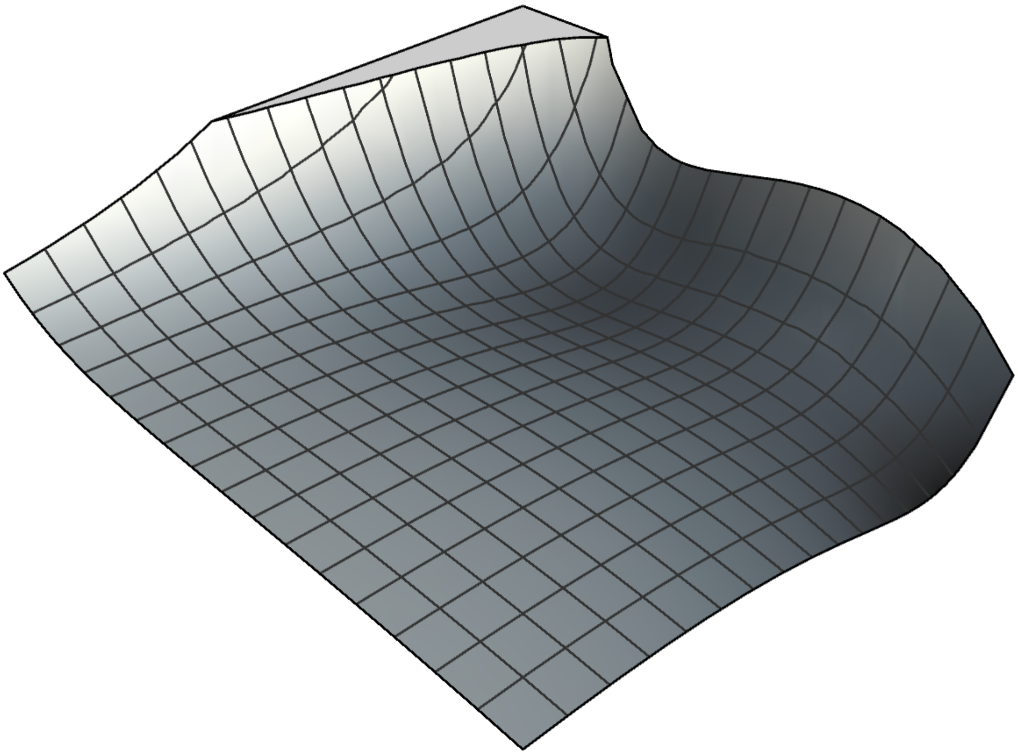

Scalar Potential of the Squashed Seven-sphere

Stability, the Swampland and Kähler Geometry

Master's thesis in Physics and Astronomy

Sebastian Bergström

Department of Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2019

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Cover: Effective 4D potential obtained by compactifying 11D supergravity on a squashed seven-sphere.

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Abstract

Finding a consistent theory of quantum gravity has been a long-standing problem in physics and all attempts made share a common feature: quantum gravitational effects become important only at very high energies, much higher than the energy scales at which current particle physics are done. Thus, connecting a particle phenomenological description to a theory of quantum gravity has proven to be incredibly difficult. A recently proposed solution to this is known as the *Swampland program* which aims to construct a set of conjectures dictating how theories consistent with quantum gravity must behave. Theories not fulfilling these conjectures are said to lie in the "swampland", while consistent theories are said to lie in the "landscape".

In this thesis the compactification of 11-dimensional supergravity on a squashed seven-sphere is studied. This scenario seems to contradict the *Non-AdS SUSY conjecture*, at least at a perturbative level, and a better of understanding of it is therefore essential for the swampland program. A thorough description of the geometry of a squashed seven-sphere is provided, which then is extended to spacetime dependent parameters of the sphere described by scalar fields. The stability in terms of these scalars is analysed, which leads to the contradiction mentioned above. Finally, an attempt to generalise the setup using complex geometry is done, treating the scalar fields as coordinates on a Kähler manifold. It is found that a simple and natural coset structure for the Kähler manifold, $SL(2,\mathbb{R})/U(1)$, might not be the correct ansatz. Hence, further studies of the full 11-dimensional theory would be beneficial in order to determine the true coset structure.

Keywords: quantum gravity, supergravity, the swampland, Kähler geometry, the seven-sphere.

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Sebastian Bergström, Gothenburg, June 2019

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1

Introduction

1.1 Quantum Gravity

What have been the greatest accomplishments of physics in the last century? Any physicist nowadays would probably give you the same answer to this question: quantum mechanics and Einstein's general theory of relativity. Out of these revolutionary theories in the beginning of the twentieth century, modern physics as we know it today grew rapidly. General relativity has given us e.g. GPS localization and a far better understanding of the dynamics of galaxies. On the other hand, lasers, modern computers and a plethora of things in our daily lives utilize quantum physics.

While the two theories have been very successful in their respective regimes, they have vastly different foundations. The computational framework of quantum physics, that is quantum field theory which is the basis for the Standard Model, is considered one of the most successful theories of all times. Quantum field theory accurately describes three of the fundamental forces in nature: the electromagnetic force, the weak nuclear force and the strong nuclear force. In this framework, a force is described by individual force-carrying particles[1] interacting with other physical particles. This means that when two particles of equal charge repel each other (two electrons, for example), quantum field theory describes the process as one of the particles emitting a photon - the quanta of the electromagnetic force - which then gets absorbed by the other particle, transferring momentum and thus causing the repulsion. This description of how forces act is not applicable to our macroscopic world, but it turns it that it is an extremely good description of subatomic phenomena. By the virtue of quantum field theory, physicists have been able to predict the properties of the electron to very high precision [1].

General relativity on the other hand is based on the geometry of spacetime. Here, the gravitational forces are not carried by particles, but is just an elegant consequence of the idea that massive objects distort spacetime and give rise to the notion of curvature in the nearby geometry. Objects in a region where space and time are curved follow the shortest path in spacetime, which manifests itself as a force to us humans constantly ”pulling” us towards a nearby heavy object[2]. This theory has also been thoroughly tested, in its early days through observations of Mercury's orbit as well as through very recent discoveries, such as the observation of gravitational waves, awarded with the Nobel prize in 2017 [2, 3].

[1] Also called quanta, since they carry ”quantized” chunks of energy. This is the etymology behind the name quantum mechanics.

[2] The widely used analogy of putting some heavy object in the middle of a trampoline and then observing how smaller objects roll towards it due to the curved surface is helpful here.

However, despite the numerous successes of these theories over the years, physicists have not succeeded in unifying these two theories in a common framework. Since three out of four forces are described by quantum mechanics, the obvious path forward would be to also quantize gravity via the introduction of a force carrying particle called the graviton. However, in quantum field theory the graviton has to be put in by hand and no matter how one tries to do it this causes the theory to break down. As of today, the most promising candidate for a quantum theory of gravity is instead string theory. In string theory, the proposed graviton is a consequence of the assumption that the most fundamental objects are not pointlike particles, but rather one-dimensional strings. All particle physics phenomena we observe in our experiments happen at low energies compared to the ”natural” energy scale in string theory, which is close to the Planck energy at $\approx 10^{19}$ GeV (see below). At low energies, the strings are very short and thus manifest themselves as pointlike particles when we study them.

While all of the above sounds promising, no one has yet been able to construct a theory describing our universe starting from string theory. Therefore, whether string theory is *the* correct description of quantum gravity or not is still up for debate. Among the biggest problems lie the prediction of extra dimensions, which we will discuss in the coming sections.

One might at this point stop and ask why physicists are so keen on finding a single theory of everything that explains all observed phenomena at the most fundamental level. Could it not simply be that we need different descriptions of reality at different scales? While there are historical situations where unification has been very successful, such as when the relationship between electricity and magnetism was discovered and gave us electromagnetism, the reason for a quantum theory of gravity is actually more fundamental. Physics is really all about model building, where we try to describe the world we live in using mathematical equations. Thus, if there exists a situation where quantum physics and gravity simultaneously are relevant, then we must have a model of quantum gravity. And in fact there is such a situation, namely in the vicinity of a black hole. There, the force of gravity is so strong that it is comparable to the the other three forces. Thanks to recent observations, both by the LIGO collaboration but also more recently the Event Horizon Telescope, we now also have compelling observational evidence of the existence of black holes [3, 4]. So, quantum gravity is not needed because it would make our equations look neat, but rather because it lies in the very core of physics to find a description of all phenomena that take place in our universe.

1.2 Energy Scales and Realistic Models

Before continuing with how one could construct quantum gravity, let us stop for a moment and consider the energy scales relevant for the physics currently under discussion. Quantum gravity is believed by most to be important only at very high energies, many orders of magnitude higher than the particle physicists of today can ever hope to achieve. For a ballpark estimation, one usually talks about the *Planck energy* , which is the energy scale derived by combining fundamental constants of the universe in order to obtain something with dimension ”energy”

$$E_{\text{Planck}} = \sqrt{\frac{\hbar c^5}{G}} \approx 10^{19}\,\text{GeV} \approx 10^9\,\text{joule} \tag{1.1}$$

where $\hbar$ is the reduced Planck's constant, c the speed of light, and G Newton's gravitational constant. Note that in natural units, energy is equivalent to mass, so this is the same as the Planck mass, M_p. One immediately sees that this energy is a huge number even in macroscopic terms, and enormous for particle physics. As a comparison, the current beam energy of the LHC is at around 10^4 GeV [5].

So given a theory of quantum gravity, such as string theory, it is in the low-energy limit where the physics we can probe happens. In this limit, the theory takes the form of a quantum field theory and we know well how to handle it. The theory will contain a graviton which might look problematic since it is non-renormalizable, but knowing that it is the limit of a consistent theory at high energies, this is no longer a problem. Field theories of this type, that is, those derived from a consistent quantum gravity theory at high energies, are said to be *UV complete*[3] .

There is a caveat here. When one tries to construct realistic models, the usual procedure is to construct a candidate field theory given some experimental constraints. After all, the physics we probe only give information regarding the low energy behaviour, so it is natural to work this way [6]. However, in order to determine if this candidate field theory is correct, it must be UV complete. And while working down from consistent high-energy theories ensures that we arrive at UV complete field theories, there is no clear way to deduce if a specific field theory of quantum gravity is UV complete. This dilemma is one of the main points in this thesis, and we will return to it at the end of this chapter. Now, we will instead turn to string theory, the challenges it brings and how one constructs field theories from it.

1.3 String Theory

Since the advent of string theory as a theory of quantum gravity in 1970's, there have been numerous efforts to connect the theoretical advancements to realistic models at low energies. The major hurdle in this procedure is the fact that string theory is only consistent in 10 dimensions. At first glance this might look like evidence for string theory being wrong, but upon further examination one realises this is not necessarily the case. The arguments for string theory, such as naturally containing gravity, are very compelling and there are realistic ways in which one can handle the extra dimensions. The procedure is called *compactification* . The basic idea is that if there are extra dimensions in our world, we would not notice their presence as long as they are compact and small enough. So, starting from a higher dimensional theory, we separate it into one part that is the universe we live in, and one part consisting of a small compact manifold that we do not perceive as part of our spacetime.

It then becomes a question what geometries these compact dimensions have. Are they curled up as tiny circles, or are they combined into more complicated geometrical objects? This is where the problems start. Even if the compact dimensions don't appear as spacetime dimensions to us, their structure is still relevant and determine properties of the particles we observe in the four-dimensional spacetime. It turns out that there is an enormous amount of possible string theory compactifications[4] and that there is no clear way of immediately figuring out which alternatives produce realistic universes. This fact

[3]UV = Ultra-Violet ≈ short wavelength ≈ high energies. It is simply a notion of high energy, and has nothing to do with the exact range of wavelengths where light is called ultra-violet light.

[4]Numbers between 10^{500} and 10^{1500} have been proposed.

is even more remarkable considering that all current variants of string theory are unique. This problem is what lies behind the commonly heard statement that string theory is unable to make predictions. We will soon address this issue and explain how the predictive power of string theory may have been underestimated. First, however, we will touch on yet another issue that comes with string theory compactifications.

When performing the compactification, we must also address the question of the geometry of the the four non-compact dimensions which are supposed to describe our four-dimensional spacetime. The vacuum of any theory must always contain a high degree of symmetry and we thus assume the vacuum spacetime to be maximally symmetric. This leads to three different choices of geometry: de Sitter (dS), Minkowski or Anti-de Sitter (AdS), characterised by positive, zero and negative curvature, respectively. It turns out that compactifying to AdS yields consistent theories, while dS vacua are very difficult to construct via string theory. In the early times of string theory research this constituted no major problem, but today there are a number of experiments showing that our universe has a small positive cosmological constant [7, 8]. This would imply dS geometry. Thus, we are faced with yet another problem. How do we construct dS solutions resembling our universe?

Before continuing, let us summarise the procedure used to construct four-dimensional models from string theory. First, one goes to the low-energy limit as described above. Then, the extra dimensions must be taken as compact, with some geometry and topology that yields the desired particle properties in four-dimensional spacetime. It is worth mentioning here that the process of compactification is also affected by working in the low-energy limit, but at a different scale. More specifically, one of the properties that is determined by the geometry of the compact dimensions is the particle masses. This means that even though we have already thrown away all massive modes when going from string theory to a low-energy theory, the four-dimensional particle spectrum obtained from compactification will contain massive states whose masses depends on the compact dimensions.

In the Standard Model, all particles we observe have zero fundamental mass, and instead obtain their masses from the Higgs mechanism [9]. The reasoning behind this is that any possible fundamentally massive states lie above the energy scales that we can probe. Thus, the most common approach is to throw away massive modes also after performing the compactification which is known as dimensional reduction. This then increases the difficulty we mentioned in the last section, namely that it is very hard to figure out whether a given field theory in at low energies in four dimensions has a UV completion or not.

In the next section, we will discuss some recently proposed solutions to this difficulty. First however, we will zoom out and look at the bigger picture of string theory. When string theory was developed during the end of the 20th century, physicists discovered that there were actually five different ways of formulating the theory. It later became clear that the seemingly independent theories were all related to each other via duality transformations. Furthermore it was found that they could all be related to a unique 11-dimensional theory called *M-theory*. The low-energy limit of M-theory is called 11-dimensional supergravity, which is a field theory that has been well understood since the 1980's. It will be this theory that is the focus of this thesis. The relations between all theories are represented schematically in figure 1.1.

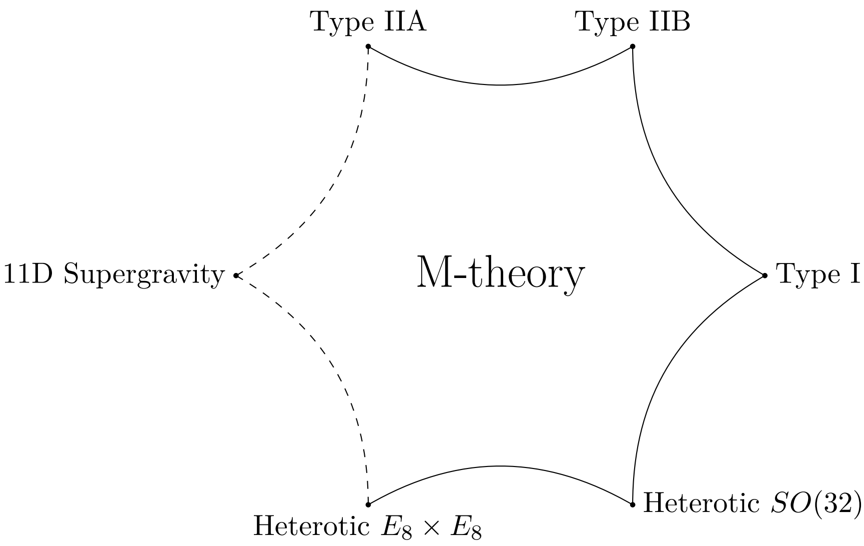


Figure 1.1: Schematic representation how all string theories are related by a set of dualities. Each solid line represents a duality transformation, while dashed lines represent compactifications.

1.4 The Swampland Program

The vast number of possible of compactifications mentioned above has given rise to the terms "landscape" and "swampland" [10]. Compactifications which give rise to consistent, UV complete, field theories are said to lie in the landscape while field theories that cannot be derived from quantum gravity are in the swampland. This leads to the very relevant question: are there theories with dS geometry that belong to the landscape? This question turns out to be very hard to answer. However, recently a number of conjectures were proposed which determine whether the vacuum of a theory is in the swampland or not. These conjectures have been formulated based on current knowledge of the behaviour of string theory compactifications in combination with ideas from black hole physics [11]. We will briefly go through some of these conjectures below, in order to get an idea about their general structure.

One conjecture is the *de-Sitter swampland conjecture* , which states that for any effective field theory obtained from string theory we must have, with some constant $c > 0$ of $\mathcal{O}(1)$

$$|\nabla V| \geq \frac{c}{M_p} V. \tag{1.2}$$

for the scalar potential V [12]. This bound excludes any stable, or meta-stable, dS vacua from the landscape. By (meta-)stable vacua we mean those where the potential is at an (local) absolute minimum, i.e. where $|\nabla V| = 0$. If the above conjecture is true this never happens for positive V. There is a modification to this conjecture known as the *refined de-Sitter conjecture* [13]. It states that either the bound in (1.2) holds or, with some constant $c' > 0$ of $\mathcal{O}(1)$

$$\min(\nabla_i \nabla_j V) \leq -\frac{c'}{M_p^2} V. \tag{1.3}$$

The indices i and j here indicate all combinations of derivatives with respect to the scalar fields in the potential. This condition then allows for dS extrema of the potential, but only if the second derivative is sufficiently negative. Thus, it still does not allow for stable de-Sitter solutions. Both of the above conjectures are automatically fulfilled in AdS-space. There is however another very recent conjecture proposed in [14], called the *Strong scalar weak gravity conjecture* which states that

$$\frac{2(V''')^2}{V''} - V'''' \geq \frac{V''}{M_p^2} \tag{1.4}$$

This conjecture is not yet formulated in terms of multiple scalar fields, thus the prime means derivatives with respect to the single scalar field. This constraint is quite interesting, as it does not depend on the potential directly and thus should be unaffected of the sign of the potential at any extrema. That is, not distinguishing between dS and AdS solutions.

Lastly, we also want to point out the the *Non-SUSY AdS conjecture* [15]. It states that any non-supersymmetric AdS theory must exhibit instability. Initially, this seems to not be very relevant to real world physics, as it does not put any constraints on de-Sitter solutions. However, the conjecture is actually a consequence of the *Weak gravity conjecture*, proposed first in [16]. The weak gravity conjecture is a much more general statement and as such it is of interest to investigate the consequence even in AdS space. We will get back to the Non-SUSY AdS conjecture in the later chapters of this thesis.

1.5 Aim

The aim of this thesis is to study the compactification of 11-dimensional supergravity to four dimensions. Specifically, we will consider first the compactification to a static squashed seven-sphere and then promote it to a dynamic one over spacetime. This will allow for analysis of the emergent four-dimensional scalar potential. The final goal is then to study the potential using complex fields, which in the future could lead to a better understanding of the stability of the solutions.

1.6 Outline

The outline of this thesis is as follows.

In Chapter 2 we will present supersymmetry and 11-dimensional supergravity together with its connections to M-theory and string theory. We will explicitly check that the SUSY variations of the theory is consistent.

Chapter 3 starts with an introduction to Freund-Rubin compactifications, followed by a geometrical discussion of the squashed seven-sphere. We end by computing the two static (Einstein metric) squashed versions of the seven-sphere.

Chapter 4 generalises the previous results to incorporate the possibility of spacetime dependent parameters for the sphere. This is necessary in order to study the stability of the solutions under spacetime variations, an analysis that will conclude this chapter.

Chapter 5 is dedicated to generalising to the previous analysis by introducing new pseudoscalar fields and further investigate the issue of stability. Kähler geometry will be discussed followed by treating the potential as a Kähler potential and extending the fields to complex fields.

Chapter 6 summarises the conclusions of the thesis, and provides some potential outlooks.

2

Supersymmetry and Supergravity

The very first step towards the final goal of this thesis will be to establish and understand 11-dimensional supergravity, the low-energy limit of M-theory that we later on will compactify to four dimensions and connect to the string theory landscape. We start this chapter by a general introduction to the general idea behind supersymmetry. Then, we proceed by introducing gravity and present the equations for 11-dimensional supergravity. In order to familiarise ourselves with the theory we also perform an explicit check of the supersymmetry invariance of the theory.

In the last part of the chapter we will also introduce compactification in some more detail, before specialising to the Freund-Rubin ansatz in the next chapter.

2.1 Supersymmetry

Supersymmetry (or SUSY for short) is the idea of a symmetry between the two fundamental types of particles that exist in our universe, bosons and fermions. It is today an essential component in almost all theories of quantum gravity. Supersymmetry in its first incarnation was proposed in 1966 [17]. A few years later it was incorporated also into the development of string theory, eventually leading up to the five superstring theories we have today [18].

While the idea of some symmetry between particle types is quite natural, it must be incorporated in a certain way to not invalidate the the *Coleman-Mandula theorem* [19]. This theorem was developed in 1967 and says in essence that no internal symmetries can be combined with Lorentz symmetries, without rendering the theory trivial. The reason that supersymmetry does not violate the theorem is that it does not have the usual Lie algebra structure of symmetries, but rather the structure of a superalgebra, instead involving an anti-commutator. This construction avoids the restriction of Coleman and Mandula's theorem. The generators of the supersymmetry algebra are fermionic and their defining property is given by [20]

$$\left\{Q_a, \bar{Q}_{\dot{b}}\right\} = 2i\sigma^{\mu}_{a\dot{b}}P_{\mu}, \tag{2.1}$$

where Q are fermionic generators of SUSY, $\sigma^{\mu} = (\mathbb{1}, \sigma^i)$, σ^i the Pauli matrices and P_{μ} the generator of momentum. As of now, no evidence of supersymmetry has been found in any particle physics experiments. Thus, it must be the case that SUSY is spontaneously broken, similar to how the Higgs potential works [21]. We will not delve deeper into this process in this thesis and simply take supersymmetry as an underlying assumption. The interested reader can consult for example [20] for a review.

Schematically, one can think of supersymmetry in terms of variations as

$$\delta(\text{boson}) = \epsilon(\text{fermion}), \qquad \delta(\text{fermion}) = \epsilon(\text{boson}), \tag{2.2}$$

where ϵ is the fermionic supersymmetry parameter. The exact appearance of these expressions depend on the theory in question. By gauging this theory, that is promoting ϵ to a local parameter $\epsilon(x)$, one obtains a supergravity theory much like how gauging the Poincaré algebra yields general relativity [22]. The exact process of going from supersymmetry to supergravity is quite complicated and we will not delve further into it in this thesis.

2.2 Supergravity in 11 Dimensions

Supergravity in eleven dimensions is the low-energy limit of M-theory, which is believed to be the unique 11-dimensional theory that unifies all string theories. It turns out that 11 dimensions is a quite special case, as it is the highest dimension[1] in which we can construct a supersymmetric theory with spin-2 or lower [24]. As for all higher-dimensional theories, in order to construct four-dimensional effective theories we perform a compactification. The procedure we will employ here is *Kaluza-Klein compactification*, or dimensional reduction. This type of compactification means that we split up the full theory as a product of two spaces, $M_D = M_d \times M_{D-d}$, where $d = 4$. The manifold M_d is taken to be our four-dimensional spacetime and M_{D-d} is the internal compact space, whose dimensions are very small so that we are not able to detect them [22].

The field content of the 4-dimensional theory is given by harmonic expansion of the 11-dimensional fields on the internal space. In Kaluza-Klein, all harmonic modes are kept, which describes the full set of fields in d dimensions. In dimensional reduction, only a finite number of modes are kept. The modes kept are those giving rise to massless or very light fields, since the heavy modes require much higher energies than we have access to in order to be excited. This process is called consistent truncation. The requirement for such a truncation is that the heavy modes that are thrown away cannot have light modes as sources. If this is fulfilled, then the truncation is consistent with the field equations.

Later on, we will compactify 11-dimensional supergravity to four dimensions. This section however, will be devoted to better understanding the 11-dimensional theory. We will do this by stating the general form of the action and explicitly check the invariance under supersymmetry transformations to linear order in the fermionic field. For a treatment of higher order fermionic terms, the reader may consult [25].

2.2.1 The Lagrangian

11D supergravity can be constructed in a number of ways. We will not do any full derivation of the equations of motion in this thesis, but refer the reader to for example [22, 26] for more details. Instead we will just state the different parts of the action, argue for their validity and then check that it is consistent by performing SUSY variations.

The bosonic part of the action is [27]

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\Big(e^{\alpha\mu}e^{\beta\nu}R_{\mu\nu\alpha\beta} - \frac{1}{48}H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma}\Big) - \frac{1}{12\kappa^2}\int B\wedge H\wedge H. \tag{2.3}$$

[1] There is an idea of a 12-dimensional theory called F-theory that has been proposed, but as of now no one really knows how to construct that theory properly [23].

Here, e^{μ}_{α} is the vielbein, $e = \det e^{\mu}_{\alpha}$, $R_{\mu\nu\alpha\beta}$ the Riemann tensor and $H = \mathrm{d}B$ a field strength. This structure is actually quite natural. First is the usual Einstein-Hilbert term and then there is a quadratic term in the field strength, much like in electromagnetism. The last term is called a Chern-Simons term. It is only topological, which means that it is fully independent of the metric g_{MN}.

