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Mirror symmetry at genus one of elliptic curves

using analytic torsion and the Kronecker limit formula

Master's thesis in Engineering Mathematics and Computational Science

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Department of Mathematical Sciences
Chalmers University of Technology
Gothenburg, Sweden 2020

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Master's Thesis 2020
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Typeset in LaTeX
Printed by the Department of Mathematical Sciences
Gothenburg, Sweden 2020

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Abstract

This master's thesis is concerned with mirror symmetry at genus one for elliptic curves. Mirror symmetry stems from string theory in physics and conjectures a relation between the symplectic (the A-model) and complex structures (the B-model) of a Calabi-Yau manifold and its mirror manifold. At genus one, the B-model calculation can be defined using analytic torsion introduced by Bershadsky, Cecotti, Ooguri and Vafa. For elliptic curves, this is calculated using the Kronecker limit formula, which is also derived in detail. The A-model is concerned with the generating series of genus one Gromov–Witten invariants which is also calculated for elliptic curves. Then the mirror symmetry correspondence is shown using the derivatives of the A- and B-model calculations.

Acknowledgements

I wish to express my gratitude to my supervisor Dennis Eriksson, who has enthusiastically guided me through this wide topic. He has helped me greatly in developing my understanding of mathematics, both in this subject and in general.

Magnus Fries, Gothenburg, Sweden 2020

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1
Introduction

The subject of mirror symmetry has been highly active in mathematical research for the past thirty years. It is originally a conjecture of string theory, but it took hold as a mathematics subject since it suggested surprising and novel relations of a specific type of Kähler manifold, namely so-called Calabi-Yau manifolds. The subject spans a great number of mathematical fields, and a wide variety of tools has been applied to it.

This text will deal with mirror symmetry at genus one when the Calabi-Yau manifold is one dimensional which can be described using an elliptic curve. The aim is to condense this case to a master's degree level and look at some details in these calculations that in other works might have been skipped over. With this as aim, extra focus will be given to some of the central objects, such as the elliptic curve.

1.1 Brief physics background and history

This section will give some background to the mirror symmetry conjecture and introduces some basic terminology.

Mirror symmetry originates from physics and the aim to find a unified theory of the fundamental forces. This led to the development of string theory where instead of seeing particles as "point-like", i.e. zero-dimensional objects, they are modeled as being "strings", i.e. one-dimensional objects. But for this theory to produce meaningful physics, the usual four dimensional space-time was not sufficient. Physicists were able to show that this theory could work given that six extra dimensions were added to space-time. That we perceive the universe as four-dimensional on a macroscopic scale is justified by the extra dimensions being "small", which mathematically should correspond to that these extra dimensions can be modeled as a compact manifold. Imposing some desired symmetries, physicists were able to show that these extra six dimensions could be endowed with a complex structure and some extra properties to make it a *Calabi-Yau manifold* with three complex dimensions, which will be called a *Calabi-Yau threefold*. Depending on the specific geometry of this Calabi-Yau threefold, the resulting physics would be different. A more elaborate physics background can be found in the introduction to [Jin18].

In addition to this, multiple models for string theory were developed. From a physics perspective it was motivated that two types of calculations would be interchanged when changing between these models and we will refer to these calculations as the A- and B-model calculations respectively. Hence, given a Calabi-Yau threefold there should also be another Calabi-Yau threefold which produces the same physics, in such a way that the A- and B-model calculations would agree. This pair of manifolds will be referred to as the *mirror pair*. From a mathematics perspective, this came as a big surprise since what these calculations depended on, the *complex moduli space* for the B-model and the *Kähler moduli space* for the A-model, do not have an apparent reason to be related in such way.

In the context of this text, this will be referred to as the *mirror symmetry conjecture*. A vague formulation can be stated as that for each Calabi-Yau manifold, there is a mirror manifold such that the A-model calculations and the B-model calculations are interchanged.

In addition to being interesting because of this somewhat "mysterious" mathematical connection, this would also have applications to some classical mathematical problems. In particular, the problem that initiated the interest in the mathematics community was a problem from enumerative geometry. One would like to count the number of *rational curves* on the quintic threefold

$$\{z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0\} \subseteq \mathbb{P}^4\mathbb{C},$$

which is information included in the A-model calculations. Here, rational *curves* have one complex dimension, and is hence topologically surfaces. Why counting such surfaces is included in a string theory model is motivated vaguely by that these are the types of surfaces that can be traced by a string traveling through space-time along with the extra six dimensions.

This mathematical problem remained unsolved for a long time, but in 1990 a group of mathematical physicists claimed to have produced a method for counting genus zero curves of any degree using the mirror symmetry conjecture and B-model calculation on a mirror manifold of the quintic threefold [COGP91]. This was much further progress than mathematical methods had been able to produce so far, but the result might not have gotten much attention by the mathematics community if not for a mathematics paper published the same year claiming to have calculated the same numbers for degrees up to three with the use of computer calculations. The physicist's series of numbers went

$$2875, 609250, 317206375, \ldots$$

whereas for the mathematicians, it went

$$2875, 609250, 2682549425.$$

This mismatch could have been regarded as a failure of the mirror symmetry conjecture, but instead, the mathematicians went over their code and found an error. When the code was corrected it produced the same number. This story is portrayed in a rather comedic way in the book on mathematical philosophy "The universe speaks in numbers: how modern math reveals nature's deepest secrets" [Far19], featuring quotes such as "To us, the physicists' methods seemed simply ridiculous" and "It was an embarrassment and a bit of a shock to [us] mathematicians that the string theorists' voodoo mathematics worked so well". Further reading regarding the background of mathematical mirror symmetry can be found in the introductions to [Mor96] and [CK99].

This was the initiation of mathematical mirror symmetry, and the subject has developed to not only include Calabi-Yau threefolds but Calabi-Yau manifolds in other dimensions as well. What will be explored in this text is the case when the Calabi-Yau manifold is one dimensional which can be described using elliptic curves.

1.2 Mathematical Mirror Symmetry

This section will describe the calculations in the A-model and B-model respectively. Mirror symmetry conjectures that these calculations are interchanged via the so-called *mirror mapping* between a mirror pair of Calabi-Yau manifolds. These definitions are not general but instead intend to give some context before we narrow the focus to the specific case of elliptic curves.

We first need to specify on which structures the A-model and B-model calculations depend. The B-model will depend on the complex structure of the Calabi-Yau manifold, as in the holomorphic atlas on the topological manifold. We parameterize the choice of such a complex structure to form the *complex moduli space*. The A-model will depend on some additional structure of the Calabi-Yau manifold, namely that we need to specify a *complexified Kähler class*. The choice of such a complexified Kähler class is similarly parameterized to form the *Kähler moduli space*.

For a Calabi-Yau manifold X with an associated complexified Kähler class ω the A-model calculations is then of the generating series of genus g Gromov–Witten invariants

$$F_g^A(X, \omega) = \sum_{\beta \in H_2(X,\mathbb{Z})} N_{g,\beta} q^\beta$$

where $q^\beta = e^{2\pi i \int_\beta \omega}$ and $N_{g,\beta}$ are the Gromov-Witten invariants. These invariants are essentially a weighted count of the number of genus g complex curves which are holomorphically mapped to a singular homology class β, up to isomorphism. This count is weighted by the number of automorphisms of each mapping. When $H_2(X, \mathbb{Z}) = \mathbb{Z}$ as in the case when X is an elliptic curve, one can see β as simply the degree of the mapping.

More precisely it is these Gromov-Witten invariants $N_{g,\beta}$ that contain the information of enumerative geometry, and supposedly they do not depend on the complex structure of X. The generating series F_g^A is also supposed to not depend on the complex structure, even though the Kähler moduli space is defined in relation to it.

If we let $(X, X^\vee)$ be a mirror pair of Calabi-Yau manifold, then the conjectured correspondence is that one should be able to calculate $F_g^A(X, \omega)$ from the B-model calculation of $\mathcal{F}_g^B(X^\vee)$ by mapping the complex moduli space of $X^\vee$ to the Kähler moduli space of X via the so-called *mirror mapping*.

The definition of $\mathcal{F}_g^B$ varies and this text will only deal with mirror symmetry *at genus one*, i.e. $g = 1$. In this case $\mathcal{F}_1^B$ can be defined with analytic torsion, introduced by Bershadsky, Cecotti, Ooguri and Vafa in [BCOV94], which can be rewritten as normalized products of ζ-regularized determinants of the Dolbeault Laplace operator $\Delta_{\bar\partial}^{p,q}$ acting on (p, q)-forms [EMM19]. In the case of an elliptic curve E, most factors vanish and we are left with

$$\mathcal{F}_1^B(E) = -\frac{1}{2} \log \det \Delta_{\bar\partial}^{1,1}.$$

The Dolbeault Laplace operator in this case can be realized as a normalization of the cartesian Laplace operator $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ acting on smooth periodic functions. Then the ζ-regularized determinant can be calculated using the Kronecker limit formula which is derived in detail in Chapter 5. This is somewhat of a novelty in this work since this does not occur in many other works.

Note that the B-model does not require that we specify a complexified Kähler class and hence does not depend on the Kähler moduli space. It is instead supposed to only depend on the complex moduli space. However, using the definition with analytic torsion, we do need to specify a Riemannian metric.

To complete the correspondence before mapping the complex moduli space of $X^\vee$ to the Kähler moduli space via the mirror mapping, one also needs to perform a so-called *holomorphic limit* of $\mathcal{F}_1^B(X^\vee)$ since this is not holomorphic whereas the generating series $F_1^A(X, \omega)$ is.

1.3 Calabi-Yau onefolds and their geometry

This text is not concerned with Calabi-Yau threefolds. Instead it is concerned with *Calabi-Yau onefolds*, meaning one complex dimension instead of three. These can be classified as compact Riemann surfaces of genus one. This means that any mirror pair in one dimension consists of two elliptic curves, which greatly simplifies this case. This is not true in higher dimensions where a mirror pair can consist of two topologically different Calabi-Yau manifolds. A short discussion of a property that a mirror pair of Calabi-Yau manifolds have, namely that they are supposed to have *mirrored Hodge diamonds*, is given in Section A.5 of the appendix. This property in one dimension is not very interesting so that discussion tries to give some small insight into higher dimension mirror symmetry.

One can describe a compact Riemann surface of genus one using an algebraic variety, specifically an elliptic curve over $\mathbb{C}$. This opens up tools from algebraic geometry and is discussed in Chapter 2. Another important description of the Calabi-Yau onefold that will be used through out the text is the Riemann surface description $\mathbb{C}/\Lambda$ where Λ is a lattice in the complex plane. This description is intimately connected to the complex moduli space on an elliptic curve described later in Section 3.1.

Chapter 3 introduces some central objects in mirror symmetry, namely the complex moduli space and the Kähler moduli space. These are central since it is these spaces that the interchanged calculations depend on respectively. One may note that these moduli spaces have structures as an orbifold and a manifold respectively, but how this structure is constructed will not be focused on.

The complex moduli space of a topological manifold can be described as the possible complex structures, meaning the choice of holomorphic atlase, that the manifold can be endowed with. The complex moduli space $\mathcal{M}_{1,1}$ for genus one curves with one fixed point, i.e. elliptic curves, is

$$\mathcal{M}_{1,1} = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$$

where $\mathrm{SL}(2,\mathbb{Z})$ is the so called the modular group and acts on the half-plane as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \text{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

This is proven and discussed in Section 3.1. The $\tau \in \mathbb{H}$ comes from the Riemann surface description $\mathbb{C}/\Lambda$ s.t. $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$.

Similarly, the Kähler moduli space of a complex manifold describes what *complexified Kähler classes* or symplectic structure that a complex manifold can be endowed with. The Kähler moduli space is hence a parametrization of the complexified Kähler cone $\mathcal{K}_{\mathbb{C}}(X)$ which is a subset of the de Rham cohomology $H^2_{\mathrm{dR}}(X,\mathbb{C})$. These objects are defined in Section 3.2 and it is shown that

$$\mathcal{K}_{\mathbb{C}}(E) \to \mathbb{H}, \quad \omega \mapsto \int_E \omega$$

is a bijection for an elliptic curve E. Hence we identify the Kähler moduli space with $\mathbb{H}$ where each point $t \in \mathbb{H}$ represents the unique choice of a complexified Kähler class ω s.t. $\int_E \omega = t$. We see here that the choice of parametrization of the complexified Kähler cone, i.e. the Kähler moduli space, is made such that it is independent of the complex moduli space.

The complex moduli space and the Kähler moduli space are seemingly very different objects, and this is why the mirror symmetry correspondence is mathematically surprising. What will eventually be done in Chapter 7 is to map the complex moduli space to the Kähler moduli space which then interchanges the B-model calculations with the A-model calculations. This mapping is called the *mirror map* and in the case of elliptic curves, this is simply $\tau \to t$ where τ is the parameter to for the complex moduli space and t for the Kähler moduli space. Hence a mirror pair of Calabi-Yau onefolds are

$$(E_\tau, \omega_t) \longleftrightarrow (E_t, \omega_\tau)$$

which will be looked at more in Section 3.3.

That one can produce such a mapping between seemingly very different objects is surprising, especially since the mirror map in higher dimensions can be more complicated. But for the case of elliptic curves, this is still surprising. On an elliptic curve, a Kähler form can be viewed as a volume form and hence a complexified Kähler class is some kind of volume measure. The $\mathrm{SL}(2,\mathbb{Z})$-relation $\tau \sim -\frac{1}{\tau}$ from the complex moduli applied on the Kähler moduli space would then interchange small volumes and big volumes. One would expect that this should drastically change the properties of the elliptic curve.

Chapter 3 also introduces the *moduli space of curve mappings* $\mathcal{M}_g(E,\beta)$ for some singular homology class $\beta \in H_2(E,\mathbb{Z})$. This moduli space consists of holomorphic mappings from curves of a genus g and curve class β to the elliptic curve E, up to isomorphism of mappings. As this text deals with mirror symmetry for $g = 1$, these curves can also be described with elliptic curves, i.e. $\mathcal{M}_1(E,\beta)$ consists of mappings between elliptic curves. This moduli space is used to define the Gromov-Witten invariants $N_{1,\beta}$ for $\beta \neq 0$ later in A-model, namely as the *orbifold Euler characteristic* of $\mathcal{M}_1(E,\beta)$. Similarly to the complex moduli space, $\mathcal{M}_1(E,\beta)$ can be described as an orbifold and entire Section 3.5 will be used to introduce a sufficient subcategory of orbifolds called *global quotients* and orbifold Euler characteristics of disjoint unions of such spaces. As it turns out, $\mathcal{M}_1(E,\beta)$ will consist of a finite number of points, and the orbifold Euler characteristic is then a weighted count of the number of curves.

1.4 Mirror symmetry at genus one of elliptic curves

Presented here are the resulting calculations of mirror symmetry at genus one for elliptic curves.

The generating series F_1^A of the A-model is calculated in Chapter 4. To do this we first calculate the Gromov-Witten invariants $N_{1,\beta}$ for an elliptic curve by counting covers using the *Galois correspondence of coverings*. The generating series will be expressed using the Dedekind Eta function

$$\eta(t) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n) \tag{1.1}$$

where $q = e^{2\pi i t}$, $q^{\frac{1}{24}} = e^{\frac{\pi}{12} i t}$ and $t \in \mathbb{H}$. This function will be reoccurring throughout this text.

Theorem A. *For a Calabi-Yau onefold E with associated complexified Kähler class $\omega_t \in \mathcal{K}_{\mathbb{C}}(E)$, the generating series at genus on is*

$$F_1^A(E, \omega_t) = -\log \eta(t)$$

where $t = \int_E \omega_t \in \mathbb{H}$ is the parameter of the Kähler moduli space, η is the Dedekind Eta function (1.1) *and* log *denotes the principal branch of the complex logarithm, i.e.* $\operatorname{Im}\log \in (-\pi, \pi)$.

Later in the text this is Theorem 4.2.

When defining the B-model calculation with analytic torsion one needs to specify a Riemannian metric to construct the Dolbeault Laplace operator $\Delta_{\bar{\partial}}$. We specify this Riemannian metric as the unique one s.t. the induced volume form ρ is the real-part of a $(1,1)$-form with f holomorphic for any local representation $f dz \wedge d\bar{z}$ aswell as requiring that $\int_E \rho = 2$. With the Riemann surface description $\mathbb{C}/\Lambda$, this Riemann metric given as

$$g_{ij} = \frac{2}{\operatorname{coVol}\Lambda} \mathbb{1}$$

where $\operatorname{coVol}\Lambda$ is the volume of a unit cell of the lattice Λ.

In Chapter 6 we calculate that

$$\mathcal{F}_1^B(E) = \frac{1}{2}\frac{d}{ds}\left((2\pi)^{-2s}E(s,\Lambda)\right)|_{s=0}$$

using the Riemann surface description $\mathbb{C}/\Lambda$, where $E(s,\Lambda)$ is the real analytic Eisenstein series, i.e. the analytic continuation of

$$E(s,\Lambda) = (\operatorname{coVol}\Lambda)^s \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} |\lambda|^{-2s} \quad \text{for} \quad \operatorname{Re}(s) > 1.$$

The expansion of $E(s,\Lambda)$ at $s = 0$, is exactly the Kronecker limit formula. Since this is central in the calculation, the entirety of Chapter 5 is devoted to a detailed derivation of the Kronecker limit formula. For a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ we let $\tau = \frac{\omega_2}{\omega_1}$, and by changing τ to $-\tau$ we can ensure that $\tau \in \mathbb{H}$. This τ is unique for the lattice Λ up to action of $\mathrm{SL}(2, Z)$ on $\mathbb{H}$.

Theorem (Kronecker limit formula)**.** *The Taylor expansion of the real analytic Eisenstein series $E(s,\Lambda)$ around $s = 0$ up to second order terms is*

$$E(s,\Lambda) = -1 - \log\left(\operatorname{Im}\tau(2\pi)^2|\eta(\tau)|^4\right)s + O(s^2)$$

where η is the Dedekind Eta function (1.1).

Later in the text this is Theorem 5.1

The result of the B-model calculation with previously specified Riemannian metric is given in the following theorem.

Theorem B. *The analytic torsion of Calabi-Yau onefold E_τ is*

$$\mathcal{F}_1^B(E_\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right)$$

where $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is the parameter of the complex moduli space and η is the Dedekind Eta function (1.1)*.*

Later in the text this is Theorem 6.1.

At the end of Chapter 6 in Section 6.5, an alternative definition of $\mathcal{F}_1^B$ is presented. There it is defined via the *holomorphic anomaly equation*. Imposing $\mathrm{SL}(2,\mathbb{Z})$-invariance coming from the complex moduli space and some boundary conditions on $\mathcal{F}_1^B$, the holomorphic anomaly equation gives the same result as the definition with analytic torsion. This does not require that we specify any Riemannian metric.

Theorem B'. *The holomorphic anomaly equation*

$$\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\mathcal{F}_1^B(\tau) = \frac{1}{8\operatorname{Im}\tau^2}$$

where $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ with boundary condition

$$\lim_{\tau\to i\infty} \mathcal{F}_1^B(\tau) - \left(-\frac{1}{2}\log\operatorname{Im}\tau + \frac{\pi}{6}\operatorname{Im}\tau\right) = 0$$

has a unique real-valued $\mathrm{SL}(2,\mathbb{Z})$-invariant solution

$$\mathcal{F}_1^B(\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right)$$

where η is the Dedekind Eta function (1.1)*.*

Later in the text this is Theorem 6.13.

This boundary condition implies that $\lim_{\tau\to i\infty}\mathcal{F}_1^B(\tau) = \infty$ and given only this we obtain the same answer up to addition of an unknown constant. To produce the correspondence later we will take the derivative of $\mathcal{F}_1^B$, so in this sense, this constant term does not matter.

The solution to the holomorphic anomaly equation then suggests an alternative proof of the Kronecker limit formula, which is explored in Section 6.6. This does indeed work as a proof, but it depends on a lot of calculations of the previous proof since the boundary condition is proven using the analytic continuation proven in Chapter 5. This proof is still interesting since it is fundamentally different in that it is based on a differential equation of the parameter τ in contrast to the proof in Chapter 5 where τ is kept constant. These derivations depend on some theory of modular forms, which connects back to the modular discriminant in Section 5.6 as well as elliptic functions and the Weierstrass equation in Section 2.3.

This text ends with Chapter 7 where the correspondence between the A-model calculations and the B-model calculations are looked at. Before we use the mirror mapping $(E_\tau, \omega_t) \longleftrightarrow (E_t, \omega_\tau)$, we are required to introduce a *holomorphic limit* since F_1^A is holomorphic while $\mathcal{F}_1^B$ is not. What is meant with a holomorphic limit is somewhat unclear, since how it is proposed in most works only produce the correspondence "up to an infinite constant" in this case. This is not satisfactory, so this text proposes that the correspondence is instead produced via the holomorphic limit of the derivatives of F_1^A and $\mathcal{F}_1^B$, i.e.

$$\lim_{\bar{\tau} \to -i\infty} \partial_\tau \mathcal{F}_1^B(E_\tau) = \partial_\tau F_1^A(E_t, \omega_\tau)$$

which indeed works in the case of elliptic curves. That one should consider the derivatives in the genus one case is also suggested in [Dij95].

2

Calabi-Yau onefolds and elliptic curves

This chapter focuses on one of the central objects of mirror symmetry, the Calabi-Yau manifold. This will mostly be concerned with the one-dimensional case. One dimensional Calabi-Yau manifolds, or Calabi-Yau *onefold*, can be classified as a compact Riemann surface of genus one, which will be shown here. We also introduce the realization as $\mathbb{C}/\Lambda$ where Λ is a complex lattice. Topologically this is a torus. Note here that one dimensional means one complex dimension, and therefore topologically a surface.

It is important to note that with this classification of Calabi-Yau onefolds, a mirror pair in one dimension has to consist of two compact Riemann surfaces of genus one, i.e. two manifolds that are topologically isomorphic. This is not true in higher dimensions, which makes the one-dimensional case substantially easier since one does not need to construct or find a mirror manifold. In Section A.5 of the appendix, a short discussion of a property that a mirror pair of Calabi-Yau manifolds have, namely that they are supposed to have *mirrored Hodge diamonds*. This property in one dimension is not very interesting so that discussion will try to give some small insight into higher dimension mirror symmetry.

A very useful tool when working with Riemann surfaces of genus one is to describe them using elliptic curves, which opens up many tools from algebraic geometry. Why this deception is possible will also be looked at in this chapter.

Although the definition varies, we will define a Calabi-Yau manifold as a compact connected Kähler manifold with *trivial canonical divisor class*. A Kähler manifold is a complex manifold on which a Kähler form can be defined, which will be looked at together with the Kähler moduli space in Section 3.2, and Definition 2.4 in the next section will define what trivial canonical divisor class means in the one-dimensional case.

2.1 Canonical divisors and Riemann-Roch Theorem

This section aims to introduce the canonical divisor class in the one-dimensional case used to define a Calabi-Yau manifold, as well as the Riemann-Roch Theorem which also depends on the definition of a canonical divisor.

These definitions and theorems will be given and used in both the category of compact Riemann surfaces and the category of smooth projective curves. One will note that these definitions and theorems are very similar for both categories, a fact that is in a more general setting referred to as *GAGA*, short for *Géometrie Algébrique et Géométrie Analytique*. It states that for a complex projective manifold most of the definitions in the categories of algebraic geometry and complex analysis coincide, a correspondence first explained by Serre in [Ser56].

We start with some basic definitions and notation.

Definition 2.1 (Riemann surface)**.** A Riemann surface is a connected complex manifold of complex dimension one.

That is, a connected second countable Hausdorff space C along with a collection of *charts* $\{(U_i, \varphi_i)\}$ called an *atlas*, where $\{U_i\}$ is an open cover of C, $\varphi_i : U_i \xrightarrow{\simeq} V_i \subseteq \mathbb{C}$ are homeomorphisms from U_i to some open $V_i \subseteq \mathbb{C}$ and the *transition functions*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are holomorphic functions between open subsets of $\mathbb{C}$.

For any point $p \in C$ we say that a chart at p is any chart $\varphi : U \to \mathbb{C}$ such that $p \in U$.

Definition 2.2. Let C be a Riemann surface.

Define a holomorphic function $f \in \mathcal{O}(C)$ on C as a function $f : C \to \mathbb{C}$ such that for any chart $\varphi : U \to \mathbb{C}$, $f \circ \varphi^{-1} : U \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic.

Define a meromorphic function $f \in \mathbb{C}(C)$ on C as a pair (g, h) of holomorphic functions $g, h \in \mathcal{O}(C)$, also denoted g/h. As a function f is seen as

$$f : \{\, p \in C \mid h(p) \neq 0 \,\} \subseteq C \to \mathbb{C}, \quad f(p) = \frac{g(p)}{h(p)}.$$

For a holomorphic function $f \in \mathcal{O}(C)$, let the order of f at a point $P \in C$, denoted $\mathrm{ord}_P(f)$, be the exponent of the lowest term in the power series expansion of $f \circ \varphi^{-1}$ at $\varphi(P)$ for any chart φ at P. This is well-defined since for two charts φ, ϕ at P the function $\varphi \circ \phi^{-1}(z) - \varphi(P)$ is zero with non-zero derivative at $\phi(P)$, using that it is bijective and holomorphic.

Extend this definition to a meromorphic function $g/h \in \mathbb{C}(C)$ as

$$\mathrm{ord}_P(g/h) = \mathrm{ord}_P(g) - \mathrm{ord}_P(h) \in \mathbb{Z}.$$

Definition 2.3 (Smooth projective curve)**.** Define a smooth projective curve C over an algebraically closed field k as a connected smooth projective variety with dimension one, i.e. transcendence degree of $k(C)$ over k is one.

If $C \subseteq \mathbb{P}^n$, denote the $k[C] := k[\mathbb{P}^n]/I(C)$ where $k[\mathbb{P}^n]$ denotes homogeneous polynomials on $\mathbb{P}^n$ and $I(C) := \{\, f \in k[\mathbb{P}^n] \mid f(P) = 0 \ \forall P \in C \,\}$. One may note that the assumption that these curves are connected implies that $I(C)$ is a prime ideal.

Let $k(C)$ be the field of fractions of $k[C]$, which will be called the *function field*, or meromorphic functions if $k = \mathbb{C}$.

