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The low-lying zeros of L-functions associated to non-Galois cubic fields

Master's thesis in Mathematics

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Abstract

We study the low-lying zeros of Artin L-functions associated to non-Galois cubic number fields through their one- and two-level densities. In particular, we find new precise estimates for the two-level density with a power-saving error term. We apply the L-functions Ratios Conjecture to study these densities for a larger class of test functions than unconditional computations allow. By reviewing a known Ratios Conjecture prediction, due to Cho, Fiorilli, Lee, and Södergren, for the one-level density, we isolate a phase transition in the lower-order terms, which reveals a striking symmetry. Our computations show that the same symmetry exists in the one-level density of several other families, that have previously been studied in the literature, and this motivates us to formulate a conjecture extending one part of the Katz–Sarnak prediction for families of symplectic symmetry type. Moreover, we isolate several phase transitions in the lower-order terms of the two-level density. To the best of our knowledge, this is the first time such phase transitions have been observed in any n-level density with $n \geq 2$.

Keywords: Mathematics, number theory, cubic fields, L-functions, low-lying zeros, one-level density, two-level density, phase transition.

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Victor Ahlquist, Gothenburg, February 2023

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1

Introduction

We begin by introducing the study of L-functions, with a particular focus on their low-lying zeros. Next, we describe the outline of the report and present our results. The chapter ends with a brief description of notation that we will be using in the sequel.

1.1 Background

An L-function is a meromorphic function associated to a mathematical object. More precisely, given complex numbers a_n related to the object of interest, we may define a corresponding generating function as the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s \in \mathbb{C},$$

as long as the series converges. Series of this form are known as Dirichlet series, and all L-functions take the form of such a series, at least when the real part of s is greater than 1. Aside from being given by a Dirichlet series, all L-functions have a meromorphic continuation to the entire complex plane, along with a functional equation. The functional equation relates the value of the L-function at a point s to the value of the dual L-function at the point $1-s$. Here, the dual L-function is simply the function obtained by taking the complex conjugate of the coefficients a_n in the Dirichlet series above. In particular, the self-dual L-functions are precisely those with real coefficients a_n.

One of the most well-known L-functions is the Riemann zeta function ζ, associated to the integers and initially defined by the Dirichlet series where one takes all a_n to equal 1. From classical analytic number theory, we know that the zeros of ζ encode information about the distribution of the primes. To make this more precise, we define a function $\pi(x)$, for real $x \geq 2$, counting the number of primes less than x. Then by studying the zeros of ζ, one can prove the Prime Number Theorem, i.e. that

$$\pi(x) = \int_2^x \frac{1}{\log t} dt + \mathcal{O}\left(x \exp(-c(\log x)^{1/2})\right),$$

for a specific constant $c > 0$, see e.g. [D3, Ch. 18]. Moreover, the value of the integral above is approximately equal to $x/(\log x)$, where $\log x$ denotes the natural logarithm of x.

The celebrated Riemann Hypothesis asserts that all the so-called nontrivial zeros of the Riemann zeta function lie on the critical line, defined as the set $\{s \in \mathbb{C} : \mathrm{Re}(s) = 1/2\}$. The Riemann Hypothesis has several important implications. In particular, it implies that the error term in the Prime Number Theorem can be taken as $\mathcal{O}\big(\sqrt{x}\log x\big)$, which is a significant improvement compared to the error term above.

The success of using the zeros of ζ to study primes indicates that one may study an object by studying the zeros of its associated L-function. We remark that all L-functions have so-called trivial zeros, which are very easy to locate. The Generalised Riemann Hypothesis (GRH) asserts that the nontrivial zeros of any L-function all lie on the critical line. If one believes in the GRH, then the point $s = 1/2$ seems to be special, in a sense being the midpoint of the critical line with respect to the transformation $s \mapsto 1-s$. Even without assuming the GRH, the point $1/2$ is the only fixed point with respect to $s \mapsto 1-s$, which indicates that it may have important properties. Indeed, The Birch and Swinnerton-Dyer Conjecture claims that this point is of special importance, at least for certain L-functions. Specifically, the conjecture asserts that if one considers the L-function associated to an elliptic curve, then the multiplicity of a potential zero at the point $s = 1/2$ is equal to the rank of the group of rational points on the elliptic curve.

Instead of directly studying zeros of L-functions at $s = 1/2$, it may be easier to study the zeros lying close to this point. Such zeros are known as low-lying zeros. Furthermore, instead of studying the zeros of a single L-function, one can study the zeros of an entire collection, or family, of L-functions. There are many examples of families of L-functions, e.g. L-functions associated to certain elliptic curves, modular forms, or number fields. The distribution of the low-lying zeros of such families of L-functions can be analysed by studying a certain average, known as the n-level density, where $n \geq 1$ is an integer.

A conjecture of Katz and Sarnak asserts that every natural family is associated to one of five symmetry types, which determines the main term of the n-level density for any n. The five different symmetry types are unitary, symplectic, and one of three different orthogonal types. The terminology comes from random matrix theory, where the same main terms arise if one studies the eigenvalues close to 1 of large random matrices. We remark that this conjecture has been partially confirmed in several different families, see e.g. [CK], [HR], [ILS], [ÖS], [Rb], [Ya], [Yo].

We will be interested in studying the n-level densities of the family $\mathcal{F}$ of L-functions associated to non-Galois cubic number fields, for $n = 1, 2$. The one-level density for this family was first studied in [Ya], who showed that the main term is the one given by the symplectic symmetry type. The main terms of the n-level densities for $n \geq 2$ were found in [CK], and as expected they were also of symplectic type. In [CFLS, Ch. 3], the one-level density was evaluated precisely, taking into account not only the main term but also several lower-order terms. We remark that in all of these cases, the n-level density was found by relating it to certain prime sums. Unfortunately, this method leads to an error term that can become quite large.

The Ratios Conjecture gives a method for evaluating certain averages of ratios of L-functions. We describe this conjecture more closely in Chapter 6. By using this conjecture to compute certain integrals, one can evaluate the n-level density by relating it to new prime sums that are more well-behaved compared to the prime sums that arise in the unconditional calculations. In particular, one obtains an error term of substantially better quality. This was done for the one-level density, once again taking into account many lower-order terms, in [CFLS, Ch. 4-5]. Aside from its application to computing the n-level density, the Ratios Conjecture has been successfully applied to finding moments of zeta functions, and it can also be used to study the Montgomery pair-correlation conjecture, see e.g. [CS].

The key to studying the n-level densities of our family $\mathcal{F}$, is a collection of precise estimates for the number of isomorphism classes of certain cubic fields, with discriminant bounded by a given magnitude. The main terms in these estimates were originally found in [DH], by relating cubic fields to certain binary cubic forms. Through recent breakthroughs, secondary terms necessary for the precise calculations in [CFLS] have been found in [BST] and [TT] independently. These proofs combine analytic and algebraic methods, and all begin by counting the corresponding cubic forms. Work has been done to improve the error terms in the estimates and the current best result is due to [BTT]. In the opposite direction, lower bounds for the error terms have been found in [CFLS] by studying the one-level density of the family $\mathcal{F}$, conditional on the GRH. This demonstrates how the zeros of L-functions provide information about the corresponding mathematical object, not only in the classical case of the Riemann zeta function.

1.2 Results and outline of the report

In Chapter 2, we begin by giving a more technical introduction to the subject, as well as describing useful properties of the L-functions we will be studying. The actual computations begin in Chapter 3, where we study the one-level density. The approach and all results here are taken from [CFLS, Ch. 3]. Moving on to Chapter 4, we obtain our first new result by extending the approach of the previous chapter to find an expression for the two-level density containing lower-order terms. The result is given in Theorem 4.1, and this is the first time an expression containing the lower-order terms has been found for the two-level density of this family.

We take a brief pause from the study of low-lying zeros in Chapter 5 to sketch the proof from [BST] of the main term in the counting function for cubic fields. Compared to the previous sections, the focus here is not to keep track of error terms, but rather to indicate the methods used for studying this counting function, as well as explain the shape of the main term. We remark that this chapter requires more algebraic knowledge compared to the other sections, where complex analysis is our main tool.

The rest of the report is concerned with applying the Ratios Conjecture. We begin in Chapter 6 by using it to find the one-level density. As in the unconditional calculations, the approach and all results are from [CFLS]. In Chapter 7, we study the expression for the prediction of the one-level density obtained in the previous chapter. First, we follow [CFLS, Ch. 5] and compare this expression with the one obtained in Chapter 3. Next, by studying the Ratios Conjecture prediction more closely, we obtain new results. In particular, we find a so-called phase transition in the secondary term, and a curious symmetry, see the remarks following Theorem 7.3. Phase transitions have been found in other families, see e.g. [FPS2], [Rk], [Wa], but the symmetry that we observe seems to have gone unnoticed in the literature. Together with a brief analysis of the one-level density of several other families, this symmetry leads us to formulate Conjecture 7.4 for a general symplectic family.

The last two chapters of the report contain our main contributions to the study of the two-level density. First, in Chapter 8, we apply the Ratios Conjecture to evaluate the two-level density and find an expression with a power-saving error term, given in Proposition 8.4. Next, in Chapter 9, we compare the expression we found conditional on the Ratios Conjecture with the unconditional expression from Chapter 4, and find that they agree quite well, see Proposition 9.1. We also explicitly find the second-order term of the Ratios Conjecture prediction and uncover several phase transitions in the lower-order terms of the two-level density, conditional on the Ratios Conjecture, see Theorem 9.3. To the best of our knowledge, this is the first time a phase transition has been observed in the lower-order terms of any n-level density with $n \geq 2$.

1.3 Notation

We briefly describe some of the notation that will be used throughout the report. First, as is usual in number theory, the letters p and q always denote primes. Sums, with index p or q, should be interpreted as sums over the set of primes with the natural ordering. Usually, these sums will be absolutely convergent, so that the ordering does not matter. Sometimes we will write the condition $p^e \mid\mid m$, for some integer m, in our sums or products. This means that we only consider the $e \geq 0$ such that p^e is the largest power of p dividing m.

We will sometimes want to study arithmetic functions, i.e. functions $f : \mathbb{Z}_+ \to \mathbb{C}$. The multiplicative arithmetic functions will be of special interest, which are the functions with $f(1) = 1$, and where $f(nm) = f(n)f(m)$ for relatively prime integers n and m. Note that a multiplicative function is completely determined by its values at prime powers.

Let $\log N$ denote the natural logarithm of the number N. Then, Euler's constant γ is defined by the limit

$$\gamma := \lim_{N\to\infty} \left(\sum_{n \leq N} \frac{1}{n} - \log N \right),$$

and we will see that this constant naturally arises from certain prime sums, and also when evaluating expressions related to the gamma function.

If $f : \mathbb{R} \to \mathbb{C}$ is an absolutely integrable function, say, then we define its Fourier transform by

$$\widehat{f}(u) := \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx.$$

The inverse Fourier transform is defined similarly, but with the sign of the exponent reversed. In particular, for even functions f, the inverse Fourier transform of f is the same as the Fourier transform of f.

When stating our results, or performing calculations, we will make use of both the big-$\mathcal{O}$ notation and Vinogradov's notation. We remind the reader that for a complex-valued function f and a nonnegative function g, we write

$$f(x) = \mathcal{O}(g(x)),$$

if there exists a constant C such that $|f(x)| \leq Cg(x)$ for all x in a set S. The constant C is often referred to as the "implied constant". The set S is usually not specified, but if x is a real variable, then S can usually be taken to be the set of all sufficiently large real numbers, often the set of all $x \geq 2$. If x is a complex variable, then S will usually be specified more explicitly. Furthermore, we write

$$f(x) = h(x) + \mathcal{O}(g(x)), \text{ when } f(x) - h(x) = \mathcal{O}(g(x)).$$

Sometimes it will be more convenient to use Vinogradov's notation and write $f(x) \ll g(x)$ when $f(x) = \mathcal{O}(g(x))$. If the implied constant depends on a variable, say ϵ, then this variable will be indicated by a subscript, i.e. as $\mathcal{O}_\epsilon$ or $\ll_\epsilon$.

Finally, we will make occasional use of little-o notation. If $g(x) > 0$ is a positive function, then by $f(x) = o(g(x))$, we mean that the quotient $f(x)/g(x)$ converges to 0 as x tends to some limit a. The limit a can usually be inferred from the context and is often equal to either 0 or ∞.

2

Preliminaries

We introduce the Dedekind zeta function, and the Artin L-function of a cubic field. The latter function will be our main object of study throughout the report. Next, we give a more technical presentation of the low-lying zeros of L-functions than in the introduction, and of the Katz-Sarnak prediction. Lastly, we define the one-level density, which will be studied in the next chapter.

2.1 The Dedekind zeta-function of a non-Galois cubic field

Let K be a field extension of $\mathbb{Q}$. We say that K is a number field if its dimension $[K : \mathbb{Q}]$ as a vector space over $\mathbb{Q}$ is finite. One may consider such a field to be a subfield of $\mathbb{C}$, and we will usually do so. To every number field, we may consider the associated Galois group $G(K/\mathbb{Q})$, which consists of all field automorphisms of K fixing $\mathbb{Q}$ pointwise. One can show that the Galois group of a number field is always finite and that its cardinality divides the dimension $[K : \mathbb{Q}]$. If the cardinality of the Galois group is equal to $[K : \mathbb{Q}]$, we say that the field extension is Galois. Every number field K is contained in a unique smallest field L, the Galois closure of K in $\mathbb{C}$, such that $\mathbb{Q} \subseteq K \subseteq L \subseteq \mathbb{C}$, where L is Galois over $\mathbb{Q}$. Furthermore, $G(L/\mathbb{Q})$ is isomorphic to a subgroup of the symmetric group S_n, $n := [K : \mathbb{Q}]$.

The discriminant D_K of a number field K will be a central object in our future studies. To define it we need some preparation, taken from [N, Ch. I]. In a number field K, one may consider the set $\mathcal{O}_K$ of all integral elements. Here, an element $x \in K$ is called integral if it is the root of some polynomial with integer coefficients. It turns out that $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of dimension $n := [K : \mathbb{Q}]$, and thus has a $\mathbb{Z}$-basis $\{\alpha_1, ..., \alpha_n\}$ of integral elements. One can then define

$$D_K := \det((\mathrm{Tr}_K(\alpha_i \alpha_j))_{i,j=1,...,n}),$$

where $\mathrm{Tr}(x)$, $x \in K$, denotes the trace of the linear map $\mathrm{Tr} \ : K \to K$ given by $y \mapsto xy$. Without much effort, one can show that the discriminant is a nonzero integer, independent of the choice of basis. With more effort, one can show that there are only finitely many number fields whose discriminant is of bounded magnitude, and that $\mathbb{Q}$ is the only field whose discriminant is strictly less than 2 in absolute value [N, Thms. III.2.16, III.2.17].

We will be interested in studying L-functions associated to cubic number fields. It turns out that the non-Galois fields are the most common, see e.g. [C], and thus we decide to focus our efforts on these fields, following [CFLS]. By the discussion above, the Galois extensions have Galois group isomorphic to the cyclic group C_3, while the non-Galois extensions have a Galois group only containing the identity element. We may sometimes refer to the non-Galois extension fields as S_3-fields as the Galois group of their Galois closures is isomorphic to S_3. For the reader unfamiliar with the results about fields stated so far, we refer to any introductory text on Galois theory.

We will now introduce the Dedekind zeta-function ζ_K associated to a number field K. We begin with the special case $\zeta_{\mathbb{Q}}$, or simply ζ, the Riemann zeta-function, defined by the equation

$$\zeta_{\mathbb{Q}}(s) = \sum_{k=1}^{\infty} \frac{1}{k^s},$$

for complex s with $\mathrm{Re}(s) > 1$. From the classical theory of the Riemann zeta-function, see e.g. [D3], we know that $\zeta(s)$ can be meromorphically continued to the entire complex plane, with a simple pole at $s = 1$ with residue equal to 1.

In the general case, we define

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}, \tag{2.1.1}$$

for s with $\mathrm{Re}(s) > 1$. Here, the sum is over all ideals I contained in the ring of integers $\mathcal{O}_K$, and $N(I)$ denotes their norm, which is defined as the cardinality of the ring $\mathcal{O}_K/I$. The norm is completely multiplicative in the sense that if one factorises I into prime ideals $I = \prod_{j=1}^{\ell} \mathfrak{p}_j$, then $N(I) = \prod_{j=1}^{\ell} N(\mathfrak{p}_j)$, where ℓ is some positive integer. We also point out that the norm of an ideal generated by a rational integer has a particularly simple expression. If $k \in \mathbb{Z}$, then $N((k)) = k^n$, where again $n = [K : \mathbb{Q}]$. For an introduction to the ring of integers, ideals, and prime factorisation in number fields, see e.g. [N, Ch. 1].

The summation in (2.1.1) has no specific ordering attached to it, which leads us to the question of convergence. We will show that for $\mathrm{Re}(s) > 1$, the series is absolutely convergent so that the ordering does not matter. Indeed, we may formally rewrite the series as

$$\sum_{k=1}^{\infty} \frac{|\{I \subseteq \mathcal{O}_K : N(I) = k|\}}{k^s}.$$

To estimate this expression we need a few facts from algebraic number theory. First, from the proof of [N, Theorem I.3.1] it follows that every nonzero prime ideal $\mathfrak{p}$ divides some ideal (p), where p is a rational prime. Thus, by multiplicativity $p \mid N(\mathfrak{p}) \mid p^n$, $n = [K : \mathbb{Q}]$. In particular, this implies that (p) splits into at most n prime factors. Combined with the fact that every integral ideal admits a unique factorisation into prime ideals, this is enough to show that the number $|\{I \subseteq \mathcal{O}_K : N(I) = k|\}$ is finite. To show convergence we follow an approach similar to [La, Ch. 8.2].

For $\mathrm{Re}\, s > 1$, consider the infinite product

$$\prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1} = \prod_{\mathfrak{p}} \left(\sum_{e=0}^{\infty} N(\mathfrak{p})^{-es} \right),$$

where $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$. For an introduction to infinite products, and some results concerning their convergence, see [A, Ch. 5.2.2], or the summary in Appendix A.1. We remark that a product over primes such as the one above is often referred to as an Euler product. Formally, multiplying out the second expression for the products yields the series defining $\zeta_K(s)$ using multiplicativity of the norm. This formal manipulation can be justified using the absolute convergence of the product, and the fact that a finite product of absolutely convergent series is itself an absolutely convergent series.

We now show that the product is absolutely convergent, which follows if we show

$$\sum_{\mathfrak{p}} \sum_{e=1}^{\infty} \frac{1}{N(\mathfrak{p})^{e\,\mathrm{Re}(s)}} < \infty.$$

Order the sum by which rational prime p that $\mathfrak{p}$ divides, and use $N(\mathfrak{p}) \geq p$ to bound the sum from above by

$$\sum_{p} \sum_{e=1}^{\infty} \frac{n}{p^{\mathrm{Re}(s)}} \leq \sum_{k=1}^{\infty} \frac{n}{k^{\mathrm{Re}\, s}} < \infty,$$

for $\mathrm{Re}(s) > 1$ as desired. In fact, the argument above also implies that the convergence is uniform on any compact subset of $\mathrm{Re}(s) > 1$, and thus that $\zeta_K(s)$ is holomorphic for such s as a uniform limit of holomorphic functions. Also, as no factor in the infinite product is 0 and the product converges absolutely, we find that $\zeta_K(s) \neq 0$ for $\mathrm{Re}(s) > 1$. We will not provide a proof, but it is possible to show that ζ_K can be meromorphically continued to $\mathbb{C}$ with a simple pole at $s = 1$ [N, Cor. VII.5.11].

Now we specialize to the case when K is a non-Galois cubic number field. The discussion below is essentially contained in [Ya, Ch. 2.2.2]. It turns out that we may write $\zeta_K(s) = \zeta(s)L(s, f_K)$, for a certain function $L(s, f_K)$. Indeed, this can be shown by considering the Euler product above. First, by letting $K = \mathbb{Q}$ above, we see that $\zeta(s)$ has the product expansion

$$\prod_p (1 - p^{-s})^{-1}.$$

Now, $\zeta_K(s)$ has a product expansion

$$\prod_p \prod_{\mathfrak{p}|(p)} (1 - N(\mathfrak{p})^{-s})^{-1}. \tag{2.1.2}$$

In particular, this shows that the Dedekind zeta function is completely determined by the splitting behaviour of the rational primes (p) in K. To find how many ways a rational prime can split over a cubic field, we consider its norm. If $(p) = \mathfrak{p}_1 ... \mathfrak{p}_k$, then $p^3 = N((p)) = N(\mathfrak{p}_1)...N(\mathfrak{p}_k)$. From the definition, we see that the norm of a prime ideal is strictly greater than 1 and thus $k \leq 3$. More precisely $N(\mathfrak{p}_i) = p^{r_i}$ and thus by considering the exponents we see that a prime p can split in five different ways in K.

To distinguish between the five different cases we assign every splitting type a symbol out of $(111), (21), (3), (1^2 1)$, and (1^3), where the symbols correspond to the splittings $(p) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_1\mathfrak{p}_2$, $\mathfrak{p}_1$, $\mathfrak{p}_1^2\mathfrak{p}_2$ and $\mathfrak{p}_1^3$ respectively. We will also use the symbols T_1,T_2,..., T_5 to refer to these splitting types. Here all $\mathfrak{p}_i$ are distinct primes. In the (111) case we see that all prime ideals have norm p, while in the (21) case, one has norm p^2 and the other norm p. The rest of the splitting types can be analysed similarly.

This allows us to find

$$\prod_{\mathfrak{p}|(p)} (1 - N(\mathfrak{p})^{-s})^{-1},$$

as a function of the splitting type of (p), and s. We call this the local factor of ζ_K at p. It turns out that every such local factor is a product of the local factor of ζ at p, with some other factor. This allows us to factor ζ_K into a product of ζ, and another function which we will call $L(s, f_K)$. It turns out that $L(s, f_K)$ is easier than ζ_K to study, and we therefore take the time to find some of the properties of $L(s, f_K)$. We first remark that we will consider the symbol $L(s, f_K)$ simply as notation for the function defined by ζ_K/ζ, but it can also be given a separate meaning as the L-function associated to a certain automorphic form f_K. Alternatively, $L(s, f_K)$ is the Artin L-function associated to the unique irreducible two-dimensional representation of the Galois group, which is isomorphic to S_3, of the Galois closure of K. We will not go into this further.

The table below summarises the different local factors we can obtain. In the third column we have removed the factor coming from ζ. Here, ω is a primitive third root of unity.

Splitting type	Local factor in ζ_K	Local factor in ζ_K/ζ
$T_1 := (111)$	$(1 - p^{-s})^{-3}$	$(1 - p^{-s})^{-2}$
$T_2 := (21)$	$(1 - p^{-s})^{-1}(1 - p^{-2s})^{-1}$	$(1 - p^{-2s})^{-1}$
$T_3 := (3)$	$(1 - p^{-3s})^{-1}$	$(1 - \omega p^{-s})^{-1}(1 - \omega^2 p^{-s})^{-1}$
$T_4 := (1^2 1)$	$(1 - p^{-s})^{-2}$	$(1 - p^{-s})^{-1}$
$T_5 := (1^3)$	$(1 - p^{-s})^{-1}$	1

2.2 General L-functions

The function ζ_K turns out to be an L-function. There is no precise definition of what constitutes an L-function, but we can give some important properties that many L-functions satisfy, see also [IK, Ch. 5.1].

First, as mentioned in the introduction, every L-function should be representable by an absolutely convergent Dirichlet series for $\text{Re}(s) > 1$, that is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

for some coefficients a_n. This has the consequence that L-functions are holomorphic for $\mathrm{Re}(s) > 1$, as a locally uniform limit of the partial series. Furthermore, this Dirichlet series should in turn be representable by an absolutely convergent Euler product, in the same domain, i.e. a product of the form

$$\prod_p \left(1 - \alpha_1(p)p^{-s}\right)^{-1} \cdot \ldots \cdot \left(1 - \alpha_d(p)p^{-s}\right)^{-1},$$

for some $|\alpha_i(p)| < p$. The number d is called the degree of the L-function.

It should be possible to meromorphically continue an L-function to the entire complex plane $\mathbb{C}$, with the only possible pole at $s = 1$. Furthermore, an L-function should satisfy a functional equation, that is an equation relating the values of the L-function at some $s \in \mathbb{C}$, to the values at $1 - s$ of either the same L-function, or of the dual L-function, mentioned in the introduction. For the self-dual L-functions, the functional equation provides a symmetry about the point $s = 1/2$. An example of a functional equation, relevant to our purposes, is given below in (2.4.1). Lastly, every L-function has a conductor, which is related to the functional equation. We will not go into the general definition of a conductor, but we mention that $L(s, f_K)$ has the conductor $|D_K|$.

We have seen that the Dedekind zeta function has a Dirichlet series representation, as well as an Euler product. One can show that it also obeys a functional equation. The same also holds for $L(s, f_K)$, which indicates that both $L(s, f_K)$ and ζ_K are L-functions, and this is indeed the case. Furthermore, it can be shown that $L(s, f_K)$ is an entire function.

The Euler product representation of an L-function ensures that it has no zeros with a real part strictly larger than 1, as an absolutely convergent product of nonzero numbers is nonzero. The functional equation then implies that an L-function does not have any zeros with a real part strictly less than 0, except for possible trivial zeros, which are a consequence of other factors present in the functional equation. The nontrivial zeros are thus all located in the critical strip, i.e. the strip in $\mathbb{C}$ where the real part is neither smaller than 0, nor larger than 1. In the introduction, we already mentioned the utility of studying zeros of L-functions, and in particular why the low-lying zeros are of interest. This brings us to the topic of the next section.

2.3 The low-lying zeros of L-functions

Any fixed L-function has a fixed set of zeros, and in particular, has a finite number of such zeros lying within distance, say 1, of the point 1/2. Thus, it is uninteresting to study the distribution of the low-lying zeros of a single L-function, and one should instead consider an entire family of L-functions. One such family is the family of Dedekind zeta functions associated to S_3-cubic fields introduced earlier in this chapter.

Katz and Sarnak conjectured that every natural family of L-functions has one of five symmetry types, which determines the distribution of the low-lying zeros, see [KS]. This conjecture is motivated by analogues in the study of the eigenvalues lying close to 1 of random matrices, and by analogues in the study of the central zeros of certain families of polynomials associated to so-called function fields. What constitutes a natural family is a deep question, see e.g. [SST], which we will not investigate further here. To make the conjecture more precise, we need some notation from [KS], which we modify slightly.

Let $\mathcal{F}$ be a family of L-functions, and $\phi : \mathbb{R} \to \mathbb{R}$ a Schwartz function. Here, a Schwartz function is an infinitely differentiable function with derivatives admitting the bound $\phi^{(k)}(x) \ll_{A,k} (1+|x|)^{-A}$. For every L-function $L(s, f) \in \mathcal{F}$ we let c_f denote the conductor of $L(s, f)$. We assume that all sets $\mathcal{F}_X := \{L(s, f) \in \mathcal{F} : c_f \leq X\}$ are finite. The distribution of low-lying zeros can then be studied through the one-level density, defined as

$$\frac{1}{|\mathcal{F}_X|} \sum_{c_f \leq X} \sum_{\gamma_f} \phi\left(\frac{\gamma_f \log c_f}{2\pi}\right), \tag{2.3.1}$$

where the inner sum is over all nontrivial zeros $\rho_f = 1/2 + i\gamma_f$ of $L(s, f)$. If the GRH is true for the family $\mathcal{F}$, then all $\gamma_f \in \mathbb{R}$ so that the expression above is well-defined for ϕ with domain $\mathbb{R}$. As ϕ decays fast, the presence of c_f in the argument of ϕ ensures that only the zeros ρ close to 1/2 give a significant contribution to the one-level density. We remark that even if one does not assume the GRH, it can still make sense to study the expression above, as long as ϕ can be extended to a function defined in the horizontal strip $\{|\mathrm{Im}(s)| \leq 1/2\}$, decaying quickly.

The conjecture of Katz and Sarnak is that as $X \to \infty$, the one-level density converges to one of the integrals

$$\int_{\mathbb{R}} \phi(x) w(G(\mathcal{F}))(x) dx.$$

Here $G(\mathcal{F})$ denotes the symmetry type of $\mathcal{F}$. The five symmetry types (or groups) are unitary, orthogonal, special orthogonal (even), special orthogonal (odd), and symplectic. The expression $w(G(\mathcal{F}))(x)$ denotes a density function associated to the symmetry type $G(\mathcal{F})$. In particular, it will be important later that the symplectic density is given by

$$1 - \frac{\sin 2\pi x}{2\pi x}.$$

This choice of naming comes from the random matrix theory analogue, mentioned earlier, where Katz and Sarnak studied the eigenvalues of random matrices from the groups mentioned above. They then found that the distribution of these eigenvalues converged to the integrals given above if one let the dimension of the matrices tend to infinity, see [KS, Ch. 2] for a summary.

Aside from the one-level density, one may also study the n-level density for any integer $n \geq 2$, by replacing the inner sum of (2.3.1) with a sum over certain n-tuples of zeros, and by replacing ϕ with an appropriate multivariate function. A similar conjecture as above can be stated for these n-level densities, but we will not do this in the general case. The important point is that once again, we expect the n-level density to converge to one of five integrals. See Chapter 4 for the study of one specific two-level density.

As we have already mentioned, the conjecture above, often called the Katz–Sarnak prediction, has been partially confirmed for several families. By the conjecture being partially confirmed, we mean that the limit of the one-level density can only be computed for a special class of Schwartz functions ϕ, with strong conditions on the support of $\widehat{\phi}$, the Fourier transform of ϕ.

2.4 The low-lying zeros of $L(s, f_K)$

We want to study the one-level density associated to the family of Dedekind zeta functions corresponding to S_3-cubic fields. It turns out that the zeros of ζ_K are difficult to study directly because of the influence of the zeros of ζ. Fortunately, these zeros can be removed by factoring ζ from ζ_K, as we saw above. As ζ is a fixed function with fixed zeros, removing these zeros from consideration will not majorly change the one-level density. In particular, it is possible to show that the symmetry type will remain unchanged, at least assuming the Riemann Hypothesis, but for the sake of brevity, we leave out this calculation.

It is not clear that the function $L(s, f_K) = \zeta_K/\zeta$ is a nice function as it could possibly have poles wherever ζ has a zero. However, as we have briefly mentioned, this is not the case and $L(s, f_K)$ is actually an L-function. Furthermore, it turns out that any Dedekind zeta function can be decomposed into a product of ζ, and other functions, known as Artin L-functions. In general, unlike $L(s, f_K)$, these Artin L-functions are not known to satisfy all of the properties for L-functions that we outlined earlier. We will not need a general definition of an Artin L-function, nor shall we provide one. We will instead think of $L(s, f_K)$ simply as the factor of ζ_K obtained by dividing with ζ. The interested reader may consult [IK, Ch. 5.13] for a general definition.

In the case of cubic non-Galois K, the Artin L-functions $L(s, f_K)$ are known to be entire. Furthermore, they satisfy the functional equation

$$\Lambda(s, f_K) = \Lambda(1 - s, f_K), \text{ where } \Lambda(s, f_K) := |D_K|^{s/2}\Gamma_{\pm}(s)L(s, f_K), \tag{2.4.1}$$

see [IK, Ch 5.10, 5.13] for all these properties. Here,

$$\Gamma_{\pm}(s) = \pi^{-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+\delta_{-}(D_K)}{2}\right), \tag{2.4.2}$$

where $\delta_{-}(x) = 0$ if $x \geq 0$, and else $\delta_{-}(x) = 1$. As usual $\Gamma(s)$, denotes the gamma function, and as before, D_K denotes the discriminant of K. For a description of some important properties of the gamma function, see Appendix A.2. By definition, the nontrivial zeros of $L(s, f_K)$ are the zeros of $\Lambda(s, f_K)$, while the nontrivial zeros are the zeros of $\Gamma_{\pm}(s)$. We see that if D_K is positive, then the nontrivial zeros are located at the non-positive even integers, while they are located at all non-positive integers if $D_K < 0$.

We are interested in studying the one-level density for the Artin L-functions corresponding to non-Galois cubic fields. In this and all following chapters, we will consider our fields to be subsets of $\mathbb{C}$. Note that by our earlier discussion $L(s, f_K)$ only depends on the splitting behaviour of the ideals generated by rational primes. In particular, $L(s, f_K)$ only depend on the isomorphism class of K.

Define two sets $\mathcal{F}^{+}(X)$ and $\mathcal{F}^{-}(X)$ by

$$\mathcal{F}^{\pm}(X) = \{\text{Non-Galois cubic } K : 0 < \pm D_K < X\},$$

where we only include one field from each isomorphism class in $\mathcal{F}^{\pm}(X)$. As the field extensions are non-Galois, each isomorphism class contains exactly three elements. Both sets are finite, by our earlier remarks about number fields of bounded discriminant. Define $N^{\pm}(X)$ to be the cardinality of $\mathcal{F}^{\pm}(X)$. For a fixed K, denote the nontrivial zeros of $L(s, f_K)$ by ρ_K, and write $\rho_K =: 1/2 + i\gamma_K$. We will let $\phi : \mathbb{R} \to \mathbb{R}$ be a fixed non-zero, real-valued, even Schwartz function whose Fourier transform has compact support, and we define $\sigma = \sup \operatorname{supp}(\widehat{\phi})$, where $\operatorname{supp}(f)$ denotes the support of a function f. As in [CFLS], we note that any such ϕ can be extended to an entire function on $\mathbb{C}$ by means of the inverse fourier transform. Holomorphicity is seen by differentiating under the integral sign, justified by the compact support of $\widehat{\phi}$. This extension of ϕ means that we can investigate our version of (2.3.1) without assuming the GRH for $L(s, f_K)$.

The one-level density is then given by

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K\right), \tag{2.4.3}$$

where $L = \log(X/(2\pi e)^2)$. Now, L is not really the logarithm of the conductor of $L(s, f_K)$, but it turns out that replacing L by the logarithm $\log|D_K|$ of the conductor yields expressions that are hard to calculate explicitly. Furthermore, for most elements in $\mathcal{F}^{\pm}(X)$, $\log|D_K|$ is approximately equal to $\log X$. We remark that this change does not alter the symmetry type, which one can confirm by summing Theorem 3.1 over dyadic intervals for X, and using that $\log x$ is nearly constant in such an interval. This particular choice of L is made in an attempt to renomalise the spacings of the γ_K to essentially be constant, cf. [IK, Thm. 5.8]. All zeros above are counted with multiplicity. The next chapter will be concerned with evaluating this one-level density.

3

The one-level density

We will now study the one-level density of the family of Artin L-functions $L(s, f_K)$ corresponding to non-Galois cubic number fields. The approach, and most of the notation, are taken from [CFLS, Ch. 3] with very few modifications. The calculation begins by relating the zeros of $L(s, f_K)$ to an expression involving certain prime sums, through an explicit formula.

The key to studying this expression is a precise estimate, given in (3.2.3), of the number of cubic fields where some prime p has a given splitting type, and where the discriminant is bounded by a given magnitude X. Using this estimate, we find the one-level density in Theorem 3.1. Next, we analyse the result of Theorem 3.1 and confirm that the main term of the one-level density is of symplectic symmetry type, at least for $\sigma = \sup \operatorname{supp}(\widehat{\phi})$ strictly less than 2/7. We end by briefly describing a result from [CFLS], bounding the error term in the counting function from (3.2.3) from below.

3.1 The explicit formula

We now begin studying the one-level density, given in (2.4.3). The approach is identical to the one found in [CFLS, Lemma 3.1]. First, as we have seen, $L(s, f_K)$ satisfies the functional equation

$$\Lambda(s, f_K) = \Lambda(1-s, f_K), \text{ where } \Lambda(s, f_K) := |D_K|^{s/2}\Gamma_\pm(s)L(s, f_K). \tag{3.1.1}$$

Here, $\Gamma_\pm(s)$ is defined in connection to (2.4.1). Now, the gamma function is meromorphic with its poles at the non-positive integers, and $L(s, f_K)$ is entire with no zeros with $\operatorname{Re}(s) > 1$. This implies that $\Lambda(s, f_K)$ is entire, and that

$$\frac{\Lambda'}{\Lambda}(s, f_K)$$

has poles precisely at the nontrivial zeros ρ_K of $L(s, f_K)$. For $T \geq 1$ let $\mathcal{R}(T)$ denote the boundary of the rectangle in the complex plane with corners at $-1/2+iT$, $-1/2-iT$, $3/2-iT$, $3/2+iT$ oriented counterclockwise. Then, by the residue theorem, we find

$$\sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K\right) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\mathcal{R}(T)} \phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right)\frac{\Lambda'}{\Lambda}(s, f_K)ds.$$

Here T is chosen to not coincide with the ordinate of any zero of $\Lambda(s, f_K)$.

Now, using the inverse Fourier transform, the support of $\widehat{\phi}$, and integration by parts, we have for any integer $k \geq 1$ and $r \neq 0$ (cf. [CFLS, Eq. (4.17)])

$$\phi\left(\frac{Lr}{2\pi i}\right) = \int_{-\sigma}^{\sigma} e^{Lrx}\widehat{\phi}(x)dx = \frac{(-1)^k}{L^k r^k}\int_{-\sigma}^{\sigma} e^{Lx\operatorname{Re}(r)+iLx\operatorname{Im}(r)}\widehat{\phi}^{(k)}(x)dx \ll_k \frac{e^{\sigma L|\operatorname{Re}(r)|}}{L^k|r|^k}. \tag{3.1.2}$$

The implied constant also depends on ϕ, but for the sake of conciseness, we do not indicate that above, nor will we do so in the rest of the report. In particular, this calculation shows that ϕ decays fast in any vertical strip.

By choosing T sufficiently far from the ordinate of any zero of $\Lambda(s, f_K)$ using [IK, Prop. 5.7, Thm. 5.8], we may bound the growth of Λ'/Λ on horizontal lines with imaginary part equal to $\pm T$ polynomially in T, where the gamma factors are bounded using Stirling's formula. Letting $T \to \infty$ above thus makes the contribution from the horizontal line segments vanish, as ϕ decays faster than any polynomial.

Thus, we have shown

$$\sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K\right) = \frac{1}{2\pi i}\left(\int_{(3/2)} - \int_{(-1/2)}\right) \phi\left(\frac{L}{2\pi i}\left(s - \frac{1}{2}\right)\right) \frac{\Lambda'}{\Lambda}(s, f_K) ds,$$

where (δ) denotes the vertical line with $\mathrm{Re}(s) = \delta$, oriented upwards. We wish to apply the change of variables $r = 1 - s$ to the second integral. We find that $dr = -ds$, and that $(-1/2)$ is mapped to $(3/2)$ with the orientation reversed. Now, by the functional equation

$$\frac{\Lambda'}{\Lambda}(s, f_K) = -\frac{\Lambda'}{\Lambda}(1 - s, f_K),$$

so that

$$\begin{aligned} \frac{1}{2\pi i}\left(\int_{(3/2)} - \int_{(-1/2)}\right)&\phi\left(\frac{L}{2\pi i}\left(s - \frac{1}{2}\right)\right) \frac{\Lambda'}{\Lambda}(s, f_K) ds \\ &= \frac{1}{\pi i}\int_{(3/2)} \phi\left(\frac{L}{2\pi i}\left(s - \frac{1}{2}\right)\right) \frac{\Lambda'}{\Lambda}(s, f_K) ds, \end{aligned} \tag{3.1.3}$$

where we also used that ϕ is even.

To continue, we need to study the integrand further. As $\mathrm{Re}(s) = 3/2 > 1$, the function $\Lambda(s, f_K)$ is nonzero and thus has a holomorphic logarithm in any simply connected set, and further, the derivative of any such logarithm is

$$\frac{\Lambda'}{\Lambda}(s, f_K),$$

which is then naturally called the logarithmic derivate of $L(s, f_K)$. By the definition of $\Lambda(s, f_K)$ we see that one such logarithm is simply

$$\frac{s}{2}\log|D_K| + \log\Gamma_\pm(s) + \log L(s, f_K),$$

where $\log f(s)$ denotes any logarithm of f. Differentiating this expression yields

$$\frac{\log|D_K|}{2} + \frac{\Gamma'_\pm(s)}{\Gamma_\pm(s)} + \frac{L'}{L}(s, f_K).$$

Before proceeding with our evaluation of the integral, we require a more explicit form of the last term. To find such a form, we begin by observing that

$$\frac{L'}{L}(s, f_K) = \frac{\zeta'_K}{\zeta_K}(s) - \frac{\zeta'}{\zeta}(s).$$

We will now expand the right-hand side into a Dirichlet series. We begin by taking the logarithm of the defining product and write

$$\log\zeta_K(s) = \log\prod_p\prod_{\mathfrak{p}|(p)}(1 - N(\mathfrak{p})^{-s})^{-1} = -\sum_p\sum_{\mathfrak{p}|(p)}\log(1 - N(\mathfrak{p})^{-s}).$$

The attentive reader may notice that the second equality is not necessarily true for an arbitrary logarithm. To solve this problem, we choose the logarithm in the two first expressions as the one defined by the right-hand side, where the logarithm in the right-hand side is the principal branch of the logarithm.

Differentiate the sum by interchanging the order of differentiation and summation. This can be justified by the uniform absolute convergence of the resulting series for $\mathrm{Re}(s) \geq 1+\epsilon$, for any $\epsilon > 0$. We find that the derivative equals

$$\frac{\zeta_K'}{\zeta_K}(s) = -\sum_p \sum_{\mathfrak{p}|(p)} \frac{N(\mathfrak{p})^{-s}\log N(\mathfrak{p})}{1-N(\mathfrak{p})^{-s}} = -\sum_p \sum_{\mathfrak{p}|(p)} \sum_{e=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{es}}, \tag{3.1.4}$$

where the last equality is the formula for a geometric sum. To continue we will need the von-Mangoldt function $\Lambda(n)$, defined on prime powers by $\Lambda(p^k) = \log p$, and else as 0. Also, recall that $N(\mathfrak{p})$ is a power of p so that we can write the sum as

$$-\sum_{n=1}^{\infty} \frac{\Lambda(n) b_K(n)}{n^s},$$

for some coefficients $b_K(n)$. Note that we need not concern ourselves with the values of $b_K(n)$ when n is not a prime power, as then $\Lambda(n) = 0$.

We first find the coefficients b_K, when $K = \mathbb{Q}$. Then the prime ideals $\mathfrak{p}$ are exactly (p) so that $N(\mathfrak{p}) = p$, and thus b_K is constantly equal to 1. We turn to the case when K is a non-Galois cubic field and study $b_K(p^e)$ for some prime p and $e \geq 1$. From the right-hand side of (3.1.4), we see that $b_K(p^e)$ only depends on the splitting type of (p) and on e.

We begin with the totally split case, when (p) has splitting type (111), and $(p) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$. Then $N(\mathfrak{p}_i) = p$, and for a fix p and e the sum over $\mathfrak{p}$ in (3.1.4) has 3 terms so that $b_K(p^e) = 3$. For the splitting type (12) the sum only has 2 terms, whence $b_K(p^e) = 2$, and similarly for the splitting type (1^3) we have $b_K(p^e) = 1$. In the case (21), we have one prime factor of norm p^2 and one of norm p. Thus, $b_K(p^e) = 1 + 2\delta_{2|e}$. Here, δ_P is a function which equals 1 if P is true, and 0 otherwise. Lastly, for splitting type (3) we find $b_K(p^e) = 3\delta_{3|e}$.

Using these results, we have shown that for $\mathrm{Re}(s) > 1$

$$\frac{L'}{L}(s, f_K) = -\sum_{n=1}^{\infty} \frac{\Lambda(n) a_K(n)}{n^s},$$

where $a_K(p^e)$ is given in the following table, see also [CFLS, Ch 2.]:

Splitting type	$a_K(p^e)$
T_1	2
T_2	$2\delta_{2\|e}$
T_3	$3\delta_{3\|e} - 1$
T_4	1
T_5	0

From these results, we can expand (3.1.3) into

$$\frac{\log|D_K|}{2\pi i} \int_{(3/2)} \phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right) ds + \frac{1}{\pi i} \int_{(3/2)} \phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right) \frac{\Gamma_\pm'}{\Gamma_\pm}(s) ds$$
$$- \frac{1}{\pi i} \int_{(3/2)} \phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right) \sum_{n=1}^{\infty} \frac{\Lambda(n) a_K(n)}{n^s} ds.$$

To proceed we will need a technique that is often referred to as 'shifting the contour'. Consider a vertical strip defined by $a \leq \mathrm{Re}(s) \leq b$, for some $a < b$, and let $f(s)$ be holomorphic in an open set containing this strip. Further, assume that $|f(s)|$ decreases to 0 uniformly in the strip, as $|\mathrm{Im}(s)| \to \infty$. Then, the conclusion is that the integral of f over (b) is equal to the integral of f over (a), assuming that one of them exists. Indeed, this follows by applying Cauchy's theorem to the rectangle with corners in $a - iT, a + iT, b - iT, b + iT$, and then letting $T \to \infty$ so that the horizontal contributions tend to 0.

We now study the third integral above. By absolute convergence, we may interchange the order of summation and integration and thus find

$$-\frac{1}{\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)a_K(n)}{\sqrt{n}}\int_{(3/2)}\phi\left(\frac{L}{2\pi i}\left(s-\frac{1}{2}\right)\right)n^{1/2-s}ds.$$

Now shift all three integrals to the line (1/2), and parametrise the integrals by $s = 1/2 + iu$. Next, make the change of variables $t = Lu/(2\pi)$. The result is

$$\frac{\log|D_K|}{L}\int_{\mathbb{R}}\phi(t)dt + \frac{2}{L}\int_{\mathbb{R}}\phi(t)\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi it}{L}\right)dt - \frac{2}{L}\sum_{n=1}^{\infty}\frac{\Lambda(n)a_K(n)}{\sqrt{n}}\int_{\mathbb{R}}\phi(t)n^{-2\pi it/L}dt.$$

To simplify further, we apply the definition of the Fourier transform to the first and third integral and find that

$$\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K\right) = \frac{\widehat{\phi}(0)\log|D_K|}{L} + \frac{2}{L}\int_{\mathbb{R}}\phi(t)\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi it}{L}\right)dt - \frac{2}{L}\sum_{n=1}^{\infty}\frac{\Lambda(n)a_K(n)}{\sqrt{n}}\widehat{\phi}\left(\frac{\log n}{L}\right), \tag{3.1.5}$$

which is also essentially the content of [CFLS, Lemma 3.1].

