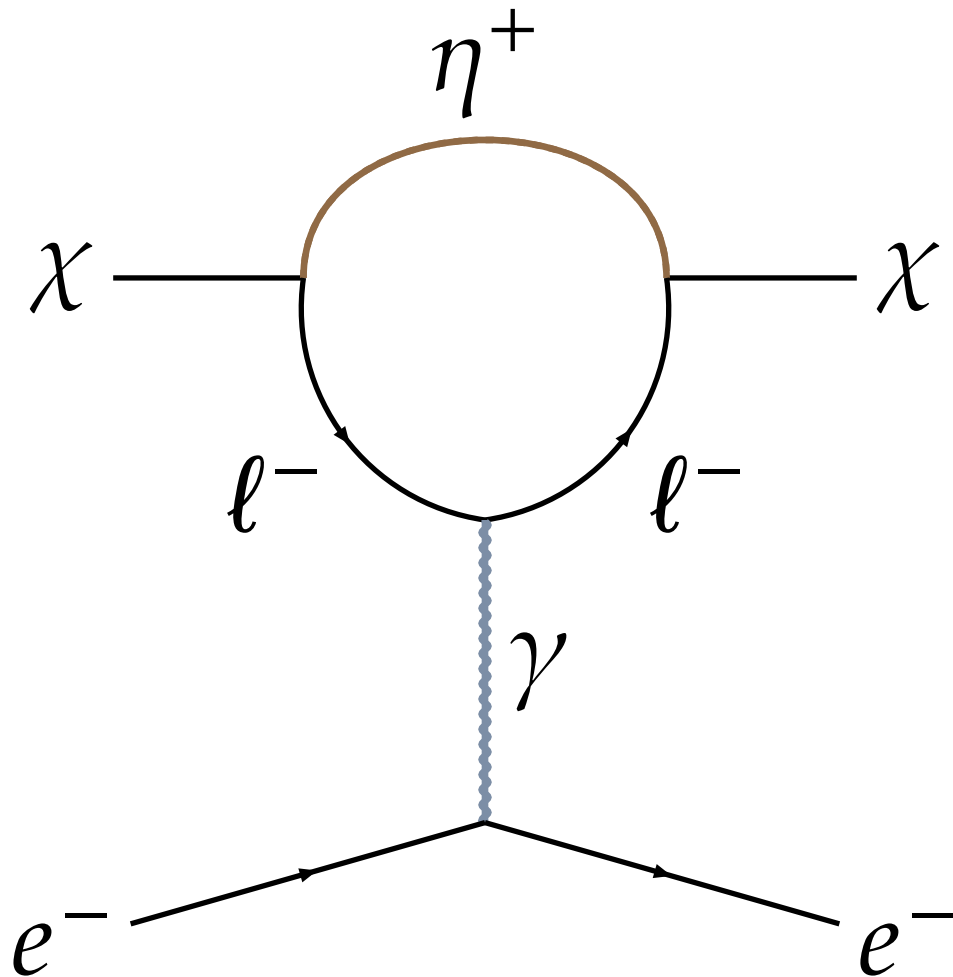




Direct Detection of Light Dark Matter in the Scotogenic Model

Master's thesis in Physics

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DEPARTMENT OF PHYSICS AND ASTRONOMY

Chalmers University of Technology

Gothenburg, Sweden 2026

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Direct Detection of Light Dark Matter in the Scotogenic Model

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CHALMERS
UNIVERSITY OF TECHNOLOGY

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Supervised by
Prof. Riccardo Catena

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Abstract

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The scotogenic model, also known as the radiative seesaw model, provides an elegant theoretical framework that simultaneously accounts for the generation of active neutrino masses and the existence of dark matter. Current realisations of the model typically focus on weakly interacting massive particle (WIMP) dark matter candidates with masses above the GeV scale. However, despite decades of direct detection searches, no conclusive evidence for WIMP dark matter has yet been found. One possible explanation for this null result is that dark matter may lie below the GeV scale, where conventional nuclear-recoil experiments lose sensitivity due to kinematic thresholds. This possibility motivates the study of sub-GeV realisations of the scotogenic model.

This thesis carries out that study for Majorana dark matter in the minimal scotogenic model. Assuming a τ -philic flavour structure, the leading-order electromagnetic interaction is a loop-induced anapole moment. Non-relativistic effective field theory then yields the inelastic scattering cross-section against bound atomic electrons in liquid xenon.

Confronting these rates with XENON1T data places upper limits on the anapole coupling, and identifies a steep loss of sensitivity below a dark matter mass $M_1 \approx 30$ MeV, driven by the quantisation of the ionisation signal. The pipeline reproduces the published XENON1T anapole constraint to within 3%. Enforcing the LEP bound on the scalar mediator mass ($m_{\eta^\pm} \gtrsim 90$ GeV), the thermal relic density requires non-perturbative couplings ($y > \sqrt{4\pi}$) below $M_1 \approx 480$ MeV, while XENON1T only excludes unphysical values ($y \gtrsim 265$).

Thermal freeze-out therefore survives only for $M_1 \gtrsim 480$ MeV, with the mediator confined to a 37 GeV window above the LEP bound and a coupling close to the perturbative ceiling. Even there the thermal target lies two orders of magnitude below the XENON1T reach in coupling. Below that window the model requires non-thermal freeze-in production, suppressing the scattering rate by roughly 45 orders of magnitude. In neither case is direct detection the discovery channel, and verification shifts to collider searches for the long-lived charged mediator.

Keywords: scotogenic model, sub-GeV dark matter, Majorana fermion, anapole moment, electron recoil, direct detection, NREFT, freeze-out, freeze-in, radiative neutrino mass.

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Ole Ellermeier, Gothenburg, June 2026

Use of AI Tools

Generative AI tools, principally Google Gemini and Anthropic Claude, were used in the preparation of this thesis. They assisted in writing and correcting the analysis code, checking derivations, researching the literature, and revising the text. Responsibility for all content rests entirely with the author. All derivations, numerical values and cited sources were verified against the primary literature, and the complete analysis pipeline reproduces the published XENON1T anapole constraint of Catena et al. [1] to within 3%.

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Chapter 1

Introduction

1.1 The missing mass of the universe

HE Standard Model (SM) of particle physics is one of the most rigorously tested frameworks in modern science [2, 3]. Decades of collider experiments have confirmed its predictions with unprecedented accuracy. At the same time, cosmological observations indicate that the SM accounts for only about 5% of the total energy budget of the universe, with the remaining 95% being made up of dark matter (DM, 27%) and dark energy (68%) [4].

The nature of dark matter has been a persistent puzzle in astrophysics for nearly a century. Ever since Fritz Zwicky's initial proposal in 1933 [5], an increasing amount of astrophysical evidence has been collected. Key examples include the anomalous velocity curves of rotating galaxies [6], the gravitational lensing effects of DM on distant galaxy clusters [7], as well as the precision anisotropies in the Cosmic Microwave Background (CMB) [4]. All of this evidence culminates in the Λ CDM model, which establishes that the cosmic relic density in the universe is dominated by cold dark matter (CDM) [8].

For decades, the most promising DM candidate was the Weakly Interacting Massive Particle (WIMP) [9]. WIMPs naturally explain the observed thermal relic abundance. Furthermore, the WIMP framework offers a straightforward direct detection scheme via the weak scattering of DM against SM nuclei. Despite its strong theoretical motivation, decades of searches with multi-tonne liquid xenon and argon time projection chambers have not been able to find conclusive evidence for its existence [10, 11]. A significant portion of the canonical WIMP parameter space has been excluded.

Driven by these persistent null results, theoretical and experimental focus has recently shifted toward the exploration of the sub-GeV dark matter regime [12, 13]. While classic WIMPs operate at the GeV-to-TeV scale, lighter DM candidates could explain the lack of events in previous experiments due to the kinematic thresholds of nuclear scattering. The sub-GeV regime therefore provides a compelling explanation for the lack of observed events in past experiments and opens the parameter space towards lower DM masses for the next generation of detectors.

1.2 The sub-GeV regime and electron recoils

The null results of past direct detection experiments are tied to the kinematics of nuclear elastic scattering. In standard direct detection experiments targeting nuclear recoils, a classical WIMP, with a mass in the 100 GeV range, carries enough momentum to scatter against a massive target nucleus (e.g., xenon, $m_N \approx 122$ GeV) and produce a detectable recoil energy of $\mathcal{O}(10 - 100)$ keV. However, if a sub-GeV DM particle scatters off a xenon nucleus, the kinematic transfer is orders of magnitude smaller, and the transferred energy thus falls well below the detector threshold [14].

To probe the sub-GeV regime, the experimental focus must shift from heavy nuclei to lighter atomic electrons ($m_e \approx 0.511$ MeV) [12, 15]. Because the electron mass is much closer to the sub-GeV scale, an incoming light DM particle can transfer a significant amount of its kinetic energy to a bound atomic electron, overcoming the atomic binding energy and ionising the atom.

In dual-phase Time Projection Chambers (TPCs) like XENON1T, a standard high-energy recoil typically produces two signals: prompt scintillation light (S1) and delayed proportional scintillation from drifted electrons (S2). However, sub-GeV electron recoils deposit so little energy that the initial S1 photon signal is often completely lost. Therefore, the detection of light dark matter relies on the amplified ionisation signal which is sensitive to single- or few-electron extractions [16].

1.3 Neutrino masses and the scotogenic solution

In addition to dark matter, the mechanism of neutrino mass generation remains a major unsolved problem in particle physics. Since neutrino oscillation experiments, such as Super-Kamiokande and SNO, proved that active neutrinos possess mass [17, 18], the Standard Model has been incomplete, as it predicts neutrinos to be strictly massless. A straightforward theoretical approach to resolve this is the Type-I seesaw mechanism, which introduces massive, right-handed sterile neutrinos [19, 20].

These heavy Majorana neutrinos can account for the non-zero masses of the active left-handed neutrinos. Because the active mass is inversely proportional to the heavy mass, a sufficiently heavy right-handed neutrino naturally results in a sub-eV active neutrino. However, the classical seesaw mechanism offers no natural explanation for the large hierarchy between the electroweak scale and the newly introduced heavy neutrino scale. Furthermore, the right-handed neutrinos are completely sterile, meaning they carry no charge under the Standard Model gauge groups and therefore remain totally disconnected from collider phenomenology.

In 2006, Ernest Ma introduced an alternative framework: the Scotogenic Model, which is also frequently referred to as the radiative seesaw model [21]. The core idea of the model is that neutrino masses are strictly forbidden at tree-level. Instead, they are generated radiatively through a quantum loop. The natural suppression of this loop-level correction accounts for the tiny scale of active neutrino masses without requiring an artificially high seesaw scale.

There are various realisations of the scotogenic model [22]. In its minimal realisation, it extends the Standard Model by adding one scalar doublet η and three heavy singlet fermions N_i , one per lepton generation. These new particles are charged under a strict, exact $\mathbb{Z}_2$ parity symmetry. All SM particles are defined as $\mathbb{Z}_2$ -even, while the new scotogenic

fields are $\mathbb{Z}_2$ -odd. This new symmetry allows for the radiative generation of neutrino masses at loop-level. It also forbids the decay of the lightest of the $\mathbb{Z}_2$ -odd particles, rendering the lightest $\mathbb{Z}_2$ -odd particle absolutely stable, and thus providing a viable dark matter candidate [21, 23].

1.4 Outline of the thesis

The primary objective of this thesis is to systematically evaluate the direct detection phenomenology of sub-GeV dark matter within the scotogenic framework. To achieve this, the fundamental quantum field theory (QFT) of loop-induced interactions is studied and connected to the macroscopic observables of dual-phase time projection chambers.

Chapter 2 introduces the formal theoretical structure of the scotogenic model. It details the extended particle content, the stabilising $\mathbb{Z}_2$ discrete symmetry, and the fundamental mechanism for radiative neutrino mass generation, while outlining the phenomenological constraints imposed by lepton flavour violation.

Chapter 3 provides the QFT evaluation of the dark matter scattering processes. The Z -boson and photon diagrams are evaluated in full, while for the Higgs the known result of Ibarra et al. [24] is quoted and shown to be negligible against an electron target. This chapter explicitly demonstrates the required divergence cancellations, verifies the charge neutrality of the DM candidate N_1 , and extracts the effective anapole form factor that governs the loop-level electromagnetic interaction. It closes by setting the three channels against one another, which establishes the photon as the dominant one.

Chapter 4 translates the relativistic scattering theory into the non-relativistic regime required for dark matter-electron direct detection. The full t -channel amplitude is systematically expanded utilising a Non-Relativistic Effective Field Theory (NR-EFT) framework [25] to isolate the dominant Galilean-invariant operators. Subsequently, the formal framework for evaluating scattering off bound atomic states in liquid xenon is established, utilising atomic response functions to model the atomic ionisation physics.

Chapter 5 is dedicated to the phenomenological results and the extraction of experimental limits. It opens with the constraints that bound the parameter space from outside: the LEP limit on the charged scalar, the ATLAS searches for direct stau production, the MEG bound on lepton flavour violation, and the Planck relic density target. The theoretical event rate is then derived by convolving the NR-EFT atomic cross-section with astrophysical velocity distributions and the discrete signal response of the XENON1T detector. After establishing an EFT-level exclusion limit on the anapole coupling, these constraints are brought together. This identifies the region in which thermal freeze-out remains perturbatively viable, and motivates the evaluation of non-thermal (FIMP) production for the mass range below it.

Chapter 6 summarises the core findings of the thesis and provides an experimental outlook, detailing the necessary shift from direct detection facilities to long-lived particle (LLP) searches at high-energy colliders. Supplementary analytical derivations are preserved in Appendices A and B: the algebraic Dirac reductions, the chiral Gordon decomposition, the reduction of the loop integral to the closed form of Kopp et al. [26], and the trace evaluation of the squared amplitude together with its matching onto the non-relativistic operator basis.

Chapter 2

The Scotogenic Model and Dark Matter Interactions

2.1 Particle content and the $\mathbb{Z}_2$ symmetry

PROPOSED as a minimal extension to the Standard Model, the scotogenic model, also known as the radiative seesaw model [21], accounts for both the generation of active neutrino masses and the observed dark matter abundance in our universe. The model expands the Standard Model field content by introducing a second scalar doublet, η , and three right-handed Majorana singlet fermions, N_i ($i = 1, 2, 3$), one per lepton generation. Only two of them are needed to reproduce the two observed neutrino mass splittings, and the analysis below makes use of that freedom: it is what allows the dark matter candidate N_1 to be decoupled from neutrino mass generation, leaving N_2 and N_3 to account for the oscillation data.