Let $M_P := \{ f \in k[C] \mid f(P) = 0 \}$ which is a maximal ideal and let $k[C]_P$ be the localization of $k[C]$ at M_P. This is a discrete valuation ring, i.e. an integral domain s.t. for any $f \in k(C), f \notin k[C]_P \implies f^{-1} \in k[C]_P$ [Sil09, Proposition 1.1, s. 17] and we define the order of a function $f \in k[C]_P$ at P as the valuation of $k[C]_P$

$$\mathrm{ord}_P : k[C]_p \to \{0, 1, 2, \dots\} \cup \{\infty\}, \ \mathrm{ord}_P(f) = \sup\{d = 0, 1, 2, \dots \mid f \in M_P^d\}.$$

We extend this notion to any $f/g \in k(C)$ as

$$\mathrm{ord}_P(f/g) = \mathrm{ord}_P(f) - \mathrm{ord}_P(g) \in \mathbb{Z}.$$

Call a function $t \in k(C)$ a *uniformizer* at P is $\mathrm{ord}_P(t) = 1$.

Definition 2.4 (Divisors and canonical divisor class)**.** For a compact Riemann surface or smooth projective curve C, the divisor group $\mathrm{Div}(C)$ is the free abelian group generated by the points of C, i.e. a divisor $D \in \mathrm{Div}(C)$ is the formal sum

$$D = \sum_{P \in C} n_P(P)$$

where all but finitely many of n_P are zero. The degree of a divisor D is defined as

$$\deg D = \sum_{P \in C} n_P.$$

For a meromorphic function $f \in \mathbb{C}(C)$, let the divisor associated with f be

$$\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f)(P).$$

Note that $\deg \mathrm{div} f = 0$ for any $f \in \mathbb{C}(C)$, which is proved directly for a compact Riemann surface case of genus one later in Lemma 2.10, and is true in general as seen in [Ful95, Corollary 19.5, p. 267] or [Sil09, Proposition 3.1 (b), p. 28].

Let ω be a meromorphic 1-form (see Appendix B.4 for definition in respective category) on C be given locally at a point P as

$$f \ dz$$

where $f \in \mathbb{C}(C)$ and in the category of Riemann surfaces dz is the basis element of $\Omega^{1,0}(C)$ and in the category of smooth projective curves z is a uniformizer at P. For such a ω, define the order at $P \in M$ as

$$\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f).$$

Let the divisor associated with a meromorphic differential 1-form ω be

$$\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega)(P).$$

We call the divisor $\mathrm{div}(\omega)$ associated to any non-zero meromorphic differential 1-form ω a *canonical divisor.*

For any two non-zero meromophic 1-forms ω_1, ω_2 on C there exists an $f \in \mathbb{C}(C)$ such that

$$\omega_1 = f\omega_2$$

(for a smooth projective curve see [Sil09, Remark 4.4, p. 32]) and therefore

$$\mathrm{div}(\omega_1) = \mathrm{div}(\omega_2) + \mathrm{div}(f).$$

We let this relation define an equivalence relation on the divisor group, i.e. for any $D_1, D_2 \in \mathrm{Div}(C)$

$$D_1 \sim D_2 \text{ if there is an } f \in \mathbb{C}(C) \text{ s.t. } D_1 = D_2 + \mathrm{div}(f).$$

Then all divisors of meromorphic 1-forms are contained in the same equivalence class, which we call the *canonical divisor class.* Hence, that C has *trivial canonical divisor class* means that there exists a non-zero meromorphic differential 1-form ω such that $\mathrm{div}(\omega) = 0$, i.e. ω is non-zero and holomorphic.

Definition 2.5. A divisor $D = \sum_{P \in C} n_P(P)$ is said to be *effective* if $n_P \geq 0$ for all $P \in C$, which is denoted $D \geq 0$. Let

$$\mathcal{L}(D) = \{f \in \mathbb{C}(C) \mid \mathrm{div}(f) + D \geq 0\}$$

which can be shown to be a finite dimensional vector space over $\mathbb{C}$ ([Ful95, Lemma 21.1 (c), p. 296] or [Sil09, Proposition 5.2, p. 34]).

Theorem 2.6 (Riemann-Roch Theorem)**.** *[Ful95, p.304] or [Sil09, p. 35] For a compact Riemann surface or a smooth projectiv curve C, there is a unique g s.t. for any canonical divisor K and any divisor $D \in \mathrm{Div}(C)$*

$$\dim \mathcal{L}(D) - \dim \mathcal{L}(K - D) = \deg D - g + 1.$$

Note that g is an invariant under isomorphism of C. It also coincides with the topological genus for orientable closed surfaces, i.e. g such that $\chi(C) = 2 - 2g$ where χ is the usual Euler characteristic.

2.2 Riemann surface description as $\mathbb{C}/\Lambda$

This section will show that a Calabi-Yau onefold can be classified as a compact Riemann surface of genus one. Then we will introduce the Riemann surface description $\mathbb{C}/\Lambda$ where Λ is a lattice in the complex plane, which then will be used throughout the text.

Theorem 2.7. *A Calabi-Yau onefold is a compact Riemann surface of genus one.*

Proof. Note that a Calabi-Yau onefold is a compact topological surface without boundary. All closed surfaces are topologically classified by *genus* and weather it is *orientable*, which is commonly shown in algebraic topology textbooks such as [Arm83, Chapter 7] or [Ful95, Chapter 17b].

In local coordinates, $\mathbb{C}$ induces a natural orientation that is preserved when changing coordinates since the transition functions are holomorphic. Hence any Riemann surface is orientable. A proof that any complex manifold is orientable is given in Section B.5 of the appendix.

That the genus is one for a Calabi-Yau onefold comes from the fact that it has trivial canonical divisor class. Then by letting $D = 0$ and $K = 0$ in Riemann-Roch Theorem (Theorem 2.6) we obtain that $g = 1$. □

Note that this classification does not mention that a Calabi-Yau onefold needs to be a Kähler manifold. Section 3.2 will define what a Kähler form is and construct such for compact Riemann surfaces of genus one. Hence any compact Riemann surface of genus one is a Calabi-Yau onefold.

To describe a compact Riemann surface of genus one we will use

$$\mathbb{C}/\Lambda \quad \text{where} \quad \Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$$

is a lattice in the complex-plane. We will assume that $\operatorname{Im}\tau > 0$ where $\tau = \frac{\omega_2}{\omega_1}$ which is looked at closely when examining the complex moduli space in Section 3.1. We will also denote $\operatorname{coVol}\Lambda = \operatorname{Im}\tau|\omega_1|^2$ as the volume of a unit cell of the lattice Λ.

That this is a Riemann surface is seen since the projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is a cover, so taking any small connected neighborhood U of a point p this gives us a chart $\varphi : U \to \mathbb{C}$ which is homeomorphic to one connected component of $\pi^{-1}(U)$. The transition functions are then all the identity or a translation by an element $\lambda \in \Lambda$.

To see that $\mathbb{C}/\Lambda$ has genus one, we see $\mathbb{C}/\Lambda$ as a parallelogram with identified edges, which is topologically a torus.

An important note regarding $\mathbb{C}/\Lambda$ is that the local 1-form dz can be seen as a global 1-form since the transition functions are all the identity or a translation. That is, we can glue them together as (with notation defined in Section B.2 of the appendix)

$$\left.\frac{\partial}{\partial z}\right|_{p,\varphi} = \left.\frac{\partial}{\partial z}\right|_{p,\phi} \quad \text{and then similarly} \quad dz|_{p,\varphi} = dz|_{p,\phi}$$

for any charts φ and ϕ at a point $p \in \mathbb{C}/\Lambda$. This would then also show that $\mathbb{C}/\Lambda$ is a Calabi-Yau manifold, since

$$C\ dz$$

is a holomorphic 1-form on $\mathbb{C}/\Lambda$ for any constant C. These are indeed the only holomorphic 1-forms since the constant functions are the only holomorphic functions on $\mathbb{C}/\Lambda$ which will be shown in Lemma 2.9.

Meromorphic functions on $\mathbb{C}/\Lambda$ are lifted to a doubly periodic meromorphic function on $\mathbb{C}$, which are also called *elliptic function*. We will now introduce some theory regarding elliptic functions that will be used when describing $\mathbb{C}/\Lambda$ as an elliptic curve.

Definition 2.8. An elliptic function is a meromorphic function f on $\mathbb{C}$ which is periodic in two directions, i.e. there is a lattice Λ in $\mathbb{C}$ s.t. $f(z+\lambda) = f(z)\ \forall \lambda \in \Lambda$. We say that f is an elliptic function relative to Λ. Such an f can also be seen as a meromorphic function on $\mathbb{C}/\Lambda$.

Lemma 2.9. *A holomorphic elliptic function, i.e. an elliptic function with no poles, is constant. Similarly, an elliptic function with no zeros is constant.*

Proof. Let f be an holomorphic elliptic function relative $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Let $D = \{\, t_1\omega_1 + t_2\omega_2 \in \mathbb{C} \mid t_1, t_2 \in [0,1] \,\}$ be the compact unit cell of the lattice. Since f is continuous it attains its maximum on D. But since f is periodic, $f(\mathbb{C}) \subseteq f(D)$, so f is bounded on $\mathbb{C}$. Liouville's theorem states that a bounded holomorphic function is constant, so f is constant.

If f is an elliptic function without zeros, then $1/f$ is holomorphic, and therefore constant. □

Lemma 2.10. *For an elliptic function f relative Λ*

$$\sum_{w \in \mathbb{C}/\Lambda} \operatorname{ord}_w f = 0$$

where $\operatorname{ord}_w f$ is the exponent in the lowest term in the Laurent expansion at w.

Proof. For an elliptic function g relative $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, choose a translated compact unit cell $D = \{z_0 + t_1\omega_1 + t_2\omega_2 \in \mathbb{C} \mid z_0 \in \mathbb{C},\ t_1, t_2 \in [0,1]\}$ of the lattice such that g has no poles on ∂D. Then the residue theorem states that

$$\sum_{w \in \mathbb{C}/\Lambda} \operatorname{res}_w g = \frac{1}{2\pi i} \int_{\partial D} g(z)dz$$

but since g is periodic, each edge of D is canceled by the opposite edge in the integral and we get $\sum_{w\in\mathbb{C}/\Lambda} \operatorname{res}_w g = 0$.

For an elliptic function f we apply this to f'/f which is also an elliptic function. This gives the desired result since $\operatorname{res}_w \left(\frac{f'}{f}\right) = \operatorname{ord}_w f$, which is seen by taking Laurent expansions at w as

$$\frac{f'(z)}{f(z)} = \frac{\sum_{n=n_0} a_n n z^{n-1}}{\sum_{n=n_0} a_n z^n} = \frac{n_0 z^{-1} + O(1)}{1 + O(z)}$$

where $n_0 = \operatorname{ord}_w f$. □

Remark 2.11. Lemma 2.10 is similar to the property of smooth projective curves in algebraic geometry that $\deg \operatorname{div} f = 0$ for any rational function f [Sil09, Proposition 3.1 (b), p. 28].

Definition 2.12. Let the Weierstrass elliptic function be

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Lemma 2.13. *The Weierstrass elliptic function is an elliptic function, i.e. doubly periodic and meromorphic.*

Proof. First we show that the sum is absolutely convergent. Fix $z \in \mathbb{C}$. For λ such that $|\lambda|$ is sufficiently large, $\left|\frac{z}{\lambda}\right| < 1$ and then

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = 2\frac{z}{\lambda^3} + O\left(\frac{z^2}{\lambda^4}\right)$$

where Taylor expansion was used. Then we can choose an big enough $M > 0$ such that

$$\sum_{\substack{\lambda\in\Lambda \\ |\lambda|>M}} \left| \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right| < \sum_{\substack{\lambda\in\Lambda \\ |\lambda|>M}} A|\lambda|^{-3}$$

which can be shown to converge. This is done in a similar case later in the proof of Lemma 5.3.

Since every term is meromorphic, $\wp(z,\Lambda)$ is meromorphic.

The periodicity is not obvious here since we are not necessarily allowed to split up the terms in the sum. To solve this we look at the derivative, and since the sum is absolutely convergent this is

$$\wp'(z,\Lambda) = \sum_{\lambda\in\Lambda} \frac{-2}{(z-\lambda)^3}.$$

By similar argument this sum is also absolutely convergent, and by changing the order of summation, $\wp'(z,\Lambda)$ is obviously periodic over the lattice. Now look at

$$\wp\left(z + \frac{\lambda}{2}, \Lambda\right) - \wp\left(z - \frac{\lambda}{2}, \Lambda\right)$$

for any fixed $\lambda \in \Lambda$. Since the derivative of the above difference is zero, it is constant. Applying this to $-z$ and using that $\wp(z,\Lambda)$ is clearly even, we obtain that the difference is zero and we are done. □

The following lemma classifies base-point preserving holomorphic mappings between two Riemann surfaces $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$. This will be used when constructing the complex moduli space of an elliptic curve in Section 3.1.

Lemma 2.14. *Any holomorphic map* $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ *conserving the base-point* 0 *is of the form* $\phi(z) = az$ *where* $a \in \mathbb{C}$ *such that* $a\Lambda \subseteq \Lambda'$.

Proof. Let $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be holomorphic such that $0 + \Lambda \mapsto 0 + \Lambda'$. We can lift this map holomorphically (since this is a local property) as

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widetilde{\phi}} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{C}/\Lambda' & \xrightarrow{\phi} & \mathbb{C}/\Lambda \end{array}$$

s.t. $\widetilde{\phi}(\Lambda) = \Lambda'$ and $\widetilde{\phi}(z+\lambda) - \widetilde{\phi}(z) \in \Lambda'$ for any $\lambda \in \Lambda$. Differentiating the last identity we get $\widetilde{\phi}'(z+\lambda) = \widetilde{\phi}'(z)$, so $\widetilde{\phi}'$ is an holomorphic elliptic function, which by Lemma 2.9 is constant. Let $\widetilde{\phi}'(z) = a$, hence we obtain that $\widetilde{\phi}(z) = az$ and $\phi(z) = az$. Furthermore, $\widetilde{\phi}(\lambda) - \widetilde{\phi}(0) = a\lambda \in \Lambda'$ for any Λ which shows that $a\Lambda \subseteq \Lambda'$.

□

2.3 Description as an elliptic curve

This section will explain why a Calabi-Yau onefold can be represented as an elliptic curve, i.e. a one-dimensional variety of genus one with a fixed base-point (see Definition 2.15). This will be done using the representation as a torus $\mathbb{C}/\Lambda$, where Λ is a lattice in the complex plane.

The object $\mathbb{C}/\Lambda$ is a compact Riemann surface, while the elliptic curve is an algebraic variety. What will be shown here is that there is an object which is both a variety and a Riemann surface, and this object is isomorphic to both a torus $\mathbb{C}/\Lambda$ and an elliptic curve in each respective category. This object is described by the Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ in $\mathbb{P}^2$.

Note that when writing the polynomial $y^2 = 4x^3 - g_2x - g_3$ and saying that it defines a variety in $\mathbb{P}^2$, we mean the variety given by homogenization of this polynomial, i.e.

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3.$$

Definition 2.15. An elliptic curve (E, P_0) over an algebraically closed field k is a one-dimensional smooth connected projective variety E of genus one together with a base-point $P_0 \in E$. Here one-dimensional means that the transcendence degree of $k(E)$ over k is one, and genus one means that $g = 1$ in Riemann-Roch Theorem (Theorem 2.6).

An important note is that when using the term elliptic curve one usually defines a base-point, which is not necessarily the case when using a manifold description.

Theorem 2.16. *A projective variety over an algebraically closed field k given by a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ in $\mathbb{P}^2$ is an elliptic curve under the condition that the polynomial discriminant $16(g_2^3 - 27g_3^2)$ is non-zero.*

The specified base-point is set to be $[0, 1, 0]$, which we will call the point at infinity.

Proof. Denote the projective variety C.

To check that C is smooth, first look for singular points in the affine space of $\mathbb{P}^2$ such that $Z \neq 0$. Setting the tangent to the equation to zero we get that $y = 0$ and $x^2 = \frac{g_2}{12}$. This point is only in C if $g_2^3 - 27g_3^2 = 0$, hence by assumption this is not a point on C. The point at infinity, i.e. $Z = X = 0$ and $Y = 1$, can similarly be shown to not be singular.

That C is one-dimensional and connected comes from that it is given by a single polynomial $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$ which is irreducible. Hence C is a smooth projective curve.

To see that C has genus one we note that

$$dx/y$$

[Sil09, Example II.4.6, p. 33] is a holomorphic differential 1-form. Hence, C has trivial canonical divisor class and by Riemann-Roch Theorem C has genus one.

□

Theorem 2.17. *An elliptic curve over a algebraically closed field k of characteristic zero is isomorphic to a variety given by a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ in $\mathbb{P}^2$.*

Proof. Let (E, P_0) be an elliptic curve. By Riemann-Roch Theorem (see Theorm 2.6), $\dim \mathcal{L}(D) = \deg D$ for any divisor D since $g = 1$ and we can let the canonical divisor be zero. So looking at $\mathcal{L}(n(P_0))$ we see that $\dim \mathcal{L}(n(P_0)) = n$.

Since $\mathcal{L}((P_0))$ is one dimensional, it is spanned by scalars. $\mathcal{L}(2(P_0))$ is two dimensional so will be spanned by the scalars together with a function $x \in k(E)$ which has a double pole in P_0. Same reasoning we get $y \in k(E)$ with a triple pole in P_0 s.t. $\mathcal{L}(3(P_0)) = \operatorname{span}(1, x, y)$. Now using that $\operatorname{ord}_{P_0}(x^2) = -4$, $\operatorname{ord}_{P_0}(xy) = -5$ and so on we obtain the following:

$$\begin{aligned} 1, x, y, x^2 &\in \mathcal{L}(4(P_0)) \\ 1, x, y, x^2, xy &\in \mathcal{L}(5(P_0)) \\ 1, x, y, x^2, xy, x^3, y^2 &\in \mathcal{L}(6(P_0)) \end{aligned}$$

But since $1, x, y, x^2, xy, x^3, y^2$ are 7 vectors while $\dim \mathcal{L}(6(P_0)) = 6$, they have to be linearly dependent over k. Since x^3 and y^2 are the only ones with a pole of degree 6 in P_0, their coefficients are either both non-zero or both zero. But they can't be both zero since then $\dim \mathcal{L}(5(P_0)) < 5$. Hence

$$y^2 + a_1yx + a_3y = a_0x^3 + a_2x^2 + a_4x + a_6$$

for some $a_i \in k$ and $a_0 \neq 0$. With the assumption that there the field is algebraically closed of characteristic zero we can complete squares and change variables simplify the expression to

$$y^2 = 4x^3 - g_2 x - g_3$$

for some $g_i \in k$, where the factor 4 is convention. From this, we can construct a mapping $\phi : E \to \mathbb{P}^2$ as

$$\phi(Q) = [x(Q), y(Q), 1] \quad \text{and} \quad \phi(P_0) = [0, 1, 0]$$

where P_0 is mapped to $[0, 1, 0]$ since y has a pole of higher order than x in P_0.

By defining a variety from the Weierstrass equation as

$$C : y^2 = 4x^3 - g_2 x - g_3 \subseteq \mathbb{P}^2$$

we see that the image of ϕ lies in C, so the map can be written as $\phi : E \to C$ instead.

This can be shown to be a surjective morphism. To show that this has degree one, one shows that $k(E) = k(x, y)$ by noting that $[k(E) : k(x, y)]$ divides both $[k(E) : k(x)] = 2$ and $[k(E) : k(y)] = 3$, and therefore $[k(E) : k(x, y)] = 1$. Then one proves that C is not singular since otherwise there is a degree one mapping to $\mathbb{P}^1$, which under composition with ϕ is a degree one mapping between smooth projective curves, E and $\mathbb{P}^1$, and therefore an isomorphism. This is a contradiction since E has genus 1 and $\mathbb{P}^1$ has genus 0. So ϕ is a degree one morphism of smooth projective curves, therefore it is an isomorphism. The details of these last steps showing that this is an isomorphism are more clearly explained in [Sil09, Proposition III.3.1a, p. 59]. □

Many of the arguments in the previous two theorems may also be used in the category of compact Riemann surfaces, such as the construction of the map from an elliptic curve to a the set given by a Weierstrass equation. Proving such theorem in the category of compact Riemann surfaces would then show that any compact Riemann surface can be given as a Weierstrass equation.

The description we will use the most in this text is the Riemann surface description $\mathbb{C}/\Lambda$. Therefore we will now prove that any $\mathbb{C}/\Lambda$ can also be descried using a Weierstrass equation. Thereafter we will prove that any Riemann surface given by a Weierstrass equation can be given a Riemann surface description $\mathbb{C}/\Lambda$, given that the polynomial discriminant $16(g_2^3 - 27g_3^2)$ is non-zero. Together with the unproven statement that previous two theorems can be proven in the category of compact Riemann surfaces, this would also show that any Riemann surface can be described as $\mathbb{C}/\Lambda$.

Theorem 2.18. *The Riemann surface $\mathbb{C}/\Lambda$ is isomorphic to the Riemann surface inscribed in $\mathbb{P}^2$ as the Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ for some or some g_1, g_2 as Riemann surfaces.*

Proof. This will be done using Weierstrass's elliptic function and its derivative,

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \quad \text{and} \quad \wp'(z,\Lambda) = \sum_{\lambda\in\Lambda} \frac{-2}{(z-\lambda)^3}.$$

As shown in Lemma 2.13 these are elliptic functions, so they can be seen as meromorphic function on $\mathbb{C}/\Lambda$. Furthermore, $\wp$ and $\wp'$ has a poles in $z = 0+\Lambda$ of degree two and three respectively.

Similar to the rational functions x and y from previous proof, it is with these functions we will define a isomorphism from $\mathbb{C}/\Lambda$ to $C = \{y^2 = 4x^3 - g_2 x - g_3\}$.

We calculate Laurent expansions at $z = 0$ as

$$\begin{aligned} \wp(z,\Lambda) &= \frac{1}{z^2} + O(z) \\ (\wp(z,\Lambda))^3 &= \frac{1}{z^6} + \frac{9}{z^2}G_4 + 15G_6 + O(z) \\ (\wp'(z,\Lambda))^2 &= \frac{4}{z^6} - \frac{24}{z^2}G_4 - 80G_6 + O(z) \end{aligned}$$

where $G_{2k} = \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \frac{1}{\lambda^{2k}}$ is Eisenstein series of weight $2k$, which will be defined properly in Definition 5.12. Note that $G_n = 0$ if n is odd, which is used to derive the expansions.

Now combining the expansions we get

$$(\wp'(z,\Lambda))^2 - 4(\wp(z,\Lambda))^3 + 60G_4\wp(z,\Lambda) + 140G_6 = O(z)$$

is holomorphic around $z = 0$ with convergence radius stretching to nearest non-zero lattice point. But this expansion can be done similarly at any lattice point, so it is entire. So it is a doubly periodic holomorphic function, then by Liouville's theorem it is constant as proven in Lemma 2.9, that is $O(z) = 0$. So by letting $g_2 = 60G_4$ and $g_3 = 140G_6$, we get

$$(\wp'(z,\Lambda))^2 = 4(\wp(z,\Lambda))^3 - g_2\wp(z,\Lambda) - g_3 \tag{2.1}$$

which is a Weierstrass equation.

Now we construct the map $\phi : \mathbb{C}/\Lambda \to \mathbb{P}^2$ by

$$\phi(z) = [\wp(z,\Lambda), \wp'(z,\Lambda), 1] \quad \text{and} \quad \phi(0) = [0,1,0].$$

By defining a Riemann surface inscribed in $\mathbb{P}^2$ from the Weierstrass equation as

$$C : y^2 = 4x^3 - g_2 x - g_3 \subseteq \mathbb{P}^2$$

we see from (2.1) that the image of ϕ lies in C and the map can be written $\phi : \mathbb{C}/\Lambda \to C$ instead.

To show that this is surjective, let $(x,y) \in C$. We want to find $a \in \mathbb{C}$ such that $\wp(a) = x$ and $\wp'(a) = y$. First note that $\wp(z) - x$ is a non constant elliptic function, so by Lemma 2.9 it has a zero which we call $a \in C$. With the equations $\wp(a) = x$, $y^2 = 4x^3 - g_2x - g_3$ and (2.1), we get $y^2 = (\wp'(a))^2$. Since $\wp$ is even while $\wp'$ is odd, we can change a to $-a$ if necessary such $y = \wp'(a)$, and we have then shown that $\phi(a) = (x, y)$.

To show that the mapping is injective, let $\phi(z_1) = \phi(z_2)$ for some $z_1, z_2 \in \mathbb{C}$. Then $\wp(z) - \wp(z_1)$ is zero at z_1, z_2 and $-z_1$, where is is zero at $-z_1$ since $\wp$ is even. Moreover, since $\wp(z)-\wp(z_1)$ has a single pole of degree two, Lemma 2.10 states it either has one zero of degree two, or two zeros of degree one. If there is one zero, then $z_1 = z_2$ in $\mathbb{C}/\Lambda$ and we are done. Otherwise, two of $z_1, z_2, -z_1$ are equal in $\mathbb{C}/\Lambda$ and the third is necessarily distinct. If $z_1 = -z_1$ in $\mathbb{C}/\Lambda$ we obtain that $\wp'(z_1) = 0$ since $\wp'$ is odd. But then $\wp(z) - \wp(z_1)$ and it's derivative is zero at z_1, so then z_1 would be a zero of degree two which is a contradiction. So we are left with the two options that $z_2 = \pm z_1$ in $\mathbb{C}/\Lambda$. To get the right sign we use that

$$\wp'(z_1) = \wp'(z_2) = \wp'(\pm z_1) = \pm\wp'(z_1)$$

since $\wp'$ is odd. This shows that ϕ is injective.

The variety C can be seen as a Riemann surface by constructing local coordinates from the implicit function theorem. Furhtermore, to see that ϕ is a biholomorphism we examine the pullback of some meromorphic differential 1-form on C. The meromorphic differential form dx/y given in 2.3 is global, holomorphic and non-zero on C and computing the pullback we get

$$\phi^* \left(\frac{dx}{y}\right) = \frac{d\wp(z)}{\wp'(z)} = dz$$

which is holomorphic and non-zero on $\mathbb{C}/\Lambda$. This shows that ϕ is a local holomorphism, and by bijectivity ϕ is a biholomorphism. □

Theorem 2.19. *For any Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ with polynomial discriminant $16(g_2^3 - 27g_3^2)$ non-zero, there exists a lattice Λ s.t.*

$$g_2 = 60G_4 \quad \textit{and} \quad g_3 = 140G_6$$

where $G_{2k} = \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \frac{1}{\lambda^{2k}}$ is Eisenstein series of weight $2k$.