3.2 Summing over K

We wish to find

$$\frac{1}{N^\pm(X)}\sum_{K\in\mathcal{F}^\pm(X)}\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K\right),$$

which means that we must sum (3.1.5) over K. The calculations are all taken from [CFLS, Ch. 3]. We begin by observing that the second term in (3.1.5) does not depend on K, and thus summing over K simply multiplies the term by $N^\pm(X)$. The only K-dependence in the first term comes from $\log|D_K|$. We use Stieltjes integration and integration by parts, see Appendix A.3, to write

$$\sum_{K\in\mathcal{F}^\pm(X)}\log|D_K| = \int_1^X \log t\, dN^\pm(t) = N^\pm(X)\log X - \int_1^X \frac{N^\pm(t)}{t}dt. \tag{3.2.1}$$

Here we made use of the fact that no nontrivial field extension of $\mathbb{Q}$ has a discriminant of modulus 1. To evaluate the integral in the right-hand side we require a precise estimate of the function $N^\pm(t)$, and therefore we postpone this calculation until later in this section.

For the sum of the third term in (3.1.5) over K, we interchange the order of summation over K and n. This is allowed as all sums only contain finitely many nonzero terms. The result is

$$-\frac{2}{L}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\widehat{\phi}\left(\frac{\log n}{L}\right)\sum_{K\in\mathcal{F}^\pm(X)}a_K(n) = -\frac{2}{L}\sum_p\sum_{e=1}^{\infty}\frac{\log p}{p^{e/2}}\widehat{\phi}\left(\frac{e\log p}{L}\right)\sum_{K\in\mathcal{F}^\pm(X)}a_K(p^e),$$

where the last step is that the summand is nonzero only when n is a prime power. By our earlier investigations, for a fixed p and $e \geq 1$, $a_K(p^e)$ only depends on how (p) splits in K, and on e. Thus, if we let $N_p^\pm(X, T_i)$ count the fields in $\mathcal{F}^\pm(X)$ where p has splitting type T_i we find that the sum above is

$$\begin{aligned}-\frac{2}{L}\sum_p\sum_{e=1}^{\infty}\frac{\log p}{p^{e/2}}\widehat{\phi}\left(\frac{e\log p}{L}\right)\Big(&2N_p^\pm(X,T_1) + 2\delta_{2|e}N_p^\pm(X,T_2)\\ &+ (3\delta_{3|e}-1)N_p^\pm(X,T_3) + N_p^\pm(X,T_4)\Big).\end{aligned} \tag{3.2.2}$$

We see that in order to study the one-level density, we need not only estimates for $N^{\pm}(X)$, but also for the more precise counts $N_p^{\pm}(X, T_i)$. Obtaining these estimates is highly nontrivial, and these results are the crux of our method for evaluating the one-level density. Using notation from [CFLS, Ch. 1], it has been shown that

$$\begin{aligned} N^{\pm}(X) &= C_1^{\pm} X + C_2^{\pm} X^{5/6} + \mathcal{O}_\epsilon(X^{\theta+\epsilon}), \\ N_p^{\pm}(X, T_i) &= A_p^{\pm}(T_i) X + B_p^{\pm}(T_i) X^{5/6} + \mathcal{O}_\epsilon(p^{\omega} X^{\theta+\epsilon}), \end{aligned} \tag{3.2.3}$$

holds for some $\theta < 5/6$, $\omega \geq 0$, and all $\epsilon > 0$. This was first shown independently in [TT] and [BST], and the current best result, proved in [BTT], allows us to take θ and ω to be 2/3. We will not fix θ or ω, but instead only require $5/6 > \theta \geq 0$ and $\omega \geq 0$, to allow our results to be improved in case (3.2.3) is proven to hold with smaller values of θ and ω. Here,

$$C_1^{+} = \frac{1}{12\zeta(3)}, \quad C_2^{+} = \frac{4\zeta\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)^3 \zeta\left(\frac{5}{3}\right)},$$

and $C_1^{-} = 3C_1^{+}$, $C_2^{-} = \sqrt{3}C_2^{+}$. We now explicitly provide the values of $A_p(T_i)$ and $B_p(T_i)$, cf. the summary in [CFLS, Ch. 2]. Let,

$$x_p = \frac{1}{1 + p^{-1} + p^{-2}}, \quad y_p = \frac{1 - p^{-1/3}}{(1 - p^{-5/3})(1 + p^{-1})}.$$

Then,

$$A_p^{\pm}(T_i) = C_1^{\pm} x_p c_i(p), \quad B_p^{\pm}(T_i) = C_2^{\pm} y_p d_i(p),$$

where $c_i(p)$ and $d_i(p)$ are given in the table below.

Splitting type	$c_i(p)$	$d_i(p)$
T_1	$\frac{1}{6}$	$\frac{(1+p^{-1/3})^3}{6}$
T_2	$\frac{1}{2}$	$\frac{(1+p^{-1/3})(1+p^{-2/3})}{2}$
T_3	$\frac{1}{3}$	$\frac{(1+p^{-1})}{3}$
T_4	$\frac{1}{p}$	$\frac{(1+p^{-1/3})^2}{p}$
T_5	$\frac{1}{p^2}$	$\frac{(1+p^{-1/3})}{p^2}$

For a proof sketch of the estimates above, see Chapter 5.

Using the estimates in (3.2.3), we can see that (3.2.1) is

$$\begin{aligned} \log X N^{\pm}(X) - \int_1^X & \frac{C_1^{\pm} t + C_2^{\pm} t^{5/6} + \mathcal{O}_\epsilon(t^{\theta+\epsilon})}{t} dt \\ &= \log X N^{\pm}(X) - C_1^{\pm} X - \frac{6C_2^{\pm}}{5} X^{5/6} + \mathcal{O}_\epsilon\left(X^{\theta+\epsilon}\right) \\ &= \log X N^{\pm}(X) - N^{\pm}(X) - \frac{C_2^{\pm}}{5} X^{5/6} + \mathcal{O}_\epsilon\left(X^{\theta+\epsilon}\right). \end{aligned} \tag{3.2.4}$$

We should also divide this expression by $N^{\pm}(X)$. We use

$$\begin{aligned} \frac{1}{N^{\pm}(X)} &= \frac{1}{C_1^{\pm} X} \cdot \frac{1}{1 + X^{-1/6} C_2^{\pm}/C_1^{\pm} + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)} \\ &= \frac{1}{C_1^{\pm} X} \left(1 - \frac{C_2^{\pm}}{C_1^{\pm}} X^{-1/6} + \left(\frac{C_2^{\pm}}{C_1^{\pm}}\right)^2 X^{-1/3} + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon} + X^{-1/2}\right)\right), \end{aligned} \tag{3.2.5}$$

and find that dividing (3.2.4) by $N^{\pm}(X)$ yields

$$\log X - 1 - \frac{C_2^{\pm}}{5C_1^{\pm}} X^{-1/6} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2} X^{-1/3} + \mathcal{O}_\epsilon(X^{\theta-1+\epsilon} + X^{-1/2}).$$

Note that if $\theta \geq 1/2$, then the second term in the error can be absorbed in the first.

The only remaining term in the one-level density is (3.2.2). By (3.2.3), and a calculation, we conclude that

$$2N_p^{\pm}(X,T_1) + 2\delta_{2|e}N_p^{\pm}(X,T_2) + (3\delta_{3|e}-1)N_p^{\pm}(X,T_3) + N_p^{\pm}(X,T_4) = C_1^{\pm}X\left(\theta_e + p^{-1}\right)x_p$$
$$+ C_2^{\pm}X^{5/6}(1+p^{-1/3})(\kappa_e(p) + p^{-1} + p^{-4/3})y_p + \mathcal{O}_\epsilon(p^\omega X^{\theta+\epsilon}), \tag{3.2.6}$$

where

$$\theta_e = \delta_{2|e} + \delta_{3|e}, \quad \kappa_e(p) = \theta_e(1+p^{-2/3}) + (1-\delta_{3|e})p^{-1/3}.$$

Set $\gamma_e(p) = (1+p^{-1/3})(\kappa_e(p) + p^{-1} + p^{-4/3})y_p$, and notice that $\theta_e, \gamma_e(p) \ll 1$.

Using the results above, we can conclude that dividing (3.2.2) by $N^{\pm}(X)$ yields

$$-\frac{2}{L}\left(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6} + \left(\frac{C_2^{\pm}}{C_1^{\pm}}\right)^2 X^{-1/3} + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon} + X^{-1/2}\right)\right)$$
$$\times \sum_p \sum_{e=1}^{\infty} \frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{e\log p}{L}\right) - \frac{2}{L}\frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\Big(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6} + \mathcal{O}_\epsilon\big(X^{\theta-1+\epsilon}$$
$$+ X^{-1/3}\big)\Big)\sum_p\sum_{e=1}^{\infty}\frac{\gamma_e(p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{e\log p}{L}\right) + \frac{1}{L}\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\sum_p\sum_{e\geq 1}\frac{\log p}{p^{e/2}}\left|\widehat{\phi}\left(\frac{\log p^e}{L}\right)\right|p^\omega\right).$$

We begin by handling the error terms above. First, $L \leq \log X$ so that we only need to consider $p^e \leq X^\sigma$ in the sums, by the support condition on $\widehat{\phi}$. The support condition also implies that $\widehat{\phi}$ is bounded, which means we may bound $\widehat{\phi}$ by the indicator function of $p^e \leq X^\sigma$ multiplied by a constant.

To estimate the last sum above we will use a version of the Prime Number Theorem, cf. [D3, Ch. 18], which asserts that

$$\theta(x) := \sum_{p\leq x}\log p = x + \mathcal{O}\left(xe^{-c'(\log x)^{1/2}}\right), \tag{3.2.7}$$

for some constant $c' > 0$. In particular, $\theta(x) \ll x$, which will be a sufficient bound for now. We then want to bound the sum

$$X^{\theta-1+\epsilon}\sum_{p^e\leq X^\sigma, e\geq 1}\frac{\log p}{p^{e/2}}p^\omega.$$

It suffices to consider the case when $e=1$, as the sub-sums obtained by fixing other values of e give a smaller contribution, and the sum over all large enough e, e.g. all $e \geq 2\omega+3$, converges even without the restriction $p^e \leq X^\sigma$ so that the contribution from such e is $\ll 1$. Thus, we find that the third sum is

$$\ll X^{\theta-1+\epsilon}\sum_{p\leq X^\sigma}\frac{\log p}{p^{1/2-\omega}} \ll X^{\theta-1+\epsilon}\int_1^{X^\sigma}\frac{1}{u^{1/2-\omega}}d\theta(u) \ll X^{\theta+\sigma(1/2+\omega)-1+\epsilon}, \tag{3.2.8}$$

where the last step is integration by parts.

To handle the rest of the error terms, we note that $\theta_1 = 0$, $\gamma_1(p) \ll p^{-1/3}$ so that all the remaining error terms are dominated by the bound we just found, or by the bound for

$$X^{-1/2}\sum_{p^e\leq X^\sigma}\frac{\gamma_e(p)\log p}{p^{e/2}} \ll X^{-1/2}\sum_{p\leq X^\sigma}\frac{\log p}{p^{5/6}} \ll X^{\sigma/6-1/2}, \tag{3.2.9}$$

where the first step is a restriction to $e=1$, and the last step is a Stieltjes integration as above.

Combining our results from this and the preceding section, and using

$$\frac{\log X}{L} = 1 + \frac{\log 4\pi^2 e^2}{L}$$

we have proven (cf. [CFLS, Theorem 1.2]).

Theorem 3.1 (CFLS)**.** *Let $5/6 > \theta \geq 1/2$, and $\omega \geq 0$ be such that* (3.2.3) *holds. Then if ϕ is a real even Schwartz function, whose Fourier transform is supported in $[-\sigma, \sigma]$, we have*

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K\right)$$
$$= \widehat{\phi}(0)\left(1 + \frac{\log(4\pi^2 e)}{L} - \frac{C_2^{\pm}}{5C_1^{\pm}}\frac{X^{-1/6}}{L} + \frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}\frac{X^{-1/3}}{L}\right) + \frac{2}{L}\int_{\mathbb{R}} \phi(t)\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}\left(\frac{1}{2} + \frac{2\pi i t}{L}\right) dt$$
$$- \frac{2}{L}\left(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6} + \frac{(C_2^{\pm})^2}{(C_1^{\pm})^2}X^{-1/3}\right)\sum_{p,e}\frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)$$
$$- \frac{2C_2 X^{-1/6}}{C_1 L}\left(1 - \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\right)\sum_{p,e}\frac{\gamma_e(p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right) + \mathcal{O}_\epsilon\left(X^{\theta - 1 + \sigma(\omega + 1/2) + \epsilon}\right).$$

We remark that $\theta - 1 + \sigma(\omega + 1/2) < 0$ is required to make the error term smaller than the main term. In particular with $\omega = \theta = 2/3$, which is currently the best value for which (3.2.3) holds, one needs $\sigma < 2/7$. The restriction to $\theta \geq 1/2$ is explained in Section 3.4.

3.3 Interpreting Theorem 3.1

We want to compare the main term of the expression above to the expected symplectic main term from the Katz-Sarnak prediction. First, the term involving the integral is $\ll 1/L$ by the fast decay of ϕ, and Stirling's formula. Estimating the rest of the terms requires a careful study of the two sums. We follow the proof of [CFLS, Lemma 3.4]. We will primarily concern ourselves with the constant terms, or the terms of size $1/L$ in the one-level density.

We begin by studying the first of the sums in Theorem 3.1. The idea is to split the sum into one 'convergent' and one 'divergent' part, where divergent means that it would diverge if the sum did not involve the compactly supported $\widehat{\phi}$. Recall that $\theta_1 = 0$, $\theta_2 = 1$, and $x_p - 1 \ll p^{-1}$, so that the divergent part is contained in the sum restricted to $e = 2$. We write the sum as

$$\sum_p \frac{\log p}{p}\widehat{\phi}\left(\frac{2\log p}{L}\right) + \sum_p \sum_{e \neq 2} \frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{e\log p}{L}\right)$$
$$+ \sum_p \frac{\log p}{p}\left(x_p\left(1 + \frac{1}{p}\right) - 1\right)\widehat{\phi}\left(\frac{2\log p}{L}\right).$$

The last two sums are estimated with a Taylor expansion of order zero of $\widehat{\phi}$, with error $\ll L^{-1}$. There is a technical subtlety involving the implied constant of the error term in this expansion. What we do is to first consider the function $\widehat{\phi}(u)$ restricted to $|u| \leq \sigma$. Then, by the compactness of the domain, a Taylor expansion of any order holds with a fixed implied constant. Now, as $\widehat{\phi}(u)$ is zero outside this domain, we may conclude that the Taylor expansion also holds in the entirety of $\mathbb{R}$, but with a possibly larger constant, as the size of the error term will dominate the size of the Taylor polynomial for large u.

To handle the remaining sum, we use the Prime Number Theorem (3.2.7), but with the error term weakened to $\mathcal{O}_A\left(x(\log x)^{-A}\right)$, which holds for any $A \geq 1$. Using Stieltjes integration, and the compact support of $\widehat{\phi}$, we calculate

$$\sum_p \frac{\log p}{p}\widehat{\phi}\left(\frac{2\log p}{L}\right) = \int_1^\infty \frac{1}{u}\widehat{\phi}\left(\frac{2\log u}{L}\right) d\theta(u)$$
$$= \int_1^\infty \frac{1}{u}\widehat{\phi}\left(\frac{2\log u}{L}\right) du + \int_1^\infty \frac{1}{u}\widehat{\phi}\left(\frac{2\log u}{L}\right) d(\theta(u) - u) = \frac{L}{2}\int_0^\infty \widehat{\phi}(u) du$$
$$- (\theta(1) - 1)\widehat{\phi}(0) - \int_1^\infty (\theta(u) - u)\left(-\frac{1}{u^2}\widehat{\phi}\left(\frac{2\log u}{L}\right) + \frac{1}{u^2 L}\widehat{\phi}'\left(\frac{2\log u}{L}\right)\right) du$$
$$= \frac{L}{4}\phi(0) + \widehat{\phi}(0) + \widehat{\phi}(0)\int_1^\infty \frac{\theta(u) - u}{u^2} du + \mathcal{O}(L^{-1}),$$

where the last step is a Taylor expansion of $\widehat{\phi}$. Note that the integral converges by the Prime Number Theorem as stated above, with say $A = 2$.

In conclusion, we have shown

$$\begin{aligned}\frac{-2}{L}\sum_p\sum_{e\geq 1}\frac{x_p(\theta_e+1/p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right) &= -\frac{\phi(0)}{2}-\frac{2}{L}\widehat{\phi}(0)\Bigg(1+\int_1^\infty\frac{\theta(u)-u}{u^2}du\\ &+\sum_p\sum_{e\neq 2}\frac{x_p(\theta_e+1/p)\log p}{p^{e/2}}+\sum_p\frac{\log p}{p}\left(x_p\left(1+\frac{1}{p}\right)-1\right)\Bigg).\end{aligned} \tag{3.3.1}$$

It is possible to estimate the second sum in Theorem 3.1 using the same method. Similar terms as those calculated above are found, but we also find one term of the form

$$L\int_0^\infty \widehat{\phi}\,(u)\,e^{Lu/6}du \ll_\epsilon X^{\sigma/6+\epsilon}. \tag{3.3.2}$$

We are content with remarking that if one carries out the calculations, then one finds the entire sum to be $\ll_\epsilon X^{\sigma/6+\epsilon}$. In particular, if $\sigma < 1$, one sees that the contribution from the term involving this sum in Theorem 3.1 is $\ll X^{-\delta}$ for some $\delta > 0$, as the sum is multiplied by $X^{-1/6}$.

As a consequence of our calculations, we can see that for $\sigma < 2/7$, the main term in Theorem 3.1 is indeed the expected symplectic main term

$$\widehat{\phi}(0)-\frac{\phi(0)}{2} \tag{3.3.3}$$

from the Katz-Sarnak prediction. Indeed, the Fourier transform of

$$\frac{\sin 2\pi x}{2\pi x},$$

is 1/2 multiplied by the characteristic function of the interval $[-1, 1]$, so that an application of Plancherel's formula, and the definition of the Fourier transform, yields

$$\int_{-\infty}^{\infty}\phi(x)\left(1-\frac{\sin 2\pi x}{2\pi x}\right)dx = \widehat{\phi}(0)-\frac{1}{2}\int_{-1}^{1}\widehat{\phi}(u)du = \widehat{\phi}(0)-\frac{\phi(0)}{2},$$

where we used $\sigma < 1$ in the last step. In Chapter 7.2, we will also analyse the secondary term of size L^{-1}, albeit in a slightly different context.

3.4 An application to counting cubic fields

We end the chapter by presenting a theorem [CFLS, Thm. 1.1], which uses the calculation of the one-level density to find a lower bound for the error term in (3.2.3). It turns out that the calculations leading up to Theorem 3.1 can be used to prove that if the estimates

$$N_p^{\pm}(X, T_i) = A_p^{\pm}(T_i)X + B_p(T_i)X^{5/6} + \mathcal{O}_\epsilon(p^\omega X^{\theta+\epsilon})$$

holds with the same value of $\omega, \theta \geq 0$ for all splitting types T_i, and all primes p, then conditional on the Generalised Riemann Hypothesis for ζ_K, one must in fact have $\theta+\omega \geq 1/2$. The idea of the proof is to show that if $\theta + \omega < 1/2$, then we may fix a Schwartz function ϕ that in a sense makes the left-hand side of (3.3.2) too large, which leads to a contradiction. We leave out the details.

This theorem is currently the only known result giving a lower bound for ω and θ. Moreover, as is pointed out in connection to [CFLS, Thm. 1.1], numerical evidence from [CFLS] indicates that this bound is sharp in the sense that (3.2.3) appears to hold with $\theta = 1/2$, and any $\omega > 0$. We remark that this result is why we restrict to $\theta \geq 1/2$, $\omega \geq 0$ in Theorem 3.1.

4

The two-level density

We now extend the results of the previous chapter to the two-dimensional case by finding the two-level density. The same two-level density has previously been studied in a more general setting in [CK, Thm. 4.26], but then only the main term was found and the error term was of size $(\log X)^{-1}$. The main result of this chapter is Theorem 4.1, where the two-level density is found with a power-saving error term. We remark that this is the first time any n-level density, with $n \geq 2$, is calculated to this precision for this family. In other families, such results are also relatively rare.

The methods of the previous chapter essentially suffice to prove Theorem 4.1. The only new theoretical result we will need is an estimate for a counting function of cubic fields, specifying the splitting behaviour of two rational primes, instead of just one. Some of the methods and notation are taken from [Rb], where the n-level density of another symplectic family is studied.

4.1 The explicit formula

Consider the zeros $\rho_K = 1/2+i\gamma_K$ of the Artin L-function $L(s, f_K)$. We want to count the zeros with multiplicity, so let m_{ρ_K} denote the multiplicity of a zero ρ_K. By the functional equation, any zero $1/2+i\gamma_k$ has a corresponding zero $1/2-i\gamma_K$, and a possible zero at $1/2$ has even multiplicity $m_{1/2}$. To every zero, we may associate a collection of pairs $(\rho_K, 1), ..., (\rho_K, m_{\rho_k})$. In the case when $\gamma_K = 0$, it is for technical reasons advantageous to instead consider the two collections of pairs $(+0, 1), ..., (+0, m_{1/2}/2)$, and $(-0, 1), ..., (-0, m_{1/2}/2)$, where $+0$, and -0 are symbols.

The two-level density is then defined as the sum

$$\frac{1}{N^{\pm}(X)} \sum_{K} \sum_{(\gamma_K, i) \neq (\pm\gamma'_K, i)} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma'_K\right).$$

In the sequel, we will simply write the condition $(\gamma_K, i) \neq (\pm\gamma'_K, i)$ as $\gamma_K \neq \pm\gamma'_K$. Here, $1/2 + i\gamma_K$ and $1/2 + i\gamma'_K$ ranges over the zeros of $L(s, f_K)$, and $1 \leq i \leq m_{\rho_K}$. Also, ϕ is a function with compactly supported Fourier transform, and with $\phi(u_1, u_2) = \phi_1(u_1)\phi(u_2)$, with ϕ_1, ϕ_2 both real, even Schwartz functions. Define the sets $B_x = \{u \in \mathbb{R}^2 : |u_1| + |u_2| \leq x\}$, and let $\sigma = \inf\{x : \text{supp}(\widehat{\phi}) \subseteq B_x\}$. The definition implies that $\widehat{\phi}(u_1, u_2)$ is nonzero only if $|u_1| + |u_2| < \sigma$, and furthermore that with $\sigma_i := \sup(\text{supp}(\widehat{\phi_i}))$, we have $\sigma_1 + \sigma_2 = \sigma$.

To study the two-level density, we begin by rewriting the inner sum above to remove the condition $\gamma_K \neq \pm\gamma'_K$, as in [Rb], which will allow us to make use of the methods from the previous chapter. By our remarks on the consequences of the functional equation above, we may write the inner sum above as

$$\sum_{\gamma_K, \gamma'_K} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma'_K\right) - 2\sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma_K\right), \tag{4.1.1}$$

by adding and removing the terms when $\gamma_K = \pm\gamma'_K$, and using that ϕ_i is even. All zeros are counted with multiplicity in the sum above.

We can already estimate

$$-\frac{2}{N^{\pm}(X)}\sum_{K}\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K,\frac{L}{2\pi}\gamma_K\right), \tag{4.1.2}$$

using the results of the previous section. Indeed, define the function $\phi_3(u)=\phi(u,u)$. Then, (4.1.2) is equal to the one-level density multiplied by -2, with the function ϕ replaced by ϕ_3. In particular, it can be estimated by using the right-hand side in Theorem 3.1, if one replaces ϕ by ϕ_3. Henceforth, we refer to this right-hand side as R_3. As the Fourier transform of a product is the convolution of the Fourier transforms, we find that $\widehat{\phi_3}(u)$ is nonzero only if $|u|<\sigma$, cf. [Rb, Claim 1]. In particular, the σ in R_3 is the same as the σ above.

Now, by the explicit formula,

$$\begin{aligned}\sum_{\gamma_K,\gamma_K'}\phi\left(\frac{L}{2\pi}\gamma_K,\frac{L}{2\pi}\gamma_K'\right)&=\sum_{\gamma_K}\phi_1\left(\frac{L}{2\pi}\gamma_K\right)\sum_{\gamma_K'}\phi_2\left(\frac{L}{2\pi}\gamma_K'\right)\\&=\left(\frac{\log|D_K|}{L}\widehat{\phi_1}(0)+\frac{2}{L}\int_{\mathbb{R}}\phi_1(t)\frac{\Gamma_\pm'}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi it}{L}\right)dt-\frac{2}{L}\sum_{n=1}^{\infty}\frac{\Lambda(n)a_K(n)}{\sqrt{n}}\widehat{\phi_1}\left(\frac{\log n}{L}\right)\right)\\&\quad\times\left(\frac{\log|D_K|}{L}\widehat{\phi_2}(0)+\frac{2}{L}\int_{\mathbb{R}}\phi_2(t)\frac{\Gamma_\pm'}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi it}{L}\right)dt-\frac{2}{L}\sum_{n=1}^{\infty}\frac{\Lambda(n)a_K(n)}{\sqrt{n}}\widehat{\phi_2}\left(\frac{\log n}{L}\right)\right).\end{aligned}$$

Similarly as in [Rb], we write this as $(C_{1,K}+E_1+D_{1,K})(C_{2,K}+E_2+D_{2,K})$, with obvious notation.

4.2 A refined counting function for the number of cubic fields

We have previously seen the estimates (3.2.3) for $N_p^{\pm}(T_i;X)$. These estimates were enough to find the one-level density but to find the two-level density we will need a slightly refined counting function. For p and q distinct primes, define $N_{p,q}^{\pm}(X,T_i,T_j)$ as the counting function for isomorphism classes of non-Galois cubic fields whose discriminant is bounded by X, where (p) has splitting type T_i, while (q) has splitting type T_j. Then, using the notation of [CFLS, Eq. (2.3)], we have

$$N_{p,q}^{\pm}(X,T_i,T_j)=A_{p,q}^{\pm}(T_i,T_j)X+B_{p,q}(T_i,T_j)X^{5/6}+\mathcal{O}_\epsilon\big(p^\omega q^\omega X^{\theta+\epsilon}\big), \tag{4.2.1}$$

with the same values of θ and ω as before, and with

$$A_{p,q}^{\pm}(T_i,T_j)=C_1^{\pm}(x_pc_i(p))(x_qc_j(q)),\ B_{p,q}^{\pm}(T_i,T_j)=C_2^{\pm}(y_pd_i(p))(y_qd_j(q)).$$

We end by remarking that one obtains similar results not only for a pair of primes but also for any n-tuple of primes, with the same θ and ω. The best such result is proven in [BTT] along with (3.2.3) and (4.2.1). Specifically, if we let $\mathbf{p}=(p_1,...,p_n)$ be an n-tuple of distinct primes, and $\mathbf{k}=(k_1,...,k_n)$ be an n-tuple of indices of splitting types, i.e. $k_i\in\{1,2,3,4,5\}$. Then we have

$$N_{\mathbf{p}}^{\pm}(X,T_{\mathbf{k}})=C_1^{\pm}X\prod_{i=1}^{n}x_{p_i}c_i(p_i)+C_2^{\pm}X^{5/6}\prod_{i=1}^{n}y_{p_i}d_i(p_i)+\mathcal{O}_\epsilon\left(X^{\theta+\epsilon}\prod_{i=1}^{n}p_i^\omega\right), \tag{4.2.2}$$

where the left-hand side counts fields where each p_i has splitting type T_i.

4.3 Finding the two-level density

The goal of this section will be to find the two-level density by proving the following theorem.

Theorem 4.1. *Let $5/6 > \theta \geq 1/2$, and $\omega \geq 0$ be such that (4.2.1) holds. Let $\phi(u_1, u_2) = \phi_1(u_1)\phi_2(u_2)$ be a product of real even Schwartz functions, with Fourier transform $\widehat{\phi}$ supported in $|u_1| + |u_2| < \sigma < (1-\theta)/(\omega + 1/2) \leq 1$, and set $\phi_3(u) = \phi_1(u)\phi_2(u)$. Then we have*

$$\begin{aligned}
&\frac{1}{N^{\pm}(X)} \sum_{K} \sum_{\gamma_K \neq \pm\gamma'_K} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma'_K\right) = \widehat{\phi_1}(0)\widehat{\phi_2}(0)\Bigg[1 + \frac{2\log(4\pi^2 e)}{L} \\
&+ \frac{4(\log^2(2\pi) + \log(2\pi)) + 2}{L^2} + F_2(X)\left(-\frac{2}{5L} + \frac{2 - 20\log(2\pi)}{25L^2}\right)\Bigg] \\
&+ \widehat{\phi_1}(0)\left(1 + \frac{2\log(2\pi e)}{L}\right)\Big(F_1(X)S_1(2) + F_2(X)S_2(2)\Big) - \frac{\widehat{\phi_1}(0)}{L}\left(F_1(X)S_1(2) + \frac{6}{5}F_2(X)S_2(2)\right) \\
&+ \widehat{\phi_2}(0)\left(1 + \frac{2\log(2\pi e)}{L}\right)\Big(F_1(X)S_1(1) + F_2(X)S_2(1)\Big) - \frac{\widehat{\phi_2}(0)}{L}\left(F_1(X)S_1(1) + \frac{6}{5}F_2(X)S_2(1)\right) \\
&+ F_1(X)\big(S_1(1)S_1(2) - S_3 + S_5\big) + F_2(X)\big(S_2(1)S_2(2) - S_4 + S_6\big) \\
&+ E_1R_2 + E_2R_1 - E_1E_2 - 2R_3 + \mathcal{O}\left(X^{\theta - 1 + \sigma(1/2+\omega) + \epsilon}\right).
\end{aligned} \tag{4.3.1}$$

Here, $F_1(X)$ and $F_2(X)$ are defined in (4.3.5), while $S_1(i)$ and $S_2(i)$ are defined in (4.3.6). The sums S_j for $j \geq 3$ are defined in (4.3.11) and (4.3.14). The integral term E_i is given by

$$E_i = \frac{2}{L}\int_{\mathbb{R}} \phi_i(t)\frac{\Gamma'_{\pm}}{\Gamma_{\pm}}\left(\frac{1}{2} + \frac{2\pi i t}{L}\right) dt.$$

Lastly, R_i is defined as the right-hand side in Theorem 3.1, excluding the error term, i.e.

$$\begin{aligned}
R_i = \widehat{\phi_i}(0)&\left(1 + \frac{\log(4\pi^2 e)}{L} - \frac{F_2(X)}{5L}\right) \\
&+ E_i + F_1(X)S_1(i) + F_2(X)S_2(i).
\end{aligned}$$

Remark. *The expression for the two-level density may seem daunting at first, and we therefore provide some advice on how to discern the largest of the terms. First, $F_2(X) \ll X^{-1/6}$ so that, at least for $\sigma < 1$, $F_2(X)$ multiplied by S_j or $S_2(1)S_2(2)$ is bounded by X to some negative power.*

Furthermore, $F_1(X) = 1 + \mathcal{O}(X^{-1/6})$ so that one may substitute every occurence of $F_1(X)$ with 1, if only the largest terms are of interest. The precise estimates of the sums $S_1(i), S_3, S_5$ are carried out in section 4.4, but we remark that all of them are $\ll 1$. Lastly, the integrals E_i are all $\ll L^{-1}$.

Theorem 4.1 is the first time the two-level density of this family has been investigated past terms of size L^{-1}. Using similar methods as in the previous chapter, the sums $S_1(1), S_1(2), S_3$ and S_5 can be expanded into ascending powers of L^{-1}. In particular, Theorem 4.1 allows us to explicitly find all terms of size $\gg L^{-k}$ for any $k \geq 1$.

To prove the theorem, we must evaluate

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} (C_{1,K} + E_1 + D_{1,K})(C_{2,K} + E_2 + D_{2,K}) \tag{4.3.2}$$

Let R_i be the right-hand side of Theorem 3.1, with $\phi = \phi_i$. Then, by the calculations in the previous chapter, and the fact that E_i does not depend on K, we see that (4.3.2) is

$$E_1R_2 + E_2R_1 - E_1E_2 + \frac{1}{N^{\pm}(X)}\left(\sum_{K \in \mathcal{F}^{\pm}(X)} (C_{1,K}C_{2,K} + D_{1,K}D_{2,K} + C_{1,K}D_{2,K} + D_{1,K}C_{2,K})\right).$$

Thus, we must evaluate four different sums to find the two-level density.

4.3.1 Summing $C_{1,K}C_{2,K}$ over K

We begin by studying the first and easiest sum above, where the calculations can essentially be reduced to integrating the function $\log^2 t := (\log t)^2$. We have

$$\sum_{K\in\mathcal{F}^{\pm}(X)} C_{1,K}C_{2,K} = \frac{\widehat{\phi_1}(0)\widehat{\phi_2}(0)}{L^2}\sum_{K\in\mathcal{F}^{\pm}(X)}\log^2|D_K| = \frac{\widehat{\phi_1}(0)\widehat{\phi_2}(0)}{L^2}\int_1^X (\log^2 t)dN^{\pm}(t).$$

To simplify, use integration by parts, and (3.2.3). The result is that the integral above equals

$$N^{\pm}(X)\log^2 X - 2\log X N^{\pm}(X) - \tfrac{2}{5}\log X C_2^{\pm}X^{5/6} + 2N^{\pm}(X) + \tfrac{22}{25}C_2^{\pm}X^{5/6} + \mathcal{O}_\epsilon(X^{\theta+\epsilon}).$$

Thus, by combining this calculation with (3.2.5) we conclude that

$$\begin{aligned}\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)} C_{1,K}C_{2,K} = \frac{\widehat{\phi_1}(0)\widehat{\phi_2}(0)}{L^2}\Bigg(&\log^2 X - 2\log X + 2\\ &+ \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\left(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\right)\left(\frac{22}{25}-\frac{2}{5}\log X\right)\Bigg) + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)\\ = \widehat{\phi_1}(0)\widehat{\phi_2}(0)\Bigg(&1+\frac{2\log(4\pi^2 e)}{L} + \frac{4\big(\log^2(2\pi)+\log(2\pi)\big)+2}{L^2}\\ &+ \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\left(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\right)\left(-\frac{2}{5L}+\frac{2-20\log(2\pi)}{25L^2}\right)\Bigg) + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right).\end{aligned}$$

4.3.2 Summing $C_{\ell,K}D_{i,K}$ over K

We interchange the order of summation to calculate

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)} C_{\ell,K}D_{i,K} = -\frac{2\widehat{\phi_\ell}(0)}{L^2N^{\pm}(X)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\widehat{\phi_i}\left(\frac{\log n}{L}\right)\sum_{K\in\mathcal{F}^{\pm}(X)} a_K(n)\log|D_K|. \tag{4.3.3}$$

As mentioned earlier $a_K(p^e)$ only depends on e, and how (p) splits in K. Thus, if we separate the summands depending on the splitting type, then we are left with the task of evaluating

$$\sum_{\substack{K\in\mathcal{F}^{\pm}(X),\\ (p)\text{ has splitting type } T_k \text{ in } K}} \log|D_K| = \int_1^X (\log t)\, dN_p^{\pm}(t,T_k),$$

for a fixed p and T_k. Applying integration by parts yields

$$N_p^{\pm}(X,T_k)\log X - N_p^{\pm}(X,T_k) - \frac{1}{5}B_p^{\pm}(T_k)X^{5/6} + \mathcal{O}_\epsilon\left(p^\omega X^{\theta+\epsilon}\right). \tag{4.3.4}$$

Before continuing, we introduce convenient notation. Define

$$F_1(X) = 1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6} + \frac{(C_2^{\pm})^2}{(C_1^{\pm})^2}X^{-1/3},\ F_2(X) = \frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\left(1-\frac{C_2^{\pm}}{C_1^{\pm}}X^{-1/6}\right), \tag{4.3.5}$$

and let

$$S_1(i) = -\frac{2}{L}\sum_{p,e}\frac{x_p\log p}{p^{e/2}}\widehat{\phi_i}\left(\frac{\log p^e}{L}\right)(\theta_e+1/p),\ S_2(i) = -\frac{2}{L}\sum_{p,e}\frac{\log p}{p^{e/2}}\widehat{\phi_i}\left(\frac{\log p^e}{L}\right)\gamma_e(p). \tag{4.3.6}$$

The point of the former notation is that

$$\frac{C_1^{\pm}X}{N^{\pm}(X)} = F_1(X) + \mathcal{O}_\epsilon(X^{\theta-1+\epsilon}) \text{ and } \frac{C_2^{\pm}X^{5/6}}{N^{\pm}(X)} = F_2(X) + \mathcal{O}_\epsilon(X^{\theta-1+\epsilon}), \tag{4.3.7}$$

at least for $\theta \geq 1/2$. After summing over K in (4.3.3), by using (3.2.6) and (4.3.4), we see that

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} C_{\ell,K} D_{i,K} = \widehat{\phi_\ell}(0) \left(1 + \frac{2\log(2\pi e)}{L}\right) \left(F_1(X)S_1(i) + F_2(X)S_2(i)\right)$$
$$- \frac{\widehat{\phi_\ell}(0)}{L} \left(F_1(X)S_1(i) + \frac{6}{5}F_2(X)S_2(i)\right) + \mathcal{O}\left(X^{\theta-1+\sigma_i(1/2+\omega)+\epsilon}\right).$$

4.3.3 Summing $D_{1,K}D_{2,K}$ over K

We now turn to the remaining sum. We write

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} D_{1,K}D_{2,K} = \frac{4}{L^2 N^{\pm}(X)}$$
$$\times \sum_{K \in \mathcal{F}^{\pm}(X)} \sum_{p,e} \sum_{q,f} \frac{(\log p)(\log q)}{p^{e/2}q^{f/2}} \widehat{\phi_1}\left(\frac{e\log p}{L}\right) \widehat{\phi_2}\left(\frac{f\log q}{L}\right) a_K(p^e)a_K(q^f). \tag{4.3.8}$$

The key to evaluating this sum is to move the sum over K inside, and then note that the coefficients are constant when both p, and q have fixed splitting types. In total, there are 16 different pairs of splitting types that produce nonzero coefficients a_K. When $p = q$, their splitting types must be equal. In this special case when $p = q$ we define

$$N^{\pm}_{p,p}(X, T_i, T_j) := N^{\pm}_p(X, T_i)\delta_{i=j}.$$

Write $a_k(p^e)$ for the value of $a_K(p^e)$ when p has splitting type T_k in K. The bound $a_K(p^e) \ll 1$ will be useful when estimating the error terms below. After separating by pairs of splitting types in (4.3.8), we are left with

$$\frac{4}{L^2 N^{\pm}(X)} \sum_{p,e} \sum_{q,f} \frac{(\log p)(\log q)}{p^{e/2}q^{f/2}} \widehat{\phi_1}\left(\frac{e\log p}{L}\right) \widehat{\phi_2}\left(\frac{f\log q}{L}\right)$$
$$\times \sum_{1 \leq k_1,k_2 \leq 4} N^{\pm}_{p,q}(X, T_{k_1}, T_{k_2}) a_{k_1}(p^e) a_{k_2}(q^f).$$

No matter if $p = q$ or not, $N^{\pm}_{p,q}(X, T_{k_1}, T_{k_2})$ has the form of an expression containing a main term, a secondary term, as well as an error that in both cases is $\ll (pq)^{\omega} X^{\theta+\epsilon}$. Thus, using the same method as when bounding the error in the previous chapter, we can split the sum over p and q and find that the contribution of the error term to (4.3.8) is

$$\ll_\epsilon X^{\theta-1+(\sigma_1+\sigma_2)(\omega+1/2)+\epsilon} = X^{\theta-1+\sigma(\omega+1/2)+\epsilon}. \tag{4.3.9}$$

Here, recall that σ_i has the property that $\widehat{\phi_i}$ is zero outside $[-\sigma_i, \sigma_i]$.

Now, the main term and secondary term of $N^{\pm}_{p,q}(X, T_i, T_j)$ are given by

$$\left(C_1^{\pm} X x_p c_i(p) x_q c_j(q) + C_2^{\pm} X^{5/6} y_p d_i(p) y_q d_j(q)\right)\delta_{p\neq q} + \left(C_1^{\pm} X x_p c_i(p) + C_2^{\pm} X^{5/6} y_p d_i(p)\right)$$
$$\times \delta_{p=q}\delta_{i=j} = \left(C_1^{\pm} X x_p c_i(p) x_q c_j(q) + C_2^{\pm} X^{5/6} y_p d_i(p) y_q d_j(q)\right) - \left(C_1^{\pm} X x_p c_i(p) x_p c_j(p)\right.$$
$$\left. + C_2^{\pm} X^{5/6} y_p d_i(p) y_p d_j(p)\right)\delta_{p=q} + \left(C_1^{\pm} X x_p c_i(p) + C_2^{\pm} X^{5/6} y_p d_i(p)\right)\delta_{p=q}\delta_{i=j}. \tag{4.3.10}$$

Thus, to find the sum of $D_{1,K}D_{2,K}$ we must evaluate three essentially different sums. Note that the first term in the left-hand side is the difference of the first and second term in the right-hand side.

To find the sum coming from the first term of (4.3.10) first separate the X and $X^{5/6}$ term, then split both sums over $p, e,\ q, f,\ k_1$ and k_2. The result is that we simply find the sums we have already studied in the previous chapter, but multiplied by each other. Specifically, also using (4.3.7), we obtain the main terms

$$F_1(X)S_1(1)S_1(2) + F_2(X)S_2(1)S_2(2).$$

We also get a contribution coming from the error terms in (4.3.7), but these can be bounded from above by the expression in (4.3.9), after splitting the sums as before.

We turn to the second term in (4.3.10). The presence of the factor $\delta_{p=q}$ means that we only have one sum over the primes. Split the inner sum over k_1, k_2. The sum of $c_i(p)a_i(p^e)$ over $i = 1, 2, 3, 4$ was calculated in the previous chapter to equal $\theta_e + 1/p$, while the sum over $d_i(p)a_i(p^e)$ equals $\gamma_e(p)$. Thus, the contribution from the second term is, excluding the error terms coming from (4.3.7) which are bounded as before,

$$\begin{aligned} -F_1(X)\frac{4}{L^2}\sum_{p,e,f}\frac{x_p^2\log^2 p}{p^{e/2+f/2}}\left(\theta_e+\frac{1}{p}\right)\left(\theta_f+\frac{1}{p}\right)\widehat{\phi_1}\left(\frac{e\log p}{L}\right)\widehat{\phi_2}\left(\frac{f\log p}{L}\right) \\ - F_2(X)\frac{4}{L^2}\sum_{p,e,f}\frac{\log^2 p}{p^{e/2+f/2}}\gamma_e(p)\gamma_f(p)\widehat{\phi_1}\left(\frac{e\log p}{L}\right)\widehat{\phi_2}\left(\frac{f\log p}{L}\right) \\ =: -F_1(X)S_3 - F_2(X)S_4, \end{aligned} \tag{4.3.11}$$

where S_3 and S_4 denote $4/L^2$ multiplied by the first and second sum respectively.

We turn to the last term of (4.3.10). Here the δ-factors restrict the sum to $p = q$ and $k_1 = k_2$. Hence, we see that to evaluate the sum we must calculate

$$\sum_{1\le k\le 4} c_k(p)a_k(p^e)a_k(p^f) = 1 + 2\delta_{2|e}\delta_{2|f} + 3\delta_{3|e}\delta_{3|f} - \delta_{3|e} - \delta_{3|f} + 1/p =: \frac{\iota_{e,f}(p)}{x_p}, \tag{4.3.12}$$

and

$$\begin{aligned} \sum_{1\le k\le 4} d_k(p)a_k(p^e)a_k(p^f) = (1+p^{-1/3})\Big(1+p^{-1/3}+p^{-2/3}+p^{-1}+p^{-4/3}+2\delta_{2|e}\delta_{2|f}(1+p^{-2/3}) \\ + (3\delta_{3|e}\delta_{3|f} - \delta_{3|e} - \delta_{3|f})(1-p^{-1/3}+p^{-2/3})\Big) =: \frac{\xi_{e,f}(p)}{y_p}. \end{aligned} \tag{4.3.13}$$

The contribution from the last sum, excluding the error term, is thus

$$\begin{aligned} F_1(X)\frac{4}{L^2}\sum_{p,e,f}\frac{\log^2 p}{p^{e/2+f/2}}\iota_{e,f}(p)\widehat{\phi_1}\left(\frac{e\log p}{L}\right)\widehat{\phi_2}\left(\frac{f\log p}{L}\right) \\ + F_2(X)\frac{4}{L^2}\sum_{p,e,f}\frac{\log^2 p}{p^{e/2+f/2}}\xi_{e,f}(p)\widehat{\phi_1}\left(\frac{e\log p}{L}\right)\widehat{\phi_2}\left(\frac{f\log p}{L}\right) \\ =: F_1(X)S_5 + F_2(X)S_6, \end{aligned} \tag{4.3.14}$$

which concludes the proof of Theorem 4.1.

