \\ \sin \theta_\beta & -\cos \theta_\beta \end{pmatrix} \tilde{\Psi}', \\ &= \tilde{\Psi}'^\dagger \left[\cos(\theta_\beta) \sigma^{z''} + \sin(\theta_\beta) \sigma^{x''} \right] \tilde{\Psi}', \end{aligned} \quad (4.25)$$

where the double prime denotes the new coordinate system, (x'', y'', z'') .

The two field components are equally relevant, with conformal dimension K , but have different physical effects. The z'' component is in the direction of the spin quantization and will destroy the quantum spin Hall state (and hence the Majorana states) much more efficiently than planar magnetic fields do [22, 33]. While the magnitude of this effect of course depends on the spin-orbit interaction strengths, we can ask whether we can define a field in the xy plane such that it does not rotate into a z'' component. The answer is yes, but it is not fully clear yet how this field would affect the phase diagram of the SAR model.

To study this case, let the field $\mathbf{B} = (B_x, B_y)$ be perpendicular to the resultant spin-orbit field, $\mathbf{B}_{SO} = (\beta_D, \alpha_R)$, formed by the Dresselhaus and Rashba fields. By rotating around an axis in the direction of $\mathbf{B}$, we can diagonalize $\mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_D$, while leaving $\mathbf{B}$ invariant. If the rotation matrix is given by

$$u = \exp \left(\frac{-i\boldsymbol{\sigma} \cdot (B_x \hat{x} + B_y \hat{y}) \theta_{SO}}{2\sqrt{B_x^2 + B_y^2}} \right), \quad (4.26)$$

the angle θ_{SO} must satisfy

$$\tan \theta_{SO} = \frac{\alpha_R}{v_F} \sqrt{1 + \frac{\beta_D^2}{\alpha_R^2}}. \quad (4.27)$$

The addition of a y component of the magnetic field is, however, a non-trivial extension of the SAR model. In the spin-chain language of section 3.2, it acquires the form

$$B_y \Psi^\dagger \sigma^y \Psi \sim B_y \sum_j (-1)^j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y). \quad (4.28)$$

When added to (3.3) it acts as a dimerization of the XYZ spin-chain. The resulting theory can be mapped to theory considered by Arlego et al. [62] in two cases, when $B_x = 0$ or when $J_z = \mu = 0$ and $B_y = v_F - \Delta$. The first case is not very interesting, as we will need an x component whenever we have the Rashba interaction present, which we expect to be the case in the relevant materials. In the second case, the Hamiltonian (3.3) takes the form

$$\begin{aligned} \mathcal{H}_j = & (v_F + \Delta) S_j^x S_{j+1}^x + (v_F - \Delta) S_j^y S_{j+1}^y \\ & + (-1)^j (v_F - \Delta) (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) - B_x S_j^z \end{aligned} \quad (4.29)$$

after a rotation by π around the x axis at every second spin site. This spin-chain theory has an Ising transition at

$$B_x = \pm \sqrt{1 - \frac{2\Delta}{v_F}}, \quad (4.30)$$

where it hosts Majorana modes [62]. The Majoranas are thus stable in the limit of no interactions, but the effect of non-zero J_z , i.e. with electron-electron interactions present, is not currently known.

To conclude, we see that the Dresselhaus spin-orbit interaction may destabilize the Majorana fermions if we take the magnetic field to be along the x axis, as it rotates into a z component. More work is required to fully understand the case of a magnetic field with components along both $\hat{x}$ and $\hat{y}$ when electron-electron interactions are present.

Chapter 5

Conclusions

We have extended the model considered by Sela, Altland and Rosch in ref. [10], to also include local Umklapp interactions from localized impurities and the Rashba and Dresselhaus spin-orbit interactions. We have shown that the Majorana states present in a system formed by a topological insulator coupled to a s-wave superconductor and an external magnetic field along the edge are stable against localized impurities and Rashba spin-orbit interaction. This stability means that the states would be quite robust in realistic experimental HgTe quantum well systems, which are likely to contain disorder and be subject to Rashba spin-orbit interactions, no matter if they are designed just to detect these states or if they are designed to use them, possibly for quantum computation.

