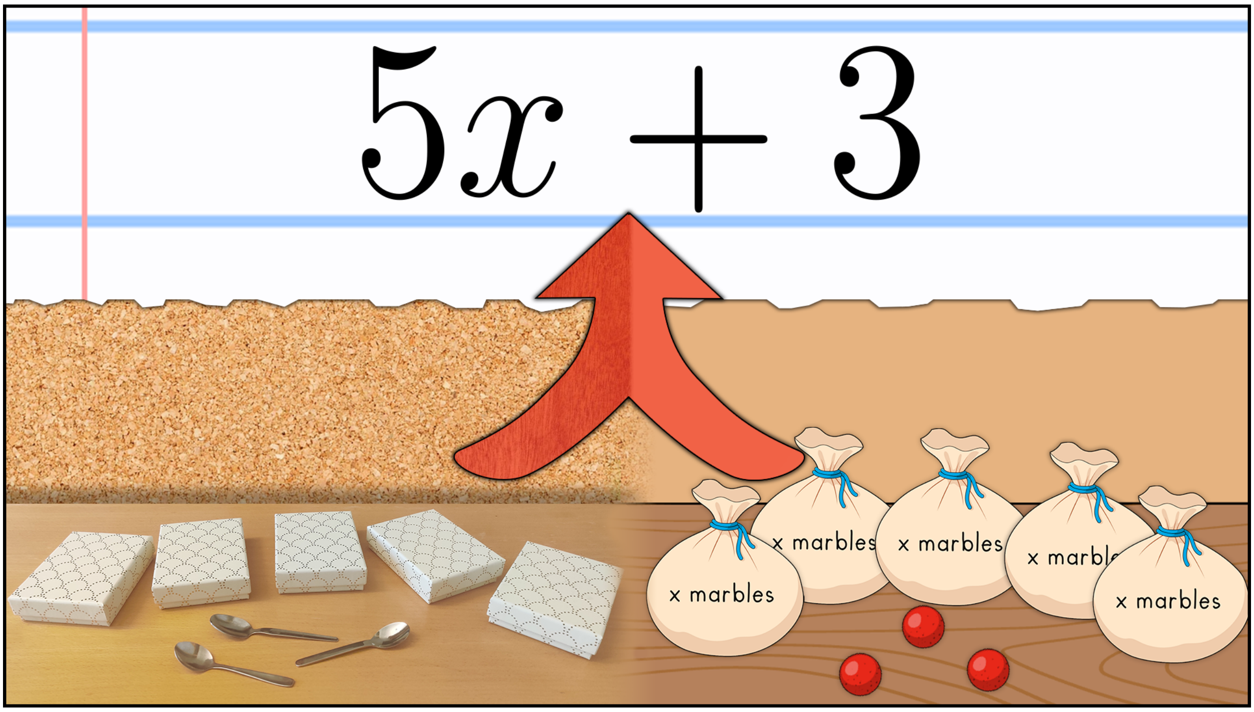

Hands-On Algebra

- An Experiential Learning Intervention in Middle School Using Physical and Virtual Manipulatives

Master's thesis in Learning and Leadership

LEO LAHTI & LISA LINDHE

DEPARTMENT OF COMMUNICATION AND LEARNING IN SCIENCE

CHALMERS UNIVERSITY OF TECHNOLOGY

Gothenburg, Sweden 2025

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Master's Thesis 2025

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Cover: An illustration symbolising the how experiential learning using physical or digital manipulatives can lead to symbolic representation.

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Abstract

Is there any difference in knowledge gained from using a virtual object rather than a physical in a learning session? Several studies show the benefit of using manipulatives for teaching. What is possible using technology changes everyday, including what learning experiences can be created. This makes new research extra relevant for this area of interest. Following an experiential learning intervention of two blocks with pre- and post-tests using virtual and/or physical manipulatives that resulted in a total of four sessions in six classes in Swedish middle school. The mean difference between post- and pre-test was 8.1 percentage units with N=62. Analysing results using a two-way and three-way ANOVA can not statistically confirm if the format of only virtual, only physical or a combination of both impacts mathematical learning differently. However, the format does affects other attributes such as engagement, accessibility and customisability. Manipulatives can be a promising way to teach, and further research with more classes is encouraged.

Keywords: digital learning, algebra, manipulatives, middle school.

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Seif, you make every situation more enjoyable with you positive energy. Your creativity, reliability and wise ideas you always bring to the table is something we will miss in up-and-coming adventures.

To all who helped us gather teaspoons from second hand stores all over - we are so thankful! They were used by the learners in the schools. Thanks to your help we saved both money and ecological resources compared to buying newly produced spoons.

To our fellow master students at Learning and Leadership - these two years together has placed you deeply in our hearts. We could not have wished for a better class.

To you reading this; we appreciate you.

Leo and Lisa, Gothenburg, May 2025

List of Acronyms

Below is the list of acronyms and domain specific words that have been used throughout this thesis listed in alphabetical order:

Akelius Math	Part of the Akelius foundation that develop digital mathematical courses.
App	Application (digital program)
ANOVA	Analysis of variance (statistical method)
Desmos	Graphing Calculator
CCSS	Common Core State Standards (mathematics curriculum)
Cuisenaire Rods	Manipulatives that consist of bars in different lengths that can be used to teach the four basic arithmetical operations and visualising fractions.
D_F	Degrees of Freedom (statistics)
GDPR	General Data Protection Regulation
Math editor	A job at Akelius Math whose central role is designing digital math courses
Pascaline	Mechanical calculator with cogwheels representing e.g. hundreds, tens and ones. Invented by Pascal in 1642 (when he was 19 years old) (Chapman, 1942).
PP	Combination of two sessions using physical manipulatives after one another.
PV	Combination of one session using physical manipulatives followed by a session using virtual manipulatives.
Score Improvement	The difference in points gained between pre- and post-tests.
VP	Combination of one session using virtual manipulatives followed by a session using physical manipulatives.
VV	Combination of two sessions using virtual manipulatives after one another.

Contents

List of Acronyms **ix**

List of Figures **xiii**

List of Tables **xv**

1 Introduction **1**
1.1 Background 1
1.2 Aim 2
1.3 Research Questions 2
1.4 Limitations 2

2 Theory **3**
2.1 Definitions and Theoretical Background 3
2.1.1 Manipulatives in Mathematics 3
2.1.2 Bruner's Theory of Representation 3
2.2 Previous Research 4
2.2.1 Learning Gains Supported by Manipulatives 4
2.2.2 Contrasts Between Physical and Virtual Manipulatives 5

3 Methods **7**
3.1 Preparation of the Study 7
3.1.1 Choice of Mathematical Topics and Literature Search 7
3.1.2 Sample 8
3.1.3 Intervention Design 8
3.2 Conducting the Study 16
3.2.1 Preparing the Learners 16
3.2.2 Pilot Sessions and Participation 16
3.3 Observation Data 17
3.3.1 Recording Learners' Interactions on Camera 17
3.3.2 Collecting Learners' Experiences through Interviews 17
3.4 Data Analysis 18
3.4.1 Assessing and Analysing Pre- and Post-tests 18
3.4.2 Themes for Analysis 18
3.4.3 Analysing Interviews 19
3.4.4 Analysing Recordings 19

3.5 AI-Usage 19

4 Results **21**
4.1 Scores on Pre- and Post-tests 21
4.1.1 Statistic Analysis of Test Scores 22
4.2 Interviews, Observations and Video Recordings 23
4.2.1 Opportunities in Virtual Manipulatives 23
4.2.2 Challenges with Virtual manipulatives 23
4.2.3 Opportunities in Physical Manipulatives 24
4.2.4 Challenges with Physical Manipulatives 24
4.2.5 Relations Between Cognitive Representations 25
4.2.6 Learners' Motivation 25

5 Discussion **27**
5.1 Discussion of the Results 27
5.1.1 Statistic Analysis Discussion 27
5.1.2 Discussion on Mathematical Outcomes 28
5.1.3 Comments on Learning Design 31
5.2 Discussion of the Data Collection 33
5.2.1 Sources of Error 33
5.2.2 Ethics and Honesty 35

6 Conclusion and Further Research **37**
6.1 Conclusion 37
6.2 Further Research 37

A Appendix 1 **I**
A.1 Specification of Requirements for Physical Manipulatives I
A.2 Interview Questions I
A.3 Mean and Difference in Results by Class II
A.4 Assessment Matrices II
A.4.1 Block 1 II
A.4.2 Block 2 VI
A.5 Pre- and Post-tests XII
A.5.1 Block 1 XII
A.5.2 Block 2 XXIII
A.6 Exercises Used in the Physical Sessions XXX
A.6.1 Physical Session 1 XXX
A.6.2 Physical Session 2 XXXV
A.6.3 Physical Session 3 XXXVIII
A.6.4 Physical Session 4 XLI
A.7 Exercises Used in the Digital Sessions XLV
A.7.1 Digital Session 1 XLV
A.7.2 Digital Session 2 LII
A.7.3 Digital Session 3 LX
A.7.4 Digital Session 4 LXVII

List of Figures

2.1 Photo of Jerome Bruner by Poughkeepsie Day School on Flickr (2013). Used under licence CC BY-NC-SA 2.0 (n.d.), no changes made. . . . 4
2.2 Flowchart describing how a virtual and physical version of the same manipulative can be combined to yield better learning. Based on Komatsu and Fujita (2024). 6

3.1 Diagram showing the design of the study. Each row represents a class with the learners used for data (two classes were used as pilot, see section 3.2.2). The letters show the sequence of manipulatives that each group was using during the four sessions; virtual manipulatives (V) or physical manipulatives (P) for the two blocks covering two different mathematical topics. This represent a different design of paring Combination (VV, VP, PP, PV) during the blocks used in this study. Note that paring of VV and VV or PP and PP is not represented in this study. 9
3.2 Learning goals for Block 1 (variables, expressions and simplifying) and Block 2 (equations and equation solving) 10
3.3 A typical slide in a digital lesson, consisting of a Desmos to provide virtual manipulatives, a question, a hint button and a textbox where learners can answer, with some supporting text, used with permission (Akelius Math, 2025). 12
3.4 Showcase of physical manipulatives in Block 1. 13
3.5 A showcase of how the learners held the plates to feel if it was balanced. 13
3.6 Learners' sets of physical assets used in Block 1 and 2. 14
3.7 Showcase of virtual manipulatives in Block 1. 15
3.8 Showcase of virtual manipulatives in Block 2. 15

5.1 An illustration of two zero-pairs with algebra tiles, one for constants and one for variables, used with permission (Akelius Math, 2025). . . 30

List of Tables

3.1 Number of learners that took both pre- and post-test in Block 1 and Block 2 respectively, divided by class. 16

4.1 Mean scores normalised from 0-100 p and Score improvement with respect to Class (A-D). "V" denotes virtual manipulatives and "P" physical. 21
4.2 Two-way ANOVA test results for the four classes in the study, analysing the effects of Class, Combination and those combined on Score improvement. 22
4.3 Three-way ANOVA statistical test results analysing the effects of Block, Class, Combination and all combinations of those on the Score improvement for the four classes in the study. 22

A.1 Mean scores and Score improvement for each class, scores normalised from 0-100 p. "V" means virtual manipulatives and "P" physical. . . II

1

Introduction

1.1 Background

Formative research in the area of education has suggested that a progression through three different types of representations (explained more thoroughly in section 2.1.2) is beneficial for teaching abstract concepts; starting with *enactive*, followed by *iconic*, and last *symbolic* (Bruner, 1966). Bruner's three representations being originally published year 1966 existed before computers were commonplace in offices and households. Computers creating possibilities to visualise and manipulate objects without them existing in the real world could be utilised for teaching abstract objects following Bruner's theories. However, a significant difference between manipulating objects on a screen and in the real world is the enactive representation, emphasising touch and movement. If virtual artefacts fulfil the enactive representation on par with physical artifacts or if it could be seen as a step in-between the enactive and iconic representations was one of the focus areas of the study.

The correlation between physical and digital learning has been explored for some time, for instance by Suh & Moyer-Packenham (2007) focusing on third-grade algebra, emphasising how using the different formats helped to strengthen the learners understanding. Another study by Zacharia & Olympiou (2011) also utilised physical and virtual objects, though in undergraduate physics tuition. The results showed how the combination between formats were equally effective in promoting students' understanding, however all groups performed better than the group that were not using manipulatives (Zacharia & Olympiou, 2011). The use of digital platforms provide vast opportunities for increasingly more realistic experiences where the learner can move, alter and manipulate objects on a screen. Akelius Math Learning Lab (Akelius Math) that initiated the thesis develop digital math courses and had an interest in exploring the combination of digital learning with physical learning, to enhance understanding. Since possibilities in digital learning changes with the pace of technology, new research about this matter becomes increasingly important.

1.2 Aim

The main focus of the study was to evaluate mathematical learning outcomes from only virtual manipulatives, only physical manipulatives or a combination of both, presented in varying order. While doing so, the study endeavoured adding to the understanding concerning if virtual manipulatives in a digital format can play a similar role as actual physical objects when it comes to learning algebra. This is connected to the *enactive* cognitive representation formulated by Bruner, which is further discussed in section 2.1.2.

1.3 Research Questions

Below is the specification of the questions that were investigated.

1. How does the combination of virtual and physical manipulatives affect mathematical learning for pre-algebra in grade 6 and 7 (in Primary school and Lower secondary school) in a Swedish context?
2. What are important aspects in designing working sessions using digital and physical manipulatives?

1.4 Limitations

The study focuses on learners in elementary school, ages 11-13, grades 6 and 7 in the Swedish education system. No comparison between different physical manipulatives for the same mathematical topic are made, only one type of physical manipulative was used for each. Only learning goals presented in Figure 3.2 were tested, even though some studies suggest that physical manipulatives can promote other skills and abilities (e.g. Hawes et al., 2022; Suh & Moyer-Packenham, 2007). The results of this study are also not compared to regular teaching without manipulatives.

2

Theory

2.1 Definitions and Theoretical Background

This section presents important definitions necessary to help understand the fundamentals of the thesis.

2.1.1 Manipulatives in Mathematics

The founder of Akelius Foundations, Mr. Akelius, have coined the term *"Green Mathematics"*. Green Mathematics refers to a context for learning mathematics done preferably outdoors, with concrete objects or through physical means representing mathematical concepts (Akelius Foundation, n.d.). This is connected to the enactive representation in Bruner's Theory of Representation, described in section 2.1.2.

Manipulatives are artefacts designed for interactive learning, in a set-up having features that can be altered by the learner. Two distinctive types of manipulatives are discussed in this study; physical and virtual manipulatives. Physical manipulatives are real objects that can be handled spatially. Virtual manipulatives are applications and digital content that the learner can interact with and modify on a screen.

2.1.2 Bruner's Theory of Representation

As mentioned in section 1.1, Bruner (see figure 2.1) presented that using several cognitive representations are beneficial for learners when faced with new abstract concepts (1966). Descriptions and examples of Bruner's three types of representations are:

- Enactive - concrete representation that can be touched (Bruner, 1966). E.g. a balance scale that the learner can put weights on representing the two sides of an equation.
- Iconic - a pictorial representation of the concrete situation (Bruner, 1966). This could for instance be pictures of marbles and bags that the learner uses to visualise a word problem in mathematics.
- Symbolic - the abstract concept in its full form (Bruner, 1966), with mathematical notation in this case. An example could be $2x + 1 = 3$.

Figure 2.1: Photo of Jerome Bruner by Poughkeepsie Day School on Flickr (2013). Used under licence CC BY-NC-SA 2.0 (n.d.), no changes made.

Bruner suggested that multiple enactive representations that are similar but highlights different aspects are beneficial for the learner (1966). He also said that a progression from enactive to iconic and then symbolic usually is recommended, but that no universal path exists for every learner and every subject (Bruner, 1966). Learners that have a strong symbolic comprehension might skip the enactive and iconic stage but that could create a lack of mental images of concrete enactments that might be helpful when tackling more difficult symbolic tasks in the future (Bruner, 1966). Bruner wrote his papers in a context where digital tools and technology was no where near today's level. This opens up for questions to if and how the virtual enactment can be effective for the learners.

2.2 Previous Research

Relevant articles and sources were gathered before the intervention to give insights into the area of study. Two important areas were found while going through the literature: the first concerning how manipulatives are used in the classroom to promote mathematical learning and the second on how digital and physical formats have been compared in research.

2.2.1 Learning Gains Supported by Manipulatives

Multiple sources suggests that the use of manipulatives enhances learning (Zacharia & Olympiou, 2011; Suh & Moyer-Packenham, 2007; Hawes et al., 2022). The process of how and if the desirable knowledge is gained seems more unclear. Some research shows that there can be a disconnect where the manipulation of objects does not have effect on the abstract knowledge (the symbolic representation). In Ball's (1992) article she stated "Although concrete materials can offer students contexts and tools for making sense of the content, mathematical ideas really do not reside in cardboard and plastic materials.". This suggests that the desired abstract ideas of math require more than actual manipulation. Thus, the use of manipulatives to achieve fundamental knowledge requires careful consideration. Suh & Moyer-Packenham (2007) highlights that cognitive load plays a role in the disconnection. The cognitive load during the activity might be overwhelming for the students, making it hard to form a connection between manipulatives and symbols (Suh & Moyer-Packenham, 2007). Another study claim that while students can fulfil tasks using manipulatives, they may do it partly by routine without considering the connection to mathematics (Gravemeijer, 1991).

The enactment of manipulatives also needs consideration. Skoumpourdi (2010) emphasised children having difficulties using cubes as manipulatives in systemetic ways to solve problems. A similar issue is shown using cuisenaire rods. Cuisenaire rods having different colors made children relate the rods' colors to their number rather than their lengths. In older ages when the rods changed colors, issues appeared (Szendrei Julianna, 1996). Even though the use of manipulatives can yield better results than traditional lecture-based teaching (Zacharia & Olympiou, 2011), how their usage is implemented is crucial (Ball, 1992).

2.2.2 Contrasts Between Physical and Virtual Manipulatives

Research regarding the use of manipulatives commonly compares different types of manipulatives against one another, for instance physical against virtual manipulatives. Alternatively, manipulatives are compared with active teaching or traditional lecture-based teaching. In Zacharia and Olympiou's (2011) study regarding experimentation using physical and virtual manipulatives for experimental physics skills at tertiary level, the outcome were the same for both formats. However, they emphasized some key differences about availability, such as portability, safety, and cost-efficiency (Zacharia & Olympiou, 2011). They also highlight that virtual experimentation is easier to replicate, distribute, takes physically less space and is easier to manage in the classroom (Zacharia & Olympiou, 2011).

Hawes et al. (2022) analysed physical and virtual material to train spatial thinking which is defined in the article as "the ability to generate, manipulate, and reason about spatial relations between and within objects". The conclusion was that concrete materials develop spatial thinking more efficiently (Hawes et al., 2022). However, the authors also stated mechanisms regarding transfer of knowledge needs to be studied more (Hawes et al., 2022).

Some studies focused on how the differences in manipulative formats could be utilised to enhance learning when using a combination of virtual and physical manipulatives. Suh & Moyer-Packenham's (2007) study about learning algebraic relationships in third grade using physical and virtual manipulatives found students translating drawings to mathematical equations and written explanations. Conclusions from that study stated that manipulative formats may have features that encourage relational thinking. The authors also compared their results to Terry's (1996) study where students in grade 2-5 who used a combination of physical and virtual manipulatives scored significantly better than the students only using one of the formats (Suh & Moyer-Packenham, 2007). Komatsu & Fujita (2024) illustrated their findings in a similar manner as in Figure 2.2, showing how a duo of manipulatives can be used together to enhance learning. Key components are *complementarities*, *redundancies* and *antagonisms* which create learning opportunities. Complementaries are aspects that only one of the versions of the manipulatives show. Redundancies are aspects that both versions of manipulatives create, that are familiar to the learner when using the second format. Antagonisms are aspects that are in contrast and might seem to contradict what the learner experienced from the first format. Instrumental genesis means using an object beyond simple actions, e.g. using boxes and spoons to represent a mathematical expression instead of just opening the box and looking what is inside.

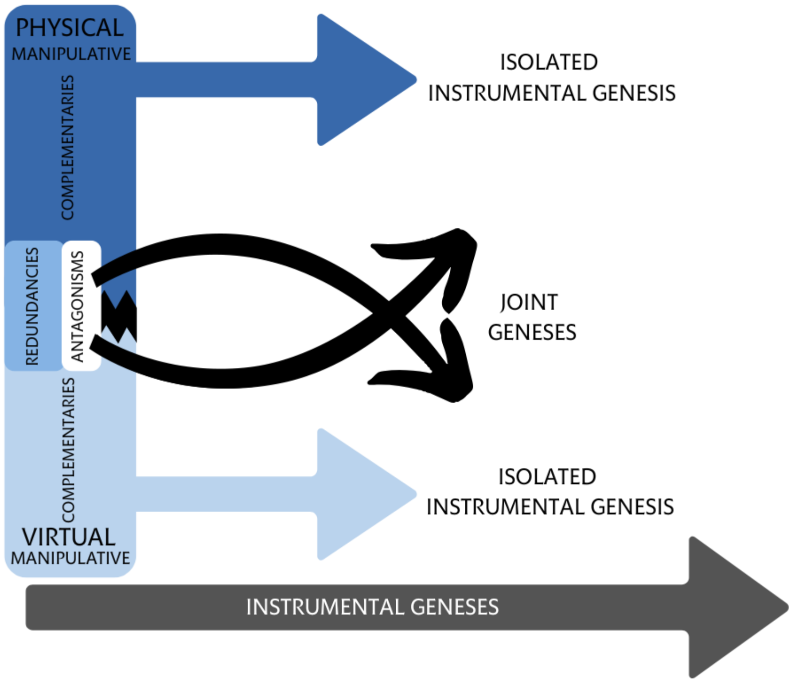


Figure 2.2: Flowchart describing how a virtual and physical version of the same manipulative can be combined to yield better learning. Based on Komatsu and Fujita (2024).

Komatsu & Fujita (2024) explains how a previous study used Pascalines and e-Pascalines, a manipulative used for adding and subtracting numbers with three cogwheels representing hundreds, tens and ones. The Pascalines had similar looks (including important features like numbers on the cogs). However, the tools acted differently, the e-Pascaline was not allowed to rotate beyond a certain notch, adding a constraint. The antagonisms between the different Pascalines make them work differently but still have some similarities which encouraged learners to think differently about the same objects, giving them opportunities to reflect and thus enhance mathematical understanding (Komatsu & Fujita, 2024).

3
Methods

3.1 Preparation of the Study

The preparatory stages of the study are described in sections 3.1.1-3.1.3, concerning how the learning intervention was designed, how the classes were chosen and which manipulatives were acquired. Other aspects relevant to the design phase are also described.

3.1.1 Choice of Mathematical Topics and Literature Search

In order to evaluate mathematical learning outcomes from using physical or virtual manipulatives, or a combination of both, mathematical topics had to be chosen. Deciding which topics to focus the study on was based on what mathematical topics were accessible in the Akelius Math platform. These were used as an inspiration. This was in order to shorten the design phase. Many mathematical topics were considered and pre-algebra was chosen. One reason for this was that the digital material had manipulatives that could also be represented physically. The outweighing factor was that some digital material was already accessible. In the end, a lot of the digital session design was still done by the researchers of this study, as described in section 3.1.3.

Relevant articles where found through searches in databases like Scopus and ERIC, using combination of keywords like "manipulatives", "algebra", "mathematics" and "middle school". More articles where found through references in the articles. Numerous articles emerged and especially influential works were identified through many articles pointing back to the same work. A common theme from researching relevant studies were articles that mainly focused on learners with disabilities. These were excluded from the study since they did not fall under the main area of investigation. Furthermore, articles that were not peer-reviewed were excluded and some books that provided relevant background were included. See chapter 2.2 for references to the literature and description of several relevant findings.

3.1.2 Sample

Schools in central Gothenburg and in the Kungsbacka muncipality were contacted, mainly by e-mail, asking whether classes would participate in the study and have sessions with manipulatives for pre-algebra. Some schools declined because lectures in algebra were planned during the same time as the study. Some classes were excluded, for instance because of having no ordinary mathematics teacher and thus having several substitute teachers coming and going. Another reason was schools having policies to not collaborate with universities. Of those that wanted to partake in the study, all relevant classes were included that had the possibility to have five sessions scheduled during the data collection period. In the end, six classes partook in the study, from four different Swedish charter schools.

3.1.3 Intervention Design

The intervention was designed with a preparatory session and four working sessions for each class, divided into two blocks (Block 1 and Block 2) with each working session being 30 minutes long. Out of these 30 minutes, 10 minutes were reserved for pre- or post-test. During the remaining 20 minutes, the learners were either using physical manipulatives or virtual manipulatives in groups of two, or sometimes three. Block 1 focused on the concept of variable as well as expressions and simplifying, while Block 2 concerned equations and equation solving. An overview of the sessions can be found in Figure 3.1. The material was reviewed by supervisors and Math editors at Akelius Math before being used to ensure tests and sessions met quality standards and were relevant for the learners.