4.4 Interpreting Theorem 4.1

In this section, we find the main term in Theorem 4.1, and confirm that it matches the expected symplectic Katz–Sarnak main term. We remark again that this has already been proven in a more general context in [CK, Theorem 4.26].

First, the main term coming from the first two rows of (4.3.1) is simply $\widehat{\phi_1}(0)\widehat{\phi_2}(0)$. Furthermore, by the comments in connection to (3.3.2), we know $F_2(X)S_2(i) \ll X^{-\delta}$, for a $\delta > 0$, if $\sigma_i < 1$. This bound also holds for $F_2(X)S_2(1)S_2(2)$ if $\sigma = \sigma_1 + \sigma_2 < 1$. Moreover, by the previous chapter, we have $F_1(X)S_1(i) = -\phi_i(0)/2 + \mathcal{O}(L^{-1})$, so that the main term coming from the third and fourth row of (4.3.1) is

$$-\frac{\phi_1(0)\widehat{\phi_2}(0)}{2} - \frac{\phi_2(0)\widehat{\phi_1}(0)}{2}.$$

In addition, the contribution from $F_1(X)S_1(1)S_1(2)$ on the fifth row is $\phi_1(0)\phi_2(0)/4$.

To continue, we need to estimate S_3, S_4, S_5 and S_6. We will actually investigate these expressions up to error $\ll L^{-2}$, as we will be interested in the term of size L^{-1} later. We begin by considering S_3, i.e

$$\frac{4}{L^2}\sum_{p,e,f}\frac{x_p^2\log^2 p}{p^{e/2+f/2}}\left(\theta_e+\frac{1}{p}\right)\left(\theta_f+\frac{1}{p}\right)\widehat{\phi_1}\left(\frac{e\log p}{L}\right)\widehat{\phi_2}\left(\frac{f\log p}{L}\right).$$

We can see that the contribution from the terms with $e+f>2$ to the sum is $\ll 1$, so that they contribute $\ll L^{-2}$ to S_3. Indeed, this follows by bounding the absolute value of the summand using the formula for a geometric sum, and the fact that the sum over $1/p^{\delta}$ is finite for $\delta>1$. However, as $\theta_1=0$, we see that the sum over p, when $e=f=1$ also contributes $\ll L^{-2}$ to S_3. Thus, we have shown $S_3\ll L^{-2}$. By a very similar method, using $\gamma_1(p)\ll p^{-1/3}$, we can show $S_4\ll L^{-2}$.

Recall that S_5 is given by

$$\frac{4}{L^2}\sum_{p,e,f}\frac{\log^2 p}{p^{e/2+f/2}}\iota_{e,f}(p)\widehat{\phi_1}\left(\frac{e\log p}{L}\right)\widehat{\phi_2}\left(\frac{f\log p}{L}\right),$$

where $\iota_{e,f}(p)$ is defined in (4.3.12). By the same reasoning as above, we only need to consider the terms where $e=f=1$. Now, $\iota_{1,1}(p)=x_p(1+p^{-1})=1+\mathcal{O}(p^{-1})$. Thus, we may replace $\iota_{1,1}(p)$ by 1 in S_5 at the cost of an error of size $\ll L^{-2}$. The only term in S_5 of interest here is therefore

$$\frac{4}{L^2}\sum_{p}\frac{\log^2 p}{p}\widehat{\phi_1}\left(\frac{\log p}{L}\right)\widehat{\phi_2}\left(\frac{\log p}{L}\right)=\frac{4}{L^2}\int_1^{\infty}\frac{\log u}{u}\widehat{\phi_1}\left(\frac{\log u}{L}\right)\widehat{\phi_2}\left(\frac{\log u}{L}\right)d\theta(u),$$

where $\theta(u)$ is as in (3.2.7). Writing $\theta(u)=u+\theta(u)-u$, and using the error term from (3.2.7), as well as integration by parts, allows us to rewrite the integral as

$$\begin{aligned}&\frac{4}{L^2}\int_1^{\infty}\frac{\log u}{u}\widehat{\phi_1}\left(\frac{\log u}{L}\right)\widehat{\phi_2}\left(\frac{\log u}{L}\right)du+\frac{4}{L^2}\int_1^{\infty}\frac{\log u}{u}\widehat{\phi_1}\left(\frac{\log u}{L}\right)\widehat{\phi_2}\left(\frac{\log u}{L}\right)d(\theta(u)-u)\\&=\frac{4}{L^2}\int_1^{\infty}\frac{\log u}{u}\widehat{\phi_1}\left(\frac{\log u}{L}\right)\widehat{\phi_2}\left(\frac{\log u}{L}\right)du+\mathcal{O}(L^{-2}).\end{aligned}$$

Finally, after making the change of variables $t=L^{-1}\log u$ and using that both $\widehat{\phi_i}$ are even, we find that

$$S_5=2\int_{-\infty}^{\infty}|u|\widehat{\phi_1}(u)\widehat{\phi_2}(u)du+\mathcal{O}(L^{-2}).$$

By using the same method, one finds $S_6\ll 1$, and thus $F_2(X)S_6\ll X^{-1/6}$.

Lastly, as $E_i\ll L^{-1}$, the only main-term contribution from the last line in (4.3.1) is from $-2R_3$. The main term from R_3 is calculated in (3.3.3), if one replaces ϕ by $\phi_3=\phi_1\phi_2$. Thus, we have shown that for ϕ, with

$$\sigma<\frac{1-\theta}{1/2+\omega}\leq 1,$$

the two-level density equals

$$\begin{aligned}&\widehat{\phi_1}(0)\widehat{\phi_2}(0)-\frac{\phi_1(0)\widehat{\phi_2}(0)}{2}-\frac{\phi_2(0)\widehat{\phi_1}(0)}{2}+\frac{\phi_1(0)\phi_2(0)}{4}\\&\quad+2\int_{-\infty}^{\infty}|u|\widehat{\phi_1}(u)\widehat{\phi_2}(u)du-2\widehat{\phi_1\phi_2}(0)+\phi_1(0)\phi_2(0)+\mathcal{O}(L^{-1}).\end{aligned}\tag{4.4.1}$$

Using the notation of [Rb], the Katz-Sarnak prediction for the main term of this symplectic family is

$$\int_{\mathbb{R}^2}\phi(x_1,x_2)\det((K_{-1}(x_i,x_j))_{i,j=1,2})dx_1dx_2,\ \text{with}\ K_{-1}(x,y)=\frac{\sin\pi(x-y)}{\pi(x-y)}-\frac{\sin\pi(x+y)}{\pi(x+y)}.$$

This is exactly (4.4.1), see e.g. [M, Thm. 5.9].

5
Counting cubic fields

In this chapter we indicate how to obtain the estimates, given in (4.2.2), for the number of cubic fields with discriminant bounded by a given magnitude and with prescribed splitting behaviour for a finite set of primes. The main terms in these estimates were found already in [DH], while the secondary terms were found independently in [BST] and [TT], through two different approaches. In this chapter essentially all arguments are taken from [BST].

To avoid too long of a digression from our main goal of studying low-lying zeros, we will focus on studying the main terms in the estimates and not the secondary terms. Despite this, we are using the method in [BST], instead of [DH], because this method lays the groundwork for the computation of the secondary term. The interested reader is therefore well-prepared for studying the rest of the argument in [BST] after having read this chapter. Furthermore, we will not focus on the error terms arising from the various calculations, as our goal is simply to motivate how one obtains the main term, as well as how to take into account the splitting types of a finite number of primes.

We begin by relating cubic fields to certain cubic rings, which are themselves related to integral cubic forms. We then show how to count the number of cubic forms with discriminant bounded by a given magnitude. Finally, we use a sieve argument to relate the estimate for the number of cubic forms to the number of cubic fields. We end the chapter by briefly describing how to generalise the calculation for the main term to also obtain a secondary term.

5.1 Cubic rings and forms

Following [BST, Ch. 2-3], we want to relate cubic fields to (binary) cubic forms. To accomplish this, we first relate cubic fields to certain cubic rings. Here, a cubic ring is a commutative ring, that is also a free $\mathbb{Z}$-module of rank 3. We say that a cubic ring R is nonmaximal at p, if it is contained in some other cubic ring R', with index divisible by p. We say that the cubic ring R is maximal if it is not nonmaximal at any p. We remark that by the theory of modules over a PID, any containment $R \subseteq R'$ of cubic rings, implies that R has finite index in R', cf. [N, Thm I.2.12]. Thus, a maximal cubic ring is not strictly contained in any cubic ring.

We define the discriminant of a cubic ring in a similar manner as the discriminant of a cubic field, i.e. as the determinant of any matrix representing the bilinear trace form $(x, y) \mapsto \mathrm{Tr}(xy)$, with respect to some $\mathbb{Z}$-basis of R. Finally, we say that a cubic ring R is a cubic order if it is an integral domain.

We are now ready to state an introductory lemma that relates cubic fields to maximal cubic rings. We remark that this lemma is not contained in [BST].

Lemma 5.1. *There is a discriminant preserving bijection between isomorphism classes of cubic fields and isomorphism classes of maximal cubic orders.*

Proof. Given a cubic field K, we consider its ring of integers $R := \mathcal{O}_K$. By algebraic number theory, we know that this is a cubic ring, see e.g. [N, Ch. I.2]. We claim that R is also maximal. Indeed, if R' is any cubic ring containing R with a finite index, then R' can naturally be considered as a subset of $\mathrm{Frac}(\mathcal{O}_K) = K$. Further, as R' is a free $\mathbb{Z}$-module of rank 3, it is in particular finitely generated over $\mathbb{Z}$, which means that all elements of R' are integral, whence $R' \subseteq \mathcal{O}_K = R$, so that R is maximal. It follows that we may associate a maximal cubic order to any cubic field.

Conversely, suppose that R is a maximal cubic order, so that we may consider $K := \mathrm{Frac}(R)$. Let x_1, x_2, x_3 generate R over $\mathbb{Z}$. Then $\mathbb{Q}[x_1, x_2, x_3]$ is a field containing R, as all x_i are algebraic, and thus it is equal to K. It follows that K is a cubic field extension of $\mathbb{Q}$, as any $\mathbb{Z}$-basis of R is a $\mathbb{Q}$-basis of K. Thus, to any maximal cubic order, we can associate a cubic field.

The maps we have constructed induce maps on the appropriate sets of isomorphism classes, and it is clear that the maps are each other's inverses. It is also clear from the definitions that the discriminant is preserved. □

We conclude that instead of counting cubic fields, we may count maximal cubic rings. Next, we relate cubic rings to cubic forms using the Delone-Faddeev correspondence. Here, an integral cubic form is a homogeneous polynomial $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, with $a, b, c, d \in \mathbb{Z}$. On the set of integral cubic forms, we define a group action of $\mathrm{GL}_2(\mathbb{Z})$ by letting $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ act on f by

$$(\gamma f)(x, y) = \frac{1}{\det(\gamma)} f((x, y)\gamma),$$

where (x, y) is considered as a row vector in the right-hand side. We now prove [BST, Thm. 9].

Theorem 5.2 (BST)**.** *There is a bijection between the set of* $\mathrm{GL}_2(\mathbb{Z})$*-orbits of integral binary cubic forms and the set of isomorphism classes of cubic rings.*

Proof. We give a direct proof from [BST], with some extra details provided. Let R be a cubic ring. We begin by finding an appropriate basis of R, which in particular will include 1. Begin by choosing any basis x_1, x_2, x_3 of R, and write $1 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, $\lambda_i \in \mathbb{Z}$. Let d_0 be the greatest integer dividing all λ_i. Then $1/d_0 \in R$, and we must thus have $d_0 = 1$, as R is a $\mathbb{Z}$-module of finite type. This implies that $R/\mathbb{Z}$ is a finitely generated and torsion-free module over the PID $\mathbb{Z}$; whence it is also free. Hence, we may pick a basis $\{1, \omega, \theta\}$ of R. Furthermore, we may require $\omega\theta \in \mathbb{Z}$ by possibly subtracting some element of $\mathbb{Z}$ from each of ω, θ. We call a basis $\{1, \omega, \theta\}$ such that $\omega\theta \in \mathbb{Z}$ a normal basis.

Using our normal basis, we write

$$\omega\theta = n, \quad \omega^2 = m - b\omega + a\theta, \quad \theta^2 = \ell - d\omega + c\theta, \tag{5.1.1}$$

for some $a, b, c, d, n, m, \ell \in \mathbb{Z}$. We then associate the binary cubic form $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ to the cubic ring R. Conversely, to a binary cubic form $f(x, y)$ as above, we associate a cubic ring R with basis $\{1, \omega, \theta\}$ with multiplication laws defined by (5.1.1). If we set

$$n = -ad, \quad m = -ac, \quad \ell = -bd, \tag{5.1.2}$$

then one can confirm that the multiplication becomes associative, so that the multiplication laws really define a ring. In particular, this is the unique choice of values for n, m, ℓ which makes the multiplication associative.

To show that this map induces a map between isomorphism classes of rings, and orbits under the $\mathrm{GL}_2(\mathbb{Z})$-action, we require another description of the map, using the wedge product. It turns out that if we start with a cubic ring R, then the associated cubic form represents the map $R/\mathbb{Z} \to \bigwedge^2(R/\mathbb{Z}) \cong \mathbb{Z}$ defined by $r \mapsto r \wedge r^2$. Indeed, if we write $r = x\omega + y\theta$, then a direct calculations shows that $r \wedge r^2 = f(x, y)(\omega \wedge \theta)$.

Now, any ring R' isomorphic to R must have elements obeying the same multiplication laws as R does. However, they need not be associated to the same cubic form, as the form depends on the choice of basis $\{\overline{\omega}, \overline{\theta}\}$ of $R/\mathbb{Z}$, where $\overline{a}$ denotes the image of $a \in R$ under the natural map $R \mapsto R/\mathbb{Z}$. Here ω, θ are chosen in the unique way making $\{1, \omega, \theta\}$ a normal basis. Any change of basis of $R/\mathbb{Z} \cong \mathbb{Z}^2$ is given by some $\gamma \in \mathrm{GL}_2(\mathbb{Z})$. Specifically, the coordinates $(\beta_1, \beta_2)^T$ with respect to the old basis are transformed into $\gamma(\beta_1, \beta_2)^T$ with respect to the new basis. Write

$$\gamma = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

Then, under the map $r \mapsto r \wedge r^2$ above, changing the basis of $R/\mathbb{Z}$ by γ results in changing $f(x,y)(\omega \wedge \theta)$ into

$$\frac{f((x,y)\gamma)}{\det \gamma}(e_1 \wedge e_2),$$

where $e_1 = \alpha_1\omega + \alpha_3\theta$, $e_2 = \alpha_2 + \alpha_4\theta$. It follows that the given map between cubic forms and cubic rings, induces a bijection between isomorphism classes of cubic rings, and $\mathrm{GL}_2(\mathbb{Z})$ orbits of cubic forms, as desired. □

The correspondence above, called the Delone-Faddeev correspondence, has several nice properties. First, it is discriminant preserving [BST, Prop 10], where we define the discriminant $\mathrm{Disc}(f)$ of the form $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ as $b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$. Indeed, this follows by simply calculating the discriminant of the ring R, with multiplication table (5.1.1). We remark that this discriminant is the usual univariate discriminant of both $f(x,1)$ and $f(1,y)$, whence it is zero precisely when $f(x,y)$ has a multiple root as a homogeneous polynomial.

Second, if we denote by $R(f)$ the cubic ring corresponding to the cubic form f, then the stabiliser of f is isomorphic to the group of ring automorphisms of $R(f)$, [BST, Prop. 12]. Indeed, this follows immediately from the connection between base changes of $R/\mathbb{Z}$, and the $\mathrm{GL}_2(\mathbb{Z})$-action on the space of binary cubic forms. Lastly, the ring $R(f)$ is an integral domain iff the cubic form f is irreducible [BST, Prop 11]. We leave out the proof for the sake of brevity. As in [BST], we remark that the Delone–Faddeev correspondence and the properties above, can be generalised to hold for other base rings than $\mathbb{Z}$. In particular, we will be interested in the base rings $\mathbb{F}_p$ and $\mathbb{R}$. We leave out the precise details of these generalisations.

We now give a criterion for the cubic ring R being maximal at p [BST, Lemma 13].

Lemma 5.3 (BST)**.** *Suppose that a cubic ring R is nonmaximal at p. Then there is a normal $\mathbb{Z}$-basis $\{1, \omega, \theta\}$ of R such that at least one of*

$$\mathbb{Z} + \mathbb{Z}(\omega/p) + \mathbb{Z}\theta \quad \textit{and} \quad \mathbb{Z} + \mathbb{Z}(\omega/p) + \mathbb{Z}(\theta/p)$$

is a ring.

Proof. As R is nonmaximal at p, there exists a cubic ring R' containing R with finite index, divisible by p. By a natural identification, we may consider R' as being contained in $(\mathbb{Z}^*)^{-1}R$, i.e. the ring of fractions created by inverting all nonzero integers in R. Next, let $S = \{p, p^2, p^3, ...\}$ and consider the ring of fractions $S^{-1}R \subseteq (\mathbb{Z}^*)^{-1}R$, i.e. the result of only inverting powers of p.

We define the ring $R_1 = R' \cap S^{-1}R$ and note that by applying the structure theorem for finite abelian groups to R'/R, we see that R_1 is nonempty, as R'/R must contain elements of order p. Clearly, R_1 contains R, and furthermore, the index $[R_1 : R]$ must equal some power of p, again by the structure theorem for finite abelian groups. Note also that R_1 is a cubic ring, as it contains the cubic ring R, and is contained in the cubic ring R'.

The existence of the Smith normal form, see [J, Thm. 3.8], implies that we may pick a basis $\{1, \omega, \theta\}$ of R so that

$$R_1 = \mathbb{Z} + \mathbb{Z}(\omega/p^i) + \mathbb{Z}(\theta/p^j), \tag{5.1.3}$$

with $i \geq j$, say. A calculation using (5.1.1) shows that the above holds even if we first normalise the basis. If $i = 1$ above, then the proof is completed, so we assume $i > 1$. We will now "reduce" the numbers i and j, until $i = 1$, which will prove the lemma.

Using the multiplication laws (5.1.1), we see that (5.1.3) being a ring is equivalent to the conditions

$$a \equiv 0 \mod p^{2i-j}, \quad b \equiv 0 \mod p^i, \quad c \equiv 0 \mod p^j, \quad d \equiv 0 \mod p^{2j-i}, \tag{5.1.4}$$

with the convention that any congruence relation holds if the exponent of p is less than or equal to 0. If $j = 0$, then we can immediately replace the pair (i, j) by $(i-1, j)$ with the conditions above still holding. If instead $j \geq 1$, then we can replace (i, j) by $(i-1, j-1)$. Applying this procedure a finite number of times, we will have reduced to $i = 1$, and $j = 0$ or $j = 1$, as desired. □

Let $R(f)$ be a cubic ring, and f the corresponding binary cubic form under the Delone–Faddeev correspondence, i.e. under the correspondence given in Theorem 5.2. Then, (5.1.4) with $i = j = 1$, or $i = 1$, $j = 0$, implies that $R(f)$ is nonmaximal at p if and only if p divides all coefficients of f, or if there is some $g(x, y) = a'x^3 + b'x^2y + c'xy^2 + dy^3 \in \mathrm{Orb}(f)$ such that $p^2 \mid a'$, and $p \mid b'$.

If we let $\mathcal{U}_p$ denote the set of cubic forms which do not fulfil any of the conditions just mentioned, then we have proved [BST, Thm. 14].

Theorem 5.4 (BST)**.** *The cubic ring $R(f)$ is maximal at p if and only if $f \in \mathcal{U}_p$. Furthermore, $R(f)$ is maximal if and only if $f \in \mathcal{U}_p$ for all primes p.*

We now introduce a connection between certain binary cubic forms and the splitting type of a prime p in the ring of integers of a cubic field. We will let $\mathbb{F}_p$ denote the field with p elements, and $\overline{\mathbb{F}}_p$ some algebraic closure of this field. Let f be any binary cubic form, which is nonzero when reduced modulo p. Then, by considering $f(x, 1)$ and $f(1, y)$, the fact that $\overline{\mathbb{F}}_p$ is algebraically closed implies that the homogeneous polynomial f has exactly three roots, counting with multiplicity, in the projective space $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$. Here, for any commutative ring R, the projective space $\mathbb{P}^1_R$ is defined as the set of equivalence classes of pairs (r_1, r_2), with $r_1, r_2 \in R$ not both 0, where (r_1, r_2) is related to (r_3, r_4) if and only if there is some unit $s \in R$ such that $(r_1 s, r_2 s) = (r_3, r_4)$. We denote the equivalence class of (r_1, r_2) by $[r_1, r_2]$.

Now, to each prime p, and cubic form f with nonzero reduction modulo p, we want to associate a symbol depending on the roots of f modulo p. If $f \bmod p$ has three distinct roots in $\mathbb{F}_p$, we define this symbol as (111), while if it has one root of multiplicity exactly equal to 2 in $\mathbb{F}_p$, we set the symbol to (1^21), and if it has a root of multiplicity 3 we define the symbol as (1^3). In the case when $f \bmod p$ has a root in some quadratic extension of $\mathbb{F}_p$, we define the symbol as (21), and lastly, if it has a root in some cubic extension, we set the symbol to (3).

The reader may note the similarity to the symbols we have used in earlier chapters to denote the splitting type of (p) in some cubic extension. The similarity is not a coincidence; indeed, the symbol we have just defined is equal to $(f_1^{e_1} f_2^{e_2} ...)$, with all $f_i, e_i \in \{1, 2, 3\}$, if and only if

$$\frac{R(f)}{(p)} \cong \frac{\mathbb{F}_{p^{f_1}}[t_1]}{(t_1^{e_1})} \oplus \frac{\mathbb{F}_{p^{f_2}}[t_2]}{(t_2^{e_2})} \oplus \cdots. \tag{5.1.5}$$

In particular, by using the Chinese Remainder Theorem and lifting representatives of the prime ideals of the right-hand side, we see that (p) splits as $\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2} \cdot ....$, where the norm of $\mathfrak{p}_i$ is given by p^{f_i}.

We illustrate the proof of (5.1.5) for the case (111). Some points of the argument are taken from [Wr, Ch. 2]. We begin with some preparation. Up to scaling, $\mathrm{GL}_2(\mathbb{Z})$ acts on the space of binary cubic forms by a change of variables. Thus, we may essentially consider the action as a linear transformation of the roots. Furthermore, for the sake of studying the ring in (5.1.5), we may study $\mathrm{GL}_2(\mathbb{F}_p)$-orbits of binary cubic forms, instead of $\mathrm{GL}_2(\mathbb{Z})$-orbits. Now, define $\mathrm{PGL}_2(\mathbb{F}_p) := \mathrm{GL}_2(\mathbb{F}_p)/D$, where D is the group of all matrices λI_2, $\lambda \in \mathbb{F}_p$. This is the projective general linear group of dimension 2. It is well-known, see e.g [DM, Exercises 2.8.4, 2.8.7], that this group is triply transitive, i.e. for any three projective points there is some element mapping these to any other three projective points. It follows that f reduced modulo p lies in the $\mathrm{GL}_2(\mathbb{F}_p)$-orbit of the form $uv(u+v)$ modulo p.

Using the above, and the Delone–Faddeev correspondence over $\mathbb{F}_p$, we see that $R(f)/(p)$ is isomorphic to $R(xy(y-x))/(p)$. Now, again by (5.1.1) and (5.1.2), the ring $R(xy(y-x))$

is determined by the multiplication laws $\omega\theta = 0$, $\theta^2 = \theta$, and $\omega^2 = \omega$. Thus,

$$\frac{R(xy(y-x))}{(p)} \cong (\mathbb{F}_p)^3,$$

as desired. An explicit isomorphism is given by $1 \mapsto (1,1,1)$, $[\omega] \mapsto (0,1,0)$ and $[\theta] \mapsto (0,0,1)$.

Before ending this section, we want to investigate how common the various splitting types are, and in particular how common they are among the maximal forms. Let T be some splitting type, and define $T_p(T)$ as the set of all integral binary cubic forms f, with nonzero reduction $\overline{f}$ modulo p, splitting according to T. We define $\mathcal{U}_p(T)$ as $\mathcal{U}_p \cap T_p(T)$. Now, if f is some integral binary cubic form, then membership in the various sets $T_p(T)$ only depends on the coefficients modulo p, while maximality at p depends on the coefficients modulo p^2. Motivated by this, we define a subset $\mathcal{U}'_p(T)$ of the set of binary cubic forms with coefficients in $\mathbb{Z}/p^2\mathbb{Z}$. Here we let $\overline{f} \in \mathcal{U}'_p(T)$ if and only if its roots modulo p splits as T, and if $\overline{f}$ is the reduction of some integral binary cubic form f modulo p^2, where f is maximal at p. We define $\mathcal{U}'_p$ as the union over i of all $\mathcal{U}'_p(T_i)$.

The set of binary cubic forms with coefficients in $\mathbb{Z}/p^2\mathbb{Z}$ is naturally identified with $(\mathbb{Z}/p^2\mathbb{Z})^4$, and on this set, we consider the measure μ_p, defined as the counting measure divided by p^8. We then have the following lemma [BST, Lemmas 18-19].

Lemma 5.5 (BST)**.** *We have*

$$\begin{aligned}
\mu_p(\mathcal{U}'_p(111)) &= \frac{1}{6} \cdot \frac{(p-1)^2 p(p+1)}{p^4}, \\
\mu_p(\mathcal{U}'_p(12)) &= \frac{1}{2} \cdot \frac{(p-1)^2 p(p+1)}{p^4}, \\
\mu_p(\mathcal{U}'_p(3)) &= \frac{1}{3} \cdot \frac{(p-1)^2 p(p+1)}{p^4}, \\
\mu_p(\mathcal{U}'_p(1^2 1)) &= \frac{(p-1)^2 (p+1)}{p^4}, \\
\mu_p(\mathcal{U}'_p(1^3)) &= \frac{(p-1)^2 (p+1)}{p^5},
\end{aligned}$$

so that we in particular have

$$\mu_p(\mathcal{U}'_p) = \frac{(p^3-1)(p^2-1)}{p^5}.$$

Proof. The last equality follows by adding together all the other densities. We show how to compute the density for $T = (111)$ and $T = (1^3)$, and refer to [BST] for a complete proof.

We begin by considering $T = (111)$. Note that the discriminant of any form f with distinct roots modulo p, cannot be divisible by p. Furthermore, if $R = R(f) \subseteq R'$, for some cubic ring R', we have $\mathrm{Disc}(R) = (R' : R)^2 \mathrm{Disc}(R')$, cf. [N, Thm I.2.12], which together with $p \nmid \mathrm{Disc}(R)$ implies that R must be maximal at p. Thus to find $\mu_p(\mathcal{U}'_p(111))$, it suffices to find the corresponding density for forms with three distinct roots. In particular, we can reduce our forms modulo p, and compute the density in $(\mathbb{F}_p)^4$. Counting such forms is the same as counting the number of ways to choose 3 different roots, and then choosing a nonzero scalar. As there are $p+1$ elements to choose from in $\mathbb{P}_{\mathbb{F}_p}$, we have

$$\mu_p(\mathcal{U}_p(111)) = \frac{1}{p^4} \cdot \binom{p+1}{3} \cdot (p-1),$$

as desired.

We turn to the case $T = (1^3)$. The density of forms in $(\mathbb{F}_p)^4$ with a triple root, is $(p-1)(p+1)/p^4$, by a similar argument as above. Now, not all forms $\overline{f}$ in $(\mathbb{Z}/p^2\mathbb{Z})^4$, whose reductions modulo p has a triple root, correspond to maximal integral forms, but as we will see most of them do. First, as the forms are nonzero modulo p, not all coefficients can be divisible by p, so we only need to check the second condition for being nonmaximal. By the action of a $\mathrm{GL}_2(\mathbb{Z})$ matrix on $\overline{f}$ (or on some lifting f), we may send the root of $\overline{f}$ mod p to $[1,0]$, which brings $\overline{f}$ to the form $ax^3 + bx^2y + cxy^3 + dy^3$, where a, b, c are divisible by p. Exactly $1/p$ of these forms also have $p^2 \mid a$, and we thus arrive at the density $(p-1)(p+1)p^{-4}(p-1)p^{-1}$, as desired. □

5.2 Counting integral binary cubic forms

We now want to find an estimate for the number of integral binary cubic forms, with discriminant bounded by X. These results will be used in the next section, where we sieve these forms for maximality.

The goal of this section will be to partially prove the following theorem [BST, Thm. 5], using the methods of [BST, Ch. 5,6].

Theorem 5.6 (BST)**.** *Let $N(\xi, \eta)$ denote the number of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of irreducible integral binary cubic forms f satisfying $\xi < Disc(f) < \eta$. Then,*

$$N(0;X) = \frac{\pi^2}{72}X + \frac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{30\Gamma(2/3)}X^{5/6} + \mathcal{O}_\epsilon\left(X^{3/4+\epsilon}\right),$$
$$N(-X,0) = \frac{\pi^2}{24}X + \frac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{10\Gamma(2/3)}X^{5/6} + \mathcal{O}_\epsilon\left(X^{3/4+\epsilon}\right).$$

Our goal will not be to rigorously prove these estimates, but rather to indicate how to obtain the main terms above.

To begin counting these forms, we need to not only consider integral binary cubic forms, but also real binary cubic forms. We denote the space of integral binary cubic forms by $V_{\mathbb{Z}}$, and the corresponding real space by $V_{\mathbb{R}}$. Further, let $V_{\mathbb{R}}^{(0)}$ be the subset of $V_{\mathbb{R}}$ consisting of forms with positive discriminant, and $V_{\mathbb{R}}^{(1)}$ be the set of forms with negative discriminant. Using the generalisation of the Delone–Faddeev correspondence mentioned in the previous section, we see that both these sets consist of one $\mathrm{GL}_2(\mathbb{R})$-orbit each, as the corresponding ring is either $\mathbb{R}^3$ or $\mathbb{R} \oplus \mathbb{C}$.

It turns out that we will need to study the group $\mathrm{GL}_2(\mathbb{R})$ a bit closer before we can start counting forms. In particular, we will need to study the natural action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})$. Our goal is to find a natural fundamental domain, i.e. some subset of $\mathrm{GL}_2(\mathbb{R})$ containing one representative from each $\mathrm{GL}_2(\mathbb{Z})$-orbit. This will be carried out in the subsection below

5.2.1 The group $\mathrm{GL}_2(\mathbb{R})$

We now study the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})$ and find a fundamental domain. First, we define

$$K_1 = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) : \gamma \text{ is orthogonal}\},$$
$$A_+ = \{a(t) : t \in \mathbb{R}_+\}, \text{ where } \quad a(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$
$$N = \{n(u) : n \in \mathbb{R}\}, \text{ where } \quad n(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$
$$\Lambda = \{L(\lambda) : \lambda \in \mathbb{R}_+\}, \text{ where } \quad L(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

We will often write λ instead of $L(\lambda)$. By considering the QR factorisation of a matrix, and taking inverses, we see that each element in $\mathrm{GL}_2(\mathbb{R})$ can be uniquely written as a product $n^T ak\lambda$, with $n \in N$, $a \in A_+$, $k \in K_1$ and $\lambda \in \Lambda$. Here n^T denotes the transpose of n.

We now find a fundamental domain, essentially following the approach in [Ls, Lemma 3.33], with some modifications. First, as the matrix

$$P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.2.1}$$

is an element of $\mathrm{GL}_2(\mathbb{Z})$ with determinant -1, we need only consider elements with a positive determinant for our fundamental domain. As an invertible matrix with integral entries, has a determinant equal to ± 1, this means that we may consider the action of $SL_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})^+$, and find a fundamental domain for this action instead. Here $\mathrm{GL}_2(\mathbb{R})^+$ denotes the elements in $\mathrm{GL}_2(\mathbb{R})$ with positive determinant, and these elements can also be written uniquely as a product $n^T ak\lambda$ as above, but now $k \in SO_2(\mathbb{R})$.

Next, we consider the group action of $\mathrm{GL}_2(\mathbb{R})^+$ on the upper half-plane $\mathbb{H}$ of $\mathbb{C}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

That this is really an action on $\mathbb{H}$ follows after a straightforward calculation, using the positivity of the determinant. Now, the matrix $(n(u))^T a(t^{-1})$ maps z to $t^2 z + u$, which shows that this is a transitive group action. Now, consider the element $i \in \mathbb{H}$. A calculation shows that this element is stabilised by all of Λ and $SO_2(\mathbb{R})$. No other element in $\mathrm{GL}_2(\mathbb{R})^+$ can stabilise i, as $(n(u))^T a(t^{-1})k\lambda$ acts on i by mapping it to $t^2 i + u$. Thus, by the orbit stabiliser theorem, we have the equivalence

$$\frac{\mathrm{GL}_2(\mathbb{R})^+}{SO_2(\mathbb{R})\Lambda} \cong \mathbb{H}, \tag{5.2.2}$$

as G-sets, where $G := \mathrm{GL}_2(\mathbb{R})^+$. Here, the left-hand side denotes the left cosets. Now, if we let I denote the identity matrix, then $\pm I \in SO_2(\mathbb{R})$. In particular, this equivalence also holds if one instead considers the action of the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/(\pm I)$ on either side of the above.

We recall that our immediate goal is to find a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})^+$. We first find a fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the left-hand side of (5.2.2), or equivalently the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. A classical result, see [S, Thm. VII.1], asserts that such a fundamental domain is given by all z with $|\mathrm{Re}(z)| < 1/2$, $|z| > 1$, at least up to a Lebesgue null-set. As the purpose of finding a fundamental domain will be to integrate over it later, we may disregard any null sets. As the isomorphism above is given by acting on i, this gives the conditions $|u| < 1/2$, $t^4 + u^2 > 1$ on u and t of $(n(u))^T a(t^{-1})$.

Lastly, we must extend the fundamental domain for the action on the cosets to a fundamental domain for the action on all of $\mathrm{GL}_2(\mathbb{R})^+$. Let K' be the subset of $SO_2(\mathbb{R})$ containing matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{5.2.3}$$

with $\pi < \theta \leq 2\pi$. This is a set of lifted representatives of the cosets $SO_2(\mathbb{R})/(\pm I)$.

We claim that up to a null set (a null set when viewing $\mathrm{GL}_2(\mathbb{R})$ as a subset of $\mathbb{R}^4$ with the Lebesgue measure), a fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})^+$, and thus also for the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})$, is given by $\mathcal{F}' = \{(n(u))^T a(t^{-1})k\lambda : k \in K', \lambda \in \Lambda, |u| < 1/2, t^4 + u^2 > 1\}$. Indeed, by our investigation of the action on cosets above, almost every element of $\mathrm{GL}_2(\mathbb{R})$ is represented by some element in $\mathcal{F}'$. Furthermore, no two elements in the set above lie in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit. We see this by noting that if $g \in \mathrm{SL}_2(\mathbb{Z})$ is such that $g(n(u_1))^T a(t_1^{-1})k_1\lambda_1 = (n(u_2))^T a(t_2^{-1})k_2\lambda_2$, then by reducing modulo $SO_2(\mathbb{R})\Lambda$, we find that $u_1 = u_2$, $t_1 = t_2$ and that g stabilises the point $(n(u_1))^T a(t_1^{-1})i \in \mathbb{H}$. It follows that $g = \pm I$ by [S, Thm. VII.1.(3)]. This shows that $\pm k_1\lambda_1 = k_2\lambda_2$; whence $\lambda_1 = \lambda_2$ and $k_1 = k_2$, as K' only contains rotations with $\pi < \theta \leq 2\pi$, which proves the claim.

The fundamental domain $\mathcal{F}'$ that we have found is not the one used in [BST]. We, therefore, show how to modify $\mathcal{F}'$ to obtain a fundamental domain more similar to the one from [BST]. Recall that we defined a matrix P in (5.2.1). As $P \in \mathrm{GL}_2(Z)$, we see that the transformed domain $\mathcal{F} := P\mathcal{F}'P^{-1}$ is also a fundamental domain. A straightforward calculation shows that $\mathcal{F}$ is the set $\{n(u)a(t)k\lambda : k \in K, \lambda \in \Lambda, |u| < 1/2, t^4+u^2 > 1\}$, where K is now the subset of $SO_2(\mathbb{R})$ containing matrices of the form (5.2.3), with $0 \leq \theta < \pi$.

We end this section by introducing a (Haar) measure dg on $\mathrm{GL}_2(\mathbb{R})$. Indeed, such a measure is given by $dg = t^{-3}\lambda^{-1}dudtdkd\lambda$, where $n(u), a(t), \lambda, k$ are as in the decomposition $n(u)a(t)k\lambda$ above. The measure dk is a measure on the real orthogonal group, induced by the parametrisation using θ, that we have seen for $SO_2(\mathbb{R})$ above. We normalise dk to give the set K measure 1. The most important property of dg is that it is $\mathrm{GL}_2(\mathbb{R})$-invariant, but we leave out the proof of this fact for the sake of brevity, see e.g. [Ls, Lemma 3.22].

5.2.2 Reducing to an integral

We now show how to apply the results of the previous subsection to count integral binary cubic forms, following [BST, Ch. 5.1]. Fix any vector $v \in V_{\mathbb{R}}^{(i)}$, $i = 0, 1$, and consider the set $\mathcal{F}_0 v$, where $\mathcal{F}_0$ is some fundamental domain of the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathrm{GL}_2(\mathbb{R})$. Recall that the set $\mathcal{F}$ is not quite an actual fundamental domain, but rather a fundamental domain up to a set of measure zero. Therefore, for these initial algebraic arguments, we work with $\mathcal{F}_0$ instead.

Let n_i be the size of the stabiliser of v in $\mathrm{GL}_2(\mathbb{R})$. By the generalised Dalone–Faddeev correspondence, $n_0 = 6$ and $n_1 = 2$. We claim that $\mathcal{F}_0 v$ is the union of n_i fundamental domains for the action of $\mathrm{GL}_2(\mathbb{Z})$ on $V_{\mathbb{R}}^{(i)}$. Indeed, this follows from the fact that $gv = g'v$ is equivalent to $g^{-1}g' \in \mathrm{Stab}(v)$, and that $\mathcal{F}_0$ is a fundamental domain for $\mathrm{GL}_2(\mathbb{Z})$ acting on $\mathrm{GL}_2(\mathbb{R})$. An element in $\mathcal{F}_0 v$ may belong to several of the n_i fundamental domains. We therefore view $\mathcal{F}_0 v$ as a multiset, where the multiplicity of a point x is given by the cardinality of $\{g \in \mathcal{F}_0 : gv = x\}$. If we let $\mathrm{GL}_2(\mathbb{R})$ act on v instead of $\mathcal{F}_0$, then every such multiplicity would be n_i. Using this fact, a calculation shows that for $x \in V_{\mathbb{Z}}$ the multiplicity above in $\mathcal{F}_0 v$ must instead be $n_i/m_i(x)$, where $m_i(x)$ is the size of the stabiliser of x in $\mathrm{GL}_2(\mathbb{Z})$. We mention that if x corresponds to an order under the Delone–Faddeev correspondence, then $m_i(x)$ either equals 1 or 3, and we have $m_i(x) = 3$ if and only if the corresponding fraction field is a C_3-field. Indeed, this follows from $\mathbb{Q}[R] = \mathrm{Frac}(R)$.

Write $N(V_{\mathbb{Z}}^{(i)}; X)$ for the number of $\mathrm{GL}_2(\mathbb{Z})$-orbits of irreducible integral binary cubic forms in $V_{\mathbb{Z}}^{(i)}$ with absolute discriminant less than X. Then, the discussion above has shown that $n_i \cdot N(V_{\mathbb{Z}}^{(i)}; X)$ is equal to the number of integral points in $\mathcal{F}_0 v$, as long as we weigh the point x with $m_i(x) = 3$ counted in $N(V_{\mathbb{Z}}^{(i)}; X)$ with a factor 1/3. This is not a problem as [BST, Lemma 22] shows that such points are very rare, and for our purposes we may therefore ignore this weighting when it is convenient to do so.

We remark as in [BST], that counting points from a single $\mathcal{F}_0 v_0$ is very hard. The method of proof will therefore instead be based on calculating averages of sets $\mathcal{F}_0 v$, where v varies over points in some compact subset of $V_{\mathbb{Z}}^{(i)}$. As mentioned earlier, in the actual calculations we will use the set $\mathcal{F}$ in place of an actual fundamental domain. Specifically, the previous discussion implies that

$$N(V_{\mathbb{Z}}^{(i)}; X) = \frac{\int_{v \in B \cap V_{\mathbb{R}}^{(i)}} \#\{x \in \mathcal{F}v \cap V_{\mathbb{Z}}^{\mathrm{irr}} : |\mathrm{Disc}(x)| < X\} |\mathrm{Disc}(v)|^{-1} dv}{n_i \int_{v \in B \cap V_{\mathbb{R}}^{(i)}} |\mathrm{Disc}(v)|^{-1} dv}, \tag{5.2.4}$$

where the set in the right-hand side is a multiset. This equality requires some explanation. First, we have identified $V_{\mathbb{R}}$ with $\mathbb{R}^4$ in the obvious way, and we have let dv denote the Lebesgue measure. The equality holds because the number $\#\{x \in \mathcal{F}v \cap V_{\mathbb{Z}}^{\mathrm{irr}} : |\mathrm{Disc}(x)| < X\}$ is equal to $n_i N(V_{\mathbb{Z}}^{(i)}; X)$, at least if the points x with $m_i(x) = 3$ are counted with a weight 1/3 in the left-hand side. Also, $V_{\mathbb{Z}}^{\mathrm{irr}}$ is the subset of $V_{\mathbb{Z}}$ consisting of irreducible forms, and B is some suitable compact subset of $V_{\mathbb{R}}$. The factor $|\mathrm{Disc}(v)|^{-1}$ is not important for the equality, but it will be important for the calculations that will follow, as we shall see in the proposition below.

The starting point for estimating $N(V_{\mathbb{Z}}^{(i)};X)$ will be (5.2.4). Our immediate goal is to replace the integral over $V_{\mathbb{R}}$ with an integral over $\mathrm{GL}_2(\mathbb{R})$, which will simplify the integration by allowing us to use the precise definition of $\mathcal{F}$ given in the previous subsection. To accomplish this, we need the following lemma [BST, Proposition 23].

Proposition 5.7 (BST)**.** *For $i=0,1$ let f be continuous on $V_{\mathbb{R}}^{i}$, and let v_i be any element of $V_{\mathbb{R}}^{(i)}$. Then, we have*

$$\int_{g\in\mathrm{GL}_2(\mathbb{R})} f(gv_1)dg = \frac{1}{2\pi}\int_{v\in\mathrm{GL}_2(\mathbb{R})v_i} f(v)|\mathrm{Disc}(v)|^{-1}dv = \frac{n_i}{2\pi}\int_{v\in V_{\mathbb{R}}^{(i)}} f(v)|\mathrm{Disc}(v)|^{-1}dv.$$

We leave out a rigorous proof, and instead give the same proof sketch as in [BST]. The first equality can be shown by changing coordinates in $\mathrm{GL}_2(\mathbb{R})$ to those induced by its natural embedding in $\mathbb{R}^4$, and then calculating the Jacobian of the map $g\mapsto gv_i$. Here, the natural embedding is given by

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto (a,b,c,d)\in\mathbb{R}^4.$$

The second equality follows from our remark above, that the multiset $\mathrm{GL}_2(\mathbb{R})v_i$ contains every point of $V_{\mathbb{R}}^{(i)}$ with multiplicity n_i. Here, the integral over a multiset is defined in the obvious way, by splitting the integral over the various multiplicities. We will make use of a slight generalisation of the proposition above, where we allow f to be multiplied by the indicator function of some reasonable subset of $V_{\mathbb{R}}^{i}$.

5.2.3 Integrating over GL_2

The content of this subsection is essentially contained in [BST, Ch. 5.3], however, we also provide some details not found in [BST]. First, we specify the compact set we are integrating over in (5.2.4) by letting $B=\{w=(a,b,c,d)\in V_{\mathbb{R}} : 3a^2+b^2+c^2+3d^2\leq C, |\mathrm{Disc}(w)|\geq 1\}$, for some fixed $C\geq 1$. The precise value of C will not matter much. A calculation confirms that this set is $SO_2(\mathbb{R})$-invariant, in the sense that $kB=B$ for all $k\in SO_2(\mathbb{R})$, which will be important later.

