On the Brunn-Minkowski and Aleksandrov-Fenchel Inequalities

Master's Thesis in Engineering Mathematics

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Abstract

The Brunn-Minkowski inequality has a wide range of generalizations and its applications spread throughout many mathematical fields. Using a inequality by Brascamp-Lieb a functional version of Brunn-Minkowski is found in Prékopa's theorem and the Prékopa-Leindler inequality. We demonstrate the wide applicability of the Brunn-Minkowski inequality and its functional counterparts.

Using basic properties of differential forms we find an alternate proof of the classical result that the volume of a Minkowksi sum is a polynomial. Further by applying techniques from the realm of differential forms an attempt is made to simplify and generalize the proof of the Aleksandrov-Fenchel inequality.

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1

Introduction

THE Brunn-Minkowski theory is central in the study of convex bodies and convex geometry but the applications of the Brunn-Minkowski inequality extends to a much larger area of mathematics. The theory has its origins in the work of Hermann Brunn and Hermann Minkowski at the end of the 19th century. The field was further developed through the work of Bonnesen, Fenchel and Aleksandrov during the 1930s. None the less the theory of convex bodies has since grown substantially, connections to other branches of mathematics have been established and new applications continue to be discovered.

The Brunn-Minkowski inequality arises when combining the notions of Minkowski addition and volume for sets in $\mathbb{R}^n$, further study of this area involves the concept of mixed volumes. In this thesis we provide a proof that the volume of a Minkowski sum is a polynomial that differs from the classical proof and uses basic properties of differential forms. Also using differential forms we attempt to construct a simpler and possibly more intuitively clear proof of the Aleksandrov-Fenchel inequality. This is done in the hope of finding a more general setting for the proof where it might be extended to a larger class of sets or functions.

In the remainder of this chapter notation and reoccurring concepts are introduced and some simple theorems are proved. We then move on to study the Brunn-Minkowski theorem its implications and present functional counterparts in Prékopa and Prékopa-Leindler's theorems. Applications of the Brunn-Minkowski inequalities and its different versions are found throughout many different fields of mathematics we reference a number of applications in text and also dedicate a section to proving a logarithmic Sobolev inequality [1] using Prékopa-Leindler's theorem. Further we show that one can obtain an inequality by Brascamp and Lieb using Prékopa's theorem.

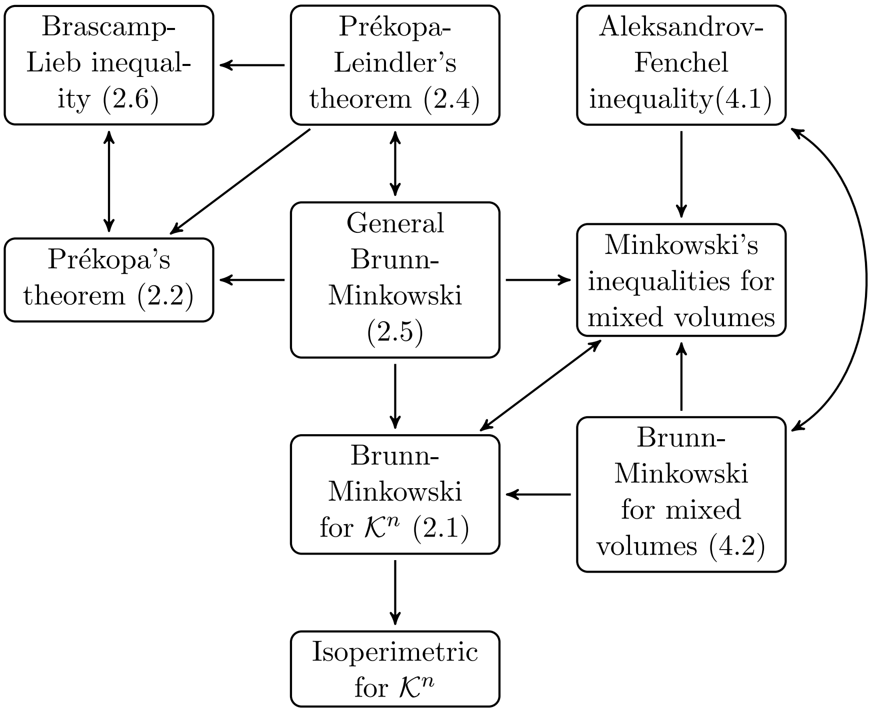

Figure 1.1: A part of the vast web of inequalities related to the Brunn-Minkowski inequality.

In chapter 3 the study of mixed volumes is presented and the fundamental properties of these quantities are discussed. The concept of mixed volumes leads us on to Minkowski's inequalities for mixed volumes and in chapter 4 we study a more general form of these inequalities namely the Alexandrov-Fenchel inequality 4.1.

Since the mathematical topics concerned in this thesis are diverse and from separate branches of mathematics it is possible that the notation used here differs from that normally used in a specific mathematical area. We therefore proceed by introducing notation and some basic concepts that are used throughout the thesis. This chapter may be found useful as a guide for the continued reading of the thesis.

1.1 Basic geometry

In n-dimensional Euclidean space ($\mathbb{R}^n$) we denote the origin by o, the unit sphere by S^{n-1} and the closed unit ball by B. An arbitrary ball of radius r centered at a point x will be denoted $B_r(x)$. The standard Euclidean norm will be denoted by $\|x\|$, and the scalar product of two elements x,y will be written as $\langle x, y \rangle$.

Definition 1.1 *The dilatate of X is the set $rX = \{rx : x \in X\}$, $r \geq 0$.*

The set $-X$ is called the reflection of X and is defined similarly. If $X = -X$ the set is called origin symmetric.

Definition 1.2 *The Minkowski sum (also known as the vector sum) of two subsets* X,Y *of* $\mathbb{R}^n$ *is the set* $X + Y = \{x + y : x \in X, y \in Y\}$.

It is noteworthy that the set $X + Y$ need not be measurable even when X,Y both are, this is a non-trivial statement due to Minkowski [2].

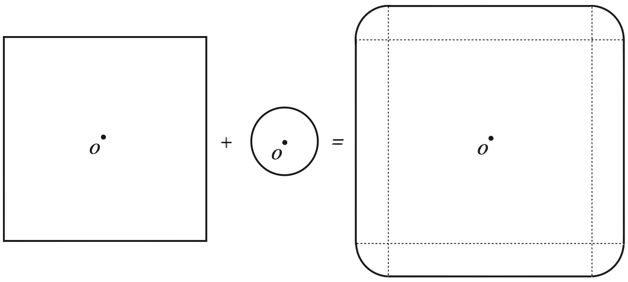

Figure 1.2: The Minkowski sum of a square and a ball, both origin symmetric, in $\mathbb{R}^2$ (Image from Gardner [3]).

A hyperplane in $\mathbb{R}^n$ is described by the scalar product and denoted

$$H_{u,\alpha} = \{x \in \mathbb{R}^n | \langle x,u \rangle = \alpha\}$$

for some non zero $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Two hyperplanes coincide if and only if $(u_1,\alpha_1) = (\lambda u_2, \lambda\alpha_2)$ for some $\lambda \in R$. Similarly we denote the (closed) halfspaces bounded by this hyperplane by

$$\begin{aligned} H^-_{u,\alpha} &= \{x \in \mathbb{R}^n | \langle x,u \rangle \leq \alpha\} \\ H^+_{u,\alpha} &= \{x \in \mathbb{R}^n | \langle x,u \rangle \geq \alpha\} \,. \end{aligned}$$

For a k-dimensional compact and measurable subset X we use $V_k(X) = \mu^k(X)$ to denote the volume of X (where μ^k is the k-dimensional Lebesgue measure). One should note the k-homogeneity of the volume which is used at several places throughout the thesis, i.e that $V_k(\lambda X) = \lambda^k V_k(X)$.

1.2 Basic convexity

The main study in this thesis is convex sets and inequalities concerning their Minkowski sums and volumes. In this section the concept of both convex sets and convex functions are introduced together with some basic properties.

1.2.1 Convex sets

Definition 1.3 *A set* $E \subset \mathbb{R}^n$ *is called convex if for all* x,$y \in E$ *and any* $\lambda \in (0,1)$ *it holds that*

$$(1 - \lambda)x + \lambda y \in E.$$

A non-empty, convex and compact subset of $\mathbb{R}^n$ is called a convex body and the class of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. We often consider two subsets of $\mathcal{K}^n$ firstly $\mathcal{K}_0^n$ is the set of all convex bodies with non-empty interior and secondly $\mathcal{K}_{reg}^n$ is the set of all convex bodies with C^∞ boundary. For these classes the following inclusions hold

$$\mathcal{K}_{reg}^n \subset \mathcal{K}_0^n \subset \mathcal{K}^n.$$

For convex bodies K,L the Minkowski sum $K + L$ is also convex furthermore the sum $aK + bK = (a + b)K$ which is not true in general for non convex bodies, for proof of this the reader is referred to Schneider [2].

Definition 1.4 *For a set $E \subset \mathbb{R}^n$ we say that a hyperplane H is a support plane at x if $x \in E \cap H$ and either $E \subseteq H^-$ or $E \subseteq H^+$, where H^- and H^+ denotes the halfspaces bounded by H.*

Note that this implies that x is a boundary point of E. If the hyperplane $H_{u,\alpha}$ supports a set E at x and $E \subseteq H_{u,\alpha}^-$ then $H_{u,\alpha}^-$ is called a supporting halfspace of E and u is an outer normal to E at x. For convex bodies it holds that for every boundary point x of E there exists at least one supporting plane. Furthermore the body E can be written as the intersection of all its supporting halfspaces, proofs of this and several other similar statements can be found in the first chapter of Schneider [2].

Since the intersection of two convex sets also is convex it follows that the restriction of a convex set to some subspace is also convex, in particular the intersection of a hyperplane and a convex set is convex.

When convergence of convex sets is considered it is always implied to be in the sense of the Hausdorff metric which is defined as

$$d(K,L) = \max\{\sup_{x\in K}\inf_{y\in L}\|x-y\|, \sup_{y\in L}\inf_{x\in K}\|x-y\|\}.$$

1.2.2 Convex functions

We proceed with the closely related area of convex functions. For convex function we let the range of the function be the extended real numbers $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\infty\}$, where we admit the following conventions. For any $\lambda \in \mathbb{R}$ let

1. $\infty + \infty = \lambda + \infty = \infty + \lambda = \infty$
2. $-\infty - \infty = -\infty + (-\infty) = \lambda - \infty = -\infty + \lambda = -\infty$ and
3. $\lambda\infty = \infty$, 0 or $-\infty$ depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

A function $f : X \to \bar{\mathbb{R}}$ is called proper if $\{f = -\infty\} = \emptyset$ and $\{f = \infty\} \neq X$.

Definition 1.5 *A function $f : X \to \mathbb{R}$ is called convex if it is proper and for all x,$y \in X$ and any $\lambda \in [0,1]$*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

If for any $\lambda \in (0,1)$ the inequality is strict then f is called strictly convex.

There are several equivalent definitions for the convexity of a function. Both the conditions that the epigraph of a function should be a convex set or that a function is equal to the supremum of its tangent planes are equivalent to the definition above. The definition implies that a convex function restricted to some subspace is again convex on that space.

A function $f : D \to \bar{\mathbb{R}}$ where $D \subset X$ is said to be convex if the extension $\hat{f}$

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in D \\ \infty & \text{if } x \in X \setminus D \end{cases}$$

is convex. Also a function f is called concave if $-f$ is convex. We state a number of theorems concerning convex functions. Several of these results will later be used without reference here, however proofs of these theorems are fairly simple and can be found in chapter 1.5 of Schneider's book Convex Bodies [2].

Theorem 1.1 *A convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is continuous on the interior of the set $\{f < \infty\}$. f is also Lipschitz on any compact subset of* $\operatorname{int}(\{f < \infty\})$.

Theorem 1.2 *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a convex function and $x \in \operatorname{int}(\{f < \infty\})$. Then if f's partial derivatives at x exists then f is differentiable at x.*

Theorem 1.3 *Let f be differentiable on an open, convex set $D \subset \mathbb{R}^n$. Then the following statements are equivalent:*

1. *f is convex,*
2. *$f(y) - f(x) \geq \langle \nabla f(x), y - x\rangle$ for all $x, y \in D$,*
3. *$\langle \nabla f(y) - \nabla f(x), y - x\rangle \geq 0$ for all $x, y \in D$.*

Theorem 1.4 *Let f be twice differentiable on an open, convex set $D \subset \mathbb{R}^n$. Then f is convex iff the Hessian of f is positive semidefinite for any $x \in D$.Similarly f is strictly convex iff the Hessian is positive definite for all $x \in D$.*

One should note that for $X = \mathbb{R}$ this is equivalent to the non-negativity respectively positivity of $f''(x)$. This theorem relates heavily to the following definition.

Definition 1.6 *A function $f : X \to \mathbb{R}$ such that for all $x, y \in X$ and any $\lambda \in [0,1]$*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) - c\frac{(1-\lambda)\lambda}{2}\|x-y\|^2 \tag{1.1}$$

for some constant $c > 0$ is called strongly convex with parameter c.

This definition implies that all the eigenvalues μ of the Hessian of f satisfy $\mu \geq c$. This follows from observing that (1.1) is equivalent to the function $g(x) = f(x) - \frac{c}{2}\|x\|^2$ being convex and thus its Hessian has no negative eigenvalues.

1.3 Generalized means

In several upcoming proofs inequalities concerning generalized means are used, therefore we in this section give a short introduction to the notion of generalized means and introduce some inequalities concerning them. For a sequence of positive numbers $a = (a_1,...,a_n)$ the generalized mean of such a function is defined as follows.

Definition 1.7 *The r-th mean of $a = (a_1,...,a_n)$ with weights $w = (w_1,...,w_n)$ is for r finite and non-zero defined by*

$$M_{r,w}(a) = \left(\sum_{i=1}^{n} w_i a_i^r\right)^{1/r},$$
$$w_i > 0 \ \forall i, \quad \sum w_i = 1.$$

For the cases $r = 0$, $r = \infty$ and $r = -\infty$ the definition is taken as the limiting value in r this yields

$$M_{0,w}(a) = \prod_{i=1}^{n} x_i^{w_i}, \quad M_{\infty,w}(a) = \max(a), \quad M_{-\infty,w}(a) = \min(a).$$

It should be noted that using the weights $w_i = 1/n$ one recovers for $r = 1$ and $r = 0$ the classical arithmetic and geometric means. The well known arithmetic-geometric mean inequality is now extended to a larger case of generalized means.

Theorem 1.5 (Generalized means inequality) *Let $a = (a_1,...,a_n)$ be a sequence of positive numbers and set $w = (w_1,...,w_n)$ such that $w_i > 0$ and $\sum w_i = 1$ then for $-\infty \leq r < s \leq \infty$*

$$M_{r,w}(a) \leq M_{s,w}(a).$$

The two means are equal if and only if $a_1 = a_2 = ... = a_n$.

For a proof of this and further inequalities concerning means the reader is referred to Hardy, Littlewood and Pólyas book Inequalities [4].

1.4 α-Concave functions

A reoccurring concept in this thesis is that of logarithmic concavity. We now introduce a more general concept of concavity that is closely related to the concept of generalized means, in particular we define the concept of logarithmic concavity.

Definition 1.8 *For $-\infty \leq \alpha \leq \infty$ a function $f : \mathbb{R}^n \to [0,\infty)$ is called α-concave if f is supported on some convex set $\Omega \subseteq \mathbb{R}^n$ and satisfies that*

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f(x)^\alpha + (1-\lambda) f(y)^\alpha]^{1/\alpha}$$

for all $\lambda \in [0,1]$ *and all* $x,y \in \Omega$. *In the cases of* $\alpha = 0$, $\alpha = -\infty$ *and* $\alpha = \infty$ *the definition should be interpreted as a limit, one then finds that* f *is* α*-concave if*

$$\begin{aligned}
&[\alpha = 0] && f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda} \\
&[\alpha = -\infty] && f(\lambda x + (1-\lambda)y) \geq \min\{f(x),f(y)\} \\
&[\alpha = \infty] && f(\lambda x + (1-\lambda)y) \geq \max\{f(x),f(y)\}.
\end{aligned}$$

For $\alpha = \infty$ it is from the definition clear that such a function is constant on its support, and thus the class of ∞-concave functions on $\mathbb{R}^n$ is the set of indicator functions of sets in $\mathcal{K}^n$. The property of a function being 0-concave is more often referred to as a function being logarithmically concave, a logarithmically concave function can always be written as $e^{-\phi}$ where $\phi : \mathbb{R}^n \to \mathbb{R}$ is convex. A related topic is the concept of α-concave measures, a measure is logarithmically concave if its density is an α-concave function. The conventions above are adapted from those used by Brascamp and Lieb in [5].