Together with the last theorem this shows that any Riemann surface given by a Weierstrass equation can be given a description as $\mathbb{C}/\Lambda$.

Proof. Let the Weierstrass equation be given as

$$y^2 = 4x^3 - \gamma_2 x - \gamma_3$$

with constants γ_2, γ_3 s.t. $\gamma_2^3 - 27\gamma_3^2 \neq 0$.

We will use the j-invariant defined as

$$j(\tau) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2}$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$ for a lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ s.t. $\tau \in \mathbb{H}$. The j-invariant is shown to be surjective to $\mathbb{C}$ in Appendix A.4 and therefore we may choose $\tau \in \mathbb{H}$ s.t.

$$1728\frac{g_2^3}{g_2^3 - 27g_3^2} = 1728\frac{\gamma_2^3}{\gamma_2^3 - 27\gamma_3^2}.$$

where γ_2 and γ_3 are the fixed coefficients of the Weierstrass equation.

Assuming that $\gamma_2 \neq 0$, then $g_2 \neq 0$ and we rewrite the last equation as

$$\frac{\gamma_3^2}{\gamma_2^3} = \frac{g_3^2}{g_2^3}.$$

Note that changing Λ to $a\Lambda$ for some non-zero constant a changes g_k to $a^{-2k}g_k$. Choose a such that $\gamma_2 = g_2$, and then we obtain $\gamma_3 = g_3$ from previous equation.

Otherwise if $\gamma_2 = 0$, then $g_2 = 0$ and we may similarly choose the constant a s.t. $\gamma_3 = g_3$.

Hence for $\Lambda = a(\mathbb{Z} + \tau\mathbb{Z})$ we have that $\gamma_2 = 60G_4$ and $\gamma_3 = 140G_6$. □

What has been shown in this section is that any projective variety given by a Weierstrass equation is an elliptic curve and any elliptic curve can be given by a Weierstrass equation. Furthermore any Riemann surface description $\mathbb{C}/\Lambda$ is isomorphic to a Riemann surface given by a Weierstrass equation as Riemann surfaces and vice versa. Hence given a Riemann surface $\mathbb{C}/\Lambda$, we can associate an elliptic curve as

$$\mathbb{C}/\Lambda \overset{\text{Riemann surfaces}}{\simeq} \{y^2 = 4x^3 + g_1x + g_3\} \subseteq \mathbb{P}^2 \overset{\text{varieties over } \mathbb{C}}{\simeq} (E, P_0)$$

for some $g_2^3 - 27g_3^2 \neq 0$. The base-point P_0 of the elliptic curve is mapped to $[0, 1, 0]$ and then $0 + \Lambda$.

Remark 2.20. The objects discussed in this section, i.e. $\mathbb{C}/\Lambda$, elliptic curves and sets given by Weierstrass equations, all have a natural group structure which the constructed maps conserve. This makes $\mathbb{C}/\Lambda$ and $\{y^2 = 4x^3 + g_1x + g_3\} \subseteq \mathbb{P}^2$ isomorphic also as Lie groups. See more in [Sil09, Proposition VI.3.6b, p. 170].

3

Geometry of a Calabi-Yau onefold

This chapter will introduce two spaces that are central in mirror symmetry, namely the *complex moduli space* and the *Kähler moduli space*. These space describe the possible complex structures, as in complex atlases, and Kähler structures, as in complexified Kähler classes, respectively. It is with respect to these spaces that the A-model and B-model calculations are made, and the mirror symmetry correspondence is made by mapping the complex moduli space of one Calabi-Yau manifold in a mirror pair to the Kähler moduli space of the other Calabi-Yau manifold. This mapping between moduli spaces are called the *mirror mapping*, which is introduced here in Section 3.3 for the case of elliptic curves.

Section 3.4 will introduce another moduli space, namely the *moduli space of curve mappings* $\mathcal{M}_g(X, \beta)$. This is used to define the *Gromov-Witten invariants* that is needed to define the generating series F_1^A of the A-model.

All these moduli spaces also have structure as manifolds or orbifolds, but how these structures come about will not be discussed. However, the definition of the Gromov-Witten invariants uses the *orbifold Euler characteristic* of both the complex moduli space and the moduli space of curve mappings. For this reason a sufficient subcategory of orbifolds will be introduced in Section 3.5 called *global quotient spaces* and orbifold Euler characteristic of a disjoint union och such spaces.

3.1 The complex moduli space

This section will introduce the complex moduli space for an elliptic curve. This will be needed later primarily in the B-model.

The complex moduli space of a manifold X is a parametrization of the possible complex structures, meaning the choice of holomorphic atlas, that can be endowed the X, up to isomorphism. The space itself also has a structure as an orbifold, but this will not be defined properly here. It will instead be apparent from the identification since it is a quotient of $\mathbb{H}$, which can be seen as a simplified case of an orbifold called a *global quotient space*. These spaces along with orbifold Euler characteristics of them are introduced later in Section 3.5. Taking the orbifold Euler characteristic we require that the number of automorphisms of each point in the complex moduli space is finite and therefore we need to fix a point of the elliptic curve. See more of this in Remark 3.2.

Claim 3.1. *The complex moduli space $\mathcal{M}_{1,1}$ for genus one curves with one fixed point can be identified as $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ where $\mathrm{SL}(2,\mathbb{Z})$ is acting on the half-plane as*

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \quad \textit{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

Proof. To identify the complex moduli space for a torus $\mathbb{C}/\Lambda$, note that Lemma 2.14 states that any holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ preserving 0 is of the form $z \mapsto az$ where $a\Lambda \subseteq \Lambda'$. This means all biholomorphic maps between elliptic curves are of the form

$$\mathbb{C}/\Lambda \to \mathbb{C}/a\Lambda, \; z \mapsto az \quad \text{for} \quad a \neq 0,$$

so what is left is to identify all Λ with $a\Lambda$ for non-zero a.

First we set one of the generators of Λ to 1, namely $a(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) = \mathbb{Z} + \tau\mathbb{Z}$ for say $\tau = \pm\frac{\omega_i}{\omega_j}$ where the sign will be absorbed by $\mathbb{Z}$. Of these four choices, only two are in $\mathbb{H}$ and are related as $-\frac{1}{\tau}$, let us specify that $\tau \in \mathbb{H}$. Furthermore, if we change τ to $\tau + 1$, the 1 is absorbed by $\mathbb{Z}$. These two actions, $\tau \mapsto -\frac{1}{\tau}$ and $\tau \mapsto \tau + 1$, generates $\mathrm{SL}(2,\mathbb{Z})$ (see Lemma A.3).

To motivate that there are no more equivalences, see each lattice as the $\mathbb{Z}$-module $\mathbb{Z}\times\mathbb{Z}$. Then since mapping Λ to $a\Lambda$ is a linear transformation it can be represented as a 2×2-matrix with values in $\mathbb{Z}$. Since multiplication with a complex number is a rotation of the complex plane this transformation has to be orientation preserving, therefore the matrix has positive determinant. Together with bijectivity, this means that determinant has to be one. This describes $\mathrm{SL}(2,\mathbb{Z})$. □

Remark 3.2. One may note that if we do not fix a point on each torus, as a set we would still get $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ since by using the automorphisms $z \to z + b$ we can ensure that our fixed point is preserved. This however would not work when taking the orbifold Euler characteristic of $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ as a global quotient space, since then there would be an uncountable number of automorphisms.

3.2 The Kähler moduli space

The given definition of a Calabi-Yau manifold in Chapter 2 assumes it is a Kähler manifold, meaning that it is possible to construct a Kähler form on the manifold. This section will introduce Kähler forms and the Kähler cone with a focus on elliptic curves. From this, we will define the complexified Kähler cone which elements are called *complexified Kähler classes*. The A-model is dependent on a choice of such a class, and we will parametrize this choice to form the Kähler moduli space. That is, the Kähler moduli space describes the possible choices of complexified Kähler classes that can be associated with a complex manifold.

The Kähler moduli space also has structure, similar to the complex moduli. It can be described as a complex manifold, but how this structure comes about will not be focused on in this text. Instead, the structure will be apparent since we will identify it with $\mathbb{H}$. The parameterization for a choice of a complexified Kähler class ω on an elliptic curve E will simply be the unique element $t \in \mathbb{H}$ s.t. $\int_E \omega = t$.

The definitions in this section will deal with de Rham cohomology both over $\mathbb{R}$ and $\mathbb{C}$ and since how these relate will be important, extensive definitions are given in Appendix B.3. There (p, q)-forms are introduced which is needed since a Kähler form will be required to be a $(1, 1)$-form. But in the case of elliptic curves, this is not as important since these are the only 2-forms.

The definition of a Kähler form will also depend on *almost complex structure* and is interlinked with the definition of a *Riemannian metric*.

Definition 3.3 (Almost complex structure)**.** Let M be a smooth real manifold. An almost complex structure I is a linear map $I_p : T_pM \to T_pM$ such that $I_p^2 = -1$ for each $p \in M$ varying smoothly on M (some explanation of varying smoothly is given in Section B.3 of the appendix).

Note that the almost complex structure in the case where the tangent space is identified as $\mathbb{C}$, as for Riemann surfaces, is simply the imaginary i of the tangent space.

Definition 3.4 (Riemannian metric)**.** Let M be a smooth real manifold. A Riemannian metric g is in each point $p \in M$ a real valued positive definite inner product varying smoothly over M. That is

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

in each point $p \in M$, such that g_p is bilinear symmetric operator and $g_p(u, u) > 0$ if $u \neq 0$.

Letting $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ be an ortonormal basis of T_pM we also define local matrix representation of the Riemannian metric g as

$$g_{ij}(p) := g_p \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right).$$

We now turn to the definition of a Kähler form.

Definition 3.5 (Kähler form and Kähler manifold)**.** A Kähler form is a real-valued $(1, 1)$-form $\omega \in \Omega^{1,1}(M)$ on a complex manifold M such that $d\omega = 0$ and the bilinear form defined as $g(u, v) := -\omega(Iu, v)$ is a Riemannian metric, where I is the almost complex structure on M.

We say that a complex manifold M is a Kähler manifold if there exists a Kähler form on M.

A Kähler form can be seen as the negative imaginary part of a non-degenerate Hermitian metric h on the manifold with the added condition that $d\omega = 0$, and then $\operatorname{Re} h(u,v) = -\omega(Iu,v)$ is also the Riemannian metric. Note that the Riemannian metric g is invariant under the almost complex structure, i.e. $g(Iu,Iv) = g(u,v)$ for all $u,v \in TM$. A Kähler form can also be seen as a symplectic form on the manifold. So the Kähler form is an additional structure on a complex manifold, and saying that the manifold is Kähler ensures that this structure exists.

Remark 3.6. That $(u,v) \mapsto -\omega(Iu,v)$ is a strictly positive definite symmetric bilinear form will in the case of an elliptic curve mean that a Kähler form integrated over the manifold is positive, which will be seen later in the proof of Theorem 3.9. In this sense, a Kähler form is a volume form, defining a size measurement of the manifold. This reasoning will be used when looking at the A-model.

That a Kähler form is a volume form can only hold in one dimension since it is a $(1,1)$-form, but in higher dimensions, a natural volume form can be constructed from a Kähler form raised to the n:th exterior power where n is the dimension of the manifold.

Definition 3.7. The Kähler cone $\mathcal{K}(M) \subseteq H^2_{\mathrm{dR}}(M,\mathbb{R})$ for a Kähler manifold M is the de Rham cohomology classes that include a Kähler form. Call a cohomology class in the Kähler cone a Kähler class.

This definition depends on the criteria that $d\omega = 0$, so that a Kähler form indeed defines a cohomology class.

One may note that $\mathcal{K}(M)$ is not a vector-subspace of $H^2_{\mathrm{dR}}(M)$, not all 2-forms in a Kähler class are Kähler forms and there is not one unique Kähler form in each Kähler class. All this will be apparent when looking at Kähler forms on elliptic curves.

To determine the Kähler classes of an elliptic curve, we will first give an elementary proof that $H^2_{\mathrm{dR}}(E,\mathbb{R}) = \mathbb{R}$ from which the Kähler classes will be apparent.

Lemma 3.8. *For an elliptic curve E,*

$$H^2_{dR}(E,\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_E \omega$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Proof. Let E be represented as $\mathbb{C}/\Lambda$ for some $\mathbb{C}$-lattice Λ.

Let ω be a differential 2-form on $\mathbb{C}/\Lambda$. Let $\phi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the projection on the torus from the plane. The pullback $\phi^*(\omega)$ is necessarily a closed form since $H^2_{\mathrm{dR}}(\mathbb{C}) = 0$, so let $\phi^*(\omega) = d\eta$ for some 1-form *eta* on $\mathbb{C}$. The pullback is necessarily Λ-periodic, but η is not, so we cannot use it as a differential form on $\mathbb{C}/\Lambda$ which would have shown that ω is closed. Instead what will be shown is that

$$d\eta = d\mu + C\ dx \wedge dy$$

where μ is periodic and $C \in \mathbb{R}$. Since this can be seen as a differential form on $\mathbb{C}/\Lambda$, this shows that $H^2_{\mathrm{dR}}(\mathbb{C}/\Lambda, \mathbb{R}) = \mathbb{R}$.

To simplify notation, let $\Lambda = \mathbb{Z} \times \mathbb{Z}$ such that a Λ-periodic, or $(1,1)$-periodic, function f means $f(x+1,y) = f(x,y+1) = f(x,y)$. This change is a diffeomorphism and will not affect cohomology.

Let $\eta = Pdx + Qdy$, then $d\eta = (\partial_x Q - \partial_y P)dx \wedge dy$ is periodic. Let $p(x,y) = \partial_x Q - \partial_y P$ and

$$\hat{P}(x,y) = \int_0^y p(x,t)dt - y\int_0^1 p(x,t)dt$$

which is $(1,1)$-periodic such that $d\eta = d(\hat{P}dx) + \left(\int_0^1 p(x,t)dt\right) dx \wedge dy$. Now let $q(x) = \int_0^1 p(x,t)dt$ and

$$\hat{Q}(x,y) = \int_0^x q(s)ds - x\int_0^1 q(s)ds$$

which is periodic in x and constant in y. With this we get that

$$d\eta = d\left(\hat{P}dx + \hat{Q}dy\right) + \left(\int_0^1\int_0^1 (\partial_x Q - \partial_y P)(t,s)dtds\right) dx \wedge dy.$$

So $[\omega] = [Cdx \wedge dy]$ in cohomology where

$$C = \int_0^1\int_0^1 (\partial_x Q - \partial_y P)(t,s)dtds = \int_{\mathbb{C}/\mathbb{Z}\times\mathbb{Z}} \omega.$$

This is a vectorspace isomorphism, simply because an integral is linear, and we are done. □

Theorem 3.9. *For an elliptic curve E, the Kähler forms are exactly the differential forms*

$$a\ dx \wedge dy$$

with strictly positive $a \in C^\infty(E)$. Further more, for any such Kähler form there is a unique $C > 0$ such that

$$[a\ dx \wedge dy] = [C\ dx \wedge dy] \quad \text{in } H^2_{dR}(E,\mathbb{R})$$

Hence the map

$$\omega \to \int_E \omega$$

is a bijection betweeen the Kähler cone $\mathcal{K}(E)$ and $\mathbb{R}_{>0}$.

Proof. Since $\dim_{\mathbb{R}}(E) = 2$, the first criteria of a Kähler form that $d\omega = 0$ is true for any differential form. For the second criteria, we note that a Riemannian metric g on E that is invariant under the almost complex structure I has to have the local matrix representation

$$g_{ij}(p) = a\mathbb{1}$$

where a is a strictly positive smooth real-valued function on E. This is the same as saying that any symmetric bilinear form on $\mathbb{R}^2$ that is invariant under 90 degree rotation has to be of the form

$$((x, y), (x', y')) \mapsto a(xx' + yy')$$

for some constant $a > 0$, given some orthonormal basis. Then since

$$dx \wedge dy \ ((x, y), (x', y')) = xy' - yx'$$

when using the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, this gives that any Kähler from on an elliptic curve is of the form

$$\omega = a \ dx \wedge dy$$

where $a : E \to \mathbb{R}$ is strictly positive smooth function.

Now the identification mapping of Lemma 3.8 is

$$a \ dx \wedge dy \mapsto \int_E a \ dx \wedge dy$$

which is strictly positive if a is. Therefore any Kähler class is mapped to $\mathbb{R}_{>0}$, and looking at any 2-forms $C \ dx \wedge dy$ for a constant $C > 0$ shows that this mapping is a bijection between $\mathcal{K}(E)$ and $\mathbb{R}_{>0}$. □

Now we turn to the complexification of the Kähler cone.

Definition 3.10. The complexified Kähler cone will be defined as

$$\mathcal{K}_{\mathbb{C}}(M) = H^2_{\mathrm{dR}}(M, \mathbb{R}) + i\mathcal{K}(M) \subseteq H^2_{\mathrm{dR}}(M, \mathbb{C})$$

for a Kähler manifold M. Call a cohomology class in the complexified Kähler cone a complexified Kähler class. When used as a paramatrization of choice of complexified Kähler class, we will refer to the complexified Kähler cone as the *Kähler moduli space.*

For an elliptic curve E, we have shown that the map $\omega \to \int_E \omega$ identifies $H^2_{\mathrm{dR}}(E, \mathbb{R})$ with $\mathbb{R}$ and $\mathcal{K}(E)$ with $\mathbb{R}_{>0}$. We get the bijection

$$\mathcal{K}_{\mathbb{C}}(E) \to \mathbb{H}, \quad \omega \mapsto \int_E \omega$$

where $\mathbb{H}$ is the upper complex half-plane. Hence the Kähler moduli space is identified as $\mathbb{H}$ where a parameter $t \in \mathbb{H}$ refers to the choice of a complexified Kähler class ω s.t. $\int_E \omega = t$. Note that this choice is then independent of the complex moduli space.

3.3 The mirror map

Now with both the complex moduli space and the Kähler moduli space introduced we can introduce the mirror map. When interchanging the A-model calculations and the B-model calculations, we are supposed to interchange the complex moduli space and Kähler moduli space. This is done via the so-called *mirror map*, which this section intends to explain in the case of elliptic curves. This case is particularly simple while in higher dimensions this might get more complected.

Let (E_τ, ω_t) be an elliptic curves together with a complexified Kähler class where $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is the parameter for the complex moduli space, i.e. E is isomorphic to the Riemann surface $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, and $t \in \mathbb{H}$ is the parameter for the Kähler moduli space, i.e. $\int_E \omega_t = t$. Then the mirror mapping which will interchange the calculations of $F_1^A(E_\tau, \omega_t)$ and $\mathcal{F}_1^B(E_\tau)$ is simply interchanging the parameters

$$(E_\tau, \omega_t) \longleftrightarrow (E_t, \omega_\tau)$$

for the complex moduli and Kähler moduli spaces.[Dij95] This interchange of calculation does however require one more step, and that is to take the so called *holomorphic limit* of $\mathcal{F}_1^B$ which will be looked at in Chapter 7.

As is said in the introduction, this is surprising since the complex moduli space if very different from the Kähler moduli space. It is also surprising since this also introduces some type of $\mathrm{SL}(2,\mathbb{Z})$ relation on the Kähler moduli space which is not immediately apparent. This will also be discussed in Chapter 7 where the calculations of F_1^A and $\mathcal{F}_1^B$ are interchanged.

3.4 Moduli space of curve mappings

This section will introduce the *moduli space of curve mappings*, which later in the A-model are used to define the *Gromov-Witten invariants*. This moduli space consists of all possible holomorphic mappings to the Calabi-Yau manifold X from a curve, i.e. a compact connected Riemann surface, of a specific genus such that the curve is mapped into some specified singular homology class. This can morally be seen as the curves *on* the Calabi-Yau manifold X of a specific genus and degree of mapping (if $H_2(X,\mathbb{Z}) = \mathbb{Z}$). To somehow count these curves is the information from enumerative geometry contained in the A-model.

Definition 3.11 (Moduli space of curve mappings)**.** The moduli space of curve mappings to an elliptic curve E of class $\beta \in H_2(E,\mathbb{Z})$ in singular homology will be defined as

$$\mathcal{M}_1(E,\beta) = \left\{ \varphi : C \to E \;\middle|\; \begin{array}{l} C \text{ connected genus one curve,} \\ \varphi \text{ holomorphic, } \varphi_*(C) = \beta \in H_2(E,\mathbb{Z}) \end{array} \right\} \Big/ \sim$$

where the equivalence relation is isomorphism of mappings as

$$\begin{array}{ccc} C_1 & \xrightarrow{\approx} & C_2 \\ & \underset{\varphi_1}{\searrow} & \downarrow{\scriptstyle \varphi_2} \\ & & E \end{array}$$

where the morphisms are holomorphic mappings. [Dij95]

Similarly to the complex moduli space, $\mathcal{M}_1(E,\beta)$ also has a structure as an orbifold. This will be used to define the Gromov-Witten invariants later in Chapter 4 as the *orbifold Euler characteristic* of $\mathcal{M}_1(E,\beta)$ for $\beta \neq 0$. Some theory of this will be introduced in Section 3.5.

3.5 Global quotient spaces and orbifold Euler characteristic

As have been mentioned, both the complex moduli space $\mathcal{M}_{1,1}$ and the moduli space of curve mappings $\mathcal{M}_g(E,\beta)$ has structures as orbifolds. In this case, orbifolds are a bit too general, since they can be seen as global quotient spaces instead. This section will introduce global quotient spaces as well as an extension of the usual Euler characteristic to these spaces which will be called *orbifold Euler characteristic*. This will be needed later in Chapter 4 where we use the orbifold Euler characteristic to define the *Gromov-Witten invariants*.

We start with some definitions.

Definition 3.12. Let X be a Hausdorff space. We say that a group G is acting on X if each element $g \in G$ is a homeomorphism of X and $(gh)(x) = g(h(x))$ for any $g, h \in G$ and $x \in X$. From this we can define the orbit space X/G which is the quotient space $X/\sim$ where $x \sim y \iff \exists g \in G : g(x) = y$, which has the quotient topology.

A group G acts *locally free* on X if for any $x \in X$ there is an open neighborhood U_x such that $g(U_x) \cap U_x = \emptyset$ if g is not the identity in G.

Definition 3.13 (Global quotient space)**.** We say that a global quotient space is a pair (X, G) where X, which we call the underlying space, is a topological manifold and G is a group acting on X, such that there is a subgroup $H \subseteq G$ acting locally free on X that has finite index in G. As a topological space (X, G) will have the topology of the orbit space X/G. By abuse of notation, we also denote the global quotient space X/G.

A global quotient space X/G is hence the orbit space of G acting on a topological manifold X except that we keep the information of the group acting on the space. We wish to extend the definition of Euler characteristic to also keep some information about the group G.

For a CW-complex Y with a k-sheeted cover $p : X \to Y$ the Euler characteristic has the property

$$\chi(X) = k\ \chi(Y). \tag{3.1}$$

If we let G be the group of deck transformations of this cover, i.e. the homeomorphism $g : X \to X$ such that $g \circ p = p$, then G is locally free, Y is homeomorphic to the orbit space X/G and $|G| = k$. Hence

$$\chi(X) = |G|\chi(X/G). \tag{3.2}$$

which holds in general as long as G is acting locally free on a CW-complex X, ensuring that X is a cover of the orbit space X/G. This theory is covered in [Hat02, p. 70]. We will use the property in (3.2) to define *orbifold Euler characteristc* of a global quotient X/G. It will be proven to be well-defined in Theorem 3.15.

Definition 3.14 (Orbifold Euler characteristic of a global quotient)**.** Let X/G be a global quotient, then if G is acting locally free on X we let the orbifold Euler characteristic $\chi(X/G)$ coincide with the usual Euler characteristic of the orbit space X/G.

If G is not acting locally free, let the orbifold Euler characteristic be

$$\chi(X/G) = [G : H]^{-1}\chi(X/H)$$

for some subgroup $H \subseteq G$ acting locally free on X and has finite index in G, which by definition of global quotient space is assumed to exist.

Furthermore, given a space Y that can be expressed as a disjoint union of finitely many global quotients, i.e.

$$Y = X_1/G_1 \sqcup \cdots \sqcup X_n/G_n,$$

we define the the orbifold Euler characteristic as

$$\chi(Y) = \sum_i^n \chi(X_i/G_i).$$

Theorem 3.15. *The definition of orbifold Euler characteristic is well-defined. That is, if G is acting on a topological manifold X then for any two subgroups $H, K \subseteq G$ of finite index in G acting locally free on X*

$$[G : H]^{-1}\chi(X/H) = [G : K]^{-1}\chi(X/K).$$

Proof. For a global quotient space X/G, let $K, H \subseteq G$ be subgroups of finite index in G that acts locally free on X.

Assume $K \subseteq H$, then $X/H \to X/K$ is a cover of degree $[H : K]$ and since index is multiplicative

$$[G : K] = [G : H][H : K]$$

we get

$$[G:H]^{-1}\chi(X/H) = [G:K]^{-1}\chi(X/K)$$

using (3.1).

Otherwise, if there is no inclusion we look at the subgroup $H \cap K$. This subgroup always has finite index in G since H and K do and it is acting locally free on X since H and K does. Then by the previous argument using any of these subgroups results in the same orbifold Euler characteristic. □

This definition will be applied on two spaces, namely the complex moduli space $\mathcal{M}_{1,1}$ and the moduli space of curve mappings $\mathcal{M}_1(E,\beta)$.

For $\mathcal{M}_1(E,\beta)$, the group acting on the topological space is the isomorphism group of mappings. As will be shown in Section 4.1, there are only finitely many mappings in $\mathcal{M}_1(E,\beta)$ up to isomorphism, so one can see this as a disjoint union of points with the group of automorphism acting on each element, and is hence a disjoint union of zero-dimensional global quotient spaces. Then the orbifold Euler characteristic becomes

$$\chi(\mathcal{M}_1(E,\beta)) = \sum_{\varphi\in\mathcal{M}_1(E,\beta)} \frac{1}{\#\mathrm{Aut}(\varphi)}$$

given that $\#\mathrm{Aut}(\varphi)$ is finite, since the Euler characteristic of a single point is 1.

The value of $\chi(\mathcal{M}_{1,1})$ will be referenced to from [HZ86], but a proof that there exists subgroups of finite index acting locally free is given in the appendix in Section A.3.