The quantum numbers of these newly introduced fields under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ are assigned as follows:

$$\eta \sim (1, \mathbf{2}, 1/2) \qquad N_i \sim (1, \mathbf{1}, 0)$$

The model imposes a new and exact symmetry: $\mathbb{Z}_2$ parity. All Standard Model fields are assigned a positive parity ($\mathbb{Z}_2$ -even, +1), while the newly introduced scotogenic fields, η and N_i , are assigned a negative parity ($\mathbb{Z}_2$ -odd, -1).

The total Lagrangian of the model can be decomposed into the unchanged SM Lagrangian and the additional scotogenic contributions,

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{kin},N} + \mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\text{kin},\eta} - V(H, \eta). \quad (2.1)$$

The fermionic sector $\mathcal{L}_{\text{kin},N}$ contains the standard kinetic terms for the new singlets, alongside a new Majorana mass term, which is permitted because the singlets carry no SM gauge charges,

$$\mathcal{L}_{\text{kin},N} = \frac{1}{2} \overline{N_i} i \not{\partial} N_i - \frac{1}{2} M_i \overline{N_i^c} P_R N_i + \text{h.c.} \quad (2.2)$$

Here $N_i^c = C\bar{N}_i^T$ is the charge conjugate of N_i , with $C = i\gamma^2\gamma^0$ the charge conjugation matrix. A mass term built from $\bar{N}^c$ and N instead of $\bar{N}$ and N is what makes the singlets Majorana, so that each is its own antiparticle. This property is used throughout Chapter 3, where it removes the vector current and the dipole moments of N_1 and leaves the anapole as the leading electromagnetic interaction.

The interaction between the SM lepton sector and the scotogenic sector is governed by the new Yukawa Lagrangian $\mathcal{L}_{\text{Yuk}}$. Due to the $\mathbb{Z}_2$ parity symmetry requirement, the standard Dirac neutrino mass term ($\bar{L}_\alpha \tilde{H} P_R N_i$) is forbidden, as the product of an even H and an odd N_i yields a parity-odd term. Instead, the only allowed Yukawa interaction must involve the odd scalar doublet η ,

$$\mathcal{L}_{\text{Yuk}} = y_{\alpha i} \bar{L}_\alpha \tilde{\eta} P_R N_i + \text{h.c.}, \quad (2.3)$$

where L_α denotes the SM left-handed lepton doublet with flavour index $\alpha \in \{e, \mu, \tau\}$, $\tilde{\eta} = i\sigma_2 \eta^*$ is the conjugate doublet, P_R is the right-handed chirality projector, and $y_{\alpha i}$ is the complex neutrino Yukawa coupling matrix.

The scalar kinetic term governs the gauge interactions of the new scalar doublet via the covariant derivative,

$$\mathcal{L}_{\text{kin},\eta} = (D_\mu \eta)^\dagger (D^\mu \eta). \quad (2.4)$$

To fully describe the dynamics and the electroweak symmetry breaking (EWSB) mechanism, the standard scalar potential must be expanded to include all renormalizable interactions between H and η that remain invariant under both the SM gauge symmetries and the $\mathbb{Z}_2$ parity. The complete scalar potential $V(H, \eta)$ is parametrised as

$$\begin{aligned} V(H, \eta) = & m_H^2 H^\dagger H + m_\eta^2 \eta^\dagger \eta + \frac{\lambda_1}{2} (H^\dagger H)^2 + \frac{\lambda_2}{2} (\eta^\dagger \eta)^2 + \lambda_3 (H^\dagger H)(\eta^\dagger \eta) \\ & + \lambda_4 (H^\dagger \eta)(\eta^\dagger H) + \frac{\lambda_5}{2} [(H^\dagger \eta)^2 + (\eta^\dagger H)^2]. \end{aligned} \quad (2.5)$$

Here, m_H^2 and m_η^2 are the bare mass parameters, while $\lambda_{1...5}$ denote the dimensionless quartic couplings. Crucially, the λ_5 term is permitted because it is quadratic in the $\mathbb{Z}_2$ -odd field η .

The parity forbids any term with an odd number of η fields, so the potential contains nothing linear in η and the point $\eta = 0$ is stationary whatever the couplings. With $m_H^2 < 0$, electroweak symmetry breaking is left to the Higgs alone, which acquires a vacuum expectation value (VEV) of $v \approx 246$ GeV, while the inert doublet does not,

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.6)$$

Were η to take a vacuum expectation value instead, the parity would be spontaneously broken, generating an unwanted tree-level Dirac mass for the neutrinos and allowing the dark matter to decay.

Because η enters the potential only in pairs, its components do not mix with the physical Higgs boson h , and the doublet decomposes with no further diagonalisation as

$$\eta = \begin{pmatrix} \eta^\pm \\ \eta^0 \end{pmatrix}, \quad \eta^0 = \frac{1}{\sqrt{2}} (\eta_R + i\eta_I), \quad (2.7)$$

a pair of charged scalars $\eta^\pm$ together with a neutral complex scalar η^0 , whose CP-even and CP-odd parts η_R and η_I are the neutral mass eigenstates. Their masses, $m_{\eta^\pm}^2 = m_\eta^2 + \frac{1}{2}\lambda_3 v^2$ and $m_{\eta_{R,I}}^2 = m_\eta^2 + \frac{1}{2}(\lambda_3 + \lambda_4 \pm \lambda_5)v^2$, must all be positive for that stationary point to be a minimum.

Because the parity is exact and remains unbroken by the vacuum, the lightest $\mathbb{Z}_2$ -odd particle has no decay channel into $\mathbb{Z}_2$ -even states alone and is therefore absolutely stable [23]. For the remainder of this thesis, a mass hierarchy is assumed wherein the lightest right-handed fermion singlet, N_1 , constitutes the dark matter candidate.

Scope: which dark matter candidate, and which scotogenic model

Two of the choices made above are our assumptions, not derived results.

The first is that the dark matter is the lightest singlet fermion and not the lightest neutral scalar. Inert-doublet dark matter is a well-studied alternative that the $\mathbb{Z}_2$ symmetry permits just as well, but LEP and electroweak precision data place the lightest neutral scalar above roughly 90 GeV (Section 5.1.1), so only the singlet fermion, whose mass is a free parameter, can be sub-GeV.

The second is that this work studies the *minimal* scotogenic model of Ma [21]. Ávila et al. [22] review the other variants: the *revamped* construction adds fermion and scalar triplets, while *scoto-seesaw* schemes combine the radiative mechanism with a tree-level seesaw. None of the results below are claimed to carry over to them.

2.2 Neutrino mass generation and lepton flavour violation

At tree level, active left-handed neutrinos remain strictly massless because the $\mathbb{Z}_2$ parity forbids a standard Dirac mass term. However, the scotogenic Lagrangian allows for the generation of a Majorana mass matrix via one-loop radiative corrections [21]. This process, mediated by the neutral scalar components η^0 and the right-handed singlets N_i , is illustrated in Figure 2.1.

The finite evaluation of this loop integral depends on the λ_5 parameter of the scalar potential,

$$V(H, \eta) \supset \frac{\lambda_5}{2} [(H^\dagger \eta)^2 + (\eta^\dagger H)^2]. \quad (2.8)$$

Following electroweak symmetry breaking ($\langle H \rangle = v/\sqrt{2}$), this term induces a mass splitting between the CP-even (η_R) and CP-odd (η_I) eigenstates of the neutral scotogenic scalar,

$$m_{\eta_R}^2 - m_{\eta_I}^2 = \lambda_5 v^2. \quad (2.9)$$

If $\lambda_5 = 0$, the loop amplitude vanishes due to an exact cancellation between the η_R and η_I propagators. For a non-zero but small mass splitting ($m_{\eta_R}^2 \approx m_{\eta_I}^2 \equiv m_{\eta^0}^2$), the resulting

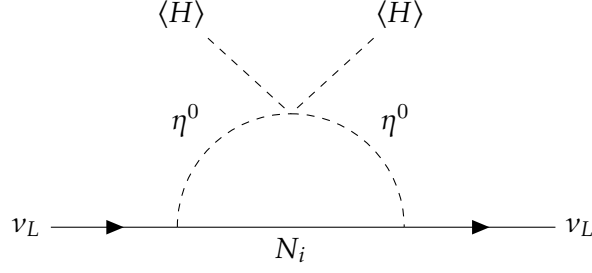


Figure 2.1: One-loop Feynman diagram generating the radiative Majorana neutrino mass in the scotogenic model. The electroweak symmetry breaking insertions from the Higgs vacuum expectation value $\langle H \rangle$ allow the λ_5 coupling to induce a mass splitting between the real and imaginary components of the scalar η^0 .

active neutrino mass matrix $(m_\nu)_{\alpha\beta}$ is given by the general expression [21],

$$(m_\nu)_{\alpha\beta} \approx \frac{\lambda_5 v^2}{32\pi^2} \sum_i \frac{y_{\alpha i} y_{\beta i} M_i}{m_{\eta^0}^2 - M_i^2} \left[1 - \frac{M_i^2}{m_{\eta^0}^2 - M_i^2} \ln \left(\frac{m_{\eta^0}^2}{M_i^2} \right) \right]. \quad (2.10)$$

Depending on the mass hierarchy between the singlet M_i and the neutral scalar mass m_{η^0} , the terms of this sum take two distinct asymptotic forms. For a heavy singlet ($M_i \gg m_{\eta^0}$), the term falls off as the inverse of its mass,

$$(m_\nu)_{\alpha\beta}^{(i)} \approx \frac{\lambda_5 v^2}{32\pi^2} \frac{y_{\alpha i} y_{\beta i}}{M_i} \left[\ln \left(\frac{M_i^2}{m_{\eta^0}^2} \right) - 1 \right]. \quad (2.11)$$

Conversely, for a singlet much lighter than the mediator ($M_1 \ll m_{\eta^0}$), the term is proportional to $M_1/m_{\eta^0}^2$ instead,

$$(m_\nu)_{\alpha\beta}^{(1)} \approx \frac{\lambda_5 v^2}{32\pi^2} \frac{y_{\alpha 1} y_{\beta 1} M_1}{m_{\eta^0}^2}. \quad (2.12)$$

This contrast is what decouples the dark matter from the neutrino sector. The ratio $M_1/m_{\eta^0}^2$, a sub-GeV mass divided by the square of an electroweak-scale one, is far smaller than the $1/M_{2,3}$ carried by the heavier singlets. N_1 therefore supplies a few per cent of the total at most, the active neutrino masses are almost entirely generated by N_2 and N_3 , and the Yukawa coupling of N_1 to the tau sector, $y_{\tau 1}$, is free to take the $\mathcal{O}(1)$ value the relic density requires, as Chapter 5 will show.

While the scotogenic model provides an elegant solution for neutrino masses, it introduces tight phenomenological constraints in the lepton sector. To reproduce the observed PMNS neutrino oscillation parameters [27, 28], the full Yukawa matrix $y_{\alpha i}$ must contain non-trivial off-diagonal elements. However, these same couplings inevitably mediate rare Lepton Flavour Violating (LFV) processes at the one-loop level, most notably $\mu \rightarrow e\gamma$ [29, 30].

Strict upper bounds on the branching ratio $\mathcal{B}(\mu \rightarrow e\gamma)$ from the MEG collaboration [31] therefore constrain the Yukawa matrix. A strictly τ -philic structure for the lightest singlet, $y_{\tau 1} \gg y_{\mu 1}, y_{e 1}$, removes N_1 from the $\mu \rightarrow e\gamma$ amplitude, leaving the bound to constrain N_2 and N_3 , which carry the mixing demanded by oscillation data [24]. This flavour structure is assumed from here on, and Section 5.1.3 makes the constraint quantitative.

Chapter 3

Quantum Field Theory of Dark Matter Scattering

3.1 Loop-induced scattering and mediator overview

BECAUSE the dark matter candidate N_1 is a Standard Model gauge singlet, it possesses no tree-level couplings to quarks or leptons. Consequently, any elastic scattering between the dark matter and the detector target material must be mediated via loop-induced effective interactions. There are three possible neutral mediators that can connect the dark sector to the Standard Model: the Higgs boson, the Z -boson, and the photon.

To systematically evaluate the scattering cross-sections induced by each of these mediators, we first define the interaction vertices and propagators derived from the scotogenic Lagrangian. The interactions between the Majorana dark matter candidate (N_1), the Standard Model lepton doublets (L_α), and the $\mathbb{Z}_2$ -odd scalar doublet (η) are dictated by the Feynman rules collected in Table 3.1.

Table 3.1: Vertices and propagators of the dark sector, as they follow from the scotogenic Lagrangian. The Yukawa coupling $y_{\alpha 1}$ connects the Majorana dark matter N_1 to the lepton doublet L_α and the $\mathbb{Z}_2$ -odd scalar doublet η , while m_f and m_S are the masses of the internal fermion and scalar.

Interaction	Diagram	Expression
Incoming DM Vertex $(N_1 \rightarrow L_\alpha + \eta)$		$-iy_{\alpha 1}P_R$

Interaction	Diagram	Expression
Outgoing DM Vertex $(L_\alpha + \eta \rightarrow N_1)$		$-iy_{\alpha 1}^* P_L$
Fermion Propagator		$\frac{i(\not{p} + m_f)}{p^2 - m_f^2}$
Scalar Propagator		$\frac{i}{k^2 - m_s^2}$

Furthermore, the couplings of the Standard Model mediators to the respective fields propagating inside the loop are given in Table 3.2.

Table 3.2: Couplings of the Higgs boson, the Z-boson and the photon to the leptons and scalars that propagate inside the loop. The momenta k and $k + q$ are those of the incoming and outgoing scalar, and the constants g_Z and λ_L are defined below.

Interaction	Diagram	Expression
Higgs coupling to Lepton		$-i \frac{m_{\ell_\alpha}}{v}$
Higgs coupling to Scalar		$-i \lambda_L v$
Z-boson coupling to Neutrino		$-i g_Z \left(\frac{1}{2}\right) \gamma^\mu P_L$
Z-boson coupling to Scalar		$-i g_Z \left(-\frac{1}{2}\right) (2k + q)^\mu$

Interaction	Diagram	Expression
Photon coupling to Lepton		$+ie\gamma^\mu$
Photon coupling to Scalar		$-ie(2k+q)^\mu$

Here, $y_{\alpha 1}$ denotes the Yukawa coupling of the dark matter to the lepton flavour α , while m_f and m_s represent the masses of the specific internal fermion (m_{ℓ_α} or $m_{\nu_\alpha} \approx 0$) and internal scalar ($m_{\eta^\pm}$ or m_{η^0}), respectively. For the momentum-dependent vertices, k and $k+q$ correspond to the incoming and outgoing scalar momenta. The chiral projection operators are defined as $P_{R,L} = \frac{1 \pm \gamma^5}{2}$. They satisfy $P_L P_L = P_L$, $P_R P_L = 0$, and the commutation relation $P_L \gamma^\mu = \gamma^\mu P_R$. For the Standard Model mediators, $g_Z = e/(\sin \theta_W \cos \theta_W)$ is the weak neutral current coupling constant, $e > 0$ is the elementary electric charge, and $v \approx 246$ GeV is the electroweak vacuum expectation value. The effective scalar quartic coupling λ_L depends on the specific scalar mass eigenstate running in the loop (e.g., $\lambda_L = \lambda_3 + \lambda_4 \pm \lambda_5$ for the neutral components $\eta_{R,I}$).