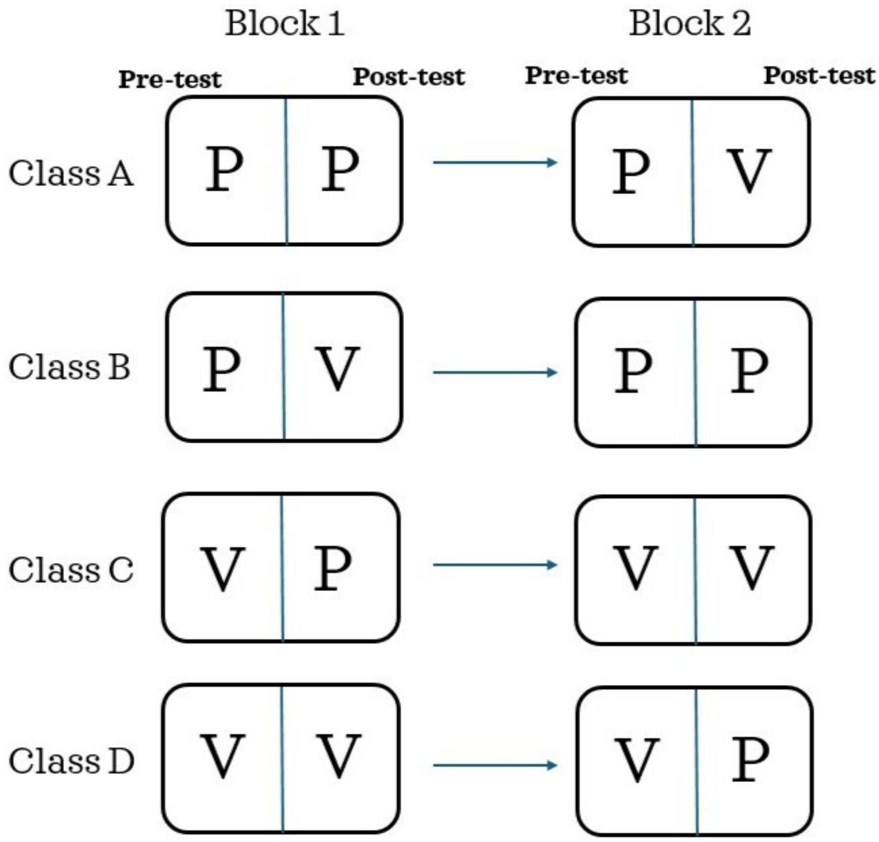


Figure 3.1: Diagram showing the design of the study. Each row represents a class with the learners used for data (two classes were used as pilot, see section 3.2.2). The letters show the sequence of manipulatives that each group was using during the four sessions; virtual manipulatives (V) or physical manipulatives (P) for the two blocks covering two different mathematical topics. This represent a different design of paring Combination (VV, VP, PP, PV) during the blocks used in this study. Note that paring of VV and VV or PP and PP is not represented in this study.

Each of the four working sessions had two versions, one with physical manipulatives and one with virtual manipulatives. There are four combinations for each block, either physical, physical (PP); physical, virtual (PV); virtual, physical (VP); or virtual, virtual (VV). Any of these four combinations (PP, PV, VP and VV) are further on referred to as the *Combination* that the class had during a block. The intervention was planned to span over a four week period for each class, in order for the learners to get some spacing in between the sessions and to minimize disrupting their regular classes. However, due to sick leave and other events the majority of the classes had their last session five weeks after the first one instead of four.

Learning Goals and Tests

Learning goals for both blocks were chosen after considering the Akelius Math educational material that lies close to the content of Common Core (CCSS), used widely in the USA (Wisevoter, n.d.). The learning goals from the digital lessons inspiring the working sessions in year 6-7 were chosen or adapted for the content. The learning goals for each block are presented in Figure 3.2.

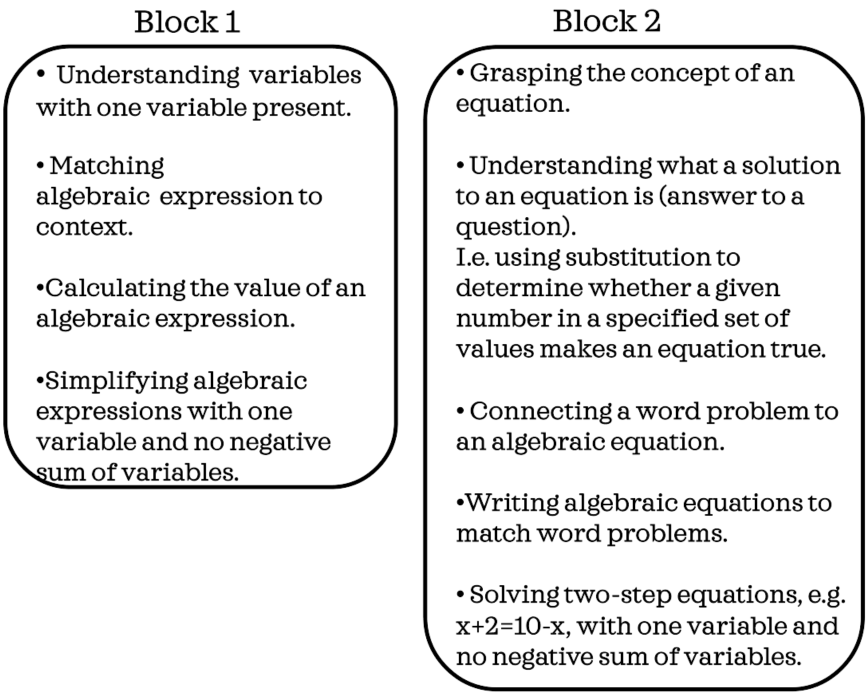


Figure 3.2: Learning goals for Block 1 (variables, expressions and simplifying) and Block 2 (equations and equation solving)

Pre- and post-tests were then designed in alignment with the learning goals, ensuring fulfilment of all learning goals could be achieved in the test and that the tests would not take longer than 10 min for the learners to complete to fit the schedule. The pre-tests' purpose is to determine the learners' knowledge before the session begins, to see the increase of mathematical learning from the sessions. If the pre-tests were not included in the study, only the learners current knowledge would have been measured. The pre- and post-tests are almost identical, consisting of the same core questions with different digits or similar everyday real life word problems. An example of this is Task 3a) in Block 1. The pre-test question being "Simplify the expression $36x - 5 - 31x + 12$" and the post-test being "Simplify the expression $46x + 20 - 42x - 14$". All questions for the pre- and post-tests in both blocks can be found in appendix A.5.

The tests were designed with grading the answers, and more importantly how the learners solved the tasks, in mind. For instance, the final question of the tests always asked learners to describe how they solved the last (usually hardest) task, this was to get an insight in how the learners think. In one of the pilot classes the test had more of these sub questions, but learners spent too much time on these subtasks that only the last question remained. Other examples includes checking if they reused information from earlier subtasks. This can be seen in the same task mentioned before, in pre-test Block 1. Task 3a) stated: “Simplify the expression $36x - 5 - 31x + 12$”, with 3b) stating: “What is the value of the expression $36x - 5 - 31x + 12$ if $x = 5$?”. Here, learners could solve 3b) by inserting $x = 5$ in the larger expression or use the already simplified expression from task 3a) which would make the calculation easier. The learners’ chosen approach were noted. All test also started of with a quick easier question, hoping all learners would be able to solve one problem. This was mainly to make the learners feel more motivated to continue with the remaining questions. Worth noting is that the first question also aligned with the learning goals. Additionally, to make the material more relevant for the learners, Swedish national tests in mathematics for grade 6 were used as inspiration to tasks and tests.

Session Design

Sessions were designed corresponding to the established learning goals, see Figure 3.2. To find exercises or inspiration for exercises the learners can perform to achieve the learning goals brainstorming and browsing already found data started. This included looking at Akelius Math lesson progression, taking inspiration from methods in articles from section 2.2, using AI and exploring Swedish national tests.

Sessions started of with an engaging task, to capture the learners interest. The remaining tasks were created in correlation with a specific learning goal. If the increase in difficulty was too steep, bridging-tasks and theory were added in-between. After a sessions’ structure was put together, the session was reviewed. Thoughts while reviewing revolved around the concept of exploration and clear instructions. Questions needed to have simple instructions so that all learners would understand what to do. If too clear instructions were given however, the learners would end up in “hands on, heads off” meaning they would do the tasks but not learn anything. Giving the learners room to experiment was valued substantially. When reviewing questions, the desired answer was written, asking if learners would write this answer to the question. If not, the question was rephrased.

Once the session’s design was finished it needed to be applied to either the physical or virtual format. Discussions on how the learners should answer in the physical format ended with writing area beside the questions which also could show the progression of sub-tasks in an intuitive way. To translate the session to the digital version, the Akelius Math system features were investigated. The system can for instance, use a graphing calculator called Desmos, check for answers on tasks using various methods such as textboxes, radio buttons and matching draggable pairs. It can also play videos, use games like memory and log data such as time taken per task and check if it was solved or not. Another notable function is that learners are not able to go backwards in the lesson unless they restart it, this is because of the reward functionality where learners receive 0-5 coins at the end of the lesson depending on the amount of correct answers. The learners also has access to a hint button which will give a hint on how to solve the question if pressed. The answers in Akelius system are also most often auto corrected. A screenshot of a typical slide from Akelius Math system used in the study can be seen in Figure 3.3.

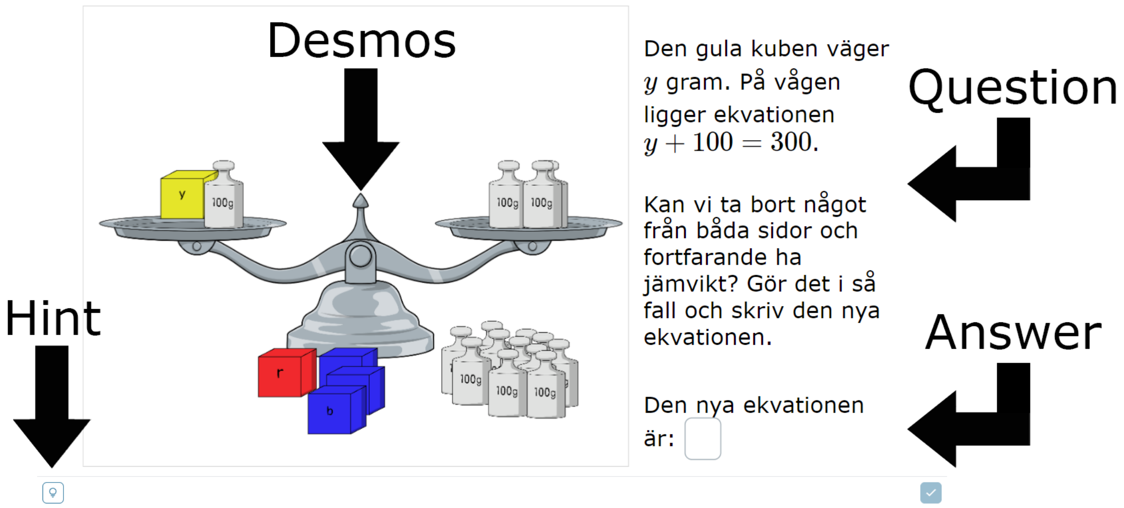


Figure 3.3: A typical slide in a digital lesson, consisting of a Desmos to provide virtual manipulatives, a question, a hint button and a textbox where learners can answer, with some supporting text, used with permission (Akelius Math, 2025).

To create virtual manipulatives in Akelius Math system, Desmos was used. Features were implemented to match the session with its physical version. Assets not found among Akelius material needed to be developed, such as images and code. Sessions of both formats were later tested by the authors from a learners' perspective (acting as if they were learners) and got reviewed by supervisors and Math editors at Akelius before used in classes.

Physical Manipulatives

A specification of requirements was made using the theoretical framework found in chapter 2 for the physical manipulatives. An excerpt of these specifications can be found in Appendix A.1. The physical manipulatives were bought or designed and manufactured. Spoons and boxes were used as physical manipulatives for Block 1 to replicate variables and constants. This is through a box having an unknown amount of spoons in it until it is opened, revealing a fixed amount, showcased in Figure 3.4. Usage of other materials was also considered, for instance bags and marbles or matchboxes and matches. Spoons and boxes were chosen due to it being familiar items learners interacts with on the daily, creating a stronger bond to green mathematics.

(a) Closed box, concealing the amount of spoons contained inside.

(b) Open box, revealing the amount of spoons contained inside.

Figure 3.4: Showcase of physical manipulatives in Block 1.

Block 2 used 3D-printed cubes with unknown weights together with metallic hundred-gram weights, as seen in Figure 3.5. To replicate equation solving with physical manipulatives the students weighted the material using their hands and balanced it on plates to distribute the pressure more evenly (making it easier to feel the weight difference) and to fit all material for the learners' tasks. A showcase of how the learners were instructed to hold the plates can be seen in Figure 3.5. Discussions regarding using real balance scales were made, however balance scales were seen as a too mechanical manipulative, thus too distant from green mathematics. Small tests in the office were conducted using boxes as plates to balance the weights in both hands. Vast majority of those who tested felt when the system was balanced with a sensitivity of about 50 grams and thus plates were used instead of a scale. The minimum difference between weights were as a buffer chosen to 100 grams making sure learners could feel the difference.

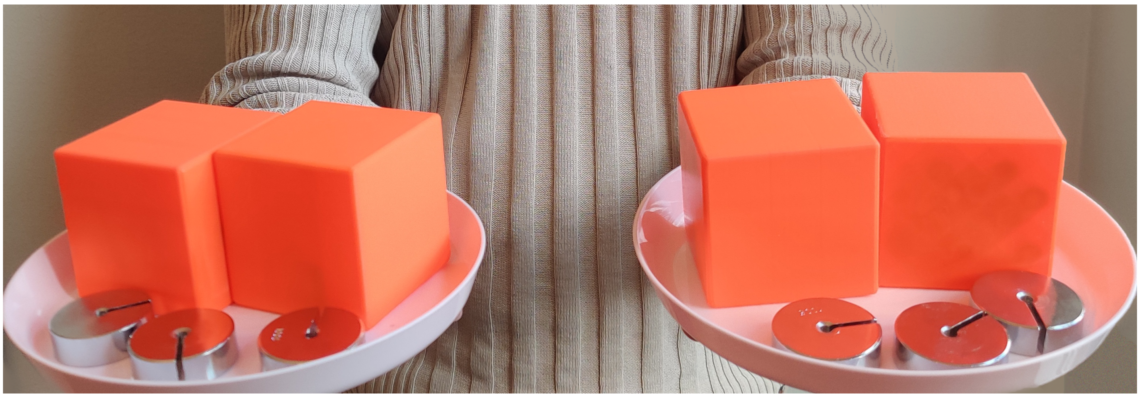
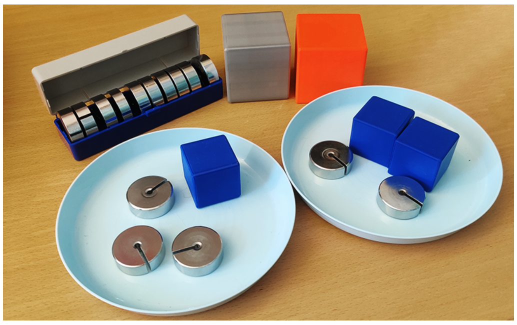


Figure 3.5: A showcase of how the learners held the plates to feel if it was balanced.

The 3D-printed cubes were designed using CAD, paused during the print and filled with hex nuts before completion, making sealed cubes with weight unknown to the learners. The cubes could also have been exchanged for other items such as food containers. Discussions regarding other solutions emerged, weighing pros and cons such as food containers could break in the classroom, the cubes could be filled with other materials like sand or gravel but it could be dangerous to air filters and to the 3D-printers. Other important factors includes that the cubes weights needed to be unknown, as well as having a low price and not being too time consuming to acquire. Considering this, 3D-printing the cubes was seen as the best solution. The set of materials from Block 1 and 2 each group of learners received can be seen in Figure 3.6 and the learners exercises used in the sessions can be found in Appendix A.6 and A.7.

(a) A learner set used for Block 1 consisting of 15 teaspoons and 3 boxes.

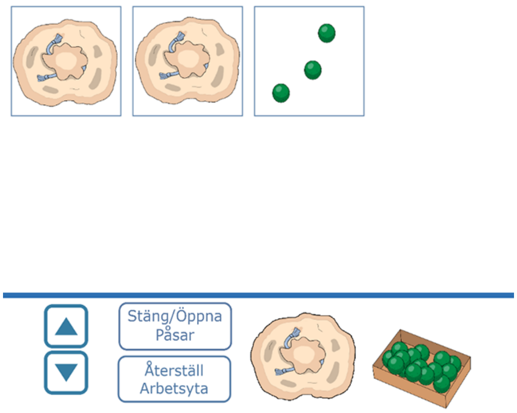


(b) A learner set used for Block 2 consisting of 15 hundred-gram weights, 3 blue hundred-gram cubes, 1 orange two-hundred-gram cube and 1 gray four-hundred-gram cube.

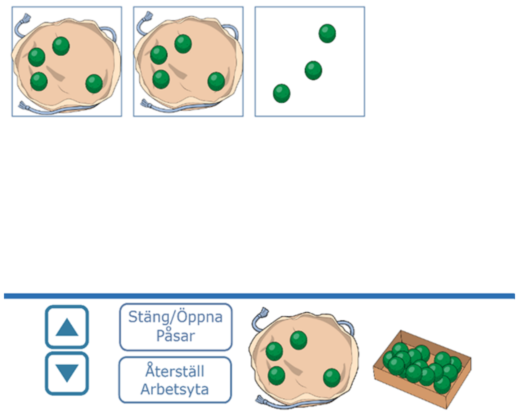
Figure 3.6: Learners' sets of physical assets used in Block 1 and 2.

Virtual Manipulatives

For Block 1, the virtual equivalent of boxes and teaspoons was marbles and bags which could be opened and closed. For this, non-available assets needed to be developed such as images of closed marble bags, buttons to alter the amounts of marbles in bags and buttons to close/open the bags. This also included developing features to check if the manipulatives were used correctly such as squares the bags can be placed into, checking if they are placed correctly as well as being opened or closed. Another menu was also incorporated; being able to swap from dragging objects to numbers instead, transitioning between concrete to abstract. A notable antagonism like the one mentioned in Komatsu & Fujita (2024) between the physical and virtual format was how the bags were filled. In the digital format, all bags were filled through pressing the arrow buttons, meaning the amount of marbles between bags could not differ. They also had a limit of 6 marbles per bag. Screenshots from Desmos, showcasing a workspace can be seen in Figure 3.7.



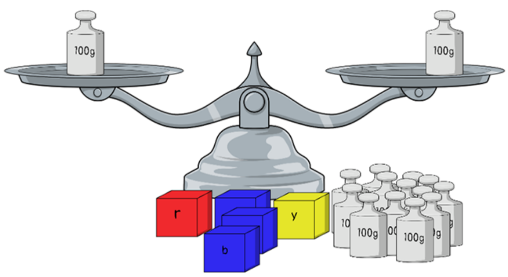
(a) Closed bags, concealing the amount of marbles contained inside, used with permission (Akelius Math, 2025).



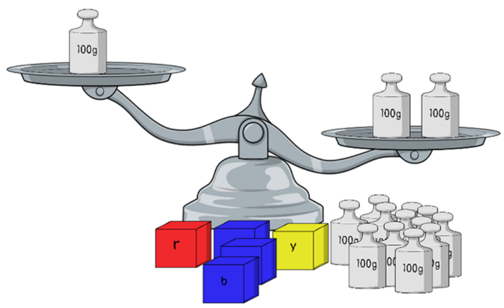
(b) Open bags, revealing the amount of marbles contained inside, used with permission (Akelius Math, 2025).

Figure 3.7: Showcase of virtual manipulatives in Block 1.

For Block 2, a digital version of a balance scale was used to replicate the learners balancing weights in their own hands. The same amount of cubes and weights were used, only having a different design. The scale was an available asset from Akelius Math, however design alterations were necessary such as changing the amount of weights and designing cubes to act as variables (the cubes weighing an unknown amount, labeled as b,y or r standing for blue, yellow or red). A notable difference between the formats is the location where the weights are placed on the balance scales. In the digital, they were called "weighing pans" and in the physical for "plates" to make the instructions clearer for the learners. Screenshots from the Desmos, showcasing the digital scale can be seen in Figure 3.8.



(a) A balanced scale in the virtual environment. Used with permission (Akelius Math, 2025).



(b) An unbalanced scale in the virtual environment. Used with permission (Akelius Math, 2025).

Figure 3.8: Showcase of virtual manipulatives in Block 2.

3.2 Conducting the Study

All learners received a concession form, holding information regarding the study and how the learners personal data would be used. Learners were pseudonymous, and their guardians' concession were required.

3.2.1 Preparing the Learners

Before the four sessions in each class an introductory session was held. The learners met the researchers, were introduced to the study, were given the opportunity to ask questions and learned how to log-in to the Akelius app. All learners got a unique log-in that they used for digital sessions and to write instead of their names on pre- and post-tests as well as on the exercise sheets. All learners received a consent form which required a sign from their guardians regarding if the learner would partake in the study, if the learner can be interviewed and/or be recorded while working with the material. Only the class teacher had the list coupling names with log-ins. This was necessary in case the learners forgot their log-in and for the teacher to know when selecting learners for interviews and recordings.

3.2.2 Pilot Sessions and Participation

Two classes participated earlier than the others and therefore they constituted the pilot study. Corrections and updates were made in the material after each pilot session to fix errors or enhance the quality. Table 3.1 shows the number of learners in each class, a total of 94 in Block 1 and 84 in Block 2, including pilot classes. All learners in the classes were assigned to work with the material as part of their education, but only those who conceded together with their guardians had their data analysed and included in this study. The number of learners presented in the table are corresponding to the number of complete pre- and post-test pairs, thus more learners were present in the classroom.

Nr. of learners	**A**	**B**	**C**	**D**	**Pilot A**	**Pilot D**
Block 1	15	10	12	25	19	13
Block 2	18	10	11	16	16	13

Table 3.1: Number of learners that took both pre- and post-test in Block 1 and Block 2 respectively, divided by class.

3.3 Observation Data

During each session, observations concerning matters like the learners' motivation, how they interacted with the material and insights gained were noted. They were then put together and documented. During sessions, more data were gathered through video recordings and afterwards through interviews as described in sections 3.3.1-3.3.2.

3.3.1 Recording Learners' Interactions on Camera

One pair of learners was usually filmed with sound during the 20-min exercise with physical or virtual manipulatives. The camera was placed high above their heads pointing to their hands and the material from behind, as not to show identifying features. Learners were informed about that the recording was only to be used for research purposes and would not be uploaded publicly. Only those who had conceded with their guardians to being recorded were filmed. The recordings' purpose was to see and hear additional information that was not noted in the classroom. Since the recordings only filmed a single pair each session they also gave another perspective than the observations that were typically remarked from the observing all the pairs during the working session. Pilot classes' recordings were also used to analyse.

3.3.2 Collecting Learners' Experiences through Interviews

Semi-structured interviews were made with some learners from each class, two by two. Interview questions can be found in Appendix A.2. The interviews were recorded with sound only, exclusively targeting learners whose guardians had consented to them being interviewed. The interviews started out with a description of their rights (GDPR) and the purpose for the recorded interview. The teacher in each class chose who would be filmed and interviewed freely from those groups. However, teachers got informed that a variation of learners were preferred. Some teachers were also encouraged to choose learners who are generally more expressive for the interviews and comfortable in the interview setting. For rather inexpressive learners, questions often got asked in relation to their ordinary tuition like about their mathematics book. These learners typically received a follow up question asking them to elaborate their answer more. More leading questions were sometimes asked to help the learners express themselves when they were not communicating much. This was attempts to receive more data regarding the material using manipulatives from the interviews. Many teachers seemed to value the sense of "fairness", spreading out the learners being interviewed and filmed since many were eager to. Some also took into consideration other things that the learners had been chosen to do during other occasions in school.

3.4 Data Analysis

This section presents the process of how the data gathered from pre- and post-tests, observations, interviews and video recordings were analysed.

3.4.1 Assessing and Analysing Pre- and Post-tests

Assessment matrices were made for the pre- and post-tests. They were designed to show not only correct or incorrect answers but also how the learners reasoned and what method they used, e.g. if they calculated the numerical value by using their simplified algebraic expression or doing the much longer calculation by inserting the value of the variable without simplifying first. Some learners only took either pre- or post-test due to absence and their tests were not included in the analysis, considering each block separately. An excerpt from the assessment matrices can be found in appendix A.4.

The scores were summarised and the mean pre- and post-test score for each class was calculated. Scores were normalised from 0 to 100 p in order to be able to make a more fair comparison of the results between the two blocks. The difference in mean score between post- and pre-test was also calculated, which is further on referred to as *Score improvement*. These numbers are shown in Table 4.1.

In order to assess the statistical effect that the combination of physical/virtual manipulatives had and the effects the group (class) had on the Score improvement a two-way ANOVA test was performed. Statistical results from that test are presented in section 4.1.1. To see if the Block affected the Score improvement a three-way ANOVA was also made with factors Block, Class and Combination. Results from that analysis are also found in section 4.1.1

3.4.2 Themes for Analysis

After the working sessions, all observations were compiled in a spreadsheet. Each observation was coded and broad themes were established. Some of the initial themes were removed since they did not relate to the research questions and others containing most of the observations got dissected into smaller themes. Themes were only created from observations and research questions due to time constraints. The themes are presented in section 4.2.

3.4.3 Analysing Interviews

Chalmers University of Technology's AI-tool, Claude Sonnet 3.5 v1 (2025) was used for transcribing and summarising the interviews according to all themes described in section 3.4.2 except number 5 *Comments on Learning Design* since that theme was not a student concern. The built-in tool was used to find relevant answers relating to the themes which was compiled in a spreadsheet. The results were later reviewed by looking through the transcriptions and listening to the interviews. Similar statements from different interviews were combined, resulting in a couple of statements for each theme.

3.4.4 Analysing Recordings

The videos from the working sessions were analysed using a matrix with themes according to section 3.4.2. Watching the videos with the themes in mind, one row for each video was filled with observations and actions relevant to the themes. Time stamps were also noted when applicable. Recurring documentation in each theme were combined to make the data more accessible. The data was later combined with the overarching themes from observations and interviews, presented in section 4.2.

3.5 AI-Usage

AI-tools were used in this study. The usage is described in falling order of influence. As described in section 3.4.3, a Chalmers adaptation of Claude Sonnet 3.5 v1 (Chalmers AI Portal, 2025) was used to transcribe and summarise interviews according to the themes. The AI made it possible to analyse interviews more rapidly, therefore all interviews could be analysed within the time-frame of the study.

The other AI used is Open AI LLM "ChatGPT" (OpenAI, 2025). ChatGPT were mostly used for grammatical and translation enhancements. Questions for grammatical input included for instance: "Should one hundred grammes be written as one-hundred-grams or any other combination?". It also assisted in finding synonyms for commonly used words in the thesis like "more" or "also" as well as short phrases such as "too far away from" that could be phrased more concisely or formal. Furthermore, the AI was sometimes used to translate words from Swedish to English, that were later reviewed by finding synonyms to it.

Outside of language, ChatGPT were used for brainstorming when designing working sessions. Towards the end of the brainstorming, it was used to potentially generate new, non-documented ideas. ChatGPT also assisted during coding in Excel, helping with debugging.