4

A-model at genus one for dimension one

This chapter will be concerned with the A-model side of mirror symmetry. These definitions are not general, rather for mirror symmetry at genus one and for the case when the Calabi-Yau manifold is one-dimensional, i.e. an elliptic curve.

The A-model is concerned with the generating series of genus one Gromov–Witten invariants F_g^A. This means computing Gromov-Witten invariants $N_{g,\beta}$ which contain information from enumerative geometry, namely that they relate to counting the number of rational curves, and sometimes coincides with this count [CK99, Introduction].

It is important to note that in contrast to the B-model, the A-model requires additional structure on the Calabi-Yau manifold, namely an associated *complexified Kähler class.* The choice of such a class is parametrized to form the Kähler moduli space and the A-model should mostly depend on the Kähler moduli space rather than the complex moduli [CK99], in the sense that the parameter τ of the complex moduli space does not appear in the final expression of F_g^A.

Definition 4.1 (Generating series)**.** For a Calabi-Yau onefold E with associated complexified Kähler class ω the generating series of genus one Gromov–Witten invariants will be defined as

$$F_1^A(E,\omega) = \sum_{\beta \in H_2(X,\mathbb{Z})} N_{1,\beta}(E)\ q^\beta$$

where $q_\beta = e^{2\pi i \int_\beta \omega}$ and $N_{1,\beta}$ are Gromov-Witten invariants at genus one, later defined in Definition 4.3 and Definition 4.12.

This chapter will culminate in the proof of the following theorem.

Theorem 4.2. *For a Calabi-Yau onefold E with associated complexified Kähler class $\omega_t \in \mathcal{K}_\mathbb{C}(E)$, the generating series at genus one is*

$$F_1^A(E,\omega_t) = -\log \eta(t)$$

where $t = \int_E \omega_t \in \mathbb{H}$ is the parameter of the Kähler moduli space, η is the Dedekind Eta function (1.1) *and* $\log$ *denotes the principal branch of the complex logarithm, i.e.* $\operatorname{Im} \log \in (-\pi, \pi)$.

4.1 Gromov-Witten invariants $N_{1,\beta}$ for $\beta \neq 0$

The Gromov-Witten invariants are a "size" measurement of the moduli space of curve mappings defined in Secition 3.4 and can be seen as a type of count of the number of curves on the Calabi-Yau manifold. The Gromov-Witten invariants depend on a singular homology class $\beta \in H_2(E, \mathbb{Z})$ and the case when $\beta = 0$ will be treated separately in Section 4.2. This section will first give definitions and then calculate the Gromov-Witten invariants of elliptic curves for genus one and $\beta \neq 0$.

This definition of the Gromov-Witten invariants is suggested by Dijkgraaf in [Dij95].

Definition 4.3 (Gromov-Witten invariants at genus one)**.** For an elliptic curve E and a singular homology class $\beta \in H_2(E, \mathbb{Z})$ s.t. $\beta \neq 0$, define Gromov-Witten invariants at genus one as

$$N_{1,\beta}(E) = \chi(\mathcal{M}_1(E, \beta))$$

where $\mathcal{M}_1(E, \beta)$ is the moduli space of curve mappings defined in Section 3.4, χ is the orbifold Euler characteristic defined in Section 3.5.

This case when $\beta \neq 0$, the choice of $N_{1,\beta}$ is more or less counting curves (orbifold Euler characteristics also divides this count by the number of automorphisms of each map) since $\mathcal{M}_1(E, \beta)$ has finite number of elements, as shown later in this section. In a more general setting, this is from the fact that the (virtual) dimension of $\mathcal{M}_g(X, \beta)$ for an arbitrary Calabi-Yau manifold X is $(\dim_{\mathbb{C}} X - 3)(1 - g)$ as stated in [Dij95], so in the case of genus one curve mappings we get $\mathrm{vdim}_{\mathbb{C}} \mathcal{M}_1(E, \beta) = 0$.

Remark 4.4. Note that in the case of a Calabi-Yau onefold E, which topologically is a torus, $H_2(E, \mathbb{Z}) = \mathbb{Z}$. We say that a map $\varphi : C \to E$ has degree d if $\varphi_*(C) = d[E] \in H_2(E, \mathbb{Z})$. So if $\beta = d[E]$ we will write $N_{1,d} := N_{1,\beta}$.

Note that we later in this section will use two separate notions of degree, namely homology degree, i.e. $\varphi_*(C) = d[E]$, and degree of a covering, i.e. $\#\varphi^{-1}(z)$ for any point $z \in E$. However, these notions coincide.

Remark 4.5. Since the mappings in $\mathcal{M}_1(E, \beta)$ are holomorphic, and therefore necessarily orientation conserving, there are no mappings with degree $d < 0$. So we only have to consider the case $d \geq 0$.

To calculate $N_{1,d}(E)$ for $d \neq 0$, we will show that each map in $\mathcal{M}_1(E, d)$ is a covering. Then we will count the number subgroups of index d in the fundamental group which corresponds to the number of covering spaces of degree d by the *Galois correspondence of coverings* in Theorem 4.7.

That each mapping in $\mathcal{M}_1(E, d)$ is a covering can be seen from Riemann-Hurwitz Theorem.

Theorem 4.6 (Riemann-Hurwitz Theorem)**.** *[Sil09, p. 35] Let $\phi : C_1 \to C_2$ be a non-constant map of smooth projective curves over $\mathbb{C}$ of genus g_1 and g_2, respectively. Then*

$$2g_1 - 2 = (2g_2 - 2)\deg\phi + \sum_{P \in C_1} (e_\phi(P) - 1)$$

where $e_\phi(P)$ is the ramification index at P, defined as $e_\phi(P) := \mathrm{ord}_P(\phi^*(z))$ for a uniformizer z at $\phi(P)$, and $\phi^* : \mathbb{C}(C_2) \to \mathbb{C}(C_1)$ is the pullback, i.e. $\phi^*(f) := f \circ \phi$.

Both spaces are indeed smooth varieties and a map of non-zero degree is non-constant. Using that both curves are genus one, we get that there are no ramification points, i.e. $e_\phi(z) = 1 \; \forall z \in E$. This means that any map between genus one curves in local coordinates has a Laurent expansion s.t. the lowest term exponent is one, so for small enough neighborhoods, they are homeomorphisms. In other words, these maps are coverings.

Theorem 4.7. *[Galois correspondence of coverings] [Hat02, Theorem 1.38, p. 67] Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of base-point-preserving isomorphism classes of path-connected covering spaces*

$$p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$$

and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

where $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is the induced homomorphism of p.

If base-points are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Since the equivalence relation in $\mathcal{M}_1(E, d)$ is not concerned with base-points, we are in the second case. But since an elliptic curve E is topologically a torus, $\pi_1(E) = \mathbb{Z} \times \mathbb{Z}$ which is abelian. So conjugacy classes of subgroups are the same as subgroups. This could be explained geometrically since we can easily shift a mapping on a torus without affecting it's homotopy class, with a map $z \to z + a$ for some constant $a \in \mathbb{C}$ in the representation of E as $\mathbb{C}/\Lambda$.

However, one may note that the morphisms for covering spaces are continuous maps and not holomorphic. But the two concepts turn out to coincide, which is expected since the A-model should not depend on the complex moduli. This makes the Gromov-Witten invariants of elliptic curves topological invariants.

More precisely, a continuous map $\varphi : C \to E$ will be homotopic to either a holomorphic or anti-holomorphic map, and therefore equivalent in the equivalence relation for covering spaces. But the equivalence relation of covering spaces is also allowed to reverse orientation since they are continuous maps and not holomorphic, and therefore we may pick a holomorphic representative for φ.

Now, to use Theorem 4.7 to calculate $N_{1,d}$ we need the following statement.

Lemma 4.8. *[Arm83, (10.14) Theorem, p. 229] Let $p : \tilde{X} \to X$ be a covering map of path-connected and locally path-connected spaces. Then for any point $x \in X$ the cardinality of the set $p^{-1}(x)$ is the index of $p_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$.*

Since E is topologically a torus, we are interested in subgroups of $\pi_1(E) = \mathbb{Z} \times \mathbb{Z}$.

Claim 4.9. *All (finite index) subgroups $H \subseteq \mathbb{Z} \times \mathbb{Z}$ are generated uniquely by $(k, 0), (l, m)$ with $0 < k, 0 < m, 0 \leq l < k$, each with index $[\mathbb{Z} \times \mathbb{Z} : H] = km$.*

Proof. To see this, pick any two generators $(x_1, y_1), (x_2, y_2)$ of a subgroup $H \subseteq \mathbb{Z} \times \mathbb{Z}$ such that they are not a integer multiple of each other. Let $d = \gcd(y_1, y_2)$ and $a, b \in \mathbb{Z}$ s.t. $ay_1 + by_2 = d$. Let $k = |\frac{y_2}{d}x_1 - \frac{y_1}{d}x_2|$ and $m = d$. Note that we are allowed to choose the sign of k. We claim that

$$(k, 0) = \frac{y_2}{d} \cdot (x_1, y_1) - \frac{y_1}{d} \cdot (x_2, y_2) \quad \text{and} \quad (l, m) = a \cdot (x_1, y_1) + b \cdot (x_2, y_2)$$

are generators of H. Moreover, by adding or subtracting $(k, 0)$ from (l, m) we can ensure that $0 \leq l < k$.

There cannot be any $(h, 0) \in H$ s.t. $0 < h < k$ since this contradicts $d = \gcd(y_1, y_2)$, which proves that $(k, 0)$ is a generator, and (l, m) is a generator since a and b are necessarily relatively prime. The uniqueness of these generators follow from the uniqueness of *greatest common divisor.*

Now to see $[\mathbb{Z} \times \mathbb{Z} : H] = km$, we note that the cosets of H correspond to lattice points in the parallelogram spanned by $(k, 0)$ and (l, m), which is the area. □

So we count the number of subgroups with index d as

$$\#\{\, k, l, m \mid 0 < k, 0 < m, 0 \leq l < k, km = d \,\} = \sum_{k|d} k = \sum_{k|d} \frac{d}{k} =: \sigma(d)$$

which then by Lemma 4.8 and Theorem 4.7 is the number of covers of degree d.

Remark 4.10. By motivating that this correspondence can be used to count elements in $\mathcal{M}_1(E, d)$, we have proven that $\mathcal{M}_1(E, d)$ consists of a finite number of elements. Or as an orbifold, $\mathcal{M}_1(E, d)$ is zero-dimensional.

The Gromov-Witten invariant $N_{1,d}$ is defined to be the orbifold Euler characteristic of $\mathcal{M}_1(E, d)$ and as discussed in Section 3.5 since $\mathcal{M}_1(E, d)$ is zero-dimensional we get that

$$N_{1,d} = \sum_{\varphi \in \mathcal{M}_1(E,d)} \frac{1}{\#\mathrm{Aut}(\varphi)}$$

where $\mathrm{Aut}(\varphi)$ is the automorphisms of an element with respect to the equivalence relation of $\mathcal{M}_1(E,d)$.

There are d automorphisms of each cover $\varphi : C \to E$, since by fixing a point $z_0 \in E$ we can shift C for each element in the fibre $\varphi^{-1}(z_0)$, namely

$$\begin{array}{ccc} C & \xrightarrow[\approx]{z \mapsto z-a} & C \\ & \searrow^{\varphi} & \downarrow \varphi \\ & & E \end{array}$$

where $a \in \varphi^{-1}(z_0)$.

So we arrive at

$$N_{1,d}(E) = \frac{\sigma(d)}{d} = \sum_{k|d} \frac{1}{k}.$$

Remark 4.11. As an alternative approach of counting the covers, one could also have constructed the maps in $\mathcal{M}_1(E,d)$ explicitly. This would be done with Lemma 2.14 showing that the number of mappings will be

$$\#\{\, \Lambda' \mid \alpha\Lambda' \subseteq \Lambda, \alpha \in \mathbb{C} \setminus \{0\} \,\}$$

which yields the same result.

4.2 Gromov-Witten invariant $N_{1,0}$

We will also need the Gromov-Witten invariant for homology class $\beta = 0$, but this choice is less intuitive. This definition is more ad hoc to make the correspondence with the B-model.

With a slight modification of Lemma 2.14 we see that any holomorphic map of degree zero between elliptic curves is constant, so maps in $\mathcal{M}_1(E,0)$ are constructed from any curve of genus one together with any point on E. The equivalence relation on $\mathcal{M}_1(E,0)$ will identify all isomorphic curves, and allow us to fix one point. The choice of curve to map from can hence be viewed as the complex moduli space of genus one curves with one point fixed $\mathcal{M}_{1,1}$ from Section 3.1. One might then consider describing $\mathcal{M}_1(E,0)$ as $\mathcal{M}_{1,1} \times E$, and then define the Gromov-Witten invariant $N_{1,0}$ to be appropriate "size" measure of this.

For the "size" of $\mathcal{M}_{1,1}$ we will use orbifold Euler characteristic similar to the definition of $N_{1,\beta}$ for $\beta \neq 0$, and as the size of E we will integrate the complexified Kähler class ω associated to E, since the A-model is supposed to be mostly dependent on the Kähler moduli of (E,ω) and not the complex moduli of E. Why the integration of a complexified Kähler class is a size measurement is motivated by that a Kähler form is a volume measure as mentioned in Remark 3.6.

For the correspondence with the B-model to be made, we will also add a normalization factor πi.

Definition 4.12 (Gromov-Witten invariants at genus one)**.** For $\beta = 0$, define the Gromov-Witten invarient of en elliptic curve E with associated complexified Kähler class ω as

$$N_{1,0}(E,\omega) = \pi i \ \chi(\mathcal{M}_{1,1}) \int_E \omega$$

where $\mathcal{M}_{1,1}$ is the complex moduli space of genus one curves with one point fixed (see Section 3.1), χ is the orbifold Euler characteristic.

Remark 4.13. Note the difference with given definitions that $N_{1,0}(E,\omega)$ depend on the complexified Kähler class, whereas $N_{1,\beta}(E)$ for $\beta \neq 0$ does not.

Note also that, even though the complex moduli space $\mathcal{M}_{1,1}$ appears here, it does not create a dependence on it. Only the geometry of the complex moduli space is used not a parameter in it.

To calculate $N_{1,0}(E,\omega)$ we need the oribifold Euler characteristic $\chi(\mathcal{M}_{1,1})$. This can be found in [HZ86] as well as in [Bro82, Chapter IX, (8.4), p. 255] and is

$$\chi(\mathcal{M}_{1,1}) = -\frac{1}{12}.$$

Note that in both [HZ86] and [Bro82] they use Euler characteristic of groups, but from discussion in the introduction to [HZ86] this should coincide with definition given in Section 3.5.

4.3 The generating series F_1^A

This section will define and calculate the generating series F_1^A (Definition 4.1) for an elliptic curve using the calculations of the Gromov-Witten invariants carried out previously in this chapter.

If we let $t \in \mathbb{H}$ be the parameter of the Kähler moduli space determining the choice of ω, then $\int_E \omega = t$ as discussed in Section 3.2. We will denote ω as ω_t to clarify. Note that with this choice, as well as the previous conclusions that the Gromov-Witten invariants for $\beta \neq 0$ are topological invariants, we get that F_1^A is independent of the complex moduli space.

As mentioned in Remark 4.4, since $H_2(E,\mathbb{Z}) = \mathbb{Z}$ we can identify $\beta = d[E]$ for some $d \in \mathbb{Z}$. Then

$$\int_{d[E]} \omega_t = d \int_E \omega_t = td$$

and the generating series is

$$F_1^A(E,\omega_t) = \sum_{d \geq 0} N_{1,d}(E) q^d \tag{4.1}$$

where $q = e^{2\pi i t}$. Note that $N_{1,d}(E) = 0$ for $d < 0$ as discussed in Remark 4.5.

Now with previous calculations in this chapter of $N_{1,d}(E)$ we get

$$F_1^A(E,\omega_t) = -\frac{\pi i}{12}t + \sum_{d>0}\sum_{k|d}\frac{1}{k}q^d.$$

To rewrite this we note that

$$\sum_{d>0}\sum_{k|d} f(d,k) = \sum_{d>0}\sum_{k>0} f(kd,k)$$

for well-behaved f, which is illustrated by Figure 4.1.

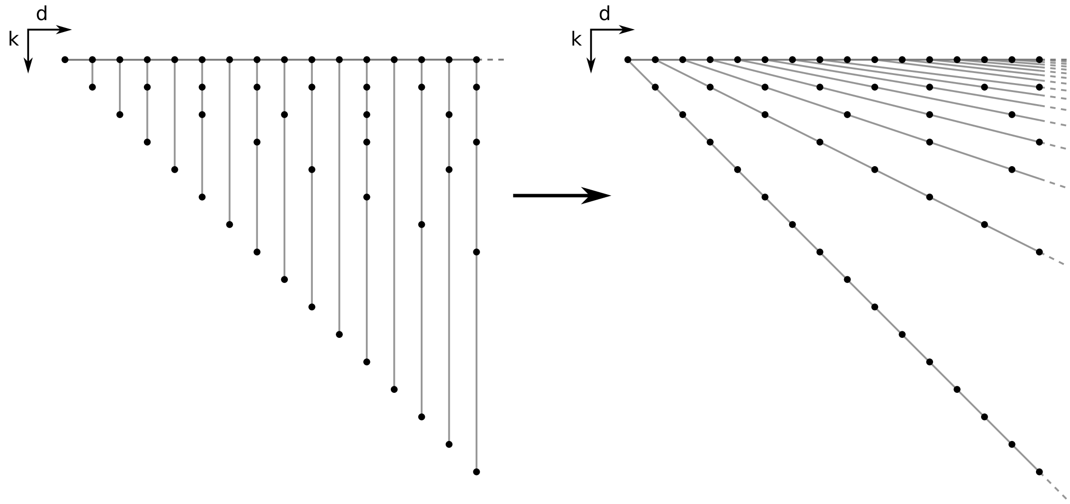

Figure 4.1: Proof by picture that $\sum_{d>0}\sum_{k|d} f(d,k) = \sum_{d>0}\sum_{k>0} f(kd,k)$. This does require some more restrictions on $f(k,d)$, but this will be ignored.

Then we use the Taylor expansion of $-\log(1-z) = \sum_{k>0}\frac{z^k}{k}$ valid for $|z| < 1$, i.e. the principal branch of the complex logarithm s.t. $\operatorname{Im}\log \in (-\pi,\pi)$, which is satisfied since $t \in \mathbb{H}$ implies that $|q^d| < 1$. Lastly using the addition rule of logarithm we end up with

$$F_1^A(E,\omega_t) = -\frac{\pi i}{12}t - \log\left(\prod_{d>0}(1-q^d)\right).$$

Using the Dedekind Eta function (1.1) we get

$$F_1^A(E,\omega_t) = -\log\eta(t).$$

which shows Theorem 4.2.

5

The Kronecker limit formula

In Chapter 6, the B-model calculations of $\mathcal{F}_1^B$ defined using analytic torsion will culminate in the Kronecker limit formula. Since this is a central part of those calculations, this entire chapter will be devoted to proving this formula.

At the end of the chapter in Section 5.6, there will also be some discussion of modular forms and the modular discriminant. This is later needed in Section 6.5 where the alternative formulation of $\mathcal{F}_1^B$, using the holomorphic anomaly equation, is presented.

The following derivation largely follows the book by Weil [And76] concerning the work of Eisenstein and Kronecker on elliptic functions. There it is stated with more generality. Another referenced derivation is given in [Lan87].

The Kronecker (first) limit formula is concerned with the real analytic Eisenstein series

$$E(s,\Lambda) = (\mathrm{coVol}\,\Lambda)^s \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} |\lambda|^{-2s} \quad \text{for} \quad \mathrm{Re}(s) > 1. \tag{5.1}$$

Specifically the analytic continuation of the series and its expansion around $s = 0$ or $s = 1$, where the sum does not converge. Here $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ is a lattice in the complex plane with $\mathrm{Im}\,\tau > 0$ assumed where $\tau = \frac{\omega_2}{\omega_1}$, and $\mathrm{coVol}\,\Lambda = \mathrm{Im}\,\tau|\omega_1|^2$ is the volume of a unit cell of the lattice. Only the expansion at $s = 0$ will be derived since this is what is needed to calculate $\mathcal{F}_1^B$, but the calculation for the expansion at $s = 1$ only differs slightly in the last calculations.

Theorem 5.1 (Kronecker limit formula)**.** *The Taylor expansion of $E(s,\Lambda)$ around $s = 0$ up to second order terms is*

$$E(s,\Lambda) = -1 - \log\left(\mathrm{Im}\,\tau(2\pi)^2|\eta(\tau)|^4\right) s + O(s^2) \tag{5.2}$$

which uses the Dedekind eta function defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n) \tag{5.3}$$

where $q = e^{2\pi i\tau}$ and the complex root means in this case $q^{\frac{1}{24}} = e^{\frac{\pi}{12}i\tau}$.

Theorem 5.2. *There exists an analytic continuation of* $E(s,\Lambda)$ *on* $\mathbb{C}\setminus\{1\}$*. It can be expressed as*

$$\begin{aligned}(\operatorname{Im}\tau)^{-s}E(s,\Lambda) &= 2\zeta(2s) \\ &+ 2\left(\frac{\pi}{\operatorname{Im}\tau}\right)^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\zeta(2-2s) \\ &+ 2\sqrt{\pi}\left(\frac{\pi}{\operatorname{Im}\tau}\right)^{s-\frac{1}{2}}\frac{1}{\Gamma(s)}G\left(s-\frac{1}{2}\right)\end{aligned} \tag{5.4}$$

with a single pole at $s=1$*. Here* ζ *is the Riemann zeta function and* Γ *is the gamma function which both are known meromorphic functions, and*

$$G(z) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\left|\frac{k}{m}\right|^{z} K_z(2\pi|km|\operatorname{Im}\tau)e^{2\pi ikm\operatorname{Re}\tau}. \tag{5.5}$$

is an entire function and uses

$$K_z(Y) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{Y}{2}\left(t+\frac{1}{t}\right)\right)t^{z-1}dt \tag{5.6}$$

which is an modified Bessel function of the second kind [1]*.*

The derivation of the analytic continuation will be done by studying $E(s,\Lambda)$ where the sum is absolutely convergent and extract the three terms seen in (5.4), each which has an analytic continuation to $s=0$. Then by the uniqueness of analytic continuation, this is also the continuation of $E(s,\Lambda)$.

5.1 Absolute convergence of $E(s,\Lambda)$

This section shows exactly where the sum expression of $E(s,\Lambda)$ is absolutely convergent and then extracts the first term of the analytic continuation.

Lemma 5.3. *The sum expression of* $E(s,\Lambda)$ *is absolutely convergent iff*

$$\operatorname{Re}(s) > 1.$$

Proof. Note that $||\lambda|^{-2s}| = |\lambda|^{-2\operatorname{Re}(s)}$.

To check the convergence, the sum will be estimated with a sum over a single index. This will be done by looking at squares given by $|m|+|n|=k$ and estimate a lower and upper bound, where k is now the summation index. This estimate times the elements in each square give the bounds.

[1] Although this integral representation does not seem to be commonly used. More commonly occurring is $K_z(Y) = \int_0^\infty \exp(-Y\cosh t)\cosh zt\; dt$.

Since circumference in two dimension scales linearly with the radius, the number of terms in each square is Ck for some C. More precisely,

$$2 + \sum_{n=-(k-1)}^{k-1} 2 = 4k$$

terms where the first "2" is for $n = \pm k$ and the second "2" is for $m = \pm k - |n|$ for each $|n| < k$.

Using this we will estimate the sum expression of $E(s, \Lambda)$ as

$$\sum_{k>0} 4k \left(\max_{|m|+|n|=k} |\lambda|^2 \right)^{-\operatorname{Re}(s)} \leq \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} |\lambda|^{-2\operatorname{Re}(s)} \leq \sum_{k>0} 4k \left(\min_{|m|+|n|=k} |\lambda|^2 \right)^{-\operatorname{Re}(s)}.$$

For the upper estimate

$$\min_{|m|+|n|=k} |\lambda|^2 = \min_{|m|+|n|=k} |\omega_1|^2((m + \operatorname{Re}\tau n)^2 + (\operatorname{Im}\tau n)^2)$$

which can be solved if m and n are continuous variables by minimizing four quadratic equations. The solution is

$$\begin{aligned} m &= \pm_1 k \pm_2 n \\ n &= \mp_1 k \frac{\operatorname{Re}\tau \pm_2 1}{\operatorname{Im}\tau^2 + (\operatorname{Re}\tau \pm_2 1)^2}, \end{aligned}$$

which yields a value of

$$k^2 |\omega_1|^2 \left(1 + \frac{(\operatorname{Re}\tau \pm_2 1)^2}{\operatorname{Im}\tau^2} \right)^{-1}.$$

In both cases this is k^2 multiplied by a positive constant not depending on k. Call this constant α^2 and we get that

$$\min_{|m|+|n|=k} |\lambda|^2 \geq (\alpha k)^2$$

.

The lower estimate is easier. Simply

$$\max_{|m|+|n|=k} |\lambda|^2 \leq k^2 |\omega_1|^2 \max(1, |\tau|^2)$$

which is also a positive constant, call it β^2.

So

$$\sum_{k>0} \frac{4k}{(\beta k)^{2\operatorname{Re}(s)}} \leq \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} |\lambda|^{-2\operatorname{Re}(s)} \leq \sum_{k>0} \frac{4k}{(\alpha k)^{2\operatorname{Re}(s)}}$$

which converges iff $2\operatorname{Re}(s) - 1 > 1$, i.e. $\operatorname{Re}(s) > 1$.

□

To extract the terms of the analytic continuation, one starts by rewriting the sum as

$$E(s,\Lambda) = (\operatorname{Im}\tau|\omega_1|^2)^s \sum_{\substack{n,m\in\mathbb{Z}\\(n,m)\neq(0,0)}} |\omega_1 n + \omega_2 m|^{-2s} = (\operatorname{Im}\tau)^s \sum_{\substack{n,m\in\mathbb{Z}\\(n,m)\neq(0,0)}} |n+\tau m|^{-2s}$$

and get the first term from the terms where $m = 0$, $\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |n|^{-2s} = 2\zeta(2s)$.

The remaining two terms will come from the partial sum

$$S(s,\nu) = \sum_{n\in\mathbb{Z}} |n+\nu|^{-2s}. \tag{5.7}$$

for $\operatorname{Im}(\nu) \neq 0$, which is extracted as

$$(\operatorname{Im}\tau)^{-s} E(s,\Lambda) = 2\zeta(2s) + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} S(s, m\tau).$$

5.2 The partial sum $S(s,\nu)$

To work with the sum $S(s,\nu)$ given in Equation (5.7) for $\operatorname{Im}(\nu) \neq 0$, Kronecker used the moderns tools of the time, namely Fourier series and the Poisson summation formula. This sum can afterward be split into the second and third term in the expression (5.4).