1.5 Characterizing functions

For sets in $\mathbb{R}^n$ there are several functions characterizing the properties of the set. The first and most commonly used is the indicator function of a set E.

Definition 1.9 *The indicator function of a set* $E \subset \mathbb{R}^n$ *is defined as the function such that*

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

In convex analysis a second characteristic function is often used called the convex indicator function.

Definition 1.10 *The convex indicator function of a set* $E \subset \mathbb{R}^n$ *is the function*

$$\mathbb{1}_E^{\infty}(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases}$$

It is clear that this function is convex if and only if the set E is convex. One might observe that the logarithmically concave function generated by the convex characteristic function of a set E is exactly the indicator function of the same set, i.e

$$\mathbb{1}_E(x) = \exp(-\mathbb{1}_E^{\infty}(x)).$$

When it comes to the topics discussed here a further function is often of more interest, namely the support function of a convex set.

Definition 1.11 *The support function of a convex body* $K \subset \mathbb{R}^n$ *is defined by*

$$h(K,u) = \sup\{\langle x,u \rangle : x \in K\}, \quad u \in \mathbb{R}^n.$$

It is clear from the definition that the $h(K,u)$ for some $K \in \mathcal{K}^n$ is 1-homogeneous and convex, further if K contains the origin its support function is non-negative. A convex set is uniquely identified by its support function and every function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

1. f is 1-homogeneous and

2. f is convex

is the support function of some $K \in \mathcal{K}^n$. This can be proved using that the $h(K,u)$ is the Legendre transform of the convex indicator function of K and basic properties of the Legendre transform.

When it comes to studying Minkowski sums the following property of supportfunctions of convex sets is extremely useful, the proof of this lemma is not provided here but is based on the definition of $H(K,u)$ and basic properties of supremums.

Lemma 1.6 *For $K_1,...,K_r \in \mathcal{K}^n$ and $\lambda_1,...,\lambda_r \geq 0$ the supportfunction satisfies the following equality*

$$h(\sum_i \lambda_i K_i,u) = \sum_i \lambda_i h(K_i,u)$$

for any $u \in \mathbb{R}^n$.

For $u \in \mathbb{R}^n \setminus \{0\}$ such that $h(K,u) < \infty$ let

$$\begin{aligned} H(K,u) =& \{x \in \mathbb{R}^n : \langle x,u \rangle = h(K,u)\}, \\ H^-(K,u) =& \{x \in \mathbb{R}^n : \langle x,u \rangle \leq h(K,u)\}. \end{aligned}$$

$H(K,u)$ and $H^-(K,u)$ are the support plane and supporting halfspace of K with outer normal u. Note that these definitions extend the corresponding definitions above.

2

The Brunn-Minkowski inequality

The Brunn-Minkowski inequality, in its original form, relates the volume of the Minkowski sum of two convex subsets of Euclidean space to the individual volumes of the two sets. The inequality was first proved in three dimensional Euclidean space by Brunn and later generalized to any dimension by Minkowski [2, 3]. The inequality states that if $K,L \in \mathcal{K}^n$ and $\lambda \in [0,1]$ then

$$V_n\left(\left(1-\lambda\right)K+\lambda L\right)^{1/n} \geq \left(1-\lambda\right)V_n\left(K\right)^{1/n}+\lambda V_n\left(L\right)^{1/n}. \tag{2.1}$$

The inequality is sharp and the theorem in its entirety includes exact conditions for equality. There are several generalizations of this inequality extending it to non-convex sets, although here the equality conditions are more complex.

The inequality is one of the most fundamental results in the theory of convex bodies and can be considered a central part of the study of geometric inequalities. We shall see that (2.1) forms a starting point towards several other related inequalities concerning the mixed volumes of convex bodies, the concept of mixed volumes will be introduced in Section 3. The inequality can for instance be found as a special case of the Aleksandrov-Fenchel inequality, which will be the main focus of the second part of this thesis.

For more on the Brunn-Minkowski inequality in convex geometry the reader is referred to Schneider [2]. However it will be demonstrated that the scope of the Brunn-Minkowski inequality goes beyond the boundaries of convex geometry. Several connections to important inequalities in analysis will be established and the inequality will be used to obtain a short and simple proof of the isoperimetric inequality for convex domains. We will also see connections to both Prékopa's theorem and its extension the Prékopa-Leindler inequality that essentially provides a functional formulation of the Brunn-Minkowski inequality.

As an illustrating example that motivates the study of (2.1) a proof of the isoperimetric inequality for convex bodies in $\mathbb{R}^n$ is presented. The isoperimetric inequality relates the n-dimensional volume of a body K to the $(n-1)$-dimensional volume of its boundary, in the plane it states that for a simple closed curve γ of length L we have that

$$L^2 \geq 4\pi A, \tag{2.2}$$

where A is the area of the domain enclosed by γ. Equality in (2.2) occurs iff γ is a circle. In arbitrary dimension the inequality can be written as

$$\left(\frac{V_n(K)}{V_n(B)}\right)^{1/n} \leq \left(\frac{S_n(K)}{S_n(B)}\right)^{1/(n-1)} \tag{2.3}$$

where $S_n(K)$ is the surface area of K, or equivalently the $(n-1)$-dimensional volume of ∂K. Using the Minkowski sum we define the surface area of a convex body [3].

Definition 2.1 *For $K \in \mathcal{K}^n$ the surface area of K is defined as*

$$S_n(K) = V_{n-1}(\partial K) = \lim_{\varepsilon \to 0} \frac{V_n(K + \varepsilon B) - V_n(K)}{\varepsilon},$$

i.e. the differential rate of the change of volume as we add a small ball to K.

It is fairly simple to see that this definition coincides with our usual definitions of length, area and volume. We now use this definition and the Brunn-Minkowski inequality to prove (2.3).

Proof We apply the Brunn-Minkowski inequality and find that

$$\begin{aligned} V_n(K + \varepsilon B) \geq& \left(V_n(K)^{1/n} + \varepsilon V_n(B)^{1/n}\right)^n \\ =& V_n(K)\left(1 + \varepsilon\left(\frac{V_n(B)}{V_n(K)}\right)^{1/n}\right)^n \\ \geq& V_n(K)\left(1 + n\varepsilon\left(\frac{V_n(B)}{V_n(K)}\right)^{1/n}\right). \end{aligned}$$

The last inequality can be found by taking the two first terms of the McLaurin expansion of $(1 + x)^n$ and noting that the remaining terms are non-negative. By the definition of surface area we have that

$$\begin{aligned} S_n(K) = V_{n-1}(\partial K) =& \lim_{\varepsilon \to 0} \frac{V_n(K + \varepsilon B) - V_n(K)}{\varepsilon} \\ \geq& \lim_{\varepsilon \to 0} \frac{V_n(K) + n\varepsilon V_n(K)\left(\frac{V_n(B)}{V_n(K)}\right)^{\frac{1}{n}} - V_n(K)}{\varepsilon} \\ =& nV_n(K)\left(\frac{V_n(B)}{V_n(K)}\right)^{\frac{1}{n}} \\ =& nV_n(K)^{\frac{n-1}{n}} V_n(B)^{\frac{1}{n}}. \end{aligned}$$

In particular if $K = B$ we have equality in (2) and we find that $V_n(B + \varepsilon B) = (1 + \varepsilon)^n V_n(B)$ and thus we can also see that $S_n(B) = nV_n(B)$. Knowing this the isoperimetric inequality as stated in (2.3) follows

$$\begin{aligned}\left(\frac{S_n(K)}{S_n(B)}\right)^{\frac{1}{n-1}} &\geq \left(\frac{nV_n(K)^{\frac{n-1}{n}} V_n(B)^{\frac{1}{n}}}{nV_n(B)}\right)^{\frac{1}{n-1}} \\ &= \left(\frac{V_n(K)}{V_n(B)}\right)^{\frac{1}{n}}.\end{aligned}$$

■

2.1 The Brunn-Minkowski inequality for sets in $\mathcal{K}^n$

We now state the complete Brunn-Minkowski theorem for convex bodies and provide a classical geometric proof due to Kneser and Süss (1932), which is also reproduced in Schneider [2]. There are several different geometrical proofs of the inequality but this proof has several similarities to proofs occurring later in the thesis. This proof also provides the full conditions of equality which is a point missed by several of the other classical proofs.

Theorem 2.1 (Brunn-Minkowski) *For K,$L \in \mathcal{K}^n$ and for $\lambda \in [0,1]$*

$$V_n\left((1-\lambda)K + \lambda L\right)^{1/n} \geq (1-\lambda)V_n(K)^{1/n} + \lambda V_n(L)^{1/n}.$$

Equality occurs for some $\lambda \in (0,1)$ iff K and L are either homothetic or lie in parallel hyperplanes.

Before a proof is presented some notation is introduced. For two bodies K_0,$K_1 \in \mathcal{K}^n$ and $\lambda \in [0,1]$ we write

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1.$$

With this notation we get for $\sigma, \tau, \lambda \in [0,1]$ that

$$\begin{aligned}(1-\lambda)K_\sigma + \lambda K_\tau =&(1-\lambda)\left[(1-\sigma)K_0 + \sigma K_1\right] + \lambda\left[(1-\tau)K_0 + \tau K_1\right] \\ =&(1 - [(1-\lambda)\sigma + \lambda\tau])K_0 + [(1-\lambda)\sigma + \lambda\tau]K_1 \\ =&(1-\alpha)K_0 + \alpha K_1 \quad \text{where} \quad \alpha = (1-\lambda)\sigma + \lambda\tau.\end{aligned}$$

Proof In the notation introduced above the inequality rewrites as

$$V_n(K_\lambda)^{1/n} \geq (1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}. \tag{2.4}$$

First consider the case where $\dim K_0 < n$ and $\dim K_1 < n$ then $V_n(K_0) = V_n(K_1) = 0$ and thus (2.4) holds trivially. We proceed by considering the case where $\dim K_0 < n$ and

$\dim K_1 = n$. In this case we have that $K_\lambda \supseteq (1-\lambda)x + \lambda K_1$, for any $x \in K_0$ implying that

$$V_n(K_\lambda) \geq V_n((1-\lambda)x + \lambda K_1) = \lambda^n V_n(K_1)$$

thus (2.4) is fulfilled. The argument works symmetrically for the case where K_0 is n-dimensional but K_1 is not.

Thus we only need to consider the case such that $\dim K_0 = \dim K_1 = n$. It is also enough to prove (2.4) for the case when both K_0 and K_1 have volume equal to 1. This follows from the observation that for

$$\tilde{K}_0 = V_n(K_0)^{-1/n} K_0 \quad \& \quad \tilde{K}_1 = V_n(K_1)^{-1/n} K_1$$

and

$$\tilde{\lambda} = \frac{\lambda V_n(K_1)^{1/n}}{(1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}}$$

we know that the inequality holds. But using the homogeneity of the volume we obtain the original inequality

$$\begin{aligned}
V_n((1-\tilde{\lambda})\tilde{K}_0 + \tilde{\lambda}\tilde{K}_1)^{1/n} &= \frac{V_n((1-\lambda)K_0 + \lambda K_1)^{1/n}}{(1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}} \\
(1-\tilde{\lambda})V_n(\tilde{K}_0)^{1/n} &= \frac{(1-\lambda)V_n(K_0)^{1/n}}{(1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}} \\
\tilde{\lambda} V_n(\tilde{K}_1)^{1/n} &= \frac{\lambda V_n(K_1)^{1/n}}{(1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}}
\end{aligned}$$

and thus (2.4) holds for arbitrary K_0 and K_1 in $\mathcal{K}_0^n$. The equality conditions can be generalized in the same manner.

We prove the theorem using induction over n. For $n = 1$ the theorem is trivial. Thus we need to prove that the statement is true for $n \geq 2$ provided that it is true for dimension $n-1$.

Begin by fixing an arbitrary $u \in S^{n-1}$ and for any $\zeta \in \mathbb{R}$ we let $H(\zeta) = H_{u,\zeta}$ and $H^-(\zeta) = H^-_{u,\zeta}$ we also define $\alpha_\lambda = -h(K_\lambda, -u)$ and $\beta_\lambda = h(K_\lambda, u)$, here H, H^- and h denotes hyperplanes, halfspaces and the support function as defined earlier. For $\zeta \in \mathbb{R}$ we define

$$\begin{aligned}
v_0(\zeta) =& V_{n-1}(K_0 \cap H(\zeta)), \\
w_0(\zeta) =& V_n(K_0 \cap H^-(\zeta))
\end{aligned}$$

and in the same manner

$$\begin{aligned}
v_1(\zeta) =& V_{n-1}(K_1 \cap H(\zeta)), \\
w_1(\zeta) =& V_n(K_1 \cap H^-(\zeta)).
\end{aligned}$$

It follows from the definitions that

$$w_0(\zeta) = \int_{\alpha_0}^{\zeta} v_0(t)dt, \quad w_1(\zeta) = \int_{\alpha_1}^{\zeta} v_1(t)dt.$$

We can see that on (α_i,β_i) $(i = \{0,1\})$ the function v_i is continuous. Thus w_i is differentiable and

$$w_i'(\zeta) = v_i(\zeta) > 0$$

for any $\zeta \in (\alpha_i, \beta_i)$. Therefore there exists an inverse function of w_i we denote it by z_i. From basic analysis one konws that

$$z_i'(\tau) = \frac{1}{v_i(z(\tau))}$$

for $0 < \tau < 1$. Define the following

$$\begin{aligned} k_i(\tau) =& K_i \cap H(z_i(\tau)), \\ z_\lambda(\tau) =& (1-\lambda)z_0(\tau) + \lambda z_1(\tau) \end{aligned}$$

then for $\lambda, \tau \in (0,1)$

$$K_\lambda \cap H(z_\lambda(\tau)) \supset (1-\lambda)k_0(\tau) + \lambda k_1(\tau).$$

By using this inclusion and the Brunn-Minkowski inequality for dimension $n-1$ we find that

$$\begin{aligned} V_n(K_\lambda) &= \int_{\alpha_\lambda}^{\beta_\lambda} V_{n-1}(K_\lambda \cap H(\zeta))d\zeta \\ &= \int_0^1 V_{n-1}(K_\lambda \cap H(z_\lambda(\tau)))z_\lambda'(\tau)d\tau \\ &\geq \int_0^1 V_{n-1}((1-\lambda)k_0(\tau) + \lambda k_1(\tau)) \left[\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right] d\tau \\ &\geq \int_0^1 \left[(1-\lambda)v_0^{1/(n-1)} + \lambda v_1^{1/(n-1)}\right]^{n-1} \left[\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right] d\tau \\ &\geq 1 \end{aligned}$$

where v_i is used to abbreviate $v_i(z_i(\tau))$. The last inequality follows from that

$$\left[(1-\lambda)v_0^{1/(n-1)} + \lambda v_1^{1/(n-1)}\right]^{n-1} \left[\frac{1-\lambda}{v_0} + \frac{\lambda}{v_1}\right] \geq 1. \tag{2.5}$$

We provide a short proof of (2.5). For $a,b,p > 0$ and $\lambda \in (0,1)$ we have that

$$\begin{aligned}
&\log\left([(1-\lambda)a^p + \lambda b^p]^{1/p}\left[\frac{1-\lambda}{a} + \frac{\lambda}{b}\right]\right) \\
&= \frac{1}{p}\log\left([(1-\lambda)a^p + \lambda b^p]\left[\frac{1-\lambda}{a} + \frac{\lambda}{b}\right]^p\right) \\
&\geq \frac{1}{p}(1-\lambda)\log\left(a^p\left[\frac{1-\lambda}{a} + \frac{\lambda}{b}\right]^p\right) + \frac{1}{p}\lambda\log\left(b^p\left[\frac{1-\lambda}{a} + \frac{\lambda}{b}\right]^p\right) \\
&= (1-\lambda)\log\left((1-\lambda) + \frac{a\lambda}{b}\right) + \lambda\log\left(\frac{b(1-\lambda)}{a} + \lambda\right) \\
&\geq (1-\lambda)\lambda\log(a/b) + \lambda(1-\lambda)\log(b/a) \\
&= (1-\lambda)\lambda\log\left(\frac{a}{b}\frac{b}{a}\right) = 0
\end{aligned}$$

where we twice use the fact that the logarithm is a concave function. Hence implying the validity of (2.5). Using the fact that the logarithm is increasing we can also see that equality holds if and only if $a = b$ or equivalently in (2.5) if $v_0 = v_1$.

Thus we have that $V_n(K_\lambda) \geq 1$ and the proof of (2.4) is complete, only the proof of the equality conditions remain.

If the bodies K_0 and K_1 lie in two parallel hyperplanes we have that K_λ also lies in a hyperplane thus implying equality in (2.4) (both sides equal to zero).

If $\dim K_0 < n$ and $\dim K_1 < n$ and we have equality then K_λ is contained in some hyperplane, thus K_0,K_1 must lie in parallel hyperplanes.

