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Non-Supersymmetric AdS Solutions in Type IIB String Theory

Using S-folds to Test Swampland Criteria for Effective Field Theories to be Consistent with Quantum Gravity

Master's thesis in Physics

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Abstract

The task of finding a satisfactory theory of quantum gravity has turned out to be extremely challenging. In the context of string theory, which is a potential framework of quantum gravity, this problem is represented by the vast number of possible string compactifications. The swampland program is an effort to sort through these possibilities and define what makes some theories of quantum gravity inconsistent. The result is a number of so-called swampland conjectures. This thesis studies an AdS vacuum in type IIB string theory that is relevant to one of these conjectures. It is explicitly shown that this vacuum, which is an S-fold of the form $\mathrm{AdS}_4 \times S^1 \times S^5$, satisfies the type IIB equations of motion. The S-fold originates from uplifting a non-compact gauging of the 4-dimensional $N = 8$ supergravity. A more simple case illustrating non-compact gaugings, related to the gauge group SO(8), is treated here. Also discussed is the topology of the S-fold, which features a non-trivial SL(2, $\mathbb{Z}$) monodromy when the S^1 is encircled, making the background non-geometric.

The connection to the swampland program appears when a 2-parameter deformation of the AdS vacuum is used to break supersymmetry. Locally, these deformations only amount to a coordinate redefinition, which protects the vacuum solution from some non-perturbative decay channels. As the non-supersymmetric S-folds are also perturbatively stable, they have been suggested as a potential challenge to the Non-SUSY AdS conjecture. However, more evidence of non-perturbative stability is likely needed to make a solid case for non-supersymmetric AdS vacua in quantum gravity.

Keywords: quantum gravity, the swampland, supergravity, type IIB string theory, S-fold, non-compact gauging

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Johan Wikström, Gothenburg, December 2022

Contents

1
Introduction

1.1 Unification and quantum gravity

Some of the most important achievements in physics can be classified as unifications. One celebrated example is the unification of electricity and magnetism, done by Maxwell in 1865. The unified theory, called electromagnetism, was able to resolve inconsistencies in previous descriptions and also predicted new phenomenon, like electromagnetic waves [1].

Today, the main theories for elementary physics are general relativity and the Standard Model. General relativity describes gravity and the Standard Model describes electroweak- and strong interactions. Although both theories have had great experimental success, each have their respective problems [2]. First of all, general relativity is not renormalisable. This means that high energy virtual processes cause ultraviolet (UV) divergences that permeate into physical predictions [1]. General relativity can thus not be seen as a consistent quantum field theory. The Standard Model on the other hand, is renormalisable and thus consistent. However, there are still phenomena it has trouble explaining, such as neutrino masses and dark matter [3].

1.2 String theory/M-theory

String theory turns out to be a framework in which gravity appears in a very natural way, while still being compatible with quantum mechanics [1]. In string theory, the concept of point-like particles are replaced by a fundamental string whose vibrational modes correspond to physical particles. The framework of string theory can be described in 5 different ways [4]. These various formulations are related by duality transformations. They are also related to the effective field theory (EFT) of M-theory, a non-perturbative theory in 11 spacetime dimensions. This connection is illustrated in figure 1.1.

The low-energy effective theory of M-theory is called 11-dimensional supergravity [4]. From there, type IIA superstring theory can be reached via compactification to 10 dimensions. This is done by making one of the space coordinates periodic, replacing its topology $\mathbb{R}$ with a circle S^1. The other string theory formulations are also 10-dimensional. This appearance of extra dimensions may seem to contradict what is obvious; there are only 4 spacetime dimensions. However, if the remaining

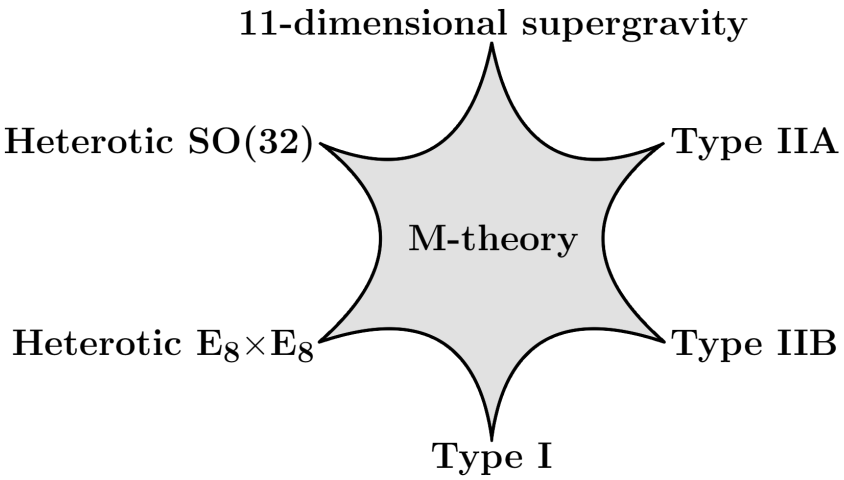

Figure 1.1: The amoeba diagram illustrates how 11-dimensional supergravity and the five string theories relate to M-theory [5]. While 11-dimensional supergravity is the low-energy limit of M-theory, the superstring theories are connected to M-theory via their low-energy supergravities [4].

extra dimensions are compact and very small, the possibility cannot be completely rejected. At length scales similar to the size of additional compact dimensions, there would be great effects on for example gravity. Gravitational experiments have not yet been able to dismiss such extra dimensions if their size is below 1µm [6]. In most contexts however, compact spaces are considered to have sizes as small as the Planck length $l_P \approx 1.6 \cdot 10^{-35}$m [4].

Other than the size, the geometry of the compact space is also of great importance. In particular, the topology is what largely determines what physics are obtained [2]. The compactification on S^1 which takes 11-dimensional supergravity to type IIA string theory is relatively straightforward since the compact space is 1-dimensional. When the compact space instead is 6- or 7-dimensional, the number of possible geometries becomes very large [1]. Finding the compactification that is in agreement with our nature turns out to be a daunting task.

1.3 The swampland program

Instead of looking for the correct compactification, recent efforts have been focused on finding more general features of background geometries. This effort, known as the swampland program, does not only encompass models in string theory, but also other theories of quantum gravity. There is however a notion of string universality, claiming that all consistent theories of quantum gravity can be described in string theory, but it is not proven [2]. Until further notice, the more general purpose of the swampland program is to form conditions that determine if an EFT can be UV completed to a consistent quantum gravity theory. If it can, it is said to be part of the landscape. EFTs that cannot are said to be in the swampland. The landscape and the swampland are distinguished by swampland constraints. These constraints should be a consequence of some properties of quantum gravity, but as these are not yet understood, there are only conjectures so far [2]. In the context of string theory, the swampland conjectures serve to classify the plausible background geometries, and some can even be proven. Being able to differentiate between EFTs in the landscape and swampland is of great importance in quantum gravity. It provides a unique way to associate physics at the Planck energy, the energy scale of quantum gravity, to physics at very low energies, even as low as the neutrino sector of the Standard Model at $\ll$ 1eV. The Standard Model possibly being incorrect is thus something that can be connected to quantum gravity.

The swampland conjectures do not only deal with compactifications as the geometry of the 4-dimensional spacetime is relevant as well. There are three types categorised by the sign of the cosmological constant Λ [7]. If $\Lambda > 0$ the spacetime is de Sitter (dS), if $\Lambda = 0$ it is Minkowski and if $\Lambda < 0$ it is anti-de Sitter (AdS). Even though Λ has been measured to be positive it is still of interest to study the other cases as they may provide clues to quantum gravity in general.

A swampland conjecture that is relevant for this thesis is the *Non-SUSY AdS conjecture* [8]. It claims that a vacuum with AdS spacetime has to be unstable unless it is supersymmetric (SUSY). Supersymmetry refers to an exchange-symmetry between fermions and bosons, which leaves the theory unchanged [9]. As this is a conjecture, it could be disproven by finding an example of a stable non-supersymmetric AdS vacuum solution in string theory. This method is however not very realistic since there could always be unknown channels through which the vacuum decays. However, it is still possible to challenge the conjecture by finding vacua that pass the known stability tests.

1.4 Outline of thesis

This thesis focuses on an AdS vacuum solution of type IIB string theory presented in [10]. Its geometry is described by an S-fold where deformations are introduced to break supersymmetry. The solution does not appear to be unstable, so it has been proposed as a challenge to the Non-SUSY AdS conjecture. The purpose of this thesis is to verify that the proposed vacuum satisfies the type IIB equations of motion, and to investigate some features of the solution. Specifically, non-compact gaugings, S-folds and stability are explored, mostly through more simple examples. Following is the thesis outline.

Chapter 2 introduces 11-dimensional supergravity and gives a first example of compactification to 4 dimensions via the Freund-Rubin ansatz. Another compactification, this time on a 7-torus, is then considered, in particular how the 4-dimensional theory can be gauged using non-compact groups while avoiding negative energies.

Chapter 3 gives a description of type IIB string theory and its equations of motion. The SL(2, $\mathbb{R}$) invariance of the field theory is also showcased. The vacuum solution to type IIB string theory that was suggested as a contradiction to the Non-SUSY AdS conjecture is then presented. In particular, the S-fold geometry and the supersymmetry breaking deformations are discussed. Finally, the equations of motion are also verified.

Chapter 4 discusses some modes of instability for vacuum solutions in string theory, and how these affect the type IIB S-fold given in [10]. The general conclusions of the thesis are then given.

Appendix A gives the definitions of the Levi Civita symbol and the generalised Kronecker delta that are used in the thesis. Differential forms are also defined and some general properties are given.

Appendix B derives the equations of motion from the Lagrangian of 11-dimensional supergravity.

Appendix C calculates the Riemann tensor for AdS_4 via Cartan's structure equations with zero torsion. This method is later used in appendix E and I.

Appendix D is based on [11] and derives $N = 8$ supergravity from 11-dimensional supergravity by dimensional reduction. Only the bosonic sector is considered.

Appendix E focuses on the Hopf fibration of S^3, which has some topological properties similar to those of the S-fold.

Appendix F derives the type field equations of type IIB string theory. This is done by varying an action that is complemented by the self-duality condition on the 5-form field strength.

Appendix G calculates the field strengths for the 2-form and 4-form fields of the type IIB AdS vacuum that is of interest in this thesis.

Appendix H shows that the AdS vacuum satisfies all type IIB equations of motion, except Einstein's equations.

Appendix I computes the Ricci tensor and curvature scalar for the AdS vacuum. These quantities are needed for appendix J.

Appendix J shows that Einstein's equations in type IIB string theory are fulfilled by the AdS vacuum.

2
11-dimensional supergravity

11-dimensional supergravity is the low-energy limit of M-theory and as the name suggests, it is a supersymmetric theory. In particular, it has $N = 1$ local supersymmetry invariance [11]. The theory obtained from dimensional reduction to $D = 4$ spacetime dimensions turns out to be more supersymmetric. The spinors in $D = 11$ are 32-dimensional, and split into 8 sets of 4-dimensional spinors, implying that the $D = 4$ theory has $N = 8$ supersymmetry [12]. Having $N > 8$ supersymmetry is not possible in 4 dimensions since it introduces fields with spin greater than 2 [9], which do not exist in nature [12]. This constraint of spin ≤ 2 in the 4-dimensional theory is what determines $D = 11$ as the highest dimension in which supergravity is consistent [12].

The field content and supersymmetry transformations of $D = 11$ supergravity are given in section 2.1, as well as the bosonic Lagrangian and the corresponding equations of motion. Section 2.2 focuses on how the 11-dimensional theory is prone to compactify spontaneously to 4 spacetime dimensions, which is motivated by the Freund-Rubin ansatz. The $N = 8$ supergravity in $D = 4$ is presented in section 2.3. Section 2.4 describes gaugings of $N = 8$ supergravity, in particular how non-compact gauge groups can be used without introducing ghosts. This is of relevance since the type IIB AdS vacuum also features a non-compact gauging.

2.1 Lagrangian and equations of motion

A virtue of doing supergravity in 11 dimensions, opposed to $D = 4$, is that the field content is much simpler. There is an elfbein ${e_M}^A$ corresponding to the graviton, a gravitino described by the vector-spinor Ψ_M and a 3-form gauge field A_{MNP} [12]. The capital latin indices M, N, P,... denote curved spacetime indices in 11 dimensions, while A, B, C,... are indices of the locally flat frame. The elfbein ${e_M}^A$ thus relates the curved spacetime metric g_{MN} to the flat Minkowski metric $\eta_{AB} = \mathrm{diag}(-1, 1,..., 1)$ via

$$g_{MN} = {e_M}^A {e_N}^B \eta_{AB}. \tag{2.1}$$

For the determinant $e = \det\left({e_M}^A\right)$ this implies that

$$g = \det(g_{MN}) = \det\left({e_M}^A {e_N}^B \eta_{AB}\right) = -e^2, \tag{2.2}$$

since $\det(\eta_{AB}) = -1$. The fields make up 128 fermionic and 128 bosonic degrees of freedom, where the bosonic are split as $44 + 84$ among ${e_M}^A$ and A_{MNP} respectively

[12]. The fields are related via the $N = 1$ local supersymmetry transformations given by

$$\delta e_M{}^A = i\bar{\epsilon}\Gamma^A\Psi_M, \tag{2.3}$$

$$\delta\Psi_M = D_M\epsilon - \frac{1}{288}(\Gamma_M{}^{PQRS} - 8\delta_M{}^P\Gamma^{QRS})(F_{PQRS} - 3\bar{\Psi}_{[P}\Gamma_{QR}\Psi_{S]})\epsilon, \tag{2.4}$$

$$\delta A_{MNP} = 3i\bar{\epsilon}\Gamma_{[MN}\Psi_{P]}, \tag{2.5}$$

where the parameter ϵ is anticommuting [13]. The Γ-matrices obey $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ and more indices are added via $\Gamma_{A_1 \ldots A_p} = \Gamma_{[A_1} \ldots \Gamma_{A_p]}$. The covariant derivative is defined as $D_M = \partial_M + \frac{1}{4}\omega_{MAB}\Gamma^{AB}$ where ω_{MAB} are the Lorentz connection coefficients [14]. The 3-form and its field strength, $F_{MNPQ} = 4\partial_{[M}A_{NPQ]}$, differs by a factor 2 from some literature, for example [12] and [14], so that $A^{\text{here}}_{MNP} = 2A^{\text{there}}_{MNP}$.

The supersymmetry transformations (2.3)-(2.5) make the Lagrangian of $D = 11$ supergravity unique up to higher order derivative terms [12]. The Lagrangian encodes how the 3 different types of fields interact. When looking for vacuum solutions to the theory, it is however necessary to set the vacuum expectation values (VEVs) of the fermionic fields to zero, $\langle\Phi_M\rangle = 0$, to obtain solutions with maximal spacetime symmetry [12, 13]. For this purpose only the bosonic Lagrangian of $D = 11$ supergravity is required. It is given by

$$\begin{aligned}\mathcal{L} = &\frac{1}{2\kappa_{11}^2}e\left[R - \frac{1}{2\cdot 4!}F_{MNPQ}F^{MNPQ}\right] + \\ &- \frac{1}{12\kappa_{11}^2}\frac{1}{3!(4!)^2}\varepsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}},\end{aligned} \tag{2.6}$$

where $\kappa_{11}^2 = 8\pi G^{(11)}$, $G^{(11)}$ being Newton's constant in 11 dimensions [13, 14]. The Levi-Civita tensor density in 11 dimensions is $\varepsilon^{M_1\ldots M_{11}}$, whereas the totally antisymmetric tensor is denoted by $\epsilon^{M_1\ldots M_{11}}$, see appendix A. The bosonic field equations, given by variation of g^{MN} and A_{MNP} respectively, are

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{12}\left(F_{MPQR}F_N{}^{PQR} - \frac{1}{8}g_{MN}F^2\right), \tag{2.7}$$

$$\nabla_M F^{MNPQ} = \frac{1}{1152}\epsilon^{NPQR_1\ldots R_8}F_{R_1R_2R_3R_4}F_{R_5R_6R_7R_8}. \tag{2.8}$$

The full derivation of the field equations is given in appendix B.

2.2 Spontaneous compactification

As was discussed in section 1.2, the 7 extra dimensions of $D = 11$ supergravity have to disappear, possibly via compactification. It is however not satisfactory to just force the 7 space dimensions to compactify. Instead, the compactification should appear spontaneously [12, 15]. In other words, the theory should have a vacuum solution with the topology $M_4 \times M_7$, where M_4 is a maximally symmetric spacetime and M_7 is compact. This turns out to be the case for $D = 11$ supergravity, which can be shown in a relatively straightforward way. First, the 11-dimensional coordinates are split as $x^M = (x^\mu,\, y^m)$, so that x^μ, $\mu = 0,..., 3$, denote 4-dimensional spacetime coordinates and y^m, $m = 1,..., 7$, denote 7-dimensional internal coordinates [12, 15]. To find a $M_4 \times M_7$-type solution the metric VEV is set as

$$\langle g_{MN} \rangle = \mathring{g}_{MN} = \begin{pmatrix} \mathring{g}_{\mu\nu}(x) & 0 \\ 0 & \mathring{g}_{mn}(y) \end{pmatrix}, \tag{2.9}$$

so that the M_4 metric VEV $\mathring{g}_{\mu\nu}$ does not depend on the internal coordinates, and vice versa [12]. The next step is to make the Freund-Rubin ansatz for the field strength VEV

$$\langle F^{\mu\nu\rho\sigma} \rangle = \mathring{F}^{\mu\nu\rho\sigma} = -6m\epsilon^{\mu\nu\rho\sigma}, \tag{2.10}$$

where $\langle F^{mnrs} \rangle = \langle F^{\mu nrs} \rangle = \langle F^{\mu\nu rs} \rangle = \langle F^{\mu\nu\rho s} \rangle = 0$ [15]. The parameter m is constant and associated with the radii of M_4 and M_7. The Freund-Rubin ansatz trivially satisfies (2.8) since the left hand side is proportional to $\nabla_M \epsilon^{MNPQ} = 0$, and the right hand side either features $\epsilon^{NPQR_1...R_8}$ with repeated indices or that $\mathring{F}_{R_1R_2R_3R_4}\mathring{F}_{R_5R_6R_7R_8} = 0$. The Bianchi identity

$$\nabla^M R_{MN} = \frac{1}{2}\nabla_N R, \tag{2.11}$$

is also fulfilled which follows from acting with ∇^M on (2.7), since the right hand side becomes zero. The Ricci scalar VEV is found by contracting (2.7) with g^{MN}

$$\mathring{R} - \frac{11}{2}\mathring{R} = \frac{1}{12}\left(\mathring{F}^2 - \frac{11}{8}\mathring{F}^2\right), \tag{2.12}$$

which implies that

$$\mathring{R} = \frac{1}{12}\frac{1-\frac{11}{8}}{1-\frac{11}{2}}\mathring{F}^2 = \frac{1}{12^2}\mathring{F}^2 = -\frac{6^2 4!}{12^2}m^2 = -6m^2, \tag{2.13}$$

where the squared field strength is given by

$$\mathring{F}^2 = \mathring{F}_{\mu\nu\rho\sigma}\mathring{F}^{\mu\nu\rho\sigma} = 6^2 m^2 \epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -6^2 4!\; m^2. \tag{2.14}$$

It is now possible to compute the Ricci tensor in the external coordinates x^μ, and the internal coordinates y^m. In four spacetime dimensions

$$\begin{aligned} \mathring{R}_{\mu\nu} &= \frac{1}{2}\mathring{g}_{\mu\nu}\mathring{R} + \frac{1}{12}\left(\mathring{g}_{\omega\nu}\mathring{F}_{\mu\rho\sigma\xi}\mathring{F}^{\omega\rho\sigma\xi} - \frac{1}{8}\mathring{g}_{\mu\nu}\mathring{F}^2\right) = \\ &= -\frac{6m^2}{2}\mathring{g}_{\mu\nu} + \frac{6^2m^2}{12}\mathring{g}_{\omega\nu}\epsilon_{\mu\rho\sigma\xi}\epsilon^{\omega\rho\sigma\xi} + \frac{6^2 4! m^2}{12\cdot 8}\mathring{g}_{\mu\nu} = \\ &= (-3 - 18 + 9)\, m^2\mathring{g}_{\mu\nu} = -12m^2\mathring{g}_{\mu\nu}, \end{aligned} \tag{2.15}$$

using that $\epsilon_{\mu\rho\sigma\xi}\epsilon^{\omega\rho\sigma\xi} = -3!\delta^\omega_\mu$, see appendix A. The Ricci tensor for the compact space is

$$\begin{aligned} \mathring{R}_{mn} &= \frac{1}{2}\mathring{g}_{mn}\mathring{R} + \frac{1}{12}\left(0 - \frac{1}{8}\mathring{g}_{mn}\mathring{F}^2\right) = \left(-\frac{6}{2} + \frac{6^2 4!}{12\cdot 8}\right)m^2\mathring{g}_{mn} = \\ &= (-3+9)m^2\mathring{g}_{mn} = 6m^2\mathring{g}_{mn}. \end{aligned} \tag{2.16}$$

The equations for the Ricci tensor on M_4 and M_7, which follow from the Freund-Rubin ansatz of (2.10), turn out to give the desired topological properties for spontaneous compactification. First off, the equation $\mathring{R}_{\mu\nu} = -12m^2\mathring{g}_{\mu\nu}$ describes AdS_4 with radius $L = \frac{2}{m}$, which is a maximally symmetric space [13]. See also appendix C for the metric and Ricci tensor of AdS_4. Furthermore, the internal space M_7 described by $\mathring{R}_{mn} = 6m^2\mathring{g}_{mn}$ has constant positive curvature and is compact [12, 16]. There are still infinitely many solutions for M_7, where one of the possibilities is the 7-sphere S^7.

As this vacuum solution satisfies the equations of motion, it shows that $D = 11$ supergravity can spontaneously compactify to a product space between AdS_4 and a compact space M_7. The compactification to 4 spacetime dimensions happens in a natural way because of the 4 indices in the field strength F_{MNPQ}, allowing for the ansatz in (2.10) [15]. It should also be noted that the ansatz $\mathring{F}^{mnpq} \sim \epsilon^{mnpq}$ is equally viable. The result is a vacuum solution on $\text{AdS}_7 \times M_4$, where M_4 is compact. There are however reasons to believe that compactification to 4-dimensional spacetime may be naturally preferred [12].

2.3 $N = 8$ supergravity

The $N = 8$ supergravity is obtained by dimensional reduction of 11-dimensional supergravity on a 7-torus [11], which is performed in appendix D. Its massless spectrum contains 1 graviton, 8 gravitinos, 28 vectors, 56 Majorana spinors and 70 scalars, 35 of which are pseudoscalars. Since the fermion VEVs are set to zero for vacuum solutions, only the bosonic part of the theory is considered. The field equations of $N = 8$ supergravity have a global invariance under $\text{E}_{7(+7)}$. The Lagrangian has a local $\text{SU}(8)/\mathbb{Z}_2$ symmetry and a global symmetry under $\text{SL}(8, \mathbb{R})$, both which are subgroups of $\text{E}_{7(+7)}$ [11].

2.3.1 The exceptional group $\text{E}_{7(+7)}$

E_7 is a 133-dimensional simple group of rank 7 [11]. The non-compact real form of E_7 considered here is $\text{E}_{7(+7)}$ whose algebra has 63 compact generators, spanning $\mathfrak{su}(8)$, and 70 non-compact generators. The signature of the Cartan-Killing form is thus $70 - 63 = +7$, consequently $\text{E}_{7(+7)}$. The Lie algebra of $\text{E}_{7(+7)}$ is denoted by $\mathfrak{e}_{7(+7)}$. A general element of the complex 56-dimensional representation of $\mathfrak{e}_{7(+7)}$ can be written as

$$\begin{pmatrix} \Lambda_{IJ}{}^{KL} & \Sigma_{IJPQ} \\ \bar{\Sigma}^{MNKL} & \bar{\Lambda}^{MN}{}_{PQ} \end{pmatrix}, \tag{2.17}$$

where $\Lambda_{IJ}{}^{KL} = \Lambda_{[I}{}^{[K}\delta_{J]}{}^{L]}$ [17]. The traceless 8×8 matrices $\Lambda_I{}^J$ are antihermitian, $\bar{\Lambda}^J{}_I = -\Lambda_I{}^J$, and the generators of the maximal compact subalgebra $\mathfrak{su}(8)$. The capital indices $I,\ J, ... = 1,..., 8$ are thus fundamental SU(8) indices. The completely antisymmetric, complex tensor Σ_{IJKL}, which is self-dual $\bar{\Sigma}^{MNPQ} = \frac{1}{4!}\eta\epsilon^{IJKLMNPQ}\Sigma_{IJKL}$, contains the 70 non-compact generators [17]. The general phase η is chosen as $+1$.

The $E_{7(+7)}$ algebra element in (2.17) reveals two 63-dimensional maximal subalgebras. The first one is $\mathfrak{su}(8)$ whose embedding in $\mathfrak{e}_{7(+7)}$ is obtained by setting $\Sigma_{IJKL} = 0$. Note however that the corresponding maximal compact subgroup of $E_{7(+7)}$ is $SU(8)/\mathbb{Z}_2$ [11]. The second subgroup is found by taking the complex generators $\Lambda_I{}^J$ and Σ_{IJKL} as real. Denoting the real-valued generators with indices i, j,... = 1,..., 8, the 28 antisymmetric Λ_i^j generate $\mathfrak{so}(8)$. The 28 Λ_i^j along with the 35 non-compact Σ_{ijkl} form the general element

$$\begin{pmatrix} \Lambda_{ij}{}^{kl} & \Sigma_{ijpq} \\ \Sigma^{mnkl} & \Lambda^{mn}{}_{pq} \end{pmatrix}, \tag{2.18}$$

of the maximal subalgebra $\mathfrak{sl}(8, \mathbb{R})$ [17]. The lower-case indices i, j,... are SO(8) vector indices.

2.3.2 The ungauged $N = 8$ supergravity

The ungauged bosonic Lagrangian of $N = 8$ supergravity takes a simple form since all interaction terms involve fermions. Schematically, it is written as

$$\mathcal{L}_0 = \frac{1}{2} e\ R + \mathcal{L}_S + \mathcal{L}_V, \tag{2.19}$$

where $e = \det\left(e_\mu{}^\alpha\right)$ is the determinant of the 4-dimensional vierbein [11, 17]. Alongside the Einstein-Hilbert term $\frac{1}{2}e\ R$, there is a scalar kinetic term $\mathcal{L}_S$ and a vector kinetic term $\mathcal{L}_V$. The 70 scalars live in the coset space $E_{7(+7)}/(SU(8)/\mathbb{Z}_2)$ parametrised by

$$\mathcal{V} = \begin{pmatrix} u_{IJ}{}^{KL} & v_{IJPQ} \\ \bar{v}^{MNKL} & \bar{u}^{MN}{}_{PQ} \end{pmatrix}, \tag{2.20}$$

in the fundamental 56 representation of $E_{7(+7)}$ [18]. Here I, J, K... = 1, ..., 8 are indices of the fundamental 8 of SU(8). It transforms like

$$\mathcal{V} \to U(x) \mathcal{V} E^{-1}, \tag{2.21}$$

under the action of $E_{7(+7)}$, where $U(x) \in SU(8)/\mathbb{Z}_2$ and $E \in E_{7(+7)}$. $\mathcal{V}$ can also be written as an exponential of the $E_{7(+7)}$ Lie algebra, $\mathcal{V} = \exp(Y)$, where

$$Y = S \begin{pmatrix} 0 & -\frac{1}{4}\sqrt{2}\varphi_{IJKL} \\ -\frac{1}{4}\sqrt{2}\bar{\varphi}^{IJKL} & 0 \end{pmatrix}. \tag{2.22}$$

The element $S \in \mathfrak{su}(8)$ can be set to unity in the so called symmetric gauge, which makes it possible to identify the 70 scalars with the 70 non-compact generators $\Sigma_{IJKL} = -\frac{1}{4}\sqrt{2}\varphi_{IJKL}$ of $\mathfrak{e}_{7(+7)}$ [18]. The scalars enter the Lagrangian via the kinetic term

$$\mathcal{L}_S = -\frac{1}{24} e\ \operatorname{tr}\left(\left[D_\mu \mathcal{V} \cdot \mathcal{V}^{-1}\right]^2\right), \tag{2.23}$$

where D_μ is a SU(8) covariant derivative. The SU(8) connection in D_μ is defined so that it cancels the contributions of the compact generators in the Lie algebra element $\mathscr{V}^{-1}\partial_\mu \mathscr{V}$ [11].

The 28 vector fields A_μ^{ij} transform in the 8 of SL(8, $\mathbb{R}$) under $E_{7(+7)}$ and are thus written with SO(8) indices [18]. Specifically, the transformation under $E_{7(+7)}$ is given by

$$\delta A_\mu^{ij} = ({\Lambda^{ij}}_{kl} - \Sigma^{ijkl}) A_\mu^{kl}. \tag{2.24}$$

The corresponding field strengths $F_{\mu\nu}^{ij} = 2\partial_{[\mu} A_{\nu]}^{ij}$ couple via a positive definite matrix $N(\varphi)^{IJ,KL}$ in the vector kinetic term $\mathcal{L}_V$ [18]. The matrix is defined as

$$N(\varphi)_{IJ,KL} = \left(\frac{\mathbf{1} + y(\varphi)}{\mathbf{1} - y(\varphi)} \right)_{IJ,KL}, \tag{2.25}$$

where $\mathbf{1}_{IJ,KL} = \frac{1}{2}(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK})$. The dependence on the scalar fields enters via $y(\varphi)_{IJ,KL} = -{(u^{-1})_{IJ}}^{MN} v_{MNKL}$. The vector kinetic term of the Lagrangian reads [17]

$$\mathcal{L}_V = -\frac{1}{8} e\; F^+_{\mu\nu ij} N(\varphi)^{ij,kl} {F^{+\mu\nu}}_{kl} + \text{h.c.}, \tag{2.26}$$

where $F^+_{\mu\nu}$ is the self-dual component of $F_{\mu\nu}$. The anti-self-dual component is denoted by $F^-_{\mu\nu}$. The full bosonic Lagrangian is then

$$\mathcal{L}_0 = \frac{1}{2} e\; R - \frac{1}{8} e\; \left(F^+_{\mu\nu ij} N(\varphi)^{ij,kl} {F^{+\mu\nu}}_{kl} + \text{h.c.} \right) - \frac{1}{24} e\; \text{tr}\left(\left[D_\mu \mathscr{V} \cdot \mathscr{V}^{-1} \right]^2 \right). \tag{2.27}$$

2.4 Gaugings of $N = 8$ supergravity

Gaugings of non-compact groups are often problematic as some of the vector particles in the gauged theory become unphysical, so-called ghosts [19]. These ghosts appear with negative kinetic energy in the standard vector kinetic term of the Lagrangian, $\mathcal{L}_V \sim -K_{AB} F^A_{\mu\nu} F^{B\mu\nu}$, since the Cartan-Killing metric K_{AB} is not positive definite. Considering the non-compact group SL(2, $\mathbb{R}$) as an example, the Cartan-Killing metric is

$$K_{AB} = \text{tr}\left(a^A a^B \right) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{2.28}$$

so that the corresponding term of the Lagrangian is

$$\mathcal{L}_V \sim -2F^1_{\mu\nu} F^{1\mu\nu} - 2F^2_{\mu\nu} F^{2\mu\nu} + 2F^3_{\mu\nu} F^{3\mu\nu}. \tag{2.29}$$

The 3 generators a^1, a^2, a^3 of $\mathfrak{sl}(2, \mathbb{R})$ are given by

$$a^1 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a^2 = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a^3 = e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.30}$$

Only a^3 is compact as it generates SO(2) which corresponds to S^1. Non-compact generators instead generate $\mathbb{R}$ [4]. Replacing K_{AB} with a positive definite metric such as δ_{AB} would break the gauge invariance. In some cases however, like the $N = 8$

supergravity, a positive definite metric can be constructed from the scalar fields so that gauge symmetry of a non-compact group still is possible [19].

The ungauged $N = 8$ supergravity has a global symmetry under SL(8, $\mathbb{R}$) transformations. Candidates for gauge symmetries are thus subgroups $K \subset$ SL(8, $\mathbb{R}$). To avoid introducing new particles, the dimension of K should be less than or equal to 28, the number of vector fields A_μ^{ij} in the theory [19].

2.4.1 The SO(8) gauging

As a starting point, the gauged theory of SO(8), which is the maximally compact subgroup of SL(8, $\mathbb{R}$), is considered. The SO(8) gauge theory is obtained by adding minimal gauge couplings to the vectors and the scalars [18]. The field strength becomes

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + g[A_\mu,\, A_\nu] = 2\partial_{[\mu}A_{\nu]} + gA^{kl}_{[\mu}A^{mn}_{\nu]}f_{kl,mn}{}^{ij}\Lambda_{ij}, \tag{2.31}$$

which amounts to replacing ∂_μ in the ungauged field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, with a SO(8) covariant derivative. Here $A_\mu = A_\mu^{ij}\Lambda_{ij}$, where the 28 antisymmetric 8×8 matrices Λ_{ij} generate $\mathfrak{so}(8)$

$$[\Lambda_{kl},\, \Lambda_{mn}] = f_{kl,mn}{}^{ij}\Lambda_{ij} = 4\Lambda_{[k[m}\delta_{n]l]}, \tag{2.32}$$

and i, j,... = 1,..., 8 are thus SO(8) vector indices. In a similar fashion, the SU(8) covariant derivative D_μ, which appears in the scalar kinetic term via $D_\mu\mathscr{V} \cdot \mathscr{V}^{-1}$, has to be changed to a SU(8)×SO(8) covariant derivative $\mathscr{D}_\mu$ given by

$$\mathscr{D}_\mu\mathscr{V} \cdot \mathscr{V}^{-1} = D_\mu\mathscr{V} \cdot \mathscr{V}^{-1} - 2g\mathscr{V} \cdot D(A_\mu) \cdot \mathscr{V}^{-1}, \tag{2.33}$$

where $D(A_\mu^{ij})$ is in the $\mathfrak{so}(8)$ subalgebra of $\mathfrak{e}_{7(+7)}$

$$D(A_\mu) = \begin{pmatrix} A_{\mu[i}{}^{[k}\delta_{j]}{}^{l]} & 0 \\ 0 & -A_{\mu[i}{}^{[k}\delta_{j]}{}^{l]} \end{pmatrix}. \tag{2.34}$$

These modifications will however break supersymmetry, which is restored by adding g-dependent terms to the supersymmetry transformations, as well as the Lagrangian [17]. By writing these terms, along with the minimal couplings, as $\mathcal{L}_g$, the Lagrangian with local SO(8) gauge symmetry can be written as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g, \tag{2.35}$$

where $\mathcal{L}_0$ is the ungauged Lagrangian of $N = 8$ supergravity. The non-compact gaugings can be obtained in a similar way, however this method depends on some complicated identities [18]. The non-compact gaugings can instead be found in a more simple manner by utilising the SO(8) gauging.

2.4.2 The non-compact gaugings

The SL(8, $\mathbb{R}$) invariance of the ungauged Lagrangian $\mathcal{L}_0$ is no longer present in the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g$ of the SO(8) model. Instead there is the local SO(8) invariance $\mathcal{L} \overset{\mathrm{SO}(8)}{\to} \mathcal{L}$. Acting with the remainder SL(8, $\mathbb{R}$)\SO(8), which is the non-compact part of SL(8, $\mathbb{R}$), results in the transformation

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g \overset{\mathrm{SL}(8,\ \mathbb{R})\backslash\mathrm{SO}(8)}{\to} \mathcal{L}' = \mathcal{L}_0 + \mathcal{L}'_g, \tag{2.36}$$

since $\mathcal{L}_0 \overset{\mathrm{SL}(8,\ \mathbb{R})}{\to} \mathcal{L}_0$ [17]. Only the terms involving gauge coupling dependent terms are modified. If the coupling constant g also is rescaled in a certain way, an equivalent gauge theory with a new gauge group is obtained [17]. The transformation in the non-compact part of SL(8, $\mathbb{R}$) is performed via the one-parameter subgroup

$$E(t) = e^{tY}, \qquad Y = \begin{pmatrix} 0 & X_{ijkl} \\ X^{ijkl} & 0 \end{pmatrix}, \tag{2.37}$$

where the real, self-dual 4-form X_{ijkl} is SO(p)×SO(q) invariant and i, j,... = 1,..., 8 again denote SO(8) vector indices. The 4-form X_{ijkl} can be constructed from the 8×8 matrix

$$X_{ab} = \begin{pmatrix} \alpha \mathbf{1}_p & 0 \\ 0 & \beta \mathbf{1}_q \end{pmatrix}, \tag{2.38}$$

where a, b,... = 1,..., 8 are right-handed SO(8) spinor indices. X_{ab} is symmetric and traceless via the condition $\alpha p + \beta q = 0$. The 4-form X_{ijkl} is then constructed from the SO(8) gamma matrices Γ_i^{ab} like [18]

$$X_{ijkl} = -\frac{1}{8}(\Gamma_{[i}\Gamma_j\Gamma_k\Gamma_{l]})^{ab} X_{ab}. \tag{2.39}$$

Considered as a 28×28 matrix, $X_{ij,kl}$ has the eigenvalues α, β and $\gamma = \frac{1}{2}(\alpha + \beta)$. Their degeneracies are $d_\alpha = \dim \mathrm{SO}(p) = \frac{1}{2}p(p-1)$, $d_\beta = \dim \mathrm{SO}(q)$ and $d_\gamma = pq$, so that $X_{ij,kl}$ is traceless via

$$\frac{1}{2}p(p-1)\alpha + \frac{1}{2}q(q-1)\beta + pq\frac{1}{2}(\alpha + \beta) = \frac{1}{2}(\alpha p + \beta q)(p + q - 1) = 0. \tag{2.40}$$

The matrix $X_{ij,kl}$ can be written in terms of the projection operators $P_\alpha^{ij,kl}$, $P_\beta^{ij,kl}$ and $P_\gamma^{ij,kl}$ associated with the eigenspaces

$$X^{ij,kl} = \alpha P_\alpha^{ij,kl} + \beta P_\beta^{ij,kl} + \gamma P_\gamma^{ij,kl}. \tag{2.41}$$

Acting on the algebra $\mathfrak{so}(8)$, P_α (P_β) projects onto the subalgebra $\mathfrak{so}(p)$ ($\mathfrak{so}(q)$) and P_γ projects onto the generators of the remainder $\mathrm{SO}(8)\backslash\{\mathrm{SO}(p) \times \mathrm{SO}(q)\}$. The scalars and vectors transform like

$$\mathcal{V} \to \mathcal{V} E(t)^{-1}, \qquad A_\mu \to e^{-tX} A_\mu, \tag{2.42}$$

which follows from (2.21) and (2.24). Since the different projections commute, $[P_x,\ P_y] = 0$ for x, $y = \alpha$, β, γ, and square to themselves, $P_x^2 = P_x$, the exponential e^{-tX} becomes

$$e^{-tX} = e^{-\alpha t} P_\alpha + e^{-\beta t} P_\beta + e^{-\gamma t} P_\gamma = e^{-\alpha t}\left(P_\alpha + \xi P_\beta + \sqrt{\xi} P_\gamma\right) = e^{-\alpha t} P, \tag{2.43}$$

where $\xi = e^{(\alpha-\beta)t}$. Similarly

$$e^{tX} = e^{\alpha t}\left(P_\alpha + \frac{1}{\xi}P_\beta + \frac{1}{\sqrt{\xi}}P_\gamma\right) = e^{\alpha t}P^{-1}. \tag{2.44}$$

The rescaling of the coupling constant is chosen as

$$g \to g' = ge^{\alpha t}, \tag{2.45}$$

in order to obtain the non-compact gaugings. In the transformation of A_μ, P acts on the $\mathfrak{so}(8)$ generators like

$$\Lambda \to \Lambda' = P\Lambda = \Lambda_\alpha + \xi\Lambda_\beta + \sqrt{\xi}\Lambda_\gamma, \tag{2.46}$$

so that

$$A_\mu \to e^{-tX}A_\mu = e^{-\alpha t}A_\mu^{ij}P^{ij,kl}\Lambda_{kl} = e^{-\alpha t}A_\mu^{ij}\Lambda'_{ij}. \tag{2.47}$$

Since the metric transforms like

$$N(\varphi)^{ij,kl} \to \left(e^{tX}\right)^{ij,mn} N(\varphi)^{mn,pq}\left(e^{tX}\right)^{pq,kl}, \tag{2.48}$$

the vector kinetic term transforms like

$$N(\varphi)^{ij,kl}F^+_{\mu\nu ij}F^{+\mu\nu}{}_{kl} + \text{h.c.} \to N(\varphi)^{ij,kl}F'^+_{\mu\nu ij}F'^{+\mu\nu}{}_{kl} + \text{h.c.}, \tag{2.49}$$

where

$$F'_{\mu\nu} = e^{tX}\left(2\partial_{[\mu}[e^{-tX}A_{\nu]}] + g'[e^{-tX}A_\mu,\, e^{-tX}A_\nu]\right). \tag{2.50}$$

The new primed field strength becomes

$$F'_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + gP^{-1}[PA_\mu,\, PA_\nu]. \tag{2.51}$$

It is now essential to figure out what happens with the second term. To do this the commutator $[\Lambda',\, \Lambda']$ that appears in $[PA_\mu,\, PA_\nu]$ is needed. First, consider the structure of the $\mathfrak{so}(8)$ algebra which can be schematically written as

$$\begin{aligned} &[\Lambda_\alpha,\, \Lambda_\alpha] \sim \Lambda_\alpha\delta, && [\Lambda_\beta,\, \Lambda_\beta] \sim \Lambda_\beta\delta, && [\Lambda_\gamma,\, \Lambda_\gamma] \sim \Lambda_\alpha\delta + \Lambda_\beta\delta, \\ &[\Lambda_\alpha,\, \Lambda_\beta] \sim 0, && [\Lambda_\alpha,\, \Lambda_\gamma] \sim \Lambda_\gamma\delta, && [\Lambda_\beta,\, \Lambda_\gamma] \sim \Lambda_\gamma\delta. \end{aligned} \tag{2.52}$$

Using this and $\Lambda'_\alpha = \Lambda_\alpha$, $\Lambda'_\beta = \xi\Lambda_\beta$, $\Lambda'_\gamma = \sqrt{\xi}\Lambda_\gamma$, the Lie algebra of the primed generators becomes

$$\begin{aligned} &[\Lambda'_\alpha,\, \Lambda'_\alpha] \sim \Lambda'_\alpha\delta, && [\Lambda'_\beta,\, \Lambda'_\beta] \sim \xi\Lambda'_\beta\delta, && [\Lambda'_\gamma,\, \Lambda'_\gamma] \sim \xi\Lambda'_\alpha\delta + \Lambda'_\beta\delta, \\ &[\Lambda'_\alpha,\, \Lambda'_\beta] \sim 0, && [\Lambda'_\alpha,\, \Lambda'_\gamma] \sim \Lambda'_\gamma\delta, && [\Lambda'_\beta,\, \Lambda'_\gamma] \sim \xi\Lambda'_\gamma\delta. \end{aligned} \tag{2.53}$$

The algebra, corresponding to the subgroup $K \subset \mathrm{SL}(8,\, \mathbb{R})$, that the Λ' generate is

$$[\Lambda'_{ij},\, \Lambda'_{kl}] = f'{}_{ij,kl}{}^{mn}\Lambda'_{mn} = 4\Lambda'_{[i[k}\eta_{l]j]}, \tag{2.54}$$

where the metric is

$$\eta_{ij} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & \xi\mathbf{1}_q \end{pmatrix}. \tag{2.55}$$

If $\xi = -1$ is chosen then $K = \mathrm{SO}(p,\, q)$ and if $\xi = 0$ then K is the group contraction $\mathrm{CSO}(p,\, q)$ [18]. Setting $\xi = 1$ returns the SO(8) model. The second term of (2.51) becomes

$$\begin{aligned} gP^{-1}[PA_\mu,\, PA_\nu] &= gP^{-1}A^{kl}_{[\mu}A^{mn}_{\nu]}[\Lambda'_{kl}, \Lambda'_{mn}] = \\ &= gA^{kl}_{[\mu}A^{mn}_{\nu]}{f'_{kl,mn}}^{ij}(P^{-1})^{ij,rs}\Lambda'_{rs} = \\ &= gA^{kl}_{[\mu}A^{mn}_{\nu]}{f'_{kl,mn}}^{ij}\Lambda_{ij}. \end{aligned} \tag{2.56}$$

The gauge coupling of the vector fields changes under a transformation parametrised by $E(t)$ and the field strength

$$F'_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + g[A_\mu,\, A_\nu]_\xi, \tag{2.57}$$

where the commutator $[.,\, .]_\xi$ is the one of (2.54), is gauge invariant under the non-compact subgroup K with parameter ξ. For the scalar kinetic term, acting with $E(t)$ yields that

$$\mathscr{D}_\mu\mathscr{V}\cdot\mathscr{V}^{-1} \to D'_\mu\mathscr{V}\cdot\mathscr{V}^{-1} - 2g\mathscr{V}E(t)^{-1}D(A'_\mu)E(t)\mathscr{V}, \tag{2.58}$$

where D'_μ denotes that the coupling constant g has transformed to $g' = e^{\alpha t}g$ in the SU(8) connection [18]. By defining

$$D'(A_\mu,\, t) = E(t)^{-1}D(A'_\mu)E(t), \tag{2.59}$$

the SU(8)×SO(8) covariant derivative $\mathscr{D}_\mu$ instead becomes SU(8)×K covariant since

$$\begin{aligned} [D'(\Lambda_1,\, t),\, D'(\Lambda_2,\, t)] &= E(t)^{-1}\left[D(\Lambda'_1),\, D(\Lambda'_2)\right]_\xi E(t) = \\ &= E(t)^{-1}D([\Lambda'_1,\, \Lambda'_2]_\xi)E(t) = \\ &= D'([\Lambda_1,\, \Lambda_2]_\xi,\, t). \end{aligned} \tag{2.60}$$

The SO(8) model is thus altered to become locally gauge invariant under the subgroup K instead. This was achieved by a transformation in the non-compact part of $\mathrm{SL}(8,\, \mathbb{R})$, that is $\mathrm{SL}(8,\, \mathbb{R})/\mathrm{SO}(8)$, via the one-parameter subgroup $E(t)$. The transformation acts on the SO(8) generators that appear in the gauge coupling terms of $\mathcal{L}$ so that the transformed generators span the algebra of K, defined in (2.54). Depending on how the parameter η is chosen, K can be non-compact. In particular, $K = \mathrm{SO}(p,\, q)$ if $\eta = -1$.