Lemma 5.4. *For* $\operatorname{Re}(s) > 1$

$$\Gamma(s)S(s,\nu) = \sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)|\eta|^{1-2s} + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} 2\pi^s \left|\frac{k}{\eta}\right|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|k\eta|)e^{2\pi i k\xi} \tag{5.8}$$

where $\nu = \xi + i\eta$, $\eta \neq 0$ *and* K_z *is the Bessel function definied in* (5.6).

Proof. Since any integer is absorbed by the summation over $n \in \mathbb{Z}$, we can see that $S(s,\nu) = \sum_{n\in\mathbb{Z}} |n+\nu|^{-2s}$ is a 1-periodic function in ξ. Let the Fourier series expansion of $S(s,\nu)$ in ξ be expressed as

$$S(s,\nu) = \sum_{k\in\mathbb{Z}} \phi(\eta, s, k)e^{2\pi i k\xi} \tag{5.9}$$

where ϕ is the Fourier coefficients. This can also be seen as applying Poisson summation formula directly on $S(s,\nu)$ by taking the Fourier transform of

$f(n) = |n+\nu|^{-2s}$, thinking of n as a continuous variable. Namely

$$\begin{aligned} S(s,\nu) &= \sum_{n\in\mathbb{Z}} |n+\nu|^{-2s} \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |n+\nu|^{-2s} e^{-2\pi i k n} dn \\ &= \sum_{k\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} |n+i\eta|^{-2s} e^{-2\pi i k n} dn \right) e^{2\pi i k \xi} \end{aligned}$$

where the last step made the change of integration variable $n+\xi \mapsto n$. Then by noting that $|n+i\eta|^2 = (n^2+\eta^2)$ we get

$$\phi(\eta,s,k) = \int_{-\infty}^{\infty} (n^2+\eta^2)^{-s} e^{-2\pi i k n} dn$$

as an expression of the Fourier coefficients.

By using the property [2]

$$\Gamma(s)\frac{1}{A^s} = \int_0^{\infty} e^{-tA} t^{s-1} dt$$

of the gamma function Γ the expression of the Fourier coefficients can be further rewritten to remove s from the exponent of n and η. This is done as follows:

$$\begin{aligned} \Gamma(s)\phi(\eta,s,k) &= \int_{-\infty}^{\infty} \Gamma(s)(n^2+\eta^2)^{-s} e^{-2\pi i k n} dn \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} e^{-t(n^2+\eta^2)} e^{-2\pi i k n} t^{s-1} dt dn \\ &= \int_0^{\infty} e^{-t\eta^2} \left(\int_{-\infty}^{\infty} e^{-tn^2} e^{-2\pi i k n} dn \right) t^{s-1} dt \\ &= \sqrt{\pi} \int_0^{\infty} \exp\left(-t\eta^2 - \frac{\pi^2 k^2}{t} \right) t^{s-\frac{1}{2}-1} dt \end{aligned}$$

where we can exchange order of the integration because

$$\int_{-\infty}^{\infty}\int_0^{\infty} \left| e^{-t(n^2+\eta^2)} e^{-2\pi i k n} t^{s-1} \right| dt dn = \int_{-\infty}^{\infty}\int_0^{\infty} e^{-t(n^2+\eta^2)} t^{\mathrm{Re}(s)-1} dt dn < \infty$$

if $\eta \neq 0$ and $\mathrm{Re}(s) > 0$, which is fine since $\mathrm{Re}(s) > 1$ is assumed.

To split $S(s,\nu)$ in to the two terms we simply look at the term $k=0$ separately, namely

$$\begin{aligned} \Gamma(s)\phi(\eta,s,0) &= \sqrt{\pi} \int_0^{\infty} \exp\left(-t\eta^2\right) t^{s-\frac{1}{2}-1} dt \\ &= \sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right) |\eta|^{1-2s}. \end{aligned}$$

[2] This is related to Mellin transform. In particular, this property is the same as saying that the Mellin transform of e^{-tA} is $\Gamma(s)\frac{1}{A^s}$.

The other term in (5.8) is given by the remeining terms where $k \neq 0$ in (5.9), and can be rewritten as

$$\Gamma(s)\phi(\eta, s, k) = 2\pi^s \left|\frac{k}{\eta}\right|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|k\eta|)$$

by preforming the variable substitution $\{t\left|\frac{\eta}{\pi k}\right| \mapsto t\}$.

This concludes the proof. $\square$

5.3 The $G(z)$ function and the Bessel function $K_z(Y)$

By summation of $S(s, m\tau)$ over m given the expression from previous section we will arrive at the second and third term of the analytic contnuation of $E(s, \Lambda)$ in (5.4). In the third term, the function $G(z)$ is used (5.5), which in turn depends on the Bessel function

$$K_z(Y) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{Y}{2}\left(t + \frac{1}{t}\right)\right) t^{z-1}dt.$$

This section will estimate $K_z(Y)$ and show that it is entire in z for $Y > 0$. This is used to show that $G(z)$ is an entire function. We will also calculate $G\left(\frac{1}{2}\right)$, since this is used in the expansion of $E(s, \Lambda)$.

Looking at the expression for $G(z)$ in (5.5), we are only interested in $K_z(Y)$ for Y real and strictly positive. For these values,

$$|K_z(Y)| \leq K_{\mathrm{Re}(z)}(Y) \tag{5.10}$$

by taking the absolute value inside the integration. This is used in following estimation which is similar to the one in [Lan87, K7, p. 271].

Lemma 5.5. *For $Y > 5Y_0 > 0$, $K_z(Y)$ can be estimated as*

$$|K_z(Y)| \leq Ce^{-Y}$$

where $C > 0$ is a constant only depending on $\mathrm{Re}(z)$ *and Y_0.*

Proof. First note that by using (5.10), z can be assumed to be real.

The intention here is to extract e^{-Y} from $e^{\frac{Y}{2}(t+\frac{1}{t})}$ in the integral. This can be done by splitting the integration as

$$\int_0^\infty = \int_{[0,\frac{1}{2}]\cup[2,\infty)} + \int_{[\frac{1}{2},2]}.$$

and estimating $t + \frac{1}{t}$ as

$$t + \frac{1}{t} \leq 2 + \frac{t}{5} + \frac{1}{5t} \quad \text{on} \quad [0, \frac{1}{2}] \cup [2, \infty)$$

and
$$t + \frac{1}{t} \leq 2 \quad \text{on} \quad [\frac{1}{2}, 2].$$
With these estimations we get
$$\begin{aligned} K_z(Y) &\leq \frac{e^{-Y}}{2}\left(\int_{[0,\frac{1}{2}]\cup[2,\infty]} \exp\left(-\frac{Y}{2}\left(\frac{t}{5}+\frac{1}{5t}\right)\right) t^{z-1}dt + \int_{[\frac{1}{2},2]} t^{z-1}dt\right) \\ &\leq \frac{e^{-Y}}{2}\left(K_z\left(\frac{Y}{5}\right) + A\right) \end{aligned}$$
where
$$A = \begin{cases} \frac{1}{z}(2^z - \frac{1}{2^z}) & , z \neq 0 \\ \log 4 & , z = 0 \end{cases}.$$

Looking at $K_z(Y)$ for real z, we see that it is always decreasing in Y. So $K_z\left(\frac{Y}{5}\right) \leq K_z(Y_0)$, which concludes the proof.

□

Lemma 5.6. *For $Y > 0$, $K_z(Y)$ is an entire function in z.*

Proof. With the estimate in Lemma 5.5 we can use Lebesgue's Dominated Convergence Theorem to differentiate under the integral sign. □

Theorem 5.7. *$G(z)$ is an entire function.*

Proof. Since $K_z(Y)$ is an entire function in z for $Y > 0$, the only thing which has to be shown is absolute convergence.

Fixing an $M > 0$, then the estimate for $K_z(Y)$ in Lemma 5.5 can be used as
$$\sum_{\substack{m,k\in\mathbb{Z} \\ |km|>M}} \left|\left|\frac{k}{m}\right|^z K_z(2\pi|km|\operatorname{Im}\tau)e^{2\pi ikm\operatorname{Re}\tau}\right| \leq \sum_{\substack{m,k\in\mathbb{Z} \\ |km|>M}} \left|\frac{k}{m}\right|^{\operatorname{Re}(z)} Ce^{-2\pi|km|\operatorname{Im}\tau}$$
which converges for any fixed z. This shows that the sum $G(z)$ is absolutely convergent. Therefore $G(z)$ is entire since $K_z(Y)$ is shown to be entire in Lemma 5.6. □

Lemma 5.8.
$$G\left(\frac{1}{2}\right) = -\frac{1}{\sqrt{\operatorname{Im}\tau}} \log(P(q)P(\bar{q})).$$
where $q = e^{2\pi i\tau}$ and $\bar{q} = \overline{e^{2\pi i\tau}} = e^{2\pi i(-\bar{\tau})}$.

Proof. By calculating that $K_{\frac{1}{2}}(Y) = \sqrt{\frac{\pi}{2Y}}e^{-Y}$ (see Appendix A.1) we can calculate
$$\begin{aligned} G\left(\frac{1}{2}\right) &= \frac{1}{2\sqrt{\operatorname{Im}\tau}} \sum_{\substack{m\in\mathbb{Z} \\ m\neq0}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq0}} \frac{1}{|m|} e^{-2\pi|km|\operatorname{Im}\tau} e^{2\pi ikm\operatorname{Re}\tau} \\ &= \frac{1}{\sqrt{\operatorname{Im}\tau}} \sum_{m>0} \sum_{k>0} \frac{1}{m}(q^{km} + \bar{q}^{km}). \end{aligned}$$

This can also be expressed with the infinite product

$$P(q) = \prod_{n>0} (1 - q^n)$$

since

$$\sum_{m>0} \sum_{k>0} \frac{1}{m} q^{km} = \sum_{k>0} - \log(1 - q^k) = - \log(P(q))$$

by using the Taylor expansion $- \log(1 - z) = \sum_{k>0} \frac{z^k}{k}$ (principal branch of complex logarithm). This expansion is valid for $|z| < 1$ and $|q^k| < 1$ since $\operatorname{Im} \tau > 0$. This concludes the proof. □

5.4 Analytic continuation of $E(s, \Lambda)$

In this section the second and third term of $E(s, \Lambda)$ in (5.4) is extracted from the expression of $S(s, m\tau)$ in (5.8). Furthermore the poles of $E(s, \Lambda)$ are identified to conclude the proof of Theorem 5.2.

From (5.8) we get the the second term of $E(s, \Lambda)$ as

$$\begin{aligned} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \phi(m \operatorname{Im} \tau, s, 0) &= \sqrt{\pi} \Gamma \left(s - \frac{1}{2} \right) \frac{1}{\Gamma(s)} \operatorname{Im} \tau^{1-2s} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} |m|^{1-2s} \\ &= 2\sqrt{\pi} \Gamma \left(s - \frac{1}{2} \right) \frac{1}{\Gamma(s)} \zeta(2s - 1) \operatorname{Im} \tau^{1-2s} \\ &= 2 \left(\frac{\pi}{\operatorname{Im} \tau} \right)^{2s-1} \frac{\Gamma(1 - s)}{\Gamma(s)} \zeta(2 - 2s) \end{aligned}$$

where the last rewrite was using the functional equation of Riemann ζ-function [Apo76, Theorem 12.7 and p. 260]

$$\sqrt{\pi} \Gamma \left(\frac{z}{2} \right) \zeta(z) = \pi^z \Gamma \left(\frac{1 - z}{2} \right) \zeta(1 - z).$$

The third term of $E(s, \Lambda)$ is

$$\sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \phi(m \operatorname{Im} \tau, s, k) e^{2\pi i k m \operatorname{Re} \tau} = 2\sqrt{\pi} \left(\frac{\pi}{\operatorname{Im} \tau} \right)^{s - \frac{1}{2}} \Gamma^{-1}(s) G \left(s - \frac{1}{2} \right)$$

where

$$G(z) = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \left| \frac{k}{m} \right|^z K_z(2\pi |km| \operatorname{Im} \tau) e^{2\pi i k m \operatorname{Re} \tau}.$$

Adding all three terms together we arrive at

$$\begin{aligned}(\operatorname{Im}\tau)^{-s}E(s,\Lambda) = 2\zeta(2s)& \\ &+2\left(\frac{\pi}{\operatorname{Im}\tau}\right)^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\zeta(2-2s)\\ &+2\sqrt{\pi}\left(\frac{\pi}{\operatorname{Im}\tau}\right)^{s-\frac{1}{2}}\frac{1}{\Gamma(s)}G\left(s-\frac{1}{2}\right)\end{aligned}$$

which is meromorphic since Γ and ζ are meromorphic and $G(z)$ showed to be entire in Theorem 5.7.

That $E(s,\Lambda)$ has a single pole at $s=1$ come from the factor $\Gamma(1-s)$ which has poles at $s=1,2,3,\dots$, but all except $s=1$ are canceled by the trivial zeros of $\zeta(2-2s)$. Note that $\zeta(2s)$ and $\zeta(2-2s)$ both have a pole at $s=\frac{1}{2}$, but these residues cancel each other and $E(s,\Lambda)$ is then holomorphic around $s=\frac{1}{2}$. Furthermore, note that the reciprocal gamma function $1/\Gamma(s)$ is entire. Many of these facts can be found in [Apo76, Chapter 12].

This concludes the proof of the analytic continuation of $E(s,\Lambda)$ in Theorem 5.2.

5.5 Expansion of $E(s,\Lambda)$ at $s=0$

In this section we will use the analytic continuation (5.4) of $E(s,\Lambda)$ to conclude the proof of Kronecker limit formula in Theorem 5.1, i.e. the expansion of $E(s,\Lambda)$ at $s=0$. Since $E(s,\Lambda)$ is holomorphic around $s=0$, we can expand it as a Taylor series.

To expand $E(s,\Lambda)$ at $s=0$ we will need the value $G(-\frac{1}{2})$. Note that $G(z)$ is even in z since we can exchange the summations and $K_z(Y)$ is also even in z by the change of variable $\{t\mapsto\frac{1}{t}\}$ in the integration. This means that $G(-\frac{1}{2})=G(\frac{1}{2})$, so the same value can also be used to expand $E(s,\Lambda)$ at $s=1$. By Theorem 5.8

$$G\left(-\frac{1}{2}\right) = -\frac{1}{\sqrt{\operatorname{Im}\tau}}\log(P(q)P(\bar{q}))$$

where $q=e^{2\pi i\tau}$.

Since $1/\Gamma(0)=0$ we get that $E(0,\Lambda)=2\zeta(0)=-1$.

Considering the derivative $\frac{dE}{ds}(s,\Lambda)$, $1/\Gamma(0)=0$ also results in most of the terms vanishing. It is calculated as

$$\begin{aligned}\frac{dE}{ds}(0,\Lambda) &= 2\log(\operatorname{Im}\tau)\zeta(0)+4\zeta'(0)+[1/\Gamma]'(0)\left(2\frac{\operatorname{Im}\tau}{\pi}\zeta(2)+2\sqrt{\operatorname{Im}\tau}G\left(\frac{1}{2}\right)\right)\\ &= -\log(\operatorname{Im}\tau)-2\log(2\pi)+\frac{1}{3}\pi\operatorname{Im}\tau-2\log(P(q)P(\bar{q}))\\ &= -\log\left(\operatorname{Im}\tau(2\pi)^2e^{-\frac{\pi}{3}\operatorname{Im}\tau}\left(P(q)P(\bar{q})\right)^2\right)\\ &= -\log\left(\operatorname{Im}\tau(2\pi)^2|\eta(\tau)|^4\right)\end{aligned}$$

where η is the Dedekind eta function defined in (5.3) and the following are used:

$$\zeta(0) = -\frac{1}{2}, \zeta'(0) = -\frac{1}{2}\log(2\pi), \zeta(2) = \frac{\pi^2}{6}, \Gamma(1) = 1, [1/\Gamma]'(0) = 1$$

where the derivative can be found in [Apo85] and the rest in [Apo76, Chapter 12].

This shows

$$E(s,\Lambda) = -1 - \log\left(\operatorname{Im}\tau(2\pi)^2|\eta(\tau)|^4\right)s + O(s^2)$$

which concludes the proof of the Kronecker limit formula.

5.6 Modular forms and the modular discriminant

This section will introduce modular forms and the modular discriminant, which will important later in Section 6.5 and also connect back to some calculations in Section 3.1.

The result of the Kroneckers limit formula was expressed using the Dedekind Eta function from (5.3). This is a holomorphic function on $\mathbb{H}$ and $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$ give the relations

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau) \quad \text{and} \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau) \tag{5.11}$$

for the two generators of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$ [Lan95, Theorem 5.1, p. 159]. Here $\sqrt{a}$ means $\sqrt{|a|e^{i\frac{\arg a}{2}}}$ where $-\pi < \arg a \leq \pi$.

Raising η to the 24:th power makes it a so-called *modular form of weight 12*, and adding a scaling factor we get the *modular discriminant*. These will now be introduced.

Definition 5.9. A modular form of weight k is a holomorphic function g on $\mathbb{H}$ s.t.

1. for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, $\quad g\left(\frac{az+b}{cz+d}\right) = (cz+d)^k g(z)$,
2. g is holomorphic at infinity, meaning $\lim_{z\to i\infty} g(z)$ exists.

Definition 5.10. Define the modular discriminant of a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ as

$$\Delta = \left(\frac{2\pi}{\omega_1}\right)^{12}\eta^{24}(\tau) \tag{5.12}$$

where $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}$. From the properties of η in (5.11) we see that this is a modular form of weight 12.

Remark 5.11. The factor ω_1 is often set to 1 as a normalization of the lattice, i.e. letting the lattice be expressed as $\mathbb{Z} + \tau\mathbb{Z}$ instead of $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. To see some motivation of this normalization one could look at the complex moduli space of an elliptic curve in Section 3.1.

The factor ω_1 will be kept throughout this text, which is also the case in for example [And76]. Some authors also seem to remove the factor 2π.

More modular forms that will be used in this text are the Eisenstein series.

Definition 5.12. The Eisenstein series of weight $2k$ for a lattice Λ is defined as

$$G_{2k}(\Lambda) = \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \frac{1}{\lambda^{2k}}.$$

That the result of Kronecker limit formula is expressed using modular forms is natural, since the real analytic Eisenstein series in (5.1) is $\mathrm{SL}(2,\mathbb{Z})$-invariant. To see this directly in the result (5.2), one notes that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

$$\operatorname{Im}\gamma\tau = |c\tau+d|^{-2}\operatorname{Im}\tau \quad \text{and} \quad |\eta(\gamma\tau)|^2 = |c\tau+d|\;|\eta(\tau)|^2.$$

The modular discriminant Δ will also appear in other contexts in this text. In Section 2.3 it appears as the polynomial discriminant of the Weierstrass equation. The polynomial discriminant of a quadratic polynomial

$$ax^3+bx^2+cx+d \quad \text{is} \quad b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd$$

and for a Weierstrass equation $y^2 = 4x^3 - g_2x - g_3$ the polynomial discriminant of the right-hand side is $16(g_2^3 - 27g_3^2)$. Letting $g_2 = 60G_4$ and $g_3 = 140G_6$ as in the proof of Claim 2.18, we get

$$16\frac{(2\pi)^{12}}{1728}\left(\left(\frac{G_4}{2\zeta(4)}\right)^3 - \left(\frac{G_6}{2\zeta(6)}\right)^2\right)$$

where we used that $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$ [Apo76, Theorem 12.17].

Indeed

$$\Delta = g_2^3 - 27g_3^2 = \frac{(2\pi)^{12}}{1728}\left(\left(\frac{G_4}{2\zeta(4)}\right)^3 - \left(\frac{G_6}{2\zeta(6)}\right)^2\right), \tag{5.13}$$

[Lan95, Theorem 4.1, p. 156] i.e. the polynomial discriminant of the Weierstrass polynomial divided by a factor 16.

Remark 5.13. The definition of the modular discriminant given in Definition 5.10 is sometimes called the q-series expansion and then the modular discriminant instead defined by (5.13). This is the case in [Lan95], along with a different normalization. So in [Lan95], (5.12) is proven from (5.13).

In sight of the discussion in Remark 5.11, (5.13) includes both the factors 2π and ω_1, where the ω_1 factor comes from G_{2k} since these are defined with ω_1. The fact that these expressions are the same for the modular discriminant will be used in Section 6.5 concerning with the *holomorphic anomaly equation*.

6

B-model at genus one for dimension one

This chapter will be concerned with the B-model calculation of mirror symmetry at genus one in the case of a Calabi-Yau onefold, which can be described with an elliptic curve. This calculation will be denoted $\mathcal{F}_1^B$. After a *holomorphic limit* described later in Section 7.1 and the mirror map described in Section 3.3, mirror symmetry conjectures that $\mathcal{F}_1^B$ should correspond to the generating series F_1^A from the A-model in Chapter 4.

The formulation of $\mathcal{F}_1^B$ used in this text will build on *analytic torsion* introduced by Bershadsky, Cecotti, Ooguri and Vafa in [BCOV94]. In the case of elliptic curves, this will result in the Kronecker limit formula in Chapter 5. One may note that the formulation with analytic torsion is specific for mirror symmetry at genus one.

The analytic torsion of a Calabi-Yau manifold X is given by

$$\mathcal{F}_1^B(X) = -\frac{1}{2}\log\left(\prod_{p,q}\left(\det \Delta_{\bar{\partial}}^{p,q}\right)^{(-1)^{p+q}pq}\right)$$

which is a rewritten form given in [EMM19], although there the $-\frac{1}{2}\log$ is not used. Here $\det \Delta_{\bar{\partial}}^{p,q}$ is the *ζ-regularized determinant* of the *Dolbeault Laplace operator* $\Delta_{\bar{\partial}}^{p,q}$ acting on (p,q)-forms $\Omega^{p,q}(X)$. Both these concepts will be introduced later in this chapter. It is important to note that to construct the Dolbeault Laplace operator it is required to specify a Riemannian metric (Definition 3.4).

In the case of an elliptic curve E, the product in definition of $\mathcal{F}_1^B(X)$ only has one term since there are only one type of (p,q)-form s.t. $pq \neq 0$, namely $(1,1)$-forms. That is

$$\mathcal{F}_1^B(E) = -\frac{1}{2}\log\left(\det \Delta_{\bar{\partial}}^{1,1}\right). \tag{6.1}$$

As will be discussed in Section 6.2, we can specifying the Riemannian metric as the unique one such that the induced volume form ρ is the real-part of a $(1,1)$-form with f holomorphic for any local representation $f dz \wedge d\bar{z}$ aswell as requiring that $\int_E \rho = 2$. Then the analytic torsion of an elliptic curve is given in following theorem.

Theorem 6.1. *The analytic torsion of Calabi-Yau onefold E_τ is*

$$\mathcal{F}_1^B(E_\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right) \tag{6.2}$$

where $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is the parameter of the complex moduli space and η is the Dedekind Eta function (5.3)*.*

In contrast to the A-model, the B-model does not require the additional structure of a complexified Kähler class and hence does not depend on the Kähler moduli space. Instead it will only depend on the complex moduli space described in Section 3.1 as $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. It is then natural that $\mathcal{F}_1^B$ is $\mathrm{SL}(2,\mathbb{Z})$-invariant.

After $\mathcal{F}_1^B$ has been calculated from analytic torsion, an alternative definition using the *holomorphic anomaly equation* will be presented in Section 6.5. With somewhat *ad hoc* boundary conditions this gives the same result. Since this is not using the Kronecker limit formula, this would suggest an alternative way of proving the Kronecker limit formula, which is explored in Section 6.6. This is indeed a proof, but it relies on many of the calculations in Chapter 5.2.

6.1 ζ-regularized determinant

This section will generalize the matrix determinant to operators acting of infinite-dimensional spaces. Similar to matrices, we would like the determinant to be the product of the eigenvalues. But for an operator on an infinite-dimensional space, this might be divergent.

Definition 6.2. For an operator L with eigenvalues $\{\lambda\}$ counting multiplicity, such that all $\lambda \neq 0$, define the the ζ-regularized determinant as

$$\det L = \exp(-\zeta'(0))$$

where ζ is the analytic continuation of

$$\zeta(s) = \sum \frac{1}{\lambda^s}.$$

This can be heuristically motivated as follows. At s where the sum $\zeta(s)$ converges absolutely,

$$\zeta'(s) = \sum \frac{-\log\lambda}{\lambda^s}.$$

So if the ζ would converge absolutely for $s = 0$ then $\zeta'(0) = \sum -\log\lambda = -\log(\prod\lambda)$ which means that $\det L = \prod\lambda$. This indeed proves that this is consistent if L operates on a finite dimensional space.

6.2 Dolbeault Laplace operator $\Delta_{\bar{\partial}}$ on an elliptic curve

This section will introduce the Dolbeault Laplace operator in general and then realizes its action on $(1,1)$-forms on an elliptic curve which is needed to calculate $\mathcal{F}_1^B$.

The definition of the Dolbeault Laplace operator depends on a Riemannian metric as well as an induced volume form, via the L^2-inner product on k-forms. On an elliptic curve this form is a Kähler form, so one might expect that this relates to the Kähler moduli space. But the Kähler moduli deals with complexified Kähler classes, whereas this definition only depends on a Kähler form. So to make the B-model independent of the Kähler moduli space, the choice of Riemannian metric, and therefore Kähler form, will be made completely separately from the choice of complexified Kähler class in the A-model. This might seem confusing, but the choice of Riemannian metric will be very natural.

In the following definitions, let M be a complex manifold of (complex) dimension n (defined in Appendix B.1) and let $g : TM \to TM$ be a Riemannian metric on M (Definition 3.4).

Definition 6.3 (Induced pointwise inner product on differential forms)**.** For a point $p \in M$, let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2n}}$ be a basis of T_pM and $dx_1, \dots, dx_{2n}$ be the dual basis of T_p^*M. Let

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

be the local matrix representation of the Riemannian metric g. Then define the induced pointwise inner product of 1-forms at p such that

$$\langle dx_i, dx_j \rangle_p := g^{ij}(p)$$

where $g^{ij}(p)$ the inverse of the matrix $g_{ij}(p)$. Note that when complex coefficients are used, the pointwise inner product of 1-forms needs to be conjugate linear in the second argument.

