



CHALMERS
UNIVERSITY OF TECHNOLOGY



An Estimator for Satellite Orbits

A Statistical Model for a Ground Based Radar System

Master's thesis in Physics and Astronomy

Enna Kukic & Carl Kylin

MASTER'S THESIS

An Estimator for Satellite Orbits

A Statistical Model for a Ground Based Radar System

Enna Kukic

&

Carl Kylin



Department of Electrical Engineering
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg, Sweden 2020

An Estimator for Satellite Orbits
A Statistical Model for a Ground Based Radar System
Enna Kukic & Carl Kylin

© Enna Kukic, 2020
© Carl Kylin, 2020

Supervisor: Anders Silander, Saab Group
Examiner: Thomas Eriksson, Department of Electrical Engineering

Master's Thesis
Department of Electrical Engineering
Chalmers University of Technology
SE-412 96 Göteborg
Telephone +46 31 772 1000

Cover: An illustration of the ground radar searching for a satellite. Images from Adobe Stock ©.

Typeset in L^AT_EX
Printed by Chalmers Reproservice
Göteborg, Sweden 2020

An Estimator for Satellite Orbits
A Statistical Model for a Ground Based Radar System
Enna Kukic & Carl Kylin
Department of Electrical Engineering
Chalmers University of Technology

Abstract

A model for the signal measured by a radar station from a satellite transiting above is investigated and presented. A simplified model consisting of one satellite in a circular equatorial orbit of known direction with a radar on the equator of a non-spinning earth is investigated. The Maximum Likelihood Estimate (MLE) and Cramér-Rao Bound (CRB) for the model are found. The orbital parameters, meaning the radius and start angle of the circular satellite orbit, are found with an average error of ± 7 m and $\pm 4 \cdot 10^{-4}^\circ$ for an incoming SNR of 10 dB and ± 7 km and $\pm 0.4^\circ$ for an incoming SNR of -20 dB. Since this variance is only slightly above the CRB, it is concluded that there is room for improvement but adding complexity to the model should be of greater interest. What should be noted is that the CRB presented in the report is an approximation of the actual CRB, since the modulated signal of choice has a difficult time derivative. By choosing a more suitable modulation it might be easier to study the relationship between the estimate and the CRB.

Keywords: Radar, Satellite orbit, Signal processing, MLE, CRB, Tracking.

Acknowledgements

We would like to begin by thanking our supervisor Anders Silander at SAAB for all the helpful insights, discussions and guidance he has provided throughout the entire project. We also wish to thank our examiner Thomas Eriksson for all the valuable input and ideas that have progressed our project further.

Finally, we want to thank family and friends for all the support through the years, and SAAB for giving us this wonderful opportunity to learn and develop at a leading radar surveillance provider.

Enna Kukic & Carl Kylin, Göteborg, June 2020

Contents

List of Figures	xi
List of Tables	xiii
1 Introduction	1
1.1 Limitations	1
1.2 Ethical aspects	2
2 Theory	3
2.1 Modelling	3
2.1.1 Describing an orbit	3
2.1.2 The rotational motion of the Earth	5
2.1.2.1 Transformation matrix, GCRS to ITRS	6
2.1.3 Rotating to a specific longitude and latitude	8
2.1.4 Radar setup and signal generation	8
2.1.5 Antenna geometry	10
2.2 Signal processing	11
2.2.1 Maximum Likelihood Estimation	11
2.2.2 Cramér-Rao Bound	13
2.2.3 Whittaker–Shannon interpolation	14
3 Methods	15
3.1 Signal generation	15
3.1.1 Signal modulation	16
3.1.2 Generation of data	17
3.2 Estimation of orbital parameters	18
3.2.1 Parameter choice and integration time	18
3.2.2 MLE of orbital parameters	19
3.2.3 Variance and Cramér-Rao Bound	20
4 Results	23
4.1 Estimation of orbital parameters	23
4.2 Detection search	23
4.2.1 Near target search	25
4.3 Variance of estimator	26
5 Discussion	29
5.1 Signal modulation and numerical CRB	29
5.2 Estimation of parameters	29

5.2.1	Estimation of angle	29
5.2.2	Estimation of orbital parameters	29
5.3	Simplifications and Assumptions	30
5.4	Complexity of model	32
5.5	Antenna geometry	32
5.6	Variance and Cramér–Rao bound	32
6	Conclusion	35
	Bibliography	37
A	Transformation matrices to relate GCRS to ITRS	I
A.1	Transformation matrix, GCRS to ITRS	I

List of Figures

2.1	A satellite orbiting the Earth will follow an elliptical path with the Earth in one of the foci. An ellipse can be characterised by its semi-major axis a and semi-minor axis b . The closest passage of the satellite is the periapsis at $\mathbf{r}_p$. The satellite location is defined by $\mathbf{r}_S$	4
2.2	The orbit of a satellite will most probably not occur in the geocentric x_2y_2 -plane, visualised by the grey plane. Therefore, the yellow orbital plane must be described. This can be done by three rotations Ω , i and ω such that the orbit occurs counter clockwise around the z_1 -axis with the periapsis occurring on the x_1 -axis.	5
2.3	The rotational motion of the Earth. There is a spin around the rotational axis tilted $\approx 23.5^\circ$ from the ecliptical pole. The rotational axis also undergoes precession around the ecliptic pole with a periodic perturbation called nutation.	6
2.4	The Geocentric Celestial Reference System GCRS with a pole at C_0 and origin at Σ_0 , and the Celestial Intermediate Reference System CIRS with a Celestial Intermediate Pole CIP at P and Celestial Intermediate Origin CIO at σ . Point P expressed in GCRS coordinates is: $x' = \sin d \cos E$, $y' = \sin d \sin E$ and $z' = \cos d$, where angle d represent the tilt of the rotation axis.	7
2.5	To the left (a) a cross sectional view of the satellite passing above the radar station. From the point of view of the radar station the satellite passes by at an angle α and distance r . The coordinate system used in this figure is the one derived in earlier sections, where the satellite orbit has been calculated from the center of the Earth. To the right (b) the view from above of the satellite passing by. Depending on the orbital parameters, the satellite need not pass straight over the radar station. Therefore another angle γ is needed to describe the transit. Finally, in relating (a) and (b) $\tilde{\mathbf{y}} = \cos \gamma \mathbf{z} + \sin \gamma \mathbf{y}$	9
2.6	An incoming plane wave reaching a row of antennas with spacing d . The incoming wave fronts will reach two adjacent antennas with a time difference $\Delta t = \frac{d}{c} \sin \alpha$, where c is the speed of light.	11
3.1	The autocorrelation peak for a AWPN-modulated signal and a sinus-modulated signal. The parameters for these modulations were chosen according to Table 3.1. What can be seen is that the AWPN-modulated signal has a sharper peak and has lower side lobes for the wrong delay than the sinus-modulated signal.	16

4.1	Correlation plot for five sequences with a sequence time of 1 ms each and a total time of 10 s. This search was done for the entire possible search area, and is referred to as a detection search. The correlation measurement, MLE, was performed for four different lengths of Inline-geometry antennas; 1, 10, 100 and 1000. When increasing the number of antennas the intensity can be seen to gather in α_0 -direction. The target value is marked with a green circle.	24
4.2	Zoomed in correlation plot for five sequences during a sequence time of 1 ms each and a total time of 10 s. This search was done for an area of $r_S \pm 15$ km and $\alpha_0 \pm 0.7^\circ$, and is referred to as a near target search. The correlation measurement, MLE, was performed for three different lengths of Inline-geometry antennas; 1, 100 and 1000. When increasing the number of antennas the intensity can be seen to gather in α_0 -direction. The target value is marked with a green circle, while the estimated value is marked as the magenta star asterisk.	25
4.3	Variance of the estimates for r_S and α_0 as a function of the SNR of the incoming signal. The variance is plotted together with the CRB for both the parameters given one antenna, represented by a blue and red line respectively. The dotted black line represent the error for the resolution limit, meaning the square step length for the grid search. This can be seen to be much lower then both the CRB and variance. Note that the scales are logarithmic.	26

List of Tables

3.1	Choice of parameters for the estimator, defined and briefly explained. . . .	18
4.1	Values for the root mean square error of the estimate, square root of the CRB and the step length for the grid search are presented for the highest and lowest value of SNR_{in}	27

1

Introduction

Today, over 2000 active satellites orbit the earth according to the UCS Satellite Database [1]. With an increasing militarisation of space the need to monitor these satellites arises. For a defence industry company like SAAB an exploration of this area is of interest in order to present viable concepts. In order to monitor the satellites one can observe the transit using a ground based radar system and estimate their orbital parameters.

Monitoring the orbit of any object presents new challenges but also significant simplifications compared to monitoring for example aircraft. Whereas aircraft can control their trajectories, objects in orbit follow well defined paths given by their orbital parameters. A radar monitoring aircraft typically estimates the distance and angle to the aircraft as well as it's velocity. For a satellite these parameters change very rapidly during the transit and are therefore difficult to estimate. On the other hand, the orbital parameters of a satellite are constant and typically do not change. This project can be seen as an early pilot study before building a facility which could actually accomplish this. The end goal of this type of system is to track if a satellite changes it's orbit in order to better monitor some geographical area.

The aim of the project is to create an estimator with which one can identify the orbital parameters of satellites using a radar system of a suitable geometry. The satellites of interest are located in the lower orbital region called Low Earth Orbit, LEO, with an altitude of 500 to 2000 km above the surface of the earth. The radar system will operate in L-band.

1.1 Limitations

In the model describing the satellite's orbit, the report will only consider satellite motion in a known direction around the equator. The satellites orbit is modeled as circular without any perturbations. It will orbit a non-rotating Earth with the radar station positioned on the equator at zero longitude. The theory for a more general orbit is presented, it was simply not executed.

There are three main perturbations of the satellite orbit: the Earth's deviation from spherical symmetry, air drag, and perturbations due to the Sun and the Moon. The most significant perturbation for a LEO satellite is the spherical deviation caused by the gravitational field, but no such effect was taken into account.

Furthermore, refraction and reflection of the signal in the atmosphere will be omitted from the scope of the project. Also, reflection, refraction, dispersion and scintillation due to the Ionosphere will be omitted.

1.2 Ethical aspects

When considering the project many ethical and societal questions can arise. The aim of the project was to create an estimator with which objects in orbit, mainly satellites, can be monitored. There was no real product developed, solemnly what one can call a pilot study was performed. The utilization of future products, benefiting from this report, is tracing satellite motion to identify suspicious behavior. Although future products would definitely be in the military realm, they are more of a surveillance nature. One could develop the product further and use it as a target finder to eliminate satellites, but this is not within the scope of the project and would need several changes and additions. Furthermore, this type of ground based radar system for tracking satellites already exists, but the technique is yet to be developed in Sweden. One example is the United States Space Surveillance Network.

When it comes to environmental impacts only a few have to be taken into consideration. The execution of the product will not impact the environment itself, but the emitted EM-waves from a radar system such as this might. Since the ground based radar system would emit high intensity radiation, more precisely microwave radiation, the area close by would have to be secluded. Another possible effect on the environment could be that it would cause issues for birds for example. Because of the signals emitted an electric field will be induced causing a magnetic field as well as radiation. This could possibly impact birds' and other flying creatures sense of direction as well as their health.

2

Theory

In this chapter, the theory behind the physical model and performed signal processing will be presented. To start with, the orbit and the rotational motion of the Earth will be introduced and subsequently described from the perspective of a ground based radar system. After this, the signal reaching the radar will be described in terms of the orbital parameters. Finally, the maximum likelihood estimate and Cramér-Rao Bound of the orbital parameters is found.

2.1 Modelling

Before going into the estimation-part of the problem one needs to fully describe the physical model of the system both regarding the orbit and the radar setup. In this section the theory behind the model of choice will be described in detail and the generation of signal data will be presented.

First off, the satellite will follow an orbit around a rotating Earth, and since the satellite has it's own plane of reference the system must be explained in the eyes of the radar. Moreover, the radar setup and the generated signal must be defined and modelled.

2.1.1 Describing an orbit

A single satellite orbiting the Earth is a classic example of a two-body problem. The solutions to these problems are widely known and are presented in for example the lecture notes written by Klioner [2]. In summary, the two bodies will orbit their common center of mass in the focal point of an ellipse. Because the Earth is several orders of magnitudes heavier than a satellite the Earth is approximately stationary in that focal point. To begin describing this orbit, consider an ellipse with semi-major axis a and semi-minor axis b as in Figure 2.1. The Earth is then located in one of the foci. In the context of orbits an important quantity is the closest passage, the periapsis $\mathbf{r}_p$. The convention when parameterising an orbital ellipse is to use the semi-major axis a and the eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$. Using this convention the coordinates of the satellite can be parameterised as

$$\mathbf{r}_{S,1} = \frac{a(1 - e^2)}{1 + e \cos \nu} (\cos \nu \hat{\mathbf{x}}_1 + \sin \nu \hat{\mathbf{y}}_1), \quad (2.1)$$

where ν is the angle between the location of the satellite $\mathbf{r}_S$ and the periapsis $\mathbf{r}_p$. This angular parameter is known as the *true anomaly* and is used to locate an object moving along a Keplerian orbit [2].

