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Where to stand when playing darts?

Questions in Probability Theory inspired by dart throwing

Master's thesis in Mathematics

Björn G. Franzén

Department of Mathematical Sciences
CHALMERS UNIVERSITY OF TECHNOLOGY
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Supervisor: Jeffrey Steif, Mathematical Sciences
Examiner: Jeffrey Steif, Mathematical Sciences

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Department of Mathematical Sciences
Chalmers University of Technology
SE-412 96 Gothenburg
Telephone +46 31 772 1000

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Abstract

This thesis investigates questions in probability theory inspired by a model of dart throwing. The deviation from where you aim is modelled as the distance to the dart board times a random vector, and in this thesis we refer to such a random vector as a *dart*. Points are then assigned by some bounded payoff function, and we study how the choice of the payoff function as well as the distribution of the dart affect the properties of the expected score when aiming in an optimal way.

Interestingly, it turns out that sometimes it can be better to move further away from the dartboard before throwing, and the main focus of this thesis is to characterise under what circumstances this is or isn't the case. For a dart X and payoff function f we call the pair (X, f) reasonable if it is always better to stand closer to the dartboard, and a dart X is called reasonable if (X, f) is reasonable for all payoff functions f.

We have found a large class of darts which are reasonable, namely those which have so-called selfdecomposable distributions, which includes many well known distributions, such as the exponential distribution, the logistic distribution, and also all stable distributions. Whether there exist reasonable darts that are not selfdecomposable remains an open question.

It turns out that when the payoff function is cosine, then reasonableness can be characterised in terms of the characteristic function of the dart, from which many different results follow. Furthermore, by studying functions of the form $e^{cx}\cos(\omega x)$, we have found that no dart with compact support is reasonable.

We have also found several sufficient conditions for a dart to be non-reasonable with respect to some continuous payoff function, one of which is the following. If a dart has a point mass, but is not a constant, then it is non-reasonable with respect to some continuous function.

Finally, we have investigated under what types of operations a set of reasonable darts may be closed, and have found two results of this nature. Firstly any independent sum of reasonable darts is reasonable. Secondly, for any $f \in C_0(\mathbb{R}^n)$, the set of darts which are reasonable with respect to f is closed with respect to convergence in distribution.

History and Collaboration of this project

Since 2015, Jeffrey Steif and Johan Wästlund have been sporadically working on a research project. I (Björn Franzén) was asked to join the project as part of my master's thesis. Since then, this project has been a collaborative effort between the three of us with each of us contributing in an essential way. The present thesis is part of a larger research paper which is actively in progress and will eventually be submitted for publication with three authors.

Acknowledgements

First of all I would like to thank my supervisor Jeffrey Steif and his colleague Johan Wästlund for letting me join this research project. They have given me an amazing opportunity to get first hand experience of mathematical research, and to conduct research together with them has been a very rewarding experience. Jeffrey Steif in particular deserves extra mention as he has contributed so much in terms of ideas, feedback and insight throughout the writing of this thesis. I would also like to thank Torbjörn Nilsson for keeping me company during quarantine, for the many discussions where I have bounced off various ideas, and for providing me with feedback and advice.

Contents

1. Introduction

In the game of darts, it is not always obvious where on the dart board one should aim, or where one should stand. This could depend on the type of game (scoring method) you play, and also on your level of skill. We study a model of the game where the dart board is $\mathbb{R}^n$, and there is some scoring- or payoff function $f : \mathbb{R}^n \to \mathbb{R}$. We always assume that you want to maximize your expected score.

1.1 A mathematical model of dart throwing

We make the following definition.

Definition 1.1. *A **payoff function** f is a bounded measurable function from $\mathbb{R}^n$ to $\mathbb{R}$.*

As a player you can aim wherever you want in $\mathbb{R}^n$, and if you aim at a point a, while standing at distance 1 from the target, then you will hit $a + X$, where X is a random vector taking values in $\mathbb{R}^n$. In this thesis we will refer to such a random vector as a *dart*.

Definition 1.2. *A **dart** is a random vector X taking values in $\mathbb{R}^n$. A dart whose law is equal to a single point mass is called **degenerate**.*

We assume that the deviation from where you aim scales linearly with the distance from the target. Thus if you stand at distance $d > 0$ and aim at a, you will hit $a + dX$ and $f(a + dX)$ is then your payoff. It is interesting to look at $Ef(a + dX)$, the expected score when standing at distance d and aiming at a using dart X, as well as $\sup_a Ef(a + dX)$, and so we introduce the following definition.

Definition 1.3. *Let X be a dart taking values in $\mathbb{R}^n$ and f be a payoff function on $\mathbb{R}^n$. The function $g_{X,f}(d)$ is defined for $d > 0$ to be*

$$g_{X,f}(d) := \sup_{a \in \mathbb{R}^n} Ef(a + dX). \tag{1.1}$$

So $g_{X,f}(d)$ is the best you can achieve with dart X, standing at distance d with payoff function f. Note that the supremum is not always assumed, so that it is sometimes best to aim arbitrarily far away.

It is natural to think that the closer you stand to the target, the better you will do; i.e., that $g_{X,f}(d)$ is a decreasing function of d. Interestingly, this is not necessarily the case as the following simple example illustrates. Let $n = 1$, X be uniform on $[0, 2]$ and f be 1 on intervals of the form $[2k, 2k + 1]$ and 0 on intervals of the form $(2k - 1, 2k)$ where k is an integer. It is then immediate to check that $g_{X,f}(1) = 1/2$ (and it doesn't matter where you aim) but $g_{X,f}(3/2) = 2/3$ (aim e.g. at 1.5). We will later see how this is related to a more general phenomenon where the behaviour of the characteristic function of X will play a central role, see Theorem 3.1.

We introduce the following concept which captures those situations where standing closer is in fact better.

Definition 1.4. *The pair (X, f) is* ***reasonable*** *if $g_{X,f}(d)$ is decreasing in d. The dart X is* ***reasonable with respect to*** *a family of payoff functions $\mathcal{F}$ if (X, f) is reasonable for all $f \in \mathcal{F}$. If (X, f) is reasonable for all payoff functions f, then X is said to be* ***reasonable****. The payoff function f is* ***reasonable with respect to*** *a family of darts $\mathcal{X}$ if (X, f) is reasonable for all $X \in \mathcal{X}$. If (X, f) is reasonable for all darts X, then f is said to be* ***reasonable****.*

In some cases it is trivial to show whether a dart or payoff function is reasonable or not with respect to some set, but often it is quite difficult. Investigating under what conditions a pair (X, f) is reasonable or not is the main focus of this thesis, and we have obtained various different interesting results which have given us a better understanding of what it means to be reasonable.

Let us first note that rescaling and shifting darts and payoff functions should not affect whether they are reasonable or not, and so we recall the following definition.

Definition 1.5. *We say that the random vectors X and Y have the same* ***type*** *if there exist $a_d > 0$ and $b_d \in \mathbb{R}^n$ such that Y and $a_d X + b_d$ have the same distribution. We say that the functions f and h have the same* ***type*** *if there exist $a_p, c_p > 0$, $b_p \in \mathbb{R}^n$, and $d_p \in \mathbb{R}$ such that $h(x) = c_p f(a_p x + b_p) + d_p$ for all x.*

If X and Y have the same type and f and h have the same type, then it will follow from Proposition 2.1 that (X, f) is reasonable if and only if (Y, h) is.

It is not immediately obvious whether there exist reasonable payoff functions and darts (other than the most trivial examples), but there are some interesting examples. We will now introduce a class of reasonable payoff functions which we choose to call *unimodal*.

Definition 1.6. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called* ***unimodal*** *if for all $x \in \mathbb{R}^n$, $f(rx)$ is decreasing in r on $[0, \infty)$.*

Proposition 2.3. *If f is a unimodal payoff function, then f is reasonable.*

Note that the preceding proposition is numbered based on in which section of the main text it is restated and proved. We will treat all other propositions and theorems in the introduction in the same way.

We now wish to introduce a large collections of darts which turn out to be reasonable, and for this reason we recall the notion of *selfdecomposable* probability measures (see [7]). However, we first need to recall what it means for a random vector to divide another random vector.

Definition 1.7. *We say a random vector X* ***divides*** *a random vector Y, written $X|Y$ if there exists a random vector Z so that if Z and X are independent, then $X + Z$ and Y have the same distribution.*

Definition 1.8. *A random vector X is* ***selfdecomposable*** *if for all $d > 1$, $X|dX$.*

Theorem 2.6. *If $X|dX$, where $d > 1$, then $g_{X,f}(s) \geq g_{X,f}(ds)$ for all f and for all s. In particular, if X is selfdecomposable, then X is reasonable.*

The notion of selfdecomposability is not very well known, but there are common examples of selfdecomposable random vectors, such as those with stable distributions, see Section 1.2.

Remark 1.9. *It is easy to check that $X|dX$ only depends on X's type and hence being selfdecomposable also only depends on the type of X.*

Remark 1.10. *It is easy to show that an independent sum of selfdecomposable random variables is selfdecomposable (which is not true in general for stable distributions). This yields a large collection of reasonable darts.*

It turns out that when the payoff function is cosine, then there is an interesting connection between $g_{X,f}$ and the characteristic function of X.

Theorem 3.1. *Let X be any dart taking values in $\mathbb{R}^n$ with characteristic function ϕ_X, and let $f(x) = \cos\left(\sum_{j=1}^n x_j\right)$. Then for any $d > 0$ we have that*

$$Ef(a + dX) = |\phi_X(d\vec{1})| \cos\left(\sum_{j=1}^n a_j + \mathrm{Arg}(\phi_X(d\vec{1}))\right), \tag{1.2}$$

where $\vec{1} = (1, 1, ..., 1)$. In particular this implies that if $f(x) = \cos\left(\sum_{j=1}^n x_j\right)$, then $g_{X,f}(d) = \left|\phi_X(d\vec{1})\right|$, so that (X, f) is reasonable if and only if $\left|\phi_X(d\vec{1})\right|$ is decreasing in $d > 0$.

From this many other results follow, including the following, which yields a type of phase transition.

Theorem 3.18. *Let X_1 be Bern(p) distributed and X_2 be N(0,σ^2) distributed. If they are independent, then $X := X_1 + X_2$ is reasonable with respect to $f(x) = \cos(x)$ if and only if*

$$\sigma^2 d\Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big) + (1-p)p\sin(d) \geq 0, \ \forall d \geq 0. \tag{1.3}$$

When $p = 1/2$, $(X, \cos(x))$ is not reasonable for any σ. But for any $p \neq 1/2$ there exists a $\sigma_p \in (0, \infty)$ such that for all $\sigma \geq \sigma_p$, $(X, \cos(x))$ is reasonable, and for any $\sigma < \sigma_p$ $(X, \cos(x))$ isn't reasonable. In addition, for $p \neq 1/2$, $\sigma_p \leq (1-p)p/(\pi|1 - 2p|^2)$.

Furthermore, we have found several sufficient conditions for a dart to not be reasonable with respect to the set of continuous payoff functions. We have, among other things, found that a non-degenerate reasonable dart does not have compact support, and does not have point masses.

Theorem 3.21. *Let X be a dart taking values in $\mathbb{R}^n$. If there is an axis such that X projected onto that axis is non-degenerate and has compact support, then there exists a continuous payoff function h with compact support such that (X, h) is not reasonable.*

In particular, if X is non-degenerate and has compact support, then there exists a continuous payoff function h with compact support such that (X, h) is non-reasonable.

Theorem 4.1. *If X is a non-degenerate dart taking values in $\mathbb{R}^n$ with a point mass, then there exists a continuous payoff function f on $\mathbb{R}^n$ such that (X, f) is not reasonable.*

Finally, we have investigated under what types of operations a set of reasonable darts may be closed, and have found that an independent sum of reasonable darts is reasonable.