If as before $\dim K_0 < n$ and $\dim K_1 = n$ we as stated above have that for all $x \in K_0$

$$V_n(K_\lambda) \geq V_n((1-\lambda)x + \lambda K_1) = \lambda^n V_n(K_1)$$

where we have equality if and only if $K_0 = \{x\}$, in which case K_0 and K_1 are homothetic. The same argument can be applied symmetrically to the case when K_0 is full-dimensional and K_1 is not.

For two sets K_0,K_1 that are homothetic we have that $K_1 = rK_0 + c$ for some $r \in \mathbb{R}$ and $c \in \mathbb{R}^n$. For convex sets Minkowski addition is distributive thus implying that

$$\begin{aligned}
(1-\lambda)K_0 + \lambda K_1 =& (1-\lambda)K_0 + \lambda(rK_0 + c) \\
=& (1-\lambda+\lambda r)K_0 + \lambda c
\end{aligned}$$

and thus by the translative invariance of V_n we have that

$$\begin{aligned}
V_n((1-\lambda)K_0 + \lambda K_1)^{1/n} =& V_n((1-\lambda+\lambda r)K_0 + \lambda c)^{1/n} \\
=& (1-\lambda+\lambda r)V_n(K_0)^{1/n} \\
=& (1-\lambda)V_n(K_0)^{1/n} + \lambda r V_n(K_0)^{1/n} \\
=& (1-\lambda)V_n(K_0)^{1/n} + \lambda V_n(K_1)^{1/n}.
\end{aligned}$$

Hence we have equality in (2.4) if K_0 and K_1 are homothetic.

In the case of fully dimensional bodies equality for some λ in (2.4) implies equality in (2.5). Thus $v_0(z_0(\tau)) = v_1(z_1(\tau))$ implying that $z_0'(\tau) = z_1'(\tau)$ for all $\tau \in [0,1]$, therefore $z_1(\tau) - z_0(\tau)$ must be constant. We assume without loss of generality, by the translation invariance of V_n, that the two bodies are centered in the origin. Thus for $i \in \{0,1\}$

$$0 = \int_{K_i} \langle x, u \rangle dx = \int_{\alpha_i}^{\beta_i} V_{n-1}(K_i \cap H(\zeta))\zeta d\zeta = \int_0^1 z_i(\tau) d\tau.$$

Therefore $z_0(\tau) = z_1(\tau)$ for all $\tau \in [0,1]$, this implies that $\beta_0 = \beta_1$ which in turn by the definition of β_i implies that $h(K_0,u) = h(K_1,u)$. This holds for any $u \in S^{n-1}$ and thus implies that $K_0 = K_1$, which before the translation and normalization of volume implies that the bodies are homothetic and the proof is complete.

■

In the upcoming sections we will introduce functional inequalities and demonstrate that using these can be used to construct simpler proofs of theorem 2.1. It will also be shown that these methods provide proofs for more general versions of the Brunn-Minkowski inequality concerning not only convex sets but more general subsets of $\mathbb{R}^n$.

2.2 Functional Brunn-Minkowski

In the following two sections Prékopa's theorem and Prékopa-Leindler's inequality are introduced and discussed as functional inequalities corresponding to different versions of the Brunn-Minkowski inequality.

2.2.1 Prékopa's theorem

The Prékopa-Leindler inequality is an extension of a simpler, but still very powerful, inequality due to Prékopa concerning log concave functions [5]. As before a function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be log concave if it satisfies the inequality

$$F((1-\lambda)x + \lambda y) \geq F(x)^{1-\lambda} F(y)^{\lambda}, \quad \forall x,y \in \mathbb{R}^n \text{ and } \forall \lambda \in [0,1].$$

It follows that F is log concave iff it can be written as $F(x) = \exp[-\phi(x)]$ where $\phi(x)$ is a convex function.

Theorem 2.2 (Prékopa's theorem) *For a log concave function $F : \mathbb{R}^{n+m} \to \mathbb{R}$ we have that*

$$G(y) = \int_{\mathbb{R}^n} F(y,z) dz$$

also is log concave. For $x \in \mathbb{R}^{m+n}$ we use the notation $x = (y,z)$ such that $y \in \mathbb{R}^m$ and $z \in R^n$.

We begin by proving theorem 2.2 in the case where $m = n = 1$ after this an inductive argument together with some basic properties of convex functions yields the result.

For $m = n = 1$ we utilize a inequality due to Brascamp and Lieb [5], this inequality can in some sense be considered a real version of Hörmander's estimate for the $\bar{\partial}$-operator.

Theorem 2.3 (The Brascamp-Lieb inequality) *For a strictly convex function $\phi \in C^2(\mathbb{R}^n)$ such that ϕ has a minimum and for any $u \in C^1(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n}(u - \widehat{u})^2 e^{-\phi} dx < \infty$ the following inequality holds*

$$\int_{\mathbb{R}^n} (u - \widehat{u})^2 e^{-\phi} dx \leq \int_{\mathbb{R}^n} \langle \nabla u, \phi_{xx}^{-1} \nabla u \rangle e^{-\phi} dx \tag{2.6}$$

where $\widehat{u} = \int_{\mathbb{R}^n} u e^{-\phi} dx / \int_{\mathbb{R}^n} e^{-\phi} dx$ and ϕ_{xx} is the Hessian matrix of ϕ.

We prove this theorem in the case where $n = 1$ which is the only case that will be used here, this proof can be found in Brascamp and Lieb's article [5] where they also extend the argument to arbitrary dimension. It will further be shown that (2.6) can be found for arbitrary value of n using Prékopa's theorem.

Proof (Brascamp-Lieb inequality in the case $n = 1$) Let ϕ obtain its minimum at the point a. We introduce a function $k(x)$ that is C^1 everywhere except possibly at a by

$$u(x) - u(a) = \phi'(x)k(x).$$

If we let $k(a) = u'(a)/\phi''(a)$ it is easy to check that k is continuous also at a.

Now in one dimension the gradient is simply the derivative and the inverse of the Hessian matrix is the reciprocal of the second derivative. Therefore one finds that

$$\begin{aligned}
\int_{\mathbb{R}^1} \langle \nabla u, \phi_{xx}^{-1} \nabla u \rangle e^{-\phi} dx &= \int_{\mathbb{R}} \frac{(u')^2}{\phi''} e^{-\phi} dx \\
&= \int_{\mathbb{R}} \frac{\left((\phi' k)'\right)^2}{\phi''} e^{-\phi} dx \\
&= \int_{\mathbb{R}} \left[\frac{(\phi' k')^2}{\phi''} + 2kk'\phi' + \phi'' k^2 \right] e^{-\phi} dx \\
&= \int_{\mathbb{R}} \left[\frac{(\phi' k')^2}{\phi''} + (k\phi')^2 \right] e^{-\phi} dx + \left[k^2 \phi' e^{-\phi} \right]_{-\infty}^{a} + \left[k^2 \phi' e^{-\phi} \right]_{a}^{\infty} \\
&= \int_{\mathbb{R}} \left[\frac{(\phi' k')^2}{\phi''} + (k\phi')^2 \right] e^{-\phi} dx \\
&= \int_{\mathbb{R}} \left[\frac{(\phi' k')^2}{\phi''} + (u(x) - u(a))^2 \right] e^{-\phi} dx \\
&\geq \int_{\mathbb{R}} (u(x) - u(a))^2 e^{-\phi} dx \\
&\geq \int_{\mathbb{R}} (u - \hat{u})^2 e^{-\phi} dx.
\end{aligned}$$

The last inequality is easily proved by expanding the squares and rearranging the terms. Thus the proof for $n = 1$ is complete.

■

Using this inequality we prove Prékopa's theorem 2.2.

Proof (Prékopa's theorem) We begin by proving the statement when ϕ is strictly convex and in C^2. Since the integral of any function can be approximated arbitrarily well with integrals of C^2 functions it follows that Prékopa's theorem also holds in the case where $\phi \notin C^2$. A similar argument lets us generalize the statement from strictly convex functions to arbitrary convex functions.

The statement that G is log concave is equivalent to saying that the function

$$g(y) = -\log\left(\int e^{-\phi(y,z)}dz\right)$$

is convex.

For $n = m = 1$ we have that g is convex iff its second derivative is non negative. However if we denote differentiation in y with subscript y one finds that

$$\begin{aligned}
g_{yy} &= \left[-\log\left(\int e^{-\phi}dz\right)\right]_{yy} \\
&= \left[\frac{\int \phi_y e^{-\phi}dz}{\int e^{-\phi}dz}\right]_y \\
&= \left[\frac{\int \phi_y e^{-\phi}dz}{\int e^{-\phi}dz}\right]^2 + \frac{\int \left(\phi_{yy} - \phi_y^2\right) e^{-\phi}dz}{\int e^{-\phi}dz} \\
&= \frac{\int \phi_{yy} e^{-\phi}dz}{\int e^{-\phi}dz} - \frac{\int \left(\phi_y - \widehat{\phi_y}\right)^2 e^{-\phi}dz}{\int e^{-\phi}dz} \\
&\geq \frac{\int \phi_{yy} e^{-\phi}dz}{\int e^{-\phi}dz} - \frac{\int \phi_{yz}^2 \phi_{zz}^{-1} e^{-\phi}dz}{\int e^{-\phi}dz} \\
&= \frac{\int \left(\phi_{yy} - \phi_{yz}^2\phi_{zz}^{-1}\right) e^{-\phi}dz}{\int e^{-\phi}dz}
\end{aligned}$$

but this is positive since

$$\phi_{yy} - \phi_{yz}^2\phi_{zz}^{-1} = \det\left(H(\phi)\right)/\phi_{zz}$$

where H denotes the Hessian matrix which is positive since ϕ is strictly convex. Thus the theorem is proved for $n = m = 1$.

We know that a function is convex if and only if it is convex on every line. This statement generalizes to log concavity, a function is log concave if and only if it is log concave on every line. But on any line in $\mathbb{R}^m$ we can use our one dimensional statement and thus find that the theorem holds for any value of m.

Now for general n we prove the theorem by iterated use of the statement in one dimension. For $\phi(z,y)$ $z \in \mathbb{R}^n$ and $y \in \mathbb{R}$ we have that

$$\int_{\mathbb{R}^n} e^{-\phi(z,y)} dz = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{-\phi(z_1,\dots,z_n,y)} dz_1 \dots dz_n$$

here we can apply our theorem to the innermost integral and find that it is a log concave function and thus for the next integral we can again apply our theorem with $n = 1$. Thus the proof is complete for arbitrary choice of m, n.

$\blacksquare$

In Brascamp and Lieb's article [5] the inequality above (2.6) in arbitrary dimension is used to prove Prékopa's theorem. We will now show that using Prékopa's theorem as stated above we can obtain the Brascamp-Lieb inequality for arbitrary value of n, the argument provided here is due to Dario Cordero-Erausquin at Université Pierre et Marie Curie.

Let $\psi_\varepsilon : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined as

$$\psi_\varepsilon(t,z) = \phi(z) + tu(z) + \frac{t^2}{2}\langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle + \varepsilon t^2 + \varepsilon \|z\|^2, \quad \varepsilon > 0$$

where we in the same manner as before have decomposed $x \in \mathbb{R}^{n+1}$ as (t,z) for $t \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Let ϕ to be strictly convex and obtain a minimum and $u \in C^1$ with compact support then one finds that this function is convex close to $t = 0$. We assume without loss of generality that the Hessian matrix of ϕ is diagonal.

$$H\psi_\varepsilon = \begin{bmatrix} \langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle + \varepsilon & \nabla \left(u + t \langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle \right)^\top \\ \nabla \left(u + t \langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle \right) & \phi_{zz} + tu_{zz} + \frac{t^2}{2} \left(\langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle \right)_{zz} + \varepsilon \end{bmatrix}$$

We must show that for t small enough this matrix is positive semi-definite. Let $\tilde{x} = (x_0,x)$ where $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$ then we must show that

$$\tilde{x}^\top (H\psi_\varepsilon)\tilde{x} \geq 0 \quad \text{for small enough } t \text{ and any } \tilde{x} \neq 0.$$

It is easy to see that it is sufficient to prove this inequality for $\tilde{x}$ such that $\|\tilde{x}\| = 1$. By simple calculations one finds that

$$\begin{aligned} \tilde{x}^\top (H\psi_\varepsilon)\tilde{x} =& x_0^2 \langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle + \varepsilon x_0^2 + 2x_0 \langle \nabla u, x \rangle \\ &+ t2x_0 \langle \nabla \left(\langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle \right), x \rangle + x^\top \phi_{zz} x \\ &+ tx^\top u_{zz} x + \frac{t^2}{2} x^\top \left(\langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle \right)_{zz} x + \varepsilon \|x\|^2 \\ =& x_0^2 \langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle + \varepsilon x_0^2 + 2x_0 \langle \nabla u, x \rangle \\ &+ x^\top \phi_{zz} x + \varepsilon \|x\|^2 + At^2 + Bt \end{aligned}$$

the exact expressions of A and B are uninteresting since we at least know they are bounded, $u \in C^1$ with compact support, $\|\tilde{x}\| = 1$ and ϕ_{zz} is positive definite. Now by assuming that ϕ_{zz} is diagonal we can further simplify this and find that

$$\begin{aligned}
\tilde{x}^\top (H\psi)\tilde{x} =& \sum_{i=1}^n x_i^2 \frac{\partial^2 \phi}{\partial z_i^2} + x_0^2 \left\{ \sum_{i=1}^n \left(\frac{\partial u}{\partial z_i}\right)^2 \left(\frac{\partial^2 \phi}{\partial z_i^2}\right)^{-1} \right\} \\
&+ 2x_0 \left(\sum_{i=1}^n x_i \frac{\partial u}{\partial z_i} \right) + \varepsilon(x_0^2 + \|x\|^2) + At^2 + Bt \\
=& \sum_{i=1}^n \left(\frac{\partial^2 \phi}{\partial z_i^2}\right)^{-1} \left\{ x_i^2 \left(\frac{\partial^2 \phi}{\partial z_i^2}\right)^2 + x_0^2 \left(\frac{\partial u}{\partial z_i}\right)^2 \right. \\
&\left. + 2x_0 x_i \frac{\partial u}{\partial z_i} \frac{\partial^2 \phi}{\partial z_i^2} \right\} + \varepsilon + At^2 + Bt \\
=& \sum_{i=1}^n \left(\frac{\partial^2 \phi}{\partial z_i^2}\right)^{-1} \left\{ x_i \left(\frac{\partial^2 \phi}{\partial z_i^2}\right) + x_0 \left(\frac{\partial u}{\partial z_i}\right) \right\}^2 \\
&+ \varepsilon + At^2 + Bt
\end{aligned}$$

thus since the sum is non-negative and A and B are bounded we must have that $\forall \varepsilon > 0$ $\exists \delta > 0$ such that for $-\delta \le t \le \delta$ this is non-negative for all $\tilde{x}$.

By letting $\psi_\varepsilon = \infty$ outside $-\delta \le t \le \delta$ we obtain a convex function. We also have that ψ_ε is C^2 with respect to t in a neighborhood of $t = 0$. By Prékopa we know that

$$\int_{\mathbb{R}^n} e^{-\psi_\varepsilon(t,z)} dz$$

is logarithmically concave and thus by the same calculations as earlier

$$\begin{aligned}
0 \le& \left[-\log \left(\int e^{-\psi_\varepsilon(t,z)} dz \right) \right]_{tt} \\
=& \frac{\int (\psi_\varepsilon)_{tt} e^{-\psi_\varepsilon} dz}{\int e^{-\psi_\varepsilon} dz} - \frac{\int \left((\psi_\varepsilon)_t - \widehat{(\psi_\varepsilon)_t} \right)^2 e^{-\psi_\varepsilon} dz}{\int e^{-\psi_\varepsilon} dz}
\end{aligned}$$

at $t = 0$ this inequality yields

$$\int (\psi_\varepsilon)_{tt}(0,z) e^{-\psi_\varepsilon(0,z)} dz \ge \int \left((\psi_\varepsilon)_t(0,z) - \widehat{(\psi_\varepsilon)_t(0,z)} \right)^2 e^{-\psi_\varepsilon(0,z)} dz$$

$$\int \left(\langle \nabla u, \phi_{zz}^{-1} \nabla u \rangle + \varepsilon \right) e^{-\phi(z) - \varepsilon\|z\|^2} dz \ge \int \left(u(z) - \widehat{u(z)} \right)^2 e^{-\phi(z) - \varepsilon\|z\|^2} dz.$$

But since this holds for arbitrary $\varepsilon > 0$ and the integrals change continuously with respect to ε it must hold also for $\varepsilon = 0$ which is exactly the Brascamp-Lieb inequality (2.6).

2.2.2 Prékopa-Leindler's inequality

A special case of the Prékopa-Leindler inequality was shown on page 24 by applying the Brunn-Minkowski inequality and some basic measure theoretical arguments. In the following section it will be shown that the general Brunn-Minkowski follows from Prékopa-Leindler's theorem, this proof can also be found in [3].

