

# From Modular Forms to Crystals and Beyond 

Bachelor's thesis in Fundamental Physics
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Cover: The imaginary part of the Eisenstein series $G_{6}$ on the unit disk [1].
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#### Abstract

In this report the connection between the partition functions for a two-dimensional model of ice crystals and the Eisenstein series that appears when studying the quantum corrections of string theory is studied. The report is mainly based on papers in which this link is briefly studied, and is meant to act as an introduction to the subject matter for future research. The text is based on literature studies combined with calculations to understand certain theories and to attain a deeper understanding of the mathematics involved. The level of mathematics in the report is enough to follow the calculations, without including anything insignificant. This is because the report is meant to appeal to both physics students and researchers interested in the, so far, unexplored area of research. The project moves from the crystals described by statistical mechanics to the realm of physics that is described by modular forms and then further onto string theory. This link has not been researched yet, which makes it an interesting subject to study.


## Sammandrag

I denna rapport undersöks kopplingen mellan partitionsfunktionerna för en tvådimensionell modell av iskristaller och de eisenstensteinserier som uppkommer när kvantkorrektioner studeras inom strängteori. Arbetet baseras främst på vetenskapliga artiklar där denna koppling studerats i korthet och är tänkt att den ska agera som en introduktion för framtida forskning. Arbetet har baserats på litteraturstudier kombinerat med egna beräkningar för att nå en djupare förståelse för teorin. Nivån på matematiken i rapporten är tillräcklig för fysikstudenter, men begränsad till den grad så att ingenting insignifikant är inkluderat. Detta eftersom rapporten ska tilltala både studenter och forskare inom, det ännu outforskade, området. Projektet rör sig från kristaller som beskrivs med hjälp av statistisk mekanik över till modulära former och sedan utforskas en möjlig koppling till strängteorin. Denna koppling har inte forskats på innan, vilket gör den mycket intressant att studera.

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## Part I

## A Summary in Swedish

## Bakgrund

Två teorier som båda revolutionerade fysiken under 1900-talet var kvantmekaniken och den allmänna relativitetsteorin. Kvantmekaniken beskriver hur materiens minsta beståndsdelar, elementarpartiklarna, bygger upp fysikens lagar, medan den allmänna relativitetsteorin främst förklarar gravitationen. Kvantmekaniken har rönt stora framgångar då den lyckats förklara ett flertal fenomen där den klassiska fysiken ej är tillräcklig. Främst rör det sig om fenomen på mycket små längdskalor, men kvantmekaniken har även varit oumbärlig för tillämpningar som lasern, transistorn och elektronmikroskopet [2]. Den allmänna relativitetsteorin publicerades av Albert Einstein år 1916. Genom att beskriva gravitationen som ett resultat av att rumtiden kröks av materia och strålning, förenar den Einsteins speciella relativitetsteori med Isaac Newtons gravitationsteori. Anmärkningsvärt nog har det visat sig att den allmänna gravitationsteorin inte är förenlig med kvantmekaniken. En teori som lyckas förena dessa båda stöttepelare inom den moderna fysiken vore således en enorm framgång. En kandidat är strängteorin, vilken har just detta syfte. Enligt denna är naturens mest fundamentala byggstenar ej punktpartiklar, utan små vibrerande strängar [3]. För de allra flesta icke-klassiska fenomen inom fysiken är det tillräckligt med antingen en kvantmekanisk eller en allmänrelativistisk beskrivning. Ett undantag är svarta hål, där effekter från både kvantmekaniken och den allmänna relativitetsteorin kan observeras. En av nycklarna för att förstå strängteorins beskrivning av svarta hål är de så kallade modulära formerna.

Modulära former är en klass av funktioner som sedan tidigt 1800-tal har varit av stor vikt inom matematiken [4]. Initialt användes de främst för att studera elliptiska funktioner, men med tiden har de visat sig ha avsevärt bredare tillämpningsområden än så. Vid mitten av 1900 -talet fann man att modulära former kunde användas för att på ett elegant sätt knyta samman tidigare åtskilda områden inom matematiken, som geometri, talteori och representationsteori [4,5]. Med tiden har modulära former även visat sig ha ett flertal tillämpningar inom fysiken. Detta faktum är av stort intresse, eftersom villkoren för att en funktion ska klassas som en modulär form inbegriper hårda krav på dess symmetriegenskaper. Inom fysiken spelar symmetrier en avgörande roll. Exempelvis säger Noethers teorem att symmetrierna hos en teori fullständigt bestämmer dess konserverade storheter [6]. Dessa ligger i sin tur till grund för teorins fundamentala lagar. I strängteorin dyker de modulära formerna upp i kvantkorrektionerna för verkan som ger upphov till Einsteins ekvationer, vilka i sin tur beskriver gravitationen. Det är just dessa korrektioner som gör den allmänna relativitetsteorin förenlig med kvantmekaniken i strängteorin. Även inom strängteorin har de modulära formerna fler tillämpningar än så. Exempelvis kan de användas för att räkna mikrotillstånden i svarta hål, vilket i sin tur kan användas för att bestämma deras entropi [5].

På senare år har kopplingar till modulära former observerats inom ytterligare ett område inom fysiken, nämligen statistisk mekanik. Det är i partitionsfunktionerna för tvådimensionella modeller av vissa kristaller, exempelvis is, som de modulära formerna dyker upp. En djupare förståelse av denna koppling skulle kunna förklara varför de modulära formerna frekvent dyker upp i olika fysiktillämpningar. Det har även potential att blottlägga ett djupgående samband mellan kristaller och svarta hål, två objekt som vid första anblick tycks vara helt avskilda från varandra.

## Syfte

Syftet med detta kandidatarbete är att identifiera och förstå den roll de modulära formerna spelar inom såväl statistisk mekanik som strängteori. För att förstå kopplingen mellan modulära former och statistisk mekanik kommer tvådimensionella modeller av kristaller att studeras. Inom strängteorin kommer kvantkorrektioner innehållande modulära former att studeras. Slutligen önskar vi även undersöka huruvida vi kan finna några samband mellan dessa, till synes, vitt skilda fält inom fysiken.

## Uppgiftsformulering

Detta projekt inbegriper tre huvudområden: modeller av kristaller inom statistisk mekanik, modulära former samt strängteori. Dels vill vi studera dessa områden separat, men vi är även speciellt intresserade av sambanden dem emellan. Projektet kan således indelas i tre delar, vilka illustreras i den schematiska bilden i figur 1.


Figur 1: En schematisk bild över projektet. Initialt undersöks kopplingen mellan modulära former och kristaller. Detta följs av en studie av de modulära formernas roll inom strängteorin. Slutligen undersöks huruvida det kan finnas någon koppling mellan strängteori och kristaller. Detta är till skillnad från övriga punkter ett tidigare helt outforskat område.

En utförligare beskrivning av projektets tre olika delar ges nedan.

1. Den första delen av projektet består av att förstå kopplingen mellan kristaller och modulära former. Denna koppling uppenbarar sig vid studium av partitionsfunktionerna som uppkommer i en tvådimensionell modell av främst iskristaller, vid namn sex-vertexmodellen. Ett naturligt första steg är därför att förstå sex-vertexmodellen samt studera vilka partitionsfunktioner den ger upphov till. Nästa steg är studier av grupp- och representationsteori. Detta eftersom det i mångt och mycket är på dessa teorier som de modulära formerna vilar, och därmed kan förstås med hjälp utav. Slutligen kan de modulära formerna studeras mer explicit, varvid kopplingarna till kristallernas partitionsfunktioner kan identifieras.
2. Nästa del i projektet är att studera kopplingen mellan modulära former och strängteori. Denna koppling utgörs av kvantkorrektionerna - d.v.s. de termer som kan försummas då man betraktar
problemet klassiskt - till den verkan som ger upphov till Einsteins ekvationer, den så kallade Einstein-Hilbert-verkan.
3. Att de modulära formerna har kopplingar till såväl kristaller som strängteori antyder att det kan finnas något samband dem emellan. Detta, tidigare outforskade, samband skulle på ett elegant sätt kunna knyta samman strängteorins abstrakta värld med något så konkret som statistisk mekanik. I projektets sista del utreder vi därför likheter och skillnader i resultaten av studierna i punkt 1 och 2 ovan.

## Avgränsningar

Arbetet under projektet kan indelas i tre huvudsakliga delar: statistisk mekanik, en matematikdel innehållande gruppteori och modulära former och slutligen strängteori. Samtliga dessa områden är mycket stora, och en utförlig utredning av dem alla vore omöjlig att åstadkomma inom projektets tidsrymd.

Inom den statistiska mekaniken har vi valt att nästintill uteslutande fokusera på de modeller av kristaller som uppvisar en koppling till modulära former. I denna del har vi varit som allra mest rigorösa och viktiga resultat har i allmänhet härletts och motiverats ordentligt. När vi studerat gruppteorin har vi avgränsat oss till att endast beröra områden som vi anser vara viktiga för förståelsen av kopplingen mellan modulära former och de fysiktillämpningar rapporten tar upp. Eftersom modulära former har starka samband med gruppen $S L(2, \mathbb{R})$ har denna grupp prioriterats.

Även i behandlingen av strängteorin har vi, med få undantag, valt att inrikta oss på de delar där kopplingen till modulära former dyker upp. Detta eftersom strängteori är ett stort och avancerat forskningsområde, varför det vore omöjligt att nå en djupare förståelse av den under projektets löptid. Att finna kopplingar mellan strängteori och kristaller är ett problem som aldrig har utforskats tidigare och ligger således på forskningsnivå. Resultatet av dessa studier har därför presenterats i form av en utblick i det sista kapitlet. Vi hoppas att denna ska fungera som en inspirationskälla och visa på de modulära formernas viktiga roll i forskningens framkant.

## Metod

Arbetsprocessen har väsentligen bestått av tre element: litteraturstudier, egna beräkningar samt föreläsningar av våra handledare Daniel Persson och Henrik Gustafsson. Litteraturstudierna har främst använts som introduktion och inspiration till de olika ämnesområdena. De egna räkningarna har använts som en metod för att befästa de via litteraturstudierna erhållna kunskaperna, samt för att härleda nya resultat och exempelräkningar där dessa saknats i litteraturen. Våra handledares föreläsningar har berört ämnen som ligger nära deras egna forskningsområden, och de har då kunnat anpassa nivån och infallsvinkeln efter våra kunskapsnivåer och projektets syfte. Det inbördes styrkeförhållandet mellan dessa tre arbetsförfaranden har varierat under projektets olika delar.

Som en introduktion och fysikalisk motivering till den statistiska mekaniken, framför allt till sexvertexmodellen, användes i stor utsträckning boken Exactly Solved Models in Statistical Mechanics av

Rodney J. Baxter [7]. För studier av kopplingen till modulära former var en rad artiklar av Benjamin Brubaker, Daniel Bump och Solomon Friedberg till stor hjälp [8, 9, 10]. På detta område fanns knappt några räknade exempel, så här utgjordes en stor del av arbetet av att vi själva beräknade partitionsfunktionerna för olika kristaller, och sedan jämförde våra resultat med teorin. I denna del av projektet ställde vi även höga krav på att själva kunna förstå och återskapa alla viktiga teoretiska resultat.

Under den mer matematiska delen av projektet användes främst litteraturstudier och föreläsningar av handledare. Vid de mer explicita studierna av modulära former var masteruppsatsen Eisenstein Series and Instantons in String Theory av Henrik Gustafsson värdefull, främst vid beräkningen av eisensteinseriens fourierutveckling [5]. Rörande grupp- och representationsteori har kandidatarbetet Group Theory and Symmetries in Particle Physics av Saladin Grebović et al. varit en viktig källa till information och inspiration [11].

I den sista delen av projektet, som behandlade strängteori, har vår handledare Daniel Persson varit till stor hjälp. Dels höll han en föreläsning om ämnet för oss och dels har vi kunnat studera hans doktorsavhandling Arithmetic and Hyperbolic Structures in String Theory [3]. Även Henrik Gustafssons masteruppsats var ånyo värdefull [5].

## Resultat

Det första området i detta arbete är statistisk mekanik. En central komponent i detta område är partitionsfunktionen som definieras. Den ismodell som ska studeras definieras också här tillsammans med de viktiga randvillkor som kommer tillämpas på alla kristaller i rapporten. Som exempel beräknas partitionsfunktionen för ett par av de enklast möjliga kristallerna i denna modell. För att kunna beräkna partitionsfunktionen för alla kristallerna i modellen definieras schurpolynomen vilka frekvent återkommer som en viktig komponent genom rapporten. Ett teorem som ger ett uttryck för partitionsfunktionerna för alla kristaller i modellen formuleras och bevisas. Det är i detta teorem som schurpolynomen för första gången spelar en viktig roll i rapporten.

Nästa centrala område i detta arbete är gruppteori. I rapporten definieras grupper utifrån gruppaxiomen och ett enkelt exempel på en grupp presenteras. Med detta definieras också liegrupper som kan ses som kontinuerliga grupper. Speciellt studeras liegruppen $S L(2, \mathbb{R})$ som är den viktigaste gruppen i denna rapport. Utifrån detta definieras liealgebror som är ett mycket viktigt koncept inom fysiken. Liealgebror studeras vidare i avsnittet om strukturteori där vi tittar på kommutatorer i liealgebran. Speciellt förklaras Chevalley-Serrerelationerna och hur cartanmatrisen hänger ihop med dessa. Denna matris innehåller all information om den tillhörande liealgebran.

Inom fysiken låter man oftast grupper representeras av matriser vilket förklaras i avsnittet om representationsteori. Det förklaras också hur representationer används för liealgebror och speciellt ges en representation för generatorerna av liealgebran $\mathfrak{s l}(2, \mathbb{R})$. Utifrån en representation definieras rötter och vikter vilka sedan används för att definiera karaktären av en representation. Med dessa verktyg kan Weyls karaktärsformel formuleras, vilken är oerhört viktig inom fysiken. Det för detta arbete mest intressanta resultatet från gruppteorin fås genom att använda Weyls karaktärsformel på $\mathfrak{s l}(2, \mathbb{R})$.

Här visar sig schurpolynomen igen. Dessa schurpolynom motsvarar kristaller med två rader i sexvertexmodellen.

Det sista stora område som gås igenom från grunden i detta arbete är modulära former. Modulära former är holomorfa funktioner på det övre halvplanet som är invarianta så när som på en viss faktor under transformation med ett element i gruppen $S L(2, \mathbb{Z})$. Släpper man kravet på holomorficitet och istället inför att funktionen ska vara en egenfunktion till Laplacianen på övre halvan av det komplexa talplanet får man istället en maassform. Som exempel på en sådan definieras eisensteinserier. Hur dessa kopplar till övriga kapitel visar sig i dess fourierutveckling. I denna finns en faktor som kan skrivas som ett schurpolynom. Dessa schurpolynom är desamma som fanns i karaktärsformeln för $\mathfrak{s l}(2, \mathbb{R})$.

Som avslutning ges en liten inblick i strängteori. Detta avsnitt tar avstamp i allmän relativitetsteori och visar hur Einstein-Hilbert-verkan ser ut. Från strängteori uppkommer kvantkorrektioner till denna verkan och i dessa korrektioner dyker eisensteinserierna upp.

## Slutsats

I denna rapport studeras till synes helt skilda områden av matematik och fysik. Vid närmare studier visar sig dock likheter, främst i form av schurpolynom. Speciellt förvånande är kanske hur dessa dyker upp i sex-vertexmodellen, en modell som inte visar något uppenbart samband med gruppen $S L(2, \mathbb{Z})$. Detta indikerar att det finns något slags samband mellan iskristaller och strängteori, ett forskningsområde som är nästan helt outforskat. Denna rapport är tänkt som en utgångspunkt för framtida forskning inom detta område.

## Part II

## The Report

## Chapter 1

## Introduction

Two theories that both revolutionized the world of physics in the 20th century were quantum mechanics and general relativity. Quantum mechanics describes how the smallest units of matter, the elementary particles, generate the laws of physics, whereas general relativity, most importantly, explains gravity. Quantum mechanics has had great success in explaining different phenomena where classical physics is not sufficient. These are mainly phenomena on very small length scales, but quantum mechanics has also been essential in applications such as the laser, transistor and electron microscope [2]. The theory of general relativity was published by Albert Einstein in 1916. By describing gravity as a result of the curvature of spacetime caused by matter and radiation, it unites Einstein's special theory of relativity with Isaac Newton's law of universal gravitation. Remarkably enough general relativity has proven to be inconsistent with quantum mechanics. Hence, a theory that unites these two mainstays of modern physics would be an enormous progress. One candidate, with this purpose, is string theory. According to string theory, the most fundamental units of the Universe are not point particles, but small vibrating strings [3]. For most non-classical phenomena in physics, either a quantum mechanical or a general relativistic description is sufficient. Black holes are exceptions. There, one can observe effects from both quantum mechanics and general relativity, making them particularly interesting in string theory. One of the keys to understanding the string theoretical description of black holes are the so-called modular forms.

Modular forms are a class of functions that since the early 19th century have played an important role in mathematics [4]. Starting as a method of studying elliptic functions, they have also proven useful in number theory and representation theory. This was for instance demonstrated by Andrew Wiles when he, after seven years of work, managed to prove Fermat's last theorem using modular forms, in what has been referred to as "the proof of the 20th century" [12]. In later years, modular forms have proven to have numerous applications in physics. This is very intriguing since a function needs to have very intricate symmetry features in order to be classified as a modular form. Symmetries are of great interest in physics. By identifying the symmetries of a theory, one can normally derive multiple important results. For example, Noether's theorem states that the symmetries define what quantities are conserved, which in turn can be used to create the fundamental laws of the theory in question [6].

One of the most recent connections to modular forms has been observed in statistical mechanics. Important work on the subject has been carried out by Benjamin Brubaker, Daniel Bump and Solomon Friedberg. They observed a connection between modular forms and the partition functions of twodimensional models of certain crystals, such as ice. This is where this thesis takes its starting point, with the primary goal to understand the mentioned connection. Repeatedly using ice as an example we have been able to understand the so-called six-vertex model, which is a two-dimensional crystal model. This model has been studied under different circumstances, e.g. in the presence of an external electric field. Thereby we have been able to determine necessary boundary conditions and a condition on the Boltzmann weights (the so-called free fermion condition) in order to derive an expression for the partition functions which contains a Schur polynomial. Schur polynomials are a class of functions with strong connections to a certain kind of modular form called the non-holomorphic Eisenstein series.

In order for the connection between modular forms and the partition functions of the crystals to become clear, a certain amount of mathematical background is required. The mathematical framework that most effectively describes symmetries is group theory. In group theory, all symmetric objects, e.g. functions, can be identified as the objects that stay invariant under all possible transformations. In order to increase comprehensibility it is preferable to also introduce the companion of group theory, representation theory, which gives a way of representing group elements, often as matrices. For this reason we have devoted chapter 3 to the foundations of these two fields. However, since group theory and representation theory are vast fields we have chosen to emphasize the areas with the deepest relations to our subject, mainly the so-called Lie groups and their structure theory. We have also focused our studies on the special linear group of order $2, S L(2, \mathbb{Z})$, since that is the group under which the Eisenstein series are invariant.

After having been acquainted with the underlying theory, we are ready for more explicit studies of modular forms in chapter 4. The emphasis will be on the earlier mentioned non-holomorphic Eisenstein series and its Fourier expansion. Studying the coefficients of this expansion, the familiar Schur polynomials will emerge. Thereby we can identify and investigate the connections between modular forms and the earlier derived partition functions. For example we can determine for what crystals - with respect to size and the choice of Boltzmann weights - the exact same Schur polynomials emerges in the partition functions as in the Fourier expansion of the Eisenstein series.

As mentioned earlier, modular forms have proven to have connections to numerous fields in physics. In the last chapter of this report we glance at the world of string theory, where the non-holomorphic Eisenstein series once again plays an important role. More precisely it appears when studying the quantum corrections to the Einstein-Hilbert action. The Einstein-Hilbert action is the action describing Einstein's equations, which in turn describe the curvature of spacetime by matter and energy [13]. According to general relativity, gravity is the result of this curvature. In addition, the Eisenstein series can also be used to count the number of microstates in black holes, from which one can easily compute their entropy [5]. Thus, one can understand the importance of the subject.

In conclusion, the primary purpose of this thesis is to identify and investigate the connection between modular forms and a certain model of crystals known as the six-vertex model. In order to do this we need a deeper understanding of group theory and representation theory. In addition, we want to briefly
investigate certain areas of string theory, where the same kind of modular forms appear. By doing so we hope to illustrate the importance of modular forms and their applications to physics.

## Reader's Guide

The report's main goal is to obtain a self-contained theory, and seeing as the subject at hand stretches over vast fields of both physics and mathematics, the extent to which all subjects have been covered has been limited. The theory included is independent, and after having read chapter 2 readers familiar with the group theory may skim through or go straight ahead to chapter 3 . This should increase the fluency in the text, and the results should appear more clearly, as well as the objective to connect two different areas of physics. Nonetheless, chapter 3 is important to help the reader understand the mathematics used in chapter 4 and 5 , and should be studied if further understanding is desired.

## Chapter 2 Statistical Mechanics and Crystals

Here, the partition functions of two-dimensional ice crystals in the so-called six-vertex model are studied, since they will further ahead reveal connections to modular forms. Moving on, both a field-free case and a case in which electric fields are present are studied and results are presented. Lastly, the dependence of the rows and weights are discussed, and a few examples are included to subsume the theory discussed.

## Chapter 3 Group and Representation Theory

In this chapter the mathematical foundation is laid with a brief review of group theory before moving onto the $\mathfrak{s l}(2)$ Lie algebra and its structure and representation theory. The roots and root space obtained when looking at the representation theory is then used in the Weyl character formula.

## Chapter 4 Modular Forms and Eisenstein Series

Modular forms are introduced and defined. This is then proceeded by a limitation of the the area studied to the non-holomorphic Eisenstein series. Further, the Fourier coefficients are rewritten as Euler products in order to show how they are connected to the partition functions in chapter 2.

## Chapter 5 Intertwining results

Most loose ends from the previous chapters are tied together and the connections promised to be delivered in the beginning are finally presented, as well as the promising direction of new research in this research area.

## Chapter 6 Conclusions and Future Direction

Here, the connection to string theory is discussed as well as how string theory might be connected to the theory discussed in earlier chapters.

## Chapter 2

## Statistical Mechanics and Crystals

It is often impossible to exactly model large systems from the physical laws that govern the system's tiniest parts. An example would be to try to model the interactions and behaviour of every individual molecule in a gas to determine how the macroscopic system will behave. Even ignoring uncertainties arising from quantum physics, the necessary computing power far exceeds anything possible today. However, we are rarely interested in the exact movement of every molecule of a macroscopic system. We instead usually deal with thermodynamic properties such as temperature, entropy and pressure. It is in this step between the microscopic and macroscopic world that statistical mechanics comes in.

A central concept in statistical mechanics is the partition function which describes a systems statistical properties. This chapter will begin by defining the partition function and make some observations regarding this. As we will get a hint at, this object is of great importance in much of statistical mechanics. We will then use this to study the six-vertex model. In the six-vertex model we have a two-dimensional lattice where each vertex can be in six different configurations, hence its name.

Another important concept that we will introduce in this chapter is the Schur polynomials which we will see can be used to calculate the partition function of a lattice in the six-vertex model. The Schur polynomials were first studied in the areas of group theory and representation theory at the beginning of the 20th century [14] and we will indeed find these again in chapter 3 when we study these areas. As we will see in later chapters it is these polynomials that make up the connection to modular forms.

### 2.1 The Partition Function

One of the most important concepts in statistical mechanics is without a doubt the partition function. This is mainly because the partition function of a system contains information about almost all ther-
modynamic properties of it, such as the free energy, heat capacity and entropy. A good introduction to statistical mechanics and the partition function can be found in An Introduction to Thermal Physics by D. V. Schroeder [15].

The expression of the partition function differs a bit depending on what kind of system is studied. One type of system that shows up a lot is the so called canonical ensemble. A system can be approximated as a canonical ensemble if it contains a constant number of particles, has a constant volume and is in thermal contact with an environment at fixed temperature. An example of systems for which these are almost always fair assumptions are crystals, which we are going to discuss in this thesis. For a canonical ensemble, the partition function $Z$ is given by

$$
\begin{equation*}
Z=\sum_{s} e^{-E_{s} /\left(k_{B} T\right)}=\sum_{s} e^{-\beta E_{s}} \tag{2.1}
\end{equation*}
$$

where $k_{B}$ denotes Boltzmann's constant, $T$ the temperature of the environment, $s$ labels the different states the system can occupy and $E_{s}$ is the energy of the corresponding state. We have also introduced the quantity $\beta=1 /\left(k_{B} T\right)$.

Perhaps the most important formula in all of statistical mechanics is

$$
\begin{equation*}
P(s)=\frac{1}{Z} e^{-\beta E_{s}} \tag{2.2}
\end{equation*}
$$

which gives the probability of finding a system in a particular state $s$ [15]. Another important use of the partition function is to calculate the Helmholtz free energy defined as $F=U-T S$ where $U$ is the total energy of the system and $S$ is the entropy. The formula is

$$
\begin{equation*}
F=-k_{B} T \ln Z \tag{2.3}
\end{equation*}
$$

This is important because of the simple connections between $F$ and other interesting thermodynamic properties such as the entropy $S$, the pressure $P$ and the chemical potential $\mu$, in accordance with

$$
\begin{equation*}
S=-\left(\frac{\partial F}{\partial T}\right)_{V, N}, \quad P=-\left(\frac{\partial F}{\partial V}\right)_{T, N}, \quad \mu=-\left(\frac{\partial F}{\partial N}\right)_{T, V} \tag{2.4}
\end{equation*}
$$

where $V$ is the volume of the system, $N$ is the number of particles and $T$ is the temperature.
Now study two systems with partition functions $Z_{1}$ and $Z_{2}$ and sets of states $\left\{s_{1}\right\}$ and $\left\{s_{2}\right\}$ respectively. The partition function of the combined system is then

$$
\begin{equation*}
Z=\sum_{s_{1}} \sum_{s_{2}} e^{-\beta\left(E_{s_{1}}+E_{s_{2}}\right)}=\sum_{s_{1}} e^{-\beta E_{s_{1}}} \sum_{s_{2}} e^{-\beta E_{s_{2}}}=Z_{1} Z_{2} \tag{2.5}
\end{equation*}
$$

As we can see, the combined partition function is simply the product of the partition functions of its constituent systems.