For k-forms of the form $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ where all v_i and w_i are 1-forms, define the induced pointwise inner product as

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle_p).$$

Extend linearly to remaining k-forms.

Definition 6.4 (L^2-inner product on differential forms)**.** For two k-forms $\nu, \nu' \in \Omega^k(M, \mathbb{C})$, define the L^2-inner product by

$$\langle \nu, \nu' \rangle_{L^2} = \int_M \langle \nu_p, \nu'_p \rangle_p \, \omega$$

where $\langle \cdot, \cdot \rangle_p$ is the pointwise inner product on the cotangent space induced by the Riemannian metric g, and $\rho(u, v) := \bigwedge^n g(Iu, v)$ is the volume form induced by g and the almost complex structure I. In the case of an elliptic curve, ρ is a Kähler form, see Remark 3.6.

Definition 6.5 (Dolbeault operators)**.** Using the Wertinger derivatives

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \text{ and } \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$$

as the basis for $TM \otimes \mathbb{C}$, the exterior derivative d acts locally on a (p, q)-form

$$\omega = \sum_{|I|=p,\ |J|=q} f_{IJ} dz_I \wedge d\bar{z}_J$$

as

$$d\omega = \sum_{m}^{n} \sum_{|I|=p,\ |J|=q} \left(\frac{\partial f_{IJ}}{\partial z_m} dz_m + \frac{\partial f_{IJ}}{\partial \bar{z}_m} d\bar{z}_m\right) \wedge dz_I \wedge d\bar{z}_J.$$

This leads to the definition of the Dolbeault operators ∂ and $\bar{\partial}$ as

$$\partial\omega = \sum_{m}^{n} \sum_{|I|=p,\ |J|=q} \frac{\partial f_{IJ}}{\partial z_m} dz_m \wedge dz_I \wedge d\bar{z}_J$$

and

$$\bar{\partial}\omega = \sum_{m}^{n} \sum_{|I|=p,\ |J|=q} \frac{\partial f_{IJ}}{\partial \bar{z}_m} d\bar{z}_m \wedge dz_I \wedge d\bar{z}_J.$$

So we see that $d = \partial + \bar{\partial}$,

$$\partial(\Omega^{p,q}(M)) \subseteq \Omega^{p+1,q}(M) \quad \text{and} \quad \bar{\partial}(\Omega^{p,q}(M)) \subseteq \Omega^{p,q+1}(M).$$

In this sense, if we let $\pi^{p,q}$ be the projection of $\Omega^{p+q}(M, \mathbb{C})$ to $\Omega^{p,q}(M)$, then

$$\partial|_{\Omega^{p,q}} = \pi^{p+1,q} \circ d|_{\Omega^{p+q}} \quad \text{and} \quad \bar{\partial}|_{\Omega^{p,q}} = \pi^{p,q+1} \circ d|_{\Omega^{p+q}}.$$

Definition 6.6 (Dolbeault Laplace operator)**.** Let the Dolbeault Laplace operator be

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

where $\bar{\partial}^*$ is the adjoint of the Dolbeault operator $\bar{\partial}$ with respect to the L^2-innerproduct, i.e. $\langle \bar{\partial}^* u, v \rangle = \langle u, \bar{\partial} v \rangle$ for any suitable differential forms u and v. Not that $\bar{\partial}^*$ takes $\Omega^{p,q}(M)$ to $\Omega^{p,q-1}(M)$, so $\Delta_{\bar{\partial}}$ takes $\Omega^{p,q}(M)$ to itself.

Denote $\Delta_{\bar{\partial}}^{p,q}$ for the restriction $\Delta_{\bar{\partial}}|_{\Omega^{p,q}(M)}$.

Now we return to the case of an elliptic curve E, for which will use the Riemann surface description $\mathbb{C}/\Lambda$. In the proof of Theorem 3.9 in Section 3.2 we saw that any Riemannian metric that is Kähler are of the form such that

$g_{ij}(p) = a\mathbb{1}$ for some strictly positive smooth function a on $\mathbb{C}/\Lambda$. Furthermore, in one dimension, the induced volume form is the Kähler form $\rho = a\ dx \wedge dy$. We also note that $g^{ij}(p) = (1/a)\mathbb{1}$.

Then going through the definitions [1], we see that for an arbitrary $(1,1)$-form $f\ dz \wedge d\bar{z}$ on E we get

$$\bar{\partial}^*(f\ dz \wedge d\bar{z}) = 2\left(\frac{\partial}{\partial \bar{z}}\frac{f}{a}\right) dz$$

and therefore

$$\Delta_{\bar{\partial}}^{1,1}(f\ dz \wedge d\bar{z}) = -2\left(\frac{\partial^2}{\partial z \partial \bar{z}}\frac{f}{a}\right) dz \wedge d\bar{z} = -\frac{1}{2}\left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{f}{a}\right)\ dz \wedge d\bar{z}.$$

We choose the function a to be constant. Specifically we choose

$$g_{ij} = \frac{2}{\operatorname{coVol}\Lambda}\ \mathbb{1} \tag{6.3}$$

where the scaling factor $\operatorname{coVol}\Lambda$ is the volume of a unit cell of the lattice Λ, using the Riemann surface description $\mathbb{C}/\Lambda$. Note that the induced volume form ρ is uniquely specified as the real-part of a $(1,1)$-form with f holomorphic for any local representation $f dz \wedge d\bar{z}$ aswell as requiring that

$$\int_E \rho = 2.$$

This is motivated by a similar discussion as for holomorphic $(1,0)$-forms in Section 2.2 and that there is only one holomorphic function on $\mathbb{C}/\Lambda$ up to some factor by Lemma 2.9.

We can then realize $\Delta_{\bar{\partial}}^{1,1}$ as

$$-\operatorname{coVol}\Lambda\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{6.4}$$

acting on smooth Λ-periodic functions on $\mathbb{C}$.

Remark 6.7. The factor 2 in (6.3) may seem misplaced. One could try to push that normalisation elsewhere, for example this 2 would not be needed if one used the Laplace-de Rham operator $\Delta = dd^* + d^*d$ instead of the Dolbeault Laplace operator $\Delta_{\bar{\partial}}$ since in the case of Kähler manifolds, $\Delta = 2\Delta_{\bar{\partial}}$ [Bal06, 5.26 Corollary, p. 64]. However this change of definitions is hard to motivate from the point of view of this text.

One may also note that the correspondence in Chapter 7 will be made with the derivative of $\mathcal{F}_1^B$, and then this factor will be removed.

[1] At one point Stokes theorem is used, i.e. $\int_M d\omega = \int_{\partial M} \omega$. But in this case, an elliptic curve has no boundary, meaning $\int_E d\omega = 0$ for any ω.

6.3 Eigenvalues of Laplace operator on a torus

In this section the eigenvalues of the Laplace operator $\Delta_{\bar{\partial}}^{1,1}$ will be derived in order to then calculate its ζ-regularized determinant. We will use the realization of $\Delta_{\bar{\partial}}^{1,1}$ stated in (6.4) acting smooth Λ-periodic functions on $\mathbb{C}$. Here Λ is a lattice in the complex plane which comes from the Riemann surface description $\mathbb{C}/\Lambda$.

Since it is of similar difficulty, we will derive the eigenvalues of the Laplace operator on an n-torus $\mathbb{R}^n/\Lambda$ where Λ is a lattice in $\mathbb{R}^n$. This is done without any scaling factor.

Proposition 6.8. *Let $\mathbb{R}^n/\Lambda$ be an n-torus where Λ is an n-lattice. The eigenvalues, counting multiplicity, for the Laplace operator $\Delta = -\sum_i^n \frac{\partial^2}{\partial x_i^2}$ are the square norm of the non-zero elements in the dual lattice $\Lambda^\vee$, namely*

$$\left\{ \|\lambda\|^2 \;\middle|\; \lambda \in \Lambda^\vee \setminus \{0\} \right\}$$

for the dual lattice

$$\Lambda^\vee := \{ x \in \mathbb{R}^n \mid \langle x, v\rangle \in 2\pi\mathbb{Z} \;\forall \text{ generators } v \text{ of } \Lambda \}$$

where $\langle\cdot,\cdot\rangle$ denotes standard scalar product on $\mathbb{R}^n$.

Proof. Let the lattice Λ be generated by vectors $v_1, \ldots, v_n \in \mathbb{R}^n$. Let u be a function on the torus, i.e. $u(x + v_i) = u(x)\ \forall i$, that is an eigensolution the Laplace-operator, i.e. $\Delta u = \lambda u$. Construct an ansatz with separation of variables,

$$u(x_1, \ldots, x_n) = \prod_k^n u_k(x_k) \quad \text{so that} \quad \lambda = \frac{\Delta u}{u} = -\sum_k^n \frac{u_k''(x_k)}{u_k(x_k)}$$

for all x such that $u_k(x_k) \neq 0$.

Since this is a sum of functions of independent variables we get that $\frac{u_k''}{u_k}$ is constant. With the argument that each u_k has to be periodic, let $\frac{u_k''}{u_k} = -c_k^2$ for some $c_k \in \mathbb{R}$ which gives the solution $u_k(x_k) = Ae^{ic_kx_k} + Be^{-ic_kx_k}$, where $\sum_k^n c_k^2 = \lambda$. Combining all these u_k together, we can express

$$u(x) = A\prod_k^n e^{ic_kx_k} + B\prod_k^n e^{-ic_kx_k}$$

since the terms mixing e^{ix_k} and e^{-ix_l} can be expressed with different c_k.

Let $\langle\cdot,\cdot\rangle$ denotes standard scalar product on $\mathbb{R}^n$. We can write is as

$$u(x) = Ae^{i\langle c,x\rangle} + Be^{-i\langle c,x\rangle} = C\cos(\langle c, x\rangle) + D\sin(\langle c, x\rangle)$$

as the general solution, where $C = A + B, D = iA - iB$.

Using periodicity, for any lattice generator v_m we get that

$$u\left(\frac{v_m}{2}\right) = u\left(-\frac{v_m}{2}\right)$$

so necessarily $\sin(\frac{1}{2}\langle c, v_m\rangle) = 0$ which is equivalent to $\langle c, v_m\rangle = 2\pi k_m$ for some $k_m \in \mathbb{Z}$. That is, c has to be in the dual lattice

$$\Lambda^\vee := \{\, x \in \mathbb{R}^n \mid \langle x, v_m\rangle \in 2\pi\mathbb{Z}\ \forall m \,\}.$$

This is a bijection to any such eigensolution, and to show that these are all the eigensolutions we note that these form a Fourier basis and therefore span the function space. So there is a bijection between eigensolutions and dual lattice points $c \in \Lambda^\vee$ each with eigenvalue $\lambda = \sum_k^n c_k^2 = \|c\|^2$. □

Now we return to the 2-torus, i.e. the elliptic curve. Note that in the following, we are using a different meaning of the imaginary i. This is because the above was over $\mathbb{R}^n$ and the following will be $\mathbb{C} \simeq \mathbb{R}^2$.

For the 2-torus $\mathbb{C}/\Lambda$ where $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ the generating vectors for the dual lattice $\Lambda^\vee = \nu_1\mathbb{Z} + \nu_2\mathbb{Z}$ can be $\nu_1 = -\frac{2\pi i}{\operatorname{coVol}\Lambda}\omega_2$ and $\nu_2 = \frac{2\pi i}{\operatorname{coVol}\Lambda}\omega_1$, where $\operatorname{coVol}\Lambda$ is the volume of the unit cell of the lattice. Together with the scaling factor of $\Delta_{\bar\partial}^{1,1}$ this gives the eigenvalues

$$\operatorname{coVol}\Lambda\ |n\nu_1 + m\nu_2|^2 = \frac{(2\pi)^2}{\operatorname{coVol}\Lambda}\ |-n\omega_2 + m\omega_1|^2$$

which are scaled square distances lattice points in Λ. So the eigenvalues of $\Delta_{\bar\partial}^{1,1}$ on $\mathbb{C}/\Lambda$ is

$$\left\{ \frac{(2\pi)^2}{\operatorname{coVol}\Lambda}|\lambda|^2 \;\middle|\; \lambda \in \Lambda \setminus \{0\} \right\}$$

counting multiplicity.

6.4 Final rewriting of $\mathcal{F}_1^B$

With the eigenvalues calculated in Section 6.3, the calculation of the ζ-regularized determinant $\det \Delta_{\bar\partial}^{1,1}$ (Definition 6.2) uses the analytic continuation of

$$\zeta(s) = \left(\frac{\operatorname{coVol}\Lambda}{(2\pi)^2}\right)^s \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} |\lambda|^{-2s}.$$

This is exactly the real analytic Eisenstein series $E(s,\Lambda)$ up to a scaling factor $(2\pi)^{-2s}$, i.e. $\zeta(s) = (2\pi)^{-2s}E(s,\Lambda)$. That this analytic continuation exists is proved in Theorem 5.2.

By using the Kronecker limit formula in Theorem 5.1 we get the expression of $\det \Delta_{\bar\partial}^{1,1}$ and therefore we obtain

$$\mathcal{F}_1^B(E_\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right)$$

from (6.1), where $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is the parameter of the complex moduli space. This concludes the proof of Theorem 6.1.

Note that $\mathcal{F}_1^B(E_\tau)$ inherits $\mathrm{SL}(2,\mathbb{Z})$-invariance from the real analytic Eisenstein series, which was discussed in Section 5.6. This is natural since $\mathcal{F}_1^B(E_\tau)$ only depends on the complex moduli of E_τ, and therefore only on the class of τ in $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$.

6.5 Holomorphic anomaly equation

An alternative way of defining $\mathcal{F}_1^B$ besides analytic torsion would be through the *holomorphic anomaly equation*

$$\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\mathcal{F}_1^B(E) = \frac{1}{8\operatorname{Im}\tau^2} \tag{6.5}$$

which in general is expressed with the dd^c-operator, as explained in [KZ14]. Here $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is the parametrization of the complex moduli space (see Section 3.1). Given that $\mathcal{F}_1^B(\tau)$ is a smooth real-valued function, $\mathrm{SL}(2,\mathbb{Z})$-invariant and satisfies the holomorphic anomaly equation with some boundary conditions, we will get a unique solution which is the same result as expression (6.2) which used analytic torsion. This will be shown here and condensed into Theorem 6.13. The $\mathrm{SL}(2,\mathbb{Z})$ invariance is natural since $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$.

As boundary conditions we say that

$$\lim_{\tau\to i\infty}\mathcal{F}_1^B(\tau) = \infty \quad \text{and} \quad \lim_{\tau\to i\infty}\mathcal{F}_1^B(\tau) - \left(-\frac{1}{2}\log\operatorname{Im}\tau + \frac{\pi}{6}\operatorname{Im}\tau\right) = 0. \tag{6.6}$$

which are decided only such that this derivation works. Especially the last condition is very *ad hoc*, and without this we would only get the correct expression up to a constant. Instead of a growth condition one could construct the second boundary condition as a value at a certain point or an integral perhaps from 0 to $i\infty$. Note that the second condition implies the first, but they will be kept separate since the first should be easier to motivate and without the second one we get the right result up to an unknown constant.

First we see that $\frac{\partial^2}{\partial\tau\partial\bar{\tau}} = \frac{1}{4}\Delta$ where $\Delta = \partial_x^2 + \partial_y^2$ expressing $\tau = x + iy$. We see that $-\frac{1}{2}\log y$ is a solution to (6.5). To get the homogeneous solution we will use following theorem that is commonly shown in a first course in complex analysis.

Theorem 6.9. *Let u be a real-valued harmonic, i.e. $\Delta u = 0$, function on an simply connected domain $\Omega \subseteq \mathbb{C}$, then there exists an holomorphic function g s.t. $u = \operatorname{Re} g$ on Ω.*

Proof. To find a candidate g by first constructing it's derivative. Let

$$h = \partial_x u - i\partial_y u$$

then $\partial_{\bar{z}} h = 0$ since u is harmonic. So h is holomorphic. The anti-derivative of h will be constructed as

$$g(z) = u(z_0) + \int_\gamma h(w)dw$$

where γ is a path from some base-point $z_0 \in \Omega$ to $z \in \Omega$. One calculates that $\lim_{z' \to z} \frac{g(z')-g(z)}{z'-z} = h(z)$ by using Cauchy's integral theorem, which requires that Ω is simply connected. This shows that g is holomorphic and with Cauchy-Riemann equations we see that $\partial_x u = \partial_x \operatorname{Re} g, \partial_y u = \partial_y \operatorname{Re} g$. Together with $g(z_0) = u(z_0)$ this shows that $\operatorname{Re} g = u$. □

Corollary 6.9.1. *Let u be a real-valued harmonic functionon an simply connected domain $\Omega \subseteq \mathbb{C}$, then*

$$u = \log|g|$$

for some non-zero holomorphic function g.

Proof. Let $\tilde{g}$ be the holomorphic function from Theorem 6.9 s.t. $\operatorname{Re} \tilde{g} = u$. Note that $\operatorname{Re} \tilde{g} = \log|e^{\tilde{g}}|$. Let $f = e^{\tilde{g}}$ which is non zero and holomorphic. □

Let $1/f$ be the function from Corollary 6.9.1, then we get

$$\mathcal{F}_1^B(\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|f|\right).$$

For the $\mathrm{SL}(2,\mathbb{Z})$-invariance first note that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

$$\operatorname{Im}\gamma\tau = |c\tau + d|^{-2} \operatorname{Im}\tau$$

which sets the requirement on f that

$$|f(\gamma\tau)| = |c\tau + d|\ |f(\tau)| \quad \text{so} \quad f(\gamma\tau) = e_\gamma(\tau)(c\tau + d)f(\tau)$$

for some $e_\gamma(\tau)$ with $|e_\gamma(\tau)| = 1$. The following lemma states that e_γ is independent of τ.

Lemma 6.10. *For a holomorphic function g, if $|g|$ is constant in an open domain Ω, g is constant on Ω.*

Proof. Let $g = u + iv$ and $u^2 + v^2 = C$. If $C = 0$ this is trivial. Else if $C \neq 0$ differentiate and get

$$u\partial_x u + v\partial_x v = 0 \quad \text{and} \quad u\partial_y u + v\partial_y v = 0.$$

Using the Cauchy Riemann equations we get

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = 0$$

where the matrix is invertible since $\begin{vmatrix} u & -v \\ v & u \end{vmatrix} = C \neq 0$, so $\partial_x u = \partial_y u = 0$. This implies that u is constant and by Cauchy Riemann equations so is v. □

Now f looks almost like a modular form (Definition 5.9), and this motivates us to remove the e_γ factor.

Proposition 6.11. *For any* $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, $e_\gamma^{12} = 1$.

Proof. This will be done by showing that $\gamma \mapsto e_\gamma$ is a group homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to S^1.

This is a homomorphism since we can ensure that $e_\gamma e_{\gamma'} = e_{\gamma\gamma'}$ by looking at $f(\gamma\gamma'\tau)$ explicitly as

$$e_{\gamma\gamma'}((ca'+dc')\tau+(cb'+dd'))f(\tau) = f(\gamma\gamma'\tau) = e_\gamma e_{\gamma'}(c(\gamma'\tau)+d)(c'\tau+d')f(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

But since S^1 is abelian, this homomorphism factorizes over the abelianization as

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{Z}) & \longrightarrow & \mathrm{SL}(2,\mathbb{Z})^{\mathrm{ab}} \\ & \overset{e_\gamma}{\searrow} & \downarrow \\ & & S^1 \end{array}$$

Now $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{ab}} \approx \mathbb{Z}_4 \times \mathbb{Z}_3 \approx \mathbb{Z}_{12}$ as proved in Corollary A.3.1, which implies that $e_\gamma^{12} = 1$ for any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

□

From the first boundary condition (6.6) we get

$$\lim_{\tau \to i\infty} \sqrt{\operatorname{Im}\tau}|f| = 0$$

which implies that $\lim_{\tau\to i\infty} f^{12} = 0$ and f^{12} is therefore *holomorphic at infinity.* Together with previous proposition which give us

$$f^{12}(\gamma\tau) = (c\tau+d)^{12} f^{12}(\tau)$$

we have shown that f^{12} is a modular form of weight 12, see Definition 5.9.

By [Lan95, Theorem 2.2, p. 10] the modular forms of weight 12 are spanned by G_4^3 and G_6^2 over $\mathbb{C}$, where G_{2k} is a Eisenstein series of weight $2k$, see Definition 5.12. At infinity we know that $f^{12}(\infty) = 0$, and since $G_{2k}(\infty) = 2\zeta(2k) \neq 0$ [Lan95, p.9] we get that f has to be in some one dimensional subspace. This subspace is spanned by the modular discriminant Δ (see Definition 5.10) since this is a modular forms of weight 12 and has a factor q meaning it is zero at infinity, so

$$f^{12} = c\Delta$$

for some $c \in \mathbb{C}$. This is also apparent since from Equation (5.13) where we see that

$$\Delta \propto \left(\frac{G_4}{2\zeta(4)}\right)^3 - \left(\frac{G_6}{2\zeta(6)}\right)^2.$$

Moreover, that this subspace is spanned by a modular form of weight 12 shows that we could not have looked at f^k for some $k < 12$, i.e. the homomorphism $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{ab}} \to S^1$ had to be injective.

Remark 6.12. From [Lan95, Theorem 2.1, p. 6] we see that that Δ is non-zero on $\mathbb{H}$, which indeed was needed for f as well. Although note that [Lan95] defines Δ in terms of G_4^3 and G_6^2 instead the q-expansion as Definition 5.10 and does not use the same normalization.

Thus we have arrived at

$$\mathcal{F}_1^B(\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|c\Delta|^{\frac{1}{12}}\right) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right) + A$$

for some constant $A \in \mathbb{R}$. Using the second boundary condition from (6.6) we get $A = 0$. So given $\mathrm{SL}(2,\mathbb{Z})$-invariance and these boundary conditions we solved the holomorphic anomaly equation uniquely, giving the same as the result of $\mathcal{F}_1^B$ as from the analytic torsion formulation in (6.2).

We summarise the result of this section as following theorem:

Theorem 6.13. *The holomorphic anomaly equation*

$$\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\mathcal{F}_1^B(\tau) = \frac{1}{8\operatorname{Im}\tau^2}$$

where $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ *with boundary condition*

$$\lim_{\tau\to i\infty} \mathcal{F}_1^B(\tau) - \left(-\frac{1}{2}\log\operatorname{Im}\tau + \frac{\pi}{6}\operatorname{Im}\tau\right) = 0$$

has a unique real-valued $\mathrm{SL}(2,\mathbb{Z})$*-invariant solution*

$$\mathcal{F}_1^B(\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right)$$

where η *is the Dedekind Eta function* (5.3).

6.6 Alternative proof for the Kronecker limit formula

We got the same result of $\mathcal{F}_1^B$ using the holomorphic anomaly equation in the previous section as when we used analytic torsion and the Kronecker limit formula to get (6.2). This suggests that the holomorphic anomaly equation would give an alternative way of proving the Kronecker limit formula. This will be proved here.

The given proof will use a lot of calculations from Chapter 5, particularly the analytic continuation in Theorem 5.2, and is therefore mostly theoretically interesting rather than a useful way of proving the Kronecker limit formula. It is interesting since this proof does not keep τ fixed (see Remark 6.15).

Let the real analytic Eisenstein series be

$$E(s,\tau) = \sum_{(n,m)\neq(0,0)} \frac{\operatorname{Im}\tau^s}{|n+m\tau|^{2s}} \quad \text{for} \quad \operatorname{Re}(s) > 1.$$

which is the same as in Chapter 5, where the absolute convergence for $\operatorname{Re}(s) > 1$ is proved in Lemma 5.3. Where this sum is absolutely convergent we can calculate a functional equations of $E(s,\tau)$ as

$$\frac{\partial^2}{\partial\tau\partial\bar{\tau}} E(s,\tau) = \frac{1}{4\operatorname{Im}\tau^2} s(s-1)E(s,\tau)$$

which we derivative in respect to s to get

$$\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\left(\frac{dE}{ds}(s,\tau)\right) = \frac{1}{4\operatorname{Im}\tau^2}\left(s(s-1)\left(\frac{dE}{ds}(s,\tau)\right) + (2s-1)E(s,\tau)\right).$$

Given that we can form an analytic continuation of $E(s,\tau)$ in respect to s on a domain connecting $s = 0$ and $\operatorname{Re}(s) > 1$, and if we calculate that $E(0,\tau) = 2\zeta(0) = -1$, then we get the holomorphic anomaly equation up to a factor $1/2$ as

$$\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\left(\frac{dE}{ds}(0,\tau)\right) = \frac{1}{4\operatorname{Im}\tau^2}.$$

All this was indeed proven in Chapter 5.

Moreover, we need to prove the boundary condition

$$\lim_{\tau\to i\infty} \frac{dE}{ds}(0,\tau) - \left(-\log\operatorname{Im}\tau - 2\log(2\pi) + \frac{\pi}{3}\operatorname{Im}\tau\right) = 0. \tag{6.7}$$

This is shown by letting it be a limit of s as

$$\lim_{\tau\to i\infty}\left(\lim_{s\to 0}\frac{dE}{ds}(s,\tau) - \left(\log\operatorname{Im}\tau E(s,\tau) + \operatorname{Im}\tau^s\left(4\zeta'(2s) + \frac{2}{\pi}\zeta(2-2s)\operatorname{Im}\tau\right)\right)\right)$$

and using the analytic continuation proven Theorem 5.2. The sum $G(z)$ is estimated as $O\left(\sqrt{\operatorname{Im}\tau}e^{-2\pi\operatorname{Im}\tau}\right)$ in a similar way to the proof of Theorem 5.7 and some known values of Γ-function and Riemann ζ-function are also needed. We get

$$\lim_{\tau\to i\infty}\left(\lim_{s\to 0}\frac{O(1)}{\Gamma(s)} + O\left(\sqrt{\operatorname{Im}\tau}e^{-2\pi\operatorname{Im}\tau}\right)\right) = 0$$

Then using Theorem 6.13 from previous section we get that

$$E(s,\tau) = -1 - \log\left(\operatorname{Im}\tau(2\pi)^2|\eta(\tau)|^4\right)s + O(s^2)$$

which is the same as the result in Chapter 5.

Remark 6.14. As previously mentioned, this proof depends on the analytic continuation of the real analytic Eisenstein series which was proven in Chapter 5. Since those calculations make up most of that proof, this is indeed not a useful way of proving the Kronecker limit formula. The calculations that is skipped by using this proof is the value of $G\left(\frac{1}{2}\right)$ proven in Lemma 5.8.