Theorem 5.1. *Assume that $X_1, ..., X_m$ are independent darts taking values in $\mathbb{R}^n$, which are all reasonable with respect to a family of payoff functions $\mathcal{F}$. If $\mathcal{F}$ is either of the following sets*

1. *The set of all payoff functions*
2. *The set of all continuous payoff functions*
3. *The set of all payoff functions of the same type as $\cos(\sum_{j=1}^{n} x_j)$*

then for any $d_1, ..., d_m, D_1, ..., D_m \geq 0$ such that $d_j \leq D_j$ for all j we have that

$$\sup_a Ef(a + \sum_{j=1}^{m} d_j X_j) \geq \sup_a Ef(a + \sum_{j=1}^{m} D_j X_j), \ \forall f \in \mathcal{F} \tag{1.4}$$

and in particular $\sum_{j=1}^{m} X_j$ is reasonable with respect to $\mathcal{F}$.

We have also found a type of closedness with respect to convergence in distribution.

Theorem 5.3. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of darts taking values in $\mathbb{R}^n$ which converges in distribution to some dart X. Then for any $f \in C_0(\mathbb{R}^n)$, $d > 0$, $\lim_j g_{X_j,f}(d) = g_{X,f}(d)$. As a consequence, for any $f \in C_0(\mathbb{R}^n)$ for which (X_j, f) is reasonable for all j, we have that (X, f) is reasonable.*

1.2 Background

In this section we will go through some of the background and definitions relevant to the text.

Let us first note that the set of selfdecomposable darts mentioned in the previous section is quite large, as it contains all darts with so-called *stable distributions.*

Definition 1.11. *A random vector X has a* ***stable distribution*** *if for independent copies X_1 and X_2 of X, and any $a, b > 0$, there exist constants $c > 0$ and $d \in \mathbb{R}^n$ such that $aX_1 + bX_2$ is equal to $cX + d$ in distribution.*

Examples of stable distributions include the normal distribution and the Cauchy distribution. [6]

Stable distributions can be best characterised in terms of their *characteristic functions.*

Definition 1.12. *Let X be a random vector taking values in $\mathbb{R}^n$. The* ***characteristic function*** *of X is a function $\phi_X : \mathbb{R}^n \to \mathbb{C}$ defined by*

$$\phi_X(t) = E\exp(i \cdot (t \cdot X)). \tag{1.5}$$

For each stable distribution there is an associated parameter $\alpha \in (0, 2]$. The value $\alpha = 2$ corresponds to a (multivariate) normal distribution with characteristic function

$$\phi_X(\vec{t}) = \exp\left(i\vec{\mu}\vec{t} - \frac{1}{2}\vec{t}^{\,T}\Sigma\vec{t}\right) \tag{1.6}$$

where $\vec{\mu}$ is a vector in $\mathbb{R}^n$ and Σ is a symmetric non-negative definite $n \times n$ matrix. For $\alpha \in (0, 2)$ [6] provides the following characterisation of the characteristic function (see p. 65).

Theorem 1.13. *Let $0 < \alpha < 2$. Then a random vector X taking values in $\mathbb{R}^n$ has an α-stable distribution if and only if there exists a finite measure Γ on the unit sphere S_n of $\mathbb{R}^n$, and some $\vec{\mu} \in \mathbb{R}^n$ such that*

a) If $\alpha \neq 1$

$$\phi_X(\vec{t}) = \exp\left(-\int_{S_n} |\vec{t}\cdot\vec{s}|^\alpha (1 - i\ \mathrm{sign}(\vec{t}\cdot\vec{s})\tan(\frac{\pi\alpha}{2}))\Gamma(\mathrm{d}\vec{s}) + i\vec{t}\cdot\vec{\mu}\right) \tag{1.7}$$

b) If $\alpha = 1$

$$\phi_X(\vec{t}) = \exp\left(-\int_{S_n} |\vec{t}\cdot\vec{s}|(1 + i\frac{2}{\pi}\mathrm{sign}(\vec{t}\cdot\vec{s})\log(|\vec{t}\cdot\vec{s}|))\Gamma(\mathrm{d}\vec{s}) + i\vec{t}\cdot\vec{\mu}\right) \tag{1.8}$$

Using this knowledge of the characteristic function we can provide a simple proof that stable distributions with $\alpha \neq 1$ are selfdecomposable. For a proof of the more general statement that all stable distributions are selfdecomposable we refer to [7], page 91.

Proposition 1.14. *Any stable distribution with $\alpha \neq 1$ is selfdecomposable.*

Proof. Let X be a random vector taking values in $\mathbb{R}^n$ with a stable distribution with $\alpha \neq 1$. As being selfdecomposable only depends on the type of X we may assume that $\vec{\mu} = 0$ in (1.6) and (1.7).

We have that

$$\phi_X(\vec{t}) = \exp(-h(\vec{t})) \tag{1.9}$$

where h is in the exponent of (1.6) and (1.7). It is easy to see that

$$\phi_{dX}(t) = \exp(-d^\alpha h(\vec{t})),\ \text{for any } d > 0 \tag{1.10}$$

and from this we get that if X_2 is an independent copy of X, and $d > 1$, then

$$\begin{aligned}\phi_{X+(d^\alpha-1)^{1/\alpha}X_2}(\vec{t}) &= \phi_X(\vec{t})\phi_{(d^\alpha-1)^{1/\alpha}X_2}(\vec{t}) = \exp(-h(\vec{t}))\exp(-(d^\alpha - 1)h(\vec{t})) \\ &= \exp(-d^\alpha h(\vec{t})) = \phi_{dX}(\vec{t})\end{aligned} \tag{1.11}$$

so that $X + (d^\alpha - 1)^{1/\alpha}X_2$ is equal to dX in distribution, thus proving that X is selfdecomposable. $\square$

Let us now define the *Bernoulli distribution* as we will refer to it several times in the text.

Definition 1.15. *A random variable X is said to have a* ***Bernoulli distribution*** *with parameter $p \in [0,1]$ if $P(X=1)=p$ and $P(X=0)=1-p$, and we say that X has a Bern(p) distribution.*

Next we will recall the concept of infinitely divisible distributions.

Definition 1.16. *A random vector X is said to have an* ***infinitely divisible*** *distribution if for all positive $m \in \mathbb{N}$, there exist m independent identically distributed random vectors $Y_1, ..., Y_m$ such that $\sum_{j=1}^{n} Y_j$ has the same distribution as X.*

It is known (see [7], p. 93) that all selfdecomposable distributions are infinitely divisible and that on $\mathbb{R}$ (see [7], p. 177) all selfdecomposable distributions are absolutely continuous as long as they are not degenerate.

The following is a concept which is used to construct examples of darts with desired properties.

Definition 1.17. *Let X and Y be darts taking values in $\mathbb{R}^n$ with laws μ_X and μ_Y. A dart Z taking values in $\mathbb{R}^n$ is a* ***convex combination*** *of X and Y if its law is given as $\mu_Z = p\mu_X + (1-p)\mu_Y$ for some $p \in [0,1]$. If $p \in (0,1)$, then Z is called a* ***nontrivial convex combination*** *of X and Y.*

Let us now recall the concept of convergence in distribution, as we will investigate questions regarding whether sets of darts that are reasonable with respect to some family of payoff functions are closed with respect to convergence in distribution. There exist several common equivalent definitions, and we will state two here which will be useful in the proof of Theorem 5.3.

Definition 1.18. *A sequence of random vectors $\{X_j\}_{j=1}^{\infty}$ taking values in $\mathbb{R}^n$* ***converges in distribution*** *to a random vector X if for any continuous and bounded function $f : \mathbb{R}^n \to \mathbb{R}$ we have that $Ef(X_n) \to Ef(X)$.*

Before stating the second definition, let us recall the concept of a continuity set.

Definition 1.19. *Let X be a random vector in $\mathbb{R}^n$, and let $B \subset \mathbb{R}^n$ be a Borel set. Then B is called a* ***continuity set*** *of X if $P(X \in \partial B) = 0$, where ∂B is the boundary of B.*

Definition 1.20. *A sequence of random vectors $\{X_j\}_{j=1}^{\infty}$ taking values in $\mathbb{R}^n$* ***converges in distribution*** *to a random vector X if for any $A \subset \mathbb{R}^n$, such that A is a continuity set of X, we have that $\lim_j P(X_j \in A) = P(X \in A)$.*

Finally, let us define two function spaces which will be used in some of our theorems. We define $C_c(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$ by

$$\begin{aligned} C_c(\mathbb{R}^n) &:= \Big\{ f : \mathbb{R}^n \to \mathbb{R} |\ f \text{ is continuous and has compact support} \Big\} \\ C_0(\mathbb{R}^n) &:= \Big\{ f : \mathbb{R}^n \to \mathbb{R} |\ f \text{ is continuous and } \lim_{||x|| \to \infty} f(x) = 0 \Big\} \end{aligned} \tag{1.12}$$

Note that $C_c(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$.

2. Darts and payoff functions

In this section we will prove some of our more fundamental results. Let us begin by stating some properties of $g_{X,f}(d)$.

2.1 Basic properties of $g_{X,f}$

Let us first recall the definition of $g_{X,f}$. Let X be a dart taking values in $\mathbb{R}^n$ and let f be a payoff function $f : \mathbb{R}^n \to \mathbb{R}$. Then the function $g_{X,f}$ is defined as

$$g_{X,f}(d) := \sup_{a \in \mathbb{R}^n} Ef(a + dX). \tag{2.1}$$

Proposition 2.1. *Let X and Y be two independent darts taking values in $\mathbb{R}^n$, and let f and h be two payoff functions on $\mathbb{R}^n$. If $a_d, a_p, c_p > 0$, $d_p \in \mathbb{R}$, and $b_d, b_p \in \mathbb{R}^n$, then the following statements hold*

1. $g_{a_d X + b_d, c_p f(a_p x + b_p) + d_p}(d) = c_p g_{X,f}(a_d a_p d) + d_p$
2. $g_{X+Y,f}(d) \le g_{X,f}(d)$
3. $g_{X,f+h}(d) \le g_{X,f}(d) + g_{X,h}(d)$
4. $\inf_x(f(x)) \le g_{X,f}(d) \le \sup_x(f(x))$

Proof. This proof only requires some simple straightforward computations.

1) We compute

$$\begin{aligned} g_{a_d X + b_d, c_p f(a_p x + b_p) + d_p}(d) &= \sup_a E\left[c_p f\left(a_p(a + d(a_d X + b_d)) + b_p\right) + d_p\right] \\ &= c_p(\sup_a E\left[f\left((a_p a + a_p d b_d + b_p) + a_d a_p d X\right)\right]) + d_p \\ &= c_p(\sup_a E\left[f\left(a + a_d a_p d X\right)\right]) + d_p = c_p g_{X,f}(a_d a_p d) + d_p. \end{aligned} \tag{2.2}$$

2) Due to the independence of X and Y we have that for any $a \in \mathbb{R}^n$

$$\begin{aligned} Ef(a + dX + dY) &= \int Ef(a + dy + dX)\mathrm{d}\mu_Y(y) \\ &\le \int \sup_a Ef(a + dX)\mathrm{d}\mu_Y(y) \\ &= \sup_a Ef(a + dX) = g_{X,f}(d) \end{aligned} \tag{2.3}$$

and thus

$$g_{X+Y,f}(d) = \sup_a \mathrm{E}\left(f(a + dX + dY)\right) \le g_{X,f}(d) \tag{2.4}$$

That 3)-4) hold can easily be shown. □

Remark 2.2. *If we let $Y = -X$, then it is easy to see that the independence assumption in Proposition 2.1 is necessary. Let for example X be uniform on $[0,1]$, and let $f(0) = 1$, $f(x) = 0$ when $x \neq 0$. Now $g_{X-X,f}(d) = 1$ but $g_{X,f}(d) = 0$.*

Next we will demonstrate that there exist both reasonable f's and reasonable X's by providing some sufficient conditions for a payoff function or dart to be reasonable.