The factor $e^{-\beta E_{s}}$ is often referred to as the Boltzmann weight of a state $s$. The Boltzmann weights are often simply denoted $w_{s}$. Using this notation the partition function can be rewritten as

$$
\begin{equation*}
Z=\sum_{s} w_{s} \tag{2.6}
\end{equation*}
$$

### 2.2 The Six-Vertex Model

The six-vertex model is a model that is used to study certain crystals containing hydrogen bonds. The most common example is ice, but it is also used to study ferroelectric materials such as $\mathrm{KH}_{2} \mathrm{PO}_{4}[7]$. In this thesis, the six-vertex model in two dimensions will be studied. The six-vertex model can be used in three dimensions as well, but then it is only exactly solvable in very special cases [7]. Exactly solvable here meaning that the partition function of the system can be found analytically.

In the two-dimensional six-vertex model of ice the oxygen atoms are often thought to form a square lattice - the model is for this reason sometimes called square ice. It is well known that each oxygen atom binds to two hydrogen atoms covalently. In the six-vertex model, it is assumed that the angle between these two bonds is either $90^{\circ}$ or $180^{\circ}$. Since oxygen has a considerably higher electronegativity than hydrogen, the electrons that are shared between the O - and H -atoms in a covalent bond will be attracted more by the O -atom than by the H -atom. As a consequence they will be located closer to the oxygen atom than to the hydrogen atom, which means the O-atom has a slightly more negative charge than the H -atom. Because of this, the edges of the O -atoms that do not bond covalently to an H -atom will bond to an H -atom in another molecule via a so called hydrogen bond (a type of dipole-dipole bond). Since the lattice is square, these bonds too must have the bonding angle $90^{\circ}$ or $180^{\circ}$.

We now understand how a two-dimensional ice crystal may look according to the six-vertex model. An example can be found in figure 2.1(a).


Figure 2.1: An example of a possible state for an ice crystal in the two-dimensional six-vertex model. The state is illustrated with three different denotations. (a) shows the orientation of the $\mathrm{H}_{2} \mathrm{O}$-molecules. (b) shows the orientations of the dipoles. The arrows are pointing towards the positive charge. (c) shows a model where an arrow pointing down or to the right in figure (b) corresponds to a positive spin and an arrow pointing up or to the left corresponds to a negative spin.

We will now introduce two other denotations of an ice crystal in the six-vertex model. First, we mark the four hydrogen bonds surrounding each oxygen atom as an arrow from the negative to the positive charge. The crystal in the example above will then look like figure 2.1(b). Note that at each vertex there must be two arrows pointing in and two pointing out. This rule is known as the ice rule and is a consequence of each O-atom bonding two H-atoms covalently (which yields inward-pointing arrows) and two with hydrogen bonds (outward-pointing arrows). Simple combinatorics shows that
this can be done in six different ways (all of which are represented in figure 2.1(b)), hence the name six-vertex model. We now introduce the third denotation by letting arrows pointing down or to the right correspond to a plus sign, and letting arrows pointing up or to the left correspond to a minus sign. The plus and minus signs can be said to correlate with spin [9]. In this notation, the crystal in our example will look like figure $2.1(\mathrm{c})^{1}$. The six possible vertex configurations are illustrated in the lattermost notation in figure 2.2


Figure 2.2: The six possible vertex configurations in the six-vertex model.

Let us now comment on the approximations that are made when using the two-dimensional six-vertex model to model ice crystals. The most crucial approximations are effectively two. The first one is made when studying a plane of the ice crystal. The molecules in an ice crystal are not arranged in distinct planes as, for example, the atoms in graphite are. In reality, each molecule has a different orientation than all its neighbours. The second approximation is made when it is assumed that the bonding angle between the two hydrogen bonds is either $90^{\circ}$ or $180^{\circ}$. In reality, this bonding angle is always $109^{\circ}[17]$. These approximations may at first sight seem rough, but the six-vertex model has proven to constitute a realistic model of ice in numerous applications [7]. One explanation of this is that the real bonding angle $109^{\circ}$ lies in between $90^{\circ}$ and $180^{\circ}$, so when we construct a large lattice using the six-vertex model the mean bonding angle will usually not be far from the value for real ice.

One may also question the treatment of the edges of the crystal in figure 2.1. Is it really fair to assume that the eight oxygen atoms at the edges all bond to four hydrogen atoms? Since this indicates that there are H -atoms bonding to $\mathrm{O}-\mathrm{atoms}$ with hydrogen bonds, without bonding covalently to another O-atom, it is of course not a fully correct model. Nevertheless it is often a fair approximation since the crystals we are going to work with almost always are much larger than the one in figure 2.1. Then, the influence of the edge vertices on the total weight of the state will be negligible, and we can treat the crystals as we did above. Furthermore, one can question whether we can choose every arrow at the edges independently of all others. The answer is generally no, but the boundary conditions are not always the same. They differ strongly depending on both which crystal one is studying and under which outer conditions, e.g. applied electric or magnetic fields [7]. We will not discuss the boundary conditions deeper for a general crystal, but they will turn out to be very strict and important for the more specific case of the six-vertex model that will be discussed in the following sections.

[^1]
### 2.2.1 The Field-Free Case

We now want to associate an energy to each of the vertex configurations in figure 2.2. These energies are of course determined by the bonding energies. It seems highly probable that under normal circumstances all vertices with a certain configuration have the same energy. This is also the case [7]. In this context, "normal circumstances" usually means that no external electric field is applied to the crystal. We define the energy of vertex $i=1,2, \ldots, 6$ from the left in figure 2.2 as $\epsilon_{i}$. Earlier studies show that, for ice, all vertex energies are equal [7], i.e.

$$
\begin{equation*}
\epsilon_{1}=\epsilon_{2}=\ldots=\epsilon_{6} \tag{2.7}
\end{equation*}
$$

This seems logical due to the symmetries of the possible vertex configurations, see figure 2.1. For ferroelectric materials such as $\mathrm{KH}_{2} \mathrm{PO}_{4}$, the relation of the vertex energies is instead

$$
\begin{equation*}
\epsilon_{3}=\epsilon_{4}=\epsilon_{5}=\epsilon_{6}>\epsilon_{1}=\epsilon_{2} \tag{2.8}
\end{equation*}
$$

Note that the vertices in figure 2.2 are pairwise symmetrical under the exchange of the plus and minus signs. Vertex configuration 1 is symmetric with vertex 2 , vertex 3 with vertex 4 and vertex 5 with vertex 6 . This fact indicates that, presuming no external electrical field is applied to the crystal, the vertex energies must satisfy the relations

$$
\begin{equation*}
\epsilon_{1}=\epsilon_{2}, \quad \epsilon_{3}=\epsilon_{4}, \quad \epsilon_{5}=\epsilon_{6} \tag{2.9}
\end{equation*}
$$

Observe that this condition is satisfied by the weights of ice in equation (2.7), as well as the weights of the ferroelectric materials in equation (2.8).
It is natural to assign a Boltzmann weight, $w_{i}=e^{-\beta \epsilon_{i}}$, to each of the possible vertex configurations. The total energy of a crystal consisting of $n_{i}$ vertices of type $i$ (for $i=1,2, \ldots, 6$ ) is of course

$$
\begin{equation*}
\epsilon=\sum_{i=1}^{6} n_{i} \epsilon_{i} \tag{2.10}
\end{equation*}
$$

From this we can conclude that the weight of a state with the energy $\epsilon$ as above must be

$$
\begin{equation*}
w_{s}=e^{-\beta \epsilon}=e^{-\beta\left(n_{1} \epsilon_{1}+\ldots+n_{6} \epsilon_{6}\right)}=e^{-\beta n_{1} \epsilon_{1}} \ldots e^{-\beta n_{6} \epsilon_{6}}=\prod_{i=1}^{6} w_{i}^{n_{i}} \tag{2.11}
\end{equation*}
$$

i.e. the product of the weights of all vertices. Let us now look at an example.

## Example 2.1

Assume we want to calculate the weight of the ice crystal in figure 2.1.
First we need to identify how many vertex configurations of each type this state consists of. We see that there is one vertex configuration of type 1, i.e. $n_{1}=1$. In the same way we identify that $n_{2}=3, n_{3}=2, n_{4}=1, n_{5}=1$ and $n_{6}=1$. The weight of the state can then be calculated using equation (2.11):

$$
\begin{equation*}
w_{s}=\prod_{i=1}^{6} w_{i}^{n_{i}}=\prod_{i=1}^{6} e^{-\beta n_{i} \epsilon_{i}}=e^{-\beta\left(\epsilon_{1}+3 \epsilon_{2}+2 \epsilon_{3}+\epsilon_{4}+\epsilon_{5}+\epsilon_{6}\right)} \tag{2.12}
\end{equation*}
$$

Since this is an ice crystal we can use the fact that $\epsilon_{1}=\epsilon_{2}=\ldots=\epsilon_{6}$ to simplify this expression. We obtain

$$
\begin{equation*}
w_{s}=e^{-9 \beta \epsilon_{1}} \tag{2.13}
\end{equation*}
$$

For a ferroelectric material, with $\epsilon_{3}=\epsilon_{4}=\epsilon_{5}=\epsilon_{6}>\epsilon_{1}=\epsilon_{2}$, we would instead obtain

$$
\begin{equation*}
w_{s}=e^{-\beta\left(4 \epsilon_{1}+5 \epsilon_{3}\right)} \tag{2.14}
\end{equation*}
$$

Remark: In the example above, the weight of one state was calculated. We still have not calculated any partition function, where we sum over all possible states. This is typically done for a given set of boundary conditions, by first constructing all possible states (with respect to the ice rule), then calculating their respective weight and finally summing these. This will be done later on, e.g. in example 2.2 and 2.3.

### 2.2.2 Row Dependent Weights and Boundary Conditions

One of the stated goals of this thesis is to find the connection between partition functions of crystals and modular forms. Earlier studies on the subject show that this connection does not exist for all choices of weights and boundary conditions [9]. One choice of boundary conditions that - with a certain choice of weights - yields a partition function in the desired form will be presented in definition 2.3. But in order to understand it, two additional definitions need to be stated.

Definition 2.1: Let $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}, \lambda_{r+1}\right)$ be a vector with $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r} \in \mathbb{Z}$ and $\lambda_{r+1}=0$, where $\lambda_{j} \geq \lambda_{j+1}$ for $j=1,2, \ldots, r$.

Definition 2.2: The Weyl vector $\rho$ is given by $\rho=(r, r-1, \ldots, 0)$.
As we will see in the following definition, the vectors $\lambda$ and $\rho$ together characterize a lattice with corresponding boundary conditions. Since $\rho$ is fully determined by $\lambda$, solely $\lambda$ is however sufficient to characterize the lattice.

Definition 2.3: For a given $\lambda$, construct a crystal $C(\lambda)$ as follows:
(i) Let the crystal have $\lambda_{1}+r+1$ columns labelled 0 to $\lambda_{1}+r$ from right to left.
(ii) Let the crystal have $r+1$ rows labelled 1 to $r+1$ from top to bottom.
(iii) Place a minus sign at the right and a plus sign at the left of each row.
(iv) Place a plus sign at the bottom of each column.
(v) Place a minus sign at the top of each column numbered by an element in the vector $\lambda+\rho$.
(vi) Place a plus sign at the top of the remaining columns.

The set of all possible states for a given vector $\lambda$ is often denoted $\mathfrak{S}_{\lambda}$. The partition function corresponding to a particular vector $\lambda$ can be found by first identifying all possible states (i.e. the states
in $\mathfrak{S}_{\lambda}$ ), then calculating the weight of each of these states and finally summing the weights. To do this concretely we need the weights of the six possible vertices in the six-vertex model, $w_{i}$. From now on - in order to obtain partition functions in the desired form - we will work with vertex weights which depend on the row of the vertex. The physical interpretation of this is that there must be an external electric field applied to the crystal. Otherwise, all vertices of the same type would contribute the same weight, independently of their position in the crystal [7]. To be rigorous, this argument only holds under the fair assumption that the whole crystal has constant temperature, since the Boltzmann weights are temperature dependent. The denotation of the different vertex weights is presented in table 2.1.

Table 2.1: The possible vertex configurations in the six-vertex model. $i$ denotes the row number of the vertex, on which the Boltzmann weights may depend.


### 2.2.3 Two Basic Examples of Partition Functions

Now, let us take a look at a pair of basic examples on how to calculate the partition function that is yielded for crystal configurations with some given vector $\lambda$.

## Example 2.2

Assume we want to calculate the partition function of the crystal constructed by the vector $\lambda=(0,0)$.

Insertion of $\lambda=(0,0)$ into definition 2.3 yields that all possible states must satisfy the boundary conditions in figure 2.3.


Figure 2.3: The boundary conditions for $\lambda=(0,0)$.
Now we study the allowed vertex configurations according to the ice rule (see table 2.1) and realize that there are two possible ways of completing this pattern. We call these states $x_{1}$ and $x_{2}$ in accordance with figure 2.4.


Figure 2.4: The two possible states for $\lambda=(0,0)$.

By studying the four vertices in each of these states we can now write down the weight of the respective state as

$$
\begin{align*}
& w\left(x_{1}\right)=b_{1}^{(1)} c_{2}^{(1)} c_{2}^{(2)} b_{2}^{(2)} \\
& w\left(x_{2}\right)=c_{2}^{(1)} a_{2}^{(1)} a_{1}^{(2)} c_{2}^{(2)} \tag{2.15}
\end{align*}
$$

where the notation from table 2.1 has been used. The partition function is found by summing these weights, which yields

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right) \tag{2.16}
\end{equation*}
$$

Let us now do the same calculation for a slightly larger lattice. We do this mainly because these two examples will prove to differ, in a crucial way, when the connection to modular forms is studied.

## Example 2.3

We now want to determine the partition function for the crystal yielded from the vector $\lambda=(1,0)$.

First note that $r=1$, in accordance with definition 2.1. After enforcing the boundary conditions in definition 2.3 one arrives at the configuration in figure 2.5.


Figure 2.5: The boundary conditions for $\lambda=(1,0)$.
Very simple combinatorics then shows that there are three possible states satisfying the ice rule, $x_{1}, x_{2}$ and $x_{3}$, which can be seen in figure 2.6.


Figure 2.6: The three possible states for $\lambda=(1,0)$.

We can now easily calculate the weight for each of these states. This yields

$$
\begin{align*}
& w\left(x_{1}\right)=b_{1}^{(1)} a_{1}^{(1)} c_{2}^{(1)} \cdot c_{2}^{(2)}\left(b_{2}^{(2)}\right)^{2} \\
& w\left(x_{2}\right)=c_{2}^{(1)} b_{2}^{(1)} a_{2}^{(1)} \cdot\left(a_{1}^{(2)}\right)^{2} c_{2}^{(2)}  \tag{2.17}\\
& w\left(x_{3}\right)=\left(c_{2}^{(1)}\right)^{2} c_{1}^{(1)} \cdot a_{1}^{(2)} c_{2}^{(2)} b_{2}^{(2)}
\end{align*}
$$

The partition function is simply the sum of these weights, i.e.

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\sum_{i=1}^{3} w\left(x_{i}\right)=c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(1)} b_{1}^{(1)}\left(b_{2}^{(2)}\right)^{2}+a_{2}^{(1)} b_{2}^{(1)}\left(a_{1}^{(2)}\right)^{2}+c_{1}^{(1)} c_{2}^{(1)} a_{1}^{(2)} b_{2}^{(2)}\right) \tag{2.18}
\end{equation*}
$$

An example for an even larger lattice - with $\lambda=(2,1,0)$ - can be found in appendix B .

### 2.3 The Connection to Schur Polynomials

In the preceding section, we learned how to calculate the partition function of a crystal configuration for a given vector $\lambda$. One of the goals of this report is to find the relation between these partition functions and modular forms. In order to do this, some constraints on the Boltzmann weights $a_{1}^{(i)}, a_{2}^{(i)} \ldots, c_{2}^{(i)}$ need to be introduced. This will soon be done. But since the connection between the partition functions and modular forms will be constituted by the so called Schur polynomials, we will begin by introducing these.

Definition 2.4: The Schur polynomial $s_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ corresponding to $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ is defined as

$$
\begin{equation*}
s_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\frac{a_{\left(\lambda_{1}+n-1, \lambda_{2}+n-2, \ldots, \lambda_{n}\right)}\left(x_{1}, x_{2}, \ldots, x_{n}\right)}{a_{(n-1, n-2, \ldots, 0)}\left(x_{1}, x_{2}, \ldots, x_{n}\right)} \tag{2.19}
\end{equation*}
$$

where

$$
a_{u}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\operatorname{det}\left(\begin{array}{cccc}
x_{1}^{u_{1}} & x_{2}^{u_{1}} & \ldots & x_{n}^{u_{1}}  \tag{2.20}\\
x_{1}^{u_{2}} & x_{2}^{u_{2}} & \ldots & x_{n}^{u_{2}} \\
\vdots & \vdots & \ddots & \vdots \\
x_{1}^{u_{n}} & x_{2}^{u_{n}} & \ldots & x_{n}^{u_{n}}
\end{array}\right)
$$

for the vector $u=\left(u_{1}, u_{2}, \ldots, u_{n}\right)$.

Let us look at an example that will prove to be very useful later on.

## Example 2.4

We claim that for $\lambda=(k, 0)$ the corresponding Schur polynomial is

$$
\begin{equation*}
s_{(k, 0)}\left(x_{1}, x_{2}\right)=x_{1}^{k}+x_{1}^{k-1} x_{2}+\ldots+x_{1} x_{2}^{k-1}+x_{2}^{k}=\sum_{i=0}^{k} x_{1}^{i} x_{2}^{k-i} \tag{2.21}
\end{equation*}
$$

From the definition of Schur polynomials we have

$$
s_{(k, 0)}\left(x_{1}, x_{2}\right)=\frac{\left|\begin{array}{cc}
x_{1}^{k+1} & x_{2}^{k+1}  \tag{2.22}\\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
x_{1}^{1} & x_{2}^{1} \\
1 & 1
\end{array}\right|}=\frac{x_{1}^{k+1}-x_{2}^{k+1}}{x_{1}-x_{2}}
$$

To show (2.21) we will use a proof of induction. Begin with the case $k=0$, then

$$
s_{(0,0)}\left(x_{1}, x_{2}\right)=\frac{\left|\begin{array}{cc}
x_{1} & x_{2}  \tag{2.23}\\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
x_{1} & x_{2} \\
1 & 1
\end{array}\right|}=1=x_{1}^{0} x_{2}^{0}
$$

So, equation (2.21) is satisfied for $k=0$. Now assume that equation (2.21) holds for some general value of $k$. Then

$$
\begin{align*}
s_{(k+1,0)}\left(x_{1}, x_{2}\right) & =\frac{\left|\begin{array}{cc}
x_{1}^{k+2} & x_{2}^{k+2} \\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
x_{1}^{1} & x_{2}^{1} \\
1 & 1
\end{array}\right|}=\frac{x_{1}^{k+2}-x_{2}^{k+2}}{x_{1}^{1}-x_{2}^{1}}=x_{1}^{k+1}+x_{2} \frac{x_{1}^{k+1}-x_{2}^{k+1}}{x_{1}-x_{2}}=  \tag{2.24}\\
& =x_{1}^{k+1}+x_{2} \sum_{i=0}^{k} x_{1}^{i} x_{2}^{k-i}=\sum_{i=0}^{k+1} x_{1}^{i} x_{2}^{k+1-i},
\end{align*}
$$

which by induction shows that (2.21) holds for all values of $k$.

We will now introduce a different way to write a Schur polynomial that we will find useful later. First use Leibniz formula for determinants which states that for a matrix $A$ of dimension $n$

$$
\begin{equation*}
\operatorname{det}(A)=\sum_{\sigma \in S_{N}} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} A_{\sigma(i), i} \tag{2.25}
\end{equation*}
$$

Apply this to the numerator in the definition of the Schur polynomials. The elements of the matrix can be written as $x_{j}^{\lambda_{i}+n-i}$, why

$$
\begin{equation*}
a_{\lambda_{1}+n-1, \lambda_{2}+n-2, \ldots, \lambda_{n}}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{\sigma \in S_{N}} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} x_{i}^{\lambda_{\sigma(i)+n-\sigma(i)}} \tag{2.26}
\end{equation*}
$$

The denominator in the definition of the Schur polynomials is known as the Vandermonde determinant and can be written as

$$
\begin{equation*}
a_{n-1, n-2, \ldots, 0}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\prod_{i<j}\left(x_{i}-x_{j}\right) \tag{2.27}
\end{equation*}
$$

The proof of this is rather simple but will not be written here. It can for example be found at [18].
Combining these two results we get the following lemma:
Lemma 2.1: Any Schur polynomial $s_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ can be written in the form

$$
\begin{equation*}
s_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{\sigma \in S_{N}} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} x_{i}^{\lambda_{\sigma(i)+n-\sigma(i)}}\left[\prod_{i<j}\left(x_{i}-x_{j}\right)\right]^{-1} \tag{2.28}
\end{equation*}
$$

With this alternative way of writing a Schur polynomial, a theorem on how the Boltzmann weights should be chosen to obtain partition functions including a Schur polynomial may be stated and proven.

Theorem 2.1: For a given $\lambda$, let $\mathfrak{S}_{\lambda}$ denote the set of possible states according to definition 2.3 and suppose that the Boltzmann weights of the vertices satisfy the so called free fermion condition

$$
\begin{equation*}
a_{1}^{(i)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(i)}=c_{1}^{(i)} c_{2}^{(i)} \tag{2.29}
\end{equation*}
$$

for all values of $i$. Then the partition function is given by

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\left[\prod_{k=1}^{r+1}\left(a_{1}^{(k)}\right)^{\lambda_{1}} c_{2}^{(k)} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\right] s_{\lambda}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}, \ldots, \frac{b_{2}^{(r+1)}}{a_{1}^{(r+1)}}\right) \tag{2.30}
\end{equation*}
$$

where $s_{\lambda}$ is the Schur polynomial corresponding to $\lambda$.
A proof of theorem 2.1 can be found in section 2.3.2.
One may wonder why it was necessary to introduce the free fermion condition in equation (2.29). In the physical interpretation, this condition is important to guarantee exact solvability even in presence of an external field [19]. This is absolutely necessary now that we are working with row-dependent weights. The free fermion condition is also important in order for the connection to modular forms to appear $[8,20]$. The essence of the free fermion condition is that it is only satisfied by systems in which there is no overlap between the wavefunctions of the fermions. Such fermions are called free, hence the name of the condition [21]. For these systems, many quantum effects may be neglected, which makes them exactly solvable [21, 22]. For an ice crystal it is mainly the electrons of the hydrogen and oxygen atoms that need to be free. This may seem odd, since overlap of the wave functions of the electrons is necessary in order for a covalent bond to arise [23]. For reasons well beyond the scope of this report, this approximation is, however, often acceptable in the six-vertex model [24].

So what weights should be chosen to be more specific than the free fermion condition? A common choice of weights is the following:

$$
\begin{equation*}
a_{1}^{(i)}=1, \quad a_{2}^{(i)}=z_{i}, \quad b_{1}^{(i)}=t_{i}, \quad b_{2}^{(i)}=z_{i}, \quad c_{1}^{(i)}=z_{i}\left(t_{i}+1\right), \quad c_{2}^{(i)}=1 \tag{2.31}
\end{equation*}
$$

Note that these weights satisfy the free fermion condition in equation (2.29). Theorem 2.1 then states that for this choice of weights, the partition function can be written as

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\prod_{i<j}\left(t_{i} z_{j}+z_{i}\right) s_{\lambda}\left(z_{1}, z_{2}, \ldots, z_{r+1}\right) \tag{2.32}
\end{equation*}
$$

There is not a physical interpretation of this particular choice of weights yet. Theoretically, one could choose any set of weights that satisfies the free fermion condition. However, the set of weights given in equation (2.31) has proven to imply especially interesting results, with deep connections to intriguing areas in mathematics such as modular forms $[8,9]$. This can be said to be a field where the mathematics anticipate the physical interpretation.

In order to fully understand the connection to modular forms we will need some mathematical background that is presented in chapter 3 and 4. An explicit investigation of the relation between crystals and modular forms is then carried out in chapter 5 .

### 2.3.1 The Examples $\lambda=(0,0)$ and $\lambda=(1,0)$ Revisited

It is now interesting to confirm that theorem 2.1 is fulfilled by the partition functions we calculated in example 2.2 and 2.3 , for $\lambda=(0,0)$ and $\lambda=(1,0)$ respectively.

## Example 2.5

For $\lambda=(0,0)$ we obtained the partition function

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right) \tag{2.33}
\end{equation*}
$$

We now want to show that, provided that the Boltzmann weights satisfy the free fermion condition, this can be written in the form of equation (2.30) in theorem 2.1.

From the definition of $\lambda$, given in definition 2.1, it is obvious that $\lambda=(0,0)$ has $\lambda_{1}=$ $\lambda_{0}=0$ and $r=1$. Inserting this into equation (2.30) yields

$$
\begin{align*}
Z\left(\mathfrak{S}_{\lambda}\right) & =\left[\prod_{k=1}^{2} c_{2}^{(k)} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\right] s_{(0,0)}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}\right) \\
& =c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right) s_{(0,0)}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}\right) \tag{2.34}
\end{align*}
$$

Using definition 2.4 of the Schur polynomials we can rewrite the Schur polynomial in this expression as

$$
s_{(0,0)}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}\right)=\frac{\left|\begin{array}{cc}
b_{2}^{(1)} / a_{1}^{(1)} & b_{2}^{(2)} / a_{1}^{(2)}  \tag{2.35}\\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
b_{2}^{(1)} / a_{1}^{(1)} & b_{2}^{(2)} / a_{1}^{(2)} \\
1 & 1
\end{array}\right|}=1
$$

So, for $\lambda=(0,0)$, equation (2.30) in theorem 2.1 can be rewritten in the form

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right) \tag{2.36}
\end{equation*}
$$

Since this is the exact same partition function as the one in equation (2.33), which was the result from example 2.2, we can conclude that theorem 2.1 really is valid for $\lambda=(0,0)$.