The type IIB AdS vacuum also features a non-compact gauging of the more complicated subgroup $[\mathrm{SO}(1,\, 1)\times\mathrm{SO}(6)]\ltimes\mathbb{R}^{12}\subset\mathrm{E}_{7(+7)}$ [20]. This gauging is also done in $N = 8$ supergravity after which it is uplifted to type IIB string theory. The $\mathrm{SO}(p,\, q)$ gaugings shown here thus provide a more simple example of how non-compact gauge groups may be used without ghosts appearing. As a final note, there is a swampland conjecture called the *Completeness Hypothesis* which implies that continuous gauge groups must be compact [2]. Hence, there is a wide belief that non-compact gaugings induce instability. This aspect of non-compact gaugings in regards to the AdS vacuum is not mentioned in [10].

3

Type IIB string theory and the AdS vacuum

Type IIB string theory is indirectly related to $D = 11$ supergravity via its T-duality with type IIA string theory. Both the type IIA and IIB theories are based on closed superstrings that exist in 10 dimensions, however type IIB is chiral while IIA is not [4]. This chapter starts off by introducing the Lagrangian and the field equations of type IIB string theory in section 3.1, where extra emphasis is put on the SL(2, $\mathbb{R}$) invariance of the field theory. This symmetry is relevant for the topology of the S-fold vacuum that is proposed to contradict the Non-AdS SUSY conjecture. This vacuum is presented in section 3.2, where its S-fold geometry and the supersymmetry breaking deformations are discussed. Section 3.2.5 is devoted to showing that the AdS vacuum satisfies the type IIB equations of motion.

3.1 Lagrangian and global SL(2, $\mathbb{R}$) symmetry

In the massless bosonic sector, both type IIA and type IIB string theory contain a graviton, a dilaton ϕ and a 2-form B_2 [4]. The remaining bosonic field content of the two theories differs however. The type IIB spectrum includes another real scalar χ, occasionally called an axion, another 2-form C_2, as well as a 4-form C_4. The field strength of C_4 is self-dual, which is a feature that cannot easily be incorporated in an action [4]. Writing down a covariant action for type IIB string theory is therefore difficult, although it can be done [21]. It is typically easier to use the action

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int d^{10}x \; e \left(R - \frac{1}{12} \mathbf{H}^T_{\mu\nu\rho} \mathcal{M} \mathbf{H}^{\mu\nu\rho} + \frac{1}{4} \operatorname{tr}\{\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}\} \right) + \\ - \frac{1}{8\kappa^2} \left(\int d^{10}x \; e \left| \tilde{F}_5 \right|^2 + \int \varepsilon_{ij} C_4 \wedge H_3^{(i)} \wedge H_3^{(j)} \right), \end{aligned} \tag{3.1}$$

given in the Einstein-frame, that together with the imposed self-duality condition

$$\tilde{F}_5 = \star \tilde{F}_5, \tag{3.2}$$

yields the correct bosonic equations of motion [4]. The spacetime indices μ, ν, ρ,... are 10-dimensional and the norm is given by $|F|^2 = \frac{1}{n!} g^{\mu_1\nu_1} ... g^{\mu_n\nu_n} F_{\mu_1...\mu_n} F_{\nu_1...\nu_n}$. In (3.1), the two 2-forms and their field strengths have been combined into 2-component vectors

$$\mathbf{B}_2 = \begin{pmatrix} B_2^{(1)} \\ B_2^{(2)} \end{pmatrix} = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \qquad \mathbf{H}_3 = d\mathbf{B}_2 = \begin{pmatrix} H_3^{(1)} \\ H_3^{(2)} \end{pmatrix} = \begin{pmatrix} H_3 \\ F_3 \end{pmatrix}, \tag{3.3}$$

and the SL(2, $\mathbb{R}$) axion-dilaton matrix that contains the scalar particles is

$$\mathcal{M} = e^{\phi}\begin{pmatrix} |\lambda|^2 & -\chi \\ -\chi & 1 \end{pmatrix}, \tag{3.4}$$

where $\lambda = \chi + ie^{-\phi}$. Under a global SL(2, $\mathbb{R}$) transformation, the complex scalar transforms via a Möbius transformation

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}, \qquad \Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \mathrm{SL}(2,\, \mathbb{R}), \tag{3.5}$$

while the axion-dilaton matrix transforms like $\mathcal{M} \to (\Lambda^{-1})^T\mathcal{M}(\Lambda^{-1})$ [4]. The combined 2-form transforms like $\mathbf{B}_2 \to \Lambda\mathbf{B}_2$, showing that the type IIB theory has a global SL(2, $\mathbb{R}$) symmetry, since the Einstein-frame metric, the 4-form C_4 and the self-dual field strength

$$\tilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 = dC_4 + \frac{1}{2}\varepsilon_{ij}B_2^{(i)} \wedge H_3^{(j)}, \tag{3.6}$$

are SL(2, $\mathbb{R}$) invariant. The global SL(2, $\mathbb{R}$) symmetry of the field theory is however broken to the discrete SL(2, $\mathbb{Z}$) in the full string theory due to various effects [4]. A special case of the SL(2, $\mathbb{Z}$) symmetry, namely when $\lambda \to -\frac{1}{\lambda}$, results in the transformation $e^{\phi} \to e^{-\phi}$ when evaluated at $\chi = 0$. This is an example of a S-duality transformation under which type IIB string theory is invariant.

3.1.1 Bosonic equations of motion

The equations of motion are found by variation with respect to each field and then imposing the self-duality condition (3.2) [21]. The full derivation is found in appendix F and provides six equations

$$\begin{aligned} \delta\phi \ &: \ \nabla^{\mu}(e^{-\phi}\partial_{\mu}e^{\phi}) - e^{2\phi}\partial_{\mu}\chi\partial^{\mu}\chi = \frac{1}{12}\Big(e^{\phi}F_{\mu\nu\rho}F^{\mu\nu\rho} - 2e^{\phi}\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad\qquad + [e^{\phi}\chi^2 - e^{-\phi}]H_{\mu\nu\rho}H^{\mu\nu\rho}\Big), \end{aligned} \tag{3.7}$$

$$\delta\chi \ : \ \nabla^{\mu}\left(e^{2\phi}\partial_{\mu}\chi\right) = \frac{1}{6}e^{\phi}\left(\chi H_{\mu\nu\rho}H^{\mu\nu\rho} - F_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{3.8}$$

$$\delta B_{\mu\nu} \ : \ \nabla_{\rho}(e^{\phi}|\lambda|^2H^{\mu\nu\rho} - e^{\phi}\chi F^{\mu\nu\rho}) = \frac{1}{3!}\tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi}, \tag{3.9}$$

$$\delta C_{\mu\nu} \ : \ \nabla_{\rho}(e^{\phi}\chi H^{\mu\nu\rho} - e^{\phi}F^{\mu\nu\rho}) = \frac{1}{3!}\tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi}, \tag{3.10}$$

$$\delta C_{\mu\nu\rho\sigma} : \ \nabla_{\xi}\left(\tilde{F}^{\mu\nu\rho\sigma\xi}\right) = \frac{1}{(3!)^2}\epsilon^{\mu\nu\rho\sigma\mu_1\ldots\mu_6}H_{\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6}, \tag{3.11}$$

$$\begin{aligned} \delta g^{\mu\nu} \ &: \ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{8}\left(\mathrm{tr}\{\partial_{\rho}\mathcal{M}\partial^{\rho}\mathcal{M}^{-1}\} - \frac{1}{3}H^T_{\rho\sigma\xi}\mathcal{M}H^{\rho\sigma\xi}\right)g_{\mu\nu} + \\ &\qquad + \frac{1}{4}\Big(H^T_{\mu\rho\sigma}\mathcal{M}H_{\nu}{}^{\rho\sigma} - \mathrm{tr}\{\partial_{\mu}\mathcal{M}\partial_{\nu}\mathcal{M}^{-1}\} + \\ &\qquad + \frac{1}{4!}\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_{\nu}{}^{\rho\sigma\xi\omega}\Big), \end{aligned} \tag{3.12}$$

along with the self-duality condition (3.2). Note that the C_4 field equations have been used to write the 2-form equations of motion on the form given in (3.9) and (3.10).

3.1.2 Compactification on $\mathbf{AdS}_5 \times M_5$

A Freund-Rubin type compactification, similar to the one in section 2.2, can be done in type IIB string theory where the resulting geometry is $\mathrm{AdS}_5 \times M^5$ [13]. The 10-dimensional spacetime indices are split as $x^\mu = (x^{\bar{\mu}},\, y^m)$, where $\bar{\mu} = 0,\ldots,\, 4$ and $m = 1,\ldots,\, 5$. The metric VEV $\mathring{g}_{\mu\nu}$ is again taken on the block diagonal form of (2.9). All field VEVs are set to zero, except for the self-dual field strength for which the ansatz

$$\mathring{\tilde{F}}_5 = 4m\,(\epsilon_5 + \star\epsilon_5)\,, \tag{3.13}$$

is made [13]. The parameter m relates to the radii of AdS_5 and M_5, and ϵ_5 is the volume form of M_5

$$(\epsilon_5)_{mnpqr} = \epsilon_{mnpqr}. \tag{3.14}$$

It obeys $\star\star\,\epsilon_5 = \epsilon_5$ so that $\star\mathring{\tilde{F}}_5 = \mathring{\tilde{F}}_5$. All field equations are trivially fulfilled, except for the type IIB Einstein's equations (3.12) that can be used to find the Ricci tensor of the external and internal manifolds. Since the 10-dimensional Ricci scalar is $R = 0$, which is a consequence of $\tilde{F}_{\mu\nu\rho\sigma\xi}\tilde{F}^{\mu\nu\rho\sigma\xi} = 0$ [22], (3.12) reduces to

$$\mathring{R}_{\mu\nu} = \frac{1}{4\cdot 4!}\mathring{\tilde{F}}_{\mu\rho\sigma\xi\omega}\mathring{\tilde{F}}_{\nu}{}^{\rho\sigma\xi\omega}. \tag{3.15}$$

The Ricci tensors are evaluated as

$$\mathring{R}_{\bar{\mu}\bar{\nu}} = \frac{1}{4\cdot 4!}\mathring{\tilde{F}}_{\bar{\mu}\rho\sigma\xi\omega}\mathring{\tilde{F}}_{\bar{\nu}}{}^{\rho\sigma\xi\omega} = \frac{1}{4\cdot 4!}16m^2\epsilon_{\bar{\mu}\bar{\rho}\bar{\sigma}\bar{\xi}\bar{\omega}}\epsilon_{\bar{\nu}}{}^{\bar{\rho}\bar{\sigma}\bar{\xi}\bar{\omega}} = -4m^2\mathring{g}_{\bar{\mu}\bar{\nu}}, \tag{3.16}$$

for the external space and

$$\mathring{R}_{mn} = \frac{1}{4\cdot 4!}\mathring{\tilde{F}}_{m\rho\sigma\xi\omega}\mathring{\tilde{F}}_{n}{}^{\rho\sigma\xi\omega} = \frac{1}{4\cdot 4!}16m^2\epsilon_{mpqrs}\epsilon_{n}{}^{pqrs} = 4m^2\mathring{g}_{mn}, \tag{3.17}$$

for the internal space. The compactification is thus on $\mathrm{AdS}_5 \times M_5$, where M_5 is compact since it has positive and constant curvature [12, 16].

3.2 The AdS vacuum

Another possible compactification of the type IIB theory is the AdS vacuum which has been proposed as a possible contradiction to the Non-SUSY AdS conjecture of the swampland program. This vacuum solution to the type IIB equations of motion is an S-fold, further discussed in section 3.2.3, and is of the form $\mathrm{AdS}_4 \times S^1_\eta \times S^5$, where η is the parameter of S^1_η [10]. The S^5 refers to a squashed 5-sphere that is parametrised by two 2-spheres connected via an angular interval such that $S^5 = \mathcal{I}_\alpha \times S^2_1 \times S^2_2$, where $\alpha \in [0, \frac{\pi}{2}]$ [23]. At the endpoints of $\mathcal{I}_\alpha$, the 2-spheres contract to become point-like. This is similar to parametrising the regular 2-sphere S^2 with the spherical coordinates θ and φ. The polar angle θ parametrises the interval $[0, \pi]$

and φ parametrises a circle S^1_φ which corresponds to the equator at $\theta = \frac{\pi}{2}$. When moving away from $\theta = \frac{\pi}{2}$, the S^1_φ shrinks and eventually contracts to a single point at $\theta = 0$ and $\theta = \pi$.

This AdS vacuum stems from an uplift of a particular non-compact gauging of the 4-dimensional $N = 8$ supergravity. As was described in section 2.4.1, g-dependent terms have to be added when gauging $N = 8$ supergravity to preserve supersymmetry. Among these terms is a complicated scalar potential that non-linearly depends on the 70 scalars and the choice of gauge group [17, 24]. Finding minima of this scalar potential is equivalent to finding vacuum solutions of gauged models in $N = 8$ supergravity [24]. The gauging for the AdS vacuum considered here uses the non-compact subgroup $[\mathrm{SO}(1,\,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ of $\mathrm{E}_{7(+7)}$ [20]. 4-dimensional vacua of this particular gauging, corresponding to extrema of the scalar potential, can then be uplifted to 10-dimensional S-fold vacua of type IIB string theory [25].

3.2.1 The metric

The metric VEV of the $\mathrm{AdS}_4 \times S^1_\eta \times S^5$ vacua is given by the line element [10]

$$ds^2_{10} = \Delta^{-1}\left[\frac{1}{2}ds^2_{\mathrm{AdS}_4} + d\eta^2 + d\alpha^2 + \frac{\cos^2\alpha}{2+\cos 2\alpha}d\Omega_1 + \frac{\sin^2\alpha}{2-\cos 2\alpha}d\Omega_2\right], \tag{3.18}$$

where Δ is a non-singular warping factor defined by

$$\Delta^{-4} = 4 - \cos^2(2\alpha). \tag{3.19}$$

With ordinary 2-spheres Ω_1 and Ω_2, this vacua is $N = 4$ supersymmetric and thus perturbatively stable [26]. However, by performing a flat deformation with parameters χ_i, $i = 1,\,2$, on the azimuthal angles

$$d\Omega_i = d\theta_i^2 + \sin^2\theta_i\; d\varphi'^2_i, \qquad d\varphi'_i = d\varphi_i + \chi_i d\eta, \tag{3.20}$$

a new type of vacua is obtained that is generally non-supersymmetric, unless $\chi_1 = -\chi_2$ for which it possesses $N = 2$ supersymmetry [10]. Written out explicitly, the metric is

$$g_{\mu\nu} = \frac{1}{\Delta}\begin{pmatrix} \frac{1}{2}\tilde{g}_{\mu\nu} & & & & & & \\ & \begin{matrix} 1 + x_1^2\chi_1^2\sin^2\theta_1 + \\ + x_2^2\chi_2^2\sin^2\theta_2 \end{matrix} & & & x_1^2\chi_1\sin^2\theta_1 & & x_2^2\chi_2\sin^2\theta_2 \\ & & 1 & & & & \\ & & & x_1^2 & & & \\ & x_1^2\chi_1\sin^2\theta_1 & & & x_1^2\sin^2\theta_1 & & \\ & & & & & y^2 & \\ & x_2^2\chi_2\sin^2\theta_2 & & & & & x_2^2\sin^2\theta_2 \end{pmatrix}, \tag{3.21}$$

where the α-dependent parameters x_1 and x_2 are given by

$$x_1^2 = \frac{\cos^2\alpha}{2+\cos(2\alpha)}, \qquad x_2^2 = \frac{\sin^2\alpha}{2-\cos(2\alpha)}. \tag{3.22}$$

The indices in (3.21) are $\mu,\ \nu = t,\, \rho,\, \theta,\, \varphi,\, \eta,\, \alpha,\, \theta_1,\, \varphi_1,\, \theta_2,\, \varphi_2$ in order, so that g_{tt} is the top left element and $g_{\varphi_2\varphi_2}$ is in the bottom right. The indices of the metric on AdS_4 with radius $L = 1$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -\cosh^2\rho & & & \\ & 1 & & \\ & & \sinh^2\rho & \\ & & & \sinh^2\rho\ \sin^2\theta \end{pmatrix}, \tag{3.23}$$

are understood to only take the values $t,\ \rho,\ \theta,\ \varphi$. The zehnbein, with the AdS_4 vierbein denoted as $\tilde{e}_\mu{}^a$, is given by

$$e_\mu{}^a = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} \frac{1}{\sqrt{2}}\tilde{e}_\mu{}^a & & & & & & \\ & 1 & & & \chi_1 x_1 \sin\theta_1 & & \chi_2 x_2 \sin\theta_2 \\ & & 1 & & & & \\ & & & x_1 & & & \\ & & & & x_1 \sin\theta_1 & & \\ & & & & & x_2 & \\ & & & & & & x_2 \sin\theta_2 \end{pmatrix}, \tag{3.24}$$

where the flat index a takes the values 0,..., 9, with $\tilde{e}_\mu{}^a$ only being non-zero for a = 0,..., 3. For $e_\mu{}^a$, the lower μ-indices are rows and the upper a-indices are columns. The inverse zehnbein $e_a{}^\mu$, where instead the flat a-indices are the rows, reads

$$e_a{}^\mu = \sqrt{\Delta} \begin{pmatrix} \sqrt{2}\tilde{e}_a{}^\mu & & & & & & \\ & 1 & & & -\chi_1 & & -\chi_2 \\ & & 1 & & & & \\ & & & \frac{1}{x_1} & & & \\ & & & & \frac{1}{x_1 \sin\theta_1} & & \\ & & & & & \frac{1}{x_2} & \\ & & & & & & \frac{1}{x_2 \sin\theta_2} \end{pmatrix}. \tag{3.25}$$

Using the notation $g = g_{\mu\nu}$, $e = e_\mu{}^a$, $e^{-1} = e_a{}^\mu$, the metric and its inverse can be calculated as

$$g = ee^T, \qquad g^{-1} = (ee^T)^{-1} = (e^{-1})^T(e^{-1}). \tag{3.26}$$

Explicitly, the inverse metric takes the form

$$g^{\mu\nu} = \Delta \begin{pmatrix} 2\ \tilde{g}^{\mu\nu} & & & & & & \\ & 1 & & & -\chi_1 & & -\chi_2 \\ & & 1 & & & & \\ & & & x_1^{-2} & & & \\ & -\chi_1 & & & \chi_1^2 + \frac{1}{x_1^2 \sin^2\theta_1} & & \chi_1\chi_2 \\ & & & & & x_2^{-2} & \\ & -\chi_2 & & & \chi_1\chi_2 & & \chi_2^2 + \frac{1}{x_2^2 \sin^2\theta_2} \end{pmatrix}. \tag{3.27}$$

Going back to the standard notation, the determinant of the zehnbein, $e = \det\left(e_\mu{}^a\right)$, is simply given by multiplying the diagonal elements of (3.24)

$$\begin{aligned} e &= \frac{1}{\sqrt{2}^4} \frac{1}{\sqrt{\Delta}^{10}} \tilde{e}\ x_1^2\ x_2^2\ \sin\theta_1\ \sin\theta_2 = \frac{1}{4}\Delta^{-5}\ \tilde{e}\ \frac{\cos^2\alpha\ \sin^2\alpha}{4-\cos^2(2\alpha)} \sin\theta_1\ \sin\theta_2 = \\ &= \frac{1}{16}\Delta^{-1}\ \tilde{e}\ \sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2, \end{aligned} \tag{3.28}$$

using the definition of the warping factor in (3.19). The metric determinant $g = \det(g_{\mu\nu})$ is then given by

$$g = -e^2 = \frac{1}{16}\ \tilde{g}\ \Delta^{-2} \cos^4\alpha\ \sin^4\alpha\ \sin^2\theta_1\ \sin^2\theta_2, \tag{3.29}$$

where $\tilde{g} = -\tilde{e}^2$ is used. While the metric is dependent on the deformation parameters χ_i, they do not appear in the determinants e and g.

3.2.2 The VEVs of the fields

The dilaton ϕ and axion χ VEVs are given by

$$e^\phi = \sqrt{2}e^{-2\eta}\frac{2-\cos(2\alpha)}{\sqrt{7-\cos(4\alpha)}} = e^{-2\eta}\sqrt{\frac{2-\cos(2\alpha)}{2+\cos(2\alpha)}}, \qquad \chi = 0. \tag{3.30}$$

The 2-forms have the VEVs

$$B_2 = -2\sqrt{2}e^{-\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}\mathrm{vol}_{\Omega_1}, \qquad C_2 = -2\sqrt{2}e^{\eta}\frac{\sin^3\alpha}{2-\cos(2\alpha)}\mathrm{vol}_{\Omega_2}, \tag{3.31}$$

where $\mathrm{vol}_{\Omega_i} = \sin\theta_i\ d\theta_i \wedge d\varphi_i'$ are the volume forms of the 2-sphere metrics. By using two new 1-forms $A_i = -\cos\theta_i d\varphi_i'$, such that $dA_i = \mathrm{vol}_{\Omega_i}$, and the 3-form ω_3 that satisfies $d\omega_3 = \mathrm{vol}_{\mathrm{AdS}_4}$ with AdS radius 1, the 4-form VEV is given by

$$C_4 = \frac{3}{2}\omega_3 \wedge \left(d\eta + \frac{2}{3}\sin(2\alpha)d\alpha\right) - \frac{1}{2}f(\alpha)d\alpha \wedge \left(A_1 \wedge \mathrm{vol}_{\Omega_2} + \mathrm{vol}_{\Omega_1} \wedge A_2\right), \tag{3.32}$$

where the function $f(\alpha)$ is

$$f(\alpha) = \sin^2(2\alpha)\frac{\cos(4\alpha)-55}{(7-\cos(4\alpha))^2}. \tag{3.33}$$

These VEVs, along with the metric (3.21), satisfy the type IIB equations of motion, which is discussed further in section 3.2.5.

3.2.3 S-folds

This type of vacuum topology, with or without the χ_i-deformations, is referred to as an S-fold [25]. The name comes from a non-trivial S-duality monodromy when encircling the S^1_η. Note that a translation in η, given by the $\mathrm{SL}(2,\mathbb{R})$ transformation

$$A(\eta) = \begin{pmatrix} e^{-\eta} & 0 \\ 0 & e^{\eta} \end{pmatrix}, \tag{3.34}$$

affects only the 2-forms and the dilaton. Since S^1_η is compact, the coordinate η must have some periodicity $\eta \simeq \eta + T$ [27]. However, going around S^1_η highlights that the monodromy

$$\mathfrak{M}_{S^1} = A^{-1}(\eta)A(\eta+T) = \begin{pmatrix} e^{-T} & 0 \\ 0 & e^{T} \end{pmatrix}, \tag{3.35}$$

arises when the periodicity is forced onto η. Although the transformation $\mathfrak{M}_{S^1}$ does not change the topological position, it alters the 2-form and dilaton fields. The background is thus globally non-geometric, although it is well-defined locally. This is reminiscent of the Hopf fibration of S^3, where $S^3 = S^2 \times S^1$ locally, but globally there is a twist of the S^1 fiber when encircling the equator of the S^2, see appendix E. The monodromy can be generalised to belong to SL(2, $\mathbb{Z}$), the global symmetry group of the full type IIB string theory, where it takes the form

$$\mathfrak{M}(n) = -\mathcal{S}\mathcal{T}^n = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{3.36}$$

Here $\mathcal{S}$, $\mathcal{T} \in$ SL(2, $\mathbb{Z}$) are the generators of inversions $z \to -\frac{1}{z}$ and translations $z \to z+1$, respectively [27]. The SL(2, $\mathbb{Z}$) monodromy is obtained by replacing the twist in (3.34) with

$$A_{(n)} = A\ h(n), \qquad h(n) = \frac{1}{(n^2-4)^{1/4}} \begin{pmatrix} \frac{1}{2}(-n+\sqrt{n^2-4}) & -1 \\ \frac{1}{2}(n+\sqrt{n^2-4}) & 1 \end{pmatrix}, \tag{3.37}$$

and restricting the periodicity to $T(n) = \ln\left(n - \sqrt{n^2-4}\right) - \ln(2)$, where $n \in \mathbb{N}$ and $n \geq 3$ [27]. The SL(2, $\mathbb{Z}$) and SL(2, $\mathbb{R}$) monodromies are then related by

$$\mathfrak{M}(n) = h^{-1}\mathfrak{M}_{S^1}h. \tag{3.38}$$

3.2.4 Flat deformations

The flat deformations appear only via the azimuthal angles in (3.20), where they can be absorbed through a change of coordinates

$$\varphi_i' = \varphi_i + \chi_i\eta. \tag{3.39}$$

In general, this coordinate redefinition is only valid locally, because φ_i' picks up a term $\chi_i T$ when going around S^1_η. Only in the special case of $\chi_i = n_i\frac{2\pi}{T}$, where $n_i \in \mathbb{Z}$, can φ_i' be considered a globally well-defined coordinate. Due to a subtlety of spinors on S^1_η however, the deformation parameters χ_i have the periodicity $\frac{4\pi}{T}$, rather than $\frac{2\pi}{T}$ [10].

The local coordinate redefinition (3.39) can still be applied to the equations of motion, since they hold in all reference frames. In other words, the flat deformations do not affect the equations of motion. The deformation parameters can therefore be set as $\chi_i = 0$ when evaluating the field equations and other covariant expressions.

3.2.5 Evaluating the field equations for the AdS vacuum

The evaluation of the field equations is essentially split into two parts; Einstein's equations (3.12) and the other ones. As showing Einstein's equations is more involved, this step is saved for last. The equations of motion are formulated in terms of the field strengths, which are computed in appendix G, where the deformation parameters are set to $\chi_i = 0$. Since the axion VEV is zero, the field equation (3.8) becomes trivial and the other equations are greatly simplified. In, particular, the B_2 and C_2 field equations become related by a SL(2, $\mathbb{Z}$) transformation, described in appendix H.4. It is thus sufficient to show only that (3.9) is satisfied, and (3.10) will then follow.

The field equations (3.7)-(3.11) are evaluated in appendix H. The scalar equations are considered first as they do not have any free indices. To evaluate the 2-form and 4-form equations, the same methodology can be used. The right hand side is only non-zero for a specific combination of free indices. The majority of the work consists of rewriting the left hand side so that it becomes clear that it equals the right hand side for this set of free indices. All other cases amount to showing that the left hand side vanishes, which always happens because the derivative acts on something constant.

This leaves only Einstein's equations. The difference compared to the other field equations is that the Ricci tensor is needed. The Ricci tensor is found using the torsion-less structure equations of Cartan and the calculations are performed via the zehnbein 1-forms $e^a = dx^\mu e_\mu{}^a$, which can be read off from (3.24). Appendix C describes this methodology in more detail, where it was used to find the Riemann tensor of AdS$_4$. Another simple example using S^2 is found in E.1.4. The Ricci tensor for the AdS vacuum is computed in appendix I. Besides the increased number of dimensions, the main complication compared to AdS$_4$ and S^2 is the α-dependent warping factor Δ, defined in (3.19), which enters all e^a. As a consequence, the Ricci tensor becomes α-dependent in a somewhat messy way. For example, the Ricci scalar reads

$$R = -24\Delta - 2\Delta^5\left[5 - 18\sin^2(2\alpha)\right] + 2\Delta^9\left[168 + 37\sin^2(2\alpha) - 5\sin^4(2\alpha)\right]. \quad (3.40)$$

The Ricci tensor R_{ab} is diagonal and fortunately, not all elements have to be calculated independently. The AdS$_4$ components are naturally the same, except the sign difference for R_{00}, which is also seen in appendix C. From the S^2 calculation in E.1.4 it is also clear that the diagonal Ricci tensor elements corresponding to the two angles of a 2-sphere are the same. As such, $R_{66} = R_{77}$, where the flat coordinates (6, 7) correspond to the 2-sphere angles (θ_1, φ_1). Similarly, $R_{88} = R_{99}$ for the other 2-sphere. The elements R_{66} and R_{88} can also be related by a shift $\alpha \to \alpha + \frac{\pi}{2}$, which exchanges the factors $x_1^2 = \cos^2\alpha/(2+\cos 2\alpha)$ and $x_2^2 = \sin^2\alpha/(2-\cos 2\alpha)$ that distinguish the 2-spheres in the metric (3.18).

With the Ricci tensor ready, Einstein's equations are then evaluated in appendix J. The equation is trivially satisfied for all off-diagonal cases of free indices $\mu\nu$, except for $\mu\nu = \eta\alpha$. The rest comes down to finding common expressions of the left- and right hand sides for the diagonal cases of the free indices. This is done so that the metric factors. Consider for example the left hand side with some specified curved index μ

$$R_{\mu\mu} - \frac{1}{2}Rg_{\mu\mu} = e_{\mu}{}^{a}e_{\mu}{}^{a}R_{aa} - \frac{1}{2}Rg_{\mu\mu} = \left(R_{aa} - \frac{1}{2}R\right) g_{\mu\mu}, \tag{3.41}$$

since $g_{\mu\nu}$ and $e_{\mu}{}^{a}$ are diagonal when $\chi_i = 0$. The flat R_{aa} can thus be used even though the evaluation is performed in a general frame.

4
Stability and conclusions

This final chapter discusses the stability of the AdS vacuum presented in 3.2, and comments on its relation to the Non-SUSY AdS conjecture. The first test is to evaluate the perturbative stability of the vacuum. This is done by finding the Kaluza-Klein spectrum in 4 dimensions and checking that the squared masses are properly bound from below [10]. The perturbative stability criterion, in the context of the Freund-Rubin compactification of $D = 11$ supergravity, is treated in section 4.1. Non-perturbative instability is also discussed in section 4.2, where one example is decay via bubbles of nothing. Finally, the conclusions regarding the thesis and the AdS vacuum are given.

4.1 The Breitenlohner-Freedman bound

Stability is guaranteed if there is an unbroken supersymmetry since the condition $H = |Q|^2 \geq 0$ follows from the SUSY algebra [28, 29]. However non-supersymmetric vacua can still be classically stable under certain conditions [13]. Consider the $\mathrm{AdS}_4 \times M_7$ vacuum obtained by Freund-Rubin compactification of $D = 11$ supergravity in section 2.2. The perturbative stability of this vacuum can be studied via the $D = 4$ mass spectrum [26]. The obtained spectrum contains spin 2, $\frac{3}{2}$, 1, $\frac{1}{2}$ and 0 states, where the non-zero spin states are classically stable if the squared mass matrix is positive semi-definite

$$M^2 \geq 0, \tag{4.1}$$

so that there are no tachyons. For spin 0 states in AdS spacetime however, the criterion is the more lenient

$$M^2 \geq -m^2, \tag{4.2}$$

where m is the constant from the Freund-Rubin ansatz $\mathring{F}_{\mu\nu\rho\sigma} = -6m\epsilon_{\mu\nu\rho\sigma}$ [26]. This is because a spin 0 field ϕ propagates in AdS spacetime according to the wave equation $-\Box\phi + \alpha\phi = 0$ where α relates to the cosmological constant Λ via $\alpha \geq 3\Lambda/4$, which is required for non-negative AdS energy [26]. Classical stability is always fulfilled for all but the $0^{+(2)}$ tower. The criterion (4.2), along with the mass operator for $0^{+(2)}$, takes the form

$$\Delta_{\mathrm{L}} \geq 3m^2, \tag{4.3}$$

where Δ_{L} is the Lichnerowicz operator acting like

$$\Delta_{\mathrm{L}} h_{ab} = -\Box h_{ab} - 2R_{acbd}h^{cd} + 2R_{(a}{}^{c}h_{b)c}, \tag{4.4}$$

on transverse, tracefree and symmetric tensors h_{ab} in the compact space [26]. General constraints like (4.3) on Δ_{L} are usually very hard to obtain.

A corollary of (4.3) is that compact product spaces $M_7 = M_{(1)} \times M_{(2)}$ are unstable, given that the Freund-Rubin constant $m \neq 0$. This is because a transverse and traceless mode

$$\tilde{h}_{ab} = \begin{pmatrix} \epsilon_1 g^{(1)}_{a_1 b_1} & 0 \\ 0 & \epsilon_2 g^{(2)}_{a_2 b_2} \end{pmatrix}, \qquad \mathrm{tr}\Big(\tilde{h}_{ab}\Big) = \epsilon_1 \dim(M_{(1)}) + \epsilon_2 \dim(M_{(2)}) = 0, \quad (4.5)$$

always can be created, where $g^{(1)}_{a_1 b_1}$ and $g^{(2)}_{a_2 b_2}$ are the metrics on $M_{(1)}$ and $M_{(2)}$ respectively [26]. The eigenvalue of the mode is $\Delta_{\mathrm{L}} \tilde{h}_{ab} = 0$, which violates the stability bound (4.3). At first glance, this seems to imply that the type IIB AdS vacuum is unstable since the compact space is of the form $S^1_\eta \times S^5$. However, the monodromy when encircling the S^1_η indicates that $S^1_\eta \times S^5$ is not a simple product space. Indeed, the Kaluza-Klein spectrum of the S-fold vacuum shows that it is perturbatively stable for all values of χ_i [10]. This stability also holds in higher dimensions [25].

4.2 Non-perturbative stability and conclusions

The classical stability covered by the Breitenlohner-Freedman bound is however only a first test, as the vacuum may decay via non-perturbative means as well. As non-perturbative string theory is not understood, it is not currently possible to fully prove stability of any vacua. Still, a number of non-perturbative decay modes are known, which can be used to argue for or against stability. A possible decay mode for non-supersymmetric vacua is via bubbles of nothing [10]. This is the case for the original Kaluza-Klein vacuum $\mathrm{M}^4 \times S^1$, where M^4 denotes 4-dimensional Minkowski space. In the decay of $\mathrm{M}^4 \times S^1$, which is perturbatively stable, a hole spontaneously appears in spacetime [30]. On the boundary of this hole, the radius of the compact S^1 shrinks to zero. This bubble does not contain some other spacetime, it is completely empty, and after a very short time it expands at the speed of light. An unstable vacuum usually falls into a stable state, but the bubble decay causes the spacetime to just vanish [30]. The deformed S-fold vacua are not prone to bubble decay since the S^1_η and S^5 of its compact space cannot collapse like the S^1 of $\mathrm{M}^4 \times S^1$ does on the boundary of the bubble [10]. There are however more intricate bubbles of nothing through which decay is more difficult to rule out. The non-supersymmetric S-folds have also been checked for standard brane-jet instabilities [10].

Although some arguments against non-perturbative decay have been made, more is needed to reach a final verdict on stability [25]. Further investigations on non-perturbative stability will either cast the S-folds into the swampland, or strengthen their case. A decent bit more evidence for stability is likely needed to pose a serious case against the Non-SUSY AdS conjecture [8]. Still, there are notable features of the S-folds, in particular how supersymmetry is broken via flat deformations. Since the deformation parameters χ_i can be locally reabsorbed into the azimuthal angles φ_i, the vacuum behaves like it is supersymmetric on a local level. The S-folds thus enjoy some benefits of supersymmetry, which saves it from some non-perturbative instability modes, while remaining non-supersymmetric globally [10]. The local coordinate redefinition also protects the S-fold solutions from higher-derivative corrections of type IIB string theory [10]. Hopefully, further research on S-folds can shed more light on their role in terms of the Non-SUSY AdS conjecture and quantum gravity in general.

A
Definitions

In this appendix the conventions of the Levi-Civita symbol and the generalised Kronecker delta are given. The Minkowski metric is always taken as ”mostly plus” in this thesis. The basics of differential forms are also given in A.2.

A.1 The Levi-Civita symbol

The Levi-Civita symbol is a tensor density that is given by

$$\varepsilon^{\mu_1...\mu_D}, \qquad \varepsilon^{012...d} = 1, \tag{A.1}$$

in $D = d + 1$ dimensions. The indices can be lowered using the metric

$$\varepsilon_{\mu_1...\mu_D} = g_{\mu_1\nu_1}...g_{\mu_D\nu_D}\varepsilon^{\nu_1...\nu_D}, \tag{A.2}$$

where

$$\varepsilon_{0...d} = \pm 1 \tag{A.3}$$

The $+$ sign is true for Riemannian manifolds and the $-$ sign is true for Lorentzian manifolds [31]. Other $\pm$ signs should also be interpreted like this. Contracting two Levi-Civita symbols yields that

$$\varepsilon^{\alpha_1...\alpha_p\beta_1...\beta_q}\varepsilon_{\alpha_1...\alpha_p\gamma_1...\gamma_q} = \pm p!\ q!\ \delta^{\beta_1...\beta_q}_{\gamma_1...\gamma_d}, \tag{A.4}$$

where the generalised Kronecker delta is defined as

$$\delta^{\alpha_1...\alpha_p}_{\beta_1...\beta_p} = \delta^{\alpha_1}_{[\beta_1}...\delta^{\alpha_p}_{\beta_p]}, \qquad \text{where} \qquad \delta^{\alpha_1...\alpha_p}_{\beta_1...\beta_p}a^{\beta_1...\beta_p} = a^{[\alpha_1...\alpha_p]}. \tag{A.5}$$

The Levi-Civita symbol can also be used to create a totally antisymmetric tensor in D dimensions

$$\epsilon_{\mu_1...\mu_D} = e\ \varepsilon_{\mu_1...\mu_D}, \qquad \epsilon^{\mu_1...\mu_D} = e^{-1}\ \varepsilon^{\mu_1...\mu_D}, \tag{A.6}$$

where $e = \sqrt{\pm g}$ and $g = \det(g_{\mu\nu})$.

A.2 Differential forms

A p-form ω_p is defined as

$$\omega_p = \frac{1}{p!}dx^{\mu_1} \wedge .. \wedge dx^{\mu_p}\ \omega_{\mu_1...\mu_p}, \tag{A.7}$$

where

$$dx^{\mu_1} \wedge ... \wedge dx^{\mu_D} = d^D x \; \varepsilon^{\mu_1 \dots \mu_D} = d^D x \; e \; \epsilon^{\mu_1 \dots \mu_D}. \tag{A.8}$$

The wedge product of a p-form α_p and a q-form β_q is

$$(\alpha_p \wedge \beta_q)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} (\alpha_p)_{[\mu_1 \dots \mu_p} (\beta_q)_{\mu_{p+1} \dots \mu_{p+q}]}. \tag{A.9}$$

It is associative

$$(\alpha_p \wedge \beta_q) \wedge \gamma_r = \alpha_p \wedge (\beta_q \wedge \gamma_r), \tag{A.10}$$

and (anti-)commutative if pq is even (odd)

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p. \tag{A.11}$$

The exterior derivative $d = dx^\mu \partial_\mu$, which satisfies $d^2 = 0$, of a p-form ω_p is defined as

$$(d\omega_p)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} (\omega_p)_{\mu_2 \dots \mu_{p+1}]}. \tag{A.12}$$

The Hodge dual of ω_p is

$$(\star \omega_p)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{D-p} \nu_1 \dots \nu_p} (\omega_p)^{\nu_1 \dots \nu_p}. \tag{A.13}$$

B

$D = 11$ supergravity field equations

In this appendix, the field equations of $D = 11$ supergravity, which is described in section 2.1, are derived. The Lagrangian of $D = 11$ supergravity is again given by [13]

$$\begin{aligned} \mathcal{L} = \frac{1}{2\kappa_{11}^2} e \left[R - \frac{1}{2 \cdot 4!} F_{MNPQ} F^{MNPQ} \right] + \\ - \frac{1}{12\kappa_{11}^2} \frac{1}{3!(4!)^2} \varepsilon^{M_1 \dots M_{11}} A_{M_1 M_2 M_3} F_{M_4 M_5 M_6 M_7} F_{M_8 M_9 M_{10} M_{11}}. \end{aligned} \tag{B.1}$$

There are only two bosonic fields g_{MN} and A_{MNP}, which can be used to vary the Lagrangian.

B.1 Varying with respect to g^{MN}

Starting off with g^{MN}, the relevant terms are

$$\delta \left(eR - \frac{1}{2 \cdot 4!} eF_{MNPQ} F^{MNPQ} \right) = \delta(eR) - \frac{1}{2 \cdot 4!} \delta(eF_{MNPQ} F^{MNPQ}). \tag{B.2}$$

The first term becomes

$$\delta(eR) = \delta e \ R + e \ \delta g^{MN} R_{MN} + e \ g_{MN} \delta R^{MN}. \tag{B.3}$$

The variation δe can be related to δg^{MN} via the identity [7]

$$\delta \det(M) = \det(M) \operatorname{tr}\{M^{-1} \delta M\}. \tag{B.4}$$

Applying this relation to the metric yields that

$$\delta g = g \ g^{MN} \delta g_{MN} = -g \ g_{MN} \delta g^{MN}, \tag{B.5}$$

where $\delta g^{MN} = -g^{MP} g^{NQ} \delta g_{PQ}$ has been used. The variation of the elfbein determinant thus becomes

$$\begin{aligned} \delta e = \delta \sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta g = -\frac{g}{2\sqrt{-g}} g^{MN} \delta g_{MN} = \\ = \frac{1}{2} e \ g^{MN} \delta g_{MN} = -\frac{1}{2} e \ g_{MN} \delta g^{MN}, \end{aligned} \tag{B.6}$$

so that

$$\begin{aligned}\delta(eR) &= \delta e\ R + e\ R_{MN}\delta g^{MN} = -\frac{1}{2}e\ R\ g_{MN}\delta g^{MN} + e\ R_{MN}\delta g^{MN} = \\ &= e\left(R_{MN} - \frac{1}{2}R\ g_{MN}\right)\delta g^{MN}.\end{aligned} \tag{B.7}$$

Now, the second term is

$$\begin{aligned}\delta(eF_{PQRS}F^{PQRS}) &= \delta e\ F_{PQRS}F^{PQRS} + 4e\ F_{MPQR}F_N{}^{PQR}\delta g^{MN} = \\ &= e\left(-\frac{1}{2}g_{MN}F_{PQRS}F^{PQRS} + 4F_{MPQR}F_N{}^{PQR}\right)\delta g^{MN},\end{aligned} \tag{B.8}$$

and thus Einstein's equations in $D = 11$ supergravity are given by

$$R_{MN} - \frac{1}{2}R\ g_{MN} = \frac{1}{12}\left(F_{MPQR}F_N{}^{PQR} - \frac{1}{8}g_{MN}F_{PQRS}F^{PQRS}\right). \tag{B.9}$$

B.2 Varying with respect to A_{MNP}

Next, variation with respect to A_{MNP} is considered. The kinetic term becomes

$$\begin{aligned}\delta(e\ F_{MNPQ}F^{MNPQ}) &= 2e\ \delta F_{MNPQ}F^{MNPQ} = 8e\ \delta(\partial_{[M}A_{NPQ]})F^{MNPQ} = \\ &= 8e\ \partial_M(\delta A_{NPQ})F^{MNPQ} = -8\partial_M(e\ F^{MNPQ})\delta A_{NPQ},\end{aligned} \tag{B.10}$$

where the boundary term is discarded. The topological term, written schematically as $\delta(\varepsilon AFF)$, becomes

$$\begin{aligned}\delta(\varepsilon AFF) &= \varepsilon^{M_1\ldots M_{11}}\delta A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} + \\ &\quad + \varepsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}\delta F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} + \\ &\quad + \varepsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}F_{M_4M_5M_6M_7}\delta F_{M_8M_9M_{10}M_{11}} = \\ &= \varepsilon^{M_1\ldots M_{11}}\delta A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} + \\ &\quad + 2\varepsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}\delta F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} = \\ &= \varepsilon^{M_1\ldots M_{11}}\delta A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} + \\ &\quad - 8\varepsilon^{M_1\ldots M_{11}}\delta A_{M_5M_6M_7}\partial_{M_4}(A_{M_1M_2M_3}F_{M_8M_9M_{10}M_{11}}) = \\ &= \varepsilon^{M_1\ldots M_{11}}\delta A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}} + \\ &\quad - 2\varepsilon^{M_1\ldots M_{11}}\delta A_{M_5M_6M_7}F_{M_4M_1M_2M_3}F_{M_8M_9M_{10}M_{11}} + \\ &\quad - 8\varepsilon^{M_1\ldots M_{11}}\delta A_{M_5M_6M_7}A_{M_1M_2M_3}\partial_{M_4}(F_{M_8M_9M_{10}M_{11}}) \\ &= 3\varepsilon^{M_1\ldots M_{11}}\delta A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}},\end{aligned} \tag{B.11}$$

after manipulating the indices a bit and noting that $\partial_{M_4}(F_{M_8M_9M_{10}M_{11}}) = 0$ in the final step. The second set of field equations are thus

$$-\frac{1}{2\cdot 4!}(-8\partial_M(e\ F^{MNPQ})) - \frac{1}{6}\frac{1}{3!(4!)^2}\left(3\varepsilon^{NPQM_4\ldots M_{11}}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}}\right) = 0, \tag{B.12}$$

which can be written as

$$\nabla_M F^{MNPQ} = \frac{1}{1152}\epsilon^{NPQR_1\ldots R_8}F_{R_1R_2R_3R_4}F_{R_5R_6R_7R_8}, \tag{B.13}$$

by using that $\nabla_M F^{MNPQ} = \frac{1}{e}\partial_M(eF^{MNPQ})$ since F^{MNPQ} is antisymmetric.