When measuring any type of signal, the incoming signal will be measured as a function of time. It is therefore important to establish how the true anomaly evolves in time. This

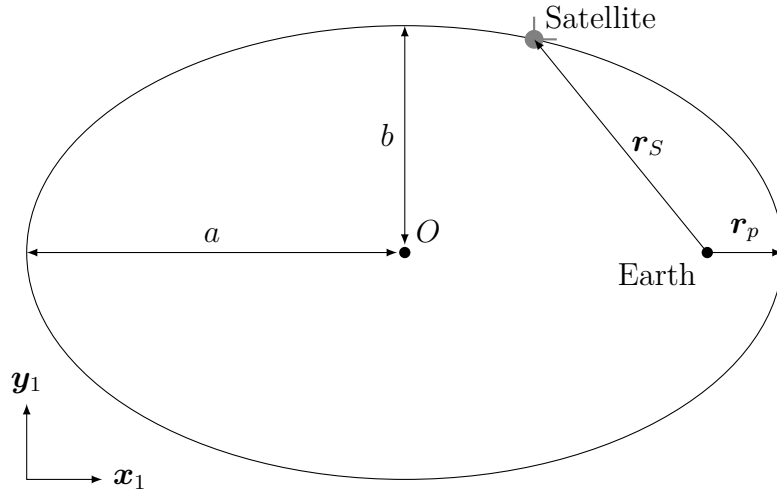


Figure 2.1: A satellite orbiting the Earth will follow an elliptical path with the Earth in one of the foci. An ellipse can be characterised by its semi-major axis a and semi-minor axis b . The closest passage of the satellite is the periapsis at r_p . The satellite location is defined by r_s .

can be done by solving the Kepler Equation [2]

$$k[a(1 - e^2)]^{-3/2}t = \int \frac{d\nu}{(1 + e \cos \nu)^2} + \text{const.}, \quad (2.2)$$

where $k = \sqrt{GM_E}$, G is the gravitational constant and M_E the mass of the Earth. This integral can not be solved analytically. To simplify the calculations a new anomaly must be introduced, the eccentric anomaly E . The true anomaly ν can be expressed in terms of the eccentric anomaly E and the eccentricity e such that [2]

$$\begin{aligned} \cos \nu &= \frac{\cos E - e}{1 - e \cos E}, \\ \sin \nu &= \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E}. \end{aligned}$$

By introducing the eccentric anomaly the Kepler Equation can be simplified to

$$E - e \sin E = M,$$

where $M = n(t - T)$ is the mean anomaly, $n = ka^{-3/2}$ is the mean velocity and T is an integration constant representing $M = 0$ [2]. By solving Equation 2.2 numerically the time dependence of the elliptical orbit can be fully described.

However, not all satellites orbit the earth in the same plane. It is thus important to describe the orientation of the orbit. This can be done in infinitely many ways, the traditional being three Euler-like angles illustrated in Figure 2.2 [2]. The orbital plane of the satellite is shown in yellow, the orientation being such that the satellite goes around the z_1 -axis with the periapsis occurring on the positive part of the x_1 -axis. To reach this system from some reference coordinate system $x_2y_2z_2$, a geocentric system, the system is first rotated Ω around the z_2 -axis so that the orbit crosses the x_2y_2 -plane at Ω and $\Omega + \pi$. After this the system is rotated an angle i about the instantaneous x' -axis in such a way that the orbit goes in a counter-clockwise motion around the z_1 -axis. The only thing

remaining is to orient the x_1 -axis such that the periapsis occurs on the positive part of the x_1 -axis. This is done by a final rotation ω around the z_1 -axis. Having decided how to describe the orbital plane, the motion in that plane can be translated to a reference system through three rotations as shown by Klioner [2]

$$\mathbf{r}_{S,2} = R_z(-\Omega)R_x(-i)R_z(-\omega)\mathbf{r}_{S,1} \quad (2.3)$$

where $R_j(\theta)$ is a rotational matrix around axis j by angle θ . The three angles Ω , i , and ω are called *longitude of the ascending node*, *inclination*, and *argument of periapsis* [2].

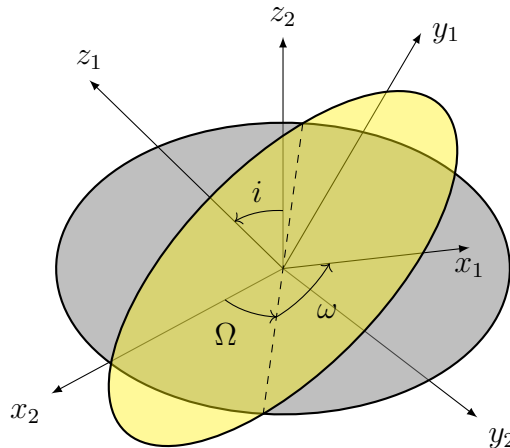


Figure 2.2: The orbit of a satellite will most probably not occur in the geocentric x_2y_2 -plane, visualised by the grey plane. Therefore, the yellow orbital plane must be described. This can be done by three rotations Ω , i and ω such that the orbit occurs counter clockwise around the z_1 -axis with the periapsis occurring on the x_1 -axis.

In summary, an orbiting object can be fully described by six parameters. These can be chosen in many different ways, the convention being the semi-major axis a , the eccentricity e , the true anomaly ν , the longitude of the ascending node Ω , the inclination i , and the argument of periapsis ω .

2.1.2 The rotational motion of the Earth

Now that the satellite orbit has been fully described and modelled one needs to address the fact that the Earth is also rotating, making the Earth's plane of reference an accelerated plane of reference.

The Earth rotates both with respect to the Sun and with respect to its own rotational axis tilted at 23.43928° [3]. During this study only the latter will be of interest, and will henceforth be referred to as the Earth's rotational motion. The Earth's rotational motion consists mainly of three components: precession and nutation, spin and polar motion. The first two components are illustrated in Figure 2.3.

The motion of precession is equivalent to the motion of a gyroscope. Precession is the rotation of the rotational axis with respect to the ecliptical pole and nutation is the periodic perturbation of the precession. The nutation mainly emerges due to solar and lunar effects [3]. The spin of the Earth is simply the spin of the rotational axis, while the polar motion is the movement of the rotational axis relative its crust.

All the motions described above imply that an observational system situated on Earth is in an accelerated frame of reference. When observing celestial or astronomical objects

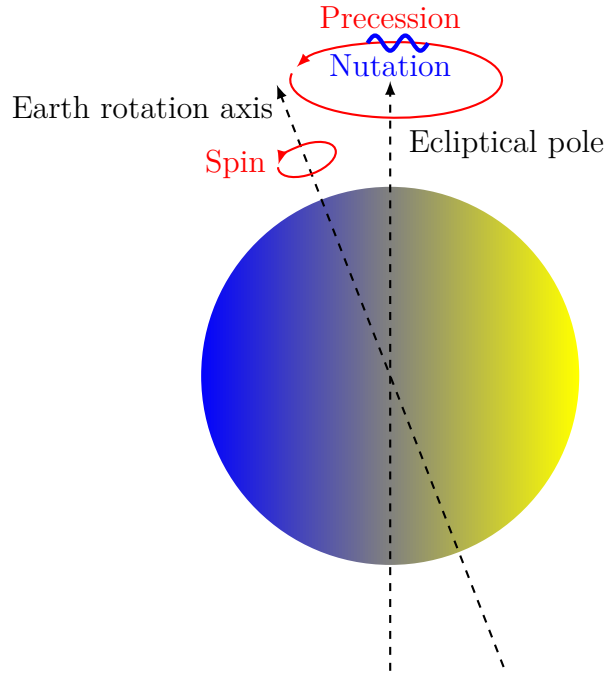


Figure 2.3: The rotational motion of the Earth. There is a spin around the rotational axis tilted $\approx 23.5^\circ$ from the ecliptical pole. The rotational axis also undergoes precession around the ecliptic pole with a periodic perturbation called nutation.

this can cause incorrect measurements of velocity, direction and position. Therefore, a reference system which rotates with the rotation of the Earth is of interest and such a reference system is the International Terrestrial Reference System, ITRS [4]. ITRS is a geocentric system which takes spin, nutation, precession and polar motion into account. The celestial object, if rotating around the Earth, moves in a different reference system called the Geocentric Celestial Reference System, GCRS [5]. The GCRS is still a geocentric reference system, but its coordinate-axes do not rotate with respect to those of the Barycentric Celestial Reference System, BCRS, a non rotating system centered at the barycenter of the Solar System [5]. Thus, the GCRS must be transformed to the ITRS and that can be done according to the technical notes published by Gérard and Luzum [6]. The transformation matrices to relate the GCRS to the ITRS can be written as

$$[\text{ITRS}] = (R_{\text{prec+nut}} R_{\text{spin}} R_{\text{polar}})^T [\text{GCRS}],$$

where $R_{\text{prec+nut}}$, R_{spin} and R_{polar} are the transformation matrices connected to the precession and nutation, spin and polar motion respectively [6].

2.1.2.1 Transformation matrix, GCRS to ITRS

The first rotation goes from GCRS to the Celestial Intermediate Reference System, CIRS, which is a geocentric coordinate system based upon current position and date taking precession and nutation into account [5]. The two reference systems are visualised in Figure 2.4. However, since the period for precession is around 26 000 years and the operating time of a generic satellite is in the magnitude of ten years this rotation would only rotate the axis in the order of $10^{-1^\circ} - 10^{-2^\circ}$ during the satellites life span making the effect insignificant [7].

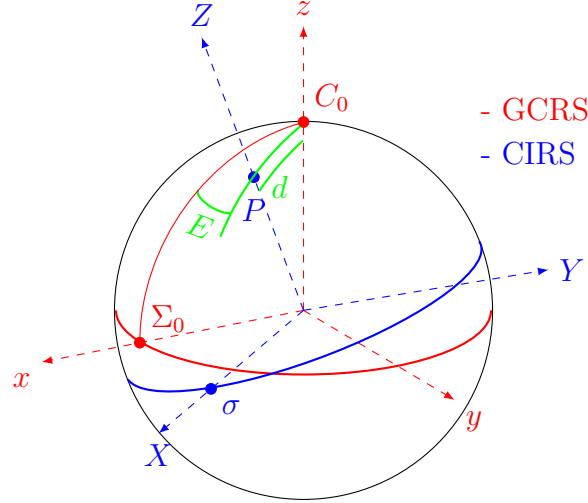


Figure 2.4: The Geocentric Celestial Reference System GCRS with a pole at C_0 and origin at Σ_0 , and the Celestial Intermediate Reference System CIRS with a Celestial Intermediate Pole CIP at P and Celestial Intermediate Origin CIO at σ . Point P expressed in GCRS coordinates is: $x' = \sin d \cos E$, $y' = \sin d \sin E$ and $z' = \cos d$, where angle d represent the tilt of the rotation axis.

The second transformation will bring CIRS to TIRS, the Terrestrial Intermediate Reference System, taking spin of the rotational axis into account. This effect has a period of one day, making it significant for an observational system. The corresponding rotational matrix is

$$R_{\text{spin}} = R_z(-ERA),$$

where ERA is the Earth Rotation Angle and spans between the CIO, the Celestial Intermediate Origin, and the TIO, the Terrestrial Intermediate Origin, along the CIRS equator [6]. The ERA can be calculated using the following equation presented by Gérard and Luzum [6]

$$ERA = 2\pi(\text{UT1 Julian day fraction} + 0.7790572732640 + 0.00273781191135448T_u),$$

where T_u is the difference between the Julian date of UT1 and the Julian date 2451545, corresponding to the day starting at 12:00 UTC on January 1, 2000. UT1 is a form of Universal Time taking polar motion into account, which differs at most with 0.9s to the Coordinated Universal Time known as UTC [8]. The transformation matrix is written in accordance to the IAU 2006 resolutions, and the transformation follows the CIO based procedure [6]. A procedure that realizes the CIRS using the Celestial Intermediate pole, CIP, as its z-axis and the CIO as its x-axis, which can be seen in Figure 2.4.

The final rotation takes the system from TIRS to ITRS by also accounting for polar motion. Values for the polar coordinates x_p and y_p are published daily by the International Earth Rotation and Reference Systems Service, IERS, and are of magnitude $10^{-2} - 10^{-1}$ arc seconds [8]. Assuming that the satellite operates for ten years and the polar motion constantly moves the CIRS pole away from the GCRS pole (see Figure 2.4), the motion would only rotate the coordinate system in the order of 10^{-4}° during the satellites life span, making this effect insignificant as well.

The entire transformation matrix now consists of

$$[\text{ITRS}] \approx R_z(-ERA)^T[\text{GCRS}] \iff \mathbf{r}_{S,3} = R_z(ERA)\mathbf{r}_{S,2} \quad (2.4)$$

To read more about the exact transformation matrices please refer to Appendix A.

2.1.3 Rotating to a specific longitude and latitude

Having reached a coordinate system centered in the core of the Earth with the x -axis pointing along longitude $\varphi = 0$ and geodetic latitude $\lambda = 0$, what remains is to rotate to a general system with the x -axis pointing in the direction of some other longitude and latitude, where the radar is located. The desired transformation to coordinate (φ, λ) can be described by

$$\mathbf{r}_{S,4} = R_y(\Lambda)R_z(\varphi)\mathbf{r}_{S,3}, \quad (2.5)$$

where Λ is the geocentric latitude. With geodetic latitude one means the angle between the normal and the equatorial plane, while geocentric latitude is the angle between a radial line and the equatorial plane [9]. Using the definitions of geodetic and geocentric latitude and an ellipsoid model of the Earth the two can be related as

$$\tan \Lambda = \frac{b_E^2}{a_E^2} \tan \lambda,$$

where a_E and b_E are the semi-major and minor axis of the ellipsoid. When taking into account the elliptic shape of the Earth, what also varies is the radius of the Earth. This can be shown to be

$$R_E^2 = \frac{a_E^4 \cos^2 \lambda + b_E^4 \sin^2 \lambda}{a_E^2 \cos^2 \lambda + b_E^2 \sin^2 \lambda}.$$

Appropriate values for a_E and b_E can be taken from the WGS84 model defined in [10], from which

$$a_E = 6\,378\,137 \text{ km}$$

$$b_E \approx 6\,356\,752.3142 \text{ km}.$$

In words, the final coordinate system is one where the x -axis points towards zenith, y -axis points east, and z -axis points north, centered in the core of the earth.

2.1.4 Radar setup and signal generation

From the previous sections a description of the satellite orbit has been established. The final step in specifying the problem is thus to describe what signal the radar station will measure. Consider a signal in time described by

$$S(t) = m(t) e^{i2\pi f_H t}, \quad (2.6)$$

where f_H is the carrier frequency and $m(t)$ is the modulated signal. Modulating the signal, either the amplitude or the phase of the waveform will make it possible to correctly determine the distance to an object. The carrier frequency f_H is the center frequency around which the radar operates, and is much higher than any frequency of the modulating signal $m(t)$ [11]. Using a L-band radar the carrier frequency would be ~ 1 GHz, whereas the modulating signal might have a bandwidth in the neighbourhood of 10-100 kHz due to the property of the carrier frequency explained above.