4

Results

4.1 Scores on Pre- and Post-tests

The results from the pre-and post-tests can be found in Table 4.1 below which the statistical analysis of the data is presented.

Combination	**PP**	**PV**	**VP**	**VV**
Block 1				
Class	**A**	**B**	**C**	**D**
Mean pre-test	32.2	40.6	44.0	40.0
Mean post-test	58.9	51.7	50.5	45.3
Score improvement	26.7	11.1	6.5	5.3
Block 2				
Class	**B**	**A**	**D**	**C**
Mean pre-test	54.8	35.4	45.8	38.5
Mean post-test	60.0	41.8	46.1	42.0
Score improvement	5.2	6.3	0.3	3.5

Table 4.1: Mean scores normalised from 0-100 p and Score improvement with respect to Class (A-D). "V" denotes virtual manipulatives and "P" physical.

It is worth noting that Combination PP for Class A in the first block resulted in a higher Score improvement (26,7) than the others. The mean Score improvement was 8.1, meaning all groups improved on average 8.1 percentage units. For following the Score improvement by Class instead of by Combination, see Table A.3 in Appendices.

4.1.1 Statistic Analysis of Test Scores

Results from the two-way ANOVA test for the four classes in the study are shown in Figure 4.2, analysing the effects of the class and the combination (PP, PV, VP or VV) on Score improvement. The distinction of the p-value for the variable Class and for the intertwined factors Class and Combination are worth nothing.

	D_F	Sum of squares	Mean square	F-statistic	p-value
C(Class)	3	3003.7	1001.2	3.900	0.011
C(Combination)	3	1462.9	487.6	1.900	0.134
C(Class):C(Comb.)	9	5163.0	573.7	2.235	0.025
Residual	109	27982.1	256.7	-	-

Table 4.2: Two-way ANOVA test results for the four classes in the study, analysing the effects of Class, Combination and those combined on Score improvement.

The results from a factorial ANOVA with three independent variables (Block, Class and Combination) are found in Table 4.3. Note that Block and Class has a p-value less than 0.05.

	D_F	Sum of squares	Mean sq.	F-statistic	p-value
C(Block)	1	1778.1	1778.1	6.926	0.010
C(Class)	3	3609.0	1203.0	4.686	0.004
C(Combination)	3	1125.5	375.2	1.461	0.229
C(Block):C(Class)	3	530.9	177.0	0.689	0.560
C(Block):C(Comb.)	3	1959.0	653.0	2.544	0.060
C(Class):C(Comb.)	9	2391.9	265.8	1.035	0.417
C(Block):C(Class):C(Comb.)	9	3558.0	395.3	1.540	0.143
Residual	109	27982.1	256.7	-	-

Table 4.3: Three-way ANOVA statistical test results analysing the effects of Block, Class, Combination and all combinations of those on the Score improvement for the four classes in the study.

4.2 Interviews, Observations and Video Recordings

This section presents the results connected to the themes using the observations supported by quotes from interviews as well as quotes and actions from video recordings.

The final seven themes are:

1. Virtual manipulatives
    (a) Opportunities virtual manipulatives create
    (b) Challenges with virtual manipulatives
2. Physical manipulatives
    (a) Opportunities physical manipulatives create
    (b) Challenges with physical manipulatives
3. Relations between cognitive representations
4. Learners' motivation
5. Comments regarding learning design

Themes 1-4 are found in subsections 4.2.1 - 4.2.4, and a discussion about Theme 5, *Comments on Learning Design*, is presented in chapter 5.

4.2.1 Opportunities in Virtual Manipulatives

The digital system used by Akelius Math has hint layers and auto correction, instantly giving learners input when necessary. This can be compared to a teacher grading the learner's work or helping them forward if they do not know what to do next. Another feature is customisability, different tasks requires a different amount of material. Multiple observations highlighted learners working with physical manipulatives having issues separating the relevant material used for each task. In the virtual environment designers can adjust the material for each exercise, solving this issue. Some learners also emphasised that the digital material felt more structured than the physical, making it easier to understand and work with. From the videos, a lot of learners worked well with the material, for instance showing understanding of the concept of equilibrium. A conversation between learners went: "-Should I put this here or there? -No, you should put this here, so it's equal". Another pair of learners also used the manipulatives in a creative way, instead of opening the bags and changing the amount of marbles, they replaced the bags with only marbles.

4.2.2 Challenges with Virtual manipulatives

Challenges with the digital material concerns intuition. Some learners had issues identifying the open area in the Desmos as place to move the material and use that space for solving the tasks. Among those learners, some tried instead to solve the tasks without moving any bags or just solve the tasks without using manipulatives in general. Learners didn't understand that they could scroll when only half of the question was exposed and had issues closing the built in keyboard when it got in the way of the question. In the video recordings from Block 2, some learners also used the balance scale by deliberately putting it notably off balance. Then, they successively added weights to one side until it was rebalanced. They also had issues identifying which of the sides was heavier by while looking at the scale. Some learners avoided working with the material all together, skipping through most of the material or visiting other websites.

Accessing the Akelius app was quite unintuitive, a lot of learners ended up on different lessons or had issues signing into the system. Entering the lesson generally took some time. The system also sometimes needed time to load in between tasks and for the balance scale to update. Additionally, through observations, interviews and learners sometimes requesting it in the classroom, the ability to go backwards in the lessons to re-enter another answer. Many learners wanted to answer correctly in the digital material, but it was not possible due to the feature. Another follow-up problem was that learners forgot what they had written in earlier sub-questions, not being able to see their earlier answers.

Pairs of learners were also confused when they received few coins while only having one question wrong. In one of the interviews a learner highlighted having issues in doing lining-up-multiplication in digital form, showcasing friction doing spontaneous math in the digital format. Learners in general showed resistance to writing notes on paper while using the digital material even if the task requested it.

4.2.3 Opportunities in Physical Manipulatives

Multiple observations suggests learners working well with the physical manipulatives. This can also be seen during videos were learners are balancing the plates according to the instructions. In interviews, the learners stated that they generally enjoyed working more with the physical manipulatives rather than the virtual. They also spoke about freedom, a learner said: "It becomes easier having those things in front of me, because then you could like experiment more" referring to having the physical material on the desk. The learners also said the physical manipulatives felt more concrete, highlighting being able to feel and see differences with the material they were working with, which did not come to light in the digital material. Examples of this includes being able to feel which side of the balance scale that was heavier and seeing a clearer connection between sub-tasks. From the videos, learners are also shown naming cubes as variables from their own initiative, calling the blue "b" and gray "g".

4.2.4 Challenges with Physical Manipulatives

As mentioned in section 4.2.1, learners had issues separating the materials used in a specific task with the remaining material. Relating to this, during an interview a learner described how the physical material felt overwhelming due to having a large amount of things in front of them. In the videos, learners also had issues using the cubes in relation to simplifying the equations, often adding more weights to find balance rather than removing. During interviews learners spoke about issues finding balance using the plates, being unsure if the plates weighed the same in their hands. This took a lot of focus away from the actual task. Some learners also said writing by hand felt exhausting and sometimes stressful.

The learners sometimes misused the manipulatives by playing with them, shaking the spoons and boxes to create loud noises or building towers from cubes. Some learners also figured out the weights of the cubes early in the session, replacing the variables in exercises with a fixed amount and thus not practising equation solving. By analysing the videos it also became apparent that some learners skipped trough the material. This was also noted in the classrooms several times.

In one of the video recordings, it was clear that the learners did not see the advantage of simplifying since they could easily calculate $2+2+3+2+4$ in their heads, making $2 \cdot 3 + 7$ (from first simplifying and then inserting the value) less advantageous.

4.2.5 Relations Between Cognitive Representations

A few learners from classes that had earlier tuition in algebra did not see the point of the manipulatives. Comments such as: "I understand the solution but not how I should do it in the program" came from these learners. In videos, we can see them solving exercises without using manipulatives since it took time and they thought that it did not support their learning. In contrast, some learners had issues understanding the material's relationship to math, some stating: "This is not even math". For instance, in the videos, learners can be seen having issues relating the enactive realisation of a plate in their left hand to the symbolic left-hand-side in an equation. The learners had the equation $2b + 4 = 6$ and were meant to place weights and cubes on plates according to the equation. However, learners divided the left hand side of the equation to the two plates, resulting in plates representing the equation $2b = 4$. Learners also had issues working the other way, interpreting the manipulatives on the plates over to an equation, but this was much less common.

During interviews, learners compared the material to their mathematics book with most stating that they preferred the book since it contained more exercises and different levels giving the learners options. However, the learners found the material more entertaining, suggesting it can be used as repetition supporting the book. Some learners also said they appreciated the progression from physical to digital, enjoying working with similar material but on the different formats. As a learner said: "It's easier to remember things if they are enjoyable".

4.2.6 Learners' Motivation

Learners motivation varied between classes and between sessions. Most often the learners worked well with the material. In interviews the learners noted how being part of a study makes them more engaged saying things like "I feel like it's new, new way to do maths". In contrast, some learners lost motivation because the study was an external project that would not influence their grades. Learners also noted that it was easy to get distracted using the material: “If you have physical things so maybe you forget a little what you are supposed to do.”. Another common topic was working in pairs. Learners in general enjoyed working in pairs, saying it’s good to have someone to talk to if you have a hard time understanding the material. But from the videos, issues were seen such as only one working while the other just followed or them having different thoughts and pulling the computer between them.

5
Discussion

5.1 Discussion of the Results

This chapter includes reasoning about the implications of the results, pointing out sources of error and going over the study's ethics.

5.1.1 Statistic Analysis Discussion

The results from the two-way ANOVA in Table 4.2 showed that both Class and the interaction effects between Class and Combination were statistically significant (p-value less than 0.05) with p-values of 0.011 and of 0.025. For the three-way ANOVA the significance of the variable Class was even higher (since the p-value was lower) with a p-value of 0.009. This shows that the group of learners highly affect the result. Class A had ordinary lessons in algebra at the time of the study, this impact is more thoroughly discussed in section 5.2.1. This is something that affect the Score improvement, which is seen in the mean results in Table 4.1, particularly for Block 1.

The variable Class in this study represents a large number of factors, both random effects and other things, concerning the learners that effects the result of this study. Factors could be, motivation to partake in the study, other mathematical instruction beside the study, or reading skills, background, previous experience with physical manipulatives, digital skills, etc. We can assume all these factors affect the results, but more detailed data collection would be needed in order to answer what factors the variable Class mainly is affected by.

That the interaction effects between Class and Combination were statistically significant (p-value of 0.025) points to the conclusion that a certain Combination (set-up) could yield different results in different groups. This could also point to the same thing mentioned before, that Class A had classes in algebra parallel to the experiential sessions and that Class D were observed to have less motivation to participate.

An interesting result is that Block seem to affect the Score improvement very drastically, looking at the results from the three-way ANOVA in Table 4.3. This might also be explained by motivation, that fewer learners completed the tests the longer the study continued. All classes had a higher Score improvement in Block 1 than in Block 2, as seen in Table 4.1. This could otherwise mean that the blocks had different levels of difficulty or that teaching using manipulatives were more suited for one block more than the other, that it might be more suited to understand terms and not processes in mathematics. It might also be worth doing further analysis of how the Block affects the Combination since a p-value of 0.060 (close to 0.05) was found in the three-way ANOVA. Maybe it is most beneficial to start with PP and then continue with PV or similarily.

The Combination in one block was not significantly affecting the Score improvement neither in the three- nor the two-way ANOVA, which points to that the order between physical and virtual manipulatives (or using only one format) might be equally efficient for learning. This is in line with the result of several other studies (Suh & Moyer-Packenham, 2007, Zacharia & Olympiou, 2011, etc.) but also goes against the conclusion from Terry (1996) and Takahasi (2002) who stated that the combination of both physical and virtual are most beneficial for learning. Since this is a limited study, more research should be conducted in order to see if some correlation exists that was not seen in the data from this intervention.

5.1.2 Discussion on Mathematical Outcomes

Here, thoughts regarding mathematical outcomes from the study are presented. This includes discussions about negative variables, notable observations in the classroom and more.

Differences in Learner Feedback

As described in section 2.2.2, the virtual and physical manipulative configurations has inherent differences. Since the virtual configuration gave immediate feedback if the answer was correct it would probably have been more comparative to assess the learners answers on exercises from physical manipulative working sessions to a higher extent than what was done in this study. For example the exercise sheets could be handed back the following session with written feedback to the learners in the second and fourth working sessions (in the middle of each block) to provide the learners with more feedback on what they understood correctly or not. Even though the researchers answered questions during the sessions not all pairs of learners asked for it. It was also seen in the videos that the learners could end up on a wrong track because of misconceptions and follow that track for a long amount of time without realising it.

The feedback given from the app with the virtual manipulatives was limited, simply stating if the answer was correct or not. Observations in the classrooms revealed many learners asking *why* they where wrong if they did not enclose the correct answer. On some occasions the learners were also right, their answers correct, but the system did not yet allow multiple correct answers in that format (e.g. equations) or there had been a mistake in producing the material. This kind of challenges were less frequent in physical configurations since the learners seemed to be more prone to ask why when discussing with the teacher/researcher about the physical exercises and not just being falsely convinced by the digital system that they were wrong. The digital system will also be developed further and a feedback layer where an explanation to why the learner was wrong pops up if they do not answer correctly is currently existing at the time of writing but was not at the time of designing the working sessions.

Scarce Enactment of Negative Variables

Since the virtual manipulatives were designed to correspond with the physical versions, and the physical manipulatives were designed to relate to everyday objects, no thorough representation of negative variables/entities were present. All the tasks in both the blocks always had a positive sum of variables and constants in the final answer. Negative terms were simply represented by removing objects, e.g. $3x - x$ was enacted by putting three closed boxes on the table and then removing one, leaving two closed boxed representing $2x$. This worked decently for pre-algebra purposes but for manipulatives covering more advanced algebra something else should be considered. Suh and Moyer-Packenham (2007) used “$-x$” balloons (filled with Helium) on their virtual balance scale that lifted equally much as an x would weigh, and likewise for constants. Another option is using different coloured algebra tiles or similar. One of the colours then symbolise positive entities and the other negative (e.g. red and blue) - providing learners an easy way to find “zero-pairs” (x combined with $-x$ equals 0, as shown in Figure 5.1) and reason about answers with negative sums of variables or constants (e.g. only red left).

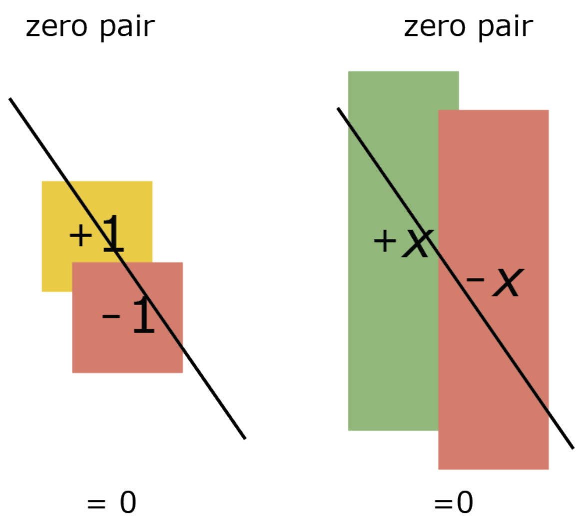


Figure 5.1: An illustration of two zero-pairs with algebra tiles, one for constants and one for variables, used with permission (Akelius Math, 2025).

Difficulties in Translating Knowledge to Different Contexts

Learners understanding a concept in one context does not automatically imply that they can translate the knowledge to a similar but yet different context. During one of the sessions in Block 1, one class had worked successfully with the spoons and boxes representing variables and constants. Most learners in the group had understood the tasks and answered correctly, as well as enacted the mathematical expressions physically. Right after our session the teacher continued the lesson by showing a picture with a closed pastille box and pastilles on the side. To our surprise, at least six learners were not able to write an expression matching that picture despite it being such a similar example to the spoons and boxes. This points to a need of guiding the learners to see similarities in different examples as well as to reflect on the core characteristics of variables and constants. It also illustrates a potential weakness in the physical manipulatives of too little variation. This can be related to variation theory, always using the same manipulatives could make the learners see the manipulative as an invariant, making it harder to differentiate the sought after critical feature. (Ling Lo, 2012). Maybe it would be more helpful for the learners if they were able to work with several physical manipulatives like spoons and boxes, pencils and pencil cases as well as matches and matchboxes.

Experiencing that Simplifying Makes Things Easier

It was suspected during the working sessions design that learners would not experience the advantage of simplifying first hand because the amount of spoons were too few. Calculations with a total of 15 spoons can most often easily be done in your head. This was observed in the classroom and seen very evidently in one of the videos described in section 4.2.4. In order to make the learners really feel the need for simplifying, more spoons could have been used, adding up to higher numbers that are harder to sum by solely using mental calculations.

Other Notes on Tasks

Learners have different preconceptions of the formats. For instance while using a computer, learners expect to receive instant feedback, design error such as coding on tasks that $3 + 6 = 14$ cannot be present and sessions should load quickly. The same goes for the physical format, having an urge to stack cubes on top of another and build large towers when the material was handed out. Counteracting these preconceptions can be challenging during a short study, but if manipulatives are used in the classroom during a longer period they might not be as prevalent.

5.1.3 Comments on Learning Design

Understanding the Target Group

A common reoccurring topic from the observations and recordings were understanding your target group, identifying what information the learners know and how to adapt the teaching to it. For instance, material had to be changed to take the learners reading comprehension into more consideration. Multiple times, learners had issues following instruction due to the language being too complex and instructions too long. Towards the later sessions, instructions got simplified which improved productivity in the classroom.

Examples of not knowing the group can be seen in exercise 1) in session 1 and session 3, see Appendix A.6 and A.7. The point of the exercises was engaging the learners with the material as to capture their interest. However, learners found the task too enjoyable, repeating it multiple times and it triggered some to play with the material instead of continuing with the tasks. Exercise 6 a) in session 4, is an advanced reading-question that learners struggled with, see appendix A.6.4 and A.7.4. Why learners had issues relating manipulatives to the question could be because there was variables on both sides of the equation, which they have not practised extensively during the sessions. There is also a high chance instructions could have been simplified more. An underlying issue connected to this could be the relation between mathematics and reading comprehension. If too clear instructions are present, for instance presenting all information in a bullet list, the learners are not given the opportunity to apply their mathematical knowledge to solve common occurring issues in real life. By making instructions clearer, learners understands the objective, but are not practising towards tasks that require a higher level of reading comprehension. Another explanation to this question could also be the term: "hands on, brains off", meaning learners have just been using the manipulatives and not been relating it to mathematics. There's also a possibility that the task was not well suited for the learners. If a better understanding of the target audience was attained during lesson designing phase these exercises would be designed differently, most likely yielding a better result in the classrooms.

Feedback to Learners

When working with digital learning, having a self correcting feature is essential since the learners are used to that. If the virtual environment should be more open and exploratory explain this very clearly and emphasise it in order to readjust the learners' expectations. Throughout the study the design shifted to more and more self correcting answers, dividing bigger tasks in so small bits where every step could be evaluated in the digital system. Sometimes the task could also be translated - instead of asking the learners to place an equation on the virtual balance scale (that could not be evaluated in the current system) they would be asked to check if an equilibrium was achieved when placing that same equation on the balance scale (answering with a yes or no that was self correcting).

When questions arise during sessions, listen to the learners question but try to interpret what the learner understands and want to understand. It's easy to just answer their question, but if the process of using manipulatives are faulty the correlation to mathematics becomes ineffective, as mentioned in section 2.2.1. The learners receiving relevant feedback is vital.

Performing Something Better than Nothing

Some learners saw the material as something you would finish, and not something to learn from. This made them take shortcuts like skipping material or to avoid using the manipulatives. Integrating a numerical answer for every task, makes the material harder to skip and directs the learners to consider an answer. After each session it is also beneficial to collect the learners' questions and answers, since it encourages learners to produce something during the session that someone else will take part of.

Avoid Hindrances

A thing that greatly helps avoiding hindrances in the sessions is preforming the session yourself from a learners point of view before handing it out in class. Small errors in instructions can lead to huge misconceptions on how the manipulatives should be used in the classroom. By preforming the lesson yourself, especially if it is the first times teaching using manipulatives the material becomes more concrete and easier to notice potential errors. Other methods includes reading the first question together in class after all instructions have been distributed. Having a controlled start at the beginning of the session with everyone's attention made focusing on the tasks at hand easier than playing with the manipulatives, minimising friction to start learning. A contrasting example to this could be the engaging exercise mentioned in previous paragraphs, were having a more playful start led to learners not focusing on tasks.

While working with virtual manipulatives it is also beneficial to have learners use the program in full screen. This makes it easier to see the material they are working with and minimises distractions from pressing other tabs that they might find more interesting. During a interview, learners mentioned: “maybe not always use marbles and bags, use some other things as well”. They followed it up by suggesting using houses with humans and goldmines containing different amounts of gold instead of marbles and bags to make the material more interesting. The suggestion is connected to gamification, a way of designing digital content that can increase the learners’ inner motivation (Seaborn & Fels, 2015). Combining that with similar ways of making the material relevant for their interests and more varied are ways their motivation might be increased and sustained over time.

5.2 Discussion of the Data Collection

In this section sources of error as well as ethical choices are discussed. This includes discussing choices made and the consequences they had in the study.

5.2.1 Sources of Error

Even though measures were taken to make the results reliable, some factors that might have made the results less accurate are discussed in this section.

Differences between Classes

Learners could not be isolated from being taught algebra from other sources during the study. Class A had lessons in algebra during the period of the study, which probably is a huge factor to their result from post-tests in Block 1 showing significantly larger improvement than the other classes. Class A were also the only class in year 7 partaking in the study, meaning more time had passed than for other classes that studied algebra more recently. According to Bjork & Allen (1970) having a longer period before repeating algebra, makes it harder to retrieve the knowledge. This could explain why learners received low points on the pre-test, since it took longer for them to receive their past knowledge than other classes, but higher on the post test, since they understood it more clearly in combination with their recent ordinary classes. However, the significant Score improvement for Class A might also point to that combining manipulatives and traditional teaching is very efficient. Class A was included in the study to fulfil the the four different combinations, two pilot classes were necessary to test both formats before the other classes.

Classes also differed in motivation to participate in the study. It was particularly visible in the observations that Class D had less motivation, especially during the pre- and post-tests. During the tests, some learners in all classes looked at other’s tests and were influenced by their answers. Learners were informed that the test should be done individually and that it wouldn’t affect their grades, to prevent dishonesty. But if a reminder was insufficient, screens were placed in-between learners to separate them from each other’s view.

Limited Thematic Analysis

The thematic analysis could have been more substantial. Themes were established from only looking at observations and research questions, not from interviews and recordings. If all data collected had been reviewed, coded and used to form the themes, other themes could have arisen. Alternatively, more qualitative interpretations of the data could have been found. The data collected from interviews could also have been improved. During interviews, learners sometimes had issues expressing their thoughts. To avoid receiving no answers from some learners, more leading questions were asked. These learners may have been influenced to say something they did not truly stand for during the interview, making the data less reliable (Wängnerud et al., 2012). Learners also performed much better while being recorded (but seemed more hesitant to ask for help when they did not understand). In at least four of the recorded sessions the learners had very high motivation placing them among the highest performing learners in the classroom. The recordings could then be skewed, showing a more positive view, once again making the data less reliable.

Circumscribed Sample

Even though six classes participated in the study the sample is too small to draw heavy conclusions from. All groups for a certain combination had less than 30 learners if you consider each block separately. This does not ensure the statistical analysis; "... sample size... based on the recommendations by Mills and Gay (2018), which advocate for a minimum of 30 participants per group to ensure the validity of statistical analyses. This approach is consistent with recent STEM education research, which often employs similar sample sizes (Yaduvanshi and Singh, 2019, p. 985)." If you consider Block 1 and 2 together still only the Combination VV has mote than 30 participants though the other Combinations are much closer to 30. Class and Block are never above 30 in each group considering the Combination.

Another aspect to consider is that only Swedish charter schools took part in the study, meaning schools in Sweden that receive government funding but runs independently. A quarter of schools in Sweden are charter schools (Friskolornas riksförbund, n.d.). Possibly they do not represent the country as a whole, which could limit the representativeness of the results.

Hindrances in Learning

Errors in the material sometimes emerged during the working sessions. This included invalid answers in the virtual material. The written instructions for physical sessions were not always matching the colour of cubes and some other errors were probably unnoticed. These blunders may have hindered learners from improving in mathematics. Other hindrances includes different wordings that were used in Block 2. In the digital format, the location where the weights are placed on the balance scales were called "weighing pans" and in the physical "plates". This was meant to make the instructions clearer for the learners. However it might make it harder for learners to transition between physical and virtual formats, affecting learning.

5.2.2 Ethics and Honesty

The sessions were designed with both formats in mind, making sure none of the variants had a clear advantage over the other. This includes having the same amount of manipulatives in both formats and not adapting the amount in the virtual format in-between tasks. Another example is mimicking the physical formats ability to see progression in the digital format, e.g in section 2, task 2 a), see appendix A.6.2 and A.7.2. In task 2 a) earners used manipulatives to symbolise $x + x + 3 + x + 4$ and in task 2 b) they wrote the numerical expression when $x = 2$, giving the answer $2 + 2 + 3 + 2 + 4$. This was mimicked in the digital format by learners changing work areas from manipulatives to numbers, with the images being located above of the numbers, as in task 2 a) and 2 b) in the physical format.

All classes were also treated equally, having the same amount of time for exercises, as well as tests. All tests were distributed with the front-page down and once a timer was started, learners were allowed to flip the papers over. If a teacher provided the study with more time for a session, the learners either continued with their ordinary mathematical education or played mathematical games provided by us. The learners also got treated the same in the classroom, helping out as soon as possible and actively supporting learners in their progress.

Learners were also often informed that none of the things they did in this project would effect their final grade in school, this was due to avoid learners feeling pressured to preform well during the sessions. Learners recorded and interviewed also got information regarding how the data would be stored and got asked if they were comfortable taking part. To avoid seizing time from the teachers ordinary mathematical lessons, both authors stayed the remaining time of the lesson and acted like supporting teachers, giving something more than only the intervention to the participating schools.