C

AdS$_4$ spacetime

This appendix focuses on anti-de Sitter spacetime in 4 dimensions which is the most relevant type for this thesis. In particular, the Riemann tensor of AdS$_4$ is calculated, from which the Ricci tensor and scalar are easily computed. The Riemann tensor is calculated via Cartan's structure equations when the torsion is zero [32]. The same method is used to find the Riemann tensor of the type IIB AdS vacuum AdS$_4 \times S^1_\eta \times S_5$ and some calculations carry over. There is however an α-dependent warping factor present for the type IIB vacuum which complicates things a little, see section 3.2. The procedure of calculating the Riemann tensor will be done explicitly here, so that other similar computations can be performed more speedily.

C.1 The Riemann tensor of AdS$_4$

The metric on AdS$_4$ is given by the line element

$$ds^2 = -\cosh^2\rho\ dt^2 + \frac{1}{a^2}d\rho^2 + \frac{1}{a^2}\sinh^2\rho\ (d\theta^2 + \sin^2\theta\ d\phi^2), \tag{C.1}$$

where the parameter a is related to the AdS radius via $L = 1/a$. The metric then reads

$$g_{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} -a^2\cosh^2\rho & & & \\ & 1 & & \\ & & \sinh^2\rho & \\ & & & \sinh^2\rho\ \sin^2\theta \end{pmatrix}. \tag{C.2}$$

The line element can also be used to read off the vierbein 1-forms e^a

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dx^\mu dx^\nu e_\mu{}^a e_\nu{}^b \eta_{ab} = e^a e^b \eta_{ab}, \tag{C.3}$$

that contract with the flat Minkowski metric. The greek μ, ν,... denote curved indices while the latin a, b,... are flat. The vierbein 1-forms for AdS$_4$ are thus

$$e^0 = \cosh\rho\ dt, \quad e^1 = \frac{1}{a}d\rho, \quad e^2 = \frac{1}{a}\sinh\rho\ d\theta, \quad e^3 = \frac{1}{a}\sinh\rho\ \sin\theta\ d\phi. \tag{C.4}$$

The first step towards the Riemann tensor is Cartan's structure equation for the torsion 2-form $T^a = \frac{1}{2}dx^\mu \wedge dx^\nu T_{\mu\nu}{}^a$ given by

$$T^a = de^a + \omega^a{}_b \wedge e^b, \tag{C.5}$$

where $\omega^a{}_b$ is the spin connection 1-form [7, 32]. Since there is always a torsion-free connection, T^a can be set to zero in (C.5) [7]. The structure equation (C.5) then uniquely defines the connection 1-forms $\omega^a{}_b$ via $de^a + \omega^a{}_b \wedge e^b = 0$. First off, the exterior derivative of the vierbein 1-forms are

$$\begin{aligned} de^0 &= d(\cosh\rho\ dt) = \sinh\rho\ d\rho \wedge dt = a\frac{\sinh\rho}{\cosh\rho}(\frac{1}{a}d\rho)\wedge(\cosh\rho\ dt) = \\ &= -a\tanh\rho\ e^0 \wedge e^1, \end{aligned} \tag{C.6}$$

$$de^1 = \frac{1}{a}dd\rho = 0, \tag{C.7}$$

$$\begin{aligned} de^2 &= \frac{1}{a}d(\sinh\rho\ d\theta) = \frac{1}{a}\cosh\rho\ d\rho\wedge d\theta = a\frac{\cosh\rho}{\sinh\rho}(\frac{1}{a}d\rho)\wedge(\frac{1}{a}\sinh\rho\ d\theta) = \\ &= a\coth\rho\ e^1\wedge e^2, \end{aligned} \tag{C.8}$$

$$\begin{aligned} de^3 &= \frac{1}{a}d(\sinh\rho\ \sin\theta\ d\phi) = \\ &= \frac{1}{a}\cosh\rho\ \sin\theta\ d\rho\wedge d\phi + \frac{1}{a}\sinh\rho\ \cos\theta\ d\theta\wedge d\phi = \\ &= a\coth\rho\ e^1\wedge e^3 + a\frac{\cot\theta}{\sinh\rho}e^2\wedge e^3. \end{aligned} \tag{C.9}$$

The torsion-free structure equation (C.5) can be rewritten as

$$de^a = \eta^{ac}\omega_{bc}\wedge e^b. \tag{C.10}$$

Since the left hand side de^a consists of terms proportional to $e^i \wedge e^j$, where i, $j ==$ 0, 1, 2, 3, the connection 1-forms are written as

$$\omega_{ab} = \omega_{iab}e^i, \tag{C.11}$$

so that the components ω_{iab} can be extracted from (C.10). The case where $a = 0$ is considered first. The relation (C.10) then reads

$$\begin{aligned} -a\tanh\rho\ e^0\wedge e^1 = \eta^{00}\omega_{ib0}e^i\wedge e^b &= \\ = -e^0\wedge e^1(\omega_{010}-\omega_{100}) &- e^0\wedge e^2(\omega_{020}-\omega_{200}) + \\ - e^0\wedge e^3(\omega_{030}-\omega_{300}) &- e^1\wedge e^2(\omega_{120}-\omega_{210}) + \\ - e^1\wedge e^3(\omega_{130}-\omega_{310}) &- e^2\wedge e^3(\omega_{230}-\omega_{320}) = \\ = -e^0\wedge e^1 2\omega_{[01]0} - e^0\wedge e^2 2\omega_{[02]0} &- e^0\wedge e^3 2\omega_{[03]0} + \\ - e^1\wedge e^2 2\omega_{[12]0} - e^1\wedge e^3 2\omega_{[13]0} &- e^2\wedge e^3 2\omega_{[23]0}. \end{aligned} \tag{C.12}$$

What follows is that $2\omega_{[01]0} = a\tanh\rho$ and all other $\omega_{[ij]0} = 0$. Next is the case where $a = 1$ which reads

$$0 = \omega_{ib1}e^i\wedge e^b, \tag{C.13}$$

implying that all $\omega_{[ij]1} = 0$. For $a = 2$

$$a\coth\rho\ e^1\wedge e^2 = \omega_{ib2}e^i\wedge e^b, \tag{C.14}$$

so that $2\omega_{[12]2} = a\coth\rho$ and all other $\omega_{[ij]2} = 0$. Lastly, $a = 3$

$$a\coth\rho\ e^1 \wedge e^3 + a\frac{\cot\theta}{\sinh\rho}e^2 \wedge e^3 = \omega_{ib3}e^i \wedge e^b, \tag{C.15}$$

yields that $2\omega_{[13]3} = a\coth\rho$, $2\omega_{[23]3} = a\frac{\cot\theta}{\sinh\rho}$ and the other $\omega_{[ij]3} = 0$. Now that the relation (C.10) is exhausted, the actual connection components ω_{abc} should be extracted from the $\omega_{[ab]c}$. This can be done via the antisymmetry properties of the connection which implies that

$$\omega_{abc} = -\omega_{acb} \qquad \Longrightarrow \qquad \omega_{abc} = \omega_{[ab]c} + \omega_{[ca]b} - \omega_{[bc]a}. \tag{C.16}$$

The non-zero ω_{abc} are thus

$$\begin{aligned}
\omega_{010} &= \omega_{[01]0} - \omega_{[10]0} = 2\omega_{[01]0} = a\tanh\rho, \\
\omega_{212} &= \omega_{[21]2} - \omega_{[12]2} = -2\omega_{[12]2} = -a\coth\rho, \\
\omega_{313} &= \omega_{[31]3} - \omega_{[13]3} = -2\omega_{[13]3} = -a\coth\rho, \\
\omega_{323} &= \omega_{[32]3} - \omega_{[23]3} = -2\omega_{[23]3} = -a\frac{\cot\theta}{\sinh\rho},
\end{aligned} \tag{C.17}$$

and the connection 1-forms are

$$\begin{aligned}
\omega_{01} &= \omega_{001}e^0 = -a\tanh\rho\ e^0, & \omega_{12} &= \omega_{212}e^2 = -a\coth\rho\ e^2, \\
\omega_{02} &= 0, & \omega_{13} &= \omega_{313}e^3 = -a\coth\rho\ e^3, \\
\omega_{03} &= 0, & \omega_{23} &= \omega_{323}e^3 = -a\frac{\cot\theta}{\sinh\rho}e^3,
\end{aligned} \tag{C.18}$$

where $\omega_{aa} = 0$ because of the antisymmetry. The goal of finding the ω_{ab} is that the curvature 2-form Θ_{ab} now can be evaluated as [32]

$$\Theta_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}. \tag{C.19}$$

Like the connection, Θ_{ab} is antisymmetric in ab which implies that $\Theta_{aa} = 0$. The curvature 2-form will later be associated with the Riemann tensor. The first term of (C.19) is the exterior derivative of ω_{ab}. A useful relation for this calculation is that

$$d(\coth\rho) = -\frac{1}{\sinh^2\rho}d\rho, \tag{C.20}$$

which follows from $\cosh^2\rho - \sinh^2\rho = 1$. The non-zero $d\omega_{ab}$ are thus

$$\begin{aligned}
d\omega_{01} &= -a\ d(\tanh\rho\ e^0) = -a(1-\tanh^2\rho)d\rho\wedge e^0 - a\tanh\rho\ de^0 = \\
&= a^2(1-\tanh^2\rho+\tanh^2\rho)e^0\wedge e^1 = a^2 e^0\wedge e^1, \\
d\omega_{12} &= \frac{a}{\sinh^2\rho}d\rho\wedge e^2 - a\coth\rho\ de^2 = \frac{a^2}{\sinh^2\rho}e^1\wedge e^2 - a^2\coth^2 e^1\wedge e^2 = \\
&= -a^2 e^1\wedge e^2, \\
d\omega_{13} &= \frac{a^2}{\sinh^2\rho}e^1\wedge e^3 - a\coth\rho\ de^3 = -a^2 e^1\wedge e^3 - a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^2\wedge e^3, \\
d\omega_{23} &= a\frac{1}{\sin^2\theta}\frac{1}{\sinh\rho}d\theta\wedge e^3 + a\cot\theta\frac{\cosh\rho}{\sinh^2\rho}d\rho\wedge e^3 - a\frac{\cot\theta}{\sinh\rho}de^3 = \\
&= \frac{a^2}{\sin^2\theta\ \sinh^2\rho}e^2\wedge e^3 + a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^1\wedge e^3 + \\
&\quad - a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^1\wedge e^3 - a^2\frac{\cot^2\theta}{\sinh^2\rho}e^2\wedge e^3 = \\
&= \frac{a^2}{\sinh^2\rho}e^2\wedge e^3.
\end{aligned} \tag{C.21}$$

The second term of (C.19) can be written as $\tilde{\omega}_{ab} = \omega_{ac}\wedge\omega_{cb}$, where $\tilde{\omega}_{aa} = 0$. The $\tilde{\omega}_{ab}$ are calculated to be

$$\begin{aligned}
\tilde{\omega}_{01} &= 0, \\
\tilde{\omega}_{02} &= \omega_{01}\wedge\omega_{12} = a^2 e^0\wedge e^2, \\
\tilde{\omega}_{03} &= \omega_{01}\wedge\omega_{13} = a^2 e^0\wedge e^3, \\
\tilde{\omega}_{12} &= \omega_{13}\wedge\omega_{32} = -a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^3\wedge e^3 = 0, \\
\tilde{\omega}_{13} &= \omega_{12}\wedge\omega_{23} = a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^2\wedge e^3, \\
\tilde{\omega}_{23} &= \omega_{21}\wedge\omega_{13} = -a^2\coth^2\rho\ e^2\wedge e^3,
\end{aligned} \tag{C.22}$$

so that the curvature 2-forms are

$$\begin{aligned}
\Theta_{01} &= a^2 e^0\wedge e^1, \\
\Theta_{02} &= a^2 e^0\wedge e^2, \\
\Theta_{03} &= a^2 e^0\wedge e^2, \\
\Theta_{12} &= -a^2 e^1\wedge e^2, \\
\Theta_{13} &= -a^2 e^1\wedge e^3 - a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^2\wedge e^3 + a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^2\wedge e^3 = \\
&= -a^2 e^1\wedge e^3, \\
\Theta_{23} &= \frac{a^2}{\sinh^2\rho}e^2\wedge e^3 - a^2\coth^2\rho\ e^2\wedge e^3 = \\
&= -a^2 e^2\wedge e^3.
\end{aligned} \tag{C.23}$$

The 2-form Θ_{ab} is related to the Riemann tensor [32]

$$\Theta_{ab} = \frac{1}{2}R_{abcd}e^c\wedge e^d, \tag{C.24}$$

which allows the non-zero R_{abcd} to be read off as

$$\begin{aligned} &R_{0101} = a^2, \quad &R_{0202} = a^2, \quad &R_{0303} = a^2 \\ &R_{1212} = -a^2, \quad &R_{1313} = -a^2, \quad &R_{2323} = -a^2. \end{aligned} \tag{C.25}$$

The Ricci tensor is then just given by $R_{bd} = \eta^{ac} R_{abcd}$

$$\begin{aligned} R_{00} &= R_{1010} + R_{2020} + R_{3030} = 3a^2, \\ R_{11} &= -R_{0101} + R_{2121} + R_{3131} = -3a^2, \\ R_{22} &= -R_{0202} + R_{1212} + R_{3232} = -3a^2, \\ R_{33} &= -R_{0303} + R_{1313} + R_{2323} = -3a^2, \end{aligned} \tag{C.26}$$

which implies that

$$R_{ab} = -3a^2 \eta_{ab}. \tag{C.27}$$

The Ricci scalar is

$$R = \eta^{ab} R_{ab} = -3a^2 \eta^{ab} \eta_{ab} = -12a^2. \tag{C.28}$$

This also shows that AdS$_4$ is a maximally symmetric space, since the Ricci and Riemann tensor can be written like

$$R_{ab} = \frac{R}{D} \eta_{ab}, \qquad R_{abcd} = \frac{R}{D(D-1)} \left(\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc} \right), \tag{C.29}$$

where $D = 4$ is the number of spacetime dimensions [33].

D

Derivation of $N = 8$ supergravity

This appendix focuses on the derivation of $N = 8$ supergravity via dimensional reduction from $D = 11$ supergravity. The appendix follows [11] closely, and focuses only on the bosonic sector. Note also that the choice of metric signature and the scaling of the 3-form fields differs from [11]. The metric signature used here is "mostly plus", in contrast with "mostly minus" which is used in [11]. The flat metric of the compact space used here is thus δ_{ab}, instead of $-\delta_{ab}$. The 3-forms are related via $A^{\text{here}}_{MNP} = 2A^{\text{there}}_{MNP}$.

D.1 Dimensional reduction of $D = 11$ supergravity

The bosonic Lagrangian of 11-dimensional supergravity is given by

$$\begin{aligned}\kappa_{11}^2\mathcal{L} = \frac{1}{2}V\ R - \frac{1}{2\cdot 48}V\ F_{MNPQ}F^{MNPQ} + \\ -\frac{1}{2\cdot 12^4}\varepsilon^{M_1\dots M_{11}}A_{M_1M_2M_3}F_{M_4M_5M_6M_7}F_{M_8M_9M_{10}M_{11}},\end{aligned} \tag{D.1}$$

where $V = \det\{e_M{}^A\}$ and M, N, P,... denote 11-dimensional spacetime indices. The gravitational constant $\kappa_{11}^2 = 8\pi G^{(11)}$ only appears as an overall factor and is thus set to 1 for the sake of convenience. The bosonic Lagrangian is invariant under general coordinate transformations in 11 dimensions, $x^M \to x^M - \xi^M$, so that the elfbein and 3-form transform like

$$\delta e_M{}^A = e_N{}^A\partial_M\xi^N + \xi^N\partial_N e_M{}^A, \tag{D.2}$$

$$\delta A_{MNP} = 3A_{Q[MN}\partial_{P]}\xi^Q + \xi^Q\partial_Q A_{MNP}, \tag{D.3}$$

local SO(1, 10) Lorentz transformations, $x^A \to x^B\Lambda_B{}^A$

$$\delta e_M{}^A = -e_M{}^B\Lambda_B{}^A, \tag{D.4}$$

$$\delta A_{MNP} = 0, \tag{D.5}$$

and abelian gauge transformations of the 3-form with parameter $\zeta_{MN} = -\zeta_{NM}$

$$\delta e_M{}^A = 0, \tag{D.6}$$

$$\delta A_{MNP} = \partial_{[M}\zeta_{NP]}. \tag{D.7}$$

$N = 8$ supergravity is the 4-dimensional theory obtained from dimensional reduction of $D = 11$ supergravity on a 7-torus. Specifically, the bosonic part of the $N = 8$

theory is found by compactifying the Lagrangian given in (D.1). For this purpose, the elfbein $e_M{}^A$ is divided into 4 blocks

$$e_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu{}^r e_r{}^a \\ e_m{}^\alpha & e_m{}^a \end{pmatrix}. \tag{D.8}$$

The different indices correspond to

	curved	flat
11-dimensional	M, N, P,...	A, B, C,...
4-dimensional spacetime	μ, ν, ρ,...	α, β, γ,...
7-dimensional compact	i, j, k,...	a, b, c,...,

(D.9)

which is the same notation as the one used in section 2.2. The off-diagonal part of the local SO(1, 10) invariance can be used to set all $e_m{}^\alpha = 0$ so that the elfbein and its inverse read

$$e_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu{}^r e_r{}^a \\ 0 & e_m{}^a \end{pmatrix}, \qquad e_A{}^M = \begin{pmatrix} e_\alpha{}^\mu & -e_\alpha{}^\mu (B_\mu{}^r e_r{}^a) e_a{}^m \\ 0 & e_a{}^m \end{pmatrix}. \tag{D.10}$$

The 55 degrees of freedom of local SO(1, 10) invariance are thus reduced to 27 by setting the 28 components of $e_m{}^\alpha = 0$. Remaining are a vierbein $e_\mu{}^\alpha$, 7 gauge fields $B_\mu{}^i$ and an internal seibenbein $e_m{}^a$, which is an element of the group GL(7) and contains 49 scalar degrees of freedom.

The 3-form A_{MNP} is divided into 4 types of fields: 35 pseudoscalars A_{ijk}, 21 pseudovectors $A_{\mu ij}$, 7 $A_{\mu\nu i}$ fields and $A_{\mu\nu\rho}$. The 4-dimensional 3-form $A_{\mu\nu\rho}$ can only appear in $\mathcal{L}$ through its field strength $F_{\mu\nu\rho\sigma}$. In 4 dimensions, $F_{\mu\nu\rho\sigma}$ is exact, and can thus be taken as the independent variable instead of $A_{\mu\nu\rho}$. However, the $F_{\mu\nu\rho\sigma}$ field does not contribute to the bosonic Lagrangian.

The dimensional reduction should now be applied to the bosonic Lagrangian in (D.1), which can be schematically written as

$$\mathcal{L} = H + H' + H'', \tag{D.11}$$

where H is the Einstein-Hilbert term, H'' is the 3-form kinetic term and H'' is the topological term. The $D = 11$ supergravity Lagrangian will thus be written in terms of the 4-dimensional field content, giving the $N = 8$ supergravity. First up is the Einstein-Hilbert term.

D.1.1 The Einstein-Hilbert term H

The Einstein-Hilbert term is given by

$$H = \frac{1}{2} V\ R = \frac{1}{2} V\ (d\omega^{AB} + \omega^A{}_C \wedge \omega^C{}_B)_{AB}. \tag{D.12}$$

The torsion-free spin connection with flat indices can be expressed as

$$\omega_{ABC} = \frac{1}{2}(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}), \tag{D.13}$$

using the anholonomy coefficients $\Omega_{ABC} = -2e_{[A}{}^{M}\partial_{B]}e_{MC}$. The symmetry properties of ω_{ABC} and Ω_{ABC} are

$$\omega_{ABC} = -\omega_{ACB}, \qquad \Omega_{ABC} = -\Omega_{BAC}. \tag{D.14}$$

To simplify the dimensional reduction, H is first rewritten in terms of the anholonomy coefficients in 11 dimensions. This is done as follows. By performing a partial integration, the first term in (D.12) becomes

$$\begin{aligned}
V\ (d\omega^{AB})_{AB} &= V\ e_{A}{}^{M}e_{B}{}^{N}(d\omega^{AB})_{MN} = 2V\ e_{A}{}^{M}e_{B}{}^{N}\partial_{[M}\omega_{N]}{}^{AB} = \{\text{P.I.}\} = \\
&= -2\partial_{[A}(V)\omega_{B]}{}^{AB} - 2V\ \partial_{[A}(e_{B}{}^{M})\omega_{M]}{}^{AB} + \\
&\quad - 2V\ \partial_{[M}(e_{A}{}^{M})\omega_{B]}{}^{AB} = \\
&= 2\partial_{B}(V)\omega_{A}{}^{AB} - 2V\ \partial_{M}(e_{[A}{}^{M})\omega_{B]}{}^{AB} - 2V\ \partial_{[A}(e_{B]}{}^{M})\omega_{M}{}^{AB} = \\
&= 2V\ \partial_{B}(\log V)\omega_{A}{}^{AB} + 2V\ \partial_{M}(e_{B}{}^{M})\omega_{A}{}^{AB} + \\
&\quad - 2V\ e_{[A}{}^{M}\partial_{B]}(e_{M}{}^{C})\omega_{C}{}^{AB} = \\
&= \{\partial_{B}\log V = \Gamma_{AB}{}^{A} = \omega_{A}{}^{A}{}_{B} - \partial_{M}e_{B}{}^{M}\} = \\
&= 2\omega_{A}{}^{A}{}_{C}\omega_{B}{}^{BC} + \Omega_{AB}{}^{C}\omega_{C}{}^{AB}.
\end{aligned} \tag{D.15}$$

The second term is then expanded

$$\begin{aligned}
(\omega^{A}{}_{C} \wedge \omega^{CB})_{AB} &= e_{A}{}^{M}e_{B}{}^{N}(\omega^{A}{}_{C} \wedge \omega^{CB})_{MN} = 2e_{A}{}^{M}e_{B}{}^{N}\omega_{[M}{}^{A}{}_{C}\omega_{N]}{}^{CB} = \\
&= \omega_{A}{}^{A}{}_{C}\omega_{B}{}^{CB} - \omega_{B}{}^{A}{}_{C}\omega_{A}{}^{CB},
\end{aligned} \tag{D.16}$$

so that the Einstein-Hilbert term can be written as

$$\begin{aligned}
H = \frac{1}{2}V\ R &= \frac{1}{2}V\ [\Omega_{AB}{}^{C}\omega_{C}{}^{AB} - \omega_{B}{}^{A}{}_{C}\omega_{A}{}^{CB} + \omega_{A}{}^{AC}\omega_{B}{}^{B}{}_{C}] = \\
&= \frac{1}{2}V\ [(\Omega_{AB}{}^{C} - \omega_{B}{}^{C}{}_{A})\omega_{C}{}^{AB} + \omega_{A}{}^{A}{}_{C}\omega_{B}{}^{BC}] = \\
&= \frac{1}{2}V\ [\omega_{AB}{}^{C}\omega_{C}{}^{AB} + \omega_{A}{}^{A}{}_{C}\omega_{B}{}^{BC}].
\end{aligned} \tag{D.17}$$

To express H in terms of the anholonomy coefficients, first note that

$$\begin{aligned}
\omega_{AB}{}^{C}\omega_{C}{}^{AB} &= \frac{1}{4}\Big[(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB})(\Omega^{CAB} - \Omega^{ABC} + \Omega^{BCA})\Big] = \\
&= \frac{1}{4}\Big[- \Omega_{ABC}\Omega^{ABC} + 2\Omega_{ABC}\Omega^{BCA}\Big] = \\
&= \frac{1}{4}\Big[- \Omega_{ABC}\Omega^{ABC} + 2\Omega_{ABC}\Omega^{CAB}\Big],
\end{aligned} \tag{D.18}$$

and secondly that

$$\begin{aligned}\omega_A{}^A{}_C\omega_B{}^{BC} &= \frac{1}{4}\Big[(\underbrace{\Omega_A{}^A{}_C}_{=0} - \underbrace{\Omega^A{}_{CA}}_{-\Omega_{CA}{}^A} + \Omega_{CA}{}^A)(\underbrace{\Omega_B{}^{BC}}_{=0} - \underbrace{\Omega^{BC}{}_B}_{-\Omega^C{}_B{}^B} + \Omega^C{}_B{}^B)\Big] = \\ &= \frac{1}{4}\Big[4\Omega_{CA}{}^A\Omega^C{}_B{}^B\Big].\end{aligned} \tag{D.19}$$

The Einstein-Hilbert term becomes

$$H = \frac{1}{2}V\ R = -\frac{1}{8}V\ [\Omega_{ABC}\Omega^{ABC} - 2\Omega_{ABC}\Omega^{CAB} - 4\Omega_{CA}{}^A\Omega^C{}_B{}^B]. \tag{D.20}$$

D.1.1.1 Weyl transformation

Before continuing with the the dimensional reduction, note that the elfbein determinant factors like

$$V = \det\{e_M{}^A\} = \det\{e_\mu{}^\alpha\}\det\{e_m{}^a\} = e\sqrt{\Delta}, \tag{D.21}$$

in the $e_m{}^\alpha = 0$ gauge. A Weyl transformation can be used to eliminate the $\sqrt{\Delta}$-factor in front of the Einstein-Hilbert term. The Weyl transformation used is

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Delta^{-1/2}g_{\mu\nu}, \qquad \Longrightarrow \qquad e_\mu{}^\alpha \to \hat{e}_\mu{}^\alpha = \Delta^{-1/4}e_\mu{}^\alpha. \tag{D.22}$$

The determinant of the vierbein then transforms like

$$e \to \hat{e} = \det\{\Delta^{-1/4}e_\mu{}^\alpha\} = \Delta^{-1}e. \tag{D.23}$$

Applying the Weyl transformation on the anholonomy coefficients $\Omega_{\alpha\beta\gamma} = 2e_\alpha{}^\mu e_\beta{}^\nu\partial_{[\mu}e_{\nu]\gamma}$ yields that

$$\begin{aligned}\Omega_{\alpha\beta\gamma} \to \hat{\Omega}_{\alpha\beta\gamma} &= 2\Delta^{1/2}e_\alpha{}^\mu e_\beta{}^\nu\partial_{[\mu}(\Delta^{-1/4}e_{\nu]}{}^\delta\eta_{\delta\gamma}) = \\ &= 2\Delta^{1/4}e_\alpha{}^\mu e_\beta{}^\nu\partial_{[\mu}e_{\nu]\gamma} + 2\Delta^{1/4}e_\alpha{}^\mu e_\beta{}^\nu e_{[\nu|}{}^\delta\eta_{\delta\gamma}\Delta^{1/4}\partial_{|\mu]}\Delta^{-1/4} = \\ &= \{\Delta^{1/4}\partial_\alpha\Delta^{-1/4} = \partial_\alpha\log\Delta^{-1/4} = -\frac{1}{4}\partial_\alpha\log\Delta\} = \\ &= \Delta^{1/4}(\Omega_{\alpha\beta\gamma} - \frac{1}{2}\eta_{\gamma[\beta}\partial_{\alpha]}\log\Delta).\end{aligned} \tag{D.24}$$

D.1.1.2 Reducing the Einstein-Hilbert term

Writing out the coefficient Ω_{ABC} with 4-dimensional indices and internal 7-dimensional indices yields three types of components that are non-zero

$$\begin{cases}\Omega_{\alpha\beta\gamma} = 2e_\alpha{}^\mu e_\beta{}^\nu\partial_{[\mu}e_{\nu]\gamma}, \\ \Omega_{\alpha\beta c} = e_{rc}G^r_{\alpha\beta}, \quad G^r_{\alpha\beta} = 2e_\alpha{}^\mu e_\beta{}^\nu\partial_{[\mu}(B_{\nu]}{}^a e_a{}^r), \\ \Omega_{\alpha bc} = e_\alpha{}^\mu e_b{}^m\partial_\mu e_{mc}.\end{cases} \tag{D.25}$$

The dimensionally reduced Einstein-Hilbert term is thus

$$\begin{aligned}\Omega^2_{ABC} - 2\Omega_{ABC}\Omega^{CAB} - 4\Omega_{CA}{}^A\Omega^C{}_B{}^B = -4R_4 + \Omega^2_{\alpha\beta c} + 2\Omega^2_{\alpha bc} - 2\Omega_{\alpha bc}\Omega^{c\alpha b} + \\ - 8\Omega_{\gamma\alpha}{}^\alpha\Omega^\gamma{}_b{}^b - 4\Omega_{\gamma a}{}^a\Omega^\gamma{}_b{}^b,\end{aligned} \tag{D.26}$$

where

$$-4R_4 = \Omega^2_{\alpha\beta\gamma} - 2\Omega_{\alpha\beta\gamma}\Omega^{\gamma\alpha\beta} - 4\Omega_{\gamma\alpha}{}^{\alpha}\Omega^{\gamma}{}_{\beta}{}^{\beta}. \tag{D.27}$$

The Weyl transformation can now be used to eliminate the persisting $\sqrt{\Delta}$-factor. The $-4R_4$ term is considered first and the relevant coefficients transform like

$$\begin{cases} \hat{\Omega}^2_{\alpha\beta\gamma} & = \Delta^{1/2}(\Omega^2_{\alpha\beta\gamma} - \Omega^{\alpha}{}_{\beta}{}^{\beta}\partial_\alpha \log\Delta + \frac{3}{8}[\partial_\alpha \log\Delta]^2), \\ \hat{\Omega}_{\alpha\beta\gamma}\hat{\Omega}^{\gamma\alpha\beta} & = \Delta^{1/2}(\Omega_{\alpha\beta\gamma}\Omega^{\gamma\alpha\beta} + \frac{1}{2}\Omega^{\alpha}{}_{\beta}{}^{\beta}\partial_\alpha \log\Delta - \frac{3}{16}[\partial_\alpha \log\Delta]^2), \\ \hat{\Omega}_{\gamma\alpha}{}^{\alpha}\hat{\Omega}^{\gamma}{}_{\beta}{}^{\beta} & = \Delta^{1/2}(\Omega_{\gamma\alpha}{}^{\alpha}\Omega^{\gamma}{}_{\beta}{}^{\beta} - \frac{3}{2}\Omega^{\alpha}{}_{\beta}{}^{\beta}\partial_\alpha \log\Delta + \frac{9}{16}[\partial_\alpha \log\Delta]^2), \end{cases} \tag{D.28}$$

which gives the Weyl transformation of R_4 as

$$\begin{aligned} -4R_4 \to -4\hat{R}_4 &= \hat{\Omega}^2_{\alpha\beta\gamma} - 2\hat{\Omega}_{\alpha\beta\gamma}\hat{\Omega}^{\gamma\alpha\beta} - 4\hat{\Omega}_{\gamma\alpha}{}^{\alpha}\hat{\Omega}^{\gamma}{}_{\beta}{}^{\beta} = \\ &= \Delta^{1/2}(-4R_4 + 4\Omega^{\alpha}{}_{\beta}{}^{\beta}\partial_\alpha \log\Delta - \frac{3}{2}[\partial_\alpha \log\Delta]^2). \end{aligned} \tag{D.29}$$

The other terms in (D.26) can be Weyl-transformed using

$$\begin{cases} \hat{\Omega}^2_{\alpha\beta c} & = \Delta(G^a_{\alpha\beta})^2, \qquad G^r_{\alpha\beta} = 2e_\alpha{}^\mu e_\beta{}^\nu \partial_{[\mu}(B_{\nu]}{}^a e_a{}^r), \\ \hat{\Omega}^2_{\alpha bc} & = \Delta^{1/2}\Omega^2_{\alpha bc}, \\ \hat{\Omega}_{\alpha bc}\hat{\Omega}^{c\alpha b} & = \Delta^{1/2}\Omega_{\alpha bc}\Omega^{c\alpha b}, \\ \hat{\Omega}_{\gamma\alpha}{}^{\alpha}\hat{\Omega}^{\gamma}{}_{b}{}^{b} & = \Delta^{1/2}\Omega^{\alpha}{}_{b}{}^{b}\Omega_{\alpha\beta}{}^{\beta} - \frac{3}{4}\Delta^{1/2}\Omega^{\alpha}{}_{b}{}^{b}\partial_\gamma \log\Delta, \\ \hat{\Omega}_{\gamma a}{}^{a}\hat{\Omega}^{\gamma}{}_{b}{}^{b} & = \Delta^{1/2}(\Omega_{\alpha b}{}^{b})^2. \end{cases} \tag{D.30}$$

The reduced and transformed Einstein-Hilbert term $H = \frac{1}{2}V\ R$ is then

$$\begin{aligned} \hat{H} = -\frac{1}{8}e\Big[&- 4R_4 + 4\Omega^{\alpha}{}_{\beta}{}^{\beta}\partial_\alpha \log\Delta - \frac{3}{2}[\partial_\alpha \log\Delta]^2 - 8\Omega^{\alpha}{}_{\beta}{}^{\beta}\Omega_{\alpha b}{}^{b} + \\ &+ 6\Omega^{\alpha}{}_{b}{}^{b}\partial_\alpha \log\Delta - \Big(4\Omega^2{}_{\alpha b}{}^{b} + 2\Omega_{\alpha bc}\Omega^{c\alpha b} - 2\Omega^2_{\alpha bc}\Big) + \sqrt{\Delta}(G^a{}_{\alpha\beta})^2\Big]. \end{aligned} \tag{D.31}$$

By noting that $\Omega_{\alpha b}{}^{b} = \frac{1}{2}\partial_\alpha \log\Delta$, which can be read off from $\partial_\alpha \log V = \omega_B{}^B{}_\alpha - \partial_\mu e_\alpha{}^\mu$ as

$$\partial_\alpha \log V = (\partial_\alpha \log e) + [\partial_\alpha \log\sqrt{\Delta}] = (\omega_\beta{}^\beta{}_\alpha - \partial_\mu e_\alpha{}^\mu) + [\Omega_{\alpha b}{}^b], \tag{D.32}$$

the terms with $\Omega^{\alpha}{}_{\beta}{}^{\beta}$ cancel. Other terms with $\Omega_{\alpha b}{}^{b}$ become linear in $[\partial_\alpha \log\Delta]^2$. The terms $2\Omega^2_{\alpha bc} - 2\Omega_{\alpha bc}\Omega^{c\alpha b}$ can also be rewritten since

$$\begin{aligned} \partial_\alpha g_{mn}\partial^\alpha g^{mn} &= \partial_\alpha(e_m{}^a e_n{}^b \eta_{ab})\partial^\alpha(\eta^{cd}e_c{}^m e_d{}^n) = \\ &= \eta_{ab}\eta^{cd}(e_m{}^a\partial_\alpha e_n{}^b + e_n{}^b\partial_\alpha e_m{}^a)(e_c{}^m\partial^\alpha e_d{}^n + e_d{}^n\partial^\alpha e_c{}^m) = \\ &= 2\partial_\alpha e_m{}^b\partial^\alpha e_b{}^m + 2\eta_{ab}\eta^{cd}e_m{}^a e_d{}^n\partial_\alpha e_n{}^b\partial^\alpha e_c{}^m = \\ &= 2\Omega_{\alpha bc}\Omega^{c\alpha b} - 2\Omega^2_{\alpha bc}. \end{aligned} \tag{D.33}$$

The Einstein-Hilbert term becomes

$$\hat{H} = \frac{1}{2}e\ R_4 - \frac{1}{8}e\sqrt{\Delta}(G^a_{\alpha\beta})^2 - \frac{1}{16}e\ [\partial_\alpha \log\Delta]^2 + \frac{1}{8}e\ \partial_\alpha g_{mn}\partial^\alpha g^{mn}. \tag{D.34}$$

Aside from the 4-dimensional Einstein-Hilbert term there is a term containing the field strength of the 7 gauge fields ${B_\mu}^r$ and two terms containing the scalar degrees of freedom. The scalars appear in the action only through g_{mn}, recall that $\Delta = \det\{g_{mn}\}$, which is symmetric with 28 degrees of freedom. The difference in dimension from the 49 of ${e_m}^a \in \mathrm{GL}(7)$ is 21, which corresponds to the local invariance group SO(7). The scalars thus live in the coset space of GL(7)/SO(7). Fixing the SO(7) gauge would, along with the condition ${e_m}^\alpha = 0$, reduce the 55 parameters of local SO(1, 10) invariance to 6, which corresponds to the local SO(1,3) invariance of the 4-dimensional theory.

D.1.2 Gauge invariance and field redefinitions

Before moving on to the other terms of the Lagrangian, how the reduced 3-form fields change under the gauge transformations should be considered. The $N = 8$ supergraviy Lagrangian should be formulated in terms of gauge invariant field strengths. The $D = 11$ Lagrangian is invariant under the transformations given in (D.2)-(D.6) under which the elfbein ${e_M}^A$, its inverse ${e_A}^M$ and the 3-form A_{MNP} transform like

$$\begin{cases} {e_M}^A & \to {e_M}^A + \partial_M \xi^N {e_N}^A + \xi^N \partial_N {e_M}^A + {e_M}^B {\Lambda_B}^A, \\ {e_A}^M & \to {e_A}^M - \partial_N \xi^M {e_A}^N + \xi^N \partial_N {e_A}^M + {\Lambda_A}^B {e_B}^M, \\ A_{MNP} & \to A_{MNP} + 3\partial_{[M} \xi^Q A_{NP]Q} + \xi^Q \partial_Q A_{MNP} + 3\partial_{[M} \zeta_{NP]}. \end{cases} \tag{D.35}$$

Here, the ${e_m}^\alpha = 0$ gauge is preserved by the condition that $\partial_m \xi^\mu = 0$. The ${B_\mu}^m$ fields transform as gauge fields

$${B_\mu}^m \to {B_\mu}^m + \partial_\mu \xi^m, \tag{D.36}$$

under coordinate transformations $x^m \to x^m - \xi^m$. When describing a 4-dimensional theory, the fields and field strengths should be independent of such internal transformations, which is why the ${B_\mu}^m$ fields should only appear as field strengths in the Lagrangian.

Out of the fields obtained by reducing A_{MNP} to 4 dimensions, the 21 pseudo vectors $A_{\mu ij}$ and the 7 $A_{\mu\nu i}$ fields have to be modified in order to transform nicely. They are not invariant under the internal coordinate change $x^m \to x^m - \xi^m$. The new fields

$$\begin{cases} A'_{\mu ij} = A_{\mu ij} - {B_\mu}^k A_{kij}, \\ A'_{\mu\nu i} = A_{\mu\nu i} - 2{B_{[\mu}}^p A_{\nu]ip} + {B_\mu}^i {B_\nu}^p A_{ipq}, \end{cases} \tag{D.37}$$

are thus defined, which are invariant under ξ^m transformaions. Under ζ_{ij} and $\zeta_{\mu i}$ gauge transformations, the primed A fields transform like

$$\begin{cases} \delta A'_{\mu ij} & = \partial_\mu \zeta_{ij}, \\ \delta A'_{\mu\nu i} & = 2\partial_{[\mu} \zeta_{\nu]i} - 2{B_{[\mu}}^p \partial_{\nu]} \zeta_{ip}. \end{cases} \tag{D.38}$$

While the field strength $F'_{\mu\nu ij}$ is gauge invariant, modifications are needed for $F'_{\mu\nu\rho i}$ since

$$\delta F'_{\mu\nu\rho i} = -3G^p_{[\mu\nu}\partial_{\rho]}\zeta_{ip}. \tag{D.39}$$

The correct modification is

$$F''_{\mu\nu\rho i} = F'_{\mu\nu\rho i} + 3G^p_{[\mu\nu}A'_{\rho]ip}, \tag{D.40}$$

since $\delta A'_{\rho ip} = \partial_\rho \zeta_{ip}$. With the gauge invariant field strengths $F'_{\mu\nu ij}$ and $F''_{\mu\nu\rho i}$ the remaining terms of the Lagrangian can be considered.