We now move the integration in (5.2.4) to $\mathrm{GL}_2(\mathbb{R})$, using Proposition 5.7. Consider the map $T:\mathrm{GL}_2(\mathbb{R})\times V_{\mathbb{R}}$ given by $T(g,v)=gv$. Fix a $v_i\in V_{\mathbb{R}}^{(i)}$, and let $H^{(i)}=T(\cdot,v_i)^{-1}(B)$, so that the multiset $H^{(i)}v_i$ contains every point of $B\cap V_{\mathbb{R}}^{(i)}$ with multiplicity n_i. Then, using Tonelli's theorem, we have that the numerator of (5.2.4) is equal to

$$\sum_{\substack{x\in V_{\mathbb{Z}}^{\mathrm{irr}}\\ |\mathrm{Disc}(x)|<X}} \int_{v\in B\cap V_{\mathbb{R}}^{(i)}} \#\{g\in\mathcal{F} : x=gv\}|\mathrm{Disc}(v)|^{-1}dv.$$

The count in this integral is constant with respect to v so that we may apply (the generalisation of) Proposition 5.7 to rewrite it as

$$\frac{2\pi}{n_i}\sum_{\substack{x\in V_{\mathbb{Z}}^{\mathrm{irr}}\\ |\mathrm{Disc}(x)|<X}} \int_{h\in H^{(i)}} \#\{g\in\mathcal{F} : x=ghv_i\}dh. \tag{5.2.5}$$

The set $H^{(i)}$ need not be very nice, which motivates us to move the integral to $\mathcal{F}$ instead. The details of this move are essentially taken from the proof of [Ls, Prop. 3.17]. Begin by rewriting the integral as

$$\int_{h\in H^{(i)}} \sum_{\substack{g\in\mathcal{F}\\ x=ghv_i}} 1\, dh = \sum_{\substack{g'\in\mathrm{GL}_2(\mathbb{R})\\ x=g'v_i}} \int_{h\in H^{(i)}} 1_{\{h\in\mathcal{F}^{-1}g'\}}dh, \tag{5.2.6}$$

where we used that multiplication by a group element is bijective, and where $\mathcal{F}^{-1}$ denotes the set of inverses of the elements in $\mathcal{F}$. Denote the inner integral above by $\mathrm{Vol}(H^{(i)} \cap \mathcal{F}^{-1}g')$, where the volume is taken with respect to the $\mathrm{GL}_2(\mathbb{R})$ invariant measure dh. Using this invariance, as well as the fact that the measure is also invariant under inversion, which can be confirmed by a computation, we see that this volume is equal to $\mathrm{Vol}(g'(H^{(i)})^{-1} \cap \mathcal{F})$. After interchanging summation and integration again, we find that the right-hand side of (5.2.6) can be written as

$$\int_{g\in\mathcal{F}} \sum_{\substack{g\in g'(H^{(i)})^{-1} \\ x=g'v_i}} 1\,dg = \int_{g\in\mathcal{F}} \sum_{\substack{g'\in gH^{(i)} \\ x=g'v_i}} 1\,dg = \int_{g\in\mathcal{F}} \#\{h\in H^{(i)} : x = ghv_i\}\,dg$$

Moving the sum from (5.2.5) back inside, we see that the expression in (5.2.5), i.e. the numerator in (5.2.4), is equal to

$$\frac{2\pi}{n_i}\int_{g\in\mathcal{F}} \#\{x\in V_{\mathbb{Z}}^{\mathrm{irr}} \cap gH^{(i)}v_i : |\mathrm{Disc}(x)| < X\}\,dg.$$

To proceed, we will analyse the integrand carefully. First, recall that by definition of $H^{(i)}$, we have $H^{(i)}v_i = B \cap V_{\mathbb{R}}^{(i)}$, where each point has multiplicity n_i. Hence, the above is equal to

$$2\pi\int_{g\in\mathcal{F}} \#\{x\in V_{\mathbb{Z}}^{\mathrm{irr}} \cap gB \cap V_{\mathbb{R}}^{(i)} : |\mathrm{Disc}(x)| < X\}\,dg, \tag{5.2.7}$$

where the set inside the integral is no longer a multiset.

Next, we write g as a product $n(u)a(t)k\lambda \in \mathcal{F}$, and use that B is $SO_2(\mathbb{R})$-invariant, to see that $gH^{(i)}v_i = gB \cap V_{\mathbb{R}}^{(i)} = n(u)a(t)\lambda B \cap V_{\mathbb{R}}^{(i)}$. We write, with a slight change in notation compared to [BST], $B(u,t,\lambda,X) = n(u)a(t)\lambda B \cap \{x \in V_{\mathbb{R}}^{(i)} : |\mathrm{Disc}(x)| < X\}$. Using this notation, we see that we should investigate the number of irreducible integral forms in $B(u,t,\lambda,X)$. This will be done using the following proposition, essentially due to Davenport [D1], [D2]. The statement is modified from [BST, Prop. 24].

Proposition 5.8 (Davenport)**.** *Let $\mathcal{R}$ be a bounded semi-algebraic subset of $\mathbb{R}^n$, defined by at most k polynomial inequalities, each having degree at most ℓ. Then the number of integral lattice points contained in $\mathcal{R}$ is*

$$\mathrm{Vol}(\mathcal{R}) + \mathcal{O}\left(\max\{\mathrm{Vol}(\overline{\mathcal{R}}), 1\}\right),$$

where $\mathrm{Vol}(\overline{\mathcal{R}})$ denotes the largest d-dimensional volume of the projection of $\mathcal{R}$ onto any d-dimensional coordinate subspace, with $d < n$. The implied constant depends only on n, k and ℓ.

Using this proposition we can count the integral lattice points in $B(u,t,\lambda,X)$. Before doing this, however, we must find a way to exclude the reducible forms. Consider a form $ax^3 + bx^2y + cxy^2 + dy^3$ in $V_{\mathbb{R}}$, and note that if $a = 0$, then the form is certainly reducible, as it has a factor y. Now, [BST, Lemma 21] asserts that reducible forms with $a \neq 0$ are very rare, and their contribution can therefore be absorbed in the error terms.

As mentioned earlier, we will not include the precise estimation of the error terms involved in the calculations. We do however mention that by only having to consider points with $|a| \geq 1$, one can control the error term arising from Proposition 5.8. We are left with the main term from (5.2.7) equalling

$$2\pi\int_{g\in\mathcal{F}}\int_{|\mathrm{Disc}(v)|<X} 1_{\{g^{-1}v\in B\cap V_{\mathbb{R}}^{(i)}\}}\,dv\,dg,$$

where the inner integral is the volume from Proposition 5.8. Interchanging the order of integration, applying Proposition 5.7, using inversion invariance of dg and then interchanging the order of integration again shows that this double integral is equal to

$$\int_{u\in B\cap V_{\mathbb{R}}^{(i)}}\int_{v\in\mathcal{F}u} 1_{\{\mathrm{Disc}(v)<X\}}\,dv\,\frac{du}{|\mathrm{disc}(u)|}.$$

Applying Proposition 5.7 to the inner integral, and making a change of variables $g \mapsto gh$ for a suitable $h \in \Lambda$ then shows that the above is equal to

$$2\pi \int_{u \in B \cap V_{\mathbb{R}^{(i)}}} \int_{g \in \mathcal{F}} 1_{\{(\det g)^2 < X\}} \cdot (\det g)^2 dg |\mathrm{Disc}(u)|^{-1} du,$$

where we also made use of the equality $\mathrm{Disc}(gv) = (\det g)^2 \mathrm{Disc}(v)$, which follows from a straightforward computation. Note that the inner integral does not depend on u, whence we can separate the two integrals. The integral over u is exactly the denominator of (5.2.4), and they thus cancel each other.

We have arrived at

$$N(V_{\mathbb{Z}}^{(i)}; X) = \frac{2\pi}{n_i} \int_{g \in \mathcal{F}} 1_{\{(\det g)^2 < X\}} \cdot (\det g)^2 dg + \text{Error}. \tag{5.2.8}$$

The error term turns out to be $\ll X^{5/6}$. It remains to compute the integral above to find the explicit value of the main term. If we write $g = n(u)a(t)k\lambda$, then $\det(g)^2 = \lambda^4$, so that we should integrate over the part of $\mathcal{F}$ where $\lambda < X^{1/4}$. From our explicit description of $\mathcal{F}$, and the definition of dg, and the fact that dk gives K measure 1, we see that the main term of (5.2.8) is

$$\frac{2\pi}{n_i} \int_0^{X^{1/4}} \int_{-1/2 \le u \le 1/2} \int_{t \ge (1-u^2)^{1/4}} \frac{\lambda^3}{t^3} dn dt d\lambda = \frac{\pi^2}{12 n_i} X.$$

As $n_0 = 6$ and $n_1 = 2$, this is indeed the desired main term.

5.3 Sieving for maximality

We now want to obtain estimates for the number of isomorphism classes of cubic fields, with discriminant bounded by X. The results of the previous section do not quite suffice to accomplish this, as we need to remove the nonmaximal forms from the count, and to accomplish this we need to place certain congruence conditions on the coefficients of the forms modulo p^2. We make this more precise below.

Recall that $N(V_{\mathbb{Z}}^{(i)}; X)$ counts irreducible, integral binary cubic forms, with discriminant less than X, up to $\mathrm{GL}_2(\mathbb{Z})$ equivalence. Similarly, for any $\mathrm{GL}_2(\mathbb{Z})$-invariant set $S \subseteq V_{\mathbb{Z}}^{(i)}$ we can let $N(S; X)$ count $\mathrm{GL}_2(\mathbb{Z})$-orbits of irreducible forms in S. We are of course particularly interested in $N\left(V_{\mathbb{Z}}^{(i)} \cap \bigcap_p \mathcal{U}_p, X\right)$, which counts orbits of forms corresponding to cubic fields. To study this counting function, it will be necessary to study the partial counts $N\left(V_{\mathbb{Z}}^{(i)} \cap \left(\bigcap_{p<Y} \mathcal{U}_p\right), X\right)$, for $Y > 2$. We will use the following theorem, which is a special case of [BST, Thm. 26].

Theorem 5.9 (BST)**.** *Let S be a $\mathrm{GL}_2(\mathbb{Z})$-invariant subset of $V_{\mathbb{Z}}$, defined by congruence conditions modulo p^2 for primes $p \in \mathcal{P}$, where $\mathcal{P}$ is a finite subset of the set of primes. Then, we have*

$$\lim_{X \to \infty} \frac{N(S \cap V_{\mathbb{Z}}^{(i)}; X)}{X} = \frac{\pi^2}{12 n_i} \prod_{p \in \mathcal{P}} \mu_p(\overline{S}_p).$$

Here μ_p denotes the counting measure on $(\mathbb{Z}/p^2\mathbb{Z})^4$, divided by p^8, and $\overline{S}_p \subseteq (\mathbb{Z}/p^2\mathbb{Z})^4$ denotes the reduction of S modulo p^2.

Proof. We will only provide a rough sketch of the proof, taken from [BST, Ch. 5.5]. The main point is that S is a union of finitely many translates $L_1, ..., L_k$ of the lattice $(\prod_{p \in \mathcal{P}} p^2) V_{\mathbb{Z}} =: mV_{\mathbb{Z}}$, by the Chinese Remainder Theorem. Again by the Chinese Remainder Theorem, we have that the number of such translates k is equal to $(\prod_{p \in \mathcal{P}} p^8 \mu_p(\overline{S}_p))$.

Next, apply the procedure of the previous section to the lattice $mV_{\mathbb{Z}}$ in place of $V_{\mathbb{Z}}$. The main difference is that the volume from Davenport's lemma gets scaled by a factor $m^{-4} = \prod_{p \in \mathcal{P}} p^{-8}$, and accordingly, the result of the previous section gets scaled by the same factor. Multiplying this by the number of translates yields the main term above. □

We remark that it is very important for the proof that the number of congruence conditions is finite, as the error term grows rapidly with the number of congruence conditions.

Now we are ready to begin the calculations for finding the main term $C_1^{\pm}X$ of (3.2.3), following [BST, Ch. 8]. We will be studying the limit of the quotient $X^{-1}N\left(V_{\mathbb{Z}}^{(i)} \cap \left(\bigcap_p \mathcal{U}_p\right); X\right)$, as $X \to \infty$. Recall that $\mathcal{U}$ is the intersection of all $\mathcal{U}_p$. We begin by bounding this from above. Trivially,

$$N\left(V_{\mathbb{Z}}^{(i)} \cap \left(\bigcap_{p<Y} \mathcal{U}_p\right); X\right) \geq N\big(V_{\mathbb{Z}}^{(i)} \cap \mathcal{U}; X\big),$$

as the former counting function imposes fewer restrictions on the forms. Next, by Theorem 5.9, the Chinese Remainder Theorem, and Lemma 5.5, we have

$$\lim_{X\to\infty} X^{-1}N\left(V_{\mathbb{Z}}^{(i)} \cap \left(\bigcap_{p<Y} \mathcal{U}_p\right); X\right) = \frac{\pi^2}{12n_i}\prod_{p<Y}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right),$$

for all Y. Letting $Y \to \infty$, and using the Euler product of $\zeta(s)$, it follows that

$$\limsup_{X\to\infty} X^{-1}N\big(V_{\mathbb{Z}}^{(i)} \cap \mathcal{U}; X\big) = \frac{\pi^2}{12n_i\zeta(2)\zeta(3)} = \frac{1}{2n_i\zeta(3)}.$$

Here we made use of the identity $\zeta(2) = \pi^2/6$, which is the conclusion of the well-known Basel problem.

We now want to show that the right-hand side above is also a lower bound for the lim inf. Let $\mathcal{W}_p$ be the set of integral forms, which are nonmaximal at p. Then, we have the inclusion

$$\bigcap_{p<Y}\mathcal{U}_p \subseteq \left(\bigcup_{p\geq Y}\mathcal{W}_p\right) \cup \mathcal{U},$$

whence

$$N\big(V_{\mathbb{Z}}^{(i)} \cap \mathcal{U}; X\big) \geq N\left(V_{\mathbb{Z}}^{(i)} \cap \left(\bigcap_{p<Y}\mathcal{U}_p\right); X\right) - \sum_{p\geq Y} N\left(V_{\mathbb{Z}}^{(i)} \cap \mathcal{W}_p; X\right).$$

We are done if we can show that the second term, when divided by X, tends to 0 as $Y \to \infty$. This follows from [BST, Prop. 29], which implies that $N\left(V_{\mathbb{Z}}^{(i)} \cap W_p; X\right) \ll Xp^{-2}$. The proof of this requires more information about subrings and overrings of cubic rings than what has been presented so far, so we omit the proof.

By using the fact that the sum over p^{-2} is summable, we can let $Y \to \infty$ and thus conclude that

$$N^{\pm}(X) = C_1^{\pm}X + o(X).$$

The attentive reader may remember that when we defined $N^{\pm}(X)$, we only included S_3-fields and not all cubic fields. However, we have a priori only shown the equality above for the counting function counting all cubic fields, but a result due to Cohn [C] asserts that the C_3-fields only contribute a term $\ll X^{1/2}$ to $N\big(V_{\mathbb{Z}}^{(i)} \cap \mathcal{U}; X\big)$, so that we may disregard this contribution.

So far we have indicated how one obtains the main term of the first equality in (3.2.3). Finally, we indicate how to obtain the main terms when imposing splitting conditions on finitely many primes. Suppose that we pick primes $p_1, ..., p_n$ and splitting types $T_{j_1}, ..., T_{j_n}$. Then, to count cubic fields satisfying these conditions, we can repeat the proof above, but replace the sets $\mathcal{U}_{p_1}, ..., \mathcal{U}_{p_n}$ by the sets $\mathcal{U}_{p_1}(T_{j_1}), ..., \mathcal{U}_{p_n}(T_{j_n})$. The argument goes through as before, but the constant $C_1^{\pm}$ should be multiplied by a factor

$$\prod_{i=1}^{n}\frac{\mu_{p_i}(\mathcal{U}_{p_i}(T_{j_i}))}{\mu_{p_i}(\mathcal{U}_{p_i})} = \prod_{i=1}^{n} x_{p_i}c_{j_i}(p_i),$$

where the equality follows from Lemma 5.5. This gives the main term of (4.2.2).

5.4 The secondary term

We provide a very brief sketch of how to modify the previous arguments to also provide a secondary term in the counting functions for cubic fields, using the approach from [BST]. A more complete description, even avoiding the study of the error terms, cannot be included without sacrificing the brevity of this chapter.

The first step is to obtain a secondary term in the counting function for irreducible binary cubic forms. Specifically, we will need an improvement of Theorem 5.9, which also provides a secondary term. The main new idea needed is a technique referred to as "slicing" in [BST]. The starting point is essentially (5.2.7), but instead of directly applying Davenport's lemma, we split the integral into two parts, depending on the size of t compared to λ. One of the integrals is then sliced into a sum of several integrals by first separating the forms in $B(u, t, \lambda, X)$ depending on the value of their first coefficient a and then summing over a. This sum over different integrals can then be transformed into a single integral that is analysed by using residue theory, which allows us to separate the secondary term from the main term.

Once we have refined the counting function for cubic forms in this way, the next step is to find second-order densities for maximal forms with different splitting types, analogously to Lemma 5.5. These second-order densities do not have the same obvious geometric interpretation as the first-order densities do, but they can be given meaning by their connection to certain lattices. We leave out the details and refer the interested reader to [BST, Ch. 7].

Finally, the last step is to sieve for maximality as above. For this, a sieve based on the inclusion-exclusion principle is used. The first step is to count all binary cubic forms. Next, for each prime p, we subtract the count for forms that are nonmaximal at p. Now, we have subtracted too large a number, as forms that are nonmaximal at more than one prime have been subtracted twice. Thus, we add back the counts for forms nonmaximal at at least two primes, and so on. This is formalised using the Möbius function and Möbius inversion but we leave out the details. The point is that maximal forms are hard to count, but it is easier to count forms nonmaximal at some finite number of primes. Applying this sieve together with the second-order densities in a refined Theorem 5.9 finally yields the secondary terms in (4.2.2). The details can be found in [BST, Ch. 9].

6

The one-level density through the Ratios Conjecture

In Chapter 3, we analysed the one-level density for relatively small values of $\sigma = \sup(\mathrm{supp}(\widehat{\phi}))$. In this chapter, we describe and apply, the L-functions Ratios Conjecture which will allow us to study the one-level density for any finite σ, conditional on this conjecture. We obtain Conjecture 6.3, which leads to the estimate of the one-level density in Proposition 6.4. Throughout the chapter, we will need to assume the Generalised Riemann Hypothesis for the Dedekind zeta functions $\zeta_K(s)$, with K a non-Galois cubic field. We remark that essentially all arguments of this chapter are taken from [CFLS, Ch. 4], but the presentation may differ.

6.1 The Ratios Conjecture recipe

In previous chapters, we have studied both the one-level density and the two-level density through a study of certain prime sums. Unfortunately, the prime sum $S_2(i)$, became harder to estimate if σ was large. The purpose of this chapter is essentially to replace S_2 with a nicer sum, which does not grow with σ, which will allow us to obtain a prediction for the one-level density for any finite σ. To accomplish this, we will need to accurately estimate certain averages of ratios, of the form

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \frac{L(1/2+\alpha_1, f_K) \cdot ... \cdot L(1/2+\alpha_n, f_K)}{L(1/2+\gamma_1, f_K) \cdot ... \cdot L(1/2+\gamma_n, f_K)}, \tag{6.1.1}$$

for some complex values $\alpha_1, ..., \alpha_n, \gamma_1, ..., \gamma_n$, whose real part is relatively small, and $n \geq 1$. Specifically, to evaluate the one-level density we will need to consider such ratios with $n = 1$, while the two-level density requires us to set $n = 2$.

Unfortunately, calculating this sum of ratios turns out to be a very hard problem, and has not been done for any family of L-functions associated to number fields. Therefore, instead of calculating the sum above rigorously, we will use a heuristic due to Conrey, Farmer and Zirnbauer [CFZ], sometimes referred to as the Ratios Conjecture recipe, to conjecture a reasonable estimate for (6.1.1). The heuristic applies to ratios of any family of L-functions, but we will only describe it for our family of Artin L-functions $L(s, f_K)$. We will also restrict to the case $n = 1$ for simplicity, but the calculations for larger n are very similar, and we will see explicit calculations for $n = 2$ in later chapters when we study the two-level density.

We will not describe the recipe exactly as given in [CFZ, Ch. 5.1], but instead describe a slightly modified version, essentially due to [CFLS]. The reason is that the secondary term in $N^{\pm}(X)$ will produce an error term that is quite large if one follows the original Ratios Conjecture recipe, a phenomenon first observed in [CFLS].

We now give the recipe for evaluating

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \frac{L(1/2+\alpha, f_K)}{L(1/2+\gamma, f_K)}. \tag{6.1.2}$$

The first step is to rewrite the summand. Begin by formally expanding the denominator into its Dirichlet series,

$$\frac{1}{L(1/2+\gamma, f_K)} = \sum_{h=1}^{\infty} \frac{\mu_K(h)}{h^{1/2+\gamma}},$$

where the explicit coefficients μ_K can be found by studying the Euler product of $L(s, f_K)$. Recall that $L(s, f_K)$ obeys the functional equation

$$\Lambda(s, f_K) = \Lambda(1-s, f_K), \text{ where } \Lambda(s, f_K) := |D_K|^{s/2}\Gamma_{\pm}(s)L(s, f_K),$$

which implies that $L(s, f_K)$ has an approximate functional equation of the form

$$L(s, f_K) = \sum_{m\leq N} \frac{\lambda_K(m)}{m^s} + \frac{|D_K|^{(1-s)/2}\Gamma_{\pm}(1-s)}{|D_K|^{s/2}\Gamma_{\pm}(s)} \sum_{m\leq M} \frac{\lambda_K(m)}{m^{1-s}} + \text{ Error },$$

for certain values of N, M, cf. [IK, Thm. 5.3]. In (6.1.2) we replace the numerator by its approximate functional equation, excluding the error term, while formally extending both summations to ∞.

The first step of the Ratios Conjecture recipe has thus transformed the summand of (6.1.2) into

$$\sum_{h,m} \frac{\lambda_K(m)\mu_K(h)}{m^{1/2+\alpha}h^{1/2+\gamma}} + |D_K|^{-\alpha}\frac{\Gamma_{\pm}(1/2-\alpha)}{\Gamma_{\pm}(1/2+\alpha)} \sum_{h,m} \frac{\lambda_K(m)\mu_K(h)}{m^{1/2-\alpha}h^{1/2+\gamma}}.$$

In Section 6.3 below, we will see that $\mu_K(n)$ and $\lambda_K(n)$ are $\ll_\epsilon n^\epsilon$. In particular, this implies that the first sum converges absolutely for $\text{Re}(\alpha), \text{Re}(\gamma) > 1/2$, while the second sum converges absolutely for $\text{Re}(\alpha) < -1/2, \text{Re}(\gamma) > 1/2$. In particular, we cannot directly guarantee that the sums converge absolutely for the same α, but this will be remedied in the last step of the recipe. Until then, we will treat both sums over h and m purely symbolically.

The next step of the recipe is to change the order of summation so that the sum over K is turned into the innermost sum. We then replace this inner sum, including the factor $(N^{\pm}(X))^{-1}$, with a suitable estimate of

$$\frac{1}{N^{\pm}(X)} \sum_{K\in N^{\pm}(X)} \lambda_K(m)\mu_K(h) \quad \text{and} \quad \frac{1}{N^{\pm}(X)} \sum_{K\in N^{\pm}(X)} |D_K|^{-\alpha}\lambda_K(m)\mu_K(h), \tag{6.1.3}$$

respectively. This leaves us with the sum

$$\sum_{h,m} \frac{M_1(X)}{m^{1/2+\alpha}h^{1/2+\gamma}} + \frac{\Gamma_{\pm}(1/2-\alpha)}{\Gamma_{\pm}(1/2+\alpha)} \sum_{h,m} \frac{M_2(X)}{m^{1/2-\alpha}h^{1/2+\gamma}},$$

where $M_1(X)$ and $M_2(X)$ denotes an estimate of the first and second term in (6.1.3) respectively.

The last step of the recipe is to find a meromorphic continuation of each term above, to a domain where both terms are well-defined at the same time, containing the point $\alpha = \gamma = 0$. We denote the terms above, and their meromorphic continuations, by $R_1(\alpha, \gamma; X)$ and $R_2(\alpha, \gamma; X)$ respectively. The Ratios Conjecture is then that (6.1.2) is equal to

$$R_1(\alpha, \gamma; X) + R_2(\alpha, \gamma; X) + \text{Error}, \tag{6.1.4}$$

in some suitable restriction of this domain. Following the original recipe, one would expect an error $\mathcal{O}_\epsilon\left(X^{-1/2+\epsilon}\right)$, see [CFZ, Ch. 5.1], but we will see that a more reasonable error in this family is $\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)$, with $\theta \geq 1/2$ as in (3.2.3).

6.2 The one-level density

We now apply the Ratios Conjecture to obtain a conjectural improvement of Theorem 3.1. This is done in [CFLS, Ch. 4-5], and essentially all arguments of this chapter are taken from there. As mentioned above, we will assume the Generalised Riemann Hypothesis for ζ_K throughout the entire chapter, in particular, this implies the regular Riemann Hypothesis.

We now begin our calculations. As $L(s, f_K)$ is entire, we have similarly as in Chapter 3 that

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K\right) = \frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \frac{1}{2\pi i} \left(\int_{(a)} - \int_{(-a)}\right) \frac{L'(1/2+r, f_K)}{L(1/2+r, f_K)} \times \phi\left(\frac{Lr}{2\pi i}\right) dr = \frac{1}{2\pi i} \left(\int_{(a)} - \int_{(-a)}\right) \phi\left(\frac{Lr}{2\pi i}\right) \frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \frac{L'(1/2+r, f_K)}{L(1/2+r, f_K)} dr. \tag{6.2.1}$$

for any $1/2 > a > 0$. Note that we made use of the Generalised Riemann Hypothesis to ensure that a can be chosen as close to 0 as we like. This motivates us to study the average

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \frac{L(1/2+\alpha, f_K)}{L(1/2+\gamma, f_K)}, \tag{6.2.2}$$

using the Ratios Conjecture. Once we have estimated this expression, we can differentiate the estimate with respect to α, and set $\alpha = \gamma = r$ to estimate the sum inside the integral in (6.2.1). The validity of differentiating an estimate will be discussed near the end of the next section.

We apply the first steps of the Ratios Conjecture recipe as described in the previous section and formally write

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \frac{L(1/2+\alpha, f_K)}{L(1/2+\gamma, f_K)} = R_1'(\alpha, \gamma; X) + R_2'(\alpha, \gamma; X) + \text{Error},$$

where

$$R_1'(\alpha, \gamma; X) = \frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \sum_{h,m} \frac{\lambda_K(m)\mu_K(h)}{m^{1/2+\alpha} h^{1/2+\gamma}},$$

and

$$R_2'(\alpha, \gamma; X) = \frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} |D_K|^{-\alpha} \frac{\Gamma_{\pm}(1/2-\alpha)}{\Gamma_{\pm}(1/2+\alpha)} \sum_{h,m} \frac{\lambda_K(m)\mu_K(h)}{m^{1/2-\alpha} h^{1/2+\gamma}}. \tag{6.2.3}$$

Recall that for now, we treat these sums purely symbolically, as they are perhaps not both convergent at the same time. To finish the recipe we need to interchange the order of summation in R_1' and R_2', and then find the estimates R_1 and R_2 from (6.1.4).

6.3 Averages over K

In this section, we will replace R_1' with an approximation R_1 which will be defined in a larger domain. The first step of finding such an approximation is to determine λ_K and μ_K explicitly. The existence of an Euler product for $L(s, f_K)$ implies that the coefficients are multiplicative, so it suffices to determine them at prime powers. As before, we will see that the splitting type influences the shape of the coefficients.

Using the table from the end of Section 2.1 we find the coefficients of $L(s, f_K)$. First, fix a prime p, $e \geq 1$, and suppose p has splitting type $T_1 = (111)$. By expanding out the Euler product we see that to find the coefficient $\lambda_K(p^e)$ we need only study the local factor at p, i.e. the factor

$$\left(1 - p^{-s}\right)^{-2} = \left(1 + p^{-s} + p^{-2s} + \ldots\right)^2.$$

A combinatorial argument then implies that the coefficient in front of p^{-es} is

$$\lambda_K(p^e) = \sum_{\ell=0}^{e} 1 = e+1.$$

One can use a similar method for the other splitting types. For the T_3 splitting type, it is useful to first write the local factor as $(1-p^{-3s})^{-1}(1-p^{-s})$. We summarise the coefficients in the table below. See also [CFLS, Ch. 2], where the same table is provided.

Splitting type	$\lambda_K(p^e)$
T_1	$e+1$
T_2	$\delta_{2\mid e}$
T_3	τ_e
T_4	1
T_5	0

Here

$$\tau_e = \begin{cases} 1 & e \equiv 0 \mod 3, \\ -1 & e \equiv 1 \mod 3, \\ 0 & e \equiv 2 \mod 3. \end{cases}$$

Finding μ_K is even simpler. Indeed, $(L(s, f_K))^{-1}$ also has an Euler product, which is simply the reciprocal of the Euler product of $L(s, f_K)$. In particular, μ_K is a multiplicative function. Furthermore, we find $\mu_K(p^j) = 0$ for $j \geq 3$. To find the behaviour for $j = 1, 2$ we study the local factors in the table from Section 2.1 more closely. If we for instance suppose that p has splitting type (111), then the local factor at p of $(L(s, f_K))^{-1}$ is $(1-p^{-s})^2 = 1 - 2p^{-s} + p^{-2s}$, whence we find that $\mu_K(p) = -2$ and $\mu_K(p^2) = 1$ for this splitting type. One handles the other splitting types analogously. We summarise the behaviour for $j = 1, 2$ in the table below.

Splitting type	$\mu_K(p)$	$\mu_K(p^2)$
T_1	-2	1
T_2	0	-1
T_3	1	1
T_4	-1	0
T_5	0	0

With these calculations completed, we are ready to find an approximation of R_1'. Write

$$R_1'(\alpha, \gamma; X) = \sum_{h,m} \frac{1}{m^{1/2+\alpha} h^{1/2+\gamma}} \cdot \frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \lambda_K(m)\mu_K(h).$$

The next step of the original Ratios Conjecture is to replace the inner sum with the main term of its average. We remark, as in [CFLS, Ch. 4], that doing so would lead to inaccurate results because of the secondary term in $N^{\pm}(X)$. Instead, we continue to follow [CFLS] and replace the inner sum with a more accurate approximation. Indeed, we have the following lemma, and proof from [CFLS, Lemma 4.1].

Lemma 6.1 (CFLS). *Let $m, h \in \mathbb{Z}_+$, and $1/2 \leq \theta < 5/6$, $\omega \geq 0$ be such that* (4.2.2) *holds. Then, for cubefree h, we have*

$$\frac{1}{N^{\pm}(X)} \sum_{K \in \mathcal{F}^{\pm}(X)} \lambda_K(m)\mu_K(h) = F_1(X) \prod_{p^e \| m, p^s \| h} f(e,s,p)x_p$$
$$+ F_2(X) \left(\prod_{p^e \| m, p^s \| h} g(e,s,p) y_p \right) + \mathcal{O}_\epsilon \left(X^{\theta-1+\epsilon} \prod_{p \mid hm, p^e \| m} (2e+5)p^{\omega} \right),$$

where

$$
\begin{aligned}
f(e,0,p) &= \frac{e+1}{6} + \frac{\delta_{2|e}}{2} + \frac{\tau_e}{3} + \frac{1}{p},\\
f(e,1,p) &= -\frac{e+1}{3} + \frac{\tau_e}{3} - \frac{1}{p},\\
f(e,2,p) &= \frac{e+1}{6} - \frac{\delta_{2|e}}{2} + \frac{\tau_e}{3},\\
g(e,0,p) &= \frac{(e+1)(1+p^{-1/3})^3}{6} + \frac{\delta_{2|e}(1+p^{-1/3})(1+p^{-2/3})}{2} + \frac{\tau_e(1+p^{-1})}{3} + \frac{(1+p^{-1/3})^2}{p},\\
g(e,1,p) &= -\frac{(e+1)(1+p^{-1/3})^3}{3} + \frac{\tau_e(1+p^{-1})}{3} - \frac{(1+p^{-1/3})^2}{p},\\
g(e,2,p) &= \frac{(e+1)(1+p^{-1/3})^3}{6} - \frac{\delta_{2|e}(1+p^{-1/3})(1+p^{-2/3})}{2} + \frac{\tau_e(1+p^{-1})}{3},
\end{aligned}
\tag{6.3.1}
$$

and where both $F_i(X)$ are defined in (4.3.5)*. In the case when both e and s equals 0, then one should consider the products as empty, equalling* 1*.*

Proof. We begin by remarking that for h that are not cubefree, the sum on the left-hand side above is simply zero. Next, write m and h as products of distinct prime powers $m = \prod_{j=1}^{J} p_j^{e_j}$ and $h = \prod_{j=1}^{J} p_j^{s_j}$, where $e_j, s_j \geq 0$. Here, $e_j = s_j = 0$ is not allowed, unless $m = h = 1$. Then, by multiplicativity

$$\sum_{K\in\mathcal{F}^{\pm}(X)} \lambda_K(m)\mu_K(h) = \sum_{K\in\mathcal{F}^{\pm}(X)} \prod_{j=1}^{J} \lambda_K(p_j^{e_j})\mu_K(p_j^{s_j}) = \sum_{\mathbf{k}} \sum_{\substack{K\in\mathcal{F}^{\pm}(X)\\ \mathbf{p}\text{ of type } T_{\mathbf{k}}}} \prod_{j=1}^{J} \lambda_K(p_j^{e_j})\mu_K(p_j^{s_j}).$$

Here, $\mathbf{k} = (k_1, ..., k_J)$ where $k_i \in \{1,2,3,4,5\}$, and the sum is over all possibilities for $\mathbf{k}$. Further $\mathbf{p} = (p_1, ..., p_J)$, and that $\mathbf{p}$ has splitting type $T_{\mathbf{k}}$ means that all p_j have splitting type T_{k_j} in K. Note that the product above is constant for every fixed $\mathbf{k}$. If p_j has splitting type T_{k_j} in K, define

$$\eta_{1,p_j}(k_j, e_j) = \lambda_K(p_j^{e_j}),\ \ \eta_{2,p_j}(k_j, e_j) = \mu_K(p_j^{e_j}),$$

and

$$\eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e}) = \prod_{j=1}^{J} \eta_{1,p_j}(k_j, e_j),\ \ \eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{e}) = \prod_{j=1}^{J} \eta_{2,p_j}(k_j, e_j).$$

It follows that

$$\sum_{K\in\mathcal{F}^{\pm}(X)} \lambda_K(m)\mu_K(h) = \sum_{\mathbf{k}} \eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s}) N_{\mathbf{p}}^{\pm}(X, T_{\mathbf{k}}).$$

Now we use (4.2.2) to write the above as

$$\sum_{\mathbf{k}} \eta_{1,\mathbf{p}}(\mathbf{k},\mathbf{e})\eta_{2,\mathbf{p}}(\mathbf{k},\mathbf{s}) \Bigg[C_1^{\pm} X \prod_{j=1}^{J} (x_{p_j} c_{k_j}(p_j)) + C_2^{\pm} X^{5/6} \prod_{j=1}^{J} \big(y_{p_j} d_{k_j}(p_j)\big) + \mathcal{O}_\epsilon\Big(X^{\theta+\epsilon} \prod_{j=1}^{J} p_j^{\omega} \Big) \Bigg].$$

Next, we separate the expression into three parts and then move the product outside. The result is

$$C_1^{\pm} X \prod_{p^e || m, p^s || h} \left(x_p \sum_{k=1}^{5} \eta_{1,p}(k,e)\eta_{2,p}(k,s) c_k(p) \right) + C_2^{\pm} X^{5/6} \times \prod_{p^e || m, p^s || h} \left(y_p \sum_{k=1}^{5} \eta_{1,p}(k,e)\eta_{2,p}(k,s) d_k(p) \right) + \mathcal{O}_\epsilon \left(X^{\theta+\epsilon} \prod_{p | hm, p^e || m} (2e+5) p^{\omega} \right),$$

where we used the explicit form of λ_K and μ_K to find the error term. The lemma follows after evaluating the sums and dividing by $N^{\pm}(X)$. The functions $f(e,s,p)$ and $g(e,s,p)$ correspond to the first and second sum above respectively. We show how to evaluate the sum defining $f(e,1,p)$, and leave the rest to the interested reader.

To find $f(e,1,p)$ we should calculate

$$\sum_{k=1}^{5} \eta_{1,p}(k,e)\eta_{2,p}(k,1)c_k(p) = \frac{-2(e+1)}{6} + \frac{0\cdot\delta_{2|e}}{2} + \frac{1\cdot\tau_e}{3} + \frac{(-1)\cdot 1}{p} + \frac{0\cdot 0}{p^2},$$

where we used the tables for μ_K and λ_K given above and the table for $c_k(p)$ given in connection to (3.2.3). After simplifying we find the desired expression. The remaining cases are handled in the same way. □

Note that the bound for the error term in Lemma 6.1 is not summable over m, h as it is not even bounded. The philosophy of the Ratios Conjecture is that one should expect enough cancellation within the error term to make it summable, and make the resulting error term small. We therefore approximate $R_1'(\alpha,\gamma;X)$ with $R_1(\alpha,\gamma;X)$, where R_1 is obtained by weighting the first two terms in Lemma 6.1 with $m^{-1/2-\alpha}h^{-1/2-\gamma}$, and then summing over m and h. After rewriting the sum as a product, we find (cf. [CFLS, Eq. (4.6)])

$$R_1(\alpha,\gamma;X) = F_1(X)R_1^M(\alpha,\gamma) + F_2(X)R_1^S(\alpha,\gamma), \tag{6.3.2}$$

where

$$R_1^M(\alpha,\gamma) = \prod_p \left(1 + \sum_{e\geq 1}\frac{x_p f(e,0,p)}{p^{e(1/2+\alpha)}} + \sum_{e\geq 0}\frac{x_p f(e,1,p)}{p^{e(1/2+\alpha)+(1/2+\gamma)}} + \sum_{e\geq 0}\frac{x_p f(e,2,p)}{p^{e(1/2+\alpha)+(1+2\gamma)}}\right), \tag{6.3.3}$$

and

$$R_1^S(\alpha,\gamma) = \prod_p \left(1 + \sum_{e\geq 1}\frac{y_p g(e,0,p)}{p^{e(1/2+\alpha)}} + \sum_{e\geq 0}\frac{y_p g(e,1,p)}{p^{e(1/2+\alpha)+(1/2+\gamma)}} + \sum_{e\geq 0}\frac{y_p g(e,2,p)}{p^{e(1/2+\alpha)+(1+2\gamma)}}\right). \tag{6.3.4}$$

Note that we are summing over all exponents giving a nonzero contribution to the left-hand side of Theorem 6.1. Similarly to the sums defining R_1', these products converges absolutely in $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 1/2$.

We will now find a meromorphic continuation of R_1^M and R_1^S, beginning with R_1^M. We assume $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/2$ so that the summands inside the products converge to 0. By (6.3.1) we have $f(1,0,p) = 1/p$, $f(2,0,p) = 1 + 1/p$ so that

$$\sum_{e\geq 1}\frac{x_p f(e,0,p)}{p^{e(1/2+\alpha)}} = \frac{1}{p^{1+2\alpha}} + \mathcal{O}\left(\frac{1}{p^{3/2+\mathrm{Re}(\alpha)}} + \frac{1}{p^{3/2+3\,\mathrm{Re}(\alpha)}}\right),$$

where we bounded the tail using a geometric series argument. In particular, if $\mathrm{Re}(\alpha) > -1/6+\delta$, $\delta > 0$, then the error is $\ll_\delta p^{-1-\delta}$. Continuing, we have $f(0,1,p) = -1/p$, $f(1,1,p) = -1 - 1/p$, and $f(0,2,p) = 0$, whence

$$\sum_{e\geq 0}\frac{x_p f(e,1,p)}{p^{e(1/2+\alpha)+(1/2+\gamma)}} = -\frac{1}{p^{1+\alpha+\gamma}} + \mathcal{O}\left(\frac{1}{p^{3/2+\mathrm{Re}(\gamma)}} + \frac{1}{p^{3/2+\mathrm{Re}(2\alpha+\gamma)}}\right),$$

and

$$\sum_{e\geq 0}\frac{x_p f(e,2,p)}{p^{e(1/2+\alpha)+(1+2\gamma)}} = \mathcal{O}\left(\frac{1}{p^{3/2+\mathrm{Re}(\alpha+2\gamma)}}\right).$$

In conclusion, if $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/6+\delta$ we have

$$R_1^M(\alpha,\gamma) = \prod_p \left(1 + \frac{1}{p^{1+2\alpha}} - \frac{1}{p^{1+\alpha+\gamma}} + \mathcal{O}_\delta\left(\frac{1}{p^{1+\delta}}\right)\right).$$

Recall the Euler product of the Riemann zeta function, $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, valid when $\mathrm{Re}(s) > 0$. Thus, for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/6+\delta$, we at least formally have

$$\begin{aligned}\frac{\zeta(1+\alpha+\gamma)}{\zeta(1+2\alpha)} &= \prod_p \frac{1-p^{-(1+2\alpha)}}{1-p^{-(1+\alpha+\gamma)}} = \prod_p \left(1-\frac{1}{p^{1+2\alpha}}\right)\left(1+\frac{1}{p^{1+\alpha+\gamma}}+\mathcal{O}\left(\frac{1}{p^{2+\mathrm{Re}(2\alpha+2\gamma)}}\right)\right)\\ &= \prod_p \left(\left(1-\frac{1}{p^{1+2\alpha}}\right)\left(1+\frac{1}{p^{1+\alpha+\gamma}}\right)+\mathcal{O}(p^{-1-\delta})\right).\end{aligned}$$

This calculation, and the formula $(1-x)(1+x) = 1-x^2$, implies that

$$A_3(\alpha,\gamma) := \frac{\zeta(1+\alpha+\gamma)}{\zeta(1+2\alpha)} R_1^M(\alpha,\gamma) = \prod_p \left(1+\mathcal{O}(p^{-1-\delta})\right), \tag{6.3.5}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/6+\delta$. In particular, the error term is absolutely summable, whence the product defining A_3 converges absolutely for these α and γ. Furthermore, the convergence is uniform for a fix $\delta > 0$, whence A_3 is holomorphic for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/6$. Moreover, we have the bound $A_3(\alpha,\gamma) \ll_\delta 1$, for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > -1/6+\delta$. We end by remarking that, by an analogous calculation, one finds that

$$A_4(\alpha,\gamma) := \frac{\zeta(5/6+\gamma)\zeta(1+\alpha+\gamma)}{\zeta(5/6+\alpha)\zeta(1+2\alpha)} R_1^S(\alpha,\gamma)$$

is holomorphic in the same set and with the same bound as A_3, see [CFLS, Eq. (4.9)].

Now that we have found meromorphic continuations of R_1^M and R_1^S, we can write (6.3.2) as

$$\begin{aligned}R_1'(\alpha,\gamma;X) \approx R_1(\alpha,\gamma;X) = F_1(X)&\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)}A_3(\alpha,\gamma)\\ + F_2(X)&\frac{\zeta(5/6+\alpha)\zeta(1+2\alpha)}{\zeta(5/6+\gamma)\zeta(1+\alpha+\gamma)}A_4(\alpha,\gamma).\end{aligned} \tag{6.3.6}$$

The symbol $\approx$ indicates that one should not expect an equality, but rather an equality up to some error. From Lemma 6.1 it appears that one can expect an error $\ll X^{\theta-1+\epsilon}$, in contrast to the bound $X^{-1/2}$ that one usually believes holds when applying the Ratios Conjecture. Nevertheless, one cannot expect equality to hold with this error for all α and γ as the right-hand side is possibly unbounded for α close to 0, or 1/6. We will therefore only assume that this bound for the error term holds when one has chosen α, γ to be positive and with a small enough real part, but not too small, in the sense that $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) \gg 1/L$. This last condition is common when applying the Ratios Conjecture, see e.g. [CS].

As we have remarked before, we wish to differentiate this estimate with respect to α. The reader may notice that one can usually not simply differentiate an estimate of a function, and expect the error term to remain unchanged. For example, it is easy to find a bounded function with an unbounded derivative. However, if we assume the equality above holds with error $\ll X^{\theta-1+\epsilon}$, then we may formally use Cauchy's formula to estimate the derivative by integrating over a circle of radius, say $\ll 1/L^2$. Thus after possibly modifying ϵ, we find that we have the same error term in the estimate for the derivative of R_1' with respect to α. Note that this argument is not rigorous, because the function $R_1'(\alpha,\gamma;X)$ is not defined for the α and γ we are considering here. The point of the argument is to indicate that a differentation is justified, and one may consider this assumption as being part of the recipe for evaluating the ratio we are studying.

6.4 Finding a differentiated estimate

We are now left with the task of differentiating the right-hand side of (6.3.6) with respect to α, and then setting $\alpha = \gamma = r$. We turn back to the case when $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 1/2$ so that we may differentiate (6.3.2) instead. To perform this differentiation we first need a few preparations.