We begin by stating the theorem and providing some motivation for the study of this inequality. Further a simple proof is provided using arguments similar to techniques used in the theory of optimal transportation.

Theorem 2.4 (Prékopa-Leindler's theorem) *Let f,g and h be non-negative and integrable functions from $\mathbb{R}^n \to \mathbb{R}$ and $\lambda \in (0,1)$ such that*

$$h((1-\lambda)x+\lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}, \quad \forall x,y \in \mathbb{R}^n. \tag{2.7}$$

Then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda}.$$

The condition (2.7) is similar to the definition of a function being log concave. If one considers a log concave function $F : \mathbb{R}^{n+m} \to \mathbb{R}$ and as in Prékopa's theorem decomposes the argument into two parts $F(z) = F(x,y)$ such that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then one can see that for $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ the following holds

$$\begin{aligned} F((1-\lambda)x_1+\lambda x_2,(1-\lambda)y_1+\lambda y_2) =& F((1-\lambda)z_1+\lambda z_2) \\ \geq& F(z_1)^{1-\lambda}F(z_2)^{\lambda} \\ =& F(x_1,y_1)^{1-\lambda}F(x_2,y_2)^{\lambda}. \end{aligned}$$

By defining $F(x,y) = F_x(y)$ and $(1-\lambda)x_1 + \lambda x_2 = w$ this can be written as

$$F_w((1-\lambda)y_1+\lambda y_2) \geq F_{x_1}(y_1)^{1-\lambda}F_{x_2}(y_2)^{\lambda}.$$

Thus (2.7) is satisfied and it is obtained that

$$\int_{\mathbb{R}^m} F_w(y)dy \geq \left(\int_{\mathbb{R}^m} F_{x_1}(y)dy\right)^{1-\lambda} \left(\int_{\mathbb{R}^m} F_{x_2}(y)dy\right)^{\lambda}.$$

Thus by defining the function

$$G(x) = \int_{\mathbb{R}^m} F_x(y)dy$$

one obtains from the above that

$$G(w) = G((1-\lambda)x_1+\lambda x_2) \geq G(x_1)^{1-\lambda}G(x_2)^{\lambda}$$

and which states precisely that the function G is log concave. Thus we have found Prékopa's theorem (theorem 2.2).

We now continue by providing a self-contained proof of (2.2). It should be noted that the proof, if restricted to indicator functions for convex sets, is very similar to the proof of Brunn-Minkowski inequality by Knesser and Süss.

Proof (Prékopa-Leindler's theorem) As before the proof is done by induction over the dimension n. We first present a proof in the case $n = 1$.

Without loss of generality we assume that the integrals of f and g are positive. We then let $u,v : (0,1) \to \mathbb{R}$ be defined as the smallest numbers such that

$$\frac{1}{F}\int_{-\infty}^{u(t)} f(x)dx = \frac{1}{G}\int_{-\infty}^{v(t)} g(x)dx = t. \tag{2.8}$$

where F, G denotes the integrals of f, g respectively. Note that u, v very well may be discontinuous but they are strictly increasing and thus almost everywhere differentiable. Also define

$$w(t) = (1-\lambda)u(t) + \lambda v(t).$$

By taking the derivative of (2.8) with respect to t the following is obtained

$$\frac{f(u(t))u'(t)}{F} = \frac{g(v(t))v'(t)}{G} = 1.$$

Then where $f(u(t)) \neq 0$ and $g(v(t)) \neq 0$ one finds that

$$\begin{aligned} w'(t) =& (1-\lambda)u'(t) + \lambda v'(t) \\ \geq& u'(t)^{1-\lambda} v'(t)^{\lambda} \\ =& \left(\frac{F}{f(u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda} \end{aligned}$$

here the first inequality is due to the arithmetic-geometric mean inequality. One obtains

$$\begin{aligned} \int_{\mathbb{R}} h(x)dx &\geq \int_0^1 h(w(t))w'(t)dt \\ &\geq \int_0^1 h((1-\lambda)u(t) + \lambda v(t))\left(\frac{F}{f(u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda} dt \\ &\geq \int_0^1 f(u(t))^{1-\lambda} g(v(t))^{\lambda}\left(\frac{F}{f(u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda} dt = F^{1-\lambda}G^{\lambda}. \end{aligned}$$

Thus the proof is complete for $n = 1$.

To proceed with the proof we assume that the theorem is proved in the case of $(n-1)$-dimensions and use this together with use of the Fubini-Tonelli theorem.

For any $s \in \mathbb{R}$ define h_s as the function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ such that $h_s(z) = h(z,s),\ \forall z \in \mathbb{R}^{n-1}$. In the same manner define both f_s and g_s. Then for any $x,y \in \mathbb{R}^{n-1}$ and $a,b,c \in \mathbb{R}^{n-1}$ such that $c = (1-\lambda)a + \lambda b$ the following holds

$$\begin{aligned} h_c((1-\lambda)x + \lambda y) =& h((1-\lambda)x + \lambda y, (1-\lambda)a + \lambda b) \\ =& h((1-\lambda)(x,a) + \lambda(y,b)) \\ \geq& f(x,a)^{1-\lambda} g(y,b)^{\lambda} \\ =& f_a(x)^{1-\lambda} g_b(y)^{\lambda}. \end{aligned}$$

Thus (2.7) holds for h_c,f_a,g_b and thus by theorem 2.4 in $(n-1)$ dimensions it holds that

$$\int_{\mathbb{R}^{n-1}} h_c(x)dx \geq \left(\int_{\mathbb{R}^{n-1}} f_a(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_b(x)dx\right)^{\lambda}.$$

Now letting

$$\begin{aligned} H(c) &= \int_{\mathbb{R}^{n-1}} h_c(x)dx \\ F(a) &= \int_{\mathbb{R}^{n-1}} f_a(x)dx \\ G(b) &= \int_{\mathbb{R}^{n-1}} g_b(x)dx \end{aligned}$$

it holds that $H(c) = H((1-\lambda)a + \lambda b) \geq F(a)^{1-\lambda} G(b)^{\lambda}$. Therefore using the Fubini-Tonelli theorem and that theorem 2.4 is true for $n = 1$

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} h_c(z)dzdc \\ &= \int_{\mathbb{R}} H(c)dc \\ &\geq \left(\int_{\mathbb{R}} F(a)da\right)^{1-\lambda} \left(\int_{\mathbb{R}} G(b)db\right)^{\lambda} \\ &= \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda} \end{aligned}$$

this concludes the induction argument and the theorem is proved.

∎

2.3 The general Brunn-Minkowski inequality

We state a generalized version of the Brunn-Minkowski inequality and provide a simple proof using Prékopa-Leindler's inequality introduced in the previous section. It is also demonstrated that the generalized Brunn-Minkowski implies Prékopa-Leindler's inequality.

Theorem 2.5 (General Brunn-Minkowski) *For K,L measurable, nonempty subsets of $\mathbb{R}^n$ such that $(1-\lambda)K+\lambda L$ is measurable then for $\lambda \in [0,1]$*

$$V_n\left((1-\lambda)K+\lambda L\right)^{1/n} \geq (1-\lambda)V_n(K)^{1/n}+\lambda V_n(L)^{1/n}.$$

One should in the statement of the theorem note that the requirement for $(1-\lambda)K+\lambda L$ to be measurable is necessary. In the case where K and L are Borel sets then the measurability of the sum follows since it is the image of the continuous map

$$\begin{aligned} T : K \times L &\to (1-\lambda)K+\lambda L \\ (x,y) &\mapsto (1-\lambda)x+\lambda y. \end{aligned}$$

In the following sections it will be shown how this powerful but simple geometric inequality is linked with inequalities from analysis. As a first example the following simple argument, suggested by Bo'az Klartag at Tel Aviv University, shows that the general Brunn-Minkowski inequality implies a special case ($\lambda = 1/2$) of Prékopa-Leindler's inequality in one dimension, to enhance this argument to any dimension and general λ is not difficult.

Suppose that for non-negative functions h,f,u from $\mathbb{R}$ to itself we have that

$$h\left(\frac{x+y}{2}\right) \geq \sqrt{f(x)g(y)} \quad \forall x,y \in \mathbb{R} \tag{2.9}$$

then it holds that

$$\{x : h(x) > t\} \supseteq \frac{\{x : f(x) > t\} + \{x : g(x) > t\}}{2}.$$

The inclusion follows from that if $y \in (\{x : f(x) > t\} + \{x : g(x) > t\})/2$ then y can be decomposed as $y = (a+b)/2$ where $a \in \{f > t\}$ and $b \in \{g > t\}$ then by the assumption (2.9) we have that

$$h(y) = h\left(\frac{a+b}{2}\right) \geq \sqrt{f(a)g(b)} > t$$

Thus we have that

$$\begin{aligned} \mu(\{x : h(x) > t\}) \geq& \mu\left(\frac{\{x : f(x) > t\} + \{x : g(x) > t\}}{2}\right) \\ \geq& \frac{1}{2}\mu(\{x : f(x) > t\}) + \frac{1}{2}\mu(\{x : g(x) > t\}). \end{aligned}$$

where the second inequality is by the Brunn-Minkowski inequality, and μ as before denotes the Lebesgue measure on $\mathbb{R}$. From basic integration theory we know that for non-negative integrable functions F

$$\int_{\mathbb{R}} \mu(\{F > t\})dt = \int F d\mu.$$

Therefore one finds that

$$\begin{aligned}\int h\,d\mu &\geq \frac{1}{2}\int f\,d\mu + \frac{1}{2}\int g\,d\mu \\ &\geq \sqrt{\int f\,d\mu \int g\,d\mu}\end{aligned}$$

which is the Prékopa-Leindler inequality. We now continye with a proof of theorem 2.5 using the Prékopa-Leindler inequality. One can therefore in some sense consider these inequalities as functional and geometrical versions of the same inequality.

Proof (General Brunn-Minkowski) Let K, L be bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)K+\lambda L$ is measurable. Then let $f = \mathbb{1}_K$, $g = \mathbb{1}_Y$ and $h = \mathbb{1}_{(1-\lambda)K+\lambda L}$, where $\mathbb{1}_X$ denotes the indicator function as defined on page 7. Then we have that the following holds

$$h((1-\lambda)x+\lambda y) \geq f(x)^{1-\lambda} g(x)^{\lambda}$$

which is exactly the condition for Prékopa-Leindler's theorem. This holds since the right hand side one has that

$$f(x)^{1-\lambda}g(y)^{\lambda} = \begin{cases} 1 & \text{if } x \in K \text{ and } y \in L \\ 0 & \text{otherwise} \end{cases}$$

but this is exactly the characteristic function of $(1-\lambda)K+\lambda L$, which is the left hand side.

Theorem 2.4 now states that

$$\begin{aligned} V_n\left((1-\lambda)K+\lambda L\right) &= \int_{\mathbb{R}^n} 1_{(1-\lambda)K+\lambda L}(x)dx \\ &\geq \left(\int_{\mathbb{R}^n} 1_K(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} 1_L(x)dx\right)^{\lambda} \\ &= V_n(K)^{1-\lambda}V_n(L)^{\lambda}. \end{aligned} \tag{2.10}$$

This is equivalent to the general Brunn-Minkowski inequality. If we apply (2.10) to the sets $V_n(K)^{-1/n}K$ and $V_n(L)^{-1/n}L$ with λ as

$$\frac{\lambda' V_n(L)^{1/n}}{(1-\lambda')V_n(K)^{1/n}+\lambda' V_n(L)^{1/n}}$$

one obtains the Brunn-Minkowski inequality.

$$\begin{aligned} V_n((1-\lambda)V_n(K)^{-1/n}K+\lambda V_n(L)^{-1/n}L) &\geq 1 \\ V_n\left(\frac{(1-\lambda')K+\lambda' L}{(1-\lambda')V_n(K)^{1/n}+\lambda' V_n(L)^{1/n}}\right) &\geq 1 \\ V_n((1-\lambda')K+\lambda' L) &\geq \left((1-\lambda')V_n(K)^{1/n}+\lambda' V_n(L)^{1/n}\right)^n \\ V_n((1-\lambda')K+\lambda' L)^{1/n} &\geq (1-\lambda')V_n(K)^{1/n}+\lambda' V_n(L)^{1/n}. \end{aligned}$$

2.4 Applications of Brunn-Minkowski

There exists a large number of applications of the Brunn-Minkowski inequality and its functional counterparts, we have already hinted towards and demonstrated several applications. In this section we use Prékopa-Leindler's inequality to obtain a logarithmic Sobolev inequality. For a more exhaustive overview of applications the reader is referred to the survey by Gardner [3].

2.4.1 The logarithmic Sobolev inequality

By applying the Prékopa-Leindler inequality a logarithmic Sobolev inequality can be deduced. In the general form that is provided here the inequality was published by Bobkov and Ledoux in the article [1]. Let E be the normed space $(\mathbb{R}^n, \|\cdot\|)$ and $E^* = (\mathbb{R}^n, \|\cdot\|_*)$ its dual, and let μ be a log-concave measure with density $\rho e^{-\phi(x)}$, where $\phi(x) : \Omega \to \mathbb{R}$ is a strongly convex function with parameter $c > 0$ defined on the open convex subset $\Omega \subset E$.

Theorem 2.6 (The logarithmic Sobolev inequality) *For $f \in C^\infty(\Omega)$ and a measure μ satisfying the above conditions it holds that*

$$\mathrm{Ent}_\mu(f^2) \leq \frac{2}{c} \int \|\nabla f\|_*^2 d\mu. \tag{2.11}$$

Ent_μ denotes the entropy of a function which is defined as

$$\mathrm{Ent}_\mu(f) = \int f \log f d\mu - \int f d\mu \cdot \log \int f d\mu.$$

Proof (Bobkov and Ledoux [1]) Assume that we can write $f^2 = e^g$ for some compactly supported $g \in C_b^\infty(\Omega)$, i.e. g smooth and all its partial derivatives are bounded in Ω. Then we find the statement by applying the Prékopa-Leindler inequality to the functions

$$\begin{aligned} u(x) &= e^{g(x)/\lambda - \phi(x)} \\ v(y) &= e^{-\phi(y)} \\ w(z) &= e^{g_\lambda(y) - \phi(z)} \end{aligned}$$

where

$$g_\lambda(z) = \sup_{x,y\in\Omega} \left\{ g(x) - (\lambda\phi(x) + (1-\lambda)\phi(y) - \phi(\lambda x + (1-\lambda)y)) : \ z = \lambda x + (1-\lambda)y \right\}.$$

From the requirement that ϕ is strongly convex one finds that

$$g_\lambda(z) \leq \sup_{x,y\in\Omega} \left\{ g(x) - \frac{c(1-\lambda)\lambda}{2} \|x-y\|^2 : \ z = \lambda x + (1-\lambda)y \right\} \tag{2.12}$$

which in turn implies that $w(\lambda x + (1-\lambda)y) \geq u(x)^\lambda v(y)^{(1-\lambda)}$. Thus we can apply the Prékopa-Leindler inequality (theorem 2.4) and find that

$$\begin{aligned}
\int e^{g_\lambda} d\mu &= \int \rho e^{g_\lambda(x)-\phi(x)} dx \\
&= \rho \int w(x) dx \\
&\geq \rho \left(\int u(x) dx \right)^\lambda \left(\int v(x) dx \right)^{(1-\lambda)} \\
&= \rho \left(\int e^{g(x)/\lambda - \phi(x)} dx \right)^\lambda \left(\int e^{-\phi(x)} dx \right)^{(1-\lambda)} \\
&= \left(\int e^{g(x)/\lambda} \rho e^{-\phi(x)} dx \right)^\lambda \left(\int \rho e^{-\phi(x)} dx \right)^{(1-\lambda)} \\
&= \left(\int e^{g(x)/\lambda} d\mu \right)^\lambda . \qquad (2.13)
\end{aligned}$$

The idea is to let $\lambda \to 1$ and obtain (2.11). We expand (2.13) around $\lambda = 1$ using Taylor's formula yielding

$$\left(\int e^{g(x)/\lambda} d\mu \right)^\lambda = h(\lambda) = h(1) + h'(1)(\lambda - 1) + O((\lambda-1)^2). \qquad (2.14)$$

If one writes h on exponential form $h(\lambda) = e^{\lambda \log \int e^{g/\lambda} d\mu}$ it is fairly easy to see that

$$h'(\lambda) = \left(\int e^{g/\lambda} d\mu \right)^\lambda \left(\log \int e^{g/\lambda} d\mu - \frac{\int e^{q/\lambda} d\mu}{\lambda \int e^{g/\lambda} d\mu} \right)$$

and with $\lambda = 1$ this yields finds that $h'(1) = -\mathrm{Ent}_\mu(e^g)$. Thus (2.14) becomes

$$\left(\int e^{g(x)/\lambda} d\mu \right)^\lambda = \int e^g d\mu + (1-\lambda)\mathrm{Ent}_\mu(e^g) + O((1-\lambda)^2). \qquad (2.15)$$