D.1.3 The 3-form kinetic term H'

The second term of the Lagrangian is

$$H' = -\frac{1}{2\cdot 48} e\sqrt{\Delta} F_{MNPQ}F^{MNPQ}. \tag{D.41}$$

Expanding the squared field strength in the flat frame yields 4 different terms

$$F^2 = F^2_{MNPQ} = F^2_{\alpha\beta\gamma\delta} + 4F^2_{\alpha\beta\gamma d} + 6F^2_{\alpha\beta cd} + 4F^2_{\alpha bcd}. \tag{D.42}$$

The formulation using curved indices is found via the definitions

$$\begin{cases} F_{\alpha bcd} = e_\alpha{}^\mu e_b{}^i e_c{}^j e_d{}^k \partial_\mu A_{ijk}, \\ F_{\alpha\beta cd} = e_\alpha{}^\mu e_\beta{}^\nu e_c{}^i e_d{}^j \tilde{F}_{\mu\nu ij}, \\ F_{\alpha\beta\gamma d} = e_\alpha{}^\mu e_\beta{}^\nu e_\gamma{}^\rho e_d{}^i \tilde{F}_{\mu\nu\rho i}, \end{cases} \tag{D.43}$$

where

$$\tilde{F}_{\mu\nu ij} = F'_{\mu\nu ij} + G^k_{\mu\nu}A_{ijk}, \qquad \tilde{F}_{\mu\nu\rho i} = F''_{\mu\nu\rho i}. \tag{D.44}$$

Using these field strengths and applying the Weyl transformation $g^{\mu\nu} \to \sqrt{\Delta}g^{\mu\nu}$ of section D.1.1.1 results in the transformation

$$\begin{aligned} F^2 \to\ & \Delta^2 F^2_{\alpha\beta\gamma\delta} + 4\Delta^{3/2} g^{\mu\sigma}g^{\nu\tau}g^{\rho\lambda}g^{ij}F_{\mu\nu\rho i}F_{\sigma\tau\lambda j} + \\ & + 6\Delta^1 g^{\mu\rho}g^{\nu\sigma}g^{ik}g^{jl}F_{\mu\nu ij}F_{\rho\sigma kl} + 4\Delta^{1/2}g^{\mu\nu}g^{il}g^{jm}g^{kn}\partial_\mu A_{ijk}\partial_\nu A_{lmn}. \end{aligned} \tag{D.45}$$

The second term of the Lagrangian becomes

$$\begin{aligned} \hat{H}' = & -\frac{1}{96}e\Delta^{3/2}F^2_{\alpha\beta\gamma\delta} - \frac{1}{24}e\Delta\tilde{F}_{\mu\nu\rho i}\tilde{F}^{\mu\nu\rho i} - \frac{1}{16}e\Delta^{1/2}\tilde{F}_{\mu\nu ij}\tilde{F}^{\mu\nu ij} + \\ & - \frac{1}{24}e\ g^{il}g^{jm}g^{kn}\partial_\mu A_{ijk}\partial^\mu A_{lmn}. \end{aligned} \tag{D.46}$$

D.1.4 The topological term H''

The third term of the Lagrangian, which is invariant under the Weyl transformation, can be written as

$$\begin{aligned} H'' &= -\frac{1}{2\cdot 12^4}\varepsilon^{M_1\ldots M_{11}}F_{M_1\ldots M_4}F_{M_5\ldots M_8}A_{M_9M_{10}M_{11}} = \\ &= \frac{1}{2\cdot 12^3}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}(4F_{\mu\nu\rho i}\partial_\sigma A_{jkl}A_{mno} - 9F_{\mu\nu ij}F_{\rho\sigma kl}A_{mno}), \end{aligned} \tag{D.47}$$

where the terms with fields of the type $A_{\mu ij}$ and $A_{\mu\nu i}$ have been integrated by parts. The topological term H'' should be expressed in terms of the field strengths $\tilde{F}_{\mu\nu ij}$ and $\tilde{F}_{\mu\nu\rho i}$. By using

$$\begin{cases} F_{\mu\nu\rho i} = \tilde{F}_{\mu\nu\rho i} - 3B_{[\mu}{}^{p}\tilde{F}_{\nu\rho]ip} + 3B_{[\mu}{}^{p}B_{\nu}{}^{q}\partial_{\rho]}A_{ipq}, \\ F_{\mu\nu ij} = \tilde{F}_{\mu\nu ij} - 2B_{[\mu}{}^{p}\partial_{\nu]}A_{ijp}, \end{cases} \tag{D.48}$$

it is found that

$$\begin{aligned} H'' = \frac{1}{2\cdot 12^3}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}\Big(&4\tilde{F}_{\mu\nu\rho i}\partial_\sigma A_{jkl}A_{mno} - 12B_\mu{}^{p}\tilde{F}_{\nu\rho ip}\partial_\sigma A_{jkl}A_{mno} + \\ &+ 12B_\mu{}^{p}B_\nu{}^{q}\partial_\rho A_{ipq}\partial_\sigma A_{jkl}A_{mno} + \\ &- 9\tilde{F}_{\mu\nu ij}\tilde{F}_{\rho\sigma kl}A_{mno} + 36\tilde{F}_{\mu\nu ij}B_\rho{}^{q}\partial_\sigma A_{klq}A_{mno} + \\ &+ 36B_\mu{}^{p}B_\nu{}^{q}\partial_\rho A_{ijp}\partial_\sigma A_{klq}A_{mno}\Big). \end{aligned} \tag{D.49}$$

The second term can be rewritten by noting that an antisymmetric tensor with more than seven internal indices is zero, meaning that $\tilde{F}_{\nu\rho[ip}\partial_\sigma A_{jkl}A_{mno]} = 0$. Note that any space-time indices $\mu, \nu, ...$ should be completely ignored in the antisymmetrisation bracket. Expanding this relation yields the result, where the antisymmetry $[ijklmno]$ is understood

$$\begin{aligned} B_\mu{}^{p}\tilde{F}_{\nu\rho ip}\partial_\sigma A_{jkl}A_{mno} &= \frac{3}{2}B_\mu{}^{q}\tilde{F}_{\nu\rho ij}\partial_\sigma A_{klq}A_{mno} - \frac{3}{2}B_\mu{}^{q}\tilde{F}_{\nu\rho ij}A_{klq}\partial_\sigma A_{mno} = \\ &= 3\tilde{F}_{\mu\nu ij}B_\rho{}^{q}\partial_\sigma A_{klq}A_{mno} + \frac{3}{2}G^p_{\mu\nu}B_\rho{}^{q}\partial_\sigma A_{ijp}A_{klq}A_{mno} + \\ &\quad - \frac{3}{4}\tilde{F}_{\mu\nu ij}G^q_{\rho\sigma}A_{klq}A_{mno}. \end{aligned} \tag{D.50}$$

The expression after the second equals sign is reached after integrating the second term by parts, where the non-trivial Bianchi identity $\partial_\sigma\tilde{F}_{\mu\nu ij} = G^p_{\sigma\mu}\partial_\nu A_{ijp}$ is used. In a similar way $\partial_\rho A_{[ip|q|}\partial_\sigma A_{jkl}A_{mno]} = 0$ can be used to find

$$\partial_\rho A_{ipq}\partial_\sigma A_{jkl}A_{mno} = -\frac{3}{2}\partial_\rho A_{ijp}\partial_\sigma A_{klq}A_{mno} + \frac{3}{2}\partial_\rho A_{ijp}A_{klq}\partial_\sigma A_{mno}. \tag{D.51}$$

Note that the left hand side and the first term are explicitly antisymmetric in p and q, which implies that the second term also is antisymmetric in p and q. Applying the relations (D.50) and (D.51) to H'' simplifies it like

$$\begin{aligned} H'' = \frac{1}{2\cdot 12^3}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}\Big(&4\tilde{F}_{\mu\nu\rho i}\partial_\sigma A_{jkl}A_{mno} - 9\tilde{F}_{\mu\nu ij}\tilde{F}_{\rho\sigma kl}A_{mno} + \\ &+ 9\tilde{F}_{\mu\nu ij}G^q_{\rho\sigma}A_{klq}A_{mno} + \\ &- 18G^p_{\mu\nu}B_\rho{}^{q}\partial_\sigma A_{ijp}A_{klq}A_{mno} + \\ &+ 18B_\mu{}^{p}B_\nu{}^{q}\partial_\rho A_{ijp}\partial_\sigma A_{klq}A_{mno} + \\ &+ 18B_\mu{}^{p}B_\nu{}^{q}\partial_\rho A_{ijp}A_{klq}\partial_\sigma A_{mno}\Big) = \\ = \frac{1}{2\cdot 12^3}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}\Big(&4\tilde{F}_{\mu\nu\rho i}\partial_\sigma A_{jkl}A_{mno} - 9\tilde{F}_{\mu\nu ij}\tilde{F}_{\rho\sigma kl}A_{mno} + \\ &+ 9\tilde{F}_{\mu\nu ij}G^q_{\rho\sigma}A_{klq}A_{mno} + \\ &- 9G^p_{\mu\nu}B_\rho{}^{q}\partial_\sigma(A_{ijp}A_{klq})A_{mno}\Big). \end{aligned} \tag{D.52}$$

The second equals sign of the above equation is reached by integrating the term with $\partial_\sigma A_{mno}$ by parts. This leaves a term including $G^p_{\mu\nu} B_\rho{}^q$ which is symmetric in p and q. All explicit dependence on the B-fields in the Lagrangian must either cancel, or be absorbed in a field strength. This is done by using the relation

$$\partial_\sigma (A_{ijp} A_{klq}) A_{mno} = 2 A_{ijp} A_{klq} \partial_\sigma A_{mno}, \tag{D.53}$$

which can be derived by expanding $A_{[ij|p|} A_{klq} \partial_\sigma A_{mno]} = 0$ whilst demanding symmetry between p and q

$$\begin{aligned} A_{ij(p|} A_{kl|q)} \partial_\sigma A_{mno} &= -\frac{2}{3} A_{i(pq)} A_{jkl} \partial_\sigma A_{mno} + A_{ij(p|} \partial_\sigma A_{kl|q)} A_{mno} = \\ &= \frac{1}{2} \partial_\sigma (A_{ijp} A_{klq}) A_{mno}. \end{aligned} \tag{D.54}$$

The term with explicit B-dependence can then be integrated by parts

$$\begin{aligned} G^p_{\mu\nu} B_\rho{}^q \partial_\sigma (A_{ijp} A_{klq}) A_{mno} &= 2 G^p_{\mu\nu} B_\rho{}^q A_{ijp} A_{klq} \partial_\sigma A_{mno} = \\ &= G^p_{\mu\nu} G^q_{\rho\sigma} A_{ijp} A_{klq} A_{mno} + \\ &\quad - 2 G^p_{\mu\nu} B_\rho{}^q \partial_\sigma (A_{ijp} A_{klq}) A_{mno}, \end{aligned} \tag{D.55}$$

by using (D.53). This implies that the $B_\rho{}^q$ is absorbed into a field strength via

$$3 G^p_{\mu\nu} B_\rho{}^q \partial_\sigma (A_{ijp} A_{klq}) A_{mno} = G^p_{\mu\nu} G^q_{\rho\sigma} A_{ijp} A_{klq} A_{mno}. \tag{D.56}$$

The Weyl-invariant topological term $H'' = \hat{H}''$ is thus found as

$$\begin{aligned} \hat{H}'' = \frac{1}{2 \cdot 12^3} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{ijklmno} \Big(& 4 \tilde{F}_{\mu\nu\rho i} \partial_\sigma A_{jkl} A_{mno} - 9 \tilde{F}_{\mu\nu ij} \tilde{F}_{\rho\sigma kl} A_{mno} + \\ & + 9 \tilde{F}_{\mu\nu ij} G^q_{\rho\sigma} A_{klq} A_{mno} - 3 G^p_{\mu\nu} G^q_{\rho\sigma} A_{ijp} A_{klq} A_{mno} \Big). \end{aligned} \tag{D.57}$$

D.1.5 Duality transformation

The field strengths $\tilde{F}_{\mu\nu ij}$ and $\tilde{F}_{\mu\nu\rho i}$ can be set as the independent variables of the Lagrangian instead of the corresponding 3-form fields. The Bianchi identities then have to be incorporated via constraints with Lagrange multipliers φ^i and $B_\mu{}^{ij} = -B_\mu{}^{ji}$. The terms H''' and H'''' given by

$$\begin{aligned} H''' &= \frac{1}{12} \varphi^i \varepsilon^{\mu\nu\rho\sigma} \partial_\mu (\tilde{F}_{\nu\rho\sigma i} - 3 G^p_{\nu\rho} A'_{\sigma ip}) = \\ &= \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} \partial_\sigma \varphi^i \tilde{F}_{\nu\rho\sigma i} - \frac{1}{8} \varphi^i \varepsilon^{\mu\nu\rho\sigma} G^p_{\rho\sigma} \tilde{F}_{\mu\nu ip}, \end{aligned} \tag{D.58}$$

and

$$\begin{aligned} H'''' &= \frac{1}{4} B^{ij}_\sigma \varepsilon^{\mu\nu\rho\sigma} \partial_\rho (\tilde{F}_{\mu\nu ij} - G^p_{\mu\nu} A_{ijp}) = \\ &= -\frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} G^{ij}_{\rho\sigma} \tilde{F}_{\mu\nu ij} - \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} B^{ij}_\sigma G^p_{\mu\nu} \partial_\rho A_{ijp}, \end{aligned} \tag{D.59}$$

thus have to be added to the Lagrangian. Here, some of the terms have been integrated by parts, and the field strength of the Lagrange multiplier B_μ^{ij} is defined as $G_{\mu\nu}^{ij} = 2\partial_{[\mu} B_{\mu]}{}^{ij}$. Finding the equations of motion for the field strengths $\tilde{F}_{\mu\nu ij}$ and $\tilde{F}_{\mu\nu\rho i}$ and inserting them back into the Lagrangian will provide the dual transformation of the Lagrangian.

D.1.5.1 The $\tilde{F}_{\mu\nu\rho i}$-terms of the Lagrangian

The part of the Lagrangian with all of the terms containing $\tilde{F}_{\mu\nu\rho i}$ is

$$\begin{aligned} K_1 = &-\frac{1}{24} e\Delta \tilde{F}_{\mu\nu\rho i}\tilde{F}^{\mu\nu\rho i} + \frac{2}{(12)^3}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}\tilde{F}_{\mu\nu\rho i}\partial_\sigma A_{jkl}A_{mno} + \\ &+ \frac{1}{12}\varepsilon^{\mu\nu\rho\sigma}\partial_\sigma\varphi^i\tilde{F}_{\mu\nu\rho i} = \\ = &-\frac{1}{24} e\Delta \tilde{F}_{\mu\nu\rho i}\tilde{F}^{\mu\nu\rho i} + \frac{1}{12}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\sigma\varphi^i + \frac{1}{6}\sqrt{\Delta} * A^{ijkl}\partial_\sigma A_{jkl}\right)\tilde{F}_{\mu\nu\rho i}, \end{aligned} \tag{D.60}$$

where

$$*A^{ijkl} = \frac{1}{12\sqrt{\Delta}}\varepsilon^{ijklmno}A_{mno}. \tag{D.61}$$

Varying with respect to $\delta\tilde{F}_{\mu\nu\rho i}$, the equations of motion are found as

$$-\frac{1}{12} e\Delta\tilde{F}^{\mu\nu\rho i} + \frac{1}{12}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\sigma\varphi^i + \frac{1}{6}\sqrt{\Delta} * A^{ijkl}\partial_\sigma A_{jkl}\right) = 0. \tag{D.62}$$

In other words

$$\begin{cases} \tilde{F}^{\mu\nu\rho i} &= \frac{1}{\Delta}\epsilon^{\mu\nu\rho\sigma}(\partial_\sigma\varphi^i + \frac{1}{6}\sqrt{\Delta} * A^{ijkl}\partial_\sigma A_{jkl}), \\ \tilde{F}_{\mu\nu\rho i} &= \frac{1}{\Delta}g_{iq}\epsilon_{\mu\nu\rho\lambda}(\partial^\lambda\varphi^q + \frac{1}{6}\sqrt{\Delta} * A^{qrst}\partial^\lambda A_{rst}). \end{cases} \tag{D.63}$$

Inserting these expressions of $\tilde{F}$ back in the Lagrangian returns

$$\begin{aligned} K_1 = -\frac{1}{4}\frac{e}{\Delta}g_{iq}g^{\mu\nu}\left(\partial_\mu\varphi^i + \frac{1}{6}\sqrt{\Delta} * A^{ijkl}\partial_\mu A_{jkl}\right) \times \\ \times \left(\partial_\nu\varphi^q + \frac{1}{6}\sqrt{\Delta} * A^{qrst}\partial_\nu A_{rst}\right). \end{aligned} \tag{D.64}$$

The duality transormation has revealed that the 3-form fields $A'_{\mu\nu i}$ describe scalars.

D.1.5.2 The $\tilde{F}_{\mu\nu ij}$-terms of the Lagrangian

The part of the Lagrangian containing $\tilde{F}_{\mu\nu ij}$ is

$$\begin{aligned}
K_2 = & -\frac{1}{16}e\Delta^{1/2}\tilde{F}_{\mu\nu ij}\tilde{F}^{\mu\nu ij} - \frac{1}{32\cdot 12}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}\tilde{F}_{\mu\nu ij}\tilde{F}_{\rho\sigma kl}A_{mno}+ \\
& + \frac{1}{32\cdot 12}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{ijklmno}\tilde{F}_{\mu\nu ij}G^q_{\rho\sigma}A_{klq}A_{mno} - \frac{1}{8}\varphi^i\varepsilon^{\mu\nu\rho\sigma}G^p_{\rho\sigma}\tilde{F}_{\mu\nu ip}+ \\
& - \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}G^{ij}_{\rho\sigma}\tilde{F}_{\mu\nu ij} = \\
= & -\frac{1}{16}e\sqrt{\Delta}\left(\tilde{F}_{\mu\nu ij}\tilde{F}^{\mu\nu ij} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} * A^{ijkl}\tilde{F}_{\mu\nu ij}\tilde{F}_{\rho\sigma kl}\right)+ \\
& - \frac{1}{8}e\ \epsilon^{\mu\nu\rho\sigma}\Big(G^{ij}_{\rho\sigma} - \frac{1}{4}\sqrt{\Delta} * A^{ijkl}A_{klq}G^q_{\rho\sigma}+ \\
& \qquad + \frac{1}{2}\left[\varphi^i G^j_{\rho\sigma} - \varphi^j G^i_{\rho\sigma}\right]\Big)\tilde{F}_{\mu\nu ij}.
\end{aligned} \tag{D.65}$$

Note that the spacetime indices are either contracted by two metrics $g^{\mu\rho}g^{\nu\sigma}$, as is the case for F^2, or by $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}$. Since $(g^{\mu\tau}g^{\nu\lambda})(g_{\tau\rho}g_{\lambda\sigma}) = \delta^\mu_\rho\delta^\nu_\sigma$ and $(\frac{1}{2}\epsilon^{\mu\nu\tau\lambda})(\frac{1}{2}\epsilon_{\tau\lambda\rho\sigma}) = -\delta^{\mu\nu}_{\rho\sigma}$, it is possible to omit the spacetime indices by making the replacements $g^{\mu\rho}g^{\nu\sigma} \to 1$ and $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \to i$, where $i^2 = -1$. In this notation K_2 reads

$$\begin{aligned}
K_2 = & -\frac{1}{16}e\sqrt{\Delta}\left(\tilde{F}_{ij}\tilde{F}^{ij} + i * A^{ijkl}\tilde{F}_{ij}\tilde{F}_{kl}\right)+ \\
& - \frac{1}{4}ie\left(G^{ij} - \frac{1}{4}\sqrt{\Delta} * A^{ijkl}A_{klq}G^q + \frac{1}{2}\left[\varphi^i G^j - \varphi^j G^i\right]\right)\tilde{F}_{ij} = \\
= & -\frac{1}{16}e\sqrt{\Delta}(\mathcal{M}^{-1})^{ij,kl}\tilde{F}_{ij}\tilde{F}_{kl}+ \\
& - \frac{1}{4}ie\left(G^{ij} - \frac{1}{4}\sqrt{\Delta} * A^{ijkl}A_{klq}G^q + \frac{1}{2}\left[\varphi^i G^j - \varphi^j G^i\right]\right)\tilde{F}_{ij},
\end{aligned} \tag{D.66}$$

where the matrix $(\mathcal{M}^{-1})^{ij,kl}$ is given by

$$\mathcal{M}_{ij,kl} = \left(g^{ij,kl} - i * A^{ijkl}\right)^{-1}, \tag{D.67}$$

where $g^{ij,kl} = \frac{1}{2}(g^{ik}g^{jl} - g^{il}g^{jk})$. Varying with respect to $\delta\tilde{F}_{ij}$ yield the equations of motion as

$$(\mathcal{M}^{-1})^{ij,kl}\tilde{F}_{kl} = -\frac{2i}{\sqrt{\Delta}}\left(G^{ij} - \frac{1}{4}\sqrt{\Delta} * A^{ijkl}A_{klq}G^q + \frac{1}{2}[\varphi^i G^j - \varphi^j G^i]\right), \tag{D.68}$$

which can also be written as

$$\tilde{F}_{ij} = -\frac{2i}{\sqrt{\Delta}}\mathcal{M}_{ij,kl}\left(G^{kl} - \frac{1}{4}\sqrt{\Delta} * A^{klmn}A_{mnp}G^p + \frac{1}{2}[\varphi^k G^l - \varphi^l G^k]\right). \tag{D.69}$$

Inserting the equations of motion back into the Lagrangian returns

$$\begin{aligned}
K_2 = & -\frac{1}{4}\frac{e}{\sqrt{\Delta}}\mathcal{M}_{ij,pq}\left(G^{ij} - \frac{1}{2}\sqrt{\Delta} * A^{ijkl}A_{klq}G^q + \frac{1}{2}[\varphi^i G^j - \varphi^j G^i]\right)\times \\
& \times\left(G^{pq} - \frac{1}{2}\sqrt{\Delta} * A^{pqrs}A_{rst}G^t + \frac{1}{2}[\varphi^p G^q - \varphi^q G^p]\right).
\end{aligned} \tag{D.70}$$

D.1.6 The reduced Lagrangian

To summarise the results thus far, the Lagrangian reduced to 4 dimensions is

$$\mathcal{L} = \frac{1}{2} eR + \mathcal{L}_S + \mathcal{L}_V, \tag{D.71}$$

having dropped the hat on the Weyl-transformed Lagrangian $\hat{\mathcal{L}}$ and the 4 on the 4-dimensional Ricci scalar R_4. The terms of $\mathcal{L}$ describing the scalars are

$$\begin{aligned} \mathcal{L}_S = \frac{1}{8} e\, \partial_\mu g_{mn} \partial^\mu g^{mn} - \frac{1}{16} e\, [\partial_\mu \log \Delta]^2 - \frac{1}{24} e g^{il} g^{jm} g^{kn} \partial_\mu A_{ijk} \partial^\mu A_{lmn} + \\ - \frac{1}{4} \frac{e}{\Delta} g_{iq} g^{\mu\nu} \left(\partial_\mu \varphi^i + \frac{1}{6} \sqrt{\Delta} * A^{ijkl} \partial_\mu A_{jkl} \right) \times \\ \times \left(\partial_\nu \varphi^q + \frac{1}{6} \sqrt{\Delta} * A^{qrst} \partial_\nu A_{rst} \right), \end{aligned} \tag{D.72}$$

where $\Delta = \det(g_{mn})$. There are 70 real scalar degrees of freedom in total. The internal metric g_{mn} contains 28, the Lagrange multipliers φ^i contain 7 and the 3-form remnant A_{ijk} contains 35. Furthermore, the A_{ijk} are pseudoscalars so that $70 = 35$ scalars $+ 35$ pseudoscalars. The 28 vector degrees of freedom are mostly coupled via the scalars in

$$\begin{aligned} \mathcal{L}_V = -\frac{1}{8} e \sqrt{\Delta} (G^a_{\alpha\beta})^2 - \frac{1}{96} \varepsilon^{\mu\nu\rho\sigma} * A^{ijkl} A_{ijp} A_{klq} G^p_{\mu\nu} G^q_{\rho\sigma} + \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} G^{ij}_{\mu\nu} G^k_{\rho\sigma} A_{ijk} + \\ - \frac{1}{4} \frac{e}{\sqrt{\Delta}} \mathcal{M}_{ij,pq} \left(G^{ij} - \frac{1}{4} \sqrt{\Delta} * A^{ijkl} A_{klq} G^q + \frac{1}{2} \left[\varphi^i G^j - \varphi^j G^i \right] \right) \times \\ \times \left(G^{pq} - \frac{1}{4} \sqrt{\Delta} * A^{pqrs} A_{rst} G^t + \frac{1}{2} \left[\varphi^p G^q - \varphi^q G^p \right] \right), \end{aligned} \tag{D.73}$$

where 7 are contained in the gauge fields $G^i_{\mu\nu}$ originating from the off-diagonal part of the elfbein $e_M{}^A$. The other 21 are from the Lagrange multipliers $G^{ij}_{\mu\nu}$, where the ij-indices are antisymmetric.

D.2 Restoring symmetry

The Lagrangian in (D.71) contains hidden symmetries. For example, since there are 28 vectors, it is tempting to extend the 7-dimensional indices i, j, k,... = 1,..., 7 to 8-dimensional ones, i', j', k' = 1,..., 8. Then $\frac{1}{2} n(n-1) = 28$ for $n = 8$. The 8-dimensional indices are also useful for dealing with the scalars.

D.2.1 The 35 true scalars

Omitting the pseudoscalars A_{ijk}, the 35 true scalars are described by the Lagrangian

$$\mathcal{L}^+_S = \frac{1}{8} e\, \partial_\mu g_{mn} \partial^\mu g^{mn} - \frac{1}{4} \frac{e}{\Delta} g_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j - \frac{1}{16} e \frac{\partial_\mu \Delta}{\Delta} \frac{\partial^\mu \Delta}{\Delta}. \tag{D.74}$$

By using the 8-dimensional indices i', j',..., it can be written in the more compact form

$$\mathcal{L}_S{}^+ = \frac{1}{8} e\, \partial_\mu S^{i'j'} \partial^\mu S_{i'j'}, \tag{D.75}$$

where the 8×8 matrix $S^{i'j'}$ and its inverse are given by

$$S^{i'j'} = \Delta^{-3/4} \begin{pmatrix} \Delta g^{ij} + \varphi^i \varphi^j & -\varphi^i \\ -\varphi^j & 1 \end{pmatrix}, \qquad S_{i'j'} = -\Delta^{-1/4} \begin{pmatrix} g_{ij} & \varphi_i \\ \varphi_j & \Delta + \varphi^2 \end{pmatrix}. \tag{D.76}$$

To show that (D.74) and (D.75) are equal, $g^{ij} \partial^\mu g_{ij} = \partial_\mu \Delta / \Delta$ is a useful relation. The matrix $S^{i'j'}$ is an elements of SL(8, $\mathbb{R}$) since $\det S^{i'j'} = 1$ follows from $\det(g_{mn}) = \Delta$. Then $\mathcal{L}_S^+$ is globally invariant under SL(8, $\mathbb{R}$) and its subgroup SO(8). The 28 scalars g_{mn} were part of the coset space GL(7, $\mathbb{R}$)/SO(7), as was discussed in section D.1.1.2. The global GL(7, $\mathbb{R}$) symmetry is however extended to SL(8, $\mathbb{R}$) when combining the true scalars.

D.2.2 The vectors, without the pseudoscalars

The terms of $\mathcal{L}_V$ that do not couple to the pseudoscalars are given by

$$\begin{aligned} \mathcal{L}_V^+ = &-\frac{1}{8} e \sqrt{\Delta} G^i_{\mu\nu} G^{\mu\nu j} g_{ij} + \\ &- \frac{1}{4} \frac{e}{\sqrt{\Delta}} \mathcal{M}^R_{ij,pq} \left(G^{ij}_{\mu\nu} + \frac{1}{2} \varphi^i G^j_{\mu\nu} - \frac{1}{2} \varphi^j G^i_{\mu\nu} \right) \times \\ &\times \left(G^{\mu\nu pq} + \frac{1}{2} \varphi^p G^{\mu\nu q} - \frac{1}{2} \varphi^q G^{\mu\nu p} \right), \end{aligned} \tag{D.77}$$

where

$$\mathcal{M}^R_{ij,pq} = \frac{1}{2} \left(g_{ip} g_{jq} - g_{iq} g_{jp} \right). \tag{D.78}$$

The vector field strengths can be combined to a common form $G^{i'j'}_{\mu\nu}$ where $i' = (i, 8)$. This is done with a rescaling

$$G^{i,8}_{\mu\nu} = -\frac{1}{2} G^i_{\mu\nu}, \qquad G^{8,i}_{\mu\nu} = \frac{1}{2} G^i_{\mu\nu}. \tag{D.79}$$

The terms in $\mathcal{L}_V^+$ can then be collected like

$$\mathcal{L}_V^+ = -\frac{1}{4} e\, \frac{1}{2} \left(S_{i'p'} S_{j'q'} - S_{i'q'} S_{j'p'} \right) G^{i'j'}_{\mu\nu} G^{\mu\nu p'q'}. \tag{D.80}$$

Again, global SL(8, $\mathbb{R}$) invariance is discovered, which suggests that the total Lagrangian $\mathcal{L}$ might also possess this symmetry.

D.2.3 Local SO(8) gauge invariance

The 28 scalars g_{mn} were concluded to live in the coset space GL(7, $\mathbb{R}$)/SO(7) in section (D.1.1.2). The reason was that the equation

$$g^{mn} e_m{}^a e_n{}^b = \delta^{ab}, \tag{D.81}$$

does not uniquely define the seibenbeins $e_m{}^a$, which are element of GL(7, $\mathbb{R}$). The flat metric is invariant under local SO(7) transformations, meaning that (D.81) only defines the class of seibenbeins $\{e_m{}^a \Lambda_a{}^b\}$, where $\Lambda_a{}^b$ is an arbitrary SO(7) rotation. The physical scalar fields of g_{mn} can thus equivalently be described using the class $\{e_m{}^a \Lambda_a{}^b\}$, which is valued in the coset space GL(7, $\mathbb{R}$)/SO(7). The only compact generators of $\mathfrak{gl}(7, \mathbb{R})$ are those that span $\mathfrak{so}(7)$. This is essential, since the Cartan-Killing metric becomes positive definite when there are only non-compact generators. As the global GL(7, $\mathbb{R}$) symmetry is extended to SL(8, $\mathbb{R}$), the local SO(7) symmetry group should also grow to SO(8), which is the maximal compact subgroup of SL(8, $\mathbb{R}$). This indeed happens, and is shown in [11]. By also noting the global $E_{7(+7)}$ invariance of the field equations, the bosonic $N = 8$ supergravity Lagrangian can be rewritten to the form given in section 2.3.2.

E

Hopf fibration of S^3

This appendix is dedicated to the Hopf fibration of S^3, which describes the 3-sphere as a fiber bundle of S^1 over S^2. The Hopf fibration is of interest since it features a twist of the S^1 when going around the equator of the S^2. This is a global property that is similar to the monodromy of the S-fold described in section 3.2.3. First off, the 2-sphere is treated in E.1, specifically the metric and the stereographic projection. The transition function defined via the stereographic projection is of particular use when the Hopf fibration is treated in E.2

E.1 The 2-sphere S^2

E.1.1 The metric

The metric on $\mathbb{R}^3$ is given by the line element

$$ds^2 = \delta_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2. \tag{E.1}$$

The S^2-metric can be found by embedding a 2-sphere in $\mathbb{R}^3$. This is done via the S^2-constraint in $\mathbb{R}^3$

$$S^2 \ : \ x^2 + y^2 + z^2 = R^2. \tag{E.2}$$

By using polar coordinates in the xy-plane, $x = r\cos\varphi$ and $y = r\sin\varphi$ so that $x^2 + y^2 = r^2$, the third coordinate is determined via the S^2-constraint as $z = \pm\sqrt{R^2 - r^2}$, where $dz = \mp r dr/\sqrt{R^2 - r^2}$. The $\mathbb{R}^3$ line element can now be rewritten in terms of r and φ

$$ds^2 = (dr^2 + r^2 d\varphi^2) + \frac{r^2}{R^2 - r^2} dr^2 = \frac{R^2}{R^2 - r^2} dr^2 + r^2 d\varphi^2. \tag{E.3}$$

By setting $r = R\sin\theta$ where $dr = R\cos\theta \ d\theta$, the line element takes the alternative form

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2, \tag{E.4}$$

which describes the geometry of S^2 with the radius R.

E.1.2 Stereographic projection

There is a useful mapping from the 2-sphere to the flat equatorial plane called the stereographic projection. Consider again the 2-dimensional sphere S^2 with radius

R. The north pole **N** and the south pole **S** of the 2-sphere are defined by

$$\mathbf{A}_i = A_i\hat{z} = (-1)^i R\hat{z}, \quad i = 1,\, 2, \qquad \Longrightarrow \qquad \mathbf{S} = \mathbf{A}_1 = -R\hat{z}, \quad \mathbf{N} = \mathbf{A}_2 = R\hat{z}. \tag{E.5}$$

To project a point **P** on S^2 to a point in the xy-plane, a line passing through **P** and either **N** or **S** is drawn. The intersection of this line with the xy-plane is then the projected point **Q**. As either **N** or **S** can be chosen, there are two projections, denoted by Φ_i. However, if the line is drawn through **N**, then **N** itself is mapped to infinity on the xy-plane, rather than a single point. The same is true for **S** if the south pole is chosen. The stereographic projections are thus

$$\Phi_i : S^2 \setminus \mathbf{A}_i \to \mathbb{R}^2. \tag{E.6}$$

The explicit projections are found by drawing the lines $\mathbf{L}_i$

$$\mathbf{L}_i = \mathbf{A}_i + \lambda_i(\mathbf{P} - \mathbf{A}_i) = \left(\lambda_i x,\, \lambda_i y,\, A_i + \lambda_i(z - A_i)\right), \tag{E.7}$$

where $\mathbf{P} = (x,\, y,\, z)$, and finding the intersections where $z = 0$. This is the case when

$$A_i + \lambda_i(z - A_i) = 0, \qquad \Longrightarrow \qquad \lambda_i = \frac{A_i}{A_i - z}. \tag{E.8}$$

The stereographic projection is thus given by

$$\Phi_i : S^2 \setminus \mathbf{A}_i \to \mathbb{R}^2 = (x,\, y,\, z) \to (\xi_i,\, \eta_i) = \frac{A_i}{A_i - z}(x,\, y), \qquad z \neq A_i, \tag{E.9}$$

where ξ_i and η_i are the coordinates of the projected plane. Similarly, the inverse projections $\Phi_i^{-1} : \mathbb{R}^2 \to S^2 \setminus \mathbf{A}_i$ are found by drawing new lines between the flat plane and the poles

$$\mathbf{K}_i = \mathbf{A}_i + \lambda_i'(\mathbf{Q}_i - \mathbf{A}_i) = \left(\lambda_i'\xi_i,\, \lambda_i'\eta_i,\, A_i - \lambda_i' A_i\right), \tag{E.10}$$

where $\mathbf{Q}_i = (\xi_i,\, \eta_i,\, 0)$. The intersection with S^2 happens when

$$\begin{aligned}
&(\lambda_i'\xi)^2 + (\lambda_i'\eta)^2 + A_i^2(1 - \lambda_i')^2 = R^2, \quad \Longrightarrow \\
&\qquad \Longrightarrow \quad \lambda_i'^2(\xi^2 + \eta^2) + R^2 + R^2\lambda_i'^2 - 2\lambda_i' R^2 = R^2, \\
&\qquad \Longrightarrow \quad \lambda_i' = \frac{2R^2}{\xi^2 + \eta^2 + R^2}.
\end{aligned} \tag{E.11}$$

The inverse projection is found as

$$\Phi_i^{-1} = (\xi_i,\, \eta_i) \to (x,\, y,\, z), \tag{E.12}$$

where

$$(x,\, y,\, z) = \left(\frac{2R^2\xi_i}{\xi_i^2 + \eta_i^2 + R^2},\ \frac{2R^2\eta_i}{\xi_i^2 + \eta_i^2 + R^2},\ A_i\left[1 - \frac{2R^2}{\xi_i^2 + \eta_i^2 + R^2}\right]\right). \tag{E.13}$$

More generally, the stereographic projection is a map onto a 2-dimensional plane from two different patches U_i on S^2. Each patch contains only one pole, $\mathbf{N} \in U_1$

and $\mathbf{S} \in U_2$, so that the projections are defined as $\Phi_i : U_i \to \mathbb{R}^2$. On the overlap $U_1 \cap U_2$, both projections Φ_1 and Φ_2 are well-defined and it is possible to define the transition function

$$\Phi_{12} = \Phi_1 \circ \Phi_2^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 = (\xi_2, \eta_2) \to (\xi_1,\, \eta_1), \tag{E.14}$$

where

$$(\xi_1,\, \eta_1) = \begin{pmatrix} \xi_1[x(\xi_2,\, \eta_2),\, y(\xi_2,\, \eta_2),\, z(\xi_2,\, \eta_2)] \\ \eta_1[x(\xi_2,\, \eta_2),\, y(\xi_2,\, \eta_2),\, z(\xi_2,\, \eta_2)] \end{pmatrix}, \tag{E.15}$$

following (E.9) and (E.13). Evaluating (E.15) and defining the complex coordinates $z_1 = \xi_1 - i\eta_1$, $z_2 = \xi_2 + i\eta_2$ allows Φ_{12} to be expressed as map from $\mathbb{C}$ to $\mathbb{C}$ where

$$z_1 = \frac{R^2}{z_2}. \tag{E.16}$$

The points $z_i = 0$ are not allowed since they map to either $\mathbf{S}$ or $\mathbf{N}$, which are not in the overlap $U_1 \cap U_2$.

E.1.3 Metric from stereographic projection

Consider $\mathbb{R}^2$ parametrised by the Cartesian coordinates $(p,\, q) \in \mathbb{R}^2$. The inverse stereographic projections $\Phi_i^{-1}(p,\, q) = (x,\, y,\, z_i)$ makes it is possible to move to U_i on S^2. Changing to polar coordinates on $\mathbb{R}^2$, so that $p = \rho \cos\alpha$ and $q = \rho \sin\alpha$, allows the patch U_i to be parametrised by

$$(x,\, y,\, z_i) = \left(\frac{2R^2 \rho \cos\alpha}{\rho^2 + R^2},\; \frac{2R^2 \rho \sin\alpha}{\rho^2 + R^2},\; A_i \left[1 - \frac{2R^2}{\rho^2 + R^2} \right] \right). \tag{E.17}$$

Using this projection, the metric on U_i can be evaluated via $ds^2(U_i) = dx^2 + dy^2 + dz_i^2$. Following

$$\begin{aligned} dx &= -2R^2 \frac{\rho}{\rho^2 + R^2} \sin\alpha \; d\alpha + 2R^2 \left(\frac{1}{\rho^2 + R^2} - \frac{2\rho^2}{(\rho^2 + R^2)^2} \right) \cos\alpha \; d\rho = \\ &= -2R^2 \frac{\rho}{\rho^2 + R^2} \sin\alpha \; d\alpha + 2R^2 \frac{R^2 - \rho^2}{(\rho^2 + R^2)^2} \cos\alpha \; d\rho, \\ dy &= 2R^2 \frac{\rho}{\rho^2 + R^2} \cos\alpha \; d\alpha + 2R^2 \frac{R^2 - \rho^2}{(\rho^2 + R^2)^2} \sin\alpha \; d\rho, \\ dz_i &= 2R^2 A_i \frac{2\rho}{(\rho^2 + R^2)^2} d\rho, \end{aligned} \tag{E.18}$$

the metric on U_i becomes

$$\begin{aligned} ds^2 &= \left(4R^4 \frac{(R^2 - \rho^2)^2}{(\rho^2 + R^2)^4} + 4R^4 A_i^2 \frac{4\rho^2}{(\rho^2 + R^2)^4} \right) d\rho^2 + 4R^4 \frac{\rho^2}{(\rho^2 + R^2)^2} d\alpha^2 = \\ &= \frac{4R^4}{(\rho^2 + R^2)^4} \left(R^4 + \rho^4 - 2R^2\rho^2 + 4R^2\rho^2 \right) d\rho^2 + \frac{4R^4}{(\rho^2 + R^2)^2} \rho^2 d\alpha^2 = \\ &= \frac{4R^4}{(\rho^2 + R^2)^2} \left(d\rho^2 + \rho^2 d\alpha^2 \right). \end{aligned} \tag{E.19}$$

Since $A_i^2 = 1$, both Φ_i^{-1} lead to the same metric which, by returning to Cartesian coordinates, can be written as

$$ds^2(S^2) = \frac{4R^4}{(p^2+q^2+R^2)^2}\left(dp^2+dq^2\right), \tag{E.20}$$

which is the metric on S^2. The $\mathbb{R}^2$ spaces that are associated with each patch are however different, and their metrics $ds^2(U_i)$ should be written using the proper ξ_i and η_i coordinates. By again using the complex coordinates $z_1 = \xi_1 - i\eta_1$ and $z_2 = \xi_2 + i\eta_2$, the metrics on U_i take form

$$ds^2(U_i) = \frac{4R^4 dz_i d\bar{z}_i}{(R^2+|z_i|^2)^2}. \tag{E.21}$$

By the rescalings $z_i \to z_i/2$ and $R \to R/2$ it can be seen that $ds^2(U_i)$ is equivalent to the Fubini-Study metric

$$ds^2 = \frac{dz d\bar{z}}{(1+|z|^2)^2}. \tag{E.22}$$

On the overlap $U_1 \cap U_2$, the transition function Φ_{12}, which exchanges coordinates via $z_1 = R^2/z_2$, does not alter the form of the metric

$$ds^2(U_1) = \frac{4R^4 dz_1 d\bar{z}_1}{(R^2+|z_1|^2)^2} = \frac{4R^4 dz_2 d\bar{z}_2}{(R^2+|z_2|^2)^2} = ds^2(U_2). \tag{E.23}$$

E.1.4 The scalar curvature of S^2

In this section, the scalar curvature of S^2 is calculated. The procedure is analogous to appendix C, where it is described in more detail. The main difference here is that the flat metric σ_{ab} on S^2 is taken to be off-diagonal.

The S^2 line element in complex coordinates is given by

$$ds^2(S^2) = \frac{4R^4 dw d\bar{w}}{(R^2+|w|^2)^2}. \tag{E.24}$$

By defining $dw^i = (dw, d\bar{w})$, the line element can be written using the S^2 metric g_{ij}

$$ds^2(S^2) = g_{ij}dw^i dw^j = e^a e^b \sigma_{ab} = 2e^1 e^2, \tag{E.25}$$

where the flat metric σ_{ab} is defined as

$$\sigma_{ab} = e_a{}^i e_b{}^j g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{E.26}$$

The zweibein 1-forms are thus

$$e^1 = \frac{\sqrt{2}R^2}{(R^2+|w|^2)}dw, \qquad e^2 = \frac{\sqrt{2}R^2}{(R^2+|w|^2)}d\bar{w}. \tag{E.27}$$

Their exterior derivatives are calculated as

$$\begin{aligned} de^1 &= -\frac{\sqrt{2}R^2}{(1+w\bar{w})^2} w\ d\bar{w} \wedge dw = \frac{w}{\sqrt{2}R^2} e^1 \wedge e^2, \\ de^2 &= -\frac{\sqrt{2}R^2}{(1+w\bar{w})^2} \bar{w}\ dw \wedge d\bar{w} = -\frac{\bar{w}}{\sqrt{2}R^2} e^1 \wedge e^2, \end{aligned} \tag{E.28}$$

so that the equation $de^a = \sigma^{ac}\omega_{ibc} e^i \wedge e^c$ can be evaluated for $a = 1,\, 2$

$$\begin{aligned} a = 1\ : &\qquad \frac{w}{\sqrt{2}R^2} e^1 \wedge e^2 = e^1 \wedge e^2 2\omega_{[12]2}, \\ a = 2\ : &\qquad -\frac{\bar{w}}{\sqrt{2}R^2} e^1 \wedge e^2 = e^1 \wedge e^2 2\omega_{[12]1}. \end{aligned} \tag{E.29}$$

The non-zero ω_{abc} are thus

$$\omega_{121} = -\frac{\bar{w}}{\sqrt{2}R^2}, \qquad \omega_{212} = -\frac{w}{\sqrt{2}R^2}, \tag{E.30}$$

which leaves the connection 1-form as

$$\omega_{12} = \omega_{112} e^1 + \omega_{212} e^2 = \frac{\bar{w}}{\sqrt{2}R^2} e^1 - \frac{w}{\sqrt{2}R^2} e^2. \tag{E.31}$$

The next step is to evaluate $\Theta_{12} = d\omega_{12} + \omega_{1a} \wedge \omega_{a2}$. The second term vanishes, $\omega_{1a} \wedge \omega_{a2} = 0$, which leaves

$$\begin{aligned} \Theta_{12} = d\omega_{12} &= \frac{1}{\sqrt{2}R^2} d\bar{w} \wedge e^1 + \frac{\bar{w}}{\sqrt{2}R^2} de^1 - \frac{1}{\sqrt{2}R^2} dw \wedge e^2 - \frac{w}{\sqrt{2}R^2} de^2 = \\ &= -\frac{R^2 + |w|^2}{2R^4} e^1 \wedge e^2 + \frac{|w|^2}{2R^4} e^1 \wedge e^2 + \\ &\quad - \frac{R^2 + |w|^2}{2R^4} e^1 \wedge e^2 + \frac{|w|^2}{2R^4} e^1 \wedge e^2 = \\ &= -\frac{1}{R^2} e^1 \wedge e^2. \end{aligned} \tag{E.32}$$

The only non-vanishing Riemann tensor component is

$$R_{1212} = -\frac{1}{R^2}, \tag{E.33}$$

and the Ricci tensor, calculated via $R_{bd} = \sigma^{ac} R_{abcd}$

$$\begin{aligned} R_{11} &= \sigma^{ac} R_{a1c1} = R_{1121} + R_{2111} = 0, \\ R_{12} &= \sigma^{ac} R_{a1c2} = R_{1122} + R_{2112} = \frac{1}{R^2}, \\ R_{22} &= \sigma^{ac} R_{a2c2} = R_{1222} + R_{2212} = 0, \end{aligned} \tag{E.34}$$

becomes

$$R_{ab} = \frac{1}{R^2} \sigma_{ab}. \tag{E.35}$$

The Ricci scalar is

$$R = \sigma^{ab} R_{ab} = \frac{2}{R^2}. \tag{E.36}$$

E.2 Hopf fibration of S^3

As a starting point, consider the 1-dimensional complex space

$$\mathbb{C}P^1 = \left\{ (z_1,\, z_2) \in \mathbb{C}^2 \setminus \{0\} \;\middle|\; (z_1,\, z_2) \equiv \lambda(z_1',\, z_2'),\;\; \lambda \in \mathbb{C} \setminus \{0\} \right\}. \tag{E.37}$$

$\mathbb{C}P^1$ is the space of all straight lines that intersect the origin in $\mathbb{C}^2$. The lines are thus characterised only by their slope, which can be defined as either z_1/z_2 or z_2/z_1. The two choices work equally well except for the two lines given by $z_1 = 0$ and $z_2 = 0$. Consequently, there are two patches on $\mathbb{C}P^1$ with different well-defined coordinates

$$\begin{aligned} U_1 \quad &: \quad z = \frac{z_1}{z_2}, \qquad\qquad z_2 \neq 0, \\ U_2 \quad &: \quad w = \frac{z_2}{z_1}, \qquad\qquad z_1 \neq 0, \end{aligned} \tag{E.38}$$

and on the overlap $U_1 \cap U_2$, the coordinates z and w are related via

$$z = \frac{1}{w}. \tag{E.39}$$

The transition function between U_1 and U_2 is the same as the one in (E.16) for the unit 2-sphere. This implies that $\mathbb{C}P^1$ and S^2 are actually the same.