The three channels are treated in turn, and they are not of equal weight. The Higgs channel (Section 3.2) is strongly suppressed for an electron target. The Z channel (Section 3.3) loses its vector coupling to the Majorana nature of N_1 , and what remains is suppressed both by the Z propagator and by a loop function that vanishes with the dark matter mass. The photon channel (Section 3.4) therefore carries the chapter, and its result, the anapole form factor f_A , is the input to every rate computed in Chapter 5.

The techniques used along the way are standard: Feynman parametrisation, momentum shifts, and dimensional regularisation in $d = 4 - 2\delta$. What is specific to this work is their application to the scotogenic topologies, and in particular the cancellation of the ultraviolet divergences between the lepton and scalar diagrams, the explicit verification of charge neutrality and of the Ward identity, and the extraction of the transverse coefficient that defines f_A . The intermediate algebra is long, so it is gathered in Appendix A: the term-by-term reduction of the finite numerators, the chiral Gordon decomposition, and the demonstration that the loop integral reduces to the closed form of Kopp et al. [26]. Only the results are needed here.

3.2 Higgs boson exchange

The one-loop exchange of the Standard Model Higgs boson (h) induces an effective scalar interaction between the dark matter candidate and the target electrons. As illustrated in Figure 3.1, there are two distinct loop topologies. The Higgs boson can couple to the scotogenic scalar multiplet η via the scalar potential, or it can couple to the charged lepton ℓ_α^- in the loop via the Standard Model Yukawa coupling.

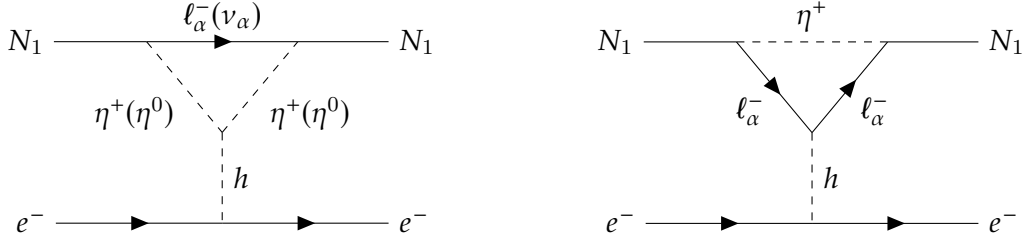


Figure 3.1: One-loop Higgs exchange diagrams mediating elastic scattering off an electron (e^-). Left: The Higgs couples to the internal scotogenic scalar multiplet η . Right: The Higgs couples to the internal charged lepton ℓ_α^- .

Following the analytical derivation by Ibarra et al. [24], the effective low-energy interaction is of the form $\mathcal{L}_{\text{eff}} \supset \Lambda_e \bar{N}_1 N_1 \bar{e} e$, where the effective scalar coupling Λ_e for a target electron is given by

$$\Lambda_e = -\frac{y_{\alpha 1}^2}{16\pi^2 m_h^2 M_1} \left[\lambda_3 G_1 \left(\frac{M_1^2}{m_{\eta^\pm}^2} \right) + \frac{\lambda_3 + \lambda_4}{2} G_1 \left(\frac{M_1^2}{m_{\eta^0}^2} \right) \right] m_e. \quad (3.1)$$

Here, $G_1(x)$ is a dimensionless loop function, and is evaluated at the mass ratio of the dark matter to the scalar mediators,

$$G_1(x) = \frac{x + (1-x) \ln(1-x)}{x}. \quad (3.2)$$

In standard WIMP searches, scalar interactions like the ones above are often the leading contribution because the Higgs couples to the heavy target nucleons, yielding an effective coupling proportional to the nucleon mass m_N . However, for sub-GeV dark matter, the target shifts to atomic electrons ($m_e = 0.511$ MeV). Consequently, the effective coupling Λ_e is directly proportional to the tiny electron Yukawa coupling ($y_e = m_e/v \approx 2 \times 10^{-6}$). Since the elastic scattering cross-section scales as $\sigma \propto \Lambda_e^2$, this scalar interaction is suppressed compared to the other gauge interactions and is therefore neglected in the subsequent numerical analysis.

3.3 Z-boson exchange

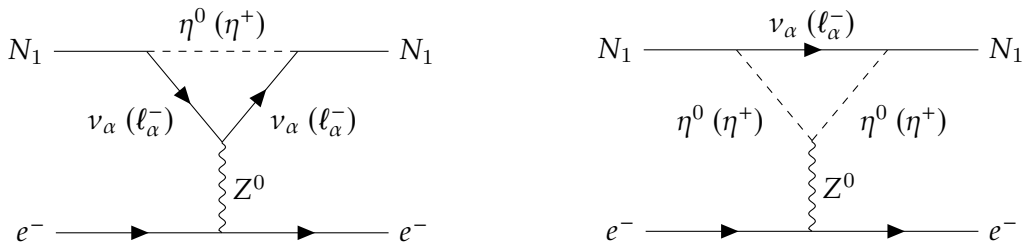


Figure 3.2: One-loop Z-boson exchange diagrams mediating elastic scattering off an electron. Left: The Z-boson couples to the internal active neutrino ν_α (charged lepton ℓ_α^-). Right: The Z-boson couples to the internal neutral scalar η^0 (charged scalar η^+).

The elastic scattering of dark matter off electrons can also be mediated by the one-loop exchange of a Z-boson. The Z-boson couples to both the neutral and charged components of the scotogenic loop, yielding four distinct topologies as illustrated in Figure 3.2. The

evaluation of these diagrams results in an effective current-current interaction between the Majorana dark matter candidate N_1 and the target electron.

While all four interaction channels contribute to the total scattering amplitude, the neutral and charged sectors share an identical Lorentz structure. In the following, the explicit evaluation is restricted to the loops containing the neutral internal fields (ν_α and η^0). The corresponding loops containing the charged fields (ℓ_α^- and η^+) follow an identical mathematical procedure, differing only in their respective coupling constants and internal masses. They are therefore omitted for brevity.

3.3.1 Z-boson coupling to the internal neutrino

We compute the one-loop vertex correction for N_1 scattering via a Z-boson coupling to the internal left-handed neutrino (ν_α), illustrated in Figure 3.3. Assigning the loop momentum k to the internal scalar propagator, momentum conservation dictates that the internal fermion propagators carry momenta $p - k$ and $p' - k$.

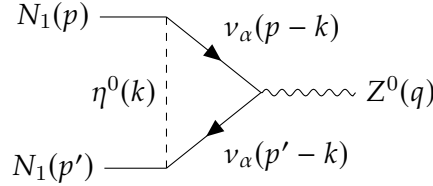


Figure 3.3: Vertex correction topology where the Z-boson couples directly to the internal neutrino line. The loop momentum k is routed through the scalar propagator.

Reading against the fermion flow (from p' to p), the unintegrated loop amplitude takes the form

$$i\mathcal{M}_a^\mu = \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-iy_{\alpha 1}^* P_L) \left[\frac{i(\not{p}' - \not{k})}{(p' - k)^2} \right] \left(\frac{-ig_Z}{2} \gamma^\mu P_L \right) \left[\frac{i(\not{p} - \not{k})}{(p - k)^2} \right] (-iy_{\alpha 1} P_R) u(p) \times \frac{i}{k^2 - m_{\eta^0}^2} \quad (3.3)$$

$$= \frac{g_Z |y_{\alpha 1}|^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') [P_L(\not{p}' - \not{k}) \gamma^\mu P_L(\not{p} - \not{k}) P_R] u(p)}{(p' - k)^2 (p - k)^2 (k^2 - m_{\eta^0}^2)}. \quad (3.4)$$

Isolating the fermion bilinear $\mathcal{N}^\mu$ in the numerator, the chiral projectors can be simplified,

$$\begin{aligned} \mathcal{N}^\mu &= P_L(\not{p}' - \not{k}) \gamma^\mu P_L(\not{p} - \not{k}) P_R \\ &= (\not{p}' - \not{k}) P_R \gamma^\mu P_L(\not{p} - \not{k}) P_R \\ &= (\not{p}' - \not{k}) \gamma^\mu P_L P_L(\not{p} - \not{k}) P_R \\ &= (\not{p}' - \not{k}) \gamma^\mu P_L(\not{p} - \not{k}) P_R \\ &= (\not{p}' - \not{k}) \gamma^\mu (\not{p} - \not{k}) P_R P_R \\ &= (\not{p}' - \not{k}) \gamma^\mu (\not{p} - \not{k}) P_R. \end{aligned} \quad (3.5)$$

Feynman parametrisation and denominator shift

To evaluate the loop integral, we must combine the product of three separate propagators in the denominator into a single expression. This is achieved using Feynman parametrisation,

which introduces auxiliary integration variables to combine the denominators. The general identity for n denominators is

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{(n-1)!}{(x_1 A_1 + x_2 A_2 + \dots + x_n A_n)^n}. \quad (3.6)$$

For our one-loop triangle diagram with three propagators ($n = 3$), we introduce the parameters x, y, z . The Dirac delta function enforces $x + y + z = 1$, allowing us to eliminate z by substituting $z = 1 - x - y$. The formula reduces to

$$\frac{1}{ABC} = \int_0^1 dx \int_0^{1-x} dy \frac{2}{(xA + yB + zC)^3}. \quad (3.7)$$

By merging the denominators, we can perform a momentum shift to complete the square, which centers the loop momentum and enables d -dimensional symmetric integration.

Assigning our specific propagators to A, B , and C , the combined denominator $D = xA + yB + zC$ is given by

$$D = x(p' - k)^2 + y(p - k)^2 + z(k^2 - m_{\eta^0}^2). \quad (3.8)$$

Expanding the squared momentum terms yields

$$D = x(k^2 - 2p' \cdot k + p'^2) + y(k^2 - 2p \cdot k + p^2) + z(k^2 - m_{\eta^0}^2). \quad (3.9)$$

We group the terms by powers of the loop momentum k ,

$$D = (x + y + z)k^2 - 2k \cdot (xp' + yp) + xp'^2 + yp^2 - zm_{\eta^0}^2. \quad (3.10)$$

Applying the parameter constraint $x + y + z = 1$ and the on-shell mass conditions for the external Dark Matter legs ($p^2 = p'^2 = M_1^2$), the expression simplifies to

$$D = k^2 - 2k \cdot (xp' + yp) + (x + y)M_1^2 - zm_{\eta^0}^2. \quad (3.11)$$

To complete the square with respect to k , we define the momentum shift vector Q_a ,

$$Q_a = xp' + yp. \quad (3.12)$$

The denominator can now be written in the completed square form $D = (k - Q_a)^2 - \Delta_a$, where $\Delta_a = Q_a^2 - C$ and $C = (x + y)M_1^2 - zm_{\eta^0}^2$.

We can evaluate Q_a^2 in terms of the momentum transfer $q = p' - p$. Squaring q gives $q^2 = 2M_1^2 - 2p' \cdot p$, which allows us to substitute $2p' \cdot p = 2M_1^2 - q^2$,

$$\begin{aligned} Q_a^2 &= (xp' + yp)^2 \\ &= x^2 M_1^2 + y^2 M_1^2 + xy(2M_1^2 - q^2) \\ &= (x^2 + 2xy + y^2)M_1^2 - xyq^2 \\ &= (x + y)^2 M_1^2 - xyq^2. \end{aligned} \quad (3.13)$$

Finally, we construct the effective mass term $\Delta_a = Q_a^2 - C$,

$$\begin{aligned}\Delta_a &= [(x+y)^2 M_1^2 - xyq^2] - [(x+y)M_1^2 - zm_{\eta^0}^2] \\ &= zm_{\eta^0}^2 + M_1^2 [(x+y)^2 - (x+y)] - xyq^2.\end{aligned}\quad (3.14)$$

Using the parameter constraint $x+y=1-z$, the bracketed coefficient of M_1^2 simplifies to: $(1-z)^2 - (1-z) = (1-z)(1-z-1) = -z(1-z)$. This yields the final effective mass denominator,

$$\Delta_a = zm_{\eta^0}^2 - z(1-z)M_1^2 - xyq^2. \quad (3.15)$$

Evaluation of the loop integral

Transitioning to the shifted loop momentum $\ell = k - Q_a$, the denominator takes the symmetric form $(\ell^2 - \Delta_a)^3$. The numerator is rewritten as

$$\mathcal{N}^\mu = (\not{p}' - \not{\ell} - \not{Q}_a)\gamma^\mu(\not{p} - \not{\ell} - \not{Q}_a)P_R. \quad (3.16)$$

In four spacetime dimensions, the UV divergence arises from the terms containing ℓ^2 in the numerator, which produce a $\int d^4\ell/\ell^4$ logarithmic singularity. Terms carrying odd powers of ℓ vanish under symmetric integration. Thus, the divergent part is

$$\mathcal{N}_{div}^\mu = \not{\ell}\gamma^\mu\not{\ell}P_R = \ell_\alpha\ell_\beta\gamma^\alpha\gamma^\mu\gamma^\beta P_R. \quad (3.17)$$

Applying d -dimensional symmetric integration ($\ell_\alpha\ell_\beta \rightarrow \frac{\ell^2}{d}g_{\alpha\beta}$) and the contraction identity $\gamma^\alpha\gamma^\mu\gamma_\alpha = (2-d)\gamma^\mu$,

$$\mathcal{N}_{div}^\mu = \ell^2 \frac{2-d}{d} \gamma^\mu P_R. \quad (3.18)$$

The divergent part of the loop integral can then be evaluated in $d = 4 - 2\delta$ dimensions,

$$i\mathcal{M}_{a,div}^\mu = \frac{g_Z|y_{\alpha 1}|^2}{2} \bar{u}(p')\gamma^\mu P_R u(p) \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\ell}{(2\pi)^d} \frac{2\left(\frac{2-d}{d}\ell^2\right)}{(\ell^2 - \Delta_a)^3}. \quad (3.19)$$