If one instead would prove the existence of the analytic continuation of $E(s,\tau)$ in a different way then in Chapter 5 and from this be able prove the boundary condition in (6.7), this proof would actually be a useful way to prove the Kronecker limit formula.

Remark 6.15. What is interesting with this proof is that it varies τ in contrast to the proof in Chapter 5 where τ is fixed. If one sees the real analytic Eisenetein series as a function of the complex moduli of an Riemann surface of genus one (see Section 3.1), then this proof if done by varying the complex structure of this surface.

Remark 6.16. A consequence of this proof is that it shows that the normalization of the modular discriminant is consistent between Definition 5.10 and (5.13), that is

$$\left(\frac{2\pi}{\omega_1}\right)^{12} q\prod_{n>0}(1-q^n)^{24} = \Delta = \frac{(2\pi)^{12}}{1728}\left(\left(\frac{G_4}{2\zeta(4)}\right)^3 - \left(\frac{G_6}{2\zeta(6)}\right)^2\right)$$

where $q = e^{2\pi i\tau}$ and G_{2k} is the Eisenstein series of weight $2k$ (Definition 5.12).

7

The mirror mapping and correspondence for dimension one

This section will explain the correspondence between the A-model and B-model at genus one in the case of elliptic curves. In this case the mirror pair consists of two elliptic curves and the mirror map simply interchanging the parameters the complex moduli space and the Kähler moduli space as

$$(E_\tau, \omega_t) \longleftrightarrow (E_t, \omega_\tau)$$

which was explained in Section 3.3.

What has been calculated is F_1^A in Theorem 4.2 and $\mathcal{F}_1^B$ in Theorem 6.1 as

$$F_1^A(E_\tau, \omega_t) = -\log \eta(t) \quad \text{and} \quad \mathcal{F}_1^B(E_\tau) = -\log\left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right)$$

where η is the Dedekind Eta function (5.3). Note that F_1^A is an holomorphic function with respect to t, while $\mathcal{F}_1^B$ with respect to τ is not. In order for them to correspond, a holomorphic limit of $\mathcal{F}_1^B$ is supposed to be constructed [Dij95, BCOV94, EMM19].

7.1 Holomorphic limit

According to [CR19, Eq. (4.3.2)], the holomorphic limit is constructed by choosing a base-point τ_0 for τ and form the limit

$$\lim_{\bar\tau \to \bar\tau_0} \mathcal{F}_1^B(\tau, \bar\tau)$$

seeing τ and $\bar\tau$ as separate variables. Similarly it is stated in [KZ14] that in local coordinates, the holomorphic limit is the degree zero part of a Taylor expansion with respect to $\bar\tau$, which results in the same thing given that the chosen base-point is in same set.

We choose the base-point $\tau_0 = i\infty$ and get

$$\lim_{\bar\tau \to -i\infty} \mathcal{F}_1^B(E_\tau) = -\log \eta(\tau) + \lim_{\bar\tau \to -i\infty} -\frac{1}{2}\log \operatorname{Im}\tau$$

since η is bounded and holomorphic, but this which does not converge. Non-rigorously one could say that this is $F_1^A(E_t, \omega_\tau)$ "up to an infinite constant". This is a similar result as stated in [CR19, p. 313] for local toric geometry.

Here we propose another construction. In [Dij95] it is said that in the case of genus one, one should rather look at the derivative of F_1^A and $\mathcal{F}_1^B$ in the genus one case. Forming the holomorphic limit of the derivatives with these calculations removes the "up to an infinite constant" since

$$\lim_{\bar{\tau}\to -i\infty} \partial_\tau \mathcal{F}_1^B(E_\tau) = \lim_{\bar{\tau}\to -i\infty} -\partial_\tau \log \eta(\tau) + \frac{1}{4\operatorname{Im}\tau} i = -\partial_\tau \log \eta(\tau)$$

so

$$\lim_{\bar{\tau}\to -i\infty} \partial_\tau \mathcal{F}_1^B(E_\tau) = \partial_\tau F_1^A(E_t, \omega_\tau). \tag{7.1}$$

This is one part of the conjectured mirror symmetry correspondence for dimension one. What is important to note here is that this implies a type of $\mathrm{SL}(2,\mathbb{Z})$-invariance on F_1^A since the $\mathrm{SL}(2,\mathbb{Z})$-invariance is natural on $\mathcal{F}_1^B$. And looking at the definition of F_1^A in (4.1), the action $t \mapsto t+1$ is obvious, but the action $t \mapsto -\frac{1}{t}$ is not. This does not have an apparent reason neither form looking at the Kähler moduli space nor the Gromov-Witten invariants. The moral would be that F_1^A is invariant when interchanging small and large scales of the Kähler moduli space. This is interesting, as also commented in [Dij95].

7.2 Counting curves using mirror symmetry

Another interesting consequence is that that one should be able to calculate $N_{1,d}(X)$ using $\mathcal{F}_1^B(X)$. In the one-dimensional case, calculating $N_{1,d}(E)$ is not difficult. This is in contrast to higher dimensions, where this is apparently more difficult. More generally one would like to be able to calculate $N_{g,d}(X)$ using $\mathcal{F}_1^B(X)$.

Using the limit in (7.1) and the definition of F_1^A in (4.1) one way of calculating $N_{1,d}$ for $d > 0$ would be

$$N_{1,d}(E_\tau) = \frac{e^{2\pi\nu d}}{2\pi i d} \int_0^1 \lim_{\bar{\tau}\to -i\infty} \partial_\tau \mathcal{F}_1^B(E_\tau)|_{\tau=\xi+i\nu} e^{2\pi i \xi d} d\xi$$

where ν is some fixed positive constant. This was a naive construction using Fourier series.

Note that the information about $N_{1,0}(E)$ is lost when constructing the holomorphic limit using a derivative. This might not be a big loss since $N_{1,0}(E)$ does not actually count covers in a classically meaning.

A

Residual theory

This chapter gives some theory that is needed in the main part of the text but is not critical for the understanding.

A.1 Calculation of $K_{\frac{1}{2}}(Y)$

Lemma A.1.

$$K_{\frac{1}{2}}(Y) = \sqrt{\frac{\pi}{2Y}}e^{-Y}$$

where $K_z(Y)$ is the Bessel function given in (5.6)*.*

Proof. The calculation involves many substitutions and goes as follows:

$$\begin{aligned}
K_{\frac{1}{2}}(2Y) &= \frac{1}{2}\int_0^\infty \exp\left(-Y\left(t+\frac{1}{t}\right)\right)\frac{1}{\sqrt{t}}dt \\
&= \int_0^\infty \exp\left(-Y\left(s^2+\frac{1}{s^2}\right)\right)ds \qquad \left\{t=s^2,\ dt=2\sqrt{t}ds\right\} \\
&= e^{-2Y}\int_0^\infty \exp\left(-Y\left(s-\frac{1}{s}\right)^2\right)ds
\end{aligned}$$

Split and at 0 to 1 substitute $\left\{s=\frac{1}{u},\ ds=-\frac{1}{u^2}du\right\}$ to get

$$\int_0^1 \exp\left(-Y\left(s-\frac{1}{s}\right)^2\right)ds = \int_1^\infty \exp\left(-Y\left(u-\frac{1}{u}\right)^2\right)\frac{1}{u^2}du$$

so renaming and adding it to the other term we get

$$\begin{aligned}
K_{\frac{1}{2}}(2Y) &= e^{-2Y}\int_1^\infty \exp\left(-Y\left(s-\frac{1}{s}\right)^2\right)\left(1+\frac{1}{s^2}\right)ds \\
&= e^{-2Y}\int_0^\infty \exp\left(-Yx^2\right)dx \qquad \left\{s-\frac{1}{s}=x,\ \left(1+\frac{1}{s^2}\right)ds=dx\right\} \\
&= \frac{1}{2}\sqrt{\frac{\pi}{Y}}e^{-2Y}
\end{aligned}$$

so we are done. □

A.2 Generators of $\mathrm{SL}(2,\mathbb{Z})$

Definition A.2. The group $\mathrm{SL}(2,\mathbb{Z})$ is 2×2-matrices with integer coefficients and determinant 1.

Lemma A.3. *The group* $\mathrm{SL}(2,\mathbb{Z})$ *is generated by*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. Let $G = \langle S, T\rangle$ be the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by S and T.

We first note that

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \quad \text{and} \quad T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}. \tag{A.1}$$

Our goal is to preform an division algorithm on an arbitrary element $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ using the identities in (A.1) to get a matrix of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Since the determinant is 1 this matrix is nessesarily of the form $\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}$ for some integer m. Multiplying with $T^{\mp m}$ we either get $S^2 = -I$ or the identity matrix, so we are done.

So only remains to design the division algorithm. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Assume $|a| > |c|$, otherwise multiply by S. Let $a = qc + r$ where $0 \leq r < |c|$ and multiply the matrix by ST^{-q} to get

$$\begin{pmatrix} -c & -d \\ r & b-qd \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

If $r = 0$, we are done, otherwise do this again to get a strictly smaller r. □

Corollary A.3.1.

$$\mathrm{SL}(2,\mathbb{Z})^{ab} \approx \mathbb{Z}_4 \times \mathbb{Z}_3$$

where $\mathrm{SL}(2,\mathbb{Z})^{ab}$ *is the abelianization of* $\mathrm{SL}(2,\mathbb{Z})$.

Proof. This follows from that S and $S^{-1}T$ generates $\mathrm{SL}(2,\mathbb{Z})$ since these have order 4 and 3 respectively. □

A.3 Some results of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$

This section will prove that $\mathrm{SL}(2,\mathbb{Z})$ has subgroups of finite index acting locally free on $\mathbb{H}$. This will be done by first proving that the only elements $\mathrm{SL}(2,\mathbb{Z})$ that fixes some point of $\mathbb{H}$ are the torsion elements. Then we also show that any torsion-free subgroup is acting locally free on $\mathbb{H}$. The smallest torsion-free subgroup of $\mathrm{SL}(2,\mathbb{Z})$ will be shown to be the commutator subgroup, and we can conclude that any subgroup of finite index acting locally free on $\mathbb{H}$ must be the commutator subgroup or a subgroup of that group. This implies a restriction on the orbifold Euler characteristic of $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ as

$$\chi(\mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) = 0 \quad \text{or} \quad |\chi(\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})| \geq \frac{1}{12}$$

since the usual Euler characteristic takes integer values and the commutator subgroup has index 12.

Lemma A.4. *The elements of* $\mathrm{SL}(2,\mathbb{Z})$ *which fixes some point of* $\mathbb{H}$ *must be torsion elements.*

Proof. An element of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ is acting on a point $z \in \mathbb{H}$ as

$$\gamma z = \frac{az+b}{cz+d} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

which by letting $z = x+iy$ and using $ad - bc = 1$ can be rewritten as

$$\gamma(x+iy) = \frac{ac(x^2+y^2) + (ad+bc)x + bd}{(cx+d)^2 + (cy)^2} + i\frac{y}{(cx+d)^2+(cy)^2}.$$

We get the equations

$$(cx+d)^2+(cy)^2 = 1 \quad \text{and} \quad (ac(x^2+y^2)+(ad+bc)x+bd)-((cx+d)^2+(cy)^2)x = 0. \tag{A.2}$$

which are satisfied if and only if γ fixes $x+iy$.

If $c = 0$ we get $d = \pm 1$ from the first equation in (A.2). Since the determinant is 1 we get $a = \pm 1$ and then the second equation in (A.2) gives that $b = 0$. Hence $\gamma = \pm I$.

Assuming $c \neq 0$, let $x + iy = \frac{t-d}{c} + i\frac{s}{|c|}$ for some $t + is \in \mathbb{H}$. Then the equations can be simplified to

$$t^2 + s^2 = 1 \quad \text{and} \quad t = \frac{a+d}{2}.$$

Solving these and changing back to $x+iy$ we get that

$$x = \frac{a-d}{2c} \quad \text{and} \quad y = \frac{1}{2|c|}\sqrt{4-(a+b)^2}$$

where we note that we must have that

$$|a + b| < 2.$$

If $a + b = 0$, then

$$\gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \text{and} \quad a^2 + bc = -1.$$

Furthermore, $x = \frac{a}{c}, y = \frac{1}{|c|}$. One can check that then $\gamma^2 = -I$ and hence a torsion element.

Similarly if $a + b = \pm 1$, then

$$\gamma = \begin{pmatrix} a & b \\ c & -a \pm 1 \end{pmatrix} \quad \text{and} \quad a^2 \mp a + bc = -1.$$

and one can check that $\gamma^3 = \pm I$ and is hence a torsion element. □

Note that the elements of $\mathrm{SL}(2, \mathbb{Z})$ mapped to non-zero elements in the abelianization are exactly the torsion elements. This implies that any subgroup not containing elements which fixes points of $\mathbb{H}$ must be a subgroup of the commutator subgroup $[\mathrm{SL}(2, \mathbb{Z}), \mathrm{SL}(2, \mathbb{Z})] = \{ ghg^{-1}h^{-1} \mid g, h \in \mathrm{SL}(2, \mathbb{Z}) \}$, since it can be defined with the short exact sequences

$$1 \longrightarrow [\mathrm{SL}(2, \mathbb{Z}), \mathrm{SL}(2, \mathbb{Z})] \hookrightarrow \mathrm{SL}(2, \mathbb{Z}) \longrightarrow \mathrm{SL}(2, \mathbb{Z})^{\mathrm{ab}} \longrightarrow 1.$$

The commutator group has index 12 in $\mathrm{SL}(2, \mathbb{Z})$ since that is the order of $\mathrm{SL}(2, \mathbb{Z})^{\mathrm{ab}}$.

Other subgroups to consider are the congruence subgroups $\Gamma(N) \subseteq \mathrm{SL}(2, \mathbb{Z})$, which are defined as the kernel of the map $\mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ and therefore consist of elements

$$\begin{pmatrix} Nk + 1 & Nl \\ Nm & Nn + 1 \end{pmatrix}$$

with determinant 1. Each of these have finite index in $\mathrm{SL}(2, \mathbb{Z})$ since $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ is a finite group. For $N \geq 4$ we can see that any such elements does not have any fixed points by using the condition A.3 in the previous proof, since $|(Nk + 1) + (Nn + 1)| \geq \min(2, |2 - N|)$. Using the same condition, one can also make sure that the same holds for $N = 3$ by trying to construct such a matrix.

We will prove that any torsion-free subgroup of $\mathrm{SL}(2, \mathbb{Z})$ is acting locally free on $\mathbb{H}$. To do this, we need the following lemma.

Lemma A.5. *Let*

$$\mathcal{D} := \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1, -\frac{1}{2} \leq \operatorname{Re} z < \frac{1}{2} \right\}$$

then for any $z \in \mathbb{H}$ there is a element $\gamma_0 \in \mathrm{SL}(2,\mathbb{Z})$ s.t. $\gamma_0 z \in \mathcal{D}$. Moreover, for any γ that is not a torsion elements, $\gamma(\mathcal{D}) \cap \mathcal{D} = \emptyset$.

Note that $\mathcal{D}$ (or the closure) is often called the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$.

Proof. Fix any element $z \in \mathbb{H}$.

We first note that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

Since there are a finite number of c, d such that $|cz+d|^2$ is less then some bound, we can choose an element $\gamma_0 \in \mathrm{SL}(2,\mathbb{Z})$ which maximizes $\mathrm{Im}(\gamma z)$ for the fixed $z \in \mathbb{H}$, i.e.

$$\mathrm{Im}(\gamma_0 z) \geq \mathrm{Im}(\gamma z).$$

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $\mathrm{Im}(Sz) = \frac{\mathrm{Im}(z)}{|z|^2}$ and $S\gamma_0 \in \mathrm{SL}(2,\mathbb{Z})$ we get that

$$\mathrm{Im}(\gamma_0 z) \geq \mathrm{Im}(S\gamma_0 z) = \frac{\mathrm{Im}(\gamma_0 z)}{|\gamma_0 z|^2}$$

which means that $\mathrm{Im}(\gamma_0 z) \geq 1$. Additionally, since $T^n z = z+n$ and $\mathrm{Im}(T^n z) = \mathrm{Im}(z)$ we can ensure that $-\frac{1}{2} \leq \mathrm{Re}(\gamma_0 z) < \frac{1}{2}$ by replacing γ_0 with $T^n\gamma_0$ for appropriate $n \in \mathbb{Z}$.

Now take any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and assume that there is a point $f \in \mathcal{D}$ s.t. $\gamma(f) \in \mathcal{D}$. We will show that γ is then a torsion element.

First look at

$$\mathrm{Im}(\gamma f) = \frac{\mathrm{Im}\, f}{|cf+d|^2}.$$

Since γ is a torsion element iff γ^{-1} is, we can choose to look at γ^{-1}. This is useful since by replacing γ with γ^{-1} and f with γf we may assume that $|cf+d|^2 \leq 1$.

If $|c| > 1$ then

$$|cf+d|^2 \geq (c\,\mathrm{Im}\, f)^2 \geq \frac{3}{4}c^2 > 1$$

using that $\mathrm{Im}\, f \geq \frac{\sqrt{3}}{2}$. Hence $|c| \leq 1$.

Similarly, if $|d| > 1$ then

$$|cf+d|^2 \geq (c\,\mathrm{Re}\, f + d)^2 = (c\,\mathrm{Re}\, f)^2 - 2c\,\mathrm{Re}\, f d + d^2 \geq 0 - |d| + d^2 > 1$$

using that $|\mathrm{Re}\, f| \leq \frac{1}{2}$ and $|c| \leq 1$. Hence $|d| \leq 1$.

Here we divide in to cases as $c = 0$ or $d = 0$ or $|c| = |d| = 1$.

If $c = 0$, then $|d| = 1$ and $a = d$ so $\gamma f = f \pm b$. Since $\mathrm{Re}\, f, \mathrm{Re}(\gamma f) \in [-\frac{1}{2}, \frac{1}{2})$ we get that $b = 0$ and hence $\gamma = \pm I$.

If $d = 0$, then $|c| = 1$ and $b = -c$ so $\gamma f = -\frac{1}{f} \pm a$. Using that $|f| \geq 1$ together with $\operatorname{Re} f \in [-\frac{1}{2}, \frac{1}{2})$ we get $\operatorname{Re}\left(-\frac{1}{f}\right) \in (-\frac{1}{2}, \frac{1}{2}]$, hence $a = 0$ or $\pm a = -1$ which in either case means that γ is a torsion element.

If $|c| = |d| = 1$, using that $|f \pm 1|^2 \geq |f|^2$ since $\operatorname{Re} f \in [-\frac{1}{2}, \frac{1}{2})$ we get

$$1 \geq |cf + d|^2 = |f \pm 1|^2 \geq |f|^2 \geq 1$$

and hence equality must hold s.t. $\operatorname{Re} f = -\frac{1}{2}$. Since $|f|^2 = 1$ we get that $f = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and then since $|cf + d|^2 = 1$ we know that if $\gamma f \in \mathcal{D}$ then $\gamma f = f$. By previous lemma this shows that γ is a torsion elements. $\square$

Corollary A.5.1. *Any torsion-free subgroup of* $\mathrm{SL}(2, \mathbb{Z})$ *is acting locally free on* $\mathbb{H}$.

Proof. Take any torsion-free subgroup H of $\mathrm{SL}(2, \mathbb{Z})$. From the previous lemma we know that $\gamma(\mathcal{D})$ are all disjoint for any $\gamma \in H$.

Furthermore, for any point in $\mathcal{D}$ one can construct a small neighborhood that only intersects finitely many such $\gamma(\mathcal{D})$ (seen by drawing them). Hence given any point $z \in \mathbb{H}$, let $\gamma_0 \in \mathrm{SL}(2, \mathbb{Z})$ be s.t. $\gamma_0 z \in \mathcal{D}$, and choose such a small neighborhood U of $\gamma_0 z$. For the neighborhood $V = \gamma_0^{-1}(U)$ of z, $\#\{\, \gamma \in \mathrm{SL}(2, \mathbb{Z}) \mid \gamma(V) \cap V \,\}$ is then finite.

Using that H has no elements fixing points of $\mathbb{H}$ we can use that $\mathbb{H}$ is Hausdorff to shrink V until $\#\{\, \gamma \in \mathrm{SL}(2, \mathbb{Z}) \mid \gamma(V) \cap V \,\} = 1$. This proves that H acts locally free on $\mathbb{H}$. $\square$

What has been done in this section is that we have now shown that $\mathrm{SL}(2, \mathbb{Z})$ has subgroups of finite index acting locally free on $\mathbb{H}$, for example, the commutator group $[\mathrm{SL}(2, \mathbb{Z}), \mathrm{SL}(2, \mathbb{Z})]$ and any congruence groups $\Gamma(N)$ for $N \geq 3$.

Furthermore, since any subgroup that has no elements fixing points in $\mathbb{H}$ must be a subgroup of the commutator group, this implies a restriction on the orbifold Euler characteristic of $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ as $\chi(\mathbb{H}/\mathrm{SL}(2, \mathbb{Z}) = 0$ or $|\chi(\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})| \geq \frac{1}{12}$ since the usual Euler characteristic takes integer values and the commutator subgroup has index 12.

A.4 The j-invariant

This section defines the j-invariant and shows that it is surjective. This is used in Theorem 2.19.

Definition A.6. Define the j-invariant as

$$j(\tau) = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2}$$

where $g_2 = 60 G_4, g_3 = 140 G_6$ and G_{2k} is Eisenstein series of weight $2k$ (Definition 5.12) for a lattice $\mathbb{Z} + \tau\mathbb{Z}$ s.t. $\tau \in \mathbb{H}$.

One may also write the j-invariant as

$$j(\tau) = 1728\frac{g_2^3}{\Delta}$$

where Δ is the modular discriminant of $\mathbb{Z} + \tau\mathbb{Z}$ (Definition 5.12) where we use the rewritten from given in (5.13). From this we see that it is indeed well defined since $\Delta \neq 0$ [Lan95, Theorem 2.1, p. 6]. Note that since $\lim_{\operatorname{Im}\tau\to\infty} G_4 = 2\zeta(4) \neq 0$ and $\lim_{\operatorname{Im}\tau\to\infty} \Delta = 0$, the j-invariant is unbounded as $\operatorname{Im}\tau \to \infty$. This means that the j-invariant is *not* a modular form (Definition 5.9), although it is $\mathrm{SL}(2,\mathbb{Z})$-invariant.

To show that the j-invariant is surjective, we will use following lemma.

Lemma A.7 (Open mapping theorem (complex analysis))**.** *If f is a holomorphic non-constant function on a domain $\Omega \subset \mathbb{C}$, then f is an open mapping, i.e. takes open sets to open sets.*

Proof. Take any w_0 in the image of f and let z_0 be s.t. $f(z_0) = w_0$. We will use Rouché's theorem to show that any w close enough w_0 is in the image of f.

Choose $\delta > 0$ s.t. the disc $|z - z_0| \leq \delta$ is in Ω and $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$. Then let $\epsilon > 0$ be s.t. $|f(z) - w_0| > \epsilon$ on the circle $|z - z_0| = \delta$. We will show that any w s.t. $|w - w_0| < \epsilon$ is in the image of f. Construct $g(z) = f(z) - w$ and write

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) - G(z).$$

We see that $|F(z)| \geq |G(z)|$ on the circle $|z - z_0| = \delta$ and by Rouché's theorem $g = F + G$ has as many zeros as F, which has at least one. This shows that w is in the image of f. □

Theorem A.8. *The j-invariant is a surjective map to $\mathbb{C}$.*

Proof. By previous lemma, $j(\mathbb{H})$ is open. We will show that it is also closed. Then since $\mathbb{C}$ is connected, $j(\mathbb{H}) = \mathbb{C}$.

Take any sequence $j(\tau_1), j(\tau_2), \ldots$ converging to some point $w \in \mathbb{C}$. Since j is $\mathrm{SL}(2,\mathbb{Z})$-invariant, we may assume that all $\tau_n \in \mathcal{D}$ where

$$\mathcal{D} = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1, -\frac{1}{2} \leq \operatorname{Re} z < \frac{1}{2} \right\}$$

is the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ acting on $\mathbb{H}$ given in Lemma A.5. But since j-invariant is unbounded as $\operatorname{Im}\tau \to \infty$ and $j(\tau_n)$ converges, the sequence $\operatorname{Im}\tau_1, \operatorname{Im}\tau_2, \ldots$ must be bounded. Then all $\tau_n \in \mathcal{D} \cap \{ z \in \mathbb{C} \mid \operatorname{Im} z \leq B \}$ for some bound B. This is a compact set so by passing to a sub-sequence we can assume that $\tau_1, \tau_2, \ldots$ converges to some τ. Then $j(\tau) = w$, and we have shown that $j(\mathbb{H})$ is closed. □

A.5 Mirror pairs and Hodge diamonds

This text is concerned with Calabi-Yau onefolds, which as shown in Chapter 2 can be described using elliptic curves. This means that any mirror pair in dimension one consists of two elliptic curves, which greatly simplifies this case. This is not true in higher dimensions where a mirror pair can consist of two topologically different Calabi-Yau manifolds.

This section will leave the case of Calabi-Yau onefolds to introduce one property that a mirror pair of Calabi-Yau manifolds should have. Namely that their *Hodge diamonds* should be a right angle rotation of each other. One says that they have *mirrored* Hodge diamonds. This criterion turns out to not be particularly interesting in one dimension.

To define the Hodge diamond we need to introduce Dolbeault cohomology which uses the Dolbeault operator $\bar{\partial}$ (Definition 6.5).

Definition A.9 (Dolbeault cohomology and Hodge numbers)**.** Let M be a complex manifold. Similarly to the exterior derivative d, the Dolbeault operator satisfies $\bar{\partial}^2 = 0$. Therefore we can construct the cochain complex

$$0 \longrightarrow \Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \Omega^{p,2}(M) \xrightarrow{\bar{\partial}} \dots$$

for each $p \geq 0$. From this cochain complex we define the Dolbeault cohomology

$$H^{p,q}(M) = \operatorname{Ker}(\bar{\partial}|_{\Omega^{p,q}(M)}) / \operatorname{Im}(\bar{\partial}|_{\Omega^{p,q-1}(M,\mathbb{R})}),$$

i.e. a quotient of a subspace of $\Omega^{p,q}(M)$. The dimension as a complex vector space will be denoted

$$h^{p,q} := \dim_{\mathbb{C}}(H^{p,q}(M))$$

which are called the *Hodge numbers.*

Definition A.10 (Hodge diamond)**.** For a complex manifold M of (complex) dimension n, the Hodge diamond is a rhombus made up of the Hodge numbers $h^{p,q}$ for $0 \leq p, q \leq n$.