Let us now consider the left hand side of (2.13). Using that $z = \lambda x + (1-\lambda)y$ and rewriting the inequality (2.12) with $h = z - y$ and $\eta = \frac{1-\lambda}{\lambda}$ yields that

$$g_\lambda(z) \leq \sup_{h \in E} \left\{ g(z + \eta h) - \frac{c\eta}{2} \|h\|^2 \right\}. \qquad (2.16)$$

Note that the supremum is taken over the entire space E which is a larger set then in inequality (2.12). Taylor's formula yields, independently of z, that

$$g(z + \eta h) = g(z) + \eta \langle \nabla g(z), h \rangle + \|h\|^2 O(\eta^2)$$

for η close to zero. We insert this into (2.16) obtaining

$$g_\lambda(z) \leq g(z) + \eta \sup_{h\in E}\left\{\langle \nabla g(z), h\rangle - (\frac{c}{2} - C\eta)\|h\|^2\right\}$$

where we utilize the fact that $|O(\eta^2)| \leq C\eta^2$ for some constant C independent of z. For any $h \in \mathbb{R}^n$ we can write it as $h = \alpha e$ where $\alpha \geq 0$ and $e \in S^{n-1}$. Writing $\theta = c - 2\eta C$ one finds that

$$\begin{aligned} g_\lambda(z) \leq & g(z) + \eta \sup_{\alpha\geq 0,\ e\in S^{n-1}} \sup \left\{\alpha\langle \nabla g(z), e\rangle - \theta\frac{\alpha^2}{2}\right\} \\ = & g(z) + \eta \sup_{\alpha\geq 0}\left\{\alpha\|\nabla g(z)\|_* - \theta\frac{\alpha^2}{2}\right\} \\ = & g(z) + \eta\frac{\|\nabla g(z)\|_*^2}{2\theta}. \end{aligned}$$

Since $\theta = c - 2C\eta > 0$ for small enough η one obtains that

$$g_\lambda(z) \leq g(z) + \frac{\eta}{2c}\|\nabla g(z)\|_*^2 + O(\eta^2)$$

uniformly for all $z \in \Omega$. Again using Taylor's formula for the exponential function

$$e^y = e^x + e^x(y - x) + O((y - x)^2)$$

one obtains that

$$\begin{aligned} e^{g_\lambda(z)} = & e^{g(z)} + e^{g(z)}(g_\lambda(z) - g(z)) + O((g_\lambda(z) - g(z))^2) \\ \leq & e^{g(z)} + \frac{\eta}{2c}\|\nabla g(z)\|_*^2 e^{g(z)} + O(\eta^2). \end{aligned}$$

Integrating both sides

$$\int e^{g_\lambda(z)} d\mu \leq \int e^{g(z)} d\mu + \frac{\eta}{2c}\int \|\nabla g(z)\|_*^2 e^{g(z)} d\mu + O(\eta^2).$$

Finally combining this with (2.13) and (2.15) yields that

$$\mathrm{Ent}_\mu(e^g) \leq \frac{1}{2\lambda c}\int \|\nabla g\|_*^2 e^g d\mu + O(1 - \lambda)$$

and letting $\lambda \to 1$

$$\mathrm{Ent}_\mu(e^g) \leq \frac{1}{2c}\int \|\nabla g\|_*^2 e^g d\mu.$$

Since $e^g = f^2$ one finds that $2\nabla f = e^{g/2}\nabla g$, inserting this into the equation above reveals the logarithmic Sobolev inequality and the proof is complete.

■

3

Mixed Volumes

THE study of Minkowski addition and volume is what leads to the Brunn-Minkowski theory, a central part of this theory revolves around the concept of mixed volumes [6]. In this chapter we introduce and prove some basic statements and properties of mixed volumes.

Definition 3.1 *The mixed volume is defined as the unique, symmetric and multilinear form $W : (\mathcal{K}^n)^n \to \mathbb{R}^+$ satisfying*

$$V_n(\sum_{i=1}^{r} \lambda_i K_i) = \sum_{i_1=1}^{r} ... \sum_{i_n=1}^{r} W(K_{i_1},...,K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

That there indeed exists such a multilinear form is not trivial and was proved by Minkowski, a proof of this will be presented in the next section. The case where only two distinct convex bodies are concerned is often of special interest and therefore the following notation is introduced

$$W_m(K_1,K_2) = W(\underbrace{K_1,...,K_1}_{n-m},\underbrace{K_2,...,K_2}_{m}). \tag{3.1}$$

For $m = 0$ it should be noted that this notation can be slightly confusing since the quantity does not depend on the second body, $W_(K_1,K_2) = W(K_1,...,K_1)$. It is in fact clear from the definition of mixed volumes that the quantity $W_0(K_1,K_2)$ equal to the volume of the first body i.e. $V_n(K_1)$.

In the notation above the volume of the Minkowski sum of two convex sets $\lambda_1 K_1$ and $\lambda_2 K_2$ takes the form

$$V(\lambda_1 K_1 + \lambda_2 K_2) = \sum_{m=0}^{n} \binom{n}{m} \lambda_1^{n-m} \lambda_2^m W_m(K_1,K_2).$$

If we again consider the definition of surface area it can be simplified using the notation and definitions above

$$\begin{aligned}
S_n(K) =& V_{n-1}(\partial K) = \lim_{\varepsilon\to 0}\frac{V_n(K+\varepsilon B)-V_n(K)}{\varepsilon}\\
=& \lim_{\varepsilon\to 0}\frac{\sum_{m=0}^{n}\binom{n}{m}\varepsilon^m W_m(K,B)-W_0(K,B)}{\varepsilon}\\
=& \lim_{\varepsilon\to 0}\sum_{m=1}^{n}\binom{n}{m}\varepsilon^{m-1}W_m(K,B)\\
=& \lim_{\varepsilon\to 0}\binom{n}{1}W_1(K,B)+\sum_{m=2}^{n}\binom{n}{m}\varepsilon^{m-1}W_m(K,B)\\
=& nW_1(K,B).
\end{aligned}$$

Thus the surface area is simply a specific mixed volume. This can be used to give an even more simplified proof of the isoperimetric inequality then that given earlier in the thesis.

3.1 The volume as a polynomial

In the following section we prove that $V_n(\sum_{i=1}^r \lambda_i K_i)$ is a homogeneous n-th degree polynomial in $\lambda_1,...,\lambda_r$ with the coefficients in the form of mixed volumes, as usual $K_1,...,K_r \in \mathcal{K}^n$ and all lambda are non-negative.

We begin by proving the non-trivial statement that $V_n(\sum \lambda_i K_i)$ is a homogeneous polynomial. Most classical proofs of this start by proving the polynomiality when restricted to convex polytopes and in a limiting argument one can conclude the proof for all convex bodies. Versions of the classical proof can be found in both Busemann and Webster [6, 7]. However the proof presented here instead approximates arbitrary $K \in \mathcal{K}^n$ using strictly convex bodies with smooth boundary and proves the statement through basic properties of the support function and differential forms.

Theorem 3.1 *For all $\lambda_1,...,\lambda_r > 0$ and any strictly convex sets $K_1,...,K_r \in \mathcal{K}^n_{reg}$ the volume $V_n(\sum_{i=1}^r \lambda_i A_i)$ is a homogeneous polynomial of degree n.*

Proof For differential forms we have that

$$d\left(\sum_{i=1}^{r} x_i\widehat{dx_i}\right) = n dx_1\wedge ...\wedge dx_n = ndx.$$

Here $\widehat{dx_i}$ denotes the wedge product of all dx_j such that $j\neq i$, i.e. $\widehat{dx_i} = (-1)^{i-1}dx_1\wedge ...\wedge dx_{i-1}\wedge dx_{i+1}\wedge ...\wedge dx_n$. By Stokes theorem one knows that

$$V_n(K) = \int_K dx = \frac{1}{n}\int_{\partial K}\sum_{i=1}^{n} x_i\widehat{dx_i}.$$

For proofs of the properties used here and definitions concerning differential forms the reader is referred to Spivak's book Calculus on Manifolds [8]. For strictly convex sets with smooth boundary the map∇h_K maps $S^{n-1} \to \partial K$ bijectively and is almost everywhere differentiable, this is stated and proved as lemma B.1 in appendix B. Using the gradient map $\nabla h_K : S^{n-1} \to \partial K$ we can pullback the integral to S^{n-1}.

$$V_n(K) = \frac{1}{n} \int_{S^{n-1}} \sum_{i=1}^{n} \frac{\partial h_K}{\partial x_i} d\widehat{\frac{\partial h_K}{\partial x_i}}.$$

We here denote the support function of K_i simply by H_i. Then using the equation above and that $h_{\sum \lambda_i K_i} = \sum \lambda_i h_{K_i}$ we obtain that

$$\int_{\sum_j \lambda_j K_j} dx = \frac{1}{n} \int_{S^{n-1}} \sum_{i=1}^{n} \left(\sum_j \lambda_j \frac{\partial H_j}{\partial x_i} \right) \left(d \sum_j \lambda_j \frac{\partial H_j}{\partial x_i} \right)^{\wedge}$$

here $(\cdot)^{\wedge}$ is used to denote $\widehat{(\cdot)}$. We therefore have that $V_n(\lambda_1 K_1 + ... + \lambda_r K_r)$ is a homogeneous n-th degree polynomial in $\lambda_1,...,\lambda_r \geq 0$. To extend this proof from the space of strictly convex bodies with smooth boundary to the entire space of non-empty convex bodies is simply a matter of approximating general convex sets with strictly convex sets.

∎

3.2 Properties of mixed volumes

In this section some important properties concerning mixed volumes are presented. We begin by proving the multilinearity of W.

Theorem 3.2 *For $K_1',K_1,K_2,...,K_n$ belong to $\mathcal{K}_0^n$ and $\alpha,\beta \geq 0$*

$$W(\alpha K_1' + \beta K_1,K_2,...,K_n) = \alpha W(K_1',K_2,...,K_n) + \beta W(K_1,K_2,...,K_n).$$

From the symmetry of W it is clear that this implies linearity of W in all of its arguments.

Proof By basic properties of the Minkowski sum it is clear that

$$V_n(\lambda_1(\alpha K_1' + \beta K_1) + \lambda_2 K_2 + ... + \lambda_n K_n) =$$
$$V_n((\lambda_1\alpha)K_1' + (\lambda_1\beta)K_1 + \lambda_2 K_2 + ... + \lambda_n K_n).$$

By equating the coefficients of the term $\lambda_1 \cdots \lambda_n$ in the polynomials corresponding to these volumes one obtains the statement

$$W(\alpha K_1' + \beta K_1,K_2,...,K_n) = \alpha W(K_1',K_2,...,K_n) + \beta W(K_1,K_2,...,K_n).$$

∎

Theorem 3.3 (Translation invariance of mixed volumes) *For sets* $K'_1,K_1,K_2,...,K_n \in \mathcal{K}^n$ *such that* $K'_1 = K_1 + a$ *any* $a \in \mathbb{R}^n$

$$W(K'_1,K_2,...,K_n) = W(K_1,K_2,...,K_n).$$

This theorem follows from the translation invariance of V_n and the fact that a translation of one of the sets K_i results in a translation of $\sum \lambda_i K_i$.

Theorem 3.4 (Continuity of mixed volumes) *For every* $j = 1,...,n$ *let* $\{K_j^i\}_{i=1}^{\infty}$ *be a sequence of sets in* $\mathcal{K}^n$ *such that* $K_j^i \to K_j \in \mathcal{K}^n$*, with respect to Hausdorff metric on* $\mathcal{K}^n$*, as* i *tends to infinity. Then*

$$W(K_1^i,...,K_n^i) \to W(K_1,...,K_n) \quad \textit{when } i \to \infty.$$

The proof of this theorem is fairly simple. The main component of the proof is the fact that if a sequence of polynomials $\{P_i\}_1^{\infty}$ converges to a polynomial Q, $\forall x \geq 0$, it follows that the coefficients of P_i converges to the coefficients of Q. We continue by stating a number of theorems concerning mixed volumes without proof, however the proofs can be found in either Schneider or Webster's books [2, 7].

Theorem 3.5 (Monotonicity) *For* $K'_1,K_1,K_2,...,K_n \in \mathcal{K}^n$ *such that* $K'_1 \subseteq K_1$ *it holds that*

$$W(K'_1,K_2,...,K_n) \leq W(K_1,K_2,...,K_n).$$

The equality cases in this inequality are not fully known and the inequality can not be made strict even if we require that $K'_1 \subsetneq K_1$. This theorem also implies the non-negativity of mixed volumes. Assume all sets contain zero and by repeated use of the above inequality one finds that $0 = W(\{0\},...,\{0\}) \leq W(K_1,...,K_n)$. The next theorem is not stated explicitly in any of the literature specified above but follows from the multilinearity of W together with the property of the Minkowski sum that

$$(K'_1 \cup K_1) + (K'_1 \cap K_1) = K'_1 + K_1$$

which is proved in Schneider [2].

Theorem 3.6 *For* $K'_1,K_1,K_2,...,K_n \in \mathcal{K}^n$ *such that* $K_1 \cup K'_1 \in \mathcal{K}^n$ *it holds that*

$$W(K'_1 \cup K_1,K_2,...,K_n) = W(K'_1,K_2,...,K_n) + W(K_1,K_2,...,K_n) - W(K'_1 \cap K_1,K_2,...,K_n).$$

The next theorem concerns the possibility of extending the mixed volume function and its properties from $\mathcal{K}^n$ to some larger space of subsets of $\mathbb{R}^n$, unfortunately it shows that this is not possible.

Theorem 3.7 (Extension of mixed volumes) *Let* $\mathcal{K}'$ *denote a class of compact subsets of* $\mathbb{R}^n$ *such that* $\mathcal{K}^n \subseteq \mathcal{K}'$ *and for any* $K,L \in \mathcal{K}'$ $K + L \in \mathcal{K}'$*. Then if a function* W' *exists such that* $W' : (\mathcal{K}')^n \to \mathbb{R}$ *and* W' *is Minkowski-additive in each of its arguments and for any* $K \in \mathcal{K}'$ $W'(K,...,K) = V_n(K)$ *then* $\mathcal{K}' = \mathcal{K}^n$*.*

Thus if one aims to extend the notion of mixed volumes to some larger space of sets one cannot preserve all of its essential properties. A complete proof of this theorem can be found in Schneider [2].

3.3 Minkowski's inequalities for mixed volumes

In this section two simple inequalities due to Minkowski concerning mixed volumes are presented. The second inequality is a stepping stone towards further inequalities of similar quadratic form, the most general such inequality is due to Aleksandrov and Fenchel and is the main focus of the next chapter. The proofs of the inequalities found here may be found in most literature treating basics of mixed volumes. For further reading Schneider [2] is highly recommended.

Theorem 3.8 (Minkowski's first inequality) *For $K,L \in \mathcal{K}^n$*

$$W_1(K,L) \geq V_n(K)^{(n-1)/n} V_n(L)^{1/n}$$

with equality if and only if K and L are homothetic.

We again use the notation introduced in (3.1). The proof of this inequality relies on the Brunn-Minkowski inequality for convex sets.

Proof Let $f(\lambda) = V_n(K_\lambda)^{1/n}$ where $K_\lambda = (1-\lambda)K_0 + \lambda K_1)$ for $K_0,K_1 \in \mathcal{K}^n$. By the Brunn-Minkowski inequality f is concave on (0,1), thus $f'(0) \geq f(1) - f(0)$.

$$f'(0) = \frac{1}{n} V_n(K_0)^{(1-n)/n} \frac{\partial}{\partial \lambda} \left[V_n(K_\lambda)\right](0).$$

Using the polynomial expression of $V_n(K_\lambda)$ we find that

$$\begin{aligned}
\frac{\partial}{\partial \lambda} \left[V_n(K_\lambda)\right](0) &= \lim_{\varepsilon \to 0} \frac{V_n(K_\varepsilon) - V_n(K_0)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\sum_{i=0}^n \binom{n}{i} (1-\varepsilon)^{n-i} \varepsilon^i W_i(K_0,K_1) - V_n(K_0)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \sum_{i=1}^n \binom{n}{i} (1-\varepsilon)^{n-i} \varepsilon^{i-1} W_i(K_0,K_1) + \frac{(1-\varepsilon)^n - 1}{\varepsilon} V_n(K_0) \\
&= n W_1(K_0,K_1) - n V_n(K_0)
\end{aligned}$$

and hence

$$f'(0) = \frac{W_1(K_0,K_1) - V_n(K_0)}{V_n(K_0)^{(n-1)/n}}.$$

Inserting the expression above into the inequality $f'(0) \geq f(1) - f(0)$ yields

$$W_1(K_0,K_1) \geq V_n(K_0)^{(n-1)/n} V_n(K_1)^{1/n}$$

which is exactly the statement. Equality implies the linearity of f and thus coincides with the equality cases in the Brunn-Minkowski's inequality.