Executing the integration over $d^d\ell$,

$$\text{Integral} = \frac{-i}{16\pi^2} (4\pi)^\delta (1-\delta)\Gamma(\delta)(\Delta_a)^{-\delta}. \quad (3.20)$$

Expanding around $\delta \rightarrow 0$ using $\Gamma(\delta) = \frac{1}{\delta} - \gamma_E + \mathcal{O}(\delta)$ and applying the Modified Minimal Subtraction ($\overline{\text{MS}}$) scheme to absorb the universal constants γ_E (Euler–Mascheroni) and $\ln(4\pi)$, we obtain the physical pole and remainder,

$$\text{Integral} \xrightarrow{\overline{\text{MS}}} \frac{-i}{16\pi^2} \left[\frac{1}{\delta} - 1 - \ln \Delta_a \right]. \quad (3.21)$$

The finite $\mathcal{O}(\ell^0)$ contribution originates from the remainder of the numerator,

$$\mathcal{N}_{fin}^\mu = [(1-x)\not{p}' - y\not{p}] \gamma^\mu [(1-y)\not{p} - x\not{p}'] P_R. \quad (3.22)$$

In the strictly non-relativistic limit ($q \rightarrow 0$), we approximate $p' \approx p$ and $x \approx y$, yielding $z = 1 - 2x$. The finite trace reduces to

$$\mathcal{N}_{fin}^\mu = z^2 \not{p}\gamma^\mu\not{p}P_R. \quad (3.23)$$

Now we can apply the Clifford algebra identity $\not{p}\gamma^\mu\not{p} = 2p^\mu\not{p} - p^2\gamma^\mu$ alongside the Dirac equation of motion ($\not{p}u(p) = M_1u(p)$ and $p^2 = M_1^2$),

$$\bar{u}(p)\mathcal{N}_{fin}^\mu u(p) = z^2 M_1 \bar{u}(p) (2p^\mu - M_1\gamma^\mu) P_R u(p). \quad (3.24)$$

Finally, using the Gordon decomposition identity $\bar{u}(p)p^\mu u(p) = M_1 \bar{u}(p)\gamma^\mu u(p)$, the finite operator matches the same structure as the divergent term,

$$\bar{u}(p)\mathcal{N}_{fin}^\mu u(p) = z^2 M_1^2 [\bar{u}(p)\gamma^\mu P_R u(p)]. \quad (3.25)$$

Because the finite part carries no loop momentum ℓ in the numerator, its integration over momentum space is governed by a straightforward scalar integral in $d = 4$ dimensions,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta_a)^3} = \frac{-i}{32\pi^2 \Delta_a}. \quad (3.26)$$

Adding up our finite and divergent results, the final effective vertex for this Diagram evaluates to

$$i\mathcal{M}_a^\mu = -i \frac{g_Z |y_{\alpha 1}|^2}{32\pi^2} [\bar{u}(p')\gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{\delta} - 1 - \ln(\Delta_a) - \frac{z^2 M_1^2}{\Delta_a} \right], \quad (3.27)$$

where the static effective mass denominator Δ_a is defined as $\Delta_a = zm_{\eta^0}^2 - z(1-z)M_1^2$, subject to the parameter constraint $z = 1 - 2x$.

3.3.2 Z-boson coupling to the neutral scalar

Feynman rules and amplitude definition

For the second topology, as shown in Figure 3.4, the Z-boson couples to the internal scalar (η^0). Because the neutrino and the scalar sit in different components of their respective weak isospin doublet, they carry opposite weak charges ($T_3 = -1/2$ for the scalar component η^0).

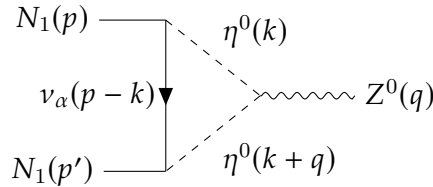


Figure 3.4: Vertex correction topology where the Z-boson couples directly to the internal scalar line.

We assign momenta similar to the previous topology. The Z-boson carries momentum q into the scalar line. The corresponding unintegrated loop amplitude is

$$i\mathcal{M}_b^\mu = \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-iy_{\alpha 1}^* P_L) \left[\frac{i(\not{p} - \not{k})}{(p-k)^2} \right] (-iy_{\alpha 1} P_R) u(p) \frac{i}{k^2 - m_{\eta^0}^2} \times \frac{i}{(k+q)^2 - m_{\eta^0}^2} \left[i \frac{g_Z}{2} (2k+q)^\mu \right] \quad (3.28)$$

$$= -\frac{g_Z |y_{\alpha 1}|^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') [P_L (\not{p} - \not{k}) P_R (2k+q)^\mu] u(p)}{(p-k)^2 (k^2 - m_{\eta^0}^2) ((k+q)^2 - m_{\eta^0}^2)}. \quad (3.29)$$

We isolate the fermion bilinear in the numerator N^μ . Because the Z-boson attaches to a scalar, there is no internal γ^μ matrix inside the fermion trace,

$$\begin{aligned} N^\mu &= P_L(\not{p} - \not{k})P_R(2k + q)^\mu \\ &= (\not{p} - \not{k})P_R P_R(2k + q)^\mu \\ &= (\not{p} - \not{k})P_R(2k + q)^\mu. \end{aligned} \quad (3.30)$$

Feynman parametrisation and integration

We introduce parameters x, y for the two scalar propagators, and z for the single fermion propagator. The combined denominator is

$$D = x(k + q)^2 - x m_{\eta_0}^2 + y k^2 - y m_{\eta_0}^2 + z(p - k)^2. \quad (3.31)$$

As justified in Figure 3.3, we take the static EFT limit ($q \rightarrow 0$) uniformly. The scalar parameters combine simply into $(x + y)$, which we can replace using $x + y = 1 - z$,

$$\begin{aligned} D &= (1 - z)(k^2 - m_{\eta_0}^2) + z(k^2 - 2p \cdot k + M_1^2) \\ &= k^2 - 2zp \cdot k + zM_1^2 - (1 - z)m_{\eta_0}^2. \end{aligned} \quad (3.32)$$

To complete the square, we define the shift vector Q_b ,

$$Q_b = zp. \quad (3.33)$$

The effective mass term $\Delta_b = Q_b^2 - C$ is evaluated using $Q_b^2 = z^2 M_1^2$,

$$\begin{aligned} \Delta_b &= z^2 M_1^2 - \left[zM_1^2 - (1 - z)m_{\eta_0}^2 \right] \\ &= (1 - z)m_{\eta_0}^2 - z(1 - z)M_1^2. \end{aligned} \quad (3.34)$$

This result is physically symmetric to Δ_a if the fermion/scalar parameter assignments $z \leftrightarrow (1 - z)$ are swapped.

$$\Delta_b = (1 - z)m_{\eta_0}^2 - z(1 - z)M_1^2. \quad (3.35)$$

The UV-divergent $\mathcal{O}(\ell^2)$ component of the integral is $\mathcal{N}_{div}^\mu = -2\ell^\mu P_R$ which can be rewritten as

$$\mathcal{N}_{div}^\mu = \frac{-2}{d} \ell^2 \gamma^\mu P_R. \quad (3.36)$$

Integrating the divergent component over $d^d \ell$ and using $\overline{\text{MS}}$ regularisation,

$$i\mathcal{M}_{b,div}^\mu = +i \frac{g_Z |y_{\alpha 1}|^2}{32\pi^2} [\bar{u}(p') \gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{\delta} - \ln \Delta_b \right]. \quad (3.37)$$

The finite $\mathcal{O}(\ell^0)$ trace $\mathcal{N}_{fin}^\mu = 2z(1 - z)p^\mu \not{p} P_R$ reduces via the Gordon decomposition to $2z(1 - z)M_1^2 \gamma^\mu P_R$. Integrating this term and combining it with the divergent component yields the final vertex structure,

$$i\mathcal{M}_b^\mu = +i \frac{g_Z |y_{\alpha 1}|^2}{32\pi^2} [\bar{u}(p') \gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{\delta} - \ln(\Delta_b) + \frac{2z(1 - z)M_1^2}{\Delta_b} \right]. \quad (3.38)$$

3.3.3 Total Z-boson vertex and divergence cancellation

The total one-loop effective vertex for the Z-boson is the sum of the two distinct topologies: $\mathcal{M}_{total}^\mu = \mathcal{M}_a^\mu + \mathcal{M}_b^\mu$.

Because the internal neutrino and scalar field occupy opposite weak isospin states, their respective couplings to the Z-boson introduce a relative minus sign between the evaluated amplitudes,

$$i\mathcal{M}_{total}^\mu = -i\frac{g_Z|y_{\alpha 1}|^2}{32\pi^2} [\bar{u}(p')\gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \left[\left(\frac{1}{\delta} - 1 - \ln \Delta_a - \frac{z^2 M_1^2}{\Delta_a} \right) - \left(\frac{1}{\delta} - \ln \Delta_b + \frac{2z(1-z)M_1^2}{\Delta_b} \right) \right]. \quad (3.39)$$

The ultraviolet poles ($\frac{1}{\delta}$) precisely cancel, showing the renormalizability of the model and proving that the loop-induced interaction is physically finite,

$$i\mathcal{M}_{total}^\mu = -i\frac{g_Z|y_{\alpha 1}|^2}{32\pi^2} [\bar{u}(p')\gamma^\mu P_R u(p)] \times \int_0^1 dx \int_0^{1-x} dy \left[-1 - \ln \left(\frac{\Delta_a}{\Delta_b} \right) - M_1^2 \left(\frac{z^2}{\Delta_a} + \frac{2z(1-z)}{\Delta_b} \right) \right]. \quad (3.40)$$

3.3.4 The Majorana vector cancellation

To interpret the physical nature of the loop-induced interaction, the chiral projector P_R in the evaluated amplitude must be expanded into its fundamental Dirac structures. Using the definition $P_R = \frac{1+\gamma^5}{2}$, the dark matter fermion bilinear decomposes into a pure vector and a pure axial-vector current,

$$\begin{aligned} \bar{u}(p')\gamma^\mu P_R u(p) &= \bar{u}(p')\gamma^\mu \left(\frac{1+\gamma^5}{2} \right) u(p) \\ &= \frac{1}{2}\bar{u}(p')\gamma^\mu u(p) + \frac{1}{2}\bar{u}(p')\gamma^\mu \gamma^5 u(p). \end{aligned} \quad (3.41)$$

However, because the dark matter candidate N_1 in the scotogenic model is a Majorana fermion, it is its own antiparticle. This fundamental property dictates that its vector current must vanish.

This cancellation is demonstrated by evaluating the transpose of the vector current. As a fermion bilinear constitutes a 1×1 matrix in spinor space, it equals its own transpose. Reversing the order of anticommuting Grassmann fields introduces a negative sign, yielding

$$\overline{N_1}\gamma^\mu N_1 = \left(\overline{N_1}\gamma^\mu N_1 \right)^T = -N_1^T (\gamma^\mu)^T \overline{N_1}^T. \quad (3.42)$$

Applying the Majorana definition $N_1 = N_1^c \equiv C\overline{N_1}^T$ implies $\overline{N_1}^T = C^{-1}N_1$. Utilising the anti-symmetric property of the charge conjugation matrix, $C^T = -C$, yields $N_1^T = -\overline{N_1}C$. Substituting these identities back into the transposed current,

$$\begin{aligned} -N_1^T (\gamma^\mu)^T \overline{N_1}^T &= -\left(-\overline{N_1}C \right) (\gamma^\mu)^T (C^{-1}N_1) \\ &= \overline{N_1} (C(\gamma^\mu)^T C^{-1}) N_1. \end{aligned} \quad (3.43)$$

Using the fundamental charge conjugation property of the Dirac algebra, $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$, the bilinear resolves to

$$\overline{N_1} \gamma^\mu N_1 = -\overline{N_1} \gamma^\mu N_1 \implies \overline{N_1} \gamma^\mu N_1 \equiv 0. \quad (3.44)$$

Consequently, the vector current of the Majorana fermion vanishes entirely, leaving only its axial-vector current. The effective Lagrangian for the Z-boson exchange therefore couples the dark matter axial current to the weak neutral current of the electron, J_μ^e . In the low-momentum transfer regime relevant for direct detection ($q^2 \ll m_Z^2$), the Z-boson propagator simplifies to $\frac{1}{q^2 - m_Z^2} \approx -1/m_Z^2$, yielding

$$\mathcal{L}_{\text{eff}} \propto \frac{1}{m_Z^2} \left(\overline{N_1} \gamma^\mu \gamma^5 N_1 \right) J_\mu^e. \quad (3.45)$$

The Z-boson amplitude is therefore suppressed by the massive electroweak scale ($1/m_Z^2$), and further by a loop function that vanishes with the dark matter mass. Together these leave the Z channel negligible against the electromagnetic anapole interaction, as quantified in Section 3.5.

3.4 Photon exchange and the anapole moment

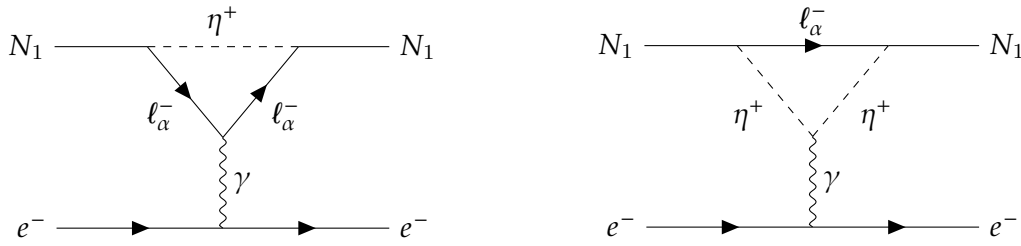


Figure 3.5: One-loop photon exchange diagrams mediating elastic scattering off an electron. Left: The photon couples to the internal charged lepton ℓ_α^- . Right: The photon couples to the internal charged scalar η^+ .

While scattering amplitudes mediated by the Z-boson and Higgs boson are suppressed by their massive propagators, dark matter can also interact with ordinary matter via massless photon exchange (see Figure 3.5). Because the Majorana dark matter candidate N_1 is electrically neutral, it possesses no tree-level coupling to photons, so its electromagnetic interactions arise purely at the one-loop level.

Due to its Majorana nature, N_1 cannot possess any vector charge, magnetic dipole moment, or electric dipole moment. Its leading electromagnetic interaction is the anapole moment, whose vertex is proportional to the transverse structure $(q^2 \gamma^\mu - q^\mu \not{q}) \gamma^5$. Because this is an $\mathcal{O}(q^2)$ effect, we must carefully retain the q^2 dependence throughout the loop calculations, unlike the Z-boson amplitude which was calculated in the static limit ($q \rightarrow 0$).