The signal presented above is emitted from the radar station, travels to the satellite, bounces off the satellite, and then travels back to the radar station. The signal is thus delayed by a factor

$$\tau(t) = \frac{2r(t)}{c} \quad (2.7)$$

where r is the distance to the satellite and c is the speed of light. This setup is illustrated in Figure 2.5. In Figure 2.5a the cross section of a satellite transit is shown. From the view of the radar station the satellite passes by at an angle α and distance r , which can be found from R_S and θ using trigonometric formulae. For example

$$r(t) = |\mathbf{r}_S - \mathbf{r}_E| = \sqrt{r_S^2 + r_E^2 - 2r_S r_E \cos \theta(t)}. \quad (2.8)$$

In Figure 2.5b the same transit is shown, but from above instead. Here the angle γ gives the final specification for the satellite position. The description of the orbit with the parameters r_S , θ , and γ is similar to that of a standard spherical coordinate system. Finally, to relate the two figures, $\tilde{\mathbf{y}} = \cos \gamma \mathbf{z} + \sin \gamma \mathbf{y}$.

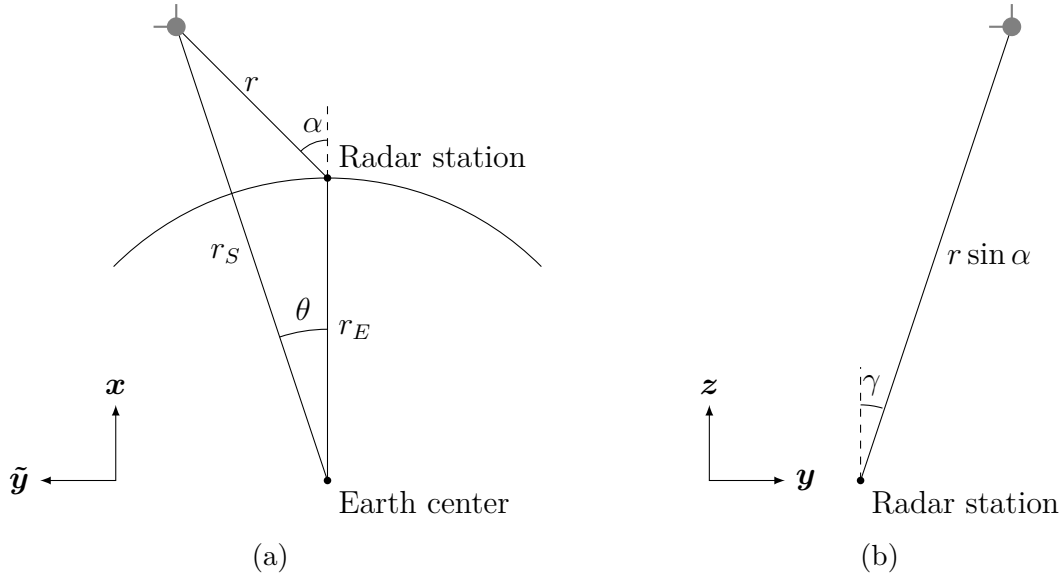


Figure 2.5: To the left (a) a cross sectional view of the satellite passing above the radar station. From the point of view of the radar station the satellite passes by at an angle α and distance r . The coordinate system used in this figure is the one derived in earlier sections, where the satellite orbit has been calculated from the center of the Earth. To the right (b) the view from above of the satellite passing by. Depending on the orbital parameters, the satellite need not pass straight over the radar station. Therefore another angle γ is needed to describe the transit. Finally, in relating (a) and (b) $\tilde{\mathbf{y}} = \cos \gamma \mathbf{z} + \sin \gamma \mathbf{y}$.

As discussed earlier, the signal that returns to the radar station is $s(t - \tau)$. What will be measured is

$$s(t; \tau(t)) = m(t - \tau(t))e^{-i2\pi f_H \tau(t)}. \quad (2.9)$$

To understand the different factors in the measured signal, they will be explained in detail. As stated earlier the first factor $m(t - \tau(t))$ comes from the modulation and the second factor $e^{-i2\pi f_H \tau(t)}$ comes from the carrier frequency, but as one can notice the pure t -dependence is gone in the second factor. When measuring a high frequency signal, that signal can be down-converted into a more easily measurable signal range. A down conversion can be explained by mixing the signal with a local oscillator [12]. Imagine a high frequency signal $\sin(\omega_H t)$ which should be measured. This signal is mixed with a sinusoidal signal generated from a local oscillator producing a signal $\sin(\omega_H t) \sin(\omega_{LO} t)$. Using standard trigonometric formulae it is easy to show that $\sin(\omega_H t) \sin(\omega_{LO} t) =$

$\sin((\omega_H + \omega_{LO})t) + \sin((\omega_H - \omega_{LO})t)$. By filtering out the high frequency the output is only the difference in frequency. In this study the high frequency signal is modeled as being mixed with a local oscillator with the same frequency, down-converting the signal to base band and making f_H the center frequency.

In this project a pure phase modulation was used, meaning that

$$s(t; \tau(t)) = e^{i\phi(t-\tau(t))} e^{-i2\pi f_H \tau(t)}. \quad (2.10)$$

There are still two effects to take into consideration. Firstly, the intensity of the reflected signal will decrease with an increasing distance proportional to $I \propto 1/\tau(t)^4$. Since the signal is modelled as having only a phase modulation the modulation will not impact the intensity. Still the intensity of the incoming signal will be very small and will depend on the distance to the object. It will therefore not be constant in time. This effect will be neglected though, because for short integration times the amplitude will approximately be constant. Secondly, the reflection of the satellite has been assumed to act like a point source. This means that the satellite is assumed to reflect the signal of one single point, and radiate that signal like a single point would. This is of course not the case; an actual target will have a complex geometry which will generate an array of reflections from different points in space. This is difficult to model without previous knowledge of the geometry. Imagine that the signal reaches two points on the satellite which are not the same distance from the radar. The reflection travelling towards the radar station will consist of a sum $s(t; \tau) + s(t; \tau + d\tau)$ of these two reflections. If the two points are relatively close to one another the reflection is approximately $s(t; \tau) + s(t; \tau + d\tau) \approx s(t; \tau)(1 + e^{-2\pi f_H d\tau})$. A simple model for an object which is not too big is thus to add an unknown phase β to the signal. The intensity will of course also be affected by the number of points generating reflections. A simplified case where the amplitude is known and set to one is considered. Since the amplitude is approximately constant for short integration times this will mostly affect the signal to noise ratio and is thus an acceptable simplification. Finally the simplified signal model can be written as

$$s(t; \tau) = e^{i\beta} e^{i\phi(t-\tau(t))} e^{-i2\pi f_H \tau(t)}. \quad (2.11)$$

2.1.5 Antenna geometry

Observing a signal from only one antenna will generally give a good estimate of the delay of a signal. If one wishes to obtain more information from the incoming signal the next step is typically to add more antennas. This will add to the amount of data available for analysis, but also encode new information. Consider the situation shown in Figure 2.6; N antennas in a row with spacing d and an incoming plane wave from some angle α measured from zenith. The wave fronts of the incoming signal will reach the antennas with a time difference $\Delta t = \frac{d}{c} \sin \alpha$, where c is the speed of light. In the case of a signal with a narrow bandwidth compared to its carrier frequency f_H , this time difference can be approximated as a difference in phase

$$\Delta\phi = 2\pi f_H \frac{d}{c} \sin \alpha.$$

This due to the fact that the emitted frequencies can be approximated to one frequency, the carrier frequency. The situation described here is almost that of the radar station, with one significant difference. The wave fronts that comes back from the satellite are not

planar, but spherical. However, the wave front is approximately planar in the far field, which is a valid approximation for a distance much greater than the Fraunhofer distance r_f [13]

$$r \gg r_f = \frac{2(Nd)^2}{\lambda}. \quad (2.12)$$

The spacing d can thus be chosen freely, provided it is not unreasonably small or so big that the plane wave approximation does not hold. A common choice is half wavelength spacing $d = \frac{\lambda}{2} = \frac{c}{2f_H}$, which gives a phase difference

$$\Delta\phi = \pi \sin \alpha. \quad (2.13)$$

This is a good choice since all angles $\alpha = [-\pi/2, \pi/2]$ give a unique phase shift. For a carrier frequency $f_H = 1$ GHz and a distance to the satellite of $r = 500$ km, half wavelength spacing would mean $d = 15$ cm and impose a limit on the number of antennas from Equation 2.12, $N \ll \sqrt{\frac{2r}{\lambda}} \approx 1800$ antennas. Neither of these are unreasonable.

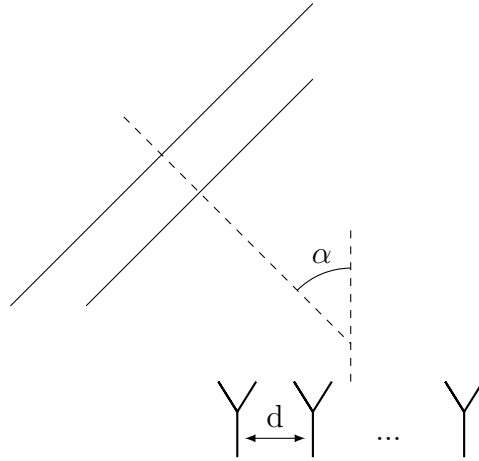


Figure 2.6: An incoming plane wave reaching a row of antennas with spacing d . The incoming wave fronts will reach two adjacent antennas with a time difference $\Delta t = \frac{d}{c} \sin \alpha$, where c is the speed of light.

2.2 Signal processing

After describing the model the next step is to review the incoming signal using signal processing. Below are the methods and estimations that are used in this study.

2.2.1 Maximum Likelihood Estimation

Having described the model in Equation 2.10, the next step is to estimate the orbital parameters from an incoming signal. One way of doing this is to use a maximum likelihood estimate. In an actual system there will be noise generated from everything from non-ideal electronic components to the cosmic microwave radiation. In this study only internally generated thermal noise will be considered, meaning that the noise can be generated as uncorrelated normal noise with mean zero. This gives a signal defined by

$$\mathbf{x} = \mathbf{s}(\boldsymbol{\theta}) + \mathbf{n} \quad (2.14)$$

where $\mathbf{n} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I})$ is a complex normal random vector with mean zero and standard deviation σ_n , $\mathbf{x} = [x(t_1), \dots, x(t_N)]^T$ is the signal that is measured by the radar station, $\mathbf{s}(\boldsymbol{\theta}) = [s(t_1; \tau(t_1; \boldsymbol{\theta})), \dots, s(t_N; \tau(t_N; \boldsymbol{\theta}))]^T$ is the signal model given by the parameters $\boldsymbol{\theta}$, and N is the length of the signal. In this study $\boldsymbol{\theta}$ are the orbital parameters. The maximum likelihood estimate of $\boldsymbol{\theta}$ is given by the max of the likelihood function $p(\mathbf{x}|\boldsymbol{\theta})$

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \arg \max_{\boldsymbol{\theta}} \{p(\mathbf{x}|\boldsymbol{\theta})\},$$

where the likelihood is given by [14]

$$p(\boldsymbol{\theta}|\mathbf{x}) = |2\pi\sigma_n^2 \mathbf{I}|^{-1/2} \exp \left\{ -\frac{1}{2} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})]^H \frac{1}{\sigma_n^2} \mathbf{I}^{-1} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})] \right\},$$

where $\mathbf{I}$ is an N by N identity matrix and H denotes the hermitian conjugate. This maximum is of course the same as the maximum of the logarithm of the likelihood, or log-likelihood, which is given by [14]

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = -\frac{N}{2} \ln 2\pi - \frac{1}{2} \ln |\sigma_n^2 \mathbf{I}| - \frac{1}{2} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})]^H \frac{1}{\sigma_n^2} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})]. \quad (2.15)$$

The two first terms are independent of $\boldsymbol{\theta}$, hence

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{\text{ML}} &= \arg \min_{\boldsymbol{\theta}} \{ [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})]^H \frac{1}{\sigma_n^2} \mathbf{I}^{-1} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})] \} \\ &= \arg \min_{\boldsymbol{\theta}} \{ [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})]^H [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})] \} \\ &= \arg \min_{\boldsymbol{\theta}} \{ |\mathbf{x}|^2 + |\mathbf{s}(\boldsymbol{\theta})|^2 - 2\text{Re}\{\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta})\} \} \\ &= \arg \max_{\boldsymbol{\theta}} \{ \text{Re}\{2\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}) - |\mathbf{s}(\boldsymbol{\theta})|^2\} \}. \end{aligned}$$

The model defined in Equation 2.10 is a complex exponential, with only varying phase. Hence $|s(t; \tau(t))| = 1$ for all t and $\tau(t)$. Thus the estimate can be simplified further, giving us

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \arg \max_{\boldsymbol{\theta}} \{ \text{Re}\{\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta})\} \}. \quad (2.16)$$

If the signal is modeled as having an unknown phase β as in Equation 2.11, this phase can also be estimated. In this case the MLE can be written as

$$\hat{\boldsymbol{\theta}}_{\text{orb,ML}} = \arg \max_{\boldsymbol{\theta}_{\text{orb}}, \beta} \{ \text{Re}\{e^{i\beta} \mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}})\} \},$$

where $\mathbf{s}(\boldsymbol{\theta}_{\text{orb}})$ is the model without the unknown phase added to it and $\boldsymbol{\theta}_{\text{orb}}$ are the orbital parameters. This is maximised when $e^{i\beta} \mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}})$ is completely real, which is true when

$$e^{i\beta} = \frac{(\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}}))^H}{|\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}})|}.$$

Applying this to the maximum likelihood estimate means that

$$\hat{\boldsymbol{\theta}}_{\text{orb,ML}} = \arg \max_{\boldsymbol{\theta}_{\text{orb}}} \left\{ \text{Re} \left\{ \frac{(\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}}))^H}{|\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}})|} \mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}}) \right\} \right\} = \arg \max_{\boldsymbol{\theta}_{\text{orb}}} |\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}_{\text{orb}})|, \quad (2.17)$$

in the case of an unknown phase.