Since the theory is supersymmetric we expect the graviton to have a superpartner, a spin-$\frac{3}{2}$ field called the gravitino. This field has a kinetic term given by the Rarita-Schwinger action [22]

$$2\kappa^2 S_{\mathrm{RS}} = \int \mathrm{d}^{11}x\, e\bar{\psi}_{\mu}{}^{a}(\Gamma^{\mu\nu\rho})_a{}^{b} D_{\nu}(\omega)\psi_{\rho b}, \tag{2.4}$$

where $\psi_{\rho b}$ is the Rarita-Schwinger vector spinor field. The spinor index b is normally not explicitly written out. There is an undetermined constant for this term relative to S_{B} that will be determined later when we check the SUSY invariance. The last part of the action is [24]

$$2\kappa^2 S_5 = \int \mathrm{d}^{11}x\, e\Big[\bar{\psi}_{\lambda}\Gamma^{\mu\nu\rho\sigma\lambda\tau}\psi_{\tau} + 12\bar{\psi}^{\mu}\Gamma^{\nu\rho}\psi^{\sigma}\Big]\Big(H_{\mu\nu\rho\sigma} + \tilde{H}_{\mu\nu\rho\sigma}\Big), \tag{2.5}$$

also with an undetermined prefactor. Here we have also introduced the supercovariant field strength

$$\tilde{H}_{\mu\nu\rho\sigma} = H_{\mu\nu\rho\sigma} - 3i\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]}. \tag{2.6}$$

Our goal now is to check that the action is invariant under local supersymmetry transformations and determine the prefactors. The full action is

$$S = S_{\mathrm{B}} + S_{\mathrm{RS}} + S_5, \tag{2.7}$$

where the relative weighting of the terms will be determined soon. The SUSY variations under consideration are [25]

$$\begin{gathered} \delta e_{\sigma}{}^{\gamma} = -2i\bar{\epsilon}\Gamma^{\gamma}\psi_{\sigma}, \qquad \delta B_{\mu\nu\rho} = 6i\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}, \\ \delta\psi = D_{\mu}\epsilon + \frac{1}{288}(-8H_{\mu\rho\sigma\tau}\Gamma^{\rho\sigma\tau} + H_{\nu\rho\sigma\tau}\Gamma_{\mu}{}^{\nu\rho\sigma\tau})\epsilon. \end{gathered} \tag{2.8}$$

A general variation of the action takes the form

$$\delta S = \frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta\psi}\delta\psi + \frac{\delta S}{\delta B}\delta B + \frac{\delta S}{\delta\omega}\frac{\delta\omega}{\delta e}\,\delta e\,. \tag{2.9}$$

In order to simplify the procedure, we may use the "1.5 order" formalism [22]. This essentially means that we think of the spin connection as a function of the vielbein, $\omega = \omega(e,\psi)$, defined by its algebraic equations of motion. Then we have that $\delta S/\delta\omega = 0$ and need not consider the last term above. Thus, before verifying the invariance we will compute the variations with respect to the spin connection as well, in order to obtain the torsion and contorsion tensor.

2.2.2 Equation of Motion for the Spin Connection

Consider the variation of the Einstein-Hilbert term with respect to the spin connection, ω. We find

$$\begin{aligned} \frac{\delta S_{\mathrm{EH}}}{\delta\omega} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{D}x\, e e^{\alpha\mu} e_{\beta}{}^{\nu}\frac{\delta R_{\mu\nu\alpha}{}^{\beta}(\omega)}{\delta\omega} = \\ &= \frac{1}{2\kappa^2}\int \mathrm{d}^{D}x\, e e^{\alpha\mu} e_{\beta}{}^{\nu}\Big[D_{\mu}\delta\omega_{\nu\alpha}{}^{\beta} - D_{\nu}\delta\omega_{\mu\alpha}{}^{\beta}\Big]. \end{aligned} \tag{2.10}$$

Now, we may rewrite the Lorentz covariant derivatives as full covariant derivatives, if we add the compensating affine connection term. The fully covariant derivative is related to the Lorentz covariant derivative by $\nabla_\mu e_\nu{}^\beta = D_\mu e_\nu{}^\beta - \Gamma_{\mu\nu}{}^\rho e_\rho{}^\beta$. This achieves two things: First, the full derivative fulfils $\nabla_\mu e_\beta{}^\nu = 0$, i.e. the metric postulate. This means that we can use partial integration and we will only pick up the terms which contain derivatives on $e = \det e_\beta{}^\nu$. Second, since the two terms are antisymmetric in μ and ν, the two added connections will combine to form the torsion, defined by

$$T_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho \tag{2.11}$$

The variation is then

$$\begin{aligned}\frac{\delta S_{\mathrm{EH}}}{\delta\omega} &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\Big(2e^{\alpha\mu}e_\beta{}^\nu\nabla_{[\mu}\delta\omega_{\nu]\alpha}{}^\beta + e^{\alpha\mu}e_\beta{}^\nu T_{\mu\nu}{}^\rho\delta\omega_{\rho\alpha}{}^\beta\Big) = \\ &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\Big(-2e^{\alpha\mu}e_\beta{}^\nu K_{\rho[\mu}{}^\rho\delta\omega_{\nu]\alpha}{}^\beta + e^{\alpha\mu}e_\beta{}^\nu T_{\mu\nu}{}^\rho\delta\omega_{\rho\alpha}{}^\beta\Big).\end{aligned} \tag{2.12}$$

Introducing the contorsion related to the torsion via

$$K_{\mu\nu\rho} \equiv -\frac{1}{2}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu}). \tag{2.13}$$

From this we can see that in our case, when the first and last indices are contracted, the last term vanish (the torsion is anti-symmetric in the two first indices, while the contorsion is anti-symmetric in the two last ones). Renaming indices together with raising and lowering of flat indices lead to the final answer

$$\begin{aligned}\frac{\delta S_{\mathrm{EH}}}{\delta\omega} = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\Bigg(\frac{1}{2}\Big(e^{\alpha\mu}e_\beta{}^\sigma(T_{\rho\mu}{}^\rho - T_{\mu\rho}{}^\rho)\delta\omega_{\sigma\alpha}{}^\beta - e^{\alpha\sigma}e_\beta{}^\nu(T_{\rho\nu}{}^\rho - T_{\nu\rho}{}^\rho)\delta\omega_{\sigma\alpha}{}^\beta\Big)+ \\ + e^{\alpha\mu}e_\beta{}^\nu T_{\mu\nu}{}^\sigma\delta\omega_{\sigma\alpha}{}^\beta\Bigg) = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\Bigg(e_\beta{}^\sigma T_{\rho\alpha}{}^\rho - e_\alpha{}^\sigma T_{\rho\beta}{}^\rho + T_{\alpha\beta}{}^\sigma\Bigg)\delta\omega_\sigma{}^{\alpha\beta}.\end{aligned} \tag{2.14}$$

The other term contributing to the variation with respect to the spin connection is the Rarita-Schwinger term. We consider it with the usual Lorentz covariant derivative, defined acting on the gravitino as

$$D_\nu(\omega)\psi_\rho = \partial_\mu\psi_\rho + \frac{1}{4}\omega_\nu{}^{\alpha\beta}\Gamma_{\alpha\beta}\psi_\rho \tag{2.15}$$

Plugging this into the RS-action and varying with respect to the spin connection we arrive at

$$\frac{\delta S_{\mathrm{RS}}}{\delta\omega} = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, ei\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\psi_\rho(\delta\omega_\nu{}^{\alpha\beta}) \tag{2.16}$$

Here we may expand the gamma matrix product according to

$$\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta} = \Gamma^{\mu\nu\rho}{}_{\alpha\beta} + 6\Gamma^{[\mu\nu}{}_{[\beta}\delta^{\rho]}_{\alpha]} + 6\Gamma^{[\mu}\delta^{\nu}_{[\beta}\delta^{\rho]}_{\alpha]} \tag{2.17}$$

The second term contains a rank 3 gamma matrix, which is antisymmetric. Thus we can use the Majorana property (see Appendix A.1) to exchange the two gravitini and then use the antisymmetry in μ and ρ to see that that term will vanish.

The equations of motion for the spin connection are then

$$\begin{aligned}\frac{\delta S_{\mathrm{RS}}+\delta S_{\mathrm{EH}}}{\delta\omega}=0 \implies\ & e_\beta{}^\nu T_{\rho\alpha}{}^\rho - e_\alpha{}^\nu T_{\rho\beta}{}^\rho + T_{\alpha\beta}{}^\nu = \\ =& -i\Big(\bar{\psi}_\mu\Gamma^{\mu\nu\rho}{}_{\alpha\beta}\psi_\rho + 2\bar{\psi}_\mu\Gamma^\mu\psi_\alpha e_\beta{}^\nu - 2\bar{\psi}_\mu\Gamma^\mu\psi_\beta e_\alpha{}^\nu + 2\bar{\psi}_\alpha\Gamma^\nu\psi_\beta\Big)\end{aligned} \tag{2.18}$$

Contracting with $e_\nu{}^\alpha$ yields $T_{\rho\beta}{}^\rho = -2i\bar{\psi}_\mu\Gamma^\mu\psi_\beta$, and then we can convert all indices to curved and obtain the torsion

$$T_{\mu\nu\rho} = i\bar{\psi}_\sigma\Gamma^{\sigma\tau}{}_{\mu\nu\rho}\psi_\tau - 2i\bar{\psi}_\mu\Gamma_\rho\psi_\nu. \tag{2.19}$$

This then leads directly to the contorsion tensor, simply using the definition

$$\begin{aligned}K_{\mu\nu\rho} \equiv& -\frac{1}{2}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu}) = \\ =& -\frac{i}{2}\bar{\psi}_\sigma\Gamma^{\sigma\tau}{}_{\mu\nu\rho}\psi_\tau + i\Big(\bar{\psi}_\mu\Gamma_\rho\psi_\nu - \bar{\psi}_\nu\Gamma_\mu\psi_\rho + \bar{\psi}_\rho\Gamma_\nu\psi_\mu\Big).\end{aligned} \tag{2.20}$$

There are two important points worth stressing in this section. First, we see explicitly that the equations of motion for the spin connection actually are algebraic and thus the "1.5 order" formalism is consistent. If the spin connection would have had differential equations of motion, this would not have been the case. Second, the torsion (and of course also the contorsion) are of second order in the fermionic field, ψ. Doing computations in differential geometry in theories with torsion is quite difficult, since many relations we are used to get extra contributions from the torsion. An example would be that the Ricci tensor is not necessarily symmetric in theories with torsion. In this case however, since the torsion is quadratic in ψ, we can actually treat the theory as torsionless as long as we are only interested in effects up to linear order of ψ. This is precisely what we will do in the following section.

2.2.3 Supersymmetric Invariance

Now we will check the SUSY variations up to linear order in ψ and determine the prefactors for S_{RS} and S_5. Even though we do not consider all orders in ψ there will be a lot of terms that must cancel each other in the following calculations. Thus we will try to employ a systematic approach. We will first treat terms with no dependence on the H-field. Then we will work our way upwards and first treat terms of order H followed by H^2. These are all the terms needed to check SUSY to linear order in ψ. Note that any quantities, such as Γ-matrices and indices, are always 11-dimensional in this section. We define the variation of the action by

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, \delta\mathcal{L}. \tag{2.21}$$

This will reduce the cluttering of the equations somewhat. We start with the Einstein-Hilbert term. The variation looks like

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{EH}} =& \delta\Big[ee^{\alpha\mu}e^{\beta\nu}R_{\mu\nu\alpha\beta}(\omega)\Big] = \\ =& \Big[(\delta e)e^{\alpha\mu}e^{\beta\nu}R_{\mu\nu\alpha\beta}(\omega) + 2e(\delta e^{\alpha\mu})e^{\beta\nu}R_{\mu\nu\alpha\beta}(\omega) + ee^{\alpha\mu}e^{\beta\nu}(\delta R_{\mu\nu\alpha\beta}(\omega))\Big].\end{aligned} \tag{2.22}$$

In the 1.5 order formalism, we throw away the last term since it is a variation only with respect to ω. It is in fact also so that $\delta R_{\mu\nu\rho\sigma}$ is a total derivative, if we consider only terms up to order ψ^2. This is also a consequence of the torsion being quadratic in ψ. Now we need the following

$$\begin{aligned}\delta e &= e e_\gamma{}^\rho(\delta e_\rho{}^\gamma),\\ \delta e^{\alpha\mu} &= \eta^{\alpha\beta}(\delta e_\beta{}^\mu) = -e^{\alpha\rho}(\delta e_\rho{}^\gamma)e_\gamma{}^\mu.\end{aligned} \tag{2.23}$$

This leads to the expression

$$\delta\mathcal{L}_{\mathrm{EH}} = e[e_\gamma{}^\rho R - 2e^{\alpha\rho}e_\gamma{}^\mu R_{\mu\alpha}(\omega)](\delta e_\rho^\gamma) = e[e_\gamma{}^\rho R - 2R_\gamma{}^\rho(\omega)](\delta e_\rho{}^\gamma). \tag{2.24}$$

We see that this is Einstein's equations, in a somewhat different form, as we would expect. Now we plug in the SUSY variation for the vielbein, which is

$$\delta e_\sigma{}^\gamma = -2i\bar{\epsilon}\Gamma^\gamma\psi_\sigma. \tag{2.25}$$

This then leads to the expression

$$\delta_\epsilon\mathcal{L}_{\mathrm{EH}} = 4ie\Big(R_\gamma{}^\rho - \frac{1}{2}e_\gamma{}^\rho R\Big)\bar{\epsilon}\Gamma^\gamma\psi_\rho = -4ie\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\bar{\psi}^\nu\Gamma^\mu\epsilon, \tag{2.26}$$

where in the last step we used the symmetry of Γ^μ to flip ϵ and ψ, producing the extra minus sign. Next, we will investigate the Rarita-Schwinger term. A general variation will be

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{RS}} =&(\delta e)\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho + e(\delta\bar{\psi}_\mu)\Gamma^{\mu\nu\rho}D_\nu\psi_\rho\\ &+e\bar{\psi}_\mu(\delta e_\alpha{}^\mu e_\beta{}^\nu e_\gamma{}^\rho)\Gamma^{\alpha\beta\gamma}D_\nu\psi_\rho + e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\delta\psi_\rho).\end{aligned} \tag{2.27}$$

Note that terms 1 and 3 will not be needed when we check SUSY to linear order, they already contain two ψ so the variation of the vielbein makes them cubic in ψ.

The transformation for the gravitino is

$$\begin{aligned}&\delta_\epsilon\psi_\mu = \tilde{D}_\mu\epsilon, \qquad \text{where}\\ &\tilde{D}_\mu\epsilon = D_\mu\epsilon + \frac{1}{288}(-8H_{\mu\rho\sigma\tau}\Gamma^{\rho\sigma\tau} + H_{\nu\rho\sigma\tau}\Gamma_\mu{}^{\nu\rho\sigma\tau})\epsilon.\end{aligned} \tag{2.28}$$

We want to check orders of H separately, so we split the variations of ψ as follows

$$\delta\mathcal{L}_{\mathrm{RS}} = \delta'\mathcal{L}_{\mathrm{RS}} + \delta''\mathcal{L}_{\mathrm{RS}}. \tag{2.29}$$

where $\delta'\mathcal{L}$ means we take only $\delta'_\epsilon\psi_\mu = D_\mu\epsilon$ and $\delta''\mathcal{L}$ means only the second term, i.e. $\delta''_\epsilon\psi_\mu = \frac{1}{288}(-8H_{\mu\rho\sigma\tau}\Gamma^{\rho\sigma\tau} + H_{\nu\rho\sigma\tau}\Gamma_\mu{}^{\nu\rho\sigma\tau})\epsilon$.

Returning to terms 2 and 4 in the general variation, equation (2.27), we may see that by using partial integration and the Majorana properties the two terms actually add. This is outlined as follows

$$\begin{aligned}\text{term } 2 = e(\delta\bar{\psi}_\mu)\Gamma^{\mu\nu\rho}D_\nu\psi_\rho &= e(D_\nu\bar{\psi}_\rho)\Gamma^{\mu\nu\rho}\delta\psi_\mu = \{\mu\leftrightarrow\rho, \Gamma^{\rho\nu\mu} = \Gamma^{\mu\nu\rho}\} =\\ = -e(D_\nu\bar{\psi}_\mu)\Gamma^{\mu\nu\rho}\delta\psi_\rho &= -D_\nu(\dots) + (D_\nu e)\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\delta\psi_\rho +\\ + e\bar{\psi}_\mu(D_\nu\Gamma^{\mu\nu\rho})\delta\psi_\rho &+ e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\delta\psi_\rho = e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\delta\psi_\rho.\end{aligned} \tag{2.30}$$

where in the last step we used that we do not consider higher order terms in ψ together with

$$\begin{aligned}D_\nu e &= 0 + \mathcal{O}\big(\psi^2\big),\\ D_\nu\Gamma^{\mu\nu\rho} &= [D_\nu(e_\alpha{}^\mu\dots)]\Gamma^{\alpha\beta\gamma} = 0 + \mathcal{O}\big(\psi^2\big).\end{aligned} \tag{2.31}$$

So, term 2 and 4 in (2.27) add. Now we proceed to computing the variation of $\mathcal{L}_{\mathrm{RS}}$ to zeroth order in H, hoping that it will cancel the Einstein-Hilbert term. We will need the definition of the Riemann tensor as a commutator of covariant derivatives acting on a spinor [24]

$$[D_\mu, D_\nu]\epsilon = \tfrac{1}{4} R_{\mu\nu\alpha\beta}\Gamma^{\alpha\beta}\epsilon. \tag{2.32}$$

Then the variation is

$$\delta'\mathcal{L}_{\mathrm{RS}} = 2e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\delta'\psi_\rho) = 2e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\frac{1}{2}[D_\nu, D_\rho]\epsilon = \frac{1}{4}e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}R_{\nu\rho}{}^{\alpha\beta}\Gamma_{\alpha\beta}\epsilon. \tag{2.33}$$

From here, we will use the identity

$$\Gamma^{\gamma\delta\epsilon}\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta}{}^{\gamma\delta\epsilon} + 6\delta^{[\gamma}_{[\alpha}\Gamma_{\beta]}{}^{\delta\epsilon]} - 6\delta^{[\gamma\delta}_{[\alpha\beta]}\Gamma^{\epsilon]}, \tag{2.34}$$

which we may use in the above equation if we convert the indices on the first gamma matrix by extracting vielbeins, $e_\gamma{}^\mu$ etc. We get

$$\delta\mathcal{L}'_{\mathrm{RS}} = \frac{1}{4}eR_{\delta\epsilon}{}^{\alpha\beta}\bar{\psi}_\gamma\Big(\Gamma_{\alpha\beta}{}^{\gamma\delta\epsilon} + 6\delta^{[\gamma}_{[\alpha}\Gamma_{\beta]}{}^{\delta\epsilon]} - 6\delta^{[\gamma\delta}_{[\alpha\beta]}\Gamma^{\epsilon]}\Big)\epsilon. \tag{2.35}$$

Here, we note that the first term vanishes, since it yields the Riemann tensor with full anti-symmetry in all indices, which is zero up to $\mathcal{O}(\psi^2)$ term. The second term is best analysed by breaking up the antisymmetry in the upper indices

$$\begin{aligned}&\frac{6}{4}e\frac{1}{3!}\Big(2R_{\delta\epsilon}{}^{\alpha\beta}\bar{\psi}_{[\alpha}\Gamma_{\beta]}{}^{\delta\epsilon} - 2\bar{\psi}_\gamma R_{\epsilon[\delta}{}^{\delta\beta}\Gamma_{\beta]}{}^{\epsilon\gamma} + 2\bar{\psi}_\gamma R_{\delta[\alpha}{}^{\alpha\beta}\Gamma_{\beta]}{}^{\gamma\delta}\Big)\epsilon = \\ =&\frac{1}{4}e\Big(0 + 4\bar{\psi}_\gamma\frac{1}{2!}(R_{\epsilon\delta}{}^{\delta\beta}\Gamma_\beta{}^{\epsilon\gamma} - R_{\epsilon\beta}{}^{\delta\beta}\Gamma_\delta{}^{\epsilon\gamma})\Big)\epsilon = 0 + \mathcal{O}\Big(\psi^2\Big).\end{aligned} \tag{2.36}$$

We put the first term to zero by the same argument as above, it is anti-symmetric in 3 index in the Riemann tensor. The remaining terms combine, and after breaking up that antisymmetry we see that we arrive at two terms with the Ricci tensor. While not fully symmetric (due to the contorsion) in our context, it is symmetric up to $\mathcal{O}(\psi^2)$ and thus vanishes when contracted with the gamma matrices. The final term in $\delta'\mathcal{L}_{\mathrm{RS}}$ is then what remains. We expand it in a similar fashion below. Note that the delta function always hits the two last indices of the Riemann tensor. The antisymmetry in those indices is manifest, so we can drop the explicit notation.

$$\begin{aligned}\delta\mathcal{L}'_{\mathrm{RS}} &= -\frac{6}{4}e\frac{1}{3!}2\Big(R_{\delta\epsilon}{}^{\gamma\delta}\bar{\psi}_\gamma\Gamma^\epsilon + R_{\delta\epsilon}{}^{\epsilon\gamma}\bar{\psi}_\gamma\Gamma^\delta + R_{\delta\epsilon}{}^{\delta\epsilon}\bar{\psi}_\gamma\Gamma^\gamma\Big)\epsilon = \\ &= -\frac{1}{2}e\Big(-R_{\epsilon\gamma}\bar{\psi}^\gamma\Gamma^\epsilon - R_{\delta\gamma}\bar{\psi}^\gamma\Gamma^\delta + R\bar{\psi}_\gamma\Gamma^\gamma\Big)\epsilon = e\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big)\bar{\psi}^\nu\Gamma^\mu\epsilon.\end{aligned} \tag{2.37}$$

We see that this result is what we obtained from the Einstein-Hilbert part, equation (2.26), apart from a factor of $4i$. These two terms are the only terms without H, so they must cancel. Thus we conclude that that the prefactor for the Rarita-Schwinger term is $4i$, relative to the Einstein-Hilbert term.

Now, we proceed to compute the variations which are linear in H and continue with the Rarita-Schwinger term and the variation $\delta''\mathcal{L}_{\mathrm{RS}}$. We start by rewriting $\delta\mathcal{L}_{\mathrm{RS}}$ so that the derivatives does not act on the variation

$$\begin{aligned}&\delta''\mathcal{L}_{\mathrm{RS}} = \\ &= 2e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu(\delta\psi_\rho) = \Big\{\text{Partial integration, dropping terms of } \mathcal{O}\Big(\psi^2\Big)\Big\} = \\ &= 2e(D_\nu\bar{\psi}_\mu)\Gamma^{\mu\nu\rho}(\delta\psi_\rho) = \frac{1}{144}e(D_\nu\bar{\psi}_\rho)\Gamma^{\mu\nu\rho}(H_{\lambda\kappa\sigma\tau}\Gamma_\mu{}^{\lambda\kappa\sigma\tau}\epsilon - 8H_{\mu\kappa\sigma\tau}\Gamma^{\kappa\sigma\tau}\epsilon) = \\ &= -\frac{3}{144}e(D_\nu\bar{\psi}_\rho)(12\Gamma^{\mu\sigma}H^{\nu\rho}{}_{\mu\sigma} + \Gamma^{\nu\rho\mu\sigma\kappa\tau}H_{\mu\sigma\kappa\tau})\epsilon.\end{aligned} \tag{2.38}$$

From where else do we get $\mathcal{O}(H)$ contributions? The kinetic term for H will contribute, which we see by writing out the general variation

$$\delta\mathcal{L}_{\text{kin}} = -\frac{1}{48}\Big((\delta e)H^2 + 2e(\delta H_{\mu\nu\rho\sigma})H^{\mu\nu\rho\sigma} + 4eH_{\mu\nu\rho\sigma}(\delta g^{\mu\mu'})H_{\mu'}{}^{\nu\rho\sigma})\Big). \tag{2.39}$$

We find that only the second term above is $\mathcal{O}(H)$. If we evaluate it using the following local SUSY variation for B

$$\delta B_{\mu\nu\rho} = 6i\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}, \tag{2.40}$$

we find

$$\begin{aligned} 2e(\delta H_{\mu\nu\rho\sigma})H^{\mu\nu\rho\sigma} &= -\frac{4}{24}e(D_\mu\delta B_{\nu\rho\sigma})H^{\mu\nu\rho\sigma} = -ie(D_\mu(\bar{\epsilon}\Gamma_{[\nu\rho}\psi_{\sigma]}))H^{\mu\nu\rho\sigma} = \\ &= -ieH^{\mu\nu\rho\sigma}\Big((D_\mu\bar{\epsilon})\Gamma_{\nu\rho}\psi_\sigma + \bar{\epsilon}\Gamma_{\nu\rho}D_\mu\psi_\sigma\Big). \end{aligned} \tag{2.41}$$

We may check that the second term here indeed cancels the first term in $\delta''\mathcal{L}_{\text{RS}}$, equation (2.38) above, if we put in the corresponding factor of $4i$. Using the Majorana property to to flip $\bar{\epsilon}$ and $D_\mu\psi_\rho$ and the symmetry of $\Gamma_{\nu\rho}$ produces the correct sign. We now collect the two remaining terms of order H that we have obtained so far

$$\delta(\mathcal{L}_{\text{RS}} + \mathcal{L}_{\text{kin}})\Big|_{\mathcal{O}(H)} = -ie\Big(\frac{1}{12}(D_\nu\bar{\psi}_\rho)\Gamma^{\nu\rho\mu\sigma\kappa\tau}\epsilon H_{\mu\sigma\kappa\tau} + H^{\mu\nu\rho\sigma}(D_\mu\bar{\epsilon})\Gamma_{\nu\rho}\psi_\sigma\Big). \tag{2.42}$$

The only remaining thing that contributes to $\mathcal{O}(H)$ terms is the fifth term, above denoted S_5:

$$\delta\mathcal{L}_5 = \delta\Big[e\bar{\psi}_\lambda\Gamma^{\mu\nu\rho\sigma\lambda\tau}\psi_\tau + 12e\bar{\psi}^\mu\Gamma^{\nu\rho}\psi^\sigma\Big]\Big(H_{\mu\nu\rho\sigma} + \tilde{H}_{\mu\nu\rho\sigma}\Big). \tag{2.43}$$

Here, $\tilde{H}$ is the supercovariant field strength, and takes the form $\tilde{H} = H + \mathcal{O}(\psi^2)$. Thus, to our current approximation we use $H + \tilde{H} \approx 2H$. Then, the variation with respect to H, e and the gamma matrices all yield terms of $\mathcal{O}(\psi^3)$, so we neglect them. Furthermore, both $\Gamma^{[2]}$ and $\Gamma^{[6]}$ are symmetric, so we have

$$(\delta\bar{\psi}^\mu)\Gamma^{\nu\rho}\psi^\sigma H_{\mu\nu\rho\sigma} = -\bar{\psi}^\sigma\Gamma^{\nu\rho}(\delta\psi^\mu)H_{\mu\nu\rho\sigma} = \bar{\psi}^\mu\Gamma^{\nu\rho}(\delta\psi^\sigma)H_{\mu\nu\rho\sigma}, \tag{2.44}$$

and analogously for $\Gamma^{[6]}$. Thus we may write the variation as

$$\delta\mathcal{L}_5 = 4e\Big[\bar{\psi}_\lambda\Gamma^{\mu\nu\rho\sigma\lambda\tau}(\delta\psi_\tau) + 12\bar{\psi}^\mu\Gamma^{\nu\rho}(\delta\psi^\sigma)\Big]H_{\mu\nu\rho\sigma}. \tag{2.45}$$

The second term in the variation of ψ will yield $\mathcal{O}(H^2)$ contributions, so we use the same notation as before to split up the variation. The linear terms in H are

$$\begin{aligned} \delta'\mathcal{L}_5 &= 4e\Big[\bar{\psi}_\lambda\Gamma^{\mu\nu\rho\sigma\lambda\tau}(D_\tau\epsilon) + 12\bar{\psi}^\mu\Gamma^{\nu\rho}(D^\sigma\epsilon)\Big]H_{\mu\nu\rho\sigma} = \{\text{p.i. in the first term}\} = \\ &= -4e(D_\tau\bar{\psi}_\lambda)\Gamma^{\mu\nu\rho\sigma\lambda\tau}\epsilon H_{\mu\nu\rho\sigma} - 48eH^{\mu\nu\rho\sigma}(D_\sigma\bar{\epsilon})\Gamma_{\nu\rho}\psi_\mu = \\ &= 48e\Big[\frac{1}{12}(D_\tau\bar{\psi}_\lambda)\Gamma^{\tau\lambda\mu\nu\rho\sigma}\epsilon H_{\mu\nu\rho\sigma} + H^{\mu\nu\rho\sigma}(D_\mu\bar{\epsilon})\Gamma_{\nu\rho}\psi_\sigma\Big], \end{aligned} \tag{2.46}$$

where we in the last step reordered and picked up an overall minus sign. The partial integration in the first step also yields extra terms, but they are of $\mathcal{O}(\psi^3)$, except for $D_\tau H_{\mu\nu\rho\sigma}$ which is zero since H is a closed 4-form. We may now see that

$$\delta'\mathcal{L}_5 = 48i\,\delta(\mathcal{L}_{\text{RS}} + \mathcal{L}_{\text{kin}})\Big|_{\mathcal{O}(H)}. \tag{2.47}$$

So all contributions of order H indeed cancel if we fix the prefactor of $\mathcal{L}_5$ to $\frac{i}{48}$.