In contrast, for the InAs/GaSb quantum well systems, we also expect the Dresselhaus spin-orbit interaction to be present. In that case the Majorana states may become destabilized. This effect can be avoided, in the case of no electron-electron interactions, if the magnetic field is placed in the plane of the topological insulator. More work is needed, however, to also treat the case when electron-electron interactions are present.

In addition, we have shown that the Luttinger parameter can be controlled via the Rashba interaction strength. This provides additional control in experiments, and makes it possible to access different parts of the phase diagram more readily.

These results do bode well for possible future detection of Majorana fermions in topological insulator systems, but it remains to be seen in which systems the Majorana fermions are first discovered. Given the considerable amount of research into the matter [4] and the recently discovered signatures of Majorana states [5, 6, 7, 8], we have certainly come a long way since Majorana's original prediction in 1937 [3]. Their existence is indeed starting to seem a foregone conclusion, yet it remains to be seen. These are interesting times.

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Appendix A

Time reversal in detail

Under time reversal, both spin and momentum are flipped. In addition, for a spin-half system one of the spin states acquires a minus sign [19]. Choosing the left mover to be the state picking up the minus sign, we have the transformations

$$\begin{aligned} T^{-1}\psi_{R\uparrow}T &= \psi_{L\downarrow}, \\ T^{-1}\psi_{L\downarrow}T &= -\psi_{R\uparrow}. \end{aligned} \quad (\text{A.1})$$

Recalling that the time reversal operator T can be written $T = U\hat{K}$, where U is a unitary matrix and $\hat{K}$ is complex conjugation [19], we now calculate

$$\begin{aligned} \psi_{L\downarrow}^\dagger &= \left(T^{-1}\psi_{R\uparrow}T\right)^\dagger = T^\dagger\psi_{R\uparrow}^\dagger T^{-1\dagger} \\ &= \hat{K}^\dagger U^\dagger \psi_{R\uparrow}^\dagger U^{-1\dagger} \hat{K}^{-1\dagger} \\ &= \hat{K}^\dagger U^{-1} \psi_{R\uparrow}^\dagger U \hat{K}^{-1\dagger}. \end{aligned} \quad (\text{A.2})$$

What is $\hat{K}^\dagger$? To answer this, let a be a c-number and consider

$$\begin{aligned} \left(\hat{K}a\right)^\dagger &= a^* \hat{K}^\dagger, \\ \left(\hat{K}a\right)^\dagger &= \left(a^* \hat{K}\right)^\dagger = \hat{K}^\dagger a, \end{aligned} \quad (\text{A.3})$$

i.e. $a^* \hat{K}^\dagger = \hat{K}^\dagger a$. But this is just the definition of $\hat{K}$ so $\hat{K}^\dagger = \hat{K}$. Obviously, we also have $\hat{K}^{-1} = \hat{K}$ and thus (A.2) becomes

$$\begin{aligned} \psi_{L\downarrow}^\dagger &= \hat{K}U^{-1}\psi_{R\uparrow}^\dagger U \hat{K} \\ &= T^{-1}\psi_{R\uparrow}^\dagger T. \end{aligned} \quad (\text{A.4})$$

Similarly, we get $\psi_{R\uparrow}^\dagger = -T^{-1}\psi_{L\downarrow}^\dagger T$. To summarize, the fields transform as

$$\begin{aligned} \psi_{L\downarrow} &\rightarrow -\psi_{R\uparrow}, \\ \psi_{R\uparrow} &\rightarrow \psi_{L\downarrow}^\dagger, \\ \psi_{L\downarrow}^\dagger &\rightarrow -\psi_{R\uparrow}^\dagger, \\ \psi_{R\uparrow}^\dagger &\rightarrow \psi_{L\downarrow}^\dagger, \end{aligned} \quad (\text{A.5})$$

under time reversal.