2.2 Reasonable payoff functions

Here we will prove that there are reasonable payoff functions. First we will show that unimodal payoff functions are reasonable.

Proposition 2.3. *If f is a unimodal payoff function, then f is reasonable.*

Proof. Fix a dart X taking values in $\mathbb{R}^n$. For any $0 < d_1 < d_2$ we have by the unimodality of f that

$$\begin{aligned} g_{X,f}(d_1) &= \sup_a Ef(a + d_1X) = \sup_a Ef\left(\frac{d_1 a}{d_2} + d_1 X\right) \\ &= \sup_a Ef\left(\frac{d_1}{d_2}(a + d_2X)\right) \geq \sup_a Ef(a + d_2X) = g_{X,f}(d_2). \end{aligned} \tag{2.5}$$

□

Remark 2.4. *If $f : \mathbb{R}^n \to \mathbb{R}$ has the property that there exists y in $\mathbb{R}^n$ such that $f(rx+y)$ is decreasing in r, where r in $[0,\infty)$, for all x in $\mathbb{R}^n$, then f is of the same type as a unimodal function, and is thus reasonable.*

Now note that if $f(x) = \arctan(||x||)$, then it is easy to check that for any dart X, $g_{X,f}(d) = \sup_x f(x)$ for all d, so that (X, f) is reasonable. This is a simple example of a second class of reasonable payoff functions, namely the set of payoff functions f such that there are arbitrarily large balls in $\mathbb{R}^n$ where $f(x)$ is arbitrarily close to $\sup_{x\in\mathbb{R}^n} f(x)$. Intuitively, this means that at any distance there is always somewhere you can aim so that the expected value will be arbitrarily close to $\sup_{x\in\mathbb{R}^n} f(x)$.

Open Question 2.5. *Are there reasonable payoff functions other than unimodal functions, and the ones that behave like arctan?*

2.3 Selfdecomposability - a sufficient condition for reasonableness

As was stated earlier, all selfdecomposable darts are reasonable. Some examples of selfdecomposable darts are listed in 2.1.

Let us now prove that all selfdecomposable darts are reasonable.

Theorem 2.6. *If $X|dX$, where $d > 1$, then $g_{X,f}(s) \geq g_{X,f}(ds)$ for all f and for all s. In particular, if X is selfdecomposable, then X is reasonable.*

Table 2.1: Table containing examples of selfdecomposable distributions.

Distribution	stable
normal distribution	yes
Cauchy distribution	yes
Lévy distribution	yes
Landau distribution	yes
Holtsmark distribution	yes
exponential distribution	no
Laplace distribution	no
Gamma distribution	no
Pareto distribution	no
F-distribution	no
log-normal	no
logistic distribution	no

Proof. Fix f and $s > 0$. From $X|dX$ it follows that $sX|dsX$. Choose a random variable Z so that if Z and X are independent, then $sX + Z$ and dsX have the same distribution. By Proposition 2.1 we have that

$$g_{X,f}(s) = g_{sX,f}(1) \geq g_{sX+Z,f}(1) = g_{dsX,f}(1) = g_{X,f}(ds). \tag{2.6}$$

□

Remark 2.7. *The point of $X|dX$ is that you can then simulate being distance ds when you are standing at distance s by randomising your target. Hence you can do at least as well at distance s as at distance ds.*

As all selfdecomposable distributions are infinitely divisible (see [7], p. 93), it is natural to ask whether all darts that have infinitely divisible distributions are reasonable, but this is not the case. It turns out that the Poisson distribution is not reasonable, which will trivially follow from Theorem 4.1.

It is furthermore known that in the one-dimensional case, all selfdecomposable distributions are unimodal (see [7], p. 404), and that as long as they are not a single point mass they are absolutely continuous (see [7], p. 177). With all this in mind it is natural to ask the following questions.

Open Question 2.8. *Is there an example of a reasonable dart which is not selfdecomposable?*

Open Question 2.9. *Is there an example of a reasonable one-dimensional dart which is not unimodal?*

Open Question 2.10. *Is there an example of a reasonable dart which is not infinitely divisible?*

Open Question 2.11. *Is there an example of a reasonable non-degenerate one-dimensional dart which is not absolutely continuous?*

Note that if the answer to Question 2.9, 2.10, or 2.11 is yes, then this would imply that the answer to Question 2.8 is also yes.

3. Trigonometric payoff

It turns out that using trigonometric functions for our payoff functions yields surprisingly many interesting results. In this section we will first demonstrate a connection between $g_{X,\cos(x)}(d)$ and the characteristic function of X, from which many other results follow. Then we will look at functions of the form $e^{cx}\cos(\omega x)$, from which we will construct payoff functions demonstrating that a dart with compact support is non-reasonable. Note that this is not a bounded function if c is non-zero, and so we will need to modify it to make it bounded.

3.1 Cosine payoff

Theorem 3.1. *Let X be any dart taking values in $\mathbb{R}^n$ with characteristic function ϕ_X, and let $f(x) = \cos\left(\sum_{j=1}^n x_j\right)$. Then for any $d > 0$ we have that*

$$Ef(a + dX) = |\phi_X(d\vec{1})| \cos\left(\sum_{j=1}^n a_j + Arg(\phi_X(d\vec{1}))\right), \tag{3.1}$$

where $\vec{1} = (1, 1, ..., 1)$.

In particular this implies that if $f(x) = \cos\left(\sum_{j=1}^n x_j\right)$, then $g_{X,f}(d) = \left|\phi_X(d\vec{1})\right|$, so that (X, f) is reasonable if and only if $\left|\phi_X(d\vec{1})\right|$ is decreasing in d on $(0, \infty)$.

Proof. We have that

$$\begin{aligned} Ef(a + dX) &= E\cos\left(\sum_{j=1}^n a_j + d\vec{1}\cdot X\right) = \mathrm{Re}\left(E\exp\left(i\sum_{j=1}^n a_j + id\vec{1}\cdot X\right)\right) \\ &= \mathrm{Re}\left(e^{i\sum_{j=1}^n a_j} Ee^{id\vec{1}\cdot X}\right) = \mathrm{Re}\left(e^{i\sum_{j=1}^n a_j}\phi_X(d\vec{1})\right) \\ &= \mathrm{Re}\left(e^{i\sum_{j=1}^n a_j}|\phi_X(d\vec{1})|e^{i\mathrm{Arg}(\phi_X(d\vec{1}))}\right) \\ &= |\phi_X(d\vec{1})|\cos\left(\sum_{j=1}^n a_j + \mathrm{Arg}(\phi_X(d\vec{1}))\right) \end{aligned} \tag{3.2}$$

and thus

$$g_{X,f}(d) = |\phi_X(d\vec{1})|. \tag{3.3}$$

□

Remark 3.2. *Note that all functions in the set $\{f : f(x) = \cos\left(\omega\sum_{j=1}^n x_j + \theta\right), \omega > 0, \theta \in [0, 2\pi]\}$ are of the same type, and thus if a dart X is reasonable with respect to any function in this set, it is reasonable with respect to all functions in the set. In particular in one dimension, if $(X, \cos(x))$ is reasonable, then $(X, \sin(x))$ is reasonable.*

Table 3.1: Table containing examples of characteristic functions and their absolute values. All of the associated distributions are non-reasonable with respect to $\cos(x)$.

Distribution	$\phi_X(t)$	$\lvert\phi_X(t)\rvert$
Bernoulli Bern(p)	$1-p+pe^{it}$	$(p^2+(1-p)^2+2p(1-p)\cos(t))^{1/2}$
Binomial B(n, p)	$(1-p+pe^{it})^n$	$(p^2+(1-p)^2+2p(1-p)\cos(t))^{n/2}$
Negative binomial NB(r, p)	$\left(\frac{1-p}{1-pe^{i\,t}}\right)^r$	$\left(\frac{\lvert 1-p\rvert}{\sqrt{1-2p\cos(t)+p^2}}\right)^r$
Poisson Pois(λ)	$e^{\lambda(e^{it}-1)}$	$\mathrm{e}^{\lambda(\cos(t)-1)}$
Uniform U(a, b)	$\frac{e^{itb}-e^{ita}}{it(b-a)}$	$\frac{\sqrt{2(1-\cos(tb-ta))}}{(b-a)\lvert t\rvert}$
Geometric Gt(p)	$\frac{p}{e^{-it}-(1-p)}$	$\frac{p}{\sqrt{1+(p-1)^2+2(p-1)\cos(t)}}$

Theorem 3.1 gives us a powerful tool to study the behaviour of $g_{X,f}(d)$, and we can immediately find several examples of common distributions that are not reasonable with respect to cosine.

Example 3.3. *The following distributions all have characteristic functions ϕ_X such that $|\phi_X(d)|$ is not decreasing for $d \in (0,\infty)$, and thus by Theorem 3.1 are not reasonable with respect to $f(x) = \cos(x)$. Their characteristic functions are listed in Table 3.1.*

- *Bernoulli distribution*
- *Binomial distribution*
- *Negative binomial distribution*
- *Poisson distribution*
- *Uniform distribution*
- *Geometric distibution*

Aside from giving us means to investigate reasonableness, Theorem 3.1 also implicitly tells us which a maximise $E\cos(a + dX)$. Note that the points of maximisation can move around in a discontinuous way, as the following example demonstrates.

Example 3.4. *Let X be a dart taking values in $\mathbb{R}$ such that $P(X = 1) = P(X = -1) = 1/2$. The characteristic function of X is $\phi_X(t) = \cos(t)$, and by Theorem 3.1 we see that X isn't reasonable. Furthermore, by the same theorem, $Ef(a + dX)$ is always optimised at $a = -Arg(\phi_X(d)) + 2k\pi$, $k \in \mathbb{Z}$, and so as d changes, the optimal place to aim switches back and forth between $2k\pi$ and $\pi + 2k\pi$.*

Remark 3.5. *(i) Note that when $d \geq 2$ the set of a's which maximises (3.1) is very large. For any fixed $a_1, ..., a_{n-1}$, there are infinitely many a_n such that $a = (a_1, ..., a_n)$ maximises $Ef(a + dX)$.*
(ii) Theorem 3.1 implies that if $\phi_X(d\vec{1}) = 0$, then it does not matter where we aim when we are at distance d.

We will now give a number of corollaries to Theorem 3.1.

There are two simple ways of combining random variables, adding independent copies, or taking convex combinations. We begin with the latter.

Corollary 3.6. *Let f be the payoff function $f(x) = \cos\left(\sum_{j=1}^{n} x_j\right)$, and let X and Y be darts taking values in $\mathbb{R}^n$ with characteristic functions ϕ_X and ϕ_Y. If ϕ_X and ϕ_Y are real-valued, and both X and Y are reasonable with respect to f, then any convex combination of X and Y is reasonable with respect to f.*

In particular, if $n = 1$, and X and Y are symmetric darts that are reasonable with respect to $\cos(x)$, then so is any convex combination of X and Y.

Proof. By Theorem 3.1, $|\phi_X(d\vec{1})|$ and $|\phi_Y(d\vec{1})|$ are decreasing in $d > 0$. Since they are real-valued, and all characteristic functions are continuous and equal to 1 at the origin, this implies that $\phi_X(d\vec{1}), \phi_Y(d\vec{1}) \geq 0$. Now let Z be a convex combination of X and Y, so that its characteristic function is given as

$$\phi_Z(\vec{t}) = p\phi_X(\vec{t}) + (1-p)\phi_Y(\vec{t}) \tag{3.4}$$

for some $p \in [0, 1]$. As $\phi_X(d\vec{1}), \phi_Y(d\vec{1})$ are decreasing, real-valued, and non-negative, it follows that $|\phi_Z(d\vec{1})| = \phi_Z(d\vec{1})$ is decreasing, and thus Z is reasonable with respect to f by Theorem 3.1. □

This corollary can be used to show that a dart X being reasonable with respect to cosine does not imply that X is reasonable, as the following example demonstrates.