What remains now is terms of $\mathcal{O}(H^2)$. This contribution will come from terms 1 and 3 in equation (2.39) i.e. $\delta\mathcal{L}_{\text{kin}}$, the δ'' variation of $\mathcal{L}_5$, and from the topological Chern-Simons term. We start by treating the topological term. A general variation is

$$\begin{aligned}\delta(B\wedge H\wedge H) &= (\delta B\wedge H\wedge H)+(B\wedge \mathrm{d}\delta B\wedge H)+(B\wedge H\wedge \mathrm{d}\delta B)=\\ &=(\delta B\wedge H\wedge H)+2(B\wedge \mathrm{d}\delta B\wedge H)\\ &=(\delta B\wedge H\wedge H)-2\,\mathrm{d}(B\wedge\delta B\wedge H)+2(\mathrm{d}B\wedge\delta B\wedge H)+2(B\wedge\delta B\wedge \mathrm{d}H)=\\ &=3(\delta B\wedge H\wedge H),\end{aligned} \tag{2.48}$$

where in the last step we dropped the total derivative and used that $\mathrm{d}H=0$. In coordinates this is

$$\delta\mathcal{L}_{\text{CS}} = -\frac{1}{6}\,\delta(B\wedge H\wedge H) = -\frac{1}{6}\frac{3}{3!4!4!}\varepsilon^{\mu_1\dots\mu_{11}}\delta B_{\mu_1\mu_2\mu_3}H_{\mu_4\dots\mu_7}H_{\mu_8\dots\mu_{11}}. \tag{2.49}$$

Here, the ε symbol indicates the metric independent Levi-Civita symbol, i.e. a tensor density. If we now plug in the variation of the B-field, $\delta B_{\mu\nu\rho}=6i\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}$ we find

$$\delta\mathcal{L}_{\text{CS}} = \frac{i}{2(24)^2}\varepsilon^{\mu\nu\rho\mu_4\dots\mu_{11}}\bar{\psi}_\rho\Gamma_{\mu\nu}\epsilon H_{\mu_4\dots\mu_7}H_{\mu_8\dots\mu_{11}}, \tag{2.50}$$

where we have dropped the explicit antisymmetrisation, it is already manifest in the ε symbol. We also picked up an extra minus sign by exchanging $\bar{\epsilon}$ and ψ_ρ. We now collect the rest of the contributing terms, starting with the kinetic term

$$\begin{aligned}\delta\mathcal{L}_{\text{kin}}\Big|_{\mathcal{O}(H^2)} &= -\frac{1}{48}e\Big(e_\alpha{}^\tau(\delta e_\tau{}^\alpha)H^2-8g^{\tau(\mu'}e_\alpha{}^{\mu)}(\delta e_\tau{}^\alpha)H_{\mu\nu\rho\sigma}H_{\mu'}{}^{\nu\rho\sigma}\Big)=\\ &=-\frac{1}{48}e\Big(-2i\bar{\epsilon}\Gamma^\tau\psi_\tau H^2+16i\bar{\epsilon}\Gamma^{(\mu}\psi^{\mu')}H_{\mu\nu\rho\sigma}H_{\mu'}{}^{\nu\rho\sigma}\Big)=\\ &=-\frac{ie}{24}\bar{\psi}_\tau\Gamma^\tau H^2+\frac{ie}{3}\bar{\psi}_\tau\Gamma^\mu\epsilon H_{\mu\nu\rho\sigma}H^{\tau\nu\rho\sigma}.\end{aligned} \tag{2.51}$$

Now we write out the $\delta''\mathcal{L}_5$ terms. These are the most complicated ones, so we will take them one at a time. The gamma matrix algebra will be done using Ulf Gran's Mathematica package "GAMMA" [28]. We start by rewriting the variation, now including the factor of $\frac{i}{48}$

$$\delta\mathcal{L}_5 = \frac{ie}{12}\Big[\bar{\psi}_\lambda\Gamma^{\mu\nu\rho\sigma\lambda\tau}(\delta\psi_\tau)+12\bar{\psi}^\mu\Gamma^{\nu\rho}(\delta\psi^\sigma)\Big]H_{\mu\nu\rho\sigma}. \tag{2.52}$$

Then we treat the first term. Inserting the δ'' variation of ψ yields

$$\begin{aligned}&\delta''\mathcal{L}_5^{(1)}=\\ &=\frac{ie}{12}\bar{\psi}_\lambda\Gamma^{\mu\nu\rho\sigma\lambda\tau}(\delta''\psi_\tau)H_{\mu\nu\rho\sigma}=\\ &=\frac{ie}{12}\bar{\psi}_\lambda\Gamma^{\mu\nu\rho\sigma\lambda\tau}\frac{1}{288}(-8H_{\tau\kappa_1\kappa_2\kappa_3}\Gamma^{\kappa_1\kappa_2\kappa_3}+H_{\kappa_1\kappa_2\kappa_3\kappa_4}\Gamma_\tau{}^{\kappa_1\kappa_2\kappa_3\kappa_4})\epsilon H_{\mu\nu\rho\sigma}=\\ &=ie\bar{\psi}_\gamma\Bigg(\frac{1}{24}\Gamma^\gamma H^2-\frac{1}{6}\Gamma^{\delta_0}H_{\delta_0\delta_1\delta_2\delta_3}H^{\gamma\delta_1\delta_2\delta_3}-\frac{7}{36}\Gamma^{\delta_3\delta_4\gamma}H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0\delta_1\delta_2}{}_{\delta_4}-\\ &-\frac{1}{8}\Gamma^{\delta_3\delta_4\delta_5}H^{\gamma\delta_1\delta_2}{}_{\delta_3}H_{\delta_1\delta_2\delta_4\delta_5}-\frac{1}{72}\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\gamma}H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0}{}_{\delta_4\delta_5\delta_6}-\\ &-\frac{1}{288}\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7}H^\gamma{}_{\delta_1\delta_2\delta_3}H_{\delta_4\delta_5\delta_6\delta_7}-\frac{1}{576}\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\gamma}H_{\delta_1\delta_2\delta_3\delta_4}H_{\delta_5\delta_6\delta_7\delta_8}\Bigg)\epsilon.\end{aligned} \tag{2.53}$$

The next term is

$$\begin{aligned}
\delta''\mathcal{L}_5^{(2)} &= ie\bar{\psi}^\mu \Gamma^{\nu\rho}(\delta''\psi^\sigma)H_{\mu\nu\rho\sigma} = \\
&= ie\bar{\psi}_\mu \Gamma_{\nu\rho}\frac{1}{288}(-8H_{\sigma\kappa_1\kappa_2\kappa_3}\Gamma^{\kappa_1\kappa_2\kappa_3} + H_{\kappa_1\kappa_2\kappa_3\kappa_4}\Gamma_\sigma{}^{\kappa_1\kappa_2\kappa_3\kappa_4})H^{\mu\nu\rho\sigma} = \\
&= ie\bar{\psi}_\gamma\Bigg(-\frac{1}{6}\Gamma^{\delta_0}H_{\delta_0\delta_1\delta_2\delta_3}H^{\gamma\delta_1\delta_2\delta_3} + \frac{1}{8}\Gamma^{\delta_3\delta_4\delta_5}H^{\gamma\delta_1\delta_2}{}_{\delta_3}H_{\delta_1\delta_2\delta_4\delta_5} \\
&\qquad + \frac{1}{288}\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7}H^\gamma{}_{\delta_1\delta_2\delta_3}H_{\delta_4\delta_5\delta_6\delta_7}\epsilon\Bigg)\epsilon.
\end{aligned} \tag{2.54}$$

To our delight, these two last terms cancel two terms in $\delta''\mathcal{L}_5^{(1)}$ and the first simply adds to another term. What then remains is

$$\begin{aligned}
&\delta''\mathcal{L}_5^{(1)} + \delta''\mathcal{L}_5^{(2)} = \\
&= ie\bar{\psi}_\gamma\Bigg(\frac{1}{24}\Gamma^\gamma H^2 - \frac{1}{3}\Gamma^{\delta_0}H_{\delta_0\delta_1\delta_2\delta_3}H^{\gamma\delta_1\delta_2\delta_3} - \frac{7}{36}\Gamma^{\delta_3\delta_4\gamma}H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0\delta_1\delta_2}{}_{\delta_4} \\
&\quad - \frac{1}{72}\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\gamma}H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0}{}_{\delta_4\delta_5\delta_6} - \frac{1}{576}\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\gamma}H_{\delta_1\delta_2\delta_3\delta_4}H_{\delta_5\delta_6\delta_7\delta_8}\Bigg)\epsilon.
\end{aligned} \tag{2.55}$$

The first two terms here cancel exactly the contribution from $\delta\mathcal{L}_{\text{kin}}$, equation (2.51).

Next note that the two partially contracted H-fields are symmetric in their remaining indices, seen by writing it as follows

$$\begin{aligned}
H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0\delta_1\delta_2}{}_{\delta_4} &= \frac{1}{2}\Big(H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0\delta_1\delta_2}{}_{\delta_4} + H^{\delta_0\delta_1\delta_2}{}_{\delta_3}H_{\delta_0\delta_1\delta_2\delta_4}\Big) = \\
&= \frac{1}{2}\Big(H_{\delta_0\delta_1\delta_2\delta_3}H^{\delta_0\delta_1\delta_2}{}_{\delta_4} + H_{\delta_0\delta_1\delta_2\delta_4}H^{\delta_0\delta_1\delta_2}{}_{\delta_3}\Big) = H_{\delta_0\delta_1\delta_2(\delta_3}H^{\delta_0\delta_1\delta_2}{}_{\delta_4)}.
\end{aligned} \tag{2.56}$$

But δ_3 and δ_4 are antisymmetrised in the gamma matrix, so the third term in (2.55) is zero. A similar argument holds for the H part of the fourth term, which is symmetric under simultaneous interchange of $1 \leftrightarrow 4, 2 \leftrightarrow 5, 3 \leftrightarrow 6$ while the gamma matrix is antisymmetric.

What remains is thus only the last term. In order to show that it cancels the Chern-Simons term we must investigate the gamma matrices further. In odd dimensions not all gamma matrices are needed to form a complete basis and some of them are related to each other. The general formula, for $D = 2m+1$ dimensions, is [22]

$$\Gamma^{\delta_1\ldots\delta_r} = (i)^{m+1}\frac{1}{e}\frac{1}{(D-r)!}\varepsilon^{\delta_1\ldots\delta_D}\Gamma_{\delta_D\ldots\delta_{r+1}}. \tag{2.57}$$

Note here the factor of $\frac{1}{e}$. It appears since the gamma matrices are expressed in flat indices. We may use this, with $D = 11$ and $r = 9$, to rewrite the last term in (2.55) as

$$\begin{aligned}
&-\frac{ie}{576}\bar{\psi}_\gamma\Gamma^{\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\gamma}\epsilon H_{\delta_1\delta_2\delta_3\delta_4}H_{\delta_5\delta_6\delta_7\delta_8} = \\
&= -\frac{i(i)^6}{(24)^2}\frac{1}{2!}\varepsilon^{\delta_1\ldots\gamma\delta_{10}\delta_{11}}\bar{\psi}_\gamma\Gamma_{\delta_{11}\delta_{10}}\epsilon H_{\delta_1\delta_2\delta_3\delta_4}H_{\delta_5\delta_6\delta_7\delta_8} \\
&= \frac{i}{2(24)^2}(-\varepsilon^{\delta_{11}\delta_{10}\gamma\delta_1\ldots\delta_8})\bar{\psi}_\gamma\Gamma_{\delta_{11}\delta_{10}}\epsilon H_{\delta_1\delta_2\delta_3\delta_4}H_{\delta_5\delta_6\delta_7\delta_8}.
\end{aligned} \tag{2.58}$$

Written in this way, it is precisely what we found in equation (2.50), but with an opposite sign. Thus we have found both the remaining prefactors of the Lagrangian and proved its SUSY invariance. We end this section by collecting the final results.

The 11-dimensional supergravity action is

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x \bigg[& eR - \frac{1}{48} eH^2 - \frac{1}{6} B \wedge H \wedge H + 4ie\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \left[\tfrac{1}{2}(\omega + \tilde{\omega}) \right] \psi_\rho + \\ & + \frac{i}{48} e \Big(\bar{\psi}_\lambda \Gamma^{\mu\nu\rho\sigma\lambda\tau} \psi_\tau + 12 \bar{\psi}^\mu \Gamma^{\nu\rho} \psi^\sigma \Big) \Big(H_{\mu\nu\rho\sigma} + \tilde{H}_{\mu\nu\rho\sigma} \Big) \bigg]. \end{aligned} \tag{2.59}$$

For the local SUSY variations

$$\begin{aligned} \delta e_\sigma{}^\gamma = -2i\bar{\epsilon}\Gamma^\gamma \psi_\sigma, \qquad \delta B_{\mu\nu\rho} = 6i\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}, \\ \delta\psi = D_\mu \epsilon + \frac{1}{288}(-8H_{\mu\rho\sigma\tau}\Gamma^{\rho\sigma\tau} + H_{\nu\rho\sigma\tau}\Gamma_\mu{}^{\nu\rho\sigma\tau})\epsilon, \end{aligned} \tag{2.60}$$

we have checked the invariance up to quadratic order in ψ

$$\delta S = 0 + \mathcal{O}\Big(\psi^2\Big). \tag{2.61}$$

This Lagrangian also leads to the Einstein equations of motion [29]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{12} \left(\tilde{H}_{\mu\rho\sigma\lambda} \tilde{H}_\nu{}^{\rho\sigma\lambda} - \frac{1}{8} g_{\mu\nu} \tilde{H}_{\rho_1\rho_2\rho_3\rho_4} \tilde{H}^{\rho_1\rho_2\rho_3\rho_4} \right). \tag{2.62}$$

2.2.4 Rescaling of the Fields

Before proceeding we will do some rescaling, in order to conform with the numerical values of a work by Duff et al. [24]. This will be helpful later when we work with the seven-sphere, since the definitions in this thesis will match theirs. We find that the following rescalings

$$B_{\mu\nu\rho} \to 2B_{\mu\nu\rho} \implies H_{\mu\nu\rho\sigma} \to 2H_{\mu\nu\rho\sigma}, \qquad \psi_\mu \to \frac{1}{\sqrt{2}}\psi_\mu, \tag{2.63}$$

leads to the following redefinition of the supercovariant field strength

$$\tilde{H}_{\mu\nu\rho\sigma} \to 2H_{\mu\nu\rho\sigma} - \frac{12}{2} i\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]} = 2(H_{\mu\nu\rho\sigma} - 3i\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]}) = 2\tilde{H}_{\mu\nu\rho\sigma}, \tag{2.64}$$

which then together with an overall factor of $\frac{1}{4}$ in the Lagrangian leads to

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int \mathrm{d}^{11}x \bigg[& \frac{1}{4} eR - \frac{1}{48} eH^2 - \frac{1}{3} B \wedge H \wedge H + \frac{i}{2} e\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \left[\tfrac{1}{2}(\omega + \tilde{\omega}) \right] \psi_\rho + \\ & + \frac{i}{192} e \Big(\bar{\psi}_\lambda \Gamma^{\mu\nu\rho\sigma\lambda\tau} \psi_\tau + 12 \bar{\psi}^\mu \Gamma^{\nu\rho} \psi^\sigma \Big) \Big(H_{\mu\nu\rho\sigma} + \tilde{H}_{\mu\nu\rho\sigma} \Big) \bigg]. \end{aligned} \tag{2.65}$$

This also causes a change in the transformation laws. Note that the fermionic parameter ϵ gets rescaled in the same way as ψ_μ, which leads to

$$\begin{aligned} \delta e_\sigma{}^\gamma = -i\bar{\epsilon}\Gamma^\gamma \psi_\sigma, \qquad \delta B_{\mu\nu\rho} = \frac{3}{2} i\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}, \\ \delta\psi = \tilde{D}_\mu \epsilon = D_\mu \epsilon + \frac{1}{144}(-8H_{\mu\rho\sigma\tau}\Gamma^{\rho\sigma\tau} + H_{\nu\rho\sigma\tau}\Gamma_\mu{}^{\nu\rho\sigma\tau})\epsilon. \end{aligned} \tag{2.66}$$

Finally, we summarize the rescaled equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{3}\Big[\tilde{H}_{\mu\rho\sigma\tau}\tilde{H}_{\nu}{}^{\rho\sigma\tau} - \frac{1}{8}g_{\mu\nu}\tilde{H}^2\Big]. \tag{2.67}$$

$$D_\mu \tilde{H}^{\mu\nu\rho\sigma} = \frac{1}{576}e\varepsilon^{\nu\rho\sigma\tau_1\ldots\tau_8}\tilde{H}_{\tau_1\ldots\tau_4}\tilde{H}_{\tau_5\ldots\tau_6}. \tag{2.68}$$

$$\Gamma^{\mu\nu\rho}\tilde{D}_\nu\psi_\rho = 0. \tag{2.69}$$

These results then match the results of Duff et.al. [24] up to signs and factors of i.

2.2.5 Supercovariant Quantities

Following this discussion, we summarize some important results here for future reference. First, we have the supercovariant connection

$$\tilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + \frac{i}{2}\bar{\psi}_\sigma\Gamma^{\sigma\tau}{}_{\mu\alpha\beta}\psi_\tau = \omega_{\mu\alpha\beta}(e) + i\Big(\bar{\psi}_\mu\Gamma_\alpha\psi_\beta - \bar{\psi}_\beta\Gamma_\mu\psi_\alpha + \bar{\psi}_\alpha\Gamma_\beta\psi_\mu\Big) \tag{2.70}$$

Any connection-dependent quantities, such as the Riemann tensor, with a tilde superscript are then meant to be defined with respect to this connection. Finally, the supercovariant field strength and the derivative is

$$\begin{aligned} \tilde{H}_{\mu\nu\rho\sigma} &= H_{\mu\nu\rho\sigma} - 3i\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]} \\ \tilde{D}_\mu &= D_\mu + \frac{1}{144}(H_{\nu\rho\sigma\tau}\Gamma_\mu{}^{\nu\rho\sigma\tau} - 8H_{\mu\nu\rho\sigma}\Gamma^{\nu\rho\sigma}) \end{aligned} \tag{2.71}$$

3

Seven-Sphere Compactification

This chapter will lay out the framework of the compactification from eleven to four dimensions. The Freund-Rubin ansatz for the compactification of the field strength will be described, followed by a thorough examination of the geometry of the squashed seven-sphere. We will see that the seven-sphere actually is a quite natural object to study in this context, as it is one of the simplest non-trivial manifolds that yields consistent compactification to four-dimensional spacetime.

3.1 The General Freund-Rubin Ansatz

Compactifying 11-dimensional supergravity to four dimensions with zero expectation value for the field strength, H_{MNPQ}, in all directions yields solutions for Ricci flat internal manifolds[24]. However, when allowing non-zero values for the expectation value of the field strength we will see that the constraint weakens and we can let the internal space admit an Einstein metric[1]. This allows for other choices of topologies, in particular higher dimensional spheres. Below we will consider a specific choice of background value for the field strength which is called the Freund-Rubin ansatz [30].

Since we will be discussing splitting of indices and gamma matrices between different manifolds, we will pass to a new notation, which differs from the one in chapter 2. Now, capital letters, $M, N, P, \ldots$ indicate 11-dimensional indices and Greek letters, $\mu, \nu, \rho, \ldots$ are 4-dimensional. This leaves the Latin indices, $m, n, p, \ldots$, for the 7-dimensional compact space. As usual, we demand vanishing fermionic expectation values, in order to preserve maximal symmetry

$$\langle \psi_M \rangle = 0. \tag{3.1}$$

Next, we will look for product solutions of the form $M_4 \times M_7$. This means that the background value of the 11-dimensional metric is

$$\langle \hat{g}_{MN}(x, y) \rangle = \begin{pmatrix} \mathring{g}_{\mu\nu}(x) & 0 \\ 0 & \mathring{g}_{mn}(y) \end{pmatrix}, \tag{3.2}$$

where the hat notation indicates 11-dimensional quantities and the superscript circle means background value. The coordinates x and y are coordinates on M_4 and M_7, respectively. Note that the x and y dependence split completely in this case, though in principle we could include a so called warp factor that incorporates x dependence in the 7-dimensional part. A maximally symmetric spacetime also demands the following form of the 4D components of the Riemann tensor

$$\mathring{R}_{\mu\nu\rho\sigma} = \frac{1}{3}\Lambda(\mathring{g}_{\mu\rho}\mathring{g}_{\nu\sigma} - \mathring{g}_{\mu\sigma}\mathring{g}_{\nu\rho}). \tag{3.3}$$

[1]An Einstein space is a space with Ricci tensor proportional to the metric.

We set the field strength to be non-zero only in the four-dimensional directions, which we write as

$$\langle H_{\mu\nu\rho\sigma} \rangle = \mathring{H}_{\mu\nu\rho\sigma}(x), \quad \langle H_{mnpq} \rangle = \mathring{H}_{mnpq}(y) = 0. \tag{3.4}$$

The x and y dependence here follow from the requirement that all "mixed" field strengths are zero and from the Bianchi identity. Now, we want the non-zero part of the field strength to be invariant under 4D spacetime symmetries, so we take it proportional to the only such four-index quantity we have available, the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$,

$$\mathring{H}_{\mu\nu\rho\sigma}(x) = -3m\mathring{\epsilon}_{\mu\nu\rho\sigma}, \tag{3.5}$$

with some constant m. This, together with setting the internal directions of the fields strength vacuum value to zero is what we call the Freund-Rubin ansatz [30]. It is worth noting that the object $\epsilon_{\mu\nu\rho\sigma}$ is the tensor, and not the symbol. From now on and until the end of this section, we drop the circle superscript notation, since we will exclusively be working with vacuum values.

Now, we will apply this ansatz to our theory. We start by recalling the field equations of 11-dimensional supergravity[2]

$$\begin{aligned} R_{MN} - \frac{1}{2}\hat{g}_{MN}R &= \frac{1}{3}\Big(H_{MPQR}H_N{}^{PQR} - \frac{1}{8}\hat{g}_{MN}H^2\Big), \\ D_M H^{MNPQ} &= -\frac{1}{4\cdot(12)^2}\epsilon^{NPQR_1\ldots R_8}H_{R_1R_2R_3R_4}H_{R_5R_6R_7R_8}. \end{aligned} \tag{3.6}$$

In the Freund-Rubin ansatz, the second equation is trivially fulfilled. The left hand side is zero since ϵ is constant, and the right hand side is zero since we would need to have repeated 4D indices in the 11-dimensional Levi-Civita tensor. The first equation then, is what will constrain our theory given the ansatz. When multiplied by $\hat{g}^{MN}$ it reduces to

$$\begin{gathered} R - \frac{1}{2}\delta^M_M R = \frac{1}{3}\Big(H^2 - \frac{1}{8}\delta^M_M H^2\Big) \implies R = \frac{1}{3}\frac{1-\frac{11}{8}}{1-\frac{11}{2}}H^2 = \frac{1}{36}H^2, \\ \text{where } H^2 = 9m^2\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -9m^2\cdot 4! \implies \boxed{R = -6m^2} \end{gathered} \tag{3.7}$$

We find the constraints on the 4- and 7-dimensional metrics by considering those two equations separately, and plugging in the value for the field strength. The 4-dimensional equation becomes

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(-6m^2) &= \frac{1}{3}\Big(9m^2\epsilon_{\mu\rho\sigma\tau}\epsilon_\nu{}^{\rho\sigma\tau} - \frac{1}{8}g_{\mu\nu}(-9\cdot 24m^2)\Big) = \\ = \{\epsilon_{\mu\rho\sigma\tau}\epsilon_\nu{}^{\rho\sigma\tau} &= -3!g_{\mu\nu}\} = 3\Big(-6m^2g_{\mu\nu} + 3m^2g_{\mu\nu}\Big) \implies \boxed{R_{\mu\nu} = -12m^2g_{\mu\nu}} \end{aligned} \tag{3.8}$$

and in an analogous way we find

$$\boxed{R_{mn} = 6m^2g_{mn}} \tag{3.9}$$

This is the constraint that we have background solutions where the internal space is an Einstein space. Note that this constraint comes purely from the 11-dimensional equations of motion and the choice of background value for the field strength. We have not yet specified the geometry of the internal manifold. In fact, one must not even specify the

[2] Here, the rescaling of the field strength discussed in the end of the last chapter has been incorporated.

dimensionality of the internal manifold. While we started from the assumption of a $M_4 \times M_7$ splitting, one can start by analysing the 11-dimensional field equations. Since the field strength has precisely four indices and should be proportional to some invariant tensor, the four-dimensional Levi-Civita tensor appears as the natural way of splitting up the theory. As a consequence, one could argue that 11-dimensional supergravity predicts the dimensionality of spacetime to be four [30, 31]. One could imagine turning things around and thinking of the four-dimensional manifold to instead be the compact one, but it turns out that this way of seeing it has some problems, described in detail in [24, 32].