3.4.1 Photon coupling to the charged lepton

In this topology, illustrated in Figure 3.6, the photon couples to the massive charged lepton (ℓ^-) running inside the loop.

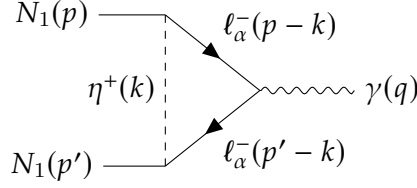


Figure 3.6: Vertex correction topology where the photon couples directly to the internal charged-lepton line. The loop momentum k is routed through the scalar propagator.

We assign momenta similarly to previous diagrams. The photon carries momentum q out of the lepton line. Reading against the fermion flow, the unintegrated loop amplitude is

$$\begin{aligned}
 i\mathcal{M}_c^\mu &= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-iy_{\alpha 1}^* P_L) \left[\frac{i(\not{p}' - \not{k} + m_\ell)}{(p' - k)^2 - m_\ell^2} \right] (+ie\gamma^\mu) \left[\frac{i(\not{p} - \not{k} + m_\ell)}{(p - k)^2 - m_\ell^2} \right] (-iy_{\alpha 1} P_R) \\
 &\quad \times u(p) \frac{i}{k^2 - m_{\eta^+}^2} \\
 &= -e|y_{\alpha 1}|^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') [P_L(\not{p}' - \not{k} + m_\ell)\gamma^\mu(\not{p} - \not{k} + m_\ell)P_R] u(p)}{((p' - k)^2 - m_\ell^2)((p - k)^2 - m_\ell^2)(k^2 - m_{\eta^+}^2)}. \tag{3.46}
 \end{aligned}$$

The presence of the lepton mass m_ℓ modifies the numerator algebra. We must evaluate the full, explicit structure. Expanding the brackets, we obtain four distinct terms,

$$\begin{aligned}
 \mathcal{N}^\mu &= P_L(\not{p}' - \not{k} + m_\ell)\gamma^\mu(\not{p} - \not{k} + m_\ell)P_R \\
 &= P_L(\not{p}' - \not{k})\gamma^\mu(\not{p} - \not{k})P_R \\
 &\quad + P_L(\not{p}' - \not{k})\gamma^\mu m_\ell P_R \\
 &\quad + P_L m_\ell \gamma^\mu(\not{p} - \not{k})P_R \\
 &\quad + P_L m_\ell \gamma^\mu m_\ell P_R. \tag{3.47}
 \end{aligned}$$

Starting with the first term (zero mass insertions),

$$\begin{aligned}
 P_L(\not{p}' - \not{k})\gamma^\mu(\not{p} - \not{k})P_R &= (\not{p}' - \not{k})P_R\gamma^\mu(\not{p} - \not{k})P_R \\
 &= (\not{p}' - \not{k})\gamma^\mu P_L(\not{p} - \not{k})P_R \\
 &= (\not{p}' - \not{k})\gamma^\mu(\not{p} - \not{k})P_R P_R \\
 &= (\not{p}' - \not{k})\gamma^\mu(\not{p} - \not{k})P_R. \tag{3.48}
 \end{aligned}$$

Evaluating the second and third terms (single mass insertions),

$$\begin{aligned}
 P_L(\not{p}' - \not{k})\gamma^\mu m_\ell P_R &= m_\ell(\not{p}' - \not{k})P_R\gamma^\mu P_R \\
 &= m_\ell(\not{p}' - \not{k})\gamma^\mu P_L P_R = 0, \tag{3.49}
 \end{aligned}$$

$$\begin{aligned}
 P_L m_\ell \gamma^\mu(\not{p} - \not{k})P_R &= m_\ell \gamma^\mu P_R(\not{p} - \not{k})P_R \\
 &= m_\ell \gamma^\mu(\not{p} - \not{k})P_L P_R = 0. \tag{3.50}
 \end{aligned}$$

Evaluating the final term (double mass insertion),

$$\begin{aligned}
 P_L m_\ell \gamma^\mu m_\ell P_R &= m_\ell^2 \gamma^\mu P_R P_R \\
 &= m_\ell^2 \gamma^\mu P_R. \tag{3.51}
 \end{aligned}$$

Because the chiral projectors annihilate the single-mass terms, only the structures with an even number of mass insertions (zero or two) survive. The simplified numerator retains the P_R structure,

$$\mathcal{N}^\mu = [(\not{p}' - \not{k})\gamma^\mu(\not{p} - \not{k}) + m_\ell^2\gamma^\mu] P_R. \quad (3.52)$$

Denominator and Momentum Shift

Using Feynman parameters x, y, z , the combined denominator is

$$D = x((p' - k)^2 - m_\ell^2) + y((p - k)^2 - m_\ell^2) + z(k^2 - m_{\eta^+}^2). \quad (3.53)$$

Expanding and grouping by powers of k (using $p^2 = p'^2 = M_1^2$ and $x + y = 1 - z$),

$$D = k^2 - 2k \cdot (xp' + yp) + (1 - z)M_1^2 - (1 - z)m_\ell^2 - zm_{\eta^+}^2. \quad (3.54)$$

Completing the square with shift vector $Q_c = xp' + yp$ (where $Q_c^2 = (1 - z)^2M_1^2 - xyq^2$), the effective mass Δ_c is

$$\begin{aligned} \Delta_c &= Q_c^2 - \left[(1 - z)M_1^2 - (1 - z)m_\ell^2 - zm_{\eta^+}^2 \right] \\ &= zm_{\eta^+}^2 + (1 - z)m_\ell^2 + M_1^2[(1 - z)^2 - (1 - z)] - xyq^2. \end{aligned} \quad (3.55)$$

Using $(1 - z)^2 - (1 - z) = -z(1 - z)$,

$$\Delta_c = zm_{\eta^+}^2 + (1 - z)m_\ell^2 - z(1 - z)M_1^2 - xyq^2. \quad (3.56)$$

Isolation of Divergence and Dimensional Regularisation

We shift the loop momentum $k = \ell + Q_c$. To isolate the ultraviolet divergence, we extract the terms quadratic in the loop momentum ℓ ,

$$\mathcal{N}_{div}^\mu = (-\not{\ell})\gamma^\mu(-\not{\ell})P_R = \not{\ell}\gamma^\mu\not{\ell}P_R. \quad (3.57)$$

Applying d -dimensional symmetric integration, we make the substitution $\not{\ell}\gamma^\mu\not{\ell} \rightarrow \frac{2-d}{d}\ell^2\gamma^\mu$. Inserting this back into the amplitude, the divergent integral becomes

$$i\mathcal{M}_{c,div}^\mu = -e|y_{\alpha 1}|^2 [\bar{u}(p')\gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \ell}{(2\pi)^d} \frac{2 \left(\frac{2-d}{d} \ell^2 \right)}{(\ell^2 - \Delta_c)^3}. \quad (3.58)$$

As derived for the Z-boson topologies, the combination of the Feynman parameter factor $2! = 2$, the Dirac trace factor $\frac{2-d}{d}$, and the d -dimensional integral measure cancels the d dependence. In $d = 4 - 2\delta$ dimensions, the integral evaluates to

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{2 \left(\frac{2-d}{d} \ell^2 \right)}{(\ell^2 - \Delta_c)^3} \xrightarrow{\overline{\text{MS}}} \frac{-i}{16\pi^2} \left[\frac{1}{\delta} - 1 - \ln \Delta_c \right]. \quad (3.59)$$

Multiplying our prefactor $(-e|y_{\alpha 1}|^2)$ by the integrated result $\left(\frac{-i}{16\pi^2} \right)$, the imaginary unit i is restored, yielding the divergent contribution,

$$i\mathcal{M}_{c,div}^\mu = +i \frac{e|y_{\alpha 1}|^2}{16\pi^2} [\bar{u}(p')\gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{\delta} - 1 - \ln \Delta_c \right]. \quad (3.60)$$

Evaluating the Finite Numerator

The remaining terms of the shifted numerator, independent of the loop momentum ℓ , constitute the finite part of the amplitude. Dropping the odd powers of ℓ (which integrate to zero due to spherical symmetry), we obtain

$$\mathcal{N}_{fin}^\mu = [(\not{p}' - \not{Q}_c)\gamma^\mu(\not{p} - \not{Q}_c) + m_\ell^2\gamma^\mu] P_R. \quad (3.61)$$

Because $\mathcal{N}_{fin}^\mu$ carries no loop momentum ℓ , its integral evaluates identically to a standard 4-dimensional scalar integral,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta_c)^3} = \frac{-i}{32\pi^2\Delta_c}. \quad (3.62)$$

Combining this finite amplitude with the previously calculated divergent amplitude, the complete equation for Figure 3.6 is

$$i\mathcal{M}_c^\mu = +i \frac{e|y_{\alpha 1}|^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \left[\bar{u}(p')\gamma^\mu P_R u(p) \left(\frac{1}{\delta} - 1 - \ln \Delta_c \right) + \frac{\bar{u}(p')\mathcal{N}_{fin}^\mu u(p)}{\Delta_c} \right], \quad (3.63)$$

where the finite numerator is $\mathcal{N}_{fin}^\mu = [(\not{p}' - \not{Q}_c)\gamma^\mu(\not{p} - \not{Q}_c) + m_\ell^2\gamma^\mu] P_R$, and the effective mass is $\Delta_c = zm_{\eta^+}^2 + (1-z)m_\ell^2 - z(1-z)M_1^2 - xyq^2$.

3.4.2 Photon coupling to the charged scalar

In this final vertex topology, illustrated in Figure 3.7, the external photon couples to the internal charged scalar (η^+). Because the scalar and the lepton must possess opposite electric charges to conserve charge at the dark matter interaction vertex, their respective photon couplings carry a relative minus sign.

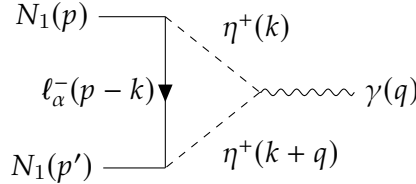


Figure 3.7: Vertex correction topology where the photon couples directly to the internal charged scalar line.

The loop momentum k is assigned to the first scalar line. The incoming photon carries momentum q into the vertex. By momentum conservation, the second scalar line carries $k+q$, and the internal massive fermion carries $p-k$. Evaluating against the fermion flow, the unintegrated loop amplitude is constructed as

$$\begin{aligned} i\mathcal{M}_d^\mu &= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-iy_{\alpha 1}^* P_L) \left[\frac{i(\not{p} - \not{k} + m_{\ell_\alpha})}{(p-k)^2 - m_{\ell_\alpha}^2} \right] (-iy_{\alpha 1} P_R) u(p) \frac{i}{k^2 - m_{\eta^+}^2} \\ &\quad \times \frac{i}{(k+q)^2 - m_{\eta^+}^2} [-ie(2k+q)^\mu] \\ &= +e|y_{\alpha 1}|^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') [P_L(\not{p} - \not{k} + m_{\ell_\alpha})P_R(2k+q)^\mu] u(p)}{(p-k)^2(k^2 - m_{\eta^+}^2)((k+q)^2 - m_{\eta^+}^2)}. \end{aligned} \quad (3.64)$$

Isolating the internal fermion bilinear, we note that the photon attaches to the scalar field. Thus, there is no internal γ^μ matrix separating the chiral projectors,

$$\mathcal{N}^\mu = P_L(\not{p} - \not{k} + m_{\ell_\alpha})P_R(2k + q)^\mu. \quad (3.65)$$

Expanding the terms and applying the chiral commutation relations ($P_L \not{V} = \not{V} P_R$),

$$\begin{aligned} \mathcal{N}^\mu &= P_L(\not{p} - \not{k})P_R(2k + q)^\mu + P_L m_{\ell_\alpha} P_R(2k + q)^\mu \\ &= (\not{p} - \not{k})P_R P_R(2k + q)^\mu + m_{\ell_\alpha} P_L P_R(2k + q)^\mu. \end{aligned} \quad (3.66)$$

Because $P_L P_R = 0$, the mass insertion term vanishes. The numerator simplifies to

$$\mathcal{N}^\mu = (\not{p} - \not{k})P_R(2k + q)^\mu. \quad (3.67)$$

Denominator and Momentum Shift

Introducing the Feynman parameters x, y for the two scalar propagators and z for the single massive fermion propagator, the combined denominator evaluates to

$$D = x(k + q)^2 - x m_{\eta^+}^2 + y k^2 - y m_{\eta^+}^2 + z(p - k)^2 - z m_{\ell_\alpha}^2. \quad (3.68)$$

Grouping the terms by powers of k , we define the shift vector $Q_d = zp - xq$. The terms independent of k are grouped as $C = xq^2 + zM_1^2 - (x + y)m_{\eta^+}^2 - z m_{\ell_\alpha}^2$.