Example 3.7. *Let X be a point mass and Y have a normal distribution, both centred at zero. Then they are both reasonable and symmetric, so by Corollary 3.6 any convex combination of X and Y is reasonable with respect to $\cos(x)$. However, as we will later see, no nontrivial convex combination of X and Y is reasonable, see Theorem 4.1.*

Remark 3.8. *If X and Y from the previous example are instead chosen to have different means, then no nontrivial convex combination of them is reasonable with respect to $\cos(x)$.*

Thanks to the properties of characteristic functions we can show that a sum of independent darts, which are reasonable with respect to cosine, is also reasonable with respect to cosine.

Corollary 3.9. *Let X and Y be two independent darts taking values in $\mathbb{R}^n$ and let $f(x) = \cos\left(\sum_{j=1}^{n} x_j\right)$. If (X, f) and (Y, f) are both reasonable, then $(X + Y, f)$ is also reasonable.*

Proof. The characteristic function $\phi_{X+Y}(d\vec{1})$ can due to independence be calculated as

$$\begin{aligned} \phi_{X+Y}(d\vec{1}) &= \mathrm{E}(e^{id\vec{1}\cdot(X+Y)}) = \mathrm{E}(e^{id\vec{1}\cdot X}e^{id\vec{1}\cdot Y}) \\ &= \mathrm{E}(e^{id\vec{1}\cdot X})\mathrm{E}(e^{id\vec{1}\cdot Y}) = \phi_X(d\vec{1})\phi_Y(d\vec{1}). \end{aligned} \tag{3.5}$$

By Theorem 3.1 a dart Z is reasonable if and only if $|\phi_Z(d\vec{1})|$ is decreasing in d, and thus $(X+Y, f)$ is reasonable. $\square$

Corollary 3.9 actually turns out to be a special case of a more general theorem, see Theorem 5.1.

Remark 3.10. *There exist (non-independent) darts X, Y taking values in $\mathbb{R}$ such that $(X, \cos(x))$ and $(Y, \cos(x))$ are reasonable, but $(X+Y, \cos(x))$ isn't reasonable, see Corollary 3.19.*

Remark 3.11. *There exist darts X, Y taking values in $\mathbb{R}$ such that $(X, \cos(x))$ is reasonable, $(Y, \cos(x))$ isn't reasonable, but an independent sum of X and Y is reasonable with respect to $\cos(x)$, see Proposition 3.18.*

Now we will give a corollary about continuous darts taking values in $\mathbb{R}$. Note that by continuous, we simply mean no atoms; we do not mean absolutely continuous.

Corollary 3.12. *Let X be a continuous dart taking values in $\mathbb{R}$, with a characteristic function ϕ_X. If $\phi_X(t)$ does not go to zero as t goes to infinity, then X isn't reasonable with respect to $\cos(x)$.*

Proof. Assume that X is reasonable with respect to $\cos(x)$. Then by Theorem 3.1 $|\phi_X(t)|$ must be decreasing in t. However, by [1] (Theorem 6.2.5, p. 164) we have that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi_X(t)|^2 \mathrm{d}t = \sum_{x\in\mathbb{R}} \mu_X(\{x\})^2, \tag{3.6}$$

where μ_X is the law of X. As X is continuous and $|\phi_X(t)|$ is decreasing this implies that $\phi_X(t)$ goes to zero, which gives us a contradiction. Thus X is not reasonable with respect to $\cos(x)$. $\square$

Corollary 3.13. *If X is a 1-dimensional dart, whose characteristic function is analytic (for example, if X is compactly supported) and has a zero, then $(X, \cos x)$ is not reasonable.*

Remark 3.14. *Analyticity in Corollary 3.13 is necessary since there is a dart whose characteristic function has a zero but such that $(X, \cos x)$ is reasonable. Namely, it is well known (see [2]) that if X has density function $\frac{1-\cos x}{\pi x^2}$, then its characteristic function is given by the tent function $\max\{1-|t|, 0\}$. By Proposition 3.1, $(X, \cos x)$ is reasonable.*

If we make certain symmetry assumptions on X, we obtain a proposition similar to Theorem 3.1 but with a different payoff function.

Proposition 3.15. *Let X be a dart taking values in $\mathbb{R}^n$ such that its law is invariant under reflections with respect to the coordinate axes, and let $f(x) = \prod_{j=1}^{n} \cos(x_j)$. Then for any $d > 0$, $g_{X,f}(d) = |\phi_X(d\vec{1})|$, where ϕ_X is the characteristic function for X. In particular, (X, f) is reasonable if and only if $|\phi_X(d\vec{1})|$ is decreasing in $d > 0$.*

Proof. Using the symmetry of X we get that

$$\begin{aligned}\phi_X(d\vec{1}) &= \int \cos(d\sum_{n=1}^{n} y_j)\mathrm{d}\mu_X(y) + i\int \sin(d\sum_{n=1}^{n} y_j)\mathrm{d}\mu_X(y) = \int \cos(d\sum_{n=1}^{n} y_j)\mathrm{d}\mu_X(y)\\ &= \int \left(\cos(dy_1)\cos(d\sum_{j=2}^{n} y_j) - \sin(dy_1)\sin(d\sum_{j=2}^{n} y_j)\right)\mathrm{d}\mu_X(y).\end{aligned} \tag{3.7}$$

Note that the term $\sin(dy_1)\sin(d\sum_{j=2}^{n} y_j)$ switches sign under the transform $(y_1, ..., y_n) \to (-y_1, y_2, ..., y_n)$, and combining this with the symmetry properties of X this gives us that the integral over this term is zero. By continuing in this fashion for each variable we get

$$\phi_X(d\vec{1}) = \int \prod_{j=1}^{n} \cos(dy_j)\mathrm{d}\mu_X(y). \tag{3.8}$$

We will now show how this relates to $g_{X,f}(d)$. We have that

$$\begin{aligned}Ef(a+dX) &= \int \prod_{j=1}^{n} \cos(a_j + dy_j)\mathrm{d}\mu_X(y)\\ &= \int \prod_{j=1}^{n} \left(\cos(a_j)\cos(dy_j) - \sin(a_j)\sin(dy_j)\right)\mathrm{d}\mu_X(y).\end{aligned} \tag{3.9}$$

Again using that $\sin(dy_j)$ switches sign under $(y_1, ..., y_j, ..., y_n) \to (y_1, ..., -y_j, ..., y_n)$ we get

$$Ef(a+dX) = \int \prod_{j=1}^{n} \cos(a_j)\cos(dy_j)\mathrm{d}\mu_X(y) = \prod_{j=1}^{n} \cos(a_j) \int \prod_{m=1}^{n} \cos(dy_m)\mathrm{d}\mu_X(y) \tag{3.10}$$

and thus $g_{X,f}(d) = |\phi_X(d\vec{1})|$. □

Remark 3.16. *Note that Theorem 3.1 together with Proposition 3.15 implies that for every dart X whose law is invariant under reflections with respect to the coordinate axes, we have that X is reasonable with respect to $f_1(x) = \cos(\sum_{j=1}^{n} x_j)$ if and only if X is reasonable with respect to $f_2(x) = \prod_{j=1}^{n} \cos(x_j)$.*

Corollary 3.17. *Let X and Y be two independent darts taking values in $\mathbb{R}^n$ such that their laws are invariant under reflections across the coordinate axes and let $f(x) = \prod_{j=1}^{n} \cos(x_j)$. Then if (X, f) and (Y, f) are both reasonable, then $(X+Y, f)$ is also reasonable.*

Proof. The proof of this is essentially the same as the proof of corollary 3.9. □

3.1.1 An example with a phase transition

It turns out that even though a Bernoulli distributed random variable isn't reasonable with respect to $\cos(x)$, an independent sum of a Bernoulli distributed random variable and a normally distributed random variable can in some cases be reasonable with respect to $\cos(x)$. The following exhibits an interesting phase transition.

Theorem 3.18. *Let X_1 be Bern(p) distributed and X_2 be N(0,σ^2) distributed. If they are independent, then $X := X_1 + X_2$ is reasonable with respect to $f(x) = \cos(x)$ if and only if*

$$\sigma^2 d\Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big) + (1-p)p\sin(d) \geq 0, \ \forall d \geq 0. \tag{3.11}$$

When $p = 1/2$, $(X, \cos(x))$ is not reasonable for any σ. But for any $p \neq 1/2$ there exists a $\sigma_p \in (0, \infty)$ such that for all $\sigma \geq \sigma_p$, $(X, \cos(x))$ is reasonable, and for any $\sigma < \sigma_p$ $(X, \cos(x))$ isn't reasonable. In addition, for $p \neq 1/2$, $\sigma_p \leq (1-p)p/(\pi|1-2p|^2)$.

Proof. By Theorem 3.1 X is reasonable with respect to f if and only if $|\phi_X(d)|$ is decreasing in d, $d > 0$. Due to independence, the characteristic function of X is

$$\phi_X(d) = \phi_{X_1}(d)\phi_{X_2}(d) = (1 - p + pe^{id})\exp(-\sigma^2 d^2/2). \tag{3.12}$$

The absolute value of this is decreasing if and only if

$$|\phi_X(d)|^2 = \Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big)\exp(-\sigma^2 d^2) \tag{3.13}$$

is decreasing. This in turn is decreasing if and only if its derivative with respect to d is non-positive on $[0, \infty)$. We have that

$$\frac{\mathrm{d}}{\mathrm{d}d}\Big(|\phi_X(d)|^2\Big) = -2\exp(-\sigma^2 d^2)\Big[\sigma^2 d\Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big) + (1-p)p\sin(d)\Big] \tag{3.14}$$

and thus X is reasonable with respect to f if and only if

$$\sigma^2 d\Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big) + (1-p)p\sin(d) \geq 0, \ \forall d > 0. \tag{3.15}$$

Note that if $p = 1/2$, then it is easy to see that $|\phi_X(\pi + 2m\pi)| = 0$ for all $m \in \mathbb{N}$, but the characteristic function is still not identically zero, and is thus not decreasing. Now assume that $p \neq 1/2$. We have that

$$p^2 + (1-p)^2 + 2(1-p)p\cos(d) = |1 - p + pe^{id}|^2 \geq |1-2p|^2 > 0, \ \forall d. \tag{3.16}$$

Thus

$$\sigma^2 d\Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big) + (1-p)p\sin(d) \geq \sigma^2 d|1-2p|^2 + (1-p)p\sin(d). \tag{3.17}$$

As $\sin(d) \geq 0$ for $d \in [0, \pi]$, it is easy to see that if $\sigma^2\pi|1-2p|^2 \geq (1-p)p$ then

$$\sigma^2 d\Big(p^2 + (1-p)^2 + 2(1-p)p\cos(d)\Big) + (1-p)p\sin(d) \geq 0, \ \forall d \geq 0. \tag{3.18}$$

Thus $p \neq 1/2$ and $\sigma^2 \geq (1-p)p/(\pi|1-2p|^2)$ is a sufficient condition for X to be reasonable with respect to f. ☐

Corollary 3.19. *There exist (dependent) darts X, Y such that $(X, \cos(x))$ and $(Y, \cos(X))$ are reasonable, but $(X + Y, \cos(x))$ is not reasonable.*

Proof. Let X_1 and X_2 be independent random variables such that X_1 is Bern(p) distributed, and X_2 is N(0,σ^2) distributed. By Theorem 3.18 we may assume that σ and p are chosen so that $(X_1 + X_2, \cos(x))$ is reasonable and $p \neq 0, 1$. Now define $X = -X_2$ and $Y = X_1 + X_2$. As X has a normal distribution, it is reasonable.

Thus $(X, \cos(x))$ and $(Y, \cos(x))$ are reasonable, but $X + Y = X_1$, which is not reasonable with respect to $\cos(x)$, since $|\phi_{X_1}(d)|$ is not decreasing as a function of $d > 0$. □

We have seen that an independent sum of a Bernoulli distribution and a normal distribution can in some cases be reasonable with respect to cosine. The following proposition demonstrates that this is not the case for a convex combination.