2.2.2 Cramér-Rao Bound

Provided an estimator satisfying certain conditions, a lower bound on the variance of the estimate can be found using the so called Cramér-Rao Bound, CRB. This is useful to visualize how good an estimate is compared to how good it could be. The CRB for a multi-variate estimator is [14]

$$C_{\text{CRB}}(\tilde{\boldsymbol{\theta}}) = J^{-1}(\tilde{\boldsymbol{\theta}}), \quad (2.18)$$

where $\tilde{\boldsymbol{\theta}}$ are the correct parameters and J is the Fisher information matrix, FIM, with elements

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = -\mathbb{E} \left[\frac{\partial^2 \mathcal{L}}{\partial \theta_i \partial \theta_j}(\tilde{\boldsymbol{\theta}}|\mathbf{x}) \right] = \mathbb{E} \left[\frac{\partial \mathcal{L}}{\partial \theta_i}(\tilde{\boldsymbol{\theta}}|\mathbf{x}) \frac{\partial \mathcal{L}}{\partial \theta_j}(\tilde{\boldsymbol{\theta}}|\mathbf{x}) \right] \quad (2.19)$$

and the lower bound is given by

$$\text{var}[\hat{\theta}_i] \geq [C_{\text{CRB}}]_{ii}.$$

It can be noted that the FIM is symmetric, $[J(\tilde{\boldsymbol{\theta}})]_{ij} = [J(\tilde{\boldsymbol{\theta}})]_{ji}$, hence only the upper or lower half and the diagonal need be computed. Equation 2.15 gives the log-likelihood

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = -\frac{N}{2} \ln 2\pi - \frac{1}{2} \ln |\sigma_n^2 I| - \frac{1}{2} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})]^H \frac{1}{\sigma_n^2} [\mathbf{x} - \mathbf{s}(\boldsymbol{\theta})].$$

Since not all the terms in the above expression depend on the parameters $\boldsymbol{\theta}$, the FIM is

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = -\mathbb{E} \left[\frac{\partial^2}{\partial \theta_i \partial \theta_j} \frac{-1}{2\sigma_n^2} \left(\mathbf{x}^H \mathbf{x} + \mathbf{s}^H(\boldsymbol{\theta}) \mathbf{s}(\boldsymbol{\theta}) - \mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}) - \mathbf{s}^H(\boldsymbol{\theta}) \mathbf{x} \right) \bigg|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}} \right].$$

Furthermore, $\mathbf{s}^H(\boldsymbol{\theta}) \mathbf{s}(\boldsymbol{\theta}) = \sum_{n=1}^N s^H(t_n; \boldsymbol{\theta}) s(t_n; \boldsymbol{\theta}) = \sum_{n=1}^N 1 = N$ is constant in all parameters. Using this the FIM simplifies further to

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = -\frac{1}{2\sigma_n^2} \mathbb{E} \left[\frac{\partial^2}{\partial \theta_i \partial \theta_j} \left(\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta}) + \mathbf{s}^H(\boldsymbol{\theta}) \mathbf{x} \right) \bigg|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}} \right].$$

The expectation value $\mathbb{E}[\]$ is taken for the random vector $\mathbf{x}$ from Equation 2.14, for which $\mathbb{E}[\mathbf{x}] = \mathbf{s}(\tilde{\boldsymbol{\theta}})$. This is independent from the parameters θ_i since $\mathbf{x}$ does not depend on $\boldsymbol{\theta}$, hence the order of the expectation value and the derivatives can be interchanged resulting in

$$\begin{aligned} [J(\tilde{\boldsymbol{\theta}})]_{ij} &= -\frac{1}{2\sigma_n^2} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \left(\mathbf{s}^H(\tilde{\boldsymbol{\theta}}) \mathbf{s}(\boldsymbol{\theta}) + \mathbf{s}^H(\boldsymbol{\theta}) \mathbf{s}(\tilde{\boldsymbol{\theta}}) \right) \bigg|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}} \\ &= -\frac{1}{\sigma_n^2} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \text{Re} \left\{ \mathbf{s}^H(\tilde{\boldsymbol{\theta}}) \mathbf{s}(\boldsymbol{\theta}) \right\} \bigg|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}}. \end{aligned} \quad (2.20)$$

The parameters that need to be estimated are the orbital parameters and the unknown phase β . Let $\boldsymbol{\theta} = [\boldsymbol{\theta}_{\text{orb}}, \beta]$, where $\boldsymbol{\theta}_{\text{orb}}$ are only the orbital parameters. Now, $\mathbf{s}(\boldsymbol{\theta}) = \mathbf{s}(\boldsymbol{\theta}_{\text{orb}}) e^{i\beta}$, meaning that $\frac{\partial \mathbf{s}}{\partial \beta}(\boldsymbol{\theta}) = i\mathbf{s}(\boldsymbol{\theta})$ and $\frac{\partial^2 \mathbf{s}}{\partial \beta^2}(\boldsymbol{\theta}) = -\mathbf{s}(\boldsymbol{\theta})$. Consider the column of the FIM given by $\theta_j = \beta$;

$$\begin{aligned} [J(\tilde{\boldsymbol{\theta}})]_{\theta_i \beta} &= -i \frac{1}{2\sigma_n^2} \frac{\partial}{\partial \theta_i} \left(\mathbf{s}^H(\tilde{\boldsymbol{\theta}}) \mathbf{s}(\boldsymbol{\theta}) - \mathbf{s}^H(\boldsymbol{\theta}) \mathbf{s}(\tilde{\boldsymbol{\theta}}) \right) \bigg|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}} \\ &= \frac{1}{\sigma_n^2} \frac{\partial}{\partial \theta_i} \text{Im} \left\{ \mathbf{s}^H(\tilde{\boldsymbol{\theta}}) \mathbf{s}(\boldsymbol{\theta}) \right\} \bigg|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}}. \end{aligned} \quad (2.21)$$

If additionally $\theta_i = \beta$, that element of the FIM will be

$$[J(\tilde{\boldsymbol{\theta}})]_{\beta\beta} = \frac{1}{2\sigma_n^2} \left(\mathbf{s}^H(\tilde{\boldsymbol{\theta}})\mathbf{s}(\tilde{\boldsymbol{\theta}}) + \mathbf{s}^H(\tilde{\boldsymbol{\theta}})\mathbf{s}(\tilde{\boldsymbol{\theta}}) \right) = \frac{N}{\sigma_n^2}. \quad (2.22)$$

Having shown these two cases, consider now the case when $\theta_i, \theta_j \in \boldsymbol{\theta} \setminus \beta = \boldsymbol{\theta}_{\text{orb}}$. The CRB is evaluated at the correct parameters $\tilde{\boldsymbol{\theta}}$ and β is not one of the parameters being differentiated, hence $\beta = \tilde{\beta}$. In this case $\mathbf{s}(\boldsymbol{\theta}) = \mathbf{s}(\boldsymbol{\theta}_{\text{orb}})e^{i\beta}$ which leads to $\mathbf{s}^H(\tilde{\boldsymbol{\theta}})\mathbf{s}(\boldsymbol{\theta}) = \mathbf{s}^H(\tilde{\boldsymbol{\theta}}_{\text{orb}})\mathbf{s}(\boldsymbol{\theta}_{\text{orb}})e^{i(\beta-\tilde{\beta})} = \mathbf{s}^H(\tilde{\boldsymbol{\theta}}_{\text{orb}})\mathbf{s}(\boldsymbol{\theta}_{\text{orb}})$. Using Equation 2.20 this means that

$$[J(\tilde{\boldsymbol{\theta}})]_{\theta_i\theta_j} = -\frac{1}{\sigma_n^2} \frac{\partial^2}{\partial\theta_i\partial\theta_j} \text{Re} \left\{ \mathbf{s}^H(\tilde{\boldsymbol{\theta}}_{\text{orb}})\mathbf{s}(\boldsymbol{\theta}_{\text{orb}}) \right\} \bigg|_{\boldsymbol{\theta}_{\text{orb}}=\tilde{\boldsymbol{\theta}}_{\text{orb}}}. \quad (2.23)$$

Using the same information for the case when $\theta_i \neq \beta, \theta_j = \beta$ simplifies Equation 2.21 to

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = \frac{1}{\sigma_n^2} \frac{\partial}{\partial\theta_i} \text{Im} \left\{ \mathbf{s}^H(\tilde{\boldsymbol{\theta}}_{\text{orb}})\mathbf{s}(\boldsymbol{\theta}_{\text{orb}}) \right\} \bigg|_{\boldsymbol{\theta}_{\text{orb}}=\tilde{\boldsymbol{\theta}}_{\text{orb}}}. \quad (2.24)$$

2.2.3 Whittaker–Shannon interpolation

If the sampling of a continuous-time band limited signal is sufficiently dense, then you can resample the signal via the Whittaker-Shannon interpolation formula to get even more samples. The meaning of sufficiently dense is that the band limit must be less than half the sampling rate

$$B < \frac{f_s}{2},$$

where $f_s/2$ is the Nyquist sampling rate and f_s is the sampling frequency [15].

Given a signal $x[n]$, sampled in discrete time n , the continuous function can be calculated to

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \text{sinc} \frac{t - nT}{T},$$

using the Whittaker-Shannon interpolation formula, also called sinc interpolation [15]. T is equal to $1/f_s$. This method is used in the project to interpolate bandlimited discrete sampled signals.

3

Methods

Having described all the theory, it is time for the estimation of orbital parameters. To simplify the estimation problem, a circular orbit around the equator with the radar station on the equator at longitude $\varphi = 0$ on a non spinning earth is considered. Additionally, the satellite is limited to spin in a known direction. Because of these simplifications the rotational matrices from Equations 2.3, 2.4 and 2.5

$$\mathbf{r}_{S,4} = R_y(\Lambda)R_z(\Phi)R_z(ERA)R_z(-\Omega)R_x(i)R_z(-\omega)\mathbf{r}_{S,1}$$

become equal to identity and $\mathbf{r}_{S,4} = \mathbf{r}_{S,1}$. The result is no rotation of coordinates. Furthermore, the circular orbit means that the eccentricity $e = 0$ which simplifies Equation 2.1 to

$$\mathbf{r}_{S,4} = \mathbf{r}_{S,1} = a(\cos \nu \hat{\mathbf{x}}_1 + \sin \nu \hat{\mathbf{y}}_1) = r_S(\cos \nu \hat{\mathbf{x}} + \sin \nu \hat{\mathbf{y}}).$$

This is similar to the case in Figure 2.5 with $\gamma = \frac{\pi}{2}$, and $\nu = \theta$. In the case of a circular orbit with a spherical gravitational field, the time evolution of the anomaly can be found from Equation 2.2:

$$\dot{\nu} = \dot{\theta} = \sqrt{\frac{GM_E}{r_S^3}}.$$

A real radar system will have some limits on what angles it can perceive incoming signals from, for example caused by surrounding terrain and limits in the lobe from the emitting and receiving antenna. To this end only satellites that start with an initial incoming angle from zenith $\alpha_0 \in [-\pi/4, \pi/4]$ rad are considered (see Figure 2.5a). Additionally, only satellites in a LEO orbit are considered, giving a limit on $r_S \in r_E + [500, 2000]$ km (see Figure 2.5a). Note that the orbital parameters that are to be estimated are radius r_S and start angle α_0 . Start angle refers to the angle at which the satellite is positioned from zenith at the start of the integration time. The radar system is assumed to have a number of antennas equally spaced with distance $d = \frac{\lambda}{2}$ along the equator, a so called Inline-geometry. Since the angle α is measured in the equatorial plane, this gives extra information about the current angle of the satellite.

All the calculations, estimations and generation of data were done in MATLAB and will be described below. Since this study is a pilot study no prior data was collected.

3.1 Signal generation

Before presenting the methods of estimation, the signal generation must be explained. When generating the signal, no prior data was collected or available. For the signal generation, first a signal modulation was established and then the generation of signal data was performed given a set of orbital parameters in MATLAB.

3.1.1 Signal modulation

First of all a suitable signal modulation was of interest. To examine this the autocorrelation was studied [14]

$$R(\tau) = \sum_{t=t_{\min}}^{t_{\max}} s(t)s(t-\tau(t))^H.$$

The main requirement was a sharp autocorrelation peak with a well-defined main lobe and low side lobes. After studying multiple modulations the most fitting modulations, due to their autocorrelation, were a sinusoidal frequency modulation and an autocorrelation optimised white phase noise modulation. The modulated signals can be written as in Equation 2.10, with the following signal modulations

$$\begin{aligned} m_{\text{AWPN}} &= e^{i\phi(t-\tau(t))} \\ m_{\text{sin}} &= e^{2\pi i B_{\text{sin}} \sin[2\pi f_{\text{PRF}}(t-\tau(t))] \cdot (t-\tau(t))}, \end{aligned}$$

where m_{AWPN} is the periodic autocorrelation optimised white phase noise modulation, henceforth referred to as AWPN-modulation, and m_{sin} is the sinusoidal modulation. The frequencies f_{PRF} and B_{sin} are the pulse repetition frequency and the band width frequency respectively. The values for these are presented in Table 3.1 in Section 3.2.1. The phase ϕ is a periodic set of data obtained after optimizing white phase noise. The data set originally had 2000 data points but was resampled using the Whittaker-Shannon interpolation formula described in Section 2.2.3, increasing the resolution by a factor $F_{\text{res}} = 100$ presented in Table 3.1.