Using the parameter constraint $x + y = 1 - z$ and the on-shell condition $p^2 = M_1^2$ (with $2p \cdot q = -q^2$), the effective mass $\Delta_d = Q_d^2 - C$ is computed,

$$\begin{aligned} \Delta_d &= (z^2 M_1^2 + zxq^2 + x^2 q^2) - (xq^2 + zM_1^2 - (1 - z)m_{\eta^+}^2 - z m_{\ell_\alpha}^2) \\ &= z m_{\ell_\alpha}^2 + (1 - z)m_{\eta^+}^2 - z(1 - z)M_1^2 + q^2(zx + x^2 - x). \end{aligned} \quad (3.69)$$

Applying the algebraic identity $zx + x^2 - x = x(z + x - 1) = -xy$, the effective mass denominator reduces to

$$\Delta_d = z m_{\ell_\alpha}^2 + (1 - z)m_{\eta^+}^2 - z(1 - z)M_1^2 - xyq^2. \quad (3.70)$$

Isolation of Divergence and Dimensional Regularisation

Applying the momentum shift $k = \ell + Q_d$ to the simplified numerator yields

$$\mathcal{N}^\mu = (\not{p} - \not{\ell} - \not{Q}_d)P_R(2\ell + 2Q_d + q)^\mu. \quad (3.71)$$

Extracting the divergent $\mathcal{O}(\ell^2)$ component,

$$\mathcal{N}_{div}^\mu = (-\not{\ell})P_R(2\ell^\mu) = -2\not{\ell}\ell^\mu P_R. \quad (3.72)$$

Under d -dimensional symmetric integration, the trace transforms as $-2\not{\ell}\ell^\mu \rightarrow \frac{-2}{d}\ell^2\gamma^\mu$. Inserting this structure into the amplitude alongside the global prefactor $+e|y_{\alpha 1}|^2$,

$$i\mathcal{M}_{d,div}^\mu = +e|y_{\alpha 1}|^2 [\bar{u}(p')\gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \ell}{(2\pi)^d} \frac{2\left(\frac{-2}{d}\ell^2\right)}{(\ell^2 - \Delta_d)^3}. \quad (3.73)$$

As previously demonstrated in the Z-boson scalar topology, combining the Feynman parameter factor ($2! = 2$) with the trace coefficient generates $-4/d$. Here, the subsequent multiplication by the d -dimensional integration measure eliminates the dimension factor d , preventing the generation of a finite -1 constant during δ -expansion,

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{2 \left(\frac{-2}{d} \ell^2 \right)}{(\ell^2 - \Delta_d)^3} \xrightarrow{\overline{\text{MS}}} \frac{-i}{16\pi^2} \left[\frac{1}{\delta} - \ln \Delta_d \right]. \quad (3.74)$$

Combining the prefactor ($+e|y_{\alpha 1}|^2$) with the computed integral $\left(\frac{+i}{16\pi^2} \right)$, the divergent contribution to the scattering amplitude is

$$i\mathcal{M}_{d,div}^\mu = -i \frac{e|y_{\alpha 1}|^2}{16\pi^2} [\bar{u}(p') \gamma^\mu P_R u(p)] \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{\delta} - \ln \Delta_d \right]. \quad (3.75)$$

Evaluating the Finite Numerator

The remaining $\mathcal{O}(\ell^0)$ components form the unintegrated finite remainder. Neglecting odd powers of ℓ , which vanish under spherical integration, the trace reduces to

$$\mathcal{N}_{fin}^\mu = (\not{p} - \not{Q}_d) P_R (2Q_d + q)^\mu. \quad (3.76)$$

Because the second term does not contain any gamma matrices, the chiral projector P_R factors out directly to the right,

$$\mathcal{N}_{fin}^\mu = (\not{p} - \not{Q}_d) (2Q_d^\mu + q^\mu) P_R. \quad (3.77)$$

Integration of this term over the $d = 4$ scalar loop introduces the standard coefficient $\frac{-i}{32\pi^2 \Delta_d}$. Combining this with the global prefactor ($+e|y_{\alpha 1}|^2$) and the Feynman parameter $2!$ yields a finite prefactor of $-i \frac{e|y_{\alpha 1}|^2}{16\pi^2}$.

To align with the global prefactor of Figure 3.6 ($+i \frac{e|y_{\alpha 1}|^2}{16\pi^2}$), a relative minus sign is factored out from both the divergent and finite components. The fully assembled equation for the scalar topology is

$$i\mathcal{M}_d^\mu = +i \frac{e|y_{\alpha 1}|^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \left[-\bar{u}(p') \gamma^\mu P_R u(p) \left(\frac{1}{\delta} - \ln \Delta_d \right) - \frac{\bar{u}(p') \mathcal{N}_{fin}^\mu u(p)}{\Delta_d} \right], \quad (3.78)$$

where $\mathcal{N}_{fin}^\mu$ is the expanded trace defined above, and the effective mass is parameterised as $\Delta_d = z m_{\ell_\alpha}^2 + (1-z) m_{\eta^+}^2 - z(1-z) M_1^2 - xyq^2$.

3.4.3 Total electromagnetic vertex and divergence cancellation

The total one-loop electromagnetic vertex is constructed by summing the individual contributions from the lepton and scalar loops: $\mathcal{M}_{total}^\mu = \mathcal{M}_c^\mu + \mathcal{M}_d^\mu$.

Factoring out the shared global coefficient $+i \frac{e|y_{\alpha 1}|^2}{16\pi^2}$, the summation of the equations derived

for both diagrams yields

$$i\mathcal{M}_{total}^\mu = +i\frac{e|y_{\alpha 1}|^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \left[\bar{u}(p') \gamma^\mu P_R u(p) \left(\frac{1}{\delta} - 1 - \ln \Delta_c \right) + \frac{\bar{u}(p') \mathcal{N}_{c,fin}^\mu u(p)}{\Delta_c} \right. \\ \left. - \bar{u}(p') \gamma^\mu P_R u(p) \left(\frac{1}{\delta} - \ln \Delta_d \right) - \frac{\bar{u}(p') \mathcal{N}_{d,fin}^\mu u(p)}{\Delta_d} \right]. \quad (3.79)$$

Grouping the divergent elements, the $+1/\delta$ pole originating from Figure 3.6 perfectly cancels the $-1/\delta$ pole from Figure 3.7. This cancellation is not an accident of the algebra. Because N_1 is a gauge singlet, the Lagrangian contains no tree-level N_1 -photon vertex, and the anapole interaction the loop generates is an operator of dimension six. Neither can serve as a counterterm in a renormalisable theory, so the divergences have nothing to be absorbed into and must cancel among themselves. Any infinity surviving here would indicate an error in the algebra rather than new physics.

The logarithmic terms combine algebraically via $\ln A - \ln B = \ln(A/B)$, yielding the finite, total observable scattering amplitude,

$$i\mathcal{M}_{total}^\mu = +i\frac{e|y_{\alpha 1}|^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \left[\bar{u}(p') \gamma^\mu P_R u(p) \left(-1 - \ln \frac{\Delta_c}{\Delta_d} \right) \right. \\ \left. + \bar{u}(p') \left(\frac{\mathcal{N}_{c,fin}^\mu}{\Delta_c} - \frac{\mathcal{N}_{d,fin}^\mu}{\Delta_d} \right) u(p) \right]. \quad (3.80)$$

3.4.4 Verification of charge neutrality

Before extracting the momentum-dependent anapole interaction, we must verify the physical consistency of our one-loop result. By definition, the Majorana dark matter candidate N_1 is an electrically neutral particle. In Quantum Field Theory, the electromagnetic vertex function evaluated at zero momentum transfer ($q \rightarrow 0$, representing the static limit) corresponds directly to the physical electric charge of the particle. Therefore, to ensure that the loop corrections do not unphysically induce a bare electric charge, the total amplitude must vanish in this static limit. We evaluate the sum of the diagrams as $q \rightarrow 0$ and demand that $\mathcal{M}_{total}^\mu(q=0) = 0$.

The $q \rightarrow 0$ Limit and Domain Simplification

In the limit $q \rightarrow 0$, the incoming and outgoing momenta are equal ($p = p'$), allowing us to factor the spinors $\bar{u}(p) \dots u(p)$ globally. Furthermore, the effective mass denominators reduce to

$$\Delta_c(0) = zm_{\eta^+}^2 + (1-z)m_\ell^2 - z(1-z)M_1^2, \quad (3.81)$$

$$\Delta_d(0) = zm_\ell^2 + (1-z)m_{\eta^+}^2 - z(1-z)M_1^2. \quad (3.82)$$

The two denominators are exchanged by the parameter reflection $z \rightarrow 1-z$, so that $\Delta_d(z) = \Delta_c(1-z)$. This identity is used repeatedly below.

The integrand now depends on the Feynman parameters only through the combination $z = 1-x-y$, which reduces the integral over the parameter triangle to a single dimension.

Changing variables from (x, y) to (x, z) leaves the domain a triangle and the Jacobian equal to unity, so the order of integration can be exchanged. Slicing the triangle at fixed z , the remaining range of x has length $1 - z$, and integrating it out leaves that length behind as the measure,

$$\int_0^1 dx \int_0^{1-x} dy f(z) = \int_0^1 dz \int_0^{1-z} dx f(z) = \int_0^1 dz (1-z) f(z). \quad (3.83)$$

The full algebraic reduction and chiral Gordon decomposition of the two numerators is shown in Appendix A. Evaluating the numerators at $q = 0$, the only surviving terms are

$$\mathcal{N}_{c,fin}^\mu(0) = z^2 M_1^2 \gamma^\mu P_L + m_\ell^2 \gamma^\mu P_R, \quad (3.84)$$

$$\mathcal{N}_{d,fin}^\mu(0) = z(1-z) M_1^2 \gamma^\mu (P_L + P_R) = z(1-z) M_1^2 \gamma^\mu. \quad (3.85)$$

Applying the 1D integral measure, we separate the total amplitude into its divergent (Part A) and finite parts (Part B), which must together sum to zero.

Part A: Evaluating the Divergent Part

The constant -1 and the logarithmic terms originate from the dimensional regularisation trace,

$$\text{Part A} = \int_0^1 dz (1-z) [-1 - \ln \Delta_c(z) + \ln \Delta_d(z)]. \quad (3.86)$$

The constant term trivially integrates to $\int_0^1 dz (1-z)(-1) = -1/2$. This constant is the physical fingerprint of the $\overline{\text{MS}}$ scheme and must be canceled by the finite fractions.

For the logarithms, we exploit the symmetry $\Delta_d(z) = \Delta_c(1-z)$. Applying the substitution $u = 1-z$ to the second integral aligns its argument with the first, transforming the measure $(1-z)$ into u . Renaming $u \rightarrow z$, the combined logarithmic integral becomes

$$\int_0^1 dz [-(1-z) + z] \ln \Delta_c(z) = \int_0^1 dz (2z-1) \ln \Delta_c(z). \quad (3.87)$$

We evaluate this using integration by parts, assigning $u = \ln \Delta_c(z)$ and $dv = (2z-1)dz$. The boundary term $uv = -z(1-z) \ln \Delta_c(z)$ vanishes at both $z = 0$ and $z = 1$. We are left only with $-\int v du$,

$$\int_0^1 dz \frac{z(1-z) \Delta'_c(z)}{\Delta_c(z)}. \quad (3.88)$$

Thus, Part A evaluates to $-\frac{1}{2} + \int_0^1 dz \frac{z(1-z) \Delta'_c(z)}{\Delta_c(z)}$.

Part B: Evaluating the Finite parts

We now construct the combined finite contribution. Utilising the exact same $z \rightarrow 1-z$ symmetry substitution, we align the scalar diagram term over the lepton diagram denominator,

$$\text{Part B} = \int_0^1 dz \frac{(1-z) \mathcal{N}_{c,fin}^\mu(0) - z \mathcal{N}_{d,fin}^\mu(0)}{\Delta_c(z)}. \quad (3.89)$$

Isolating the P_L terms from the numerators, we find the following cancellation,

$$(1 - z) [z^2 M_1^2] - z [z(1 - z) M_1^2] = 0. \quad (3.90)$$

The surviving P_R piece of Part B is $\int_0^1 dz \frac{(1-z)m_\ell^2 - z^2(1-z)M_1^2}{\Delta_c(z)}$. We now add this to the result of the integration by parts in Part A. Factoring out the common $(1 - z)$ factor,

$$\text{Combined Numerator} = (1 - z) [z \Delta'_c(z) + m_\ell^2 - z^2 M_1^2]. \quad (3.91)$$

Expanding the internal bracket using the derivative $\Delta'_c(z) = m_{\eta^+}^2 - m_\ell^2 - M_1^2 + 2zM_1^2$,

$$\begin{aligned} &= (1 - z) [z(m_{\eta^+}^2 - m_\ell^2 - M_1^2 + 2zM_1^2) + m_\ell^2 - z^2 M_1^2] \\ &= (1 - z) [zm_{\eta^+}^2 + m_\ell^2(1 - z) - M_1^2(z - z^2)] \\ &= (1 - z) [zm_{\eta^+}^2 + (1 - z)m_\ell^2 - z(1 - z)M_1^2]. \end{aligned} \quad (3.92)$$

Through algebraic cancellation, the reconstructed bracket is exactly equal to the effective mass denominator $\Delta_c(z)$. Thus, the entire combined integral reduces to just the integration measure itself,

$$\int_0^1 dz \frac{(1 - z)\Delta_c(z)}{\Delta_c(z)} = \int_0^1 dz(1 - z) = +\frac{1}{2}. \quad (3.93)$$

Combining this result with the constant inherited from the dimensional regularisation trace in Part A, the total zero-momentum amplitude collapses,

$$\mathcal{M}_{total}^\mu(q = 0) \propto -\frac{1}{2} + \frac{1}{2} = 0. \quad (3.94)$$

Therefore, the effective electric charge of the dark matter vanishes, confirming that the dark matter candidate is electrically neutral.

3.4.5 Extraction of the anapole form factor

Following the verification of the gauge invariance of the one-loop electromagnetic vertex, the leading-order physical observable is extracted: the anapole moment.

In the fundamental theoretical calculation, the loop amplitude yields a momentum-dependent form factor $f_A(q^2)$. However, to connect this theoretical result to phenomenological direct detection experiments, the amplitude must be evaluated in the appropriate kinematic limit. By matching the full high-energy theory onto a low-energy Effective Field Theory (EFT), we can identify the specific interaction structures that survive at non-relativistic momentum transfers.

The Non-Relativistic Limit and Momentum Taylor Expansion

In standard direct detection kinematics for sub-GeV dark matter (e.g., for the XENON1T or LZ experiments), the target xenon atoms are effectively at rest, and the assumed dark matter particles scatter from the local galactic halo with non-relativistic velocities ($v \sim 10^{-3}c$).

Because the scattering occurs primarily off bound atomic electrons, the characteristic momentum transfer in these collisions is restricted to the $q \sim \mathcal{O}(\text{keV} - \text{MeV})$ regime.

At the same time, the masses of the virtual particles propagating inside the electromagnetic loop, namely the charged scotogenic scalar m_{η^+} and the internal charged lepton m_{ℓ_α} , reside at far higher energy scales. Enforcing the LEP collider bounds and the tau-philic flavour structure, these loop masses are bounded by $m_{\eta^+} \gtrsim 90 \text{ GeV}$ [32] and $m_\tau \approx 1.77 \text{ GeV}$ [27]. This hierarchy between the momentum transfer and the internal loop masses defines a physically infinitesimal expansion parameter,

$$\frac{q^2}{m_{\text{loop}}^2} \lesssim \mathcal{O}(10^{-7}) \ll 1. \quad (3.95)$$

Because the $\mathcal{O}(q^0)$ electric charge terms vanish to satisfy charge neutrality, the leading non-zero contribution to the scattering amplitude arises at $\mathcal{O}(q^2)$. Higher-order interaction terms, such as those proportional to $\mathcal{O}(q^4)$, are kinematically suppressed by additional powers of this infinitesimal parameter and are therefore negligible for direct detection experiments.