Proposition 3.20. *Let X be a Bern(p) distributed random variable and let Y be an absolutely continuous random variable. Then no nontrivial convex combination of X and Y is reasonable with respect to* $\cos(x)$*.*

Proof. Let Z be a convex combination of X and Y. We have that the characteristic function of Z satisfies

$$\phi_Z(d) = p\phi_X(d) + (1-p)\phi_Y(d) \tag{3.19}$$

for some $p \in [0, 1]$. By the triangle inequality we have that

$$p|\phi_X(d)| - (1-p)|\phi_Y(d)| \leq |\phi_Z(d)| \leq p|\phi_X(d)| + (1-p)|\phi_Y(d)|. \tag{3.20}$$

As Y is absolutely continuous, its characteristic function is the fourier transform of a function which is L^1 integrable on $\mathbb{R}$, which by the Riemann–Lebesgue lemma implies that $\lim_{k\to\infty} |\phi_Y(d)| = 0$

However, as $|\phi_X(d)|$ is a periodic function oscillating between two different values, this implies that $|\phi_Z(d)|$ isn't decreasing. Thus by Theorem 3.1, $(Z, \cos(x))$ isn't reasonable. □

3.2 Compact darts

It turns out that no non-degenerate dart with compact support is reasonable. In fact, we will see that an even stronger claim holds.

Theorem 3.21. *Let X be a dart taking values in $\mathbb{R}^n$. If there is an axis such that X projected onto that axis is non-degenerate and has compact support, then there exists a continuous payoff function h with compact support such that (X, h) is not reasonable.*

In particular, if X is non-degenerate and has compact support, then there exists a continuous payoff function h with compact support such that (X, h) is non-reasonable.

In order to prove that this theorem holds in more than one dimension, we will first need the following proposition.

Proposition 3.22. *Let X be a dart taking values in $\mathbb{R}^n$, and let Y be a random variable which is the projection of X onto an axis. If there is an f on $\mathbb{R}$ such that (Y, f) is non-reasonable, then there exists an f' on $\mathbb{R}^n$ such that (X, f') is non-reasonable. If f is continuous, then f' can be chosen to be continuous, if f has compact support then f' can be chosen to have compact support, and if f is in $C_c(\mathbb{R})$ then f' can be chosen to be in $C_c(\mathbb{R}^n)$.*

Proof. Let f be a payoff function from $\mathbb{R}$ to $\mathbb{R}$ such that (Y, f) is non-reasonable. Now assume that we chose our basis in $\mathbb{R}^n$ so that $Y = X_1$, and define the payoff function $f' : \mathbb{R}^n \to \mathbb{R}$ by

$$f'(x) = f(x_1). \tag{3.21}$$

It is easy to see that (X, f') is non-reasonable, and that if f is continuous, then so is f'.

Now assume that f has compact support. Note that (Y, f) is reasonable if and only if $(Y, f + a)$ is reasonable, where $a \in \mathbb{R}$, and so we may assume $f \geq 0$. Now define a payoff function $f_B : \mathbb{R}^n \to \mathbb{R}$ by

$$f_B(x) = f(x_1) \prod_{j=2}^{n} h_B(x_j), \tag{3.22}$$

where $B > 0$, and h_B is the continuous function on $\mathbb{R}$ such that $h_B(y) = 1$ for $|y| \leq B$, $h_B(y) = 0$ for $B + 1 \leq |y|$, and in between h_B is a straight line. Note that if f is continuous, then so is f_B, and if f has compact support, then so does f_B.

We now have that for all $a = (a_1, ...a_n)$ in $\mathbb{R}^n$

$$Ef_B(a + dX) \leq Ef(a_1 + dY) \tag{3.23}$$

so that

$$g_{X,f_B}(d) \leq g_{Y,f}(d), \ \forall B, d > 0. \tag{3.24}$$

As (Y, f) is non-reasonable there exists $0 < d_1 < d_2$ such that $g_{Y,f}(d_1) < g_{Y,f}(d_2)$. By the monotone convergence theorem, we have that for any $a \in \mathbb{R}^n, \ d > 0$

$$\lim_{B \to \infty} Ef_B(a + dX) = Ef(a_1 + dY) \tag{3.25}$$

and so for any a such that $Ef(a_1 + d_2 Y) > g_{Y,f}(d_1)$ this implies that for any sufficiently large B

$$Ef_B(a + d_2 X) > g_{Y,f}(d_1). \tag{3.26}$$

Thus $g_{X,f_B}(d_2) > g_{Y,f}(d_1) \geq g_{X,f_B}(d_1)$, and so (X, f_B) is non-reasonable. □

Now let us prove Theorem 3.21.

Proof of Theorem 3.21. By Proposition 3.22 it suffices to show that this theorem holds in one dimension.

By [4] (see Theorem 7.2.3, p. 202) we have that the characteristic function of X, ϕ_X, is an entire function with infinitely many zeros. Let z_0 be any zero of ϕ_X. As the characteristic function is entire, all of its zeros are isolated, and thus there exists a $d_0 > 1$ such that $\phi_X(d_0 z_0) \neq 0$.

Now let $c, \omega \in \mathbb{R}$ be defined so that

$$z_0 = \omega - ic, \tag{3.27}$$

and define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = e^{cx} \cos(\omega x) = \mathrm{Re}\left(e^{(c+i\omega)x}\right) = \mathrm{Re}\left(e^{iz_0 x}\right). \tag{3.28}$$

Note that this is not a payoff function as it is not bounded. For any $d > 0$ and $a \in \mathbb{R}$ we have that

$$Ef(a + dX) = e^{ca}\mathrm{Re}\left(e^{i\omega a} E e^{iz_0 dX}\right) = e^{ca}\mathrm{Re}\left(e^{i\omega a}\phi_X(dz_0)\right) \tag{3.29}$$

and so by taking $d = 1$ we get

$$Ef(a + X) = 0, \ \forall a. \tag{3.30}$$

Furthermore, for $a_0 = -\mathrm{Arg}\left(\phi_X(d_0 z_0)\right)/\omega$ this gives us

$$Ef\left(a_0 + d_0 X\right) = \exp\left(-\frac{c\mathrm{Arg}\left(\phi_X(d_0 z_0)\right)}{\omega}\right)|\phi_X(d_0 z_0)| > 0 \tag{3.31}$$

Since $d_0 > 1$, this gives us the type of non-reasonable behaviour we are after. We now however have to modify f so that it is bounded while maintaining this behaviour.

As X is bounded, there is a $B > 0$ such that $P(d_0|X| \leq B) = 1$. Now let us define the payoff function h by $h(x) = f(x)$ for $|x| \leq |a_0| + B$, $h(x) = -\sup_{|y|\leq|a_0|+10B}(|f(y)|)$ for $|a_0| + 2B \leq |x|$, and for $|a_0| + B \leq |x| \leq |a_0| + 2B$ it is defined as

$$h(x) = \frac{|x| - (|a_0| + B)}{B}\left(-f(x) - \sup_{|y|\leq|a_0|+10B}(|f(y)|)\right) + f(x). \tag{3.32}$$

Note that h is continuous and that

$$\begin{aligned} h(x) &\leq f(x), \ |x| \leq |a_0| + 10B \\ h(x) &\leq 0, \ |a_0| + 2B \leq |x|. \end{aligned} \tag{3.33}$$

To see the first of these inequalities, note that for $|a_0| + B \leq |x| \leq |a_0| + 2B$, $h(x)$ is equal to $f(x)$ plus a non-positive term.

With this definition we will have that

$$Eh(a_0 + d_0 X) = Ef(a_0 + d_0 X) > 0 \tag{3.34}$$

and

$$\begin{aligned} Eh(a + X) &\leq Ef(a + X) = 0, \ 0 \leq |a| \leq |a_0| + 9B \\ Eh(a + X) &\leq 0, \ |a_0| + 9B \leq |a|. \end{aligned} \tag{3.35}$$

Thus $g_{X,h}(1) \leq 0$, $g_{X,h}(d_0) > 0$, so that (X, h) isn't reasonable. Finally, since h is constant outside a finite region, we can clearly add a constant to it so that is has compact support, and this will not affect whether it is reasonable with respect to X. $\square$

Remark 3.23. *Note that in* (3.29)*, if c and $\phi_X(dz_0)$ are non-zero, then one can make $Ef(a + dX)$ arbitrarily large by choosing a appropriately. Thus if we allowed for unbounded payoff functions, we would have that $g_{X,f}(d) \in \{0, \infty\}$ for all $d > 0$.*

Now that we know that compact support implies non-reasonableness, it is interesting to note that this gives us an example of an absolutely continuous dart which is non-reasonable, but which is reasonable with respect to $\cos(x)$.

In [8] it is proven that there exists $f \in C^\infty$ which is real, non-negative, symmetric, supported on $[-1, 1]$, and not identically equal to zero, such that its fourier transform $\hat{f}(d)$ is monotone decreasing for $d \geq 0$ (and hence non-negative). After possibly rescaling, any such f is the probability density function of some absolutely continuous random variable, which by Theorem 3.1 is reasonable with respect to $\cos(x)$, but by Theorem 3.21 isn't reasonable.

4. Continuous payoff functions

This section aims to further characterise reasonableness by providing examples of dart properties which imply that a dart isn't reasonable with respect to the family of continuous payoff functions.

Let X be a dart taking values in $\mathbb{R}^n$ with law μ_X, and characteristic function ϕ_X. If any of the following statements hold, then there is a continuous payoff function f such that (X, f) isn't reasonable.

- $|\phi_X(d\vec{1})|$ is not decreasing as a function of $d \in (0, \infty)$.
- The projection of X onto some axis is non-degenerate and has compact support
- X is non-degenerate and has a point mass.
- μ_X is absolutely continuous, and X isn't reasonable with respect to some (not necessarily continuous) payoff function.
- There exists a payoff function h such that (X, h) isn't reasonable, and h has at most countably many discontinuities.

The first two of these statements come from Theorem 3.1 and 3.21 respectively. The fact that the rest also imply the existence of a continuous f such that (X, f) isn't reasonable will be proven in the rest of this section.

4.1 Non-degenerate darts with point masses

In this section we wish to show that no non-degenerate dart with a point mass is reasonable with respect to the set of continuous payoff functions. The proof of this is rather technical, but if we only wanted to show that such a dart is not reasonable with respect to some (not necessarily continuous) payoff function, then we could provide a much simpler proof. Let us sketch the proof of this simpler case, as the more technical proof will be easier to understand after having seen an easier version with the same basic idea behind it.

Let X be a non-degenerate dart taking values in $\mathbb{R}^n$ with at least one point mass. As being reasonable only depends on type, we may assume that $P(X = x)$ is maximised at $x = 0$. Now define the payoff function

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & 0 < ||x|| < 1 \\ q, & 1 \leq ||x||, \end{cases} \tag{4.1}$$

for some $1 > q > 0$. If q is chosen to be sufficiently small, it is always best for any distance d to aim so that one of the larger point masses can hit 0, i.e. it is optimal to choose an $a \in \mathbb{R}^n$ so that $a + dX$ has one of its largest point masses at 0. It can be shown that if q is chosen appropriately, then $\lim_{d\to 0} g_{X,f}(d) = P(X = 0)$, whereas $\lim_{d\to\infty} g_{X,f}(d) = P(X = 0) + q(1 - P(X = 0))$, which then implies that (X, f) is not reasonable. The intuition behind this is the following. For small values of d, most of the law of dX is focused within a ball with a really small radius, and so if you aim so that one of the larger point masses can hit 0, then almost all of the probability mass of $a + dX$ will be focused in the region where $||x|| < 1$, and so $Ef(a + dX) \approx P(a + dX = 0) \leq P(X = 0)$. But by making d large, it is possible to hit 0 with one of the larger point masses, while having almost no probability mass focused inside the region where $0 < ||x|| < 1$, and so for $a = 0$ we get that $Ef(dX) \approx P(X = 0) + q(1 - P(X = 0))$.