Let us now explicitly study the time reversal symmetry of the interactions. We begin by checking how eq. (2.21) transforms under time reversal:

$$\begin{aligned}
\mathcal{H}_{fw} &= g_2 \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \psi_{R\uparrow} + \frac{g_4}{2} \left[\left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right)^2 + \left(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} \right)^2 \right] \\
&\rightarrow g_2 \left(-\psi_{R\uparrow}^\dagger \right) \left(-\psi_{R\uparrow} \right) \psi_{L\downarrow}^\dagger \psi_{L\downarrow} \\
&+ \frac{g_4}{2} \left[\left(\left(-\psi_{R\uparrow}^\dagger \right) \left(-\psi_{R\uparrow} \right) \right)^2 + \left(\psi_{L\downarrow}^\dagger \psi_{L\downarrow} \right)^2 \right] \\
&= \mathcal{H}_{fw}.
\end{aligned} \tag{A.6}$$

Similarly, for the Umklapp term of eq. (2.19):

$$\begin{aligned}
\mathcal{H}_{um} &= g_u \left[\psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \partial_x \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \partial_x \psi_{R\uparrow} \psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow} \right] \\
&\rightarrow g_u \left[-\psi_{R\uparrow}^\dagger \partial_x \left(-\psi_{R\uparrow} \right) \psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow} \right. \\
&\quad \left. + \psi_{L\downarrow}^\dagger \partial_x \psi_{L\downarrow} \left(-\psi_{R\uparrow} \right) \partial_x \left(-\psi_{R\uparrow} \right) \right] \\
&= \mathcal{H}_{um}.
\end{aligned} \tag{A.7}$$

We also have the Rashba Hamiltonian of eq. (2.25) (assuming $\alpha_R(x) \in \mathbb{R}$):

$$\begin{aligned}
\mathcal{H}_R &= \alpha_R(x) \left(\psi_{L\downarrow}^\dagger \partial_x \psi_{R\uparrow} - \psi_{R\uparrow}^\dagger \partial_x \psi_{L\downarrow} \right) \\
&\rightarrow \alpha_R(x) \left[-\psi_{R\uparrow}^\dagger \partial_x \psi_{L\downarrow} - \psi_{L\downarrow}^\dagger \partial_x \left(-\psi_{R\uparrow} \right) \right] \\
&= \mathcal{H}_R,
\end{aligned} \tag{A.8}$$

and finally the Dresselhaus Hamiltonian of eq. (2.27) (assuming $\beta_D(x) \in \mathbb{R}$):

$$\begin{aligned}
\mathcal{H}_D &= -i\beta_D(x) \left(\psi_{R\uparrow}^\dagger \partial_x \psi_{L\downarrow} + \psi_{L\downarrow}^\dagger \partial_x \psi_{R\uparrow} \right) \\
&\rightarrow +i\beta_D(x) \left[-\psi_{L\downarrow}^\dagger \partial_x \psi_{R\uparrow} - \psi_{R\uparrow}^\dagger \partial_x \psi_{L\downarrow} \right] \\
&= \mathcal{H}_D.
\end{aligned} \tag{A.9}$$

These are the four time-reversal invariant interactions in our model. The couplings to external magnetic fields and superconductors in eq. (3.1) obviously break the time reversal symmetry of the system (the external field is fixed no matter the sign of the momentum of the electrons, and so is the directions into and out of the nearby superconductor). Explicitly we get

$$\begin{aligned}
\delta\mathcal{H} &= B \left(\psi_{L\downarrow}^\dagger \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow} \right) + \Delta \left(\psi_{L\downarrow} \psi_{R\uparrow} + \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger \right) \\
&\rightarrow B \left[-\psi_{R\uparrow}^\dagger \psi_{L\downarrow} + \psi_{L\downarrow}^\dagger \left(-\psi_{R\uparrow} \right) \right] + \Delta \left[-\psi_{R\uparrow} \psi_{L\downarrow} + \psi_{L\downarrow}^\dagger \left(-\psi_{R\uparrow}^\dagger \right) \right] \\
&= -\delta\mathcal{H},
\end{aligned} \tag{A.10}$$

which is the expected result.

Appendix B

Ghost site approach to the local Umklapp interaction

The idea here is to put the local Umklapp interaction treated in section 3.2.2 on a ghost site inbetween two sites in the main lattice, in order to see if the impurity then separates from the bulk spin chain Hamiltonian. If the idea of an impurity squeezed into the space between two lattice points seems hard to accept, think of it as an external impurity physically situated next to our 1D lattice, as in fig. B.1. Either way, we number this “ghost” site $j = \frac{1}{2}$ ¹.