Furthermore, we see that in this ansatz the seven-dimensional Ricci tensor actually corresponds to a compact space, since the constant $6m^2$ is positive, which is precisely what we desire. The four-dimensional part instead has a negative prefactor, $-12m^2$, and thus would imply AdS-space if we also want the background to be maximally symmetric.

We now proceed to see how the supercovariant derivatives, $\tilde{D}_M$ look in this ansatz. It will be important to us since they appear in the Killing spinor equation, which determines the number of surviving supersymmetries upon compactification. We will address this more carefully shortly. The derivatives are obtained by plugging in the definition of the field strength and using the following decomposition of the gamma matrices

$$\hat{\Gamma}_A = \begin{cases} \gamma_\alpha \otimes \mathbb{1} & A = \alpha = 0,1,2,3 \\ \gamma_5 \otimes \Gamma_a & A = a = 4,5,6,7,8,9,10 \end{cases} \tag{3.10}$$

where Γ_a are the seven-dimensional gamma matrices. Recall the supercovariant derivative

$$\tilde{D}_M = D_M + \frac{1}{144}\Big(H_{NPQR}\hat{\Gamma}_M{}^{NPQR} - 8H_{MNPQ}\hat{\Gamma}^{NPQ}\Big) \tag{3.11}$$

We first investigate the case when $M = m$. The last term then vanishes from the ansatz. The remaining term contains a rank 5 gamma matrix, with one 7-dimensional index and four 4-dimensional ones. The matrix γ_5 anti-commutes with all rank 1 gamma matrices and is defined as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta} = -\frac{i}{4!}\epsilon_{\nu\rho\sigma\tau}\gamma^{\nu\rho\sigma\tau} \tag{3.12}$$

Note that in the last equality the Levi-Civita symbol is changed to the tensor. The relevant identities for the Levi-Civita tensor can be found in appendix A.2. Using this we can rewrite the remaining term as

$$-\frac{3m}{144}\epsilon_{\nu\rho\sigma\tau}\hat{\Gamma}_m{}^{\nu\rho\sigma\tau} = -\frac{m}{48}\epsilon_{\nu\rho\sigma\tau}\gamma_5\gamma^{\nu\rho\sigma\tau}\otimes\Gamma_m = -\frac{mi4!}{48}\gamma_5\gamma_5\otimes\Gamma_m = -\frac{i}{2}m\Gamma_m \tag{3.13}$$

For the case $M = \mu$, it is instead the first term that vanish, since it will contain a 4-dimensional gamma matrix with five indices. Writing out the second term yields

$$\frac{8\cdot 3m}{144}\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}\otimes\mathbb{1} = \left\{\gamma_\mu\gamma_5 = -\frac{i}{4!}\epsilon_{\nu\rho\sigma\tau}\underbrace{\gamma_\mu\gamma^{\nu\rho\sigma\tau}}_{=4\delta_\mu^\nu\gamma^{\rho\sigma\tau}} = -\frac{i}{3!}\epsilon_{\mu\rho\sigma\tau}\gamma^{\rho\sigma\tau}\right\} = im\frac{3!}{6}\gamma_\mu\gamma_5 \tag{3.14}$$

Summarizing the answers we have

$$\begin{aligned} \tilde{D}_\mu &= D_\mu + im\gamma_\mu\gamma_5 \\ \tilde{D}_m &= D_m - \frac{i}{2}m\Gamma_m \end{aligned} \tag{3.15}$$

3.1.1 The Killing Spinor Equation

As mentioned earlier, we assume the background solutions we are currently considering to have a high degree of symmetry. Thus, it is natural to also demand preserved supersymmetry after compactification [33]. This requirement can be formulated in terms of the supersymmetry parameter ϵ as

$$\delta\psi_M = \tilde{D}_M \epsilon = 0 \tag{3.16}$$

This is the *Killing spinor equation*, and we say that those ϵ which are solutions to this equations are Killing spinors. Much like Killing vectors determine ordinary (bosonic) symmetries in other theories, this equation determines the number of supersymmetries [2, 22]. The Killing spinors are taken to separate as

$$\epsilon(x,y) = \epsilon(x)\eta(y) \tag{3.17}$$

and we may then look for solutions on the internal and external manifolds separately. Focusing on the constraints on the background solution for the internal manifold we thus look for solutions to the equation

$$\tilde{D}_m \eta = D_m \eta - \frac{im}{2}\Gamma_m \eta = 0 \tag{3.18}$$

Acting with another supercovariant derivative and taking the commutator then yields

$$\left[\tilde{D}_m, \tilde{D}_n\right]\eta = 0 \tag{3.19}$$

This is called the *integrability condition.* The name stems from the fact that if this condition is fulfilled, there may exist solutions obtainable by integrating the equation. Though this fact is not obvious when written in the current way. Writing the integrability condition out explicitly yields

$$\begin{aligned} 0 = [D_m, D_n]\eta + \left[\tfrac{im}{2}\Gamma_m, \tfrac{im}{2}\Gamma_n\right]\eta = \tfrac{1}{4}R_{mnab}\Gamma^{ab}\eta - \tfrac{m^2}{4}[\Gamma_m, \Gamma_n]\eta = \\ = \tfrac{1}{4}R_{mnab}\Gamma^{ab}\eta - \tfrac{m^2}{2}e_{ma}e_{nb}\Gamma^{ab}\eta = \tfrac{1}{4}C_{mn}{}^{ab}\Gamma_{ab}\eta, \end{aligned} \tag{3.20}$$

where $C_{mn}{}^{ab}$ is the Weyl tensor, defined in general dimensions d, by

$$C_{\mu\nu}{}^{\rho\sigma} = R_{\mu\nu}{}^{\rho\sigma} - \tfrac{4}{d-2}R_{[\mu}{}^{[\rho}g_{\nu]}{}^{\sigma]} + \tfrac{2}{(d-1)(d-2)}g_{[\mu}{}^{[\rho}g_{\nu]}{}^{\sigma]}. \tag{3.21}$$

Thus, in order to determine the number of supersymmetries surviving after compactifying the theory, we have to count the number of solutions to the equation

$$C_{mn}{}^{ab}\Gamma_{ab}\eta = 0. \tag{3.22}$$

At this point, if we specify the geometry of the internal manifold we could explicitly count the number of supersymmetries. We will not do this now, but return with a qualitative analysis of the equation later on in the chapter. In upcoming section, we will focus on understanding the properties of the seven-sphere and define what is meant by ”squashing” a sphere.

3.2 A Squashed Sphere in Higher Dimensions

Perhaps the best way to think of a round seven-sphere is as *the set of all points at a given distance from the origin in an eight-dimensional space.* Understanding the squashed sphere is more difficult but can be done in a number of ways. Here we will take the viewpoint of it as a fibre bundle. Another common approach is via quaternions[3] which is well described in [24].

3.2.1 Fibre Bundles

The fibre bundle structure is used to describe spaces which locally has the structure of a direct product of two other spaces, but not necessary so globally. It has many applications in physics, especially in gauge theories [34], although here we will mainly use it to better understand the structure of the squashed seven-sphere.

Definition 1 *A fibre bundle is defined by the following elements [34]:*

(i) The total space (differentiable manifold) E

(ii) The base space (differentiable manifold) M

(iii) The typical fibre (differentiable manifold) F

(iv) A surjective projection, π, from the total space to the base space. The inverse $\pi^{-1}(p) = F_p$ is called the fibre at p.

(v) The structure group G, which is a Lie group acting on the fibre

(vi) An open covering $\{U_i\}$ of M, with the mapping $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. ϕ_i is called the local trivialisation.

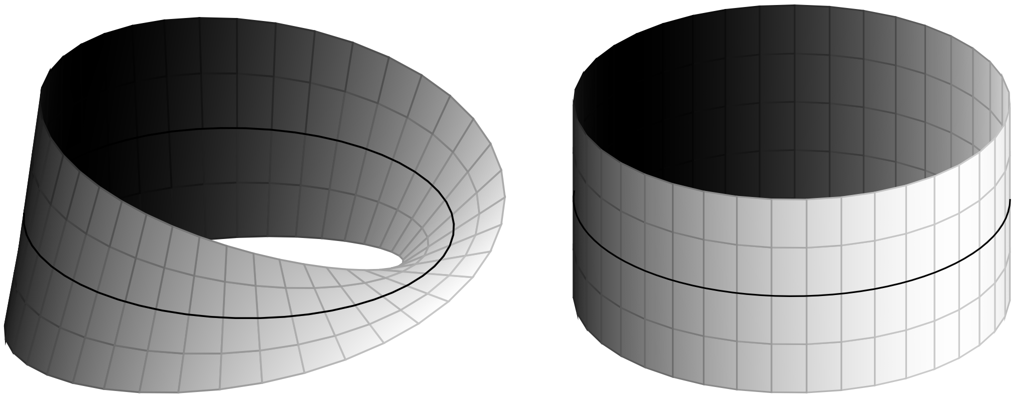
(vii) For some $p \in M$ define $\phi_i(p, f)$ by $\phi_{i,p}(f)$. Define the transition function between two patches by $t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$. On any overlap of patches, we demand that $t_{ij}(p) \in G$

This description is quite mathematical, and we will not need it in its entirety to understand the squashed seven-sphere. In fact, the terminology to be used when referring to a fibre bundle will be "*E is an F bundle over M*" without specifying the rest of the structure. The main point is what we stated above, the manifold E looks locally like $M \times F$. The projection π and the local trivialisations ϕ_i ensure that this local behaviour is consistent in all patches U_i and the constraint that the transition functions $t_{ij} \in G$ ensures that the patches can be "glued together" in a consistent way. If all transition functions are identity maps the bundles is called a *trivial bundle* and the total space, E is globally a direct product of $M \times F$.

A great example of where fibre bundles are useful is the Möbius band. First, define a cylinder as an $\mathbb{R}$ bundle over S^1. This means, that base space is a circle and the typical fibres are taken to be one-dimensional lines extending out from the circle. The cylinder is a trivial bundle; it is also globally described by a product of a circle and a line segment. In order to construct a Möbius band from a cylinder one makes a cut along the vertical direction, twists one end and glues it back together. In the language of fibre bundles, this construction corresponds to a a non-trivial transition function. Define the patches U_i to be two semicircles overlapping at their endpoints, and make the cut at the overlap and then "glue" it together. The transition function between the two patches is no longer the identity, instead it changes the orientation of fibre. The Möbius band is no longer globally

[3] Quaternions are, very simplified, a generalisation of complex numbers with three instead of one imaginary unit.

Figure 3.1: A Möbius band and a cylinder. The cylinder is both globally and locally the direct product $S^1 \times \mathbb{R}$, but the Möbius band only has this structure locally. In particular, it has only one edge.



a direct product space, but locally one cannot distinguish it from a cylinder.

3.2.2 The Seven-sphere as a Fibre Bundle

In the formalism developed above, the seven-sphere is an S^3 bundle over S^4. It does not have this structure globally however, which is why the fibre bundle description is necessary. The standard metric on S^4 is [24]

$$\mathrm{d}s^2(S^4) = \mathrm{d}\mu^2 + \frac{1}{4}\sin^2(\mu)\tilde{\Sigma}_i^2, \tag{3.23}$$

where the coordinate $\mu \in [0, \pi]$ and the $\tilde{\Sigma}_i$ are generators of $SU(2)$. If we wanted to construct a direct product of $S^4 \times S^3$ from this, we would add a new term similar to the second term but with a new set of $SU(2)$ generators. The procedure to obtain a fibre bundle structure is similar, but we also add a term inside the square of the new $SU(2)$ generators. In order for this to yield something non-trivial, we must take it to depend on the S^4 coordinates. Following Duff et al. [24] we take the new term to be

$$A_i = \sin^2(\tfrac{1}{2}\mu)\tilde{\Sigma}_i. \tag{3.24}$$

Introducing a local S^4 dependence on the S^3 is the same as specifying some gauge symmetry in the system, so we call this new term the gauge potential. The fibre bundle metric on S^7 is then

$$\mathrm{d}s^2(S^7) = \mathrm{d}\mu^2 + \frac{1}{4}\sin^2(\mu)\tilde{\Sigma}_i^2 + \lambda^2(\sigma_i - A_i)^2, \tag{3.25}$$

where the σ_i are a new set of $SU(2)$ generators. The squashing of the sphere is given by the parameter λ, which can be interpreted as changing the S^3 part relative to the S^4.

This way of seeing the seven-sphere is quite instructive, but not the best way when computing the Riemann tensor of the sphere. Therefore, we recast the metric to another form. First, we define the following linear combination

$$\sigma_i = \frac{1}{2}(\nu_i + \omega_i). \tag{3.26}$$

Plugging these into (3.25) yields

$$\begin{aligned} ds^2(S^7) =& d\mu^2 + \frac{1}{4}\sin^2(\mu)\tilde{\Sigma}_i^2 + \lambda^2\Big(\frac{1}{2}(\nu_i+\omega_i) - \sin^2(\frac{1}{2}\mu)\tilde{\Sigma}_i\Big)^2 \\ =& d\mu^2 + \frac{1}{4}\sin^2(\mu)\tilde{\Sigma}_i^2 + \lambda^2\Big(\frac{1}{2}(\nu_i+\omega_i) + \frac{1}{2}\cos(\mu)\tilde{\Sigma}_i - \frac{1}{2}\tilde{\Sigma}_i\Big)^2 \end{aligned} \tag{3.27}$$

Finally, we set $\tilde{\Sigma}_i = \omega_i$ and two terms in the last parentheses cancel. This leads to the metric as presented in [24] and is the one we will use in the following section.

3.3 Riemann Tensor of the Squashed Seven-sphere

In this section we will derive the Riemann curvature tensor for the squashed seven-sphere. At the end of the section we will see why this manifold is interesting when compactifying 11-dimensional supergravity.

Our starting point for the following calculation will be the defining equation for vanishing torsion, also known as the first Maurer-Cartan equation [34].

$$T^a = de^a + \omega^a{}_b \wedge e^b = 0, \tag{3.28}$$

where e^a are the frame 1-forms and $\omega^a{}_b$ is the connection 1-form. The metric on the squashed seven sphere is given by [24]

$$ds^2 = d\mu^2 + \frac{1}{4}\sin^2\mu\omega_i^2 + \frac{1}{4}\lambda^2(\nu_i + \cos\mu\omega_i)^2. \tag{3.29}$$

It is worth mentioning the notation here. When writing $\cos\mu\omega_i$ we always mean that the only argument of cosine is the symbol directly following, i.e., $\cos\mu\omega_i = \cos(\mu)\omega_i$. If there is more than one argument, we would always write out the parentheses explicitly. From the above metric we can read of the frame 1-forms

$$e^0 = d\mu, \qquad e^i = \frac{1}{2}\sin\mu\omega_i, \qquad e^{\hat{\imath}} = \frac{1}{2}\lambda(\nu_i + \cos\mu\omega_i). \tag{3.30}$$

It will be useful later on to also have these expression inverted

$$\omega_i = \frac{2}{\sin\mu}e^i, \qquad \nu_i = \frac{2}{\lambda}e^{\hat{\imath}} - \cot\mu e^k. \tag{3.31}$$

The forms ω_i and ν_i are related by

$$\nu_i = \sigma_i + \Sigma_i, \qquad \omega_i = \sigma_i - \Sigma_i, \tag{3.32}$$

where σ and Σ are left-invariant one-forms satisfying the $SU(2)$ algebra

$$d\sigma_i = -\frac{1}{2}\varepsilon_{ijk}\sigma_j\wedge\sigma_k, \qquad d\Sigma_i = -\frac{1}{2}\varepsilon_{ijk}\Sigma_j\wedge\Sigma_k. \tag{3.33}$$

In order to find the curvature tensor we will first have to solve for all components of the connection 1-form. Note the notation introduced above where we split up the 7-dimensional indices as $a = (0, i, \hat{\imath})$, where $i = 1, 2, 3$ and $\hat{\imath} = 4, 5, 6 = \hat{1}, \hat{2}, \hat{3}$. Thus, we will rewrite the connections in terms of their components in the frame fields as follows

$$\omega_{ab} = \omega_{0ab}e^0 + \omega_{iab}e^i + \omega_{\hat{\imath}ab}e^{\hat{\imath}}. \tag{3.34}$$

But first, we write out the solutions to the Maurer-Cartan equation, $\mathrm{d}e^a = \omega_{ba} \wedge e^b$ explicitly for the three cases. We start with the simplest case and do it explicitly for clarity. We have

$$\mathrm{d}e^0 = \mathrm{d}\mathrm{d}\mu = 0 = \omega_{00} \wedge e^0 + \omega_{i0} \wedge e^i + \omega_{\hat{\imath}0} \wedge e^{\hat{\imath}}. \tag{3.35}$$

The symmetry $\omega_{(ab)} = 0$ means that the first term vanish. For the remaining two terms we split up the connection in components. Then we find

$$0 = \omega_{0i0}e^0 \wedge e^i + \omega_{ji0}e^j \wedge e^i + \omega_{\hat{\jmath}i0}e^{\hat{\jmath}} \wedge e^i + \omega_{0\hat{\imath}0}e^0 \wedge e^{\hat{\imath}} + \omega_{j\hat{\imath}0}e^j \wedge e^{\hat{\imath}} + \omega_{\hat{\jmath}\hat{\imath}0}e^{\hat{\jmath}} \wedge e^{\hat{\imath}}. \tag{3.36}$$

Since the left-hand side is zero, all these terms must vanish separately. However, due to the anti-symmetric property of the wedge product we only know that the component with antisymmetry in the first two indices vanish. Summarizing the information obtained from the equation above then leads to

$$\begin{aligned} &\omega_{[0i]0} = 0 \implies \boxed{\omega_{0i0} = 0} \qquad \omega_{[ji]0} = 0, \qquad \omega_{[\hat{\jmath}i]0} = 0, \\ &\omega_{[0\hat{\imath}]0} = 0 \implies \boxed{\omega_{0\hat{\imath}0} = 0} \qquad \omega_{[\hat{\jmath}\hat{\imath}]0} = 0. \end{aligned} \tag{3.37}$$

The boxed equations follow from the antisymmetry of the connection in the two last indices, which implies $\omega_{ab}{}^b = 0$. The remaining three equations will allow us to deduce the remaining components later on, but first we need to solve the remaining Maurer-Cartan equations.

$$\begin{aligned} \mathrm{d}e^i &= \tfrac{1}{2}(\mathrm{d}\sin\mu)\omega_i + \tfrac{1}{2}\sin\mu\mathrm{d}\omega_i = \tfrac{1}{2}\cos\mu e^0 \wedge \omega_i - \tfrac{1}{4}\sin\mu(\varepsilon_{ijk}(\sigma_j \wedge \sigma_k - \Sigma_j \wedge \Sigma_k)) \\ &= \Big\{\omega_j \wedge \nu_k = \sigma_j \wedge \sigma_k + \sigma_j \wedge \Sigma_k - \Sigma_j \wedge \sigma_k - \Sigma_j \wedge \Sigma_k \implies \\ &\varepsilon_{ijk}\omega_j \wedge \nu_k = \varepsilon_{ijk}\sigma_j \wedge \sigma_k - \varepsilon_{ijk}\Sigma_j \wedge \Sigma_k\Big\} = \tfrac{1}{2}\cos\mu e^0 \wedge \omega_i - \tfrac{1}{4}\sin\mu\varepsilon_{ijk}\omega_j \wedge \nu_k \\ &= \cot\mu e^0 \wedge e^i - \tfrac{1}{2}\varepsilon_{ijk}e^j \wedge \nu_k = \cot\mu e^0 \wedge e^i - \tfrac{1}{\lambda}\varepsilon_{ijk}e^j \wedge e^{\hat{k}} + \cot\mu\varepsilon_{ijk}e^j \wedge e^k. \end{aligned} \tag{3.38}$$

Here we have used the expressions in (3.31) in order to express everything as exterior products of the frame 1-forms. This will allows us to read off the connection components from the right hand side of the Maurer-Cartan equation

$$\begin{aligned} \mathrm{d}e^i = {} & \omega_{j0i}e^j \wedge e^0 + \omega_{\hat{\jmath}0i}e^{\hat{\jmath}} \wedge e^0 + \\ & + \omega_{0ji}e^0 \wedge e^j + \omega_{kji}e^k \wedge e^j + \omega_{\hat{k}ji}e^{\hat{k}} \wedge e^j + \\ & + \omega_{0\hat{\jmath}i}e^0 \wedge e^{\hat{\jmath}} + \omega_{k\hat{\jmath}i}e^k \wedge e^{\hat{\jmath}} + \omega_{\hat{k}\hat{\jmath}i}e^{\hat{k}} \wedge e^{\hat{\jmath}}. \end{aligned} \tag{3.39}$$

We find that there is five different terms above, since $e^j \wedge e^0 = -e^0 \wedge e^j$ and so on. Comparing antisymmetrised coefficients then yields the following

$$\begin{aligned} &2\omega_{[0j]i} = \cot\mu\delta_{ij} \implies \boxed{\omega_{j0i} = -\cot\mu\delta_{ij}} \qquad \omega_{[0\hat{\jmath}]i} = 0, \\ &\omega_{[jk]i} = \varepsilon_{ijk}\cot\mu \implies \boxed{\omega_{kij} = \varepsilon_{ijk}\cot\mu} \qquad 2\omega_{[j\hat{k}]i} = -\frac{1}{\lambda}\varepsilon_{ijk}, \qquad \omega_{[\hat{k}\hat{\jmath}]i} = 0. \end{aligned} \tag{3.40}$$

The boxed equations follow immediately from the symmetry of δ_{ij} and the antisymmetry of ε_{ijk}, respectively. Next, we solve the last Maurer-Cartan equation

$$\begin{aligned} \mathrm{d}e^{\hat{\imath}} &= \tfrac{1}{2}\lambda(\mathrm{d}\nu_i + \mathrm{d}(\cos\mu\omega_i)) = \tfrac{1}{2}\lambda\Big(- \tfrac{1}{2}\varepsilon_{ijk}(\sigma_j \wedge \sigma_k + \Sigma_j \wedge \Sigma_k) - \sin\mu e^0 \wedge \omega_i + \\ &\quad + \cos\mu(-\tfrac{1}{2}\varepsilon_{ijk}\omega_j \wedge \nu_k)\Big) = \Big\{\nu_j \wedge \nu_k + \omega_j \wedge \omega_k = 2(\sigma_j \wedge \sigma_k + \Sigma_j \wedge \Sigma_k)\Big\} = \\ &= \Big(- \frac{\lambda}{2\sin^2\mu}\varepsilon_{ijk} - \frac{\lambda\cot^2\mu}{2}\varepsilon_{ijk} + \lambda\cot^2\mu\varepsilon_{ijk}\Big)e^j \wedge e^k - \frac{1}{2\lambda}\varepsilon_{ijk}e^{\hat{\jmath}} \wedge e^{\hat{k}} - \lambda e^0 \wedge e^i \end{aligned} \tag{3.41}$$

Here we have again used the results of (3.31) to rewrite the results. Note that there is two $e^{\hat{\jmath}} \wedge e^k$ terms in the above calculation that we have not written out since they cancel each other. Now we use that

$$\cot^2\mu - \frac{\cot^2\mu}{2} - \frac{1}{2\sin^2\mu} = \frac{1}{2\sin^2\mu}(\cos^2\mu - 1) = -\frac{1}{2}, \tag{3.42}$$

to obtain

$$\mathrm{d}e^{\hat{\imath}} = -\frac{\lambda}{2}\varepsilon_{ijk}e^j \wedge e^k - \frac{1}{2\lambda}\varepsilon_{ijk}e^{\hat{\jmath}} \wedge e^{\hat{k}} - \lambda e^0 \wedge e^i. \tag{3.43}$$

Comparing this to the right hand side

$$\begin{aligned}\mathrm{d}e^{\hat{\imath}} = {}& \omega_{j0\hat{\imath}}e^j \wedge e^0 + \omega_{\hat{\jmath}0\hat{\imath}}e^{\hat{\jmath}} \wedge e^0 + \\ & + \omega_{0j\hat{\imath}}e^0 \wedge e^j + \omega_{kj\hat{\imath}}e^k \wedge e^j + \omega_{\hat{k}j\hat{\imath}}e^{\hat{k}} \wedge e^j + \\ & + \omega_{0\hat{\jmath}\hat{\imath}}e^0 \wedge e^{\hat{\jmath}} + \omega_{k\hat{\jmath}\hat{\imath}}e^k \wedge e^{\hat{\jmath}} + \omega_{\hat{k}\hat{\jmath}\hat{\imath}}e^{\hat{k}} \wedge e^{\hat{\jmath}},\end{aligned} \tag{3.44}$$

gives the following results

$$\begin{aligned}\omega_{[0j]\hat{\imath}} = -\tfrac{1}{2}\lambda\delta_{ij}, \qquad \omega_{[0\hat{\jmath}]\hat{\imath}} = 0, \qquad \omega_{[jk]\hat{\imath}} = -\varepsilon_{ijk}\tfrac{\lambda}{2} \implies \boxed{\omega_{jk\hat{\imath}} = -\tfrac{\lambda}{2}\varepsilon_{ijk}} \\ \omega_{[j\hat{k}]\hat{\imath}} = 0, \qquad \omega_{[\hat{\jmath}\hat{k}]\hat{\imath}} = -\tfrac{1}{2\lambda}\varepsilon_{ijk} \implies \boxed{\omega_{\hat{\jmath}\hat{k}\hat{\imath}} = -\tfrac{1}{2\lambda}\varepsilon_{ijk}}\end{aligned} \tag{3.45}$$

From the above analysis we have determined six components of ω_{abc} completely, the boxed equations. Since the 1-form ω_{bc} is antisymmetric in its last two indices, each taking three values, $0, i, \hat{\imath}$, there are five such omegas. Each having three components then amounts to a total of 15 components to compute. To figure out the remaining components we will use the identity

$$\omega_{abc} = \omega_{[ab]c} + \omega_{[ca]b} - \omega_{[bc]a}, \tag{3.46}$$

which follows from the fact that ω_{abc} is anti-symmetric in its last two indices. We then find

$$\begin{aligned}&\omega_{\hat{\jmath}0i} = 0 + 0 + \tfrac{1}{2}\lambda\delta_{ij} = \tfrac{1}{2}\lambda\delta_{ij}, && \\ &\omega_{j0\hat{\imath}} = \tfrac{1}{2}\lambda\delta_{ij} + 0 - 0 = \tfrac{1}{2}\lambda\delta_{ij}, && \omega_{\hat{\jmath}0\hat{\imath}} = 0 + 0 - 0 = 0, \\ &\omega_{0ij} = \tfrac{1}{2}\cot\mu\delta_{ij} - \tfrac{1}{2}\cot\mu\delta_{ij} - 0 = 0, && \omega_{\hat{k}ij} = \tfrac{1}{2\lambda}\varepsilon_{jik} - \tfrac{1}{2\lambda}\varepsilon_{ijk} + \varepsilon_{kij}\tfrac{\lambda}{2} = \varepsilon_{ijk}\Big(\tfrac{\lambda}{2} - \tfrac{1}{\lambda}\Big), \\ &\omega_{0i\hat{\jmath}} = -\tfrac{1}{2}\lambda\delta_{ij} + 0 - 0 = -\tfrac{1}{2}\lambda\delta_{ij}, && \omega_{\hat{k}i\hat{\jmath}} = 0 + 0 - 0 = 0, \\ &\omega_{0\hat{\imath}\hat{\jmath}} = 0 + 0 - 0 = 0, && \omega_{k\hat{\imath}\hat{\jmath}} = 0 + 0 - 0 = 0.\end{aligned}$$