To prove that X is not reasonable with respect to some continuous function we will use a similar payoff function, but there will be some crucial differences. We will still get a situation where $\lim_{d\to\infty} g_{X,f}(d) = P(X = 0) + q(1 - P(X = 0))$, but it can be shown that for any dart X and any continuous payoff function f, $\lim_{d\to 0} g_{X,f}(d) = \sup_x f(x) = \sup_d g_{X,f}(d)$, and so we cannot use exactly the same argument. Instead f will be constructed in such a way so that $g_{X,f}(1) \approx P(X = 0)$ and $\lim_{d\to\infty} g_{X,f}(d) = P(X = 0) + q(1 - P(X = 0))$ for some $q > 0$, and so (X, f) will not be reasonable.

Theorem 4.1. *If X is a non-degenerate dart taking values in $\mathbb{R}^n$ with a point mass, then there exists a continuous payoff function f on $\mathbb{R}^n$ such that (X, f) is not reasonable.*

To prove this theorem we will need the following lemma, which will also be used in the proof of Theorem 4.4.

Lemma 4.2. *If μ is a finite measure on $(\mathbb{R}^n, B(\mathbb{R}^n))$, where $B(\mathbb{R}^n)$ is the Borel σ-algebra on $\mathbb{R}^n$, then for any $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\mu\left(\{y \in \mathbb{R}^n : ||x - y|| < \delta\}\right) < \epsilon + \max_y \mu(\{y\}),\ \forall x \in \mathbb{R}^n. \tag{4.2}$$

Proof. Let us first prove the statement for the case where μ is continuous.

Assume that the statement is false. Then there exists an $\epsilon > 0$ and a sequence $\{x_m\}_{m=1}^{\infty}$ such that

$$\mu\left(\{y \in \mathbb{R}^n : ||x_m - y|| < \frac{1}{m}\}\right) \geq \epsilon,\ \forall m. \tag{4.3}$$

Now note that since $\lim_{m\to\infty} \mu(\mathbb{R}^n \backslash \{y : ||y|| \leq m\}) = 0$ we must have that the sequence $\{x_m\}_{m=1}^{\infty}$ lies in some compact set, and thus there exists a convergent subsequence $\{x_{m_k}\}_{k=1}^{\infty}$. Denote the limit by $x = \lim_k x_{m_k}$. We will now show that $\mu(x) \geq \epsilon$, thus giving us a contradiction, by first showing that for any $\delta > 0$ we have that $\mu\left(\{y \in \mathbb{R}^n : ||x - y|| < \delta\}\right) \geq \epsilon$. Fix $\delta > 0$. As $\lim x_{m_k} = x$, there exists a $K > 0$ such that $||x - x_{m_k}|| < \delta/2$ for all $k \geq K$. Now if we choose $s > 0$ so that $s > K$ and $1/m_s < \delta/2$ we get that $\{y \in \mathbb{R}^n : ||x_{m_s} - y|| < 1/m_s\} \subseteq \{y \in \mathbb{R}^n : ||x - y|| < \delta\}$ so that

$$\mu\left(\{y \in \mathbb{R}^n : ||x - y|| < \delta\}\right) \geq \mu\left(\{y \in \mathbb{R}^n : ||x_{m_s} - y|| < 1/m_s\}\right) \geq \epsilon. \tag{4.4}$$

And since this can be done for any $\delta > 0$, we must have that $\mu(x) \geq \epsilon$, which gives us our contradiction, and so we have proven the lemma for the case where μ is continuous.

It is now trivial to extend this argument from the continuous case to the case where μ has exactly one point mass, and so let us assume that μ has more than one point mass. Fix $\epsilon > 0$, and divide μ into its discrete and continuous parts, μ_d and μ_c. By what we have already shown, there exists a $\delta_1 > 0$ such that

$$\mu_c\left(\{y \in \mathbb{R}^n : ||x-y|| < \delta_1\}\right) < \frac{\epsilon}{2},\ \forall x \in \mathbb{R}^n. \tag{4.5}$$

Now let $\{x_j\}_{j=1}^N$ be points in $\mathbb{R}^n$ corresponding to the N largest point masses of μ, ordered so that $\mu(\{x_1\}) \geq \mu(\{x_j\})$ for all j, with $N > 1$ chosen so that

$$\mu_d(\mathbb{R}^n) - \sum_{j=1}^N \mu_d(\{x_j\}) < \frac{\epsilon}{2}. \tag{4.6}$$

Now for any $\delta_2 > 0$ such that

$$\delta_2 < \min\left(\left\{\frac{||x_i - x_j||}{2} : i, j \in \{1, ..., N\}, i \neq j\right\}\right) \tag{4.7}$$

we have that

$$\mu_d\left(\{y \in \mathbb{R}^n : ||x-y|| < \delta_2\}\right) < \frac{\epsilon}{2} + \mu(\{x_1\}),\ \forall x \in \mathbb{R}^n. \tag{4.8}$$

Thus if we choose $\delta \leq \min(\delta_1, \delta_2)$ we get

$$\mu\left(\{y \in \mathbb{R}^n : ||x-y|| < \delta\}\right) < \epsilon + \max_y \mu(\{y\}),\ \forall x \in \mathbb{R}^n. \tag{4.9}$$

as desired. $\square$

We now give a proof of Theorem 4.1.

Proof of Theorem 4.1. As being reasonable only depends on type, we may assume that $P(X = x)$ is maximised at $x = 0$.

Now let us look at continuous payoff functions of the form

$$f(x) = \begin{cases} (1 - \frac{1}{\delta}||x||),\ ||x|| < \delta \\ 0,\ \delta \leq ||x|| \leq S \\ q(\frac{1}{S}||x|| - 1),\ S \leq ||x|| \leq 2S \\ q,\ 2S \leq ||x|| \end{cases} \tag{4.10}$$

where $\delta, q, S > 0$. Our goal is to show that if we choose these parameters appropriately, then $g_{X,f}(d) - g_{X,f}(1)$ will be positive for all d sufficiently large, and thus (X, f) is non-reasonable. We will choose these parameters in the following order, first q, then δ, and finally S. We begin by choosing

$$q < P(X = 0). \tag{4.11}$$

Now note that

$$Ef(a+X) \leq qP(||a+X|| > S) + P(||a+X|| < \delta) \leq q + P(||a|| - \delta < ||X||) \quad (4.12)$$

and since

$$\lim_{||a|| \to \infty} P(||a|| - \delta < ||X||) = 0 \quad (4.13)$$

this implies that there is a $D > 0$ such that

$$Ef(a+X) < P(X=0), \ \forall \ ||a|| \geq D. \quad (4.14)$$

As $g_{X,f}(1) \geq Ef(X) \geq P(X=0)$, this implies that $g_{X,f}(1) = \sup_{||a||<D} Ef(a+X)$. Thus

$$g_{X,f}(1) \leq \sup_{||a||<D} \left(qP(||a+X|| > S) + P(||a+X|| < \delta)\right). \quad (4.15)$$

By Lemma 4.2 we can choose δ so that

$$P(||a+X|| < \delta) \leq P(X=0) + \frac{q(1-P(X=0))}{4}, \ \forall \ a \in \mathbb{R}^n. \quad (4.16)$$

Furthermore, for any $||a|| < D$ we have that $P(||a+X|| > S) \leq P(||X|| > S-D)$, which goes to zero as S goes to infinity, and so we can then choose S so that

$$P(||a+X|| > S) < \frac{1-P(X=0)}{4}, \ \forall \ ||a|| < D \quad (4.17)$$

and now we get

$$g_{X,f}(1) \leq P(X=0) + \frac{q(1-P(X=0))}{2} \quad (4.18)$$

Now let us look at $g_{X,f}(d)$. We have that

$$\begin{aligned} g_{X,f}(d) &\geq Ef(dX) \geq P(X=0) + qP(||dX|| > 2S) \\ &\geq P(X=0) + qP\left(||X|| > \frac{2S}{d}\right). \end{aligned} \quad (4.19)$$

Thus

$$g_{X,f}(d) - g_{X,f}(1) \geq qP\left(||X|| > \frac{2S}{d}\right) - \frac{q(1-P(X=0))}{2} \quad (4.20)$$

and since

$$\lim_{d\to\infty} qP\left(||X|| > \frac{2S}{d}\right) = q(1 - P(X = 0)) \tag{4.21}$$

and we know that X is non-degenerate, this implies that (X, f) isn't reasonable. $\square$

4.2 Absolutely continuous darts

It turns out that for absolutely continuous darts, being reasonable is equivalent to being reasonable with respect to the set of continuous payoff functions. The proof of this is a bit difficult and somewhat technical, and requires some knowledge of measure theory. The basic idea is the following.

Let X be an absolutely continuous dart which is not reasonable with respect to some payoff function f. If f is vanishing outside of a set of finite Lebesgue measure, then we can use Lusin's Theorem to approximate f with some continuous payoff function in a way which preserves the non-reasonableness. To extend this result to any payoff function f we will also need the Tietze extension theorem.

Theorem 4.3. *Assume that X is a dart taking values in $\mathbb{R}^n$ with an absolutely continuous law μ_X, and f is a payoff function on $\mathbb{R}^n$ such that (X, f) isn't reasonable. Then there exists a continuous payoff function h on $\mathbb{R}^n$ such that (X, h) isn't reasonable. Furthermore, if f vanishes outside of a set of finite Lebesgue measure (but does not necessarily have compact support), then h can be chosen to have compact support.*

Proof. Let us first assume that f vanishes outside of a set of finite Lebesgue measure.

Let m denote Lebesgue measure. By Lusin's Theorem (see [3], p. 217), for any $\epsilon > 0$ there exists a measurable set $A \subseteq \mathbb{R}^n$ and a continuous $h_\epsilon \in C_c(\mathbb{R}^n)$, such that $f = h_\epsilon$ on A, $m(A^c) < \epsilon$, and $\sup_x |h_\epsilon(x)| \leq \sup_x |f(x)|$.

For any $d > 0$ we have that

$$\begin{aligned}|Ef(a + dX) - Eh_\epsilon(a + dX)| &\leq \int_{\{x:a+dx\in A^c\}} |f(a + dx) - h_\epsilon(a + dx)| \mathrm{d}\mu_X(x)\\ &\leq 2\sup_y |f(y)|\mu_X(\{x : a + dx \in A^c\})\end{aligned} \tag{4.22}$$

and thus

$$\begin{aligned}g_{X,h_\epsilon}(d) &\leq \sup_a \left(Ef(a + dX) + 2\sup_y |f(y)|\mu_X(\{x : a + dx \in A^c\})\right)\\ g_{X,h_\epsilon}(d) &\geq \sup_a \left(Ef(a + dX) - 2\sup_y |f(y)|\mu_X(\{x : a + dx \in A^c\})\right)\end{aligned} \tag{4.23}$$

If (X, f) isn't reasonable, then there exist $d_1, d_2 > 0$ such that $d_1 < d_2$ and $g_{X,f}(d_1) < g_{X,f}(d_2)$. As μ_X is an absolutely continuous finite measure there exists a $\delta > 0$ such that for any measurable set E with $m(E) < \delta$, we have that

$$\mu_X(E) < \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{8 \sup_y |f(y)|}. \tag{4.24}$$

Now note that by the properties of Lebesgue measure, $m(\{x : a + dx \in A^c\}) = m((A^c - a)/d) = m(A^c)/d^n < \epsilon/d^n$, and now choose $\epsilon > 0$ so that $\epsilon/d_1^n < \delta$. For all $d \geq d_1$ we now get

$$\begin{aligned} g_{X,h_\epsilon}(d) &\leq \sup_a \left(Ef(a + dX) + 2 \sup_y |f(y)| \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{8 \sup_y |f(y)|} \right) \\ &= g_{X,f}(d) + \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{4} \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} g_{X,h_\epsilon}(d) &\geq \sup_a \left(Ef(a + dX) - 2 \sup_y |f(y)| \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{8 \sup_y |f(y)|} \right) \\ &= g_{X,f}(d) - \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{4} \end{aligned} \tag{4.26}$$

We now get that

$$\begin{aligned} g_{X,h_\epsilon}(d_2) - g_{X,h_\epsilon}(d_1) &\geq g_{X,f}(d_2) - g_{X,f}(d_1) - \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{2} \\ &= \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{2} > 0 \end{aligned} \tag{4.27}$$

and thus (X, h_ϵ) isn't reasonable.