We can also confirm that equation (2.32) is fulfilled if we insert the set of weights from equation (2.31). Insertion of these weights into equation (2.34) yields

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\left(t_{1} z_{2}+z_{1}\right) s_{(0,0)}\left(z_{1}, z_{2}\right) \tag{2.37}
\end{equation*}
$$

which can easily be seen to satisfy equation (2.32) with $r=1$. However, since we showed in equation (2.35) that the Schur polynomial

$$
\begin{equation*}
s_{(0,0)}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}\right)=1, \tag{2.38}
\end{equation*}
$$

independently of the choice of weights, this is preferably rewritten as

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=t_{1} z_{2}+z_{1} \tag{2.39}
\end{equation*}
$$

Remark: In this example, theorem 2.1 was fulfilled independently of the choice of weights. Thus, we did not have to use the free fermion condition. This is a consequence of the Schur polynomial being independent of the weights (we had $s_{(0,0)}=1$ ). As we have mentioned before, the free fermion condition is necessary in order for the connection to modular forms to appear. And since this connection is constituted by the Schur polynomials, no weight dependence for the Schur polynomials means that the connection appears independently of the choice of weights. Thus, the free fermion condition is not necessary. However, in the following example for $\lambda=(1,0)$ and other more complex examples, the free fermion condition is crucial.

## Example 2.6

For $\lambda=(1,0)$ we obtained the partition function

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(1)} b_{1}^{(1)}\left(b_{2}^{(2)}\right)^{2}+a_{2}^{(1)} b_{2}^{(1)}\left(a_{1}^{(2)}\right)^{2}+c_{1}^{(1)} c_{2}^{(1)} a_{1}^{(2)} b_{2}^{(2)}\right) \tag{2.40}
\end{equation*}
$$

To show that this expression can be written in the form of equation (2.30) in theorem 2.1, provided that the weights satisfy the free fermion condition, we note that $\lambda=(1,0)$ has $r=1$ and $\lambda_{1}=1$. Insertion of this into equation (2.30) yields

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=a_{1}^{(1)} c_{2}^{(1)} a_{1}^{(2)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right) s_{(1,0)}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}\right) \tag{2.41}
\end{equation*}
$$

Now we use the definition of the Schur polynomials to evaluate the Schur polynomial in this expression. We obtain

$$
s_{(1,0)}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}\right)=\frac{\left|\begin{array}{cc}
\left(b_{2}^{(1)} / a_{1}^{(1)}\right)^{2} & \left(b_{2}^{(2)} / a_{1}^{(2)}\right)^{2}  \tag{2.42}\\
1 & 1
\end{array}\right|}{\left|\begin{array}{cc}
b_{2}^{(1)} / a_{1}^{(1)} & b_{2}^{(2)} / a_{1}^{(2)} \\
1 & 1
\end{array}\right|}=\frac{a_{1}^{(2)} b_{2}^{(1)}+a_{1}^{(1)} b_{2}^{(2)}}{a_{1}^{(1)} a_{1}^{(2)}}
$$

after some simple algebra. Insertion of this result into equation (2.41) yields

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right)\left(a_{1}^{(2)} b_{2}^{(1)}+a_{1}^{(1)} b_{2}^{(2)}\right) \tag{2.43}
\end{equation*}
$$

This expression is not the same as the one in equation (2.40), which depends on that we still have not used the free fermion condition. This condition tells us that all weights must satisfy $a_{1}^{(i)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(i)}=c_{1}^{(i)} c_{2}^{(i)}$. Applying this on the product $c_{1}^{(1)} c_{2}^{(1)}$ in the last term in equation (2.40) tells us that equation (2.40) can be rewritten as

$$
\begin{aligned}
Z\left(\mathfrak{S}_{\lambda}\right) & =c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(1)} b_{1}^{(1)}\left(b_{2}^{(2)}\right)^{2}+a_{2}^{(1)} b_{2}^{(1)}\left(a_{1}^{(2)}\right)^{2}+a_{1}^{(1)} a_{2}^{(1)} a_{1}^{(2)} b_{2}^{(2)}+b_{1}^{(1)} b_{2}^{(1)} a_{1}^{(2)} b_{2}^{(2)}\right) \\
& =c_{2}^{(1)} c_{2}^{(2)}\left(a_{1}^{(2)} a_{2}^{(1)}+b_{1}^{(1)} b_{2}^{(2)}\right)\left(a_{1}^{(2)} b_{2}^{(1)}+a_{1}^{(1)} b_{2}^{(2)}\right)
\end{aligned}
$$

This is the exact same expression as the one in equation (2.43). Thus, we have shown that the partition function for $\lambda=(1,0)$ can be written in the form of equation (2.30) in theorem 2.1 if and only if the weights satisfy the free fermion condition.

We may also insert the common choice of weights given in equation (2.31), either directly into equation (2.43) obtaining

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=t_{1} z_{2}^{2}+z_{1}^{2}+z_{1} z_{2}\left(t_{1}+1\right)=\left(t_{1} z_{2}+z_{1}\right)\left(z_{1}+z_{2}\right) \tag{2.44}
\end{equation*}
$$

or into the more general formula given in equation (2.32) obtaining

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\left(t_{1} z_{2}+z_{1}\right) s_{(1,0)}\left(z_{1}, z_{2}\right) \tag{2.45}
\end{equation*}
$$

These expressions are of course equal, since we have shown that equality for general weights satisfying the free fermion condition above.

Remark: In both of the examples in this section we were able to work exclusively with arbitrary weights satisfying the free fermion condition. Thus, one can question why it was necessary to introduce the set of weights in equation (2.31). One reason has already been mentioned; it will make the connection to modular forms more clear. Another reason is that the simplifications that this choice entails, e.g. equation (2.30) simplifies to (2.32), is crucial when handling more complex systems. This becomes obvious for the case $\lambda=(2,1,0)$ which can be found in appendix B .

### 2.3.2 A Proof of Theorem 2.1

Before actually proving theorem 2.1 we need to introduce some identities, lemmas and corollaries that will be used in the proof.

First we introduce the star-triangle identity which is the following identity between partition func-
tions:

where $\tau, \sigma, \beta, \theta, \rho$ and $\alpha$ are fixed boundary conditions with values + or $-. R, S$ and $T$ can be viewed as sets from which the weights of the corresponding vertex is chosen. $S$ and $T$ could for example denote different rows. In this identity we have allowed a new set of configurations that are rotated in regards to the other configurations. These new configuration will soon prove useful in the proof of theorem 2.1.

The star-triangle identity is also known as the Yang-Baxter equation. This equation is very important in many different fields of physics and mathematics. Some of these are quantum field theory, quantum groups and $\mathrm{C}^{*}$-algebras [25]. Equation (2.46) is rewritten in a form more easily recognizable as the Yang-Baxter equation in appendix A.

The identity does not hold for an arbitrary choice of weights, however in Schur Polynomials and the Yang-Baxter Equation by Ben Brubaker, Daniel Bump and Solomon Friedberg [8] conditions are given for $S$ and $T$ such that an $R$ can be found so the identity is correct. We are satisfied by stating that given weights $S$ from row $i$ and $T$ from row $j$ the identity holds for the set of weights $R$ given in table 2.2. This is also shown in appendix A

Table 2.2: A set of weights for which the star-triangle identity hold.


Lemma 2.2: The expression

$$
\begin{equation*}
\left(a_{1}^{(i)} a_{2}^{(i+1)}+b_{1}^{(i+1)} b_{2}^{(i)}\right) Z\left(\mathfrak{S}_{\lambda}\right) \tag{2.47}
\end{equation*}
$$

is symmetric with regards to exchanging $i$ and $i+1$.
Proof: Study a lattice as before but with slightly modified boundary conditions. Connect row $i$ and $i+1$ on the left side with a tilted vertex with weights chosen as above. Let the two leftmost edges of this vertex have spin + . The only allowed configurations of this vertex is then with both right-hand spin being + as well. An example can be seen below in a lattice defined by $\lambda=(3,1,0)$ with a extra vertex connecting rows $i=2$ and $j=3$


This lattice now has a partition function given by equation (2.47). Now apply the star triangle identity and we get the following lattice:


This lattice has the same partition function as before. Repeat the star-triangle identity until the diagonal vertex reaches the right edge. We then have the following system:


The only allowed configuration of the diagonal vertex is now with all surrounding spins equal to -. Notice that this configuration have the same weight as the one with all spins being + but with $i$ and $j$ switched. Also notice how the row numbers of $i$ and $j$ have switched. Since the partition function is still unchanged we have now shown the lemma.

Corollary 2.1: The expression

$$
\begin{equation*}
\prod_{i<j}\left(a_{1}^{(i)} a_{2}^{(j)}+b_{1}^{(j)} b_{2}^{(i)}\right) Z\left(\mathfrak{S}_{\lambda}\right) \tag{2.48}
\end{equation*}
$$

is symmetric with regards to any pair of spectral parameters $k$ and $l$.
Proof: Apply lemma 2.2 multiple times.

Lemma 2.3: Study just one row of a lattice. Let the left edge have positive spin. Denote by $m$ the number of negative spins on the upper edge and $m^{\prime}$ the number of negative spins on the lower edge.

If the right edge has positive spin then $m^{\prime}=m$ and if it has negative spin then $m^{\prime}=m-1$
Proof: Since we are only working with one row we can neglect writing out the index specifying the row. We try to fill the row from the left side. We are interested in the difference between the number of negative spins on the upper and lower edge. By adding a weight $c_{1}$ we change the rightmost spin from negative to positive and by adding a weight $c_{2}$ we change it from positive to negative. For each $c_{1}$ we add one negative spin to the lower edge and for each $c_{2}$ we add one negative spin to the upper edge. None of the other weights change the rightmost spin nor do they change the difference between the number of negative spins between the upper and lower edge. We know that the leftmost edge has positive spin. If the number of $c_{1}$ is equal to the number of $c_{2}$ the row must end with a positive spin and there is the same amount of negative spins on the lower and upper edge. If we have one more of $c_{2}$ than $c_{1}$ the row must end with a negative spin and we have one more negative spin on the upper edge.

With these results formulated and proven, we are now ready to prove the important theorem 2.1.
Proof of theorem 2.1: Begin by using the free fermion condition in equation (2.29) to make the substitution $c_{1}^{(i)}=\left(a_{1}^{(i)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(i)}\right) / c_{2}^{(i)}$. The resulting partition function is then independent of $c_{1}^{(i)}$ and we may thus take it to be 0 . This makes it so that the only states we need to consider in the calculation of the partition function are the ones that do not contain the vertex configuration corresponding to $c_{1}^{(i)}$.

With the choice of boundary conditions according to lemma 2.3 and that we do not need to look at the states with a configuration $c_{1}^{(i)}$ we understand that each row has exactly one vertex with configuration $c_{2}^{(i)}$. This means that each term of $Z\left(\mathfrak{S}_{\lambda}\right)$ contains the factor

$$
\begin{equation*}
\prod_{i=1}^{r+1} c_{2}^{(i)} \tag{2.49}
\end{equation*}
$$

We find this exact factor in the formula we wish to show. For simplicity we let $c_{2}^{(i)}=1$ for the rest of the proof.

An identity we soon will find useful is the following:

$$
\begin{equation*}
\prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)=\prod_{i<j} a_{1}^{(j)} b_{1}^{(i)}\left(\frac{b_{2}^{(j)}}{a_{1}^{(j)}}-\frac{b_{2}^{(i)}}{a_{1}^{(i)}}\right)=\prod_{i=1}^{r+1}\left(a_{1}^{(j)}\right)^{i-1}\left(b_{1}^{(i)}\right)^{r+1-i} \prod_{i<j}\left(\frac{b_{2}^{(j)}}{a_{1}^{(j)}}-\frac{b_{2}^{(i)}}{a_{1}^{(i)}}\right) \tag{2.50}
\end{equation*}
$$

where we in the first equality have used the free fermion condition.
We now have $Z\left(\mathfrak{S}_{\lambda}\right)$ as a function of $a_{1}^{(i)}, a_{2}^{(i)}, b_{1}^{(i)}$ and $b_{2}^{(i)}$. Using that each row contains exactly one vertex with configuration $c_{2}^{(i)}$ we conclude that all but one of the minus spins on the upper edge of the row $i$ are from vertices with the configurations $a_{2}^{(i)}$ or $b_{1}^{(i)}$. From the boundary conditions we know that on the top boundary the number of minus spins is $r+1$. Using lemma 2.3 we conclude that in row $i$ there are $r+1-i$ number of vertices with configuration $a_{2}^{(i)}$ or $b_{1}^{(i)}$.

Now take $Z\left(\mathfrak{S}_{\lambda}\right)$ and make the substitution $a_{2}^{(i)}=-b_{1}^{(i)} z_{i}$. Call the resulting function $N_{\mathfrak{S}_{\lambda}}$. Note that $Z\left(\mathfrak{S}_{\lambda}\right)=N_{\mathfrak{S}_{\lambda}}$ so each term of $N_{\mathfrak{S}_{\lambda}}$ still corresponds to a specific state. In each term of $N_{\mathfrak{S}_{\lambda}}$ the
power of $b_{1}^{(i)}$ is now equal to $r+1-i$. We may write

$$
\begin{equation*}
N_{\mathfrak{S}_{\lambda}}=N_{\mathfrak{S}_{\lambda}}^{\prime} \prod_{i=1}^{r+1}\left(b_{1}^{(i)}\right)^{r+1-i} \tag{2.51}
\end{equation*}
$$

where $N_{\mathfrak{S}_{\lambda}}^{\prime}$ is a polynomial in the variables $a_{1}^{(i)}, b_{2}^{(i)}$ and $z_{i}$.
We know that the total number of columns is $\lambda_{1}+r+1$. Using what we know of the number of configurations $a_{2}^{(i)}, b_{1}^{(i)}, c_{1}^{(i)}$ and $c_{2}^{(i)}$ we conclude that the total number of vertices $a_{1}^{(i)}$ and $b_{2}^{(i)}$ in each row is equal to $\lambda_{1}+r+1-1-(r+1-i)=\lambda_{1}+i-1$. Write

$$
\begin{equation*}
N_{\mathfrak{S}_{\lambda}}^{\prime}=N_{\mathfrak{S}_{\lambda}}^{\prime \prime} \prod_{i=1}^{r+1}\left(a_{1}^{(i)}\right)^{\lambda_{1}+i-1} \tag{2.52}
\end{equation*}
$$

where $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ is a polynomial in the variables $b_{2}^{(i)} / a_{1}^{(i)}$ and $z_{i}$.
From our condition with $c_{1}^{i}=0$ we know that

$$
\begin{equation*}
\frac{a_{2}^{(i)}}{b_{1}^{(i)}}=-\frac{b_{2}^{(i)}}{a_{1}^{(i)}}=-z_{i} \tag{2.53}
\end{equation*}
$$

Using this we know that $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ is a polynomial only in the variable $z_{i}$.
Putting this together we have

$$
\begin{align*}
Z\left(\mathfrak{S}_{\lambda}\right) & =N_{\mathfrak{S}_{\lambda}}^{\prime} \prod_{i=1}^{r+1}\left(b_{1}^{(i)}\right)^{r+1-i}=N_{\mathfrak{S}_{\lambda}}^{\prime \prime} \prod_{i=1}^{r+1}\left(b_{1}^{(i))^{r+1-i}}\left(a_{1}^{(i)}\right)^{\lambda_{1}+i-1}\right. \\
& =N_{\mathfrak{S}_{\lambda}}^{\prime \prime} \prod_{i=1}^{r+1}\left(a_{1}^{(i)}\right)^{\lambda_{1}} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\left[\prod_{i<j}\left(z_{j}-z_{i}\right)\right]^{-1} . \tag{2.54}
\end{align*}
$$

Then, rewrite this as

$$
\begin{equation*}
\prod_{i<j}\left(a_{1}^{(i)} a_{2}^{(j)}+b_{1}^{(j)} b_{2}^{(i)}\right) Z\left(\mathfrak{S}_{\lambda}\right)=N_{\mathfrak{S}_{\lambda}}^{\prime \prime} \prod_{i=1}^{r+1}\left(a_{1}^{(i)}\right)^{\lambda_{1}} \prod_{i \neq j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\left[\prod_{i<j}\left(z_{j}-z_{i}\right)\right]^{-1} \tag{2.55}
\end{equation*}
$$

By lemma 2.2 the left hand side is symmetric with regard to the spectral parameters. Since the denominator of the right hand side is antisymmetric and the other explicit terms are symmetric we conclude that $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ is antisymmetric.
The power of $z_{i}$ in a term of $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ is equal to the sum of the power of $a_{2}^{(i)}$ and $b_{2}^{(i)}$ in the corresponding term of $Z\left(\mathfrak{S}_{\lambda}\right)$. As we recently stated, in each row there is exactly one vertex configuration $c_{2}^{(i)}$. By looking at the left and right spin of the admissible states we can conclude that all vertices left of $c_{1}^{(i)}$ have state $a_{1}^{(i)}$ or $b_{1}^{(i)}$. Similarly we conclude that all vertices right of $c_{1}^{(i)}$ have state $a_{2}^{(i)}$ or $b_{2}^{(i)}$. If we know the positions of the weights $c_{1}^{(i)}$ in each row for a specific state we therefore know the power of $z_{i}$ in the corresponding term in $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$.
Since $c_{1}^{(i)}=0$, all negative spins for a state of a lattice must be in columns that stretch from the top of the lattice and end in a vertex configuration $c_{2}^{(i)}$. From the boundary conditions the indices of these


Figure 2.7: A state of the lattice corresponding to $\lambda=(3,2,0)$. The state corresponds to $\sigma(i)=i$
columns must be from the vector $l_{i}=\lambda_{i}+\rho_{i}$ with $1 \leq i \leq r+1$. That means that the column indices of the $c_{2}^{(i)}$-configurations can be written as $l_{\sigma(i)}$ where $\sigma$ is a permutation in the symmetric group, $\sigma \in S_{r+1}$. From $\sigma$ we know all the positions of negative spins in columns for a specific state. This is enough information to uniquely fill the rest of the spins in the lattice. In other words there exists a bijection between $\sigma$ and states of the lattice.

Using this we can write each term of $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ as

$$
\begin{equation*}
\pm \prod_{i=0}^{r} z_{i}^{l_{\sigma(i)}} \tag{2.56}
\end{equation*}
$$

Since we know that $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ is antisymmetric with regard to the spectral parameters we can write it as

$$
\begin{equation*}
N_{\mathfrak{S}_{\lambda}}^{\prime \prime}= \pm \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma) \prod_{i=0}^{r} z_{i}^{l_{\sigma(i)}} \tag{2.57}
\end{equation*}
$$

We now need to determine the remaining overall sign. To do this it is enough to determine the sign of just one state. From how the partition function is defined we conclude that all terms in $Z\left(\mathfrak{S}_{\lambda}\right)$ have sign + . By looking at how $N_{\mathfrak{S}_{\lambda}}^{\prime \prime}$ is defined from $Z\left(\mathfrak{S}_{\lambda}\right)$ we can see that a term changes sign once for each configuration $a_{2}^{(i)}$ the state contains. Now study a state corresponding to $\sigma(i)=i$. For this $\sigma$ we have $\operatorname{sgn}(\sigma)=1$. An example of such a state with $\lambda=(3,2,0)$ can be seen in figure 2.7 .

Note how every column indexed by $l$ only has spin + in the bottom part and only spin - in the upper part with a configuration $c_{2}^{(i)}$ in between. All other columns only have spin + . Right of the $c_{2}^{(i)}-$ configuration there are only spin - . This means that in a certain column where the $c_{2}^{(i)}$-configuration is in row $i$, the number of configurations $a_{2}^{(j)}$ are $i-1$. Since there is exactly one $c_{2}^{(i)}$-configuration in each row the total number of $a_{2}^{(2)}$-configurations is

$$
\begin{equation*}
\sum_{i=1}^{r+1}(i-1)=\frac{1}{2} r(r+1) \tag{2.58}
\end{equation*}
$$

Using this we now have

$$
\begin{equation*}
N_{\mathfrak{S}_{\lambda}}^{\prime \prime}=(-1)^{\frac{1}{2} r(r+1)} \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma) \prod_{i=0}^{r} z_{i}^{l_{\sigma(i)}} \tag{2.59}
\end{equation*}
$$

Now study the term

$$
\begin{equation*}
\prod_{i<j}\left(z_{j}-z_{i}\right)=\prod_{j=2}^{r+1} \prod_{i=1}^{j-1}(-1) \prod_{i<j}\left(z_{i}-z_{j}\right)=(-1)^{\sum_{j=2}^{r+1}(j-1)} \prod_{i<j}\left(z_{i}-z_{j}\right)=(-1)^{\frac{1}{2} r(r+1)} \prod_{i<j}\left(z_{i}-z_{j}\right) . \tag{2.60}
\end{equation*}
$$

Combining equation (2.54) with (2.59) and (2.60) yields

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\prod_{i=1}^{r+1}\left(a_{1}^{(i)}\right)^{\lambda_{1}} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\left[\prod_{i<j}\left(z_{i}-z_{j}\right)\right]^{-1} \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma) \prod_{i=0}^{r} z_{i}^{l_{\sigma(i)}} \tag{2.61}
\end{equation*}
$$

Now apply lemma 2.1 and we get

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\prod_{i=1}^{r+1}\left(a_{1}^{(i)}\right)^{\lambda_{1}} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right) s_{\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r+1}\right)}\left(z_{1}, z_{2}, \ldots, z_{r+1}\right) \tag{2.62}
\end{equation*}
$$

To obtain the exact same formula as in theorem 2.1, we will use that $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r+1}\right)$ and the fact that equation (2.53) tells us that $z_{i}=b_{2}^{(i)} / a_{i}^{(i)}$. We also include the factor from equation (2.49) that we, without loss of generality, neglected throughout the proof by letting $c_{2}^{(i)}=1$. Equation (2.62) may then be rewritten as

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\left[\prod_{i=1}^{r+1}\left(a_{1}^{(i)}\right)^{\lambda_{1}} c_{2}^{(i)} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\right] s_{\lambda}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}, \ldots, \frac{b_{2}^{(r+1)}}{a_{1}^{(r+1)}}\right) \tag{2.63}
\end{equation*}
$$

which is in the exact same form as equation (2.30) in theorem 2.1.

## Chapter 3

## Group and Representation Theory

Group theory is utterly important for many physicists, as groups often represent symmetries of different sorts in physical theories. Steven Weinberg is a physicist who won the Nobel Prize in Physics in 1979 together with Sheldon Glashow and Abdus Salam "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current." [26]. He famously said that "The universe is an enormous direct product of representations of symmetry groups." [27] which proves how crucial group theory is when using physics to understand the universe.

This chapter mostly contains the mathematics needed to proceed to chapter 4 and onward. Seeing this is a mathematics chapter and that the mathematics used is well-recognized, further reading can be found in textbooks like Groups, Representations and Physics by H F Jones[28] or Mathematical Methods for Physicists by Arfken, Weber and Harris[29]. The masters thesis Eisenstein Series and Instantons in String Theory by Henrik Gustafsson [5] has also proven useful for the writing of this chapter. If something particular stands out, then Wikipedia is a great place to find more information.

### 3.1 Definitions and Examples

Definition 3.1: A group is the set of objects or operations, called the elements, of $G$ that may be combined through an operation, $\star$, to satisfy the following four axioms:

1. Closure - for two elements in $G, g$ and $h$, there exists a unique element $k$ such that:

$$
g \star h=k \in G
$$

2. Associativity - for all elements $g$, $h$ and $k \in G$, chosen arbitrarily, it follows that:

$$
(g \star h) \star k=g \star(h \star k)
$$

3. Unit element There exists a unique unit element $e$ in $G$ such that

$$
e \star g=g \star e=g, \forall g \in G
$$

4. Inverse element Each element $g$ in $G$ has an inverse $g^{-1} \in G$ where it follows that

$$
g \star g^{-1}=g^{-1} \star g=e, \forall g \in G
$$

It can also be useful to define what a subgroup is, as this will be of avail later.
Definition 3.2: Let $H$ be a subset of $G, H \subseteq G$. Then it follows for $H$ that:

1. The identity element, e, lies in $H$.
2. If $h_{1}, h_{2} \in H$, then $h_{1} \star h_{2} \in H$
3. If $h \in H$, then $h^{-1} \in H$

All of these ramifications are somewhat self-evident as the subgroup $H$ must fulfil the requirements of a group, in definition 3.1.

Now, to get a better understanding of what a group is, let us have a look at an example.

## Example 3.1

When looking at the equilateral triangle below, one can identify a group with six elements that would leave the triangle unchanged.


Figure 3.1: Equilateral triangle with symmetry axes $a, b$ and $c$.

- The identity, e, leaving the triangle as it is.
- A rotation of $120^{\circ}\left(1 / 3\right.$ of a revolution) around the centre, $C_{1}$.
- A rotation of $240^{\circ}(2 / 3$ of a revolution $)$ around the centre, $C_{2}$
- An operation in which the triangle is reflected through axis a, $C_{a}$
- An operation in which the triangle is reflected through axis b, $C_{b}$
- An operation in which the triangle is reflected through axis $\mathrm{c}, C_{c}$

The multiplication table of the group is presented in table 3.1 below.

Table 3.1: Table depicting the products of the different elements, the rotations of an equilateral triangle. The table entry on row a, column b is the product $\mathrm{ab}, C_{a} C_{b}=C_{1}$.

|  | e | $C_{1}$ | $C_{2}$ | $C_{a}$ | $C_{b}$ | $C_{c}$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| e | e | $C_{1}$ | $C_{2}$ | $C_{a}$ | $C_{b}$ | $C_{c}$ |
| $C_{1}$ | $C_{1}$ | $C_{2}$ | $e$ | $C_{c}$ | $C_{a}$ | $C_{b}$ |
| $C_{2}$ | $C_{2}$ | $e$ | $C_{1}$ | $C_{b}$ | $C_{c}$ | $C_{a}$ |
| $C_{a}$ | $C_{a}$ | $C_{b}$ | $C_{c}$ | $e$ | $C_{1}$ | $C_{2}$ |
| $C_{b}$ | $C_{b}$ | $C_{c}$ | $C_{a}$ | $C_{2}$ | $e$ | $C_{1}$ |
| $C_{c}$ | $C_{c}$ | $C_{a}$ | $C_{b}$ | $C_{1}$ | $C_{2}$ | $e$ |

In this group, the operation $\star$ purports multiplication. Checking the axioms in definition 3.1:

1. Closure - looking at table 3.1, one clearly sees that no elements multiplied together make a new element - the group is closed.
2. Associativity - take the example
$C_{1} \star\left(C_{2} \star C_{a}\right)=C_{1} \star C_{b}=C_{a}=e \star C_{a}=\left(C_{1} \star C_{2}\right) \star C_{a}$
which applies to any combination of elements. This shows associativity of the group.
3. Identity element - only one element leaves another unchanged, in this example this is $e$.
4. Inverse - as seen in Table 3.1 each element has its own, unique, inverse element.