As in eqs. (3.11) and (3.13), we approximate the fields as slow, i.e. we assume $\psi_{R\uparrow}(x + \frac{\alpha}{2}) \approx \psi_{R\uparrow}(x)$ and $\psi_{L\downarrow}(x + \frac{\alpha}{2}) \approx \psi_{L\downarrow}(x)$. The expansions for the lattice fermions become

$$a_j \sim e^{i\frac{\pi}{2}x} \psi_{R\uparrow}(x) + e^{-i\frac{\pi}{2}x} \psi_{L\downarrow}(x), \quad (\text{B.1})$$

$$a_{j+\frac{1}{2}} \sim e^{i\frac{\pi}{2}(x+\frac{\alpha}{2})} \psi_{R\uparrow}(x + \frac{\alpha}{2}) + e^{-i\frac{\pi}{2}(x+\frac{\alpha}{2})} \psi_{L\downarrow}(x + \frac{\alpha}{2}) \quad (\text{B.2})$$

Setting set the lattice spacing $\alpha = 1$ we have

$$a_{j+\frac{1}{2}} \approx e^{i\frac{\pi}{2}x} e^{i\frac{\pi}{4}} \psi_{R\uparrow}(x) + e^{-i\frac{\pi}{2}x} e^{-i\frac{\pi}{4}} \psi_{L\downarrow}(x). \quad (\text{B.3})$$

¹In order to keep the structure of the Hamiltonian for the free theory and the other interactions, we must use the same numbering scheme for the ordinary sites as before. The other conceivable idea, to number the ordinary sites by $j = 0, 2, 4, 6, \dots$ and let the impurity be located at $j = 1$, will fail in the inverse Jordan-Wigner transformation to follow.

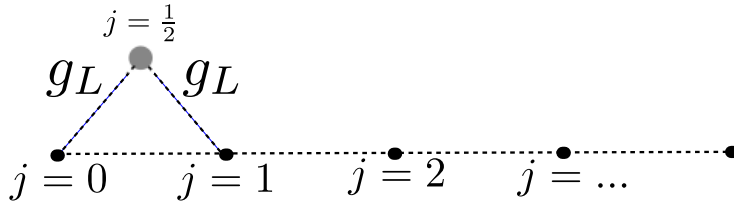


Figure B.1 – Sketch of the ghost site as located next to the main lattice chain. The g_L term is only active in the links to sites $j = 0$ and $j = 1$.

The fields can then be expressed as

$$\psi_{L\downarrow}(x) \sim \frac{1}{\sqrt{2}} e^{i\frac{\pi}{2}x} \left(e^{-i\frac{\pi}{4}} a_j + i a_{j+\frac{1}{2}} \right), \quad (\text{B.4})$$

$$\psi_{R\uparrow}(x) \sim \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{2}x} \left(e^{i\frac{\pi}{4}} a_j - i a_{j+\frac{1}{2}} \right) \quad (\text{B.5})$$

and the local Umklapp interaction (3.10),

$$\begin{aligned} \mathcal{H}_{um}^{local} = g_L & \left[\psi_{R\uparrow}^\dagger(0) \psi_{R\uparrow} \left(\frac{\alpha}{2} \right) \psi_{L\downarrow} \left(\frac{\alpha}{2} \right) \psi_{L\downarrow}(0) \right. \\ & \left. + \psi_{L\downarrow}^\dagger(0) \psi_{L\downarrow} \left(\frac{\alpha}{2} \right) \psi_{R\uparrow} \left(\frac{\alpha}{2} \right) \psi_{R\uparrow}(0) \right], \end{aligned} \quad (\text{B.6})$$

can be rewritten in terms of the lattice fermions. It becomes

$$\begin{aligned} \mathcal{H}_{um}^{local} = \frac{ig_L}{4} & \left[-\sqrt{2} a_0^\dagger a_{\frac{1}{2}}^\dagger a_1 a_0 + 2i a_0^\dagger a_{\frac{1}{2}}^\dagger a_1 a_{\frac{1}{2}} \right. \\ & - \sqrt{2} a_0^\dagger a_1^\dagger a_{\frac{1}{2}} a_0 + 2i a_0^\dagger a_1^\dagger a_1 a_0 - \sqrt{2} a_0^\dagger a_1^\dagger a_1 a_{\frac{1}{2}} \\ & \left. + 2i a_{\frac{1}{2}}^\dagger a_1^\dagger a_{\frac{1}{2}} a_0 - \sqrt{2} a_{\frac{1}{2}}^\dagger a_1^\dagger a_1 a_0 \right]. \end{aligned} \quad (\text{B.7})$$