To illustrate, if $n = 3$ the Hodge diamond is

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

For two complex manifolds M and N, we say that they have *mirrored* Hodge diamonds if

$$h_M^{p,q} = h_N^{n-q,p}.$$

which can be seen as if their Hodge diamonds are related by a 90-degree rotation about the Hodge diamonds center.

Note that for a connected complex manifold the exterior derivative on $\Omega^{0,0}(M)$, i.e. smooth functions, is zero only if the function is constant, which means that

$$h^{0,0} = 1.$$

Similarly, using that M is connected and Poincaré duality we see that

$$h^{n,n} = 1.$$

Together with the mirror relation of the Hodge diamonds of the mirror pair, this require that both manifolds have $h^{0,0} = h^{0,n} = h^{n,0} = h^{n,n} = 1$ in their respective Hodge diamonds. This is restrictive to what manifolds mirror symmetry could be applied to, in particular, the criteria

$$h^{n,0} = 1$$

is implied from being a Calabi-Yau manifold, which is shown using Dolbeault's theorem.

We add two more relations, namely Hodge symmetry

$$h^{p,q} = h^{q,p}$$

explained from that $\Omega^{p,q}(M)$ are related to $\Omega^{q,p}(M)$ by conjugation, and

$$h^{p,q} = h^{n-p,n-q}$$

coming from Serre duality.

Combining all of these means that for a mirror pair of Calabi-Yau threefolds, their respective Hodge diamonds have to be

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & c & & c & & \\ & d & & a & & d & \\ 1 & & b & & b & & 1 \\ & d & & a & & d & \\ & & c & & c & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & d & & d & & \\ & c & & b & & c & \\ 1 & & a & & a & & 1 \\ & c & & b & & c & \\ & & d & & d & & \\ & & & 1 & & & \end{array}$$

for some non-negative integers a, b, c and d.

For a mirror pair of Calabi-Yau onefolds, this simply means that both of them have the Hodge diamond as

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

which can only happen for elliptic curves since

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

is the Hodge diamond of any compact genus g curve C and the mirror relation gives that $h_M^{0,0} = h_N^{1,0}$. This can be seen using Hodge decomposition

$$H^{1,0}(C) \oplus H^{0,1}(C) = H^1_{\mathrm{dR}}(C, \mathbb{C})$$

and Poincaré duality implying

$$H^1_{\mathrm{dR}}(C, \mathbb{C}) = H_1(C, \mathbb{Z}) \otimes \mathbb{C},$$

then by knowing that $\operatorname{rank} H_1(C, \mathbb{Z}) = 2g$ we are done.

These relations of the Hodge diamond can be found in the introduction to [CK99], although there a Calabi-Yau manifold is defined more restrictive such that $h^{p,0} = h^{p,n} = h^{0,p} = h^{n,p} = 0$ for $1 \leq p < n$.

B

Prerequisite theory

This chapter will give some definitions for manifolds and differential forms needed in the text. At the end of the chapter a proof that any complex manifold can be orientable is given in arbitrary dimensions.

B.1 Manifolds and functions on manifolds

This section will give some basic definitions for manifolds.

Definition B.1 (Topological manifold)**.** A topological manifold M is a second countable Hausdorff space that is locally homeomorphic to Euclidean space, i.e. each point has a neighborhood isomorphic to an open set of $\mathbb{R}^n$. We say that M has real dimension n.

Definition B.2 (Smooth manifold)**.** A smooth manifold M is a topological manifold along with a collection of *charts* $\{(U_i, \varphi_i)\}$ called an *atlas*, where $\{U_i\}$ is an open cover of M, $\varphi_i : U_i \xrightarrow{\simeq} V_i \subseteq \mathbb{R}^n$ are homeomorphisms from U_i to some open $V_i \subseteq \mathbb{R}^n$ and the *transition functions*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subseteq \mathbb{R}^n \to \varphi_i(U_i \cap U_j) \subseteq \mathbb{R}^n$$

are infinitely differential functions. Two atlases on a smooth manifold are said to be equivalent if their union is also an atlas.

For any point $p \in M$ we say that a chart at p is any chart $\varphi : U \to \mathbb{R}^n$ such that $p \in U$.

Definition B.3 (Smooth function on a manifold)**.** A smooth function $f \in C^\infty(M)$ on a smooth manifold M is function $f : M \to \mathbb{R}$ such that for any chart $\varphi : U \to \mathbb{R}^n$, $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable.

Definition B.4 (Smooth function between manifolds)**.** Let M and N be smooth manifolds of dimension m and n respectively. A smooth function $h \in C^\infty(M, N)$ between M and N is function $h : M \to N$ such that for any point $p \in M$, any chart $\varphi_M : U \subset M \to \mathbb{R}^m$ at p and any chart $\varphi_N : V \subset N \to \mathbb{R}^n$ at $f(p)$

$$\varphi_N \circ h \circ \varphi_M^{-1} : \varphi_M(U) \subseteq \mathbb{R}^m \to \varphi_N(V) \subseteq \mathbb{R}^n$$

is infinitely differentiable.

Let the *pullback* of a function $f \in C^\infty(N)$ with respect to h be

$$h_*(f) = f \circ h \in C^\infty(M).$$

Definition B.5 (Holomorphic function in several variables)**.** A function $f : U \subseteq \mathbb{C}^n \to \mathbb{C}$ is said to be holomorphic if

$$z \mapsto f(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n)$$

is holomorphic for any fixed $a_1, \ldots, a_n$.

A function $f = (f_1, \ldots, f_m) : U \subseteq \mathbb{C}^n \to \mathbb{C}^m$ is said to be holomorphic if each f_j is.

Definition B.6 (Complex manifold)**.** A complex manifold M is a smooth manifold of real dimension $2n$ such that the *transition functions*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subseteq \mathbb{C}^n \to \varphi_i(U_i \cap U_j) \subseteq \mathbb{C}^n$$

are holomorphic functions when identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. We say that M has complex dimension n.

Definition B.7 (Holomorphic function on a manifold)**.** A holomorphic function $f \in \mathcal{O}(M)$ on a complex manifold M is function $f : M \to \mathbb{C}$ such that for any chart $\varphi : U \to \mathbb{C}^n$, $f \circ \varphi^{-1} : U \subseteq \mathbb{C}^n \to \mathbb{C}$ is holomorphic.

Define a holomorphic map between complex manifolds and its pullback similar to the smooth case, except holomorphic instead of smooth.

Definition B.8 (Meromorphic function on a manifold)**.** A meromorphic function $f \in \mathbb{C}(M)$ is a pair (g, h) of holomorphic functions $g, h \in \mathcal{O}(M)$. As a function f is seen as

$$f : \{\, p \in M \mid h(p) \neq 0 \,\} \subseteq M \to \mathbb{C}, \quad f(p) = \frac{g(p)}{h(p)}.$$

Another consistent definition would be that a meromorphic function is a holomorphic function between M and the Riemann sphere $\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$ seen as a complex manifold, such that it is not constantly ∞.

B.2 Tangent space and cotangent space

This section gives definitions of the tangent space and the cotangent space of a smooth manifold. This is needed later to define differential forms.

Definition B.9 (Tangent and cotangent space)**.** Let M be a smooth manifold. Define a derivation at $p \in M$ as a linear map

$$D_p : \{\, f \in C^\infty(V) \mid V \text{ is some neighborhood of } p \,\} \to \mathbb{R}$$

such that

$$D_p(fg) = f(p)D_p(g) + D_p(f)g(p).$$

The derivations at $p \in M$ form a $\mathbb{R}$-vector space T_pM called the tangent space of M at p.

For any $p \in M$, let $\varphi : U \to \mathbb{R}^n$ be a chart at p, then we define the local coordinate basis of T_pM with respect to φ as

$$\left.\frac{\partial}{\partial x_k}\right|_{p,\varphi}(f) := \frac{\partial}{\partial x_k}(f \circ \varphi^{-1})(\varphi(p)).$$

Given another chart ϕ at p we see that

$$\left.\frac{\partial}{\partial x_k}\right|_{p,\phi} = \sum_{l}^{n} \left[\frac{\partial}{\partial x_k}\varphi \circ \phi^{-1}\right]_l (\phi(p)) \left.\frac{\partial}{\partial x_l}\right|_{p,\varphi}.$$

To simplify notation we will use the Jacobian matrix of the transition function $\phi \circ \varphi^{-1}$ denoted as

$$J_{kl}^{\phi\circ\varphi^{-1}}(p) = \left[\frac{\partial}{\partial x_k}\phi \circ \varphi^{-1}\right]_l (\varphi(p)).$$

Strictly speaking this is the Jacobian matrix of $\phi\circ\varphi^{-1}$ composed with φ, which is easier to use in this context.

This square matrix is always invertible since $\phi \circ \varphi^{-1}$ is a smooth homeomorphism between open subsets of $\mathbb{R}^n$. By the inverse function theorem we see that

$$(J^{\phi\circ\varphi^{-1}}(p))^{-1} = J^{\varphi\circ\phi^{-1}}(p). \tag{B.1}$$

The relation for changing basis in the tangent space is then

$$\left.\frac{\partial}{\partial x_k}\right|_{p,\phi} = \sum_{l}^{n} (J^{\phi\circ\varphi^{-1}}(p))^{-1}_{kl} \left.\frac{\partial}{\partial x_l}\right|_{p,\varphi}. \tag{B.2}$$

Define the smooth vector field X on M as a collection of tangent vectors

$$X|_p \in T_pM$$

for each point $p \in M$ such that then map $p \mapsto D_p(f)$ is in $C^\infty(M)$ for any fixed $f \in C^\infty(M)$.

For a fixed chart $\varphi : U \to \mathbb{R}^n$, we can write a smooth vector field X locally for any $p \in U$ as

$$X|_p = \sum_{k}^{n} f_k^{\varphi} \circ \varphi(p) \left.\frac{\partial}{\partial x_k}\right|_{p,\varphi}$$

where f_k^{φ} is a smooth function on $\varphi(U) \subseteq \mathbb{R}^n$. For this local representation to be well defined we see that for another chart ϕ at p

$$f_l^{\phi} \circ \phi(p) = \sum_{k}^{n} J_{kl}^{\phi\circ\varphi^{-1}}(p)\ f_k^{\varphi} \circ \varphi(p). \tag{B.3}$$

One may note that if $X|_p(f) = 0$ for all $f \in C^\infty(M)$, then $f_k^\varphi = 0$ for any local representation since the Jacobi matrix is inevitable.

By usual abuse of notation we do not write out p and φ, so with $\frac{\partial}{\partial x_i}$ we mean $\frac{\partial}{\partial x_i}\Big|_{p,\varphi}$ for some choice of p and φ. We write

$$X = \sum_k^n f_k \frac{\partial}{\partial x_k}$$

Definition B.10 (Cotangent space)**.** The cotangent space at a point $p \in M$ is the dual $\mathbb{R}$-vector space T_p^*M to the tangent space at p. We define the local basis of the cotagnet space as the dual basis to the local basis of the tangent space, i.e. for some $p \in M$ and chart $\varphi : U \to \mathbb{R}^n$ at p

$$dx_l|_{p,\varphi}\left(\frac{\partial}{\partial x_k}\bigg|_{p,\varphi}\right) = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}.$$

Then for another carts ϕ at p

$$dx_l|_{p,\phi}\left(\frac{\partial}{\partial x_k}\bigg|_{p,\varphi}\right) = J_{kl}^{\phi\circ\varphi^{-1}}(p)$$

so

$$dx_l|_{p,\phi} = \sum_k^n J_{kl}^{\phi\circ\varphi^{-1}}(p)\ dx_k|_{p,\varphi}. \tag{B.4}$$

B.3 Smooth differential forms and de Rham cohomology

This section will introduce differential forms and de Rham cohomology over both $\mathbb{R}$ and $\mathbb{C}$. It will also introduce (p,q)-forms.

Definition B.11 (Real smooth differential forms)**.** Let M be a smooth real manifold. Define real smooth differential k-form $\omega \in \Omega^k(M,\mathbb{R})$ as collection of alternating multilinear maps

$$\omega_p : \bigoplus^k T_pM \to \mathbb{R}$$

for each point $p \in M$, varying smoothly over M. Here T_pM denotes the tangent space of M at p and varying smoothly means that the function $p \mapsto \omega_p(X_1|_p, \ldots, X_k|_p)$ is in $C^\infty(M)$ for any smooth vector fields X_i on M.

So for each point $p \in M$, ω_p is an element in the antisymmetric tensor product of the cotangent space, i.e. $\bigwedge^k T_p^*M$. Hence for a fixed chart $\varphi : U \to \mathbb{R}^n$, we can write a real smooth differential k-form ω locally as

$$\omega_p = \sum_{i_1<\cdots<i_k}^n f_{i_1,\ldots,i_k}^\varphi \circ \varphi(p)\ dx_{i_1}|_{p,\varphi} \wedge \cdots \wedge dx_{i_k}|_{p,\varphi}$$

for any $p \in U$ where $f^{\varphi}_{i_1,\dots,i_k}$ is a smooth function on $\varphi(U) \subseteq \mathbb{R}^n$ and $dx_{i_1}|_{p,\varphi}$ denotes the cotangent basis. Here the $f^{\varphi}_{i_1,\dots,i_k}$ are under some condition similar to (B.3) for smooth vector fields, but this is messy to write down for a k-form where $k > 1$. For a 1-form

$$\omega_p = \sum_i^n f_i^{\varphi} \circ \varphi(p)\ dx_i|_{p,\varphi}$$

the condition given another chart ϕ at p is that

$$f_k^{\phi} \circ \phi(p) = \sum_i^n (J^{\phi\circ\varphi^{-1}}(p))^{-1}_{ki}\ f_i^{\varphi} \circ \varphi(p). \tag{B.5}$$

By usual abuse of notation we do not write out p and φ, so with dx_i we mean $dx_i|_{p,\varphi}$ for some choice of p and φ. We also introduce multi-index notation such that any real smooth differential k-form ω can locally be written as

$$\omega = \sum_{|I|=k}^n f_I\ dx_I$$

Furthermore, let $\Omega^0(M,\mathbb{R}) = C^\infty(M)$ i.e. real smooth functions on M.

Note that $\Omega^k(M,\mathbb{R})$ is locally a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} \Omega^0(M,\mathbb{R}) = n$ and therefore $\dim_{\mathbb{R}} \Omega^k(M,\mathbb{R}) = \binom{n}{k}$ where n is the dimension of M. In this sense for $k > 1$

$$\Omega^k(M,\mathbb{R}) = \underbrace{\Omega^1(M,\mathbb{R}) \wedge \cdots \wedge \Omega^1(M,\mathbb{R})}_{\#k}.$$

Definition B.12 (Exterior derivative)**.** Define the exterior derivative d acting on a 0-form $f \in C^\infty(M)$ as

$$df = \sum_j^n \frac{\partial}{\partial x_j}(f)\ dx_j \in \Omega^1(M,\mathbb{R})$$

which extends to k-form locally for a chart φ at $p \in M$ as

$$d\left(\sum_{|I|=k}^n f_I^{\varphi} dx_I|_{p,\varphi}\right) = \sum_{|I|=k}^n d(f_I^{\varphi} \circ \varphi^{-1}) \wedge dx_I|_{p,\varphi}.$$

This can be rewritten simply as

$$d\left(\sum_{|I|=k}^n f_I dx_I\right) = \sum_j^n \sum_{|I|=k}^n \frac{\partial f_I}{\partial x_j}\ dx_j \wedge dx_I$$

which hence takes $\Omega^k(M,\mathbb{R})$ to $\Omega^{k+1}(M,\mathbb{R})$ and satisfies $d^2 = 0$ since partial derivatives commute.

This is well-defined, i.e. not dependent on the local representation, since for any 0-form $f \in C^\infty(M)$ and any charts φ and ϕ at a point p

$$\sum_j^n \frac{\partial}{\partial x_j}\bigg|_{p,\phi}(f)\ dx_j|_{p,\phi} = \sum_j^n \frac{\partial}{\partial x_j}\bigg|_{p,\varphi}(f)\ dx_j|_{p,\varphi}$$

which is shown by using (B.2) and (B.4). This relies on the inverse function theorem which is used to show (B.1).

Call $\operatorname{Im} d$ exact forms and $\operatorname{Ker} d$ closed forms. Note that for two differential forms $\omega \in \Omega^k(M, \mathbb{R})$ and $\sigma \in \Omega^l(M, \mathbb{R})$, the exterior derivative is acting on $\omega \wedge \sigma \in \Omega^{(k+l)}(M, \mathbb{R})$ as

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

which is in some sense similar to the product rule of a classical derivative.

Definition B.13 (de Rham cohomology over $\mathbb{R}$)**.** For a smooth real manifold M, the de Rham cochain complex is constructed as

$$0 \longrightarrow \Omega^0(M, \mathbb{R}) \xrightarrow{\ d\ } \Omega^1(M, \mathbb{R}) \xrightarrow{\ d\ } \Omega^2(M, \mathbb{R}) \xrightarrow{\ d\ } \dots$$

which gives the definition of the de Rham cohomology as

$$H^k_{\mathrm{dR}}(M, \mathbb{R}) = \operatorname{Ker}(d|_{\Omega^k(M,\mathbb{R})}) / \operatorname{Im}(d|_{\Omega^{k-1}(M,\mathbb{R})}).$$

So then de Rham cohomology classes in $H^k_{\mathrm{dR}}(M, \mathbb{R})$ are closed k-forms up to exact k-forms.

Definition B.14 (Complex smooth differential forms)**.** For a complex manifold M, one often looks at the complex tangent space defined as $T_pM \otimes \mathbb{C}$, forming the tangent space T_pM while seeing M as a real manifold. Similarly to the real case, define a complex smooth differential k-form $\omega \in \Omega_k(M, \mathbb{C})$ as a collection of alternating multilinear maps

$$\omega_p : \bigoplus^k (T_pM \otimes \mathbb{C}) \to \mathbb{C}$$

for each point $p \in M$ varying smoothly over M [1]. Since tensor products distributes over direct sums we can see ω_p as a element of $(\bigwedge^k T_p^*M) \otimes \mathbb{C}$ [2]. In that sense, $\Omega^k(M, \mathbb{C}) = \Omega^k(M, \mathbb{R}) \otimes \mathbb{C}$.

Definition B.15 (de Rham cohomology over $\mathbb{C}$)**.** Define the exterior derivative d and the de Rham cohomology $H^k_{\mathrm{dR}}(M, \mathbb{C})$ similarly as in the real case. Since tensor product commutes with cohomology of a chain complex [Mat87, Exercise 7.6, p. 53], one can see that

$$H^k_{\mathrm{dR}}(M, \mathbb{C}) = H^k_{\mathrm{dR}}(M, \mathbb{R}) \otimes \mathbb{C}.$$

[1] Note that the complex differential are only smooth, not holomorphic.

[2] Not $\bigwedge^k (T_p^*M \otimes \mathbb{C})$ since it has higher dimension.

Definition B.16 (Differential (p,q)-forms)**.** For a complex manifold M of (complex) dimension n, denote the local coordinates $z_j = x_j + iy_j$. As a basis for the real vector space T^*M we used the basis $dx_1, dy_1, \dots, dx_n, dy_n$ but when at the complex vector space $T^*M \otimes \mathbb{C}$ it is natural to change basis to $dz_1, d\bar{z}_1, \dots, dz_n, d\bar{z}_n$. Then a complex differential form $\omega \in \Omega^k(M, \mathbb{C})$ can locally be written as

$$\omega = \sum_{|I|+|J|=k}^{n} f_I \; dz_I \wedge d\bar{z}_J.$$

From this we define a differential (p,q)-form $\omega \in \Omega^{p,q}(M)$ to be locally of the form

$$\omega = \sum_{|I|=p,\ |J|=q} f_{IJ} dz_I \wedge d\bar{z}_J$$

where $f_{IJ} \in C^\infty(M) \otimes \mathbb{C}$.

One needs to ensure that this is well-defined, that a (p,q)-form with respect to one chart is necessarily a (p,q)-form with the respect to another chart with the same p and q. This is true since for two charts φ and ϕ at a point p the cotangent basis elements $dz_i|_{p,\varphi}$ is a necessarily a linear combination of $dz_j|_{p,\phi}$ and no $d\bar{z}_j|_{p,\phi}$ by (B.4) since the transition functions are holomorphic, and similarly $d\bar{z}_i|_{p,\varphi}$.

We can also see that

$$\Omega^r(M, \mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}(M)$$

and

$$\Omega^{p,q}(M) = \underbrace{\Omega^{1,0}(M) \wedge \cdots \wedge \Omega^{1,0}(M)}_{\#p} \wedge \underbrace{\Omega^{0,1}(M) \wedge \cdots \wedge \Omega^{0,1}(M)}_{\#q}.$$

B.4 Meromorphic differential forms

In this section, we define meromorphic differential 1-forms for both complex manifolds and smooth projective curves.

Definition B.17 (Meromorphic differential 1-forms on complex manifolds)**.** Let M be a complex manifold. Define a meromorphic differential 1-form ω locally as

$$\omega_p = \sum_{i}^{n} f_i dz_i$$

for some meromorphic function $f_p \in \mathbb{C}(M)$ where dz_i is a basis element of $T^*M \otimes \mathbb{C}$, such that ω_p varies holomorphically over M.

Let φ and ϕ be charts at a point p. Since the transition function $\phi \circ \varphi^{-1}$ is holomorphic, many matrix entries of the Jacobian matrix vanishes if one uses the basis $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ for $TM \otimes \mathbb{C}$. The condition of the local representation in (B.5) is then

$$f_k^{\phi}(p) = \sum_i^n \left[\frac{\partial}{\partial z_k} \varphi \circ \phi^{-1}\right]_i (\phi(p))\ f_i^{\varphi}(p).$$

Definition B.18 (Meromorphic 1-forms on smooth algebraic curves)**.** On a smooth algebraic curve C over an algebraically closed field k, the meromorphic 1-forms Ω_C is a $k(C)$-vector space generated by formal symbols dx for $x \in k(C)$ with following relations

$$\begin{aligned} d(x+y) &= dx + dy \\ d(xy) &= ydx + xdy \\ da &= 0 \end{aligned}$$

for any $x, y \in k(C)$ and $a \in k$.

B.5 Orientability of a complex manifold

In this section, we define what is meant for a smooth manifold to be orientable and prove that any complex manifold is orientable.

Definition B.19. We say that a smooth manifold is orientable if it has an atlas such that the Jacobian determinant for any transition function is positive in every point.

Theorem B.20. *Any complex manifold is orientable.*

Proof. Let M be a complex manifold of dimension n. Since we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ we will denote the local coordinates as $x_1, y_1, \ldots, x_n, y_n$.

Given two charts φ and ϕ at a point $p \in M$ the Jabobian matrix with this notation for the local coordinates is then

$$J_{(2k-1)l}^{\phi\circ\varphi^{-1}}(p) = \left[\frac{\partial}{\partial x_k} \phi \circ \varphi^{-1}\right]_l (\varphi(p)) \quad \text{and} \quad J_{(2k)l}^{\phi\circ\varphi^{-1}}(p) = \left[\frac{\partial}{\partial y_k} \phi \circ \varphi^{-1}\right]_l (\varphi(p))$$

for $k = 1, \ldots, n$ and $l = 1, \ldots, 2n$. For clearer notation, we will drop the notation $\phi \circ \varphi^{-1}$, i.e. $J := J^{\phi\circ\varphi^{-1}}$.

Since the transition functions of a complex manifold are holomorphic they satisfies the Cauchy-Riemann equations in each coordinate

$$\left[\frac{\partial}{\partial x_k} \phi \circ \varphi^{-1}\right]_{(2l-1)} = \left[\frac{\partial}{\partial y_k} \phi \circ \varphi^{-1}\right]_{(2l)}$$

and

$$\left[\frac{\partial}{\partial y_k} \phi \circ \varphi^{-1}\right]_{(2l-1)} = -\left[\frac{\partial}{\partial x_k} \phi \circ \varphi^{-1}\right]_{(2l)}$$

which in terms of the Jacobian matrix means

$$J_{(2k-1)(2l-1)} = J_{(2k)(2l)} \quad \text{and} \quad J_{(2k)(2l-1)} = -J_{(2k-1)(2l)} \tag{B.6}$$

for $k, l = 1, \ldots, n$.

To better understand the structure of J we see that is has a block matrix structure as

$$J = \begin{pmatrix} J^{1,1} & \ldots & J^{1,n} \\ \vdots & \ddots & \vdots \\ J^{n,1} & \ldots & J^{n,n} \end{pmatrix} \tag{B.7}$$

where each block matrix is of the form

$$J^{i,j} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{B.8}$$

and hence $\det(J^{i,j}) = a^2 + b^2 \geq 0$. Note that a block matrix on such form is closed under both addition and multiplication, which will be important later.

Moreover, look at the adjunct matrix of J, i.e. the matrix

$$\operatorname{adj}(J)_{ij} = (-1)^{i+j} J_{ji}.$$

From this we see that it satisfies the same properties (B.6) as J, and hence the inverse of J have the same block matrix form (B.7) as J since

$$A^{-1} = \det(A)^{-1} \operatorname{adj}(A)$$

for any invertible matrix A.

We will use induction to see that the determinant of J is positive, where the base case is when $\det(J) = \det(J^{1,1}) \geq 0$ and non-zero since J is invertible. To do this we will use Schur's determinant identity for block matrices

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B)$$

assuming the block matrix A is invertible. This identity is shown by the factorization of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$

So

$$J = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where D is a 2×2-matrix of the form (B.8) and we split A, B and C into 2×2-matrices B^j and C^i similarly to (B.7). Note that since J is invertible, A is invertible and hence by the induction hypothesis $\det(A) > 0$ since it has the same properties at J. Furthermore, A^{-1} also have those properties from adjunct facts, and we split A^{-1} into 2×2-matrices $E^{i,j}$. Now using Schur's determinant identity and block matrix multiplication we get

$$\det(J) = \det(A)\det(D - CA^{-1}B) = \det(A)\det(D - \sum_{i}^{n-1}\sum_{j}^{n-1} C^i E^{i,j} B^j).$$

Since 2×2-matrices of the form (B.7) are closed under multiplication and addition we get that both those determinants are non-negative. They are also non-zero since J is invertible, so we have shown that

$$\deg(J) > 0$$

for any transition function at any point. □

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