This group is a Dihedral Group, $D_{3}$ which also, in fact, is isomorphic to the cyclical group $S_{3}$. Names for the group elements in other literature include:

$$
\begin{aligned}
& S_{3}=\left\{(),\left(\begin{array}{ll}
1 & 2
\end{array}\right),\left(\begin{array}{ll}
1 & 3
\end{array}\right),\left(\begin{array}{l}
1
\end{array}\right),\left(\begin{array}{ll}
1 & 3
\end{array}\right),\left(\begin{array}{l}
2
\end{array}\right)\right\} \\
& D_{3}
\end{aligned}=\left\{e, c, c^{2}, b_{1}, b_{2}, b_{3}\right\}
$$

Definition 3.3: A group $G$ is Abelian if $g \star h=h \star g, \forall g, h \in G$.
Remark: The group in example 3.1 is not abelian.

### 3.2 Lie Theory

In example 3.1, the equilateral triangle served to show what symmetries within geometrical objects are. The transformations left the object, here triangle, unchanged. Other types of groups reveal symmetries within a theory. The so-called Lie groups are for example used in particle physics where the combination of the three Lie groups, $\mathrm{SU}(3), \mathrm{SU}(2), \mathrm{U}(1)$ corresponds to the composition of the Standard Model.

Other examples include the Poincaré groups which express physical symmetry in special relativity and the Point groups which are used in chemistry to understand molecular symmetries.

Definition 3.4: A Lie group is an infinitely differentiable topological space that is locally Euclidian obeying the properties of the group and that satisfies the condition of the group elements being differentiable.

An example of a Lie group is the rotation of a circle - one can rotate it by any angle and regain the same circle. The symmetries of the triangle, in example 3.1, were in contrast not continuous, but required rotations of particular angle.

## Example 3.2

Let $\mathbf{v}$ be a vector in $\mathbb{R}^{2}$. Use a matrix $\mathbf{R}(\theta)$ to rotate $\mathbf{v}$ by the angle $\theta$ to get the vector $\mathbf{v}^{\prime}$. Seeing a rotation of a vector does not change the length of that vector, the norm of $\mathbf{v}$ and $\mathbf{v}^{\prime}$ must agree; $\mathbf{v}^{T} \mathbf{v}=\mathbf{v}^{\prime T} \mathbf{v}^{\prime}$. It then follows that

$$
\begin{equation*}
\mathbf{v}^{\prime T} \mathbf{v}^{\prime}=(\mathbf{R} \mathbf{v})^{T}(\mathbf{R} \mathbf{v})=\mathbf{v}^{T} \mathbf{R}^{T} \mathbf{R} \mathbf{v}=\mathbf{v}^{T} \mathbb{I} \mathbf{v}=\mathbf{v}^{T} \mathbf{v} \tag{3.1}
\end{equation*}
$$

In order to keep the length of $\mathbf{v}$ constant through a rotation, we now have that $\mathbf{R}$ has to be orthogonal, i.e. $\mathbf{R}^{T} \mathbf{R}=\mathbb{I}$. The representation of rotations in $\mathbb{R}^{2}$ is the rotational matrix

$$
\mathbf{R}(\theta)=\left(\begin{array}{cc}
\cos (\theta) & -\sin (\theta)  \tag{3.2}\\
\sin (\theta) & \cos (\theta)
\end{array}\right)
$$

For this rotation, matrix is orthogonal which specifies that $\mathbf{R}^{T} \mathbf{R}=\mathbb{I} \Rightarrow \operatorname{det} R= \pm 1$. When restricting the matrices to those of determinant +1 the special orthogonal group is defined. Since $\theta$ can take any value, the group is said to be continuous. It is then concluded that this rotation lies in the $S O(2)$ group. The 2 merely specifies that the representation matrix is $2 \times 2$.

There are many more Lie groups, such as the Euclidian space with addition $\mathbb{R}^{n}$, and the General Linear group, $\operatorname{GL}(n, \mathbb{R})$, consisting of $n \times n$ real matrices etc. However, the most interesting group which will be the main focus henceforth is $S L(2, \mathbb{R})$.

Definition 3.5: The group $S L(n, \mathbb{R})$ is the Special Linear group and consists of all real $n \times n$ matrices with determinant $=1$.

A Lie algebra is a vector space together with a Lie bracket $[x, y]$ which is a non-associative multiplication. In older texts, a Lie algebra is most often referred to as an "infinitesimal group" as it was introduced to study the concept of infinitesimal transformations.

Definition 3.6: The Lie algebra that generates $S L(n, \mathbb{R}), \mathfrak{s l}(n, \mathbb{R})$, is defined as all traceless $n \times n$ matrices and has the Lie bracket $[X, Y]:=X Y-Y X$.

The Lie bracket will be defined in definition 3.7, shortly. We start off with an example:

## Example 3.3

Let us look at the rotation matrix, viewed in example 3.2. Here we have a matrix $\mathbf{R}(\theta)$ where $\theta$ can take any (real) value. We can, for small rotations, use series expansion to get the first order term

$$
\mathbf{R}(\theta)=\left(\begin{array}{cc}
\cos (\theta) & -\sin (\theta)  \tag{3.3}\\
\sin (\theta) & \cos (\theta)
\end{array}\right) \approx\left(\begin{array}{cc}
1 & -\theta \\
\theta & 1
\end{array}\right)
$$

This can be written as a combination of the identity matrix, $\mathbb{I}$, and another matrix in order to emphasize what the transformation does, i.e. a slight shift from the original position.

$$
\begin{gather*}
\mathbf{R}(\theta)=\left(\begin{array}{cc}
1 & -\theta \\
\theta & 1
\end{array}\right)=\mathbb{I}+\left(\begin{array}{cc}
0 & -\theta \\
\theta & 0
\end{array}\right)=\mathbb{I}-i \theta \mathbf{T}  \tag{3.4}\\
\text { where } \mathbf{T} \text { is the matrix }\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right) \tag{3.5}
\end{gather*}
$$

Do keep in mind that this only applies to small rotations. However, if we wish to rotate by a bigger angle, let us say $\phi$, we can simply split the rotation into $N$ smaller parts and let $N$ approach infinity when Taylor expanding, obtaining

$$
\begin{equation*}
\mathbf{R}(\phi)=\lim _{N \rightarrow \infty}\left(\mathbb{I}-i \mathbf{T} \frac{\phi}{N}\right)^{N}=e^{-i \mathbf{T} \phi} \tag{3.6}
\end{equation*}
$$

As a matrix once again we regain the usual rotational matrix:

$$
\mathbf{R}(\phi)=e^{-i \mathbf{T} \phi}=\left(\begin{array}{cc}
\cos (\phi) & -\sin (\phi)  \tag{3.7}\\
\sin (\phi) & \cos (\phi)
\end{array}\right) .
$$

As we made many infinitesimal rotations from default, the rotational matrix was regained. Remembering that the rotational matrix belongs to the $S O(2, \mathbb{R})$ group, see example 3.2 , we can categorize the Lie algebra in the example.

Remark 3.1: Worth mentioning is that the matrix $\mathbf{T}$ is a generator of the $S O(2, \mathbb{R})$ group, see definition 3.8.

Remark 3.2: When dealing with the Lie algebra of a Lie group one uses lower case gothic letters. For example, the Lie algebra of $S L(2, \mathbb{R})$ is written as $\mathfrak{s l}(2, \mathbb{R})$. The $\mathbb{R}$ merely describes the fact that there are only real numbers in the group.

Looking at (3.4) in example 3.7 one clearly sees that the infinitesimal step away from the identity is $i \mathbf{T} \phi$. This then gives us the following Lie algebra:

$$
\begin{equation*}
\mathfrak{s o}(2, \mathbb{R})=\{i \mathbf{T} \phi, \phi \in \mathbb{R} \text { and } \mathbf{T} \text { as defined in example 3.3 }\} \tag{3.8}
\end{equation*}
$$

As the case with Lie groups, Lie algebras also have axioms that define whether of not they fulfil the requirements of being called a Lie algebra. The axioms follow in definition 3.7.

Definition 3.7: A Lie algebra is a vector space $\mathfrak{g}$ over some field $F$ with the Lie bracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. The Lie bracket satisfies the following axioms:

1. Bilinearity $-\forall$ scalars $a, b$ in $F$ and $\forall$ elements $x, y, z \in \mathfrak{g}$ it follows that:

$$
\begin{aligned}
{[a x+b y, z] } & =a[x, z]+b[y, z], \\
\text { and }[z, a x+b y] & =a[z, x]+b[z, y],
\end{aligned}
$$

2. Alternating on $\mathfrak{g}$

$$
[x, x]=0, \forall x \in \mathfrak{g}
$$

3. The Jacobi identity:

$$
[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0, \quad \forall x, y, z \in \mathfrak{g}
$$

Let $\mathfrak{g}$ be a Lie algebra with generators $x, y$. If these generators exist in the smallest subalgebra of $\mathfrak{g}$, then they are the elements of said Lie algebra. As for the sub groups, the subalgebras must be closed under the Lie bracket, i.e.:

## Example 3.4

Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$. In order for $\mathfrak{h}$ to be a subspace, it must be closed. I.e.

$$
[x, y] \in \mathfrak{h}
$$

for all $x, y \in \mathfrak{h}$

### 3.3 Structure Theory

Lie algebras can be divided into different classes, depending on the structure of the algebra. In this section, the different classes are described and discussed. But before continuing, let us take a look at the definitions of structure constants and generators, they will come in handy later.

Definition 3.8: The generator of a Lie algebra is defined as the elements of a Lie algebra $\mathfrak{g}$ if the smallest sub algebra containing the elements is $\mathfrak{g}$ itself.

As stated earlier, the matrix $\mathbf{T}$ in example 3.3 is for instance a generator for $S O(2)$. Also, we later see that the generators of $\mathfrak{s l}(2, \mathbb{R})$ for instance are the Chevalley triples in equation (3.13).

Definition 3.9: The structure constant, fabc, of a Lie algebra describes how the Lie bracket combines two generators of a given Lie algebra as a linear combination of generators in that Lie algebra.

Let $T^{i} \in \mathfrak{g}$

$$
\begin{equation*}
\left[T^{a}, T^{b}\right]=f^{a b c} T^{c} \tag{3.9}
\end{equation*}
$$

The Lie brackets of the elements are completely determined by the structure constants and the group structure of the Lie group is therefore also effectively determined by the structure constant.

Examples in which these terms, among others, appear can be found in section 3.3.3.

### 3.3.1 Different Lie Algebra Classifications

When classifying a Lie algebra, one looks at the structure constants of said Lie algebra. The structure constants outline the commutator relations. It can be very useful to, via change of basis or through transformation, change all commutators to a standard form simultaneously. By looking at these standard forms, one can find the Lie algebra's structure. The different kinds of structures are found below:

1. Abelian The Abelian Lie algebra is that in which the lie bracket vanishes, $[a, b]=0, \forall a, b \in \mathfrak{g}$, this is the trivial Lie bracket. The bracket vanishes as the generators commute. All Abelian Lie algebras are n-dimensional, with the trivial lie bracket and are of the form $\mathfrak{t}^{n}$.
2. Nilpotent For a nilpotent Lie algebra $\mathfrak{g}$, the lower central series terminates. This means that $\exists \mathfrak{g}_{i}=0 \in \mathfrak{g}$. The lower case series is the series of descending subalgebras.

Definition 3.10: The lower central series is defined as:

$$
\begin{equation*}
\mathfrak{g}>[\mathfrak{g}, \mathfrak{g}]>[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}]>\ldots \tag{3.10}
\end{equation*}
$$

Without going into too much detail, an important characteristics of all nilpotent Lie algebras are that they are solvable, which is useful as it is much easier to prove that a Lie algebra is nilpotent than solvable.
3. Solvable A Solvable Lie algebra has a derived series which terminates.

Definition 3.11: The derived series is defined as:

$$
\begin{equation*}
\mathfrak{g} \geq[\mathfrak{g}, \mathfrak{g}] \geq[[\mathfrak{g}, \mathfrak{g}],[\mathfrak{g}, \mathfrak{g}]] \geq[[[\mathfrak{g}, \mathfrak{g}],[\mathfrak{g}, \mathfrak{g}]],[[\mathfrak{g}, \mathfrak{g}],[\mathfrak{g}, \mathfrak{g}]]] \geq \ldots \tag{3.11}
\end{equation*}
$$

Moreover, all nilpotent algebras are solvable, but on the contrary all solvable algebras are not nilpotent.
4. Simple A Lie algebra which lacks non-trivial ideals, defined below, and is not abelian is said to be simple. A direct sum of many simple Lie algebras combines to a semisimple Lie algebra.

Definition 3.12: Let $I$ be a subspace of $\mathfrak{g}, I \subseteq \mathfrak{g}$, which satisfies

$$
\begin{equation*}
[\mathfrak{g}, I] \subseteq I \tag{3.12}
\end{equation*}
$$

Then $I$ is said to be an ideal of the Lie algebra $\mathfrak{g}$.
5. Semisimple As mentioned above, a direct sum of simple Lie algebras add up to a semisimple Lie algebra. A semisiple Lie algebra's only ideals are $\{0\}$ and $\mathfrak{g}$.

The representation of a semisimple Lie algebra can be fully reduced. A fully reduced representation is in fact a direct sum of irreducible representations.

The main objective of this report lies within the semisimple and simple Lie algebras as they contain the Lie algebra $\mathfrak{s l}(2, \mathbb{Z})$. The $\mathfrak{s l}(2, \mathbb{Z})$, or $\mathfrak{s l}_{2}(\mathbb{Z})$, algebra is in fact found to be a central part of special relativity, general relativity and super symmetry. To gain a better understanding of this central group and its connection to both crystals and, later on, modular forms and sting theory, the structure of the group must be studied. This is done in the following section through the Chevalley-Serre relations and the Cartan matrix.

### 3.3.2 The Chevalley-Serre Relations and the Cartan Matrix

Take $r$ sets of generator triplets that each span $\mathfrak{s l}(2, \mathbb{R})$

$$
\begin{equation*}
\left(e_{i}, f_{i}, h_{i}\right), \quad i=1,2, \ldots, r \tag{3.13}
\end{equation*}
$$

and interact via the Chevalley relations

$$
\left\{\begin{array}{l}
{\left[h_{i}, h_{j}\right]=0}  \tag{3.14}\\
{\left[h_{i}, e_{j}\right]=A_{i j} e_{j}} \\
{\left[h_{i}, f_{j}\right]=-A_{i j} f_{j}} \\
{\left[e_{i}, f_{j}\right]=\delta_{i j} h_{j}}
\end{array}\right.
$$

where $A_{i j}$ is an $r$-by- $r$ matrix containing integers, called the Cartan matrix. The Cartan matrix holds all information about the corresponding Lie algebra. This yields an infinite-dimensional Lie algebra as $\left[e_{i}, e_{j}\right]$ and $\left[f_{i}, f_{j}\right]$ are unconstrained, which is undesirable. Thus, additional constraints are required. By utilizing the adjoint action, $\operatorname{ad}_{X}(Y)=[X, Y]$ (the adjoint action is defined more thoroughly in definition 3.14 on page 36) we create the Serre relations

$$
\left\{\begin{array}{l}
\operatorname{ad}_{e_{i}}^{1-A_{i j}}\left(e_{j}\right)=0  \tag{3.15}\\
\operatorname{ad}_{f_{i}}^{1-A_{i j}}\left(f_{j}\right)=0
\end{array}\right.
$$

The definition of $\operatorname{ad}_{e_{i}}^{p}\left(e_{j}\right)=\left[e_{i},\left[e_{i}, \ldots\left[e_{i}, e_{j}\right]\right] \ldots\right]$ is $p$ commutators. This reduces the number of dimensions of the Lie algebra from infinity to a finite dimension.

### 3.3.3 Dynkin Diagrams

The Dynkin diagrams are a very useful way to visualize simple Lie algebras. In the simple Lie algebras the rank, that is the $r$ in (3.13), determines the number of dots in the Dynkin diagram and the lines between the dots show how the different Lie algebras are connected. From this we see that the Dynkin diagrams contain all the vital information on the Lie algebra and from this, the Cartan matrix can be read. As seen previously, the Cartan matrix is the starting point from which one constructs the Chevalley-Serre form of the Lie algebra.

All simple Lie algebras are categorized into different classes using Dynkin diagrams. The Dynkin diagrams are the same as the Chevalley-Serre representation but in another format. The components, the dots, are the $\mathfrak{s l}(2)$ Lie algebras and has a set of the triplets in (3.13). The line, or lines, between each dot is the summation relation among them given by the Cartan matrix.


Figure 3.2: Dynkin diagram of the simple Lie algebra $\mathfrak{s l}(n+1)$. Here, n is the number of components.

A few examples of different Lie algebras and their Dynkin notation are listed in table 3.2

Table 3.2: Table of a few Lie algebras and their Dynkin notation.

$$
\begin{array}{l|l}
A_{n} & \mathfrak{s l}_{n+1}, \text { the special linear Lie algebra } \\
B_{n} & \mathfrak{s o}_{2 n+1}, \text { the odd-dimensional special orthogonal Lie algebra } \\
C_{n} & \mathfrak{s p}_{2 n}, \text { the symplectic Lie algebra. } \\
D_{n} & \mathfrak{s o}_{2 n}, \text { the even-dimensional special orthogonal Lie algebra. }
\end{array}
$$

All Lie algebras can be constructed by directly adding simple Lie algebras and divided into the four families above, $A_{n}, B_{n}, C_{n}$ and $D_{n}$, except five exceptions: $E_{6}, E_{7}, E_{8}, F_{4}$, and $G_{2}$. Although these families are interesting, they will not be studied further as the main focus of this thesis lies on the the $\mathfrak{s l}(2)$ Lie algebra which belongs to the $A_{1}$ family. Two examples follow below to conclude this section on one note. First, here is an example dealing with $\mathfrak{s l}(2, \mathbb{R})$.

## Example 3.5

Let $(e, h, f)$ be the Chevalley basis for the Lie algebra $\mathfrak{s l}(2, \mathbb{R})$. Here we see that the rank $=1$ since we only have one set of Chevalley triples,

$$
\left\{\begin{array}{l}
{[e, f]=h}  \tag{3.16}\\
{[h, e]=2 e} \\
{[h, f]=-2 f}
\end{array}\right.
$$

Seeing the Chevalley relations in (3.14) describe how the Cartan matrix interact with the basis, one finds that the matrix can be described by a 2 or by $2 \mathbb{I}$. The Serre-relations does not help here as $a d_{e_{1}}^{1-A_{11}}\left(e_{1}\right)$ is nonsense as $a d^{-1}$ does not mean anything. We can
then conclude that $[e, e]=0$ and that the same goes for $h$ and $f$, so no new commutators are created. The dimension of $\mathfrak{s l}(2, \mathbb{R})$ is then 3 . Since the rank is 1 the Dynkin diagram would consist of just one dot.
$A_{1}$ :

To illustrate the procedure for a lie algebra with a higher rank we will now study $\mathfrak{s l}(3, \mathbb{R})$.

## Example 3.6

For $\mathfrak{s l}(3, \mathbb{R})$ the Chevalley triples $\left(e_{1}, h_{1}, f_{1}\right)$ and $\left(e_{2}, h_{2}, f_{2}\right)$ make up the basis. Here, we have the rank= 2 due to the two sets of Chevalley triples. For this Lie algebra, the Cartan matrix $\mathbf{A}$ is found, for this representation, to look like this:

$$
\mathbf{A}=\left(\begin{array}{cc}
2 & -1  \tag{3.17}\\
-1 & 2
\end{array}\right)
$$

The Chevalley relations then give us:

$$
\left\{\begin{array}{l}
{\left[h_{1}, e_{1}\right]=2 e_{1}}  \tag{3.18}\\
{\left[h_{1}, f_{1}\right]=-2 f_{1}} \\
{\left[h_{1}, f_{2}\right]=-\mathbf{A}_{12} f_{2}=f_{2}} \\
{\left[h_{1}, e_{2}\right]=\mathbf{A}_{12} e_{2}=-e_{2}} \\
{\left[e_{1}, f_{1}\right]=0} \\
{\left[h_{1}, h_{2}\right]=0}
\end{array}\right.
$$

However, now the commutators containing the same kind of generators, such as $\left[e_{1}, e_{2}\right]$, are not constrained. This is fixed using the Serre relations in (3.15):

$$
\begin{align*}
\operatorname{ad}_{e_{1}}^{1-A_{12}}\left(e_{2}\right) & =0  \tag{3.19}\\
\Rightarrow \operatorname{ad}_{e_{1}}^{2}\left(e_{2}\right)=[e_{1}, \underbrace{\left[e_{1}, e_{2}\right]}_{e_{3}}] & =0 \tag{3.20}
\end{align*}
$$

Since $\left[e_{1}, e_{2}\right]$ in the equation above is unconstrained, this creates a new generator. We define $e_{3}:=\left[e_{1}, e_{2}\right]$. In the same way, we find that $f_{3}:=\left[f_{1}, f_{2}\right]$ also belongs to the Lie algebra. Now, we have explored all different combinations of generators and found that there are 8 generators of $\mathfrak{s l}(2, \mathbb{R})$, namely:

$$
\begin{equation*}
\left\{h_{1}, h_{2}, e_{1}, e_{2}, e_{3}, f_{1}, f_{2}, f_{3}\right\} \tag{3.21}
\end{equation*}
$$

This means that the dimension of $\mathfrak{s l}(3, \mathbb{R})$ is 8 . Here, the Dynkin diagram would be two dots connected by one line:

$$
A_{2}
$$



A great way to define a Lie algebra dots and lines is through the Chavalley-Serre representation. The Chevalley-Serre representation contains the same information as the Dynkin diagrams, but expressed in a different manner. This is as important to the representation of Lie algebras, but the ChevalleySerre representation will be more useful when studying the important Weyl character formula in section 3.5.

### 3.4 Representation Theory

In short, a representation is a mapping of an abstract group or group element to a set of linear operators acting on a vector space, which respects the multiplication table of said group. In physics, representations - often matrix representations - are studied and not "actual groups". Consequently the subject is crucial to understanding the Universe. This reduces abstract problems to linear algebra; a widely more understood subject [30].

Definition 3.13: A representation of a Lie group $G$ is a group homomorphism and consists of a realization $\Pi$ and a module $V$ which the realization acts on,

$$
\Pi: G \rightarrow G L(V)
$$

For all $g_{1}, g_{2} \in G: \Pi\left(g_{1}\right) \Pi\left(g_{2}\right)=\Pi\left(g_{1} g_{2}\right)$.
Note in the definition above that $\Pi\left(g_{1}\right) \Pi\left(g_{2}\right)$ is normal matrix multiplication while $g_{1} g_{2}=g_{1} \star g_{2}$ uses the group operation of $G$. The definition of a representation of a Lie algebra is analogous but denoted with lower case letters, for example $\pi$.

## Example 3.7

Here follows two examples of representations.

## 1. The trivial representation

We can define a trivial representation of a Lie group $G$ on a one-dimensional complex vector space $\mathbb{C}$ by

$$
\begin{gathered}
\Pi: G \rightarrow \mathrm{GL}(1, \mathbb{C}) \\
\Pi(g)=I
\end{gathered}
$$

for all $g$ in $G$, where $I$ is the identity element of $\mathbb{C}$. For the Lie algebra $\mathfrak{g}$ we can define the trivial representation as

$$
\begin{gathered}
\pi: \mathfrak{g} \rightarrow \operatorname{gl}(1, \mathbb{C}) \\
\pi(x)=0
\end{gathered}
$$

for all $x$ in $\mathfrak{g}$.

## 2. The defining representation

Let $G$ be a Lie group. Then the defining representation of $G$ is

$$
\Pi: G \rightarrow \Pi(G)=G \subset \mathrm{GL}(n, \mathbb{C}) .
$$

For $\operatorname{SU}(3)$ the defining representation would be all unitary $3 \times 3$ matrices on $\mathbb{C}^{3}$ with the determinant 1 . This is analogous for a Lie algebra.

Before a third representation is mentioned another definition is needed.
Definition 3.14: Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. We define the mapping

$$
\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g})
$$

by the formula

$$
\begin{equation*}
\operatorname{Ad}_{g}(t)=g t g^{-1}, \quad g \in G, t \in \mathfrak{g} . \tag{3.22}
\end{equation*}
$$

Additionally we define the Lie algebra homomorphism

$$
\mathrm{ad}: \mathfrak{g} \rightarrow \mathfrak{g l}(g)
$$

by

$$
\begin{equation*}
\operatorname{ad}_{x}(y)=[x, y], \quad x, y \in \mathfrak{g} . \tag{3.23}
\end{equation*}
$$

This is called the adjoint action.
With definition 3.14 we can construct the adjoint representation of $G(\mathrm{Ad})$ with the module being the vector space formed by the Lie algebra or of $\mathfrak{g}$ where the module is the Lie algebra itself. This representation is very important, as it can be used to derive the Lie algebra of a Lie group [11].
Definition 3.15: If a representation is bijective it is called faithful.
Definition 3.16: Let $G$ be a group and $\Pi$ a representation of $G$ on the vector space $V$, with the non-trivial subspace $W \neq V, W \neq\{0\}$. If such a subspace exists that is invariant, i.e. satisfies $\Pi\left(g_{i}\right) w \in W$ for all $g_{i} \in G$ and $w \in W$, then the representation is reducible. If no such subspace exists it is irreducible.

Remark 3.3: The trivial representation from example 3.7 is an irreducible representation, as the one dimensional vector space $\mathbb{C}$ has no non-trivial subspaces.
Definition 3.17: Let $G$ be a Lie group, $\Pi$ a representation of $G$ acting on $V$ and $\Sigma$ a representation of $G$ acting on $W$. A linear map

$$
\varphi: V \rightarrow W
$$

is called a morphism of representations if

$$
\begin{equation*}
\varphi(\Pi(g) v)=\Sigma(g) \varphi(v) \quad \forall g \in G, v \in V . \tag{3.24}
\end{equation*}
$$

If $\varphi$ is invertible it is called an isomorphism of representations. If an isomorphism exists between $V$ and $W$ they are said to be isomorphic, or equivalent.