Now it is time to determine the form of the inverse Jordan-Wigner transformation. To the best of my knowledge, applications of the Jordan-Wigner technology to half-integer sites have not been discussed in the literature. To what extent the procedure I suggest here is mathematically well-defined and unambiguous, requires further study. However, the important thing seems to be that there is a unique path traversing all sites, in order to define the operator string that appears in the Jordan-Wigner transformation [21]. Here, it is easy to define the path by the consecutive steps $j = 0, \frac{1}{2}, 1, 2, 3, \dots$

We will use the following Jordan-Wigner transformations:

$$S_j^z = a_j^\dagger a_j - \frac{1}{2}, \quad j = 0, \frac{1}{2}, 1, 2, \dots, \quad (\text{B.8})$$

$$S_j^+ = a_j^\dagger (-1)^j e^{i\pi \sum_{l=0}^{j-1} a_l^\dagger a_l + i\pi \theta(j-1) a_{\frac{1}{2}}^\dagger a_{\frac{1}{2}}}, \quad j = 0, 1, 2, \dots, \quad (\text{B.9})$$

$$S_{\frac{1}{2}}^+ = i a_{\frac{1}{2}}^\dagger e^{i\pi a_0^\dagger a_0}, \quad (\text{B.10})$$

where θ is the step function. The inverse transformations then become

$$a_j^\dagger a_j = S_j^z + \frac{1}{2}, \quad j = 0, \frac{1}{2}, 1, 2, \dots, \quad (\text{B.11})$$

$$a_j^\dagger = (-1)^j S_j^+ e^{-i\pi \sum_{l=0}^{j-1} a_l^\dagger a_l - i\pi \theta(j-1) a_{\frac{1}{2}}^\dagger a_{\frac{1}{2}}}, \quad j = 0, 1, 2, \dots, \quad (\text{B.12})$$

$$a_{\frac{1}{2}}^\dagger = -i S_{\frac{1}{2}}^+ e^{-i\pi a_0^\dagger a_0}. \quad (\text{B.13})$$

The local Umklapp term in the Hamiltonian can now be written

$$\begin{aligned}
\mathcal{H}_{um}^{local} = \frac{gL}{4} & \left[S_0^z \left(\sqrt{2} S_{\frac{1}{2}}^+ S_1^- - \sqrt{2} S_1^+ S_{\frac{1}{2}}^- \right) + \frac{\sqrt{2}}{2} \left(S_{\frac{1}{2}}^+ S_1^- - S_1^+ S_{\frac{1}{2}}^- \right) \right. \\
& + S_{\frac{1}{2}}^z \left(2 S_0^+ S_1^- + 2 S_1^+ S_0^- \right) + \left(S_0^+ S_1^- + S_1^+ S_0^- \right) \\
& + S_1^z \left(\sqrt{2} S_0^+ S_{\frac{1}{2}}^- - \sqrt{2} S_{\frac{1}{2}}^+ S_0^- \right) + \frac{\sqrt{2}}{2} \left(S_0^+ S_{\frac{1}{2}}^- - S_{\frac{1}{2}}^+ S_0^- \right) \\
& \left. - S_0^z S_1^z - S_0^z - S_1^z \right]. \tag{B.14}
\end{aligned}$$

This Hamiltonian is Hermitian, but unfortunately it looks quite complicated to work with. In fact, there seems to be no attempt in the literature, neither analytical or numerical, to treat this type of spin chain model.

Appendix C

The cumulant expansion

To prove the formula

$$\langle e^\Omega \rangle = e^{\langle \Omega \rangle + \frac{1}{2}(\langle \Omega^2 \rangle - \langle \Omega \rangle^2) + \dots} \quad (\text{C.1})$$

first recall a few definitions from probability theory. These definitions can be found in any good book on the subject, e.g. ref. [63].