The metric on S^3 can be inbedded into $\mathbb{C}^2$, similarly to how S^2 was inbedded in $\mathbb{R}^3$ in appendix E.1.1. The S^3 constraint in $\mathbb{C}^2$ is

$$S^3 \;:\; z_1\bar{z}_1 + z_2\bar{z}_2 = R^2. \tag{E.40}$$

This condition is invariant under U(1) transformations $z_i \to e^{i\varphi} z_i$, which alludes to a $S^1 \equiv$U(1) structure of S^3. The S^1 structure will be shown more explicitly later. Now, by using the z coordinate from (E.38), the constraint can be written as

$$R^2 = \left(1 + |z|^2\right) |z_2|^2, \tag{E.41}$$

however, this is only valid on U_1 where $z_2 \neq 0$. The line element of S^3 is thus found via the flat metric on $\mathbb{C}^2$

$$\begin{aligned} ds^2(S^3,\, U_1) &= dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 = (\bar{z} d\bar{z}_2 + \bar{z}_2 d\bar{z}) + dz_2 d\bar{z}_2 = \\ &= \left(1 + |z|^2\right) dz_2 d\bar{z}_2 + z\bar{z}_2 dz_2 d\bar{z} + z_2 \bar{z} dz d\bar{z}_2 + |z_2|^2 dz d\bar{z} = \\ &= R^2 \left(\frac{|dz_2|^2}{|z_2|^2} + \frac{|dz|^2}{1 + |z|^2} \right) + \left(z_2 d\bar{z}_2 \bar{z} dz + \text{c.c.} \right), \end{aligned} \tag{E.42}$$

where $dz_1 = z dz_2 + z_2 dz$ was used. An equivalent formulation is given by

$$\begin{aligned} ds^2(S^3,\, U_1) &= \left| R \frac{dz_2}{z_2} + \frac{|z_2|^2}{R} \bar{z} dz \right|^2 + R^2 \left(\frac{|dz|^2}{1 + |z|^2} - \frac{|z dz|^2}{(1 + |z|^2)^2} \right) = \\ &= \left| R \frac{dz_2}{z_2} + \frac{|z_2|^2}{R} \bar{z} dz \right|^2 + R^2 \frac{dz d\bar{z}}{(1 + |z|^2)^2}, \end{aligned} \tag{E.43}$$

where the last term corresponds to the Fubini-Study metric on the unit 2-sphere

$$ds^2(S^2) = \frac{dzd\bar{z}}{(1+|z|^2)^2}, \tag{E.44}$$

which makes it clear that there is a connection between S^3 and S^2. The coordinate z which parametrises the 2-sphere is however also present in the first term of $ds^2(S^3, U_i)$, which makes the relation non-trivial. To further investigate this term, z_2 can be written in polar coordinates $z_2 = se^{i\psi}$. The S^3 contraint $(1+|z|^2)|z_2|^2 = R^2$ then implies that

$$s^2 = \frac{R^2}{1+|z|^2}, \qquad \Longrightarrow \qquad \frac{ds}{s} = -\frac{1}{2}\frac{zd\bar{z}+\bar{z}dz}{1+|z|^2}. \tag{E.45}$$

Then

$$\frac{dz_2}{z_2} = \frac{1}{se^{i\psi}}\left(e^{i\psi}ds + id\psi\ se^{i\psi}\right) = \frac{ds}{s} + id\psi, \tag{E.46}$$

which can be used to rewrite the inside of the first square as

$$\begin{aligned} R\frac{dz_2}{z_2} + \frac{|z_2|^2}{R}\bar{z}dz &= R\frac{ds}{s} + iRd\psi + \frac{s^2}{R}\bar{z}dz = \\ &= -\frac{R}{2}\frac{zd\bar{z}+\bar{z}dz}{1+|z|^2} + iRd\psi + R\frac{\bar{z}dz}{1+|z|^2} = \\ &= iR\left(d\psi - \frac{i}{2}\frac{\bar{z}dz - zd\bar{z}}{1+|z|^2}\right). \end{aligned} \tag{E.47}$$

The metric on S^3 it thus

$$ds^2(S^3, U_1) = R^2\left|d\psi - \frac{i}{2}\frac{\bar{z}dz - zd\bar{z}}{1+|z|^2}\right|^2 + R^2\frac{dzd\bar{z}}{(1+|z|^2)^2}. \tag{E.48}$$

Note that the S^3 constraint also can be written as $R^2 = \left(1+|w|^2\right)|z_1|^2$ on U_2. This implies that the metrics $ds^2(S^3, U_1)$ and $ds^2(S^3, U_2)$ will take the same form since the calculation is completely analogous. The metric $ds^2(S^3, U_2)$ is therefore just

$$ds^2(S^3, U_2) = R^2\left|d\psi - \frac{i}{2}\frac{\bar{w}dw - wd\bar{w}}{1+|w|^2}\right|^2 + R^2\frac{dwd\bar{w}}{(1+|w|^2)^2}. \tag{E.49}$$

Now, by expressing the S^2 metric in terms of the angular coordinates (θ, ϕ), $ds^2(S^3, U_1)$ can be rewritten as

$$ds^2(S^3, U_1) = R^2\left(d\psi + \frac{1}{2}(1-\cos\theta)d\phi\right)^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{E.50}$$

The second term is clearly the S^2 metric obtained in (E.4) with radius R. As for the first term, it contains the coordinate ψ which parametrises the fiber S^1. The 3-sphere thus contains both S^1 and S^2, and can be described as the fiber bundle of S^1 over S^2. This implies that $S^3 = S^1 \times S^2$ locally, which is not globally true since there is a 1-form gauge field for a monopole on U_1 that also enters the metric (E.50). The gauge field is

$$A_1(U_1) = \frac{1}{2}(1-\cos\theta)d\phi = \frac{i}{2}\frac{zd\bar{z}-\bar{z}dz}{1+|z|^2} \xrightarrow[z\to 0]{} 0. \tag{E.51}$$

It is well-defined as $z \to 0$, which is the preferred coordinate on U_1, the patch that corresponds to the upper half-sphere of S^2 which contains the north pole $\mathbf{N}$. Note however that on the overlap $U_1 \cap U_2$, the transition function $z = 1/w$ implies that

$$A_1(U_2) = -\frac{i}{2}\frac{\frac{d\bar{w}}{w}-\frac{dw}{\bar{w}}}{1+|w|^2} \xrightarrow[w\to 0]{} \infty. \tag{E.52}$$

The monopole field that is well-defined on the lower half-sphere U_2 instead takes the form

$$A_2(U_2) = \frac{1}{2}(-1-\cos\theta)d\phi = \frac{i}{2}\frac{wd\bar{w}-\bar{w}dw}{1+|w|^2} \xrightarrow[w\to 0]{} 0. \tag{E.53}$$

On $U_1 \cap U_2$, A_1 and A_2 are connected via a gauge transformation $A_1 - A_2 = d\phi$, which means that the field strengths still are equal everywhere on S^2, $F_1 - F_2 = dd\phi = 0$.

The field strength F on S^2 is thus not exact, meaning that it cannot be written as $F = dA$ since there is no field A that is defined globally on S^2. This has consequences for a topoloical number on S^2, namely the first Chern number C_1. It is calculated as

$$\begin{aligned} C_1 &= -\frac{1}{2\pi}\int_{S^2} F = -\frac{1}{2\pi}\left(\int_{U_1} dA_1 + \int_{U_2} dA_2\right) = \\ &= -\frac{1}{2\pi}\int_{S^1}(A_1 - A_2) = -\frac{1}{2\pi}\int_{S^1} d\phi = -1, \end{aligned} \tag{E.54}$$

where the patches U_1 and U_2 are chosen so that their boundaries are the equator of S^2. The implication of $C_1 = -1$ is that traversing around the equator on the S^2 also corresponds to circuiting the S^1 once. This is the global property of the Hopf fibration that is not encoded in the local expression $S^3 = S^1 \times S^2$. A more simple example of a similar phenomenon is the Möbius strip. Locally, the Möbius strip is a cylinder $S^1 \times \mathbb{R}$. However, if one starts on the inside of the Möbius strip, encircling the S^1 once leads to the outside of the strip. A Möbius strip is thus not $S^1 \times \mathbb{R}$ globally.

This kind of global feature is also present in the S-fold vacuum $\mathrm{AdS}_4 \times S^1_\eta \times S^5$, which is described in section 3.2.3. In particular, there is a non-trivial S-duality monodromy when encircling S^1_η.

F

Type IIB field equations

This appendix is dedicated to deriving the type IIB field equations that are given in section 3.1.1. The action for type IIB string theory is

$$S = \frac{1}{2\kappa^2}\int d^{10}x\ e\left(R - \frac{1}{12}\mathbf{H}^T_{\mu\nu\rho}\mathcal{M}\mathbf{H}^{\mu\nu\rho} + \frac{1}{4}\operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\}\right) + \\ - \frac{1}{8\kappa^2}\left(\int d^{10}x\ e\left|\tilde{F}_5\right|^2 + \int \varepsilon_{ij}C_4 \wedge H_3^{(i)} \wedge H_3^{(j)}\right), \tag{F.1}$$

where the axion-dilaton matrix and its inverse are given by

$$\mathcal{M} = e^\phi \begin{pmatrix} |\lambda|^2 & -\chi \\ -\chi & 1 \end{pmatrix}, \qquad \mathcal{M}^{-1} = e^\phi \begin{pmatrix} 1 & \chi \\ \chi & |\lambda|^2 \end{pmatrix}, \tag{F.2}$$

and $\lambda = \chi + ie^{-\phi}$ [4]. The equations of motion are found by varying the action with respect to the different fields and then applying the self-duality condition $\tilde{F}_5 = \star\tilde{F}_5$ [21]. Some terms of the action can be written out more explicitly using

$$H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho} = e^\phi(|\lambda|^2 H_{\mu\nu\rho}H^{\mu\nu\rho} - 2\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + F_{\mu\nu\rho}F^{\mu\nu\rho}) \tag{F.3}$$

$$\frac{1}{4}\operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\} = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi), \tag{F.4}$$

$$\varepsilon_{ij}C_4 \wedge H_3^{(i)} \wedge H_3^{(j)} = 2C_4 \wedge H_3 \wedge F_3. \tag{F.5}$$

The self-dual field strength is again given by

$$\tilde{F}_{\mu\nu\rho\sigma\xi} = F_{\mu\nu\rho\sigma\xi} - 5C_{\mu\nu}H_{\rho\sigma\xi} + 5B_{\mu\nu}F_{\rho\sigma\xi}, \tag{F.6}$$

and its norm is $\left|\tilde{F}_5\right|^2 = \frac{1}{5!}g^{\mu_1\nu_1}\ldots g^{\mu_5\nu_5}\tilde{F}_{\mu_1\ldots\mu_5}\tilde{F}_{\nu_1\ldots\nu_5}$.

F.1 Varying with respect to ϕ

Starting off with the dilaton ϕ, there are 2 terms that contribute:

$$\begin{aligned}\delta\left(\frac{1}{4}e\ \operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\}\right) &= -\frac{1}{2}\delta\left(e\ [\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi]\right) = \\ &= -\frac{1}{2}\left(2e\ \partial_\mu\phi\partial^\mu\delta\phi + 2e\ e^{2\phi}\partial_\mu\chi\partial^\mu\chi\delta\phi\right) = \\ &= \left(\partial^\mu(e\partial_\mu\phi) - e\ e^{2\phi}\partial_\mu\chi\partial^\mu\chi\right)\delta\phi,\end{aligned} \tag{F.7}$$

and

$$\begin{aligned}\delta\left(-\frac{1}{12}e\ H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho}\right) &= -\frac{1}{12}e\ \delta\Big(e^{\phi}\Big[|\lambda|^2 H_{\mu\nu\rho}H^{\mu\nu\rho} - 2\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad + F_{\mu\nu\rho}F^{\mu\nu\rho}\Big]\Big) = \\ &= -\frac{1}{12}e\ \delta\Big(\Big[e^{\phi}\chi^2 + e^{-\phi}\Big]H_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad - 2e^{\phi}\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + e^{\phi}F_{\mu\nu\rho}F^{\mu\nu\rho}\Big) = \\ &= \frac{1}{12}e\ \Big(\Big[e^{-\phi} - e^{\phi}\chi^2\Big]H_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad + 2e^{\phi}\chi F_{\mu\nu\rho}H^{\mu\nu\rho} - e^{\phi}F_{\mu\nu\rho}F^{\mu\nu\rho}\Big)\delta\phi.\end{aligned} \tag{F.8}$$

The field equation for ϕ is thus given by

$$\begin{aligned}0 &= \partial^\mu(e\partial_\mu\phi) - e\ e^{2\phi}\partial_\mu\chi\partial^\mu\chi + \\ &\quad + \frac{1}{12}e\ \Big(\Big[e^{-\phi} - e^{\phi}\chi^2\Big]H_{\mu\nu\rho}H^{\mu\nu\rho} + 2e^{\phi}\chi F_{\mu\nu\rho}H^{\mu\nu\rho} - e^{\phi}F_{\mu\nu\rho}F^{\mu\nu\rho}\Big).\end{aligned} \tag{F.9}$$

The first term can alternatively be written as

$$\partial^\mu(e\partial_\mu\phi) = \nabla^\mu(e^{-\phi}\partial_\mu e^{\phi}), \tag{F.10}$$

so that the equation of motion for ϕ reads

$$\begin{aligned}\nabla^\mu(e^{-\phi}\partial_\mu e^{\phi}) - e^{2\phi}\partial_\mu\chi\partial^\mu\chi &= \frac{1}{12}\Big(e^{\phi}F_{\mu\nu\rho}F^{\mu\nu\rho} - 2e^{\phi}\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad + [e^{\phi}\chi^2 - e^{-\phi}]H_{\mu\nu\rho}H^{\mu\nu\rho}\Big).\end{aligned} \tag{F.11}$$

F.2 Varying with respect to χ

The χ-dependent terms of the action are the same as the ones that were considered when varying with respect to ϕ. Varying with respect to χ is however slightly easier since

$$\begin{aligned}\delta\left(\frac{1}{4}e\ \operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\}\right) &= -\frac{1}{2}\delta\left(e\ [\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi]\right) = \\ &= -\frac{1}{2}2e\ e^{2\phi}\partial_\mu\chi\partial^\mu\delta\chi = \\ &= \partial^\mu\left(e\ e^{2\phi}\partial_\mu\chi\right)\delta\chi,\end{aligned} \tag{F.12}$$

and

$$\begin{aligned}\delta\left(-\frac{1}{12}e\ H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho}\right) &= -\frac{1}{12}e\ \delta\Big(e^{\phi}\Big[|\lambda|^2 H_{\mu\nu\rho}H^{\mu\nu\rho} - 2\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad + F_{\mu\nu\rho}F^{\mu\nu\rho}\Big]\Big) = \\ &= -\frac{1}{12}e\ e^{\phi}\delta\left(\chi^2 H_{\mu\nu\rho}H^{\mu\nu\rho} - 2\chi F_{\mu\nu\rho}H^{\mu\nu\rho}\right) = \\ &= -\frac{1}{6}e\ e^{\phi}\left(\chi H_{\mu\nu\rho}H^{\mu\nu\rho} - F_{\mu\nu\rho}H^{\mu\nu\rho}\right)\delta\chi.\end{aligned} \tag{F.13}$$

The field equation for χ becomes

$$\partial^\mu \left(e\ e^{2\phi}\partial_\mu \chi\right) - \frac{1}{6} e\ e^{\phi}\left(\chi H_{\mu\nu\rho}H^{\mu\nu\rho} - F_{\mu\nu\rho}H^{\mu\nu\rho}\right) = 0, \tag{F.14}$$

which can be written as

$$\nabla^\mu \left(e^{2\phi}\partial_\mu \chi\right) = \frac{1}{6} e^{\phi}\left(\chi H_{\mu\nu\rho}H^{\mu\nu\rho} - F_{\mu\nu\rho}H^{\mu\nu\rho}\right). \tag{F.15}$$

F.3 Varying with respect to C_4

Varying with respect to C_4 involves the $\left|\tilde{F}_5\right|^2$-term, which becomes

$$\begin{aligned}\delta(e\left|\tilde{F}_5\right|^2) &= \frac{1}{5!}\delta(e\ \tilde{F}_{\mu\nu\rho\sigma\xi}\tilde{F}^{\mu\nu\rho\sigma\xi}) = \frac{2}{5!} e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\delta \tilde{F}_{\mu\nu\rho\sigma\xi} = \\ &= \frac{2}{5!} 5e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\partial_\mu \delta C_{\nu\rho\sigma\xi} = -\frac{1}{2\cdot 3!}\partial_\mu\left[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\right]\delta C_{\nu\rho\sigma\xi} = \\ &= -\frac{1}{2\cdot 3!}\partial_\xi\left[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\right]\delta C_{\mu\nu\rho\sigma},\end{aligned} \tag{F.16}$$

as well as the topological term

$$\delta\left(2\ C_4 \wedge H_3 \wedge F_3\right) = 2\ \delta C_4 \wedge H_3 \wedge F_3. \tag{F.17}$$

Written in terms of component the variation of the topological term becomes

$$2\ \delta C_4 \wedge H_3 \wedge F_3 = \frac{2}{3!\cdot 3!\cdot 4!} d^{10}x\ \varepsilon^{\mu\nu\rho\sigma\mu_1\ldots\mu_6} H_{\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6}\delta C_{\mu\nu\rho\sigma}, \tag{F.18}$$

so that the field equations for C_4 read

$$-\frac{1}{2\cdot 3!}\partial_\xi\left[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\right] + \frac{2}{3!\cdot 3!\cdot 4!}\varepsilon^{\mu\nu\rho\sigma\mu_1\ldots\mu_6} H_{\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6} = 0, \tag{F.19}$$

which simplifies to

$$\nabla_\xi\left[\tilde{F}^{\mu\nu\rho\sigma\xi}\right] = \frac{1}{(3!)^2}\epsilon^{\mu\nu\rho\sigma\mu_1\ldots\mu_6} H_{\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6}. \tag{F.20}$$

F.4 Varying with respect to B_2

The 2-form B_2 is involved in 3 terms of the action, the first being

$$\begin{aligned}\delta\left(-\frac{1}{12} e\ H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho}\right) &= -\frac{1}{12} e\ \delta\Big(e^{\phi}\Big[|\lambda|^2 H_{\mu\nu\rho}H^{\mu\nu\rho} - 2\chi F_{\mu\nu\rho}H^{\mu\nu\rho} + \\ &\qquad\qquad + F_{\mu\nu\rho}F^{\mu\nu\rho}\Big]\Big) = \\ &= -\frac{1}{12} e\ e^{\phi}\left(6|\lambda|^2 H^{\mu\nu\rho}\partial_\mu \delta B_{\nu\rho} - 6\chi F^{\mu\nu\rho}\partial_\mu\delta B_{\nu\rho}\right) = \\ &= \frac{1}{2}\left(\partial_\mu\left[e\ e^{\phi}|\lambda|^2 H^{\mu\nu\rho}\right] - \partial_\mu\left[e\ e^{\phi}\chi F^{\mu\nu\rho}\right]\right)\delta B_{\nu\rho} = \\ &= \frac{1}{2}\left(\partial_\rho\left[e\ e^{\phi}|\lambda|^2 H^{\mu\nu\rho}\right] - \partial_\rho\left[e\ e^{\phi}\chi F^{\mu\nu\rho}\right]\right)\delta B_{\mu\nu},\end{aligned} \tag{F.21}$$

the second is

$$\begin{aligned}\delta(-\frac{1}{4}e\left|\tilde{F}_5\right|^2) &= -\frac{1}{4\cdot 5!}\delta(e\ \tilde{F}_{\mu\nu\rho\sigma\xi}\tilde{F}^{\mu\nu\rho\sigma\xi}) = -\frac{2}{4\cdot 5!}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\delta\tilde{F}_{\mu\nu\rho\sigma\xi} = \\ &= -\frac{2}{4\cdot 5!}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\ 5(-3C_{\mu\nu}\partial_\rho\delta B_{\sigma\xi} + \delta B_{\mu\nu}F_{\rho\sigma\xi}) = \\ &= -\frac{1}{8\cdot 3!}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}(-3C_{\rho\sigma}\partial_\xi\delta B_{\mu\nu} + \delta B_{\mu\nu}F_{\rho\sigma\xi}) = \\ &= -\frac{1}{8\cdot 3!}(3\partial_\xi[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}]C_{\rho\sigma} + e\ \tilde{F}^{\mu\nu\rho\sigma\xi}F_{\xi\rho\sigma}+ \\ &\qquad + e\ \tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi})\delta B_{\mu\nu} = \\ &= (-\frac{1}{16}\partial_\xi[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}]C_{\rho\sigma} - \frac{1}{24}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi})\delta B_{\mu\nu},\end{aligned} \tag{F.22}$$

and the final is the topological term

$$\begin{aligned}\delta\left(-\frac{1}{2}C_4\wedge H_3\wedge F_3\right) &= -\frac{1}{2}C_4\wedge\delta H_3\wedge F_3 = -\frac{1}{2}C_4\wedge d\delta B_2\wedge F_3 = \\ &= \frac{1}{2}dC_4\wedge\delta B_2\wedge F_3 = -\frac{1}{2}\delta B_2\wedge F_3\wedge F_5,\end{aligned} \tag{F.23}$$

which becomes

$$\begin{aligned}-\frac{1}{2}\delta B_2\wedge F_5\wedge F_3 &= -\frac{1}{2}\frac{1}{10!}d^{10}x\ \varepsilon^{\mu\nu\mu_1\ldots\mu_8}\frac{10!}{2!\cdot 5!\cdot 3!}\delta B_{\mu\nu}F_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8} = \\ &= -\frac{1}{4\cdot 5!\cdot 3!}d^{10}x\ \varepsilon^{\mu\nu\mu_1\ldots\mu_8}F_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8}\delta B_{\mu\nu},\end{aligned} \tag{F.24}$$

in component form. The field equations for B_2 are thus

$$\begin{aligned}0 = \frac{1}{2}\left(\partial_\rho\left[e\ e^\phi|\lambda|^2H^{\mu\nu\rho}\right] - \partial_\rho\left[e\ e^\phi\chi F^{\mu\nu\rho}\right]\right) - \frac{1}{16}\partial_\xi[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}]C_{\rho\sigma}+ \\ -\frac{1}{24}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi} - \frac{1}{4\cdot 5!\cdot 3!}\varepsilon^{\mu\nu\mu_1\ldots\mu_8}F_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8}.\end{aligned} \tag{F.25}$$

By using the covariant derivative they instead read

$$\begin{aligned}\nabla_\rho\left[e^\phi|\lambda|^2H^{\mu\nu\rho} - e^\phi\chi F^{\mu\nu\rho}\right] = \frac{1}{8}\nabla_\xi\left[\tilde{F}^{\mu\nu\rho\sigma\xi}\right]C_{\rho\sigma} + \frac{1}{12}\tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi}+ \\ + \frac{1}{2\cdot 5!\cdot 3!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}F_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8}.\end{aligned} \tag{F.26}$$

The B_2 equations of motion can however be simplified further by using the C_4 field equations on the $\nabla_\xi \tilde{F}^{\mu\nu\rho\sigma\xi}$-term and then rearranging some indices

$$
\begin{aligned}
\nabla_\rho \left[e^\phi |\lambda|^2 H^{\mu\nu\rho} - e^\phi \chi F^{\mu\nu\rho}\right] &= \frac{1}{8}\nabla_\xi \left[\tilde{F}^{\mu\nu\rho\sigma\xi}\right] C_{\rho\sigma} + \frac{1}{12}\tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi} + \\
&\quad + \frac{1}{2\cdot 5!\cdot 3!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}F_{\mu_4\dots\mu_8} = \\
&= \frac{1}{8\cdot(3!)^2}\epsilon^{\mu\nu\rho\sigma\mu_1\dots\mu_6}C_{\rho\sigma}H_{\mu_1\mu_2\mu_3}F_{\mu_4\mu_5\mu_6} + \\
&\quad + \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\rho\sigma\xi\mu_1\dots\mu_5}F_{\rho\sigma\xi}\tilde{F}_{\mu_1\dots\mu_5} + \\
&\quad + \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}F_{\mu_4\dots\mu_8} = \\
&= -\frac{1}{12\cdot 4!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}C_{\mu_4\mu_5}H_{\mu_6\mu_7\mu_8} + \\
&\quad + \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}\tilde{F}_{\mu_4\dots\mu_8} + \\
&\quad + \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}F_{\mu_4\dots\mu_8} = \\
&= \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}(-5C_{\mu_4\mu_5}H_{\mu_6\mu_7\mu_8} + \\
&\qquad + \tilde{F}_{\mu_4\dots\mu_8} + F_{\mu_4\dots\mu_8}).
\end{aligned}
\tag{F.27}
$$

Since $\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}(5B_{\mu_4\mu_5}F_{\mu_6\mu_7\mu_8}) = 0$ the right hand side can be written as

$$
\begin{aligned}
\textbf{r.h.s.} &= \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}(\tilde{F}_{\mu_4\dots\mu_8} + F_{\mu_4\dots\mu_8} + \\
&\qquad - 5C_{\mu_4\mu_5}H_{\mu_6\mu_7\mu_8} + 5B_{\mu_4\mu_5}F_{\mu_6\mu_7\mu_8}) = \\
&= \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}F_{\mu_1\mu_2\mu_3}(2\tilde{F}_{\mu_4\dots\mu_8}) = \frac{1}{6}(\frac{1}{5!}\epsilon^{\mu\nu\mu_1\dots\mu_8}\tilde{F}_{\mu_4\dots\mu_8})F_{\mu_1\mu_2\mu_3} = \\
&= \frac{1}{6}\tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi}.
\end{aligned}
\tag{F.28}
$$

The final expression for the B_2 equations of motion is

$$
\nabla_\rho \left[e^\phi |\lambda|^2 H^{\mu\nu\rho} - e^\phi \chi F^{\mu\nu\rho}\right] = \frac{1}{3!}\tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi}. \tag{F.29}
$$

F.5 Varying with respect to C_2

The C_2-dependent terms are the same as for B_2. Under variation with respect to C_2 they become

$$\begin{aligned}
\delta\Big(-\frac{1}{12}e\ H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho}\Big) &= -\frac{1}{12}e\ \delta\Big(e^{\phi}\Big[|\lambda|^2 H_{\mu\nu\rho}H^{\mu\nu\rho} - 2\chi F_{\mu\nu\rho}H^{\mu\nu\rho}+ \\
&\qquad + F_{\mu\nu\rho}F^{\mu\nu\rho}\Big]\Big) = \\
&= -\frac{1}{12}e\ e^{\phi}\left(-6\chi H^{\mu\nu\rho}\partial_\mu\delta C_{\nu\rho} + 6F^{\mu\nu\rho}\partial_\mu\delta C_{\nu\rho}\right) = \\
&= \frac{1}{2}\Big(-\partial_\mu\Big[e\ e^{\phi}\chi H^{\mu\nu\rho}\Big] + \partial_\mu\Big[e\ e^{\phi}F^{\mu\nu\rho}\Big]\Big)\delta C_{\nu\rho} = \\
&= \frac{1}{2}\Big(-\partial_\rho\Big[e\ e^{\phi}\chi H^{\mu\nu\rho}\Big] + \partial_\rho\Big[e\ e^{\phi}F^{\mu\nu\rho}\Big]\Big)\delta C_{\mu\nu},
\end{aligned} \tag{F.30}$$

and

$$\begin{aligned}
\delta(-\frac{1}{4}e\left|\tilde{F}_5\right|^2) &= -\frac{1}{4\cdot 5!}\delta(e\ \tilde{F}_{\mu\nu\rho\sigma\xi}\tilde{F}^{\mu\nu\rho\sigma\xi}) = \frac{2}{4\cdot 5!}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\delta\tilde{F}_{\mu\nu\rho\sigma\xi} = \\
&= -\frac{2}{4\cdot 5!}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}\ 5(3B_{\mu\nu}\partial_\rho\delta C_{\sigma\xi} - \delta C_{\mu\nu}H_{\rho\sigma\xi}) = \\
&= -\frac{1}{8\cdot 3!}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}(3B_{\rho\sigma}\partial_\xi\delta C_{\mu\nu} - \delta C_{\mu\nu}H_{\rho\sigma\xi}) = \\
&= -\frac{1}{8\cdot 3!}(-3\partial_\xi[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}]B_{\rho\sigma} - e\ \tilde{F}^{\mu\nu\rho\sigma\xi}H_{\xi\rho\sigma}+ \\
&\qquad - e\ \tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi})\delta C_{\mu\nu} = \\
&= (\frac{1}{16}\partial_\xi[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}]B_{\rho\sigma} + \frac{1}{24}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi})\delta C_{\mu\nu}
\end{aligned} \tag{F.31}$$

and finally

$$\begin{aligned}
\delta\Big(-\frac{1}{2}C_4\wedge H_3\wedge F_3\Big) &= -\frac{1}{2}C_4\wedge H_3\wedge\delta F_3 = -\frac{1}{2}C_4\wedge H_3\wedge d\delta C_2 = \\
&= -\frac{1}{2}dC_4\wedge H_3\wedge\delta C_2 = \frac{1}{2}\delta C_2\wedge H_3\wedge F_5,
\end{aligned} \tag{F.32}$$

where

$$\frac{1}{2}\delta C_2\wedge H_3\wedge F_5 = \frac{1}{4\cdot 5!\cdot 3!}d^{10}x\ \varepsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8}\delta C_{\mu\nu}. \tag{F.33}$$

The field equations for C_2 become

$$\begin{aligned}
0 = \frac{1}{2}\Big(-\partial_\rho\Big[e\ e^{\phi}\chi H^{\mu\nu\rho}\Big] + \partial_\rho\Big[e\ e^{\phi}F^{\mu\nu\rho}\Big]\Big) + \frac{1}{16}\partial_\xi[e\ \tilde{F}^{\mu\nu\rho\sigma\xi}]B_{\rho\sigma}+ \\
+ \frac{1}{24}e\ \tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi} + \frac{1}{4\cdot 5!\cdot 3!}\varepsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8},
\end{aligned} \tag{F.34}$$

or equivalently

$$\begin{aligned}
\nabla_\rho\Big[e^{\phi}\chi H^{\mu\nu\rho} - e^{\phi}F^{\mu\nu\rho}\Big] = \frac{1}{8}\nabla_\xi[\tilde{F}^{\mu\nu\rho\sigma\xi}]B_{\rho\sigma} + \frac{1}{12}\tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi}+ \\
+ \frac{1}{2\cdot 5!\cdot 3!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8}
\end{aligned}. \tag{F.35}$$

Analogously to the B_2 case, the C_4 equations of motion can again be used to simplify the right hand side

$$\begin{aligned}
\textbf{r.h.s.} &= \frac{1}{8}\nabla_\xi[\tilde{F}^{\mu\nu\rho\sigma\xi}]B_{\rho\sigma} + \frac{1}{12}\tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi} + \\
&\quad + \frac{1}{2\cdot 5!\cdot 3!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8} = \\
&= \frac{1}{8\cdot(3!)^2}\epsilon^{\mu\nu\mu_1\ldots\mu_8}B_{\mu_1\mu_2}H_{\mu_3\mu_4\mu_5}F_{\mu_6\mu_7\mu_8} + \\
&\quad + \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}\tilde{F}_{\mu_4\ldots\mu_8} + \\
&\quad + \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_8} = \\
&= \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}(5B_{\mu_4\mu_5}F_{\mu_6\mu_7\mu_8} + \tilde{F}_{\mu_4\ldots\mu_8} + F_{\mu_4\ldots\mu_8}) = \\
&= \frac{1}{12\cdot 5!}\epsilon^{\mu\nu\mu_1\ldots\mu_8}H_{\mu_1\mu_2\mu_3}(\tilde{F}_{\mu_4\ldots\mu_8} + F_{\mu_4\ldots\mu_8} + \\
&\qquad\qquad - 5C_{\mu_4\mu_5}H_{\mu_6\mu_7\mu_8} + 5B_{\mu_4\mu_5}F_{\mu_6\mu_7\mu_8}) = \\
&= \frac{1}{6}\tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi},
\end{aligned} \tag{F.36}$$

which results in the C_2 field equations taking the form

$$\nabla_\rho\left[e^\phi\chi H^{\mu\nu\rho} - e^\phi F^{\mu\nu\rho}\right] = \frac{1}{3!}\tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi}. \tag{F.37}$$

F.6 Varying with respect to $g^{\mu\nu}$

Finally, Einstein's equations for type IIB string theory are derived. Varying the action with respect to $g^{\mu\nu}$ yields that

$$\delta\left(e\left[R - \frac{1}{12}H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho} + \frac{1}{4}\operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\} - \frac{1}{4}\left|\tilde{F}_5\right|^2\right]\right) = 0. \tag{F.38}$$

The first term is

$$\delta(eR) = e(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}, \tag{F.39}$$

the second term is

$$\begin{aligned}
\delta\left(-\frac{1}{12}e\ H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho}\right) &= -\frac{1}{12}\Big(\delta e\ H^T_{\mu\nu\rho}\mathcal{M}H^{\mu\nu\rho} + \\
&\qquad + 3e\ H^T_{\mu\rho\sigma}\mathcal{M}H_\nu{}^{\rho\sigma}\delta g^{\mu\nu}\Big) = \\
&= -\frac{1}{12}e\Big(- \frac{1}{2}g_{\mu\nu}H^T_{\rho\sigma\xi}\mathcal{M}H^{\rho\sigma\xi} + \\
&\qquad + 3H^T_{\mu\rho\sigma}\mathcal{M}H_\nu{}^{\rho\sigma}\Big)\delta g^{\mu\nu},
\end{aligned} \tag{F.40}$$

the third term is

$$\begin{aligned}\delta(\frac{1}{4}e\ \operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\}) &= \frac{1}{4}\Big(\delta e\ \operatorname{tr}\{\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}\}+\\ &\qquad + e\ \operatorname{tr}\{\partial_\mu\mathcal{M}\partial_\nu\mathcal{M}^{-1}\}\delta g^{\mu\nu}\Big) =\\ &= \frac{1}{4}e\Big(-\frac{1}{2}g_{\mu\nu}\operatorname{tr}\{\partial_\rho\mathcal{M}\partial^\rho\mathcal{M}^{-1}\}+\\ &\qquad + \operatorname{tr}\{\partial_\mu\mathcal{M}\partial_\nu\mathcal{M}^{-1}\}\Big)\delta g^{\mu\nu},\end{aligned} \tag{F.41}$$

and the fourth term is

$$\begin{aligned}\delta(-\frac{1}{4}e\left|\tilde{F}_5\right|^2) &= -\frac{1}{4\cdot 5!}(\delta e\ \tilde{F}_{\mu\nu\rho\sigma\xi}\tilde{F}^{\mu\nu\rho\sigma\xi} + 5e\ \tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega}\delta g^{\mu\nu}) =\\ &= -\frac{1}{4\cdot 5!}e\ (-\frac{1}{2}g_{\mu\nu}\tilde{F}_{\mu_1\ldots\mu_5}\tilde{F}^{\mu_1\ldots\mu_5} + 5\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega})\delta g^{\mu\nu}.\end{aligned} \tag{F.42}$$

The field equations for $g^{\mu\nu}$ become

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \frac{1}{12}\left(-\frac{1}{2}g_{\mu\nu}H^T_{\rho\sigma\xi}\mathcal{M}H^{\rho\sigma\xi} + 3H^T_{\mu\rho\sigma}\mathcal{M}H_\nu{}^{\rho\sigma}\right)+\\ &\quad -\frac{1}{4}\left(-\frac{1}{2}g_{\mu\nu}\operatorname{tr}\{\partial_\rho\mathcal{M}\partial^\rho\mathcal{M}^{-1}\} + \operatorname{tr}\{\partial_\mu\mathcal{M}\partial_\nu\mathcal{M}^{-1}\}\right)+\\ &\quad +\frac{1}{4\cdot 5!}\left(-\frac{1}{2}g_{\mu\nu}\tilde{F}_{\mu_1\ldots\mu_5}\tilde{F}^{\mu_1\ldots\mu_5} + 5\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega}\right) =\\ &= \frac{1}{8}\left(-\frac{1}{3}H^T_{\rho\sigma\xi}\mathcal{M}H^{\rho\sigma\xi} + \operatorname{tr}\{\partial_\rho\mathcal{M}\partial^\rho\mathcal{M}^{-1}\} - \left|\tilde{F}_5\right|^2\right)g_{\mu\nu}+\\ &\quad +\frac{1}{4}\left(H^T_{\mu\rho\sigma}\mathcal{M}H_\nu{}^{\rho\sigma} - \operatorname{tr}\{\partial_\mu\mathcal{M}\partial_\nu\mathcal{M}^{-1}\} + \frac{1}{4!}\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega}\right),\end{aligned} \tag{F.43}$$

which can be written as

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \frac{1}{8}\left(\operatorname{tr}\{\partial_\rho\mathcal{M}\partial^\rho\mathcal{M}^{-1}\} - \frac{1}{3}H^T_{\rho\sigma\xi}\mathcal{M}H^{\rho\sigma\xi}\right)g_{\mu\nu}+\\ &\quad +\frac{1}{4}\left(H^T_{\mu\rho\sigma}\mathcal{M}H_\nu{}^{\rho\sigma} - \operatorname{tr}\{\partial_\mu\mathcal{M}\partial_\nu\mathcal{M}^{-1}\} + \frac{1}{4!}\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega}\right),\end{aligned} \tag{F.44}$$

using the self-duality condition $\tilde{F}_5 = \star\tilde{F}_5$.

G

The B_2, C_2 and C_4 field strengths

This appendix focuses on calculating the field strengths of B_2, C_2 and C_4, which are needed when evaluating the equations of motion in appedix H and J. First off, the VEVs of the field content are restated. In the background, the dilaton and axion are

$$e^{\phi} = \sqrt{2}e^{-2\eta}\frac{2-\cos(2\alpha)}{\sqrt{7-\cos(4\alpha)}} = e^{-2\eta}\sqrt{\frac{2-\cos(2\alpha)}{2+\cos(2\alpha)}}, \qquad \chi = 0. \tag{G.1}$$

The VEVs of the 2-form fields are given by

$$B_2 = -2\sqrt{2}e^{-\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}\mathrm{vol}_{\Omega_1}, \qquad C_2 = -2\sqrt{2}e^{\eta}\frac{\sin^3\alpha}{2-\cos(2\alpha)}\mathrm{vol}_{\Omega_2}. \tag{G.2}$$

The deformation parameters can be set as $\chi_i = 0$, see section 3.2.4, so that the volume forms of the 2-spheres read

$$\mathrm{vol}_{\Omega_i} = \sin\theta_i \; d\theta_i \wedge d\varphi_i. \tag{G.3}$$

The 4-form VEV is

$$C_4 = \frac{3}{2}\omega_3 \wedge \left(d\eta + \frac{2}{3}\sin(2\alpha)d\alpha\right) - \frac{1}{2}f(\alpha)d\alpha \wedge (A_1 \wedge \mathrm{vol}_{\Omega_2} + \mathrm{vol}_{\Omega_1} \wedge A_2). \tag{G.4}$$

G.1 2-form field strengths H_3 and F_3

To tackle the IIB equations of motion, the field strengths of the 2-forms are needed. Starting with $H_3 = dB_2$

$$\begin{aligned} H_3 = \partial_\eta &\left(-2\sqrt{2}e^{-\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}\right) d\eta \wedge \mathrm{vol}_{\Omega_1} + \\ &+ \partial_\alpha \left(-2\sqrt{2}e^{-\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}\right) d\alpha \wedge \mathrm{vol}_{\Omega_1}. \end{aligned} \tag{G.5}$$

The derivative with respect to α is evaluated with some useful double angle identities

$$\begin{aligned}
\partial_\alpha\left(-\frac{2\sqrt{2}e^{-\eta}\cos^3\alpha}{2+\cos(2\alpha)}\right) &= -2\sqrt{2}e^{-\eta}\Bigg(-\frac{3\cos^2\alpha\ \sin\alpha}{2+\cos(2\alpha)}+\\
&\qquad -\frac{\cos^3\alpha\ (-2\sin(2\alpha))}{(2+\cos(2\alpha))^2}\Bigg)=\\
&= -\sqrt{2}e^{-\eta}\frac{\sin(2\alpha)\cos\alpha}{2+\cos(2\alpha)}\left(-3+\frac{4\cos^2\alpha}{2+\cos(2\alpha)}\right)=\\
&= -\sqrt{2}e^{-\eta}\frac{\sin(2\alpha)\ \cos\alpha}{(2+\cos(2\alpha))^2}\times\\
&\quad\times\left(-3(2+\cos(2\alpha))+4\cos^2\alpha\right)=\\
&= \sqrt{2}e^{-\eta}\sin(2\alpha)\ \cos\alpha\ \frac{4+\cos(2\alpha)}{(2+\cos(2\alpha))^2}.
\end{aligned} \tag{G.6}$$

The field strength of the 2-form is then

$$\begin{aligned}
H_3 &= 2\sqrt{2}e^{-\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}d\eta\wedge\mathrm{vol}_{\Omega_1}+\\
&\quad+\sqrt{2}e^{-\eta}\sin(2\alpha)\ \cos\alpha\ \frac{4+\cos(2\alpha)}{(2+\cos(2\alpha))^2}d\alpha\wedge\mathrm{vol}_{\Omega_1}.
\end{aligned} \tag{G.7}$$

The calculation of the other field strength F_3 is analogous and yields that

$$\begin{aligned}
F_3 &= -2\sqrt{2}e^{\eta}\frac{\sin^3\alpha}{2-\cos(2\alpha)}d\eta\wedge\mathrm{vol}_{\Omega_2}+\\
&\quad-\sqrt{2}e^{\eta}\sin(2\alpha)\ \sin\alpha\ \frac{4-\cos(2\alpha)}{(2-\cos(2\alpha))^2}d\alpha\wedge\mathrm{vol}_{\Omega_2}.
\end{aligned} \tag{G.8}$$

Their components then read

$$\begin{cases}
H_{\eta\theta_1\varphi_1} &= 2\sqrt{2}e^{-\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}\sin\theta_1,\\
H_{\alpha\theta_1\varphi_1} &= \sqrt{2}e^{-\eta}\sin(2\alpha)\ \cos\alpha\ \frac{4+\cos(2\alpha)}{(2+\cos(2\alpha))^2}\sin\theta_1,
\end{cases} \tag{G.9}$$

and

$$\begin{cases}
F_{\eta\theta_2\varphi_2} &= -2\sqrt{2}e^{\eta}\frac{\sin^3\alpha}{2-\cos(2\alpha)}\sin\theta_2,\\
F_{\alpha\theta_2\varphi_2} &= -\sqrt{2}e^{\eta}\sin(2\alpha)\ \sin\alpha\ \frac{4-\cos(2\alpha)}{(2-\cos(2\alpha))^2}\sin\theta_2.
\end{cases} \tag{G.10}$$

The square of H_3 is calculated as

$$\begin{aligned}
H^2 &= H_{\mu\nu\rho}H^{\mu\nu\rho} = 3!(H_{\eta\theta_1\varphi_1}H^{\eta\theta_1\varphi_1} + H_{\alpha\theta_1\varphi_1}H^{\alpha\theta_1\varphi_1} + H_{\eta\alpha\theta_1}H^{\eta\alpha\theta_1}) = \\
&= 3!\left(H_{\eta\theta_1\varphi_1}H_{\eta\theta_1\varphi_1}g^{\eta\eta}g^{\theta_1\theta_1}g^{\varphi_1'\varphi_1'} + H_{\alpha\theta_1\varphi_1}H_{\alpha\theta_1\varphi_1}g^{\alpha\alpha}g^{\theta_1\theta_1}g^{\varphi_1'\varphi_1'}\right) = \\
&= 3!\frac{\Delta^3}{x^4\sin^2\theta_1}\Bigg(8e^{-2\eta}\frac{\cos^6\alpha}{(2+\cos(2\alpha))^2}\sin^2\theta_1 + \\
&\qquad + 2e^{-2\eta}\sin^2(2\alpha)\ \cos^2\alpha\ \frac{(4+\cos(2\alpha))^2}{(2+\cos(2\alpha))^4}\sin^2\theta_1\Bigg) = \\
&= 8\cdot 3!\frac{\Delta^3}{x^4}e^{-2\eta}\frac{\cos^4\alpha}{(2+\cos(2\alpha))^2}\left(\cos^2\alpha + \sin^2(\alpha)\frac{(4+\cos(2\alpha))^2}{(2+\cos(2\alpha))^2}\right) = \\
&= 8\cdot 3!\Delta^3 e^{-2\eta}\left(\cos^2\alpha + \sin^2(\alpha)\frac{(4+\cos(2\alpha))^2}{(2+\cos(2\alpha))^2}\right) = \qquad\qquad \text{(G.11)}\\
&= \frac{8\cdot 3!\Delta^3 e^{-2\eta}}{(2+\cos(2\alpha))^2}\Big(\cos^2\alpha\ \Big[4+\cos^2(2\alpha)+4\cos(2\alpha)\Big] + \\
&\qquad + \sin^2\alpha\ \Big[16+\cos^2(2\alpha)+8\cos(2\alpha)\Big]\Big) = \\
&= \frac{8\cdot 3!\Delta^3 e^{-2\eta}}{(2+\cos(2\alpha))^2}\Big(4+\cos^2(2\alpha)+4\cos(2\alpha) + \\
&\qquad + 12\sin^2\alpha + 4\sin^2\alpha\ \cos(2\alpha)\Big) = \\
&= 24\Delta^3 e^{-2\eta}\frac{19-\cos(4\alpha)}{(2+\cos(2\alpha))^2}.
\end{aligned}$$

The calculation for F^2 is similar and the results of the squared 2-form field strengths are

$$H^2 = 24\Delta^3 e^{-2\eta}\frac{19-\cos(4\alpha)}{(2+\cos(2\alpha))^2}, \qquad F^2 = 24\Delta^3 e^{2\eta}\frac{19-\cos(4\alpha)}{(2-\cos(2\alpha))^2}. \tag{G.12}$$

G.2 Self-dual field strength $\tilde{F}_5$

The field strength of the 4-form C_4 is given by

$$F_5 = dC_4 = \frac{3}{2}d\omega_3\wedge d\eta + \sin(2\alpha)d\omega_3\wedge d\alpha + f(\alpha)d\alpha\wedge \mathrm{vol}_{\Omega_1}\wedge \mathrm{vol}_{\Omega_2}. \tag{G.13}$$

The AdS_4 volume form is

$$d\omega_3 = \mathrm{vol}_{\mathrm{AdS}_4} = \tilde{e}\ dt\wedge dx\wedge dy\wedge dz, \qquad \tilde{e} = \sqrt{-\det(\tilde{g}_{\mu\nu})}, \tag{G.14}$$

where $\tilde{g}_{\mu\nu}$ is the metric on AdS_4. The F_5 components are

$$\begin{cases} F_{txyz\eta} = \frac{3}{2}\ \tilde{e}, \\ F_{txyz\alpha} = \sin(2\alpha)\ \tilde{e}, \\ F_{\alpha\theta_1\varphi_1\theta_2\varphi_2} = f(\alpha)\sin\theta_1\ \sin\theta_2. \end{cases} \tag{G.15}$$

The relevant field strength for the equations of motion however, is the self-dual

$$\tilde{F}_{\mu\nu\rho\sigma\xi} = F_{\mu\nu\rho\sigma\xi} - 5C_{[\mu\nu}H_{\rho\sigma\xi]} + 5B_{[\mu\nu}F_{\rho\sigma\xi]}. \tag{G.16}$$

The components with AdS$_4$-indices are the same as the ones for F_5 since there is no contribution from the 2-forms. The other $\tilde{F}_5$-components can be calculated using (G.16), but are more easily obtained from $F_{txyz\eta}$ and $F_{txyz\alpha}$ via the self-duality condition. There are two other components, the first being

$$\begin{aligned}
\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2} &= \frac{1}{5!}\epsilon_{\alpha\theta_1\varphi_1\theta_2\varphi_2\mu_1\ldots\mu_5}g^{\mu_1\nu_1}\ldots g^{\mu_5\nu_5}\tilde{F}_{\nu_1\ldots\nu_5} = \\
&= \frac{5!}{5!}e\ \varepsilon_{\alpha\theta_1\varphi_1\theta_2\varphi_2 txyz\eta}g^{tt}g^{xx}g^{yy}g^{zz}g^{\eta\eta}\tilde{F}_{txyz\eta} = \\
&= -e\ \varepsilon_{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2}g^{tt}g^{xx}g^{yy}g^{zz}g^{\eta\eta}\tilde{F}_{txyz\eta} = \\
&= \left(\frac{1}{16}\tilde{e}\ \Delta^{-1}\sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2\right)\left(16\Delta^5\frac{1}{\tilde{g}}\right)\left(\frac{3}{2}\tilde{e}\right) = \\
&= \frac{3}{2}\Delta^4\sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2\frac{\tilde{e}^2}{g(\mathrm{AdS}_4)} = \{\tilde{e}^2 = -\tilde{g}\} \\
&= -\frac{3}{2}\Delta^4\sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2,
\end{aligned} \tag{G.17}$$

and the second being

$$\begin{aligned}
\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2} &= \frac{1}{5!}\epsilon_{\eta\theta_1\varphi_1\theta_2\varphi_2\mu_1\ldots\mu_5}g^{\mu_1\nu_1}\ldots g^{\mu_5\nu_5}\tilde{F}_{\nu_1\ldots\nu_5} = \\
&= \frac{5!}{5!}e\ \varepsilon_{\eta\theta_1\varphi_1\theta_2\varphi_2 txyz\alpha}g^{tt}g^{xx}g^{yy}g^{zz}g^{\alpha\alpha}\tilde{F}_{txyz\alpha} = \\
&= e\ \varepsilon_{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2}g^{tt}g^{xx}g^{yy}g^{zz}g^{\alpha\alpha}\tilde{F}_{txyz\alpha} = \\
&= -\left(\frac{1}{16}\tilde{e}\Delta^{-1}\sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2\right)\left(16\Delta^5\frac{1}{\tilde{g}}\right)(\sin(2\alpha)\tilde{e}) = \\
&= -\Delta^4\sin^3(2\alpha)\ \sin\theta_1\ \sin\theta_2\frac{\tilde{e}^2}{\tilde{g}} = \{\tilde{e}^2 = -\tilde{g}\} \\
&= \Delta^4\sin^3(2\alpha)\ \sin\theta_1\ \sin\theta_2.
\end{aligned} \tag{G.18}$$

The components of the self-dual $\tilde{F}_5$ are

$$\begin{cases}
\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2} = -\frac{3}{2}\Delta^4\sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2, \\
\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2} = \Delta^4\sin^3(2\alpha)\ \sin\theta_1\ \sin\theta_2, \\
\tilde{F}_{txyz\eta} = \frac{3}{2}\ \tilde{e}, \\
\tilde{F}_{txyz\alpha} = \sin(2\alpha)\ \tilde{e}.
\end{cases} \tag{G.19}$$

H

Evaluation of the scalar, 2-form and 4-form field equations

In this appendix all of the type IIB equations of motion, except Einstein's equations which are treated in appendix J, are evaluated for the AdS vacuum given in 3.2. A brief description of the procedure is given in section 3.2.5. The deformation parameters are set to $\chi_i = 0$ as the field equations are local, see section 3.2.4.