∎

Further if one considers the second derivative of this function one finds a further statement that is Minkowski's second inequality.

Theorem 3.9 (Minkowski's second inequality) *For* $K,L \in \mathcal{K}^n$

$$W_1(K,L)^2 \geq V_n(K)W_2(K,L). \tag{3.2}$$

No exact conditions for equality are known, but as above equality holds in the case of homothetic bodies.

Proof As above let $f(\lambda) = V_n(K_\lambda)^{1/n}$ where $K_\lambda = (1-\lambda)K_0 + \lambda K_1)$ for $K_0,K_1 \in \mathcal{K}^n$. By the Brunn-Minkowski inequality f is concave on (0,1), this implies that $f''(0) \leq 0$. Using a similar procedure as above the second derivative is calculated and one obtains the expression

$$f''(0) = -(n-1)V_n(K_0)^{(1-2n)/n}\left(W_1(K_0,K_1)^2 - W_2(K_0,K_1)V_n(K_0)\right) \leq 0.$$

By the non-negativity of the volume this yields the statement

$$W_1(K_0,K_1)^2 \geq V_n(K_0)W_2(K_0,K_1)$$

which completes our proof.

■

In the appendix A of Hörmanders Notions of Convexity [9] further inequalities concerning general homogeneous polynomials of degree $n \geq 3$ and their polarized forms are stated and proved. Using these one can state further inequalities similar to those by Minkowski above.

4

The Aleksandrov-Fenchel inequality

The previous chapter was concluded with quadratic inequalities due to Minkowski. In the following chapter it will be proven that Minkowski's second inequality is a special case of a more general inequality satisfied by mixed volumes. Aleksandrov and Fenchel independently proved this theorem, although Fenchel's proof is mostly a sketch and to the authors knowledge no detailed version has appeared [6]. Aleksandrov second proof (found in [10]) of this theorem is based on the same idea as Hilbert's proof of Brunn-Minkowski's inequality, which can be found in [11].

Theorem 4.1 (Aleksandrov-Fenchel's inequality) *For $K_1,...,K_n \in \mathcal{K}^n$ the mixed volume satisfies the following inequality*

$$W^2(K_1,...,K_n) \geq W(K_1,K_1,K_3,...,K_n)W(K_2,K_2,K_3,...,K_n). \tag{4.1}$$

If K_1 and K_2 are homothetic equality holds, but general conditions for equality are unknown. It is clear that Minkowski's second inequality for mixed volumes (3.2) is a special case of this. Further a generalized Brunn-Minkowski inequality can be obtained as a special case, a simple proof of this can be found on pages 49-50 in Busemann [6].

Theorem 4.2 (Brunn-Minkowski for mixed volumes) *For sets $K,L,C_1,...,C_{n-2} \in \mathcal{K}^n$ with $K_\lambda = (1-\lambda)K + \lambda L$ and for $m \geq 2$ the function*

$$f(\lambda) = W^{1/m}(C_1,...,C_{n-m},\underbrace{K_\lambda,...,K_\lambda}_{m})$$

is concave. For $m = n$ this is the classical Brunn-Minkowski inequality.

By induction Aleksandrov deduced the following corollary of (4.1).

Corollary 4.3 *If $K_1,...,K_n \in \mathcal{K}^n$ then*

$$W(K_1,...,K_n)^i \geq \prod_{j=1}^{i} W(\underbrace{K_j,...,K_j}_{i},K_{i+1},...,K_n).$$

For $n = 2$ this is exactly the Aleksandrov-Fenchel inequality also for $n = m$ one obtains that

$$W^n(K_1,...,K_n) \geq V_n(K_1)V_n(K_2)\cdots V_n(K_n).$$

The proof provided here will cover the upcoming sections and is along the same lines as Aleksandrov's second proof but formulated using the concept of positive differential forms. See appendix A for a short introduction to the concept of positive forms and their properties. Throughout the remainder of this chapter we for sets $K_i \in \mathcal{K}^n_{reg}$ assume that the origin is contained in K_i. Furthermore when there is no risk of confusion we denote the support function of K_i simply by H_i, for such a body the support function is a smooth function everywhere except at the origin.

4.1 Preparatory lemmas

We begin with a lemma connecting positive forms to the concept of the determinants and in extension also the concept of discriminants which form the foundation for Aleksandrov's proof of the theorem.

For $A_1,...,A_r$ $n \times n$-matrices we have that the determinant of $\lambda_1 A_1 + ... + \lambda_r A_r$ is an n-th degree homogeneous polynomial in $\lambda_1,...,\lambda_r > 0$, the coefficients of this polynomial are called the polarized form of the determinant and have much in common with the mixed volumes. For a n-tuple of matrices we denote it by $\det(A_1,...,A_n)$.

Lemma 4.4 *For each of the $n \times n$-matrices $A_1,...,A_n$ associate the following $(1,1)$-form*

$$A_k = [a^{(k)}_{i,j}]_{i,j} \sim \sum a^{(k)}_{i,j} dx_i \wedge d\xi_j = \alpha_k$$

it then holds that

$$\frac{\alpha_k^n}{n!} = \frac{\alpha_k \wedge ... \wedge \alpha_k}{n!} = \det(A_k) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n.$$

Furthermore

$$\frac{\alpha_1 \wedge ... \wedge \alpha_n}{n!} = \det(A_1,...,A_n) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n$$

where $\det(A_1,...,A_n)$ again is the polarized form of the determinant, i.e. the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial $\det(\lambda_1 A_1 + ... + \lambda_n A_n)$.

The proof of this lemma is omitted here but can be found in Appendix B.

In the following proofs and results we will often evaluate certain expressions on the sphere in what will be referred to as our normal coordinates. For a point $x \in S^{n-1}$ we perform a orientation preserving linear change of coordinates such that the origin is kept fix and x is mapped to $u = (0,...,0,1)$. For any k-homogeneous function we know that its first-order partial derivatives are $(k-1)$-homogeneous. For 1-homogeneous, that will be studied here, this implies that in our standard coordinates it $\forall k \in \{1,...,n\}$ and any $\alpha > 0$ holds that

$$\frac{\partial f(0,...,0,\alpha)}{\partial u_k} = \frac{\partial f(0,...,0,1)}{\partial u_k}$$

and hence

$$\frac{\partial^2 f(0,...,0,\alpha)}{\partial u_k \partial u_n} = 0.$$

By the above we observe that the Hessian matrix of such a function will in our normal coordinates have only zeros in the n-th row and column.

The following theorem will be used when studying the eigenvalues of a second order differential operator defined using the forms considered here. The theorem implies an Aleksandrov type inequality for our differential forms.

Theorem 4.5 (Aleksandrov's pointwise inequality) *For $\omega_1,...,\omega_{n-1}$ postive (1,1)-forms generated by positive definite matrices and for any symmetric (1,1)-form α we have that*

$$\omega_1 \wedge ... \wedge \omega_{n-1} \wedge \alpha = 0 \tag{4.2}$$

implies that

$$\omega_1 \wedge ... \wedge \omega_{n-2} \wedge \alpha \wedge \alpha \leq 0. \tag{4.3}$$

Furthermore equality holds if and only if $\alpha = 0$.

By the theorem above it follows that for $f_1,...,f_{n-2}$ 1-homogeneous convex functions, strictly convex in all directions but radially, and some 1-homogeneous ϕ it holds that

$$dd^\# f_1 \wedge ... \wedge dd^\# f_{n-2} \wedge dd^\# \phi \wedge d|x|^2/2 \wedge d^\# |x|^2/2 = 0 \quad \text{implies that}$$
$$dd^\# f_1 \wedge ... \wedge dd^\# f_{n-3} \wedge dd^\# \phi \wedge dd^\# \phi \wedge d|x|^2/2 \wedge d^\# |x|^2/2 \leq 0.$$

In the proof of theorem 4.5 we will use the following inequality that holds for any bilinear form having Lorentz signature, i.e. any bilinear form having one unique negative eigenvalue and all other eigenvalues positive.

Lemma 4.6 *Let $a : E \times E \to \mathbb{R}$ be a symmetric bilinear form with Lorentz signature. Then*

$$a(e_1,e_1) < 0 \quad \text{and} \quad a(e_1,e_2) = 0$$

implies that

$$a(e_2,e_2) \geq 0$$

with equality if and only if e_2 is the zero element.

Proof Let $e_1,e_2 \in E$ satisfy the above assumptions, then for any $\lambda \in \mathbb{R}$

$$\begin{aligned} a(e_1 + \lambda e_2, e_1 + \lambda e_2) &= a(e_1,e_1) + 2\lambda a(e_1,e_2) + \lambda^2 a(e_2,e_2) \\ &= a(e_1,e_1) + \lambda^2 a(e_2,e_2) \end{aligned}$$

this is clearly a continuous function in λ which is negative close to $\lambda = 0$. It is clear that this function changes sign for some $\lambda \in \mathbb{R}$ iff $a(e_2,e_2) > 0$.

We will now argue that if this function does not change sign our bilinear form will not have Lorentz signature, which is a contradiction.

If the function $a(e_1 + \lambda e_2, e_1 + \lambda e_2)$ is always negative we have that for any vector $x \in \text{span}(e_1,e_2)$ we have that $a(x,x) < 0$, if $e_2 \neq 0$ this space is two dimensional. Having such a two dimensional subspace of E implies the existence of a second negative eigenvalue of our form a, thus a cannot have Lorentzian signature which contradicts our assumption. Hence the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(\lambda) = a(e_1 + \lambda e_2, e_1 + \lambda e_2)$ must if $e_2 \neq 0$ change sign as lambda increases from zero and thus $a(e_2,e_2) > 0$. This also implies that equality holds iff $e_2 = 0$.

■

We move on to proving the pointwise inequality stated in theorem 4.5. The proof of this theorem is in theory very similar to the main proof of Aleksandrov-Fenchel's inequality and can be seen as preparation for what is to come. The proof given here is adapted form Aleksandrov's proof of the corresponding inequality for mixed discriminants, which can be found in [10].

Proof (Aleksandrov's pointwise inequality) We prove the statement by induction over the dimension and using lemma 4.6. We throughout the proof use the following notation to denote the coefficient of the standard volume element of a (n,n)-form by $[\omega]$ that is for any (n,n)-form ω

$$\omega = [\omega] dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n$$

note that $[\omega]$ is a function of x. We begin by proving the statement for $n = 2$. Since we by a linear change of variables can simultaneously diagonalize the forms ω_1 and α we find that (4.2) simplifies to

$$[\omega_1 \wedge \alpha] = \omega_{11}^{(1)} \alpha_{22} + \omega_{22}^{(1)} \alpha_{11} = 0$$

where $\omega_{ij}^{(1)}$ and α_{ij} again denote the coefficients of our forms. Since $\omega_{ii}^{(1)} > 0$ this implies that α_{11} and α_{22} have opposite sign. For (4.3) we find that

$$[\alpha \wedge \alpha] = 2\alpha_{11}\alpha_{22} \leq 0$$

with equality if and only if both coefficients are zero. Thus the statement is true when $n = 2$.

Assume that the statement is true for dimension $n-1$ where $n > 2$. We define the following bilinear form on the space of (1,1)-forms

$$\langle \alpha_1, \alpha_2 \rangle_n = [\omega_1 \wedge ... \wedge \omega_{n-2} \wedge \alpha_1 \wedge \alpha_2]$$

where ω_i are (1,1)-forms corresponding in the usual sense to positive definite matrices. It is clear from the definition of the wedge product that this function is bilinear and symmetric.

We claim that such a bilinear form has negative Lorentzian signature, that is one positive eigenvalue and the remaining ones negative. If we can prove this then we are done by lemma 4.6. If $\langle \cdot, \cdot \rangle$ is a negative Lorentzian form it follows that $-\langle \cdot, \cdot \rangle$ is Lorentzian and thus the lemma states that if

$$-\langle \alpha_1, \alpha_2 \rangle < 0, \quad -\langle \alpha_2, \alpha_1 \rangle = 0 \quad \Rightarrow \quad -\langle \alpha_1, \alpha_1 \rangle \geq 0$$

with equality if and only if α_1 is the zero form, which is exactly what we aim to find.

Claim I: The form $\langle \cdot, \cdot \rangle_n$ has no eigenvalue equal to zero.

That a bilinear form has an eigenvalue zero is equivalent to the statement that the linear function defined by $\langle \alpha_0, \cdot \rangle_n$ is identically zero for some nontrivial (1,1)-form α_0. This statement is equivalent to that

$$\omega_1 \wedge ... \wedge \omega_{n-2} \wedge \alpha_0 = 0$$

for some nontrivial α_0. By a linear change of variables we can make both ω_{n-2} and α_0 diagonal. By the above we have that

$$[dx_i \wedge d\xi_i \wedge \omega_1 \wedge ... \wedge \omega_{n-2} \wedge \alpha_0] = 0 \quad \forall i = 1,...,n.$$

For every $i = 1,...,n$ this is the restriction of a form $\langle \cdot, \cdot \rangle_{n-1}$ generated by positive forms to the subspace orthogonal to $dx_i \wedge d\xi_i$. By our inductive assumption this, for each such form, implies that

$$\langle \alpha_0, \alpha_0 \rangle_{n-1} = [dx_i \wedge d\xi_i \wedge \omega_1 \wedge ... \wedge \omega_{n-3} \wedge \alpha_0 \wedge \alpha_0] \leq 0 \tag{4.4}$$

with equality iff α_0 is identically zero. Since we have that ω_{n-2} is diagonalized and positive we have that its coefficients $\omega_{ii}^{(n-1)} > 0$ yielding that

$$\sum_{i=1}^{n} \omega_{ii}^{(n-2)} [dx_i \wedge d\xi_i \wedge \omega_1 \wedge ... \wedge \omega_{n-3} \wedge \alpha_0 \wedge \alpha_0] = [\omega_1 \wedge ... \wedge \omega_{n-2} \wedge \alpha_0 \wedge \alpha_0] \leq 0.$$

By our choice of α_0 as the eigenform corresponding to the eigenvalue zero we must have equality. Equality must therefore also hold in (4.4) implying that α_0 restricted to each of the subspace must be identically zero and thus it must be everywhere identically zero hence claim I is true.

Claim II: The form $\langle\cdot,\cdot\rangle_n$ has only one positive eigenvalue.

We prove the claim by first concluding that in the special case of

$$\omega_1 = ... = \omega_{n-1} = \omega = \sum dx_i \wedge d\xi_i$$

the claim holds. In this case the only positive eigenvalue is 1 with corresponding eigenform ω. The proof continues by continuously deforming these ω into arbitrary positive forms.

We consider the special case $\omega_i = \omega$ for all i then for $\alpha = \sum_{i,j} \alpha_{ij} dx_i \wedge d\xi_j$ such that

$$\langle\omega, \alpha\rangle_n = \left[\omega^{n-1} \wedge \alpha\right] = (n-1)! \sum_{i=1}^{n} \alpha_{ii} = 0 \tag{4.5}$$

one finds that

$$\langle\alpha, \alpha\rangle_n = \left[\omega^{n-2} \wedge \alpha^2\right] = 2(n-2)! \sum_{i<k} \left\{\alpha_{ii}\alpha_{kk} - \alpha_{ik}^2\right\}. \tag{4.6}$$

By condition (4.5) and the fact that

$$2\sum_{i<k} a_i a_k = \left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2$$

equation (4.6) simplifies to

$$\langle\alpha, \alpha\rangle_n = -(n-2)! \left\{\sum_{i=1}^{n} \alpha_{ii}^2 + 2\sum_{i<k} \alpha_{ik}^2 \right\} \leq 0$$

with equality if and only if $\alpha = 0$. If we disregard the side-condition (4.5) the value of (4.6) is maximized if $\alpha_{ik} = 0$ when $i \neq k$ and $\alpha_{ii} = \alpha_{kk}$ for any i, k, i.e. α is some multiple of ω. By inserting one finds that this corresponds to an eigenvalue equal to 1. But the condition set for α expresses the weighted orthogonality to this eigenform. Since our bilinear form $\langle\cdot, \cdot\rangle_n$ is negative definite under this condition we know that this must be the only positive eigenvalue.

What remains to prove is that for any choice of ω_1,...,ω_{n-2} (1,1)-forms generated by positively definite matrices we always only have one positive eigenvalue. Let the form $\omega_i^{(t)} = (1-t)\omega + t\omega_i$ where ω is defined as above and t is in the unit interval. We know that this form is always associated with positive definite matrix, the set of such matrices is a convex cone. Therefore by claim I we know that the bilinear form defined by

$$\langle\cdot,\cdot\rangle_n^{(t)} = \left[\omega_1^{(t)} \wedge ... \wedge \omega_{n-2}^{(t)} \wedge \cdot \wedge \cdot\right]$$

has no eigenvalue equal to zero for any value of $t \in [0,1]$. Furthermore the bilinear form is continuously transformed implying that also the eigenvalues must change continuously.