6
Conclusion and Further Research

6.1 Conclusion

Results from the two-way and three-way ANOVA showed no significance for the variable Combination (representing the combination of physical (P) and virtual (V) manipulatives; PP, PV, VV, VP) affecting the results, with p-values of 0.134 and 0.229. Therefore, the set-up of only virtual, only physical or a combination does not impact mathematical learning in a statistically significant way in this study. However, uneffected by format, the mean Score improvement of all classes was 8.1 percentage units, indicating that learning occurred when using manipulatives (this is not compared with having an "ordinary" lesson without manipulatives).
The formats do present different opportunities and challenges. Some notable for the digital format is being easily customisable and great for simple feedback but has risks of being non-intuitive for learners and having less room to explore. Opportunities for the physical format include being more enjoyable to work with, encouraging exploration and being concrete. But, it has some notable challenges being distractive and easily misused. These factors along with other factors outside of learning, mentioned by Zacharia & Olympiou (2011), such as availability, portability, safety, and cost-efficiency are therefore important when choosing the type of format used in the learning session. All of these factors are relevant when designing interactive learning that spurs motivation and encourage usage of manipulatives that can lead learners to understand abstract mathematical concepts in a deeper way.

6.2 Further Research

Since the sessions were 30 minutes long and 10 minutes were dedicated to either pre- or post-tests, learners only had 40 minutes in each block to work with the material. Throughout the study, making these 40 minutes as effective as possible was a high priority, however extra time might be necessary for the learners to show a significant change in their mathematical knowledge. A solution for this could be combining Block 1 and 2, removing one pre-test and one post-test in the process, giving an extra 20 minutes for learning. Another solution could be extending the study's volume, for instance to eight weeks. Using all classes of the schools ordinary algebra teaching could be another option. This would also solve the issue of learners learning algebra outside of the study while it was conducted. By either changing the algebra tuition to use manipulatives throughout or conducting the study before learners encounter algebra in year 6, like during the first term or in year 5, the issue would be solved. But, examining if Class A:s huge Score improvement is due to having normal teaching in combination with manipulatives could be worth exploring further.

Additional adjustments includes having more learners and classes partake in the study to have more data points on different combinations. Helping keep learners motivated for the entire study, avoiding notable shifts in quality between blocks would also be helpful. Optimising each type of manipulative could be achieved by not limiting the manipulative formats potential to be equal, such as including more manipulatives in the digital version or altering features like presented by Komatsu & Fujita (2024) in subsection 2.2.2. Using the context of this thesis, this could be implemented by utilising more of the opportunities created and addressing the challenges. In the digital, the app could be designed differently. Examples of this is pressing a single button and the correct tasks appeared and the bugs could be sorted out. This would give the learners a smoother experience. A feature for notes can be implemented if learners need to do some spontaneous calculations or write mental notes, and so on. The same could also be said about the physical material, using other less noisy options like sand or padded boxes to restore the learning environment. Taking advantage of real scales or having a larger difference in weight to minimise the precision needed to feel balance. Altering the cubes features, giving the possibility to change the weight of the cubes to avoid learners reusing the fixed amount and so on. Further research could also emphasise the difference between algebra using manipulatives compared to standard education in a Swedish context, since similar research was not found in the pre-study.

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A

Appendix 1

A.1 Specification of Requirements for Physical Manipulatives

- Could be used to transfer knowledge, not only for moving around or playing with.
- Wish: could be used for more than one mathematical topic (relevant for Akelius Math).
- Wish: not too familiar, quite abstract. It is easier to relate to mathematics and not get stuck in the details of the object.
- Affordable.
- Not too heavy or bulky, easy to store and transport.

A.2 Interview Questions

- What do you think about math in general?
- How did you experienced the activities during the last two lessons?
  - What has been good/bad? Why?
- Do you feel that you’ve learned something when using the material?
  - What have you learned?
  - What helped you learn that?
- How fun was it to work with the material?
  - What made it fun or not so fun?
  - What could have made it even more fun?
- Do you have any thoughts on how the app works?
  - How do the boxes and spoons work?
- Did you understand what you were supposed to do in the tasks?
  - Was there any task that was hard to understand?
- What do you think about doing the tests?
- How did you like working in pairs?

Extra interview questions – when both physical and digital versions have been used:

- Which version (virtual [V] / physical [P]) did you find better?
  - Which one did you learn the most from?

  - Which one was more fun, and why?

  For those who had two different versions in the same block:

- What did you think about the setup? (V/P)
  - How did you experience having the first lesson digital and the second one with physical materials?
  - Did the physical materials become easier to use after having worked with the digital version?
- What did you think about the setup? (P/V)
  - How did you experience having the first lesson with physical materials and the second one digital?
  - Did the digital materials become easier to use after having worked with the physical version?
- Do you have any thoughts on how the app works?

A.3 Mean and Difference in Results by Class

Class [Combination]	**A [PP]**	**B [PV]**	**C [VP]**	**D [VV]**
Block 1				
Mean pre-test	32.2	40.6	44.0	40.0
Mean post-test	58.9	51.7	50.5	45.3
Score improvement	26.7	11.1	6.5	5.3
Class [Combination]	**A [PV]**	**B [PP]**	**C [VV]**	**D [VP]**
Block 2				
Mean pre-test	35.4	54.8	38.5	45.8
Mean post-test	41.8	60.0	42.0	46.1
Score improvement	6.3	5.2	3.5	0.3

Table A.1: Mean scores and Score improvement for each class, scores normalised from 0-100 p.
"V" means virtual manipulatives and "P" physical.

A.4 Assessment Matrices

Filled assessment matrices for some learners (with removed log-ins and classes) for Block 1 and 2.

A.4.1 Block 1

		Uppgift	1 a) och b) Pennfack		2 a) Vad är en variabel?		
Klass	Löpnummer	Inloggning	[0-2 p]	[0-2 p]	"tal som kan variera" [1 p]	"okänt tal" [1 p]	"x/y/z" [1p]
[redacted]	1	[redacted]	1	2		1	1
[redacted]	2	[redacted]	2	2	-		
[redacted]	3	[redacted]	2	2		1	1
[redacted]	4	[redacted]	2	2	-		
[redacted]	5	[redacted]	2	2	0		
[redacted]	6	[redacted]	2	2		1	1
[redacted]	7	[redacted]	2	2			1
[redacted]	8	[redacted]	2	2	0		
[redacted]	9	[redacted]	2	2	-		
[redacted]	10	[redacted]	2	2	-		
[redacted]	11	[redacted]	2	2	-		
[redacted]	12	[redacted]	2	-	0		
[redacted]	13	[redacted]	2	2	-		
[redacted]	14	[redacted]	2	2			1
[redacted]	15	[redacted]	2	2			1
[redacted]	16	[redacted]	2	2			1
[redacted]	17	[redacted]	2	2		0	
[redacted]	18	[redacted]	2	2			1

2 b) Vad kan man använda en variabel till?	3a) Förenkla uttrycket 36x – 5 – 31x + 12	
"skriva ett okänt tal", "räkna ut något", "skriva uttryck", "lösa problem"? [1 p]	Förenklar antingen konstanttermerna eller variabeltermerna korrekt [1p]	5x+7, korrekt förenkling [1 p]
1	1	1
-	-	
1	1	1
-	0	
1	1	
1	1	1
1	1	
-	0	
-	0	
0	1	
-	-	
0	-	
-	1	
1	1	
1	1	
1	0	
0	-	
1	0	

3b) Vad är värdet av uttrycket 36x – 5 – 31x + 12 om x=3?		4 a) Bullar till utställningen
22, brutforce [1p]	22, använder förenklingen [2p]	Skriver någon mer persons bidrag än Robins korrekt, "Alice: 2x / Samira: 3x / Viktor: -2" [1p]
-		-
-		0
1		1
0		1
0		1
0		1
-		1
-		0
0		1
0		1
-		1
-		1
0		1
1		1
0		1
0		1
-		-
0		1

		b)	
x+2x+3x-2, korrekt uttryck [1 p]	6x-2, korrekt förenklat uttryck [1 p]	28, korrekt svar [1p]	28, använder förenklade uttrycket från a) [2p]
		-	
		-	
0		0	
		-	
		1	
1	1	-	
1		-	
		-	
1		1	
1	1	1	
1	1	-	
1	1	-	
		1	
1		1	
		0	
		-	
		-	
		0	

Beskriv hur du löste 4)		SUMMA
"vet ej"/"räknade bara"	"ställde upp uttryck, förenklade och satte sedan in värdet på x"	
-		8
-		4
-		11
-		5
	X	8
	x	13
-		10
-		4
-		8
-		10
-		8
x		6
-		7
	x	12
x		8
-		8
-		4
	x	7

A.4.2 Block 2

Upplägg	Löpnr	Inloggningsnummer	Uppgift Inloggning	1) Handdukar 3, [1 p]	Bara uttryck [-1 p]
[redacted]	1	[redacted]	[redacted]	1	
[redacted]	2	[redacted]	[redacted]	1	
[redacted]	3	[redacted]	[redacted]	1	
[redacted]	4	[redacted]	[redacted]	1	
[redacted]	5	[redacted]	[redacted]	1	
[redacted]	6	[redacted]	[redacted]	1	
[redacted]	7	[redacted]	[redacted]	1	
[redacted]	8	[redacted]	[redacted]	1	
[redacted]	9	[redacted]	[redacted]	1	
[redacted]	10	[redacted]	[redacted]	1	-
[redacted]	11	[redacted]	[redacted]	1	
[redacted]	12	[redacted]	[redacted]	1	
[redacted]	13	[redacted]	[redacted]	0	
[redacted]	14	[redacted]	[redacted]	0	
[redacted]	15	[redacted]	[redacted]	1	
[redacted]	16	[redacted]	[redacted]	1	
[redacted]	17	[redacted]	[redacted]	1	
[redacted]	18	[redacted]	[redacted]	0	

2) Ringa in ekvationerna Både uttryck och ekv. [0 p]	Inte alla ekvationer, inga uttryck [1 p]
0	
0	
0	
-	-
	1
	1
	1
	1
	1

	3) Vilken har lösningen x=11?	
Alla ekvationer, inga uttryck [2 p]	x+1=13-1, korrekt [1 p]	Ringar in uttrycket [-1 p]
2	1	
2	1	
2	1	
2	1	
2	1	
	1	
	1	
	1	
2	1	
-	1	
2	0	-1
		-1
	1	
	1	
	1	
2	0	
2	1	
	1	

4 a) Lös ekvationen x + 2 = 10

Skriver ut mellansteg/redovisar, t.ex. x+2-2=10-2, visar att de kan lösa algebraiskt [1 p]	x=8, korrekt [1 p]
1	0
1	1
1	1
1	1
0	1
1	1
1	1
0	0
-	-
0	0
1	1
-	-
1	1
-	-
1	1
	1
1	1
1	1

4 b) Lös ekvationen 2x – 5 = x + 10

Skriver ut mellansteg/redovisar [1p]	x=15, korrekt [1 p]
-	-
-	-
0	0
0	1
0	1
0	0
0	1
-	-
-	-
1	0
-	-
-	-
-	-
-	-
-	-
-	-
1	0
1	0

4) Balansvåg [max 3 p totalt]

"lika på båda sidor" [3 p] /	"jämvikt" [3 p] /	"VL=HL" [3 p]
-	-	-
-	-	-
		3
		3
3		
1.5	1.5	
3		
3		
-	-	-
3		
0	0	0
-	-	-
0	0	0
3		
3		
3		
3		
3		

5) a) Jovan och chokladbollarna		5 b) Ekvation för chokladbollarna
4+x=10, korrekt [2p]	Valt något felaktigt [-1p]	n=5*16-3, [3 p]
-	-1	0
	-1	-
-	-	-
2		3
2		0
2		0
2		0
2		-
	-1	-
-	-	-
	-1	0
-	-	-
	-1	-
	-1	0
	-1	0
2		3
2		3
	-1	-

5 c) Totalt antal chokladbollar efter bakning	
77, [2 p]	Använder sin ekvation från b), "n=77" [1 p]
-	-
-	-
-	-
2	1
2	0
2	0
2	0
2	
2	0
-	-
0	0
-	-
0	-
0	0
0	0
-	-
2	0
-	-

Beskriv hur du löste 5) "vet ej"/"räknade bara"	**summa poäng [max 19]**
	4
	5
	9
	18
	13
	11
	12
	9
	5
	6
	3
	1
	3
	4
	7
	12
Jag försökte ta information till variabler och siffror.	17
	7

A.5 Pre- and Post-tests

All pre- and post-tests used in the study, separated into different blocks.

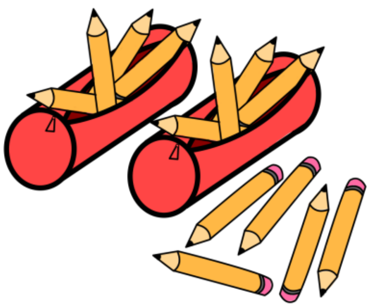
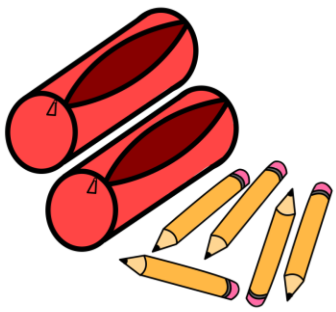
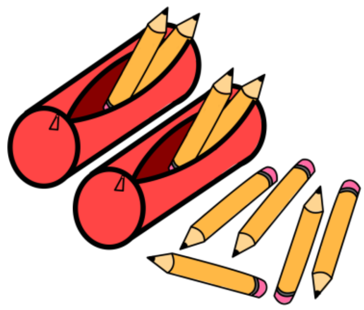
A.5.1 Block 1

Pre- and post-test for block 1.

Förtest 1 Inloggning: ____________

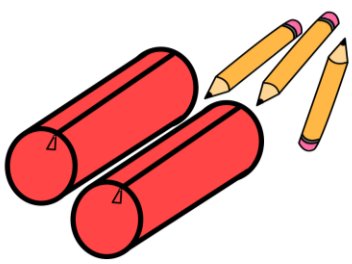
1a) Nedan finns bilder på pennor och pennfack. Dra streck mellan korrekt bild och uttryck.

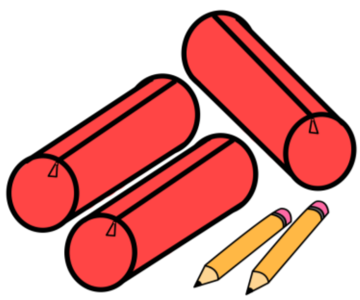
$2\cdot0+5$ $2\cdot4+5$ $2\cdot2+5$

1b) Nedan finns bilder på fler pennor och pennfack. Dra ett streck mellan korrekt bild och uttryck då antalet pennor i pennfacket kallas för *x*.

$2\cdot x+3$ $3\cdot x+2$ $x+5$





2a) Vad är en variabel? Förklara med 1 mening.

2b) Vad kan man använda en variabel till? Ge ett exempel. Svara med en mening.

3a) Förenkla uttrycket $36x - 5 - 31x + 12$

3b) Vad är värdet av uttrycket $36x - 5 - 31x + 12$ om $x=3$?

4) Alice, Samira, Viktor och Robin bakar bullar till en utställning

- Robin bakar x bullar
- Alice bakar dubbelt så många som Robin
- Samira bakar 3 gånger fler bullar än Robin
- Viktor äter upp 2 bullar

4a) Skriv ett uttryck för hur många bullar de tar med till utställningen.

4b) Hur många bullar tar de med till utställningen om Robin har bakat 5 bullar? Skriv ditt svar:

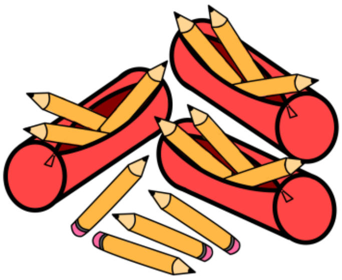
4c) Beskriv hur du tänkte när du löste uppgift 4a) och 4b), svara med max 3 meningar.

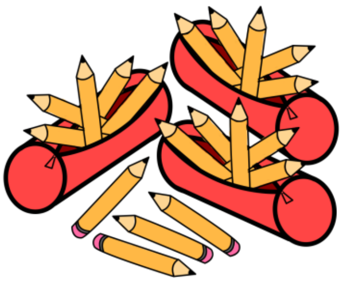
Bra jobbat! Du är färdig med testet!

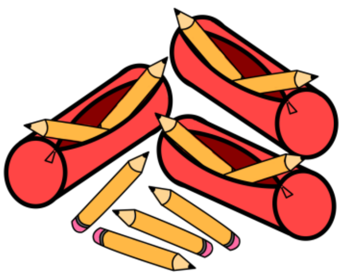
Eftertest 1 Inloggning: ___________

1a) Nedan finns bilder på pennor och pennfack. Dra streck mellan korrekt bild och uttryck.

$3 \cdot 5+4$ $3 \cdot 2+4$ $3 \cdot 3+4$

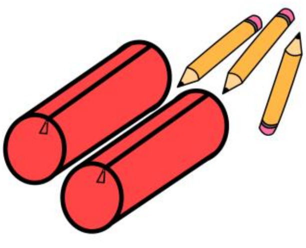


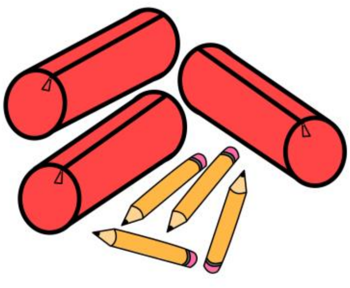


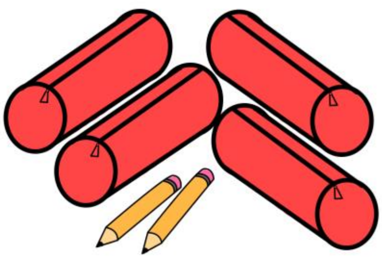


1b) Nedan finns bilder på fler pennor och pennfack. Dra ett streck mellan korrekt bild och uttryck då antalet pennor i pennfacket kallas för *x*.

$3 \cdot x+4$ $2 \cdot x+3$ $4 \cdot x+2$







2a) Vad är en variabel? Förklara med en mening.

2b) Vad kan man använda en variabel till? Ge ett exempel. Svara med en mening.

3a) Förenkla uttrycket 46x + 20 – 42x – 14.

3b) Vad är värdet av uttrycket 46x + 20 – 42x - 14 om x=5?

4) Bjarne, Ali, Julia och Elina samlar pantburkar till sin förening

- Ali samlar x pantburkar
- Julia samlar 4 gånger fler pantburkar än Ali
- Bjarne samlar dubbelt så många pantburkar som Ali
- Elina samlar 17 pantburkar

4a) Skriv ett uttryck för hur många pantburkar de har samlat till föreningen.

4b) Hur många pantburkar har de totalt samlat till föreningen om Ali har samlat 10 burkar? Skriv ditt svar:

4c) Beskriv hur du tänkte när du löste uppgift 4a) och 4b), svara med max 3 meningar.

Bra jobbat! Du är färdig med testet!

A.5.2 Block 2

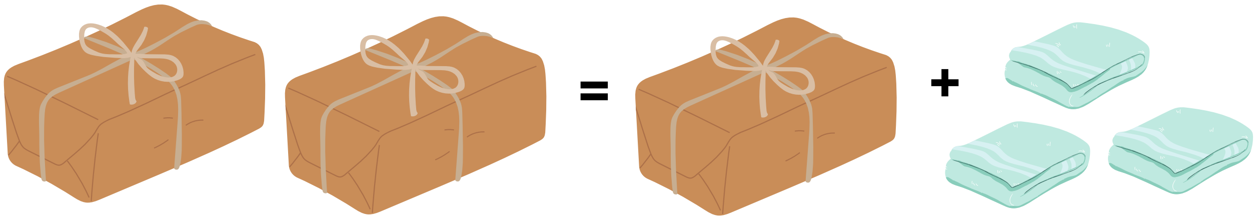
Pre- and post-test for block 2.

Inloggning: ________________

Svara på så många uppgifter du kan.

Gör ditt bästa!

1 Det är lika många handdukar i varje paket. Hur många handdukar finns i ett paket?



Svar: ______________________________________

2 Ringa in ekvationerna:

$3x - x = x + 1$ $x + 8$ $8 - x = 7$

$15x + 4 = 34$ $15x + 4$

3 En av ekvationerna har lösningen x = 11. Vilken?
Ringa in ditt svar.

$22 - 11 = x + 3$ $22 - x$ $x + 1 = 13 - 1$

4 a) Lös ekvationen x + 2 = 10. *Redovisa din lösning.*

b) Lös ekvationen 2x − 5 = x + 10. *Redovisa din lösning.*

4 En ekvation brukar ibland jämföras med en balansvåg.
Varför det, tror du? Skriv en till tre meningar.

5 Jovan vill baka fem satser chokladbollar. För att göra en sats med 16 chokladbollar behövs 2 dl socker. Han har bara 4 dl socker men annars har han alla ingredienser hemma.

a) Vilken av ekvationerna beskriver hur mycket socker han måste köpa för att baka satserna? x är antalet dl socker han behöver köpa.
Ringa in ditt svar.

$$2 + 4 + x = 16$$

$$\frac{16}{2} = x$$

$$16 \cdot 5 = x$$

$$4 + x = 10$$

$$16 = 2x$$

b) Mitt i bakningen av de fem satserna så blir Jovan hungrig och äter upp tre chokladbollar. Skriv en ekvation för hur många chokladbollar Jovan har då *n* är antalet chokladbollar efter bakningen.

c) Hur många chokladbollar har Jovan efter bakningen?

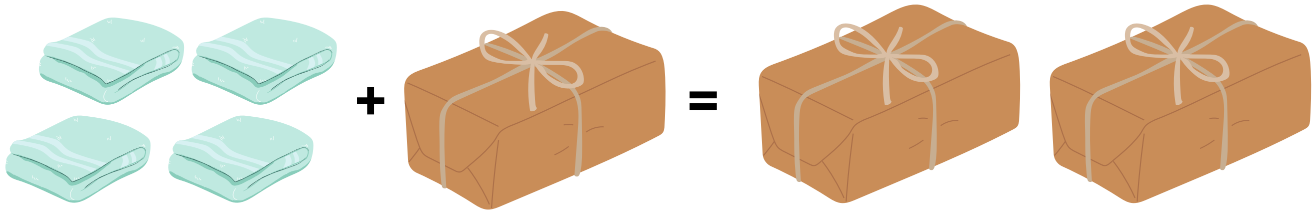
d) Hur tänkte du när du löste uppgift b) och c)?

Inloggning: ______________

Svara på så många uppgifter du kan.

Gör ditt bästa!

1 Det är lika många handdukar i varje paket. Hur många handdukar finns i ett paket?



Svar: ___________________________________

2 Ringa in ekvationerna:

$4x + x = 1 - x$ $10x + 5 = 25$ $10x + 5$

$3x + 2$ $13 + x = 14$

3 En av ekvationerna har lösningen x = 8. Vilken?
Ringa in ditt svar.

$16 - x$ $x - 3 = 11 - 6$ $13 - 5 = x + 4$

4 a) Lös ekvationen x − 3 = 11. *Redovisa din lösning.*

b) Lös ekvationen 2x + 5 = 15 + x. *Redovisa din lösning.*

4 En ekvation brukar ibland jämföras med en balansvåg. Varför det, tror du? Skriv en till tre meningar.

5 Johanna ska plantera 14 fruktträd hemma. För att plantera 2 träd behövs 4 säckar planteringsjord. Hon har dock bara 3 säckar planteringsjord hemma.

a) Vilken av ekvationerna beskriver hur många säckar planteringsjord hon måste köpa för att plantera alla träden? x är antalet säckar jord hon behöver köpa.
Ringa in ditt svar.

$$3 + x = 14 \cdot 2$$

$$\frac{14}{2} = x$$

$$2 + 3 + x = 14$$

$$14 = 3x$$

$$14 \cdot 2 = x$$

b) Varje träd ger 5 frukter första året. Tyvärr blir 7 frukter skadade av insekter och kan inte ätas. n är antalet frukter Johanna kan äta. Skriv en ekvation som beskriver antalet frukter hon kan äta.

c) Hur många av frukterna kan Johanna äta?

d) Hur tänkte du när du löste uppgift b) och c)?

A.6 Exercises Used in the Physical Sessions

All physical sessions used in the study.

A.6.1 Physical Session 1

Pass 1

1a) En av er gömmer hur många skedar du vill i en av askarna utan att den andra vet. Den andra gissar hur många skedar som ligger i asken. Hade du rätt?

1b) Byt roller och gör det igen.

Antalet skedar i asken är en *variabel*.

En variabel kan ha **olika** värden.

2a) En av er gömmer valfritt antal skedar i asken. Placera 9 skedar bredvid. Be kompisen skriva ett uttryck för hur många skedar ni har använt med en variabel och siffror.

2b) Öppna asken och skriv ett numeriskt uttryck (bara med siffror) för hur många skedar ni totalt har använt.

3a) Antalet skedar som ligger i asken kan vi kalla för *x*. Den andra av er gömmer valfritt antal skedar i asken. Placera asken och skedar framför er så att det motsvarar uttrycket x+5.	$x + 5$
3b) Öppna asken och skriv ner det numeriska uttrycket som det motsvarar.	
3c) Ta bort en sked från asken. Skriv ner vad uttrycket är nu.	
3d) Hur förändras uttrycken när vi har en stängd eller öppen ask? Hur förändras uttrycken när antalet i asken ändras? Skriv ned vad du tänker.	
3e) Vilket värde har *x* om det är 5 skedar i asken och 7 bredvid? Tips: kolla uppgift 3 a)-c) eller lägg upp uttrycket framför er.	

Nu kommer ni använda er av flera askar som alla motsvarar samma variabel.

Eftersom de motsvarar samma variabel **måste** det vara **lika många** skedar i varje ask.

4) Först använder Nina en ask och lägger två skedar i den. Sedan använder Nina en ask till. Hur många skedar måste Nina då lägga i den andra asken för att det ska bli rätt? Skriv ned ditt svar:

5a) Lägg upp askar och skedar som motsvarar uttrycket x+x+4.	$x + x + 4$
5b) Om x=3, vad blir det numeriska uttrycket? Skriv ned ditt svar.	
5c) Vad är uttryckets värde? Hur många skedar har ni totalt använt?	