A straightforward calculation confirms that

$$\begin{aligned}
&f(e,0,p)+f(e-1,1,p)+f(e-2,2,p)=g(e,0,p)+g(e-1,1,p)+g(e-2,2,p)=0,\\
&f(1,0,p)+f(0,1,p)=g(1,0,p)+g(0,1,p)=0,
\end{aligned}$$

where the first equality holds for $e \geq 2$. We use these relations to study the products (6.3.3) and (6.3.4) defining $R_1^M(r,r)$ and $R_1^S(r,r)$. By grouping together the term corresponding to $e \geq 2$ from the first sum, with the term corresponding to $e-1$ from the second sum, and the term for $e-2$ from the third sums, we see using the relations on the first row above that the sum of all these terms is 0. The relations on the second row allow us to group together the rest of the terms and see that these equal 0 as well. It follows that $R_1^M(r,r)$ and $R_1^S(r,r)$ equal 1, by virtue of every factor equalling 1, and also that

$$A_3(r,r)=A_4(r,r)=1. \tag{6.4.1}$$

Next, if f is a function, then the logarithmic derivative is given by

$$\left.\frac{d}{dx}\right|_{x=a} \log f(x) = \frac{f'(a)}{f(a)},$$

which in turn is simply $f'(a)$ if $f(a)=1$. Thus, instead of differentiating R_1^M and R_1^S, we can differentiate their logarithms. Now,

$$\log R_1^M(\alpha,\gamma)=\sum_p \log\left(1+\sum_{e\geq 1}\frac{x_p f(e,0,p)}{p^{e(1/2+\alpha)}}+\sum_{e\geq 0}\frac{x_p f(e,1,p)}{p^{e(1/2+\alpha)+(1/2+\gamma)}}+\sum_{e\geq 0}\frac{x_p f(e,2,p)}{p^{e(1/2+\alpha)+(1+2\gamma)}}\right).$$

As we remarked, setting $\alpha=\gamma$ makes the argument inside every logarithm equal to 1, and thus

$$\begin{aligned}
&\left.\frac{\partial}{\partial\alpha}\right|_{\alpha=\gamma=r}\sum_p \log\left(1+\sum_{e\geq 1}\frac{x_p f(e,0,p)}{p^{e(1/2+\alpha)}}+\sum_{e\geq 0}\frac{x_p f(e,1,p)}{p^{e(1/2+\alpha)+(1/2+\gamma)}}+\sum_{e\geq 0}\frac{x_p f(e,2,p)}{p^{e(1/2+\alpha)+(1+2\gamma)}}\right)\\
&=\sum_p \left.\frac{\partial}{\partial\alpha}\right|_{\alpha=\gamma=r}\left(1+\sum_{e\geq 1}\frac{x_p f(e,0,p)}{p^{e(1/2+\alpha)}}+\sum_{e\geq 0}\frac{x_p f(e,1,p)}{p^{e(1/2+\alpha)+(1/2+\gamma)}}+\sum_{e\geq 0}\frac{x_p f(e,2,p)}{p^{e(1/2+\alpha)+(1+2\gamma)}}\right)\\
&=-\sum_p \frac{x_p\log p}{p^{e(1/2+r)}}\left(\sum_{e\geq 1} f(e,0,p)e+\sum_{e\geq 1} f(e-1,1,p)(e-1)+\sum_{e\geq 2} f(e-2,2,p)(e-2)\right)\\
&=-\sum_p\left(\frac{x_p\log p}{p^{1/2+r}}\cdot\frac{1}{p}+\sum_{e\geq 2}\frac{x_p(f(e,0,p)-f(e,2,p))\log p}{p^{e(1/2+r)}}\right).
\end{aligned}$$

In the last step, we used that

$$\begin{aligned}
&ef(e,0,p)+(e-1)f(e-1,1,p)+(e-2)f(e-2,2,p)\\
&\qquad=(e-1)\big(f(e,0,p)+f(e-1,1,p)+f(e-2,0,p)\big)+\big(f(e,0,p)-f(e-2,2,p)\big),
\end{aligned}$$

for $e \geq 2$, together with $f(1,0,p)=1/p$. Next, one can check that $f(e,0,p)-f(e-2,2,p)=\theta_e+1/p$ so that we can write the result as

$$R_{1,\alpha}^M(r,r) := \left.\frac{\partial}{\partial\alpha}\right|_{\alpha=\gamma=r} R_1(\alpha,\gamma) = -\sum_p\sum_{e\geq 1}\frac{x_p\log p}{p^{e(1/2+r)}}\left(\theta_e+\frac{1}{p}\right). \tag{6.4.2}$$

This sum converges absolutely for $\mathrm{Re}(r)>0$, and thus by the uniqueness of analytic continuations, this formula for the derivative is valid for all such r.

To differentiate R_1^S, one proceeds in precisely the same manner, using g instead of f. Applying the equality

$$y_p\big(g(e,0,p)-g(e-2,2,p)\big)=\gamma_e(p),$$

one finds

$$\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\gamma=r}R_1^S(\alpha,\gamma)=-\sum_p\sum_{e\geq 1}\frac{\gamma_e(p)\log p}{p^{e(1/2+r)}}.$$

Now, $\gamma_1(p)=p^{-1/3}+\mathcal{O}(p^{-2/3})$ so that the expression above converges absolutely for $\mathrm{Re}(r)>1/6$. By using the Dirichlet series for the logarithmic derivative of the Riemann zeta function, we can subtract an appropriate term to extend the domain of definition. Indeed, we have

$$\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\gamma=r}R_1^S(\alpha,\gamma)=-\sum_p\sum_{e\geq 1}\frac{(\gamma_e(p)-p^{-e/3})\log p}{p^{e(1/2+r)}}+\frac{\zeta'}{\zeta}(5/6+r),$$

which is now valid for $\mathrm{Re}(r)>0$, $r\neq 1/6$.

In total, we have now managed to differentiate R_1 with respect to α, and have found

$$R_{1,\alpha}(r,r;X):=\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\gamma=r}R_1(\alpha,\gamma;X)=-F_1(X)\sum_p\sum_{e\geq 1}\frac{x_p\log p}{p^{e(1/2+r)}}\left(\theta_e+\frac{1}{p}\right)$$
$$-F_2(X)\sum_p\sum_{e\geq 1}\frac{(\gamma_e(p)-p^{-e/3})\log p}{p^{e(1/2+r)}}+F_2(X)\frac{\zeta'}{\zeta}(5/6+r).$$

6.5 Estimating R_2'

We now turn to the estimate of R_2', defined in (6.2.3). We begin with an extension of Lemma 6.1 from [CFLS, Corollary 4.2].

Corollary 6.2 (CFLS). *Let $m,h\in\mathbb{Z}_+$, and $1/2\leq\theta<5/6$, $\omega\geq 0$ be such that* (4.2.2) *holds. Then for cubefree h, and α such that $0<\mathrm{Re}(\alpha)<1/2$, we have*

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\alpha}\lambda_K(m)\mu_K(h)=\frac{F_1(X)X^{-\alpha}}{1-\alpha}\prod_{p^e||m,p^s||h}f(e,s,p)x_p+\frac{F_2(X)X^{-\alpha}}{1-6\alpha/5}$$
$$\times\left(\prod_{p^e||m,p^s||h}g(e,s,p)y_p\right)+\mathcal{O}_\epsilon\left((1+|\alpha|)X^{\theta-1-\mathrm{Re}(\alpha)+\epsilon}\prod_{p|hm,p^e||m}(2e+5)p^\omega\right).$$

Proof. We choose to only provide the idea of the proof, as the calculations are very similar to calculations that we have performed several times already. Indeed, one proves the formula above by using Stieltjes integration

$$\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\alpha}\lambda_K(m)\mu_K(h)=\int_1^X u^{-\alpha}d\left(\sum_{K\in\mathcal{F}^{\pm}(u)}\lambda_K(m)\mu_K(h)\right).$$

The result then follows by using integration by parts, as well as the estimate from Lemma 6.1. □

We use this corollary to find a reasonable estimate of R_2', after having interchanged the order of summation so that the sum over K is the innermost sum. Indeed, by similar reasoning as when we estimated R_1', a reasonable estimate is $R_2(\alpha,\gamma;X)$, defined as the expression

$$\frac{\Gamma_\pm(1/2-\alpha)}{\Gamma_\pm(1/2+\alpha)}\left(\frac{X^{-\alpha}}{1-\alpha}F_1(X)R_1^M(-\alpha,\gamma)+\frac{X^{-\alpha}}{1-6\alpha/5}F_2(X)R_1^S(-\alpha,\gamma)\right)=\frac{\Gamma_\pm(1/2-\alpha)}{\Gamma_\pm(1/2+\alpha)}$$
$$\times\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)}\left(\frac{X^{-\alpha}}{1-\alpha}F_1(X)A_3(-\alpha,\gamma)+\frac{X^{-\alpha}}{1-6\alpha/5}F_2(X)\frac{\zeta(5/6-\alpha)}{\zeta(5/6+\gamma)}A_4(-\alpha,\gamma)\right).$$

We expect this to be a good approximation of R_2' for the same range of α and γ as when handling R_1'. Note that we must work with A_3 and A_4 instead of R_1^M and R_1^S, as $\mathrm{Re}(-\alpha), \mathrm{Re}(\gamma)$ cannot both be greater than 1/2 if $\alpha = \gamma$.

Now we wish to differentiate R_2 with respect to α. The trick to avoiding laborious calculations is to write R_2 as a product, where one of the factors is

$$\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)},$$

and then apply the product rule for differentiation. As we want to set $\alpha = \gamma = r$ after differentiating, we can use that $\zeta(1-r+r)^{-1} = \zeta(1)^{-1} = 0$, combined with

$$\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\gamma=r} \frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)} = \frac{\zeta'(1-\alpha+\gamma)\zeta(1-2\alpha) - 2\zeta'(1-2\alpha)\zeta(1-\alpha+\gamma)}{\zeta(1-\alpha+\gamma)^2}\bigg|_{\alpha=\gamma=r} = -\zeta(1-2r).$$

For both these results, we used that $\zeta(s)$ is meromorphic with a simple pole at $s = 1$ with residue equal to 1, see e.g. [D3, p.32]. We thus find

$$\frac{\partial}{\partial\alpha}\bigg|_{\alpha=\gamma=r} R_2(\alpha,\gamma;X) = -\zeta(1-2r)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}$$
$$\times\left(\frac{X^{-r}}{1-r}F_1(X)A_3(-r,r) + \frac{X^{-r}}{1-6r/5}F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)\right).$$

6.6 Finding the one-level density

Using the calculations of the preceding sections we can formulate a Ratios Conjecture for ratios of Artin L-functions, [CFLS, Conjecture 4.3].

Conjecture 6.3 (CFLS). *Let* $1/2 \le \theta < 5/6$, $\omega \ge 0$ *be such that* (4.2.2) *holds. Then there is some* $\delta < 1/6$ *such that for any fixed* $\epsilon > 0$, *and* $r \in \mathbb{C}$ *with* $1/L \ll \mathrm{Re}(r) < \delta$, *and* $|r| \le X^{\epsilon/2}$, *we have*

$$\frac{1}{N^\pm(X)}\sum_{K\in\mathcal{F}^\pm(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)} = -F_1(X)\sum_p\sum_{e\ge1}\frac{x_p\log p}{p^{e(1/2+r)}}\left(\theta_e+\frac{1}{p}\right)$$
$$-F_2(X)\sum_p\sum_{e\ge1}\frac{(\gamma_e(p)-p^{-e/3})\log p}{p^{e(1/2+r)}} + F_2(X)\frac{\zeta'}{\zeta}(5/6+r) - \zeta(1-2r)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}$$
$$\times\left(\frac{X^{-r}}{1-r}F_1(X)A_3(-r,r) + \frac{X^{-r}}{1-6r/5}F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)\right) + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right).$$

Remark. *Some of the conditions above require explanation. First, as we noted above, we want to keep away from any poles of the expressions we are working with. Thus, we assume that* r *has real part smaller than* $1/6$, *as well as a real part which is not too close to* 0. *Further, as a safety measure, we only require our estimates to be true for* $\mathrm{Re}(r) < \delta$, *where* δ *is some number less than* $1/6$. *Lastly, the condition on* $|r|$ *is to make sure that the factor* $(1+|\alpha|)$ *from Corollary 6.2 is not too large.*

The conditions on r may seem overly restrictive, but we will see that Conjecture 6.3 suffices to evaluate the one-level density. Indeed, we have the following proposition and proof, [CFLS, Proposition 4.4].

Proposition 6.4 (CFLS). *Let* $1/2 \le \theta < 5/6$, $\omega \ge 0$ *be such that* (4.2.2) *holds. Assume the Generalised Riemann Hypothesis for all* $\zeta_K(s)$ *with* $K \in \mathcal{F}^\pm(X)$ *and Conjecture 6.3. Then, if* ϕ *is a real, even Schwartz function whose Fourier transform is compactly supported, we* *have*

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K\right)=\widehat{\phi}(0)\bigg(1+\frac{\log(4\pi^2e)}{L}-\frac{C_2^{\pm}}{5C_1^{\pm}}\frac{X^{-1/6}}{L}+\frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}\frac{X^{-1/3}}{L}\bigg)$$
$$+\frac{2}{L}\int_{\mathbb{R}}\phi(t)\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}\left(\frac{1}{2}+\frac{2\pi it}{L}\right)dt-\frac{2}{L}F_1(X)\sum_{p,e}\frac{x_p(\theta_e+1/p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)$$
$$-\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)\gamma_e(p)+\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{5e/6}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)$$
$$-\frac{1}{\pi i}\int_{(1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\bigg[\zeta(1-2r)\frac{\Gamma_{\pm}(1/2-r)}{\Gamma_{\pm}(1/2+r)}\bigg(\frac{X^{-r}}{1-r}F_1(X)A_3(-r,r)+\frac{X^{-r}}{1-6r/5}F_2(X)$$
$$\times\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)\bigg)-F_2(X)\frac{\zeta'}{\zeta}(5/6+r)\bigg]dr+\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)$$

Remark. *Note that the only condition on the support $[-\sigma,\sigma]$ of $\widehat{\phi}$ is that $\sigma<\infty$.*

Proof. We recall that from (6.2.1), we know that under the Generalised Riemann Hypothesis, we have

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K\right)$$
$$=\frac{1}{2\pi i}\left(\int_{(1/L)}-\int_{(-1/L)}\right)\phi\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}dr, \tag{6.6.1}$$

where we have replaced a with $1/L$. We would like to replace the inner sum with the expression from Conjecture 6.3, but this requires some preparation. First, we can only apply the conjecture when the real part of r is greater than 0. Therefore, we begin the calculations by modifying the integral over $(-1/L)$.

First, by logarithmically differentiating the functional equation (3.1.1), we find that

$$-\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}=\log|D_K|+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(1/2+r)+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(1/2-r)+\frac{L'(1/2-r,f_K)}{L(1/2-r,f_K)}, \tag{6.6.2}$$

so that

$$-\frac{1}{2\pi i}\int_{(-1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}dr$$
$$=\frac{1}{2\pi i}\int_{(1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}dr$$
$$+\frac{1}{2\pi i}\int_{(-1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\left(\log|D_K|+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(1/2+r)+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(1/2-r)\right)dr.$$

To analyse the last integral we begin by shifting the contour to (0). A straightforward calculation, very similar to calculations which we have already performed when studying similar terms in the proof of Theorem 3.1, then shows that up to an error $\ll X^{\theta-1+\epsilon}$, we have

$$\frac{1}{2\pi i}\int_{(-1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\left(\log|D_K|+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(1/2+r)+\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}(1/2-r)\right)$$
$$=\widehat{\phi}(0)\left(1+\frac{\log(4\pi^2e)}{L}-\frac{C_2^{\pm}}{5C_1^{\pm}}\frac{X^{-1/6}}{L}+\frac{(C_2^{\pm})^2}{5(C_1^{\pm})^2}\frac{X^{-1/3}}{L}\right)+\frac{2}{L}\int_{\mathbb{R}}\phi(t)\frac{\Gamma_{\pm}'}{\Gamma_{\pm}}\left(\frac{1}{2}+\frac{2\pi it}{L}\right)dt.$$

It remains to evaluate

$$\frac{1}{\pi i}\int_{(1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}dr.$$

We are almost ready to apply Conjecture 6.3, except for the fact that this is only possible for relatively small $|r|$. However, as L'/L grows slowly on $(1/2+1/L)$ assuming GRH, see [IK, Cor. 5.18], while $\phi(Lr/(2\pi i))$ decays very fast in vertical strips by (3.1.2), we may restrict the integral to $|\mathrm{Im}(r)| \leq X^{\epsilon/2}$ at the cost of an insignificant error $\ll_\epsilon X^{-1}$, say. Indeed, in (3.1.2), pick $k \geq N/\epsilon$, for some large enough N.

Next, in this restricted integral we may substitute the inner sum by the expression from the Ratios Conjecture. We now wish to extend this integral back to $(1/L)$. This can once again be done after estimating the integrand and using the fast decay of ϕ. Indeed, we have already shown that both A_3 and $A_4 \ll 1$ for the r we are integrating over. Using Stirling's formula we can find a polynomial bound for the ratio of gamma functions, while the various zeta functions are estimated by using the Riemann Hypothesis, see [MV, Thm. 13.18, 13.23]. Lastly, to find a polynomial bound for the various sums, use

$$\sum_{p,e\geq 2} \frac{\log p}{p^{e(1/2+1/L)}} \ll \frac{1}{1+1/L-1} = L \leq \log X,$$

which follows from integration by parts using the Prime Number Theorem. We are left with

$$\begin{aligned} \frac{1}{\pi i}\int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right) \frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)} dr = \frac{1}{\pi i}\int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right) \\ \times \Bigg[- F_1(X)\sum_p\sum_{e\geq 1} \frac{x_p \log p}{p^{e(1/2+r)}}\left(\theta_e + \frac{1}{p}\right) - F_2(X)\sum_p\sum_{e\geq 1}\frac{(\gamma_e(p)-p^{-e/3})\log p}{p^{e(1/2+r)}} \\ + F_2(X)\frac{\zeta'}{\zeta}(5/6+r) - \zeta(1-2r)\frac{\Gamma_{\pm}(1/2-r)}{\Gamma_{\pm}(1/2+r)}\Bigg(\frac{X^{-r}}{1-r}F_1(X)A_3(-r,r) \\ + \frac{X^{-r}}{1-6r/5}F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)\Bigg) + \mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)\Bigg] dr + \mathcal{O}_\epsilon\left(X^{-1}\right). \end{aligned} \tag{6.6.3}$$

The error term inside the integral simply integrates to $\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)$. To handle the first two integrands, involving sums, we proceed as in the proof of Theorem 3.1 and shift the contour to (0) to obtain

$$-\frac{2}{L}F_1(X)\sum_{p,e}\frac{x_p\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)(\theta_e+1/p) - \frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)(\gamma_e(p)-p^{-e/3}).$$

The proposition follows after splitting the last sum into two parts. □

7

Interpreting the Ratios Conjecture prediction for the one-level density

We will now study the result in Proposition 6.4 in two different ways. We begin by comparing the result to what we obtained unconditionally in Theorem 3.1, for small σ. This has been done in [CFLS, Ch. 5], and we essentially follow their approach. Once this has been done, we will study the expression in Proposition 6.4 more closely to extract an explicit main and secondary term, using methods from [FPS2]. This also uncovers a phase transition in the lower-order terms of the Ratios Conjecture prediction when σ exceeds 1. This has previously been done in other families, see Section 7.3 for several examples, but our Theorem 7.3 is the first result in this direction for the family of Artin L-functions associated to non-Galois cubic fields.

We end the section by pointing out a relation between the coefficients of the terms of size L^{-1} in the one-level density, for the family we have been studying. We show that the same relation can be found in several other symplectic families, and this evidence leads us to formulate Conjecture 7.4.

7.1 Two expressions for the one-level density

We have calculated the one-level density in two different ways. First, by unconditional calculations, but with significant restrictions on the support $[-\sigma, \sigma]$ of $\widehat{\phi}$, in Theorem 3.1. In the previous chapter, we instead calculated the one-level density conditional on the Generalised Riemann Hypothesis and the Ratios Conjecture, with a relaxed condition on the support. Naturally, one would expect that the two different results agree with each other. In fact, one usually expects the Ratios Conjecture to be able to accurately predict terms up to size $\ll_\epsilon X^{-1/2+\epsilon}$, see [CFZ, Ch. 5.1].

As we can only obtain a nontrivial error term in Theorem 3.1 when

$$\sigma < \frac{1-\theta}{\omega + 1/2} \leq 1,$$

it only makes sense to compare the two expressions for $\sigma < 1$. Note that the second inequality follows from the result $\theta + \omega \geq 1/2$ that we mentioned at the end of Chapter 3.1. By comparing the two expressions for the one-level density, excluding the error terms, we see that they are equal, except for a term

$$\begin{aligned} J(X) := \frac{2}{L} F_2(X) \sum_{p,e} \frac{\log p}{p^{5e/6}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) - \frac{1}{\pi i} \int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right) \left[\zeta(1-2r) \frac{\Gamma_\pm(1/2 - r)}{\Gamma_\pm(1/2 + r)} \right. \\ \times \left. \left(\frac{X^{-r}}{1-r} F_1(X) A_3(-r, r) + \frac{X^{-r}}{1 - 6r/5} F_2(X) \frac{\zeta(5/6 - r)}{\zeta(5/6 + r)} A_4(-r, r) \right) - F_2(X) \frac{\zeta'}{\zeta}(5/6 + r) \right] dr. \end{aligned} \tag{7.1.1}$$

Somewhat surprisingly, it was shown in [CFLS, Ch. 5] that $J(X) \gg X^{-1/3}$, even for small values of σ. As the error term in Theorem 3.1 has the form

$$\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon+\sigma(1/2+\omega)}\right),$$

we remark, as in [CFLS], that if one was able to improve the results in (3.2.3) to hold with $\theta < 2/3$, then there would be a discrepancy between the Ratios Conjecture prediction and the actual one-level density, of a larger size than expected. As we have mentioned, numerical results from [CFLS] indicates that (3.2.3) may hold with $\theta = 1/2$ and any $\omega > 0$. However, the current best values θ, $\omega = 2/3$ do not suffice to show that a discrepancy exists.

The rest of this section will be dedicated to the proof of the following proposition, which is essentially [CFLS, Lemma 5.4]. The proof is taken from [CFLS, Ch. 5].

Proposition 7.1 (CFLS)**.** *For $\sigma < 1$, the difference $J(X)$ from* (7.1.1) *between the Ratios Conjecture prediction in Proposition 6.4, and the actual one-level density in Proposition 3.1, satisfies*

$$J(X) \ll_\epsilon X^{\sigma/6-1/3+\epsilon} + X^{\sigma/2-1/2+\epsilon}. \tag{7.1.2}$$

In particular, for sufficiently small σ, the two expressions differ by $J(X) \ll_\epsilon X^{-1/3+\epsilon}$.

To begin estimating $J(X)$ we focus on the part of the integrand in (7.1.1) involving $A_4(-r,r)$. As $A_4(-r,r)$ is holomorphic for $|\mathrm{Re}(r)| < 1/6$, we may shift

$$-\frac{1}{\pi i}\int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right)\zeta(1-2r)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\frac{X^{-r}}{1-6r/5}F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)dr$$

to the line $(1/6-\epsilon)$. Recall that on this line $A_4 \ll_\epsilon 1$. Using $F_2(X) \ll X^{-1/6}$, $|X^{-r}| = X^{-\mathrm{Re}(r)}$, the decay of ϕ specified in (3.1.2), together with a polynomial estimate of the rest of the factors, as in the proof of Proposition 6.4, we can see that the shifted integral is

$$\ll_\epsilon X^{\sigma/6-1/3+\epsilon}.$$

To estimate the remaining part of $J(X)$ we will need to shift the integrand involving $A_3(-r,r)$ further to the right. This will require us to first find a meromorphic continuation of A_3. We prove the following result [CFLS, Lemma 5.1].

Lemma 7.2 (CFLS)**.** *The expression*

$$A_3(-r,r) = \zeta(3)\zeta\left(\tfrac{3}{2}-3r\right)\prod_p\left(1-\frac{1}{p^{3/2+r}}+\frac{1}{p^{5/2-r}}-\frac{1}{p^{5/2-3r}}-\frac{1}{p^{3-4r}}+\frac{1}{p^{9/2-5r}}\right), \tag{7.1.3}$$

furnishes a meromorphic continuation of $A_3(-r,r)$ to $|Re(r)| < 1/2$, with a simple pole at $r = 1/6$ with residue

$$-\frac{\zeta(3)}{3\zeta(5/3)\zeta(2)}.$$

Furthermore, this is the only pole for $|\mathrm{Re}\, r| < 1/2$.

Proof. Recall that A_3 is defined in (6.3.5) by the product formula

$$\begin{aligned}
A_3(-r,r) &= \prod_p\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p^{1-2r}}\right)\\
&\times\left(1+\sum_{e\geq 1}\frac{x_p f(e,0,p)}{p^{e(1/2-r)}}+\sum_{e\geq 0}\frac{x_p f(e,1,p)}{p^{e(1/2-r)+(1/2+r)}}+\sum_{e\geq 0}\frac{x_p f(e,2,p)}{p^{e(1/2-r)+(1+2r)}}\right)\\
&= \zeta(3)\prod_p\left(1-\frac{1}{p^{1-2r}}\right)\\
&\times\left(1+\frac{1}{p}+\frac{1}{p^2}+\sum_{e\geq 1}\frac{f(e,0,p)}{p^{e(1/2-r)}}+\sum_{e\geq 0}\frac{f(e,1,p)}{p^{e(1/2-r)+(1/2+r)}}+\sum_{e\geq 0}\frac{f(e,2,p)}{p^{e(1/2-r)+(1+2r)}}\right),
\end{aligned}$$

where we used $x_p = (1 + 1/p + 1/p^2)^{-1}$. As $1 + 1/p$ is equal to the expression for $f(e, 0, p)$ evaluated at $e = 0$, we can write this as

$$\zeta(3) \prod_p \left(1 - \frac{1}{p^{1-2r}}\right) \left(\frac{1}{p^2} + \sum_{e \geq 0} \frac{1}{p^{e(1/2-r)}} \left(f(e, 0, p) + \frac{f(e, 1, p)}{p^{1/2+r}} + \frac{f(e, 2, p)}{p^{1+2r}}\right)\right).$$

The attentive reader may remember that in Lemma 6.1, we said that $f(0, 0, p)$ should be interpreted as 1. In the expression above, we have used the convention $f(0, 0, p) = 1 + 1/p$, which better agrees with the formula for $f(e, 0, p)$ given in the mentioned lemma. This is simply a matter of notational convenience. From the definition of $f(e, s, p)$, we then see that the sum equals

$$\begin{aligned} &\frac{1}{6}\left(1 - \frac{1}{p^{1/2+r}}\right)^2 \sum_{e \geq 0} \frac{e+1}{p^{e(1/2-r)}} + \frac{1}{2}\left(1 - \frac{1}{p^{1+2r}}\right) \sum_{e \geq 0} \frac{\delta_{2|e}}{p^{e(1/2-r)}} \\ &\quad + \frac{1}{3}\left(1 + \frac{1}{p^{1/2+r}} + \frac{1}{p^{1+2r}}\right) \sum_{e \geq 0} \frac{\tau_e}{p^{e(1/2-r)}} + \frac{1}{p}\left(1 - \frac{1}{p^{1/2+r}}\right) \sum_{e \geq 0} \frac{1}{p^{e(1/2-r)}}. \end{aligned}$$

These sums can all be explicitly evaluated by appealing to the formula for a geometric series, as well as using

$$\sum_{e \geq 0} (e+1) x^e = \frac{d}{dx} \sum_{e \geq 0} x^{e+1} = \frac{1}{(1-x)^2},$$

and the definition $\tau_e = \delta_{3|e} - \delta_{3|(e-1)}$. The result is

$$\frac{1}{6} \cdot \frac{\left(1 - p^{-1/2-r}\right)^2}{(1 - p^{-1/2+r})^2} + \frac{1}{2} \cdot \frac{1 - p^{-1-2r}}{1 - p^{-1+2r}} + \frac{1}{3} \cdot \frac{1 + p^{-1/2-r} + p^{-1-2r}}{1 + p^{-1/2+r} + p^{-1+2r}} + \frac{1}{p} \cdot \frac{1 - p^{-1/2-r}}{1 - p^{-1/2+r}}. \quad (7.1.4)$$

Using this expression for the sum, one can simplify and confirm that (7.1.3) holds. This can be done quickly using a computer algebra system, or more laboriously by hand. The point is that multiplying by

$$\frac{1}{\zeta(3/2 - 3r)} = \prod_p \left(1 - \left(\frac{1}{p^{1/2-r}}\right)^3\right),$$

essentially clears the denominators in (7.1.4). As the product in (7.1.3) converges absolutely for $|\mathrm{Re}(r)| < 1/2$, we have obtained the desired continuation.

To find the residue at $r = 1/6$, use that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, see [D3, p. 32], to write

$$\zeta(3/2 - 3r) = -\frac{1}{3} \cdot \frac{1}{r - 1/6} + \mathcal{O}(1),$$

and set $r = 1/6$ in the rest of (7.1.3). □

We now estimate the rest of $J(X)$. We need to study

$$\begin{aligned} &\frac{2}{L} F_2(X) \sum_{p,e} \frac{\log p}{p^{5e/6}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) - \frac{1}{\pi i} \int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right) \left[\zeta(1-2r) \frac{\Gamma_\pm(1/2 - r)}{\Gamma_\pm(1/2 + r)}\right. \\ &\left. \times \frac{X^{-r}}{1-r} F_1(X) A_3(-r, r) - F_2(X) \frac{\zeta'}{\zeta}(5/6 + r)\right] dr. \end{aligned}$$

The shape of the pole at $s = 1$ of $\zeta(s)$ implies that $\zeta'(s)/\zeta(s)$ has a simple pole with residue -1 at $s = 1$. Thus, if we shift the contour to the line $(1/2 - \epsilon)$, we pick up the negative of the residue

$$-2\phi\left(\frac{L}{12\pi i}\right) \left(-\frac{2\zeta(2/3)\Gamma_\pm(1/3)\zeta(3) X^{-1/6}}{5\Gamma_\pm(2/3)\zeta(5/3)\zeta(2)} F_1(X) + F_2(X)\right),$$

at $s = 1/6$.

To simplify this expression, we will need the functional equation of $\zeta(s)$, i.e. the relation [D3, p. 59]

$$\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s).$$

Combining this with the definition of $\Gamma_\pm$ and $C_i^\pm$, and using the reflection and duplication formula for the Gamma function, see Appendix A.2, we see that

$$\frac{2\zeta(2/3)\Gamma_\pm(1/3)\zeta(3)}{5\Gamma_\pm(2/3)\zeta(5/3)\zeta(2)} = \frac{C_2^\pm}{C_1^\pm}.$$

Also, as

$$X^{-1/6}\frac{C_2^\pm}{C_1^\pm}F_1(X) = F_2(X) + \mathcal{O}\left(X^{-1/2}\right),$$

we see that the residue is $\ll_\epsilon X^{\sigma/6-1/2}$ by using (3.1.2).

It remains to estimate

$$\begin{aligned}&\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{5e/6}}\widehat{\phi}\left(\frac{\log p^e}{L}\right) - \frac{1}{\pi i}\int_{(1/2-\epsilon)}\phi\left(\frac{Lr}{2\pi i}\right)\left[\zeta(1-2r)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\right.\\&\left.\times\frac{X^{-r}}{1-r}F_1(X)A_3(-r,r) - F_2(X)\frac{\zeta'}{\zeta}(5/6+r)\right]dr.\end{aligned} \tag{7.1.5}$$

The integral of the term above is estimated by estimating each factor separately and then using the rapid decay of ϕ, cf. the proof of Proposition 6.4. As $|X^{-r}| = X^{-1/2+\epsilon}$, we can bound the first integral by $X^{\sigma/2-1/2+\epsilon}$ multiplied by some appropriate constant, possibly depending on ϵ.

To handle the second integrand, expand ζ'/ζ into its Dirichlet series and interchange the order of summation and integration. Then, similarly as in the proof of Theorem 3.1, move the integral to (0) and use the definition of Fourier transform. The result is simply

$$-\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{5e/6}}\widehat{\phi}\left(\frac{\log p^e}{L}\right),$$

which cancels against the sum in (7.1.5). This concludes the proof of Proposition 7.1.

7.2 A phase transition in the one-level density

In the previous section, we studied the expression from Proposition 6.4 for $\sigma < 1$, essentially following [CFLS], and found that it agreed quite well with the unconditional result from Theorem 3.1. In this section, we will drop the condition on σ, and we will see that for $\sigma > 1$ one finds new main terms in the one-level density coming from the term $J(X)$. These terms have not previously been isolated in this family. More precisely we will prove the following theorem.

Theorem 7.3. *Assume the Generalised Riemann Hypothesis for $\zeta_K(s)$, and Conjecture 6.3. Then, if ϕ is a real, even Schwartz function whose Fourier transform is compactly supported, we have*

$$\begin{aligned}&\frac{1}{N^\pm(X)}\sum_{K\in\mathcal{F}^\pm(X)}\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K\right)\\&= \left(\widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(r)dr\right) + \frac{\widehat{\phi}(0)}{L}\left(1 - 4\log 2 - \pi\delta_+ - C\right)\\&+ \frac{\widehat{\phi}(1)}{L}\left(-1 + 4\log 2 + \pi\delta_+ + C\right) + \mathcal{O}\left(\frac{1}{L^2}\right),\end{aligned}$$

where

$$C = 2\sum_{p}(\log p)\frac{2p^{5/2}+2p^2+p^{3/2}-p-p^{1/2}-1}{(p^3-1)(p+p^{1/2}+1)},$$

and where $\delta_+ = 1$ *if we are considering positive discriminants, and else it equals* 0.

Remark. *The main term is the symplectic main term that we expect from the Katz–Sarnak prediction. The secondary term involves* $\widehat{\phi}(1)$*, which is possibly nonzero only if* $\sigma > 1$*. Such behaviour in the one-level density is sometimes referred to as a phase transition, see e.g. [FPS2], and has been observed in several other families, see the next section for several examples. The term involving* $\widehat{\phi}(1)$ *is entirely contained within the term* $J(X)$ *from* (7.1.1)*, which explains why we could only obtain a power-saving bound for this term when* $\sigma < 1$.

Remark. *The coefficient in front of* $L^{-1}\widehat{\phi}(0)$ *is the negative of the coefficient in front of* $L^{-1}\widehat{\phi}(1)$*. We will return to this phenomenon in the next section.*

The rest of the section is dedicated to the proof of this theorem. The strategy used in the proof is essentially that of [FPS2], but with the details modified to fit this particular family. By Proposition 6.4, the one-level density is given by

$$\begin{aligned}
&\widehat{\phi}(0)\left(1+\frac{\log(4\pi^2 e)}{L}\right)+\frac{2}{L}\int_{\mathbb{R}}\phi(t)\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi i t}{L}\right)dt\\
&-\frac{2}{L}F_1(X)\sum_{p,e}\frac{x_p\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)(\theta_e+1/p)-\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)\gamma_e(p)\\
&+\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{5e/6}}\widehat{\phi}\left(\frac{\log p^e}{L}\right)\\
&-\frac{1}{\pi i}\int_{(1/L)}\phi\left(\frac{Lr}{2\pi i}\right)\left[\zeta(1-2r)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\left(\frac{X^{-r}}{1-r}F_1(X)A_3(-r,r)\right.\right.\\
&\left.\left.+\frac{X^{-r}}{1-6r/5}F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)\right)-F_2(X)\frac{\zeta'}{\zeta}(5/6+r)\right]dr+\mathcal{O}(L^{-2}).
\end{aligned}$$

We will estimate this expression one term at a time.

7.2.1 An integral of Gamma functions

We begin by estimating

$$\frac{2}{L}\int_{\mathbb{R}}\phi(t)\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi i t}{L}\right)dt.$$

After using Stirling's formula to bound the factor involving $\Gamma_\pm$ polynomially we may use a cutoff argument similar to the ones we have already used to restrict the integral to $|t| \leq L$, with an error that is, say $\ll L^{-2}$. The precise exponent is not important, as ϕ decays very fast. In fact, we could have bounded the tail of the integral by any $\ll_N L^{-N}$. Next, we perform a zeroth order Taylor expansion of

$$\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}+\frac{2\pi i t}{L}\right),$$

to write the cut-off integral as

$$\frac{2}{L}\cdot\frac{\Gamma'_\pm}{\Gamma_\pm}(1/2)\int_{-L}^{L}\phi(t)dt+\mathcal{O}\left(\frac{1}{L^2}\int_{-L}^{L}|t\phi(t)|dt\right)=\frac{2}{L}\cdot\frac{\Gamma'_\pm}{\Gamma_\pm}(1/2)\widehat{\phi}(0)+\mathcal{O}\left(L^{-2}\right),$$

where we extended the integral back to $(-\infty,\infty)$ in the last step.

Now,

$$\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}\right)=-\log\pi+\frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right)+\frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{3-2\delta_+}{4}\right).$$

By logarithmically differentiating the product definition of $\Gamma(s)$, one can find $\Gamma'(1)/\Gamma(1) = -\gamma$, where γ is Euler's constant. Then using this, combined with the reflection formula and duplication formula, it follows that

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - \log 8,\ \frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - \log 8.$$

Thus, we have that

$$\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2}\right) = -\log 8\pi - \gamma - \frac{\pi}{2}\delta_+, \tag{7.2.1}$$

whence

$$\frac{2}{L}\int_{\mathbb{R}} \phi(t)\frac{\Gamma'_\pm}{\Gamma_\pm}\left(\frac{1}{2} + \frac{2\pi i t}{L}\right)dt = \frac{\widehat{\phi}(0)}{L}\left(-2\log 8\pi - 2\gamma - \pi\delta_+\right).$$

7.2.2 Estimating prime sums

We turn to the estimation of the sums over primes. From (3.3.1) we already know

$$\begin{aligned}
&\frac{-2}{L}\sum_p\sum_{e\geq 1}\frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}}\widehat{\phi}\left(\frac{\log p^e}{L}\right) = -\frac{\phi(0)}{2} \\
&- \frac{2}{L}\widehat{\phi}(0)\left[1 + \int_1^\infty \frac{\theta(u) - u}{u^2}du + \sum_p\sum_{e\neq 2}\frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}} + \sum_p \frac{\log p}{p}\left(x_p\left(1 + \frac{1}{p}\right) - 1\right)\right] \\
&+ \mathcal{O}(L^{-2}),
\end{aligned}$$

where $\theta(u) = \sum_{p\leq u}\log p$. To simplify, we take a look at the integral. We have

$$\int_1^\infty \frac{\theta(u) - u}{u^2}du = \lim_{N\to\infty}\int_1^N \frac{\theta(u) - u}{u^2}du = \lim_{N\to\infty}\left(\int_1^N \frac{\theta(u)}{u^2}du - \log N\right). \tag{7.2.2}$$

Further, we know that this limit exists, as the original integral converges by the Prime Number Theorem. Using integration by parts, and the Prime Number Theorem, we obtain

$$\int_1^N \frac{\theta(u)}{u^2}du = -\int_1^N \theta(u)d\left(\frac{1}{u}\right) = -\frac{\theta(N)}{N} + \sum_{p\leq N}\frac{\log p}{p} = \sum_{p\leq N}\frac{\log p}{p} - 1 + o(1).$$

To study this sum, we use an argument from [Lu, p. 199-200]. Instead of working with $\theta(n)$, it will be more convenient to use the von-Mangoldt function $\Lambda(n)$. We write

$$\sum_{n\leq N}\frac{\theta(n)}{n} = -\sum_{n\leq N}\frac{\Lambda(n) - \theta(n)}{n} + \sum_{n\leq N}\frac{\Lambda(n) - 1}{n} + \sum_{n\leq N}\frac{1}{n}.$$

By definition of γ,

$$\sum_{n\leq N}\frac{1}{n} = \log N + \gamma + o(1).$$

Also by definition, we have

$$\lim_{N\to\infty}\sum_{n\leq N}\frac{\Lambda(n) - \theta(n)}{n} = \sum_{p, e\geq 2}\frac{\log p}{p^e} = \sum_p \frac{\log p}{p(p-1)}.$$

In particular, these calculations, together with the existence of the integral in the left-hand side of (7.2.2), shows that

$$\lim_{N\to\infty}\sum_{n\leq N}\frac{\Lambda(n) - 1}{n}$$

exists. Using summation by parts, one can show that if a Dirichlet series converges at some, say real, point x_0, then it converges uniformly in the set $\{x \in \mathbb{R} : x \geq x_0\}$. Thus, we have

$$\sum_n \frac{\Lambda(n) - 1}{n} = \lim_{s \to 1^+} \sum_n \frac{\Lambda(n) - 1}{n^s} = \lim_{s \to 1^+} \left(-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) \right). \tag{7.2.3}$$

It is well-known, see e.g. [D3, p. 81], that

$$\zeta(s) = \frac{1}{s-1} + \gamma + \mathcal{O}(s-1) \tag{7.2.4}$$

near $s = 1$, whence (7.2.3) equals -2γ. Putting together all our calculations, we have shown that

$$\int_1^\infty \frac{\theta(u) - u}{u^2} du = -1 - \gamma - \sum_p \frac{\log p}{p(p-1)}.$$

Lastly, we should estimate the two sums

$$-\frac{2}{L} F_2(X) \sum_{p,e} \frac{\log p}{p^{e/2}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) \gamma_e(p) + \frac{2}{L} F_2(X) \sum_{p,e} \frac{\log p}{p^{5e/6}} \widehat{\phi}\left(\frac{\log p^e}{L}\right).$$

As $\gamma_1(p) = p^{-1/3} + \mathcal{O}(p^{-2/3})$, we have

$$\sum_{p,e} \frac{\log p}{p^{e/2}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) \gamma_e(p) = \sum_{p,e} \frac{\log p}{p^{5e/6}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) + \mathcal{O}\left(\sum_{p \leq X^\sigma} \frac{\log p}{p}\right).$$

An application of Stieltjes integration shows that the error term is $\ll L$. Thus, as $F_2(X) \ll X^{-1/6}$, we find that

$$-\frac{2}{L} F_2(X) \sum_{p,e} \frac{\log p}{p^{e/2}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) \gamma_e(p) + \frac{2}{L} F_2(X) \sum_{p,e} \frac{\log p}{p^{5e/6}} \widehat{\phi}\left(\frac{\log p^e}{L}\right) \ll \frac{1}{L^2}. \tag{7.2.5}$$

In fact, the exponent 2 can be made as large as we wish.

7.2.3 Estimating an integral

It remains to estimate

$$-\frac{1}{\pi i} \int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right) \Bigg[\zeta(1-2r) \frac{\Gamma_\pm(1/2 - r)}{\Gamma_\pm(1/2 + r)} \Bigg(\frac{X^{-r}}{1-r} F_1(X) A_3(-r, r) + \frac{X^{-r}}{1 - 6r/5} F_2(X) \frac{\zeta(5/6 - r)}{\zeta(5/6 + r)} A_4(-r, r)\Bigg) - F_2(X) \frac{\zeta'}{\zeta}(5/6 + r)\Bigg] dr.$$

Using (3.1.2), $F_2(X) \ll X^{-1/6}$, a Laurent expansion of $\zeta(1-2r)$, as well as estimates for the other factors that we have made use of before, e.g. when proving Proposition 6.4, we can directly see that the only relevant term for our purposes is

$$-\frac{1}{\pi i} \int_{(1/L)} \phi\left(\frac{Lr}{2\pi i}\right) \zeta(1-2r) \frac{\Gamma_\pm(1/2 - r)}{\Gamma_\pm(1/2 + r)} \frac{X^{-r}}{1-r} F_1(X) A_3(-r, r) dr.$$

Indeed, the other terms are $\ll_\epsilon X^{-1/6+\epsilon}$. To estimate this integral, we follow the approach of [FPS2, Lemma 4.6]. As $F_1(X) = 1 + \mathcal{O}(X^{-1/6})$ we may also replace this factor by 1.

First, we make the change of variables $s = Lr/(2\pi i)$, which transforms the integral into

$$-\frac{2}{L} \int_{\mathcal{L}} \phi(s) \zeta\left(1 - \frac{4\pi i s}{L}\right) \frac{\Gamma_\pm(1/2 - 2\pi i s/L)}{\Gamma_\pm(1/2 + 2\pi i s/L)} \frac{X^{-2\pi i s/L}}{1 - 2\pi i s/L} A_3\left(-\frac{2\pi i s}{L}, \frac{2\pi i s}{L}\right) ds,$$

where $\mathcal{L}$ is the horizontal line given by $\mathrm{Im}(s) = -1/2\pi$, oriented from left to right. Next, shift the contour from $(1/L)$ to $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$, defined by

$$\mathcal{C}_0 = \{s \in \mathbb{R} : |s| \geq L^{\epsilon}\},\ \mathcal{C}_1 = \{s \in \mathbb{R} : \eta \leq |s| \leq L^{\epsilon}\},\ \mathcal{C}_2 = \{s = \eta e^{it} : -\pi \leq t \leq 0\},$$

with $\eta > 0$, and some fixed $\epsilon > 0$. Note that this contour avoids the pole of ζ. Further, by a cutoff argument that we have seen several times, we may disregard the contribution from $\mathcal{C}_0$.

Write

$$X^{-2\pi is/L} = e^{-2\pi is \log X/L} = e^{-2\pi is} e^{-4\pi is \log(2\pi e)/L}.$$

On $\mathcal{C}_1$ and $\mathcal{C}_2$, s is small enough that we may expand every factor of the integrand, except $\phi(s)$ and $e^{-2\pi is}$, into its corresponding Laurent series. For our purposes, it will be sufficient to consider the expansion

$$-\frac{2}{L}\int_{\mathcal{C}_1\cup\mathcal{C}_2} \phi(s)\, e^{-2\pi is}\left(-\frac{L}{4\pi is} + \gamma + \mathcal{O}\left(\frac{|s|}{L}\right)\right)\left(1 - \frac{4\pi is}{L}\cdot\frac{\Gamma'_{\pm}}{\Gamma_{\pm}}\left(\frac{1}{2}\right) + \mathcal{O}\left(\frac{|s|^2}{L^2}\right)\right)$$
$$\times\left(1 - \frac{4\pi is\log(2\pi e)}{L} + \mathcal{O}\left(\frac{|s|^2}{L^2}\right)\right)\left(1 + \frac{2\pi is}{L} + \mathcal{O}\left(\frac{|s|^2}{L^2}\right)\right)\left(1 + \frac{2\pi iCs}{L} + \mathcal{O}\left(\frac{|s|^2}{L^2}\right)\right) ds,$$

with

$$C := \left.\frac{d}{dr}\right|_{r=0} A_3(-r, r). \tag{7.2.6}$$

In the Taylor expansion of $A_3(-r, r)$ we made use of (6.4.1), which implies that $A_3(0, 0) = 1$. Now, one may multiply together all the factors, and use (7.2.1), to see that the above is

$$\int_{\mathcal{C}_1\cup\mathcal{C}_2}\left[\phi(s)\frac{e^{-2\pi is}}{2\pi is}\left(1 + \frac{2\pi is}{L}\left(-1 + 4\log 2 + \pi\delta_+ + C\right)\right) + \mathcal{O}\left(\frac{|s|^2}{L^2}\right)\right] ds.$$

We can immediately see that the error term integrates to be $\ll L^{-2}$, whence it can be disregarded. As ϕ is a Schwartz function, the integral over the remaining terms can be extended to include $\mathcal{C}_0$ once again. We now estimate this extended integral.