Summarizing our results we then obtain

$$\begin{aligned}&\omega_{0i} = -\cot\mu e^i + \tfrac{\lambda}{2}e^{\hat{\imath}}, \qquad \omega_{0\hat{\imath}} = \tfrac{\lambda}{2}e^i, \qquad \omega_{ij} = \varepsilon_{ijk}\cot\mu e^k + \varepsilon_{ijk}\Big(\tfrac{\lambda}{2} - \tfrac{1}{\lambda}\Big)e^{\hat{k}}, \\ &\omega_{i\hat{\jmath}} = -\tfrac{\lambda}{2}\delta_{ij}e^0 - \tfrac{\lambda}{2}\varepsilon_{ijk}e^k, \qquad \omega_{\hat{\imath}\hat{\jmath}} = -\tfrac{1}{2\lambda}\varepsilon_{ijk}e^{\hat{k}}.\end{aligned} \tag{3.47}$$

3.3.1 Computing the Curvature Tensor

Having determined all components of the connection we are now ready to compute the curvature 2-form, Θ_{ij}. Writing out the curvature 2-form explicitly will then let us read off the components of the Riemann tensor. The curvature 2-form is defined by

$$\Theta_{ab} \equiv \mathrm{d}\omega_{ab} + \omega_{ac} \wedge \omega_{cb}, \tag{3.48}$$

and related to the Riemann tensor as

$$\Theta_{ab} = \tfrac{1}{2} R_{abcd} e^c \wedge e^d. \tag{3.49}$$

The components are computed by taking the results in (3.47), plugging them into (3.48) and computing each component at a time. By using equations (3.38) and (3.43) we are left with only algebra that needs to be done, eventually leading up to the following result

$$\begin{aligned}
\Theta_{0i} &= e^0 \wedge e^i \Big(1 - \tfrac{3}{4}\lambda^2\Big) + \varepsilon_{ijk} e^{\hat{\jmath}} \wedge e^{\hat{k}} \tfrac{1}{4}\Big(1 - \lambda^2\Big), \\
\Theta_{0\hat{\imath}} &= e^0 \wedge e^{\hat{\imath}} \tfrac{1}{4}\lambda^2 + \varepsilon_{ijk} e^j \wedge e^{\hat{k}} \tfrac{1}{4}\Big(\lambda^2 - 1\Big), \\
\Theta_{ij} &= e^i \wedge e^j \Big(1 - \tfrac{3}{4}\lambda^2\Big) + e^{\hat{\imath}} \wedge e^{\hat{\jmath}} \tfrac{1}{2}\Big(1 - \lambda^2\Big), \\
\Theta_{i\hat{\jmath}} &= \varepsilon_{ijk} e^0 \wedge e^{\hat{k}} \tfrac{1}{4}\Big(\lambda^2 - 1\Big) + e^i \wedge e^{\hat{\jmath}} \tfrac{1}{4}\lambda^2 + e^j \wedge e^{\hat{\imath}} \tfrac{1}{4}\Big(\lambda^2 - 1\Big) + e^m \wedge e^{\hat{m}} \delta_{ij} \tfrac{1}{4}\Big(1 - \lambda^2\Big), \\
\Theta_{\hat{\imath}\hat{\jmath}} &= \varepsilon_{ijk} e^0 \wedge e^k \tfrac{1}{2}\Big(1 - \lambda^2\Big) + e^i \wedge e^j \tfrac{1}{2}\Big(1 - \lambda^2\Big) + e^{\hat{\imath}} \wedge e^{\hat{\jmath}} \frac{1}{4\lambda^2}.
\end{aligned} \tag{3.50}$$

This fully determines the Riemann tensor. For example, if one seeks R_{0i0j}, we turn to the first line and read off to find

$$R_{0i0j} = \delta_{ij}\Big(1 - \tfrac{3}{4}\lambda^2\Big). \tag{3.51}$$

Terms like $R_{ki\ell j}$ are somewhat trickier to find immediately, the procedure is

$$\begin{aligned}
\Theta_{ki} = \cdots + e^k \wedge e^i \Big(1 - \tfrac{3}{4}\lambda^2\Big) = \cdots + \delta^k_\ell \delta^i_j e^\ell \wedge e^j \Big(1 - \tfrac{3}{4}\lambda^2\Big) = \tfrac{1}{2} R_{kiab} e^a \wedge e^b \\
\implies \tfrac{1}{2} R_{ki[\ell j]} = \delta^k_{[\ell}\delta^i_{j]}\Big(1 - \tfrac{3}{4}\lambda^2\Big) \implies R_{ki\ell j} = 2\delta^{ki}_{\ell j}\Big(1 - \tfrac{3}{4}\lambda^2\Big).
\end{aligned} \tag{3.52}$$

The most complicated terms to read off are those mixing hatted and unhatted coordinates, partly because in those cases we cannot naively impose the antisymmetry as we did above. Let us consider an example. In order to find $R_{k\hat{\imath}\ell\hat{\jmath}}$ we need to do the following

$$\begin{aligned}
\Theta_{k\hat{\imath}} = \cdots + \Big(\delta^k_\ell \delta^i_j \tfrac{1}{4}\lambda^2 + \delta^i_\ell \delta^k_j \tfrac{1}{4}\Big(\lambda^2 - 1\Big) + \delta_{\ell j}\delta^{ik}\tfrac{1}{4}\Big(1 - \lambda^2\Big)\Big) e^\ell \wedge e^{\hat{\jmath}} = \tfrac{1}{2} R_{k\hat{\imath}ab} e^a \wedge e^b \\
\implies R_{k\hat{\imath}[\ell\hat{\jmath}]} = R_{k\hat{\imath}\ell\hat{\jmath}} = \delta^k_\ell \delta^i_j \tfrac{1}{4}\lambda^2 + \delta^i_\ell \delta^k_j \tfrac{1}{4}\Big(\lambda^2 - 1\Big) + \delta_{\ell j}\delta^{ik}\tfrac{1}{4}\Big(1 - \lambda^2\Big).
\end{aligned} \tag{3.53}$$

Note that the factor of $\frac{1}{2}$ is not there in this case, it vanishes since the exterior product appears in two ways related by antisymmetry when we expand the sum in a and b. Furthermore, we do not impose explicit antisymmetrisation on the right hand side. The reason is actually due to our notation. Consider the last term, where we actually have mixed indices in the Kronecker deltas, e.g. $\delta_{\ell\hat{\jmath}}$. We have defined this quantity in the following way

$$\delta_{\ell\hat{\jmath}} = \begin{cases} 1, & \ell = 1 \text{ and } \hat{\jmath} = \hat{1} = 4 \\ 1, & \ell = 2 \text{ and } \hat{\jmath} = \hat{2} = 5 \\ 1, & \ell = 3 \text{ and } \hat{\jmath} = \hat{3} = 6 \\ 0, & \text{else.} \end{cases} = \begin{pmatrix} & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}. \tag{3.54}$$

So, while written as a δ, it is not really symmetric under interchange of hatted and unhatted indices. This becomes evident when viewing it in the matrix form presented above. So, we will leave the expression in equation (3.53) as it is of now, and then it will simplify when we compute the Ricci tensor later on.

Now, we are ready to use the above result to compute the Ricci tensor on the squashed seven sphere. It is as usual defined by

$$R_{ab} = R^{c}{}_{acb} = R_{cadb}\delta^{dc}. \tag{3.55}$$

It is worth noting that upstairs or downside indices do not matter here, since both (hatted and unhatted) flat metrics are Euclidean. The following identity will come in handy in the calculations to follow

$$\delta^{ki}_{\ell j}\delta^{\ell}_{k} = \frac{D-1}{2}\delta^{i}_{j}. \tag{3.56}$$

For $D = 3$, which is the case for the hatted and unhatted metrics, the prefactor is just 1. We now start by computing the zero-components of the Ricci tensor

$$R_{00} = R_{i0j0}\delta^{j}_{i} + R_{\hat{\imath}0\hat{\jmath}0}\delta^{j}_{i} = \delta_{ij}\Big(1 - \tfrac{3}{4}\lambda^2\Big)\delta^{j}_{i} + \delta_{ij}\tfrac{1}{4}\lambda^2\delta^{j}_{i} = 3 - \tfrac{9}{4}\lambda^2 + \tfrac{3}{4}\lambda^2 = 3\Big(1 - \tfrac{1}{2}\lambda^2\Big). \tag{3.57}$$

$$R_{0i} = R_{j0ki}\delta^{j}_{k} + R_{\hat{\jmath}0\hat{k}i}\delta^{j}_{k} = 0 + \varepsilon_{jik}\tfrac{1}{4}\Big(\lambda^2 - 1\Big)\delta^{j}_{k} = \{\varepsilon_{kik} = 0\} = 0. \tag{3.58}$$

$$R_{0\hat{\imath}} = R_{j0k\hat{\imath}}\delta^{j}_{k} + R_{\hat{\jmath}0\hat{k}\hat{\imath}}\delta^{j}_{k} = 0 + 0. \tag{3.59}$$

Then, proceeding with R_{ij} we will use (3.52), (3.53) and the identity in (3.56). This leads to

$$\begin{aligned} R_{ij} =& R_{0i0j} + R_{ki\ell j}\delta^{\ell}_{k} + R_{\hat{k}i\hat{\ell}j}\delta^{\ell}_{k} = \delta_{ij}\Big(1 - \tfrac{3}{4}\lambda^2\Big) + 2\delta^{ki}_{\ell j}\Big(1 - \tfrac{3}{4}\lambda^2\Big)\delta^{\ell}_{k} \\ &+ \Big(\delta^{k}_{\ell}\delta^{i}_{j}\tfrac{1}{4}\lambda^2 + \delta^{i}_{\ell}\delta^{k}_{j}\tfrac{1}{4}\Big(\lambda^2 - 1\Big) + \delta_{\ell j}\delta^{ik}\tfrac{1}{4}\Big(1 - \lambda^2\Big)\Big)\delta^{\ell}_{k} = \\ =& 3\delta_{ij}\Big(1 - \tfrac{3}{4}\lambda^2\Big) + \delta^{i}_{j}\tfrac{3}{4}\lambda^2 + \delta^{i}_{j}\tfrac{1}{4}\Big(\lambda^2 - 1\Big) - \delta^{i}_{j}\tfrac{1}{4}\Big(\lambda^2 - 1\Big) = \delta_{ij}3\Big(1 - \tfrac{1}{2}\lambda^2\Big), \end{aligned} \tag{3.60}$$

where we also used the fact that $\delta^{i}_{\ell}\delta^{k}_{j}\delta^{\ell}_{k} = \delta^{i}_{j}$ and $\delta^{k}_{\ell}\delta^{\ell}_{k} = 3$. Next is the mixed term, which we immediately find to be vanishing

$$R_{i\hat{\jmath}} = R_{0i0\hat{\jmath}} + R_{ki\ell\hat{\jmath}}\delta^{l}_{k} + R_{\hat{k}i\hat{\ell}\hat{\jmath}}\delta^{\ell}_{k} = 0 + 0 + 0. \tag{3.61}$$

Then finally we compute $R_{\hat{\imath}\hat{\jmath}}$, making use of the results in (3.60) to find

$$R_{\hat{\imath}\hat{\jmath}} = R_{0\hat{\imath}0\hat{\jmath}} + R_{k\hat{\imath}\ell\hat{\jmath}}\delta^{\ell}_{k} + R_{\hat{k}\hat{\imath}\hat{\ell}\hat{\jmath}}\delta^{l}_{k} = \delta_{ij}\tfrac{1}{4}\lambda^2 + \delta_{ij}\tfrac{3}{4}\lambda^2 + \delta^{ki}_{\ell j}\frac{1}{2\lambda^2}\delta^{\ell}_{k} = \delta_{ij}\left(\lambda^2 + \frac{1}{2\lambda^2}\right). \tag{3.62}$$

So we find that the Ricci tensor is indeed diagonal. In order for it to be an Einstein metric we require it to be proportional to the metric and thus we solve

$$3 - \tfrac{3}{2}\lambda^2 = \lambda^2 + \frac{1}{2\lambda^2} \implies \lambda^2 = \frac{3}{5} \pm \frac{2}{5}. \tag{3.63}$$

That is, apart from the round sphere with $\lambda^2 = 1$ (which we knew was an Einstein metric, since it is maximally symmetric) the squashed sphere with squashing parameter $\lambda^2 = \frac{1}{5}$ also admits an Einstein metric.

It is quite interesting that there is only one specific configuration where we can compactify 11-dimensional supergravity on a squashed seven-sphere in the Freund-Rubin ansatz. Having determined the geometry of the internal manifold we return to the Killing spinor equation

$$C_{mn}{}^{ab}\Gamma_{ab}\eta = 0. \tag{3.64}$$

Consider first the case when $\lambda = 1$ and the round seven-sphere. Since it is a maximally symmetric space the Weyl tensor is zero and the number of solutions is equal to the number of components of the spinor[2]. In $d = 7$ dimensions, a spinor has $2^{(d-1)/2} = 8$ components and thus we have 8 preserved supersymmetries, which means we have an $\mathcal{N} = 8$ supergravity theory in four dimensions. In the case of the squashed seven-sphere, we do not have maximal symmetry and the Weyl tensor is non-zero. Using the results of Awada et al. [35], the Weyl tensor on the squashed S^7 can be seen to have 14 independent components. One can then go to a specific representation of the seven-dimensional gamma matrices and solve the Killing spinor equation. It turns out that there exists exactly one solution for the squashed seven-sphere [35] in the Freund-Rubin ansatz. More importantly, Duff et al. [24] have also showed that if one changes the orientation of the sphere, by a parity transformation of the internal metric, there are no solutions to the Killing spinor equation. We will return to this fact later on, and discuss it in the context of the Swampland.

4

Stability of the S^7 Scale and Squashing Modes

In this chapter, we will perform the actual compactification on the seven-sphere and analyse the effective four-dimensional theory it gives rise to. First however, it is worth taking a step back and summarise what we have done so far.

Our starting point has been 11-dimensional supergravity, the low-energy limit of M-theory which unifies all five versions of string theory. Due to the theory's higher-dimensional nature, we must treat the extra dimensions as directions on a compact manifold. In the previous chapter we saw that the theory spontaneously compactifies to exactly four dimensions, due to the four index structure of the field strength. We chose the background ansatz of Freund-Rubin and found that this demanded a hyperbolic four-dimensional spacetime (AdS) and a compact seven-dimensional Einstein space. This is fulfilled by a round seven-sphere, but we showed that a certain distortion of the S^7 metric also yields an Einstein space. We stated that the main difference between these two solutions is the number of supersymmetries in the four-dimensional resulting theory.

We start this chapter by letting the squashing of the sphere be spacetime dependent, in order to obtain a scalar potential in the four-dimensional theory. Then, we compute the new curvature tensor and find the equations of motion. Lastly, we will use these equations to determine the Lagrangian and the effective four-dimensional scalar potential. We also comment on the solutions linearised around the squashed ground state.

4.1 Spacetime Scalars in the Metric

We begin by writing down a general 11-dimensional metric that is the product of an arbitrary spacetime metric and a squashed S^7. However, we now want to promote the squashing parameter to a spacetime dependent Lorentz scalar and introduce a so called *scaling parameter* which controls the overall size of the S^7. We will denote these parameters $v(x)$ and $u(x)$, respectively. In general, when we want distinguish between coordinate dependencies we take x to be coordinates on spacetime and y to be coordinates on S^7. The squashing parameter will relate to the now x-dependent λ from the previous chapter as

$$\lambda^2(x) = e^{-7v(x)}. \tag{4.1}$$

We will also introduce the following combinations of the scalar fields u and v

$$A \equiv \tfrac{7}{2}u, \qquad B \equiv -u - \tfrac{3}{2}v, \qquad C \equiv -u + 2v. \tag{4.2}$$

The reason for the above definitions is only to write the metric in a convenient form. Note especially that $2(B-C)=7v$. We follow [36] and write the general 11-dimensional metric as

$$\begin{aligned} \mathrm{d}\hat{s}^2 = \hat{\eta}_{AB}\mathrm{d}X^A\mathrm{d}X^B =& e^{-2A}\eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta \\ &+ e^{-2B}\left[\frac{1}{4}\mathrm{d}\mu^2 + \frac{1}{16}\sin^2\mu\sum_{i=1}^{3}\omega_i^2\right] \\ &+ e^{-2C}\sum_{i=1}^{3}\frac{1}{16}(\nu_i+\cos\mu\omega_i)^2. \end{aligned} \tag{4.3}$$

The prefactor e^{-2A} in front of the four-dimensional metric is purely there for convenience. The scalars u and v are parameters of the S^7 but we are free to also choose any overall scale of the metric. This specific choice will eliminate any u-dependent factors in the Ricci scalar term in the resulting Lagrangian. In terms of the seven-sphere we looked at in the last chapter the metric is

$$\mathrm{d}\hat{s}^2 = e^{-2A}\eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta + \frac{e^{-2B}}{4}\times\mathrm{d}s^2\Big(S^7\Big)\Big|_{\lambda=\lambda(x)} \tag{4.4}$$

where $\mathrm{d}s^2(S^7)$ is the same as in equation (3.29). Our goal is now to find the Ricci tensor of this metric. Thanks to our notation, we will be able to reuse our results from the previous section. In order to do this we must reverse engineer this metric from a direct product of spacetime and S^7. We will first define the frame fields in the following way

$$e^\alpha = e^\alpha(x), \qquad e^0 = e^0(y) = \tfrac{1}{2}\mathrm{d}\mu, \tag{4.5}$$

$$e^i = e^i(y) = \tfrac{1}{4}\sin\mu\omega^i, \qquad e^{\hat{\imath}} = e^{\hat{\imath}}(x,y) = \tfrac{1}{4}e^{-\frac{7}{2}v(x)}(\nu_i+\cos\mu\omega^i). \tag{4.6}$$

Note that these definitions are very similar to what we used earlier, but now with an additional factor of $\frac{1}{2}$. This is to absorb the $\frac{1}{4}$ in $\mathrm{d}\hat{s}^2$. Since we aim to reuse our previous results we must note that this change will increase the previous answers for the Ricci tensor on S^7 by a factor of 4. In fact, it will be useful to recall the previous results rewritten in terms of the scalar v. We have

$$R_{ab} = \delta_{ab}\Big(12-6e^{-7v}\Big), \qquad R_{\hat{\imath}\hat{\jmath}} = \delta_{ij}\Big(4e^{-7v}+2e^{7v}\Big). \tag{4.7}$$

Here, the index $a=(0,i)$ is on S^4 while $\hat{\imath}$ is on S^3 when we write S^7 as a twisted product of $S^4\times S^3$. Now, in terms of these frame fields we see that the metric can be written as

$$\mathrm{d}\hat{s}^2 = e^{-2B}\Big[e^{-2(A-B)}e^\alpha e^\alpha + e^a e^a + e^{\hat{\imath}}e^{\hat{\imath}}\Big]. \tag{4.8}$$

By introducing the rescaled frame field $\tilde{e}^\alpha = e^{-(A-B)}e^\alpha$, we note that this metric is related by a Weyl rescaling with e^{-2B} to the metric

$$\mathrm{d}\bar{s}^2 = \tilde{e}^\alpha\tilde{e}^\alpha + e^a e^a + e^{\hat{\imath}}e^{\hat{\imath}}. \tag{4.9}$$

This metric is very close to a direct product of an arbitrary spacetime metric and an S^7 metric. However, since there is an x-dependent factor in $e^{\hat{\imath}}$ we cannot directly reuse our old results. However, it is straightforward to compute the Ricci tensor of this metric in terms of our old results. In order to simplify the upcoming computations we will introduce the notation were we split up the exterior derivative as

$$\mathrm{d} = \mathrm{d}_4 + \mathrm{d}_7. \tag{4.10}$$

The first term acts only on spacetime dependent quantities and the second terms acts only on the internal directions. Thus, terms like $\mathrm{d}_7 e^{\hat{\imath}}$ will produce exactly the results we have obtained earlier, since the x-dependent prefactor is a constant with respect to the internal coordinates. Again we will start from the first Maurer-Cartan equation

$$\mathrm{d}\tilde{e}^A = \bar{\omega}_{[BC]}{}^A \tilde{e}^B \wedge \tilde{e}^C. \tag{4.11}$$

We use the $\bar{\omega}$ notation here to denote that this ω receive corrections from the $\lambda(x)$ prefactor in $e^{\hat{\imath}}$. However, since λ does not depend on the internal coordinates, any new components of $\bar{\omega}$ must contain a mix of indices, and such connection components were all zero for constant λ. Note that all quantities now are with respect to the scaled framefields, $\tilde{e}^\alpha$, so we will have to rescale these coordinates in the end. We see that the only term yielding a new contribution is given by $\mathrm{d}e^{\hat{\imath}}$ and thus find

$$\mathrm{d}\tilde{e}^{\hat{\imath}} = \mathrm{d}e^{\hat{\imath}} = \mathrm{d}_4 e^{\hat{\imath}} + \mathrm{d}_7 e^{\hat{\imath}} = \frac{1}{\lambda}\tilde{\partial}_\alpha \lambda \tilde{e}^\alpha \wedge e^{\hat{\imath}} + \omega_{[IJ]}{}^{\hat{\imath}} e^I \wedge e^J. \tag{4.12}$$

Capital indices $I, J, \ldots$ denote all indices on S^7, i.e. $I = (0, i, \hat{\imath})$. We find that almost all components looks the same as in the previous case with λ now equal to $\lambda(x)$. We will from now on sometimes write $\lambda(x)$ in terms of $v(x)$ to clean up expressions. The new components obtained from (4.11) are

$$\bar{\omega}_{[\alpha\hat{\jmath}]}{}^{\hat{\imath}} = -\frac{7}{2\cdot 2}\tilde{\partial}_\alpha v \delta^i_j \implies \begin{cases} \bar{\omega}_{\alpha\hat{\jmath}\hat{\imath}} = \bar{\omega}_{[\alpha\hat{\jmath}]\hat{\imath}} + \bar{\omega}_{[\hat{\imath}\alpha]\hat{\jmath}} - \bar{\omega}_{[\hat{\jmath}\hat{\imath}]\alpha} = 0 \\ \bar{\omega}_{\hat{\jmath}\alpha\hat{\imath}} = \bar{\omega}_{\alpha\hat{\jmath}\hat{\imath}} - 2\bar{\omega}_{[\alpha\hat{\jmath}]\hat{\imath}} = \frac{7}{2}\tilde{\partial}_\alpha v \delta_{ji} \end{cases} \tag{4.13}$$

So, the 1-form $\bar{\omega}_{\alpha\hat{\imath}}$ is

$$\bar{\omega}_{\alpha\hat{\imath}} = \frac{7}{2}\tilde{\partial}_\alpha v \delta_{ij} e^{\hat{\jmath}} = \frac{7}{2}\tilde{\partial}_\alpha v e_{\hat{\imath}}. \tag{4.14}$$

We may now turn to compute the new 11-dimensional curvature two-forms, coming from the $\mathrm{d}\bar{s}^2$ metric. Since the curvature two-form contains quadratic terms in $\bar{\omega}$ it is possible that even the two-form in the internal directions receives corrections. We will now use the notation $\bar{\Theta}_{\alpha\beta}$ to denote 11-dimensional 2-forms with $\lambda = \lambda(x)$. In contrast $\tilde{\Theta}_{\alpha\beta}$ will denote 11-dimensional 2-forms but with constant λ. All quantities are with respect to the rescaled spacetime frame fields $\tilde{e}^\alpha$.

We start with the 2-form with space-time indices

$$\bar{\Theta}_{\alpha\beta} = \mathrm{d}\bar{\omega}_{\alpha\beta} + \bar{\omega}_{\alpha C}\wedge\bar{\omega}_{C\beta} = \mathrm{d}\tilde{\omega}_{\alpha\beta} + \tilde{\omega}_{\alpha\gamma}\wedge\tilde{\omega}_{\gamma\beta} - \bar{\omega}_{\alpha\hat{\imath}}\wedge\bar{\omega}_{\beta\hat{\imath}} = \tilde{\Theta}_{\alpha\beta} - \bar{\omega}_{\alpha\hat{\imath}}\wedge\bar{\omega}_{\beta\hat{\imath}} = \tilde{\Theta}_{\alpha\beta}. \tag{4.15}$$

Here we used that $\bar{\omega}_{\alpha\beta} = \tilde{\omega}_{\alpha\beta}$. This allows us to rewrite the right hand side in terms of the $\tilde{\Theta}_{\alpha\beta}$ Furthermore, the extra term contains only a term of the form $e_{\hat{\imath}} \wedge e_{\hat{\imath}}$ and is thus zero. Continuing with the other components we find

$$\bar{\Theta}_{\alpha 0} = \mathrm{d}\bar{\omega}_{\alpha 0} + \bar{\omega}_{\alpha B}\wedge\bar{\omega}_{B0} = -\bar{\omega}_{\alpha\hat{\imath}}\wedge\bar{\omega}_{0\hat{\imath}} = \frac{7}{2}\tilde{\partial}_\alpha v e_{\hat{\imath}} \wedge \frac{1}{2}e^{-\frac{7}{2}v} e_i = \frac{7}{4}e^{-\frac{7}{2}v}\tilde{\partial}_\alpha v e^{\hat{\imath}} \wedge e^i. \tag{4.16}$$

$$\bar{\Theta}_{\alpha i} = \mathrm{d}\bar{\omega}_{\alpha i} + \bar{\omega}_{\alpha B}\wedge\bar{\omega}_{Bi} = -\bar{\omega}_{\alpha\hat{\jmath}}\wedge\bar{\omega}_{i\hat{\jmath}} = \frac{7}{4}e^{-\frac{7}{2}v}\tilde{\partial}_\alpha v \delta_{ij} e^{\hat{\jmath}} \wedge e^0 + \frac{7}{4}e^{-\frac{7}{2}v}\tilde{\partial}_\alpha v \varepsilon_{ijk} e^{\hat{\jmath}} \wedge e^k. \tag{4.17}$$

$$\begin{aligned} \bar{\Theta}_{\alpha\hat{\imath}} = \mathrm{d}\bar{\omega}_{\alpha\hat{\imath}} + \bar{\omega}_{B\hat{\imath}} = \tilde{e}^\gamma \wedge e^{\hat{\jmath}}\frac{7}{2}\delta_{ji}\Big(\tilde{\partial}_\gamma\tilde{\partial}_\alpha v + \tilde{\omega}_{\gamma\alpha}{}^\beta\tilde{\partial}_\beta v - \frac{7}{2}(\tilde{\partial}_\alpha v)(\tilde{\partial}_\gamma v)\Big) + \\ + e^J \wedge e^K \frac{7}{2}\omega_{[JK]\hat{\imath}}\tilde{\partial}_\alpha v - e^{\hat{\jmath}} \wedge e^{\hat{k}}\frac{7}{4}e^{\frac{7}{2}v}\varepsilon_{ijk}\tilde{\partial}_\alpha v = \\ = \tilde{e}^\beta \wedge e^{\hat{\jmath}}\frac{7}{2}\delta_{ij}\Big(\tilde{D}_\beta\tilde{\partial}_\alpha v - \frac{7}{2}(\tilde{\partial}_\beta v)(\tilde{\partial}_\alpha v)\Big) \\ - e^0 \wedge e^j \delta_{ji}\frac{7}{2}\tilde{\partial}_\alpha v - e^j \wedge e^k \frac{7}{4}\varepsilon_{ijk} e^{-\frac{7}{2}v}\tilde{\partial}_\alpha v. \end{aligned} \tag{4.18}$$

In the last equation the term containing $e^{\hat{\jmath}} \wedge e^{\hat{k}}$ from $\mathrm{d}_7 e^{\hat{\imath}}$ cancelled to a similar term coming from $\omega_{\alpha\hat{\jmath}} \wedge \omega_{\hat{\jmath}\hat{\imath}}$.