6a) En av er gömmer samma antal skedar i alla askarna. Tänk på att alla askar måste ha **lika många** skedar i sig. Placera ut askarna och dela upp resten av skedarna i olika stora grupper bredvid. Be kompisen skriva ett uttryck från det du har lagt fram.	
6b) Öppna nu askarna och skriv ett numeriskt uttryck för det som ligger framför er.	
6c) Skriv ner uttryckets värde. 6d) Byt roller och gör det igen.	

7) Leo tycker väldigt mycket om ägg. Han köper två äggkartonger i affären på måndagen. Nästa dag går han förbi affären igen och köper ett till paket. Han använder 5 ägg till att steka pannkakor. Sedan går han ut och hämtar 8 ägg från hönshuset. Antalet ägg i äggkartongerna när man köper dem i affären kallas för x. Skriv ett uttryck för hur många ägg Leo har.

Tips: askarna och skedarna kan vara ett bra hjälpmedel.

8) Lisa ska grilla paprika under en grillfest. Hemma i kylskåpet har hon en förpackning med paprikor samt två extra paprikor. Lisa går till affären och köper två förpackningar till med paprikor. När Lisa skär upp paprikorna märker hon att en är dålig och måste tyvärr slänga den. Antalet paprikor Lisa har kan beskrivas med uttrycket: $2 \cdot x + x + 2 - 1$, vad motsvarar x i detta uttryck?

Tips: askarna och skedarna kan vara ett bra hjälpmedel.

Bra jobbat med uppgifterna!

Om ni har tid över, gör uppgift 6 igen eller skriv egna textuppgifter till varandra.

A.6.2 Physical Session 2

Pass 2

Antalet skedar i en ask kallar vi x. Om flera askar motsvarar x så ska det vara lika många skedar inuti.

GRÖN algebra

1 a) Placera 3 stängda askar framför er med mellanrum och skriv ett uttryck som innehåller additionstecken.

1 b) Stapla askarna på varandra och skriv ett uttryck som innehåller ett multiplikationstecken.

2 a) Lägg upp askarna och skedarna utifrån uttrycket x+x+3+x+4.

b) Öppna askarna och placera 2 skedar i varje ask. Skriv det numeriska uttrycket.

c) Beräkna uttryckets värde. Tips: uttryckets värde är samma sak som antalet skedar i uttrycket.

$x + x + 3 + x + 4$

3 a) Låt askarna och lådorna vara som i uppgift 2 a). **Sortera** askarna för sig och skedarna för sig.

b) Skriv uttrycket som det är nu. Att skriva upp ett sorterat uttryck är en del av det som kallas att *förenkla* uttrycket.

c) Öppna askarna och skriv det numeriska uttrycket.

d) Beräkna uttryckets värde.

e) Jämför med uppgift 2 b). Var det någon skillnad när ni beräknade uttryckets värde? I vilken av uppgifterna var det lättast att beräkna antalet skedar, och varför? Skriv ner era tankar.

$x + x + 3 + x + 4$

4 Uttryck som $2 \cdot x+3$ skrivs ofta utan multiplikationstecknet, såsom 2x+3. Detta är eftersom matematiker föredrar att skriva mindre. Nu ska vi bli matematiker!

a) Skriv om $7 \cdot x - 11$ utan multiplikationstecknet.

b) Skriv om $9 \cdot x + 5$ utan multiplikationstecknet.

5. Vad är 3x – x? Använd er gärna av askarna och skedarna för att lösa problemet.

5 a) Vad är värdet av uttrycket 2x + x + 5 – 1 om x=2? Här kan ni använda er av askarna eller papper och penna.

b) Beskriv stegen för hur du löste uppgiften.

6 Lägg upp uttrycket 2x + 6 – x – 3 + 1.

$2x + 6 - x - 3 + 1$

a) Förenkla uttrycket.

b) Beräkna uttryckets värde när x = 7.

7 Du har uttrycket 16x + 7 – 2x – 5x.

$16x + 7 - 2x - 5x$

a) Förenkla uttrycket.

b) Beräkna uttryckets värde när x = 3.

8 Din granne, Zac, är 2 år yngre än vad du är. Din syster Hanane är fyra år äldre än du.
Variabeln z beskriver Zacs ålder.
Skriv ett uttryck hur gamla ni tre är tillsammans.

Tips: Askarna och skedarna kan vara ett bra hjälpmedel.

A.6.3 Physical Session 3

Pass 3

GRÖN algebra

1 Ta 7 av era vikter och en låda.
Lägg allt ni inte använder åt sidan.

a) En gömmer valfritt antal vikter i lådan och lägger de andra bredvid.
Den andra gissar hur många som ligger i asken.

b) Byt roller och gör det igen.

c) Går det att veta hur många vikter det är i lådan utan att gissa? Hur?

2 Lägg fyra vikter på ena plattan. Lägg sedan en blå kub och en vikt på den andra. Lägg plattorna i dina händer.

a) Lägg 100 grams vikter på den lättare plattan tills plattorna väger lika mycket.
Hur många gram lade ni till på plattan?

När båda plattorna väger lika mycket kallas det för jämvikt.

b) Är plattorna i jämvikt? ☐ Ja ☐ Nej

c) Ifall vi tar bort en vikt från ena plattan, är de fortfarande i jämvikt? ☐ Ja ☐ Nej

När två uttryck är lika med varandra kallas det för en ekvation.
I en ekvation ska vänsterledet (VL) alltid vara samma som högerledet (HL).
Det kan jämföras med jämvikt.

3 Lisa har en leksaksbil som kan lastas med 400 gram av packning. Bilen och packningen väger tillsammans 800 gram, vilken av kuberna motsvarar bilens vikt?

Tips: lägg upp ekvationen med hjälp av vikterna och testa att använda olika kuber.

Så länge vi gör samma sak på båda sidor är ekvationen fortfarande i jämvikt!

4 En blå kub väger b gram.
Lägg upp vikter på plattorna som motsvarar ekvationen:
2b + 400 = 600.

a) Stämmer ekvationen?

☐ Ja ☐ Nej

b) Få de blå kuberna ensamma medan det fortfarande är jämvikt. Skriv ekvationen nu.

c) Hur många gram väger en blå kub?
Detta är lösningen till ekvationen.

$2b + 400 = 600$

Bra jobbat på passet!
Om ni har tid så gör egna textuppgifter med ekvationer som ni kan balansera i händerna.

Exempel:

$y + 200 + 100 = 600 - 100$

A.6.4 Physical Session 4

Pass 4

Inloggning: ____________________

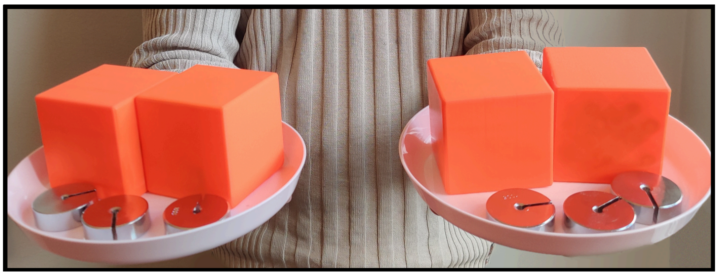
När båda plattorna väger lika mycket kallas det för jämvikt.

1 Lägg sex vikter på ena plattan. Lägg sedan en orange kub och en vikt på den andra. Lägg plattorna i dina händer.

a) Lägg 100 grams vikter på den lättare plattan tills plattorna väger lika mycket.
Hur många gram lade ni till på plattan?

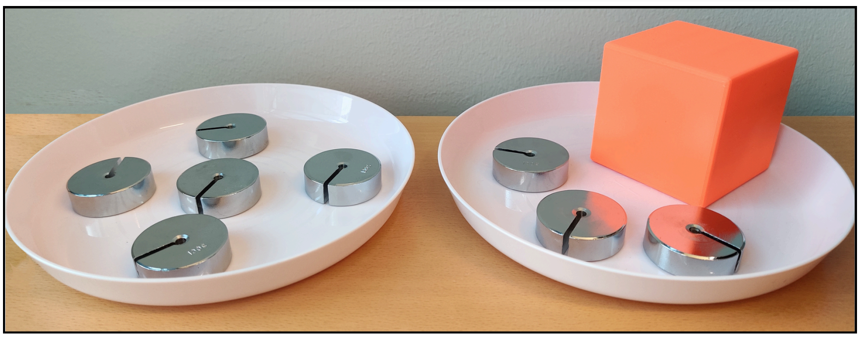
I en ekvation finns det två uttryck som är lika med varandra.
Vänsterledet (VL) ska alltid vara samma som högerledet (HL).
Det kan liknas med jämvikt.

VL = HL



När det är jämvikt kan vi sätta likhetstecken mellan uttrycken.

2 a) Skriv en ekvation för det du ser på plattorna nedan. Antalet gram som den orangea kuben väger kallar vi y.



b) Lägg upp ekvationen ni skrev i 2 a) på plattorna. Få den orangea kuben ensam på sin platta med jämvikt.

Skriv ner varje steg.

c) Hur många gram väger den orangea kuben? Det är lösningen till ekvationen.

$y =$

3 a) Antalet gram som den blå kuben väger kallar vi b.
Ni har ekvationen 3b + 400 = 700.
Stämmer ekvationen, är det jämvikt?
☐ Ja ☐ Nej

b) Få de tre blå kuberna ensamma på sin platta medan det fortfarande är jämvikt.
Skriv upp den förenklade ekvationen.

c) Hur mycket väger en blå kub? Detta är lösningen på ekvationen.

d) Hur kan ni kontrollera lösningen att lösningen stämmer? Det finns flera sätt.

$3b + 400 = 700$
$b =$

4 Lägg upp ekvationen 2b + 200 = b + 300 på plattorna.

a) Få de blå kuberna ensamma på en sida.
Kan ni lösa det utan att lägga till fler vikter?
Hur ser ekvationen ut nu?

$2b + 200 = b + 300$

5 Lägg en grå kub, en blå kub och 200 g på ena plattan. På andra plattan lägger ni en blå kub och 600 g.

a) Skriv en ekvation för det ni lagt upp.

b) Hur många gram väger den gråa kuben? Lös ekvationen.

6 a) Fotbollstränaren har en säck med okänt antal bollar. Dessutom finns det femhundra bollar i klubbhuset och trehundra i förrådet. tvåhundra av dessa är för slitna och behöver slängas.

Det totala antalet bollar är dubbelt så många som det finns i säcken. Skriv upp en ekvation som motsvarar detta.

Om ni vill kan ni använda vikterna och kuberna som hjälp.

b) Lös ekvationen.

Ni kan använda vikterna och kuberna om ni vill.

Bra jobbat på passet! Om ni har tid över så gör egna textuppgifter till varandra som liknar uppgift 7

Skriv gärna era textuppgifter här :)

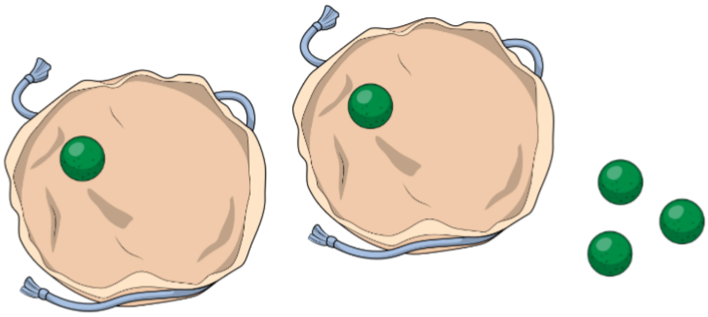
A.7 Exercises Used in the Digital Sessions

All digital sessions used in the study.

A.7.1 Digital Session 1

Jag har lagt lika många kulor i varje påse.

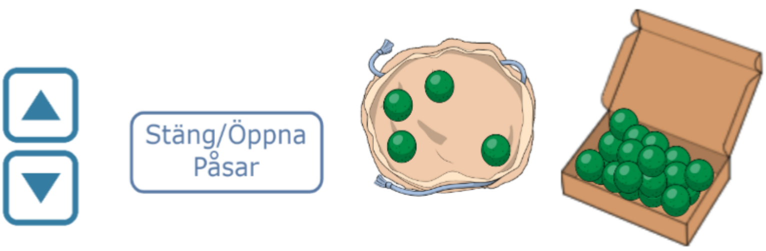
Ska vi utforska hur vi kan skriva uttryck för hur många det är?



$$2 \cdot 1 + 3$$

Antalet kulor i påsarna kan variera.

Klicka så att det är 5 kulor i varje påse.



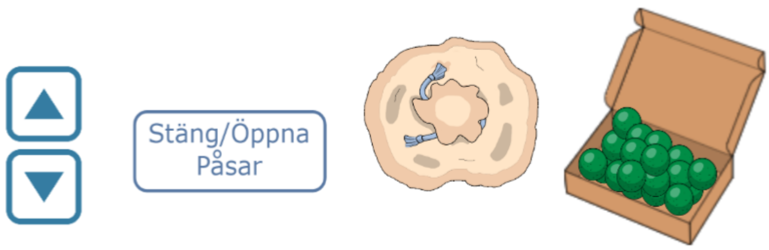
Vi kan dra in fler kulor.

Ändra antalet kulor i påsen med pilarna.

En av er, lägg hur många kulor du vill i påsen och stäng utan att den andra ser. Kompisen ska gissa hur många kulor som ligger i påsen.

Byt roller och gör det igen.

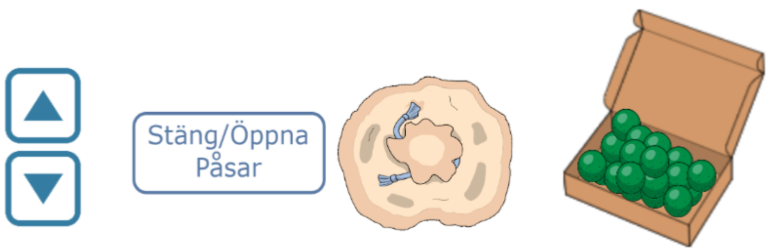
Antalet kulor i påsen är en variabel. En variabel kan ha **olika** värden.



En av er gömmer valfritt antal kulor i påsen. Placera 9 kulor bredvid.

a) Be kompisen skriva ett uttryck för hur många kulor ni använt med en variabel och siffror.

b) Öppna sedan påsarna och skriv ett nytt uttryck med siffror för hur många kulor ni totalt har framför er.



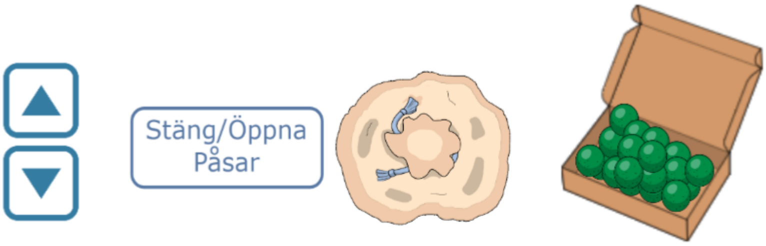
Antalet kulor i påsen kan vi kalla x.

a) Den andra fyller påsen med hemligt antal. Placera ut så att det motsvarar uttrycket $x + 5$. Skriv ned uttrycket på era papper.

b) Öppna påsen och skriv det numeriska uttrycket.

c) Behåll påsen öppen och minska antalet

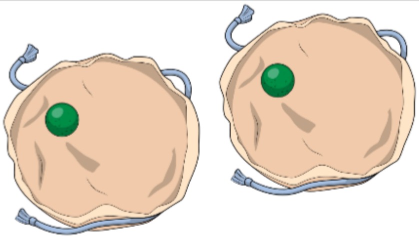
Hur förändras uttrycken när vi har en stängd eller öppen ask? Hur förändras uttrycken när antalet i asken ändras? Skriv ner vad du tänker.



Vilket värde har x om det är 2 kulor i påsen och 5 bredvid?

Skriv svar här:

Nu kommer ni att använda er av flera påsar, som alla motsvarar samma variabel. Eftersom de motsvarar samma variabel kommer det vara **lika många** kulor i varje påse.

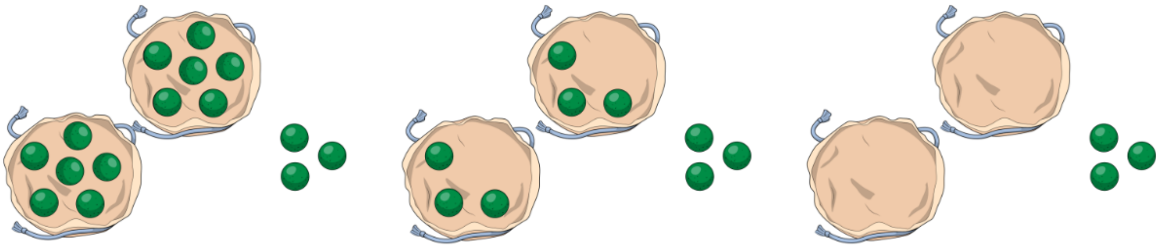


$$2 \cdot 1$$

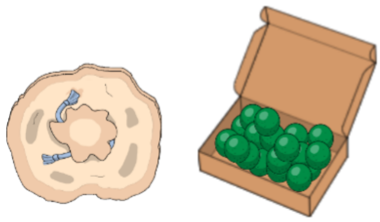
Varje påse fylls alltid med lika många kulor, varför händer det? Skriv ner vad du tänker.

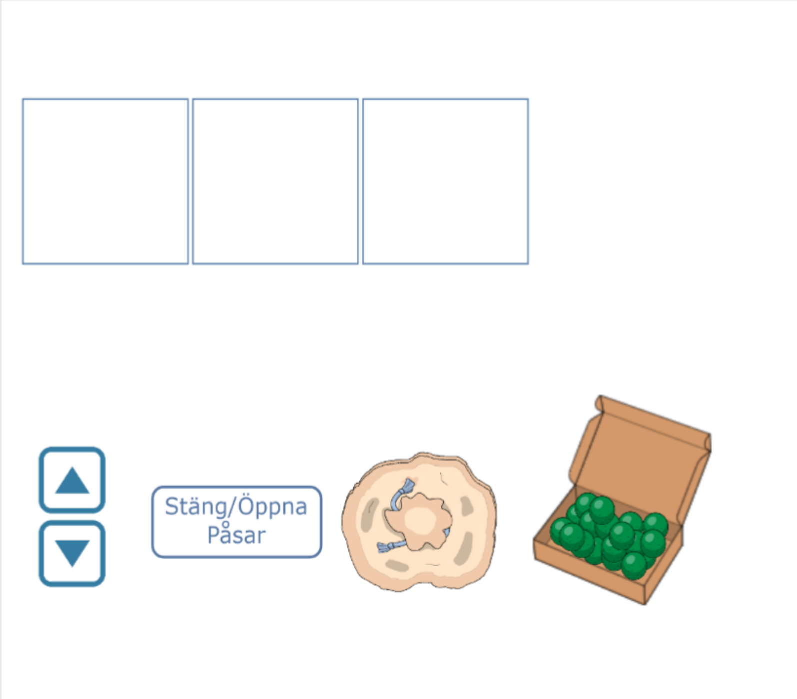
Klicka och dra för att para ihop uttrycken med motsvarande bild.

$2 \cdot 0 + 3$	$2 \cdot 3 + 3$	$2 \cdot 6 + 3$

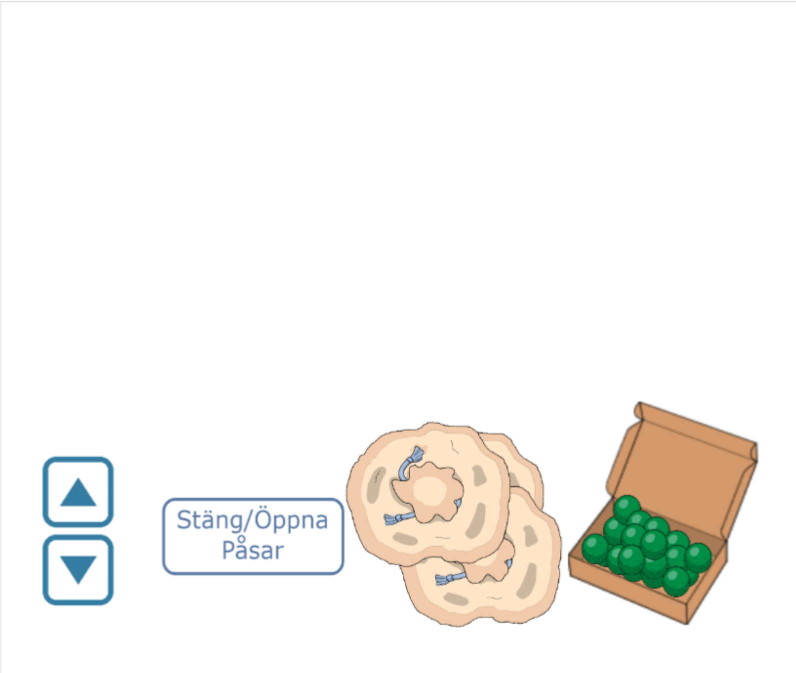


Lägg upp påsar och kulor i rutorna så det motsvarar uttrycket $x + 5 + 4 + x$





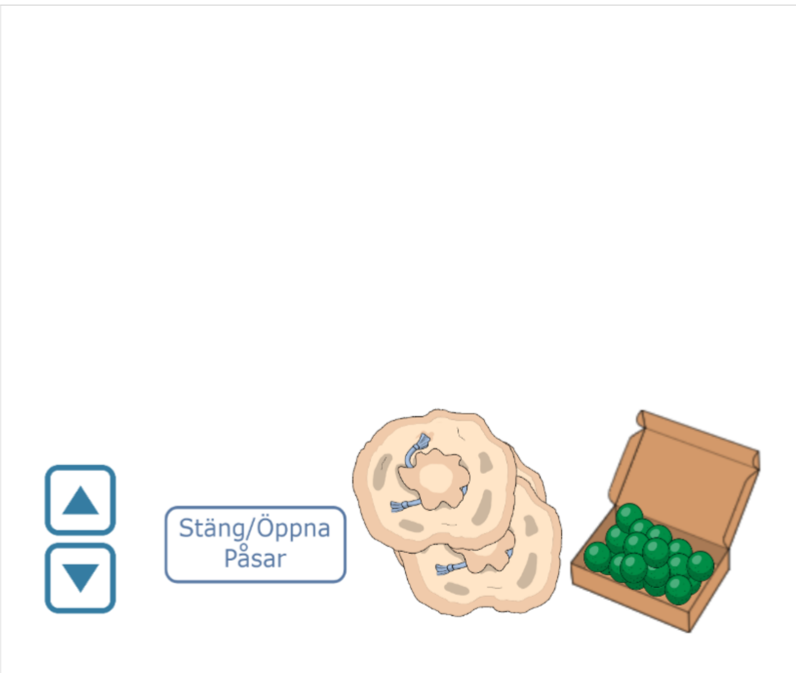
Lägg upp påsar och kulor i rutorna så det motsvarar uttrycket $x + x + 4$ då $x = 3$



a) En av er, fyll en eller flera påsar med valfritt antal kulor och dra resterande kulor till olika stora högar bredvid. Be kompisen skriva ett uttryck som motsvarar det du har placerat ut.

b) Öppna nu pås(arna) och skriv ett uttryck för det som finns på skärmen.

c) Skriv ner hur



Leo tycker väldigt mycket om ägg.

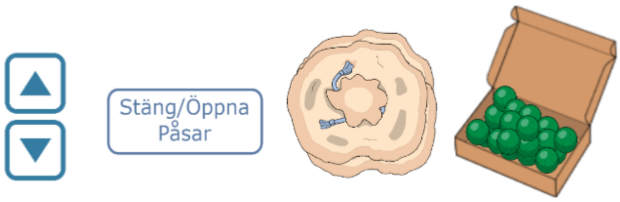
Han köper två äggkartonger i affären på måndagen. Nästa dag går han förbi affären igen och köper ett till paket.

Han använder 5 ägg till att steka pannkakor. Sedan går han ut och hämtar 8 ägg från hönshuset.

Antalet ägg i

Lisa ska grilla paprika under en grillfest.

I kylskåpet har hon en förpackning med paprikor samt 2 extra paprikor. Lisa går till affären och köper två



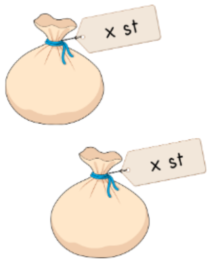
- ◯ 2
- ◯ en förpackning med paprikor
- ◯ antalet paprikor i en förpackning
- ◯ en paprika

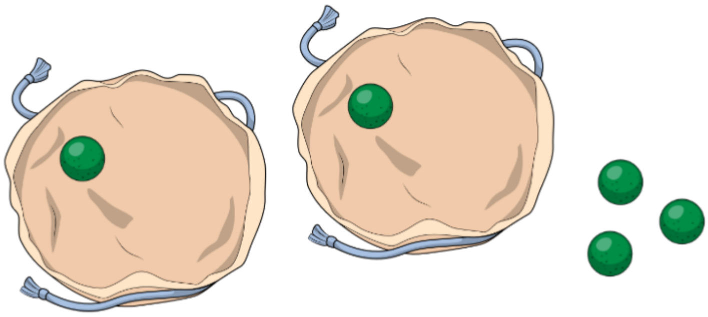
A.7.2 Digital Session 2

Jag har hört att det finns ett enklare sätt att räkna ut värdet av ett uttryck.

Det skulle vara skönt om det går snabbare.

Ja, det gör det! Vi kan lära oss om det tillsammans.





$$2 \cdot 1 + 3$$

Antalet kulor i påsarna kan variera.

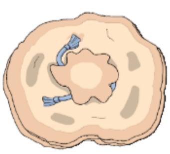
Klicka så att det är 4 kulor i varje påse.