We begin with the secondary term, which is the easiest to study. Indeed, we have

$$\frac{(-1 + 4\log 2 + \pi\delta_+ + C)}{L}\int_{\mathcal{C}}\phi(s)\, e^{-2\pi is} ds = \frac{(-1 + 4\log 2 + \pi\delta_+ + C)}{L}\int_{-\infty}^{\infty}\phi(t)\, e^{-2\pi it} dt$$
$$+ o_\eta(1) = \frac{(-1 + 4\log 2 + \pi\delta_+ + C)\,\widehat{\phi}(1)}{L} + o_\eta(1),$$

where the last step is the definition of the Fourier transform. The term $o_\eta(1)$ should be interpreted as a term converging to 0 when $\eta \to 0^+$, corresponding to the contribution from $\mathcal{C}_2$.

We turn to the main term, which requires slightly more effort. We write it as

$$\int_{|t|\geq\eta}\phi(t)\frac{\cos(2\pi t) - i\sin(2\pi t)}{2\pi it}dt + \frac{1}{2\pi i}\int_{\mathcal{C}_2}\phi(s)\frac{e^{-2\pi is}}{s}ds.$$

The last integral is over a half-circle around a pole. One may Taylor expand $\phi(s)$ and $e^{-2\pi is}$ close to 0 and then parameterise the contour to see that the contribution from the last integral is

$$\frac{\phi(0)}{2} + o_\eta(1).$$

In the other integral, we may disregard the term containing $\cos(2\pi t)$, as this integrand is odd, as ϕ is even. Then, by the continuity of the integrand involving $\sin(2\pi t)$ at 0, what remains is

$$-\int_{|t|\geq\eta}\phi(t)\frac{\sin(2\pi t)}{2\pi t}dt = -\int_{-\infty}^{\infty}\phi(t)\frac{\sin(2\pi t)}{2\pi t}dt + o_\eta(1) = -\frac{1}{2}\int_{-1}^{1}\widehat{\phi}(u)du + o_\eta(1).$$

In the last step, we used Plancherel's theorem, and the fact that the Fourier transform of the indicator function of $[-1,1]$ is

$$2 \cdot \frac{\sin(2\pi t)}{2\pi t}.$$

Letting $\eta \to 0^+$ finishes the analysis.

7.2.4 Completing the proof of Theorem 7.3

Collecting the results of the previous sections, we have proven that the one-level density is given by

$$\begin{aligned}
&\widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(u) du \\
&\quad + \frac{\widehat{\phi}(0)}{L}\Bigg(2\sum_p \frac{\log p}{p(p-1)} - 2\sum_p\sum_{e\neq 2} \frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}} - 2\sum_p \frac{\log p}{p}\left(x_p\left(1+\frac{1}{p}\right)-1\right) \\
&\quad + 1 - 4\log 2 - \pi\delta_+\Bigg) + \frac{\widehat{\phi}(1)}{L}\left(-1 + 4\log 2 + \pi\delta_+ + C\right) + \mathcal{O}(L^{-2}).
\end{aligned}$$

Everything is given quite explicitly above, except for the constant C, defined in (7.2.6). By Lemma 7.2, $A_3(-r,r)$ is given by

$$\zeta(3)\zeta\left(\tfrac{3}{2} - 3r\right)\prod_p\left(1 - \frac{1}{p^{3/2+r}} + \frac{1}{p^{5/2-r}} - \frac{1}{p^{5/2-3r}} - \frac{1}{p^{3-4r}} + \frac{1}{p^{9/2-5r}}\right). \tag{7.2.7}$$

As $A_3(0,0) = 1$, we may calculate its logarithmic derivative to find C. Taking the logarithm of (7.2.7), differentiating with respect to r, and then setting $r = 0$, yields

$$C = 2\sum_p \frac{(\log p)(2p^{5/2} + 2p^2 + p^{3/2} - p - p^{1/2} - 1)}{(p^3-1)(p + p^{1/2} + 1)}.$$

It remains to show that

$$2\sum_p \frac{\log p}{p(p-1)} - 2\sum_p\sum_{e\neq 2}\frac{x_p(\theta_e + 1/p)\log p}{p^{e/2}} - 2\sum_p\frac{\log p}{p}\left(x_p\left(1+\frac{1}{p}\right) - 1\right) = -C. \tag{7.2.8}$$

This is done by evaluating the sums explicitly, using $x_p = (1+1/p+1/p^2)^{-1}$, $\theta_e = \delta_{2|e} + \delta_{3|e}$. In particular, as θ_e only depends on the congruence class of e modulo 6, one can split every sum involving θ_e into six different sums, depending on the congruence class. Together with the formula for a geometric series, this shows that the part of the second sum in (7.2.8) involving θ_e, including the factor -2, is equal to

$$-2\sum_p \frac{x_p \log p}{p^3-1}\left(2 + 1/p + p^{3/2} + p\right),$$

while the part of the second sum involving $1/p$ is equal to

$$-2\sum_p \frac{x_p(\log p)(p^{3/2}+1)}{p^2(p-1)}.$$

Adding the two other sums from (7.2.8), and simplifying using $x_p = \left(1 + 1/p + 1/p^2\right)^{-1}$ proves the equality in (7.2.8). Specifically, the pth term in the sum defining C is equal to the negative of the pth term in the sum obtained by adding together all the sums from (7.2.8). A computer algebra system may be of use in proving this equality.

7.3 A symplectic phenomenon

A curious detail of Theorem 7.3 is that the coefficient of $\widehat{\phi}(0)/L$ is the negative of the coefficient of $\widehat{\phi}(1)/L$. It is not clear why this should be the case, and in the proof above, showing this required quite some effort. The point of this section is to motivate the following conjecture, extending one part of the Katz–Sarnak prediction.

Conjecture 7.4. *Let $\mathcal{F}$ be a natural family of L-functions whose symmetry type is symplectic. Then, the one-level density corresponding to this family has the form*

$$\widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(u)du + \frac{C_{\mathcal{F}}}{M}\left(\widehat{\phi}(1) - \widehat{\phi}(0)\right) + o(M^{-1}),$$

where $C_{\mathcal{F}}$ is a constant depending on the family $\mathcal{F}$, and M is an analogue of our L.

We will present a few different symplectic families where the one-level density has been found explicitly, with error term $o\left(M^{-1}\right)$, and show that in these cases, the conjecture above holds. In all the examples that we will study, the relation between the coefficients cannot be found without some additional effort, which may explain why the phenomenon in Conjecture 7.4 appears to not have been pointed out before in the literature. When studying the examples below, we omit most of the details and background and instead focus on proving the conjecture in these special cases.

We remark that the conjecture does not hold for all families. For example in [DFS], the one-level density for an SO(even) family is studied, and then the coefficients are not each other's negatives, but rather equal to each other. However, in the same article, a related SO(odd) family is studied, and then the same relation as in the conjecture above holds. It would be interesting to see if similar relations continue to hold in other families of the same symmetry type, but that is beyond the scope of this thesis.

7.3.1 Quadratic Dirichlet characters

In [FPS2], a symplectic family related to certain multiplicative functions, known as Dirichlet characters, was investigated. The results are conditional on the Generalised Riemann Hypothesis for this family, but not on the Ratios Conjecture, see [FPS2, Theorem 1.1]. The one-level density was found to equal

$$\begin{aligned}
&\widehat{\phi}(0) + \int_{1}^{\infty}\widehat{\phi}(u)du + \frac{\widehat{\phi}(0)}{L}\left(\log(2e^{1-\gamma}) + \frac{2}{\widehat{w}(0)}\int_{0}^{\infty} w(x)(\log x)dx\right)\\
&+\frac{1}{L}\int_{0}^{\infty}\frac{e^{-x/2}+e^{-3x/2}}{1-e^{-2x}}\left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{x}{L}\right)\right)dx - \frac{2}{L}\sum_{p>2, j\geq 1}\frac{\log p}{p^j}\left(1+\frac{1}{p}\right)^{-1}\widehat{\phi}\left(\frac{2j\log p}{L}\right)\\
&+\frac{\widehat{\phi}(1)}{L}\left(-\frac{7}{3}\log 2 - 1 - \gamma + 2\frac{\zeta'}{\zeta}(2) - \frac{2}{\widehat{w}(0)}\int_{0}^{\infty} w(x)(\log x)dx\right) + \mathcal{O}(L^{-2}).
\end{aligned}$$

Here, the function ϕ is a real, even Schwartz function, and the corresponding σ is assumed to be less than 2. The function $w(x)$ is a weight function, but it will not be important for our analysis. Also, the L in the expression above is not quite the same L as we have used, but they have the same role. This expression was not simplified further in [FPS2], so we sketch a simplification here.

A Taylor expansion shows that the integral containing the exponential functions is $\ll L^{-2}$. Thus, the only term that needs estimation is the sum over primes. We omit the details, but the calculations are very similar to those of Section 3.3, and those of the previous section. Carrying out the calculations shows that the expression above is

$$\begin{aligned}
&\widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(u)du + \frac{\widehat{\phi}(0)}{L}\left(\frac{7}{3}\log 2 + 1 + \gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{2}{\widehat{w}(0)}\int_{0}^{\infty} w(x)(\log x)dx\right)\\
&\quad+\frac{\widehat{\phi}(1)}{L}\left(-\frac{7}{3}\log 2 - 1 - \gamma + 2\frac{\zeta'}{\zeta}(2) - \frac{2}{\widehat{w}(0)}\int_{0}^{\infty} w(x)(\log x)dx\right) + \mathcal{O}(L^{-2}),
\end{aligned} \tag{7.3.1}$$

which means that the expected relation between the two coefficients holds.

7.3.2 Hecke characters

The next symplectic family that we shall concern ourselves with is related to so-called Hecke characters. As before we will not elaborate further on the setup. In [Wa, Conjecture 1.1], the one-level density of a certain symplectic family was found, conditional on the Generalised Riemann Hypothesis and the Ratios Conjecture for the relevant L-functions. The one-level density in question equals

$$\widehat{\phi}(0) - \int_{-1}^{1} \widehat{\phi}(u)du + \frac{1}{M}\big(c\widehat{\phi}(0) - d\widehat{\phi}(1)\big).$$

Here, M plays the same role as L does in our analysis, and c, d are some constants. The relation

$$c = d - c_1 - \gamma, \tag{7.3.2}$$

is also given, where

$$c_1 = 1 + \int_1^\infty \frac{\psi(t) - t}{t^2} dt$$

and $\psi(t) := \sum_{n\leq t} \Lambda(n)$ is the summatory function of the von-Mangoldt function.

The relation (7.3.2) is not simplified further in [Wa], but it is possible to accomplish such a simplification using the methods of the previous section. Indeed, by the same arguments as when we found the value of

$$\int_1^\infty \frac{\theta(t) - t}{t^2} dt,$$

one finds that $c_1 = -\gamma$ so that in fact $c = d$, as expected.

7.3.3 A polynomial analogue

The last example we shall study is slightly different from the examples above, in that the L-functions are not related to number fields. Instead, the L-functions are related to certain polynomial rings, but we will not go into this further.

In [Rk, Cor. 3], an analogue of the one-level density is studied, and a symplectic main term is found. Further, a secondary term is found, with coefficient

$$\widehat{\phi}(0)\Bigg(\sum_{P \text{ monic, irred}} \frac{\deg P}{|P|^2 - 1} + \frac{1}{2}\Bigg) - \widehat{\phi}(1)\left(\frac{1}{q-1} + \frac{1}{2}\right),$$

for $\sigma < 2$. Here, $q > 1$ is a prime power, $|P| = q^{\deg P}$, and the sum is over all monic, irreducible polynomials in the finite field $\mathbb{F}_q$. We remark that the expression above is not the expression given in [Rk], but instead a slight reformulation taken from [FPS1, Eq. 1.5].

We will show that the two coefficients are related by simplifying the sum. First, by the formula for a geometric series,

$$\sum_{P \text{ monic, irred}} \frac{\deg P}{|P|^2 - 1} = \sum_{P \text{ irred, monic}} \sum_{j=1}^{\infty} \frac{\deg P}{|P|^{2j}} = \sum_{\text{monic } f} \frac{\Lambda(f)}{|f|^2}.$$

Here, $\Lambda(f)$ is the polynomial von-Mangoldt function defined as being 0 unless $f = P^k$ is the power of an irreducible polynomial, and in this case $\Lambda(f) = \deg P$. To simplify the last sum, apply the equality

$$\sum_{\text{monic } f \text{ of degree } n} \Lambda(f) = q^n,$$

see e.g. [Ro, Prop 2.1]. Then, by splitting the sum over each degree, we find that it equals

$$\sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q-1},$$

as desired.

8

The Ratios Conjecture prediction for the two-level density

In this chapter, we obtain a Ratios Conjecture prediction for the two-level density in the form of Proposition 8.4, which gives an expression for the two-level density for any $\sigma < \infty$. The main difficulty in proving this theorem is that the products that arise are harder to evaluate explicitly than in the one-level case. As is usual when applying the Ratios Conjecture, we assume the Generalised Riemann Hypothesis for $\zeta_K(s)$ throughout the chapter. We emphasise that the results of this chapter are new.

8.1 Sums over K

Recall that by (4.1.1), the two-level density is given by

$$\frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \sum_{\gamma_K,\gamma_K'} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma_K'\right) - \frac{2}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \sum_{\gamma_K} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma_K\right). \tag{8.1.1}$$

The second sum is just the one-level density, where the function $\phi_3(u) = \phi(u,u) = \phi_1(u)\phi_2(u)$ is used in place of the "ϕ" from (2.4.3). The one-level density has already been predicted using the Ratios Conjecture in Proposition 6.4, so we need not consider this sum for now. We write the first sum as

$$\frac{1}{N^{\pm}(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \left(\sum_{\gamma_K} \phi_1\left(\frac{L}{2\pi}\gamma_K\right)\right)\left(\sum_{\gamma_K'} \phi_2\left(\frac{L}{2\pi}\gamma_K'\right)\right),$$

which by the Residue theorem, and the Generalised Riemann Hypothesis for $\zeta_K(s)$, equals

$$\begin{aligned}\frac{1}{(2\pi i)^2}&\left(\int_{(1/L)} - \int_{(-1/L)}\right)\left(\int_{(1/L)} - \int_{(-1/L)}\right)\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\\ &\times \frac{1}{N(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\frac{L'(1/2+s,f_K)}{L(1/2+s,f_K)} drds.\end{aligned} \tag{8.1.2}$$

We are thus motivated to study

$$\frac{1}{N(X)} \sum_{K\in\mathcal{F}^{\pm}(X)} \frac{L(1/2+\alpha,f_K)}{L(1/2+\gamma,f_K)}\frac{L(1/2+\beta,f_K)}{L(1/2+\delta,f_K)},$$

using the Ratios Conjecture. We will then differentiate the result with respect to α and β, and set $\alpha=\gamma=r$, $\beta=\delta=s$ so that we can estimate the sum above with the result. We proceed very similarly to the one-level density case. The numerators are rewritten using the approximate functional equation, while the denominators are expanded into their Dirichlet series. The resulting expression will contain four different terms, instead of only two, as in Chapter 6. Write

$$\frac{1}{N(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L(1/2+\alpha,f_K)}{L(1/2+\gamma,f_K)}\frac{L(1/2+\beta,f_K)}{L(1/2+\delta,f_K)}=R'_{1,1}(\alpha,\beta,\gamma,\delta;X)$$
$$+R'_{1,2}(\alpha,\beta,\gamma,\delta;X)+R'_{2,1}(\alpha,\beta,\gamma,\delta;X)+R'_{2,2}(\alpha,\beta,\gamma,\delta;X)+\text{Error},$$

where

$$R'_{1,1}(\alpha,\beta,\gamma,\delta;X)=\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\sum_{h_1,h_2,m_1,m_2}\frac{\lambda_K(m_1)\mu_K(h_1)\lambda_K(m_1)\mu_K(h_2)}{m_1^{1/2+\alpha}h_1^{1/2+\gamma}m_2^{1/2+\beta}h_2^{1/2+\delta}},$$
$$R'_{2,1}(\alpha,\beta,\gamma,\delta;X)=\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\alpha}\frac{\Gamma_{\pm}(1/2-\alpha)}{\Gamma_{\pm}(1/2+\alpha)}$$
$$\times\sum_{h_1,h_2,m_1,m_2}\frac{\lambda_K(m_1)\mu_K(h_1)\lambda_K(m_1)\mu_K(h_2)}{m_1^{1/2-\alpha}h_1^{1/2+\gamma}m_2^{1/2+\beta}h_2^{1/2+\delta}},$$
$$R'_{1,2}(\alpha,\beta,\gamma,\delta;X)=\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\beta}\frac{\Gamma_{\pm}(1/2-\beta)}{\Gamma_{\pm}(1/2+\beta)}$$
$$\times\sum_{h_1,h_2,m_1,m_2}\frac{\lambda_K(m_1)\mu_K(h_1)\lambda_K(m_1)\mu_K(h_2)}{m_1^{1/2+\alpha}h_1^{1/2+\gamma}m_2^{1/2-\beta}h_2^{1/2+\delta}},$$
$$R'_{2,2}(\alpha,\beta,\gamma,\delta;X)=\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}|D_K|^{-\alpha-\beta}\frac{\Gamma_{\pm}(1/2-\alpha)}{\Gamma_{\pm}(1/2+\alpha)}\frac{\Gamma_{\pm}(1/2-\beta)}{\Gamma_{\pm}(1/2+\beta)}$$
$$\times\sum_{h_1,h_2,m_1,m_2}\frac{\lambda_K(m_1)\mu_K(h_1)\lambda_K(m_1)\mu_K(h_2)}{m_1^{1/2-\alpha}h_1^{1/2+\gamma}m_2^{1/2-\beta}h_2^{1/2+\delta}}.$$

We will now find suitable approximations for each of the terms above. We begin by stating a generalisation of Lemma 6.1.

Lemma 8.1. *Let $m_1, m_2, h_1, h_2 \in \mathbb{Z}_+$, and $1/2 \le \theta < 5/6$, $\omega \ge 0$ be such that* (4.2.2) *holds. Then for cubefree h_1, h_2 we have*

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\lambda_K(m_1)\mu_K(h_1)\lambda_K(m_2)\mu_K(h_2)=F_1(X)P_1+F_2(X)P_2$$
$$+\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\prod_{p|m_1h_1m_2h_2,\,p^{e_1}||m_1,\,p^{e_2}||m_2}\big(4(e_1+1)(e_2+1)+3\big)p^{\omega}\right),$$

where

$$P_1=\prod_{p^{e_1}||m_1,\,p^{s_1}||h_1,\,p^{e_2}||m_2,\,p^{s_2}||h_2}f(e_1,e_2,s_1,s_2,p)x_p,$$
$$P_2=\prod_{p^{e_1}||m_1,\,p^{s_1}||h_1,\,p^{e_2}||m_2,\,p^{s_2}||h_2}g(e_1,e_2,s_1,s_2,p)y_p,$$

and

$$\begin{aligned}
f(e_1,e_2,0,0,p) &= \frac{(e_1+1)(e_2+1)}{6} + \frac{\delta_{2|e_1}\delta_{2|e_2}}{2} + \frac{\tau_{e_1}\tau_{e_2}}{3} + \frac{1}{p},\\
f(e_1,e_2,0,1,p) &= -\frac{(e_1+1)(e_2+1)}{3} + \frac{\tau_{e_1}\tau_{e_2}}{3} - \frac{1}{p},\\
f(e_1,e_2,0,2,p) &= \frac{(e_1+1)(e_2+1)}{6} - \frac{\delta_{2|e_1}\delta_{2|e_2}}{2} + \frac{\tau_{e_1}\tau_{e_2}}{3},\\
f(e_1,e_2,1,1,p) &= \frac{2(e_1+1)(e_2+1)}{3} + \frac{\tau_{e_1}\tau_{e_2}}{3} + \frac{1}{p},\\
f(e_1,e_2,1,2,p) &= -\frac{(e_1+1)(e_2+1)}{3} + \frac{\tau_{e_1}\tau_{e_2}}{3},\\
f(e_1,e_2,2,2,p) &= \frac{(e_1+1)(e_2+1)}{6} + \frac{\delta_{2|e_1}\delta_{2|e_2}}{2} + \frac{\tau_{e_1}\tau_{e_2}}{3},\\
g(e_1,e_2,0,0,p) &= \frac{(e_1+1)(e_2+1)(1+p^{-1/3})^3}{6} + \frac{\delta_{2|e_1}\delta_{2|e_2}(1+p^{-1/3})(1+p^{-2/3})}{2}\\
&\quad + \frac{\tau_{e_1}\tau_{e_2}(1+p^{-1})}{3} + \frac{(1+p^{-1/3})^2}{p},\\
g(e_1,e_2,0,1,p) &= -\frac{(e_1+1)(e_2+1)(1+p^{-1/3})^3}{3} + \frac{\tau_{e_1}\tau_{e_2}(1+p^{-1})}{3} - \frac{(1+p^{-1/3})^2}{p},\\
g(e_1,e_2,0,2,p) &= \frac{(e_1+1)(e_2+1)(1+p^{-1/3})^3}{6} - \frac{\delta_{2|e_1}\delta_{2|e_2}(1+p^{-1/3})(1+p^{-2/3})}{2}\\
&\quad + \frac{\tau_{e_1}\tau_{e_2}(1+p^{-1})}{3},\\
g(e_1,e_2,1,1,p) &= \frac{2(e_1+1)(e_2+1)(1+p^{-1/3})^3}{3} + \frac{\tau_{e_1}\tau_{e_2}(1+p^{-1})}{3} + \frac{(1+p^{-1/3})^2}{p},\\
g(e_1,e_2,1,2,p) &= -\frac{(e_1+1)(e_2+1)(1+p^{-1/3})^3}{3} + \frac{\tau_{e_1}\tau_{e_2}(1+p^{-1})}{3},\\
g(e_1,e_2,2,2,p) &= \frac{(e_1+1)(e_2+1)(1+p^{-1/3})^3}{6} + \frac{\delta_{2|e_1}\delta_{2|e_2}(1+p^{-1/3})(1+p^{-2/3})}{2}\\
&\quad + \frac{\tau_{e_1}\tau_{e_2}(1+p^{-1})}{3}.
\end{aligned} \tag{8.1.3}$$

Further, both $f(e_1,e_2,s_1,s_2,p)$ and $g(e_1,e_2,s_1,s_2,p)$ are symmetric in the sense that they are both invariant under a permutation of e_1 and e_2, or s_1 and s_2. In the case when all of e_1,e_2, s_1, and s_2 equal 0 *one should consider the products as empty, equalling* 1.

Proof. The proposition is proven in the same manner as Lemma 6.1, with the main difference being that one has to evaluate slightly different sums to find the functions f and g. We leave out the calculations. □

Note that we used the same symbols f and g for the functions above as we did in Lemma 6.1. This will hopefully not lead to any confusion, as the functions have a different number of arguments.

8.2 Estimating $R'_{1,1}$

Lemma 8.1 indicates that a reasonable approximation of $R'_{1,1}$ is given by

$$R_{1,1}(\alpha,\beta,\gamma,\delta;X) := F_1(X)R^M_{1,1}(\alpha,\beta,\gamma,\delta) + F_2(X)R^S_{1,1}(\alpha,\beta,\gamma,\delta), \tag{8.2.1}$$

with

$$R^M_{1,1}(\alpha,\beta,\gamma,\delta) = \prod_p \left(1 + \Sigma^M_0 + \Sigma^M_1 + \Sigma^M_2\right),$$
$$\Sigma^M_i = \frac{1}{p^{i(1/2+\gamma)}}\left(\sum_{e_1,e_2} \frac{x_p f(e_1,e_2,i,0,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+\beta)}} + \sum_{e_1,e_2} \frac{x_p f(e_1,e_2,i,1,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+\beta)+(1/2+\delta)}}\right.$$
$$\left. + \sum_{e_1,e_2} \frac{x_p f(e_1,e_2,i,2,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+\beta)+2(1/2+\delta)}}\right), \tag{8.2.2}$$

and

$$R^S_{1,1}(\alpha,\beta,\gamma,\delta) = \prod_p \left(1 + \Sigma^S_0 + \Sigma^S_1 + \Sigma^S_2\right),$$
$$\Sigma^M_i = \frac{1}{p^{i(1/2+\gamma)}}\left(\sum_{e_1,e_2} \frac{x_p g(e_1,e_2,i,0,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+\beta)}} + \sum_{e_1,e_2} \frac{x_p g(e_1,e_2,i,1,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+\beta)+(1/2+\delta)}}\right.$$
$$\left. + \sum_{e_1,e_2} \frac{x_p g(e_1,e_2,i,2,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+\beta)+2(1/2+\delta)}}\right).$$

The summations over e_1, e_2 should be interpreted as summing over all $e_1, e_2 \geq 0$, except that the term containing $f(0,0,0,0,p)$, or $g(0,0,0,0,p)$, should not be included. We remark that both $R^M_{1,1}$ and $R^S_{1,1}$ are invariant under permuting α and β, or γ and δ, by the invariance of f and g.

It is apparent that the products above converge as long as e.g. the real parts of all variables are greater than 1/2. Just as in the one-level calculations, we begin by extending the domain of these products further. Indeed, by arguments analogous to those establishing the meromorphic continuation of R^M_1 and R^S_1, one finds that the functions

$$A_1(\alpha,\beta,\gamma,\delta) := \frac{\zeta(1+\alpha+\delta)\zeta(1+\beta+\delta)\zeta(1+\alpha+\gamma)\zeta(1+\beta+\gamma)}{\zeta(1+\alpha+\beta)\zeta(1+2\alpha)\zeta(1+2\beta)\zeta(1+\gamma+\delta)} R^M_{1,1}(\alpha,\beta,\gamma,\delta),$$
$$A_2(\alpha,\beta,\gamma,\delta) := \frac{\zeta(5/6+\delta)\zeta(5/6+\gamma)}{\zeta(5/6+\alpha)\zeta(5/6+\beta)}$$
$$\times \frac{\zeta(1+\alpha+\delta)\zeta(1+\beta+\delta)\zeta(1+\alpha+\gamma)\zeta(1+\beta+\gamma)}{\zeta(1+\alpha+\beta)\zeta(1+2\alpha)\zeta(1+2\beta)\zeta(1+\gamma+\delta)} R^S_{1,1}(\alpha,\beta,\gamma,\delta), \tag{8.2.3}$$

are holomorphic for $\alpha,\beta,\gamma,\delta$ with real part greater than $-1/6$. Further by studying the defining product, we have that both $A_1(\alpha,\beta,\gamma,\delta)$ and $A_2(\alpha,\beta,\gamma,\delta)$ are $\ll_\epsilon 1$, when $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta) > -1/6+\epsilon$.

Now we want to differentiate these expressions with respect to α and β, but we first need some preparation. One can confirm that

$$\begin{aligned} &f(e_1,1,i,0,p) + f(e_1,0,i,1,p) = g(e_1,1,i,0,p) + g(e_1,0,i,1,p) = 0,\\ &f(e_1,e_2,i,0,p) + f(e_1,e_2-1,i,1,p) + f(e_1,e_2-2,i,2,p) = 0,\\ &g(e_1,e_2,i,0,p) + g(e_1,e_2-1,i,1,p) + g(e_1,e_2-2,i,2,p) = 0,\\ &f(e_1,0,i,0,p) = f(e_1,i,p), \quad g(e_1,0,i,0,p) = g(e_1,i,p), \end{aligned} \tag{8.2.4}$$

where $f(e,s,p)$ and $g(e,s,p)$ are the functions from Lemma 6.1. We use the first two rows above to evaluate $R^M_{1,1}(r,s,r,s)$, by matching terms in each Σ^M_i, as in the argument showing $A_3(r,r) = 1$. These equalities suffices to eliminate all terms from each Σ^M_i, except for the terms involving $f(e_1,0,i,0)$, so that

$$\left(1 + \Sigma^M_0 + \Sigma^M_1 + \Sigma^M_2\right) = 1 + \sum_{e_1\geq 1} \frac{x_p f(e_1,0,0,0,p)}{p^{e_1(1/2+r)}} + \sum_{e_1\geq 0} \frac{x_p f(e_1,0,1,0,p)}{p^{(e_1+1)(1/2+r)}}$$
$$+ \sum_{e_1\geq 0} \frac{x_p f(e_1,0,2,0,p)}{p^{(e_1+2)(1/2+r)}}.$$

As $f(e_1,0,i,0)=f(e_1,i,p)$, this is exactly the pth factor of the product defining $R_1^M(r,r)$, which we have already shown is equal to 1. Proceeding similarly to evaluate $R_{1,1}^S$, we find that

$$R_{1,1}^M(r,s,r,s)=R_{1,1}^S(r,s,r,s)=A_1(r,s,r,s)=A_2(r,s,r,s)=1. \tag{8.2.5}$$

By a similar calculation, we can also show that

$$R_{1,1}^M(r,\alpha,r,\gamma)=R_{1,1}^M(\alpha,s,\gamma,s)=R_1^M(\alpha,\gamma), \tag{8.2.6}$$

with R_1 as in the Ratios Conjecture calculation for the one-level density. A similar result holds for $R_{1,1}^S$. This equality holds factorwise, in the sense that the pth factors of the defining products are all equal. We also have

$$A_1(r,\alpha,r,\gamma)=A_1(\alpha,s,\gamma,s)=A_3(\alpha,\gamma), \tag{8.2.7}$$

by a similar calculation, and an analogous relation holds between A_2 and A_4.

Instead of calculating the derivative of $R_{1,1}^M$ and $R_{1,1}^S$ directly we will calculate the logarithmic derivative. We begin by studying $R_{1,1}^M$. Now, using (8.2.5) we have

$$\left.\frac{\partial^2}{\partial\alpha\partial\beta}\right|_{\alpha=\gamma=r,\,\beta=\delta=s}\log R_{1,1}^M(\alpha,\beta,\gamma,\delta)=R_{1,1,\alpha,\beta}^M(r,s,r,s)-R_{1,1,\alpha}^M(r,s,r,s)R_{1,1,\beta}^M(r,s,r,s),$$

where the subscripts denote that we have differentiated with respect to the indicated variable. Thus, to find the derivative $R_{1,1,\alpha,\beta}^M$, we need to add a correction term to the logarithmic derivative. By (8.2.6), this correction term is equal to

$$R_{1,1,\alpha}^M(r,s,r,s)R_{1,1,\beta}^M(r,s,r,s)=R_{1,\alpha}^M(r,r)R_{1,\alpha}^M(s,s),$$

and the right-hand side is given explicitly in (6.4.2).

We are left with the task of calculating

$$\sum_p\left.\frac{\partial^2}{\partial\alpha\partial\beta}\right|_{\alpha=\gamma=r,\,\beta=\delta=s}\log w(\alpha,\beta,\gamma,\delta),$$

where we have written

$$w(\alpha,\beta,\gamma,\delta)=1+\Sigma_0^M+\Sigma_1^M+\Sigma_2^M,$$

and suppressed the p-dependence. Now, as $w(r,s,r,s)=1$

$$\left.\frac{\partial^2}{\partial\alpha\partial\beta}\right|_{\alpha=\gamma=r,\,\beta=\delta=s}\log w(\alpha,\beta,\gamma,\delta)=w_{\alpha,\beta}(r,s,r,s)-w_\alpha(r,s,r,s)w_\beta(r,s,r,s),$$

so that in total

$$R_{1,1,\alpha,\beta}^M(r,s,r,s)=\sum_p w_{\alpha,\beta}(r,s,r,s)-\sum_p w_\alpha(r,s,r,s)w_\beta(r,s,r,s)+R_{1,\alpha}^M(r,r)R_{1,\alpha}^M(s,s).$$

To evaluate the expression above we must differentiate the function w, and we begin by finding the derivative with respect to α evaluated at (r,s,r,s). As $w(\alpha,s,\gamma,s)$ is equal to the pth factor defining $R_1^M(\alpha,\gamma)$, we have already found this derivative. Indeed, by (6.4.2), we have

$$w_\alpha(r,s,r,s)=-\sum_{e\ge1}\frac{x_p\log p}{p^{e(1/2+r)}}\left(\theta_e+\frac{1}{p}\right).$$

We remark that there is no dependence on s. By symmetry, we find a very similar equality for the derivative with respect to β, by simply substituting r for s. As before these equalities are valid for $\mathrm{Re}(r)>0$ and $\mathrm{Re}(s)>0$ respectively.

Now we want to differentiate with respect to both α and β. We begin by differentiating w with respect to β, set $\beta=\delta=s$, and find that

$$w_\beta(\alpha,s,\gamma,s)=\left(\Sigma_{0,\beta}^M+\Sigma_{1,\beta}^M+\Sigma_{2,\beta}^M\right),$$

where

$$\Sigma_{i,\beta}^{M} := \frac{d}{d\beta}\Big|_{\beta=\delta=s} \Sigma_i^M = -\frac{x_p \log p}{p^{i(1/2+\gamma)}}\Bigg(\sum_{e_1,e_2} \frac{e_2 f(e_1,e_2,i,0,p)}{p^{e_1(1/2+\alpha)+e_2(1/2+s)}} + \sum_{e_1,e_2} \frac{e_2 f(e_1,e_2,i,1,p)}{p^{e_1(1/2+\alpha)+(e_2+1)(1/2+s)}} + \sum_{e_1,e_2} \frac{e_2 f(e_1,e_2,i,2,p)}{p^{e_1(1/2+\alpha)+(e_2+2)(1/2+s)}}\Bigg) = -\frac{x_p \log p}{p^{i(1/2+\gamma)}} \times \Bigg(\sum_{e_1 \geq 0} \frac{f(e_1,1,i,0,p)}{p^{e_1(1/2+\alpha)+(1/2+s)}} + \sum_{e_1\geq 0, e_2 \geq 2} \frac{\big(f(e_1,e_2,i,0,p) - f(e_1,e_2-2,i,2,p)\big)}{p^{e_1(1/2+\alpha)+e_2(1/2+s)}}\Bigg),$$

where we used (8.2.4) in the last step, similarly as in the one-level density computations.

To simplify the notation, we define a function h by

$$h(e_1,e_2,i) = f(e_1,e_2,i,0,p) - f(e_1,e_2-2,i,2,p),$$

where we have suppressed the p-dependence in the left-hand side. A straightforward calculation yields

$$\begin{aligned} h(e_1,e_2,0) &= \frac{e_1+1}{3} + \delta_{2|e_1}\delta_{2|e_2} + \frac{\tau_{e_1}(3\delta_{3|e_2}-1)}{3} + \frac{1}{p}, \\ h(e_1,e_2,1) &= -\frac{2(e_1+1)}{3} + \frac{\tau_{e_1}(3\delta_{3|e_2}-1)}{3} - \frac{1}{p}, \\ h(e_1,e_2,2) &= \frac{e_1+1}{3} - \delta_{2|e_1}\delta_{2|e_2} + \frac{\tau_{e_1}(3\delta_{3|e_2}-1)}{3}. \end{aligned}$$

Now we differentiate with respect to α, and set $\alpha = \gamma = r$ to find

$$\begin{aligned} \Sigma_{i,\alpha,\beta}^{M} &= \frac{x_p \log^2 p}{p^{i(1/2+r)}}\Bigg(\sum_{e_1\geq 0} \frac{e_1 f(e_1,1,i,0,p)}{p^{e_1(1/2+r)+(1/2+s)}} + \sum_{e_1 \geq 0, e_2\geq 2} \frac{e_1 h(e_1,e_2,i)}{p^{e_1(1/2+r)+e_2(1/2+s)}}\Bigg) \\ &=: \Sigma_{i,\alpha,\beta}^{M,1} + \Sigma_{i,\alpha,\beta}^{M,2}. \end{aligned}$$

To find $w_{\alpha,\beta}(r,s,r,s)$ we must sum all the $\Sigma_{i,\alpha,\beta}^{M}$ over i and we begin by summing the $\Sigma_{i,\alpha,\beta}^{M,1}$. This is done using a special case of the second equality of (8.2.4) with the first and second, and third and fourth arguments interchanged, using symmetry. Specifically, we use

$$f(e_1,1,0,0,p) + f(e_1-1,1,1,0,p) + f(e_1-2,1,2,0,p) = 0.$$

We find

$$\Sigma_{0,\alpha,\beta}^{M,1} + \Sigma_{1,\alpha,\beta}^{M,1} + \Sigma_{2,\alpha,\beta}^{M,1} = \frac{x_p \log^2 p}{p^{1/2+s}}\Bigg(\frac{f(1,1,0,0,p)}{p^{1/2+r}} + \sum_{e_1\geq 2} \frac{f(e_1,1,0,0,p) - f(e_1-2,1,2,0,p)}{p^{e_1(1/2+r)}}\Bigg).$$

Now, by the symmetry of f and the definition of h, we have $f(e_1,1,0,0,p) - f(e_1-2,1,2,0,p) = h(1,e_1,0) = 1 - \delta_{3|e_1} + 1/p$. Also, we have $f(1,1,0,0,p) = 1 + 1/p = 1 - \delta_{3|1} + 1/p$.

To simplify the sum of $\Sigma_{i,\alpha,\beta}^{M,2}$, over i, we need a few identities for the function h. Similarly as with f, we have

$$h(e_1,e_2,0) + h(e_1-1,e_2,1) + h(e_1-2,e_2,2) = 0.$$

This implies that

$$\Sigma_{0,\alpha,\beta}^{M,2} + \Sigma_{1,\alpha,\beta}^{M,2} + \Sigma_{2,\alpha,\beta}^{M,2} = x_p \log^2 p \Bigg(\sum_{e_2\geq 2} \frac{h(1,e_2,0)}{p^{(1/2+r)+e_2(1/2+s)}} + \sum_{e_1\geq 2, e_2\geq 2} \frac{h(e_1,e_2,0) - h(e_1-2,e_2,2)}{p^{e_1(1/2+r)+e_2(1/2+s)}}\Bigg).$$

A calculation confirms that $h(e_1, e_2, 0) - h(e_1 - 2, e_2, 2) = \iota_{e_1,e_2}(p)/x_p$, with ι from (4.3.12). Furthermore, $\iota_{e_1,1}(p)/x_p = 1 - \delta_{3|e_1} + 1/p$, and similarly for $\iota_{1,e_2}(p)/x_p$ by symmetry. Thus, we have shown that

$$\sum_p w_{\alpha,\beta}(r,s,r,s) = \sum_p \sum_{e,f\geq 1} \frac{x_p \iota_{e,f}(p)\log^2 p}{p^{e(1/2+r)+f(1/2+s)}},$$

valid for $\mathrm{Re}(r), \mathrm{Re}(s) > 0$. Finally, by putting together all of our calculations, we can conclude that

$$R^M_{1,1,\alpha,\beta}(r,s,r,s) = \sum_p \sum_{e,f\geq 1} \frac{\iota_{e,f}(p)\log^2 p}{p^{e(1/2+r)+f(1/2+s)}} - \sum_p \sum_{e,f\geq 1} \frac{x_p^2(\theta_e + 1/p)(\theta_f + 1/p)\log^2 p}{p^{e(1/2+r)+f(1/2+s)}}$$
$$+ \left(\sum_p \sum_{e\geq 1} \frac{x_p(\theta_e + 1/p)\log p}{p^{e(1/2+r)}}\right)\left(\sum_q \sum_{f\geq 1} \frac{x_q(\theta_f + 1/q)\log q}{q^{f(1/2+s)}}\right). \tag{8.2.8}$$

Without any significant additional difficulties, one may employ the same method of calculation using the function g, instead of f, to find $R^S_{1,1,\alpha,\beta}(r,s,r,s)$. We leave out the details, but the resulting expression is given by

$$R^S_{1,1,\alpha,\beta}(r,s,r,s) = \sum_p \sum_{e,f\geq 1} \frac{\xi_{e,f}(p)\log^2 p}{p^{e(1/2+r)+f(1/2+s)}} - \sum_p \sum_{e,f\geq 1} \frac{x_p^2\gamma_e(p)\gamma_f(p)\log^2 p}{p^{e(1/2+r)+f(1/2+s)}}$$
$$+ \left(\sum_p \sum_{e\geq 1} \frac{(\gamma_e(p) - p^{-e/3})\log p}{p^{e(1/2+r)}} - \frac{\zeta'}{\zeta}(\tfrac{5}{6}+r)\right)\left(\sum_q \sum_{f\geq 1} \frac{(\gamma_f(q) - q^{-f/3})\log q}{q^{f(1/2+s)}} - \frac{\zeta'}{\zeta}(\tfrac{5}{6}+s)\right), \tag{8.2.9}$$

with $\xi_{e,f}(p)$ from (4.3.13). This calculation is also valid for $\mathrm{Re}(r), \mathrm{Re}(s) > 0$, and this finishes the estimation of $R_{1,1,\alpha,\beta}(r,s,r,s;X)$.

8.3 Finishing the estimates

We now turn to the task of estimating the derivatives of $R'_{1,2}$, $R'_{2,1}$ and $R'_{2,2}$. First, we must find appropriate estimates of all these terms, using a generalisation of Corollary 6.2. Indeed, we have the following corollary of Lemma 8.1.

Corollary 8.2. *Let $m_1, m_2, h_1, h_2 \in \mathbb{Z}_+$, and $1/2 \leq \theta < 5/6$, $\omega \geq 0$ be such that* (4.2.2) *holds. Then for cubefree h_1, h_2, and κ with $0 < \mathrm{Re}(\kappa) < 1/2$, we have*

$$\frac{1}{N(X)} \sum_K |D_K|^{-\kappa} \lambda_K(m_1)\mu_K(h_1)\lambda_K(m_2)\mu_K(h_2) = F_1(X)\frac{X^{-\kappa}}{1-\kappa}P_1 + F_2(X)\left(\frac{X^{-\kappa}}{1-6\kappa/5}\right)P_2$$
$$+\mathcal{O}\left((1+|\kappa|)\prod_{p|h_1h_2m_1m_2,\, p^{e_1}||m_1,\, p^{e_2}||m_2} p^{\omega}\big(4(e_1+1)(e_2+1)+3\big)X^{\theta-\mathrm{Re}\,\kappa-1+\epsilon}\right),$$

with P_1 and P_2 as in Lemma 8.1.

Proof. This is proven from Lemma 8.1 in the same way as Corollary 6.2 is proven from Lemma 6.1. □

The point of the corollary will be to apply it with $\kappa = \alpha, \beta$ or $\alpha + \beta$ to the estimation of $R'_{2,1}$, $R'_{1,2}$ and $R'_{2,2}$ respectively. We consider one product at a time.

First, the corollary indicates that a reasonable estimate of $R'_{1,2}(\alpha,\beta,\gamma,\delta;X)$ is

$$R_{1,2}(\alpha,\beta,\gamma,\delta;X) := \frac{\Gamma_\pm(1/2-\beta)}{\Gamma_\pm(1/2+\beta)}\biggl(F_1(X)\frac{X^{-\beta}}{1-\beta}R^M_{1,1}(\alpha,-\beta,\gamma,\delta) + F_2(X)\frac{X^{-\beta}}{1-6\beta/5}R^S_{1,1}(\alpha,-\beta,\gamma,\delta)\biggr).$$

In order to handle this expression, we will need to work with the products A_1 and A_2 instead of $R^M_{1,1}$ and $R^S_{1,1}$. We use (8.2.3) to write

$$R_{1,2}(\alpha,\beta,\gamma,\delta;X) = \frac{\Gamma_\pm(1/2-\beta)}{\Gamma_\pm(1/2+\beta)}\frac{\zeta(1+\alpha-\beta)\zeta(1+2\alpha)\zeta(1-2\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1-\beta+\delta)\zeta(1+\alpha+\gamma)\zeta(1-\beta+\gamma)} \times\biggl(F_1(X)\frac{X^{-\beta}}{1-\beta}A_1(\alpha,-\beta,\gamma,\delta) + \frac{\zeta(5/6+\alpha)\zeta(5/6-\beta)}{\zeta(5/6+\delta)\zeta(5/6+\gamma)}F_2(X)\frac{X^{-\beta}}{1-6\beta/5}A_2(\alpha,-\beta,\gamma,\delta)\biggr).$$

Next, we differentiate this expression with respect to β, set $\beta=\delta=s$ and obtain

$$R_{1,2,\beta}(\alpha,s,\gamma,s;X) = -\frac{\Gamma_\pm(1/2-s)}{\Gamma_\pm(1/2+s)}\frac{\zeta(1+2\alpha)\zeta(1+\gamma+s)\zeta(1+\alpha-s)\zeta(1-2s)}{\zeta(1+\alpha+s)\zeta(1+\alpha+\gamma)\zeta(1-s+\gamma)} \times\biggl(F_1(X)\frac{X^{-s}}{1-s}A_1(\alpha,-s,\gamma,s) + \frac{\zeta(5/6+\alpha)\zeta(5/6-s)}{\zeta(5/6+s)\zeta(5/6+\gamma)}F_2(X)\frac{X^{-s}}{1-6s/5}A_2(\alpha,-s,\gamma,s)\biggr), \tag{8.3.1}$$

at least assuming $\alpha\neq s$. As usual, a variable as a subscript denotes differentiation with respect to the indicated variable. We will see later that the apparent singularity at $\alpha=s$ will not be important. Next, we differentiate with respect to α and set $\alpha=\beta=r$ and find

$$\begin{aligned} R_{1,2,\alpha,\beta}(r,s,r,s;X) = &-\frac{\Gamma_\pm(1/2-s)}{\Gamma_\pm(1/2+s)}\zeta(1-2s)\left(\frac{\zeta'}{\zeta}(1+r-s)+\frac{\zeta'}{\zeta}(1+2r)-\frac{\zeta'}{\zeta}(1+r+s)\right)\\ &\times\left(F_1(X)\frac{X^{-s}}{1-s}A_1(r,-s,r,s)+\frac{\zeta(5/6-s)}{\zeta(5/6+s)}F_2(X)\frac{X^{-s}}{1-6s/5}A_2(r,-s,r,s)\right)\\ &-\frac{\Gamma_\pm(1/2-s)}{\Gamma_\pm(1/2+s)}\zeta(1-2s)\biggl[F_1(X)\frac{X^{-s}}{1-s}A_{1,\alpha}(r,-s,r,s)+\frac{\zeta(5/6-s)}{\zeta(5/6+s)}\frac{\zeta'}{\zeta}\left(\tfrac{5}{6}+r\right)\\ &\times F_2(X)\frac{X^{-s}}{1-6s/5}A_2(r,-s,r,s)+\frac{\zeta(5/6-s)}{\zeta(5/6+s)}F_2(X)\frac{X^{-s}}{1-6s/5}A_{2,\alpha}(r,-s,r,s)\biggr]. \end{aligned} \tag{8.3.2}$$

Note that it is possible to simplify the result slightly by using (8.2.7), and the corresponding equality for A_2.