Next, we turn to computing the curvature 2-form components in the internal directions. We still denote 11-dimensional quantities with $\lambda = \lambda(x)$ by $\bar{\Theta}_{IJ}$. However, there is no need to use the notation $\tilde{\Theta}_{IJ}$ since for constant λ the metric is block diagonal and the S^7 is unaffected by rescalings in the spacetime framefields. We find

$$\bar{\Theta}_{0i} = \Theta_{0i} + \tilde{e}^{\alpha} \wedge e^{\hat{\jmath}} \delta_{ij} \tilde{\partial}_{\alpha} \lambda. \tag{4.19}$$

$$\bar{\Theta}_{0\hat{\imath}} = \Theta_{0\hat{\imath}} + \frac{1}{2} \tilde{e}^{\alpha} \wedge e^{j} \delta_{ij} \tilde{\partial}_{\alpha} \lambda. \tag{4.20}$$

$$\bar{\Theta}_{ij} = \Theta_{ij} + \tilde{e}^{\alpha} \wedge e^{\hat{k}} \varepsilon_{ijk} \tilde{\partial}_{\alpha} \lambda. \tag{4.21}$$

$$\bar{\Theta}_{i\hat{\jmath}} = \Theta_{i\hat{\jmath}} - \tilde{e}^{\alpha} \wedge e^{0} \frac{1}{2} \delta_{ij} \tilde{\partial}_{\alpha} \lambda - \tilde{e}^{\alpha} \wedge e^{k} \frac{1}{2} \varepsilon_{ijk} \tilde{\partial}_{\alpha} \lambda. \tag{4.22}$$

$$\bar{\Theta}_{\hat{\imath}\hat{\jmath}} = \Theta_{\hat{\imath}\hat{\jmath}} - e^{\hat{k}} \wedge e^{\hat{\ell}} \frac{1}{\lambda^2} (\tilde{\partial}\lambda)^2 \delta_{ij}^{k\ell} = \Theta_{\hat{\imath}\hat{\jmath}} - e^{\hat{k}} \wedge e^{\hat{\ell}} \frac{49}{4} (\tilde{\partial} v)^2 \delta_{ij}^{k\ell}. \tag{4.23}$$

4.1.1 Computing the Ricci Tensor

From the curvature 2-forms above we find the relevant Riemann tensor components to obtain the Ricci tensor. It is as previously defined by

$$\bar{\Theta}_{AB} = \frac{1}{2} \bar{R}_{ABCD} \tilde{e}^{C} \wedge \tilde{e}^{D}. \tag{4.24}$$

The process is analogous to the previously described one of the squashed seven-sphere. We start with the space-time directions. The Ricci tensor is

$$\bar{R}_{\alpha\beta} = \bar{R}_{\alpha C \beta D} \eta^{CD}, \tag{4.25}$$

and the relevant Riemann components are thus immediately found from the curvature 2-forms to be

$$\bar{R}_{\alpha\gamma\beta\delta} = \tilde{R}_{\alpha\gamma\beta\delta}, \qquad \bar{R}_{\alpha c \beta d} = 0, \qquad \bar{R}_{\alpha\hat{\imath}\beta\hat{\jmath}} = \frac{7}{2} \delta_{ij} \Big(\tilde{D}_{\beta} \tilde{\partial}_{\alpha} v - \frac{7}{2} (\tilde{\partial}_{\beta} v)(\tilde{\partial}_{\alpha} v) \Big), \tag{4.26}$$

and thus we find

$$\bar{R}_{\alpha\beta} = \tilde{R}_{\alpha\beta} + \frac{21}{2} \Big(\tilde{D}_{\beta} \tilde{\partial}_{\alpha} v - \frac{7}{2} (\tilde{\partial}_{\beta} v)(\tilde{\partial}_{\alpha} v) \Big). \tag{4.27}$$

For the internal S^4 directions we have

$$\bar{R}_{ab} = \bar{R}_{aCbD} \eta^{CD}, \tag{4.28}$$

so we need the following components

$$\bar{R}_{a\alpha b\beta} = 0, \qquad \bar{R}_{acbd} = R_{acbd}, \qquad \bar{R}_{a\hat{\imath}b\hat{\jmath}} = R_{a\hat{\imath}b\hat{\jmath}}. \tag{4.29}$$

And thus we see that

$$\bar{R}_{ab} = R_{ab}, \tag{4.30}$$

and for the S^3

$$\bar{R}_{\hat{\imath}\hat{\jmath}} = \bar{R}_{\hat{\imath}C\hat{\jmath}D} \eta^{CD}, \tag{4.31}$$

we need the components

$$\bar{R}_{\hat{\imath}\alpha\hat{\jmath}\beta} = \frac{7}{2}\delta_{ij}\Big(\tilde{D}_\beta\tilde{\partial}_\alpha v - \frac{7}{2}(\tilde{\partial}_\beta v)(\tilde{\partial}_\alpha v)\Big), \quad \bar{R}_{\hat{\imath}a\hat{\jmath}b} = R_{\hat{\imath}a\hat{\jmath}b}, \quad \bar{R}_{\hat{\imath}\hat{k}\hat{\jmath}\hat{\ell}} = R_{\hat{\imath}\hat{k}\hat{\jmath}\hat{\ell}} - \frac{49}{2}(\tilde{\partial}v)^2\delta^{j\ell}_{ik}, \tag{4.32}$$

which upon contraction yields the Ricci tensor

$$\bar{R}_{\hat{\imath}\hat{\jmath}} = R_{\hat{\imath}\hat{\jmath}} + \frac{7}{2}\delta_{ij}\tilde{\Box}_4 v - \delta_{ij}\frac{3\cdot 49}{4}(\tilde{\partial}v)^2. \tag{4.33}$$

It is straightforward to verify, from the components in $\bar{\Theta}_{AB}$, that the Ricci tensor is still diagonal so all other components are zero.

4.1.2 Rescaling the Ricci Tensor

Now we have found the Ricci tensor components of the metric

$$\mathrm{d}\bar{s}^2 = \tilde{e}^\alpha(x)\tilde{e}^\alpha(x) + e^a(y)e^a(y) + e^{\hat{\imath}}(x,y)e^{\hat{\imath}}(x,y). \tag{4.34}$$

In order to find the Ricci tensor of the metric in equation (4.3), we will go through the procedure we outlined previously. First we will rescale the metric by an overall factor of $e^{-2B(x)}$. Then, we will have our answer expressed in terms of $\tilde{e}^\alpha$ and thus we need to also transform all spacetime quantities. We start by stating the transformation rules that we will need. First of all we need the Weyl rescaling of the Ricci tensor in flat indices. For a general rescaling of the form $\hat{e}^A = e^{-s\gamma(X)}e^A$ we have [22]

$$\begin{aligned}\hat{R}_{AB} = e^{2s\gamma}\Big(&R_{AB} + s[\eta_{AB}\Box_{11}\gamma + (D-2)D_B\partial_A\gamma] \\ &- s^2(D-2)\Big[\eta_{AB}(\partial\gamma)^2 - (\partial_A\gamma)(\partial_B\gamma)\Big]\Big).\end{aligned} \tag{4.35}$$

Note that the $\Box$ is with respect to all 11 dimensions, this will be important later. Next, we need to know how to transform from $\tilde{e}^\alpha$ quantities to e^α. If we start from the rescaling $\tilde{e}^\alpha = e^{-s\gamma}e^\alpha$ we may derive the following transformation rules

$$\tilde{e}^\alpha = \tilde{e}_\mu{}^\alpha\mathrm{d}x^\mu = e^{-s\gamma}e_\mu{}^\alpha\mathrm{d}x^\mu \implies \tilde{e}_\mu{}^\alpha = e^{-s\gamma}e_\mu{}^\alpha, \qquad \tilde{e}_\alpha{}^\mu = e^{s\gamma}e_\alpha{}^\mu. \tag{4.36}$$

$$\det\tilde{e}^\alpha_\mu = e^{-sD\gamma}\det e^\alpha_\mu. \tag{4.37}$$

$$\tilde{g}^{\mu\nu} = \tilde{e}_\alpha{}^\mu\tilde{e}_\beta{}^\nu\eta^{\alpha\beta} = e^{2s\gamma}g^{\mu\nu}. \tag{4.38}$$

$$\tilde{\partial}_\alpha = \tilde{e}_\alpha{}^\mu\partial_\mu = e^{s\gamma}e_a{}^\mu\partial_\mu = e^{s\gamma}\partial_\alpha. \tag{4.39}$$

$$\tilde{\omega}_{\gamma\alpha\beta} = e^{s\gamma}(\omega_{\gamma\alpha\beta} - s\eta_{\gamma\alpha}\partial_\beta\gamma + s\eta_{\gamma\beta}\partial_\alpha\gamma). \tag{4.40}$$

$$\begin{aligned}\tilde{D}_\beta\tilde{\partial}_\alpha B &= \tilde{\partial}_\beta(e^{s\gamma}\partial_\alpha B) + e^{s\gamma}\tilde{\omega}_{\beta\alpha}{}^\gamma\partial_\gamma B = \\ &= e^{2s\gamma}\Big(D_\beta\partial_\alpha B + s\Big[2\partial_{(\alpha}\gamma\partial_{\beta)}B - \eta_{\alpha\beta}(\partial\gamma)(\partial B)\Big]\Big).\end{aligned} \tag{4.41}$$

$$\begin{aligned}\tilde{\Box}_4 B &= \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\Big(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu B\Big) = \\ &= e^{sD\gamma}\Big(\Big(\partial_\mu e^{-s(D-2)\gamma}\Big)g^{\mu\nu}\partial_\nu B + e^{-s(D-2)\gamma}\Box B\Big) = \\ &= e^{2s\gamma}\Big(-s(D-2)(\partial\gamma)(\partial B) + \Box_4 B\Big).\end{aligned} \tag{4.42}$$

Finally we will need the transformation of the rescaled Ricci tensor $\tilde{R}_{\alpha\beta}$. It is given in (4.35) if we let $A = \alpha, B = \beta$ and change the $\Box$ to be in 4 dimensions.

Having these matters settled we are now ready to compute the final Ricci tensor, $\hat{R}_{AB}$. We will start with the ab-component, which is fairly simple but illustrative. From (4.30) we note that there are no extra terms appearing. This means that we can immediately plug in the result in (4.35) with $\gamma = B$. Two of the terms vanish, since B is independent of the internal directions. We find that

$$\hat{R}_{ab} = e^{2B}\Big(R_{ab} + \eta_{ab}\tilde{\Box}_{11}B - 9\eta_{ab}(\tilde{\partial}B)^2\Big). \tag{4.43}$$

The final step is now to use the transformations derived above to write the expression in terms of e^α rather than $\tilde{e}^\alpha$. The partial derivatives are straightforward and only pick up a factor of $e^{2(A-B)}$. The 11-dimensional box however, should be rewritten in terms of a four-dimensional, which is not trivial. It actually picks up an extra term, which we see from

$$\begin{aligned}\tilde{\Box}_{11}B = \eta^{AB}\tilde{D}_A\tilde{\partial}_B B = \eta^{AB}\Big(\tilde{\partial}_A\tilde{\partial}_B B + \tilde{\omega}_{AB}{}^C\tilde{\partial}_C B\Big) =\\ = \eta^{\alpha\beta}\tilde{\partial}_\alpha\tilde{\partial}_\beta B + \eta^{\alpha\beta}\tilde{\omega}_{\alpha\beta}{}^\gamma\tilde{\partial}_\gamma B + \eta^{\hat{\imath}\hat{\jmath}}\tilde{\omega}_{\hat{\imath}\hat{\jmath}}{}^\gamma\tilde{\partial}_\gamma B = \tilde{\Box}_4 B - \frac{21}{2}(\tilde{\partial}^\gamma v)(\tilde{\partial}_\gamma B).\end{aligned} \tag{4.44}$$

Now we may simply plug in the expressions for the tilde-quantities and the value for R_{ab} to find the final answer. We can also use that $v = -\frac{2}{21}(7B + 2A)$ to more easily see the cancellation of the contracted derivatives.

$$\begin{aligned}\hat{R}_{ab} &= e^{2B}R_{ab} + e^{2A}\eta_{ab}\Big(\Box_4 B - 2\partial_\gamma(A-B)\partial^\gamma B + \partial_\gamma(7B+2A)\partial^\gamma B - 9(\partial B)^2\Big)\\ &= e^{2B}R_{ab} + e^{2A}\eta_{ab}\Box_4 B = \eta_{ab}\Big(12e^{-2u-3v} - 6e^{-2u-10v} - e^{7u}(\Box_4 u + \tfrac{3}{2}\Box_4 v)\Big).\end{aligned} \tag{4.45}$$

We perform a similar procedure for $\hat{R}_{\hat{\imath}\hat{\jmath}}$. First we have

$$\hat{R}_{\hat{\imath}\hat{\jmath}} = e^{2B}\Big(\bar{R}_{\hat{\imath}\hat{\jmath}} + \eta_{ij}\tilde{\Box}_{11}B + 9\tilde{D}_{\hat{\jmath}}\tilde{\partial}_{\hat{\imath}}B - 9\eta_{ij}(\tilde{\partial}B)^2\Big). \tag{4.46}$$

We need to keep the $D\partial B$ term here, since the connection term in the covariant derivative contributes to the result. If we split up the 11-dimensional box as before and plug in $\bar{R}_{\hat{\imath}\hat{\jmath}}$ we arrive at

$$\begin{aligned}\hat{R}_{\hat{\imath}\hat{\jmath}} = e^{2B}\Big(R_{\hat{\imath}\hat{\jmath}} + \frac{7}{2}\eta_{ij}\tilde{\Box}_4 v - \eta_{ij}\frac{3\cdot 49}{4}(\tilde{\partial}v)^2\Big) +\\ + e^{2B}\eta_{ij}\Big(\tilde{\Box}_4 B - \frac{21}{2}(\tilde{\partial}^\gamma v)(\tilde{\partial}_\gamma B) + 9\bar{\omega}_{\hat{\jmath}\hat{\imath}}{}^\gamma\tilde{\partial}_\gamma B - 9\eta_{ij}(\tilde{\partial}B)^2\Big) =\\ = e^{2B}R_{\hat{\imath}\hat{\jmath}} + e^{2A}\eta_{ij}\Big(\Box_4 C - \Box_4 B + 6\partial_\gamma(B + \tfrac{1}{2}C)\partial^\gamma(C-B) - 3(\partial(C-B))^2 +\\ + \Box_4 B + 6\partial_\gamma(B + \tfrac{1}{2}C)\partial^\gamma B - 3\partial^\gamma(C-B)\partial_\gamma B - 9\partial^\gamma(C-B)\partial_\gamma B - 9(\partial B)^2\Big).\end{aligned} \tag{4.47}$$

where we used the fact that $A - B = -3(B + \frac{1}{2}C)$ and rewrote $7v = 2(C - B)$. Now, a little bit of algebra shows that almost all of the terms cancel and we are left with

$$\hat{R}_{\hat{\imath}\hat{\jmath}} = e^{2B}R_{\hat{\imath}\hat{\jmath}} + e^{2A}\eta_{ij}\Box_4 C = \eta_{ij}\Big(4e^{-2u-10v} + 2e^{-2u+4v} - e^{7u}(\Box_4 u - 2\Box_4 v)\Big). \tag{4.48}$$

Finally, we compute $\hat{R}_{\alpha\beta}$. This is somewhat more complicated since we must perform an additional Weyl rescaling of the $\tilde{R}_{\alpha\beta}$ term in $\bar{R}_{\alpha\beta}$. Using equation (4.35) we see that

$\tilde{e}^\alpha = e^{-(A-B)}e^\alpha$ yields

$$\begin{aligned}\tilde{R}_{\alpha\beta} = e^{2(A-B)}\Big(&R_{\alpha\beta} + \eta_{\alpha\beta}\Box_4(A-B) + 2D_\beta\partial_\alpha(A-B) - \\ &- 2\eta_{\alpha\beta}\partial_\gamma(A-B)\partial^\gamma(A-B) + 2\partial_\alpha(A-B)\partial_\beta(A-B)\Big).\end{aligned} \tag{4.49}$$

As before, we start by writing out $\hat{R}_{\alpha\beta}$ from (4.35)

$$\begin{aligned}\hat{R}_{\alpha\beta} = e^{2B}\Big(&\tilde{R}_{\alpha\beta} + \frac{21}{2}\tilde{D}_\beta\tilde{\partial}_\alpha v - \frac{63}{4}(\tilde{\partial}_\beta v)(\tilde{\partial}_\alpha v) + \\ &+ \eta_{\alpha\beta}\tilde{\Box}_4 B - \frac{21}{2}(\tilde{\partial}^\gamma v)(\tilde{\partial}_\gamma B) \\ &+ 9\tilde{D}_\beta\tilde{\partial}_\alpha B - 9\eta_{\alpha\beta}\tilde{\partial}_\gamma B\tilde{\partial}^\gamma B + 9\tilde{\partial}_\alpha B\tilde{\partial}_\beta B\Big).\end{aligned} \tag{4.50}$$

Now all we need to do is to plug in the tilde transformations and perform the algebra. After a mass slaughter of terms we are left with

$$\hat{R}_{\alpha\beta} = e^{7u}\Big(R_{\alpha\beta} + \frac{7}{2}\eta_{\alpha\beta}\Box_4 u - \frac{63}{2}(\partial_\alpha u)(\partial_\beta u) - 21(\partial_\alpha v)(\partial_b v)\Big). \tag{4.51}$$

In curved indices this is

$$\hat{R}_{\mu\nu} = \hat{e}_\mu{}^\alpha\hat{e}_\nu{}^\beta\hat{R}_{\alpha\beta} = R_{\mu\nu} + \frac{7}{2}g_{\mu\nu}\Box_4 u - \frac{63}{2}(\partial_\mu u)(\partial_\nu u) - 21(\partial_\mu v)(\partial_\nu v), \tag{4.52}$$

since

$$\hat{e}_\mu{}^\alpha = e^{-\frac{7}{2}u}e_\mu{}^a \quad \text{and} \quad e_\mu{}^\alpha e_\nu{}^\beta\eta_{\alpha\beta} \equiv g_{\mu\nu}, \tag{4.53}$$

and we have obtained the desired expressions for the case of a seven-sphere with spacetime dependent parameters.

4.2 Effective 4D Lagrangian

We will now use the results for the Ricci tensor to obtain the effective 4D Lagrangian of the theory. The idea is that since we have written the spacetime components of the 11-dimensional Ricci tensor in terms of the scalar fields and the 4-dimensional Ricci tensor, we will be able to obtain equations of motions for the three dynamical quantities, R, u, v, if we solve these equations. We start from the Einstein equations in 11D supergravity with the fermion field set to zero

$$\hat{R}_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{3}\Big(\hat{H}_{MPQR}\hat{H}_N{}^{PQR} - \frac{1}{8}g_{MN}\hat{H}^2\Big), \tag{4.54}$$

which we rewrite as

$$\hat{R}_{MN} = \frac{1}{3}\Big(\hat{H}_{MPQR}\hat{H}_N{}^{PQR} - \frac{1}{12}\hat{g}_{MN}\hat{H}^2\Big). \tag{4.55}$$

The hatted notation refers to the metric in (4.3). We will employ the Freund-Rubin ansatz, but now allowing for the constant of proportionality m to be dependent on the spacetime. Thus, we write $f(x) = -3m$. The ansatz is then

$$\hat{H}_{MNPQ} = \hat{H}_{\mu\nu\rho\sigma} = f\hat{\epsilon}_{\mu\nu\rho\sigma} = f\hat{e}_4\varepsilon_{\mu\nu\rho\sigma} = fe^{-14u}\epsilon_{\mu\nu\rho\sigma}. \tag{4.56}$$

Then inserting into the field equation for H

$$\hat{\nabla}_M\hat{H}^{MNPQ} = 0, \tag{4.57}$$

together with

$$\hat{H}^{MNPQ} = fe^{-14u}\hat{g}^{MM'}\hat{g}^{NN'}\hat{g}^{PP'}\hat{g}^{QQ'}\epsilon_{M'N'P'Q'} = -3me^{(-14+28)u}\epsilon^{\mu\nu\rho\sigma}, \tag{4.58}$$

yields

$$\hat{\nabla}_M\hat{H}^{MNPQ} = \frac{1}{\hat{e}}\hat{\partial}_M(\hat{e}fe^{14u}\epsilon^{MNPQ}) = \frac{1}{\hat{e}}\hat{\partial}_M(fe^{7u}\varepsilon^{MNPQ}). \tag{4.59}$$

Here we have used the notation where $\hat{e} = e^{-7u}e$ is the 11-dimensional determinant of the vielbein and $\hat{e}_4 = e^{-14u}e_4$ is the 4-dimensional part. Furthermore, in the last equality we have absorbed the determinant in the Levi-Civita tensor to write it as the tensor density instead. Since the tensor density is constant, we find that

$$\hat{\partial}_M(fe^{7u})\varepsilon^{MNPQ} = \hat{\partial}_\mu(fe^{7u})\varepsilon^{\mu\nu\rho\sigma} \implies Q = fe^{7u} = \text{constant}. \tag{4.60}$$

Thus, we may rewrite

$$\hat{H}_{\mu\nu\rho\sigma} = Qe^{-21u}\epsilon_{\mu\nu\rho\sigma}. \tag{4.61}$$

Inserting this into Einstein's equations then yields, for the spacetime components

$$\begin{aligned}\hat{R}_{\mu\nu} &= \frac{1}{3}\Big(\hat{H}_{\mu\rho\sigma\tau}\hat{H}_\nu{}^{\rho\sigma\tau} - \frac{1}{12}\hat{g}_{\mu\nu}\hat{H}^2\Big) = \frac{1}{3}\Big(Q^2e^{-21u}\epsilon_{\mu\rho\sigma\tau}\epsilon_\nu{}^{\rho\sigma\tau} - \frac{1}{12}e^{-7u}g_{\mu\nu}Q^2e^{-14u}|\epsilon|^2\Big)\\ &= \frac{1}{3}Q^2e^{-21u}\Big(-3!g_{\mu\nu} + \frac{4!}{12}g_{\mu\nu}\Big) = -\frac{4}{3}Q^2e^{-21u}g_{\mu\nu} = -\frac{4}{3}Q^2e^{-14u}\hat{g}_{\mu\nu},\end{aligned} \tag{4.62}$$

and for the internal components

$$\hat{R}_{mn} = \frac{1}{3}Q^2e^{-14u}\frac{4!}{12}\hat{g}_{mn} = \frac{2}{3}Q^2e^{-14u}\hat{g}_{mn}. \tag{4.63}$$

Equating these results with the ones obtained in the previous section will now yield equations of motion for $R_{\mu\nu}$, u and v. Note that changing between flat and curved indices simply means changing $\hat{g} \leftrightarrow \eta$ since

$$\hat{R}_{\alpha\beta} = \hat{e}_\alpha{}^\mu\hat{e}_\beta{}^\nu\hat{R}_{\mu\nu} = -\frac{4}{3}Q^2e^{-14u}\eta_{\alpha\beta}, \tag{4.64}$$

and similarly for $\hat{R}_{mn}$. Starting with the internal parts, converting (4.45) to curved indices and equating with (4.63) yields

$$\frac{2}{3}Q^2e^{-14u} = 12e^{-2u-3v} - 6e^{-2u-10v} - e^{7u}\Box u - \frac{3}{2}e^{7u}\Box v, \tag{4.65}$$

and when comparing (4.48) and (4.63) we find in the same fashion

$$\frac{2}{3}Q^2e^{-14u} = 4e^{-2u-10v} + 2e^{-2u+4v} - e^{7u}\Box u + 2e^{7u}\Box v. \tag{4.66}$$

Since we work in four dimensions now, we denote $\Box \equiv \Box_4$. We obtain the field equations for the scalar fields by adding these equations together. We find

$$\begin{aligned}\Box v &= -\tfrac{4}{7}e^{-9u+4v} + \tfrac{24}{7}e^{-9u-3v} - \tfrac{20}{7}e^{-9u-10v},\\ \Box u &= \tfrac{6}{7}e^{-9u+4v} + \tfrac{48}{7}e^{-9u-3v} - \tfrac{12}{7}e^{-9u-10v} - \tfrac{2}{3}Q^2e^{-21u}.\end{aligned} \tag{4.67}$$