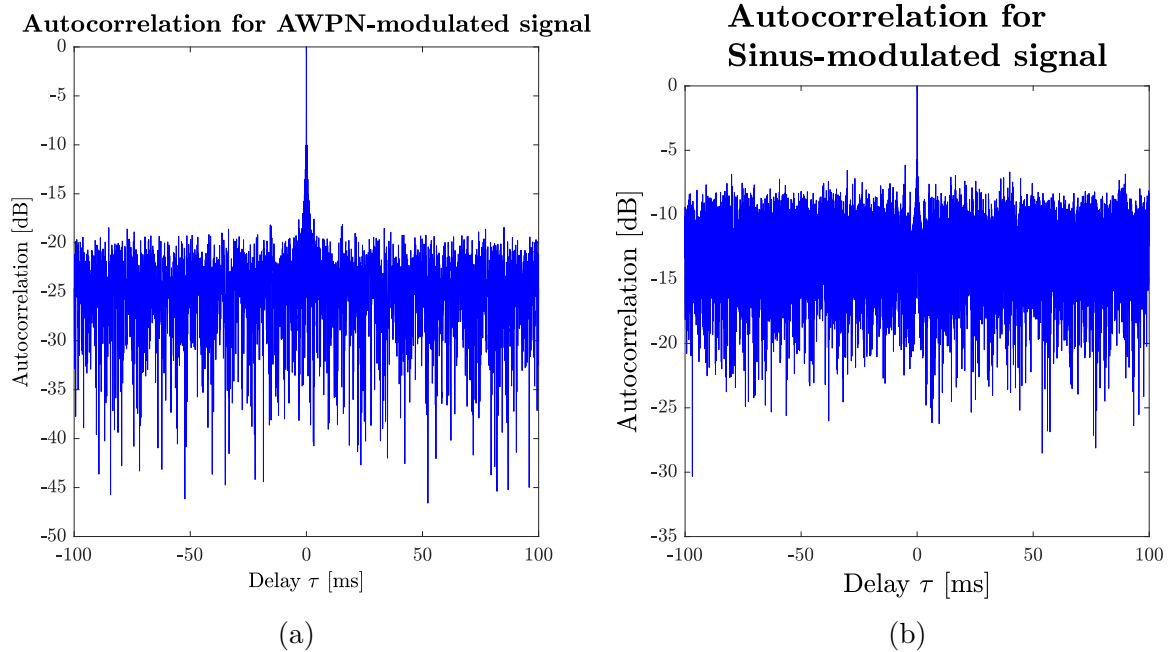


Figure 3.1: The autocorrelation peak for a AWPN-modulated signal and a sinus-modulated signal. The parameters for these modulations were chosen according to Table 3.1. What can be seen is that the AWPN-modulated signal has a sharper peak and has lower side lobes for the wrong delay than the sinus-modulated signal.

In Figure 3.1 the autocorrelation for the two modulations is presented. As can be seen in the figure both of the modulations result in a well-defined main lobe at $\tau = 0$,

but the AWPB-modulation has lower side lobes for the wrong delay than the sinusoidal modulation, meaning it is better at rejecting incorrect correlation matches.

A sharp autocorrelation peak will likely lead to a sharp peak when determining the parameters. This means incorrect parameter choices will be rejected. When considering a detection problem a very sharp peak can be difficult to find via a grid search. Hence both the modulations suit near target searches better than detection problems, due to their sharp peaks. Since near target searches are more relevant for the purpose of tracking known satellites this is no issue. Moving forward the AWPB-modulation will be used since it has lower side lobes.

3.1.2 Generation of data

The orbital parameters are uniform randomly generated within the limits $\alpha_0 \in [-\pi/4, \pi/4]$ rad and $r_S \in r_E + [500, 2000]$ km. With these parameters, a given integration time and time step defined in detail in the next section, a corresponding incoming signal can be generated as described in Section 2.1.4. The randomly generated orbital parameters give a unique delay $\tau(t)$ which can be found from Equation 2.7 using the expression for the distance from the radar to the satellite $r(t)$ in Equation 2.8. For more details see Section 2.1.4. The incoming signal data is generated according to Equation 2.14 using the signal model described in Equation 2.11.

The unknown phase β is not randomly generated due to the fact that its value does not need to be estimated. In the MLE, the unknown phase is what leads to the absolute value of the correlation measurement instead of the real part as described in Section 2.2.1. Furthermore, the FIM in Section 2.2.2 does not have any explicit dependence on β , see Equations 2.20-2.24.

When adding noise to the incoming signal $\mathbf{x}$ this is done in MATLAB according to

$$\mathbf{x} = \mathbf{s}(\boldsymbol{\theta}) + \mathbf{n}, \quad \mathbf{n}_i = \frac{10^{-\text{SNR}_{\text{in}}/10}}{\sqrt{2}}(u + iv) = \frac{\sigma_n}{\sqrt{2}}(u + iv), \quad (3.1)$$

where $u, v \sim \mathcal{N}(0, 1)$ are normally distributed numbers with standard deviation one, and SNR_{in} is the signal to noise ratio of the incoming signal and noise.

Having generated the signal for one antenna, consisting of N samples in time, the signal for all the other antennas need to be generated as well. Suppose the radar consists of N_{antenna} . The model for the incoming signal can be organized into a $N \times N_{\text{antenna}}$ matrix:

$$S = \begin{bmatrix} s(t_1; \tau(t_1))e^{0 \cdot \Delta\phi(t_1)} & \dots & s(t_1; \tau(t_1))e^{(N_{\text{antenna}}-1) \cdot \Delta\phi(t_1)} \\ \vdots & \ddots & \vdots \\ s(t_N; \tau(t_N))e^{0 \cdot \Delta\phi(t_N)} & \dots & s(t_N; \tau(t_N))e^{(N_{\text{antenna}}-1) \cdot \Delta\phi(t_N)} \end{bmatrix}.$$

Here $s(t; \tau(t))$ is the signal at the first antenna and $\Delta\phi(t) = \Delta\phi(\alpha(t; \boldsymbol{\theta}_{\text{orb}}))$ is the phase difference from Equation 2.13 which depends on both time and the orbital parameters. To add noise to this model a matrix of noise N is generated instead of a vector as in Equation 3.1 and the signal $\mathbf{x}$ is a matrix X instead of a vector. This need not confuse; the data could be organised into a vector, the matrix is used simply because it simplifies later calculations.

Note that the random generation of parameters is done by performing a grid search in the relevant area of interest.

3.2 Estimation of orbital parameters

In this section the estimation of orbital parameters will be presented. First the choice of estimator parameters will be motivated followed by the calculations.

3.2.1 Parameter choice and integration time

In Table 3.1 all the used parameters are defined and briefly explained. To understand the choices they will be motivated below.

Table 3.1: Choice of parameters for the estimator, defined and briefly explained.

Parameter	Value	Description
dt	0.1 μ s	Time step for all estimations
T_{seq}	1 ms	Sequence time
T_{tot}	10 s	Total time
N_{seq}	5	Number of sequences
N_{grid}	200	Number of grid points in each direction
T_p	0.2 s	Period of AWPEN-modulated signal
f_H	1 GHz	Carrier frequency for both modulations
B_{sin}	100 kHz	Bandwidth of sinus-modulation
f_{PRF}	30 Hz	Pulse repetition frequency for sinus-modulation
B_{AWPN}	10 kHz	Bandwidth of AWPEN-modulation
F_{res}	100	Resolution factor for Whittaker–Shannon interpolation
r_{lim}	[500,2000] km	Maximum radius from the radar
α_{lim}	[-45,45] $^\circ$	Maximum angle from the radar

Firstly, the time step dt was chosen so that it could dissolve the Doppler shift properly. Meaning

$$dt < \frac{1}{f_H \dot{\tau}_{\text{max}}}.$$

And, when calculating the correlation measurements the grid points were chosen to have both a reasonable resolution and a reasonable computation time. However, when choosing the period of the AWPEN-modulated signal T_p the width of the correlation peak was taken into account. For a shorter T_p , meaning a faster modulation, the bandwidth will be increased and the correlation peak will be more defined in radial-direction making it harder to detect and therefore demanding a better grid resolution. For the period to give an unambiguous range of 2000 km back and forth, it has to be at least

$$T_p > \frac{2r_{\text{max}}}{c} \approx 0.01 \text{ s}.$$

The period was therefore chosen to be greater than 0.01 s, but large enough to not demand a higher grid resolution.

Secondly, to completely comprehend the orbit one needs to follow the satellite during a longer time T_{tot} of a couple of seconds to collect enough data. This size of integration time causes a high computation time though, and eliminates the possibility of assuming a constant amplitude and incoming angle during the integration time. To go around these issues the integration time will be divided into multiple sequences equally distributed.

During a shorter sequence time the delay will not change much, it behaves linearly, and the incoming angle and amplitude will therefore be approximately constant. Instead of looking at data from the entire integration time, N_{seq} sequences of length T_{seq} during a total time T_{tot} are selected. See Table 3.1 for exact values.

Finally, the frequencies were chosen to fit the radar specifications and the sampling of the AWPN-modulation.

3.2.2 MLE of orbital parameters

The approach for the MLE of the orbital parameters was to measure $|\mathbf{x}^H \mathbf{s}(\boldsymbol{\theta})|$ for different parameters surrounding the correct set of parameters; to perform a grid search. This was done according to Section 3.1.2. By iteratively changing the parameters for the incoming signal and matching it with the emitted signal described in Equation 2.6, according to the maximum likelihood estimate described in Equation 2.17, most of the intensity gathered near the correct answer resulting in a peak. This peak could then be identified by simply finding the maximum of the grid search.

The approach was done both for detection of a satellite, where no prior orbital information was assumed, and for a near target search where the approximate orbit was known prior to the search. The grid search consisted of 200×200 points for the radius and start angle respectively as stated in Table 3.1. However, for both these searches the integration time was divided into shorter sequences as explained and motivated in Section 3.2.1. This will effect the likelihood function and therefore also the MLE. If the data from the two sequences are assumed to be independent the likelihood functions for the individual events will multiply [16]. The two sequences are assumed to be generated from the same orbital parameters but with different unknown phases β . The likelihood function for two events $\mathbf{x}_1$ and $\mathbf{x}_2$ therefore result in

$$p(\boldsymbol{\theta}|\mathbf{x}_1 \cup \mathbf{x}_2) = p(\boldsymbol{\theta}|\mathbf{x}_1)p(\boldsymbol{\theta}|\mathbf{x}_2).$$

Due to this the log-likelihood function will sum up for the sequences. The orbital parameters are identical for each sequence and the unknown phase β is not. This combined with the logic that gave the MLE in Equation 2.17 means that the MLE is given by

$$\hat{\boldsymbol{\theta}}_{ML} = \arg \max_{\boldsymbol{\theta}_{\text{orb}}, \beta_1, \dots, \beta_N} \{\text{Re}\{e^{i\beta_1} \mathbf{x}_1^H \mathbf{s}_1(\boldsymbol{\theta}_{\text{orb}}) + \dots + e^{i\beta_N} \mathbf{x}_N^H \mathbf{s}_N(\boldsymbol{\theta}_{\text{orb}})\}\} \quad (3.2)$$

$$= \arg \max_{\boldsymbol{\theta}_{\text{orb}}} \{|\mathbf{x}_1^H \mathbf{s}_1(\boldsymbol{\theta}_{\text{orb}})| + \dots + |\mathbf{x}_N^H \mathbf{s}_N(\boldsymbol{\theta}_{\text{orb}})|\}, \quad (3.3)$$

where $\mathbf{x}_n$, $\mathbf{s}_n$, and β_n , are the data, model, and unknown phase for sequence n . The measure that is maximised to find the MLE will be referred to as the correlation measure, since it is essentially a measure of how correlated the model $\mathbf{s}(\boldsymbol{\theta}_{\text{orb}})$ is with the data $\mathbf{x}$ for a parameter choice $\boldsymbol{\theta}_{\text{orb}}$.

As mentioned in Section 3.1.2 the data is organised into an $N \times N_{\text{antenna}}$ matrix, where N is the number of samples in time from one antenna element. Consider the correlation measure $|\mathbf{x}_n^H \mathbf{s}_n(\boldsymbol{\theta}_{\text{orb}})|$ for sequence n ; this is not well defined when both $\mathbf{x}_n$ and $\mathbf{s}_n$ are matrices. The correct measure would instead be to take the absolute value of the trace of this matrix. A simplification is made to reduce the time needed to compute the estimate; the incoming angle $\alpha(t)$ is assumed to be constant in time during a sequence. Given a set of orbital parameters where the correlation measure is to be evaluated, the incoming angle is assumed to be equal to what it would be in the middle of the sequence $\alpha(t) \approx \alpha(t_{N/2})$.

The correlation measure with this approximation can be simplified to

$$\left| \left[e^{0 \cdot \Delta\phi(\alpha(t_{N/2}))}, \dots, e^{(N_{\text{antenna}}-1) \cdot \Delta\phi(\alpha(t_{N/2}))} \right] \mathbf{x}_n^H \mathbf{s}_n(\boldsymbol{\theta}_{\text{orb}}) \right|. \quad (3.4)$$

Here $\mathbf{x}_n$ is the previously mentioned $N \times N_{\text{antenna}}$ matrix whereas $\mathbf{s}_n$ is the model of the signal that reaches the first antenna, a column vector of length N . The factor $\left[e^{0 \cdot \Delta\phi(\alpha(t_{N/2}))}, \dots, e^{(N_{\text{antenna}}-1) \cdot \Delta\phi(\alpha(t_{N/2}))} \right]$ describes the difference in phase at the different antenna elements and is a row vector with length N_{antenna} . The impact of this approximation has been investigated by comparing the results with and without the approximation. For the chosen sequence time T_{seq} the impact of the approximation was not noticeable. The advantage of this simplification can be explained by considering the number of operations needed to calculate the correlation measure for some orbital parameters $\boldsymbol{\theta}_{\text{orb}}$. Without the simplification two matrices would be generated and then the trace would be calculated for this matrix product. A more computationally effective way is to look at the data as a vector of length $N \cdot N_{\text{antenna}}$ and multiply this with the corresponding model of equal length, since this effectively calculates only the trace. The computational complexity of this operation is $\mathcal{O}(N^2 N_{\text{antenna}}^2)$ [17]. Using the simplification, a $1 \times N_{\text{antenna}}$ vector is multiplied with a $N_{\text{antenna}} \times N$ matrix and a $N \times 1$ vector. The computational complexity of this operation is either $\mathcal{O}(N N_{\text{antenna}}) + \mathcal{O}(N_{\text{antenna}}^2)$ or $\mathcal{O}(N N_{\text{antenna}}) + \mathcal{O}(N^2)$ depending on the order of operations [17]. If we assume that MATLAB chooses the computationally most effective way of doing this, i.e. takes into account which of N and N_{antenna} is largest, the complexity of the simplified case is $\sim \mathcal{O}(N N_{\text{antenna}})$. The non-simplified case thus has a computational complexity which is roughly squared compared to that of the simplified case.