To obtain an estimate for $R'_{2,1,\alpha,\beta}$, we can simply exchange the role of r and s above. This is possible by the symmetry of $R^M_{1,1}$ and $R^S_{1,1}$, induced by the symmetry of the functions f and g, see (8.2.1), (8.2.2) and the remark after (8.1.3). We find that an appropriate estimate is

$$\begin{aligned} R_{2,1,\alpha,\beta}(r,s,r,s;X) = &-\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\left(\frac{\zeta'}{\zeta}(1+s-r)+\frac{\zeta'}{\zeta}(1+2s)-\frac{\zeta'}{\zeta}(1+r+s)\right)\\ &\times\left(F_1(X)\frac{X^{-r}}{1-r}A_1(-r,s,r,s)+\frac{\zeta(5/6-r)}{\zeta(5/6+r)}F_2(X)\frac{X^{-r}}{1-6r/5}A_2(-r,s,r,s)\right)\\ &-\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\biggl[F_1(X)\frac{X^{-r}}{1-r}A_{1,\beta}(-r,s,r,s)+\frac{\zeta(5/6-r)}{\zeta(5/6+r)}\frac{\zeta'}{\zeta}\left(\tfrac{5}{6}+s\right)\\ &\times F_2(X)\frac{X^{-r}}{1-6r/5}A_2(-r,s,r,s)+\frac{\zeta(5/6-r)}{\zeta(5/6+r)}F_2(X)\frac{X^{-r}}{1-6r/5}A_{2,\beta}(-r,s,r,s)\biggr]. \end{aligned} \tag{8.3.3}$$

Finally, we turn to $R'_{2,2}$. We define the estimate $R_{2,2}(\alpha,\beta,\gamma,\delta;X)$ by

$$\frac{\Gamma_\pm(1/2-\alpha)\Gamma_\pm(1/2-\beta)}{\Gamma_\pm(1/2+\alpha)\Gamma_\pm(1/2+\beta)}\Bigg(F_1(X)\frac{X^{-\alpha-\beta}}{1-\alpha-\beta}R^M_{1,1}(-\alpha,-\beta,\gamma,\delta) + F_2(X)\frac{X^{-\alpha-\beta}}{1-6(\alpha+\beta)/5}R^S_{1,1}(-\alpha,-\beta,\gamma,\delta)\Bigg),$$

which is motivated by Corollary 8.2, with $\kappa=\alpha+\beta$. Rewriting this in terms of A_1 and A_2, we find

$$R_{2,2}(\alpha,\beta,\gamma,\delta;X)=\frac{\Gamma_\pm(1/2-\alpha)\Gamma_\pm(1/2-\beta)}{\Gamma_\pm(1/2+\alpha)\Gamma_\pm(1/2+\beta)} \times \frac{\zeta(1-\alpha-\beta)\zeta(1-2\alpha)\zeta(1-2\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\alpha+\delta)\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)\zeta(1-\beta+\gamma)}\Bigg(F_1(X)\frac{X^{-\alpha-\beta}}{1-\alpha-\beta} \times A_1(-\alpha,-\beta,\gamma,\delta)+\frac{\zeta(5/6-\alpha)\zeta(5/6-\beta)}{\zeta(5/6+\delta)\zeta(5/6+\gamma)}F_2(X)\frac{X^{-\alpha-\beta}}{1-6(\alpha+\beta)/5}A_2(-\alpha,-\beta,\gamma,\delta)\Bigg).$$

We now differentiate with respect to β and set $\beta=\delta=s$. It will suffice to differentiate the fraction of zeta functions, which can be done by appealing to (8.3.1), and substituting α by $-\alpha$. The result is that

$$R_{2,2,\beta}(\alpha,s,\gamma,s;X)=-\frac{\Gamma_\pm(1/2-\alpha)\Gamma_\pm(1/2-s)}{\Gamma_\pm(1/2+\alpha)\Gamma_\pm(1/2+s)}\frac{\zeta(1-2\alpha)\zeta(1+\gamma+s)\zeta(1-\alpha-s)\zeta(1-2s)}{\zeta(1-\alpha+s)\zeta(1-\alpha+\gamma)\zeta(1-s+\gamma)} \times\Bigg(F_1(X)\frac{X^{-\alpha-s}}{1-\alpha-s}A_1(-\alpha,-s,\gamma,s) + F_2(X)\frac{\zeta(5/6-\alpha)\zeta(5/6-s)}{\zeta(5/6+\gamma)\zeta(5/6+s)}\frac{X^{-\alpha-s}}{1-6(\alpha+s)/5}A_2(-\alpha,-s,\gamma,s)\Bigg).$$

Next, we differentiate this with respect to α and set $\alpha=\gamma=r$. We find that

$$R_{2,2,\alpha,\beta}(r,s,r,s;X)=\frac{\Gamma_\pm(1/2-r)\Gamma_\pm(1/2-s)}{\Gamma_\pm(1/2+r)\Gamma_\pm(1/2+s)}\frac{\zeta(1-r-s)\zeta(1-2r)\zeta(1-2s)\zeta(1+r+s)}{\zeta(1-r+s)\zeta(1-s+r)} \times\Bigg(F_1(X)\frac{X^{-r-s}}{1-r-s}A_1(-r,-s,r,s) + F_2(X)\frac{\zeta(5/6-r)\zeta(5/6-s)}{\zeta(5/6+r)\zeta(5/6+s)}\frac{X^{-r-s}}{1-6(r+s)/5}A_2(-r,-s,r,s)\Bigg). \tag{8.3.4}$$

8.4 Finding the two-level density

With the calculations complete, we have found the following Ratios Conjecture:

Conjecture 8.3. *Let $1/2\le\theta<5/6$, $\omega\ge0$ be such that (4.2.2) holds. Then there is some $\delta<1/6$ such that for any fixed $\epsilon>0$, and $r,s\in\mathbb{C}$ with $\mathrm{Re}(r),\mathrm{Re}(s)>0$, $1/L\ll\mathrm{Re}(r),\mathrm{Re}(s)<\delta$, and $|r|,|s|\le X^{\epsilon/3}$, we have that*

$$\frac{1}{N(X)}\sum_{K\in\mathcal{F}^\pm(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\frac{L'(1/2+s,f_K)}{L(1/2+s,f_K)}=F_1(X)R^M_{1,1,\alpha,\beta}(r,s,r,s) + F_2(X)R^S_{1,1,\alpha,\beta}(r,s,r,s)+R_{1,2,\alpha,\beta}(r,s,r,s;X)+R_{2,1,\alpha,\beta}(r,s,r,s;X) + R_{2,2,\alpha,\beta}(r,s,r,s;X)+\mathcal{O}\left(X^{\theta-1+\epsilon}\right),$$

where $R^M_{1,1,\alpha,\beta}$, $R^S_{1,1,\alpha,\beta}$, $R_{1,2,\alpha,\beta}$, $R_{2,1,\alpha,\beta}$, and $R_{2,2,\alpha,\beta}$ are given in (8.2.8), (8.2.9), (8.3.2), (8.3.3), *and* (8.3.4) *respectively.*

Remark. *From the expressions* (8.3.2) *and* (8.3.3) *it may appear that the right-hand side is ill-defined at every point where* $r = s$*. The existence of such singularities would be odd, as the left-hand side is finite, at least assuming the Generalised Riemann Hypothesis, as a finite sum of finite numbers. However, by a careful study of the sum of* (8.3.2) *and* (8.3.3) *we can see that there is in fact no singularity. Indeed, this follows by first making the change of variables* $z = s - r$*,* $w = s + r$*, a Laurent expansion of* $\zeta'(1 \pm z)/\zeta(1 \pm z)$*, and a complex multivariate Taylor expansion of the rest of the expression around any point* $(z, w) = (0, w_0)$*. For a proof of the multivariate Taylor expansion, see e.g. [Le, Ch. 1.2].*

Using this conjecture, we can find a prediction for the two-level density. Indeed, we will prove the following proposition:

Proposition 8.4. *Let* $1/2 \le \theta < 5/6$*,* $\omega \ge 0$ *be such that* (4.2.2) *holds. Assume the Generalised Riemann Hypothesis for the functions* $\zeta_K(s)$*, with* $K \in \mathcal{F}^{\pm}(X)$*, Conjecture 6.3 and Conjecture 8.3. Let* $\phi(u_1, u_2) = \phi_1(u_1)\phi_2(u_2)$ *be a product of real, even Schwartz functions whose Fourier transform has compact support. Then, we have*

$$\begin{aligned}
&\frac{1}{N^{\pm}(X)} \sum_{K} \sum_{\gamma_K \neq \pm\gamma_K'} \phi\left(\frac{L}{2\pi}\gamma_K, \frac{L}{2\pi}\gamma_K'\right) = \\
&\widehat{\phi_1}(0)\widehat{\phi_2}(0)\left[1 + \frac{2\log(4\pi^2 e)}{L} + \frac{4\left(\log^2(2\pi) + \log(2\pi)\right) + 2}{L^2} + F_2(X)\left(-\frac{2}{5L} + \frac{2 - 20\log(2\pi)}{25L^2}\right)\right] \\
&+ \widehat{\phi_1}(0)\left(1 + \frac{2\log(2\pi e)}{L}\right)\left(F_1(X)S_1(2) + F_2(X)S_2(2)\right) - \frac{\widehat{\phi_1}(0)}{L}\left(F_1(X)S_1(2) + \frac{6}{5}F_2(X)S_2(2)\right) \\
&+ \widehat{\phi_2}(0)\left(1 + \frac{2\log(2\pi e)}{L}\right)\left(F_1(X)S_1(1) + F_2(X)S_2(1)\right) - \frac{\widehat{\phi_2}(0)}{L}\left(F_1(X)S_1(1) + \frac{6}{5}F_2(X)S_2(1)\right) \\
&+ \left(\widehat{\phi_2}(0)J_1(X) + \widehat{\phi_1}(0)J_2(X)\right)\left(1 + \frac{2\log(2\pi e) - 1}{L}\right) + \frac{\widehat{\phi_2}(0)}{L}J_1'(X) + \frac{\widehat{\phi_1}(0)}{L}J_2'(X) \\
&+ E_1 J_2(X) + E_2 J_1(X) + J''(X) \\
&+ F_1(X)\left(S_1(1)S_1(2) - S_3 + S_5\right) + F_2(X)\left(S_2(1)S_2(2) - S_4 + S_6\right) \\
&+ E_1 R_2 + E_2 R_1 - E_1 E_2 - 2R_3 - 2J_3(X) + \mathcal{O}_\epsilon\left(X^{\theta - 1 + \epsilon}\right),
\end{aligned} \tag{8.4.1}$$

where

$$\begin{aligned}
J_i'(X) = &-\frac{2}{L} \cdot \frac{1}{5} F_2(X) \sum_{p,e} \frac{\log p}{p^{5e/6}} \widehat{\phi_i}\left(\frac{\log p^e}{L}\right) - \frac{2}{2\pi i} \cdot \frac{F_2(X)}{5} \int_{(1/L)} \phi_i\left(\frac{Lr}{2\pi i}\right) \frac{\zeta'}{\zeta}\left(\tfrac{5}{6} + r\right) dr \\
&+ \frac{2}{2\pi i} \int_{(1/L)} \phi_i\left(\frac{Lr}{2\pi i}\right) \zeta(1 - 2r) \frac{\Gamma_{\pm}(1/2 - r)}{\Gamma_{\pm}(1/2 + r)} \Bigg(F_1(X) \frac{rX^{-r}}{(1 - r)^2} A_3(-r, r) \\
&\quad + F_2(X) \cdot \frac{6}{5} \cdot \frac{(1/6 + r)X^{-r} A_4(-r, r)}{(1 - 6r/5)^2} \cdot \frac{\zeta(5/6 - r)}{\zeta(5/6 + r)} \Bigg) dr,
\end{aligned} \tag{8.4.2}$$

and

$$J''(X) = F_2(X)\left(S_2(1) + \frac{2}{L}\sum_{p,e\geq 1}\frac{\log p}{p^{5e/6}}\widehat{\phi_1}\left(\frac{\log p^e}{L}\right) + \frac{2}{2\pi i}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\frac{\zeta'}{\zeta}\left(\tfrac{5}{6}+r\right)dr\right)$$
$$\times\left(S_2(2) + \frac{2}{L}\sum_{q,f\geq 1}\frac{\log q}{q^{5f/6}}\widehat{\phi_2}\left(\frac{\log q^f}{L}\right) + \frac{2}{2\pi i}\int_{(1/L)}\phi_2\left(\frac{Lr}{2\pi i}\right)\frac{\zeta'}{\zeta}\left(\tfrac{5}{6}+s\right)ds\right)$$
$$- F_2(X)S_2(1)S_2(2)$$
$$+ \frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\Big(R_{1,2,\alpha,\beta}(r,s,r,s;X)+$$
$$R_{2,1,\alpha,\beta}(r,s,r,s;X) + R_{2,2,\alpha,\beta}(r,s,r,s;X)\Big)drds. \tag{8.4.3}$$

Here, S_i, R_i and $F_i(X)$ are defined in Chapter 4. Also, $J_i(X)$ is defined as the expression (7.1.1)*, with 'ϕ' replaced by ϕ_i, with the convention that $\phi_3(u) = \phi(u,u) = \phi_1(u)\phi_2(u)$.*

Proof. We begin by using (4.1.1) to see that the two-level density is equal to

$$\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\sum_{\gamma_K,\gamma_K'}\phi\left(\frac{L}{2\pi}\gamma_K,\frac{L}{2\pi}\gamma_K'\right) - \frac{2}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\sum_{\gamma_K}\phi\left(\frac{L}{2\pi}\gamma_K,\frac{L}{2\pi}\gamma_K\right).$$

The second sum can be found directly using Proposition 6.4, with the test function ϕ replaced by $\phi_3(u)$. By (8.1.2), the first sum is given by

$$\frac{1}{(2\pi i)^2}\cdot\frac{1}{N(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\left(\int_{(1/L)} - \int_{(-1/L)}\right)\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\phi_1\left(\frac{Lr}{2\pi i}\right)dr$$
$$\times\left(\int_{(1/L)} - \int_{(-1/L)}\right)\frac{L'(1/2+s,f_K)}{L(1/2+s,f_K)}\phi_2\left(\frac{Ls}{2\pi i}\right)ds, \tag{8.4.4}$$

which will now be our object of study.

As in the proof of Proposition 6.4, we must first prepare the integral so that the conditions of Conjecture 8.3 apply. Using (6.6.2), we immediately find that

$$-\frac{1}{2\pi i}\int_{(-1/L)}\phi_i\left(\frac{Lt}{2\pi i}\right)\frac{L'(1/2+t,f_K)}{L(1/2+t,f_K)}dt = \frac{1}{2\pi i}\int_{(1/L)}\phi_i\left(\frac{Lt}{2\pi i}\right)\frac{L'(1/2+t,f_K)}{L(1/2+t,f_K)}dt$$
$$+ \frac{1}{2\pi i}\int_{(-1/L)}\phi_i\left(\frac{Lt}{2\pi i}\right)\left(\log|D_K| + \frac{\Gamma_\pm'}{\Gamma_\pm}(1/2+t) + \frac{\Gamma_\pm'}{\Gamma_\pm}(1/2-t)\right)dt,$$

with $i = 1,2$. The second integral can be shifted to (0), and we find that it equals $C_{i,K} + E_i$, with notation from Chapter 4. This implies that (8.4.4) equals

$$\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\frac{L'(1/2+s,f_K)}{L(1/2+s,f_K)}drds$$
$$+ \frac{2}{2\pi i}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}(C_{2,K}+E_2)dr$$
$$+ \frac{2}{2\pi i}\int_{(1/L)}\phi_2\left(\frac{Ls}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+s,f_K)}{L(1/2+s,f_K)}(C_{1,K}+E_1)ds$$
$$+ \frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}(C_{1,K}+E_1)(C_{2,K}+E_2). \tag{8.4.5}$$

The last term above has already been simplified in Chapter 3 and 4. The second and third terms are identical up to a permutation of the variables, so we concentrate on the second term. The sum involving E_2 is easy to calculate, as E_2 is independent of both K and r, so that we can directly apply

$$\begin{aligned}&\frac{2}{2\pi i}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}dr\\&=F_1(X)S_1(1)+F_2(X)S_2(1)+J_1(X)+\mathcal{O}_\epsilon\left(X^{\theta-1+\epsilon}\right)\end{aligned}$$

from (6.6.3), with the sums $S_i(j)$ as in Chapter 4.

The sum involving $C_{2,K}$ is harder to estimate. Recall that

$$C_{2,K}=\widehat{\phi_2}(0)\frac{\log|D_K|}{L}.$$

Let $u\geq 1$ be a real variable. We will make use of Stieltjes integration and the result

$$\begin{aligned}&\sum_{K\in\mathcal{F}^{\pm}(u)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\\&=-C_1^{\pm}u\sum_p\sum_{e\geq1}\frac{x_p\log p}{p^{e(1/2+r)}}\left(\theta_e+\frac{1}{p}\right)-C_2^{\pm}u^{5/6}\sum_p\sum_{e\geq1}\frac{(\gamma_e(p)-p^{-e/3})\log p}{p^{e(1/2+r)}}\\&+C_2^{\pm}u^{5/6}\frac{\zeta'}{\zeta}(5/6+r)-\zeta(1-2r)\frac{\Gamma_{\pm}(1/2-r)}{\Gamma_{\pm}(1/2+r)}\left(\frac{u^{-r}}{1-r}C_1^{\pm}uA_3(-r,r)\right.\\&\left.+\frac{u^{-r}}{1-6r/5}C_2^{\pm}u^{5/6}\frac{\zeta(5/6-r)}{\zeta(5/6+r)}A_4(-r,r)\right)+\mathcal{O}_\epsilon\left(u^{\theta+\epsilon}\right),\end{aligned}\tag{8.4.6}$$

again from (6.6.3). Note that (6.6.3) can only be used when u is large compared to $|r|$. We ignore this minor technical point for now and discuss it later. We compute

$$\begin{aligned}&\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\log|D_K|=\int_1^X(\log u)\,d\left(\sum_{K\in\mathcal{F}^{\pm}(u)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}\right)\\&=\log X\sum_{K\in\mathcal{F}^{\pm}(X)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}-\int_1^X\frac{1}{u}\sum_{K\in\mathcal{F}^{\pm}(u)}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}du.\end{aligned}$$

Using the same cutoff argument as in the proof of Proposition 6.4, and the result above, we find that

$$\begin{aligned}&\frac{2}{2\pi i}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\frac{1}{N^{\pm}(X)}\sum_{K\in\mathcal{F}^{\pm}(X)}C_{2,K}\frac{L'(1/2+r,f_K)}{L(1/2+r,f_K)}dr\\&=\frac{\widehat{\phi_2}(0)}{L}\Bigg[(\log X)\big(F_1(X)S_1(1)+F_2(X)S_2(1)+J_1(X)\big)\\&-\left(F_1(X)S_1(1)+\frac{6}{5}F_2(X)S_2(1)+J_1(X)\right)-\frac{2}{L}\cdot\frac{F_2(X)}{5}\sum_{p,e}\frac{\log p}{p^{5e/6}}\widehat{\phi_1}\left(\frac{\log p^e}{L}\right)\\&-\frac{2}{2\pi i}\cdot\frac{F_2(X)}{5}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\frac{\zeta'}{\zeta}\left(\frac{5}{6}+r\right)dr\\&+\frac{2}{2\pi i}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\zeta(1-2r)\frac{\Gamma_{\pm}(1/2-r)}{\Gamma_{\pm}(1/2+r)}\left(F_1(X)\frac{rX^{-r}}{(1-r)^2}A_3(-r,r)\right.\\&\left.+F_2(X)\cdot\frac{6}{5}\cdot\frac{(1/6+r)X^{-r}A_4(-r,r)}{(1-6r/5)^2}\cdot\frac{\zeta(5/6-r)}{\zeta(5/6+r)}\right)dr\Bigg].\end{aligned}$$

We now discuss the technical issue raised above. As we mentioned, to find (8.4.6), one needs r to be small in comparison to u so that Conjecture 6.3 can be applied. Hence, we cannot be sure that this really holds unless $u \geq X^{1/2}$, $|r| \leq X^{\epsilon/3}$, say. However, using e.g [IK, Thm. 5.17] to bound the size of the summand, we can see that the terms involving $u \leq X^{1/2}$ will not be important to the final result, as they can be absorbed in the error term so that there is no problem. The cutoff argument also ensures that we only need to consider small $|r|$.

The only term left to analyse is the first term of (8.4.5). We use a cutoff argument so that Conjecture 8.3 can be applied, similar to the argument from Proposition 6.4. For technical reasons, one may want to move one of the integrals to the line $(2/L)$ before trying to make this argument, so that it is possible to estimate

$$\frac{\zeta'}{\zeta}\left(1 \pm (r-s)\right) \ll L,$$

and then move the integral back once Conjecture 8.3 has been applied. We leave out the details for bounding the rest of the integrand for large $|r|, |s|$, as they have been given before in similar situations.

The result is that the first term of (8.4.5) equals

$$\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)} \phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\bigg(F_1(X)R^M_{1,1,\alpha,\beta}(r,s,r,s) + F_2(X)R^S_{1,1,\alpha,\beta}(r,s,r,s)$$
$$+ R_{1,2,\alpha,\beta}(r,s,r,s;X) + R_{2,1,\alpha,\beta}(r,s,r,s;X) + R_{2,2,\alpha,\beta}(r,s,r,s;X)\bigg)drds + \mathcal{O}_\epsilon(X^{\theta-1+\epsilon}).$$

We will only simplify the integral over the first two terms. Recall that these terms are equal to certain sums, given in (8.2.8) and (8.2.9). To simplify, interchange the order of summation and integration, move both contours to (0), and apply the definition of the Fourier transform. We find a total contribution of $F_1(X)\big(S_1(1)S_1(2) - S_3 + S_5\big) + F_2(X)\big(S_2(1)S_2(2) - S_4 + S_6\big) + J''(X)$. The proposition follows. □

9

Interpreting the Ratios Conjecture prediction for the two-level density

In this chapter, we will study the Ratios Conjecture prediction of the two-level density from Proposition 8.4. We first focus on a comparison to the unconditional result from Theorem 4.1 for $\sigma < 1$, and in this direction, Proposition 9.1 implies that the two results agree quite well. Next, we will study the Ratios Conjecture prediction in the extended range $\sigma < \infty$. For such σ, we explicitly find the main term and the secondary term of the predicted two-level density in Theorem 9.3. In particular, we detect phase transitions when σ_1, σ_2, and σ exceed 1 respectively.

9.1 Two expressions for the two-level density

We now let

$$\sigma = \sigma_1 + \sigma_2 < 1,$$

and compare the results of Theorem 4.1 and Proposition 8.4. In the notation of Proposition 8.4, the two expressions for the two-level density, excluding the error terms, differ by

$$\begin{aligned} I(X) := \left(\widehat{\phi_2}(0)J_1(X) + \widehat{\phi_1}(0)J_2(X)\right)\left(1 + \frac{\log(4\pi^2 e)}{L}\right) + \frac{\widehat{\phi_2}(0)}{L}J_1'(X) + \frac{\widehat{\phi_1}(0)}{L}J_2'(X) \\ + E_1 J_2(X) + E_2 J_1(X) + J''(X) - 2J_3(X). \end{aligned}$$

We then have the following proposition:

Proposition 9.1. *For $\sigma < 1$, the difference $I(X)$ between the non-error terms in the Ratios Conjecture prediction from Proposition 8.4, and the actual two-level density given in Theorem 4.1, admits the bound*

$$I(X) \ll_\epsilon X^{\sigma/6-1/3+\epsilon} + X^{\sigma/2-1/2+\epsilon}. \tag{9.1.1}$$

In particular, for sufficiently small σ, depending on ϵ, we have $I(X) \ll_\epsilon X^{-1/3+\epsilon}$.

The rest of this section will be dedicated to proving Proposition 9.1. First, we already have a bound for $J_i(X)$, given in (7.1.2). Recalling that $\widehat{\phi_3}$ is supported in $[-\sigma, \sigma]$, we thus find that

$$I(X) = \mathcal{O}_\epsilon\left(X^{\sigma/6-1/3+\epsilon} + X^{\sigma/2-1/2+\epsilon}\right) + \frac{\widehat{\phi_2}(0)}{L}J_1'(X) + \frac{\widehat{\phi_1}(0)}{L}J_2'(X) + J''(X). \tag{9.1.2}$$

We turn our attention to $J_i'(X)$. Note that the definition of $J_i'(X)$ is very similar to the definition of $J(X)$, and we can bound $J_i'(X)$ in almost the same way as we bounded $J(X)$, so we do not provide all the details. First, by using the calculations from when we bounded $J(X)$, it is not hard to see that the residue at $r = 1/6$ coming from the integrand of the first integral in (8.4.2) cancels against the residue from the first integrand of the second integral, up to an error $\ll X^{\sigma/6-1/2}$. Thus, if we shift the contour to $(1/2-\epsilon)$ when integrating these terms, the contribution from the residue can be absorbed in the error term from (9.1.2). Once the first integral has been moved, we may cancel it against the sum in (8.4.2), while the second integral that was moved can be bounded as $\ll X^{\sigma/2-1/2+\epsilon}$. Finally, we shift the contour of the remaining integral in (8.4.2) to $(1/6-\epsilon)$ and estimate the resulting integral. The result is that

$$J_i'(X) \ll_\epsilon X^{\sigma/6-1/3+\epsilon} + X^{\sigma/2-1/2+\epsilon}.$$

It remains to estimate $J''(X)$. We begin by handling the first three rows of the definition (8.4.3). First, we may move the integral

$$\frac{2}{2\pi i}\int_{(1/L)} \phi_i\left(\frac{Lr}{2\pi i}\right)\frac{\zeta'}{\zeta}(5/6+r)dr$$

to the line $(1/2-\epsilon)$. We pick up the negative of the residue at $r=1/6$, as we are shifting the contour to the right, and the above thus equals

$$2\phi_i\left(\frac{L}{12\pi i}\right) - \frac{2}{L}\sum_{p,e\geq 1}\frac{\log p}{p^{5e/6}}\widehat{\phi_i}\left(\frac{\log p^e}{L}\right).$$

It follows that the sum of the first three rows of (8.4.3) equal

$$F_2(X)\left(4\phi_1\left(\frac{L}{12\pi i}\right)\phi_2\left(\frac{L}{12\pi i}\right) + 2S_2(2)\phi_1\left(\frac{L}{12\pi i}\right) + 2S_2(1)\phi_2\left(\frac{L}{12\pi i}\right)\right). \tag{9.1.3}$$

This expression cannot be absorbed in the error term of (9.1.2) directly, but we will see that other terms from $J''(X)$ cancels these terms.

It remains to estimate the rest of $J''(X)$, namely the integral of $R_{1,2,\alpha,\beta}, R_{2,1,\alpha,\beta}$ and $R_{2,2,\alpha,\beta}$. We begin by estimating $R_{2,2,\alpha,\beta}$, as this will be quite simple. First, recall that both $A_1(\alpha,\beta,\gamma,\delta)$ and $A_2(\alpha,\beta,\gamma,\delta)$ are $\ll_\epsilon 1$, when $|\mathrm{Re}(\alpha)|, |\mathrm{Re}(\beta)|, |\mathrm{Re}(\gamma)|, |\mathrm{Re}(\delta)| < 1/6-\epsilon$. Next, note that in (8.3.4), the expression for $R_{2,2,\alpha,\beta}$ contains a factor X^{-r-s}. Thus, moving both contours to $(1/6-\epsilon/2)$, we can bound

$$\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)} R_{2,2,\alpha,\beta}(r,s,r,s;X)\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)drds \ll_\epsilon X^{\sigma/6-1/3+\epsilon},$$

where we also used (3.1.2) and $\sigma=\sigma_1+\sigma_2$.

It remains to consider the term involving $R_{1,2,\alpha,\beta}$ and $R_{2,1,\alpha,\beta}$. The first part of $R_{2,1,\alpha,\beta}$ from (8.3.3) is given by

$$-\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\left(\frac{\zeta'}{\zeta}(1+s-r)+\frac{\zeta'}{\zeta}(1+2s)-\frac{\zeta'}{\zeta}(1+r+s)\right)$$
$$\times\left(F_1(X)\frac{X^{-r}}{1-r}A_1(-r,s,r,s)+\frac{\zeta(5/6-r)}{\zeta(5/6+r)}F_2(X)\frac{X^{-r}}{1-6r/5}A_2(-r,s,r,s)\right).$$

We consider the integral of this term together with the integral of the corresponding term from $R_{1,2,\alpha,\beta}$, and merge them into a single integral. The reason for this is to remove the singularities at $s=r$.

We begin by separating the terms involving A_2 from the terms involving A_1, so that the integral of this part of $R_{1,2,\alpha,\beta}+R_{2,1,\alpha,\beta}$ is split into two different double-integrals. We then begin by considering the integral of the two terms containing a factor A_2. The first of these terms, coming from $R_{2,1,\alpha,\beta}$, contains a factor $F_2(X)X^{-r}$, while the term coming from $R_{1,2,\alpha,\beta}$ contains a factor $F_2(X)X^{-s}$. We shift the contour for the r-integral to $(1/6-\epsilon)$ and the contour for the s-integral to $(1/6-2\epsilon)$. The reason for shifting to different lines is that we can bound the integral over each term separately without worrying about the pole at $s=r$. The result is that we may bound the integral of these two terms as being $\ll_\epsilon X^{\sigma/6-1/3+\epsilon}$, after possibly modifying ϵ.

Next, we consider the integral of the terms containing A_1. Recall that $A_1(-r,s,r,s) = A_3(-r,r)$, and $A_1(r,-s,r,s) = A_3(-s,s)$ so that we may write this integral as

$$-\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\Bigg[\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\bigg(\frac{\zeta'}{\zeta}(1+s-r) + \frac{\zeta'}{\zeta}(1+2s) - \frac{\zeta'}{\zeta}(1+r+s)\bigg)F_1(X)\frac{X^{-r}}{1-r}A_3(-r,r) + \frac{\Gamma_\pm(1/2-s)}{\Gamma_\pm(1/2+s)}\zeta(1-2s) \times\left(\frac{\zeta'}{\zeta}(1+r-s) + \frac{\zeta'}{\zeta}(1+2r) - \frac{\zeta'}{\zeta}(1+r+s)\right)F_1(X)\frac{X^{-s}}{1-s}A_3(-s,s)\Bigg]drds.$$

We shift the integral over r to $(1/2-\epsilon)$ and afterwards, we shift the integral over s to $(1/2-2\epsilon)$. The shifted integral itself can be absorbed in the error term above, as the integral contains the factors X^{-r} and X^{-s}. What is more interesting is the residues we pick up at $r = 1/6$ and $s = 1/6$ respectively, coming from the pole of A_3.

Using the calculation of the residue from the one-level computation, and the relation

$$F_1(X)X^{-1/6}\cdot\frac{2\zeta(2/3)\Gamma_\pm(1/3)\zeta(3)}{5\Gamma_\pm(2/3)\zeta(5/3)\zeta(2)} = F_1(X)X^{-1/6}\frac{C_2^\pm}{C_1^\pm} = F_2(X) + \mathcal{O}(X^{-1/2}), \tag{9.1.4}$$

we see that the contribution from the residues is, up to an acceptable error term, equal to

$$-F_2(X)\phi_1\left(\frac{L}{12\pi i}\right)\frac{4}{2\pi i}\int_{(1/L)}\phi_2\left(\frac{Ls}{2\pi i}\right)\left(\frac{\zeta'}{\zeta}(5/6+s) + \frac{\zeta'}{\zeta}(1+2s) - \frac{\zeta'}{\zeta}(7/6+s)\right)ds$$
$$-F_2(X)\phi_2\left(\frac{L}{12\pi i}\right)\frac{4}{2\pi i}\int_{(1/2-\epsilon)}\phi_1\left(\frac{Lr}{2\pi i}\right)\left(\frac{\zeta'}{\zeta}(5/6+r) + \frac{\zeta'}{\zeta}(1+2r) - \frac{\zeta'}{\zeta}(7/6+r)\right)dr.$$

The second integral can be simplified by expanding ζ'/ζ into its Dirichlet series, exchanging the order of summation and integration, shifting the integral to (0), making a change of variables, and then applying the definition of the Fourier transform. The first integral is simplified by first shifting to $1/6+\epsilon$, picking up a residue at $s = 1/6$. The shifted integral is then simplified by the procedure described above for the second integral.

We have studied the integral of the first two rows in (8.3.2) and (8.3.3), coming from $R_{1,2,\alpha,\beta}$ and $R_{2,1,\alpha,\beta}$ respectively. The result is that this integral gives, up to terms $\ll_\epsilon X^{\sigma/6-1/3+\epsilon} + X^{\sigma/2-1/2+\epsilon}$, a contribution

$$-4F_2(X)\phi_1\left(\frac{L}{12\pi i}\right)\phi_2\left(\frac{L}{12\pi i}\right) - 2F_2(X)\phi_1\left(\frac{L}{12\pi i}\right)\cdot\left(-\frac{2}{L}\right)\Bigg(\sum_{p,e\geq1}\frac{\log p}{p^{5e/6}}\widehat{\phi_2}\left(\frac{\log p^e}{L}\right) + \sum_{p,e\geq1}\frac{\log p}{p^e}\widehat{\phi_2}\left(\frac{\log p^{2e}}{L}\right) - \sum_{p,e\geq1}\frac{\log p}{p^{7e/6}}\widehat{\phi_2}\left(\frac{\log p^e}{L}\right)\Bigg) - 2F_2(X)\phi_2\left(\frac{L}{12\pi i}\right)\cdot\left(-\frac{2}{L}\right) \times\left(\sum_{p,e\geq1}\frac{\log p}{p^{5e/6}}\widehat{\phi_1}\left(\frac{\log p^e}{L}\right) + \sum_{p,e\geq1}\frac{\log p}{p^e}\widehat{\phi_1}\left(\frac{\log p^{2e}}{L}\right) - \sum_{p,e\geq1}\frac{\log p}{p^{7e/6}}\widehat{\phi_1}\left(\frac{\log p^e}{L}\right)\right). \tag{9.1.5}$$

Note that the first term above cancels the first term of (9.1.3).

To finish our calculations we must consider the integral of the rest of terms coming from $R_{1,2,\alpha,\beta}$ and $R_{2,1,\alpha,\beta}$. By symmetry, it will be sufficient to study the terms coming from $R_{2,1,\alpha,\beta}$. The integral in question is then

$$-\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\Bigg[F_1(X)\frac{X^{-r}}{1-r}A_{1,\beta}(-r,s,r,s) + F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}\frac{\zeta'}{\zeta}\left(\tfrac{5}{6}+s\right)\frac{X^{-r}}{1-6r/5}A_2(-r,s,r,s) + F_2(X)\frac{\zeta(5/6-r)}{\zeta(5/6+r)}\frac{X^{-r}}{1-6r/5}A_{2,\beta}(-r,s,r,s)\Bigg]drds.$$

As usual, the integrands involving $F_2(X)X^{-r}$ will be unimportant and can be absorbed into the error term. Here we also make use of the fact that on the contour we are integrating over, we have that $A_{2,\beta} \ll 1$, which follows from Cauchy's integral formula for the derivative of a holomorphic function. We are left with

$$-\frac{4F_1(X)}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\frac{X^{-r}}{1-r}A_{1,\beta}(-r,s,r,s)drds. \tag{9.1.6}$$

We want to shift the integral above to the right, but to accomplish this, we must first find a meromorphic continuation for $A_{1,\beta}(-r,s,r,s)$. Recall that $A_1(-r,\beta,r,s)$ is a priori only defined when the real part of β and s are both greater than $-1/6$, and the modulus of the real part of r is less than $1/6$. Therefore, we begin by presenting an analogue of Lemma 7.2, extending the domain of definition.

Lemma 9.2. *Let* $|\text{Re}(r)| < 1/6$*, and* $\text{Re}(\beta), \text{Re}(s) > -1/6$ *. Then, we have*

$$\begin{aligned}
A_1(-r,\beta,r,s) &= \zeta(3)\prod_p\left(1-\frac{1}{p^{1+\beta-r}}\right)\left(1-\frac{1}{p^{1-2r}}\right)\left(1-\frac{1}{p^{1+2\beta}}\right)\left(1-\frac{1}{p^{1+r+s}}\right)\\
&\times\left(1-\frac{1}{p^{1+s-r}}\right)^{-1}\left(1-\frac{1}{p^{1+\beta+s}}\right)^{-1}\left(1-\frac{1}{p^{1+\beta+r}}\right)^{-1}\\
&\times\left[\frac{1}{p^2}+\frac{1}{6}\frac{\left(1-p^{-(1/2+r)}\right)^2\left(1-p^{-(1/2+s)}\right)^2}{\left(1-p^{-(1/2-r)}\right)^2\left(1-p^{-(1/2+\beta)}\right)^2}+\frac{1}{2}\frac{\left(1-p^{-(1+2r)}\right)\left(1-p^{-(1+2s)}\right)}{\left(1-p^{-(1-2r)}\right)\left(1-p^{-(1+2\beta)}\right)}\right.\\
&+\frac{1}{3}\frac{\left(1+p^{-(1/2+r)}+p^{-(1+2r)}\right)\left(1+p^{-(1/2+s)}+p^{-(1+2s)}\right)}{\left(1+p^{-(1/2-r)}+p^{-(1-2r)}\right)\left(1+p^{-(1/2+\beta)}+p^{-(1+2\beta)}\right)}\\
&\left.+\frac{1}{p}\frac{\left(1-p^{-(1/2+r)}\right)\left(1-p^{-(1/2+s)}\right)}{\left(1-p^{-(1/2-r)}\right)\left(1-p^{-(1/2+\beta)}\right)}\right].
\end{aligned}$$

For the sake of brevity, we leave out the proof, but we remark that it is very similar to the proof of Lemma 7.2.

Suppose that both $\text{Re}(\beta), \text{Re}(s) > 1/2+\epsilon$. Then, for $|\text{Re}(r)| < 1/2$ we could write the expression above as

$$\begin{aligned}
&\zeta(3)\prod_p\left(\left(1-\frac{1}{p^{1-2r}}\right)\left[\frac{1}{p^2}+\frac{1}{6}\frac{\left(1-p^{-(1/2+r)}\right)^2}{\left(1-p^{-(1/2-r)}\right)^2}+\frac{1}{2}\frac{\left(1-p^{-(1+2r)}\right)}{\left(1-p^{-(1-2r)}\right)}\right.\right.\\
&\left.\left.+\frac{1}{3}\frac{\left(1+p^{-(1/2+r)}+p^{-(1+2r)}\right)}{\left(1+p^{-(1/2-r)}+p^{-(1-2r)}\right)}+\frac{1}{p}\frac{\left(1-p^{-(1/2+r)}\right)}{\left(1-p^{-(1/2-r)}\right)}\right]+\mathcal{O}\left(p^{-1-\epsilon}\right)\right).
\end{aligned} \tag{9.1.7}$$

Comparing this to the proof of Lemma 7.2, we see that if we multiply this product by the Euler product of $\zeta(3/2-3r)^{-1}$, then the result converges absolutely. Thus, we have proven that $A_1(-r,\beta,r,s)$ is meromorphic in $\text{Re}(\beta), \text{Re}(s) > 1/2+\epsilon$, $|\text{Re}(r)| < 1/2-\epsilon$, with a singularity whenever $r = 1/6$.

With this result in mind, we move the integral over s in (9.1.6) to $(1/2+2\epsilon)$, which yields

$$\begin{aligned}
-\frac{4F_1(X)}{(2\pi i)^2}\int_{(1/2+2\epsilon)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)&\\
\times\frac{X^{-r}}{1-r}A_{1,\beta}(-r,s,r,s)drds.&
\end{aligned} \tag{9.1.8}$$

The next step will be to move the inner integral over r to $(1/2-\epsilon)$. The shifted integral can then be estimated as usual, and we can see that it will give no significant contribution to the result. We remark that on the line $(1/2-\epsilon)$ the factor $A_{1,\beta}$ is estimated by combining

Cauchy's integral formula for the derivative with an estimate of A_1. The estimate for A_1 is obtained by using (9.1.7) together with an estimate for $\zeta(3/2-3r)$ when the real part of r is close to $1/2$. As the shifted integral is small, it is more interesting to study the residue we pick up at $r = 1/6$.

To find the residue at $r = 1/6$ we will first need to study $A_{1,\beta}$ more closely. Thus, we will first need to differentiate $A_1(-r, \beta, r, s)$ with respect to β, and then set $\beta = s$. We will accomplish this by logarithmically differentiating the expression in Lemma 9.2. First, we multiply by $\zeta(3/2-3r)^{-1}$ to ensure that all technical conditions for moving differentiation inside a sum are fulfilled. Next, recall that the pole at $r = 1/6$ of $A_1(-r, \beta, r, s)$ is simple, and evidently, it remains so after differentiation with respect to β. Define

$$g(r, \beta, s) := A_1(-r, \beta, r, s)\zeta(3/2-3r)^{-1},$$

which is then an absolutely convergent product in $|\mathrm{Re}(r)| < 1/2$, $\mathrm{Re}(\beta), \mathrm{Re}(s) > 1/2 + \epsilon$. Recall that $A_1(-r, s, r, s) = A_3(-r, r)$ so that

$$\left.\frac{d}{d\beta}\right|_{\beta=s} \log g(r, \beta, s) = \frac{A_{1,\beta}(-r, s, r, s)}{A_3(-r, r)}.$$

The residue is then equal to

$$\begin{aligned}\lim_{r\to 1/6}(r-1/6)A_{1,\beta}(-r, s, r, s) &= \lim_{r\to 1/6}(r-1/6)A_3(-r, r)\left.\frac{d}{d\beta}\right|_{\beta=s}\log g(r, \beta, s)\\ &= -\frac{\zeta(3)}{3\zeta(5/3)\zeta(2)}\left.\frac{d}{d\beta}\right|_{\beta=s}\log g(1/6, \beta, s),\end{aligned}$$

by Lemma 7.2.

We turn to the task of calculating the logarithmic derivative. First,

$$\begin{aligned}\log g(1/6, \beta, s) = \log\zeta(3) + \sum_p \Bigg(&\log\left(1-\frac{1}{p}\right) + \log\left(1-\frac{1}{p^{5/6+\beta}}\right) + \log\left(1-\frac{1}{p^{2/3}}\right)\\ &+ \log\left(1-\frac{1}{p^{1+2\beta}}\right) + \log\left(1-\frac{1}{p^{7/6+s}}\right) - \log\left(1-\frac{1}{p^{5/6+s}}\right)\\ &- \log\left(1-\frac{1}{p^{1+\beta+s}}\right) - \log\left(1-\frac{1}{p^{7/6+\beta}}\right)\\ &+ \log\Bigg[\frac{1}{p^2} + \frac{1}{6}\frac{\left(1+p^{-1/3}\right)^2\left(1-p^{-(1/2+s)}\right)^2}{\left(1-p^{-(1/2+\beta)}\right)^2} + \frac{1}{2}\frac{\left(1+p^{-2/3}\right)\left(1-p^{-(1+2s)}\right)}{\left(1-p^{-(1+2\beta)}\right)}\\ &+ \frac{1}{3}\frac{\left(1-p^{-1/3}+p^{-2/3}\right)\left(1+p^{-(1/2+s)}+p^{-(1+2s)}\right)}{\left(1+p^{-(1/2+\beta)}+p^{-(1+2\beta)}\right)} + \frac{1}{p}\frac{\left(1+p^{-1/3}\right)\left(1-p^{-(1/2+s)}\right)}{\left(1-p^{-(1/2+\beta)}\right)}\Bigg]\Bigg).\end{aligned}$$

Next, we differentiate this with respect to β and set $\beta = s$. Note that if we set $\beta = s$ in the argument of the last logarithm, then this argument is precisely equal to the reciprocal of $y_p(1+p^{-1/3})$. Using this, the result after differentiation is

$$\begin{aligned}\sum_p \Bigg(&\frac{\log p}{p^{5/6+s}-1} + \frac{\log p}{p^{1+2s}-1} - \frac{\log p}{p^{7/6+s}-1}\\ &+ y_p(1+p^{-1/3})\Bigg(-\frac{(1+p^{-1/3})^2\log p}{3(p^{1/2+s}-1)} - \frac{(1+p^{-2/3})\log p}{p^{1+2s}-1}\\ &+ \frac{(1-p^{-1/3}+p^{-2/3})\log p}{3}\left(\frac{1}{p^{1/2+s}-1} - \frac{3}{p^{3/2+3s}-1}\right) - \frac{(1+p^{-1/3})\log p}{p(p^{1/2+s}-1)}\Bigg)\Bigg).\end{aligned}$$

Simplifying, and then applying the formula for a geometric series shows that the above is equal to

$$-\sum_p \log p\Bigg(-\sum_{e\geq 1}\frac{1}{p^{5e/6+es}} - \sum_{e\geq 1}\frac{1}{p^{e+2es}} + \sum_{e\geq 1}\frac{1}{p^{7e/6+es}} + y_p(1+p^{-1/3})$$
$$\times \left((1+p^{-2/3})\sum_{e\geq 1}\frac{\delta_{2|e}+\delta_{3|e}}{p^{e/2+es}} + p^{-1/3}\sum_{e\geq 1}\frac{1-\delta_{3|e}}{p^{e/2+es}} + (p^{-1}+p^{-4/3})\sum_{e\geq 1}\frac{1}{p^{e/2+es}}\right)\Bigg)$$
$$= -\sum_p \log p\Bigg(-\sum_{e\geq 1}\frac{1}{p^{5e/6+es}} - \sum_{e\geq 1}\frac{1}{p^{e+2es}} + \sum_{e\geq 1}\frac{1}{p^{7e/6+es}} + \sum_{e\geq 1}\frac{\gamma_e(p)}{p^{e/2+es}}\Bigg).$$

Let us denote the expression above by $R(s)$.