Equating (4.62) and (4.51) and plugging in (4.67) yields

$$\begin{aligned} R_{\mu\nu} =& -\tfrac{7}{2}g_{\mu\nu}\Box u + \tfrac{63}{2}(\partial_\mu u)(\partial_\nu u) + 21(\partial_\mu v)(\partial_\nu v) - \frac{4}{3}Q^2 e^{-21u} g_{\mu\nu} \\ \Longrightarrow\ R_{\mu\nu} =& \tfrac{63}{2}(\partial_\mu u)(\partial_\nu u) + 21(\partial_\mu v)(\partial_\nu v) + \\ &+ g_{\mu\nu}\Big(-3e^{-9u+4v} - 24e^{-9u-3v} + 6e^{-9u-10v} + Q^2 e^{-21u}\Big). \end{aligned} \tag{4.68}$$

We are now ready to construct the effective 4D Lagrangian from these equations of motion. Our starting point will be a Lagrangian of the form

$$\mathcal{L} = \sqrt{-g}\Big(R + h(u,v)\Big), \tag{4.69}$$

where $h(u,v)$ is some function of the scalar fields. Then the variation is

$$\delta\mathcal{L} = \delta(\sqrt{-g})\,(R + h(u,v)) + \sqrt{-g}\delta R + \sqrt{-g}\delta h(u,v). \tag{4.70}$$

The equations of motion for v and u must come from the $\delta h(u,v)$ part. Starting with v we may see that

$$\delta((\partial^\mu v)(\partial_\mu v)) = -2(\delta v)\Box v + (\partial_\mu v)(\partial_\nu v)\delta g^{\mu\nu}, \tag{4.71}$$

where we have dropped the boundary terms. Also note that we have

$$\delta e^{-9u+av} = (\delta v) a e^{-9u+av} - (\delta u) 9 e^{-9u+av}, \tag{4.72}$$

for some constant a. Thus, in order to obtain the $\Box v$ equation in (4.67) we set

$$h(u,v) = \tfrac{A}{2}(\partial v)^2 - \tfrac{A}{7}e^{-9u+4v} - \tfrac{8A}{7}e^{-9u-3v} + \tfrac{2A}{7}e^{-9u-10v} + \ldots \tag{4.73}$$

where A is an undetermined prefactor. Doing the same procedure with the u equation tells us that we must add two terms to the above expression, $(\partial u)^2$ and $Q^2 e^{-21u}$. This leads to the final expression

$$h(u,v) = \tfrac{3A}{4}(\partial u)^2 + \tfrac{A}{21}Q^2 e^{-21u} + \tfrac{A}{2}(\partial v)^2 - \tfrac{A}{7}e^{-9u+4v} - \tfrac{8A}{7}e^{-9u-3v} + \tfrac{2A}{7}e^{-9u-10v}. \tag{4.74}$$

We can now write out the variation with respect to the metric

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} =& -\frac{1}{2}\sqrt{-g}g_{\mu\nu}(R + h(u,v)) + \sqrt{-g}R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\frac{\delta R_{\mu\nu}}{\delta g^{\mu\nu}} + \\ & + \sqrt{-g}\Big(\tfrac{3A}{4}(\partial_\mu u)(\partial_\nu u) + \frac{A}{2}(\partial_\mu v)(\partial_\nu v)\Big), \end{aligned} \tag{4.75}$$

where we may drop $\frac{\delta R_{\mu\nu}}{\delta g^{\mu\nu}}$ because it is a total derivative. Uncontracted derivatives on u and v appear only in the last parentheses and we may immediately compare this result to (4.68) to find that $A = -42$. This choice reproduces the equations of motion for $R_{\mu\nu}$, which can be seen by contracting (4.68) with $g^{\mu\nu}$ to find

$$R = \tfrac{63}{2}(\partial u)^2 + 21(\partial v)^2 - 12e^{-9u+4v} - 96e^{-9u-3v} + 24e^{-9u-10v} + 4Q^2 e^{-21u}. \tag{4.76}$$

and then carrying out the algebra in (4.75). Thus, the effective 4-dimensional Lagrangian is

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\Big(R - \tfrac{63}{2}(\partial u)^2 - 21(\partial v)^2 - V(u,v)\Big), \\ V &= 2Q^2 e^{-21u} - 6e^{-9u+4v} - 48e^{-9u-3v} + 12e^{-9u-10v}. \end{aligned} \tag{4.77}$$

The potential as a function of the two scalar fields can be seen in figure 4.1

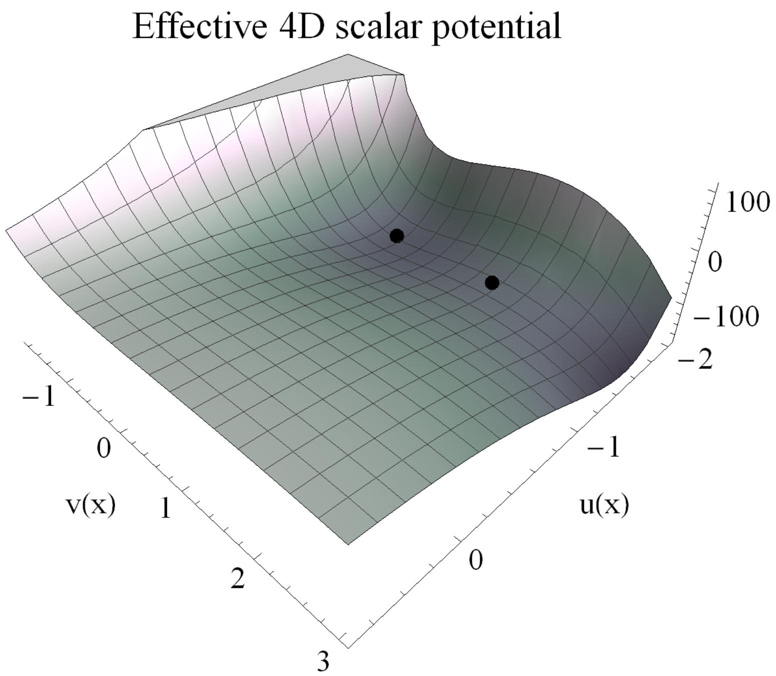


Figure 4.1: The effective 4D potential obtained by compactifying 11D supergravity on the squashed S^7 in the Freund-Rubin ansatz. Here, the conserved charge has been set to $Q = 1$. The two points correspond to the two ground-state solutions, round S^7 with $v = 0$ and squashed S^7 with $v = \frac{1}{7}\log 5$. While not easily seen in this figure, the extrema also have different values of u.

4.2.1 AdS Solutions and Linear Stability

Now we will specialise to anti-de Sitter space. Such a space is maximally symmetric and thus fulfil [2]

$$R_{\mu\nu\rho\sigma} = \tfrac{1}{3}\Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{4.78}$$

We now look for the ground-state solution, in which the scalars u and v should be constant. Looking back at (4.67) we find from the equation for $\Box v_1 = 0$ that there are two possible solutions for constant v. Setting $v_1 = 0$ is trivially a solution and corresponds to the sphere being the normal round S^7. However, there is another solution given by

$$\begin{aligned} 0 = -4e^{4v_1} + 24e^{-3v_1} - 20e^{-10v_1} &= -4e^{7v_1} + 24 - 20e^{-7v_1} \\ \implies e^{7v_1} = 5 \implies v_1 &= \tfrac{1}{7}\log 5. \end{aligned} \tag{4.79}$$

which is precisely the squashed solution with an Einstein metric. Indeed, setting $\lambda^2 = e^{-7v_1}$ yields precisely the results we obtained for the squashed sphere with no spacetime dependent parameters. The round sphere will not provide much insight to us, so we will instead focus on the squashed version. For other studies regarding the round sphere and stability, consult for example [24, 36]. Now, we plug this solution into the $\Box u = 0$ equation which yields

$$e^{12u_1} = \frac{e^{10v_1}}{3^4}Q^2. \tag{4.80}$$

Now, we may contract two indices in (4.78) to obtain $R_{\mu\nu} = \Lambda g_{\mu\nu}$. If we equate this with (4.68) and set $v = v_1$ and $u = u_1$ we get

$$\Lambda = -\frac{2916\sqrt{5}}{125}\frac{1}{Q^{3/2}}. \tag{4.81}$$

Note that this is also equivalent to

$$e^{-9u_1} = -\frac{e^{10v_1}}{108}\Lambda. \tag{4.82}$$

We now want to be able to read off the scalar potential, V from our Lagrangian. The form of the Lagrangian with a cosmological constant that we seek is

$$\frac{1}{\sqrt{-g}}\mathcal{L} = R - 2\Lambda + (\text{derivatives on scalar fields}) - V(\text{scalars}). \tag{4.83}$$

So, from equation (4.77) we write

$$\mathcal{L} = \sqrt{-g}\Big(R - 2\Lambda - \tfrac{63}{2}(\partial u)^2 - 21(\partial v)^2 - V(u,v)\Big), \tag{4.84}$$

where

$$\begin{aligned} V(u,v) &= -2\Big(\Lambda - Q^2 e^{-21u} + 3e^{-9u+4v} + 24e^{-9u-3v} - 6e^{-9u-10v}\Big) = \\ &= -2\Lambda - 2\Lambda e^{-9(u-u_1)}\Big(\tfrac{3}{4}e^{-12(u-u_1)} - \tfrac{1}{36}e^{14v_1}e^{4(v-v_1)} \\ &\qquad - \tfrac{2}{9}e^{7v_1}e^{-3(v-v_1)} + \tfrac{1}{18}e^{-10(v-v_1)}\Big). \end{aligned} \tag{4.85}$$

Using that $e^{7v_1} = 5$ we rewrite this as

$$V(u,v) = -2\Lambda - \tfrac{2}{36}\Lambda e^{-9(u-u_1)}\Big(27e^{-12(u-u_1)} - 25e^{4(v-v_1)} - 40e^{-3(v-v_1)} + 2e^{-10(v-v_1)}\Big). \tag{4.86}$$

We may now find and characterise the stationary points of this potential which leads to information regarding stability with the respect to the scalars u and v. We do this by looking at the linearised field equations directly. By linearising around the stationary points $u = u_1$ and $v = v_1$ corresponding to the ground state we may immediately extract information about the behaviour of the scalars. Starting with the field equation for u in (4.67), the linear approximation is given by

$$\Box u \approx \Box u\Big|_{\substack{u=u_1\\ v=v_1}} + \frac{\partial \Box u}{\partial u}\Big|_{\substack{u=u_1\\ v=v_1}}(u-u_1) + \frac{\partial \Box u}{\partial v}\Big|_{\substack{u=u_1\\ v=v_1}}(v-v_1). \tag{4.87}$$

It turns out that $\frac{\partial \Box u}{\partial v} = 0$ at $u = u_1$ and $v = v_1$ so we find that

$$\Box(u-u_1) \approx -5^{-\frac{10}{7}}648e^{-9u_1} = -6\Lambda(u-u_1). \tag{4.88}$$

Since $\Lambda < 0$ in AdS space, the coefficient on the right hand side is positive, indicating that $u = u_1$ corresponds to a stable minimum.

Next, we investigate the $\Box v$ equation in (4.67). Again we find that $\frac{\partial \Box v}{\partial u} = 0$ so after some algebra we arrive at

$$\Box(v-v_1) \approx \frac{20}{27}\Lambda(v-v_1). \tag{4.89}$$

Here the situation is reversed, and perturbations around v_1 seems to be unstable. However, the answer is actually much more intricate. It can been shown that the squashed seven-sphere is in fact stable to perturbations in v based on an argument first proposed by Breitenlohner and Freedman [37]. In fact, the Breitenlohner-Freedman bound states that stability holds as long as the second derivative for some scalar field ϕ is bounded from below by (recall that $\Lambda < 0$)

$$\frac{1}{\phi}\Box\phi \geq \frac{3}{4}\Lambda. \tag{4.90}$$

Qualitatively, this bound relates to the kinetic terms of the scalar field. The hyperbolic geometry of AdS space means that a dynamical change in a scalar field is not energetically favourable due to the kinetic term of that field. In certain cases, this unfavourable energy is enough to change an unstable extremum in the potential to a stable one when considering the whole theory. The Breitenlohner-Freedman bound specifies precisely when this happens. If we let v be a small perturbation around the ground state we write $v = v_1 + \Delta v$ and find

$$\frac{1}{\Delta v}\Box\Delta v \approx \frac{20}{27}\Lambda = \frac{80}{108}\Lambda > \frac{81}{108}\Lambda = \frac{3}{4}\Lambda. \tag{4.91}$$

The inequality holds since $\Lambda < 0$. We note that in the light of this argument, the squashed seven-sphere is indeed stable, although not by much. What is even more remarkable is that this argument still holds if the orientation of the sphere is reversed. Upon performing a parity transformation, both v and Λ are unchanged. It is only the conserved charge Q that changes sign, and if we had any pseudoscalar fields in the theory they would too [24]. Recalling our discussion in 3.3.1 regarding the maximum number of supersymmetries from the Killing spinor equation we may now understand the contradiction with the Non-AdS SUSY conjecture:

Conjecture 1 (Non-SUSY AdS Conjecture) *There are no non-supersymmetric stable solutions in AdS space*

The orientation-reversed squashed seven-sphere has no supersymmetries but still exhibits perturbative stability, as we just saw. This leads to one of two possible conclusions. Either non-perturbative effects are needed in order to see the instability of the squashed seven-sphere, or the above conjecture is wrong. Investigating non-perturbative corrections are very complicated, but an important first step would be to understand the perturbative behaviour of the theory in more detail. Our current analysis is actually quite limited, as we have only looked at two specific scalars in four dimensions. The full 11-dimensional theory yields a large amount of fields in four dimensions upon compactification. The reason for only considering these two scalars are of course that if one includes more fields, the analysis can get very complicated.

Having more knowledge about pseudoscalar fields in four dimensions would however be very interesting, as they transform under orientation-reversing. In principle one could repeat the procedure of this chapter, but include also pseudoscalars in the beginning. This might be a feasible strategy, although complicated, but we will instead go with another method in order to find the pseudoscalar fields. The basic idea is that the pseudoscalar fields always appear in a certain way together with the scalar fields and form complex scalars in the full theory. Thus, one could view our current results as coming from that theory, but in the limit where the pseudoscalars are zero. With some knowledge of the general structure of the theory one could then hope to be able to re-introduce the pseudoscalars from our current position, without having to go back to the 11-dimensional theory. This will be the subject of the next and last chapter of this thesis.

5

$\mathcal{N} = 1$ Supergravity in 4D

This chapter will conclude the thesis and present some analysis of the obtained four-dimensional potential and its connection to the Swampland. At first, we will introduce relevant theory of complex manifolds which will be the framework we use when trying to construct the scalar potential including pseudoscalar fields. After reviewing a simple example from type IIB string theory the method will be applied to our potential.

5.1 Moduli Spaces

When compactifying string theory or supergravity to four dimensions on Ricci flat compact internal manifolds, one generally finds a number of solutions for a given manifold. These solutions arise since for a manifold with a specified topology, there is freedom to distort the geometry of the manifold while preserving the topological properties. These distortions are usually described by a set of parameters which are taken as scalar fields on the external space time. If we describe the space of solutions obtained by varying the values of the scalar fields, we obtain what is called a *moduli space*. The scalars are called *moduli* and are regarded as coordinates on the moduli space.

It turns out that these moduli spaces often are described by a specific class of complex manifolds called *Kähler manifolds*. A common example is the compactification of type IIB string theory a torus T^2, where the torus is described by two scalar fields that form a complex coordinate on the coset space $SU(1,1)/U(1)$. More important for this thesis is that scalar fields of four-dimensional supergravity theories with one supersymmetry, $\mathcal{N} = 1$, also parametrise Kähler manifolds [22]. In the case of S^7 compactifications the ”size” and ”squashing” parameters on the S^7 are moduli. For certain values, these moduli give an internal Einstein space and $\mathcal{N} = 1$ supergravity in four dimensions. Therefore, we provide a short introduction to complex manifolds and the definition of Kähler manifolds. For a more thorough explanation, the reader may consult for [20, 22].

5.1.1 Complex Manifolds

Complex manifolds are locally $2n$-dimensional real manifolds described instead by n complex coordinates. We may express the coordinates as

$$\begin{aligned} z^{\alpha} &= \phi^{\alpha} + i\phi^{\alpha+n}, \quad \alpha = 1, \dots, n \\ \bar{z}^{\bar{\alpha}} &= \phi^{\alpha} - i\phi^{\alpha+n} = \bar{z}^{\alpha}, \end{aligned} \tag{5.1}$$

where ϕ^a is real and the complex manifold has a full set of coordinates $z^a = (z^{\alpha}, \bar{z}^{\alpha})$, $a = 1, \dots, 2n$. In terms of the real coordinates, the manifold behaves as if a standard

Riemannian manifold. In complex coordinates, the situation changes slightly. We can still write out the metric as

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}z^a\mathrm{d}z^b, \tag{5.2}$$

and we can go from the real coordinates to the complex ones via the transformations $\phi^i \to z^a$ as we are used to, e.g. the metric is

$$g_{ab} = g_{ij}\frac{\partial\phi^i}{\partial z^a}\frac{\partial\phi^j}{\partial z^b}, \quad \text{where} \quad \mathrm{d}s^2 = g_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j. \tag{5.3}$$

The important difference arises when we split the complex coordinate z^a into z^α and $\bar{z}^\alpha$. The set of coordinates z^α is not invariant under general coordinate transformations, but instead we must require that they transform holomorphically, that is $z^\alpha \to z'^\alpha = f^\alpha(z)$ where f is a holomorphic function. Knowing this, we can split up the metric as

$$\mathrm{d}s^2 = 2g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha\mathrm{d}\bar{z}^{\bar{\beta}} + g_{\alpha\beta}\mathrm{d}z^\alpha\mathrm{d}z^\beta + g_{\bar{\alpha}\bar{\beta}}\mathrm{d}\bar{z}^\alpha\mathrm{d}\bar{z}^\beta. \tag{5.4}$$

Next, we say that the metric admits a hermitian structure if $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$. The important part of this definition is that the structure is preserved under holomorphic transformations of the coordinates. We now want to define a Kähler manifold. First, from a hermitian metric we may construct the fundamental 2-form, also called the *Kähler form*

$$\Omega = -2ig_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}}. \tag{5.5}$$

The Kähler form is real, since

$$\bar{\Omega} = 2i\bar{g}_{\alpha\bar{\beta}}\mathrm{d}\bar{z}^{\bar{\alpha}} \wedge \mathrm{d}z^\beta = \left\{\bar{g}_{\alpha\bar{\beta}} = g_{\beta\bar{\alpha}}\right\} = -2ig_{\beta\bar{\alpha}}\mathrm{d}z^\beta \wedge \mathrm{d}\bar{z}^{\bar{\alpha}} = \Omega. \tag{5.6}$$

We now state the following definition

Definition 2 *A complex manifold that admits a hermitian structure and has a closed Kähler form,* $\mathrm{d}\Omega = 0$*, is called a Kähler manifold.*

5.1.2 Kähler Geometry

We now explore the Kähler manifolds further. First, we start from the definition and compute the exterior derivative of the Kähler form

$$0 = \mathrm{d}\Omega = -i(\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}})\mathrm{d}z^\gamma \wedge \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}} + \text{c.c.} \implies \partial_\gamma g_{\alpha\bar{\beta}} = \partial_\alpha g_{\gamma\bar{\beta}} \tag{5.7}$$

From the last equality we may deduce that the Kähler metric can be written as a partial derivative in its first index, i.e. $g_{\alpha\bar{\beta}} = \partial_\alpha C_{\bar{\beta}}$ for some function $C_{\bar{\beta}}$. Then, from the condition $\bar{g}_{\alpha\bar{\beta}} = g_{\beta\bar{\alpha}}$ we see that this condition must hold for both indices and thus we can write

$$g_{\alpha\bar{\beta}} = \frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial z^{\bar{\beta}}}K(z,\bar{z}) \tag{5.8}$$

where $K(z,\bar{z})$ is a scalar called the Kähler potential. It is invariant under so called Kähler transformations

$$K \to K' = K + f(z) + \bar{f}(\bar{z}) \tag{5.9}$$

Note that the above definition of the Kähler potential is not always globally defined. Then the transition function between different patches of the manifold is given by the Kähler transformation.

5.1.3 Example: Type IIB String Theory Scalar Moduli

We will now study the moduli space of the scalars in type IIB string theory. It will prove instructive to guide us in our understanding of the Kähler potential. The scalar part of the type IIB supergravity action is [27]

$$\mathcal{L} = -\Big(\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi\Big). \tag{5.10}$$

Our goal will be to express this via the Kähler metric, g_{ij}, and some complex coordinate τ as

$$\mathcal{L} = -g_{\tau\bar{\tau}}\partial_\mu\tau\partial^\mu\bar{\tau}. \tag{5.11}$$

From the appearance of ϕ in the exponential in (5.10) we are led to an ansatz for τ of the form

$$\tau = \chi + ie^{-\phi}, \tag{5.12}$$

which leads to

$$\begin{aligned}\mathcal{L} &= -g_{\tau\bar{\tau}}\Big((\partial_\mu\chi - ie^{-\phi}\partial_\mu\phi)(\partial^\mu\chi + ie^{-\phi}\partial^\mu\phi)\Big) = \\ &= -g_{\tau\bar{\tau}}\Big(\partial_\mu\chi\partial^\mu\chi + e^{-2\phi}\partial_\mu\phi\partial^\mu\phi\Big) \implies g_{\tau\bar{\tau}} = e^{2\phi}.\end{aligned} \tag{5.13}$$

Then, we use the definition of the Kähler metric to find

$$K(\tau,\bar{\tau}) = \int \mathrm{d}\tau\mathrm{d}\bar{\tau}e^{2\phi} = \int \mathrm{d}\tau\mathrm{d}\bar{\tau}\frac{-4}{(\tau-\bar{\tau})^2}, \tag{5.14}$$

since $\tau - \bar{\tau} = 2ie^{-\phi}$. Carrying out the integration yields

$$\int \mathrm{d}\tau\mathrm{d}\bar{\tau}\frac{-4}{(\tau-\bar{\tau})^2} = \int \mathrm{d}\tau\left(\frac{-4}{\tau-\bar{\tau}} + f'(\tau)\right) = -4\log|\tau-\bar{\tau}| + f(\tau) + g(\bar{\tau}). \tag{5.15}$$

As we might expect we determine the Kähler potential up to a Kähler transformation. Note that

$$|\tau - \bar{\tau}| = \frac{\tau-\bar{\tau}}{i}, \tag{5.16}$$

and we take the integration constant to be $4\log 2$ which means we can take the Kähler potential as

$$K = -4\log\frac{\tau-\bar{\tau}}{2i} = 4\phi. \tag{5.17}$$

5.2 Kähler Description of the Effective 4D Lagrangian

We now turn to the four-dimensional Lagrangian we obtained in chapter 4. It is written only in terms of the Ricci tensor and the two real scalar fields u and v. We want to study the behaviour of the Lagrangian under parity transformations, since the orientation-reversing of the seven-sphere is the phenomena we are interested in. In other words, we would like to reintroduce pseudoscalar fields into the Lagrangian, as they transform with a sign under parity. Recall, we have been very selective when choosing the two specific scalars u and v to study. The full 11-dimensional theory will in general yield many fields in four dimensions when compactified where some of them are pseudoscalars. We view the

Lagrangian we obtained in (4.77) as the limit when all fields except u and v are zero. Denoting the pseudoscalar fields by χ_i, $i = 1, 2$ we schematically have

$$\begin{aligned} \mathcal{L}\Big|_{\chi_i=0} &= \sqrt{-g}\Big(R - \frac{63}{2}(\partial u)^2 - 21(\partial v)^2 - V(u,v)\Big), \\ V(u,v) &= 2Q^2 e^{-21u} - 6e^{-9u+4v} - 48e^{-9u-3v} + 12e^{-9u-10v}. \end{aligned} \tag{5.18}$$

Of course all other fields that could appear at the four-dimensional level are set to zero above as well, but we do not explicitly write that out since we want to focus on pseudoscalars. Reintroducing the pseudoscalars at this point would be impossible if we had no further information. We will here utilise the fact that we mentioned earlier, that complex scalars in $\mathcal{N} = 1$ supergravity theories are coordinates on Kähler manifolds. From this we then combine the real scalars u and v with corresponding pseudoscalars χ_1 and χ_2 to form complex scalar fields, similarly to the example regarding type IIB string theory above. It is this structure that gives us clues to how the pseudoscalar fields must appear, only from information of the scalar fields.

Denote the complex scalar fields by τ_i. As in the above example, we seek a Lagrangian of the form

$$\frac{1}{\sqrt{-g}}\mathcal{L} = R - g_{ij}\partial\tau^i\partial\bar{\tau}^j - \mathcal{V}, \qquad i,j = 1,2 \tag{5.19}$$

with the difference that we now have included the Ricci term and a scalar potential. In supergravity, the scalar potential is usually divided into two parts called an F-term and a D-term, that is $\mathcal{V} = V_F + V_D$. A more thorough description on how to obtain the potentials from a general supergravity multiplet can be found in [22]. The main point for this thesis is to think of them as distinct parts of the scalar potential. We will mainly focus on the F-term, V_F, for reasons that will become clear later on. In term of the Kähler potential it is given by

$$V_F = e^K\Big(W_i\bar{W}_j g^{ij} - 3|W|^2\Big), \tag{5.20}$$

where K is the Kähler potential, W the superpotential, and subscripted indices denote Kähler covariant derivatives, defined by

$$W_i = \frac{\partial W}{\partial\tau_i} + W\frac{\partial K}{\partial\tau_i}, \tag{5.21}$$

for the complex fields τ_i. The superpotential, W, is a holomorphic function of the complex fields. It is this superpotential that we will be able to tune in order to match the scalar potential we obtained earlier to V_F. There is one last preliminary step that needs to be done in order to use this formalism. The kinetic terms in the Lagrangian must be canonically normalised relative to the Einstein-Hilbert term in order for the above expressions to be valid. At first this looks quite simple, and can be achieved by simply redefining the fields. However, upon doing this redefinition the exponents in the potential get scaled by $\sqrt{63}$ and $\sqrt{42}$ for u and v respectively. This actually sets some constraint on what ansatz we can make for the complex scalar and the superpotential. Due to the fact that the superpotential must be holomorphic, it can only contain integer powers of τ_i. In order for this to yield exponents with irrational coefficients for u and v, one then has to include some parameter in the definition of τ.