### 3.4.1 Representation Theory for $\mathfrak{s l}(2, \mathbb{R})$

The group $\mathrm{SU}(n)$ is the special unitary group in $n$ dimensions. The defining representation for $S U(2)$, for example, would be all $2 \times 2$ unitary matrices with the determinant 1 . If we then define the matrices

$$
\begin{equation*}
T_{1}=\frac{1}{2} i \sigma_{1}, \quad T_{2}=\frac{1}{2} i \sigma_{2}, \quad T_{3}=\frac{1}{2} i \sigma_{3} \tag{3.25}
\end{equation*}
$$

from the Pauli matrices

$$
\sigma_{1}=\left(\begin{array}{cc}
0 & 1  \tag{3.26}\\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

any matrix $U$ of the group $S U(2)$ can be parameterized as

$$
\begin{equation*}
U=e^{a_{1} T_{1}+a_{2} T_{2}+a_{3} T_{3}}, \quad a_{1}, a_{2}, a_{3} \in \mathbb{R} \tag{3.27}
\end{equation*}
$$

Subsequently the Lie algebra can be expressed as $\mathfrak{s u}(2)=\mathbb{R} T_{1} \oplus \mathbb{R} T_{2} \oplus \mathbb{R} T_{3}=\operatorname{span}_{\mathbb{R}}\left(T_{1}, T_{2}, T_{3}\right)$. By allowing complex linear combinations of the $T$ matrices the general unitarity is lost but they instead span $\mathfrak{s l}(2, \mathbb{C})$ - as all $T$ matrices are traceless and $\mathfrak{s l}(2, \mathbb{C})=\left\{X \in \mathbb{C}^{2}: \operatorname{tr}(X)=0\right\}$.

Within $\mathfrak{s l}(2, \mathbb{C})$ there are three linear combinations of the $T$ operators that are of particular interest. Here they are denoted

$$
\begin{align*}
& e=\quad T_{2}-i T_{1}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \\
& f=-\left(T_{2}+i T_{1}\right)=\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)  \tag{3.28}\\
& h=\quad-2 i T_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
\end{align*}
$$

This is a representation of the generators in section 3.3.2 - thus the same notation. These operators satisfy the Chevalley relations for $\mathfrak{s l}(2, \mathbb{C})$

$$
\begin{equation*}
[e, f]=h, \quad[h, e]=2 e, \quad[h, f]=-2 f \tag{3.29}
\end{equation*}
$$

Remark 3.4: From this we see that the Cartan matrix for $\mathfrak{s l}(2)$ is a one-dimensional matrix containing only the element 2 , which was used in example 3.5.

With them a triangular decomposition of $\mathfrak{s l}(2, \mathbb{C})$ can be formed and $(e, f, h)$ is therefore called the Chevalley basis. Both the Chevalley basis and its commutators are real, hence $\mathfrak{s l}(2, \mathbb{R})$ is easily obtained by only allowing real linear combinations.

### 3.4.2 Triangular Decomposition

A Lie algebra can be decomposed into its so-called triangular decomposition which proves helpful in a number of cases.

Definition 3.18: The triangular decomposition of a Lie algebra:

$$
\begin{equation*}
\mathfrak{s l}(n, \mathbb{R})=\underbrace{\operatorname{span}_{\mathbb{R}}\left(f_{1}, f_{2}, \ldots, f_{i}\right)}_{\text {Lower triangular }=\mathfrak{n}_{-}} \oplus \underbrace{\operatorname{span}_{\mathbb{R}}\left(h_{1}, \ldots, h_{k}\right)}_{\text {Diagonal }=\mathfrak{h}} \oplus \underbrace{\operatorname{span}_{\mathbb{R}}\left(e_{1}, e_{2}, \ldots, e_{i}\right)}_{\text {Lower triangular }=\mathfrak{n}_{+}}=\mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+} \tag{3.30}
\end{equation*}
$$

Here, the different Chevalley-Serre generators are divided into three different compositions: $\mathfrak{n}_{-}, \mathfrak{h}$ and $\mathfrak{n}_{+}$. Here, the $\mathfrak{n}_{-}$are the lowering generators, the $\mathfrak{n}_{+}$are the raising generators and the $\mathfrak{h}$ are the level generators. They are called this due to the fact that they are either upper- or lower triangular or neither. Also, the roots of $f_{i} \in \mathfrak{n}_{-}$and $e_{i} \in \mathfrak{n}_{+}$combined with $h_{i} \in \mathfrak{h}$ are either positive or negative, respectively.

## Example 3.8

The triangular decomposition of $\mathfrak{s l}(3, \mathbb{R})$ is

$$
\begin{equation*}
\mathfrak{s l}(3, \mathbb{R})=\underbrace{\operatorname{span}_{\mathbb{R}}\left(f_{1}, f_{2}, f_{3}\right)}_{\text {Lower triangular }=\mathfrak{n}_{-}} \oplus \underbrace{\operatorname{span}_{\mathbb{R}}\left(h_{1}, h_{2}\right)}_{\text {Diagonal=h}} \oplus \underbrace{\operatorname{span}_{\mathbb{R}}\left(e_{1}, e_{2}, e_{3}\right)}_{\text {Lower triangular }=\mathfrak{n}_{+}}=\mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+} \tag{3.31}
\end{equation*}
$$

When using this definition one can find the so-called roots of a representation, which will be defined in the following section. These root and weights can then be used in order to characterize the different groups which will become clear later on, in section 3.5.

### 3.4.3 Roots and Weights

Using a set of $r$ Chevalley-Serre bases the following definition can be made.
Definition 3.19: Let $h \in \operatorname{span}_{\mathbb{R}}\left(h_{i}\right)=\mathfrak{h}, i=1, \ldots, r$ and

$$
\begin{equation*}
\left[h, e_{i}\right]=\alpha_{i}(h) e_{i} \tag{3.32}
\end{equation*}
$$

where $\alpha_{i}: \mathfrak{h} \rightarrow \mathbb{R}$. These $\alpha_{i}(h) \in \mathbb{R}$ are called simple roots.
Definition 3.20: Let $h \in \mathfrak{h}, x \in \operatorname{span}_{\mathbb{R}}\left(e_{i}\right)=\mathfrak{n}_{+}, i=1, \ldots, r$ and

$$
\begin{equation*}
[h, x]=\alpha(h) x \tag{3.33}
\end{equation*}
$$

where $\alpha: \mathfrak{h} \rightarrow \mathbb{R}$. Then $\alpha(h)$ is called a root.
Claim: Taking multiple commutators of $e_{i}$ 's correspond to summing the simple roots.
The proof will be given for the case with two commutators, for readability. The general proof is, however, obtained by simply repeating this proof for all commutators.

Proof for two commutators: Use the Jacobi identity on $\left[h,\left[e_{i}, e_{j}\right]\right]$. This yields:

$$
\begin{align*}
{\left[h,\left[e_{i}, e_{j}\right]\right] } & =-\left[e_{i},\left[e_{j}, h\right]\right]-\left[e_{j},\left[h, e_{i}\right]\right] \\
& =\left[e_{i},\left[h, e_{j}\right]\right]-\left[e_{j},\left[h, e_{i}\right]\right] \\
& =\alpha_{j}(h)\left[e_{i}, e_{j}\right]-\alpha_{i}(h)\left[e_{j}, e_{i}\right]  \tag{3.34}\\
& =\left(\alpha_{i}(h)+\alpha_{j}(h)\right)\left[e_{i}, e_{j}\right]
\end{align*}
$$

From definition 3.19 and 3.20 one can construct so-called root spaces.
Definition 3.21: The root space $\Phi$ of some representation of a Lie algebra is defined as the set of all roots of said representation, i.e. $\Phi=\left\{\alpha_{1}(h), \alpha_{2}(h), \ldots, \alpha_{n}(h)\right\}$, where $\alpha_{i}(h)$ are the roots.

Definition 3.22: The simple root space $\Phi^{+}$is defined as the set of all simple roots of the representation of the Lie algebra. Note that $\Phi^{+}$is a subset of $\Phi$.

With the roots, a lattice can be built since all the roots are integers. An example of the roots of $\mathfrak{s l}(2, \mathbb{R})$ is presented in order to make the idea of what roots are more tangible.

## Example 3.9

For $\mathfrak{s l}(2, \mathbb{R})$ we have rank $=1$. This means that the $i$ describing the number of different Chevalley-Serre generators is $i=1$, so we have only one set of Chevally-Serre generators.

$$
\left\{\begin{array}{l}
{[h, e]=2 e}  \tag{3.35}\\
{[h, f]=-2 f} \\
{[e, f]=h}
\end{array}\right.
$$

The simple root in this case is $\alpha(h)=2$. However, the root $-\alpha(h)$ also exist, although it is not a simple root, defined in definition 3.20 above.

Are there any other roots for this representation? Well, when investigating whether $\alpha+\alpha$ is a root one sees that this would have to correspond to $x_{\alpha+\alpha}$ which is generated by $[e, e]=0$, so $2 \alpha$ cannot be a root. The root lattice that is formed from these roots is therefore the even integers on the real axis, $Q=\mathbb{Z} \alpha(h)$ :


The root system for this representation of $\mathfrak{s l}(2, \mathbb{R})$ is thus given by $\Phi=\{-2,2\}$, whereas the simple root space is given by $\Phi^{+}=\{2\}$.

The root system calculated in the example above will prove to be of great importance when studying the so-called Weyl character formula in section 3.5.1.

Definition 3.23: Let $(\pi, V)$ be a representation of $\mathfrak{s l}(2, \mathbb{C})$ and $v \in V$ satisfies

$$
\begin{equation*}
\pi(h) v=\lambda v \tag{3.36}
\end{equation*}
$$

i.e. is an eigenvector with eigenvalue $\lambda \in \mathbb{C}$. Such a vector is called a weight vector and $\lambda$ its weight.

This definition can, naturally, be restricted to the real case, thus applying to $\mathfrak{s l}(2, \mathbb{R})$.
Lemma 3.1: Any representation $(\pi, V)$ can be decomposed as

$$
\begin{equation*}
V=\bigoplus_{\lambda} V_{\lambda} \tag{3.37}
\end{equation*}
$$

where $V_{\lambda} \subset V$ contains all weight vectors of weight $\lambda$, i.e.

$$
\begin{equation*}
V_{\lambda}=\{v \in V: \pi(h) v=\lambda v\} \tag{3.38}
\end{equation*}
$$

This is called a weight decomposition.

### 3.5 Characters of Representations and the Weyl Character Formula

As mentioned before, there are many ways to represent a given Lie algebra. One way to distinguish these representations from each other is to study their so-called characters. This is most effectively done with the use of the Weyl character formula. But before being able to state this crucial formula, we need to introduce a few more concepts.

As we saw in lemma 3.1, any module $V$ of a representation of $\mathfrak{s l}(2, \mathbb{C})$ can be decomposed as

$$
\begin{equation*}
V=\bigoplus_{\lambda \in \Lambda} V_{\lambda} \tag{3.39}
\end{equation*}
$$

where $V_{\lambda}$ denotes the subspace of $V$ which consists of all vectors of weight $\lambda$ and $\Lambda$ is the set of weights in $V$.

Definition 3.24: The highest weight $\lambda_{h}$ of a vector space $V$ is defined as the weight that satisfies $\operatorname{Re}\left(\lambda_{h}\right) \geq \operatorname{Re}\left(\lambda^{\prime}\right)$ for all weights $\lambda_{h}, \lambda^{\prime} \in \Lambda$. The corresponding vectors $v_{h} \in V_{\lambda_{h}}$ are called the highest weight vectors of $V$.

Lemma 3.2: Let $v_{h} \in V_{\lambda_{h}}$ be a highest weight vector of a vector space $V$ representing $\mathfrak{s l}(2, \mathbb{C})$ and let

$$
\begin{equation*}
v^{k}=\frac{f^{k}}{k!} v_{h}, \quad k \geq 0 \tag{3.40}
\end{equation*}
$$

Then

$$
\begin{align*}
h v^{k} & =\left(\lambda_{h}-2 k\right) v^{k} \\
f v^{k} & =(k+1) v^{k+1}  \tag{3.41}\\
e v^{k} & =\left(\lambda_{h}-k+1\right) v^{k-1}
\end{align*}
$$

where $e, f$ and $h$ denotes the generators of the Lie algebra.

Theorem 3.1: Let $V_{n}$ be a finite vector space spanned by the basis $\left\{v^{0}, v^{1}, \ldots, v^{n}\right\}$ defined in lemma 3.2 and define the action of the generators as

$$
\begin{align*}
h v^{k} & =(n-2 k) v^{k} \\
f v^{k} & =(k+1) v^{k+1}, \quad k<n \\
f v^{n} & =0  \tag{3.42}\\
e v^{k} & =(n+1-k) v^{k-1}, \quad k>0 \\
e v^{0} & =0 .
\end{align*}
$$

Then $V_{n}$ is an irreducible representation of $\mathfrak{s l}(2, \mathbb{C})$, which is often referred to as the irreducible representation of highest weight $n$.

In accordance with the theorem above one can always find an irreducible representation from a vector space $V$ representing $\mathfrak{s l}(2, \mathbb{C})$. This is done in terms of its highest weight $\lambda_{h}$ (denoted $n$ in the theorem). We denote this representation as $V\left(\lambda_{h}\right)$. An important remark is that there are similar ways of doing this for other Lie algebras than $\mathfrak{s l}(2, \mathbb{C})$. In fact, one can find an irreducible representation of some highest weight $n$ for all semi-simple Lie algebras [31].

Now, define $\mathbb{Z}[\Lambda]$ as all linear combinations of the base elements $\left\{e^{\lambda}, \lambda \in \Lambda\right\}$ with integer coefficients. These base elements are so-called formal exponentials which satisfy the multiplication rule $e^{\mu} e^{\lambda}=e^{\mu+\lambda}$. $\mathbb{Z}[\Lambda]$ is a type of mathematical object called a ring.

Definition 3.25: Define the character of a representation $V(\lambda)$ as the element in $\mathbb{Z}[\Lambda]$ given by

$$
\begin{equation*}
\chi_{\lambda}=\sum_{\mu \in \Lambda} m_{\lambda}(\mu) e^{\mu} \tag{3.43}
\end{equation*}
$$

where $m_{\lambda}(\mu)$ denotes the dimension of $V_{\mu}$ in this representation.
Remark: Since a representation consists of both a realization and a module (often a vector space), it would be more correct to talk about "the character of the module of a representation" but one normally is not so precise.

Before stating the Weyl character formula we need to introduce the denotation

$$
\epsilon_{\lambda}(\mu)= \begin{cases}1, & \text { if } \mu=\lambda  \tag{3.44}\\ 0, & \text { otherwise }\end{cases}
$$

We also need to define the reflection $\sigma$ in a root $\alpha(h)$ as well as the important Weyl group.
Definition 3.26: Let $E$ be a finite, real vector space with an inner product $(\cdot, \cdot)$. The linear map $E \rightarrow E$ known as the reflection in $\alpha \in E$ of some $\beta$ is then given by

$$
\begin{equation*}
\sigma_{\alpha}(\beta)=\beta-\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \tag{3.45}
\end{equation*}
$$

Note that when $E$ is a one-dimensional vector space - e.g. the one spanned by the roots of $\mathfrak{s l}(2, \mathbb{R})$ in the representation used in this thesis - the inner product in equation (3.45) turns into common multiplication.

Definition 3.27: For a root space $\Phi$, the corresponding Weyl group is given by the set $\left\{\sigma_{\alpha}: \alpha \in \Phi\right\}$.
Remark: In these two definitions, and henceforth, we denote the root $\alpha(h)$ simply as $\alpha$, although this denotation earlier has been used for the linear map from $\mathfrak{h}$ to $\mathbb{R}$ that the root $\alpha(h)$ corresponds to.

We are now ready to state the important Weyl character formula, which holds for irreducible representations.

Theorem 3.2: Weyl character formula states that all irreducible representations satisfy the relation

$$
\begin{equation*}
\left(\sum_{\sigma \in W} \operatorname{sgn}(\sigma) \epsilon_{\sigma \rho}\right) * \chi_{\lambda}=\sum_{\sigma \in W} \operatorname{sgn}(\sigma) \epsilon_{\sigma(\lambda+\rho)} \tag{3.46}
\end{equation*}
$$

where $\lambda$ denotes the highest weight of the representation, sgn denotes the sign function,

$$
\begin{equation*}
\rho=\frac{1}{2} \sum_{\alpha \in \Phi^{+}} \alpha \tag{3.47}
\end{equation*}
$$

is the Weyl vector and $*$ denotes the convolution given by

$$
\begin{equation*}
f * g(\mu)=\sum_{\nu+\theta=\mu} f(\nu) g(\theta) \tag{3.48}
\end{equation*}
$$

For a proof of Weyl character formula, see e.g. [32].

### 3.5.1 Applying the Weyl Character Formula to $\mathfrak{s l}(2, \mathbb{R})$

What can we learn from the Weyl character formula when applying it to $\mathfrak{s l}(2, \mathbb{R})$ ? The representation of $\mathfrak{s l}(2, \mathbb{R})$ that consequently has been studied in this report is the one that is generated by the matrices

$$
e=\left(\begin{array}{ll}
0 & 1  \tag{3.49}\\
0 & 0
\end{array}\right), \quad f=\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right), \quad h=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

As we saw in example $3.9, \mathfrak{s l}(2, \mathbb{R})$ has only one simple root and that is $\alpha=2$. Thus, equation (3.47) for the Weyl vector yields

$$
\begin{equation*}
\rho=\frac{1}{2} \sum_{\alpha \in \Phi^{+}} \alpha=1 \tag{3.50}
\end{equation*}
$$

We now need to find the Weyl group of this representation, which in accordance with definition 3.27 is given by the set $\left\{\sigma_{\alpha}: \alpha \in \Phi\right\}$. We learned from example 3.9 that the root space of $\mathfrak{s l}(2, \mathbb{R})$ is given by $\Phi=\{-2,2\}$, so using definition 3.26 we now want to calculate the reflections of some $\beta$ in the two roots. Since the vector space spanned by the roots of $\mathfrak{s l}(2, \mathbb{R})$ is one-dimensional in our representation, equation (3.45) yields

$$
\begin{equation*}
\sigma_{\alpha}(\beta)=\beta-\frac{2 \beta \alpha}{\alpha^{2}} \alpha=-\beta \tag{3.51}
\end{equation*}
$$

Thus we see that the reflection $\sigma_{\alpha}$ is independent of the root $\alpha$. Hence $\sigma_{-2}(\beta)=\sigma_{2}(\beta)=-\beta$. Since the product of two group elements must always belong to the group - see the definition of a group,
given in definition 3.1 - we are also allowed to perform multiple reflections. Performing two reflections, we obtain

$$
\begin{equation*}
\sigma_{\alpha}\left(\sigma_{\alpha}(\beta)\right)=\sigma_{\alpha}(-\beta)=\beta \tag{3.52}
\end{equation*}
$$

So this map turned out to correspond to the identity element $e$. This is good, since a group must always contain the identity element (again, see definition 3.1). It is easy to realize that an even number of reflections in some weight $\alpha$ always returns the original element, i.e. $\sigma_{\alpha}^{2 n}(\beta)=\beta$ whereas an odd number of reflections yields $\sigma_{\alpha}^{2 n+1}(\beta)=-\beta$, if $n \in \mathbb{N}$. Hence the Weyl group $W$ of this representation of $\mathfrak{s l}(2, \mathbb{R})$ contains only two elements and is given by $W=\{e, \sigma\}$, where $\sigma(\beta)=-\beta$. Since these two elements correspond to the identity respectively a change of the sign, one may - when representing the elements in the Weyl group with numbers - rewrite this as $W=\{1,-1\}$.

We are now ready to apply the Weyl character formula, which was given in Theorem 3.2. Since we know that $\rho=1$ and $W=\{1,-1\}$, the left hand side of the Weyl character formula can be simplified to

$$
\begin{equation*}
\text { LHS }=\left(\sum_{\sigma \in W} \operatorname{sgn}(\sigma) \epsilon_{\sigma \rho}\right) * \chi_{\lambda}=\left(\operatorname{sgn}(-1) \cdot \epsilon_{-1 \cdot 1}+\operatorname{sgn}(1) \cdot \epsilon_{1 \cdot 1}\right) * \chi_{\lambda}=\left(\epsilon_{1}-\epsilon_{-1}\right) *\left(\sum_{n \in \mathbb{Z}} m_{\lambda}(n) \epsilon_{n}\right) \tag{3.53}
\end{equation*}
$$

where we in the last step used two properties that are proven in [32] but not will be proven here. The first one being that $\Lambda=\mathbb{Z}$ for our representation of $\mathfrak{s l}(2, \mathbb{R})$ (this is a consequence of the roots being integers and is not too hard to prove) and the second one being that we can replace the formal exponential in the definition of $\chi_{\lambda}$ with an $\epsilon$ under the reigning conditions. Now, let us evaluate the right hand side of the Weyl character formula. This yields

$$
\begin{equation*}
\mathrm{RHS}=\sum_{\sigma \in W} \operatorname{sgn}(\sigma) \epsilon_{\sigma(\lambda+\rho)}=\epsilon_{\lambda+1}+\epsilon_{-\lambda-1} \tag{3.54}
\end{equation*}
$$

Before equating the left and the right hand side, we want to evaluate them for some integer $k$. It need to be an integer since $\Lambda=\mathbb{Z}$ and $\lambda \in \Lambda$. This yields

$$
\begin{equation*}
\operatorname{LHS}(k)=\sum_{p+q=k}\left(\epsilon_{1}-\epsilon_{-1}\right)(p)\left(\sum_{n \in \mathbb{Z}} m_{\lambda}(n) \epsilon_{n}(q)\right) \tag{3.55}
\end{equation*}
$$

by the definition of the convolution. Since $\left(\epsilon_{1}-\epsilon_{-1}\right)(p)=0$ unless $p=-1$ (which implies $q=k+1$ ) or $p=1$ (which implies $q=k-1$ ), one obtains

$$
\begin{equation*}
\operatorname{LHS}(k)=\sum_{n \in \mathbb{Z}} m_{\lambda}(n)\left(\epsilon_{n}(k-1)-\epsilon_{n}(k+1)\right)=m_{\lambda}(k-1)-m_{\lambda}(k+1) \tag{3.56}
\end{equation*}
$$

Evaluation of RHS in equation (3.54) for the same integer $k$ gives

$$
\operatorname{RHS}(k)=\left(\epsilon_{\lambda+1}+\epsilon_{-\lambda-1}\right)(k)= \begin{cases}1, & \text { if } k=\lambda+1  \tag{3.57}\\ -1, & \text { if } k=-\lambda-1 \\ 0, & \text { otherwise }\end{cases}
$$

Equating equation (3.56) and (3.57) one obtains

$$
m_{\lambda}(k-1)-m_{\lambda}(k+1)= \begin{cases}1, & \text { if } k=\lambda+1  \tag{3.58}\\ -1, & \text { if } k=-\lambda-1 \\ 0, & \text { otherwise }\end{cases}
$$

Note that if $k \neq \pm(\lambda+1)$, then $m_{\lambda}(k-1)=m_{\lambda}(k+1)$. Letting $k=\lambda+3, \lambda+5, \ldots$ respectively, this relation yields

$$
\begin{equation*}
m_{\lambda}(\lambda+2)=m_{\lambda}(\lambda+4)=m_{\lambda}(\lambda+6)=\ldots \tag{3.59}
\end{equation*}
$$

Instead letting $k=-\lambda-3,-\lambda-5,-\lambda-6, \ldots$ respectively, one obtains

$$
\begin{equation*}
m_{\lambda}(-\lambda-2)=m_{\lambda}(-\lambda-4)=m_{\lambda}(-\lambda-6)=\ldots \tag{3.60}
\end{equation*}
$$

For $k=\ldots, \lambda-2, \lambda, \lambda+2, \ldots$ respectively, one obtains

$$
\begin{equation*}
\ldots=m_{\lambda}(\lambda-3)=m_{\lambda}(\lambda-1)=m_{\lambda}(\lambda+1)=m_{\lambda}(\lambda+3)=\ldots \tag{3.61}
\end{equation*}
$$

Now note that there are an infinite number of entries in equation (3.59), (3.60) and (3.61). Since we are only studying finite-dimensional representations of Lie algebras, all entries in these three equations must equal zero.

There are still some choices of $k$ in equation (3.58) that haven't been evaluated. First, let $k=$ $-\lambda+1,-\lambda+3, \ldots, \lambda-3, \lambda-1$. This gives us

$$
\begin{equation*}
m_{\lambda}(-\lambda)=m_{\lambda}(-\lambda+2)=\ldots=m_{\lambda}(\lambda-2)=m_{\lambda}(\lambda) . \tag{3.62}
\end{equation*}
$$

Please note that since $\lambda$ is finite, this equation has a finite number of entries. Now letting $k=\lambda+1$, equation (3.58) yields

$$
\begin{equation*}
m_{\lambda}(\lambda)-m_{\lambda}(\lambda+2)=1 \tag{3.63}
\end{equation*}
$$

Since we know from equation (3.59) that $m_{\lambda}(\lambda+2)=0$, then $m_{\lambda}(\lambda)=1$. Equation (3.62) then yields

$$
\begin{equation*}
m_{\lambda}(-\lambda)=m_{\lambda}(-\lambda+2)=\ldots=m_{\lambda}(\lambda-2)=m_{\lambda}(\lambda)=1 \tag{3.64}
\end{equation*}
$$

We have now calculated $m_{\lambda}(n)$ for all possible arguments $n$ and we can conclude that

$$
m_{\lambda}(n)= \begin{cases}1, & \text { if } n \in\{-\lambda,-\lambda+2, \ldots, \lambda-2, \lambda\}  \tag{3.65}\\ 0, & \text { otherwise }\end{cases}
$$

Note that we did not need to use the case $k=-\lambda-1$ in equation (3.58), but that it simply yields $m_{\lambda}(-\lambda-2)-m_{\lambda}(-\lambda)=-1$, which agrees with our result.

Inserting the result from equation (3.65) into the definition of the character $\chi_{\lambda}$, given in definition 3.25, one obtains

$$
\begin{equation*}
\chi_{\lambda}=\sum_{\mu \in \Lambda} m_{\lambda}(\mu) e^{\mu}=e^{-\lambda}+e^{-\lambda+2}+\ldots+e^{\lambda-2}+e^{\lambda} \tag{3.66}
\end{equation*}
$$

Using what we learned from example 2.4 we may write this as the Schur polynomial

$$
\begin{equation*}
\chi_{\lambda}=s_{(\lambda, 0)}\left(e^{-1}, e\right) \tag{3.67}
\end{equation*}
$$

This equation fully characterizes the representation of $\mathfrak{s l}(2, \mathbb{R})$ we are consequently working with in this report, i.e. the one generated by the matrices $e, f$ and $h$ in equation (3.49). It is easy to realize that the same character formula applies to $\mathfrak{s l}(2, \mathbb{C})$ (in the representation used throughout this thesis). This formula will prove to be crucial, for example in section 5.1 where its connection to modular forms will be investigated. But first, a chapter introducing modular forms is needed to proceed.