To analytically extract this leading $\mathcal{O}(q^2)$ contribution, a Taylor expansion of the loop integrands around $q^2 = 0$ is performed. The explicit momentum dependence is hidden within the effective mass denominators of the Feynman parameter integrals,

$$\Delta_{c,d}(q^2) = \Delta_{c,d}(0) - xyq^2. \quad (3.96)$$

Treating the momentum transfer as a perturbation, the propagators are expanded to first order in q^2 ,

$$\frac{1}{\Delta_{c,d}(q^2)} \approx \frac{1}{\Delta_{c,d}(0)} + \frac{xyq^2}{\Delta_{c,d}^2(0)} + \mathcal{O}(q^4). \quad (3.97)$$

Similarly, the dimensional regularisation procedure generated logarithmic terms dependent on these effective masses. Expanding these logarithms yields an analogous $\mathcal{O}(q^2)$ shift,

$$\ln \Delta_{c,d}(q^2) \approx \ln \Delta_{c,d}(0) - \frac{xyq^2}{\Delta_{c,d}(0)} + \mathcal{O}(q^4). \quad (3.98)$$

Applying these expansions systematically shifts the $\mathcal{O}(q^2)$ components out of the loop denominators and into the numerator, allowing for the analytical collection of all kinematic interaction terms.

Momentum Decomposition of the Finite Numerators

To construct the total expanded amplitude, the denominator expansions must be combined with the finite numerators derived via the Gordon decomposition (see Appendix A). Because these numerators contain momentum terms of varying orders, we first group them by their momentum transfer dependence q ,

$$\mathcal{N} = \mathcal{N}^{(0)} + \mathcal{N}^{(1)} + \mathcal{N}^{(2)}, \quad (3.99)$$

where $\mathcal{N}^{(0)}$ is independent of q (e.g., mass-induced currents), $\mathcal{N}^{(1)}$ contains linear q^μ or $\not{q}$ terms, and $\mathcal{N}^{(2)}$ contains quadratic q^2 or $q^\mu \not{q}$ terms.

Multiplying this numerator by the expanded propagator, and truncating any cross-terms of $\mathcal{O}(q^3)$ or higher, the general structure of the loop integrands is given by

$$\frac{\mathcal{N}}{\Delta(q^2)} \approx \underbrace{\frac{\mathcal{N}^{(0)}}{\Delta(0)}}_{\text{Category 1: } \mathcal{O}(q^0)} + \underbrace{\frac{\mathcal{N}^{(1)}}{\Delta(0)}}_{\text{Category 2: } \mathcal{O}(q^1)} + \underbrace{\left[\frac{\mathcal{N}^{(2)}}{\Delta(0)} + \frac{\mathcal{N}^{(0)}xyq^2}{\Delta^2(0)} \right]}_{\text{Category 3 \& 4: } \mathcal{O}(q^2)} + \mathcal{O}(q^3). \quad (3.100)$$

For Dirac fermions, linear momentum structures ($\mathcal{O}(q^1)$) within the effective vertex correspond to magnetic or electric dipole moments. However, by definition of charge conjugation symmetry, a Majorana fermion is its own antiparticle and is strictly forbidden from possessing any static electromagnetic moments. Consequently, any isolated $\mathcal{O}(q^1)$ terms must either vanish or reorganise at higher orders to preserve the Majorana condition.

There are two explicit $\mathcal{O}(q^1)$ numerators, $\mathcal{N}^{(1)}$, after Gordon decomposition,

$$\mathcal{N}_c^{(1)} = -xzM_1\gamma^\mu \not{q} P_R + yzM_1\not{q}\gamma^\mu P_L, \quad (3.101)$$

$$\mathcal{N}_d^{(1)} = -zyM_1\not{q}\gamma^\mu P_L - zxM_1\not{q}\gamma^\mu P_R + (1-z)(1-2x)M_1q^\mu P_L. \quad (3.102)$$

To evaluate these structures, the chiral projectors are first decomposed into their vector and axial-vector components ($P_{R,L} = \frac{1}{2} \pm \frac{1}{2}\gamma^5$).

Applying the vector projection ($P_{R,L} \rightarrow 1/2$), the products of Dirac matrices can be expanded into their symmetric and antisymmetric parts using the Clifford algebra,

$$\not{q}\gamma^\mu = q^\mu - i\sigma^{\mu\nu}q_\nu \quad \text{and} \quad \gamma^\mu\not{q} = q^\mu + i\sigma^{\mu\nu}q_\nu. \quad (3.103)$$

When evaluated between Majorana spinors, the antisymmetric tensor structures ($\bar{u}\sigma^{\mu\nu}u$) must identically vanish due to charge conjugation symmetry. Therefore, within the bilinear, we can effectively substitute $\not{q}\gamma^\mu \rightarrow q^\mu$ and $\gamma^\mu\not{q} \rightarrow q^\mu$. Applying this Majorana simplification to the explicit numerators yields

$$\mathcal{N}_c^{(1)} \Big|_{\text{Vector}} = \frac{1}{2}M_1z(-x+y)q^\mu, \quad (3.104)$$

$$\mathcal{N}_d^{(1)} \Big|_{\text{Vector}} = \frac{1}{2}M_1 \left[-z(y+x) + (1-z)(1-2x) \right] q^\mu. \quad (3.105)$$

Summing these two contributions to find the total vector amplitude, the y terms cancel ($zy - zy = 0$). Combining the remaining terms and factoring out the common vector structure reduces the total $\mathcal{O}(q^1)$ vector component to a simple polynomial,

$$\frac{1}{2}M_1q^\mu(1-z-2x)\bar{u}u. \quad (3.106)$$

This term does not vanish algebraically. Instead, it integrates to zero over the Feynman parameters. Integrating over the inner Feynman parameter $x \in [0, 1-z]$, the polynomial evaluates to zero,

$$\int_0^{1-z} (1-z-2x)dx = \left[(1-z)x - x^2 \right]_0^{1-z} = 0. \quad (3.107)$$

Thus, the parameter space of the quantum loop enforces the vanishing of the scalar dipole moment.

Applying the axial projection ($P_R \rightarrow +\frac{1}{2}\gamma^5$, $P_L \rightarrow -\frac{1}{2}\gamma^5$) to the same explicit numerators, the linear polynomials in x vanish identically upon executing the inner Feynman parameter integral over dx ,

$$\int_0^{1-z} dx \left(\mathcal{N}_c^{(1)} + \mathcal{N}_d^{(1)} \right) \Big|_{\text{Axial}} = -\frac{1}{2}z(1-z)^2 M_1 q^\mu \gamma^5. \quad (3.108)$$

While this pseudo-vector term ($\bar{u} q^\mu \gamma^5 u$) appears to be $\mathcal{O}(q^1)$, it carries an intrinsic kinematic suppression. Utilising the free Dirac equations of motion for the external spinors ($\not{p}u = M_1 u$ and $\bar{u}\not{p}' = M_1 \bar{u}$), we can derive the pseudo-scalar Gordon identity. By inserting the momentum transfer $q = p' - p$, the bare γ^5 matrix is related to a momentum contraction,

$$\bar{u}(p')\gamma^5 u(p) = \frac{1}{2M_1}\bar{u}(p') [\not{p}'\gamma^5 + \gamma^5\not{p}] u(p) = \frac{1}{2M_1}\bar{u}(p')\not{q}\gamma^5 u(p). \quad (3.109)$$

Substituting this identity back into our integrated numerator mathematically promotes the seemingly linear term to $\mathcal{O}(q^2)$,

$$-\frac{1}{2}z(1-z)^2 M_1 q^\mu \left[\frac{1}{2M_1} \not{q}\gamma^5 \right] = -\frac{1}{4}z(1-z)^2 q^\mu \not{q}\gamma^5. \quad (3.110)$$

Because the vector component integrates to zero and the axial component is promoted to quadratic order, we conclude there are zero surviving physical interactions at $\mathcal{O}(q^1)$. All remaining kinematic terms evaluate at $\mathcal{O}(q^2)$ or higher.

Gauge Invariance and the Ward Identity

The derived $\mathcal{O}(q^2)$ terms can be separated into two kinematic categories based on their dependence on the momentum transfer vector q^μ . Terms proportional to $q^2 \gamma^\mu \gamma^5$ are denoted as the transverse component, while terms proportional to $q^\mu \not{q}\gamma^5$ are denoted as the longitudinal component. To satisfy $U(1)_{\text{em}}$ gauge invariance, these terms cannot exist independently. The Ward identity ($q_\mu \mathcal{M}^\mu = 0$) dictates that they must dynamically balance one another to form the gauge-invariant anapole tensor structure: $(q^2 \gamma^\mu - q^\mu \not{q})\gamma^5$.

Parameterising the total $\mathcal{O}(q^2)$ amplitude with integral coefficients C_T (transverse) and C_L (longitudinal),

$$i\mathcal{M}_{(q^2)}^\mu = i [C_T q^2 \gamma^\mu \gamma^5 + C_L q^\mu \not{q}\gamma^5]. \quad (3.111)$$

Contraction with the external photon momentum q_μ yields $i(C_T + C_L)q^2 \not{q}\gamma^5$. For the amplitude to vanish and preserve gauge invariance, we require the integral identity,

$$\int_0^1 dx \int_0^{1-x} dy C_L(x, y, z) = - \int_0^1 dx \int_0^{1-x} dy C_T(x, y, z). \quad (3.112)$$

At the integrand level, the algebraic function $C_L(x, y, z)$ does not trivially equal $-C_T(x, y, z)$. However, shifts in the internal loop momentum correspond to integration by parts within the Feynman parameter space. The Ward identity mathematically ensures that the boundary terms of this integration evaluate to zero, forcing the two coefficients to be perfectly equal and opposite once integrated. Because gauge invariance guarantees this relationship, explicitly calculating the complicated longitudinal component is redundant. The total anapole form factor is uniquely determined by extracting only the transverse coefficient, C_T .

Extracting the Transverse Coefficient

To isolate the anapole form factor, every transverse $q^2\gamma^\mu$ contribution generated by the previous Taylor expansions must be collected and evaluated under the Majorana condition.

First, we verify that the vector components of these explicit $\mathcal{O}(q^2)$ terms vanish. Applying the vector projection ($P_{R,L} \rightarrow 1/2$) to the transverse numerators yields terms proportional to a single, unpaired Dirac matrix,

$$\mathcal{N}_c^{(2)} \Big|_{\text{Vector}} = \frac{xy}{2\Delta_c(0)} M_1 q^2 \gamma^\mu, \quad (3.113)$$

$$\mathcal{N}_d^{(2)} \Big|_{\text{Vector}} = 0. \quad (3.114)$$

When evaluated between Majorana spinors, the vector bilinear identically vanishes due to charge conjugation symmetry ($\bar{u}(p')\gamma^\mu u(p) = 0$). Consequently, the entire vector component of the transverse amplitude vanishes independently of the Feynman parameter integration, proving that the physical interaction is purely axial-vector.

Applying the axial projection ($P_R \rightarrow +\frac{1}{2}\gamma^5$, $P_L \rightarrow -\frac{1}{2}\gamma^5$), the surviving $q^2\gamma^\mu\gamma^5$ coefficients are gathered from the three perturbative expansion sources:

1. The Explicit Transverse Numerators:

The Gordon decomposition of the lepton diagram (Figure 3.6) natively produced a transverse term, while the scalar diagram (Figure 3.7) produced none. Extracting the axial component yields

$$\frac{xyq^2\gamma^\mu P_R}{\Delta_c(0)} \xrightarrow{\text{Axial}} \left[\frac{xy}{2\Delta_c(0)} \right] q^2\gamma^\mu\gamma^5. \quad (3.115)$$

2. The Denominator Taylor Expansion:

Expanding the mass denominators generates the kinematic shift $+xyq^2/\Delta^2(0)$, which multiplies the static $\mathcal{O}(q^0)$ numerators. For the scalar diagram, the static numerator is proportional to $(P_L + P_R) = 1$, forming a pure vector current that vanishes under the Majorana condition. For the lepton diagram, applying the axial substitution yields

$$\left[z^2 M_1^2 \gamma^\mu P_L + m_{\ell_\alpha}^2 \gamma^\mu P_R \right] \frac{xyq^2}{\Delta_c^2(0)} \xrightarrow{\text{Axial}} \left[\frac{xy(m_{\ell_\alpha}^2 - z^2 M_1^2)}{2\Delta_c^2(0)} \right] q^2\gamma^\mu\gamma^5. \quad (3.116)$$

3. The Logarithmic Taylor Expansion:

The dimensional regularisation procedure produced logarithms that, when expanded, generate an $\mathcal{O}(q^2)$ shift attached to the global prefactor $\gamma^\mu P_R$. Projecting out the axial part,

$$\gamma^\mu P_R \left[xyq^2 \left(\frac{1}{\Delta_c(0)} - \frac{1}{\Delta_d(0)} \right) \right] \xrightarrow{\text{Axial}} \left[\frac{xy}{2} \left(\frac{1}{\Delta_c(0)} - \frac{1}{\Delta_d(0)} \right) \right] q^2\gamma^\mu\gamma^5. \quad (3.117)$$

Summing these three evaluated components, the common tensor structure $\frac{1}{2}xyq^2\gamma^\mu\gamma^5$ is factored out. The composite transverse integrand C_T is

$$C_T = \frac{1}{2}xy \left[\frac{1}{\Delta_c(0)} + \frac{m_{\ell_\alpha}^2 - z^2 M_1^2}{\Delta_c^2(0)} + \frac{1}{\Delta_c(0)} - \frac{1}{\Delta_d(0)} \right] \quad (3.118)$$

$$= \frac{1}{2}xy \left[\frac{2}{\Delta_c(0)} - \frac{1}{\Delta_d(0)} + \frac{m_{\ell_\alpha}^2 - z^2 M_1^2}{\Delta_c^2(0)} \right]. \quad (3.119)$$

The Final Anapole Form Factor (f_A)

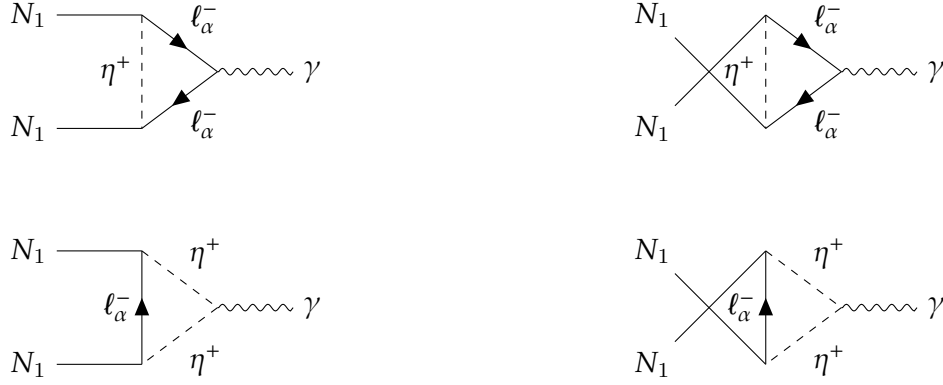


Figure 3.8: The one-loop diagrams generating the effective dark matter-photon coupling. For Majorana dark matter, all four diagrams contribute. The diagrams on the right represent the crossed topologies, which double the axial-vector contribution while mathematically canceling the vector contribution. For Dirac dark matter, only the uncrossed diagrams on the left exist.