Let us now deal with the case where f isn't vanishing outside of a set of finite Lebesgue measure. For any $\epsilon > 0$ we can chose a sequence of positive numbers $\{r_m\}_{m=1}^\infty$ such that $r_m < 1$ for all m and

$$m(\{y : m < ||y|| < m + r_m\}) < \frac{\epsilon}{2^{m+1}}, \; m \in \mathbb{N}. \tag{4.28}$$

Now define the sets

$$\begin{aligned} C_0 &:= \{y : ||y|| \leq 1\} \\ C_m &:= \{y : m + r_m \leq ||y|| \leq m + 1\}, \; m \in \mathbb{N}. \end{aligned} \tag{4.29}$$

The C_m's cover almost all of space in that

$$m((\bigcup_{m=0}^\infty C_m)^c) = m(\bigcup_{m=1}^\infty \{y : m < ||y|| < m + r_m\}) = \sum_{m=1}^\infty \frac{\epsilon}{2^{m+1}} = \frac{\epsilon}{2}. \tag{4.30}$$

Now we define a sequence of payoff functions $\{f_m\}_{m=0}^\infty$ by

$$f_m(x) := f(x)\chi_{C_m}(x), \; m = 0, 1, 2... \tag{4.31}$$

where χ_{C_m} is the indicator function of C_m.

By Lusin's Theorem we have that for each f_m there exists a continuous function h_m and a measurable set A_m such that $f_m = h_m$ on A_m, $m(A_m^c) < \epsilon/2^{m+2}$, and $\sup_x |h_m(x)| \leq \sup_x |f_m(x)| \leq \sup_x |f(x)|$.

Now define the function $h_\epsilon : \bigcup_{m=0}^\infty C_m \to \mathbb{R}^n$ as $h_\epsilon(x) = \sum_{m=0}^\infty h_m(x)\chi_{C_m}(x)$. Now h_ϵ is a continuous function on the closed set $\bigcup_{m=0}^\infty C_m$, and it satisfies $\sup_x |h_\epsilon(x)| \leq \sup_x |f(x)|$, and so by the Tietze extension theorem there exists a continuous map $h'_\epsilon : \mathbb{R}^n \to \mathbb{R}$ such that $h_\epsilon = h'_\epsilon$ on $\bigcup_{m=0}^\infty C_m$ and $\sup_x |h'_\epsilon(x)| \leq \sup_x |f(x)|$.

We now have that h'_ϵ is equal to f everywhere except on a subset of

$$\left(\bigcup_{m=0}^\infty A_m^c\right) \bigcup \left(\bigcup_{m=0}^\infty C_m\right)^c, \tag{4.32}$$

which has a total Lebesgue measure less than or equal to ϵ. With this we are in the same situation as we were at the beginning of the proof in the case where f was vanishing outside of a set of finite Lebesgue measure, except that the function we have found this time does not necessarily have compact support. As the compact support was not used in any of the remaining steps, the rest of this proof proceeds in the same way as in the first case. □

4.3 Countably discontinuous payoff

In this section we will prove a result which is similar to the result in the previous section, but which does not make any assumptions on the dart. For all darts X, being reasonable with respect to the set of continuous payoff functions is equivalent to being reasonable with respect to the set of payoff functions with at most countably many discontinuities. The proof of this is a bit difficult, and we will again make use of the Tietze extension theorem. The basic idea of the proof is the following.

Given a dart X and a payoff function f with at most a countable number of discontinuities, we can use Lemma 4.2 to show the following. For any $\epsilon > 0$ and $d_0 > 0$, there is an open set A such that f is continuous outside of A, and $\mu_X(\{x : a+dX \in A\}) < \epsilon$ for all $d \geq d_0$ and $a \in \mathbb{R}^n$. Using the Tietze extension theorem we can then find a continuous payoff function h_ϵ which is equal to f on A^c. Now if $g_{X,f}(d)$ is non-decreasing on $[d', \infty)$ for some $d' > 0$, then by setting $d_0 = d'$ and choosing ϵ to be small enough, we will get that $g_{X,h_\epsilon}(d)$ is non-decreasing on $[d', \infty)$.

Theorem 4.4. *Assume that X is a dart taking values in $\mathbb{R}^n$ and f is a payoff function on $\mathbb{R}^n$ such that (X, f) isn't reasonable. Then if f has at most a countable number of discontinuities, then there exists a continuous payoff function h on $\mathbb{R}^n$ such that (X, h) isn't reasonable.*

Proof. X cannot be degenerate, as it would then be reasonable with respect to all functions. If X is non-degenerate and has a point mass, then by Proposition 4.1 there is a continuous payoff function h such that (X, h) isn't reasonable.

Assume now instead that μ_X is continuous (but not necessarily absolutely continuous). Let $\{x_m\}_{m=1}^\infty$ be the discontinuity points of f.

As (X, f) isn't reasonable, there exists $d_1, d_2 \geq 0$ such that $d_1 < d_2$ and $g_{X,f}(d_1) < g_{X,f}(d_2)$. By Lemma 4.2 for any $\epsilon > 0$ there exists a sequence of positive numbers $\{\delta_m\}_{m=1}^{\infty}$ such that for all m we have

$$\mu_X\left(\{y \in \mathbb{R}^n : ||x - y|| < \frac{\delta_m}{d_1}\}\right) < \frac{\epsilon}{2^m},\ \forall x \in \mathbb{R}^n. \tag{4.33}$$

Now define the set

$$A = \bigcup_{m=1}^{\infty} \{z \in \mathbb{R}^n : ||x_m - z|| < \delta_m\}. \tag{4.34}$$

Note that A is an open set which contains all the discontinuities of f. Thus f is continuous on the closed set A^c, and by the Tietze extension theorem there exists a continuous function h_ϵ on $\mathbb{R}^n$ such that h_ϵ is equal to f on A^c and $\sup_x |h_\epsilon(x)| \leq \sup_x |f(x)|$.

Let us now look at

$$\begin{aligned}
|Ef(a + dX) - Eh_\epsilon(a + dX)| &= \int_{\{x:a+dx\in A\}} |f(a + dx) - h_\epsilon(a + dx)| \mathrm{d}\mu_X(x) \\
&\leq 2 \sup_y |f(y)| \mu_X(\{x : a + dx \in A\}) \\
&\leq 2 \sup_y |f(y)| \sum_{m=1}^{\infty} \mu_X(\{x : a + dx \in \{z \in \mathbb{R}^n : ||x_m - z|| < \delta_m\}\}) \\
&\leq 2 \sup_y |f(y)| \sum_{m=1}^{\infty} \mu_X\left(\{x \in \mathbb{R}^n : ||\frac{x_m - a}{d} - x|| < \frac{\delta_m}{d}\}\right).
\end{aligned}$$

Equation (4.33) now gives us that for all $d \geq d_1$

$$|Ef(a + dX) - Eh_\epsilon(a + dX)| \leq 2 \sup_y |f(y)| \epsilon. \tag{4.35}$$

From this we get that for all $d \geq d_1$

$$\begin{aligned}
g_{X,h_\epsilon}(d) &\leq g_{X,f}(d) + 2 \sup_y |f(y)| \epsilon \\
g_{X,h_\epsilon}(d) &\geq g_{X,f}(d) - 2 \sup_y |f(y)| \epsilon
\end{aligned} \tag{4.36}$$

which implies

$$g_{X,h_\epsilon}(d_2) - g_{X,h_\epsilon}(d_1) \geq g_{X,f}(d_2) - g_{X,f}(d_1) - 4 \sup_y |f(y)| \epsilon. \tag{4.37}$$

Thus if we choose

$$\epsilon < \frac{g_{X,f}(d_2) - g_{X,f}(d_1)}{4 \sup_y |f(y)|} \tag{4.38}$$

then we see that (X, h_ϵ) is not reasonable. $\square$

Open Question 4.5. *Is there a dart which is reasonable w.r.t. all continuous payoff functions, but not all payoff functions?*

5. Closure properties

It is interesting to ask whether the sets of reasonable/non-reasonable darts/payoff functions are closed under various operations and limiting procedures. In this section we demonstrate two results of this nature.

5.1 Independent sum of reasonable darts

Let $\mathcal{F}$ be some set of payoff functions, and let $\mathcal{X}_{\mathcal{F}}$ be the set of darts which are reasonable with respect to $\mathcal{F}$. It is natural to ask whether $X + Y \in \mathcal{X}_{\mathcal{F}}$ for any independent $X, Y \in \mathcal{X}_{\mathcal{F}}$, and we will see that this at least holds for certain specific $\mathcal{F}$.

The proof of this uses the fact that for some sets of payoff functions $\mathcal{F}$ we have that

$$\left\{h : \mathbb{R}^n \to \mathbb{R} \mid h(x) = Ef(x + \sum_{i=1}^{n} d_i X_i),\ d_i \geq 0\ \forall i, \quad f \in \mathcal{F}\right\} \subseteq \mathcal{F}, \tag{5.1}$$

for any independent darts $X_1, ..., X_m$. As this is not true for all $\mathcal{F}$, this proof can only be used for certain specific sets of payoff functions, for example the set of all payoff functions. As a result of this, any independent sum of reasonable darts is reasonable.

Theorem 5.1. *Assume that $X_1, ..., X_m$ are independent darts taking values in $\mathbb{R}^n$, which are all reasonable with respect to a family of payoff functions $\mathcal{F}$. If $\mathcal{F}$ is either of the following sets*

1. *The set of all payoff functions*
2. *The set of all continuous payoff functions*
3. *The set of all payoff functions of the same type as $\cos(\sum_{j=1}^{n} x_j)$,*

then for any $d_1, ..., d_m, D_1, ..., D_m \geq 0$ such that $d_j \leq D_j$ for all j we have that

$$\sup_a Ef(a + \sum_{j=1}^{m} d_j X_j) \geq \sup_a Ef(a + \sum_{j=1}^{m} D_j X_j),\ \forall f \in \mathcal{F} \tag{5.2}$$

and in particular $\sum_{j=1}^{m} X_j$ is reasonable with respect to $\mathcal{F}$.

Proof. We begin by showing that this holds for $m = 2$. Let X, Y be independent darts which are reasonable with respect to one of the three $\mathcal{F}$'s above. Fix an $f \in \mathcal{F}$, and choose $d_1, d_2, D_1, D_2 \geq 0$ such that $d_1 \leq D_1, d_2 \leq D_2$ and define the function

$$h(x) = Ef(x + d_1 X). \tag{5.3}$$

It can be shown that $h \in \mathcal{F}$, so that we have that Y is reasonable with respect to h, and thus

$$\sup_a Eh(a + d_2 Y) \geq \sup_a Eh(a + D_2 Y) \tag{5.4}$$

and from this we get, using the independence of X and Y, that

$$\begin{aligned}\sup_a Ef(a+d_1X+d_2Y) &= \sup_a \int Ef(a+d_1X+d_2y)\mathrm{d}\mu_Y(y)\\ &= \sup_a \int h(a+d_2y)\mathrm{d}\mu_Y(y) = \sup_a Eh(a+d_2Y)\\ &\geq \sup_a Eh(a+D_2Y) = \sup_a \int h(a+D_2y)\mathrm{d}\mu_Y(y) \qquad (5.5)\\ &= \sup_a \int Ef(a+d_1X+D_2y)\mathrm{d}\mu_Y(y)\\ &= \sup_a Ef(a+d_1X+D_2Y).\end{aligned}$$

And by using the same argument again it follows that

$$\sup_a Ef(a+d_1X+d_2Y) \geq \sup_a Ef(a+D_1X+D_2Y) \tag{5.6}$$

as desired. This clearly implies that $X+Y$ is reasonable with respect to $\mathcal{F}$. The more general statement follows by induction. ☐

Open Question 5.2. *Are there independent darts X, Y and a payoff function f, such that (X, f) and (Y, f) are both reasonable, but $(X+Y, f)$ is not reasonable?*

5.2 Convergence in distribution

It turns out that the set of darts which are reasonable with respect to $C_0(\mathbb{R}^n)$ is closed with respect to convergence in distribution. The proof of this is a bit difficult, and uses two equivalent definitions of convergence in distribution (see Section 1.2).