## Chapter 4

## Modular Forms and Eisenstein Series

Modular forms are, roughly speaking, complex-valued functions with a discrete symmetry. They have applications in many fields of mathematics, such as number and group theory, as well as appearing frequently in string theory. Famously, they were a key in proving Fermat's last theorem [33]. A generalization of modular forms are so-called automorphic forms.

A slight variation of the modular forms are the Maass forms. They have a symmetry similar to modular forms, but differ in some properties. In this chapter, we will study a certain Maass form, namely the non-holomorphic Eisenstein series, and connect it to previous subjects in this report through Schur polynomials by studying its Fourier expansion. The focus of this chapter will only be to show the results needed to make the connection. The discussion will be left to chapter 5 .

### 4.1 An Introduction to Modular and Maass Forms

For the definition of a modular form, consider a matrix in $S L(2, \mathbb{Z})$ acting on the upper half of the complex plane, $\mathbb{H}=\{z \in \mathbb{C} \mid \operatorname{Im} z>0\}$, through the action

$$
\gamma(\tau)=\frac{a \tau+b}{c \tau+d}, \gamma=\left(\begin{array}{ll}
a & b  \tag{4.1}\\
c & d
\end{array}\right) \in S L(2, \mathbb{Z}), \tau \in \mathbb{H} .
$$

The domain $\mathbb{H}$ is closed under this action, that is, the image of $\tau$ with $\operatorname{Im}(\tau)>0$ also has a positive imaginary part, which can be seen from

$$
\begin{align*}
\operatorname{Im}(\gamma(\tau)) & =\operatorname{Im}\left(\frac{a \tau+b}{c \tau+d}\right)=\operatorname{Im}\left(\frac{(a \tau+b)(c \bar{\tau}+d)}{(c \bar{\tau}+d)(c \bar{\tau}+d)}\right) \\
& =\operatorname{Im}\left(\frac{a d \tau+b c \bar{\tau}}{|c \tau+d|^{2}}\right)=\frac{\operatorname{Im}(\tau) \overbrace{(a d-b c)}^{=1}}{|c \tau+d|^{2}}=\frac{\operatorname{Im}(\tau)}{|c \tau+d|^{2}}, \tag{4.2}
\end{align*}
$$

where $a d-b c=1$ since $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in S L(2, \mathbb{Z})$ implies that $\operatorname{det}(\gamma)=a d-b c=1$.
We are now ready to define a modular form.
Definition 4.1: A holomorphic modular form of weight $k$ is a holomorphic function $f: \mathbb{H} \rightarrow \mathbb{C}$ satisfying the condition

$$
\begin{equation*}
f(\gamma(\tau)):=f\left(\frac{a \tau+b}{c \tau+d}\right)=(c \tau+d)^{k} f(\tau) \tag{4.3}
\end{equation*}
$$

We note that the action of the matrices $T=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ and $U=\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)$ and the condition above imply the properties

$$
\begin{align*}
& f(T(\tau))=f(\tau+1)=f(\tau), \text { so } \mathrm{f} \text { is periodic in the real part }  \tag{4.4}\\
& f(U(\tau))=f \underbrace{\left(\frac{-\tau}{-1}\right)}_{=\tau}=(-1)^{k} f(\tau), \text { which requires even } k \text { for } f \neq 0 \tag{4.5}
\end{align*}
$$

The automorphic forms previously mentioned differ from modular forms in that they may be symmetric under the action of other groups than $S L(2, \mathbb{Z})$.

Remark 4.1: The only modular form of weight $k=0$ is the zero function.
An example of a modular form is the holomorphic Eisenstein series.

## Example 4.1

The holomorphic Eisenstein series of weight $k \geq 2$ is given by

$$
\begin{equation*}
E_{k}(\tau)=\sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{1}{(m \tau+n)^{k}} \tag{4.6}
\end{equation*}
$$

where the primed sum indicates that the origin $(m, n)=(0,0)$ is excluded. Let us confirm that this series transforms as required by definition 4.1 under the previously defined action of $S L(2, \mathbb{Z})$ :

$$
\begin{align*}
E_{k}(\gamma(\tau)) & =E_{k}\left(\frac{a \tau+b}{c \tau+d}\right)=\sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{1}{\left(m \frac{a \tau+b}{c \tau+d}+n\right)^{k}} \\
& =(c \tau+d)^{k} \sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{1}{(\underbrace{(a m+c n)}_{=m^{\prime}} \tau+\underbrace{(b m+d n)}_{=n^{\prime}})^{k}} . \tag{4.7}
\end{align*}
$$

We have a bijection between all pairs of integers $(m, n)$ and $\left(m^{\prime}, n^{\prime}\right)$ since

$$
\left\{\begin{array} { l } 
{ m ^ { \prime } = a m + c n }  \tag{4.8}\\
{ n ^ { \prime } = b m + d n }
\end{array} \Leftrightarrow \left\{\begin{array}{l}
m=d m^{\prime}-c n^{\prime} \\
n=a n^{\prime}-b m^{\prime}
\end{array}\right.\right.
$$

where we have used that $a d-c b=1$. Note that $(m, n)=(0,0)$ corresponds to $\left(m^{\prime}, n^{\prime}\right)=$ $(0,0)$. We therefore obtain

$$
\begin{equation*}
E_{k}(\gamma(\tau))=(c \tau+d)^{k} \sum_{\left(m^{\prime}, n^{\prime}\right) \in \mathbb{Z}^{2}}^{\prime} \frac{1}{\left(m^{\prime} \tau+n^{\prime}\right)^{k}} \tag{4.9}
\end{equation*}
$$

which is what we wanted to show.

For our purposes, however, a variation of modular forms called Maass forms are more interesting.
Definition 4.2: A Maass form of weight 0 is a function $f: \mathbb{H} \rightarrow \mathbb{C}$ which satisfies

$$
f\left(\frac{a \tau+b}{c \tau+d}\right)=f(\tau), \text { with } \gamma=\left(\begin{array}{ll}
a & b  \tag{4.10}\\
c & d
\end{array}\right) \in S L(2, \mathbb{Z})
$$

while also being an eigenfunction to the Laplacian on $\mathbb{H}$ :

$$
\begin{equation*}
\Delta_{\mathbb{H}} f(\tau)=\lambda f(\tau), \tag{4.11}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta_{\mathbb{H}}:=y^{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right) . \tag{4.12}
\end{equation*}
$$

This definition drops the requirement that $f$ is a holomorphic function, and instead demands that it is an eigenfunction to the Laplacian on $\mathbb{H}$. Just as the modular forms, all Maass forms are periodic in the real part as the definition requires

$$
\begin{equation*}
f(T(\tau))=f(\tau+1)=f(\tau) \tag{4.13}
\end{equation*}
$$

again with $T=\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$. This periodicity implies the existence of a Fourier expansion for the Maass forms. In the coming sections we will study the coefficients in this expansion for a certain Maass form: the non-holomorphic Eisenstein series.

### 4.2 Properties of the Non-Holomorphic Eisenstein Series

We begin by defining the object we intend to study, the non-holomorphic Eisenstein series. We will use the convention that primed sums leave out the origin.

Definition 4.3: The non-holomorphic Eisenstein series with $z=x+i y \in \mathbb{H}$ and $s \in \mathbb{C}$ is given by

$$
\begin{equation*}
\mathcal{E}(z, s)=\sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{y^{s}}{|m z+n|^{2 s}} \tag{4.14}
\end{equation*}
$$

This series converges for $\operatorname{Re}(s)>1$. Compare this to the holomorphic Eisenstein series in example 4.1, which is similar except for the explicit $y$-dependence.

In order to show that this is indeed a Maass form, the conditions given in definition 4.2 must be verified. That it is invariant under the action of $S L(2, \mathbb{Z})$ is demonstrated below:

$$
\begin{align*}
\mathcal{E}(\gamma(z), s) & =\mathcal{E}\left(\frac{a z+b}{c z+d}, s\right)=\sum_{(m, n) \in \mathbb{Z}^{2}} \frac{y^{s} /|c z+d|^{2 s}}{\left|m \frac{a z+b}{c z+d}+n\right|^{2 s}}=\sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{\left(y^{s} /|c z+d|^{2 s}\right)|c z+d|^{2 s}}{\mid(m a+c n)} z+\left.\underbrace{(m b+d n)}_{=m \prime}\right|^{2 s}  \tag{4.15}\\
& =\sum_{(m \prime, n \prime) \in \mathbb{Z}^{2}}^{\prime} \frac{y^{s}}{|(m \prime) z+(n \prime)|^{2 s}}=\mathcal{E}(z, s)
\end{align*}
$$

where the second to last step is similar to the end of example 4.1 where we showed the bijection between pairs of integers $(m, n)$ and $\left(m^{\prime}, n^{\prime}\right)$. Note that this series has weight 0 , meaning it is wholly invariant under the action of $S L(2, \mathbb{Z})$.

That it is an eigenfunction to $\Delta_{\mathbb{H}}$ will not be shown here, but a proof may be found in [5]. The corresponding eigenvalue can, however, be identified in the relation

$$
\begin{equation*}
\Delta_{\mathbb{H}} \mathcal{E}(z, s)=s(s-1) \mathcal{E}(z, s) \tag{4.16}
\end{equation*}
$$

as $s(s-1)$.
To determine the coefficients of the Fourier expansion of the non-holomorphic Eisenstein series, we will use so-called Poisson resummation. To make the coming calculations easier to follow, this result will be proven in this section.

Theorem 4.1: With $f: \mathbb{R} \rightarrow \mathbb{C}$ being a Schwarz function (for our purposes, we need only note that this requirement guarantees convergence properties) with a Fourier transform $\hat{f}(k)=\int_{-\infty}^{\infty} f(x) e^{-2 \pi i k x} \mathrm{~d} x$. Then,

$$
\begin{equation*}
\sum_{n \in \mathbb{Z}} f(n)=\sum_{k \in \mathbb{Z}} \hat{f}(k) \tag{4.17}
\end{equation*}
$$

This is known as Poisson resummation.
Proof: Define the function F as

$$
F(x):=\sum_{n \in \mathbb{Z}} f(x+n)
$$

This function is clearly periodic, with $F(x)=F(x+1)$, and can thus be written as a Fourier series:

$$
\begin{equation*}
F(x)=\sum_{k \in \mathbb{Z}} c_{k} e^{2 \pi i k x} \tag{4.18}
\end{equation*}
$$

The coefficients for this series are calculated as follows:

$$
\begin{align*}
c_{k} & =\int_{0}^{1} F(x) e^{-2 \pi i k x} \mathrm{~d} x=\int_{0}^{1} \sum_{n \in \mathbb{Z}} f(x+n) e^{-2 \pi i k x} \mathrm{~d} x \\
& =\sum_{n \in \mathbb{Z}} \int_{0}^{1} f(\underbrace{x+n}_{=x^{\prime}}) e^{-2 \pi i k x} \mathrm{~d} x=\sum_{n \in \mathbb{Z}} \int_{0}^{1} f\left(x^{\prime}\right)_{=x^{\prime}}) e^{-2 \pi i k x^{\prime}} \mathrm{d} x^{\prime}  \tag{4.19}\\
& =\int_{-\infty}^{\infty} f\left(x^{\prime}\right) e^{-2 \pi i k x^{\prime}} \mathrm{d} x^{\prime}=\hat{f}(k) .
\end{align*}
$$

So, we have

$$
\begin{equation*}
F(x):=\sum_{n \in \mathbb{Z}} f(x+n)=\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2 \pi i k x} \tag{4.20}
\end{equation*}
$$

and with $x=0$ this gives the desired result:

$$
\begin{equation*}
\sum_{n \in \mathbb{Z}} f(n)=\sum_{k \in \mathbb{Z}} \hat{f}(k) \tag{4.21}
\end{equation*}
$$

### 4.3 Fourier Expansion of the Non-Holomorphic Eisenstein Series

We want to find the Fourier expansion of the non-holomorphic Eisenstein series given by

$$
\begin{equation*}
\mathcal{E}(z, s):=\sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{y^{s}}{|m z+n|^{2 s}} \tag{4.22}
\end{equation*}
$$

This expansion exists since the Eisenstein series is periodic in $x \rightarrow x+1$, as demonstrated in [4.4]. The purpose of this is to find a connection to crystals by identifying Schur polynomials in the coefficients of this expansion. We will first calculate the $y$-dependence, using only the fact that the Eisenstein series is an eigenfunction to the Laplacian. We will then calculate the $s$-dependence using the explicit form of the series, given in [4.22].

### 4.3.1 The $y$-dependence of the Fourier Coefficients

We wish to write the series as

$$
\begin{equation*}
\mathcal{E}(z, s)=\sum_{n \in \mathbb{Z}} a_{n}(y) e^{2 \pi i n x} \tag{4.23}
\end{equation*}
$$

As mentioned before, the Eisenstein series is an eigenfunction to the Laplace operator $\Delta=y^{2}\left(\partial_{x}^{2}+\partial_{y}^{2}\right)$, with the eigenvalue $s(s-1)$. Hence it satisfies the equation $\Delta \mathcal{E}(z, s)=s(s-1) \mathcal{E}(z, s)$. Applying this to (4.23), we obtain

$$
\begin{equation*}
y^{2}\left(-4 \pi^{2} \sum_{n \in \mathbb{Z}} n^{2} a_{n}(y) e^{2 \pi i n x}+\sum_{n \in \mathbb{Z}} a_{n}^{\prime \prime}(y)\right)=s(s-1) \sum_{n \in \mathbb{Z}} a_{n}(y) e^{2 \pi i n x} \tag{4.24}
\end{equation*}
$$

Separating the equation into two parts, one with $n=0$ and one with $n \neq 0$, and noting that the coefficients preceding $e^{2 \pi i n x}$ must be equal for all n , we get the following differential equations:

$$
\begin{array}{ll}
y^{2} a_{o}^{\prime \prime}(y)=s(s-1) a_{o}(y) & n=0 \\
y^{2}\left(a_{n}^{\prime \prime}(y)-4 \pi^{2} n^{2} a_{n}(y)\right)=s(s-1) a_{n}(y) & n \neq 0 \tag{4.26}
\end{array}
$$

It is easy to verify that differential equation (4.25) has the solution $a_{0}(y)=C_{1}(s) y^{s}+C_{2}(s) y^{1-s}$. To solve differential equation (4.26), we will use the substitutions $t=2 \pi|n| y$ and $a_{n}=\sqrt{y} f_{n}(t)$.

We finally get

$$
\begin{equation*}
t^{2} f_{n}^{\prime \prime}(t)+t f_{n}^{\prime}(t)-\left(t^{2}+\left(s-\frac{1}{2}\right)^{2}\right) f_{n}(t) \tag{4.27}
\end{equation*}
$$

This is known as the modified Bessel differential equation of order $s-\frac{1}{2}$, having the known solution

$$
\begin{equation*}
f_{n}(t)=A_{n}(s) K_{s-1 / 2}(t)+B_{n}(s) I_{s-1 / 2}(t) \tag{4.28}
\end{equation*}
$$

where $K_{s-1 / 2}$ and $I_{s-1 / 2}$ are modified Bessel functions. However, the $I_{s-1 / 2}(t)$ part will be neglected, since we will let $n \rightarrow \infty$, in turn implying that $t \rightarrow \infty$, a limit in which $I_{s-1 / 2}(t)$ is divergent.

### 4.3.2 The $s$-dependence of the Fourier Coefficients

We have now found the $y$-dependence of the Fourier coefficients, and the Fourier expansion expressed in terms of $a_{n}$ and $y$ is

$$
\begin{equation*}
\mathcal{E}(z, s)=C_{1}(s) y^{s}+C_{2}(s) y^{1-s}+\sqrt{y} \sum_{n \in \mathbb{Z}}^{\prime} a_{n}(s) K_{s-1 / 2}(2 \pi|n| y) e^{2 \pi i n x} \tag{4.29}
\end{equation*}
$$

The arguments made thus far apply generally to all modular forms ${ }^{1}$, since we have only used periodicity and the fact that the Eisenstein series is an eigenfunction of the Laplace operator. In order to find the dependence on $s$, we will need to return to the expression for the Eisenstein series found in (4.23). First, we will rewrite the sum, separating the $m=0$ terms, aiming to manipulate the rest of the sum. All sums are over $\mathbb{Z}$ unless otherwise specified. The convention of primed sums excluding the origin (which is simply 0 when summing over all integers) will once again be used. We obtain

$$
\begin{align*}
\mathcal{E}(z, s) & :=\sum_{(m, n) \in \mathbb{Z}^{2}}^{\prime} \frac{y^{s}}{|m z+n|^{2 s}}=\sum_{n}^{\prime} \frac{y^{s}}{|n|^{2 s}}+\sum_{m}^{\prime} \sum_{n} \frac{y^{s}}{|m z+n|^{2 s}}  \tag{4.30}\\
& =2 \zeta(2 s) y^{s}+\sum_{m}^{\prime} \sum_{n} \frac{y^{s}}{|m z+n|^{2 s}},
\end{align*}
$$

where $\zeta(s)$ is the Riemann-Zeta function. For the intended rewriting of the sum for $m \neq 0$, we will use the Gamma function $\Gamma(s)$, defined as

$$
\begin{equation*}
\Gamma(s)=\int_{0}^{\infty} e^{-x} x^{s-1} \mathrm{~d} x=\varepsilon^{s} \int_{0}^{\infty} \frac{e^{-\varepsilon / t}}{t^{s+1}} \mathrm{~d} t \tag{4.31}
\end{equation*}
$$

where we made the substitution $x=\varepsilon / t$. This gives

$$
\begin{equation*}
\frac{1}{\varepsilon^{s}}=\frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{e^{-\varepsilon / t}}{t^{s+1}} \mathrm{~d} t \tag{4.32}
\end{equation*}
$$

Letting $\varepsilon=\pi|m z+n|^{2}$ and summing over $n$, we thus get

$$
\begin{align*}
\sum_{n} \frac{1}{|m z+n|^{2}} & =\frac{\pi^{s}}{\Gamma(s)} \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1}} \sum_{n} e^{-\frac{\pi}{t}|m z+n|^{2}} \mathrm{~d} t \\
& =\frac{\pi^{s}}{\Gamma(s)} \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1}} e^{-\frac{\pi}{t} m^{2} y^{2}} \sum_{n} e^{-\frac{\pi}{t}(m x+n)^{2}} \mathrm{~d} t \tag{4.33}
\end{align*}
$$

where we in the last step used that $z=x+i y$, separating the $y$-dependent parts from the sum over $n$.

[^2]We will now use Poisson resummation, found in theorem 4.1, which states that

$$
\begin{equation*}
\sum_{n} f(n)=\sum_{k} \hat{f}(k) \tag{4.34}
\end{equation*}
$$

where $\hat{f}$ is the Fourier transform of $f$. Applying this to $f(n)=\exp \left(-\frac{\pi}{t}(m x+n)^{2}\right)$, we can write the sum over $n$ from (4.33) as

$$
\begin{equation*}
\sum_{n} e^{-\frac{\pi}{t}(m x+n)^{2}}=\sum_{k} \sqrt{t} e^{-\pi k^{2} t+2 \pi i k m x} \tag{4.35}
\end{equation*}
$$

where we on the right-hand side sum over the Fourier transform of the function in the left-hand sum. Inserting this back in (4.33) and prime-summing over $m$, for the moment leaving out the factor preceding the integral, we get

$$
\begin{align*}
& \sum_{m}^{\prime} \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1}} e^{-\frac{\pi}{t} m^{2} y^{2}} \sum_{k} \sqrt{t} e^{-\pi k^{2} t+2 \pi i k m x}=\sum_{k} \sum_{m}^{\prime} e^{2 \pi k m x} \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1 / 2}} e^{-\pi k^{2} t-\frac{\pi}{t} m^{2} y^{2}}  \tag{4.36}\\
&=\sum_{m}^{\prime} \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1 / 2}} e^{-\frac{\pi}{t} m^{2} y^{2}}+\sum_{k}^{\prime} \sum_{m}^{\prime} e^{2 \pi k m x} \int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1 / 2}} e^{-\pi k^{2} t-\frac{\pi}{t} m^{2} y^{2}}
\end{align*}
$$

Solutions to the two obtained integrals can be determined for $\operatorname{Re}(s)>1 / 2$, required for convergence. The solutions are

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1 / 2}} e^{-\frac{\pi}{t} m^{2} y^{2}}=\left(\frac{1}{\pi m^{2} y^{2}}\right)^{s-1 / 2} \Gamma(s-1 / 2) \tag{4.37}
\end{equation*}
$$

for the first one, and

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\mathrm{d} t}{t^{s+1 / 2}} e^{-\pi k^{2} t-\frac{\pi}{t} m^{2} y^{2}}=2\left|\frac{k}{m y}\right|^{s-1 / 2} K_{s-1 / 2}(2 \pi|k m| y) \tag{4.38}
\end{equation*}
$$

for the second one.
Returning to (4.30), using the information found in (4.33), (4.36) and the integral solutions found above we have that

$$
\begin{align*}
\mathcal{E}(z, s)= & 2 \zeta(2 s) y^{s}+\frac{y^{s} \pi^{s}}{\Gamma(s)}\left(\sum_{m}^{\prime}\left(\frac{1}{\pi m^{2} y^{2}}\right)^{s-1 / 2} \Gamma(s-1 / 2)\right. \\
& \left.+2 \sum_{k}^{\prime} \sum_{m}^{\prime} e^{2 \pi i k m x}\left|\frac{k}{m y}\right|^{s-1 / 2} K_{s-1 / 2}(2 \pi|k m| y)\right)  \tag{4.39}\\
= & 2 \zeta(2 s) y^{s}+2 \sqrt{\pi} y^{1-s} \frac{\Gamma(s-1 / 2)}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{1}{m^{2 s-1}} \\
& +2 \sqrt{y} \frac{\pi^{s}}{\Gamma(s)} \sum_{k}^{\prime} \sum_{m}^{\prime}\left|\frac{k}{m}\right|^{s-1 / 2} K_{s-1 / 2}(2 \pi|k m| y) e^{2 \pi i k m x}
\end{align*}
$$

However, since our aim is to find the coefficients in (4.29), which told us that

$$
\begin{equation*}
\mathcal{E}(z, s)=C_{1}(s) y^{s}+C_{2}(s) y^{1-s}+\sqrt{y} \sum_{n}^{\prime} a_{n}(s) K_{s-1 / 2}(2 \pi|n| y) e^{2 \pi i n x} \tag{4.40}
\end{equation*}
$$

we will substitute $n=k m$ and sum over n and m to get the required form. Given that $k=n / m \in \mathbb{Z}$, we should sum only over the $m$ which are divisors of $n$, denoted $m \mid n$. Looking only at the sums in the
last term of (4.39), we get

$$
\begin{equation*}
\sum_{k}^{\prime} \sum_{m}^{\prime}\left|\frac{k}{m}\right|^{s-1 / 2} K_{s-1 / 2}(2 \pi|k m| y) e^{2 \pi i k m x}=\sum_{n}^{\prime} \sum_{\substack{m \mid n \\ m>0}}\left|\frac{n}{m^{2}}\right|^{s-1 / 2} K_{s-1 / 2}(2 \pi|n| y) e^{2 \pi i n x} \tag{4.41}
\end{equation*}
$$

Lastly, we will introduce the divisor function $\sigma_{s}(n)$ (also called the instanton measure, for reasons we will understand in chapter 6) as

$$
\begin{equation*}
\sigma_{s}(n):=\sum_{\substack{m \mid n \\ m>0}} m^{s} \tag{4.42}
\end{equation*}
$$

Using this to rewrite (4.39), we have

$$
\begin{align*}
\mathcal{E}(z, s)= & 2 \zeta(2 s) y^{s}+2 \sqrt{\pi} \zeta(2 s-1) \frac{\Gamma(s-1 / 2)}{\Gamma(s)} y^{1-s} \\
& +\sqrt{y} \sum_{n}^{\prime} 4 \frac{\pi^{s}}{\Gamma(s)}|n|^{s-1 / 2} \sigma_{1-2 s}(n) K_{s-1 / 2}(2 \pi|n| y) e^{2 \pi i n x} \tag{4.43}
\end{align*}
$$

By simply comparing this expression to (4.29), we get the $s$-dependence of the Fourier coefficients:

$$
\begin{align*}
& C_{1}(s)=2 \zeta(2 s) \\
& C_{2}(s)=2 \sqrt{\pi} \zeta(2 s-1) \frac{\Gamma(s-1 / 2)}{\Gamma(s)}  \tag{4.44}\\
& a_{n}(s)=4 \frac{\pi^{s}}{\Gamma(s)}|n|^{s-1 / 2} \sigma_{1-2 s}(n)
\end{align*}
$$

thus completing the work of finding the Fourier expansion of the non-holomorphic Eisenstein series. We will now take a closer look at $a_{n}(s)$, which is the numerical (i.e. $z$-independent) part of the Fourier coefficient for $n \neq 0$, and show that it contains hidden Schur polynomials.

### 4.4 Euler Products

As stated when introducing modular forms, we intend to relate them to the six-vertex model by finding Schur polynomials, earlier seen in the partition functions of the aforementioned model, in the Fourier coefficients of the Eisenstein series. To do this, we will need the following results, proving that the $n$-dependent parts of the Fourier coefficients can be written as Euler products.