Def. 1. *The r :th moment, μ_r , is the expectation value of the real-valued random variable X^r with probability density function $f_X(x)$ (note the difference between the random variable X and its realization x), i.e.*

$$\mu_r \equiv \langle X^r \rangle \equiv \int_{-\infty}^{\infty} x^r f_X(x) \, dx, \quad r = 0, 1, 2, \dots \quad (\text{C.2})$$

Def. 2. *The moment-generating function is defined by*

$$M_X(t) \equiv \langle e^{tX} \rangle, \quad t \in \mathbb{R}. \quad (\text{C.3})$$

The moment-generating function gets its name from the fact that we can expand the exponential around $t = 0$, and then find any moment by differentiation with respect to t . Unfortunately, this expansion is not well-defined for all random distributions. One can, however, always form the characteristic function. If the probability density function $f_X(x)$ exists, the characteristic function is simply its Fourier transform¹.

Def. 3. *The characteristic function is defined by*

$$\varphi_X(t) \equiv \langle e^{itX} \rangle = \int_{-\infty}^{\infty} e^{itx} f_X(x) \, dx. \quad (\text{C.4})$$

If the moment-generating function exists, then it can be related to the characteristic function by $\varphi_X(-it) = M_X(t)$. The Taylor expansion (around $t = 0$) of the characteristic function is

$$\varphi_X(t) = 1 + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \mu_k. \quad (\text{C.5})$$

¹Note that one can define the characteristic function even for probability distributions without probability density functions, but we will not discuss such cases here.

Def. 4. The cumulant-generating function is defined as the logarithm of the characteristic function:

$$\kappa_X(t) \equiv \log \varphi_X(t). \quad (\text{C.6})$$

Def. 5. The cumulants χ_k ($k = 1, 2, 3, \dots$) are defined through the expansion of κ_X as follows:

$$\kappa_X(t) \equiv \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \chi_k. \quad (\text{C.7})$$

To calculate the cumulants in terms of the moments, the idea is to calculate the logarithm of the expansion of the characteristic function, and to compare it with the expansion of the cumulant-generating function. First, recall the expansion of the logarithm:

$$\log(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(z-1)^k}{k}. \quad (\text{C.8})$$

Using this expansion, we find

$$\begin{aligned} \kappa_X(t) &= \log \left[\langle e^{itX} \rangle \right] \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \left[\langle e^{itX} \rangle - 1 \right]^k \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} (-1)^k \left[1 - \langle e^{itX} \rangle \right]^k \\ &= - \sum_{k=1}^{\infty} \frac{1}{k} [1 - \varphi_X(t)]^k \\ &= - \sum_{k=1}^{\infty} \frac{1}{k} \left(- \sum_{j=1}^{\infty} \frac{(it)^j}{j!} \right) \mu_k. \end{aligned} \quad (\text{C.9})$$

By computing the first set of terms one easily finds

$$\begin{aligned} \chi_1 &= \langle x \rangle, \\ \chi_2 &= \langle x^2 \rangle - \langle x \rangle^2, \\ \chi_3 &= \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3 \\ &\dots \end{aligned} \quad (\text{C.10})$$

Finally, we use the above to calculate the average $\langle e^\Omega \rangle$:

$$\begin{aligned} \langle e^\Omega \rangle &= \varphi_\Omega(-i) = e^{\log \varphi_\Omega(-i)} = e^{\kappa_\Omega(-i)} \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{1}{k!} \chi_k \right) \\ &= e^{\langle \Omega \rangle + \frac{1}{2}(\langle \Omega^2 \rangle - \langle \Omega \rangle^2) + \dots} \end{aligned} \quad (\text{C.11})$$

Q.E.D.