H.1 The axion equation of motion

The background value of the axion being $\chi = 0$ drastically simplifies its equation of motion to

$$F_{\mu\nu\rho} H^{\mu\nu\rho} = 0. \tag{H.1}$$

This is satisfied since there are no three common indices for which both $H_{\mu\nu\rho}$ and $F_{\mu\nu\rho}$ are non-vanishing.

H.2 The dilaton equation of motion

Since $\chi = 0$ the dilaton equation of motion reads

$$\nabla^\mu (e^{-\phi} \partial_\mu e^\phi) = \frac{1}{12}(e^\phi F^2 - e^{-\phi} H^2), \tag{H.2}$$

where

$$e^\phi = \sqrt{2} e^{-2\eta} \frac{2 - \cos(2\alpha)}{\sqrt{7 - \cos(4\alpha)}}. \tag{H.3}$$

Starting with the right hand side of (H.2), it is simplified as

$$
\begin{aligned}
\textbf{r.h.s.} &= \frac{1}{12}(e^{\phi}F^2 - e^{-\phi}H^2) = \\
&= 2\Delta^3(19-\cos(4\alpha))\left[\frac{e^{2\phi}e^{\phi}}{(2-\cos(2\alpha))^2} - \frac{e^{-2\phi}e^{-\phi}}{(2+\cos(2\alpha))^2}\right] = \\
&= 2\Delta^3(19-\cos(4\alpha))\left[\frac{\sqrt{2}(2-\cos(2\alpha))}{\sqrt{7-\cos(4\alpha)}(2-\cos(2\alpha))^2} + \right. \\
&\qquad\qquad \left. - \frac{1}{\sqrt{2}}\frac{\sqrt{7-\cos(4\alpha)}}{(2-\cos(2\alpha))(2+\cos(2\alpha))^2}\right] = \\
&= \sqrt{2}\Delta^3\frac{19-\cos(4\alpha)}{2-\cos(2\alpha)}\left[\frac{2}{\sqrt{7-\cos(4\alpha)}} - \frac{\sqrt{7-\cos(4\alpha)}}{(2+\cos(2\alpha))^2}\right] = \\
&= \sqrt{2}\Delta^3\frac{19-\cos(4\alpha)}{2-\cos(2\alpha)}\;\frac{2(2+\cos(2\alpha))^2-(7-\cos(4\alpha))}{\sqrt{7-\cos(4\alpha)}(2+\cos(2\alpha))^2} = \\
&= \sqrt{2}\Delta^3\frac{19-\cos(4\alpha)}{2-\cos(2\alpha)}\;\frac{8+2\cos^2(2\alpha)+8\cos(2\alpha)-7+\cos(4\alpha)}{\sqrt{7-\cos(4\alpha)}(2+\cos(2\alpha))^2} = \\
&\overset{(1)}{=} \sqrt{2}\Delta^3\frac{19-\cos(4\alpha)}{2-\cos(2\alpha)}\;\frac{2+2\cos(4\alpha)+8\cos(2\alpha)}{\sqrt{7-\cos(4\alpha)}(2+\cos(2\alpha))^2} = \\
&\overset{(2)}{=} 2\sqrt{2}\Delta^3\frac{19-\cos(4\alpha)}{2-\cos(2\alpha)}\;\frac{2\cos^2(2\alpha)+4\cos(2\alpha)}{\sqrt{7-\cos(4\alpha)}(2+\cos(2\alpha))^2} = \\
&= 4\sqrt{2}\Delta^3\cos(2\alpha)\frac{19-\cos(4\alpha)}{2-\cos(2\alpha)}\;\frac{2+\cos(2\alpha)}{\sqrt{7-\cos(4\alpha)}(2+\cos(2\alpha))^2} = \\
&= 4\sqrt{2}\Delta^3\frac{19-\cos(4\alpha)}{\sqrt{7-\cos(4\alpha)}}\;\frac{\cos(2\alpha)}{(2-\cos(2\alpha))(2+\cos(2\alpha))} = \\
&= 4\sqrt{2}\Delta^3\cos(2\alpha)\frac{19-\cos(4\alpha)}{\sqrt{7-\cos(4\alpha)}}\;\frac{1}{4-\cos^2(2\alpha)} = \\
&= 4\sqrt{2}\Delta^7\cos(2\alpha)\frac{19-\cos(4\alpha)}{\sqrt{7-\cos(4\alpha)}},
\end{aligned}
\tag{H.4}
$$

where the identity $2\cos^2(2\alpha) = \cos(4\alpha)+1$ is used at (1) and $\cos(4\alpha) = 2\cos^2(2\alpha)-1$ is used at (2). As for the left hand side, since e^{ϕ} only depends on η and α it becomes

$$
\begin{aligned}
\textbf{l.h.s.} = \nabla^{\mu}(e^{-\phi}\partial_{\mu}e^{\phi}) &= \frac{1}{e}\partial_{\mu}(e\ e^{-\phi}g^{\mu\nu}\partial_{\nu}e^{\phi}) = \\
&= \frac{1}{e}\left[\partial_{\eta}(e\ e^{-\phi}g^{\eta\eta}\partial_{\eta}e^{\phi}) + \partial_{\alpha}(e\ e^{-\phi}g^{\alpha\alpha}\partial_{\alpha}e^{\phi})\right].
\end{aligned}
\tag{H.5}
$$

Since $e^{-\phi}\partial_{\eta}e^{\phi} = -2$ and the metric is independent of η, the first term vanishes. The left hand side is thus

$$
\textbf{l.h.s.} = \nabla^{\mu}(e^{-\phi}\partial_{\mu}e^{\phi}) = \frac{1}{e}\partial_{\alpha}(e\ \Delta\ e^{-\phi}\partial_{\alpha}e^{\phi}) = \frac{\partial_{\alpha}(e\ \Delta)}{e}e^{-\phi}\partial_{\alpha}e^{\phi} + \Delta\partial_{\alpha}(e^{-\phi}\partial_{\alpha}e^{\phi}).
\tag{H.6}
$$

Breaking this calculation down, the first factor is

$$\begin{aligned}\frac{\partial_\alpha(e\ \Delta)}{e} &= \frac{\partial_\alpha(\cos^2\alpha\ \sin^2\alpha)}{\Delta^{-1}\cos^2\alpha\ \sin^2\alpha} = 2\Delta\frac{\cos^3\alpha\ \sin\alpha - \cos\alpha\ \sin^3\alpha}{\cos^2\alpha\ \sin^2\alpha} = \\ &= 2\Delta\frac{\cos^2\alpha - \sin^2\alpha}{\cos\alpha\ \sin\alpha} = 4\Delta\frac{\cos(2\alpha)}{\sin(2\alpha)} = 4\Delta\cot(2\alpha).\end{aligned} \tag{H.7}$$

Next, the derivative of the dilaton is

$$\begin{aligned}\partial_\alpha e^\phi &= \sqrt{2}e^{-2\eta}\left[\frac{2\sin(2\alpha)}{\sqrt{7-\cos(4\alpha)}} - \frac{1}{2}\frac{2-\cos(2\alpha)}{(7-\cos(4\alpha))^{3/2}}(4\sin(4\alpha))\right] = \\ &= 2\sqrt{2}\frac{e^{-2\eta}\sin(2\alpha)}{\sqrt{7-\cos(4\alpha)}}\left[1 - \frac{2\cos(2\alpha)(2-\cos(2\alpha))}{7-\cos(4\alpha)}\right] = \\ &= 2\sqrt{2}\frac{e^{-2\eta}\sin(2\alpha)}{(7-\cos(4\alpha))^{3/2}}\left[7-\cos(4\alpha)-4\cos(2\alpha)+2\cos^2(2\alpha)\right] = \\ &= 2\sqrt{2}\frac{e^{-2\eta}\sin(2\alpha)}{(7-\cos(4\alpha))^{3/2}}\left[8-4\cos(2\alpha)\right] = \\ &= 8\sqrt{2}e^{-2\eta}\sin(2\alpha)\frac{2-\cos(2\alpha)}{(7-\cos(4\alpha))^{3/2}},\end{aligned} \tag{H.8}$$

which implies that

$$e^{-\phi}\partial_\alpha e^\phi = \frac{8\sin(2\alpha)}{7-\cos(4\alpha)}. \tag{H.9}$$

Finally the derivative of $e^{-\phi}\partial_\alpha e^\phi$ is

$$\begin{aligned}\partial_\alpha(e^{-\phi}\partial_\alpha e^\phi) &= \frac{16\cos(2\alpha)}{7-\cos(4\alpha)} - \frac{8\sin(2\alpha)}{(7-\cos(4\alpha))^2}(4\sin(4\alpha)) = \\ &= 16\frac{\cos(2\alpha)}{7-\cos(4\alpha)}\left[1-\frac{4\sin^2(2\alpha)}{7-\cos(4\alpha)}\right] = \\ &= 16\frac{\cos(2\alpha)}{(7-\cos(4\alpha))^2}\left[7-\cos(4\alpha)-4\sin^2(2\alpha)\right] = \\ &= 16\cos(2\alpha)\frac{5+\cos(4\alpha)}{(7-\cos(4\alpha))^2}.\end{aligned} \tag{H.10}$$

The left hand side becomes to

$$\begin{aligned}\textbf{l.h.s.} &= \Delta\left[4\cot(2\alpha)\frac{8\sin(2\alpha)}{7-\cos(4\alpha)} + 16\cos(2\alpha)\frac{5+\cos(4\alpha)}{(7-\cos(4\alpha))^2}\right] = \\ &= 16\Delta\frac{\cos(2\alpha)}{(7-\cos(4\alpha))^2}\left[2(7-\cos(4\alpha))+(5+\cos(4\alpha))\right] = \\ &= 16\Delta\cos(2\alpha)\frac{19-\cos(4\alpha)}{(7-\cos(4\alpha))^2},\end{aligned} \tag{H.11}$$

leaving the dilaton equation of motion as

$$\textbf{l.h.s.} = 16\Delta\cos(2\alpha)\frac{19-\cos(4\alpha)}{(7-\cos(4\alpha))^2} \stackrel{?}{=} 4\sqrt{2}\Delta^7\cos(2\alpha)\frac{19-\cos(4\alpha)}{\sqrt{7-\cos(4\alpha)}} = \textbf{r.h.s.}, \tag{H.12}$$

which reduces to

$$\frac{4}{(7-\cos(4\alpha))^{3/2}} \stackrel{?}{=} \sqrt{2}\Delta^6. \tag{H.13}$$

This is true since

$$\Delta^{-4} = 4-\cos^2(2\alpha) = \frac{1}{2}(7-\cos(4\alpha)), \quad \Longrightarrow \quad \Delta^6 = \frac{2\sqrt{2}}{(7-\cos(4\alpha))^{3/2}}, \tag{H.14}$$

which shows that the dilaton equation of motion is fulfilled.

H.3 The C_4 equations of motion

The equations of motion for C_4 are

$$\nabla_\xi \tilde{F}^{\mu\nu\rho\sigma\xi} = \frac{1}{(3!)^2}\epsilon^{\mu\nu\rho\sigma\mu_1\ldots\mu_6} H_{\mu_1\mu_2\mu_3} F_{\mu_4\mu_5\mu_6}. \tag{H.15}$$

This equation should be satisfied for all combinations of its four free indices $\mu\nu\rho\sigma$. The order of the indices is not important however, since all indices are completely antisymmetric. Because of H_3 and F_3, the right hand side is only non-zero if $\mu\nu\rho\sigma = txyz$. This case will be dealt with first. All other cases are trivial in the sense that they amount to $0 = 0$. They are thus validated by showing that the left hand side vanishes unless $\mu\nu\rho\sigma = txyz$.

H.3.1 The non-trivial case $\mu\nu\rho\sigma = txyz$

The non-vanishing right hand side is

$$\begin{aligned}
\textbf{r.h.s.} &= \frac{1}{(3!)^2}\epsilon^{txyz\mu_1\ldots\mu_6} H_{\mu_1\mu_2\mu_3} F_{\mu_4\mu_5\mu_6} = \\
&= \epsilon^{txyz\alpha\theta_1\varphi_1\eta\theta_2\varphi_2} H_{\alpha\theta_1\varphi_1} F_{\eta\theta_2\varphi_2} + \epsilon^{txyz\eta\theta_1\varphi_1\alpha\theta_2\varphi_2} H_{\eta\theta_1\varphi_1} F_{\alpha\theta_2\varphi_2} = \\
&= \epsilon^{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2}\left(-H_{\alpha\theta_1\varphi_1} F_{\eta\theta_2\varphi_2} + H_{\eta\theta_1\varphi_1} F_{\alpha\theta_2\varphi_2}\right) = \\
&= \frac{2}{e}\sin^2(2\alpha)\Bigg(\frac{\sin^2\alpha\ (4+\cos(2\alpha))}{(2-\cos(2\alpha))(2+\cos(2\alpha))^2} + \\
&\qquad\qquad - \frac{\cos^2\alpha\ (4-\cos(2\alpha))}{(2+\cos(2\alpha))(2-\cos(2\alpha))^2}\Bigg)\sin\theta_1\ \sin\theta_2 = \\
&= \frac{2}{e}\Delta^8\sin^2(2\alpha)\Big[\sin^2\alpha\ (4+\cos(2\alpha))(2-\cos(2\alpha)) + \\
&\qquad\qquad - \cos^2\alpha\ (4-\cos(2\alpha))(2+\cos(2\alpha))\Big]\sin\theta_1\ \sin\theta_2 = \\
&= \frac{2}{e}\Delta^8\sin^2(2\alpha)\Big[-2\cos(2\alpha) - (8-\cos^2(2\alpha))\cos(2\alpha)\Big]\sin\theta_1\ \sin\theta_2 = \\
&= -\frac{2}{e}\Delta^8\sin^2(2\alpha)\cos(2\alpha)\Big[10-\cos^2(2\alpha)\Big]\sin\theta_1\ \sin\theta_2 = \\
&= -\frac{1}{e}\Delta^8\sin^2(2\alpha)\cos(2\alpha)\left[19-\cos(4\alpha)\right]\sin\theta_1\ \sin\theta_2.
\end{aligned} \tag{H.16}$$

The left hand side for $\mu\nu\rho\sigma = txyz$ becomes

$$\textbf{l.h.s.} = \nabla_\xi \tilde{F}^{txyz\xi} = \nabla_\eta \tilde{F}^{txyz\eta} + \nabla_\alpha \tilde{F}^{txyz\alpha} = \nabla_\alpha \tilde{F}^{txyz\alpha}, \tag{H.17}$$

since $\tilde{F}^{txyz\eta}$ and the metric are independent of η. Evaluating further

$$\begin{aligned}
\textbf{l.h.s.} = \nabla_\alpha \tilde{F}^{txyz\alpha} &= \frac{1}{e}\partial_\alpha \left(e\ \tilde{F}_{txyz\alpha} g^{tt} g^{xx} g^{yy} g^{zz} g^{\alpha\alpha}\right) = \\
&= \frac{1}{e}\partial_\alpha \left([\frac{1}{16}\tilde{e}\ \Delta^{-1}\sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2][\tilde{e}\ \sin(2\alpha)][\Delta^5 16\frac{1}{\tilde{g}}]\right) = \\
&= \frac{1}{e}\partial_\alpha \left(-\Delta^4 \sin^3(2\alpha)\right)\sin\theta_1\ \sin\theta_2 = \\
&= \frac{1}{e}(-4\Delta^3\partial_\alpha\Delta\ \sin^3(2\alpha) - 6\Delta^4\sin^2(2\alpha)\cos(2\alpha))\sin\theta_1\ \sin\theta_2 = \\
&= \frac{2}{e}\Delta^3\sin^2(2\alpha)(-2\partial_\alpha\Delta\ \sin(2\alpha) - 3\Delta\cos(2\alpha))\sin\theta_1\ \sin\theta_2.
\end{aligned} \tag{H.18}$$

Using the derivative of the warping factor

$$\begin{aligned}
\partial_\alpha\Delta = \partial_\alpha\left((4-\cos^2(2\alpha))^{-1/4}\right) &= \\
&= -\frac{1}{4}(4-\cos^2(2\alpha))^{-5/4}(4\cos(2\alpha)\sin(2\alpha)) = \\
&= -\Delta^5\sin(2\alpha)\cos(2\alpha),
\end{aligned} \tag{H.19}$$

the left hand side becomes

$$\begin{aligned}
\textbf{l.h.s.} &= \frac{2}{e}\Delta^3\sin^2(2\alpha)(2\Delta^5\sin^2(2\alpha)\cos(2\alpha) - 3\Delta\cos(2\alpha))\sin\theta_1\ \sin\theta_2 = \\
&= \frac{2}{e}\Delta^8\sin^2(2\alpha)\cos(2\alpha)(2\sin^2(2\alpha) - 3(4-\cos^2(2\alpha)))\sin\theta_1\ \sin\theta_2 = \\
&= -\frac{1}{e}\Delta^8\sin^2(2\alpha)\cos(2\alpha)[19-\cos(4\alpha))]\sin\theta_1\ \sin\theta_2,
\end{aligned} \tag{H.20}$$

which makes it clear that

$$\textbf{l.h.s.} = -\frac{1}{e}\Delta^8\sin^2(2\alpha)\cos(2\alpha)[19-\cos(4\alpha))]\sin\theta_1\ \sin\theta_2 = \textbf{r.h.s.}. \tag{H.21}$$

H.3.2 The trivial case $\mu\nu\rho\sigma \neq txyz$

If the free indices $\mu\nu\rho\sigma \neq txyz$, the left hand side should vanish. There are four non-vanishing $\tilde{F}_5$-components. The two that appear when $\mu\nu\rho\sigma = txyz$ are $\tilde{F}_{txyz\eta}$ and $\tilde{F}_{txyz\alpha}$, where the derivative contracted with the indices η and α respectively. If the derivative contracts with t however, the left hand side becomes

$$\textbf{l.h.s.} = \nabla_t \tilde{F}^{txyz\mu} = 0, \qquad \mu = \eta,\, \alpha, \tag{H.22}$$

since the metric and all $\tilde{F}_5$-components are independent of t. If t is not among the free indices the left hand side vanishes. The same is true for x, y and z

$$\textbf{l.h.s.} = \nabla_x \tilde{F}^{txyz\mu} = \frac{1}{e}\partial_x(e\ \tilde{F}_{txyz\mu} g^{tt} g^{xx} g^{yy} g^{zz} g^{\mu\mu}) \sim \partial_x(\tilde{e}\ \tilde{e}\ \tilde{e}^{-2}) = 0, \tag{H.23}$$

since $e \sim \tilde{F}_{txyz\mu} \sim \tilde{e}$ (for $\mu = \eta,\, \alpha$) and $g^{tt}g^{xx}g^{yy}g^{zz} \sim \tilde{g}^{-1} \sim \tilde{e}^{-2}$ because all x, y, z-dependence is encoded in $\tilde{e}$. The two remaining $\tilde{F}_5$-components are $\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}$ and $\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}$. Similar to the case with t, there is no dependence on η, φ_1 or φ_2, implying that if they are not among the free indices, the left hand side is zero. To show that this is also the case for θ_1 and θ_2, the zehnbein determinant, the $\tilde{F}_5$-components and the relevant inverse metric components are considered

$$e \sim \tilde{F}_{\mu\theta_1\varphi_1\theta_2\varphi_2} \sim \sin\theta_1\ \sin\theta_2, \quad g^{\varphi_1\varphi_1}g^{\varphi_2\varphi_2} \sim \frac{1}{\sin^2\theta_1\ \sin^2\theta_2}, \quad \mu = \eta,\, \alpha. \tag{H.24}$$

This results a vanishing the left hand side

$$\mathbf{l.h.s.} = \nabla_{\theta_i}\tilde{F}^{\mu\theta_1\varphi_1\theta_2\varphi_2} \sim \partial_{\theta_i}(e\ \tilde{F}_{\mu\theta_1\varphi_1\theta_2\varphi_2}g^{\varphi_1\varphi_1}g^{\varphi_2\varphi_2}) = 0, \qquad \mu = \eta,\, \alpha, \tag{H.25}$$

leaving $\mathbf{l.h.s.} = \nabla_\alpha\tilde{F}^{\alpha\theta_1\varphi_1\theta_2\varphi_2}$ as the only case left to check. The α-dependence of the relevant quantities are given by

$$\begin{aligned} &e \sim \Delta^{-1}\sin^2(2\alpha), \qquad \tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2} \sim \Delta^4\sin^2(2\alpha), \\ &g^{\alpha\alpha}g^{\theta_1\theta_1}g^{\varphi_1\varphi_1}g^{\theta_2\theta_2}g^{\varphi_2\varphi_2} \sim \Delta^5\frac{1}{x_1^4x_2^4} \sim \Delta^5\left(\Delta^{-8}\frac{1}{\sin^4(2\alpha)}\right) \sim \Delta^{-3}\frac{1}{\sin^4(2\alpha)}. \end{aligned} \tag{H.26}$$

Again the α-dependence cancels out and the left hand side vanishes

$$\mathbf{l.h.s.} = \nabla_\alpha\tilde{F}^{\alpha\theta_1\varphi_1\theta_2\varphi_2} \sim \partial_\alpha(e\ \tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}g^{\alpha\alpha}g^{\theta_1\theta_1}g^{\varphi_1\varphi_1}g^{\theta_2\theta_2}g^{\varphi_2\varphi_2}) = 0, \tag{H.27}$$

which finally shows that the C_4 equations of motions are satisfied by the AdS vacuum.

H.4 The B_2 and C_2 equations of motion

The B_2 and C_2 equations of motion with $\chi = 0$ are given by

$$\nabla_\rho(e^{-\phi}H^{\mu\nu\rho}) = \frac{1}{3!}\tilde{F}^{\mu\nu\rho\sigma\xi}F_{\rho\sigma\xi}, \qquad \nabla_\rho(e^{\phi}F^{\mu\nu\rho}) = -\frac{1}{3!}\tilde{F}^{\mu\nu\rho\sigma\xi}H_{\rho\sigma\xi}. \tag{H.28}$$

However, these equations are actually S-dual to eachother. This can be seen by first writing the field equations in terms of differential forms [22]

$$d(e^{-\phi}\star H_3) = F_3 \wedge \tilde{F}_5, \qquad d(e^{\phi}\star F_3) = -H_3 \wedge \tilde{F}_5. \tag{H.29}$$

Consider the transformation given by the SL(2, $\mathbb{Z}$) element

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{H.30}$$

which transforms the 2-forms and their field strengths like

$$\Lambda\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_2 \\ -B_2 \end{pmatrix}, \qquad \Longrightarrow \qquad \Lambda\begin{pmatrix} H_3 \\ F_3 \end{pmatrix} = \begin{pmatrix} F_3 \\ -H_3 \end{pmatrix}. \tag{H.31}$$

The dilaton transforms like $e^{\phi} \to e^{-\phi}$ according to (3.5), and all other fields are unchanged. The equations in (H.29) are dual under this SL(2, $\mathbb{Z}$) transformation and it is thus sufficient to only show that only

$$\nabla_\rho (e^{-\phi} H^{\mu\nu\rho}) = \frac{1}{3!} \tilde{F}^{\mu\nu\rho\sigma\xi} F_{\rho\sigma\xi}, \tag{H.32}$$

is satisfied. The right hand side is only non-zero when the free indices are $\mu\nu = \theta_1\varphi_1$. This case is treated first.

H.4.1 The non-trivial case $\mu\nu = \theta_1\varphi_1$

The right hand side is

$$\begin{aligned}
\textbf{r.h.s.} &= \frac{1}{3!} \tilde{F}^{\mu\nu\rho\sigma\xi} F_{\rho\sigma\xi} = \tilde{F}^{\theta_1\varphi_1\eta\theta_2\varphi_2} F_{\eta\theta_2\varphi_2} + \tilde{F}^{\theta_1\varphi_1\alpha\theta_2\varphi_2} F_{\alpha\theta_2\varphi_2} = \\
&= \frac{1}{5!} \epsilon^{\theta_1\varphi_1\eta\theta_2\varphi_2\mu_1\ldots\mu_5} \tilde{F}_{\mu_1\ldots\mu_5} F_{\eta\theta_2\varphi_2} + \frac{1}{5!} \epsilon^{\theta_1\varphi_1\alpha\theta_2\varphi_2\mu_1\ldots\mu_5} \tilde{F}_{\mu_1\ldots\mu_5} F_{\alpha\theta_2\varphi_2} = \\
&= \epsilon^{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2} \left(\tilde{F}_{txyz\alpha} F_{\eta\theta_2\varphi_2} - \tilde{F}_{txyz\eta} F_{\alpha\theta_2\varphi_2} \right) = \\
&= \frac{1}{e} \left(\tilde{F}_{txyz\alpha} F_{\eta\theta_2\varphi_2} - \tilde{F}_{txyz\eta} F_{\alpha\theta_2\varphi_2} \right) = \\
&= \frac{16\Delta}{\tilde{e}\ \sin^2(2\alpha)\ \sin\theta_1\ \sin\theta_2} \tilde{e} \left(\sin(2\alpha) F_{\eta\theta_2\varphi_2} - \frac{3}{2} F_{\alpha\theta_2\varphi_2} \right) = \\
&= -\frac{16\sqrt{2} e^{\eta} \Delta \sin\alpha}{\sin(2\alpha)\ \sin\theta_1\ (2 - \cos(2\alpha))^2} \times \\
&\qquad \times \left(2\sin^2\alpha\ (2 - \cos(2\alpha)) - \frac{3}{2}(4 - \cos(2\alpha)) \right) = \\
&= \frac{4\sqrt{2}\Delta\ e^{\eta}}{\sin\theta_1\ \cos\alpha\ (2 - \cos(2\alpha))^2} (7 + 3\cos(2\alpha) - \cos(4\alpha))).
\end{aligned} \tag{H.33}$$

Next, dealing with the left hand side of the equation

$$\begin{aligned}
\textbf{l.h.s.} &= \nabla_\rho \left[e^{-\phi} H^{\theta_1\varphi_1\rho} \right] = \frac{1}{e} \partial_\eta \left[e\ e^{-\phi} H^{\eta\theta_1\varphi_1} \right] + \frac{1}{e} \partial_\alpha \left[e\ e^{-\phi} H^{\alpha\theta_1\varphi_1} \right] = \\
&= \frac{1}{e} \partial_\eta \left[e\ e^{-\phi} H_{\eta\theta_1\varphi_1} g^{\eta\eta} g^{\theta_1\theta_1} g^{\varphi_1\varphi_1} \right] + \\
&\qquad + \frac{1}{e} \partial_\alpha \left[e\ e^{-\phi} H_{\alpha\theta_1\varphi_1} g^{\alpha\alpha} g^{\theta_1\theta_1} g^{\varphi_1\varphi_1} \right] = \\
&= \frac{1}{e} \partial_\eta \left[e\ e^{-\phi} H_{\eta\theta_1\varphi_1} \frac{\Delta^3}{x_1^4 \sin^2\theta_1} \right] + \frac{1}{e} \partial_\alpha \left[e\ e^{-\phi} H_{\alpha\theta_1\varphi_1} \frac{\Delta^3}{x_1^4 \sin^2\theta_1} \right],
\end{aligned} \tag{H.34}$$

where the only η dependence in the first term is $e^{-\phi} H_{\eta\theta_1\varphi_1} \sim e^{\eta}$, making the partial derivative ∂_η redundant. To continue evaluating the left hand side, the derivative

$$\frac{\partial_\alpha(\Delta\ e)}{e} = 4\Delta \cot(2\alpha), \tag{H.35}$$

which was calculated in (H.7), will be needed. The left hand side becomes

$$\begin{aligned}
\textbf{l.h.s.} &= \frac{\Delta^3}{x_1^4 \sin^2\theta_1} e^{-\phi} H_{\eta\theta_1\varphi_1} + \frac{1}{e}\partial_\alpha \left[e\ e^{-\phi} H_{\alpha\theta_1\varphi_1} \frac{\Delta^3}{x_1^4 \sin^2\theta_1}\right] = \\
&= \frac{\Delta^3}{x_1^4 \sin^2\theta_1} e^{-\phi} H_{\eta\theta_1\varphi_1} + \frac{\Delta^2}{x_1^4 \sin^2\theta_1} e^{-\phi} H_{\alpha\theta_1\varphi_1} \frac{\partial_\alpha(\Delta\ e)}{e} + \\
&\quad + \frac{\Delta}{\sin^2\theta_1}\partial_\alpha \left[\frac{\Delta^2}{x_1^4} e^{-\phi} H_{\alpha\theta_1\varphi_1}\right] = \\
&= \frac{\Delta^3}{x_1^4 \sin^2\theta_1} e^{-\phi} H_{\eta\theta_1\varphi_1} + \frac{4\Delta^3\cot(2\alpha)}{x_1^4 \sin^2\theta_1} e^{-\phi} H_{\alpha\theta_1\varphi_1} + \\
&\quad + \frac{\Delta}{\sin^2\theta_1}\partial_\alpha \left[\frac{\Delta^2}{x_1^4} e^{-\phi} H_{\alpha\theta_1\varphi_1}\right] ..
\end{aligned} \tag{H.36}$$

The derivative in the last term is

$$\begin{aligned}
\partial_\alpha\left(\frac{\Delta^2}{x_1^4} e^{-\phi} H_{\alpha\theta_1\varphi_1}\right) &= \partial_\alpha \Bigg(\Delta^2 \frac{(2+\cos(2\alpha))^2}{\cos^4\alpha} e^\eta \frac{\sqrt{7-\cos(4\alpha)}}{2-\cos(2\alpha)} \times \\
&\qquad \times \sin(2\alpha)\cos\alpha\ \frac{4+\cos(2\alpha)}{(2+\cos(2\alpha))^2}\sin\theta_1 \Bigg) = \\
&= e^\eta \sin\theta_1\ \partial_\alpha \left(\Delta^2 \frac{4+\cos(2\alpha)}{\cos^3\alpha}\frac{\sqrt{7-\cos(4\alpha)}}{2-\cos(2\alpha)}\sin(2\alpha)\right) = \\
&= \{\sqrt{7-\cos(4\alpha)} = \sqrt{2}\Delta^{-2}\} = \\
&= 2\sqrt{2} e^\eta \sin\theta_1\ \partial_\alpha \left(\frac{\sin\alpha\ (4+\cos(2\alpha))}{\cos^2\alpha\ (2-\cos(2\alpha))}\right) = \\
&= 2\sqrt{2} e^\eta \sin\theta_1 \Bigg(\frac{(4+\cos(2\alpha))}{\cos\alpha\ (2-\cos(2\alpha))} + \\
&\qquad - \frac{2\sin\alpha\ \sin(2\alpha)}{\cos^2\alpha\ (2-\cos(2\alpha))} + \\
&\qquad + \frac{2\sin^2\alpha\ (4+\cos(2\alpha))}{\cos^3\alpha\ (2-\cos(2\alpha))} + \\
&\qquad - \frac{2\sin\alpha\ \sin(2\alpha)\ (4+\cos(2\alpha))}{\cos^2\alpha\ (2-\cos(2\alpha))^2}\Bigg) = \\
&= \frac{2\sqrt{2}e^\eta\sin\theta_1}{\cos^3\alpha\ (2-\cos(2\alpha))^2}\Bigg((4+\cos(2\alpha))\times \\
&\qquad \times \left[-\sin^2(2\alpha)(\cos^2\alpha + 2\sin^2\alpha)(2-\cos(2\alpha))\right] + \\
&\qquad - \sin^2(2\alpha)(2-\cos(2\alpha))\Bigg) = \\
&= \frac{1}{\sqrt{2}} e^\eta \sin\theta_1 \frac{35 - 27\cos(2a) + 11\cos(4\alpha) + \cos(2\alpha)\cos(4\alpha)}{\cos^3\alpha\ (2-\cos(2\alpha))^2}.
\end{aligned} \tag{H.37}$$

The left hand side becomes

$$\begin{aligned}
\textbf{l.h.s.} = & \left(\frac{\Delta^3(2+\cos(2\alpha))^2}{\cos^4\alpha\ \sin^2\theta_1}\right)\left(\frac{e^{2\eta}}{\sqrt{2}}\frac{\sqrt{7-\cos(4\alpha)}}{2-\cos(2\alpha)}\right)\times \\
& \times\left(\frac{2\sqrt{2}}{e^\eta}\frac{\cos^3\alpha}{2+\cos(2\alpha)}\sin\theta_1\right)+ \\
& + 4\Delta\cot(2\alpha)\left(\frac{\Delta^2(2+\cos(2\alpha))^2}{\cos^4\alpha\ \sin^2\theta_1}\right)\left(\frac{e^{2\eta}}{\sqrt{2}}\frac{\sqrt{7-\cos(4\alpha)}}{2-\cos(2\alpha)}\right)\times \\
& \quad \times\left(\frac{\sqrt{2}}{e^\eta}\sin(2\alpha)\ \cos\alpha\ \frac{4+\cos(2\alpha)}{(2+\cos(2\alpha))^2}\sin\theta_1\right)+ \\
& + \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{35-27\cos(2a)+11\cos(4\alpha)+\cos(2\alpha)\cos(4\alpha)}{\cos^3\alpha\ (2-\cos(2\alpha))^2} = \\
= & \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{4(2+\cos(2\alpha))}{\cos\alpha\ (2-\cos(2\alpha))}+ \\
& + \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{8\sin(2\alpha)\ \cot(2\alpha)\ (4+\cos(2\alpha))}{\cos^3\alpha\ (2-\cos(2\alpha))}+ \\
& + \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{35-27\cos(2a)+11\cos(4\alpha)+\cos(2\alpha)\cos(4\alpha)}{\cos^3\alpha\ (2-\cos(2\alpha))^2} = \\
= & \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{1}{\cos^3\alpha\ (2-\cos(2\alpha))^2}\Big(4(4-\cos^2(2\alpha))+ \\
& + 8\sin(2\alpha)\cot(2\alpha)(4+\cos(2\alpha))(2-\cos(2\alpha))+ \\
& + [35-27\cos(2a)+11\cos(4\alpha)+\cos(2\alpha)\cos(4\alpha)]\Big) = \\
= & \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{1}{\cos^3\alpha\ (2-\cos(2\alpha))^2}\times \\
& \times\Big([7+7\cos(2\alpha)-\cos(4\alpha)-\cos(2\alpha)\cos(4\alpha)]+ \\
& \quad + 4\,[-2+15\cos(2\alpha)-2\cos(4\alpha)-\cos(2\alpha)\cos(4\alpha)]+ \\
& \quad + [35-27\cos(2a)+11\cos(4\alpha)+\cos(2\alpha)\cos(4\alpha)]\Big) = \\
= & \frac{\Delta\ e^\eta}{\sqrt{2}\sin\theta_1}\frac{34+40\cos(2\alpha)+2\cos(4\alpha)-4\cos(2\alpha)\cos(4\alpha)}{\cos^3\alpha\ (2-\cos(2\alpha))^2} = \\
= & \frac{4\sqrt{2}\Delta\ e^\eta}{\sin\theta_1\ \cos\alpha\ (2-\cos(2\alpha))^2}(7+3\cos(2\alpha)-\cos(4\alpha)).
\end{aligned} \tag{H.38}$$

The same expression is finally found for both sides of the equation

$$\textbf{l.h.s.} = \frac{4\sqrt{2}\Delta\ e^\eta}{\sin\theta_1\ \cos\alpha\ (2-\cos(2\alpha))^2}\,(7+3\cos(2\alpha)-\cos(4\alpha)) = \textbf{r.h.s.}, \tag{H.39}$$

showing that the equations of motion for the 2-forms are satisfied.

H.4.2 The trivial case $\mu\nu \neq \theta_1\varphi_1$

What remains now is to show that the left hand side vanishes when $\mu\nu \neq \theta_1\varphi_1$. First note that the only non-trivial H_3-components are $H_{\eta\theta_1\varphi_1}$ and $H_{\alpha\theta_1\varphi_1}$. Both feature θ_1 and φ_1 indices and if $\mu\nu \neq \theta_1\varphi_1$, then either θ_1 or φ_1 must contract with the derivative in the left hand side. However

$$\mathbf{l.h.s.} = \nabla_{\varphi_1}[e^{-\phi}H^{\mu\theta_1\varphi_1}] = 0, \qquad \mu = \eta,\, \alpha, \tag{H.40}$$

since neither the metric or any fields depend on φ_1. If the derivative instead contracts with θ_1, the only θ_1-dependent components are

$$e \sim \sin\theta_1, \qquad H_{\eta\theta_1\varphi_1} \sim\ H_{\alpha\theta_1\varphi_1} \sim \sin\theta_1, \qquad g^{\varphi_1\varphi_1} \sim \frac{1}{\sin^2\theta_1}, \tag{H.41}$$

implying that

$$\mathbf{l.h.s.} = \nabla_{\theta_1}(e^{-\phi}H^{\mu\theta_1\varphi_1}) \sim \partial_{\theta_1}(e\ H_{\mu\theta_1\varphi_1}g^{\varphi_1\varphi_1}) \sim \partial_{\theta_1}(1) = 0, \qquad \mu = \eta,\, \alpha. \tag{H.42}$$

This shows that the 2-form field equations are satisfied.

I

Ricci tensor of the type IIB AdS vacuum

The Ricci tensor of the AdS vacuum given in section 3.2 with the metric

$$ds^2_{10} = \Delta^{-1}\left[\frac{1}{2}ds^2_{\mathrm{AdS}_4} + d\eta^2 + d\alpha^2 + \frac{\cos^2\alpha}{2+\cos 2\alpha}d\Omega_1 + \frac{\sin^2\alpha}{2-\cos 2\alpha}d\Omega_2\right], \tag{I.1}$$

is calculated using the method explained in appendix C. The Ricci tensor is needed to evaluate Einstein's equations in appendix J. The deformation parameters are set to $\chi_i = 0$.