Theorem 4.2: The Riemann-zeta function $\zeta(s)$ can, with $\operatorname{Re}(s)>1, n \in \mathbb{N}$ and $s \in \mathbb{C}$ be written as

$$
\begin{equation*}
\zeta(s):=\sum_{n=1}^{\infty} \frac{1}{n^{s}}=\prod_{p \text { prime }} \frac{1}{1-p^{-s}} \tag{4.45}
\end{equation*}
$$

where the last product is taken over all prime numbers.
Proof: First, consider one term in the product and expand it as an infinite geometric series.

$$
\begin{equation*}
\frac{1}{1-p^{-s}}=1+\frac{1}{p^{s}}+\frac{1}{p^{2 s}}+\ldots \tag{4.46}
\end{equation*}
$$

Now, consider a product of two terms.

$$
\begin{equation*}
\frac{1}{1-p^{-s}} \frac{1}{1-q^{-s}}=\left(1+\frac{1}{p^{s}}+\frac{1}{p^{2 s}}+\ldots\right)\left(1+\frac{1}{q^{s}}+\frac{1}{q^{2 s}}+\ldots\right)=1+\frac{1}{p^{s}}+\frac{1}{(q p)^{s}}+\ldots+\frac{1}{p^{2 s}}+\frac{1}{(q p)^{2 s}}+\ldots \tag{4.47}
\end{equation*}
$$

The rearrangement of the terms is legitimate given that the series converge for $\operatorname{Re}(s)>1$. We remark that for any $n$ with prime factorization $n=p^{a} q^{b}$ for some $a, b \in \mathbb{N}, \frac{1}{n^{s}}$ occurs exactly once in this sum. Now, set some upper limit $r$ and consider

$$
\begin{equation*}
\zeta(s)-\prod_{p<r} \frac{1}{1-p^{-s}} \tag{4.48}
\end{equation*}
$$

where the product is over all primes smaller than r . By the reasoning above, this product can be written as a sum where, for all natural numbers that can be factored into primes smaller than $r, \frac{1}{n^{s}}$ occurs exactly once. Thus, the expression above is a sum of $\frac{1}{n^{s}}$ where the $n$ 's have a prime factor of $r$ or bigger, a set of numbers we will call $S$. To complete the proof, it is enough to show that this sum goes to zero as $r$ goes to infinity. This can easily be done, since

$$
\begin{equation*}
\left|\zeta(s)-\prod_{p<r} \frac{1}{1-p^{-s}}\right|=\left|\sum_{n \in S} \frac{1}{n^{s}}\right| \leq \sum_{n \in S}\left|\frac{1}{n^{s}}\right| \leq \sum_{n \geq r}\left|\frac{1}{n^{s}}\right|=\sum_{n \geq r} \frac{1}{n^{\mathrm{Re}(s)}} \longrightarrow 0, \quad \text { as } r \longrightarrow \infty \tag{4.49}
\end{equation*}
$$

where we in the last equality used that $\left|n^{a+i b}\right|=\left|e^{(a+i b) \log n}\right|=e^{a \log n}=n^{a}$. Thus, we have verified the stated theorem.

We now turn to the $n$-dependent part of the coefficient $a_{n}(s)$, which is where we will find the Schur polynomials.

### 4.4.1 Schur Polynomials

We will now, for the first time, catch a glimpse of the promised connection to the six-vertex model by proving that the numerical part of the Fourier coefficient, i.e. $a_{n}(s)$, can be written as a product of Schur polynomials. Since we will focus only on the part of $a_{n}(s)$ that depends on the summation variable $n$ in the Fourier expansion, we introduce

$$
\begin{equation*}
\kappa_{s}(n):=|n|^{s} \sigma_{-2 s}(n) \tag{4.50}
\end{equation*}
$$

Doing so, we may rewrite the Fourier coefficient from equation (4.44) as

$$
\begin{equation*}
a_{n}\left(s+\frac{1}{2}\right)=4 \frac{\pi^{s+1 / 2}}{\Gamma\left(s+\frac{1}{2}\right)} \kappa_{s}(n) \tag{4.51}
\end{equation*}
$$

The reason we study the coefficient for the parameter $s+\frac{1}{2}$ instead of simply $s$ is that the expression for the $n$-dependent part, i.e. $\kappa_{s}(n)$, becomes slightly easier to work with.

Remark: In some literature on the subject, e.g. [9], the so-called normalized Eisenstein series is introduced as

$$
\begin{equation*}
\mathcal{E}^{*}(z, s)=\frac{1}{2} \pi^{-s} \Gamma(s) \mathcal{E}(z, s) \tag{4.52}
\end{equation*}
$$

Then, the numerical part of the Fourier coefficient, $a_{n}^{*}(s)$, satisfies the convenient expression

$$
\begin{equation*}
a_{n}^{*}\left(s+\frac{1}{2}\right)=2 \kappa_{s}(n) \tag{4.53}
\end{equation*}
$$

We will now show how $\kappa_{s}(n)$ can be written as a Schur polynomial for an arbitrary value of $n$. The fundamental theorem of arithmetic states that every integer greater than 1 either is a prime itself or uniquely can be written as a product of prime numbers, i.e. $n=\prod_{i<\infty} p_{i}^{a_{i}}$ where $p_{i}$ are primes with the multiplicity $a_{i}$ in the factorization of $n$.

Theorem 4.3: The product $\kappa_{s}(n):=|n|^{s} \sigma_{-2 s}(n)$ may for an arbitrary $n=\prod_{i<\infty} p_{i}^{a_{i}}$ be written as the Schur polynomial

$$
\begin{equation*}
\kappa_{s}(n)=\prod_{i<\infty} s_{\left(a_{i}, 0\right)}\left(p_{i}^{s}, p_{i}^{-s}\right) \tag{4.54}
\end{equation*}
$$

Proof: We will begin by calculating $\kappa_{s}\left(p^{a}\right)$, with $p$ being a prime number. Since the positive divisors of $p^{a}$ are simply $1, p, p^{2}, \ldots, p^{a-1}, p^{a}$, we have

$$
\begin{align*}
\kappa_{s}\left(p^{a}\right) & =\left|p^{a}\right|^{s} \sigma_{-2 s}\left(p^{a}\right)=p^{a s}\left(1+p^{-2 s}+p^{-4 s}+\ldots+p^{-2 a s}\right)=p^{a s} \frac{1-p^{-(a+1) 2 s}}{1-p^{-2 s}}  \tag{4.55}\\
& =\frac{p^{a s}-p^{-s(2+a)}}{1-p^{-2 s}}
\end{align*}
$$

This can be written as a Schur polynomial, since we can see from definition 2.4 that

$$
s_{(a, 0)}\left(p^{s}, p^{-s}\right)=\frac{\operatorname{det}\left(\begin{array}{cc}
\left(p^{s}\right)^{a+1} & \left(p^{-s}\right)^{a+1}  \tag{4.56}\\
1 & 1
\end{array}\right)}{\operatorname{det}\left(\begin{array}{cc}
p^{s} & p^{-s} \\
1 & 1
\end{array}\right)}=\frac{p^{(a+1) s}-p^{-(a+1) s}}{p^{s}-p^{-s}}=\frac{p^{a s}-p^{-s(2+a)}}{1-p^{-2 s}} .
$$

Next, we will verify that $\kappa_{s}\left(p^{a} q^{b}\right)=\kappa_{s}\left(p^{a}\right) \kappa_{s}\left(q^{b}\right)$ with $p$ and $q$ being prime numbers, i.e. that $\kappa(n)$ is a multiplicative function. Since the positive divisors of $p^{a} q^{b}$ are $p^{i} q^{j}$, with $i=0,1, \ldots, a$ and $j=0,1, \ldots, b$, we can write

$$
\begin{align*}
\kappa_{s}\left(p^{a} q^{b}\right) & =\left|p^{a} q^{b}\right|^{s} \sigma_{-2 s}\left(p^{a} q^{b}\right) \\
& =\left|p^{a} q^{b}\right|^{s}\left(1+p^{-2 s}+p^{-4 s}+\ldots+p^{-2 a s}+q+p^{-2 s} q+p^{-4 s} q+\ldots+p^{-2 a s} q^{-2 b s}\right) \\
& =\left|p^{a}\right|^{s}\left(1+p^{-2 s}+\ldots+p^{-2 a s}\right)\left|q^{b}\right|^{s}\left(1+q^{-2 s}+\ldots+q^{-2 b s}\right)  \tag{4.57}\\
& =\left|p^{a}\right|^{s} \sigma_{-2 s}\left(p^{a}\right)\left|q^{b}\right|^{s} \sigma_{-2 s}\left(q^{b}\right) \\
& =\kappa_{s}\left(p^{a}\right) \kappa_{s}\left(q^{b}\right),
\end{align*}
$$

which we wanted to show. Combining the two results above for an arbitrary $n=p_{1}^{a_{1}} p_{2}^{a_{2}} \ldots$, applying the multiplicative property repeatedly, we obtain the desired result

$$
\begin{align*}
\kappa_{s}(n) & =\left|p_{1}^{a_{1}} p_{2}^{a_{2}} \ldots\right|^{s} \sigma_{-2 s}\left(p_{1}^{a_{1}} p_{2}^{a_{2}} \ldots\right)=\left|p_{1}^{a_{1}}\right|^{s}\left|p_{2}^{a_{2}}\right|^{s} \ldots \sigma_{-2 s}\left(p_{1}^{a_{1}}\right) \sigma_{-2 s}\left(p_{2}^{a_{2}}\right) \ldots \\
& =\prod_{i<\infty}\left(\left|p_{i}^{a_{i}}\right|^{s} \sigma_{-2 s}\left(p_{i}^{a_{i}}\right)\right)=\prod_{i<\infty} \kappa_{s}\left(p_{i}^{a_{i}}\right)=\prod_{\left(a_{i}, 0\right)}\left(p_{i}^{s}, p_{i}^{-s}\right) \tag{4.58}
\end{align*}
$$

Remark 4.2: When writing the divisor sum as an Euler product, the so-called p-adic norm is conventionally used. For our purposes, we can use the following simplified definition of the $p$-adic norm, which is equivalent to the actual definition when only integers are considered.

Definition 4.4: The $p$-adic norm of an integer $n$, denoted $|n|_{p}$, is defined as

$$
|n|_{p}=\left\{\begin{array}{l}
p^{-a}, \text { if } p \text { is a prime factor of } \mathrm{n} \text { with multiplicity } a  \tag{4.59}\\
1, \text { otherwise } .
\end{array}\right.
$$

Using the $p$-adic norm, the Euler product for $\kappa_{s}(n)$ is

$$
\begin{equation*}
\kappa_{s}(n)=\prod_{p<\infty} \frac{|n|_{p}^{-s}-p^{-2 s}|n|_{p}^{s}}{1-p^{-2 s}} . \tag{4.60}
\end{equation*}
$$

We note that, if a given prime $p$ is not a prime factor of $n$, the $p$-adic norm will simply be 1 , whereby the corresponding factor in the product will be

$$
\frac{1-p^{-2 s}}{1-p^{-2 s}}=1,
$$

whereas if $p$ is a prime factor of n with multiplicity $a$, the $p$-adic norm is $p^{-a}$, thereby contributing a factor

$$
\frac{p^{a s-p^{-s(2+a)}}}{1-p^{-2 s}}=s_{\left(a_{i}, 0\right)}\left(p_{i}^{s}, p_{i}^{-s}\right) .
$$

Thus, this version of the Euler product equals the one in theorem 4.3.
We can now rewrite the Fourier coefficients from (4.44) by substituting $|n|^{s-1 / 2} \sigma_{1-2 s}(n)=\kappa_{s-1 / 2}(n)$, $\zeta(2 s)$ and $\zeta(2 s-1)$ according to the results in theorems 4.2 and 4.3. This yields

$$
\begin{align*}
& C_{1}(s)=2 \prod_{p \text { prime }} \frac{1}{1-p^{-2 s}}  \tag{4.61}\\
& C_{2}(s)=2 \sqrt{\pi} \frac{\Gamma(s-1 / 2)}{\Gamma(s)} \prod_{p \text { prime }} \frac{1}{1-p^{-2 s+1}}  \tag{4.62}\\
& a_{n}(s)=4 \frac{\pi^{s}}{\Gamma(s)} \prod_{i<\infty} s_{\left(a_{i}, 0\right)}\left(p_{i}^{s-1 / 2}, p_{i}^{1 / 2-s}\right), \tag{4.63}
\end{align*}
$$

showing that the Schur polynomials once again appear. This will make it possible to connect the non-holomorphic Eisenstein series to the partition functions from the six-vertex model, as we desire. This will be done in the next chapter.

## Chapter 5

## Intertwining Results

In this chapter we will connect the subjects that have been studied in the earlier chapters. First of all we will illustrate how the Eisenstein series can be elegantly related to the seemingly more abstract studies on group- and representation theory from chapter 3. A key role in this process will be played by the character formula. Further on, we will investigate the connection between the partition functions of crystals and the Eisenstein series, which is one of the main purposes of this thesis.

### 5.1 Eisenstein Series and the Weyl Character Formula

The goal of this section is to study the connection between the non-holomorphic Eisenstein series and the character formula for $\mathfrak{s l}(2)$. This connection appears when studying $\kappa_{s}(n)$, which was defined in equation (4.50). $\kappa_{s}(n)$ is the $n$-dependent part of $a_{n}(s)$, which in turn is the numerical part of the Fourier coefficient of the aforementioned Eisenstein series, where $n$ is the summation variable in the Fourier expansion. Since the parameter $s$ is fixed for a given Eisenstein series, $\kappa_{s}(n)$ is sometimes referred to as the non-constant part of $a_{n}(s)$ [9]. In theorem 4.3 we proved that, for an arbitrary $n=\prod_{i<\infty} p_{i}^{k_{i}}$ where all $p_{i}$ are primes, $\kappa_{s}(n)$ can be written as a product of Schur polynomials, according to

$$
\begin{equation*}
\kappa_{s}(n):=|n|^{s} \sigma_{-2 s}(n)=\prod_{i<\infty} s_{\left(k_{i}, 0\right)}\left(p_{i}^{s}, p_{i}^{-s}\right) \tag{5.1}
\end{equation*}
$$

(The notation of the multiplicity of the primes has been changed from $a_{i}$ to $k_{i}$ in order for upcoming results to coincide with customary notation.) We are now going to prove how this result can also be reached by the use of our studies on characters of representations, and particularly the Weyl character formula, from section 3.5. In order to complete the derivation we will, however, need to introduce a few additional concepts.

From section 3.5 we know that we can always find an irreducible representation $V_{n}$ of $\mathfrak{s l}(2)$, by finding the irreducible representation of some highest weight $n$ in accordance with theorem 3.1. However, these are not the only irreducible representations. For every $V_{n}$ there are also an infinite number of so-called symmetric powers, which are also irreducible representations. The symmetric power of $V_{n}$
of order $k=2,3, \ldots$ is denoted $\operatorname{Sym}^{k}\left(V_{n}\right)$. Exactly how these are defined and how they look in detail is not of importance for this thesis, but they have one significant property that we will need to be familiar with.

Lemma 5.1: If the matrix $A \in S L(2, \mathbb{C})$ has the eigenvalues $\alpha$ and $\beta$ in the standard representation $V_{n}$, then the matrix $\operatorname{Sym}^{k}(A)$ - i.e. the "same" group element but in the representation $\operatorname{Sym}^{k}\left(V_{n}\right)-$ has the eigenvalues $\alpha^{k}, \alpha^{k-1} \beta, \ldots, \alpha \beta^{k-1}$ and $\beta^{k}$.

To reach the desired result we also need to give an alternative definition of the character of a representation.

Definition 5.1: The character $\chi_{\rho}$ of a representation $\rho$ is defined as

$$
\chi_{\rho}(g):=\operatorname{tr}(\rho(g))
$$

where $g$ is some group element and $\operatorname{tr}$ denotes the trace.
One can naturally show that this definition is equivalent to the one given in definition 3.25 , in section 3.5.

Now, we can write down an expression for the character of an arbitrary irreducible representation $\operatorname{Sym}^{k}(A)$ of some matrix in $\mathfrak{s l}(2, \mathbb{C})$. Here, $k=1$ corresponds to the standard representation we have been working with exclusively in this report, whereas $k=2,3, \ldots$ corresponds to a symmetric power of that representation. From the definition given above we know that

$$
\begin{equation*}
\chi_{k}(A)=\operatorname{tr}\left(\operatorname{Sym}^{k}(A)\right)=\sum_{k_{1}+k_{2}=k} \alpha^{k_{1}} \beta^{k_{2}} \tag{5.2}
\end{equation*}
$$

assuming the matrix $A$ has the eigenvalues $\alpha$ and $\beta$. In the last step we used lemma 5.1 and the wellknown fact from linear algebra that the trace of a matrix is equal to the sum of its eigenvalues.

We now want to relate this result to $\kappa_{s}(n)$, the non-constant part of the numerical Fourier coefficient, given in equation (5.1). Simply expanding the sum that the divisor function $\sigma_{-2 s}(n)$ in $\kappa_{s}(n)$ represents yields, for $n=p^{k}$, that

$$
\begin{align*}
\kappa_{s}\left(p^{k}\right) & =\left|p^{k}\right|^{s} \sigma_{-2 s}\left(p^{k}\right)=p^{k s} \cdot \sum_{K \mid p^{k}} K^{-2 s}=p^{k s}\left(1^{-2 s}+p^{-2 s}+\ldots+\left(p^{(k-1)}\right)^{-2 s}+\left(p^{k}\right)^{-2 s}\right)=  \tag{5.3}\\
& =p^{k s}+p^{(k-2) s}+\ldots+p^{(2-k) s}+p^{-k s}
\end{align*}
$$

If we now set $\alpha=p^{s}$ and $\beta=p^{-s}$ in the expression for the character in equation (5.2) we obtain

$$
\begin{aligned}
\chi_{k}(A) & =\sum_{k_{1}+k_{2}=k} p^{s k_{1}} p^{-s k_{2}}=p^{s k}+p^{s(k-1)} p^{-s}+\ldots+p^{s} p^{-s(k-1)}+p^{-k s}= \\
& =p^{k s}+p^{(k-2) s}+\ldots+p^{(2-k) s}+p^{-k s}
\end{aligned}
$$

Note that this is exactly equal to $\kappa_{s}\left(p^{k}\right)$ in equation (5.3). Hence, we can conclude that if $A$ is a matrix with eigenvalues $p^{s}$ and $p^{-s}$ we can write

$$
\kappa_{s}\left(p^{k}\right)=\chi_{k}(A)=\chi_{k}\left(\left(\begin{array}{ll}
p^{s} &  \tag{5.4}\\
& p^{-s}
\end{array}\right)\right)
$$

The expression in the right-hand side simply denotes that we take the character of some matrix with the eigenvalues $p^{s}$ and $p^{-s}$.

For an arbitrary $n=\prod_{i<\infty} p_{i}^{k_{i}}$, we use that $\kappa_{s}(n)$ is a multiplicative function (which was shown in equation (4.57)) to write

$$
\kappa_{s}(n)=\prod_{i<\infty} \kappa_{s}\left(p_{i}^{k_{i}}\right)=\prod_{i<\infty} \chi_{k_{i}}\left(\left(\begin{array}{cc}
p_{i}^{s} &  \tag{5.5}\\
& p_{i}^{-s}
\end{array}\right)\right)
$$

where we in the last step used the result from equation (5.4). Equation (5.5) is an important result relating the Fourier coefficients of the non-holomorphic Eisenstein series to the character of an arbitrary irreducible representation of $\mathfrak{s l}(2)$. However, we still have not used what we learned on irreducible representations from the Weyl character formula in section 3.5.1. Doing so we can simplify equation (5.5) substantially, and thereby relate it to equation (5.1) which we derived in section 4.3 by Fourier expansion of the Eisenstein series.

For $\mathfrak{s l}(2)$ the Weyl character formula yielded

$$
\begin{equation*}
\chi_{\lambda}=e^{\lambda}+e^{\lambda-2}+\ldots+e^{2-\lambda}+e^{-\lambda}=s_{(\lambda, 0)}\left(e^{-1}, e\right) \tag{5.6}
\end{equation*}
$$

Simply applying this formula to each of the characters in the product in equation (5.5) yields

$$
\begin{equation*}
\kappa_{s}(n)=\prod_{i<\infty} s_{\left(k_{i}, 0\right)}\left(p_{i}^{s}, p_{i}^{-s}\right) \tag{5.7}
\end{equation*}
$$

Note that this is the same expression as the one in equation (5.1), which was obtained in section 4.4.1 by Fourier expansion of the non-holomorphic Eisenstein series. Thus, this calculation shows how the Fourier coefficients of these Eisenstein series are related to the character formula for $\mathfrak{s l}(2)$. More precisely, one can identify $\kappa_{s}(n)$, i.e. the non-constant part of the numerical Fourier coefficient, by the character of the representation of $\mathfrak{s l}(2)$ one is using, as in equation (5.5). This is an important and intriguing property of the non-holomorphic Eisenstein series.

Deeper studies in the subject show that the result reached above is only a special case of a more general result called the Casselman-Shalika formula. In order to remotely understand this formula we need to be aware of the so-called metaplectic Eisenstein series. These are generalizations of the Eisenstein series we have been working with, from $S L(2)$ to all of the so-called reductive group [9, 16]. In the case of Lie groups, a reductive group is a Lie group whose corresponding Lie algebra has a fully reducible adjoint representation (see chapter 3 for definitions of these notions). This condition is for instance satisfied by all semisimple Lie algebras, e.g. the general linear groups $G L(n)$ which include the special linear groups $S L(n)$. In the generalization of the Eisenstein series, the notion of Fourier coefficients is replaced by that of the so-called Whittaker coefficients. The Casselman-Shalika formula states that the values of these Whittaker coefficients may be captured by characters of representations, for all reductive groups. For $G L(n)$, this implies that the characters may be expressed in terms of Schur polynomials $[9,16]$. It is this property that we proved for $S L(2)$ above.

Finally, it is interesting to make some observations on what the different quantities in the character formula correspond to in the expression for $\kappa_{s}(n)$ from the Fourier expansion. Comparing equation (5.6) with (5.7) for some fixed number $i$, we see that the basis $e$ in the formal exponentials in the character
formula corresponds to $p_{i}^{s}$ in the Fourier coefficient expression, where $p_{i}$ is a prime and $s$ is the parameter of the Eisenstein series. Furthermore, we see that $\lambda$ - i.e. the weight of the representation of which we take the character - corresponds to the multiplicity $k_{i}$ of the prime $p_{i}$ in the prime factorization of $n$. These are non-trivial connections which would have been very hard to find without performing the calculation above.

### 5.2 The Connection Between Crystals and Modular Forms

We are now interested in combining our results on the partition functions of crystals from chapter 2 with the results from chapter 4, on Eisenstein series. As we have already hinted, this connection is constituted by the Schur polynomials. We now want to investigate for which crystals, with respect to size and choice of Boltzmann weights, the exact same Schur polynomials appear as in the Fourier expansion of the non-holomorphic Eisenstein series.

In section 4.4.1 we found that the non-constant part of the numerical Fourier coefficients of the Eisenstein series is given by

$$
\begin{equation*}
\kappa_{s}(n)=\prod_{i<\infty} s_{\left(k_{i}, 0\right)}\left(p_{i}^{s}, p_{i}^{-s}\right) \tag{5.8}
\end{equation*}
$$

where $n=\prod_{i<\infty} p_{i}^{k_{i}}$.
From theorem 2.1 we know that the partition function of any crystal constructed from a vector $\lambda$ can be written as

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\left[\prod_{k=1}^{r+1}\left(a_{1}^{(k)}\right)^{\lambda_{1}} c_{2}^{(k)} \prod_{i<j}\left(a_{1}^{(j)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(j)}\right)\right] s_{\lambda}\left(\frac{b_{2}^{(1)}}{a_{1}^{(1)}}, \frac{b_{2}^{(2)}}{a_{1}^{(2)}}, \ldots, \frac{b_{2}^{(r+1)}}{a_{1}^{(r+1)}}\right) \tag{5.9}
\end{equation*}
$$

provided that the Boltzmann weights satisfy the free fermion condition $a_{1}^{(i)} a_{2}^{(i)}+b_{1}^{(i)} b_{2}^{(i)}=c_{1}^{(i)} c_{2}^{(i)}$. We now want to relate this result to equation (5.8). Since the interesting connection is constituted by the Schur polynomial, the product of Boltzmann weights in front of the Schur polynomial is disregarded in this section. One can, however, control the value of this product with the choice of the Boltzmann weights that do not appear in the Schur polynomial, i.e. $a_{2}, b_{1}$ and $c_{2}$.

The first observation we make is that in order for the Schur polynomial in equation (5.9) to be able to be the same as the one in (5.8), we must have $r=1$ and $\lambda=\left(k_{i}, 0\right)$. The former of these conditions makes sure that the Schur polynomials have the same number of arguments, but note that this condition actually follows from the second one since the number of entries in the vector $\lambda$ is always $r+1$, according to definition 2.1. From definition 2.3 , where we learned how to construct a crystal corresponding to a certain $\lambda$, it follows that $\lambda=\left(k_{i}, 0\right)$ yields a crystal with two rows and $k_{i}+2$ columns. Hence, we can conclude that the connection between crystals and the non-holomorphic Eisenstein series only appears for crystals with two rows.

This result agrees with earlier studies on the subject that have tackled it from a more group theoretical point of view. For example, the authors of [9] conclude that for an arbitrary value of $r$, which corresponds to crystals with $r+1$ rows, connections can always be found with modular forms that are invariant under the group $S L(r+1)$. Since the non-holomorphic Eisenstein series that we are working
with is invariant under $S L(2)$ (this was shown in equation (4.15)), we find connections to crystals with two rows. Crystals with more than two rows instead have connections to the aforementioned metaplectic Eisenstein series, which are generalizations of the Eisenstein series that we have been studying. A crystal with $r^{\prime}+1$ rows (where $r^{\prime}>1$ ) simply has connections to the metaplectic Eisenstein series that are invariant under $S L\left(r^{\prime}+1\right)$ [9].

After having concluded what dimensions the crystals should have in order for the connection to modular forms to possibly appear, we now also want to investigate what Boltzmann weights we should choose. In order for the Schur polynomials in equation (5.8) and (5.9) to be equal, we need to have

$$
\begin{equation*}
\frac{b_{2}^{(1)}}{a_{1}^{(1)}}=p_{i}^{s}, \quad \frac{b_{2}^{(2)}}{a_{1}^{(2)}}=p_{i}^{-s} \tag{5.10}
\end{equation*}
$$

These are important and non-trivial conditions on the Boltzmann weights. Note, however, that equation (5.8) contains a product of Schur polynomials, whereas (5.9) contains only one Schur polynomial. Thus, these conditions are not sufficient in order to obtain the exact same expressions. Then we need to form the product of a countably infinite number of partition functions, each with the weights chosen differently. More precisely, the weights should be chosen in accordance with equation (5.10) with $i=1,2, \ldots$ respectively. Also note that the crystal corresponding to partition function $i$ should have 2 rows and $k_{i}+2$ columns. Thus, the crystals may be of different dimension. Finding a physical interpretation of this far from trivial connection between crystals and the Fourier expansion of the non-holomorphic Eisenstein series is most probably a difficult task. This is something we will not be able to accomplish in this thesis, but we encourage further research on the subject.