3.2.3 Variance and Cramér-Rao Bound

In Section 2.2.2 the concept of a lower bound for the variance of an estimate, the Cramér-Rao Bound, was introduced. To evaluate how good an estimator is, it can be compared to the CRB. If it is close, it is close to the limit of how good any estimator can be. It is therefore important to calculate the variance of the estimates for different signal to noise ratios, $\text{SNR}_{\text{in}} = -10 \log_{10} \sigma_n$ [dB], and compare this to the CRB. In this study the variance was calculated for a given SNR_{in} 20 times in order to get a relatively good estimate of the variance while not taking too long to perform. The SNR_{in} ranged between -20 dB to 10 dB. The variance could then be approximated according to

$$\text{var}[\hat{\theta}_i] \approx \sum_{i=1}^{20} \frac{(\hat{\theta}_i - \tilde{\theta}_i)^2}{20},$$

where $\hat{\theta}_i$ is the estimate and $\tilde{\theta}_i$ is the correct value. The CRB is calculated as the inverse of the FIM according to Equation 2.18 and thus the only remaining issue is to calculate the FIM. According to Equation 2.19 the elements of the FIM can be calculated according to

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = -\mathbb{E} \left[\frac{\partial^2 \mathcal{L}}{\partial \theta_i \partial \theta_j}(\tilde{\boldsymbol{\theta}}|\mathbf{x}) \right] = \mathbb{E} \left[\frac{\partial \mathcal{L}}{\partial \theta_i}(\tilde{\boldsymbol{\theta}}|\mathbf{x}) \frac{\partial \mathcal{L}}{\partial \theta_j}(\tilde{\boldsymbol{\theta}}|\mathbf{x}) \right].$$

The correlation function is measured using short sequences during a longer integration time as mentioned before. Since the data produced during this time is considered to be IID except for the unknown phase β , the log-likelihood will split into terms for each

sequence which will add up to the total log-likelihood as explained in the previous section. Considering two sequences $\mathbf{x}_1$ and $\mathbf{x}_2$, the FIM can be written as

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = -\mathbb{E} \left[\frac{\partial^2 \mathcal{L}}{\partial \theta_i \partial \theta_j}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \beta_1 | \mathbf{x}_1) + \frac{\partial^2 \mathcal{L}}{\partial \theta_i \partial \theta_j}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \beta_2 | \mathbf{x}_2) \right].$$

The two random phases β_1 and β_2 are not assumed to be the same. This means that for every extra sequence considered an extra parameter β_n is added to the problem. One might ask what the FIM element for two different phases is. Let's consider $\theta_i = \beta_1, \theta_j = \beta_2$;

$$\begin{aligned} [J(\tilde{\boldsymbol{\theta}})]_{\beta_1 \beta_2} &= -\mathbb{E} \left[\frac{\partial^2 \mathcal{L}}{\partial \beta_1 \partial \beta_2}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_1 | \mathbf{x}_1) + \frac{\partial^2 \mathcal{L}}{\partial \beta_1 \partial \beta_2}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_2 | \mathbf{x}_2) \right] \\ &= \left\{ \mathcal{L}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_1 | \mathbf{x}_1) \text{ is independent of } \beta_2 \right\} \\ &= -\mathbb{E} \left[\frac{\partial^2 \mathcal{L}}{\partial \beta_1 \partial \beta_2}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_2 | \mathbf{x}_2) \right] = -\mathbb{E} \left[\frac{\partial^2 \mathcal{L}}{\partial \beta_2 \partial \beta_1}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_2 | \mathbf{x}_2) \right] \\ &= \left\{ \mathcal{L}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_2 | \mathbf{x}_2) \text{ is independent of } \beta_1 \right\} \\ [J(\tilde{\boldsymbol{\theta}})]_{\beta_1 \beta_2} &= 0. \end{aligned}$$

The FIM entries for $\theta_j = \beta_n$ and $\theta_i \in \boldsymbol{\theta}_{\text{orb}}$ can be calculated with Equation 2.24

$$[J(\tilde{\boldsymbol{\theta}})]_{\theta_i \beta_n} = \frac{1}{\sigma_n^2} \frac{\partial}{\partial \theta_i} \text{Im} \left\{ \mathbf{s}_n^H(\tilde{\boldsymbol{\theta}}_{\text{orb}}) \mathbf{s}_n(\boldsymbol{\theta}_{\text{orb}}) \right\} \bigg|_{\boldsymbol{\theta}_{\text{orb}} = \tilde{\boldsymbol{\theta}}_{\text{orb}}},$$

where $\mathbf{s}_n$ is the model for the data from sequence n . The data from the other sequences are not included since $\mathcal{L}(\tilde{\boldsymbol{\theta}}_{\text{orb}}, \tilde{\beta}_m | \mathbf{x}_m)$ is independent of β_n for $m \neq n$. The FIM entries for $\theta_i = \theta_j = \beta_n$ can similarly be calculated with Equation 2.22

$$[J(\tilde{\boldsymbol{\theta}})]_{\beta_n \beta_n} = \frac{N}{\sigma_n^2},$$

where $N = \frac{T_{\text{seq}}}{dt} + 1$ is the number of samples in one sequence. Finally, the FIM entries for $\theta_i, \theta_j \in \boldsymbol{\theta}_{\text{orb}}$ can be calculated from Equation 2.23

$$[J(\tilde{\boldsymbol{\theta}})]_{ij} = -\frac{1}{\sigma_n^2} \sum_n \frac{\partial^2}{\partial \theta_i \partial \theta_j} \text{Re} \left\{ \mathbf{s}_n^H(\tilde{\boldsymbol{\theta}}_{\text{orb}}) \mathbf{s}_n(\boldsymbol{\theta}_{\text{orb}}) \right\} \bigg|_{\boldsymbol{\theta}_{\text{orb}} = \tilde{\boldsymbol{\theta}}_{\text{orb}}}.$$

In the case of the two parameters r_S and α_0 the FIM can be written in matrix form as

$$J(\tilde{\boldsymbol{\theta}}) = \begin{bmatrix} J_{r_S r_S} & J_{r_S \alpha_0} & J_{r_S \beta_1} & \cdots & \cdots & J_{r_S \beta_N} \\ J_{\alpha_0 r_S} & J_{\alpha_0 \alpha_0} & J_{\alpha_0 \beta_1} & \cdots & \cdots & J_{\alpha_0 \beta_N} \\ J_{\beta_1 r_S} & J_{\beta_1 \alpha_0} & \frac{N}{\sigma_n^2} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ J_{\beta_N r_S} & J_{\beta_N \alpha_0} & 0 & \cdots & 0 & \frac{N}{\sigma_n^2} \end{bmatrix},$$

where $J_{ij} = [J(\tilde{\boldsymbol{\theta}})]_{ij}$. Since the chosen AWP-modulated signal can not be described analytically, neither can the derivatives in the equations above. These are instead calculated numerically using a finite difference. Additionally, because of the random nature of the

chosen modulation, calculating derivatives numerically at each sample is not a good idea since they may be very big for individual points. It is therefore better to calculate these derivatives after calculating the scalar products. This is done using MATLABs function `gradient` which is used to calculate the numerical gradient of the real and imaginary part of the correlation field $\mathbf{s}_n^H(\tilde{\boldsymbol{\theta}}_{\text{orb}})\mathbf{s}_n(\boldsymbol{\theta}_{\text{orb}})$. The step lengths for derivatives in each direction were adapted so that good estimates for the second order derivatives could be made, meaning that they were small enough to generate low error, but not so small as to hit the limit in resolution generated by the discrete sampling of the AWPn-signal. The values were chosen to $dr_S = 60 \text{ m}$, $d\alpha_0 \approx 4.000 \cdot 10^{-3^\circ}$.

4

Results

For all the figures and numbers presented in this section, uniformly generated parameters in the limits for α_0 and r_S have been studied. However, for the sake of comparison all figures in this Chapter will have the same orbital values; $r_S \approx 8300$ km and $\alpha_0 \approx 36^\circ$. These values give as representative an illustration of the results as any other. The radar setup has Inline-geometry antennas of various lengths. No noise was added to the received signal until looking at the variance, to better visualise the results.

The results will consist of two main parts; the estimation of orbital parameters and the variance of the estimator.

4.1 Estimation of orbital parameters

Two approaches were conducted when estimating the orbital parameters. The first was to look at a detection problem, meaning there was no prior information assumed regarding the orbit of the satellite when proceeding with the grid search. The entire possible area of interest was searched. The second approach was to perform a near target search, meaning that prior information of the orbit was assumed when doing a grid search. Thus, only a small selected area around the parameters of interest was searched.

4.2 Detection search

For the detection search a correlation measurement was done for four different lengths of Inline-geometry antennas; 1, 10, 100 and 1000. The intensity of the correlation can be visualised in Figure 4.1. What can be seen from the four subfigures is that for a higher number of antennas the intensity of the correlation measurement seems to gather in angular direction. The difference especially stands out for the two later cases using 100 and 1000 antennas. In the subfigures the target value, meaning the correct value, is marked with a green circle. Inside this circle one can see that where the intensity of the correlation measurement peaks, the area becomes yellow. For the first two cases, using one and ten antennas, the intensity does gather but not enough to produce a difference to the width of the yellow area. For 100 antennas the yellow area can be seen to shrink, but for 1000 antennas it shrinks to an unnoticeable size. The peak however can still be located close to the target value. Even though there is no visual difference between one and ten antennas, ten antennas will produce ten times the intensity which of courses increases the signal to noise ratio.

What also should be mentioned is that the estimated value is as close as it can be, given that one does not sample the grid search at the correct value. The estimated value is within half the step length of the grid search from the correct value. Even for one

antenna this is true. Using 200×200 grid points the step length becomes 0.45° for α_0 and 7.5 km for r_S .

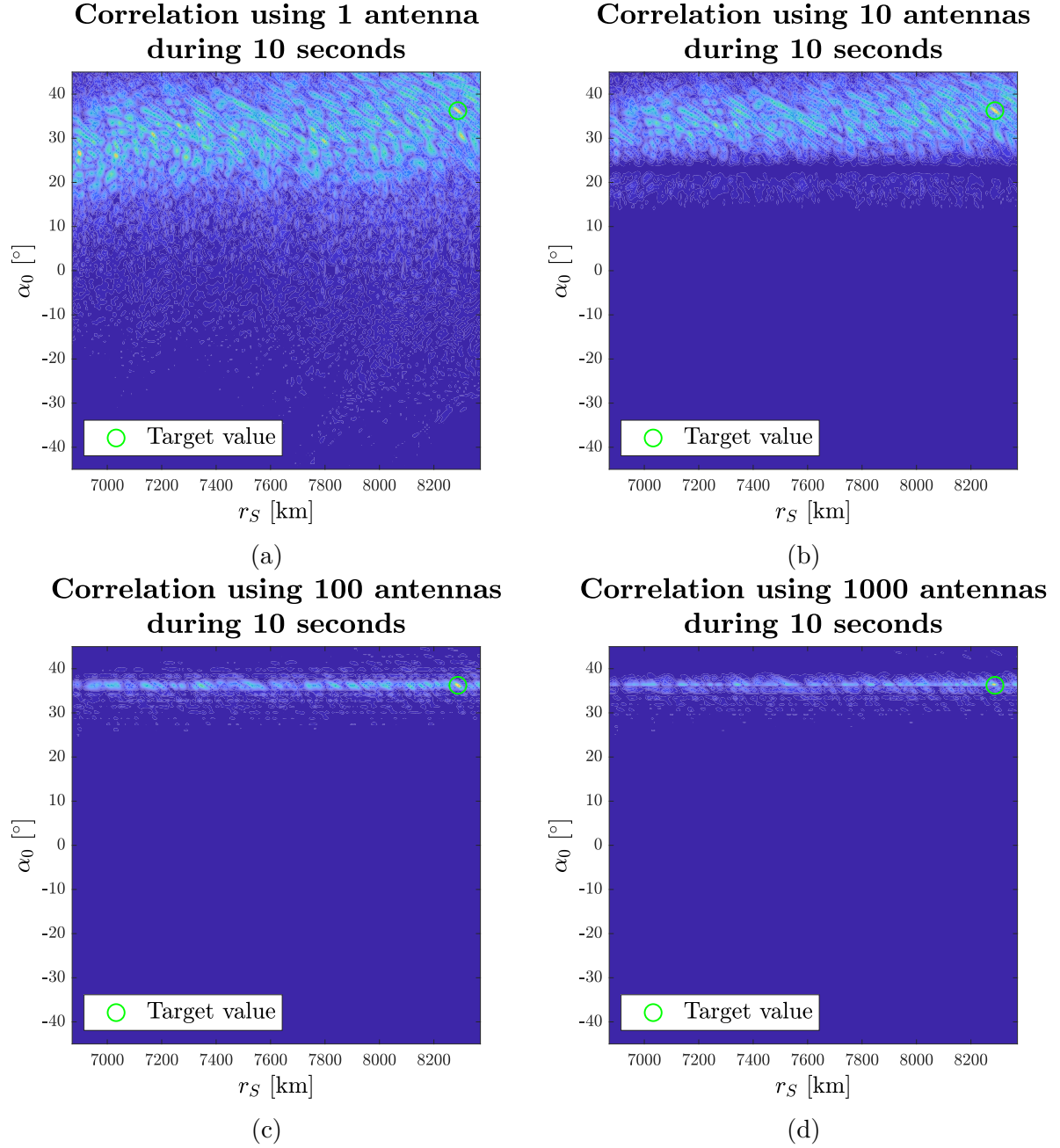


Figure 4.1: Correlation plot for five sequences with a sequence time of 1 ms each and a total time of 10 s. This search was done for the entire possible search area, and is referred to as a detection search. The correlation measurement, MLE, was performed for four different lengths of Inline-geometry antennas; 1, 10, 100 and 1000. When increasing the number of antennas the intensity can be seen to gather in α_0 -direction. The target value is marked with a green circle.

However, in Figure 4.1 the area of the correlation peak in each of the subfigures is quite small and hard to properly see. To better resolve and visualise the correlation peak for the different cases one can zoom in around the target value. In the next section the correlation measure is shown for the same case as in Figure 4.1, only zoomed in. This is referred to as a near target search.

4.2.1 Near target search

For the near target search a new correlation measurement using the same number of grid points was done for three different lengths of Inline-geometry antennas; 1, 100 and 1000, visualised in Figure 4.2. By using the same number of grid points on an area of $r_S \pm 15$ km and $\alpha_0 \pm 0.7^\circ$, the step length becomes 0.007° for α_0 and 150 m for r_S .