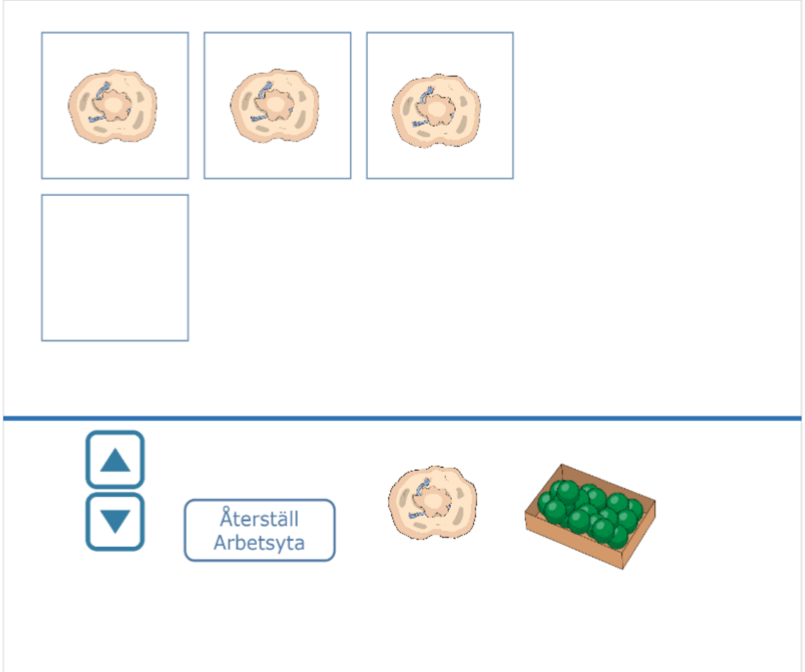
Placera en stängd påse i varje ruta och skriv ett motsvarande uttryck som innehåller additionstecken.

Svar:

Stäng/Öppna Påsar

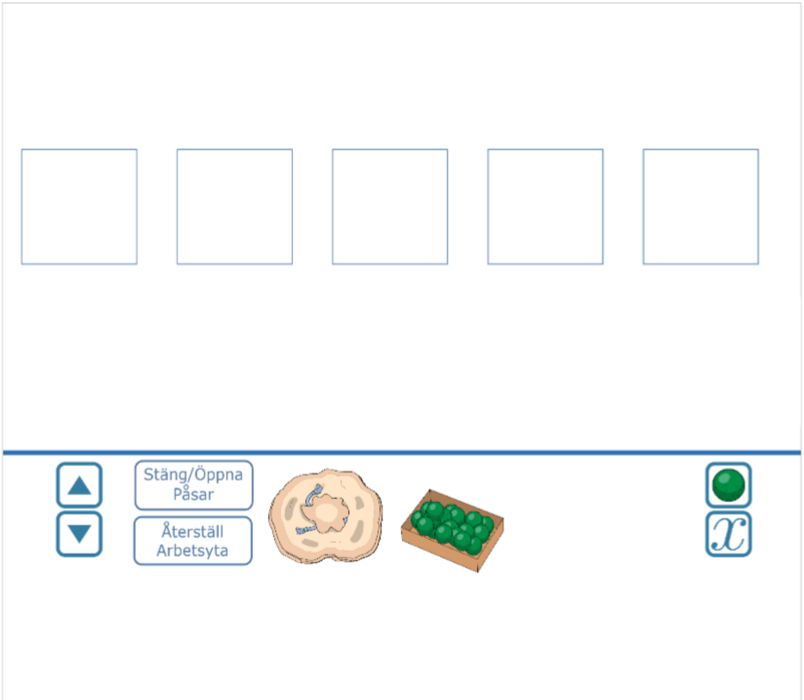
Återställ Arbetsyta





Placera påsarna på varandra i rutan nedanför och skriv ett uttryck som innehåller ett multiplikationstecken.

Svar: ☐ · ☐



Steg 1: Lägg påsarna och kulorna i rutorna så det motsvarar uttrycket $x + x + 2 + x + 4$.

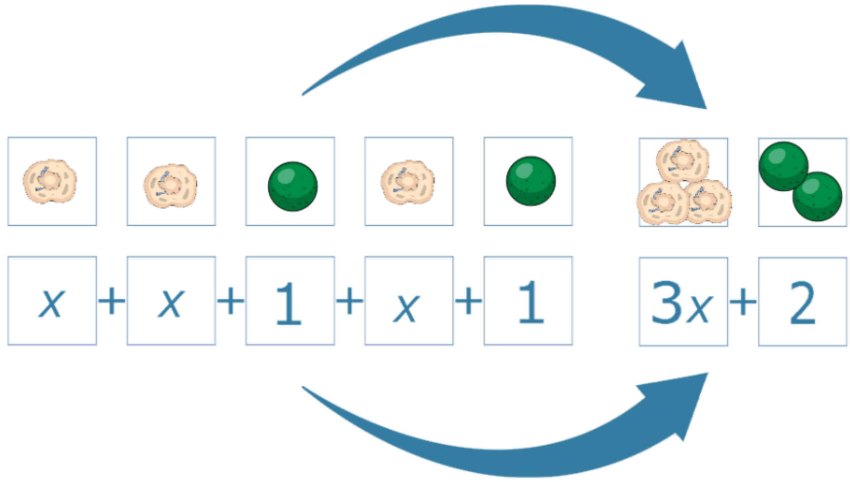
Steg 2: Öppna påsarna och ändra antalet kulor med pilarna till en kula. Byt meny genom att klicka på knappen med ett blått x längst ner till höger.

1 a) Skriv det numeriska uttrycket.

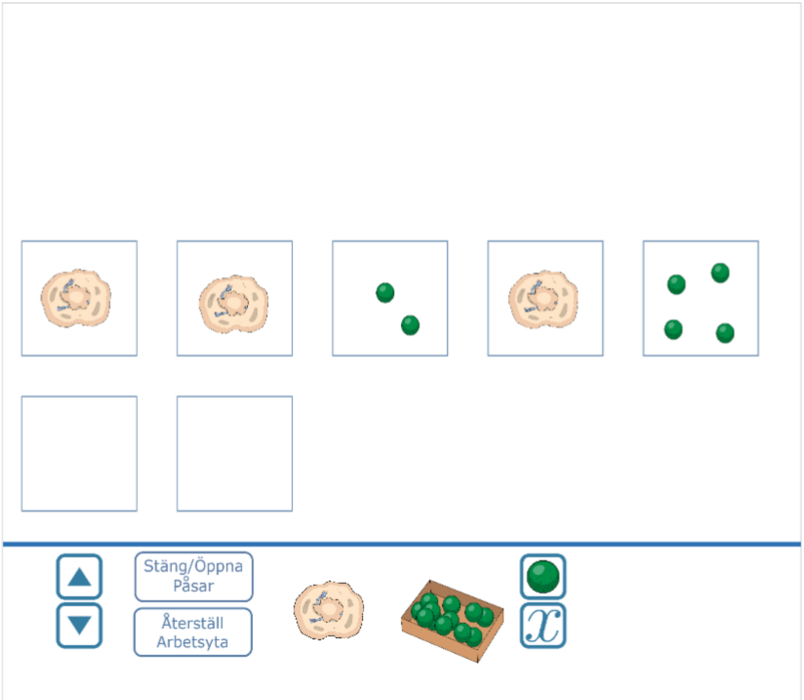
1 d) Beräkna uttryckets värde.

1 + 1 + 2 + 1 + 4

Svar: ☐

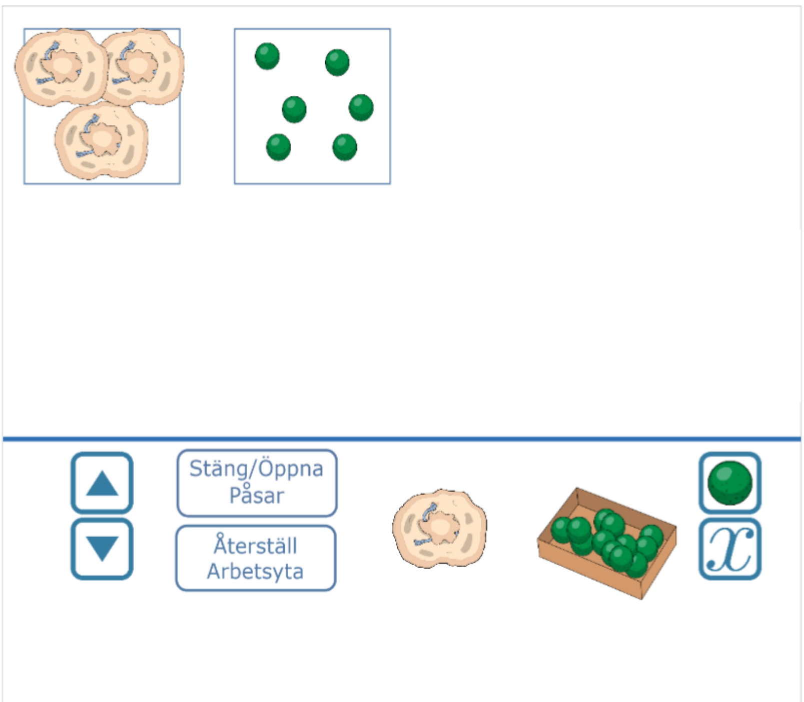


Att skriva upp ett sorterat uttryck är en del av det som kallas att *förenkla* uttrycket.



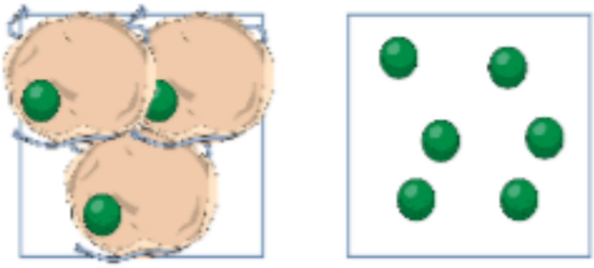
2 a) Sortera påsarna och kulorna som ligger i rutorna ovan för sig.

2 b) Dra sedan in siffror och variabel i rutorna så att du får uttrycket som motsvarar det du sorterat



Det förenklade uttrycket är $3x + 6$.

2 c) Öppna påsarna, byt meny och skriv det numeriska uttrycket.



3 + 6

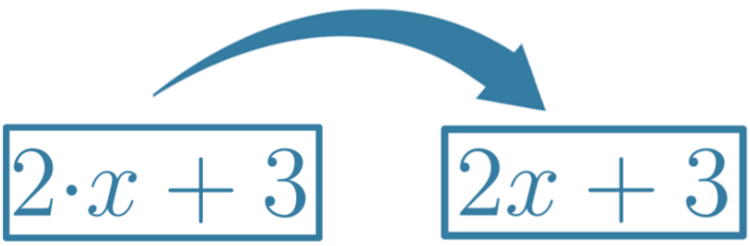
2 d) Beräkna uttryckets värde.

Svar:

Jämför med uppgift 1. Var det någon skillnad när ni beräknade uttryckets värde? I vilken av uppgifterna var det lättast att beräkna antalet kulor, och varför? Skriv ner era tankar. Vill du se bild av uttrycken igen, tryck på hint.

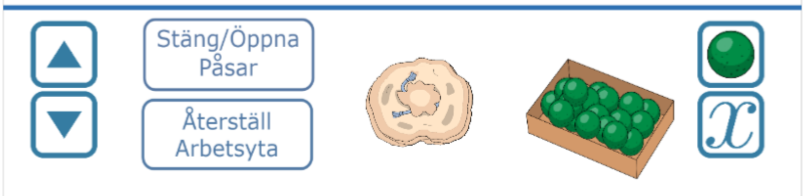
Uttryck som $2 \cdot x + 3$ skrivs ofta utan multiplikationstecknet, såsom $2x + 3$. Det är för att matematiker föredrar att skriva mindre.

Nu ska vi bli matematiker!



Para ihop uttrycken som är ekvivalenta (alltså motsvarar varandra)

$7 \cdot x - 11$		$7x - 11$
$9 \cdot x + 5$		$25x - 0,5$
$25 \cdot x - 0,5$		$9x + 5$

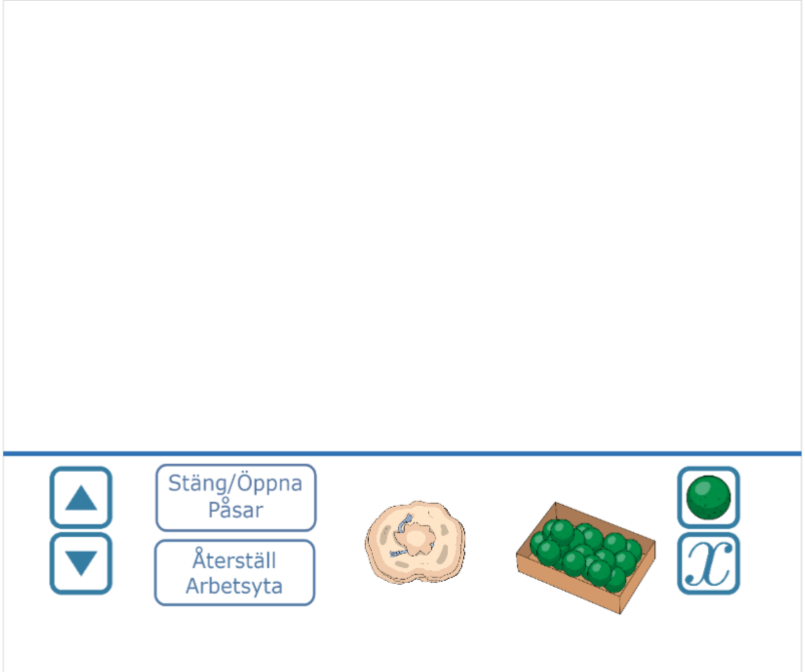


Vad är värdet av uttrycket
$2x + x + 5 - 1$
om x=2?

Här kan ni använda er av påsarna eller papper och penna.

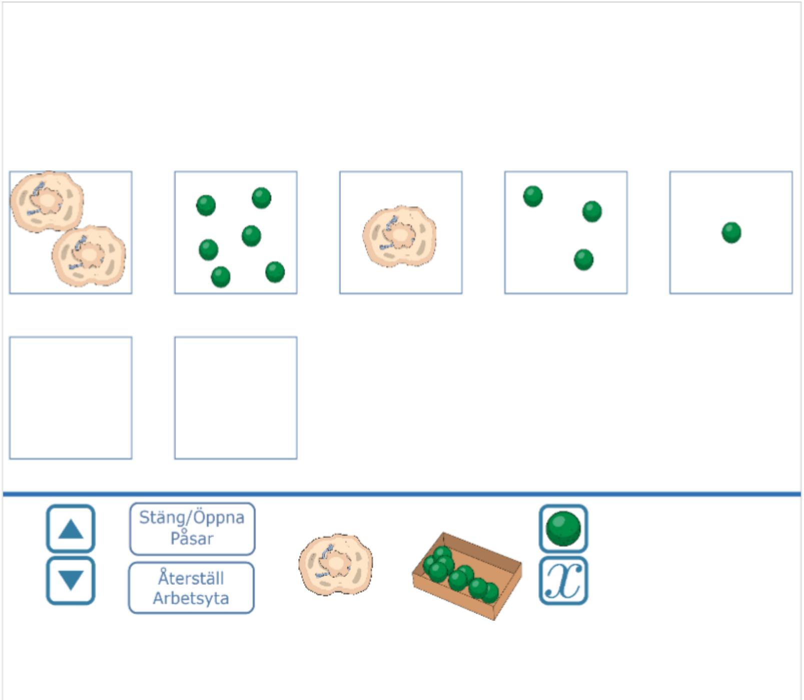
Svar:

Beskriv stegen för hur du löste uppgiften
"Vad är värdet av uttrycket 2x + x + 5 – 1 om x=2?"



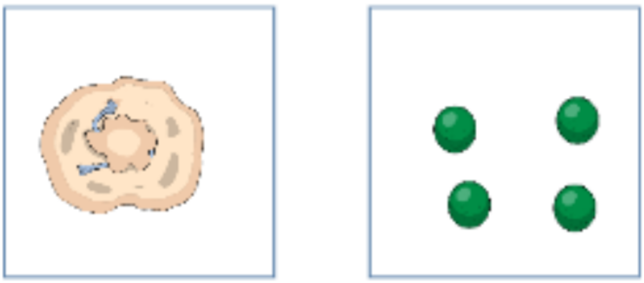
Vad är $3x - x$?
Använd er gärna av påsarna för att lösa problemet.

Svar:



Du har uttrycket $2x + 6 - x - 3 + 1$ med påsar och kulor.

Förenkla uttrycket genom att sortera påsarna och kulorna för sig

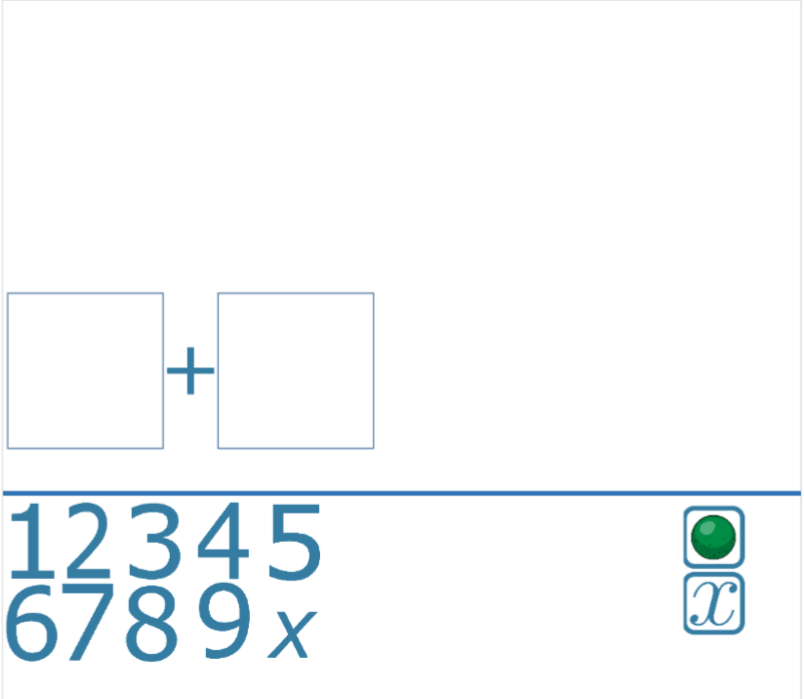


x + 4

Det förenklade uttrycket är $x + 4$.

b) Beräkna uttryckets värde när $x = 7$.

Svar:



Du har uttrycket
$16x + 7 - 2x - 5x.$

a) Förenkla uttrycket genom att dra rätt siffror och variabel till rutorna.

Du har uttrycket
$16x + 7 - 2x - 5x.$

b) Beräkna uttryckets värde när $x = 3$

Svar:

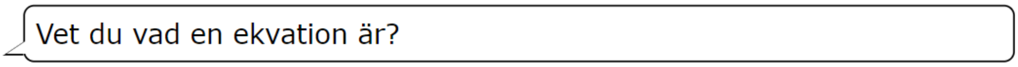
Din granne, Zac, är 2 år yngre än vad du är. Din syster Hanane är fyra år äldre än du.

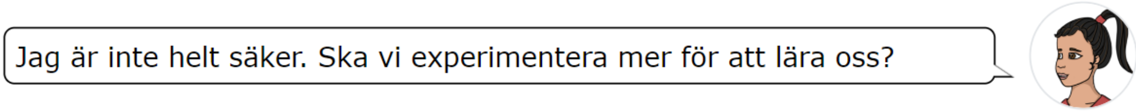
Variabeln z beskriver Zacs ålder.

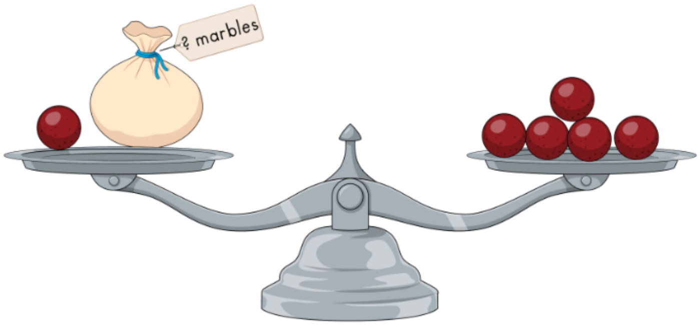
Skriv ett förenklat uttryck hur gamla ni tre är tillsammans.

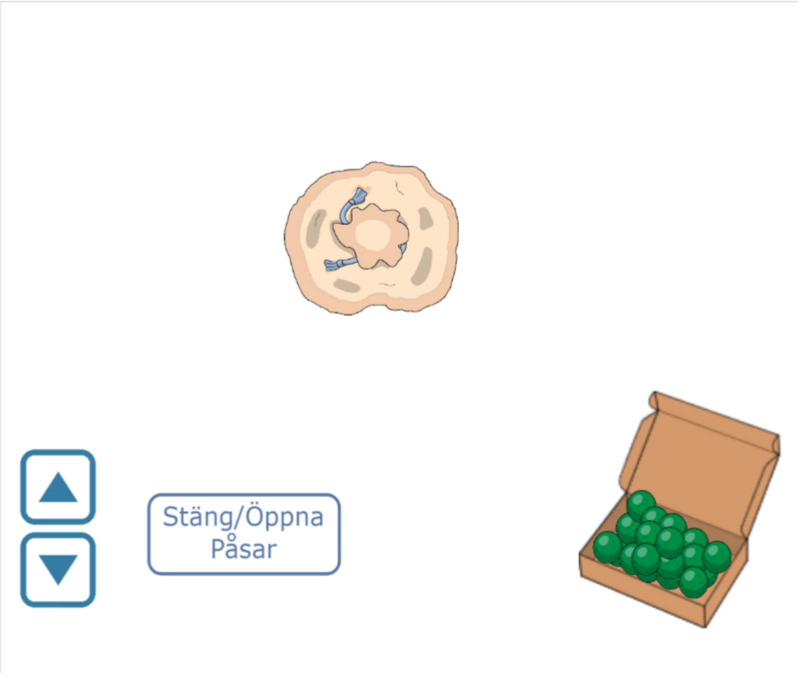
Svar:

A.7.3 Digital Session 3







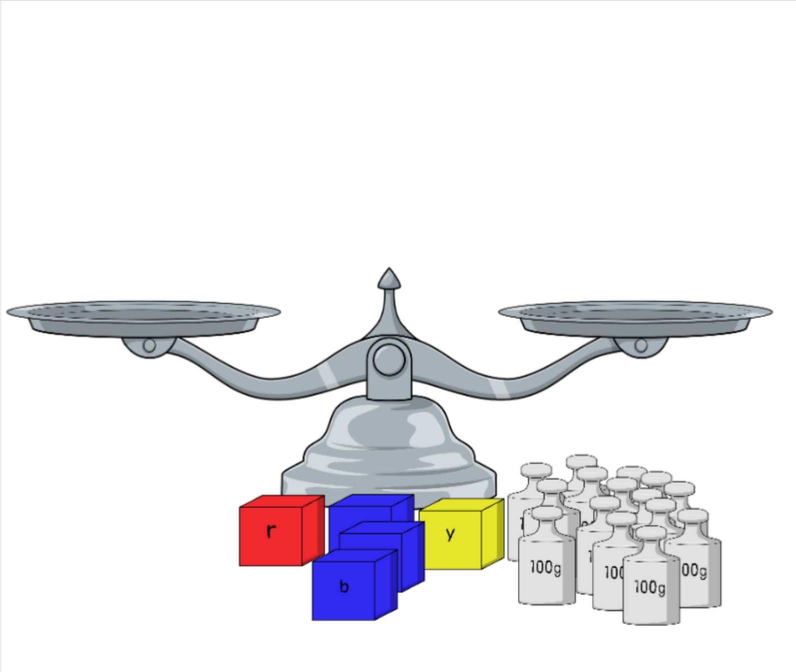


Använd sju kulor totalt. En av er gömmer valfritt antal i en av påsarna med hjälp av pilarna och drar de andra kulorna så att de ligger bredvid påsen.

Klasskamraten gissar hur många kulor som ligger i påsen.

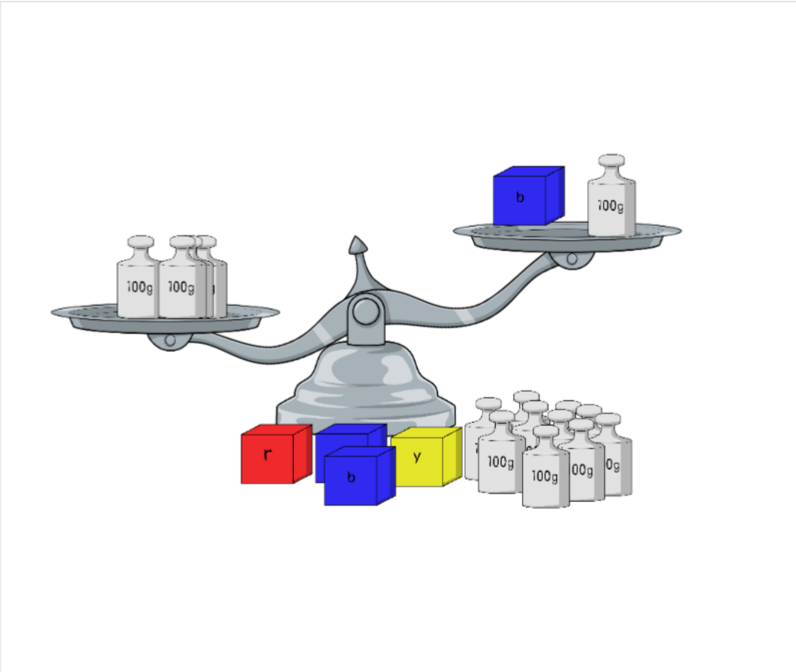
Byt roller och gör det igen.

Går det att veta hur många kulor det är i påsen utan att gissa? Hur? Testa att använda er av tipset.



Testa att dra vikter och kuber till vågskålarna (vågens plattor).