Although we could not find a physically realistic situation where we obtain partition functions on the exact same form as $\kappa_{s}(n)$ (the non-constant part of the numerical Fourier coefficients of the Eisenstein series), we have still been able to expose a connection between crystals and modular forms, constituted by the Schur polynomials. This is due to the $S L(2)$-invariance of the Eisenstein series, and have deep connections to group theory. For an investigation on the subject from a more group theoretical point of view, we recommend the work of Brubaker, Bump and Friedberg [8, 9, 10].

## Chapter 6

## Conclusions and Future Directions

String theory is an attempt to unify many of today's leading physical theories. Among these are general relativity and quantum mechanics, and it is this connection that is of interest in this thesis. As stated in earlier chapters, modular forms have important applications in string theory. In this chapter we will study this connection a bit closer. We will also summarize our work and give some directions for future research in the field.

### 6.1 An Outlook to String Theory

The following calculation will outline how the Eisenstein series appear in string theory. It is not intended to be a rigorous derivation, nor an entry point to string theory. We will glance over large areas of both physics and mathematics, and equations will show up without much motivation. Still it may be enough to get some understanding of how Eisenstein series can be useful in string theory.

In special relativity an important object is the Minkowski metric $\eta_{\mu \nu}$ where $\mu$ and $\nu$ range from 1 to 4: 3 spatial dimensions and 1 time dimension. It is a constant matrix which is used similarly to the unity matrix. The metric is needed in the unification of space and time. The Minkowski metric is applicable in a flat space, that is, one that is not distorted by gravity.

In general relativity however, the metric is no longer constant but a function of the spacetime coordinates $x$. The metric is then written as $g_{\mu \nu}(x)$. There is no need to restrict this to 4 dimensions, so instead let $\mu$ and $\nu$ range from 0 to $d$. The theory then operates in a spacetime with $d+1$ dimensions, out of which one is a time dimension.
$g_{\mu \nu}$ is restricted by Einstein's equation

$$
\begin{equation*}
R_{\mu \nu}\left(g_{\mu \nu}\right)-\frac{1}{2} g_{\mu \nu} R\left(g_{\mu \nu}\right)=16 \pi G T_{\mu \nu} \tag{6.1}
\end{equation*}
$$

where $R_{\mu \nu}$ is the Riemann tensor, $R$ is the Ricci tensor, $G$ is Newtons constant and $T_{\mu \nu}$ specifies a distribution of mass. When a distribution of mass is introduced, this alters $g_{\mu \nu}$ according to this equation. This is called that the spacetime curves and gives rise to gravity.

Instead of working with this equation, it is often more useful to use the action principle. To understand this it is first necessary to define a functional. A functional $S$ is a map

$$
\begin{equation*}
S:\{\text { fields }\} \rightarrow \mathbb{R} \tag{6.2}
\end{equation*}
$$

## Example 6.1

An example of a functional $J$ is

$$
\begin{equation*}
J(y)=\int_{x_{1}}^{x_{2}} f\left(y(x), \frac{\mathrm{d} y(x)}{\mathrm{d} x}, x\right) \mathrm{d} x \tag{6.3}
\end{equation*}
$$

where $f$ is a fixed function and $x_{1}$ and $x_{2}$ are constant. The argument of the functional is the function $y$.

Functionals can be derivated similarly to ordinary functions. However, to discern these cases, functional derivatives are written as

$$
\frac{\delta J}{\delta y}
$$

using $\delta$ instead of d. Functions $y$ that gives $\delta J / \delta y=0$ are called stationary points analogous to ordinary functions. This is most often simply written as $\delta J=0$.

Functionals are very important in physics because many theories can be expressed using stationary points of functionals. One extremely important case comes from the Lagrangian $L=T-V$, where $T=\frac{1}{2} m \dot{x}$ is the kinetic energy and $V(x)$ is the potential energy [29]. It can then be shown that Newton's laws of motion are equivalent to stationary points of the functional $S(x(t))$, i.e.

$$
\begin{equation*}
\delta S=\delta \int_{t_{1}}^{t_{2}} L(x, \dot{x}) d t=0 \tag{6.4}
\end{equation*}
$$

This case is known as Hamilton's principle.
One reason that functionals are useful is that they only take scalar values. This means that they are invariant under all symmetries. In general relativity the action that we are interested in is

$$
\begin{equation*}
S\left(g_{\mu \nu}\right)=\int \mathrm{d}^{D} x \sqrt{-g} R(g) \tag{6.5}
\end{equation*}
$$

where $g=\operatorname{det}\left(g_{\mu \nu}\right)$. This is known as the Einstein-Hilbert action and is equivalent to equation (6.1).

The remainder of this calculation is based on [5]. From string theory, which operates in $D=10$ dimensions, one can derive quantum corrections to this action. The first non-trivial one of these is

$$
\begin{equation*}
\int \mathrm{d}^{10} x \sqrt{-g} f(\phi, \chi) R^{4}(g) \tag{6.6}
\end{equation*}
$$

where $\phi$ and $\chi$ are scalar fields corresponding to the particles dilaton and axion respectively. Now define

$$
\begin{equation*}
g_{s}=e^{\phi} \tag{6.7}
\end{equation*}
$$

which is known as the string coupling. It is now possible to calculate $f(\phi, \chi)$ as a series in $g_{s}$.
The first two terms in the weak-coupling limit $g_{s} \rightarrow 0$ can be shown to be

$$
2 \zeta(3) g_{s}^{-3 / 2}
$$

and

$$
2 \zeta(2) g_{s}^{1 / 2}
$$

It turns out that there are no higher order terms. These two terms are called perturbative effects because they can be written as a series in $g_{s}$. However, there can be non-perturbative effects which can not be found like the first two. That is because $f$ is somewhat analogous to the function $e^{1 / x}$. If you try to do a Taylor expansion of this function around 0 you will see that this is not possible since all derivatives vanish. We can conclude that $f$ must be on the form

$$
\begin{equation*}
f(\phi, \chi)=2 \zeta(3) g_{s}^{-3 / 2}+2 \zeta(2) g_{s}^{1 / 2}+\mathcal{O}\left(e^{-1 / g_{s}}\right) \tag{6.8}
\end{equation*}
$$

To further determine $f$, a different approach must be taken. It turns out that string theory has an exact $S L(2, \mathbb{Z})$-symmetry. For this symmetry to apply to equation (6.6), $f$ must be invariant under $S L(2, \mathbb{Z})$. It is also possible to show that $f$ must be an eigenfunction to $\nabla_{\mathbb{H}}$ with eigenvalue $3 / 4$. There is a unique such function, namely

$$
\begin{equation*}
f(\phi, \chi)=f(\tau)=\sum_{(m, n) \in \mathcal{Z}^{2}}^{\prime} \frac{y^{3 / 2}}{|m z+n|^{3}} \tag{6.9}
\end{equation*}
$$

where $\tau=\chi+i e^{-\phi}=x+i y$. Note that $y=g_{s}^{-1}$. We recognize this function as the Eisenstein series $\mathcal{E}(\tau, 3 / 2)$. From equation (4.43) we thus get the Fourier expansion of $f$ as

$$
\begin{equation*}
f(\tau)=2 \zeta(3) y^{3 / 2}+\sqrt{\pi} y^{-1 / 2} \frac{\Gamma(1)}{\Gamma(3 / 2)} \zeta(2)+\frac{4 \pi^{3 / 2} \sqrt{y}}{\Gamma(3 / 2)} \sum_{n \in \mathbb{Z}}^{\prime}|n| \sigma_{-2}(n) K_{1}(2 \pi|n| y) e^{2 \pi i n x} \tag{6.10}
\end{equation*}
$$

Using $\Gamma(x+1)=x \Gamma(x)$ and $\Gamma(1 / 2)=\sqrt{\pi}$ we get $\Gamma(3 / 2)=\frac{1}{2} \sqrt{\pi}$ [34]. Inserting this into the above equation yields

$$
\begin{equation*}
f(\tau)=2 \zeta(3) y^{3 / 2}+2 \zeta(2) y^{-1 / 2}+8 \pi \sqrt{y} \sum_{n \in \mathbb{Z}}^{\prime}|n| \sigma_{-2}(n) K_{1}(2 \pi|n| y) e^{2 \pi i n x} \tag{6.11}
\end{equation*}
$$

We recognize the first two terms from equation (6.8).
Let us now look at the remaining terms in the weak-coupling limit $g_{s} \rightarrow 0$, corresponding to $y \rightarrow \infty$. In this limit, the Bessel function expands to

$$
\begin{equation*}
K_{1}(2 \pi|n| y) \approx \frac{e^{-2 \pi|n| y}}{2 \sqrt{|n| y}}(1+\mathcal{O}(1 / y)) \tag{6.12}
\end{equation*}
$$

Insert this into (6.11) and exchange $y$ for $g_{s}$ and we get

$$
\begin{equation*}
f(\tau)=2 \zeta(3) g_{s}^{-3 / 2}+2 \zeta(2) g_{s}^{1 / 2}+4 \pi \sum_{n \in \mathbb{Z}}^{\prime} \sqrt{|n|} \sigma_{-2}(n) e^{2 \pi\left(i n x-|n| / g_{s}\right)}\left(1+\mathcal{O}\left(g_{s}\right)\right) \tag{6.13}
\end{equation*}
$$

Note that the sum is suppressed by the term $e^{2 \pi|n| / g_{s}}$, just as stated in (6.8). These are the nonperturbative terms.

It turns out that the non-perturbative terms corresponds to the effects of particles known as instantons. In each term the number $n$ is called the instanton charge and the divisor function $\sigma_{-2}(n)$ is called the instanton measure. The instanton measure counts the number of microstates the instanton can be in.

### 6.2 Concluding Remarks

In this thesis we have primarily investigated how modular forms are connected to the partition functions of two-dimensional models of certain crystals, such as ice. By studying the Fourier expansion of the non-holomorphic Eisenstein series, we have been able to identify the connection between this particular modular form and the two-dimensional crystals. In doing so, we were able to determine conditions on the size and Boltzmann weights of the crystals in order for the sought-after connection to appear. For example, we proved that the connection to the non-holomorphic Eisenstein series only appear for crystals consisting of two rows. We have also glanced at the world of string theory, where modular forms once again are of great importance. In this part, we have illustrated how the Eisenstein series emerge when studying the quantum corrections to the Einstein-Hilbert action.

By doing the above we have hopefully demonstrated the importance of modular forms in physics, both today and tomorrow. In string theory, they will keep playing a key role in the search for unification of quantum mechanics and general relativity. But as we have illustrated in this thesis, they also appear in other fields, such as statistical mechanics. Thus, our studies indicate that there may exist some hidden connection between the seemingly well separated fields of crystals and string theory. In the future search for this connection, we would recommend a thorough investigation of how the partition functions of crystals of arbitrary size are related to modular forms. This should partly include further studies of the metaplectic Eisenstein series on higher-dimensional groups, but more importantly it should also include a clarification of the physical interpretation. In string theory, numerous interpretations of the different parts of the Fourier expansion of the Eisenstein series have already been identified, as we saw in section 6.1. The divisor function $\sigma_{s}(n)$ is the instanton measure which counts the microstates, $n$ is the instanton charge and the parameter $s$ of the Eisenstein series need to be chosen to $3 / 2$ in order for the sought-after connection to appear. Finding similar interpretations and conditions for the partition functions of crystals might be the key to understanding their connection to string theory.

Whether a connection between crystals and string theory exists or not will for the time being remain uncertain. The importance of modular forms in future mathematical physics is, however, not.

## Appendix A

## Star-Triangle Identity

The weights of the six-vertex model can be written as $R_{\nu \beta}^{\theta \gamma}$ where connection between the weights and the indices can be seen below:


Using this we may rewrite (2.46) as

$$
\begin{equation*}
\sum_{\delta \phi \psi} T_{\tau \beta}^{\psi \delta} S_{\sigma \delta}^{\psi \alpha} R_{\phi \psi}^{\theta \rho}-\sum_{\gamma, \mu, \nu} R_{\sigma \tau}^{\nu \mu} S_{\nu \beta}^{\theta \gamma} T_{\mu \gamma}^{\rho \alpha}=0 \tag{A.1}
\end{equation*}
$$

Now arrange the weights in a matrix as following:

$$
R=\left(\begin{array}{llll}
a_{1} & & &  \tag{A.2}\\
& b_{1} & c_{1} & \\
& c_{2} & b_{2} & \\
& & & a_{2}
\end{array}\right)
$$

We may view $R$ as an endomorphism on the vector space $V \otimes V$ where $V$ is a vector space spanned by $v_{+}$and $v-$. The ordering of the base vectors is then $v_{+} \otimes v_{+}, v_{+} \otimes v_{-}, v_{-} \otimes v_{+}$and $v_{-} \otimes v_{-} . R$ then operates on a state by

$$
\begin{equation*}
R\left(v_{\nu} \otimes v_{\beta}\right)=\sum_{\theta, \gamma} R_{\nu \beta}^{\theta \gamma} v_{\theta} \otimes v_{\gamma} \tag{A.3}
\end{equation*}
$$

Now define $R_{i j}$ as an endomorphism on the space $V \otimes V \otimes V$ where $R$ acts on the components $i$ and $j$. For example if $R=R^{\prime} \otimes R^{\prime \prime}$ then $R_{13}=R^{\prime} \otimes I \otimes R^{\prime \prime}$. This also means that $R_{12}=R \otimes I$ and
$R_{23}=I \otimes R$. We have

$$
\begin{align*}
& R_{12}\left(v_{\nu} \otimes v_{\beta} \otimes v_{\mu}\right)=\sum_{\theta, \gamma} R_{\nu \beta}^{\theta \gamma} v_{\theta} \otimes v_{\gamma} \otimes v_{\mu} \\
& R_{13}\left(v_{\nu} \otimes v_{\beta} \otimes v_{\mu}\right)=\sum_{\theta, \gamma} R_{\nu \mu}^{\theta \gamma} v_{\theta} \otimes v_{\beta} \otimes v_{\gamma}  \tag{A.4}\\
& R_{23}\left(v_{\nu} \otimes v_{\beta} \otimes v_{\mu}\right)=\sum_{\theta, \gamma} R_{\beta \mu}^{\theta \gamma} v_{\nu} \otimes v_{\theta} \otimes v_{\gamma}
\end{align*}
$$

Define the Yang-Baxter commutator as

$$
\begin{equation*}
\llbracket \phi, \psi, \chi \rrbracket=\phi_{12} \psi_{13} \chi_{23}-\chi_{23} \psi_{13} \phi_{12} \tag{A.5}
\end{equation*}
$$

We now claim that the vanishing of $\llbracket R, S, T \rrbracket$ is equivalent to the star-triangle identity.

$$
\begin{gather*}
R_{12} S_{13} T_{23}\left(v_{\sigma} \otimes v_{\tau} \otimes v_{\beta}\right)-T_{23} S_{13} R_{12}\left(v_{\sigma} \otimes v_{\tau} \otimes v_{\beta}\right)= \\
R_{12} S_{13} \sum_{\psi, \delta} T_{\tau \beta}^{\psi \delta} v_{\sigma} \otimes v_{\psi} \otimes v_{\delta}-T_{23} S_{13} \sum_{\nu, \mu} R_{\sigma \tau}^{\nu \mu} v_{\nu} \otimes v_{\mu} \otimes v_{\beta}= \\
R_{12} \sum_{\psi, \delta, \phi, \alpha} T_{\tau \beta}^{\psi \delta} S_{\sigma \delta}^{\phi \alpha} v_{\phi} \otimes v_{\psi} \otimes v_{\alpha}-T_{23} \sum_{\nu, \mu, \theta, \gamma} R_{\sigma \tau}^{\nu \mu} S_{\nu \beta}^{\theta \gamma} v_{\theta} \otimes v_{\mu} \otimes v_{\gamma}= \\
\sum_{\psi, \delta, \phi, \theta, \rho, \alpha} T_{\tau \beta}^{\psi \delta} S_{\sigma \delta}^{\phi \alpha} R_{\phi \psi}^{\theta \rho} v_{\theta} \otimes v_{\rho} \otimes v_{\alpha}-\sum_{\nu, \mu, \gamma, \theta, \rho, \alpha} R_{\sigma \tau}^{\nu \mu} S_{\nu \beta}^{\theta \gamma} T_{\mu \gamma}^{\rho \alpha} v_{\theta} \otimes v_{\rho} \otimes v_{\alpha}= \\
\sum_{\theta, \rho, \alpha}\left(\sum_{\psi, \delta, \phi} T_{\tau \beta}^{\psi \delta} S_{\sigma \delta}^{\phi \alpha} R_{\phi \psi}^{\theta \rho}-\sum_{\nu, \mu, \gamma} R_{\sigma \tau}^{\nu \mu} S_{\nu \beta}^{\theta \gamma} T_{\mu \gamma}^{\rho \alpha}\right) v_{\theta} \otimes v_{\rho} \otimes v_{\alpha}=0 \\
\sum_{\psi, \delta, \phi} T_{\tau \beta}^{\psi \delta} S_{\sigma \delta}^{\phi \alpha} R_{\phi \psi}^{\theta \rho}-\sum_{\nu, \mu, \gamma} R_{\sigma \tau}^{\nu \mu} S_{\nu \beta}^{\theta \gamma} T_{\mu \gamma}^{\rho \alpha}=0 \tag{A.6}
\end{gather*}
$$

This is exactly how we wrote the star-triangle identity in equation (A.1).

To study the vanishing of $\llbracket R, S, T \rrbracket$ we want to write $R_{i j}$ as a matrix. Begin by determining $R_{23}=$ $I \otimes R$.

$$
I \otimes R=\left(\begin{array}{ll}
1 &  \tag{A.7}\\
& 1
\end{array}\right) \otimes\left(\begin{array}{lllllll}
a_{1} & & & & \\
& b_{1} & c_{1} & \\
& c_{2} & b_{2} & \\
& & & & a_{2}
\end{array}\right)=\left(\begin{array}{lllllll}
a_{1} & & & & & & \\
& b_{1} & c_{1} & & & & \\
& c_{2} & b_{2} & & & & \\
\\
& & & a_{2} & & & \\
\\
& & & & a_{1} & & \\
\\
& & & & & b_{1} & c_{1} \\
\\
& & & & & c_{2} & b_{2} \\
\\
& & & & & & \\
& & & & & & \\
& & & & & \\
& & & & & & \\
& & & & & \\
& & & & & \\
& & &
\end{array}\right)
$$

Similarly for $R_{12}$

$$
R \otimes I=\left(\begin{array}{lllll}
a_{1} & & &  \tag{A.8}\\
& b_{1} & c_{1} & \\
& c_{2} & b_{2} & \\
& & & a_{2}
\end{array}\right) \otimes\left(\begin{array}{ll}
1 & \\
& 1
\end{array}\right)=\left(\begin{array}{lllllll}
a_{1} & & & & & & \\
& a_{1} & & & & & \\
& & b_{1} & & c_{1} & & \\
& & & b_{1} & & c_{1} & \\
\\
& & c_{2} & & b_{2} & & \\
\\
& & & c_{2} & & b_{2} & \\
\\
& & & & & & a_{2}
\end{array}\right]
$$

To determine $R_{13}$ we will need the twist map $\tau$ defined by $\tau(v \otimes w)=w \otimes v$. That means that $\tau$ acts on $R$ by rearranging the basis vectors as

$$
\left\{\begin{array} { l } 
{ v _ { + } \otimes v _ { + } }  \tag{A.9}\\
{ v _ { + } \otimes v _ { - } } \\
{ v _ { - } \otimes v _ { + } } \\
{ v _ { - } \otimes v _ { - } }
\end{array} \quad \rightarrow \left\{\begin{array}{l}
v_{+} \otimes v_{+} \\
v_{-} \otimes v_{+} \\
v_{+} \otimes v_{-} \\
v_{-} \otimes v_{-}
\end{array}\right.\right.
$$

Note that this can be seen as changing the order of the second and third basis vector. We can then write

$$
\tau(R)=\left(\begin{array}{llll}
1 & & &  \tag{A.10}\\
& & 1 & \\
& 1 & & \\
& & & 1
\end{array}\right) \times R \times\left(\begin{array}{llll}
1 & & & \\
& & 1 & \\
& 1 & & \\
& & & 1
\end{array}\right)=T \times R \times T
$$

Using this we can write $R_{13}$ as $\tau_{23}\left(R_{12}\right)$

$$
R_{13}=(I \otimes T)(R \otimes I)(I \otimes T)=\left(\begin{array}{llllllll}
a_{1} & & & & & & &  \tag{A.11}\\
& b_{1} & & & c_{1} & & & \\
& & a_{1} & & & & & \\
& & & b_{1} & & & c_{1} & \\
& c_{2} & & & b_{2} & & & \\
& & & & & a_{2} & & \\
& & & c_{2} & & & b_{2} & \\
& & & & & & & a_{2}
\end{array}\right)
$$

We now have all we need to check the vanishing of $\llbracket R, S, T \rrbracket$. Choose the weights in $R$ as in table 2.2 and let $S$ contain the weights of row $i$ and $T$ the weights of row $j$. Insert these into

$$
\llbracket R, S, T \rrbracket=R_{12} S_{13} T_{23}-T_{23} S_{13} R_{12}
$$

and we find that we obtain 0 . That means that the Star-Triangle Identity holds for the set of weights in table 2.2.

## Appendix B

## Partition Function for $\lambda=(2,1,0)$

Assume we want to calculate the partition function corresponding to $\lambda=(2,1,0)$.
First, we use definition 2.3 to find the boundary conditions that all states must satisfy. This yields the crystal in figure B.1.


Figure B.1: The boundary conditions for $\lambda=(2,1,0)$.

We continue by identifying all states that satisfy these boundary conditions. This gives a total number of 26 states. Two of them are illustrated in figure B.2. The particularly interested reader is encouraged to contact any of the authors for a list of all possible states.

In order to obtain manageable equations we now make the choice to work with the weights given in equation (2.31). We then calculate the weights of each of the 26 states by simply multiplying the Boltzmann weights of the 15 vertices in each crystal. For the states in figure B. 2 this yields

$$
\begin{aligned}
& w\left(x_{1}\right)=\left(a_{1}^{(1)}\right)^{2}\left(b_{1}^{(1)}\right)^{2} c_{2}^{(1)} \cdot b_{1}^{(2)} a_{1}^{(2)} c_{2}^{(2)}\left(b_{2}^{(2)}\right)^{2} \cdot c_{2}^{(3)}\left(b_{2}^{(3)}\right)^{4}=t_{1}^{2} t_{2} z_{2}^{2} z_{3}^{4} \\
& w\left(x_{2}\right)=b_{1}^{(1)} a_{1}^{(1)} c_{2}^{(1)} b_{2}^{(1)} a_{2}^{(1)} \cdot b_{1}^{(2)}\left(a_{1}^{(2)}\right)^{3} c_{2}^{(2)} \cdot c_{2}^{(3)}\left(b_{2}^{(3)}\right)^{4}=t_{1} z_{1}^{2} t_{2} z_{3}^{4} .
\end{aligned}
$$

After calculating the weights of all 26 states, we simply summarize them to obtain the partition function $Z\left(\mathfrak{S}_{\lambda}\right)$. With the help of the command Factor in Mathematica we then simplify the obtained partition function, yielding

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\left(z_{1}+z_{2}\right)\left(z_{1}+z_{3}\right)\left(z_{2}+z_{3}\right)\left(z_{1}+t_{1} z_{2}\right)\left(z_{1}+t_{1} z_{3}\right)\left(z_{2}+t_{2} z_{3}\right) \tag{B.1}
\end{equation*}
$$



Figure B.2: Two of the 26 possible states for $\lambda=(2,1,0)$.

We may also want to control that theorem 2.1 is satisfied. Since the vector $\lambda=(2,1,0)$ has $r=2$, equation (2.32) - which is equation (2.30) in theorem 2.1 with our choice of Boltzmann weights inserted - yields

$$
\begin{equation*}
Z\left(\mathfrak{S}_{\lambda}\right)=\prod_{i<j}\left(t_{i} z_{j}+z_{i}\right) s_{\lambda}\left(z_{1}, z_{2}, z_{3}\right)=\left(t_{1} z_{2}+z_{1}\right)\left(t_{1} z_{3}+z_{1}\right)\left(t_{2} z_{3}+z_{2}\right) s_{(2,1,0)}\left(z_{1}, z_{2}, z_{3}\right) \tag{B.2}
\end{equation*}
$$

The Schur polynomial can be found from the definition as

$$
s_{(2,1,0)}\left(z_{1}, z_{2}, z_{3}\right)=\frac{\left|\begin{array}{ccc}
z_{1}^{4} & z_{2}^{4} & z_{3}^{4} \\
z_{1}^{2} & z_{2}^{2} & z_{3}^{2} \\
1 & 1 & 1
\end{array}\right|}{\left|\begin{array}{ccc}
z_{1}^{2} & z_{2}^{2} & z_{3}^{2} \\
z_{1} & z_{2} & z_{3} \\
1 & 1 & 1
\end{array}\right|}=\frac{\left(z_{1}^{2}-z_{2}^{2}\right)\left(z_{1}^{2}-z_{3}^{2}\right)\left(z_{2}^{2}-z_{3}^{2}\right)}{\left(z_{1}-z_{2}\right)\left(z_{1}-z_{3}\right)\left(z_{2}-z_{3}\right)}=\left(z_{1}+z_{2}\right)\left(z_{1}+z_{3}\right)\left(z_{2}+z_{3}\right)
$$

where we first used equation (2.27) to simplify the denominator and then used that we can see in the determinant expression that the nominator must be equal to the denominator after the substitution $z_{i} \rightarrow z_{i}^{2}$ for $i=1,2,3$. In the last step we used the formula for the difference of squares. Insertion of this result into equation (B.2) yields

$$
Z\left(\mathfrak{S}_{\lambda}\right)=\left(z_{1}+z_{2}\right)\left(z_{1}+z_{3}\right)\left(z_{2}+z_{3}\right)\left(z_{1}+t_{1} z_{2}\right)\left(z_{1}+t_{1} z_{3}\right)\left(z_{2}+t_{2} z_{3}\right)
$$

which is identical with the expression in equation (B.1). Hence, we have shown that theorem 2.1 is satisfied.

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[^1]:    ${ }^{1}$ This notation with plus and minus signs is taken from articles by Brubaker et al. [9, 10, 16]. The remaining figures in this chapter are to a varying extent inspired by figures from these articles.

[^2]:    ${ }^{1}$ apart from the fact that our eigenvalue to the Laplacian, $s(s+1)$, would have to be replaced with an arbitrary number.