In light of this argument, we take the ansatz for the complex fields to be

$$\tau_1 = \chi_1 + ie^{-\alpha u}, \qquad \tau_2 = \chi_2 + ie^{-\beta v}. \tag{5.22}$$

We also rescale the fields u and v to get the Lagrangian into the canonical form. With $u \to \frac{1}{\sqrt{63}}u$ and $v \to \frac{1}{\sqrt{42}}v$, the potential takes the form

$$V = 2Q^2 e^{-a_1 u} - e^{-a_2 u} \times \left(6e^{b_1 v} + 48e^{-b_2 v} - 12e^{-b_3 v}\right), \tag{5.23}$$

with the coefficients

$$\begin{aligned} a_1 &= \frac{21}{\sqrt{63}} = 7\frac{1}{\sqrt{7}}, \quad a_2 = \frac{9}{\sqrt{63}} = 3\frac{1}{\sqrt{7}}, \\ b_1 &= \frac{4}{\sqrt{42}} = 2\sqrt{\frac{2}{3}}\frac{1}{\sqrt{7}}, \quad b_2 = \frac{3}{\sqrt{42}} = \sqrt{\frac{3}{2}}\frac{1}{\sqrt{7}}, \quad b_3 = \frac{10}{\sqrt{42}} = 5\sqrt{\frac{2}{3}}\frac{1}{\sqrt{7}}, \end{aligned} \tag{5.24}$$

and the kinetic terms are in the desired form.

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{2}(\partial u)^2 - \tfrac{1}{2}(\partial v)^2. \tag{5.25}$$

We now want to compute the Kähler potential. The first step is to differentiate the complex field τ_1, which yields

$$\partial\tau_1 \partial\bar{\tau}_1 = (\partial\chi_1)^2 + \alpha^2 e^{-2\alpha u}(\partial u)^2. \tag{5.26}$$

The result for τ_2 is similar. Comparing equations (5.25), (5.19) and last term above leads to

$$g_{11} = \frac{1}{2\alpha^2}e^{2\alpha u}, \qquad g_{22} = \frac{1}{2\beta^2}e^{2\beta v}, \qquad g_{12} = g_{21} = 0. \tag{5.27}$$

We can rewrite this in terms of the complex fields, using that $e^{2\alpha u} = (\operatorname{Im}\tau_1)^{-2}$, to get

$$g_{11} = -\frac{2}{\alpha^2}\frac{1}{(\tau_1 - \bar{\tau}_1)^2}, \qquad g_{22} = -\frac{2}{\beta^2}\frac{1}{(\tau_2 - \bar{\tau}_2)^2}. \tag{5.28}$$

The Kähler potential can be found by integrating this expression, which follows from the definition of the Kähler metric

$$g_{ij} \equiv \frac{\partial}{\partial\tau_i}\frac{\partial}{\partial\bar{\tau}_j}K. \tag{5.29}$$

Then we find the expression

$$K = -\frac{2}{\alpha^2}\log(\tau_1 - \bar{\tau}_1) - \frac{2}{\beta^2}\log(\tau_2 - \bar{\tau}_2) + (\text{Kähler transformation}). \tag{5.30}$$

As mentioned in the above example, we have the freedom to change the potential by Kähler transformations. We will do this in order to obtain a simpler expression. Note that

$$\log(\tau_1 - \bar{\tau}_1) = \log(2i) + \log(e^{-\alpha u}) = \log(2i) - \alpha u. \tag{5.31}$$

Since constant shifts are contained in the set of Kähler transformations, we can throw away the $\log 2i$ factor and then we get

$$K = \frac{2u}{\alpha} + \frac{2v}{\beta}, \tag{5.32}$$

and the prefactor in the expression for the scalar potential is

$$e^K = e^{\frac{2u}{\alpha} + \frac{2v}{\beta}}. \tag{5.33}$$

Next, we need one derivative on the Kähler potential, as it appears in the Kähler covariant derivative. We have that

$$\partial_1 K \equiv \frac{\partial K}{\partial \tau_1} = -\frac{2}{\alpha^2}\frac{1}{\tau_1-\bar{\tau}_1} = \frac{i}{\alpha^2}e^{\alpha u} \qquad \partial_2 K = -\frac{2}{\alpha^2}\frac{1}{\tau_2-\bar{\tau}_2} = \frac{i}{\beta^2}e^{\beta v}. \tag{5.34}$$

The scalar potential, V_F, is thus

$$\begin{aligned} V_F = e^{\frac{2u}{\alpha}+\frac{2v}{\beta}}\Big(&2\alpha^2 e^{-2\alpha u}|\partial_1 W + (\partial_1 K)W|^2 \\ &+2\beta^2 e^{-2\beta v}|\partial_2 W + (\partial_2 K)W|^2 - 3W\bar{W}\Big). \end{aligned} \tag{5.35}$$

This expression is quite complicated, and thus we will start to investigate it piece by piece. Before doing that however, it is instructive to comment on our ansatz. The expressions in (5.22) both parametrise the upper complex half-plane. This corresponds to the symmetry group $SL(2,\mathbb{R})/U(1)$. Since we have two complex coordinates, the manifold is described by the direct product $SL(2,\mathbb{R})/U(1)\times SL(2,\mathbb{R})/U(1)$. Note that at this point this is merely an assumption, indeed we do not know the structure of the kinetic terms for the pseudoscalar fields which dictate the full structure of the Kähler manifold. This is one of the places where we have to guess, and we take this structure as it is fairly simple, yet non-trivial and will point us to interesting results.

5.2.1 The Superpotential for $v=0$, $Q=0$

The first step we want to take is to check the case when both Q and v are set to zero. In this limit, the scalar potential is

$$V = -42e^{-a_2 u}. \tag{5.36}$$

We make the simplest possible polynomial ansatz for the superpotential

$$W = a\tau_1^m \tag{5.37}$$

with some coefficient a and integer power m. Then we compute the Kähler covariant derivative

$$W_1 = am\tau_1^{m-1} + \frac{i}{\alpha^2}e^{\alpha u}a\tau_1^m. \tag{5.38}$$

In the expression for V_F we need the absolute value of this, which is

$$\begin{aligned} |W_1|^2 =& a^2m^2(\tau_1\bar{\tau}_1)^{m-1} + \frac{e^{2\alpha u}}{\alpha^4}a^2(\tau_1\bar{\tau}_1)^m \\ &-\frac{ia^2m}{\alpha^2}e^{\alpha u}\bar{\tau}_1(\tau_1\bar{\tau}_1)^{m-1} + \frac{ia^2m}{\alpha^2}e^{\alpha u}\tau_1(\tau_1\bar{\tau}_1)^{m-1} \end{aligned} \tag{5.39}$$

We want to make sure that we obtain the correct scalar potential in the limit when $\chi_i = 0$, where $\tau_1 = ie^{-\alpha u}$ and $\tau_1\bar{\tau}_1 = e^{-2\alpha u}$. Then we find

$$\begin{aligned} |W_1|^2\Big|_{\chi_1=0} =& a^2m^2e^{-2\alpha u(m-1)} + \frac{a^2}{\alpha^4}e^{-2\alpha u(m-1)} \\ &-\frac{a^2m}{\alpha^2}e^{-2\alpha u(m-1)} - \frac{a^2m}{\alpha^2}e^{-2\alpha u(m-1)} \end{aligned} \tag{5.40}$$

We can now plug this into the expression for the scalar potential and then we obtain

$$\begin{aligned} V_F \Big|_{\chi_1=0} &= e^K \Big(g^{11}|W_1|^2 - 3W\bar{W} \Big) \Big|_{\chi_1=0} = \\ &= e^{2u(\frac{1}{\alpha}-\alpha m)} \left(2a^2\alpha^2 m^2 + \frac{2a^2}{\alpha^2} - 4a^2 m - 3a^2 \right) \end{aligned} \tag{5.41}$$

It is promising that this procedure generates exactly one unique exponential in u. However, if we would not have considered $Q = 0$ this superpotential cannot generate the two different exponential terms needed. We will come back to this issue later on. The next step is to compare this result to the potential $V = -42e^{-a_2 u}$. The exponent yields the equation

$$2\left(\frac{1}{\alpha} - \alpha m\right) = -a_2 \implies m = \frac{a_2}{2\alpha} + \frac{1}{\alpha^2}. \tag{5.42}$$

For $a_2 = 9/\sqrt{63}$ this yields infinitely many solutions for α if we demand m to be integer valued. We also get an equation for the coefficient, in which we can insert the above equation for m in terms of α

$$a^2(2\alpha^2 m^2 + \frac{2}{\alpha^2} - 4m - 3) = -42 \implies a^2(\frac{a_2^2}{2} - 3) = -42. \tag{5.43}$$

Interestingly, the dependence on α vanishes. The constant a^2 controls the overall scale, and in this scenario where $a_2 = 9/\sqrt{63}$ the quantity in the parenthesis is negative and we find

$$a^2 = 42 \times \frac{14}{33} = \frac{196}{11}. \tag{5.44}$$

So, we see that it is possible to find a superpotential consistent with our ansatz in the limit $v = 0$ and $Q = 0$. However, generating also the Q-term would require extending the superpotential. Adding yet another polynomial term in τ_1 is close at hand and we will do precisely this in the next section, but for τ_2 instead. The results obtained there are however analogous to those for τ_1 and we will discuss the Q-term in more detail later on.

We proceed our analysis by focusing on the three exponents in v by keeping $Q = 0$ and instead setting $u = 0$. If this proves consistent one could combine that solution with the one obtained in this section to possibly form an ansatz for the full superpotential of the theory.

5.2.2 The Superpotential for $u = 0$, $Q = 0$

In this limit, the scalar potential is

$$V = -6e^{b_1 v} - 48e^{-b_2 v} + 12e^{-b_3 v}. \tag{5.45}$$

In order to generate three terms, we include one extra term in the ansatz for the superpotential

$$W = a\tau^m + b\tau^n, \qquad \tau = \chi + ie^{-\beta v}. \tag{5.46}$$

We have dropped the subscript 2 on τ here in order to reduce cluttering. The procedure now is similar to that in the last subsection. First, the Kähler covariant derivative is

$$W_2 = am\tau^{m-1} + bn\tau^{n-1} + \frac{i}{\beta^2} e^{\beta v}(a\tau^m + b\tau^n), \tag{5.47}$$

and squaring it yields

$$\begin{aligned}|W_2|^2 =& a^2m^2(\tau\bar{\tau})^{m-1} + b^2n^2(\tau\bar{\tau})^{n-1} + \frac{e^{2\beta v}}{\beta^4}|W|^2 + abmn(\tau\bar{\tau})^{m-1}(\bar{\tau}^{n-m} + \tau^{n-m}) \\ &+ i\frac{am}{\beta^2}e^{\beta v}(\tau\bar{\tau})^{m-1}\big(\tau(a + b\tau^{n-m}) - \bar{\tau}(a + b\bar{\tau}^{n-m})\big) \\ &+ i\frac{bn}{\beta^2}e^{\beta v}(\tau\bar{\tau})^{n-1}\big(\tau(b + a\tau^{m-n}) - \bar{\tau}(b + a\bar{\tau}^{m-n})\big).\end{aligned} \tag{5.48}$$

We now want to evaluate this expression when $\chi = 0$. We have that $\tau\bar{\tau} = e^{-2\beta v}$ and

$$|W|^2\bigg|_{\chi=0} = a^2e^{-2\beta vm} + b^2e^{-2\beta vn} + abe^{-\beta v(n+m)}i^{n-m}(1 + (-1)^{n-m}). \tag{5.49}$$

The potential is as stated earlier given by

$$V_F = e^{\frac{2v}{\beta}}\Big(2\beta^2e^{-2\beta v}|W_2|^2 - 3|W|^2\Big). \tag{5.50}$$

We can utilise that $|W|^2$ appears also in the derivative term, with exponential prefactor such that it combines with the last term above. Evaluating the expression in parentheses at $\chi = 0$ yields

$$\begin{aligned}&2\beta^2e^{-2\beta v}|W_2|^2 - 3|W|^2\bigg|_{\chi=0} = \left[\frac{2}{\beta^2} - 3\right]|W|^2 \\ &+ 2\beta^2\Big[a^2m^2e^{-2\beta vm} + b^2n^2e^{-2\beta vn} + abmne^{-\beta v(m+n)}i^{n-m}(1 + (-1)^{n-m}) \\ &- \frac{2a^2m}{\beta^2}e^{-2\beta vm} - \frac{2abm}{\beta^2}e^{-\beta v(n+m)}i^{n-m}(1 + (-1)^{n-m}) \\ &- \frac{2b^2n}{\beta^2}e^{-2\beta vn} - \frac{2abn}{\beta^2}e^{-\beta v(n+m)}i^{m-n}(1 + (-1)^{m-n})\Big].\end{aligned} \tag{5.51}$$

Note that if $n - m$ is odd in the above expression, all terms with $e^{-\beta v(n+m)}$ vanish and the potential simplifies to

$$\begin{aligned}\tilde{V}_F = e^{\frac{2v}{\beta}}\Bigg(&e^{-2\beta vm}a^2\left[\frac{2}{\beta^2} - 3 + 2\beta^2m^2 - 4m\right] \\ &+e^{-2\beta vn}b^2\left[\frac{2}{\beta^2} - 3 + 2\beta^2n^2 - 4n\right]\Bigg)\end{aligned} \tag{5.52}$$

This solution is however not sufficient to reconstruct our potential for v, since we require three unique exponents. For the case when $n - m$ is even, we get any extra term to the above expression. We also have that $(-1)^{n-m} = (-1)^{m-n}$ and thus that $i^{m-n} = i^{n-m}$. The scalar potential is then

$$V_F = \tilde{V}_F + e^{\frac{2v}{\beta} - \beta v(m+n)}2i^{n-m}ab\left[\frac{2}{\beta^2} - 3 + 2\beta^2mn - 4m - 4n\right]. \tag{5.53}$$

The next step is to match these exponents to b_1, b_2 and b_3 respectively, in order to determine m, n and β. The scalar potential is invariant under simultaneous exchange of $m \leftrightarrow n$ and $a \leftrightarrow b$ so there are three possible ways of assigning the exponents

$$\begin{cases}\frac{2}{\beta} - \beta(m+n) = b_1 \\ \frac{2}{\beta} - 2\beta n = -b_2 \\ \frac{2}{\beta} - 2\beta m = -b_3\end{cases} \text{or} \begin{cases}\frac{2}{\beta} - \beta(m+n) = -b_2 \\ \frac{2}{\beta} - 2\beta n = b_1 \\ \frac{2}{\beta} - 2\beta m = -b_3\end{cases} \text{or} \begin{cases}\frac{2}{\beta} - \beta(m+n) = -b_3 \\ \frac{2}{\beta} - 2\beta n = -b_2 \\ \frac{2}{\beta} - 2\beta m = b_1\end{cases} \tag{5.54}$$

It turns out however that for integer n and m there are no solutions for any of the above cases and the previously specified values of b_1, b_2 and b_3. Being unable to recover our four-dimensional scalar potential indicates that something is wrong with our assumptions. There are essentially two possibilities:

1. The proposed superpotential is incorrect.
2. The coset structure for the v-field, $SL(2,\mathbb{R})/U(1)$, is incorrect.

Alternative 1 is quite unlikely, since it would be hard to modify the current ansatz. Adding more terms is not an option as it would produce new exponents in the scalar potential. Other functional dependence on τ in the superpotential would be possible, but in order to preserve the holomorphic structure the options are quite limited. The most probable explanation is thus alternative 2. As mentioned earlier, the specific structure of the Kähler manifold depends on the kinetic terms of both the scalar and pseudoscalar field. In our ansatz with $\tau_2 = \chi + ie^{-\beta v}$ we assume the kinetic term for the pseudoscalar to be of the same form as in the type IIB example in 5.1.3. Since this assumption does not reproduce our four-dimensional scalar potential, the kinetic term for the pseudoscalar is probably more complicated.

5.2.3 The Q-term

We now return to the case where we set $v = 0$ and try to reproduce both exponents in u in the scalar potential. As we saw in the previous section in equation (5.52), it is possible to generate only two distinct exponents from a superpotential with two terms. Starting from the expression in (5.52) and changing $v \to u$ and $\beta \to \alpha$ corresponds to instead considering the case where $v = 0$ and the superpotential ansatz to be

$$W = a\tau_1^m + b\tau_1^n. \tag{5.55}$$

The scalar potential is

$$V = 2Q^2 e^{-\frac{21}{\sqrt{63}}u} - 42e^{-\frac{9}{\sqrt{63}}u}, \tag{5.56}$$

so matching the exponents yields the equations

$$\begin{cases} \frac{2}{\alpha} - 2\alpha m = -\frac{21}{\sqrt{63}} \\ \frac{2}{\alpha} - 2\alpha n = -\frac{9}{\sqrt{63}} \end{cases} \tag{5.57}$$

Solving for α in the first equation and substituting in the second then leads to two solutions for n in terms of m

$$n = \frac{2m\left(8m + 3 \pm \frac{3\sqrt{16m+7}}{\sqrt{7}}\right)}{\left(\sqrt{16m+7} \pm \sqrt{7}\right)^2}, \tag{5.58}$$

which needs to be solved for integer m and n. It is not obvious whether there exists integer solutions to this or not, but using any numerical software it can be found that there are actually a number of solutions. For the case where $\pm = +$ the three lowest integer solutions (m, n) are: $(21, 18), (35, 31), (98, 91)$. For $\pm = -$ we have: $(21, 25), (35, 40), (98, 106)$. Recall that we have already assumed that $m - n$ is odd, and thus the lowest possible integer solution to the above equation is $m = 21$, $n = 18$. As we saw in equation (5.43), the prefactor is independent of m and n and we must thus also verify that the correct signs are reproduced in this setting. As we have already concluded we have that $a_2^2/2 - 3 < 0$

which gives us the negative term in the scalar potential. Fortunately, $a_1^2/2 - 3 = 1/2 > 0$, so we also get the correct sign of the Q^2-term in (5.56). The overall scaling of the terms can be determined individually by tuning a and b. We find that

$$\begin{aligned} b^2\left(\frac{a_2^2}{2} - 3\right) &= -42 \implies b^2 = \frac{196}{11}, \\ a^2\left(\frac{a_1^2}{2} - 3\right) &= 2Q^2 \implies a^2 = 4Q^2. \end{aligned} \tag{5.59}$$

So, in contrast to the case where $v \neq 0$, $u = 0$, it seems like we can obtain exactly our desired four-dimensional scalar potential for the u-field.

5.3 Structure of the Kähler Manifold and the Swampland

In light of our findings in this section, it seems like our ansatz must be revised in order to produce the desired results. The ansatz for u, that is $\tau_1 = \chi_1 + ie^{-\alpha u}$, which parametrises the upper complex half plane corresponds to the coset structure presented in 5.1.3. While this ansatz yielded the desired results, we cannot be sure whether it is the correct structure. The reason is that since the ansatz for v, $\tau_2 = \chi_2 + ie^{-\beta v}$, is incorrect we do not know the overall structure of the Kähler manifold parametrised by the two complex coordinates. A direct product structure, such as $SL(2,\mathbb{R})/U(1) \times G_v/H_v$ where G_v/H_v is a consistent coset structure for v, would imply that the solution we obtained for u could be used. However, the Kähler manifold could also be such that the u and v parts are not independent. For example, one could imagine that the Kähler metric had one component dependent on both fields, such as $g_{11}(u)$ but $g_{22}(u, v)$. Then the Kähler potential contains mixed terms of u and v and the above analysis would have to be performed much more carefully.

Unfortunately, the inability to reintroduce the pseudoscalars in the theory also hinders any further investigation of the *Non-AdS SUSY conjecture*. One could try to experiment with other complex structures for the v-field, but finding the correct one could be very difficult. A better approach for future work would be to instead find the kinetic terms for the pseudoscalars in some other way. Hopefully, one could do this by going back to the full theory. Indeed, if one introduces the pseudoscalars already at the 11-dimensional level, as we did for u and v, one could find the corresponding four-dimensional equations of motion and from there construct the terms in the Lagrangian.

6

Conclusions

In this thesis the compactification of 11-dimensional supergravity on the squashed seven-sphere has been studied in the context of the Swampland program. After providing a brief overview of the 11-dimensional theory, the geometry of a squashed seven-sphere was reviewed. The compactification was done in the Freund-Rubin ansatz, which was shown to lead to an AdS background geometry in four dimensions. By taking the scale and squashing parameters of sphere to be spacetime dependent, we constructed an effective four-dimensional supergravity Lagrangian containing a scalar potential. It was shown that the potential features two perturbatively stable extrema, corresponding to the round and squashed seven-sphere. It was argued that the extremum for the squashed seven-sphere retains stability after orientation-reversing, a process which breaks all supersymmetry. This contradicts the *Non-AdS SUSY conjecture* at a perturbative level.

In order to better understand this effect, an attempt to reintroduce pseudoscalar fields coming from the 11-dimensional theory was done. The basic idea behind this attempt rested upon the fact that complex scalars in supergravity parametrise Kähler manifolds. The size and squashing scalars were combined with pseudoscalars to form two complex coordinates, each parametrising the upper complex half-plane, which allowed for construction of the metric and potential of the Kähler manifold. Taking a polynomial ansatz for the superpotential and computing the scalar potential on the Kähler manifold allowed for comparison with the scalar potential from the compactification. The parametrisation of the upper complex half-plane, corresponding to the coset structure $SL(2,\mathbb{R})/U(1)$, was seen to be consistent for the size parameter, but unable to reproduce the desired scalar potential for the squashing parameter. Thus, it appears as one would need further information of the structure of the Kähler manifold in order to proceed with this method, as the $SL(2,\mathbb{R})/U(1) \times SL(2,\mathbb{R})/U(1)$ structure used in this thesis is not correct.

Obtaining a better understanding of the squashed seven-sphere and any possible contradictions with the Swampland program is highly interesting. Further studies of the perturbative properties, as done in this work, is one possible way. Along the lines of this thesis one could explore different coset structures of the Kähler manifold or non-polynomial forms of the superpotential. A somewhat different approach would be to introduce the pseudoscalars already at the 11-dimensional level, and explicitly see how they appear in four dimensions. Such an approach could possibly be combined with the findings of this thesis in order to provide a complete understanding of the structure of the Kähler manifold.

A

Conventions

A.1 Indices and Gamma matrices

We use Greek letters for spacetime indices. Letters from the beginning of the alphabet, $\alpha, \beta, \ldots$ are flat indices, those from the middle of the alphabet, $\mu, \nu, \ldots$ are curved. Spinor indices are written with Latin letters, $a, b, \ldots$ for flat and $m, n, \ldots$ for curved. Note that this means that $m, n, \ldots$ will almost never be used in these context. Instead they will have a more standard meaning in compactifications, by describing the coordinates on the internal manifold. Gamma matrices in eleven dimensions are defined by

$$\left\{\hat{\Gamma}^{\alpha}, \hat{\Gamma}^{\beta}\right\} = 2\eta^{\alpha\beta} \tag{A.1}$$

In the most general case, $\hat{\Gamma}$ are 11-dimensional, Γ are seven-dimensional and γ are four-dimensional. However, in parts where the work is done exclusively in 11 dimensions, Γ (no hat) is also used for the 11-dimensional ones. Antisymmetrised products of gamma matrices are written

$$\Gamma^{[n]} = \Gamma^{\alpha_1 \ldots \alpha_n} \equiv \Gamma^{[\alpha_1}\Gamma^{\alpha_2} \ldots \Gamma^{\alpha_n]} \tag{A.2}$$

In four dimensions, γ_5 is defined as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.3}$$

The symmetries of 11-dimensional gamma matrices are taken so that

$$\begin{aligned} \bar{\chi}\hat{\Gamma}^{[m]}\psi &= -\bar{\psi}\hat{\Gamma}^{[m]}\chi, \qquad \text{for } m = 1, 2, 5 \\ \bar{\chi}\hat{\Gamma}^{[m]}\psi &= +\bar{\psi}\hat{\Gamma}^{[m]}\chi, \qquad \text{for } m = 3, 4 \end{aligned} \tag{A.4}$$

for Majorana spinors. Since the flip of the spinors produce a minus sign from the anti-commuting nature we will, for example, call $\Gamma^{\alpha\beta\delta}$ antisymmetric while Γ^{α} is said to be symmetric.

A.2 Tensors

The Riemann tensor is defined by

$$R_{\mu\nu\alpha}{}^{\beta}(\omega) \equiv 2\partial_{[\mu}\omega_{\nu]\alpha}{}^{\beta} + 2\omega_{[\mu|\alpha|}{}^{\gamma}\omega_{\nu]\gamma}{}^{\beta} \tag{A.5}$$

Where ω is the spin connection. If we solve the equation of motion for the spin connection we will find

$$\omega_{\mu}{}^{\alpha\beta} = \omega_{\mu}{}^{\alpha\beta}(e) + K_{\mu}{}^{\alpha\beta} \tag{A.6}$$

where K is what we define as the contorsion, and $\omega(e)$ is the torsion-free connection.

The Levi-Civita symbol is defined as positive with upper indices[1]. Explicitly in four dimensions

$$\varepsilon^{0123} = 1, \qquad \varepsilon_{0123} = -1 \tag{A.7}$$

The Levi-Civita symbol can then be used to define the determinant of any matrix as

$$\det M = -\frac{1}{D!}\varepsilon^{b_1\ldots b_D}\varepsilon_{a_1\ldots a_D}M^{a_1}{}_{b_1}\ldots M^{a_D}{}_{b_D} \tag{A.8}$$

These definitions are however only true if written in the local frame. In order to express the Levi-Civita symbol in curved indices we use the following definitions

$$\begin{aligned}\varepsilon_{\mu_1\ldots\mu_D} &\equiv e^{-1}\varepsilon_{a_1\ldots a_D}e_{\mu_1}{}^{a_1}\ldots e_{\mu_D}{}^{a_D}\\ \varepsilon^{\mu_1\ldots\mu_D} &\equiv e\varepsilon^{a_1\ldots a_D}e_{a_1}{}^{\mu_1}\ldots e_{a_D}{}^{\mu_D}\end{aligned} \tag{A.9}$$

with $e = \det e_\mu{}^a$ and $e^{-1} = \det e_a{}^\mu$. Note that the symbol is not a tensor. It is sometimes convenient to use the tensor description and thus we define

$$\epsilon_{\mu_1\ldots\mu_D} = e\varepsilon_{\mu_1\ldots\mu_D} \qquad \epsilon^{\mu_1\ldots\mu_D} = e^{-1}\varepsilon^{\mu_1\ldots\mu_D} \tag{A.10}$$

Note that this means that the usual contraction identities can be used also for the tensor, since the factors of e cancel. A general formula for contractions is

$$\varepsilon_{a_1\ldots a_n}b_1\ldots b_p\varepsilon^{a_1\ldots a_n c_1\ldots c_p} = -p!n!\delta^{c_1\ldots c_p}_{b_1\ldots b_p} \tag{A.11}$$

The Kronecker delta with multiple indices is defined by

$$\delta^{\rho\sigma}_{\mu\nu} = \delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} \tag{A.12}$$

The covariant d'Alembertian operator acting on scalars is defined by

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \tag{A.13}$$

[1] Comparing to Supergravity [22], this corresponds to $s_5 = -1$

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