Thus since we never have any eigenvalues equal to zero we know that our unique positive eigenvalue must remain positive and all others must remain negative. Hence our bilinear form has negative Lorentzian signature and the proof is complete.

■

Lemma 4.7 (Integral formula for mixed volumes) *For strictly convex bodies $K_1,...,K_n \in \mathcal{K}^n_{reg}$ we have that*

$$W(K_1,...,K_n) = \frac{1}{n!}\int_{S^{n-1}\times\mathbb{R}^n} H_n dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-1} \wedge d^{\#}|x|^2/2. \tag{4.7}$$

Proof This formula follows from repeated application of Stokes' theorem and an equality between two differential forms on S^{n-1} stated as lemma B.2 in appendix B.

Let K be a arbitrary body in $\mathcal{K}^n_{reg}$ we denote its support function as H which is smooth, 1-homogeneous and convex outside of the origin. By H^ε we denote a function that is everywhere smooth and agrees with H everywhere except in $B_\varepsilon(0)$.

$$\begin{aligned}
V(K) =& \int_K dx \\
=& \frac{1}{n}\int_{\partial K}\sum_{i=1}^n x_i \widehat{dx_i} \\
=& \frac{1}{n}\int_{S^{n-1}}\sum_{i=1}^n \frac{\partial H}{\partial x_i} \widehat{d\frac{\partial H}{\partial x_i}} \\
=& \frac{1}{n}\int_{S^{n-1}}\sum_{i=1}^n \frac{\partial H^\varepsilon}{\partial x_i}\left(\widehat{d\frac{\partial H^\varepsilon}{\partial x_i}}\right) \\
=& \frac{1}{n}\sum_{i=1}^n\int_B d\left(\frac{\partial H^\varepsilon}{\partial x_i}\right)\wedge\left(\widehat{d\frac{\partial H^\varepsilon}{\partial x_i}}\right) \\
=& \frac{1}{n}\int_B\left(\sum_{k=1}^n \frac{\partial^2 H^\varepsilon}{\partial x_i \partial x_k}dx_i\right)\wedge\prod_{j\neq i}\left(\sum_{k=1}^n\frac{\partial^2 H^\varepsilon}{\partial x_j\partial x_i}dx_i\right) \\
=& \frac{1}{n}\sum_{i=1}^n\int_B\sum_{\sigma\in S_n}\prod_{j=1}^n\frac{\partial^2 H^\varepsilon}{\partial x_j \partial x_{\sigma(j)}}dx_{\sigma(1)}\wedge ... \wedge dx_{\sigma(n)} \\
=& \int_B \sum_{\sigma\in S_n}\prod_{j=1}^n \frac{\partial^2 H^\varepsilon}{\partial x_j\partial x_{\sigma(j)}}\mathrm{sgn}(\sigma)d\mathbf{x}.
\end{aligned}$$

But this is exactly the determinant of the Hessian times the volume element. Applying lemma 4.4 yields that

$$\begin{aligned}\int_B \det(H^\varepsilon_{xx})d\mathbf{x} =& \frac{1}{n!}\int_{B\times\mathbb{R}^n}\left(dd^\# H^\varepsilon\right)^n \\ =& \frac{1}{n!}\int_{S^{n-1}\times\mathbb{R}^n} d^\# H^\varepsilon \wedge \left(dd^\# H^\varepsilon\right)^{n-1} \\ =& \frac{1}{n!}\int_{S^{n-1}\times\mathbb{R}^n} d^\# H \wedge \left(dd^\# H\right)^{n-1} \\ =& \frac{1}{n!}\int_{S^{n-1}\times\mathbb{R}^n} H d^\# |x|^2/2 \wedge \left(dd^\# H\right)^{n-1}\end{aligned} \tag{4.8}$$

where the last equality is exactly the statement of lemma B.2. Lettin $K = \lambda_1 K_1 + ... + \lambda_m K_m$ where K_1,...,K_m are arbitrary sets in $\mathcal{K}^n_{reg}$ and λ_1,...,$\lambda_m > 0$ one finds that

$$V(\lambda_1 K_1 + ... + \lambda_m K_m) = \frac{1}{n!}\int_{S^{n-1}\times\mathbb{R}^n}\left(\sum_{i=1}^m \lambda_i H_i\right) d^\#|x|^2/2 \wedge \left(dd^\#\left(\sum_{i=1}^m \lambda_i H_i\right)\right)^{n-1}$$

this is clearly an n-th degree homogeneous polynomial in λ_1,...,λ_m, comparing the terms of this polynomial with that of our polynomial of mixed volumes it is clear that

$$W(K_1,...,K_n) = \frac{1}{n!}\int_{S^{n-1}\times\mathbb{R}^n} H_n d^\#|x|^2/2 \wedge dd^\# H_1 \wedge ... \wedge dd^\# H_{n-1}$$

which is equal to our expression above, moving a (0,1)-form past any (1,1)-form yields no change of sign. Having the $d^\#|x|^2/2$ term last will be slightly more natural when we are working in our normal coordinates since this will result in a $d\xi_n$ term. Furthermore since the left-hand side is invariant under a permutation of the arguments this must also be true for the right-hand side, this is also intuitively clear from the first equality in (4.8).

∎

It is worth mentioning that these calculations could be done without the smoothing of H but this would result in slightly more complex calculations since we would have what corresponds to currents in the complex case instead of the smooth forms above. Parts of the corresponding calculations with currents is done in Lagerberg's article [12], further he shows how such currents are closely related to the Monge-Ampere measure.

We provide a lemma that will allow us to express the integral representations of mixed volumes above as an integral over the usual surface measure on S^{n-1}.

Lemma 4.8 *Let M be a hypersurface in $\mathbb{R}^n$ locally defined by the equation*

$$\{x \in \mathbb{R}^n \ : \ g(x) = 0\},$$

where g is a map from some domain $U \subset \mathbb{R}^n$ to $\mathbb{R}$ with surjective differential for all x in U. For any $x \in M$ and any $(n-1)$-form η on M we have that for some constant c

$$\eta = c\omega^M$$

at x, where ω^M is the volume form on M. If we let the orientation of M be chosen such that $dg \wedge \eta$ is positive on $\mathbb{R}^n$ we find that

$$\frac{dg}{|dg|} \wedge \eta = cdx_1 \wedge ... \wedge dx_n.$$

Proof We denote the tangent plane of the hypersurface M at x by T_x and let ν be a unit normal vector to this plane, with the property that $dg(\nu) > 0$. Further let $e_1,...,e_{n-1}$ denote an orthonormal basis of T_x, this implies that $\nu,e_1,...,e_{n-1}$ is an orthonormal basis of $\mathbb{R}^n$ with positive orientation. Thus we have that

$$\left(\frac{dg}{|dg|} \wedge \eta\right)(\nu,e_1,...,e_{n-1}) = \tilde{c}$$

for some constant $\tilde{c}$. However by the definition of the wedge product we have that

$$\left(\frac{dg}{|dg|} \wedge \eta\right)(a_1,...,a_n) = \frac{1}{(n-1)!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{dg}{|dg|}(a_{\sigma(1)})\eta(a_{\sigma(2)},...,a_{\sigma(n)}).$$

inserting $(a_1,...,a_n) = (\nu,e_1,...,e_{n-1})$ this simplifies to

$$\begin{aligned}\left(\frac{dg}{|dg|} \wedge \eta\right)(\nu,e_1,...,e_{n-1}) &= \frac{1}{(n-1)!} \frac{dg}{|dg|}(\nu) \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma)\eta(e_{\sigma(1)},...,e_{\sigma(n-1)}) \\ &= \frac{dg}{|dg|}(\nu)\eta(e_1,...,e_{n-1})\end{aligned}$$

where we used that $dg/|dg|$ is zero restricted to T_x. Further since ν is normalized $dg/|dg|(\nu) = 1$ and $e_1,...,e_{n-1}$ is a orthonormal basis for T_x the last expression is exactly our constant c. This implies that $\tilde{c} = c$ and the proof is complete.

■

Thus our integral representation of mixed volumes can be represented as the following surface area integral

$$\frac{1}{n!} \int_{S^{n-1}} H_n \left[dd^\# H_1 \wedge ... \wedge dd^\# H_{n-1} \wedge d|x|^2/2 \wedge d^\# |x|^2/2\right] dS.$$

The main idea in Hilbert's proof of Brunn-Minkowski's inequality is to reduce the proof to solving an eigenvalue problem for an elliptic operator on S^{n-1}. We present a definition of the ellipticity of a differential operator on a manifold X that is equivalent to the classical definition but more suitable in our setting, this definition can be found in Warner's book [13].

Definition 4.1 *A differential operator L of order l is elliptic at $x \in X$ if and only if*

$$L(\phi^l u)(x) \neq 0$$

for each C^∞-function $u : X \to \mathbb{C}^n$ such that $u(x) \neq 0$ and any smooth function $\phi : X \to \mathbb{R}$ satisfying $\phi(x) = 0$ and $d\phi \neq 0$. An operator is said to be elliptic on a manifold X if it is elliptic at every $x \in X$.

Lemma 4.9 *For H_1,...,H_{n-2} support functions of sets in $\mathcal{K}^n_{reg}$ the differential operator on the space of smooth 1-homogeneous functions*

$$\phi \mapsto L(\phi)$$

where L is defined by the form

$$L(\phi) = \left[dd^{\#}H_1 \wedge dd^{\#}H_2 \wedge ... \wedge dd^{\#}H_{n-2} \wedge dd^{\#}\phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2\right]$$

is elliptic operator at every $\zeta \in S^{n-1}$. Furthermore a function ϕ is mapped to zero if and only if ϕ is linear.

Proof We begin by proving the statement of ellipticity. Assume that ϕ and u satisfy the assumptions above then

$$\begin{aligned} L(\phi^2 u)dx_1 \wedge d\xi_1 \wedge ... \wedge dx_{n-1} \wedge d\xi_{n-1} \wedge d\xi_n \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 = \\ dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-2} \wedge dd^{\#}(\phi^2 u) \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \end{aligned} \tag{4.9}$$

now by a linear, orientation preserving change of our standard variables that fixes x_n we can simultaneously diagonalize $dd^{\#}H_1$ and $dd^{\#}\left(\phi^2 u\right)$. Then (4.9) simplifies to

$$\sum_{i=1}^{n-1} \frac{\partial^2 H_1}{\partial x_i^2} dx_i \wedge d\xi_i \wedge dd^{\#}H_2 \wedge ... \wedge dd^{\#}H_{n-2} \wedge dd^{\#}(\phi^2 u) \wedge dx_n \wedge d\xi_n$$

$$= 2u \sum_{i=1}^{n} \sum_{j \neq i} \frac{\partial^2 H_1}{\partial x_i^2} \left(\frac{\partial \phi}{\partial x_j}\right)^2 dx_i \wedge d\xi_i \wedge dx_j \wedge d\xi_j \wedge dd^{\#}H_2 \wedge ... \wedge dd^{\#}H_{n-2} \wedge dx_n \wedge d\xi_n$$

which is non zero since all $\partial^2 H_1/\partial x_i^2$ are positive and our assumptions for u and ϕ. Since this holds at any point on S^{n-1} we have that our operator is elliptic.

It is trivial to see that our operator maps all linear functions to zero since $dd^{\#}\phi = 0$ for any linear function ϕ. The opposite direction of our statement follows from a simple argument using theorem 4.5. We need to find all ϕ 1-homogeneous, smooth functions such that

$$dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-2} \wedge dd^{\#}\phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 = 0$$

for arbitrary choices of K_1,...,$K_{n-2} \in \mathcal{K}^n_{reg}$. By theorem 4.5 we have that this implies

$$dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-3} \wedge dd^{\#}\phi \wedge dd^{\#}\phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \leq 0 \tag{4.10}$$

with equality if and only iff ϕ is linear. But by our representation formula for mixed volumes and the invariance under permutation of the arguments we have that

$$
\begin{aligned}
0 &= \int_{S^{n-1}} \phi \left[dd^{\#} H_1 \wedge ... \wedge dd^{\#} H_{n-2} \wedge dd^{\#} \phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \right] dS \\
&= \int_{S^{n-1}} H_{n-2} \left[dd^{\#} H_1 \wedge ... \wedge dd^{\#} H_{n-3} \wedge dd^{\#} \phi \wedge dd^{\#} \phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \right] dS.
\end{aligned}
$$

But since $H_{n-2} > 0$ on S^{n-1} (we have assumed all bodies contain the origin) we find that we must have equality in (4.10) and thus ϕ must be linear.

∎

4.2 Hilbert's method using positive forms

We will now combined the previous results to complete the proof of theorem 4.1. In our proof all convex sets are assumed to be strictly convex and in $\mathcal{K}^n_{reg}$ with the origin as an innerpoint. Since mixed volumes are translation invariant and all convex sets can be approximated with strictly convex sets in $\mathcal{K}^n_{reg}$ it is clear that the general statement follows. The proof involves a fair amount of theory for elliptic differential operators on S^{n-1}. Most of this theory is stated here without proof however for an introduction to this theory we recommend [14] alternatively Hilbert's pioneering book on the subject [11].

We extend our notion of the mixed volume functional to the space of support functions of sets in $\mathcal{K}^n_{reg}$ by the rule

$$W(H_1,...,H_n) = W(K_1,...,K_n).$$

The function W now extends naturally to the vector space generated by such functions. Using this formalism theorem 4.1 can be stated as follows: For bodies $K_1,...,K_{n-1}$ and any ϕ that is the difference between support functions of convex bodies we have that

$$W^2(H_1,...,H_{n-1},\phi) \geq W(H_1,...,H_{n-1},H_{n-1}) W(H_1,...,H_{n-2},\phi,\phi). \tag{4.11}$$

It is clear that (4.11) implies that

$$W(H_1,...,H_{n-2},\phi,\phi) \leq 0 \quad \text{if} \quad W(H_1,...,H_{n-1},\phi) = 0. \tag{4.12}$$

However the two statements (4.11) and (4.12) are in fact equivalent, to simplify proving this we introduce the following notation

$$W_p(H,\phi) = W(H_1,...,H_{n-p},\underbrace{\phi,...,\phi}_{p}).$$

Let λ be such that $W_1(H,\phi) = \lambda W_1(H,H_{n-1})$. Then we have that

$$W_1(H,\phi - \lambda H_{n-1}) = W_1(H,\phi) - \lambda W_1(H,H_{n-1}) = 0$$

thus by (4.12) yields that $W_2(H,\phi - \lambda H_{n-1}) \leq 0$. Expanding this expression gives us that

$$\begin{aligned} W_2(H,\phi - \lambda H_{n-1}) =& W_2(H,\phi) - W_2(H,\lambda H_{n-1}) \\ =& W_2(H,\phi) - \lambda^2 W_1(H,H_{n-1}) \\ =& W_2(H,\phi) - \frac{W_1^2(H,\phi)}{W_1(H,H_{n-1})} \leq 0 \end{aligned}$$

and we have obtained (4.11). To prove theorem 4.1 it therefore suffices to prove that (4.12) holds for arbitrary choice of $K_1,...,K_{n-1}$ and ϕ.

Writing this on the form of (4.7) we must prove that

$$\int_{S^{n-1}} \phi \left[dd^{\#} H_1 \wedge ... \wedge dd^{\#} H_{n-1} \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \right] dS = 0.$$

implies that

$$\int_{S^{n-1}} \phi \left[dd^{\#} H_1 \wedge ... \wedge dd^{\#} H_{n-2} \wedge dd^{\#}\phi \wedge x_n \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \right] dS \leq 0 \quad (4.13)$$

Proving this is done using Hilbert's method consisting of solving the extremal value problem for the following functional

$$\int_{S^{n-1}} \phi \left[dd^{\#} H_1 \wedge ... \wedge dd^{\#} H_{n-2} \wedge dd^{\#}\phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \right] dS \quad (4.14)$$

under the condition that

$$\int_{S^{n-1}} \frac{\phi^2}{H_{n-1}} \left[dd^{\#} H_1 \wedge ... \wedge dd^{\#} H_{n-1} \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2 \right] dS = 1.$$

From lemma 4.9 we know that this is an elliptic problem. We also have that on S^{n-1}

$$\frac{1}{H_{n-1}} L(H_{n-1}) > 0$$

thus by the theory of Hilbert [11] the problem reduces to solving the eigenvalue problem

$$L(\phi) + \lambda \frac{\phi}{H_{n-1}} L(H_{n-1}) = 0.$$

Since this is an elliptic problem we know that there only exists finitely many negative eigenvalues, we will now prove that there only exists one. It is easily verified that H_{n-1} is an eigenfunction with eigenvalue $\lambda = -1$. By the second part of lemma 4.9 we know that $\lambda = 0$ is an eigenvalue of multiplicity n corresponding to all linear ϕ.