In conclusion, our calculations show that the contribution from the residue we pick up from (9.1.8) is, up to an acceptable error term, equal to

$$-2F_2(X)\phi_1\left(\frac{L}{12\pi i}\right)\cdot\frac{2}{2\pi i}\int_{(1/2+2\epsilon)}\phi_2\left(\frac{Ls}{2\pi i}\right)R(s)ds, \tag{9.1.9}$$

where we made use of (9.1.4). Note that we picked up the negative of the residue as we shift to the right. Next, interchange the order of integration and summation to move the integral inside every sum of $R(s)$. Then shift all contours to (0) and apply the definition of the Fourier transform, after making a change of variables. We find that (9.1.9) is equal to

$$-2F_2(X)\phi_1\left(\frac{L}{12\pi i}\right)\cdot\left(-\frac{2}{L}\right)\Bigg(-\sum_{p,e\geq 1}\frac{\log p}{p^{5e/6}}\widehat{\phi_2}\left(\frac{\log p^e}{L}\right) - \sum_{p,e\geq 1}\frac{\log p}{p^{e}}\widehat{\phi_2}\left(\frac{\log p^{2e}}{L}\right)$$
$$+\sum_{p,e\geq 1}\frac{\log p}{p^{7e/6}}\widehat{\phi_2}\left(\frac{\log p^e}{L}\right)\Bigg) - 2F_2(X)\phi_1\left(\frac{L}{12\pi i}\right)S_2(2).$$

This cancels against the corresponding terms from (9.1.3) and (9.1.5). By symmetry, the terms coming from $R_{1,2,\alpha,\beta}$ will cancel the rest of the terms of (9.1.3) and (9.1.5). Thus, all our calculations have shown that

$$J''(X) \ll_\epsilon X^{\sigma/6-1/3+\epsilon} + X^{\sigma/2-1/2+\epsilon},$$

as desired. This proves Proposition 9.1.

9.2 Phase transitions in the two-level density

We now turn to the problem of investigating the Ratios Conjecture prediction for the two-level density for any σ. When we calculated the one-level density, we found phase transitions in both the main term and the secondary term, when the support σ reached 1. In this section, we will see that for the two-level density, there are several phase transitions, one when each σ_i reaches 1, and another when $\sigma = \sigma_1 + \sigma_2$ does. Specifically, the purpose of this section is to prove the following theorem:

Theorem 9.3. *Assume the Generalised Riemann Hypothesis for $\zeta_K(s)$, and Conjecture 8.3.*

Then, we have the estimate

$$\begin{aligned}\frac{1}{N^{\pm}(X)}\sum_{K}\sum_{\gamma_K\neq\pm\gamma_K'}\phi\left(\frac{L}{2\pi}\gamma_K,\frac{L}{2\pi}\gamma_K'\right)&=\widehat{\phi_1}(0)\widehat{\phi_2}(0)-\frac{\widehat{\phi_1}(0)}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du\\
-\frac{\widehat{\phi_2}(0)}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du&+\frac{1}{4}\int_{-1}^{1}\widehat{\phi_1}(u)du\int_{-1}^{1}\widehat{\phi_2}(t)dt+2\int_{-1}^{1}|u|\widehat{\phi_1}(u)\widehat{\phi_2}(u)du\\
+4\int_0^1\widehat{\phi_1}(u)\int_0^{1-u}\widehat{\phi_2}(t)dtdu&-2\int_{-1}^{1}\widehat{\phi_1}(u)\widehat{\phi_2}(u)du+\frac{D^{\pm}_{S_3}}{L}\Bigg(-2\widehat{\phi_1}(0)\widehat{\phi_2}(0)\\
+\frac{\widehat{\phi_1}(0)-\widehat{\phi_1}(1)}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du&+\frac{\widehat{\phi_2}(0)-\widehat{\phi_2}(1)}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du+\widehat{\phi_1}(0)\widehat{\phi_2}(1)\\
+\widehat{\phi_2}(0)\widehat{\phi_1}(1)+2\int_{-1}^{1}\widehat{\phi_1}(u)\widehat{\phi_2}(u)du&-4\int_0^1\widehat{\phi_1}(u)\widehat{\phi_2}(1-u)du\Bigg)+\mathcal{O}\left(\frac{1}{L^2}\right),\end{aligned}\tag{9.2.1}$$

where

$$D^{\pm}_{S_3}:=-1+4\log 2+\pi\delta_+ + C.$$

Here, $\delta_+=1$ *if we are considering positive discriminants, and else it equals* 0.

Remark. *The main term is the result of integrating* $\widehat{\phi_1}(u_1)\widehat{\phi_2}(u_2)$ *against*

$$\big(\delta(u_1)-\tfrac{1}{2}\chi_{[-1,1]}(u_1)\big)\big(\delta(u_2)-\tfrac{1}{2}\chi_{[-1,1]}(u_2)\big)-2\left((1-|u_1|)\delta(u_1+u_2)\chi_{[-1,1]}(u_1)-\frac{1}{2}\chi_B(u_1,u_2)\right),$$

where B *denotes the unit* L^1*-ball, and* δ *the Dirac delta distribution. This is exactly the expected Katz-Sarnak main term, see [M, Ch. 5], where the calculations leading up to Theorem 5.9 can be used, also for support* $\sigma>1$.

Remark. *It is interesting to see that the secondary term essentially consists of integral linear combinations of* $D^{\pm}_{S_3}/L$ *multiplied with appropriate transforms of* ϕ_1 *and* ϕ_2*, which can be compared to the result of Proposition 7.2. Also, note that the very last term of* (9.2.1) *is zero if* $\phi(u_1,u_2)$ *is supported inside the* L^1*-unit-ball, whence this term gives a phase transition at* $\sigma=1$.

The rest of the section will be focused on proving this theorem. The starting point of all our calculations is Proposition 8.4. Unlike the case when $\sigma<1$, the sum $S_2(i)$ may be very large, in fact as large as some power of X. Hence, we will need to eliminate a significant part of these sums. We remark that in this section, an "acceptable error" will always mean an error that is $\ll L^{-2}$.

We now study the Ratios Conjecture prediction in (8.4.1), one row at a time. The second row of (8.4.1) is already simplified. The relevant terms here are

$$\widehat{\phi_1}(0)\widehat{\phi_2}(0)\left(1+\frac{2\log(4\pi^2e)}{L}\right).$$

We turn to the third row. To eliminate the influence of the S_2 terms, we add the term

$$\widehat{\phi_1}(0)J_2(X)\left(1+\frac{\log(4\pi^2e)}{L}\right)+\frac{\widehat{\phi_1}(0)}{L}J_2'(X),\tag{9.2.2}$$

from the fifth row to the third row. We now study the expression that we just added. By our work in Section 7.2, we know that

$$J_i(X)=\frac{2}{L}F_2(X)\sum_{p,e}\frac{\log p}{p^{5e/6}}\widehat{\phi_i}\left(\frac{\log p^e}{L}\right)+\frac{\phi_i(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_i}(u)du+D^{\pm}_{S_3}\frac{\widehat{\phi_i}(1)}{L}+\mathcal{O}\left(\frac{1}{L^2}\right).$$

We turn to the definition (8.4.2) of J_i'. More specifically we take a look at the integrals involved in the definition. First, by a similar argument as in the one-level calculations, we may disregard the contribution of any integral involving $F_2(X)$. The only remaining integral is

$$\frac{2}{2\pi i}\int_{(1/L)}\phi_i\left(\frac{Lr}{2\pi i}\right)\zeta(1-2r)\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}F_1(X)\frac{rX^{-r}}{(1-r)^2}A_3(-r,r)dr.$$

We can handle this just as we originally handled $J(X)$, by Taylor expanding all relevant factors of the integrand. The only difference between this integral and the corresponding integral from $J_i(X)$ is the presence of the factor $r(1-r)^{-2}$. As this factor is equal to 0 when $r=0$, a calculation reveals that this integral term is $\ll L^{-1}$. As there is already another factor L^{-1} in front of $J_2'(X)$ in (9.2.2), we can completely disregard all integral terms from $J_2'(X)$.

To finish the estimation of the third row, we use (7.2.5), $F_1(X)=1+\mathcal{O}(X^{-1/6})$, the estimates in (3.3.1) and Section 7.2 for S_1 to see that the result after adding (9.2.2) to the third row of (8.4.1) is

$$\widehat{\phi_1}(0)\left(1+\frac{\log(4\pi^2 e)}{L}\right)\left((2\gamma-C)\frac{\widehat{\phi_2}(0)}{L}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du+D^{\pm}_{S_3}\frac{\widehat{\phi_2}(1)}{L}\right)+\mathcal{O}\left(\frac{1}{L^2}\right).$$

By symmetry, one has a similar result for the fourth row, and this also eliminates all remaining terms of the fifth row.

We skip the sixth row for now and turn to the seventh row of (8.4.1). We begin by studying the term containing the factor $F_2(X)$. By our calculations in Chapter 4, $S_4, S_6 \ll 1$, so that the only relevant part of this term is $F_2(X)S_2(1)S_2(2)$. To this term, we add the first three rows of the definition (8.4.3) of $J''(X)$, recalling that a term $J''(X)$ is present on row six of (8.4.1). Appealing to (7.2.5), we see that the result of this addition is certainly $\ll L^{-2}$.

We move on to the term $F_1(X)(S_1(1)S_1(2)-S_3+S_5)$, which up to acceptable error equals

$$\left(-\frac{\phi_1(0)}{2}+(2\gamma-C)\frac{\widehat{\phi_1}(0)}{L}\right)\left(-\frac{\phi_2(0)}{2}+(2\gamma-C)\frac{\widehat{\phi_2}(0)}{L}\right)+2\int_{-\infty}^{\infty}|u|\widehat{\phi_1}(u)\widehat{\phi_2}(u)du,$$

using (3.3.1) and the computations from Sections 7.2 and 4.4.

Now, we look at the last row of (8.4.1). It will be convenient to add the two first terms of the sixth row to this row. Using $E_i \ll L^{-1}$, we are, up to an error $\ll L^{-2}$, left with

$$E_1(R_2+J_2(X))+E_2(R_1+J_1(X))-2(R_3+J_3(X)).$$

We may use Theorem 7.3, which tells us the value of $R_i+J_i(X)$, and the calculation of E_i in Section 7.2.1 to see that this is

$$\frac{(-2\log 8\pi-2\gamma-\pi\delta_+)}{L}\left(\widehat{\phi_1}(0)\left(\widehat{\phi_2}(0)-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du\right)+\widehat{\phi_2}(0)\left(\widehat{\phi_1}(0)-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right)\right)$$
$$-2\widehat{\phi_1\phi_2}(0)+\int_{-1}^{1}\widehat{\phi_1\phi_2}(u)du+\frac{2D^{\pm}_{S_3}\widehat{\phi_1\phi_2}(0)}{L}-\frac{2D^{\pm}_{S_3}\widehat{\phi_1\phi_2}(1)}{L}+\mathcal{O}\left(L^{-2}\right).$$

The only thing that remains is to simplify all rows except the first three of $J''(X)$, i.e. we need to study

$$\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\Big(R_{1,2,\alpha,\beta}(r,s,r,s;X)+R_{2,1,\alpha,\beta}(r,s,r,s;X)$$
$$+R_{2,2,\alpha,\beta}(r,s,r,s;X)\Big)drds.$$

The methods used for studying this integral are very similar to the methods we used for studying the integral in $J(X)$ in Section 7.2.3, but this case will require quite some additional effort.

We begin by focusing on the integral of $R_{2,2,\alpha,\beta}$, defined in (8.3.4). As usual, we may discard the term containing $F_2(X)$, and replace $F_1(X)$ with 1. We make a change of variables $s' = Ls/(2\pi)$, $r' = Lr/(2\pi)$, and move both of the resulting integrals to (δ), for some small, fixed $\delta > 0$. We also rename r' as r, and s' as s. Using that ϕ_1 and ϕ_2 are even, we are left with the integral

$$\begin{aligned}\frac{4}{(Li)^2}\int_{(\delta)}\int_{(\delta)}\phi_1(ir)\,\phi_2(is)\,&\frac{\Gamma_\pm(1/2-2\pi s/L)}{\Gamma_\pm(1/2+2\pi s/L)}\frac{\Gamma_\pm(1/2-2\pi r/L)}{\Gamma_\pm(1/2+2\pi r/L)}\\ &\times\frac{\zeta(1-2\pi(r+s)/L)\zeta(1+2\pi(r+s)/L)}{\zeta(1-2\pi(r-s)/L)\zeta(1+2\pi(r-s)/L)}\zeta(1-4\pi r/L)\zeta(1-4\pi s/L)\\ &\times\frac{X^{-2\pi(r+s)/L}}{1-2\pi(r+s)/L}A_1(-2\pi r/L,-2\pi s/L,2\pi r/L,2\pi s/L)drds.\end{aligned}$$

We also rewrite

$$X^{-2\pi(r+s)/L} = e^{-2\pi(r+s)}e^{-2\pi(r+s)\log(4\pi^2e^2)/L}.$$

Next, we proceed similarly to earlier integral calculations. A cutoff argument shows that we may integrate over a truncated (δ), where the imaginary part of the variable is bounded in absolute value by L^ϵ, say. On this truncated contour, we Taylor expand every factor in the integrand about $(r,s)=(0,0)$, except for ϕ_1, ϕ_2 and $e^{-2\pi(r+s)}$. Then, by another cutoff argument, we may extend both contours back to the entire (δ).

We turn to the actual calculations. First, we find the relevant Taylor expansion. This has been done before, so we leave out most of the details, but some words need to be said about the function A_1. To find the Taylor expansion, we must find the partial derivatives, with respect to both r and s at the origin. By symmetry the partial derivatives are equal, so it suffices to find one. We know that $A_1(-r,0,r,0) = A_3(-r,r)$, so that this partial derivative is the constant C, by definition. As $A_1(r,s,r,s)=1$, we also know that $A_1(0,0,0,0)=1$. Thus, by the complex multivariate version of Taylor's theorem,

$$A_1(-2\pi r/L,-2\pi s/L,2\pi r/L,2\pi s/L) = 1+\frac{2\pi(r+s)C}{L}+\mathcal{O}\left(\frac{|r|^2+|s|^2}{L^2}\right).$$

See [Le, Thm. 1.2.1] for statement and proof of a complex multivariate power series expansion. The proof can be modified to produce an error term for the truncated series.

Taylor expanding the rest of the integrand, except for the factors mentioned above, shows that the integral in question is the integral of $\phi_1(ir)\phi_2(is)$ against

$$\frac{e^{-2\pi(r+s)}(r-s)^2}{(2\pi i)^2rs(r+s)^2}\left(1+\frac{2\pi D^{\pm}_{S_3}(r+s)}{L}\right)+\mathcal{O}_\delta\left(\frac{P(|r|,|s|)}{L^2}\right), \tag{9.2.3}$$

where P is a polynomial and where the factor $-4/L^2$ in front of the integral has been included. The specific definition of P does not matter, as (3.1.2) shows that integrating any polynomial against ϕ_1, and ϕ_2 gives a result of size $\ll 1$.

We now write

$$\frac{(r-s)^2}{rs(r+s)^2} = \frac{1}{rs}-\frac{4}{(r+s)^2}.$$

Then, we see that to integrate (9.2.3) above, it suffices to integrate $\phi_1(ir)\phi_2(is)$ against the four expressions

$$\frac{e^{-2\pi(r+s)}}{rs},\ \frac{e^{-2\pi(r+s)}}{(r+s)^2},\ \frac{e^{-2\pi(r+s)}}{r}+\frac{e^{-2\pi(r+s)}}{s},\ \frac{re^{-2\pi(r+s)}}{(r+s)^2}+\frac{se^{-2\pi(r+s)}}{(r+s)^2}, \tag{9.2.4}$$

over $r,s\in(\delta)$ and then multiplying by appropriate constants. The first of these expressions is easily evaluated using

$$\int_{(\delta)}\frac{e^{-2\pi r}}{r}\phi_i(ir)dr = (2\pi i)\left(\frac{\phi_i(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_i}(u)du\right),$$

a result we found in Section 7.2.3, albeit formulated slightly differently. The third expression can be integrated by combining the result above with

$$\int_{(\delta)} e^{-2\pi r}\phi_i(ir)dr = \int_{(0)} e^{-2\pi r}\phi_i(ir)dr = i\int_{\mathbb{R}} e^{-2\pi it}\phi_i(t)dt = \frac{2\pi i}{2\pi}\widehat{\phi}_i(1).$$

The second and fourth expressions are harder to evaluate. We will use a method very similar to the one used in [MS, Ch. 3] to calculate these integrals. We begin with the second expression and write

$$\int_{(\delta)}\int_{(\delta)} \frac{e^{-2\pi(r+s)}}{(r+s)^2}\phi_1(ir)\phi_2(ir)drds = \int_{(\delta)}\int_{(\delta)} \frac{e^{-2\pi(r+s)}}{(r+s)^2}\phi_1(ir)\int_{-\infty}^{\infty}\widehat{\phi_2}(u)e^{2\pi us}dudrds,$$

where we used the inverse Fourier transform. We also replaced u by $-u$, which is allowed as ϕ_2 is even. Next, we note that all integrals converge absolutely, including the integral over s, so that we may use Fubini's theorem to interchange the order of integration. The result is

$$\int_{-\infty}^{\infty}\widehat{\phi_2}(u)\int_{(\delta)}\phi_1(ir)e^{-2\pi r}\int_{(\delta)}\frac{e^{2\pi(u-1)s}}{(r+s)^2}dsdrdu.$$

These integrals are all integrals in the Lebesgue sense, so a set of measure zero is not important. In particular, we will ignore the set where $u=1$. For $u<1$, we may move the innermost integral over s to the line (N), and let $N\to\infty$, so that by using the fast decay of the exponential function here, we see that the inner integral is equal to 0 for such u. If instead $u>1$ we move the integral to the line $(-N)$ and let $N\to\infty$. The shifted integral also vanishes, but we pick up the residue at the point $s=-r$ so that the above is equal to

$$(2\pi i)2\pi\int_1^{\infty}(u-1)\widehat{\phi_2}(u)\int_{(\delta)}\phi_1(ir)e^{-2\pi r}e^{-2\pi(u-1)r}drdu = (2\pi i)^2\int_1^{\infty}(u-1)\widehat{\phi_1}(u)\widehat{\phi_2}(u)du.$$

Finally, we turn to the fourth and last expression in (9.2.4). The reason we have split this expression into two parts is so that the integral over each of these terms converges absolutely. We concentrate on the term $re^{-2\pi(r+s)}(r+s)^{-2}$, as we can find the integral of the other term using symmetry. The procedure is very similar to the calculation we just performed. Indeed, the only difference between the integrands is a factor r, so that we may perform the same steps as above until we arrive at the integral

$$(2\pi i)2\pi\int_1^{\infty}(u-1)\widehat{\phi_2}(u)\int_{(\delta)}\phi_1(ir)re^{-2\pi ru}drdu.$$

We move the inner integral to (0), and let $r=it$ to see that this integral is

$$-\int_{-\infty}^{\infty}t\phi_1(t)e^{-2\pi itu}dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\phi_1(t)\frac{d}{du}\left(e^{-2\pi itu}\right)dt = \frac{1}{2\pi i}\widehat{\phi_1}'(u).$$

Thus, by symmetry

$$\int_{\delta}\int_{\delta}\phi_1(ir)\phi_2(is)\left(\frac{re^{-2\pi(r+s)}}{(r+s)^2}+\frac{se^{-2\pi(r+s)}}{(r+s)^2}\right)drds = 2\pi\int_1^{\infty}(u-1)\frac{d}{du}\left(\widehat{\phi_1}(u)\widehat{\phi_2}(u)\right)du,$$

which we see is equal to

$$-2\pi\int_1^{\infty}\widehat{\phi_1}(u)\widehat{\phi_2}(u)du,$$

after integrating by parts.

Putting our results together, we have shown that up to an error $\ll L^{-2}$, we have

$$\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)R_{2,2,\alpha,\beta}(r,s,r,s;X)drds = \left(\frac{\phi_1(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right)$$
$$\times\left(\frac{\phi_2(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du\right)-4\int_1^{\infty}(u-1)\widehat{\phi_1}(u)\widehat{\phi_2}(u)du+\frac{D_{S_3}^{\pm}}{L}\left[\widehat{\phi_2}(1)\right.$$
$$\left.\times\left(\frac{\phi_1(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right)+\widehat{\phi_1}(1)\left(\frac{\phi_2(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du\right)-4\int_1^{\infty}\widehat{\phi_1}(u)\widehat{\phi_2}(u)du\right],$$

It remains to calculate the integral of $R_{1,2,\alpha,\beta}$ and $R_{2,1,\alpha,\beta}$. Once again, we need not take the integrands involving $F_2(X)$ into account. We would like to treat $R_{1,2}$ and $R_{2,1}$ separately, but recall that there is a possible pole at $r = s$, unless we treat terms involving $(\zeta'/\zeta)(1 \pm (r-s))$ together.

We begin with the terms not involving any logarithmic derivative of ζ, which allows us to consider the terms from $R_{1,2,\alpha,\beta}$ and $R_{2,1,\alpha,\beta}$ separately. We begin by focusing on the terms from $R_{2,1,\alpha,\beta}$. We make the change of variables $s' = Ls/(2\pi)$, $r' = Lr/(2\pi)$, move both contours to (δ), and then rename s' to s and r' to r. We also replace $F_1(X)$ by 1, and absorb the difference in the error term. We should then compute the integral

$$\frac{4}{L^2}\int_{(\delta)}\int_{(\delta)}\phi_1(ir)\phi_2(is)\frac{\Gamma_\pm(1/2-2\pi r/L)}{\Gamma_\pm(1/2+2\pi r/L)}\zeta(1-4\pi r/L) \times \frac{X^{-2\pi r/L}}{1-2\pi r/L}A_{1,\beta}(-2\pi r/L, 2\pi s/L, 2\pi r/L, 2\pi s/L)drds.$$

As before we may use a Taylor expansion to see

$$\frac{4}{L^2}\frac{\Gamma_\pm(1/2-2\pi r/L)}{\Gamma_\pm(1/2+2\pi r/L)}\zeta(1-4\pi r/L)\frac{X^{-2\pi r/L}}{1-2\pi r/L} = -\frac{e^{-2\pi r}}{\pi r L} + \mathcal{O}\left(\frac{1}{L^2}\right).$$

We will also need a zeroth order Taylor expansion of $A_{1,\beta}$, which means we must calculate $A_{1,\beta}(0,0,0,0)$. Now, we have $A_{1,\beta}(0,0,0,0) = A_{3,\alpha}(0,0)$, as $A_1(r,\beta,r,\delta) = A_3(\beta,\delta)$. Furthermore, by the definition (6.3.5) and (6.4.2)

$$A_{3,\alpha}(s,s) = -\sum_p\sum_{e\geq 1}\frac{x_p(\theta_e+1/p)\log p}{p^{e(1/2+s)}} - \frac{\zeta'}{\zeta}(1+2s) = -\sum_p\sum_{e\geq 1}\frac{(x_p(\theta_e+1/p)-\delta_{2|e})\log p}{p^{e(1/2+s)}},$$

which is now valid when $\mathrm{Re}(s) > -1/2$, as $\theta_1 = 0$, $\theta_2 = 1$, $x_p = 1 + \mathcal{O}(p^{-1})$. Setting $s = 0$ and comparing with the left-hand side of (7.2.8) shows that $A_{3,\alpha}(0,0) = -C/2$.

From the calculations above, we see that we need to integrate

$$\frac{C}{L}\int_{(\delta)}\int_{(\delta)}\phi_1(ir)\phi_2(is)\frac{e^{-2\pi r}}{2\pi r}drds = -\frac{C}{L}\widehat{\phi_2}(0)\left(\frac{\phi_1(0)}{2} - \frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(t)dt\right),$$

where the equality follows from our earlier integral calculations in this section. The corresponding term from $R_{1,2,\alpha,\beta}$ can be found using symmetry.

We now continue with the other terms of $R_{2,1,\alpha,\beta}$, i.e. the terms coming from

$$-\frac{\Gamma_\pm(1/2-r)}{\Gamma_\pm(1/2+r)}\zeta(1-2r)\frac{X^{-r}}{1-r}A_3(-r,r)\left(\frac{\zeta'}{\zeta}(1+s-r) + \frac{\zeta'}{\zeta}(1+2s) - \frac{\zeta'}{\zeta}(1+r+s)\right), \tag{9.2.5}$$

where we used $A_1(-r,s,r,s) = A_3(-r,r)$. We will first focus on the term in the middle. Making the usual change of variables in the integral, and Taylor expanding, yields the integral

$$-\frac{2}{L}\int_{(\delta)}\int_{(\delta)}\phi_1(ir)\phi_2(is)\frac{e^{-2\pi r}}{2\pi r}\left(1+\frac{2\pi D^{\pm}_{S_3} r}{L}\right)\frac{\zeta'}{\zeta}(1+4\pi s/L)drds.$$

Next, by (7.2.4), we have

$$\frac{\zeta'}{\zeta}(1+z) = -\frac{1}{z} + \gamma + \mathcal{O}(|z|).$$

Thus, the integral in question is

$$\int_{(\delta)}\int_{(\delta)}\phi_1(ir)\phi_2(is)\frac{e^{-2\pi r}}{(2\pi)^2 rs}\left(1+\frac{2\pi D^{\pm}_{S_3} r}{L} - \frac{4\pi\gamma s}{L}\right)drds. \tag{9.2.6}$$

This can be computed by using our previous calculations, and the result

$$\int_{(\delta)} \frac{\phi_2(is)}{s} ds = \frac{2\pi i}{2}\phi_2(0),$$

which can be shown by a calculation similar to the calculation of one of the integrals in Section 7.2.3, where we had an additional factor $e^{-2\pi s}$. We conclude that (9.2.6) equals

$$-\frac{\phi_2(0)}{2}\left(\frac{\phi_1(0)}{2} - \frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right) + \frac{1}{L}\left(2\gamma\widehat{\phi_2}(0)\left(\frac{\phi_1(0)}{2} - \frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right) - D_{S_3}^{\pm}\frac{\phi_2(0)}{2}\widehat{\phi_1}(1)\right).$$

It remains to integrate the rest of the terms in (9.2.5) against ϕ_1 and ϕ_2. Here, we must add the corresponding terms from $R_{1,2,\alpha,\beta}$ and integrate these at the same time, as otherwise there is a pole at $r = s$. We then make the usual change of variables and shift the integral over r to $(\delta/2)$ and the integral over s to (δ). We point out that if the same contour had been chosen, then it would not have been possible to control the error in the Taylor expansion, as it would not have been possible to bound $|r - s|$ from below.

Taylor expanding as before shows that we need to evaluate

$$-\frac{2}{L}\int_{(\delta/2)}\int_{(\delta)} \phi_1(ir)\phi_2(is)\left[\frac{e^{-2\pi r}}{2\pi r}\left(1 + \frac{2\pi D_{S_3}^{\pm} r}{L}\right)\left(\frac{\zeta'}{\zeta}(1 + 2\pi(s-r)/L) - \frac{\zeta'}{\zeta}(1 + 2\pi(r+s)/L)\right)\right.$$
$$\left. + \frac{e^{-2\pi s}}{2\pi s}\left(1 + \frac{2\pi D_{S_3}^{\pm} s}{L}\right)\left(\frac{\zeta'}{\zeta}(1 + 2\pi(r-s)/L) - \frac{\zeta'}{\zeta}(1 + 2\pi(r+s)/L)\right)\right] dsdr.$$

Also, by the Laurent expansion of ζ'/ζ that we found above, we have

$$\frac{\zeta'}{\zeta}(1 + 2\pi(s-r)/L) - \frac{\zeta'}{\zeta}(1 + 2\pi(r+s)/L) = -\frac{2rL}{2\pi(s-r)(r+s)}\left(1 + \mathcal{O}_\delta\left(\frac{Q_1(|r|,|s|)}{L^2}\right)\right),$$

for some polynomial Q_1. Hence, the integral under consideration is, up to the usual error term, equal to

$$2\int_{(\delta/2)}\int_{(\delta)} \phi_1(ir)\phi_2(is)\frac{2}{(2\pi)^2(s-r)(s+r)}\left[e^{-2\pi r}\left(1 + \frac{2\pi D_{S_3}^{\pm} r}{L}\right) - e^{-2\pi s}\left(1 + \frac{2\pi D_{S_3}^{\pm} s}{L}\right)\right] dsdr. \tag{9.2.7}$$

We first calculate the main term coming from the expression above, i.e. the integral

$$\frac{4}{(2\pi)^2}\int_{(\delta/2)}\int_{(\delta)} \phi_1(ir)\phi_2(is)\frac{\left(e^{-2\pi r} - e^{-2\pi s}\right)}{(s-r)(s+r)} dsdr.$$

We begin by expanding $\phi_1(ir)$ into an integral involving its Fourier transform, and interchange the order of integration to find

$$\frac{4}{(2\pi)^2}\int_{-\infty}^{\infty}\widehat{\phi_1}(u)\int_{(\delta)}\int_{(\delta/2)} \phi_2(is)\frac{\left(e^{2\pi r(u-1)} - e^{2\pi ru - 2\pi s}\right)}{(s-r)(s+r)} drdsdu. \tag{9.2.8}$$

For $u < 0$, we may move the integral over r very far to the right (i.e. to the line (N), and then let $N \to \infty$), which makes the shifted integral vanish. The contour for r lies to the left of the contour for s, so there is a possible pole where $r = s$, but the residue here is 0, so this is in fact not a pole, whence we can discard all contributions from $u < 0$.

For $u > 1$, we instead move the integral over r very far to the left. We pick up the residue at $r = -s$, which then gives the contribution

$$\frac{2i}{(2\pi)}\int_{1}^{\infty}\widehat{\phi_1}(u)\int_{(\delta)} \phi_2(is)\frac{\left(e^{-2\pi s(u-1)} - e^{-2\pi s(u+1)}\right)}{s} dsdu.$$

As there is no pole at $s=0$, we can move the integral over s to the imaginary axis and write $s=it$, and use that ϕ_2 is even, whence the above equals

$$\begin{aligned}2\int_1^\infty \widehat{\phi_1}(u)\int_{-\infty}^\infty \phi_2(t)\left(\frac{\sin(2\pi t(u-1))}{2\pi t}-\frac{\sin(2\pi t(u+1))}{2\pi t}\right)dtdu\\ =-2\int_1^\infty \widehat{\phi_1}(u)\int_{u-1}^{u+1}\widehat{\phi_2}(t)dtdu,\end{aligned}$$

where we used Plancherel's theorem and the scaling property of the Fourier transform.

We have yet to analyse (9.2.8) when $0<u<1$. We handle the two terms separately. First, shift the contour over r far to the right to find

$$\frac{4}{(2\pi)^2}\int_0^1 \widehat{\phi_1}(u)\int_{(\delta)}\int_{(\delta/2)}\frac{\phi_2(is)e^{2\pi r(u-1)}}{(s-r)(s+r)}drdsdu=\frac{2i}{2\pi}\int_0^1\widehat{\phi_1}(u)\int_{(\delta)}\phi_2(is)\frac{e^{2\pi s(u-1)}}{s}dsdu,$$

where we picked up the negative of the residue at $r=s$, as we shifted the contour to the right. Next, by instead shifting to the far left we see that

$$\frac{-4}{(2\pi)^2}\int_0^1 \widehat{\phi_1}(u)\int_{(\delta)}\int_{(\delta/2)}\frac{\phi_2(is)e^{2\pi(ru-s)}}{(s-r)(s+r)}drdsdu=\frac{-2i}{2\pi}\int_0^1\widehat{\phi_1}(u)\int_{(\delta)}\frac{\phi_2(is)e^{-2\pi s(u+1)}}{s}dsdu,$$

after picking up the residue at $r=-s$. We add together both of these integrals, shift the contour to (0), and apply Plancherel's formula to see that their sum is equal to

$$2\int_0^1\widehat{\phi_1}(u)\int_0^{1-u}\widehat{\phi_2}(t)dtdu-2\int_0^1\widehat{\phi_1}(u)\int_0^{u+1}\widehat{\phi_2}(t)dtdu.$$

Thus, after simplifying, we see that the terms of constant size coming from (9.2.7) is

$$4\int_0^1\widehat{\phi_1}(u)\int_0^{1-u}\widehat{\phi_2}(t)dtdu-2\int_0^\infty\widehat{\phi_1}(u)\int_{u-1}^{u+1}\widehat{\phi_2}(t)dtdu.$$

We turn to the rest of (9.2.7), i.e.

$$\frac{4D^{\pm}_{S_3}}{2\pi L}\int_{(\delta/2)}\int_{(\delta)}\phi_1(ir)\phi_2(is)\frac{re^{-2\pi r}-se^{-2\pi s}}{(s-r)(s+r)}dsdr. \tag{9.2.9}$$

We will need to treat each term separately in order to make use of absolute convergence.

To calculate the integral over the first term, we expand ϕ_2 into its Fourier transform and find that this integral equals

$$\frac{4D^{\pm}_{S_3}}{2\pi L}\int_{-\infty}^\infty\widehat{\phi_2}(u)\int_{(\delta/2)}\int_{(\delta)}\phi_1(ir)\frac{re^{2\pi(su-r)}}{(s-r)(s+r)}dsdrdu.$$

If $u<0$, we can shift the integral over s to the right, to see that it is zero. We encounter no poles, as the s-contour lies to the right of the r-contour. For $u>0$, we instead shift the contour to the left picking up the poles at $s=r$ and $s=-r$, which leaves us with

$$\frac{2iD^{\pm}_{S_3}}{L}\int_0^\infty\widehat{\phi_2}(u)\int_{(\delta/2)}\phi_1(ir)\left(e^{2\pi r(u-1)}-e^{-2\pi r(u+1)}\right)drdu.$$

By shifting the contour to (0), we see that this is simply

$$-\frac{2D^{\pm}_{S_3}}{L}\int_0^\infty\widehat{\phi_2}(u)\left(\widehat{\phi_1}(u-1)-\widehat{\phi_1}(u+1)\right)du.$$

Now we calculate the integral over the other term in (9.2.9) by expanding ϕ_1 into its Fourier transform. We get

$$-\frac{4D^{\pm}_{S_3}}{2\pi L}\int_{-\infty}^\infty\widehat{\phi_1}(u)\int_{(\delta)}\int_{(\delta/2)}\phi_2(is)\frac{se^{2\pi(ru-s)}}{(s-r)(s+r)}drdsdu. \tag{9.2.10}$$

For $u<0$, we shift the integral over r to the right, and pick up the negative of the residue at $r=s$, while for $u>0$, we shift the integral to the left and pick up the residue at $r=-s$. In total, we obtain

$$-\frac{2iD^{\pm}_{S_3}}{L}\int_{-\infty}^{0}\widehat{\phi_1}(u)\int_{(\delta)}\phi_2(is)e^{2\pi s(u-1)}dsdu-\frac{2iD^{\pm}_{S_3}}{L}\int_{0}^{\infty}\widehat{\phi_1}(u)\int_{(\delta)}\phi_2(is)e^{-2\pi s(u+1)}dsdu,$$

which we can simplify further by moving the contour to (0). We obtain

$$\frac{2D^{\pm}_{S_3}}{L}\int_{-\infty}^{0}\widehat{\phi_1}(u)\widehat{\phi_2}(u-1)du+\frac{2D^{\pm}_{S_3}}{L}\int_{0}^{\infty}\widehat{\phi_1}(u)\widehat{\phi_2}(u+1)du.$$

Adding together the results above, and using that both Fourier transforms are even, we see that (9.2.9) is equal to

$$\frac{2D^{\pm}_{S_3}}{L}(\widehat{\phi_1}*\widehat{\phi_2})(1)-\frac{4D^{\pm}_{S_3}}{L}\int_{0}^{1}\widehat{\phi_1}(u)\widehat{\phi_2}(1-u)du,$$

where $*$ denotes the convolution operator.

With all integral calculations finished, we may conclude that

$$\begin{aligned}&\frac{4}{(2\pi i)^2}\int_{(1/L)}\int_{(1/L)}\phi_1\left(\frac{Lr}{2\pi i}\right)\phi_2\left(\frac{Ls}{2\pi i}\right)\big(R_{2,1,\alpha,\beta}(r,s,r,s;X)+R_{1,2,\alpha,\beta}(r,s,r,s;X)\big)drds\\
&=-\frac{C}{L}\widehat{\phi_2}(0)\left(\frac{\phi_1(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(t)dt\right)-\frac{C}{L}\widehat{\phi_1}(0)\left(\frac{\phi_2(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(t)dt\right)\\
&-\frac{\phi_2(0)}{2}\left(\frac{\phi_1(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right)+\frac{1}{L}\left[2\gamma\widehat{\phi_2}(0)\left(\frac{\phi_1(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_1}(u)du\right)-D^{\pm}_{S_3}\frac{\phi_2(0)}{2}\widehat{\phi_1}(1)\right]\\
&-\frac{\phi_1(0)}{2}\left(\frac{\phi_2(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du\right)+\frac{1}{L}\left[2\gamma\widehat{\phi_1}(0)\left(\frac{\phi_2(0)}{2}-\frac{1}{2}\int_{-1}^{1}\widehat{\phi_2}(u)du\right)-D^{\pm}_{S_3}\frac{\phi_1(0)}{2}\widehat{\phi_2}(1)\right]\\
&+4\int_{0}^{1}\widehat{\phi_1}(u)\int_{0}^{1-u}\widehat{\phi_2}(t)dtdu-\int_{-1}^{1}\widehat{\phi_1\phi_2}(u)du\\
&+\frac{2D^{\pm}_{S_3}}{L}(\widehat{\phi_1}*\widehat{\phi_2})(1)-\frac{4D^{\pm}_{S_3}}{L}\int_{0}^{1}\widehat{\phi_1}(u)\widehat{\phi_2}(1-u)du+\mathcal{O}\left(\frac{1}{L^2}\right).\end{aligned}$$

Here, we used the identity

$$2\int_{0}^{\infty}\widehat{\phi_1}(u)\int_{u-1}^{u+1}\widehat{\phi_2}(t)dtdu=\int_{-1}^{1}\widehat{\phi_1\phi_2}(u)du,$$

which follows by applying Plancherel's theorem twice, using the fact that a convolution of even functions is even, and

$$\int_{u-1}^{u+1}\widehat{\phi_2}(t)dt=(\widehat{\phi_2}*\chi_{[-1,1]})(u),$$

where χ_A is the indicator function of the set A. Theorem 9.3 now follows from combining all results of this section.

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A

Infinite products, the gamma function and integration

We present a few results that are used throughout the report, usually without proofs.

A.1 Infinite products

We give a brief description of infinite products, following [A, Ch. 5.2.2]. These products are frequently used throughout the report.

Let a_n be complex numbers, whose magnitude is less than 1/2, say for large enough n. We are then interested in the infinite product

$$\prod_{n=1}^{\infty}(1+a_n). \tag{A.1.1}$$

To give this expression meaning, we consider the Nth partial product P_N and say that the infinite product converges to $P \neq 0$ if the partial products converge to P. We will often want to take logarithms of infinite products, and this is the reason for excluding the value $P = 0$. In this direction, we have the following result, which is essentially [A, Thm. 5.5].

Theorem A.1. *Let a_n be complex numbers such that $|a_n| < 1/2$ for all n. The infinite product*

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if and only if the series

$$\sum_{n=1}^{\infty}\log(1+a_n)$$

converges, where log *denotes the principal branch of the logarithm. Further, if the series converges to S, then the product converges to e^S.*

We say that an infinite product is absolutely convergent if the product obtained by replacing a_n by $|a_n|$ is convergent. We have a very simple criterion for determining absolute convergence, as a consequence of the theorem above [A, Thm. 5.6].

Theorem A.2. *Let a_n be complex numbers with $|a_n| < 1/2$ for all n. Then, the infinite product* (A.1.1) *converges absolutely if and only if the sum*

$$\sum_{n=1}^{n}|a_n|$$

converges.

In particular, by combining these theorems with a Taylor expansion, we see that an absolutely convergent product is convergent. We remark that if a_n is large in magnitude for small n, then we can simply exclude these terms when treating convergence.

We are mostly interested in the case when the coefficients a_n depend on some complex variable s, and we then write $a_n(s)$. In particular, if the $a_n(s)$ are holomorphic functions, we are interested in knowing whether the function defined by the infinite product (A.1.1) is also holomorphic. A well-known result from complex analysis asserts that this is the case if the convergence of the partial products is locally uniform, or equivalently uniform on compact sets. Both theorems above can be extended to relate the uniform convergence of a product to the uniform convergence of a series, but we leave out the details. See the remarks after [A, Thm. 5.6].

A.2 The Gamma function

Now that we have introduced infinite products, we are ready to define the Gamma function $\Gamma(s)$, following [A, Ch. 5.2.4]. We define

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n},$$

where γ is Euler's constant. A Taylor expansion, combined with the mentioned extensions of the theorems above, shows that the product is absolutely uniformly convergent on compact subsets of $\mathbb{C} \setminus \{0, -1, -2, ...\}$. In particular, $\Gamma(s)$ is a meromorphic function, with poles precisely at the non-positive integers, as $(1 + s/n)$ is only zero on the negative integers. Furthermore, $\Gamma(s)$ has no zeros. The gamma function is related to the factorial function by the relation $\Gamma(n+1) = n!$ for integers $n \geq 0$.

We now list several important properties of the Gamma function. First, we have the relation $s\Gamma(s) = \Gamma(s+1)$, which is an extension of the relation $n! = n \cdot (n-1)!$. Moreover, we have the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

and the duplication formula

$$\Gamma(s)\Gamma\left(s + \tfrac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

Lastly, we will need a growth estimate for $\Gamma(s)$ given by Stirling's formula. We use the form from [D3, Ch. 10], namely

$$\log \Gamma(s) = \left(s - \tfrac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \mathcal{O}_\delta\left(|s|^{-1}\right),$$

for s with $-\pi + \delta < \arg s < \pi - \delta$, with $\delta > 0$. Here $\log \Gamma(s)$ denotes a fixed logarithm of $\Gamma(s)$, and $\log s$ is the principal branch of the logarithm applied to s. Fixing $\delta < \pi/2$ gives an estimate that is valid in the right half-plane for $|\Gamma(s)|$. In particular, we have the equality

$$|\Gamma(s)| = \sqrt{2\pi}|s|^{\mathrm{Re}(s)-1/2} e^{-\mathrm{Re}(s)} e^{-\mathrm{Im}(s)\mathrm{Arg}(s)} \left(1 + \mathcal{O}\left(|s|^{-1}\right)\right),$$

which is useful when estimating an L-function through its functional equation. Note in particular that $\Gamma(s)$ decays fast in any vertical strip, as $\mathrm{Im}(s)$ and $\mathrm{Arg}(s)$ have the same sign.

A.3 Stieltjes integration

We now introduce Stieltjes integration, essentially following [Rd, Ch. 6], which is an invaluable tool for studying sums. One may consider the quite general Lebesgue-Stieltjes integral, but for our purposes, the Riemann-Stieltjes integral will be enough.

Let f be a monotonically increasing function, and let g be a function bounded on $[a, b]$, with only finitely many discontinuities, none of which coincides with the discontinuities of f. Then, it is possible to define the Riemann-Stieltjes integral

$$\int_a^b g(x)df(x).$$

We will not write out the construction here, but it is defined very similarly to the usual Riemann integral, by considering partitions $a = a_0 < ... < a_n = b$. Instead of using Riemann sums where the values of g are weighed against the lengths of the subintervals, one weighs the values of g against the differences $f(a_{i+1}) - f(a_i)$. We remark that the definition can readily be extended to functions f of bounded variation, by using linearity, as such functions are differences of non-decreasing functions.

The first important equality we will need is that

$$\int_a^b g(x)df(x) = \int_a^b g(x)f'(x)dx \tag{A.3.1}$$

if f is continuously differentiable.

The reason we are interested in these integrals is because of their connections to sums. Let a_n be a sequence of real numbers, and define the function

$$f(x) = \sum_{1\le n\le x} a_n.$$

Then, assuming that g is continuous, say, it should be intuitively clear from the definition that we have

$$\sum_{a<n\le x} a_n g(n) = \int_a^x g(x)df(x),$$

as the function f changes value precisely at the integers. If g is in addition continuously differentiable, we have the summation by parts formula

$$\sum_{a<n\le t} a_n f(n) = \int_a^t g(x)df(x) = f(t)g(t) - f(a)g(a) - \int_a^t g'(x)f(x)dx.$$

That the left-hand side is equal to the right-hand side can actually be proven quite easily without any Stieltjes integration, by using that f is piecewise constant.

Lastly, we mention that the more general formula for integration by parts

$$\int_a^t f(x)dg(x) = f(t)g(t) - f(a)g(a) - \int_a^t g(x)df(x),$$

holds for a wider class of functions than the ones above. However, for our purposes, we will only need to use it in the case described above, or when both f and g are continuously differentiable. The latter case reduces to the familiar formula for integration by parts by using (A.3.1). The point of using Stieltjes integration at all is that it provides a convenient formalism for manipulating the various sums we encounter throughout the report.