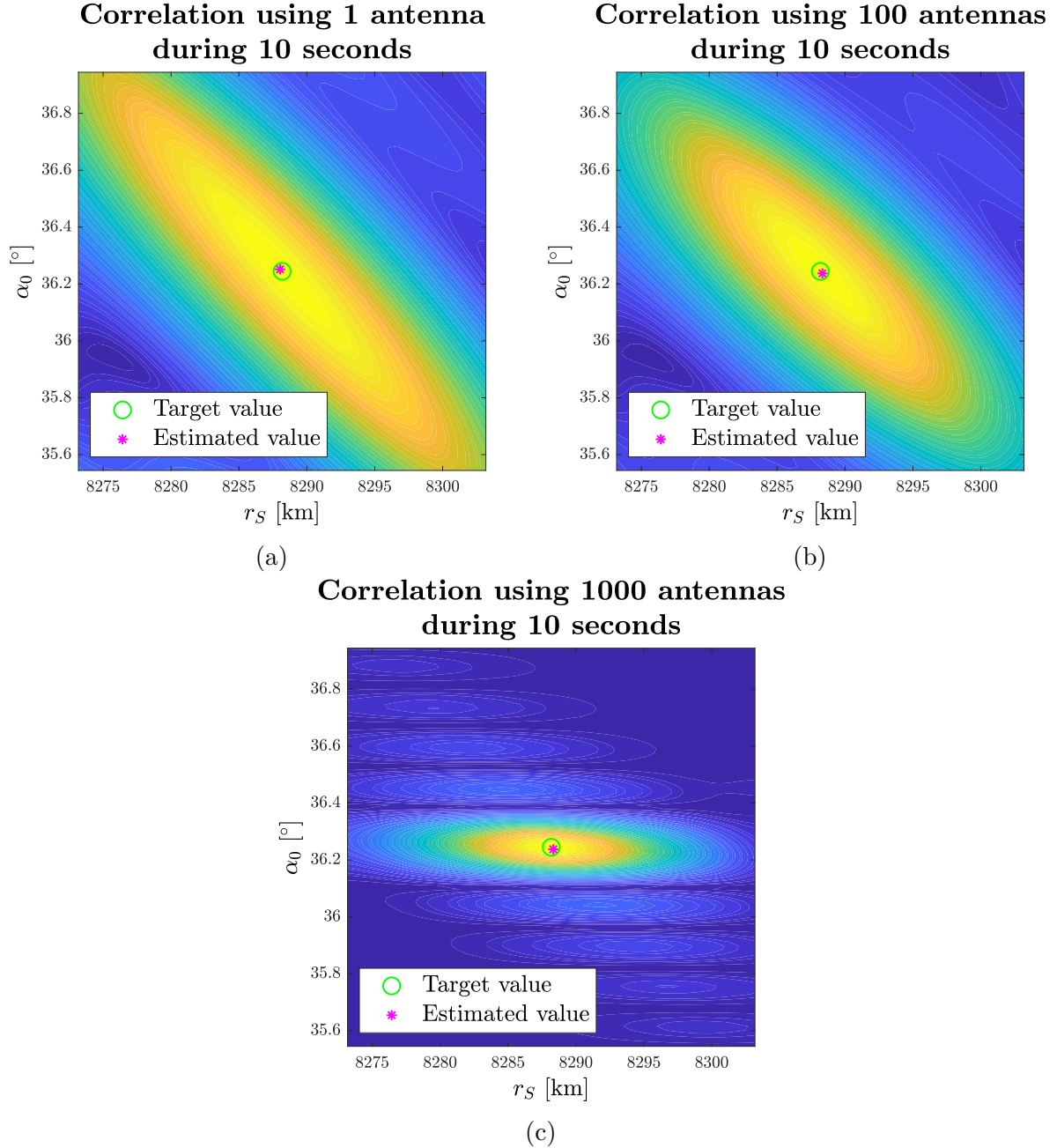


Figure 4.2: Zoomed in correlation plot for five sequences during a sequence time of 1 ms each and a total time of 10 s. This search was done for an area of $r_S \pm 15$ km and $\alpha_0 \pm 0.7^\circ$, and is referred to as a near target search. The correlation measurement, MLE, was performed for three different lengths of Inline-geometry antennas; 1, 100 and 1000. When increasing the number of antennas the intensity can be seen to gather in α_0 -direction. The target value is marked with a green circle, while the estimated value is marked as the magenta star asterisk.

Now, the difference between using one and 100 antennas appears to not be that significant. However, when using 1000 antennas the intensity gathers quite a bit in angular direction making the correlation peak more well-defined. The number of antennas still do not change the estimated values for the orbital parameters, these are still limited by the step length of the grid. In Figure 4.2 the estimated value is marked with a magenta asterisk, while the target value is marked with a green circle. The yellow areas indicate high intensity.

Like in the previous section, more antennas will generate more data and will always increase the SNR. For 1000 antennas there is a significant amount of information about the angle encoded in the signal. Increasing the number of antennas of course comes at the cost of increased computational time.

4.3 Variance of estimator

Until now the result of the correlation measurements have only been shown for no noise. To analyse how well the estimator performs for different noise levels a sweep over different SNR_{in} was done for the near target search. The variance was then calculated for one antenna with SNR_{in} ranging between -20 to 10 dB. It was then averaged over 20 estimations for each SNR, as explained in Section 3.2.3. To compare the results of the estimator the CRB was calculated. Both the variance and the CRB for the two orbital parameters can be seen in Figure 4.3. A setup with only one antenna was chosen to be able to get a good estimate of the variance within a reasonable time.

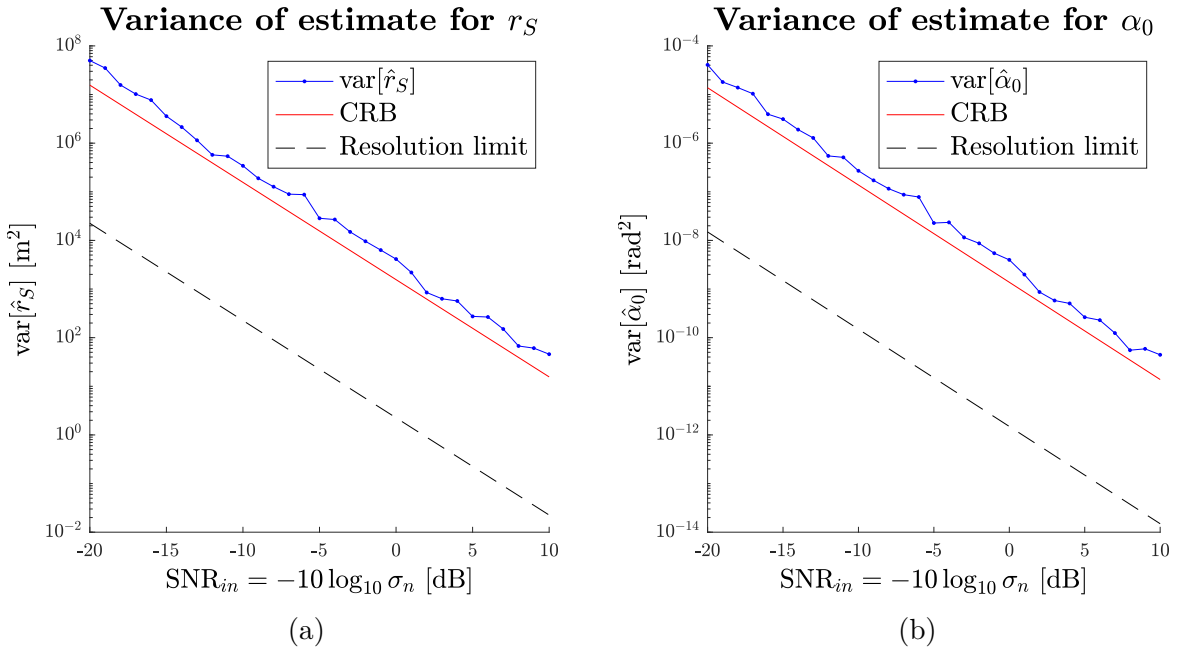


Figure 4.3: Variance of the estimates for r_S and α_0 as a function of the SNR of the incoming signal. The variance is plotted together with the CRB for both the parameters given one antenna, represented by a blue and red line respectively. The dotted black line represent the error for the resolution limit, meaning the square step length for the grid search. This can be seen to be much lower than both the CRB and variance. Note that the scales are logarithmic.

In Figure 4.3 the variance of the estimates for r_S and α_0 are shown as a function of the

SNR of the incoming signal. To avoid being limited by the resolution of the grid search the resolution was increased with a factor $\propto 1/\sigma_n$ for ranging SNR_{in} , where σ_n is defined in Section 3.1.2, Equation 3.1. A dashed line can be seen in the subfigures which gives the error for the resolution limit. The variance and the CRB are represented by a blue and red line respectively. Note that both the scales are logarithmic. The variance for the estimate, the CRB and the error for the resolution limit at maximum and minimum SNR_{in} are presented in Table 4.1.

Table 4.1: Values for the root mean square error of the estimate, square root of the CRB and the step length for the grid search are presented for the highest and lowest value of SNR_{in} .

SNR_{in} [dB]	RMSE		Square root of CRB		Step length	
	r_S	α_0	r_S	α_0	r_S	α_0
$\text{SNR}_{\text{in}} = -20$	7.07 km	0.366°	3.95 km	0.213°	1.5 km	0.07°
$\text{SNR}_{\text{in}} = 10$	6.76 m	$3.82 \cdot 10^{-4^\circ}$	3.95 m	$0.213 \cdot 10^{-4^\circ}$	1.50 m	$7.00 \cdot 10^{-5^\circ}$

The CRB is calculated according to the method explained in Section 3.2.3, and is based on numerical derivatives which are only approximations of the actual derivative. The CRB shown in Figure 4.3 is thus only an approximation of the actual CRB. Since the variance of both the estimate for r_S in Figure 4.3a and α_0 in Figure 4.3b have the same slope as the CRB they seem to perform well. There is a small offset to the CRB, perhaps indicating a small bias.

To conclude this section the estimator performs well, as good as ± 7 m and $\pm 4 \cdot 10^{-4^\circ}$ for good incoming SNR and ± 7 km and $\pm 0.4^\circ$ for poorer incoming SNR.

5

Discussion

In this chapter a discussion will be presented, covering the signal modulation, parameter properties, variance and Cramér–Rao bound, future applications and improvements. The chapter will mainly focus on what can be done in future studies.

5.1 Signal modulation and numerical CRB

The signal was modulated with the AWPN-modulation which has a bandwidth of 10 kHz. This signal modulation seems to perform well for this problem. What could be improved is the signal bandwidth, which might be made larger in order to incorporate more information. The one disadvantage of the AWPN-modulation is the fact that it requires the CRB to be calculated numerically. The numerical derivatives for the CRB are quite sensitive to the step length in each direction, meaning it is difficult to know what a good step length would be. For future studies, one might look at using some other sort of modulation which has a closed analytical form and for which the derivatives can be calculated analytically. This could be used to see if the estimate is close to the CRB, while an AWPN-like signal could be used to produce lower variance but without the possibility of comparing to the CRB.

5.2 Estimation of parameters

This section will discuss the estimation of the two orbital parameters; start angle α_0 and radius r_S . The discussion will touch upon both how well the estimator performs and what could be improved.

5.2.1 Estimation of angle

In this very simplified model, one antenna is sufficient to determine the incoming angle. This is unique to this simplified model and will not be the case for a more detailed model. Still, this is very different from a regular radar station and highlights the difference between this problem and a regular radar.

5.2.2 Estimation of orbital parameters

The estimation of the orbital parameters performs well, as well as the grid search allows it for zero noise and not far off the CRB for more noise cases as can be seen in Figure 4.3. The estimation works both for detection and for near target search, but could of course be improved. One thing that could be improved is to avoid being dependent on a rigorous grid search in order to obtain a good estimate, and this could be done by taking

advantage of the shape of a rough surface. For example, one could do either a gradient search or fit a 2D-Gaussian surface to the peak, but there are some difficulties. The orbital parameters have a large difference when it comes to their order of magnitude. r_S is much larger than α_0 and that introduces many difficulties regarding analysis of the correlation peak. When attempting to do a gradient search close by the peak or to fit a 2D-Gaussian surface to the peak, this difference causes problems. A gradient search would be a great option to avoid rigorous sampling of grid points, but since r_S is so much larger than the angle, the gradient search algorithm avoids steps in r_S -direction. Also, a 2D-Gaussian fit would serve well as a lowpass filter of the surface, damping the surrounding, but it experiences the same difficulties. The fitted surface is sensitive to the exact choice of standard deviation σ in r_S -direction. However, both these methods are very interesting to study further but a normalisation of the coordinates would have to be done.

Also, it would be of interest in future studies to investigate how to properly sample the grid. There might be an optimised way of distributing the grid points, perhaps having more in areas where the intensity is high. One idea is to randomly distribute the grid points after an initial correlation has been calculated for the first sequence, and to have greater probability of grid point in an area of high correlation intensity.

Except for improving the current estimation, one might also consider a different estimation method. One estimate that could be of interest is a Maximum a posteriori estimate, MAP, which uses a priori knowledge to find the next estimate. This would mean that while tracking the satellite this estimate would benefit from prior estimates made.

5.3 Simplifications and Assumptions

This report has considered a simplified version of the actual problem. Apart from not considering the complete set of parameters the simplifications are listed below;

- **Uncorrelated noise:** In Section 2.2.1 the noise which reaches an antenna is assumed to be completely uncorrelated in time. When only looking at one antenna this is a good approximation. When looking at several antennas the only noise that would be completely uncorrelated is the thermal noise generated in the electrical components. The noise generated from the outside, e.g. from the cosmic microwave radiation, will be correlated between the different antennas. This needs to be taken into account into the model.

Another external effect that will generate time correlated noise is the reflections from the Ionosphere. The Ionosphere consists of three main layers: D, E and F [18]. It is characterized by its ionization from the solar and cosmic radiation, a factor which effects radio waves. The D layer at 70 to 90 km is only active at day time, its free electrons recombine with oxygen ions during the night, but even then the reflections it causes are small. The E layer at 90 to 160 km functions as the D layer but does not lose all its free electrons during night, and therefore causes some signal loss during the night as well. Still, the reflections are considered to be negligible. The F layer though, at 160 to around 500 km, consists of the greatest concentration of free electrons which almost do not reduce over the night, they only redistribute. The F layer reflects radio waves with frequencies up to about 35 MHz, hence it should not pose an issue. Still there might some noise caused from these reflections.