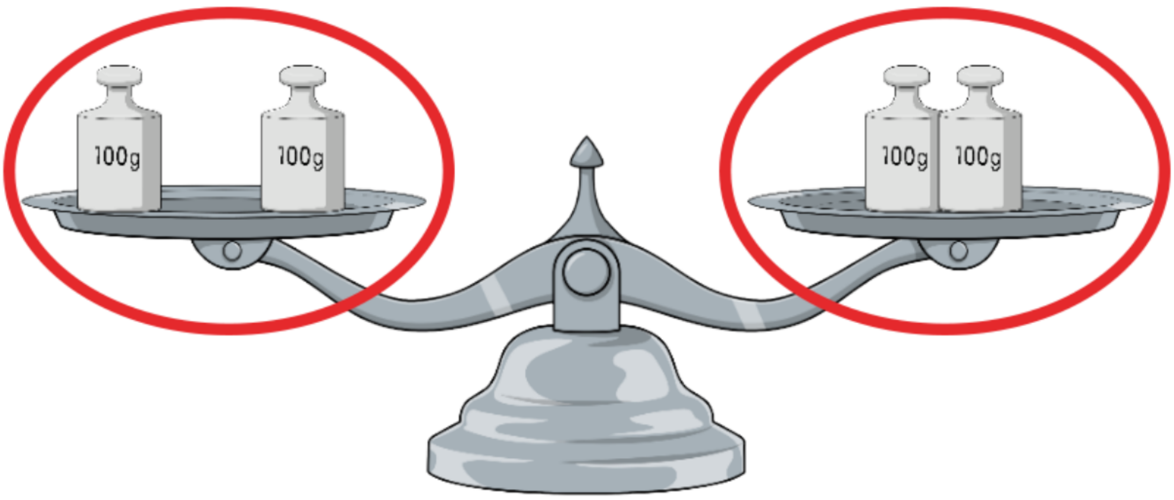
När ni känner er nöjda, gå vidare.



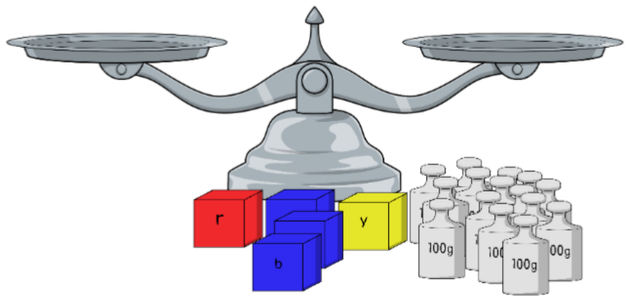
Lägg 100 grams vikter på den lättare sidan av balansvågen tills vågskålarna väger lika mycket.

Hur många gram la ni till på plattan?

Svar:



När båda vågskålarna väger lika mycket kallas det för **jämvikt**.

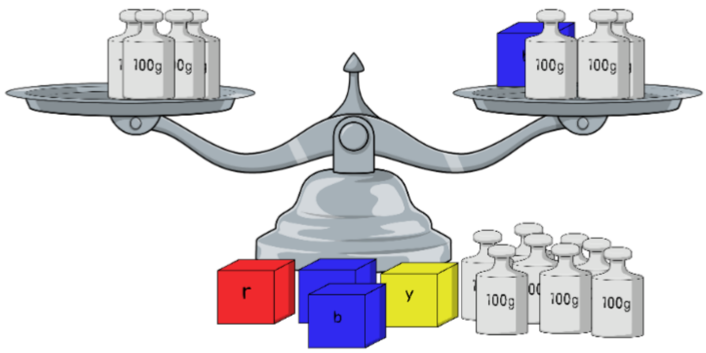


Dra fyra 100 grams vikter till ena vågskålen.
Dra tre 100 grams vikter och ett b till andra vågskålen.

Är plattorna i jämvikt?

◯ ja

◯ nej

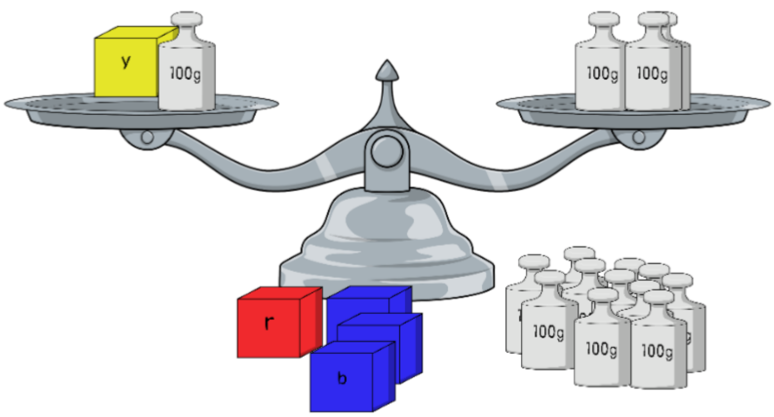


Ifall vi tar bort en vikt från vågen, är den fortfarande i jämvikt?

Tips: testa att ta bort en vikt från vågen.

◯ ja

◯ nej

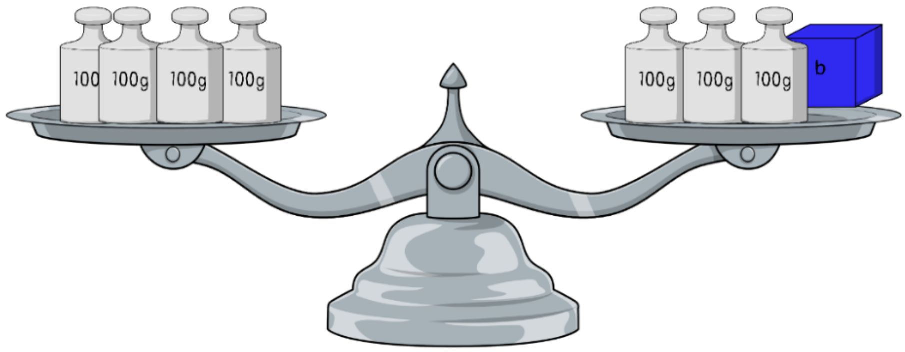


Den gula kuben väger y gram. På vågen ligger ekvationen $y + 100 = 300.$

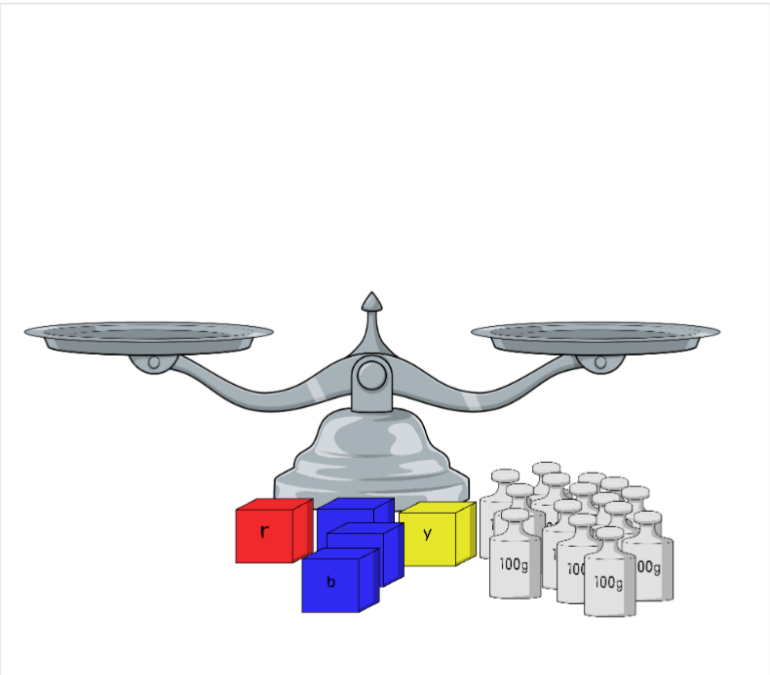
Kan vi ta bort något från båda sidor och fortfarande ha jämvikt? Gör det i så fall och skriv den nya ekvationen.

Den nya ekvationen är: ☐

VL = HL



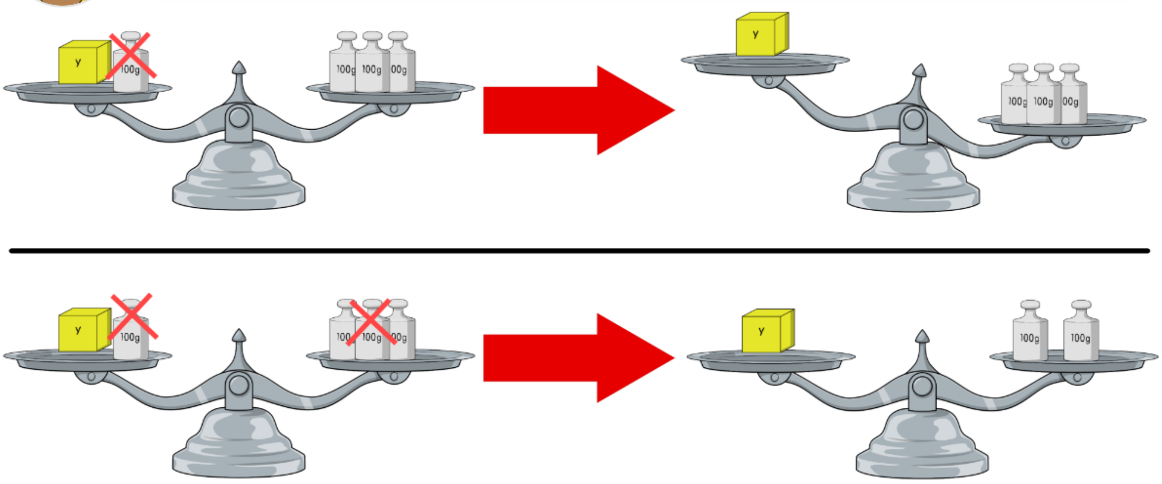
När två uttryck är lika med varandra kallas det för en ekvation. I en ekvation ska vänsterledet (VL) alltid vara samma som högerledet (HL). Det kan jämföras med jämvikt.

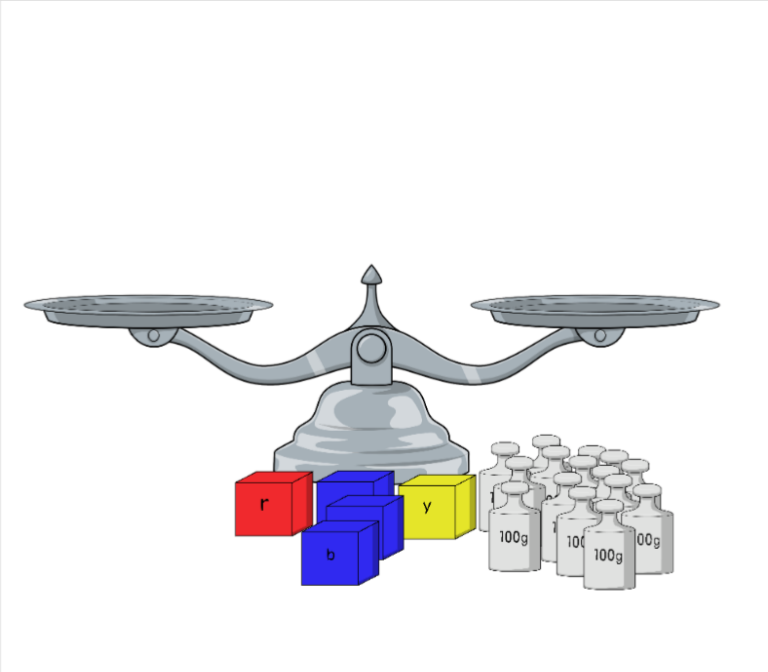


Lisa har en leksaksbil. som kan lastas med 400 gram av packning. Bilen och packningen väger tillsammans 800 gram, vilken av kuberna motsvarar bilens vikt?

- ○ gul
- ○ röd
- ○ blå

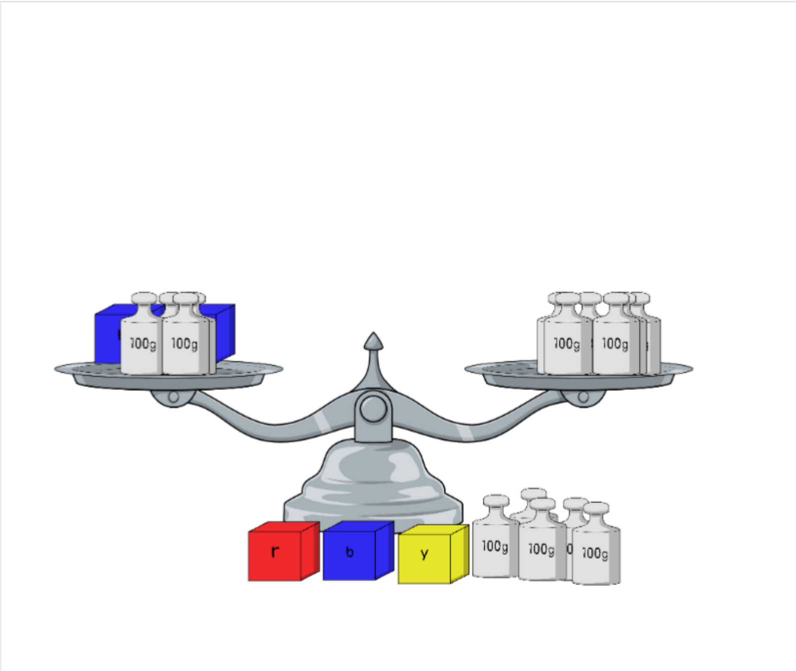
Så länge vi gör samma sak på båda sidor är ekvationen fortfarande i jämvikt.





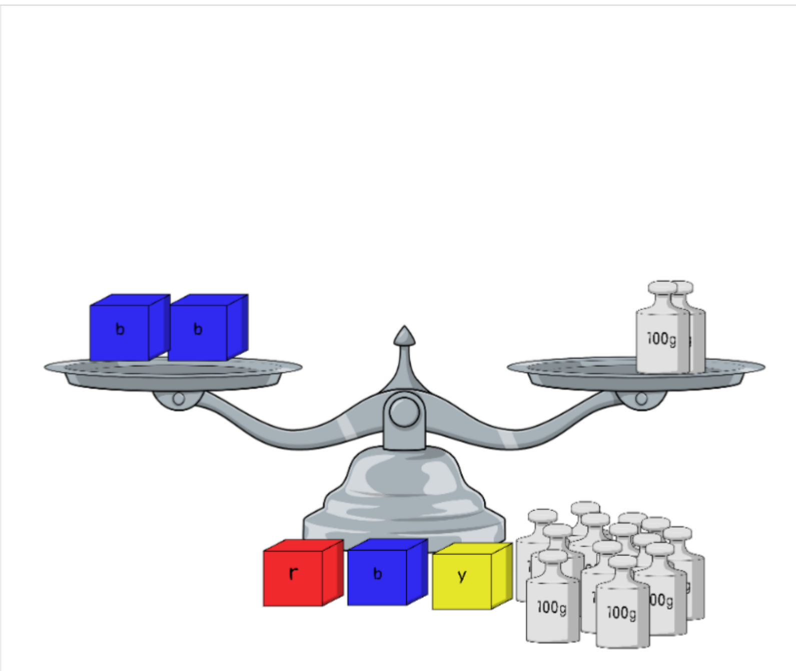
En blå kub väger b gram. Lägg upp vikter som motsvarar ekvationen $2b + 400 = 600$.
Stämmer ekvationen?

- ◯ Ja
- ◯ Nej



Få de blå kuberna ensamma medan det fortfarande är jämvikt.

Skriv upp den förenklade ekvationen: ☐

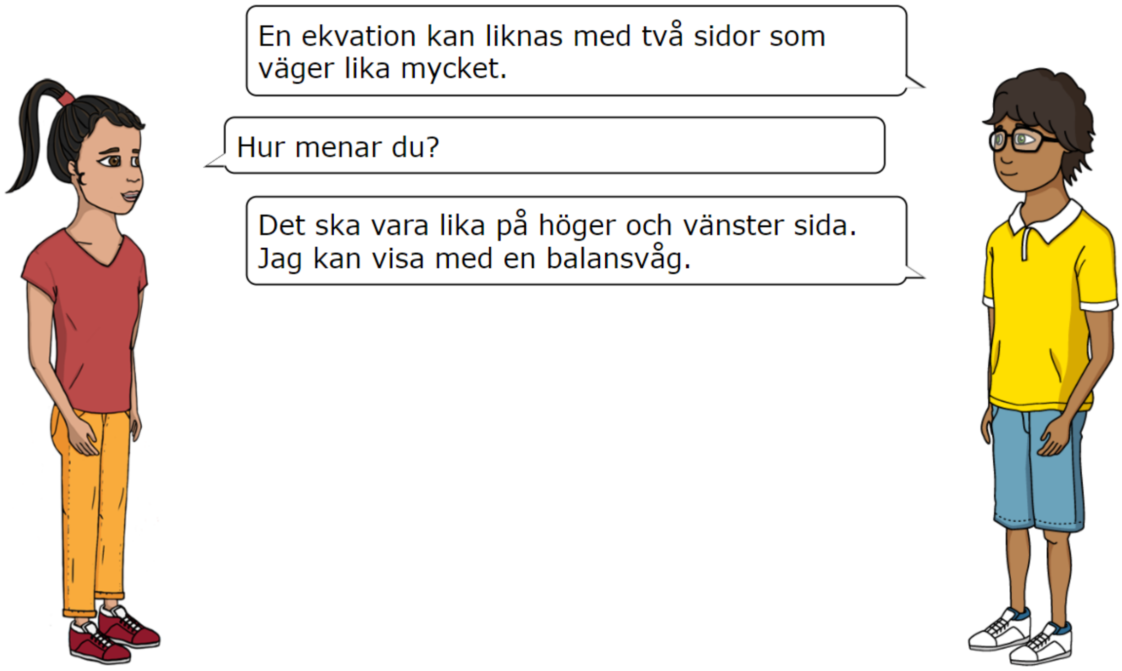


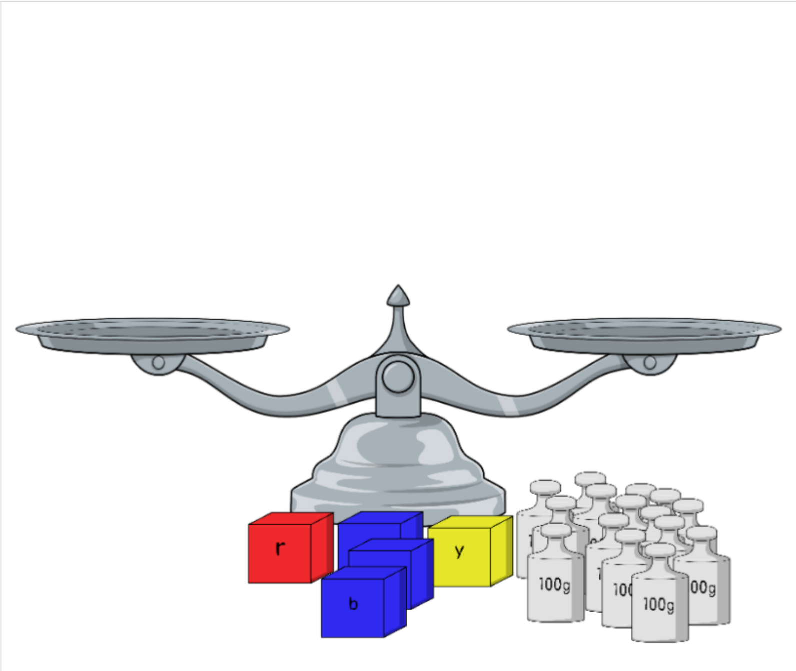
Hur många gram väger en blå kub? Detta är lösningen till ekvationen.

b= ☐

Bra jobbat gänget! Ni är klara med materialet!

A.7.4 Digital Session 4

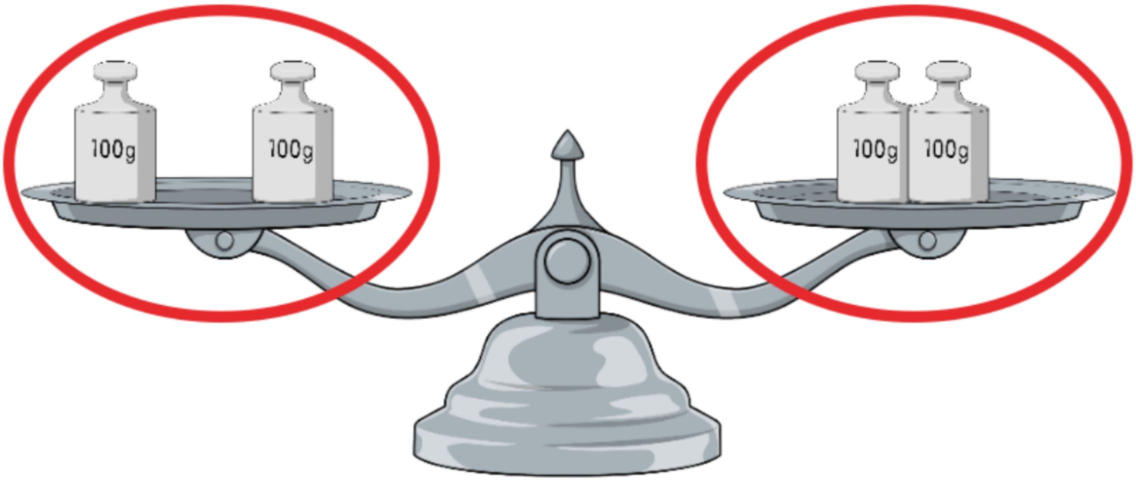


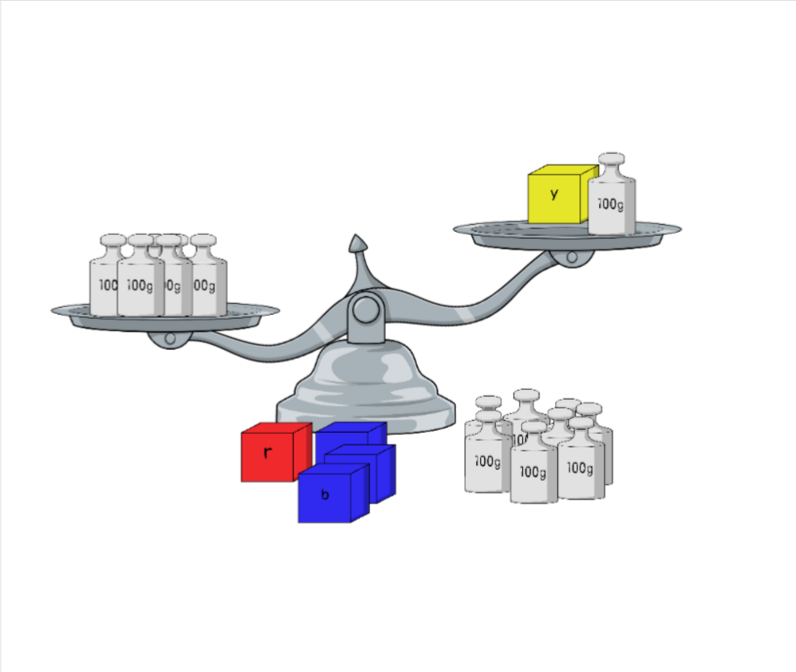


Testa att flytta på vikter och kuber till vågskålarna (vågens plattor).

När ni känner er nöjda, gå vidare.

När det är lika mycket vikt i vågskålarna kallas det **jämvikt**.



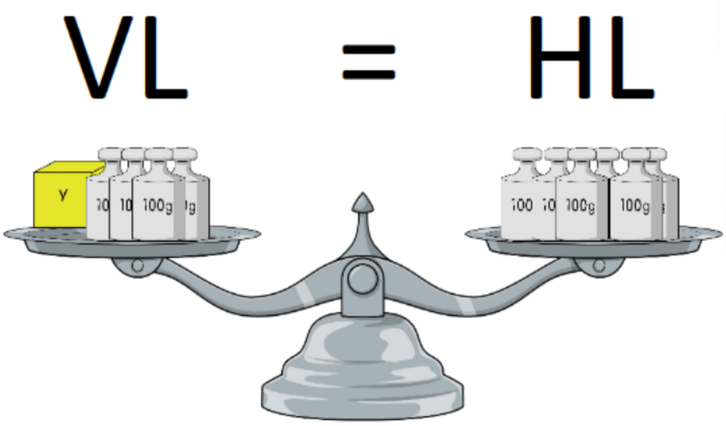


Lägg 100 grams vikter på den lättare sidan av balansvågen tills vågskålarna väger lika mycket.

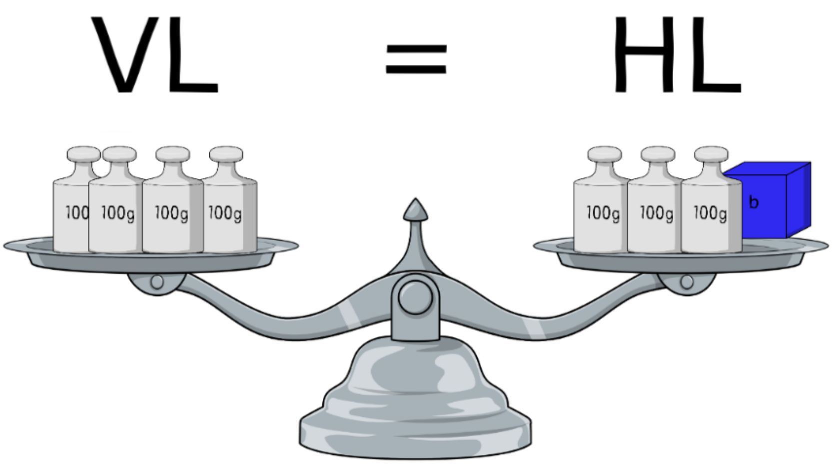
Hur många gram la ni till på plattan?