I.1 The Ricci tensor

The zehnbein 1-forms of the AdS vacuum can be read off from (3.24), where the ones related to AdS_4 are

$$\begin{aligned} e^0 &= \frac{1}{\sqrt{2\Delta}}\cosh\rho\ dt, & e^1 &= \frac{1}{\sqrt{2\Delta}}\frac{1}{a}d\rho, \\ e^2 &= \frac{1}{\sqrt{2\Delta}}\frac{1}{a}\sinh\rho\ d\theta, & e^3 &= \frac{1}{\sqrt{2\Delta}}\frac{1}{a}\sinh\rho\ \sin\theta\ d\varphi. \end{aligned} \tag{I.2}$$

The 1-forms related to S^1_η and the interval $\mathcal{I}_\alpha$ of the squashed 5-sphere are

$$e^4 = \frac{1}{\sqrt{\Delta}}d\eta, \qquad e^5 = \frac{1}{\sqrt{\Delta}}d\alpha, \tag{I.3}$$

and for the two 2-spheres, the zehnbein 1-forms are

$$\begin{aligned} e^6 &= \frac{1}{\sqrt{\Delta}}x_1 d\theta_1, & e^7 &= \frac{1}{\sqrt{\Delta}}x_1\sin\theta_1\ d\varphi_1, \\ e^8 &= \frac{1}{\sqrt{\Delta}}x_2 d\theta_2, & e^9 &= \frac{1}{\sqrt{\Delta}}x_2\sin\theta_2\ d\varphi_2, \end{aligned} \tag{I.4}$$

where the α-dependent x_1 and x_2 are defined in (3.22).

I.1.1 The connection ω_{ab}

To find the 1-form connection ω_{ab}, the exterior derivatives de^a are needed. To calculate these, it is useful to first note that

$$d\left(\frac{1}{\sqrt{\Delta}}\right) = -\frac{1}{2}\frac{\Delta'}{\Delta^{3/2}}d\alpha = -\frac{1}{2}\frac{\Delta'}{\Delta}e^5 = -\frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^5\left(\frac{1}{\sqrt{\Delta}}\right). \tag{I.5}$$

Only $de^5 = 0$, and the non-vanishing de^a are found as

$$\begin{aligned}
de^0 &= \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^0 \wedge e^5 - \sqrt{2\Delta}a\tanh\rho\; e^0 \wedge e^1, \\
de^1 &= \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^1 \wedge e^5, \\
de^2 &= \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^2 \wedge e^5 + \sqrt{2\Delta}a\coth\rho\; e^1 \wedge e^2, \\
de^3 &= \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^3 \wedge e^5 + \sqrt{2\Delta}a\coth\rho\; e^1 \wedge e^3 + \sqrt{2\Delta}a\frac{\cot\theta}{\sinh\rho}e^2 \wedge e^3, \\
de^4 &= \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^4 \wedge e^5, \\
de^6 &= \sqrt{\Delta}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^5 \wedge e^6, \\
de^7 &= \sqrt{\Delta}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^5 \wedge e^7 + \sqrt{\Delta}\frac{\cot\theta_1}{x_1}e^6 \wedge e^7, \\
de^8 &= \sqrt{\Delta}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^5 \wedge e^8, \\
de^9 &= \sqrt{\Delta}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^5 \wedge e^9 + \sqrt{\Delta}\frac{\cot\theta_2}{x_2}e^8 \wedge e^9.
\end{aligned} \tag{I.6}$$

The non-zero $\omega_{[ij]a}$ can then be found using the equation $de^a = \eta^{ac}\omega_{ijc}e^i \wedge e^j$.

$$\begin{aligned}
a = 0: &\quad 2\omega_{[01]0} = \sqrt{2\Delta}a\tanh\rho, \qquad 2\omega_{[05]0} = -\frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}, \\
a = 1: &\quad 2\omega_{[15]1} = \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}, \\
a = 2: &\quad 2\omega_{[12]2} = \sqrt{2\Delta}a\coth\rho, \qquad 2\omega_{[25]2} = \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}, \\
a = 3: &\quad 2\omega_{[13]3} = \sqrt{2\Delta}a\coth\rho, \qquad 2\omega_{[23]3} = \sqrt{2\Delta}a\frac{\cot\theta}{\sinh\rho}, \\
&\quad 2\omega_{[35]3} = \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}, \\
a = 4: &\quad 2\omega_{[45]4} = \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}, \\
a = 5: &\quad \text{all } \omega_{[ij]5} = 0, \\
a = 6: &\quad 2\omega_{[56]6} = \sqrt{\Delta}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right), \\
a = 7: &\quad 2\omega_{[57]7} = \sqrt{\Delta}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right), \quad 2\omega_{[67]7} = \sqrt{\Delta}\frac{\cot\theta_1}{x_1}, \\
a = 8: &\quad 2\omega_{[58]8} = \sqrt{\Delta}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right), \\
a = 9: &\quad 2\omega_{[59]9} = \sqrt{\Delta}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right), \quad 2\omega_{[89]9} = \sqrt{\Delta}\frac{\cot\theta_2}{x_2},
\end{aligned} \tag{I.7}$$

Because of the antisymmetry in the last indices, $\omega_{abc} = \omega_{a[bc]}$, the 1-form connections are

$$\begin{aligned}
&\omega_{01} = \omega_{001}e^0 = -\sqrt{2\Delta}a\tanh\rho\; e^0, && \omega_{12} = \omega_{212}e^2 = -\sqrt{2\Delta}a\coth\rho\; e^2,\\
&\omega_{13} = \omega_{313}e^3 = -\sqrt{2\Delta}a\coth\rho\; e^3, && \omega_{23} = \omega_{323}e^3 = -\sqrt{2\Delta}a\frac{\cot\theta}{\sinh\rho}\; e^3,\\
&\omega_{05} = \omega_{005}e^0 = \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^0, && \omega_{i5} = \omega_{ii5}e^i = -\frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}e^i, \quad i = 1,\ldots 4,\\
&\omega_{5j} = \omega_{j5j}e^j = -\sqrt{\Delta}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)e^j, && j = 6,\, 7,\\
&\omega_{5k} = \omega_{k5k}e^k = -\sqrt{\Delta}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)e^k, && k = 8,\, 9,\\
&\omega_{67} = \omega_{767}e^7 = -\sqrt{\Delta}\frac{\cot\theta_1}{x_1}e^7, && \omega_{89} = \omega_{989}e^9 = -\sqrt{\Delta}\frac{\cot\theta_2}{x_2}e^9.
\end{aligned} \tag{I.8}$$

The ω_{ab} can now be used to find the curvature 2-form via $\Theta_{ab} = d\omega_{ab} + \omega_{ac}\wedge\omega_{cb}$.

I.1.2 The $d\omega_{ab}$ term of Θ_{ab}

The exterior derivative of the non-zero ω_{ab} are calculated. First

$$\begin{aligned}
d\omega_{01} &= -\sqrt{2}a\; d\left(\sqrt{\Delta}\tanh\rho\; e^0\right) =\\
&= \frac{\Delta'}{\sqrt{2}}a\tanh\rho\; e^0\wedge e^5 + 2\Delta a^2\left(1-\tanh^2\rho\right)e^0\wedge e^1 +\\
&\quad - \sqrt{2\Delta}a\tanh\rho\; de^0 =\\
&= \left[\frac{\Delta'}{\sqrt{2}}a\tanh\rho - \frac{\Delta'}{\sqrt{2}}a\tanh\rho\right]e^0\wedge e^5 +\\
&\quad + 2\Delta a^2\left[1-\tanh^2\rho+\tanh^2\rho\right]e^0\wedge e^1 =\\
&= 2\Delta a^2\; e^0\wedge e^1,\\
d\omega_{12} &= -\sqrt{2}a\; d\left(\sqrt{\Delta}\coth\rho\; e^2\right) =\\
&= \frac{\Delta'}{\sqrt{2}}a\coth\rho\; e^2\wedge e^5 - 2\Delta a^2\left(1-\coth^2\rho\right)e^1\wedge e^2 +\\
&\quad - \sqrt{2\Delta}a\coth\rho\; de^2 =\\
&= -2\Delta a^2\; e^1\wedge e^2,\\
d\omega_{13} &= -\sqrt{2}a\; d\left(\sqrt{\Delta}\coth\rho\; e^3\right) =\\
&= \frac{\Delta'}{\sqrt{2}}a\coth\rho\; e^3\wedge e^5 - 2\Delta a^2\left(1-\coth^2\rho\right)e^1\wedge e^3 +\\
&\quad - \sqrt{2\Delta}a\coth\rho\; de^3 =\\
&= -2\Delta a^2\; e^1\wedge e^3 - 2\Delta a^2\frac{\coth\rho\;\cot\theta}{\sinh\rho}e^2\wedge e^3,
\end{aligned} \tag{I.9}$$

$$
\begin{aligned}
d\omega_{23} &= -\sqrt{2}a\ d\left(\sqrt{\Delta}\frac{\cot\theta}{\sinh\rho}e^3\right) = \\
&= \frac{\Delta'}{\sqrt{2}}a\frac{\cot\theta}{\sinh\rho}e^3\wedge e^5 + 2\Delta a^2\frac{\coth\rho\ \cot\theta}{\sinh\rho}e^1\wedge e^3 + \\
&\quad + 2\Delta a^2\frac{1+\cot^2\theta}{\sinh^2\rho}e^2\wedge e^3 - \sqrt{2\Delta}a\frac{\cot\theta}{\sinh\rho}de^3 = \\
&= 2\Delta a^2\frac{1}{\sinh^2\rho}e^2\wedge e^3,
\end{aligned}
\tag{I.10}
$$

then

$$
\begin{aligned}
d\omega_{05} &= \frac{1}{2}d\left(\frac{\Delta'}{\sqrt{\Delta}}e^0\right) = \frac{1}{2}\left(\frac{\Delta''}{\sqrt{\Delta}} - \frac{1}{2}\frac{(\Delta')^2}{\Delta^{3/2}}\right)d\alpha\wedge e^0 + \frac{1}{2}\frac{\Delta'}{\sqrt{\Delta}}de^0 = \\
&= \frac{1}{2}\left(-\Delta'' + \frac{1}{2}\frac{(\Delta')^2}{\Delta} + \frac{1}{2}\frac{(\Delta')^2}{\Delta}\right)e^0\wedge e^5 - \frac{\Delta'}{\sqrt{2}}a\ e^0\wedge e^1 = \\
&= \frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right)e^0\wedge e^5 - \frac{\Delta'}{\sqrt{2}}a\tanh\rho\ e^0\wedge e^1, \\
d\omega_{15} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right)e^1\wedge e^5, \\
d\omega_{25} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right)e^2\wedge e^5 - \frac{\Delta'}{\sqrt{2}}a\coth\rho\ e^1\wedge e^2, \\
d\omega_{35} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right)e^3\wedge e^5 - \frac{\Delta'}{\sqrt{2}}a\coth\rho\ e^1\wedge e^3 - \frac{\Delta'}{\sqrt{2}}a\frac{\cot\theta}{\sinh\rho}e^2\wedge e^3, \\
d\omega_{45} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right)e^4\wedge e^5,
\end{aligned}
\tag{I.11}
$$

then

$$
\begin{aligned}
d\omega_{56} &= -d\left(\sqrt{\Delta}\left[\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right]e^6\right) = \\
&= -\left(\frac{1}{2}\Delta'\frac{x_1'}{x_1} + \frac{1}{4}\frac{(\Delta')^2}{\Delta} + \Delta\frac{x_1''}{x_1} - \Delta\frac{(x_1')^2}{x_1^2} - \frac{1}{2}\Delta''\right)e^5\wedge e^6 + \\
&\quad - \sqrt{\Delta}\left[\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right]de^6 = \\
&= \frac{1}{2}\left(\Delta'\frac{x_1'}{x_1} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_1''}{x_1} + \Delta''\right)e^5\wedge e^6, \\
d\omega_{57} &= \frac{1}{2}\left(\Delta'\frac{x_1'}{x_1} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_1''}{x_1} + \Delta''\right)e^5\wedge e^7 + \\
&\quad - \Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\frac{\cot\theta_1}{x_1}e^6\wedge e^7,
\end{aligned}
\tag{I.12}
$$

$$
\begin{aligned}
d\omega_{67} &= -d\left(\frac{\sqrt{\Delta}}{x_1}\cot\theta_1 e^7\right) = \\
&= \frac{\Delta}{x_1}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\cot\theta_1\; e^5\wedge e^7 + \frac{\Delta}{x_1^2}\left(1+\cot^2\theta_1\right)e^6\wedge e^7 + \\
&\quad - \frac{\sqrt{\Delta}}{x_1}\cot\theta_1\; de^7 = \\
&= \frac{\Delta}{x_1^2}e^6\wedge e^7,
\end{aligned}
\tag{I.13}
$$

and finally

$$
\begin{aligned}
d\omega_{58} &= \frac{1}{2}\left(\Delta'\frac{x_2'}{x_2} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_2''}{x_2} + \Delta''\right)e^5\wedge e^8, \\
d\omega_{59} &= \frac{1}{2}\left(\Delta'\frac{x_2'}{x_2} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_2''}{x_2} + \Delta''\right)e^5\wedge e^9 + \\
&\quad - \Delta\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\frac{\cot\theta_2}{x_2}e^8\wedge e^9, \\
d\omega_{89} &= \frac{\Delta}{x_2^2}e^8\wedge e^9.
\end{aligned}
\tag{I.14}
$$

I.1.3 The $\omega_{ac}\wedge\omega_{cb}$ term of Θ_{ab}

The final step to find the curvature 2-form is to calculate the second term $\tilde{\omega}_{ab} = \omega_{ac}\wedge\omega_{cb}$. First off

$$
\begin{aligned}
\tilde{\omega}_{01} &= \omega_{05}\wedge\omega_{51} = \frac{1}{4}\frac{(\Delta')^2}{\Delta}e^0\wedge e^1, \\
\tilde{\omega}_{02} &= \omega_{01}\wedge\omega_{12} + \omega_{05}\wedge\omega_{52} = \left(2\Delta a^2 + \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right)e^0\wedge e^2, \\
\tilde{\omega}_{03} &= \omega_{01}\wedge\omega_{13} + \omega_{05}\wedge\omega_{53} = \left(2\Delta a^2 + \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right)e^0\wedge e^3, \\
\tilde{\omega}_{04} &= \omega_{05}\wedge\omega_{54} = \frac{1}{4}\frac{(\Delta')^2}{\Delta}e^0\wedge e^4, \\
\tilde{\omega}_{05} &= \omega_{01}\wedge\omega_{15} = \frac{\Delta'}{\sqrt{2}}a\tanh\rho\; e^0\wedge e^1, \\
\tilde{\omega}_{06} &= \omega_{05}\wedge\omega_{56} = -\frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)e^0\wedge e^6, \\
\tilde{\omega}_{07} &= \omega_{05}\wedge\omega_{57} = -\frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)e^0\wedge e^7, \\
\tilde{\omega}_{08} &= \omega_{05}\wedge\omega_{58} = -\frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)e^0\wedge e^8, \\
\tilde{\omega}_{09} &= \omega_{05}\wedge\omega_{59} = -\frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)e^0\wedge e^9,
\end{aligned}
\tag{I.15}
$$

then

$$\begin{aligned}
&\tilde{\omega}_{12} = \omega_{15} \wedge \omega_{52} = -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^1 \wedge e^2, \\
&\tilde{\omega}_{13} = \omega_{15} \wedge \omega_{53} + \omega_{12} \wedge \omega_{23} = -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^1 \wedge e^3 + 2\Delta a^2 \frac{\coth\rho \ \cot\theta}{\sinh\rho} e^2 \wedge e^3, \\
&\tilde{\omega}_{14} = \omega_{15} \wedge \omega_{54} = -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^1 \wedge e^4, \\
&\tilde{\omega}_{15} = 0, \\
&\tilde{\omega}_{1j} = \omega_{15} \wedge \omega_{5j} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^j, \qquad j = 6,\ 7, \\
&\tilde{\omega}_{1k} = \omega_{15} \wedge \omega_{5k} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^k, \qquad k = 8,\ 9,
\end{aligned} \tag{I.16}$$

then

$$\begin{aligned}
&\tilde{\omega}_{23} = \omega_{25} \wedge \omega_{53} + \omega_{21} \wedge \omega_{13} = \left(-\frac{1}{4}\frac{(\Delta')^2}{\Delta} - 2\Delta a^2 \coth^2\rho\right) e^2 \wedge e^3, \\
&\tilde{\omega}_{24} = \omega_{25} \wedge \omega_{54} = -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^2 \wedge e^4, \\
&\tilde{\omega}_{25} = \omega_{21} \wedge \omega_{15} = \frac{\Delta'}{\sqrt{2}} a \coth\rho \ e^1 \wedge e^2, \\
&\tilde{\omega}_{26} = \omega_{25} \wedge \omega_{56} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^6, \\
&\tilde{\omega}_{27} = \omega_{25} \wedge \omega_{57} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^7, \\
&\tilde{\omega}_{28} = \omega_{25} \wedge \omega_{58} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^8, \\
&\tilde{\omega}_{29} = \omega_{25} \wedge \omega_{59} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^9,
\end{aligned} \tag{I.17}$$

then

$$\begin{aligned}
&\tilde{\omega}_{34} = \omega_{35} \wedge \omega_{54} = -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^3 \wedge e^4, \\
&\tilde{\omega}_{35} = \omega_{31} \wedge \omega_{15} + \omega_{32} \wedge \omega_{25} = \frac{\Delta'}{\sqrt{2}} a \coth\rho \ e^1 \wedge e^3 + \frac{\Delta'}{\sqrt{2}} a \frac{\cot\theta}{\sinh\rho} e^2 \wedge e^3, \\
&\tilde{\omega}_{36} = \omega_{35} \wedge \omega_{56} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^6, \\
&\tilde{\omega}_{37} = \omega_{35} \wedge \omega_{57} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^7, \\
&\tilde{\omega}_{38} = \omega_{35} \wedge \omega_{58} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^8, \\
&\tilde{\omega}_{39} = \omega_{35} \wedge \omega_{59} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^9,
\end{aligned} \tag{I.18}$$

then

$$\begin{aligned}
\tilde{\omega}_{45} &= 0, \\
\tilde{\omega}_{46} &= \omega_{45} \wedge \omega_{56} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^6, \\
\tilde{\omega}_{47} &= \omega_{45} \wedge \omega_{57} = \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^7, \\
\tilde{\omega}_{48} &= \omega_{45} \wedge \omega_{58} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^8, \\
\tilde{\omega}_{49} &= \omega_{45} \wedge \omega_{59} = \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^9,
\end{aligned} \tag{I.19}$$

then

$$\begin{aligned}
\tilde{\omega}_{56} &= 0, \\
\tilde{\omega}_{57} &= \omega_{56} \wedge \omega_{67} = \frac{\Delta}{x_1}\cot\theta_1\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^6 \wedge e^7, \\
\tilde{\omega}_{58} &= 0, \\
\tilde{\omega}_{59} &= \omega_{58} \wedge \omega_{89} = \frac{\Delta}{x_2}\cot\theta_2\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^8 \wedge e^9,
\end{aligned} \tag{I.20}$$

then

$$\begin{aligned}
\tilde{\omega}_{67} &= \omega_{65} \wedge \omega_{57} = -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 e^6 \wedge e^7, \\
\tilde{\omega}_{68} &= \omega_{65} \wedge \omega_{58} = -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^6 \wedge e^8, \\
\tilde{\omega}_{69} &= \omega_{65} \wedge \omega_{59} = -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^6 \wedge e^9,
\end{aligned} \tag{I.21}$$

then

$$\begin{aligned}
\tilde{\omega}_{78} &= \omega_{75} \wedge \omega_{58} = -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^7 \wedge e^8, \\
\tilde{\omega}_{79} &= \omega_{75} \wedge \omega_{59} = -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^7 \wedge e^9,
\end{aligned} \tag{I.22}$$

and finally

$$\tilde{\omega}_{89} = \omega_{85} \wedge \omega_{59} = -\Delta\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 e^8 \wedge e^9. \tag{I.23}$$

I.1.4 The curvature 2-form Θ_{ab}

The curvature 2-form components read

$$
\begin{aligned}
\Theta_{01} &= \left(2\Delta a^2 + \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right) e^0 \wedge e^1,\\
\Theta_{02} &= \left(2\Delta a^2 + \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right) e^0 \wedge e^2,\\
\Theta_{03} &= \left(2\Delta a^2 + \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right) e^0 \wedge e^3,\\
\Theta_{04} &= \frac{1}{4}\frac{(\Delta')^2}{\Delta} e^0 \wedge e^4,\\
\Theta_{05} &= \frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right) e^0 \wedge e^5,\\
\Theta_{06} &= -\frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^0 \wedge e^6,\\
\Theta_{07} &= -\frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^0 \wedge e^7,\\
\Theta_{08} &= -\frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^0 \wedge e^8,\\
\Theta_{09} &= -\frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^0 \wedge e^9,
\end{aligned}
\tag{I.24}
$$

$$
\begin{aligned}
\Theta_{12} &= \left(-2\Delta a^2 - \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right) e^1 \wedge e^2,\\
\Theta_{13} &= \left(-2\Delta a^2 - \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right) e^1 \wedge e^3,\\
\Theta_{14} &= -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^1 \wedge e^4,\\
\Theta_{15} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right) e^1 \wedge e^5,\\
\Theta_{16} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^6,\\
\Theta_{17} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^7,\\
\Theta_{18} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^8,\\
\Theta_{19} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^9,
\end{aligned}
\tag{I.25}
$$

$$\begin{aligned}
\Theta_{23} &= \left(-2\Delta a^2 - \frac{1}{4}\frac{(\Delta')^2}{\Delta}\right) e^2 \wedge e^3, \\
\Theta_{24} &= -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^2 \wedge e^4, \\
\Theta_{25} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right) e^2 \wedge e^5, \\
\Theta_{26} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^6, \\
\Theta_{27} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^7, \\
\Theta_{28} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^8, \\
\Theta_{29} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^2 \wedge e^9,
\end{aligned} \tag{I.26}$$

$$\begin{aligned}
\Theta_{34} &= -\frac{1}{4}\frac{(\Delta')^2}{\Delta} e^3 \wedge e^4, \\
\Theta_{35} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right) e^3 \wedge e^5, \\
\Theta_{36} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^6, \\
\Theta_{37} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^7, \\
\Theta_{38} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^8, \\
\Theta_{39} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^3 \wedge e^9,
\end{aligned} \tag{I.27}$$

$$\begin{aligned}
\Theta_{45} &= -\frac{1}{2}\left(\frac{(\Delta')^2}{\Delta} - \Delta''\right) e^4 \wedge e^5, \\
\Theta_{46} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^6, \\
\Theta_{47} &= \frac{\Delta'}{2}\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^7, \\
\Theta_{48} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^8, \\
\Theta_{49} &= \frac{\Delta'}{2}\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^4 \wedge e^9,
\end{aligned} \tag{I.28}$$

$$\begin{aligned}
\Theta_{56} &= \frac{1}{2}\left(\Delta'\frac{x_1'}{x_1} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_1''}{x_1} + \Delta''\right) e^5 \wedge e^6, \\
\Theta_{57} &= \frac{1}{2}\left(\Delta'\frac{x_1'}{x_1} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_1''}{x_1} + \Delta''\right) e^5 \wedge e^7, \\
\Theta_{58} &= \frac{1}{2}\left(\Delta'\frac{x_2'}{x_2} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_2''}{x_2} + \Delta''\right) e^5 \wedge e^8, \\
\Theta_{59} &= \frac{1}{2}\left(\Delta'\frac{x_2'}{x_2} - \frac{(\Delta')^2}{\Delta} - 2\Delta\frac{x_2''}{x_2} + \Delta''\right) e^5 \wedge e^9,
\end{aligned} \tag{I.29}$$

$$\begin{aligned}
\Theta_{67} &= \frac{\Delta}{x_1^2} e^6 \wedge e^7 - \Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 e^6 \wedge e^7, \\
\Theta_{68} &= -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^6 \wedge e^8 = \\
&= -\left(\Delta\frac{x_1' x_2'}{x_1 x_2} + \frac{1}{4}\frac{(\Delta')^2}{\Delta} - \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right)\right) e^6 \wedge e^8, \\
\Theta_{69} &= -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^6 \wedge e^9 = \\
&= -\left(\Delta\frac{x_1' x_2'}{x_1 x_2} + \frac{1}{4}\frac{(\Delta')^2}{\Delta} - \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right)\right) e^6 \wedge e^9,
\end{aligned} \tag{I.30}$$

$$\begin{aligned}
\Theta_{78} &= -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^7 \wedge e^8 = \\
&= -\left(\Delta\frac{x_1' x_2'}{x_1 x_2} + \frac{1}{4}\frac{(\Delta')^2}{\Delta} - \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right)\right) e^7 \wedge e^8, \\
\Theta_{79} &= -\Delta\left(\frac{x_1'}{x_1} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^7 \wedge e^9 = \\
&= -\left(\Delta\frac{x_1' x_2'}{x_1 x_2} + \frac{1}{4}\frac{(\Delta')^2}{\Delta} - \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right)\right) e^7 \wedge e^9,
\end{aligned} \tag{I.31}$$

$$\Theta_{89} = \frac{\Delta}{x_2^2} e^8 \wedge e^9 - \Delta\left(\frac{x_2'}{x_2} - \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 e^8 \wedge e^9. \tag{I.32}$$

I.1.5 A first expression of R_{ab}

Since all components of the curvature 2-form can be written like $\Theta_{ab} = f_{ab}\,(a,\, \alpha)\ e^a \wedge e^b$, the non-zero Riemann tensor components can be directly read off as $R_{abab} =$

$f_{ab}(a,\,\alpha)$. The result is

$$
\begin{aligned}
R_{00} &= \eta^{aa} R_{0a0a} = 6\Delta a^2 + \frac{5}{2}\frac{(\Delta')^2}{\Delta} - \frac{\Delta''}{2} - \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right),\\
R_{11} &= \eta^{aa} R_{1a1a} = -6\Delta a^2 - \frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right),\\
R_{22} &= \eta^{aa} R_{2a2a} = -6\Delta a^2 - \frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right),\\
R_{33} &= \eta^{aa} R_{3a3a} = -6\Delta a^2 - \frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right),\\
R_{44} &= \eta^{aa} R_{4a4a} = -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right),\\
R_{55} &= \eta^{aa} R_{5a5a} = -\frac{9}{2}\frac{(\Delta')^2}{\Delta} + \frac{9}{2}\Delta'' + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right) - 2\Delta\left(\frac{x_1''}{x_1} + \frac{x_2''}{x_2}\right) =\\
&= -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right) +\\
&\quad + 4\Delta'' - 2\frac{(\Delta')^2}{\Delta} - 2\Delta\left(\frac{x_1''}{x_1} + \frac{x_2''}{x_2}\right),\\
R_{66} &= \eta^{aa} R_{6a6a} = -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right) +\\
&\quad + 4\Delta'\frac{x_1'}{x_1} + \Delta\left(\frac{1}{x_1^2} - \frac{x_1''}{x_1} - \frac{{x_1'}^2}{x_1^2} - 2\frac{x_1'x_2'}{x_1x_2}\right),\\
R_{77} &= \eta^{aa} R_{7a7a} = -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right) +\\
&\quad + 4\Delta'\frac{x_1'}{x_1} + \Delta\left(\frac{1}{x_1^2} - \frac{x_1''}{x_1} - \frac{{x_1'}^2}{x_1^2} - 2\frac{x_1'x_2'}{x_1x_2}\right),\\
R_{88} &= \eta^{aa} R_{8a8a} = -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right) +\\
&\quad + 4\Delta'\frac{x_2'}{x_2} + \Delta\left(\frac{1}{x_2^2} - \frac{x_2''}{x_2} - \frac{{x_2'}^2}{x_2^2} - 2\frac{x_1'x_2'}{x_1x_2}\right),\\
R_{99} &= \eta^{aa} R_{9a9a} = -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right) +\\
&\quad + 4\Delta'\frac{x_2'}{x_2} + \Delta\left(\frac{1}{x_2^2} - \frac{x_2''}{x_2} - \frac{{x_2'}^2}{x_2^2} - 2\frac{x_1'x_2'}{x_1x_2}\right).
\end{aligned}
\tag{I.33}
$$

There is a common part of all R_{ab}-components that can be identified as

$$
v(\alpha) = -\frac{5}{2}\frac{(\Delta')^2}{\Delta} + \frac{\Delta''}{2} + \Delta'\left(\frac{x_1'}{x_1} + \frac{x_2'}{x_2}\right). \tag{I.34}
$$

The unique parts of the Ricci tensor can then be defined as $\tilde{R}_{aa} = R_{aa} - \eta_{aa} v(\alpha)$. The diagonal elements of $\tilde{R}_{ab}$ are then

$$
\begin{aligned}
\tilde{R}_{00} &= R_{00} + v(\alpha) = +6\Delta a^2, \\
\tilde{R}_{11} &= R_{11} - v(\alpha) = -6\Delta a^2, \\
\tilde{R}_{22} &= R_{22} - v(\alpha) = -6\Delta a^2, \\
\tilde{R}_{33} &= R_{33} - v(\alpha) = -6\Delta a^2 \\
\tilde{R}_{44} &= R_{44} - v(\alpha) = 0, \\
\tilde{R}_{55} &= R_{55} - v(\alpha) = 4\Delta'' - 2\frac{(\Delta')^2}{\Delta} - 2\Delta\left(\frac{x_1''}{x_1} + \frac{x_2''}{x_2}\right), \\
\tilde{R}_{66} &= R_{66} - v(\alpha) = 4\Delta'\frac{x_1'}{x_1} + \Delta\left(\frac{1}{x_1^2} - \frac{x_1''}{x_1} - \frac{{x_1'}^2}{x_1^2} - 2\frac{x_1'x_2'}{x_1x_2}\right), \\
\tilde{R}_{77} &= R_{77} - v(\alpha) = 4\Delta'\frac{x_1'}{x_1} + \Delta\left(\frac{1}{x_1^2} - \frac{x_1''}{x_1} - \frac{{x_1'}^2}{x_1^2} - 2\frac{x_1'x_2'}{x_1x_2}\right), \\
\tilde{R}_{88} &= R_{88} - v(\alpha) = 4\Delta'\frac{x_2'}{x_2} + \Delta\left(\frac{1}{x_2^2} - \frac{x_2''}{x_2} - \frac{{x_2'}^2}{x_2^2} - 2\frac{x_1'x_2'}{x_1x_2}\right), \\
\tilde{R}_{99} &= R_{99} - v(\alpha) = 4\Delta'\frac{x_2'}{x_2} + \Delta\left(\frac{1}{x_2^2} - \frac{x_2''}{x_2} - \frac{{x_2'}^2}{x_2^2} - 2\frac{x_1'x_2'}{x_1x_2}\right).
\end{aligned}
\tag{I.35}
$$

I.1.6 Calculating the Ricci tensor explicitly

The Ricci tensor only depends on α and must be expressed explicitly in terms of α in order to evaluate Einstein's equations. The current expressions of $v(\alpha)$ and $\tilde{R}_{aa}$ are written using Δ, x_1, x_2 and up to second order derivatives of these. First off, the derivative of the warping factor, defined by $\Delta = (4 - \cos^2(2\alpha))^{-1/4}$, was calculated in (H.19) as

$$\Delta' = -\Delta^5 \sin(2\alpha)\cos(2\alpha). \tag{I.36}$$

This yields that

$$\frac{(\Delta')^2}{\Delta} = \Delta^9 \sin^2(2\alpha)\cos^2(2\alpha), \tag{I.37}$$

and the second order derivative is

$$
\begin{aligned}
\Delta'' &= -\partial_\alpha\left(\Delta^5 \sin(2\alpha)\cos(2\alpha)\right) = \\
&= -5\Delta^4\Delta' \sin(2\alpha)\cos(2\alpha) - 2\Delta^5\cos^2(2\alpha) + 2\Delta^5\sin^2(2\alpha) = \\
&= 5\Delta^9\sin^2(2\alpha)\cos^2(2\alpha) - 2\Delta^5\cos(4\alpha).
\end{aligned}
\tag{I.38}
$$

Next, the parameters x_1 and x_2 are given by

$$x_1 = \frac{\cos\alpha}{\sqrt{2+\cos(2\alpha)}}, \qquad x_2 = \frac{\sin\alpha}{\sqrt{2-\cos(2\alpha)}}. \tag{I.39}$$

The derivative of x_1 is

$$\begin{aligned} x_1' &= -\frac{\sin\alpha}{\sqrt{2+\cos(2\alpha)}} - \frac{1}{2}\frac{\cos\alpha}{(2+\cos(2\alpha))^{3/2}}\left(-2\sin(2\alpha)\right) = \\ &= \frac{\sin\alpha}{(2+\cos(2\alpha))^{3/2}}\Big(-(2+\cos(2\alpha)) + 2\cos^2\alpha\Big) = \\ &= -\frac{\sin\alpha}{(2+\cos(2\alpha))^{3/2}}, \end{aligned} \tag{I.40}$$

and x_2' is found in a similar way so that

$$x_1' = -\frac{\sin\alpha}{(2+\cos(2\alpha))^{3/2}}, \qquad x_2' = \frac{\cos\alpha}{(2-\cos(2\alpha))^{3/2}}. \tag{I.41}$$

The second order derivatives are given by

$$\begin{aligned} x_1'' &= -\frac{\cos\alpha}{(2+\cos(2\alpha))^{3/2}} + \frac{3}{2}\frac{\sin\alpha}{(2+\cos(2\alpha))^{5/2}}\left(-2\sin(2\alpha)\right) = \\ &= -\frac{\cos\alpha}{(2+\cos(2\alpha))^{5/2}}\Big(2+\cos(2\alpha) + 6\sin^2\alpha\Big) = \\ &= -\frac{\cos\alpha\ \left(5-2\cos(2\alpha)\right)}{(2+\cos(2\alpha))^{5/2}}, \end{aligned} \tag{I.42}$$

and

$$\begin{aligned} x_2'' &= -\frac{\sin\alpha}{(2-\cos(2\alpha))^{3/2}} - \frac{3}{2}\frac{\cos\alpha}{(2-\cos(2\alpha))^{5/2}}\left(2\sin(2\alpha)\right) = \\ &= -\frac{\sin\alpha}{(2-\cos(2\alpha))^{5/2}}\Big(2-\cos(2\alpha) + 6\cos^2\alpha\Big) = \\ &= -\frac{\sin\alpha\ \left(5+2\cos(2\alpha)\right)}{(2-\cos(2\alpha))^{5/2}}. \end{aligned} \tag{I.43}$$

Some useful expressions involving x_1 and x_2 are then

$$\frac{x_1'}{x_1} = -\frac{\tan\alpha}{2+\cos(2\alpha)}, \qquad \frac{x_2'}{x_2} = \frac{\cot\alpha}{2-\cos(2\alpha)}, \tag{I.44}$$

which yields that

$$\begin{aligned} \frac{x_1'}{x_1} + \frac{x_2'}{x_2} &= -\frac{\tan\alpha}{2+\cos(2\alpha)} + \frac{\cot\alpha}{2-\cos(2\alpha)} = \\ &= \Delta^4\left(\cot\alpha\ \left(2+\cos(2\alpha)\right) - \tan\alpha\ \left(2-\cos(2\alpha)\right)\right) = \\ &= 3\Delta^4\left(\cot\alpha - \tan\alpha\right) = 6\Delta^4\frac{\cos(2\alpha)}{\sin(2\alpha)} = \\ &= 6\Delta^4\cot(2\alpha), \end{aligned} \tag{I.45}$$

and lastly

$$
\begin{aligned}
\frac{x_1''}{x_1}+\frac{x_2''}{x_2} &= -\frac{5-2\cos(2\alpha)}{\left(2+\cos(2\alpha)\right)^2}-\frac{5+2\cos(2\alpha)}{\left(2-\cos(2\alpha)\right)^2}= \\
&= -\Delta^8\Big[\left(5-2\cos(2\alpha)\right)\left(2-\cos(2\alpha)\right)^2+ \\
&\qquad\qquad +\left(5+2\cos(2\alpha)\right)\left(2+\cos(2\alpha)\right)^2\Big]= \\
&= -\Delta^8\left[10\left(4+\cos^2(2\alpha)\right)+16\cos^2(2\alpha)\right]= \\
&= -2\Delta^8\left[20+13\cos^2(2\alpha)\right].
\end{aligned}
\tag{I.46}
$$

The function $v(\alpha)$ can now explicitly be expressed as

$$
\begin{aligned}
v(\alpha) &= -\frac{5}{2}\frac{\left(\Delta'\right)^2}{\Delta}+\frac{\Delta''}{2}+\Delta'\left(\frac{x_1'}{x_1}+\frac{x_2'}{x_2}\right)= \\
&= -\frac{5}{2}\Delta^9\sin^2(2\alpha)\cos^2(2\alpha)+\frac{5}{2}\Delta^9\sin^2(2\alpha)\cos^2(2\alpha)+ \\
&\quad -\Delta^5\cos(4\alpha)-6\Delta^9\sin(2\alpha)\cos(2\alpha)\cot(2\alpha)= \\
&= -\Delta^5\cos(4\alpha)-6\Delta^9\cos^2(2\alpha)= \\
&= \{\text{Rewriting in terms of } \sin(2\alpha)\}= \\
&= -\Delta^5\left[1-2\sin^2(2\alpha)\right]-6\Delta^9\left[1-\sin^2(2\alpha)\right],
\end{aligned}
\tag{I.47}
$$

which directly yields the first five $\tilde{R}_{aa}$, $a=0,\dots,4$. Next up is $\tilde{R}_{55}$ which is calculated as

$$
\begin{aligned}
\tilde{R}_{55} &= 4\Delta''-2\frac{\left(\Delta'\right)^2}{\Delta}-2\Delta\left(\frac{x_1''}{x_1}+\frac{x_2''}{x_2}\right)= \\
&= 20\Delta^9\sin^2(2\alpha)\cos^2(2\alpha)-8\Delta^5\cos(4\alpha)-2\Delta^9\sin^2(2\alpha)\cos^2(2\alpha)+ \\
&\qquad +4\Delta^9\left[20+13\cos^2(2\alpha)\right]= \\
&= -8\Delta^5\cos(4\alpha)+\Delta^9\left[80+52\cos^2(2\alpha)+18\sin^2(2\alpha)\cos^2(2\alpha)\right]= \\
&= \{\text{Rewriting in terms of } \sin(2\alpha)\}= \\
&= -8\Delta^5\left[1-2\sin^2(2\alpha)\right]+\Delta^9\left[132-34\sin^2(2\alpha)-18\sin^4(2\alpha)\right].
\end{aligned}
\tag{I.48}
$$

The $\tilde{R}_{66} = \tilde{R}_{77}$ element is given by

$$
\begin{aligned}
\tilde{R}_{66} &= 4\Delta' \frac{x_1'}{x_1} + \Delta \left(\frac{1}{x_1^2} - \frac{x_1''}{x_1} - \frac{{x_1'}^2}{{x_1}^2} - 2\frac{x_1' x_2'}{x_1 x_2} \right) = \\
&= 4\Delta^5 \sin(2\alpha)\cos(2\alpha)\,\frac{\tan\alpha}{2+\cos(2\alpha)} + \\
&\quad + \Delta \Bigg(\frac{2+\cos(2\alpha)}{\cos^2\alpha} + \frac{5 - 2\cos(2\alpha)}{(2+\cos(2\alpha))^2} - \frac{\tan^2\alpha}{(2+\cos(2\alpha))^2} + 2\Delta^4 \Bigg) = \\
&= 2\Delta^5 + 4\Delta^5 \sin(2\alpha)\cos(2\alpha)\,\frac{\tan\alpha}{2+\cos(2\alpha)} + \\
&\quad + \frac{\Delta}{\cos^2\alpha\ \ (2+\cos(2\alpha))^2} \times \\
&\qquad \times \left((2+\cos(2\alpha))^3 + \cos^2\alpha\ \ (5 - 2\cos(2\alpha)) - \sin^2\alpha \right) = \\
&= 2\Delta^5 + 4\Delta^5 \sin(2\alpha)\cos(2\alpha)\,\frac{\tan\alpha}{2+\cos(2\alpha)} + \\
&\quad + 2\Delta \frac{7 + 4\cos^2\alpha + 4\cos^4\alpha}{(2+\cos(2\alpha))^2} = \\
&= 2\Delta^5 + \Delta^9 \Big(8\sin^2\alpha\ \ \cos(2\alpha)\,[2-\cos(2\alpha)] + \\
&\qquad\qquad + 2\left[7 + 4\cos^2\alpha + 4\cos^4\alpha\right][2-\cos(2\alpha)]^2 \Big) = \\
&= 2\Delta^5 + 2\Delta^9 \Big[51 - 4\cos^2\alpha - 32\cos^4\alpha - 16\cos^6\alpha + 16\cos^8\alpha\Big] = \\
&= 2\Delta^5 + 2\Delta^9 \Big[40 - 20\cos(2\alpha) - 8\cos^2(2\alpha) + \\
&\qquad\qquad + 2\cos^3(2\alpha) + \cos^4(2\alpha)\Big] = \\
&= 2\Delta^5 + 2\Delta^9 \left[33 + 6\sin^2(2\alpha) + \sin^4(2\alpha)\right] + \\
&\quad - 4\Delta^9 \cos(2\alpha)\left[9 + \sin^2(2\alpha)\right].
\end{aligned} \tag{I.49}
$$

For the last element, $\tilde{R}_{88} = \tilde{R}_{99}$, note that the expression for it in (I.35) is the same as for $\tilde{R}_{66}$, if x_1 is replaced by x_2. It is possible to get $\tilde{R}_{88}$ from $\tilde{R}_{66}$ by using a shift in α. Denoting this shift with Γ, it is defined as $\Gamma\alpha = \alpha + \frac{\pi}{2}$. It has the advantageous features

$$\Gamma \sin\alpha = \cos\alpha, \qquad \Gamma\cos\alpha = -\sin\alpha, \tag{I.50}$$

as well as

$$\Gamma\sin(2\alpha) = -\sin(2\alpha), \qquad \Gamma\cos(2\alpha) = -\cos(2\alpha). \tag{I.51}$$

For x_1 and x_2 this implies that

$$\Gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \qquad \Longrightarrow \qquad \Gamma \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} x_2^2 \\ x_1^2 \end{pmatrix}. \tag{I.52}$$

All sign differences appearing from a Γ transformation cancel in the expression for $\tilde{R}_{66}$ either via squares or fractions. Moreover, $\Gamma\Delta = \Delta$ and since Γ also commutes with ∂_α, it is true that

$$\begin{aligned}\tilde{R}_{88} = \Gamma\left(\tilde{R}_{66}\right) = \Gamma\Big(2\Delta^5 + 2\Delta^9 \left[33 + 6\sin^2(2\alpha) + \sin^4(2\alpha)\right] + \\ - 4\Delta^9 \cos(2\alpha)\left[9 + \sin^2(2\alpha)\right]\Big) = \\ = 2\Delta^5 + 2\Delta^9 \left[33 + 6\sin^2(2\alpha) + \sin^4(2\alpha)\right] + \\ + 4\Delta^9 \cos(2\alpha)\left[9 + \sin^2(2\alpha)\right].\end{aligned} \tag{I.53}$$

The sum between the two is then

$$\tilde{R}_{66} + \tilde{R}_{88} = 4\Delta^5 + 4\Delta^9 \left[33 + 6\sin^2(2\alpha) + \sin^4(2\alpha)\right]. \tag{I.54}$$

I.2 The Ricci scalar

The AdS radius for the vacuum is 1 which implies that $a = 1$. The Ricci scalar can then be calculated as

$$\begin{aligned}R = \eta^{ab}R_{ab} = 10v(\alpha) - 4\cdot 6\Delta + \tilde{R}_{55} + 2\left(\tilde{R}_{66} + \tilde{R}_{88}\right) = \\ = -10\Delta^5\left(1 - 2\sin^2(2\alpha)\right) - 60\Delta^9\left(1 - \sin^2(2\alpha)\right) - 24\Delta + \\ - 8\Delta^5\left(1 - 2\sin^2(2\alpha)\right) + \Delta^9\left[132 - 34\sin^2(2\alpha) - 18\sin^4(2\alpha)\right] + \\ + 8\Delta^5 + 8\Delta^9\left[33 + 6\sin^2(2\alpha) + \sin^4(2\alpha)\right] = \\ = -24\Delta - 2\Delta^5\left[5 - 18\sin^2(2\alpha)\right] + \\ + 2\Delta^9\left[168 + 37\sin^2(2\alpha) - 5\sin^4(2\alpha)\right].\end{aligned} \tag{I.55}$$

J

Einstein's equations

In this appendix it is shown that the AdS vacuum of section 3.2 fulfills Einstein's equations in type IIB string theory. The deformation parameters are set to $\chi_i = 0$. Using that the axion VEV is $\chi = 0$, Einstein's equations read

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = &- \left[\frac{1}{4}\partial_\rho\phi\partial^\rho\phi + \frac{1}{24}\left(e^{-\phi}H^2 + e^{\phi}F^2\right)\right]g_{\mu\nu} + \\ &+ \frac{1}{4}\left(e^{-\phi}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + e^{\phi}F_{\mu\rho\sigma}F_\nu{}^{\rho\sigma}\right) + \\ &+ \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4\cdot 4!}\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega} \end{aligned} \tag{J.1}$$

First off, the term on the right hand side that is proportional to the metric can be evaluated. By using

$$e^{-\phi}\partial_\alpha e^{\phi} = \frac{8\sin(2\alpha)}{7-\cos(4\alpha)} = 4\Delta^4\sin(2\alpha)\,, \tag{J.2}$$

which was shown in (H.9), the dilaton kinetic term can be written as

$$\begin{aligned} \frac{1}{4}\partial_\rho\phi\partial^\rho\phi &= \frac{1}{4}\left(e^{-\phi}\partial_\rho e^{\phi}\right)\left(e^{-\phi}\partial_\sigma e^{\phi}\right)g^{\rho\sigma} = \\ &= \frac{1}{4}\left(e^{-\phi}\partial_\eta e^{\phi}\right)\left(e^{-\phi}\partial_\eta e^{\phi}\right)g^{\eta\eta} + \frac{1}{4}\left(e^{-\phi}\partial_\alpha e^{\phi}\right)\left(e^{-\phi}\partial_\alpha e^{\phi}\right)g^{\alpha\alpha} = \\ &= \frac{1}{4}\Delta\left[4 + \left(4\Delta^4\sin(2\alpha)\right)^2\right] = \\ &= \Delta\left[1 + 4\Delta^8\sin^2(2\alpha)\right] \end{aligned} \tag{J.3}$$

The other term is

$$\begin{aligned} \frac{1}{24}\left(e^{-\phi}H^2 + e^{\phi}F^2\right) = &\frac{1}{24}24\Delta^3\left(19-\cos(4\alpha)\right)\left[\frac{\Delta^2}{2-\cos(2\alpha)} + \right. \\ &\left. + \frac{1}{\Delta^2\left(2+\cos(2\alpha)\right)^2\left(2-\cos(2\alpha)\right)}\right] = \\ &= \Delta^5\left(19-\cos(4\alpha)\right)\left[\frac{1}{2+\cos(2\alpha)} + \frac{1}{2-\cos(2\alpha)}\right] = \\ &= 4\Delta^9\left(19-\cos(4\alpha)\right) = \\ &= \{\text{Rewrite in terms of } \sin(2\alpha)\} = \\ &= 8\Delta^9\left[9+\sin^2(2\alpha)\right]. \end{aligned} \tag{J.4}$$

This yields that

$$\frac{1}{4}\partial_\rho\phi\partial^\rho\phi+\frac{1}{24}\left(e^{-\phi}H^2+e^\phi F^2\right)=\Delta+4\Delta^9\left[18+2\sin^2(2\alpha)+\sin^2(2\alpha)\right]=\\=\Delta+12\Delta^9\left[6+\sin^2(2\alpha)\right],\tag{J.5}$$

and Einstein's equations read

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=&-\left(\Delta+12\Delta^9\left[6+\sin^2(2\alpha)\right]\right)g_{\mu\nu}+\\&+\frac{1}{4}\left(e^{-\phi}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}+e^\phi F_{\mu\rho\sigma}F_\nu{}^{\rho\sigma}\right)+\\&+\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{1}{4\cdot4!}\tilde{F}_{\mu\rho\sigma\xi\omega}\tilde{F}_\nu{}^{\rho\sigma\xi\omega}\end{aligned}\tag{J.6}$$

The only off-diagonal case where the right hand side is not trivially zero is when $\mu\nu=\eta\alpha$. This case is dealt with first, whereafter the diagonal cases are treated.