Theorem 5.3. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of darts taking values in $\mathbb{R}^n$ which converges in distribution to some dart X. Then for any $f \in C_0(\mathbb{R}^n)$, $d > 0$, $\lim_j g_{X_j,f}(d) = g_{X,f}(d)$. As a consequence, for any $f \in C_0(\mathbb{R}^n)$ for which (X_j, f) is reasonable for all j, we have that (X, f) is reasonable.*

Proof. Let $f \in C_0(\mathbb{R}^n)$ and fix $d > 0$. We will begin by showing that $\liminf_{j\to\infty} g_{X_j,f}(d) \geq g_{X,f}(d)$. Fix a. Then

$$\liminf_{j\to\infty} g_{X_j,f}(d) \geq \liminf_{j\to\infty} Ef(a+dX_j) = Ef(a+dX) \tag{5.7}$$

Since this is true for every a, we have

$$\liminf_{j\to\infty} g_{X_j,f}(d) \geq g_{X,f}(d) \tag{5.8}$$

Now assume that $g_{X_j,f}(d)$ does not go to $g_{X,f}(d)$. Then there must exist some $\epsilon > 0$ and some subsequence $g_{X_{j_k},f}(d)$ such that

$$g_{X_{j_k},f}(d) - g_{X,f}(d) > \epsilon \text{ for all } k. \tag{5.9}$$

As $\{g_{X_{j_k},f}(d)\}_{j=1}^{\infty}$ is bounded, we may furthermore assume that our subsequence was chosen so that it has a limit.

As $f \in C_0(\mathbb{R}^n)$ we have that for any $\epsilon' > 0$ there is a D such that $|f(x)| < \epsilon'$ when $||x|| > D$, and so for any dart Z we have that

$$\begin{aligned} |Ef(a+dZ)| &\leq \epsilon' P(||a+dZ|| > D) + \sup_x |f(x)| P(||a+dZ|| \leq D) \\ &\leq \epsilon' + \sup_x |f(x)| P\left(\frac{||a|| - D}{d} \leq ||Z||\right). \end{aligned} \tag{5.10}$$

As the final term goes to zero when $||a|| \to \infty$, this implies that $g_{Z,f}(d) \geq 0$, and that if $g_{Z,f}(d) > 0$, then there must exist a bounded sequence $\{a_{Z,j}\}$ such that $\lim_j Ef(a_{Z,j} + dZ) = g_{Z,f}(d)$. For any subsequence converging to some $a_{Z,\infty}$, the fact that $f \in C_0(\mathbb{R}^n)$ implies that $Ef(a_{Z,\infty} + dZ) = g_{Z,f}(d)$. Hence the optimum is always achieved.

Thus by (5.9), $g_{X_{j_k},f}(d) \geq \epsilon$ for all k, and there exists a sequence $\{a_{j_k}\}_{k=1}^{\infty}$ such that $g_{X_{j_k},f}(d) = Ef(a_{j_k} + dX_{j_k})$ for all k. We will now show that $\{a_{j_k}\}_{k=1}^{\infty}$ must be bounded. Assume that $\{a_{j_k}\}_{k=1}^{\infty}$ isn't bounded, and choose a subsequence $\{a_{j_{k_t}}\}_{t=1}^{\infty}$ such that $\lim_t ||a_{j_{k_t}}|| = \infty$. For any $D' > 0$ such that $|f(x)| < \epsilon/4$ for all $||x|| > D'$ we have that

$$Ef(a_{j_{k_t}} + dX_{j_{k_t}}) \leq \frac{\epsilon}{4} + \sup_x |f(x)| P\left(\frac{||a_{j_{k_t}}|| - D'}{d} \leq ||X_{j_{k_t}}||\right). \tag{5.11}$$

Now choose $D'' > 0$ so that

$$P\left(\frac{D'' - D'}{d} \leq ||X||\right) < \frac{\epsilon}{4\sup_x |f(x)|} \tag{5.12}$$

and so that $\{x : \frac{D''-D'}{d} \leq ||x||\}$ is a continuity set for the law of X. As $\lim_t ||a_{j_{k_t}}|| = \infty$, for any sufficiently large t we have that

$$Ef(a_{j_{k_t}} + dX_{j_{k_t}}) \leq \frac{\epsilon}{4} + \sup_x |f(x)| P\left(\frac{D'' - D'}{d} \leq ||X_{j_{k_t}}||\right). \tag{5.13}$$

As the right hand side converges to something less than or equal to $\epsilon/2$, this gives us a contradiction, as $Ef(a_{j_{k_t}} + dX_{j_{k_t}}) = g_{X_{j_{k_t}},f}(d) \geq \epsilon$ for all t. Thus $\{a_{j_k}\}_{k=1}^{\infty}$ must be bounded, and so there exists a subsequence $\{a_{j_{k_t}}\}_{t=1}^{\infty}$ which converges to some $a_\infty \in \mathbb{R}^n$.

Since f is in $C_0(\mathbb{R}^n)$, it is uniformly continuous, so there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all $||x - y|| < \delta$. As $a_{j_{k_t}} \to a_\infty$, there is a $T > 0$ such that $||a_{j_{k_t}} - a_\infty|| < \delta$ for all $t > T$, and so we have that for any $t > T$

$$g_{X_{j_{k_t}},f}(d) = Ef(a_{j_{k_t}} + dX_{j_{k_t}}) \leq Ef(a_\infty + dX_{j_{k_t}}) + \frac{\epsilon}{2}. \tag{5.14}$$

Taking the limit now gives us

$$\lim_{t\to\infty} g_{X_{j_{k_t}},f}(d) \leq \lim_{t\to\infty} Ef(a_\infty + dX_{j_{k_t}}) + \frac{\epsilon}{2} = Ef(a_\infty + dX) + \frac{\epsilon}{2} \leq g_{X,f}(d) + \frac{\epsilon}{2}. \tag{5.15}$$

which contradicts (5.9). Thus $g_{X_j,f}(d)$ does converge to $g_{X,f}(d)$.

If (X_j, f) is reasonable for all j, then $g_{X,f}$ is the limit of a sequence of decreasing functions, and is therefore also decreasing, which implies that (X, f) is reasonable. $\square$

6. Discussion

We have found many results which partially characterise reasonableness, but there is still much which is unknown. Here we will try to summarise and discuss some interesting remaining open questions.

One of our main results is that all selfdecomposable darts are reasonable, and so it is natural to ask whether there exists a reasonable dart which is not selfdecomposable. In one dimension it is known that, except in the degenerate case, a selfdecomposable dart is unimodal and absolutely continuous. We are therefore interested in looking for reasonable darts among the random variables which do not exhibit both of these properties.

We know that a non-degenerate reasonable dart cannot have point masses, but we still do not know whether it can have a singular component. In one dimension, we know that a reasonable continuous dart must have a characteristic function $\phi_X(t)$ which goes to zero as t goes to infinity (see Corollary 3.12), which rules out many singular measures. However, there do exist singular random variables whose characteristic function goes to zero, see [5], and so it is feasible that a dart with such a component may be reasonable.

As for unimodalness, we do not yet know whether it is a necessary condition for a dart to be reasonable, but it seems likely that it is, especially for absolutely continuous darts. Let f_X be a bounded probability density function of some one-dimensional dart X, and study whether (X, f_X) is reasonable. For distance 1 it can be shown that $g_{X,f_X}(1)$ is equal to the L^2 norm of f_X, but for $1 \approx d < 1$ it seems plausible that f_X and f_{dX} will not match as well as for distance 1, so that $g_{X,f_X}(d) < g_{X,f_X}(1)$. Whether this argument holds, in general or in certain specific situations, is yet to be investigated, but it does provide us with some intuition on this subject.

As for payoff functions, we know that unimodalness is a sufficient but not a necessary condition for reasonableness. It might however still be the case that given certain conditions on the payoff function, unimodalness is necessary. The examples we know of which are reasonable but not unimodal all have the property that one can make $Ef(a + dX)$ arbitrarily close to $\sup_x f(x)$ for any d by aiming arbitrarily far away, and so it might be worth further investigating what happens when this is not the case. For example, we could look at continuous payoff functions f for which there is a $D > 0$ such that $\sup_{||x||>D} f(x) < \sup_x f(x)$. It is conceivable for such functions that unimodalness is a necessary condition for reasonableness. If we restrict ourselves even further to payoff functions f which are of the same type as some probability density function f_X, then we can again study whether (X, f_X) is reasonable, and from this try to gain insight regarding f.

We have characterised reasonableness with respect to $\cos(x)$ in terms of the absolute value of the characteristic function. For this reason it is interesting to study exactly under what conditions the absolute value of the characteristic function is or isn't decreasing on the positive part of $\mathbb{R}$. As the concept of characteristic functions is very well studied, it seems likely that there exist more known results that are relevant which we have not yet found.

One result which we expect holds, but have yet not shown, is the following. In one dimension, is a dart with more than one point mass not reasonable with respect to $\cos(x)$? The intuition behind this is as follows. We divide the law of the dart into a discrete and a continuous part, and then by (3.6), the contribution to the characteristic function from the continuous part should not matter "on average" for large distances, whereas we expect the contribution from the discrete part to not converge to anything. For certain arrangements of the point masses, the contribution from the discrete part will be periodic, which simplifies the analysis, but this does not hold in general.

We have showed that non-degenerate compact darts are non-reasonable. A related result which we tried to show is that any dart such that its characteristic function is entire and has a zero is non-reasonable. The proof of Theorem 3.21 demonstrates that this is the case if we allow for unbounded payoff functions. For the case where the dart is compact we managed to modify the function $f(x) = e^{cx}\cos(\omega x)$ to be bounded, while maintaining the non-reasonableness, but we don't know if this is possibly to do in general.

We have found that if (X, f) is non-reasonable, and it is the case that X is absolutely continuous, or that f has at most a countable number of discontinuities, then there is a continuous h such that (X, h) is non-reasonable. It is now interesting to ask whether there exists a dart X which is non-reasonable, but which is reasonable with respect to all continuous payoff functions. Even if there are such examples, it might be difficult to find one. By Theorem 4.1, 4.3, and 4.4, X would have to be continuous but not absolutely continuous, be reasonable with respect to all payoff functions with at most countably many discontinuities, but still not be reasonable with respect to some payoff function.

As for closure properties, we have found some interesting results, but much remains yet unknown. One of the more interesting questions of this nature is whether there exist independent darts X, Y and a payoff function f such that (X, f) and (Y, f) is reasonable, but $(X + Y, f)$ is not reasonable.

We have demonstrated that if $f \in C_0(\mathbb{R}^n)$, $\{X_j\}$ converges in distribution to some X, and (X_j, f) is reasonable for all j, then (X, f) is also reasonable. It would be interesting to see if this holds for some larger set than $C_0(\mathbb{R}^n)$, for example if it holds for any continuous payoff function. It would also be interesting to find examples where it does not hold.

In summary, we have found many different results related to the concept of reasonableness, but there are still a great number of open questions remaining.

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