What remains to prove is that our negative eigenvalue $\lambda = -1$ is unique for all choices of sets $K_1,...,K_{n-1}$. This follows from completely solving the system in the case where $K_1 = ... = K_{n-2} = B$ and arguing that the eigenvalues of an elliptic operator change continuously under a continuous transformation of the coefficients, in our case this corresponds to continuously changing the bodies $K_1,...,K_{n-2}$.

We begin by exploring the case where all of our bodies are equal to B. In our standard coordinates one finds that

$$dd^{\#}h_B = \sum_{i=1}^{n-1} dx_i \wedge d\xi_i$$

thus implying

$$dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-2} \wedge dd^{\#}\phi \wedge dx_n \wedge d\xi_n = \\ \sum_{i=1}^{n-1} \frac{\partial^2\phi}{\partial x_i^2} dx_1 \wedge d\xi_1 \wedge ... \wedge dx_{n-1} \wedge d\xi_{n-1} \wedge dx_n \wedge d\xi_n$$

but this is exactly the Laplace operator on the S^{n-1}. We now from basic theory of elliptic operators on S^{n-1} that we only have one unique negative eigenvalue of this operator, namely $\lambda = -1$. It also still holds that we have $\lambda = 0$ as an eigenvalue of multiplicity n corresponding to the linear functions, all other eigenvalues are positive.

Let $H^{(t)}$ for $t \in [0,1]$ denote the following continuous transformation from B to a set $K \in \mathcal{K}^n_{reg}$

$$H^{(t)} = (1-t)B + tH.$$

Thus we have that as t moves from 0 to 1 the differential operator $L^{(t)}$ defined by

$$L^{(t)}(\phi) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_{n-1} \wedge d\xi_{n-1} \wedge dx_n \wedge d\xi_n = \\ dd^{\#}H_1^{(t)} \wedge ... \wedge dd^{\#}H_{n-2}^{(t)} \wedge dd^{\#}\phi \wedge d|x|^2/2 \wedge d^{\#}|x|^2/2$$

is continuously transformed from the Laplacian on the sphere to the general operator in (4.14). Throughout this transformation we know that $\lambda = 0$ remains an eigenvalue with multiplicity n corresponding to all linear functions, we have also proved that $\lambda = -1$ is an eigenvalue. By the theory of elliptic operators we know that the eigenvalues of these operators change continuously during this transformation and thus it must hold that $\lambda = -1$ remains the unique negative eigenvalue for any value of t.

Since the side condition in (4.13) excludes the eigenfunction corresponding to our unique negative eigenvalue we find that

$$W(H_1,...,H_{n-2},\phi,\phi) \leq 0$$

since the extrema of (4.14) correspond our eigenvalues. This conclude the proof of theorem 4.1.

∎

With the proof given above we aim to provide a more general setting where one might could generalize the theorem to larger classes of functions or possibly derive exact conditions of equality. With the current proofs, including the one presented here, exact equality conditions are not clear largely because of the limiting procedure when approximating arbitrary convex sets with either polytopes or as in our case strictly convex sets in $\mathcal{K}^n_{reg}$. It is possible that one may be able to adapt the proof given above by introducing currents instead of forms and thus possibly eliminate the need for approximation when one deals with the integral representation of mixed volumes.

A

Positive super forms

The concept of positive super forms is an adaption of the formalism for positive forms in $\mathbb{C}^n$. For an introduction to positive forms and the further concept of positive super currents the reader is referred to Aron Lagerberg's article [12]. In the remainder of this section we restrict ourselves to a short introduction of the concepts used in this thesis.

Let V and W denote two n-dimensional vector spaces over the reals, with coordinates $x = (x_1,...,x_n) \in V$ and $\xi = (\xi_1,...,\xi_n) \in W$. Fix an isomorphism $J : V \to W$ such that $J(x) = \xi$, we also denote its inverse by J so that $J(\xi) = x$. We extend J to the space $E = V \times W = \{(x,\xi) : \ x \in V, \ \xi \in W\}$ by letting $J(x,\xi) = (J(x),J(\xi))$. Consider the space of smooth differential forms on E whose coefficients only depend on x, α is such a form if

$$\alpha = \sum_{K,L} \alpha_{K,L}(x) dx_K \wedge d\xi_L \tag{A.1}$$

where $\alpha_{K,L}$ are smooth functions and K, L are multi-indices of lengths p and q. Also dx_K denotes $dx_{k_1} \wedge ... \wedge dx_{k_p}$ when $K = (k_1,...,k_p)$ and the same for $d\xi_L$. We use the convention that the sum only goes over multi-indices K, L such that $k_1 < ... < k_p$ and $\xi_1 < ... < \xi_q$. A form as in (A.1) is called a (p,q)-form.

By identifying J with the corresponding isomorphism J^* we have that $J(dx_i) = d\xi_i$ and extend it to arbitrary (p,q)-forms by

$$J\left(\sum_{K,L} \alpha_{K,L}(x) dx_K \wedge d\xi_L\right) = \sum_{K,L} \alpha_{K,L}(x) d\xi_K \wedge dx_L.$$

We say that a (n,n)-form α is positive if

$$\alpha = f dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n$$

for some function $f \geq 0$. There are several notions of the positivity of a form, the one considered here is the weak positivity of a (p,p)-form α.

Definition A.1 *The* (p,p)*-form* ω *is weakly positive if for any* (1,0)*-forms* α_1,...,α_{n-p} *the* (n,n)*-form*

$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p} \wedge J(\alpha_{n-p}) \wedge \omega$$

is positive.

Throughout this thesis when a form is said to be positive it is meant in the weak sense. We continue by proving two simple properties of weakly positive forms that will be used in the proof of the Aleksandrov-Fenchel theorem 4.1 above.

Theorem A.1 *A* (1,1)*-form*

$$\omega = \sum_{i,j} \omega_{i,j}(x) dx_i \wedge d\xi_j$$

is positive if and only if the matrix $[\omega_{i,j}(x)]_{i,j}$ *is a positive semidefinite matrix.*

Proof By the definition of a weakly positive form we have that for any (1,0)-forms α_1,...α_{n-1} the form

$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-1} \wedge J(\alpha_{n-1}) \wedge \omega$$

is positive. But the system of α_i must be linearly independent, otherwise the product would be zero. Thus we can change coordinates by a transform T such that α_i is sent to dx_i. Thus we have that ω is positive if and only if

$$dx_1 \wedge d\xi_1 \wedge ... \wedge dx_{n-1} \wedge d\xi_{n-1} \wedge T(\omega)$$

is positive. But this is equivalent to that the coefficient of $dx_n \wedge d\xi_n$ in $T(\omega)$ is non-negative. But if T is determined by the matrix $[t_{i,j}]_{i,j}$ we find that

$$\begin{aligned} T(\omega) &= \sum_{i,j} \omega_{i,j} \left(\sum_{k=1}^{n} t_{i,k} dx_k \right) \wedge \left(\sum_{l=1}^{n} t_{j,l} d\xi_l \right) \\ &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \omega_{i,j} t_{i,k} t_{j,l} dx_k \wedge d\xi_l. \end{aligned}$$

The condition that the coefficient of $dx_n \wedge d\xi_n$ is positive now translates to that

$$\sum_{i=1}^{n} t_{i,n} \sum_{j=1}^{n} \omega_{i,j} t_{j,n} \geq 0$$

which is true if and only if $[\omega_{i,j}]_{i,j}$ is positive semidefinite.

∎

Theorem A.2 *If Ω is a positive $(p{,}p)$-form then for any positive $(1{,}1)$-form ω the $(p+1{,}p+1)$-form defined by*

$$\omega \wedge \Omega$$

is positive.

Proof For any (1,0)-forms $\alpha_1,...,\alpha_{n-p}$ we know that the form

$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p} \wedge J(\alpha_{n-p}) \wedge \Omega$$

is positive. We must show that the form

$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p-1} \wedge J(\alpha_{n-p-1}) \wedge \omega \wedge \Omega$$

is positive. But if $\omega = \sum_{i,j} \omega_{i,j} dx_i \wedge d\xi_j$ we know that the matrix $[\omega_{i,j}]_{i,j}$ is positive semidefinite. But then there is a linear change of variables T such that this matrix is diagonal with non negative elements. Since $T(\Omega)$ is positive and that T maps (1,0)-forms to (1,0)-forms we find that the form

$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p} \wedge J(\alpha_{n-p}) \wedge T(\Omega)$$

is positive for all (1,0)-forms α_i. But then we must have that the form

$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p-1} \wedge J(\alpha_{n-p-1}) \wedge dx_{n-p} \wedge d\xi_{n-p} \wedge T(\Omega) =$$
$$\alpha_1 \wedge J(\alpha_1) \wedge ... \wedge \alpha_{n-p-1} \wedge J(\alpha_{n-p-1}) \wedge T(\omega \wedge \Omega)$$

is positive. Thus we have that $\omega \wedge \Omega$ is positive.

■

Let $\alpha_1,...,\alpha_{n-1}$ be (1,1)-forms where the associcated matrices are positively definite it holds that for any (1,0)-form $\alpha \neq 0$

$$\alpha_1 \wedge ... \wedge \alpha_{n-1} \wedge \alpha \wedge J(\alpha) > 0.$$

We introduce the notion of integration of a positive $(n{,}n)$-form in the following manner which allows us to move back from our extended space to our original real setting.

Definition A.2 *For a $(n{,}n)$-form we define integration in the following manner*

$$\int_{\Omega \times \mathbb{R}^n} \omega dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n = \int_{\Omega} \omega dx.$$

We introduce a differential operator similar to the outer derivative d that acts on a smooth function $f : \mathbb{R}^n \to \mathbb{C}$ mapping it to a (0,1)-form.

Definition A.3 *Let the operator $d^{\#}$ be defined as*

$$d^{\#} = J \circ d \circ J$$

where d is the outer derivative and J is defined as above. For a smooth function $f : \mathbb{R}^n \to \mathbb{C}$ we have that

$$d^{\#} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d\xi_i.$$

The composition of this operator with the outer derivative will be used in the proof of Aleksandrov-Fenchel's inequality.

Definition A.4 *By $dd^{\#}$ we denote the composition of d and $d^{\#}$ that is*

$$dd^{\#} = d(J \circ d \circ J).$$

For a function $f : \mathbb{R}^n \to \mathbb{C}$ this operator acts in the following manner

$$dd^{\#} f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge d\xi_j.$$

It follows from our theorem above that the (1,1)-form $dd^{\#} f$ is positive if and only if f is convex.

B

Technical proofs

THIS appendix provides proofs of some lemmas that were left unproven in the main text. The lemmas were stated without proof since the proofs do not significantly enhance the understanding of this thesis and would only obstruct the flow of the main text. The first lemma was used when proving that the volume of a Minkowski sum is a polynomial.

Lemma B.1 *The map ∇h_K is for any strictly convex $K \in \mathcal{K}^n_{reg}$ a bijective, differentiable map from $S^{n-1} \to \partial K$.*

Proof For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ we know that $\nabla f(x)$ is the unique vector $y \in \mathbb{R}^n$ satisfying the following inequality

$$f(z) \geq f(x) + \langle y, z - x \rangle \quad \forall z \in \mathbb{R}^n. \tag{B.1}$$

What is to be proven is that for $f = h_K$ ($K \in \mathcal{K}^n_{reg}$ that contains the origin) we have that this unique y is the vector $H_K(x) \cap K \in \partial K$.

For $y = H_K(x) \cap K$, which is uniquely defined for $K \in \mathcal{K}^n_{reg}$, we have that $\langle x,y \rangle = h_K(x)$ (definition 1.5 of the support plane H_K) thus

$$h_K(x) + \langle y,z - x\rangle = h_K(x) - h_K(x) + \langle y,z \rangle = \langle y,z \rangle$$

which is obviously less than $h_K(z)$ and thus (B.1) holds implying that y is the gradient of h_K at x.

Thus we have proved that this gradient map indeed maps every $x \in S^{n-1}$ to a point on the boundary of K. The bijectiveness follows from that the boundary of K is equal to $\cup_{x \in S^{n-1}} H_K(x) \cap K$ and from that each such set is a unique singleton when K is regular.

The differentiability follows from the fact that h_K is for regular bodies smooth everywhere outside the origin. This completes the proof of our lemma.

∎

Lemma 4.4 (page 35) For each of the $n \times n$-matrices $A_1,...,A_n$ associate the following (1,1)-form

$$A_k = [a_{i,j}^{(k)}]_{i,j} \sim \sum a_{i,j}^{(k)} dx_i \wedge d\xi_j = \alpha_k$$

it then holds that

$$\frac{\alpha_k^n}{n!} = \frac{\alpha_k \wedge ... \wedge \alpha_k}{n!} = \det(A_k) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n.$$

Furthermore

$$\frac{\alpha_1 \wedge ... \wedge \alpha_n}{n!} = \det(A_1,...,A_n) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n$$

where $\det(A_1,...,A_n)$ again is the polarized form of the determinant, i.e. the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial $\det(\lambda_1 A_1 + ... + \lambda_n A_n)$.

Proof We start by proving the first statement and then the second will follow by simple properties of wedge products and determinants. Let A be a $n \times n$-matrix with elements $a_{i,j}$ and let α denote its associated form then

$$\begin{aligned}
\underbrace{\alpha \wedge ... \wedge \alpha}_{n} &= \left(\sum_{i,j} a_{i,j} dx_i \wedge d\xi_j \right) \wedge ... \wedge \left(\sum_{i,j} a_{i,j} dx_i \wedge d\xi_j \right) \\
&= \sum_{\sigma,\gamma \in S_n} \left(\prod_{i=1}^n a_{\sigma(i),\gamma(i)} \right) dx_{\sigma(1)} \wedge d\xi_{\gamma(1)} \wedge ... \wedge dx_{\sigma(n)} \wedge d\xi_{\gamma(n)} \\
&= n! \sum_{\gamma \in S_n} \left(\prod_{i=1}^n a_{i,\gamma(i)} \right) dx_1 \wedge d\xi_{\gamma(1)} \wedge ... \wedge dx_n \wedge d\xi_{\gamma(n)} \\
&= n! \sum_{\gamma \in S_n} \mathrm{sgn}(\gamma) \left(\prod_{i=1}^n a_{i,\gamma(i)} \right) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n \\
&= n! \det(A) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n.
\end{aligned}$$

Thus the first part of the lemma holds true, for the second part we recall by the definition of the polarized form that we for matrices $A_1,...,A_r$ have

$$\det(\lambda_1 A_1 + ... \lambda_r A_r) = \sum_{i_1,...,i_n=1}^{r} \lambda_{i_1} \cdots \lambda_{i_n} \det(A_{i_1},...,A_{i_n})$$

and also by our previous calculation

$$\det(\lambda_1 A_1 + ... \lambda_r A_r) dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n = \frac{1}{n!} (\lambda_1 \alpha_1 + ... + \lambda_r \alpha_r)^n$$

this is clearly a homogeneous polynomial of degree n where the coefficient of some $\lambda_{i_1} \cdots \lambda_{i_n}$ is

$$\frac{1}{n!} \alpha_{i_1} \wedge ... \wedge \alpha_{i_n}.$$

Thereby for any choice of n-tuple $(i_1,...,i_n)$ it must hold that

$$\det(A_{i_1},...,A_{i_n})dx_1 \wedge d\xi_1 \wedge ... \wedge dx_n \wedge d\xi_n = \frac{1}{n!}\alpha_{i_1} \wedge ... \wedge \alpha_{i_n}.$$

∎

Lemma B.2 *For a smooth and* 1*-homogeneous function* ϕ *we have that on* S^{n-1}

$$d^{\#}\phi \wedge dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-1} = \phi d^{\#}|x|^2/2 \wedge dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-1} \tag{B.2}$$

here $H_1,...,H_{n-1}$ *are as ususal supportfunctions of sets in* K^n_{reg}*.*

Proof Let ω be the $(n-1,n-1)$-form $dd^{\#}H_1 \wedge ... \wedge dd^{\#}H_{n-1}$ then we have that by a change of variables to our normal coordinates (B.2) becomes

$$\begin{aligned}\left(\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}d\xi_i\right)\wedge\omega =& \frac{\partial\phi}{\partial x_n}d\xi_n \wedge \omega \\ =& \phi d\xi_n \wedge \omega \\ =& \phi d^{\#}|x|^2/2 \wedge \omega\end{aligned}$$

where the first inequality follows from that ω is of the form $\omega(x)dx_1 \wedge d\xi_1 \wedge ... \wedge dx_{n-1} \wedge d\xi_{n-1}$ and second follows from that ϕ is 1-homogeneous and that we only concern ourselves with S^{n-1}.

∎

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