Appendix D

The Hubbard-Stratonovich transformation

The following step in (3.74),

$$\begin{aligned}
& -\frac{g_L}{(2\pi)^2} \int d\tau \left(e^{i\sqrt{16\pi}\phi_s} \int \mathcal{D}\phi_f \right. \\
& \quad \times \exp \left[\int_f \frac{d\omega}{2\pi} \left(i\sqrt{16\pi} e^{i\omega\tau} \phi_f(\omega) - \frac{|\omega|}{2K} |\phi_f|^2 \right) \right] + \text{h.c.} \Big) \\
& = -\frac{g_L}{(2\pi)^2} \int d\tau \left(e^{i\sqrt{16\pi}\phi_s} \exp \left[- \int_f \frac{d\omega}{2\pi} \frac{16\pi}{2} \frac{K}{|\omega|} \right] + \text{h.c.} \right), \quad (\text{D.1})
\end{aligned}$$

deserves a closer derivation. First, to clean up the expression we remove the prefactors and the integration over τ , as the crucial step is found in the functional integration. We have

$$\begin{aligned}
& \int \mathcal{D}\phi_f \exp \left[\int_f \frac{d\omega}{2\pi} \left(i\sqrt{16\pi} e^{i\omega\tau} \phi_f(\omega) - \frac{|\omega|}{2K} |\phi_f|^2 \right) \right] \\
& = \exp \left[- \int_f \frac{d\omega}{2\pi} \frac{16\pi}{2} \frac{K}{|\omega|} \right]. \quad (\text{D.2})
\end{aligned}$$

This step can be considered an inverse Hubbard-Stratonovich transformation. The general form of the H-S transformation is [11]:

$$\exp[-\rho_m V_{mn} \rho_n] = \int \mathcal{D}q \exp \left[-\frac{1}{4} q_m V_{mn}^{-1} q_n - i q_m \rho_m \right] \quad (\text{D.3})$$

for a general bilinear form $\rho_m V_{mn} \rho_n$, where V is a positive definite matrix and q is some auxiliary field (often a physical, non-microscopic quantity). Identifying

$$\rho_\omega V_{\omega\omega'} \rho_{\omega'} = \int_f \frac{d\omega}{2\pi} 16\pi \frac{K}{2|\omega|}, \quad (\text{D.4})$$

we see that there must be a delta function $\delta_{\omega\omega'}$ in V , so V will be non-singular and invertible. In particular, we can choose

$$\begin{aligned} V_{\omega\omega'} &= \frac{K}{2|\omega|} \delta_{\omega\omega'}, \\ \Rightarrow V_{\omega\omega'}^{-1} &= \frac{2|\omega|}{K} \delta_{\omega\omega'}, \\ \rho_w &= \sqrt{16\pi} e^{-i\omega\tau}, \\ q &= \phi_f. \end{aligned}$$

With these choices we have

$$\begin{aligned} \rho_\omega V_{\omega\omega'} \rho_{\omega'} &= \int_f \frac{d\omega}{2\pi} \int_f \frac{d\omega'}{2\pi} \delta_{\omega\omega'} \frac{16\pi K}{2|\omega|} e^{i\tau(\omega-\omega')} \\ &= \int_f \frac{d\omega}{2\pi} \frac{16\pi K}{2|\omega|}, \end{aligned} \quad (D.5)$$

$$\begin{aligned} -\frac{1}{4} \phi_{f,\omega} V_{\omega\omega'}^{-1} \phi_{f,\omega'} &= -\frac{1}{4} \int_f \frac{d\omega}{2\pi} \int_f \frac{d\omega'}{2\pi} \phi_f^*(\omega) \frac{2|\omega|}{K} \delta_{\omega\omega'} \phi_f(\omega') \\ &= -\int_f \frac{d\omega}{2\pi} \frac{|\omega|}{2K} \phi_f^*(\omega) \phi_f(\omega) \\ &= -\int_f \frac{d\omega}{2\pi} \frac{|\omega|}{2K} |\phi_f|^2 \end{aligned} \quad (D.6)$$

and

$$\begin{aligned} -i \phi_{f,\omega} \rho_\omega &= -i \int_f \frac{d\omega}{2\pi} \phi_f^*(\omega) \sqrt{16\pi} e^{-i\omega\tau} \\ &= +i \int_f \frac{d\omega}{2\pi} \phi_f^*(-\omega) \sqrt{16\pi} e^{i\omega\tau} \\ &= i\sqrt{16\pi} \int_f \frac{d\omega}{2\pi} e^{i\omega\tau} \phi_f(\omega). \end{aligned} \quad (D.7)$$

Inserting these equations in (D.3) we get just (D.2). Thus (3.74) holds. Q.E.D.