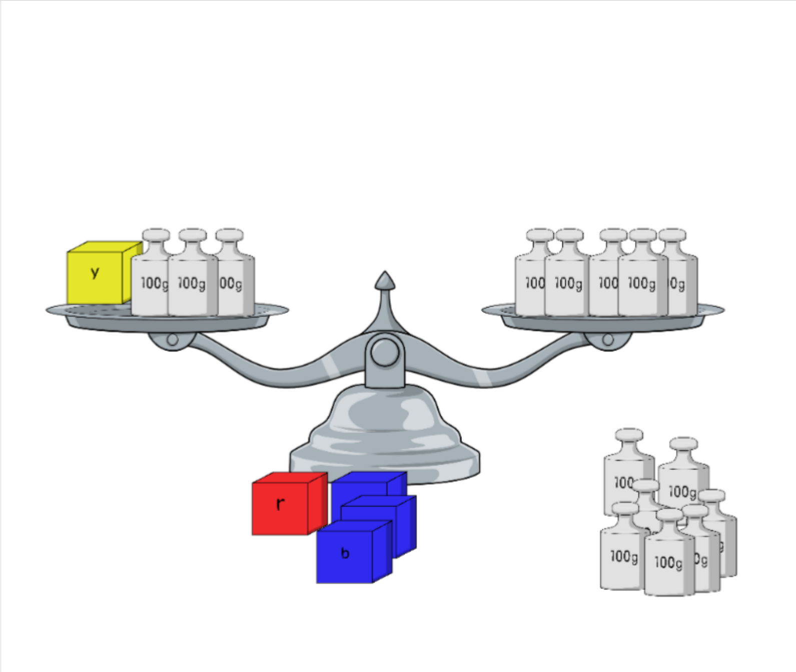
Svar:

I en ekvation finns det två uttryck som är lika med varandra. Vänsterledet (VL) ska alltid vara samma som högerledet (HL). Det kan liknas med jämvikt.



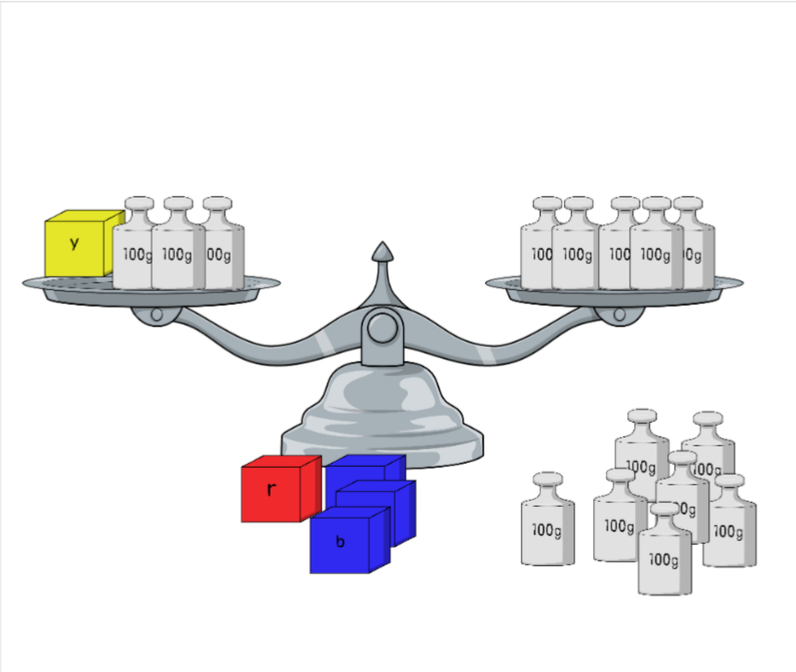
När det är jämvikt kan vi sätta likhetstecken mellan uttrycken.



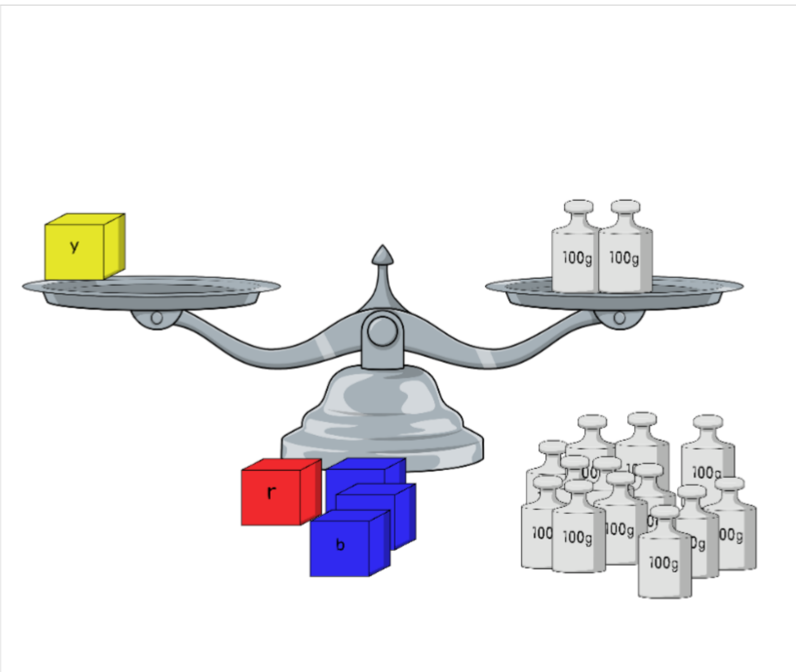


Skriv en ekvation för det du ser på vågen.

Det finns ledtrådar (knappen längst ner till vänster).

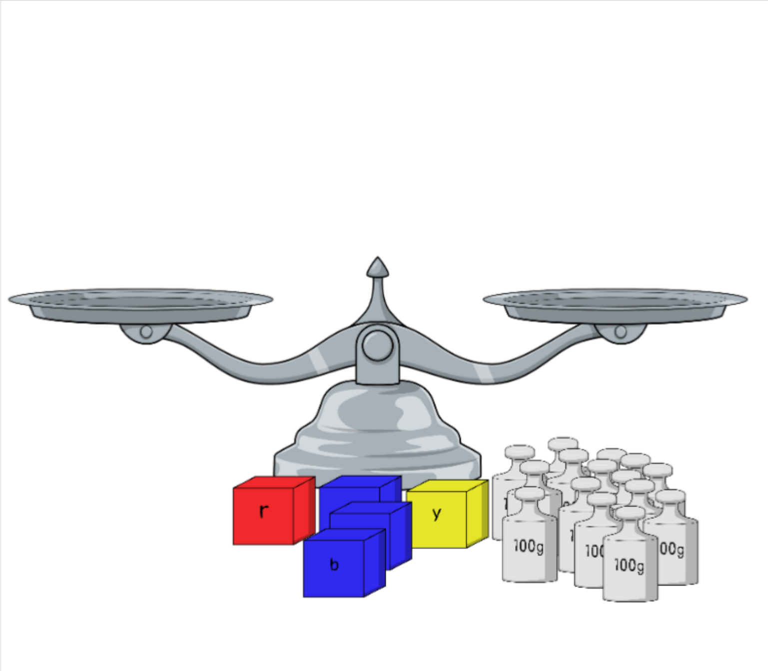


Få den gula kuben ensam i sin vågskål med jämvikt.



Hur många gram väger den gula kuben? Det är lösningen till ekvationen.

y=

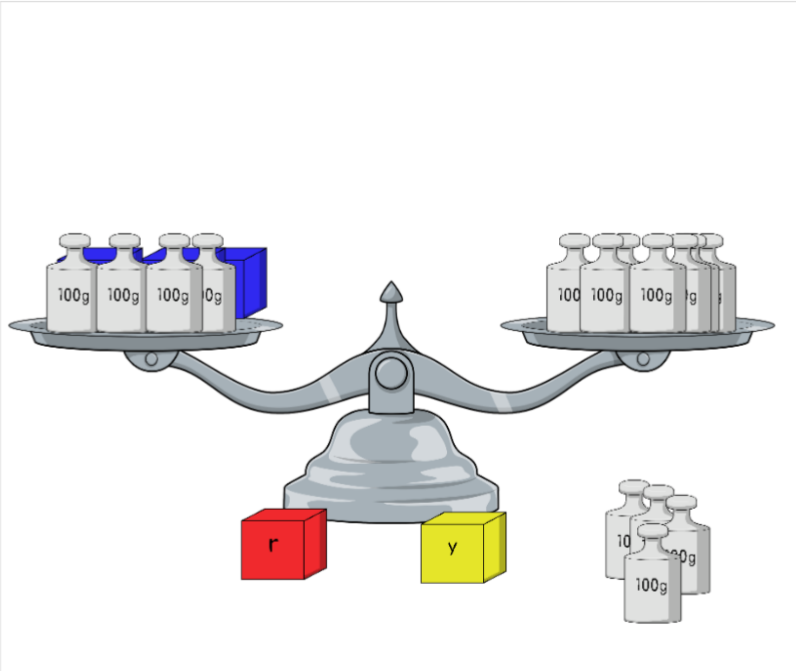


Antalet gram som den blå kuben väger kallar vi b. Ni har ekvationen

$3b + 400 = 700$.

Stämmer ekvationen, är det jämvikt?

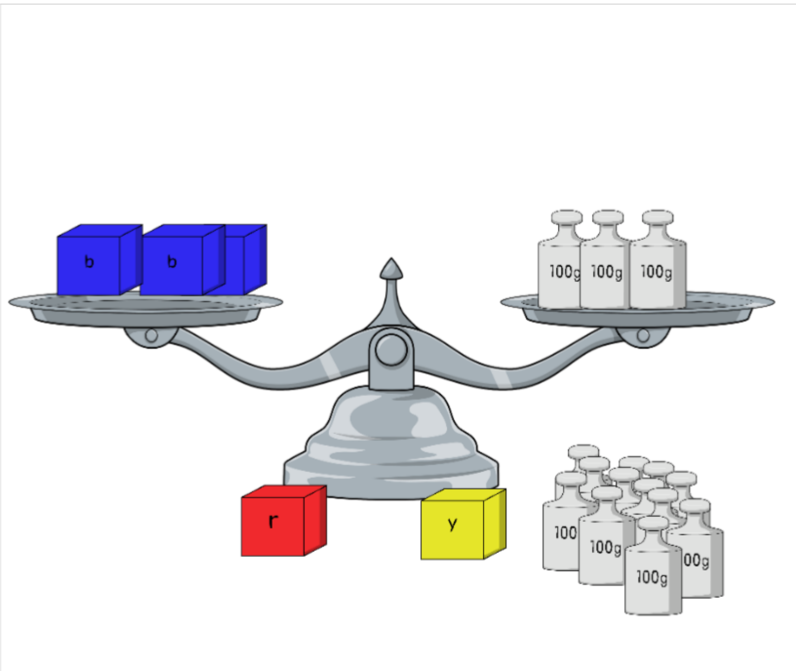
- ja
- nej



Få de tre blå kuberna ensamma i sin vågskål medan det fortfarande är jämvikt.

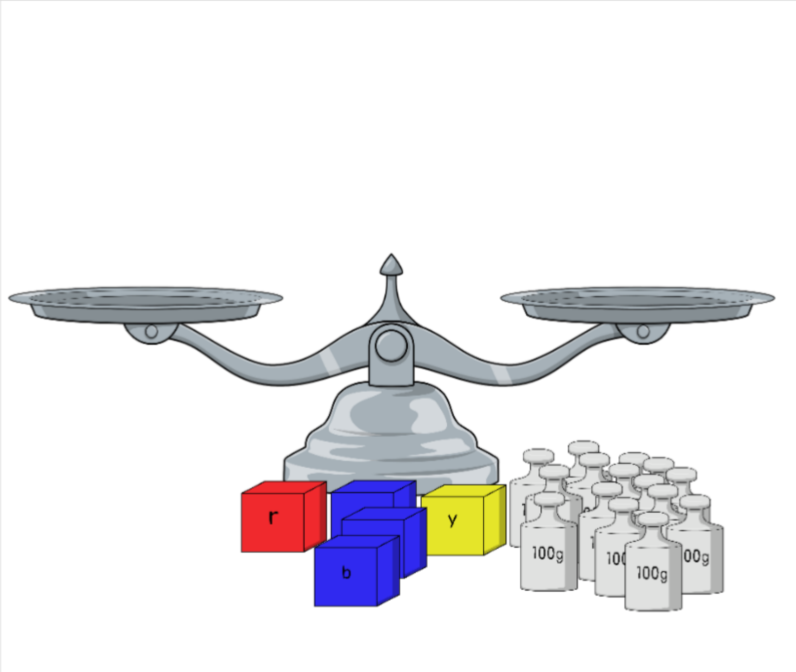
Skriv upp den förenklade ekvationen.

$3b$ = ☐



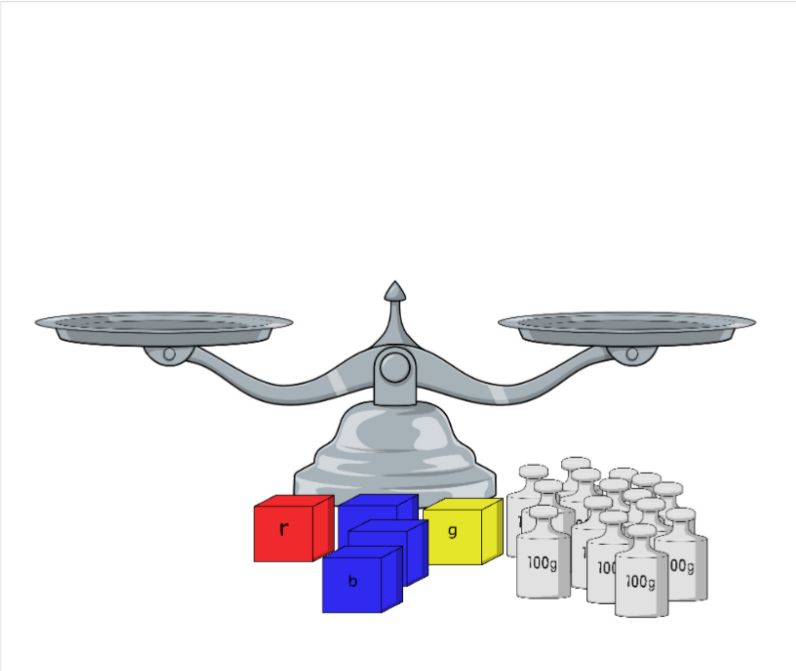
Hur mycket väger en blå kub? Detta är lösningen på ekvationen.

b = ☐



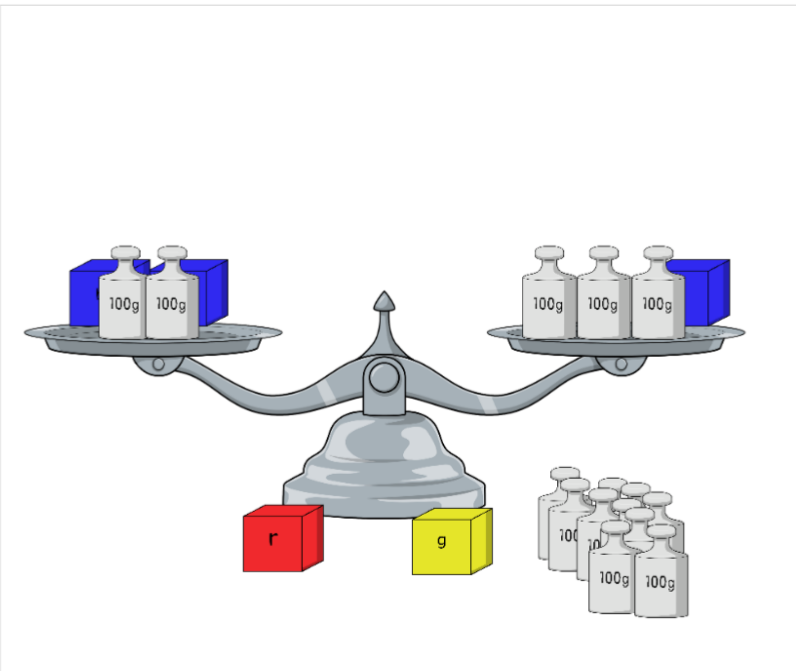
Ekvationen vi började med var $3b + 400 = 700$ och lösningen är $b = 100$.

Hur kan ni kontrollera lösningen att lösningen stämmer? Det finns flera sätt.



Lägg upp ekvationen $2b + 200 = b + 300$ på vågen.

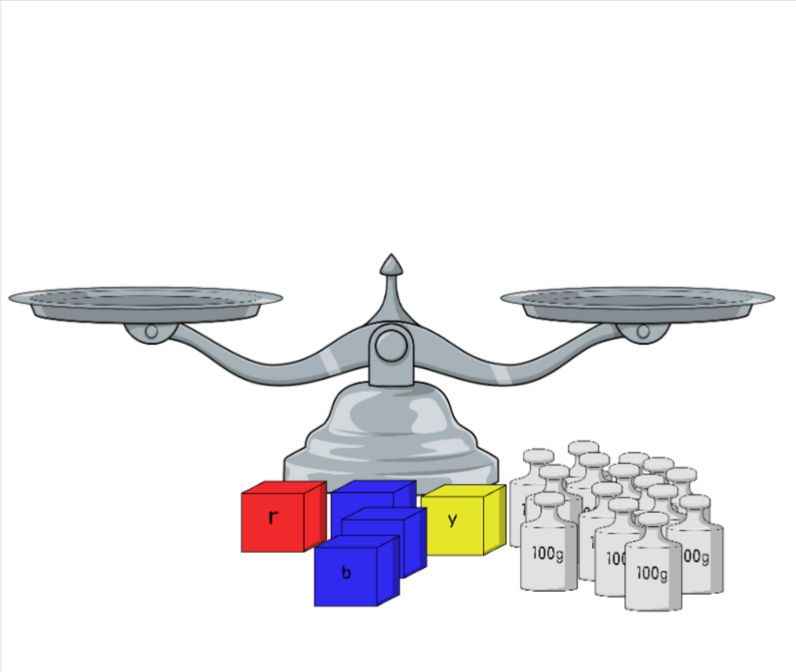
Gå sedan vidare.



Ekvationen är $2b + 200 = b + 300$.

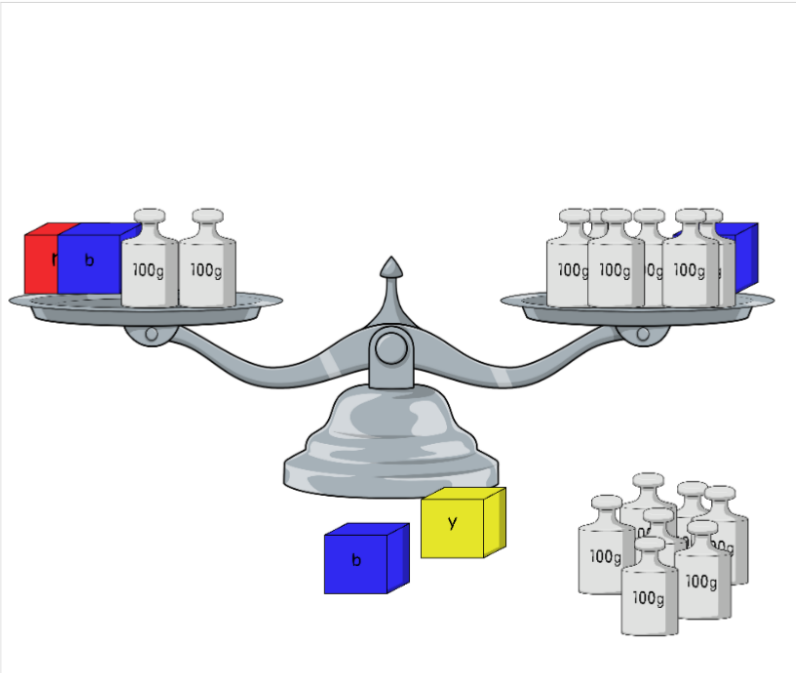
Få de blå kuberna ensamma på en sida.

Kan ni göra det utan att lägga till fler vikter?



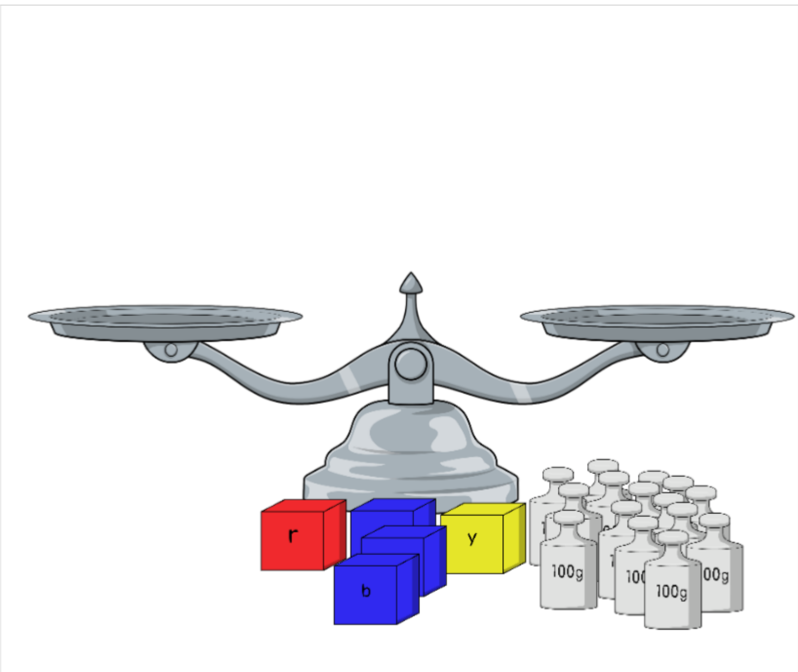
Lägg en röd kub, en blå kub och 200 g i ena vågskålen. I andra vågskålen lägger ni en blå kub och 600 g.

Skriv en ekvation för det ni lagt upp.



Hur många gram väger den röda kuben? Lös ekvationen.

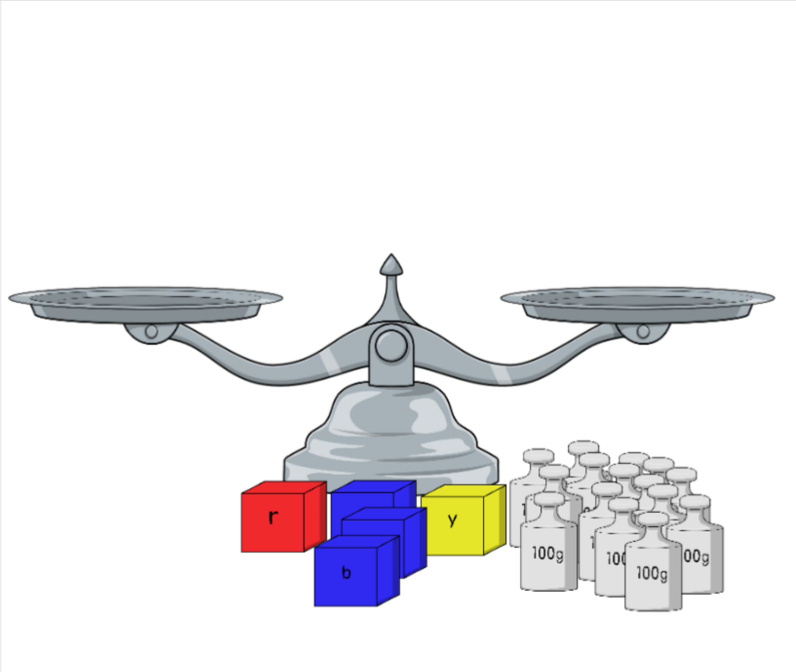
r =



Fotbollstränaren har en säck med okänt antal bollar. Dessutom finns det femhundra bollar i klubbhuset och trehundra i förrådet. Sjuhundra av dessa är för slitna och behöver slängas.

Det totala antalet bollar är dubbelt så många som det finns i säcken. Skriv upp en ekvation som motsvarar detta.

Om ni vill kan ni



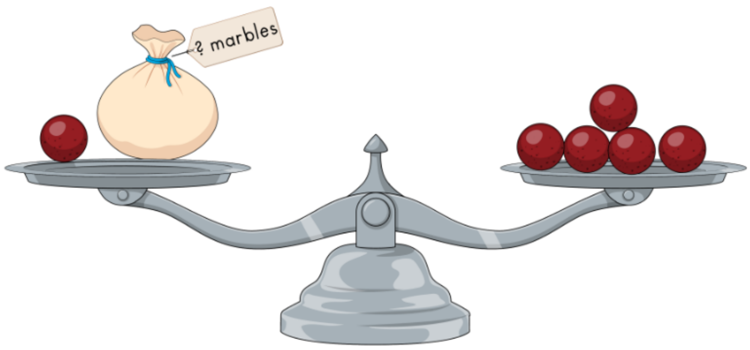
Ekvationen som motsvarar fotbollstränarens antal bollar är $b + 500 + 300 - 700 = 2b$.

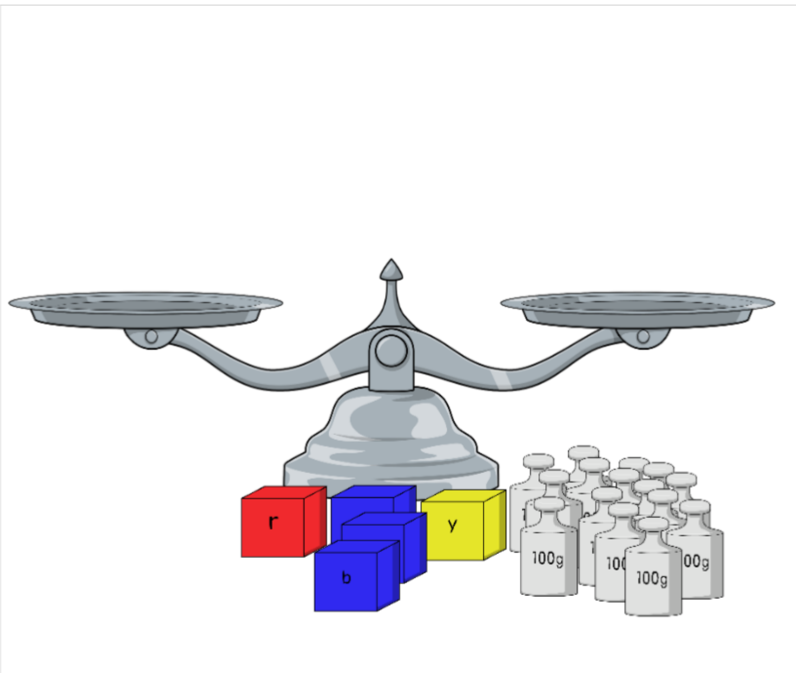
Lös ekvationen.

Ni kan använda vikterna och kuberna om ni vill.

Svar:

Gör en egen ekvation eller textuppgift som din klasskompis kan lösa med kuberna (eller arbeta vidare i matteboken).





Den här kan ni använda om ni ska lösa varandras textuppgifter.

Extra fråga: vad är $x - x$? Svar:

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