J.1 The case $\mu\nu=\eta\alpha$

Since the metric and Ricci tensor are diagonal, the left hand side is zero. The term proportional to the metric in the right hand side is also zero, so the remaining condition is

$$0=\underbrace{\frac{1}{4}\left(e^{-\phi}H_{\eta\rho\sigma}H_\alpha{}^{\rho\sigma}+e^\phi F_{\eta\rho\sigma}F_\alpha{}^{\rho\sigma}\right)}_{I}+\frac{1}{2}\partial_\eta\phi\partial_\alpha\phi+\frac{1}{4\cdot4!}\tilde{F}_{\eta\rho\sigma\xi\omega}\tilde{F}_\alpha{}^{\rho\sigma\xi\omega}.\tag{J.7}$$

The first term is evaluated as

$$\begin{aligned}I=&\frac{1}{2}e^{-\phi}H_{\eta\theta_1\varphi_1}H_{\alpha\theta_1\varphi_1}g^{\theta_1\theta_1}g^{\varphi_1\varphi_1}+\frac{1}{2}e^\phi F_{\eta\theta_2\varphi_2}F_{\alpha\theta_2\varphi_2}g^{\theta_2\theta_2}g^{\varphi_2\varphi_2}=\\=&\,2e^{-\phi}e^{-2\eta}\sin(2\alpha)\frac{4+\cos(2\alpha)}{2+\cos(2\alpha)}x^4\sin^2\theta_1\frac{\Delta^2}{x^4\sin^2\theta_1}+\\&+2e^\phi e^{2\eta}\sin(2\alpha)\frac{4-\cos(2\alpha)}{2-\cos(2\alpha)}y^4\sin^2\theta_2\frac{\Delta^2}{y^4\sin^2\theta_2}=\\=&\,2\sin(2\alpha)\left(\frac{4+\cos(2\alpha)}{4-\cos^2(2\alpha)}+\Delta^4\left(4-\cos(2\alpha)\right)\right)=\\=&\,16\Delta^4\sin(2\alpha),\end{aligned}\tag{J.8}$$

the second term becomes

$$\frac{1}{2}\partial_\eta\phi\partial_\alpha\phi=\frac{1}{2}\left(e^{-\phi}\partial_\eta e^\phi\right)\left(e^{-\phi}\partial_\alpha e^\phi\right)=-4\Delta^4\sin(2\alpha),\tag{J.9}$$

and the third term is

$$\begin{aligned}\frac{1}{4\cdot 4!}\tilde{F}_{\eta\rho\sigma\xi\omega}\tilde{F}_{\alpha}{}^{\rho\sigma\xi\omega} &= \frac{1}{4}\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}^{\alpha\theta_1\varphi_1\theta_2\varphi_2}g_{\alpha\alpha} + \frac{1}{4}\tilde{F}_{\eta txyz}\tilde{F}^{\alpha txyz}g_{\alpha\alpha} = \\ &= \frac{1}{4}g_{\alpha\alpha}\Big(\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\epsilon^{\alpha\theta_1\varphi_1\theta_2\varphi_2 txyz\eta}\tilde{F}_{txyz\eta}+ \\ &\qquad + \tilde{F}_{txyz\eta}\epsilon^{txyz\alpha\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\Big) = \\ &= -\frac{1}{4}\frac{2}{\Delta\ e}\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{txyz\eta} = \\ &= -\frac{3}{4}\frac{1}{\Delta\ e}\Delta^4\tilde{e}\sin^3(2\alpha)\ \sin\theta_1\ \sin\theta_2 = \\ &= -12\Delta^4\sin(2\alpha)\,.\end{aligned} \tag{J.10}$$

The three terms of the right hand side are then added

$$\begin{aligned}\textbf{r.h.s.} &= \frac{1}{4}\left(e^{-\phi}H_{\eta\rho\sigma}H_{\alpha}{}^{\rho\sigma} + e^{\phi}F_{\eta\rho\sigma}F_{\alpha}{}^{\rho\sigma}\right) + \\ &\qquad + \frac{1}{2}\partial_\eta\phi\partial_\alpha\phi + \frac{1}{4\cdot 4!}\tilde{F}_{\eta\rho\sigma\xi\omega}\tilde{F}_{\alpha}{}^{\rho\sigma\xi\omega} = \\ &= (16-4-12)\,\Delta^4\sin(2\alpha) = 0,\end{aligned} \tag{J.11}$$

and the case where $\mu\nu = \eta\alpha$ is thus satisfied.

J.1.1 The case $\mu\nu = ii$ with $i = t,\ x,\ y,\ z$

The first set of diagoal cases is when $\mu\nu = ii$, where $i = t,\ x,\ y,\ z$. The right hand side then reads

$$\textbf{r.h.s.} = -\left(\Delta + 12\Delta^9\left[6+\sin^2(2\alpha)\right]\right)g_{ii} + \frac{1}{4\cdot 4!}\tilde{F}_{i\rho\sigma\xi\omega}\tilde{F}_{i}{}^{\rho\sigma\xi\omega}. \tag{J.12}$$

The second term is evaluated as

$$\begin{aligned}\frac{1}{4\cdot 4!}\tilde{F}_{i\rho\sigma\xi\omega}\tilde{F}_{i}{}^{\rho\sigma\xi\omega} &= \frac{1}{4}\left(\tilde{F}_{txyz\eta}\tilde{F}^{txyz\eta} + \tilde{F}_{txyz\alpha}\tilde{F}^{txyz\alpha}\right)g_{ii} = \\ &= \frac{1}{4}\epsilon^{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2}\Big(\tilde{F}_{txyz\eta}\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}+ \\ &\qquad - \tilde{F}_{txyz\alpha}\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\Big)g_{ii} = \\ &= -\frac{1}{4}\frac{\tilde{e}}{e}\Delta^4\sin^2(2\alpha)\sin\theta_1\ \sin\theta_2\ \left(\frac{9}{4}+\sin^2(2\alpha)\right)g_{ii} = \\ &= -\Delta^5\left[9+4\sin^2(2\alpha)\right]g_{ii},\end{aligned} \tag{J.13}$$

which yields that

$$\begin{aligned}\textbf{r.h.s.} &= -\left(\Delta + 12\Delta^9\left[6+\sin^2(2\alpha)\right]\right)g_{ii} - \Delta^5\left(9+4\sin^2(2\alpha)\right)g_{ii} = \\ &= -\left(\Delta + \Delta^5\left[9+4\sin^2(2\alpha)\right] + 12\Delta^9\left[6+\sin^2(2\alpha)\right]\right)g_{ii}.\end{aligned} \tag{J.14}$$

Next, the left hand side is

$$\textbf{l.h.s.} = R_{ii} - \frac{1}{2}Rg_{ii} = e_i{}^a e_i{}^a R_{aa} - \frac{1}{2}Rg_{ii}, \tag{J.15}$$

where $\left(e_i{}^a\right)^2 = \eta_{aa} g_{ii}$ for $i = t,\, x,\, y,\, z$. The left hand side becomes

$$\begin{aligned}
\mathbf{l.h.s.} &= \left(\eta_{aa} R_{aa} - \frac{1}{2} R\right) g_{ii} = \\
&= \Big(v\left(\alpha\right) - 6\Delta + 12\Delta + \Delta^5 \left(5 - 18\sin^2\left(2\alpha\right)\right) + \\
&\qquad - \Delta^9 \left(168 + 37\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right)\Big) g_{ii} = \\
&= \Big(6\Delta + 4\Delta^5 \left[1 - 4\sin^2\left(2\alpha\right)\right] + \\
&\qquad - \Delta^9 \left[174 + 31\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right]\Big) g_{ii}.
\end{aligned} \tag{J.16}$$

It is now possible to check if **l.h.s.** $\stackrel{?}{=}$ **r.h.s.**. The metric factor g_{ii} cancels and the condition reads

$$\begin{aligned}
&6\Delta + 4\Delta^5 \left[1 - 4\sin^2\left(2\alpha\right)\right] - \Delta^9 \left[174 + 31\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right] \stackrel{?}{=} \\
&\quad \stackrel{?}{=} -\Delta - \Delta^5 \left[9 + 4\sin^2\left(2\alpha\right)\right] - 12\Delta^9 \left[6 + \sin^2\left(2\alpha\right)\right],
\end{aligned} \tag{J.17}$$

which can be rewritten as

$$\begin{aligned}
0 &\stackrel{?}{=} 7\Delta + \Delta^5 \left[13 - 12\sin^2\left(2\alpha\right)\right] - \Delta^9 \left[102 + 19\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right] = \\
&= \Delta^9 \Big[7\left(4 - \cos^2\left(2\alpha\right)\right)^2 + \left(13 - 12\sin^2\left(2\alpha\right)\right)\left(4 - \cos^2\left(2\alpha\right)\right) + \\
&\qquad - \left(102 + 19\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right)\Big] = \\
&= \Delta^9 \Big[7\left(3 + \sin^2\left(2\alpha\right)\right)^2 + \left(13 - 12\sin^2\left(2\alpha\right)\right)\left(3 + \sin^2\left(2\alpha\right)\right) + \\
&\qquad - \left(102 + 19\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right)\Big] = \\
&= \Delta^9 \Big[\left(7\cdot 9 + 3\cdot 13 - 102\right) + \left(7\cdot 6 + 13 - 3\cdot 12 - 19\right)\sin^2\left(2\alpha\right) + \\
&\qquad + \left(7 - 12 + 5\right)\sin^4\left(2\alpha\right)\Big] = \\
&= \Delta^9 \Big[0 + 0\cdot\sin^2\left(2\alpha\right) + 0\cdot\sin^4\left(2\alpha\right)\Big] = \\
&= 0.
\end{aligned} \tag{J.18}$$

Einstein's equations are thus satisfied for $\mu\nu = tt,\, xx,\, yy,\, zz$.

J.2 The case $\mu\nu = \eta\eta$

The next case is $\mu\nu = \eta\eta$ and the right hand side is

$$\begin{aligned}
\mathbf{r.h.s.} = &- \left(\Delta + 12\Delta^9 \left[6 + \sin^2\left(2\alpha\right)\right]\right) g_{\eta\eta} + \\
&+ \frac{1}{4}\left(e^{-\phi} H_{\eta\rho\sigma} H_\eta{}^{\rho\sigma} + e^{\phi} F_{\eta\rho\sigma} F_\eta{}^{\rho\sigma}\right) + \\
&+ \frac{1}{2}\partial_\eta\phi\partial_\eta\phi + \frac{1}{4\cdot 4!}\tilde{F}_{\eta\rho\sigma\xi\omega}\tilde{F}_\eta{}^{\rho\sigma\xi\omega}.
\end{aligned} \tag{J.19}$$

The second term, $I = \frac{1}{4}\left(e^{-\phi}H_{\eta\rho\sigma}H_{\eta}{}^{\rho\sigma} + e^{\phi}F_{\eta\rho\sigma}F_{\eta}{}^{\rho\sigma}\right)$, becomes

$$\begin{aligned}
I &= \frac{1}{2}\left(e^{-\phi}H_{\eta\theta_1\varphi_1}H^{\eta\theta_1\varphi_1} + e^{\phi}F_{\eta\theta_2\varphi_2}F^{\eta\theta_1\varphi_1}\right)g_{\eta\eta} = \\
&= \frac{1}{2}\left(e^{-\phi}H^2_{\eta\theta_1\varphi_1}\frac{\Delta^2}{x_1^4\sin^2\theta_1} + e^{\phi}F^2_{\eta\theta_2\varphi_2}\frac{\Delta^2}{x_2^4\sin^2\theta_2}\right) = \\
&= 4\left(\frac{\cos^2\alpha}{2-\cos(2\alpha)} + \Delta^4\sin^2\alpha\ \left[2-\cos(2\alpha)\right]\right) = \\
&= 4\Delta^4\left(\cos^2\alpha\ \left[2+\cos(2\alpha)\right] + \sin^2\alpha\ \left[2-\cos(2\alpha)\right]\right) = \\
&= 4\Delta^4\left[2+\cos^2(2\alpha)\right] = 4\Delta^5\left[3-\sin^2(2\alpha)\right]g_{\eta\eta}.
\end{aligned} \tag{J.20}$$

The two last terms are evaluated as

$$\frac{1}{2}\partial_\eta\phi\partial_\eta\phi = \frac{1}{2}\left(e^{-\phi}\partial_\eta e^{\phi}\right)^2 = 2 = 2g^{\eta\eta}g_{\eta\eta} = 2\Delta g_{\eta\eta}, \tag{J.21}$$

and

$$\begin{aligned}
\frac{1}{4\cdot 4!}\tilde{F}_{\eta\rho\sigma\xi\omega}\tilde{F}_{\eta}{}^{\rho\sigma\xi\omega} &= \frac{1}{4}\left(\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}^{\eta\theta_1\varphi_1\theta_2\varphi_2} + \tilde{F}_{txyz\eta}\tilde{F}^{txyz\eta}\right)g_{\eta\eta} = \\
&= \frac{1}{4}\epsilon^{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2}\Big(\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{txyz\alpha} + \\
&\qquad + \tilde{F}_{txyz\eta}\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}\Big)g_{\eta\eta} = \\
&= \frac{1}{4}\frac{\tilde{e}}{e}\Delta^4\sin^2(2\alpha)\sin\theta_1\ \sin\theta_2\ \left(-\frac{9}{4}+\sin^2(2\alpha)\right)g_{\eta\eta} = \\
&= -\Delta^5\left[9-4\sin^2(2\alpha)\right]g_{\eta\eta}.
\end{aligned} \tag{J.22}$$

The right hand side thus becomes

$$\begin{aligned}
\textbf{r.h.s.} &= \Big(-\Delta - 12\Delta^9\left[6+\sin^2(2\alpha)\right] + 4\Delta^5\left[3-\sin^2(2\alpha)\right] + \\
&\qquad + 2\Delta - \Delta^5\left[9-4\sin^2(2\alpha)\right]\Big)g_{\eta\eta} = \\
&= \left(\Delta + 3\Delta^5 - 12\Delta^9\left[6+\sin^2(2\alpha)\right]\right)g_{\eta\eta}.
\end{aligned} \tag{J.23}$$

The left hand side is

$$\begin{aligned}
\textbf{l.h.s.} &= R_{\eta\eta} - \frac{1}{2}Rg_{\eta\eta} = e_\eta{}^4 e_\eta{}^4 R_{44} - \frac{1}{2}Rg_{\eta\eta} = \left(R_{44}-\frac{1}{2}R\right)g_{\eta\eta} = \\
&= \left(v(\alpha) - \frac{1}{2}R\right)g_{\eta\eta} = \\
&= \Big(-\Delta^5\left[1-2\sin^2(2\alpha)\right] - 6\Delta^9\left[1-\sin^2(2\alpha)\right] + 12\Delta + \\
&\qquad + \Delta^5\left[5-18\sin^2(2\alpha)\right] + \\
&\qquad - \Delta^9\left[168+37\sin^2(2\alpha)-5\sin^4(2\alpha)\right]\Big)g_{\eta\eta} = \\
&= \Big(12\Delta + 4\Delta^5\left[1-4\sin^2(2a)\right] + \\
&\qquad - \Delta^9\left[174+31\sin^2(2\alpha)-5\sin^4(2\alpha)\right]\Big)g_{\eta\eta}
\end{aligned} \tag{J.24}$$

The equation is fulfilled if **l.h.s.** $\stackrel{?}{=}$ **r.h.s.** which reads

$$\begin{aligned}12\Delta + 4\Delta^5\left[1 - 4\sin^2(2a)\right] - \Delta^9\left[174 + 31\sin^2(2\alpha) - 5\sin^4(2\alpha)\right] \stackrel{?}{=} \\ \stackrel{?}{=} \Delta + 3\Delta^5 - 12\Delta^9\left[6 + \sin^2(2\alpha)\right].\end{aligned} \tag{J.25}$$

Combining all the terms on one side yields that

$$\begin{aligned}0 &\stackrel{?}{=} 11\Delta + \Delta^5\left[1 - 16\sin^2(2a)\right] - \Delta^9\left[102 + 19\sin^2(2\alpha) - 5\sin^4(2\alpha)\right] = \\ &= \Delta^9\Big(11\left[4 - \cos^2(2\alpha)\right]^2 + \left[1 - 16\sin^2(2a)\right]\left[4 - \cos^2(2\alpha)\right] + \\ &\qquad - 102 - 19\sin^2(2\alpha) + 5\sin^4(2\alpha)\Big) = \\ &= \Delta^9\Big(11\left[3 + \sin^2(2\alpha)\right]^2 + \left[1 - 16\sin^2(2a)\right]\left[3 + \sin^2(2\alpha)\right] + \\ &\qquad - 102 - 19\sin^2(2\alpha) + 5\sin^4(2\alpha)\Big) = \\ &= \Delta^9\Big([9\cdot 11 + 3 - 102] + [6\cdot 11 + 1 - 3\cdot 16 - 19]\sin^2(2\alpha) + \\ &\qquad + [11 - 16 + 5]\sin^4(2\alpha)\Big) = \\ &= \Delta^9\left(0 + 0\cdot\sin^2(2\alpha) + 0\cdot\sin^4(2\alpha)\right) = \\ &= 0,\end{aligned} \tag{J.26}$$

and Einstein's equations are fulfilled when $\mu\nu = \eta\eta$.

J.3 The case $\mu\nu = \alpha\alpha$

For the $\mu\nu = \alpha\alpha$ case, the right hand side reads

$$\begin{aligned}\textbf{r.h.s.} = &- \left(\Delta + 12\Delta^9\left[6 + \sin^2(2\alpha)\right]\right) g_{\alpha\alpha} + \\ &+ \underbrace{\frac{1}{4}\left(e^{-\phi} H_{\alpha\rho\sigma} H_{\alpha}{}^{\rho\sigma} + e^{\phi} F_{\alpha\rho\sigma} F_{\alpha}{}^{\rho\sigma}\right)}_{I} + \\ &+ \frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi + \frac{1}{4\cdot 4!}\tilde{F}_{\alpha\rho\sigma\xi\omega}\tilde{F}_{\alpha}{}^{\rho\sigma\xi\omega}.\end{aligned} \tag{J.27}$$

The last two terms are evaluated as

$$\frac{1}{2}\partial_\alpha\phi\partial_\alpha\phi = \frac{1}{2}\left(e^{-\phi}\partial_\alpha e^{\phi}\right)^2 = 8\Delta^8\sin^2(2\alpha) = 8\Delta^9\sin^2(2\alpha)\, g_{\alpha\alpha}, \tag{J.28}$$

and

$$\begin{aligned}
\frac{1}{4\cdot 4!}\tilde{F}_{\alpha\rho\sigma\xi\omega}\tilde{F}_{\alpha}{}^{\rho\sigma\xi\omega} &= \frac{1}{4}\left(\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}^{\alpha\theta_1\varphi_1\theta_2\varphi_2} + \tilde{F}_{\alpha txyz}\tilde{F}^{\alpha txyz}\right) g_{\alpha\alpha} = \\
&= \frac{1}{4}\Big(\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}\epsilon^{\alpha\theta_1\varphi_1\theta_2\varphi_2 txyz\eta}\tilde{F}_{txyz\eta} + \\
&\qquad + \tilde{F}_{txyz\alpha}\epsilon^{txyz\alpha\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\Big) g_{\alpha\alpha} = \\
&= -\frac{1}{4}\frac{1}{e}\left(\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{txyz\eta} + \tilde{F}_{txyz\alpha}\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\right) g_{\alpha\alpha} = \\
&= -\frac{1}{4}\frac{\tilde{e}}{e}\Delta^4 \sin\theta_1\ \sin\theta_2\ \left(-\frac{9}{4}\sin^2(2\alpha) + \sin^4(2\alpha)\right) g_{\alpha\alpha} = \\
&= \Delta^5\left[9 - 4\sin^2(2\alpha)\right] g_{\alpha\alpha}.
\end{aligned} \tag{J.29}$$

The second term becomes

$$\begin{aligned}
I &= \frac{1}{2}\left(e^{-\phi}H_{\alpha\theta_1\varphi_1}H^{\alpha\theta_1\varphi_1} + e^{\phi}F_{\alpha\theta_2\varphi_2}F^{\alpha\theta_2\varphi_2}\right) g_{\alpha\alpha} = \\
&= \frac{1}{2}\left(e^{-\phi}H^2_{\alpha\theta_1\varphi_1}\frac{\Delta^2}{x_1^4\sin^2\theta_1} + e^{\phi}F^2_{\alpha\theta_2\varphi_2}\frac{\Delta^2}{x_2^4\sin^2\theta_2}\right) = \\
&= \frac{\sin^2(2\alpha)\cos^2\alpha}{\Delta^2(2-\cos(2\alpha))}\frac{(4+\cos(2\alpha))^2}{(2+\cos(2\alpha))^4}\frac{\Delta^2(2+\cos(2\alpha))^2}{\cos^4\alpha} + \\
&\quad + \Delta^2(2-\cos(2\alpha))\sin^2(2\alpha)\sin^2\alpha\frac{(4-\cos(2\alpha))^2}{(2-\cos(2\alpha))^4}\frac{\Delta^2(2-\cos(2\alpha))^2}{\sin^4\alpha} = \\
&= \Delta^4\sin^2(2\alpha)\left(\frac{(4+\cos(2\alpha))^2}{\cos^2\alpha\ \ (2+\cos(2\alpha))} + \frac{(4-\cos(2\alpha))^2}{\sin^2\alpha\ \ (2-\cos(2\alpha))}\right) = \\
&= \Delta^8\frac{\sin^2(2\alpha)}{\sin^2\alpha\ \cos^2\alpha}\Big(\sin^2\alpha\ \ [2-\cos(2\alpha)]\,[4+\cos(2\alpha)]^2 + \\
&\qquad + \cos^2\alpha\ \ [2+\cos(2\alpha)]\,[4-\cos(2\alpha)]^2\Big) = \\
&= 4\Delta^8\Big(\sin^2\alpha\ \left[32 - 6\cos^2(2\alpha) - \cos^3(2\alpha)\right] + \\
&\qquad + \cos^2\alpha\ \left[32 - 6\cos^2(2\alpha) + \cos^3(2\alpha)\right]\Big) = \\
&= 4\Delta^8\Big(32 - 6\cos^2(2\alpha) + \cos^4(2\alpha)\Big) = \\
&= \{\text{Rewrite in terms of } \sin(2\alpha)\} = \\
&= 4\Delta^9\Big[27 + 4\sin^2(2\alpha) + \sin^4(2\alpha)\Big] g_{\alpha\alpha}.
\end{aligned} \tag{J.30}$$

In total, the right hand side is

$$\begin{aligned}
\textbf{r.h.s.} &= \Big(-\Delta - 12\Delta^9\left[6 + \sin^2(2\alpha)\right] + 4\Delta^9\left[27 + 4\sin^2(2\alpha) + \sin^4(2\alpha)\right] + \\
&\qquad + 8\Delta^9\sin^2(2\alpha) + \Delta^5\left[9 - 4\sin^2(2\alpha)\right]\Big) g_{\alpha\alpha} = \\
&= \left(-\Delta + \Delta^5\left[9 - 4\sin^2(2\alpha)\right] + 4\Delta^9\left[9 + 3\sin^2(2\alpha) + \sin^4(2\alpha)\right]\right) g_{\alpha\alpha}.
\end{aligned} \tag{J.31}$$

The left hand side is

$$\begin{aligned}
\mathbf{l.h.s.} &= R_{\alpha\alpha} - \frac{1}{2}Rg_{\alpha\alpha} = e_{\alpha}{}^{5}e_{\alpha}{}^{5}R_{55} - \frac{1}{2}Rg_{\alpha\alpha} = \left(R_{55} - \frac{1}{2}R\right)g_{\alpha\alpha} = \\
&= \left(v\left(\alpha\right) + \tilde{R}_{55} - \frac{1}{2}R\right)g_{\alpha\alpha} = \\
&= \Big(- \Delta^5\left[1 - 2\sin^2\left(2\alpha\right)\right] - 6\Delta^9\left[1 - \sin^2\left(2\alpha\right)\right] + \\
&\quad - 8\Delta^5\left[1 - 2\sin^2\left(2\alpha\right)\right] + \Delta^9\left[132 - 34\sin^2\left(2\alpha\right) - 18\sin^4\left(2\alpha\right)\right] + \\
&\quad + 12\Delta + \Delta^5\left[5 - 18\sin^2\left(2\alpha\right)\right] + \\
&\quad - \Delta^9\left[168 + 37\sin^2\left(2\alpha\right) - 5\sin^4\left(2\alpha\right)\right]\Big)g_{\alpha\alpha} = \\
&= \left(12\Delta - 4\Delta^5 - \Delta^9\left[42 + 65\sin^2\left(2\alpha\right) + 13\sin^4\left(2\alpha\right)\right]\right)g_{\alpha\alpha},
\end{aligned} \tag{J.32}$$

and the condition $\mathbf{l.h.s.} \stackrel{?}{=} \mathbf{r.h.s.}$ thus reads

$$\begin{aligned}
&12\Delta - 4\Delta^5 - \Delta^9\left[42 + 65\sin^2\left(2\alpha\right) + 13\sin^4\left(2\alpha\right)\right] \stackrel{?}{=} \\
&\quad \stackrel{?}{=} -\Delta + \Delta^5\left[9 - 4\sin^2\left(2\alpha\right)\right] + 4\Delta^9\left[9 + 3\sin^2\left(2\alpha\right) + \sin^4\left(2\alpha\right)\right],
\end{aligned} \tag{J.33}$$

which can be rewritten as

$$\begin{aligned}
0 &\stackrel{?}{=} 13\Delta - \Delta^5\left(13 - 4\sin^2\left(2\alpha\right)\right) - \Delta^9\left[78 + 77\sin^2\left(2\alpha\right) + 17\sin^4\left(2\alpha\right)\right] = \\
&= \Delta^9\Big(13\left[4 - \cos^2\left(2\alpha\right)\right]^2 - \left[13 - 4\sin^2\left(2\alpha\right)\right]\left[4 - \cos^2\left(2\alpha\right)\right] + \\
&\qquad - 78 - 77\sin^2\left(2\alpha\right) - 17\sin^4\left(2\alpha\right)\Big) = \\
&= \Delta^9\Big(13\left[3 + \sin^2\left(2\alpha\right)\right]^2 - \left[13 - 4\sin^2\left(2\alpha\right)\right]\left[3 + \sin^2\left(2\alpha\right)\right] + \\
&\qquad - 78 - 77\sin^2\left(2\alpha\right) - 17\sin^4\left(2\alpha\right)\Big) = \\
&= \Delta^9\Big(\left[9\cdot 13 - 3\cdot 13 - 78\right] + \left[6\cdot 13 - 13 + 12 - 77\right]\sin^2\left(2\alpha\right) + \\
&\qquad + \left[13 + 4 - 17\right]\sin^4\left(2\alpha\right)\Big) = \\
&= \Delta^9\left(0 + 0\cdot\sin^2\left(2\alpha\right) + 0\cdot\sin^4\left(2\alpha\right)\right) = \\
&= 0.
\end{aligned} \tag{J.34}$$

Einstein's equations are fulfilled when $\mu\nu = \alpha\alpha$.

J.4 Case: $\mu\nu = ii$ for $i = \theta_1,\ \varphi_1$

Since $R_{66} = R_{77}$, the cases $\mu\nu = \theta_1\theta_1$ and $\mu\nu = \varphi_1\varphi_1$ can be evaluated simultaneously. The right hand side is

$$\begin{aligned}
\mathbf{r.h.s.} &= -\left(\Delta + 12\Delta^9\left[6 + \sin^2\left(2\alpha\right)\right]\right)g_{ii} + \\
&\quad + \frac{1}{4}e^{-\phi}H_{i\rho\sigma}H_{i}{}^{\rho\sigma} + \frac{1}{4\cdot 4!}\tilde{F}_{i\rho\sigma\xi\omega}\tilde{F}_{i}{}^{\rho\sigma\xi\omega}
\end{aligned} \tag{J.35}$$

Treating the terms one by one yields that

$$\begin{aligned}\frac{1}{4}e^{-\phi}H_{i\rho\sigma}H_i{}^{\rho\sigma} &= \frac{1}{2}e^{-\phi}\left[H_{\eta\theta_1\varphi_1}H^{\eta\theta_1\varphi_1}+H_{\alpha\theta_1\varphi_1}H^{\alpha\theta_1\varphi_1}\right]g_{ii} = \\ &= \frac{1}{2\cdot 3!}e^{-\phi}H^2 g_{ii} = 2\Delta\frac{19-\cos(4\alpha)}{(2-\cos(2\alpha))(2+\cos(2\alpha))^2} = \\ &= 2\Delta^5\frac{19-\cos(4\alpha)}{2+\cos(2\alpha)} = 2\Delta^9\left[2-\cos(2\alpha)\right]\left[19-\cos(4\alpha)\right] = \\ &= 4\Delta^9\left[2-\cos(2\alpha)\right]\left[10-\cos^2(2\alpha)\right] = \\ &= 4\Delta^9\left[20-10\cos(2\alpha)-2\cos^2(2\alpha)+\cos^3(2\alpha)\right] = \\ &= 8\Delta^9\left[9+\sin^2(2\alpha)\right]-4\Delta^9\cos(2\alpha)\left[9+\sin^2(2\alpha)\right],\end{aligned} \tag{J.36}$$

and

$$\begin{aligned}\frac{1}{4\cdot 4!}\tilde{F}_{i\rho\sigma\xi\omega}\tilde{F}_i{}^{\rho\sigma\xi\omega} &= \frac{1}{4}\Big(\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}^{\eta\theta_1\varphi_1\theta_2\varphi_2}+\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}^{\alpha\theta_1\varphi_1\theta_2\varphi_2}\Big)g_{ii} = \\ &= \frac{1}{4}\epsilon^{txyz\eta\alpha\theta_1\varphi_1\theta_2\varphi_2}\Big(\tilde{F}_{\eta\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{txyz\alpha}+ \\ &\qquad\qquad -\tilde{F}_{\alpha\theta_1\varphi_1\theta_2\varphi_2}\tilde{F}_{txyz\eta}\Big)g_{ii} = \\ &= \frac{1}{4}\frac{\tilde{e}}{e}\Delta^4\sin^2(2\alpha)\sin\theta_1\ \sin\theta_2\ \Big(\sin^2(2\alpha)+\frac{9}{4}\Big)g_{ii} = \\ &= \Delta^5\left[9+4\sin^2(2\alpha)\right]g_{ii}.\end{aligned} \tag{J.37}$$

The right hand side becomes

$$\begin{aligned}\mathbf{r.h.s.} &= \Big(-\Delta-12\Delta^9\left[6+\sin^2(2\alpha)\right]+8\Delta^9\left[9+\sin^2(2\alpha)\right]+ \\ &\quad -4\Delta^9\cos(2\alpha)\left[9+\sin^2(2\alpha)\right]+\Delta^5\left[9+4\sin^2(2\alpha)\right]\Big)g_{ii} = \\ &= \Big(-\Delta+\Delta^5\left[9+4\sin^2(2\alpha)\right]-4\Delta^9\sin^2(2\alpha)+ \\ &\quad -4\Delta^9\cos(2\alpha)\left[9+\sin^2(2\alpha)\right]\Big)g_{ii}.\end{aligned} \tag{J.38}$$

The left hand side is

$$\begin{aligned}\mathbf{l.h.s.} &= R_{ii}-\frac{1}{2}Rg_{ii} = e_i{}^a e_i{}^a R_{aa}-\frac{1}{2}Rg_{ii} = \Big(R_{aa}-\frac{1}{2}R\Big)g_{ii} = \\ &= \Big(\upsilon(\alpha)+\tilde{R}_{66}-\frac{1}{2}R\Big)g_{ii} = \\ &= \Big(-\Delta^5\left[1-2\sin^2(2\alpha)\right]-6\Delta^9\left[1-\sin^2(2\alpha)\right]+2\Delta^5+ \\ &\quad +2\Delta^9\left[33+6\sin^2(2\alpha)+\sin^4(2\alpha)\right]+ \\ &\quad -4\Delta^9\cos(2\alpha)\left[9+\sin^2(2\alpha)\right]+12\Delta+\Delta^5\left[5-18\sin^2(2\alpha)\right]+ \\ &\quad -\Delta^9\left[168+37\sin^2(2\alpha)-5\sin^4(2\alpha)\right]\Big)g_{ii} = \\ &= \Big(12\Delta+2\Delta^5\left[3-8\sin^2(2\alpha)\right]+ \\ &\quad -\Delta^9\left[108+19\sin^2(2\alpha)-7\sin^4(2\alpha)\right]+ \\ &\quad -4\Delta^9\cos(2\alpha)\left[9+\sin^2(2\alpha)\right]\Big)g_{ii}.\end{aligned} \tag{J.39}$$

Note that the term $-4\Delta^9 \cos(2\alpha)\,[9+\sin^2(2\alpha)]$ appears on both sides of the equation and thus cancels when checking that **l.h.s.** $\stackrel{?}{=}$ **r.h.s.**

$$\begin{aligned} 12\Delta + 2\Delta^5 \left[3 - 8\sin^2(2\alpha)\right] - \Delta^9 \left[108 + 19\sin^2(2\alpha) - 7\sin^4(2\alpha)\right] \stackrel{?}{=} \\ \stackrel{?}{=} -\Delta + \Delta^5 \left[9 + 4\sin^2(2\alpha)\right] - 4\Delta^9 \sin^2(2\alpha)\,. \end{aligned} \tag{J.40}$$

This can be rewritten as

$$\begin{aligned} 0 &\stackrel{?}{=} 13\Delta - \Delta^5 \left(3 + 20\sin^2(2\alpha)\right) - \Delta^9 \left[108 + 15\sin^2(2\alpha) - 7\sin^4(2\alpha)\right] = \\ &= \Delta^9 \Big(13\left[4 - \cos^2(2\alpha)\right]^2 - \left[3 + 20\sin^2(2\alpha)\right]\left[4 - \cos^2(2\alpha)\right] + \\ &\qquad - 108 - 15\sin^2(2\alpha) + 7\sin^4(2\alpha)\Big) = \\ &= \Delta^9 \Big(13\left[3 + \sin^2(2\alpha)\right]^2 - \left[3 + 20\sin^2(2\alpha)\right]\left[3 + \sin^2(2\alpha)\right] + \\ &\qquad - 108 - 15\sin^2(2\alpha) + 7\sin^4(2\alpha)\Big) = \\ &= \Delta^9 \Big([9\cdot 13 - 9 - 108] + [6\cdot 13 - 3 - 3\cdot 20 - 15]\sin^2(2\alpha) + \\ &\qquad + [13 - 20 + 7]\sin^4(2\alpha)\Big) = \\ &= \Delta^9 \Big(0 + 0\cdot\sin^2(2\alpha) + 0\cdot\sin^4(2\alpha)\Big) = \\ &= 0, \end{aligned} \tag{J.41}$$

which shows that Einstein's equations are fulfilled when $\mu\nu = \theta_1\theta_1,\ \varphi_1\varphi_1$.

J.5 The case $\mu\nu = ii$ for $i = \theta_2,\ \varphi_2$

The last case where $\mu\nu = ii$ for $i = \theta_2,\ \varphi_2$ is similar to the previous one, and the right hand side reads

$$\begin{aligned} \textbf{r.h.s.} = -\left(\Delta + 12\Delta^9\left[6 + \sin^2(2\alpha)\right]\right) g_{ii} + \\ + \frac{1}{4} e^{\phi} F_{i\rho\sigma} F_i{}^{\rho\sigma} + \frac{1}{4\cdot 4!} \tilde{F}_{i\rho\sigma\xi\omega} \tilde{F}_i{}^{\rho\sigma\xi\omega}. \end{aligned} \tag{J.42}$$

The last two terms are

$$\begin{aligned} \frac{1}{4} e^{\phi} F_{i\rho\sigma} F_i{}^{\rho\sigma} &= \frac{1}{2} e^{\phi} \left[F_{\eta\theta_2\varphi_2} F^{\eta\theta_2\varphi_2} + F_{\alpha\theta_2\varphi_2} F^{\alpha\theta_2\varphi_2}\right] g_{ii} = \\ &= \frac{1}{2\cdot 3!} e^{\phi} F^2 g_{ii} = 2\Delta^5 \frac{19 - \cos(4\alpha)}{(2 - \cos(2\alpha))} = \\ &= 2\Delta^9 \left[2 + \cos(2\alpha)\right]\left[19 - \cos(2\alpha)\right] = \\ &= 4\Delta^9 \left[2 + \cos(2\alpha)\right]\left[10 - \cos^2(2\alpha)\right] = \\ &= 4\Delta^9 \left[20 + 10\cos(2\alpha) - 2\cos^2(2\alpha) - \cos^3(2\alpha)\right] = \\ &= 8\Delta^9 \left[9 + \sin^2(2\alpha)\right] + 4\Delta^9 \cos(2\alpha)\left[9 + \sin^2(2\alpha)\right], \end{aligned} \tag{J.43}$$

and

$$\frac{1}{4\cdot 4!} \tilde{F}_{i\rho\sigma\xi\omega} \tilde{F}_i{}^{\rho\sigma\xi\omega} = \Delta^5 \left[9 + 4\sin^2(2\alpha)\right] g_{ii}, \tag{J.44}$$

so that the right hand side becomes

$$\begin{aligned}\textbf{r.h.s.} = \Big(&- \Delta - 12\Delta^9 \Big[6 + \sin^2(2\alpha)\Big] + 8\Delta^9 \Big[9 + \sin^2(2\alpha)\Big] + \\ &+ 4\Delta^9 \cos(2\alpha) \Big[9 + \sin^2(2\alpha)\Big] + \Delta^5 \Big[9 + 4\sin^2(2\alpha)\Big]\Big) g_{ii} = \\ = \Big(&- \Delta + \Delta^5 \Big[9 + 4\sin^2(2\alpha)\Big] - 4\Delta^9 \sin^2(2\alpha) + \\ &+ 4\Delta^9 \cos(2\alpha) \Big[9 + \sin^2(2\alpha)\Big]\Big) g_{ii}.\end{aligned} \tag{J.45}$$

The left hand side is

$$\begin{aligned}\textbf{l.h.s.} &= R_{ii} - \frac{1}{2} R g_{ii} = e_i{}^a e_i{}^a R_{aa} - \frac{1}{2} R g_{ii} = \Big(R_{aa} - \frac{1}{2} R\Big) g_{ii} = \\ &= \Big(v(\alpha) + \tilde{R}_{88} - \frac{1}{2} R\Big) g_{ii} = \\ &= \Big(- \Delta^5 \Big[1 - 2\sin^2(2\alpha)\Big] - 6\Delta^9 \Big[1 - \sin^2(2\alpha)\Big] + 2\Delta^5 + \\ &\quad + 2\Delta^9 \Big[33 + 6\sin^2(2\alpha) + \sin^4(2\alpha)\Big] + \\ &\quad + 4\Delta^9 \cos(2\alpha) \Big[9 + \sin^2(2\alpha)\Big] + 12\Delta + \Delta^5 \Big[5 - 18\sin^2(2\alpha)\Big] + \\ &\quad - \Delta^9 \Big[168 + 37\sin^2(2\alpha) - 5\sin^4(2\alpha)\Big]\Big) g_{ii} = \\ &= \Big(12\Delta + 2\Delta^5 \Big[3 - 8\sin^2(2\alpha)\Big] + \\ &\quad - \Delta^9 \Big[108 + 19\sin^2(2\alpha) - 7\sin^4(2\alpha)\Big] + \\ &\quad + 4\Delta^9 \cos(2\alpha) \Big[9 + \sin^2(2\alpha)\Big]\Big) g_{ii}.\end{aligned} \tag{J.46}$$

The equation is fulfilled if **l.h.s.** $\stackrel{?}{=}$ **r.h.s.** which reads

$$\begin{aligned}&12\Delta + 2\Delta^5 \Big[3 - 8\sin^2(2\alpha)\Big] - \Delta^9 \Big[108 + 19\sin^2(2\alpha) - 7\sin^4(2\alpha)\Big] \stackrel{?}{=} \\ &\quad \stackrel{?}{=} -\Delta + \Delta^5 \Big[9 + 4\sin^2(2\alpha)\Big] - 4\Delta^9 \sin^2(2\alpha),\end{aligned} \tag{J.47}$$

and can be rewritten as

$$\begin{aligned}0 &\stackrel{?}{=} 13\Delta - \Delta^5 \Big(3 + 20\sin^2(2\alpha)\Big) - \Delta^9 \Big[108 + 15\sin^2(2\alpha) - 7\sin^4(2\alpha)\Big] = \\ &= \Delta^9 \Big(13 \Big[4 - \cos^2(2\alpha)\Big]^2 - \Big[3 + 20\sin^2(2\alpha)\Big] \Big[4 - \cos^2(2\alpha)\Big] + \\ &\qquad - 108 - 15\sin^2(2\alpha) + 7\sin^4(2\alpha)\Big) = \\ &= \Delta^9 \Big(13 \Big[3 + \sin^2(2\alpha)\Big]^2 - \Big[3 + 20\sin^2(2\alpha)\Big] \Big[3 + \sin^2(2\alpha)\Big] + \\ &\qquad - 108 - 15\sin^2(2\alpha) + 7\sin^4(2\alpha)\Big) = \\ &= \Delta^9 \Big([9 \cdot 13 - 9 - 108] + [6 \cdot 13 - 3 - 3 \cdot 20 - 15] \sin^2(2\alpha) + \\ &\qquad + [13 - 20 + 7] \sin^4(2\alpha)\Big) = \\ &= \Delta^9 \Big(0 + 0 \cdot \sin^2(2\alpha) + 0 \cdot \sin^4(2\alpha)\Big) = \\ &= 0.\end{aligned} \tag{J.48}$$

As this was the last case, the AdS vacuum of section 3.2 satisfies the type IIB Einstein's equations.

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