The current model is still a reasonable first step, but more information needs to be added to account for all of the phenomena.

- **Constant amplitude:** In Section 2.1.4 the amplitude $\frac{A}{\tau(t)^2}$ of the signal is considered constant during a short sequence. In fact it is also considered constant during all of the sequences. Neither of these is correct, although the first is probably an acceptable simplification. This needs to be investigated further by adding the amplitude to the set of parameters that need to be estimated. Incorporating the amplitude into the parameters also means that the FIM matrix which is used to calculate the CRB will change.
- **Independent sequences:** In Section 3.2.2 the data from two sequences are assumed to be independent. The signal which is generated from the reflection is assumed to have some added uncorrelated noise. The viability of this assumption is discussed above, with the conclusion that the noise should be independent in time but not between the different antennas. Hence the data in two sequences should generate independent data.
- **Independent unknown phase:** In Section 3.2.2 the unknown phase β is assumed to be completely independent between the different sequences. The unknown phase β depends on the geometry of the satellite, more specifically on the part facing the radar station. A satellite will probably face the surface of the earth or at least not rotate with respect to the surface of the earth. Imagine that the phase is measured during an entire transit. It will likely change during the transit and might be used to further identify a particular satellite or if that satellite is facing a different direction than before. If the phase were to be assumed to be constant during all of the sequences the maximum likelihood estimate in Equation 3.3 would go from $\arg \max_{\theta_{\text{orb}}} \sum_n |\mathbf{x}_n^H \mathbf{s}_n(\theta_{\text{orb}})|$ to $\arg \max_{\theta_{\text{orb}}} |\sum_n \mathbf{x}_n^H \mathbf{s}_n(\theta_{\text{orb}})|$. Having tried both ways, the latter seems to displace the maximum away from the true maximum when the true maximum is not sampled. This is probably because the $e^{-2\pi f_H \tau}$ term of the model rotates the phase very fast as a function of the parameters. The same offset in parameters θ might produce different offsets in phase for the correlation measure of two sequences. Hence adding these and then calculating the absolute value does not seem to generate the correct answer.
- **Constant angle:** In Section 3.2.2 the incoming angle is assumed to be constant during a sequence. This is done to reduce the computational complexity as discussed in Section 3.2.2. As discussed the validity of this approximation was checked and worked well during the short sequences.
- **Non-spinning earth:** In the beginning of Chapter 3 the Earth is considered to be non-spinning. This is obviously something which will need to be added to the model. However, the rotation is highly deterministic. Adding this effect would likely only generate more information about the orbit.
- **Spherically symmetrical gravitational field:** In the model described in the beginning of Chapter 3 the gravitational field of the earth is assumed to be spherically symmetrical. In reality the earth is a spheroid; it is slightly pressed together at the poles. The gravitational field is therefore not spherically symmetrical. This will cause the orbital parameters to drift over time and is the largest effect on the orbit between 300 km and 30 000 km [2]. For satellites with a lower or higher orbit other effect may be important to study as well, such as air drag and perturbations due to the Sun and the Moon.

5.4 Complexity of model

The model that is considered in this report is a simplified case of the actual orbit of a satellite around the Earth. Except for the simplifications regarding the model covered in the previous section, the model that is considered is one where most of the parameters are set to zero. The next natural step would be to introduce an elliptical orbit with some orientation given by the angle ω in Figure 2.2. After this the two other angles Ω and i might be introduced and a radar system which is not located on the equator might be considered. This however, takes the problem from estimating two parameters to having to estimate four and then six, i.e. increasing the dimension of the problem. It might be beneficial to further study this simplified case, removing the simplifications from the previous section first. This since the problem would still only be two dimensional, making it easier to study for example the correlation measures in Figures 4.1 and 4.2.

5.5 Antenna geometry

The antenna geometry considered is one where the antennas are standing in a line along the equator. It is obvious from Figures 4.1 and 4.2 that more antennas give more information regarding the orbit. In this simplified case this merely adds information about the incoming angle. When increasing the complexity and considering an elliptical orbit around the equator, one antenna will still be sufficient to estimate the parameters. When increasing the complexity one step more and estimating the angles Ω and i , an array of antennas will be needed to estimate the parameters. In this case an inline array of antennas is probably not the optimal antenna geometry. One might investigate different geometries and their effect on for example the CRB for different parameters given a set number of antennas.

5.6 Variance and Cramér–Rao bound

As can be seen in Figure 4.3 the variance for the estimates of r_S and α_0 are close to the CRB. As mentioned previously, the CRB shown can be thought of as an estimate of the actual CRB. There might be a slightly different offset to the CRB, since the numerical derivatives might not produce exactly the correct answers, but the effect is negligible. Using a different modulation the numerical derivatives could be avoided, making the calculations more precise and computationally effective. What can be seen without any doubt is that the variance is decreasing with the same slope as the CRB. This indicates that the estimator performs quite well.

The offset to the CRB, the bias, could come from the number of grid points or some other unknown factor. When stating that the bias could depend from the number of grid points, this is obviously not from the resolution limit in Figure 4.3. Instead, imagine for example that with the current resolution the MLE is some distance away from the correct answer. There might be a set of parameters between two grid points that better explain the data and if the correlation was sampled in this point it would produce a lower variance. The solution to this problem would be to increase resolution. However, increasing resolution of course also means that the computational time would increase. By experimenting with different resolutions, it is possible that a good trade-off could be found.

In Section 4.3 it is explained that the resolution factor of the grid search is increased with increasing SNR in order not to hit the limit imposed by a finite resolution. Increasing the resolution also meant decreasing the limits which the correlation is calculated within, essentially looking in a smaller area. A reasonable question is thus if the decreasing limit is perhaps 'pushing' down the variance of the estimate. The variance has been calculated without increasing the resolution. The only difference from the current result is that the variance hits the resolution limit around $\text{SNR}_{\text{in}} = -2$ dB, hence the decreasing limit is probably not causing this behavior. The value for the variance was averaged over 20 iterations for each SNR. The number 20 was chosen by iteratively adding more iterations. The variance seen in Figure 4.3 seems to follow a clear linear trend, hence the decision to stop at 20 iterations was taken.

6

Conclusion

The conclusion of this study is that the chosen method was suitable for the scope of the project. The estimator performs well for both detection and near target search, and seems to come close to the CRB. The average error of the parameters is ± 7 m and $\pm 4e-4^\circ$ for good incoming SNR of 10 dB and ± 7 km and $\pm 0.4^\circ$ for poorer incoming SNR of -20 dB. Only a simplified case is considered, hence future studies might first focus on removing these simplifications. To be able to better study the effect that certain simplifications might have the signal modulation should be changed to one where the CRB is not based on numerical derivatives. After these simplifications have been removed, one should focus on increasing the complexity of the orbital model.

Bibliography

- [1] “UCS Satellite Database,” Apr. 1, 2020. Accessed on: June 1, 2020. [Online] Available: <https://www.ucsusa.org/resources/satellite-database>.
- [2] S. A. Klioner, “Basic Celestial Mechanics,” 2016. Accessed on: Feb. 16, 2020. [Online] Available: <https://arxiv.org/pdf/1609.00915.pdf>.
- [3] S.-H. Na, “Earth Rotation – Basic Theory and Features,” in *Geodetic Sciences - Observations, Modeling and Applications* (Shuanggen Jin, ed.), ch. 9, Rijeka, Croatia: InTech, 2013.
- [4] “The International Terrestrial Reference System (ITRS),” 2013. Accessed on: Mar. 12, 2020. [Online] Available: <https://www.iers.org/iers/EN/Science/ITRS/ITRS.html>.
- [5] I. Ridpath, *A Dictionary of Astronomy*. Oxford Paperback Reference, New York, NY, USA: OUP Oxford, 2012.
- [6] P. Gérard and B. Luzum, “IERS Conventions (2010),” Tech. Note No. 36, IERS Convention Centre, Frankfurt, Germany, 2010. Accessed on: Feb. 26, 2020. [Online] Available: <http://www.iers.org/TN36/>.
- [7] D. P. Stern, “Precession,” Oct. 10, 2016. Accessed on: Mars 22, 2020. [Online] Available: <https://www-istp.gsfc.nasa.gov/stargaze/Sprecess.htm>.
- [8] “EOP of today,” 2013. Accessed on: Mars 19, 2020. [Online] Available: https://www.iers.org/iers/EN/DataProducts/tools/eop_of_today/eop_of_today_tool.html.
- [9] “Latitude,” 2020. Accessed on: June 15, 2020. [Online] Available: <https://en.wikipedia.org/wiki/Latitude>.
- [10] “United States Department of Defense, World Geodetic System 1984: Its Definition and Relationships with Local Geodetic Systems,” Tech. Rep. TR8350.2, National Imagery and Mapping Agency, St. Louis, MO, USA, 2000. Accessed on: May 20, 2020. [Online] Available: <https://earth-info.nga.mil/GandG/publications/tr8350.2/wgs84fin.pdf>.
- [11] “Carrier wave,” 2020. Accessed on: June 15, 2020. [Online] Available: https://en.wikipedia.org/wiki/Carrier_wave.
- [12] A. M. Niknejad, “Lecture 15: Introduction to mixers.” EECS142, Lecture Notes, University of California, Berkeley, USA, 2015. Accessed on: June 15, 2020. [Online] Available: http://rfic.eecs.berkeley.edu/~niknejad/ee142_fa05lects/pdf/lect15.pdf.
- [13] T. L. Singal, *Wireless communications*, pp. 79–80. Tata McGraw-Hill Education, 2010.
- [14] U. Spagnolini, *Statistical signal processing in engineering*, pp. 64–65, 117–120, 143–146. John Wiley & Sons Ltd, 1 ed., 2017.

- [15] “Whittaker–Shannon interpolation formula,” 2020. Accessed on: June 1, 2020. [Online] Available: https://en.wikipedia.org/wiki/Whittaker%E2%80%9993Shannon_interpolation_formula.
- [16] “Likelihood function,” 2020. Accessed on: June 20, 2020. [Online] Available: https://en.wikipedia.org/wiki/Likelihood_function#Products_of_likelihooods.
- [17] “Computational complexity of mathematical operations,” 2020. Accessed on: June 26, 2020. [Online] Available: https://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations#Matrix_algebra.
- [18] M. B. McElroy, “Ionosphere and magnetosphere,” Aug. 8, 2012. Accessed on: Apr. 11, 2020. [Online] Available: <https://www.britannica.com/science/ionosphere-and-magnetosphere>.
- [19] United States Naval Observatory. Nautical Almanac Office, Great Britain. Nautical Almanac Office, *The Astronomical Almanac*. St. Louis, MO, USA: [Department of Defense], Navy Department, Naval Observatory, Nautical Almanac Office, 2011. Accessed on: May 20, 2020. [Online] Available: <https://books.google.se/books?id=dtKGqkDBCaIC>.

A

Transformation matrices to relate GCRS to ITRS

The transformation matrices to relate the GCRS to the ITRS can be written as

$$[\text{ITRS}] = (R_{\text{prec+nut}} R_{\text{spin}} R_{\text{polar}})^T [\text{GCRS}], \quad (\text{A.1})$$

where $R_{\text{prec+nut}}$, R_{spin} and R_{polar} are the transformation matrices connected to the precession and nutation, spin and polar motion respectively [6]. The different transformation matrices will transform the system into different reference systems, with the first transition going from GCRS to the Celestial Intermediate Reference System, CIRS, which is a geocentric coordinate system based upon current position and date taking precession and nutation into account [5]. The second transformation will bring CIRS to TIRS, the Terrestrial Intermediate Reference System, taking spin into account. Finally, the last transition takes the system from TIRS to ITRS by also accounting for polar motion.

Note that the transformation matrices are written in accordance to the IAU 2006 resolutions and that the transformation will follow the Celestial Intermediate Origin, CIO, based procedure [6].

A.1 Transformation matrix, GCRS to ITRS

The first transformation needed is a transformation matrix taking the system from the GCRS to the CIRS. This is done by taking the precession and nutation into account into account and the transformation matrix can be written as the corresponding transformation matrix can be written as

$$R_{\text{prec+nut}} = \begin{bmatrix} 1 - aX^2 & -aXY & -X \\ -aXY & 1 - aY^2 & -Y \\ X & Y & 1 - a(X^2 + Y^2) \end{bmatrix} R_z(s), \quad (\text{A.2})$$

where $X = \sin d \cos E$ and $Y = \sin d \sin E$ are the coordinates of the celestial intermediate pole, CIP, the angles E and d are marked in Figure 2.4, $a = 1/(1 + \cos d) = 1/(1 + \sqrt{1 - X^2 - Y^2})$, rotational matrix $R_z(s)$ and s is the CIO locator [6][3]. The CIO locator positions the CIO σ on the CIRS equator. It is the angle corresponding to the difference in right ascension of the equators node due to precession and nutation [19].

The second transformation needed is a transformation matrix taking the system from CIRS to TIRS, accounting for the spin of the rotational axis. This effect has a period of one day, making it significant for an observational system. The corresponding rotational

matrix is

$$R_{\text{spin}} = R_z(-ERA), \quad (\text{A.3})$$

where ERA is the Earth Rotation Angle and spans between the CIO and the TIO, the Terrestrial Intermediate Origin along the CIRS equator [6]. ERA can be calculated using the following equation

$$ERA = 2\pi(\text{UT1 Julian day fraction} + 0.7790572732640 + 0.00273781191135448T_u), \quad (\text{A.4})$$

where T_u is the difference between the Julian date of UT1 and the Julian date 2451545, corresponding to the day starting at 12:00 UTC on January 1, 2000. UT1 is a form of Universal Time taking polar motion into account, which differs at most with 0.9s to the Coordinated Universal Time known as UTC [8].

Finally, all that is left is to account for the polar motion moving the reference system from TIRS to ITRS with the corresponding rotation matrix

$$R_{\text{polar}} = R_z(-s')R_y(x_p)R_x(y_p), \quad (\text{A.5})$$

where x_p and y_p are the polar coordinates of the CIP, R_i are rotational matrices and s' is the TIO locator which positions the Terrestrial Intermediate Origin, TIO, on the TIRS equator [6]. It is the angle corresponding to the difference in right ascension of the equators ITRS and TIRS node due to polar motion.