To define the final physical observable, the full Majorana nature of the dark matter candidate must be accounted for. The preceding derivations evaluated the uncrossed loop topologies. However, a Majorana fermion permits crossed diagrams where the external lines are exchanged. Due to charge conjugation symmetry, the vector components of these crossed diagrams acquire a relative minus sign and cancel, while the axial-vector components acquire a relative plus sign, yielding an identical contribution. This multiplies a symmetry factor of 2 to the Majorana anapole moment (cancelling the $1/2$ from the C_T polynomial). The crossed and uncrossed topologies are illustrated in Figure 3.8.

Factoring the fully assembled, gauge-invariant anapole tensor $(q^2\gamma^\mu - q^\mu\cancel{q})\gamma^5$ out of the integration measure, the total effective electromagnetic vertex for the scotogenic Majorana dark matter is parameterised as

$$i\mathcal{M}_{total}^\mu = i f_A \cdot \bar{u}(p')(q^2\gamma^\mu - q^\mu\cancel{q})\gamma^5 u(p), \quad (3.120)$$

where the static anapole form factor $f_A \equiv f_A(q^2 \rightarrow 0)$ is defined by the finite two-dimensional Feynman parameter integral,

$$f_A = \frac{e|y_{\alpha 1}|^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \, xy \left[\frac{2}{\Delta_c(0)} - \frac{1}{\Delta_d(0)} + \frac{m_{\ell_\alpha}^2 - z^2 M_1^2}{\Delta_c^2(0)} \right]. \quad (3.121)$$

The static effective mass denominators evaluated at zero momentum transfer are

$$\Delta_c(0) = z m_{\eta^+}^2 + (1-z) m_{\ell_\alpha}^2 - z(1-z) M_1^2, \quad (3.122)$$

$$\Delta_d(0) = z m_{\ell_\alpha}^2 + (1-z) m_{\eta^+}^2 - z(1-z) M_1^2, \quad (3.123)$$

with the dependent parameter defined as $z = 1 - x - y$.

This framework establishes the leading-order electromagnetic interaction of the dark sector. The finite integral depends purely on the fundamental model masses ($M_1, m_{\eta^+}, m_{\ell_\alpha}$) and serves as the direct phenomenological input for calculating the differential scattering cross-section.

Analytical Evaluation and Comparison with Literature

To validate this fundamental derivation, the momentum-space amplitude is mapped to the effective field theory (EFT) representations standard in the literature. In the analysis of leptophilic dark matter by Kopp et al. (2014) [26], the general effective dimension-6 anapole interaction for a neutral fermion N_1 is defined as

$$\mathcal{L}_{\text{eff}} \supset A \bar{N}_1 \gamma^\mu \gamma^5 N_1 (\partial^\nu F_{\mu\nu}), \quad (3.124)$$

where A represents the anapole moment. Executing a Fourier transform into momentum space ($\partial_\mu \rightarrow -iq_\mu$) and utilising the definition of the electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the derivative operator maps to the momentum-space structure,

$$\partial^\nu F_{\mu\nu} \longrightarrow q^2 A_\mu - q_\mu (q^\nu A_\nu). \quad (3.125)$$

To extract the corresponding Feynman vertex rule from this Lagrangian, the classical fields must be evaluated as external quantum states. Multiplying by i and factoring out the external photon polarisation vector $\epsilon_\mu(q)$, the effective Lagrangian generates the exact gauge-invariant momentum tensor derived from the full scotogenic loop calculation,

$$iA \bar{u}(p') (q^2 \gamma^\mu - q^\mu \not{q}) \gamma^5 u(p) \epsilon_\mu(q). \quad (3.126)$$

Therefore, factoring out the external spinors and polarisation, the coefficient A from the coordinate-space EFT is mathematically identical to the derived momentum-space form factor f_A .

To prove the equivalence of this result with the literature, the Feynman parameter integral is analytically evaluated and compared to the closed-form expression derived in [26].

The parameter integral is reduced as in Section 3.4.4, by changing variables to (x, z) and integrating out x at fixed z . The mass denominators depend on z alone and pass through the inner integral untouched, but here the numerator carries an explicit factor xy , so what is left behind is no longer simply the length of the slice. Substituting $y = 1 - z - x$,

$$\int_0^{1-z} x(1-z-x) dx = \frac{(1-z)^3}{6}. \quad (3.127)$$

The measure is therefore $(1-z)^3/6$ instead of the $(1-z)$ found there. The $1/6$ multiplies the global prefactor $1/(16\pi^2)$ to yield $1/(96\pi^2)$,

$$f_A = \frac{e|y_{\alpha 1}|^2}{96\pi^2} \int_0^1 dz (1-z)^3 \left[\frac{2}{\Delta_c(0)} - \frac{1}{\Delta_d(0)} + \frac{m_{\ell_\alpha}^2 - z^2 M_1^2}{\Delta_c^2(0)} \right]. \quad (3.128)$$

Defining the dimensionless mass ratios $\mu = m_{\eta^+}^2/M_1^2$ and $\epsilon = m_{\ell_\alpha}^2/M_1^2$, and factoring $1/M_1^2$ out of the mass denominators, the 1D integral is evaluated in Mathematica [33]. The exact

symbolic integration yields

$$\begin{aligned} \mathcal{I}_{\text{raw}} = & \frac{1}{12\sqrt{-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu)}} \times \\ & \left[(2 - 6\epsilon + 6\mu) \arctan \left(\frac{1 + \epsilon - \mu}{\sqrt{-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu)}} \right) \right. \\ & + (2 - 6\epsilon + 6\mu) \arctan \left(\frac{1 - \epsilon + \mu}{\sqrt{-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu)}} \right) \\ & \left. + 3\sqrt{-(-1 + \epsilon)^2 + 2(1 + \epsilon)\mu} \ln \left(\frac{\epsilon}{\mu} \right) \right]. \end{aligned} \quad (3.129)$$

As shown in Appendix A, this result is equivalent to Equation (3.2) presented by Kopp et al. for the non-relativistic regime $|q^2| \ll m_{\ell_\alpha}^2$: [26]

$$f_A = -\frac{e|y_{\alpha 1}|^2}{96\pi^2 M_1^2} \left[\frac{3}{2} \ln \left(\frac{\mu}{\epsilon} \right) - \frac{1 + 3\mu - 3\epsilon}{\sqrt{(\mu - 1 - \epsilon)^2 - 4\epsilon}} \operatorname{arctanh} \left(\frac{\sqrt{(\mu - 1 - \epsilon)^2 - 4\epsilon}}{\mu - 1 + \epsilon} \right) \right]. \quad (3.130)$$

By independently deriving this final result for the anapole form factor from the fundamental Lagrangian and achieving an exact analytical match with the established literature, the theoretical framework is fully validated. This final expression serves as the phenomenological input for calculating the differential scattering cross-section of DM and electrons in the subsequent chapter.

3.5 Relative size of the three channels

The three preceding sections each produce an effective contact interaction between the dark matter and an atomic electron. Chapters 4 and 5 retain only the anapole, and we now make that choice quantitative instead of leaving it to the scaling arguments given above.

The anapole vertex carries two powers of the momentum transfer and the photon propagator carries $1/q^2$, so the two cancel, as Section 4.1 shows explicitly once the amplitude is assembled. Photon exchange therefore leaves a contact interaction and not the long-range force a massless mediator would ordinarily give, which is what allows the three channels to be set side by side,

$$\mathcal{L}_\gamma \supset e f_A (\bar{N}_1 \gamma^\mu \gamma^5 N_1) (\bar{e} \gamma_\mu e), \quad (3.131)$$

$$\mathcal{L}_Z \supset \xi_e (\bar{N}_1 \gamma^\mu \gamma^5 N_1) (\bar{e} \gamma_\mu \gamma^5 e), \quad (3.132)$$

$$\mathcal{L}_h \supset \Lambda_e (\bar{N}_1 N_1) (\bar{e} e), \quad (3.133)$$

with f_A from Equation (3.121) and Λ_e from Equation (3.1). The Z coefficient is taken from Ibarra et al. [24], whose quark expression we evaluate for an electron target,

$$\xi_e = \frac{|y_{\alpha 1}|^2 a_e}{32\pi^2 m_Z^2} \left[(v_\ell + a_\ell) G_2 \left(\frac{M_1^2}{m_{\eta^\pm}^2} \right) + (v_\nu + a_\nu) G_2 \left(\frac{M_1^2}{m_{\eta^0}^2} \right) \right], \quad (3.134)$$

where $a_e = -1/2$ is the weak isospin of the electron, $a_\ell = -\frac{g_Z}{2} \frac{1}{2}$, $v_\ell = -\frac{g_Z}{2} \left(\frac{1}{2} - 2 \sin^2 \theta_W \right)$, $v_\nu = a_\nu = \frac{g_Z}{2} \frac{1}{2}$, with g_Z as in Section 3.1, and the loop function is

$$G_2(x) = -1 + \frac{2[x + (1-x) \ln(1-x)]}{x^2}. \quad (3.135)$$

All three couplings scale as $|y_{\alpha 1}|^2$, so the Yukawa coupling cancels from every ratio below.

Both massive mediators are suppressed at low dark matter mass, and for the same reason. N_1 is a gauge singlet: it carries no weak charge and no Higgs coupling of its own, so neither coupling exists until the loop generates it. What the loop generates switches off as the dark matter gets light, since G_1 and G_2 both vanish as $M_1/m_\eta \rightarrow 0$, so both channels are weakest in exactly the mass range studied here. The Higgs is further held down by a second factor, because it couples in proportion to mass and the electron has very little of it, a factor $y_e = m_e/v \approx 2 \times 10^{-6}$.

The anapole is not suppressed in this way, despite appearances. The factor of M_1^2 in the denominator of Equation (3.121) suggests that f_A should grow as the dark matter gets lighter, but the bracket cancels it exactly, leaving a form factor set by the mediator instead of the dark matter,

$$f_A \xrightarrow{M_1 \ll m_\eta} -\frac{e|y_{\alpha 1}|^2}{16\pi^2 m_\eta^2} \times \mathcal{O}(1). \quad (3.136)$$

At $m_\eta = 100$ GeV this gives $f_A = 4.2 \times 10^{-7} \text{ GeV}^{-2}$, flat across the entire sub-GeV range.

Table 3.3: Ratio of the anapole coupling to the Z- and Higgs-mediated couplings for an electron target, at $m_\eta = 100$ GeV with $\lambda_3 = \lambda_4 = 1$. Event rates scale as the square of these ratios. The final column sits at $M_1 = m_\eta$, the edge of the fermionic branch of the model, and is included to show that the ordering does not reverse.

M_1	100 MeV	500 MeV	1 GeV	100 GeV
anapole / Z	1.2×10^7	4.8×10^5	1.2×10^5	50
anapole / Higgs	6.2×10^7	1.2×10^7	6.2×10^6	3.8×10^5

Table 3.3 collects the result. At the benchmark $M_1 = 100$ MeV the anapole exceeds the Z contribution by seven orders of magnitude in the coupling and the Higgs contribution by even more. In event rate, which goes as the square, the margins are 10^{14} and 10^{15} . Discarding both channels is therefore not an approximation that requires defending numerically. The conclusion also does not depend on the chosen benchmark quartics: λ_3 and λ_4 set the strength of the Higgs coupling to η and enter Λ_e linearly, so raising them from unity to the perturbative ceiling of 4π would strengthen the Higgs channel by about an order of magnitude and still leave it six orders below the anapole.

The comparison above is specific to an electron target and to sub-GeV masses, and this is worth emphasising. Ibarra et al. [24] and the review of Ávila et al. [22] both present the Z-mediated spin-dependent cross-section as the leading contribution for this model, but their treatments consider dark matter at the electroweak scale scattering off nuclei. Because the Z coupling grows as M_1^2 , what is negligible here is not negligible there: across Table 3.3 the gap closes by more than five orders of magnitude between 100 MeV and 100 GeV. There is no disagreement with the work presented here. The ordering of the three channels depends on the target and on the mass, and for electron recoils below a GeV the anapole is the only channel that matters.

Chapter 4

Dark Matter-Electron Scattering and the Non-Relativistic Limit

4.1 The full t -channel scattering amplitude



WITH the effective electromagnetic vertex of the scotogenic Majorana dark matter defined, we now proceed to construct the full physical observable: the scattering amplitude for a dark matter particle colliding with a Standard Model electron.

We consider the t -channel scattering process $N_1(p) + e^-(k) \rightarrow N_1(p') + e^-(k')$, where p, p' and k, k' are the initial and final four-momenta of the dark matter and electron, respectively. The momentum transfer carried by the virtual photon is defined as $q = p - p' = k' - k$.

The complete relativistic scattering amplitude is assembled by contracting the effective dark matter anapole current with the standard quantum electrodynamic (QED) electron current, mediated by the photon propagator. The three fundamental components of this interaction are:

1. **The Dark Matter Vertex:** As derived in Section 3.4.5, the effective anapole vertex is given by:

$$i\Gamma_{\text{DM}}^\mu = if_A(q^2\gamma^\mu - q^\mu \not{q})\gamma^5,$$

2. **The Photon Propagator:** In the Feynman gauge, the massless photon mediating the electromagnetic interaction is:

$$\Pi_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2},$$

3. **The Electron Vertex:** The standard QED interaction vertex for an electron (with charge $-e$) is:

$$i\Gamma_e^\nu = -ie\gamma^\nu.$$

Stitching these components together, the full matrix element $i\mathcal{M}$ is:

$$i\mathcal{M} = [\bar{u}(p')i\Gamma_{\text{DM}}^\mu u(p)] \Pi_{\mu\nu} [\bar{u}(k')i\Gamma_e^\nu u(k)].$$

Substituting the explicit forms of the vertices and the propagator into the expression, we obtain:

$$i\mathcal{M} = [\bar{u}(p')if_A(q^2\gamma^\mu - q^\mu \not{q})\gamma^5 u(p)] \left(\frac{-ig_{\mu\nu}}{q^2} \right) [-ie\bar{u}(k')\gamma^\nu u(k)].$$

Pulling the scalar constants and the $1/q^2$ pole to the front, the amplitude simplifies to:

$$i\mathcal{M} = -ie f_A \frac{1}{q^2} [\bar{u}(p')(q^2 \gamma^\mu - q^\mu \not{q}) \gamma^5 u(p)] [\bar{u}(k') \gamma_\mu u(k)].$$

At this stage a simplification occurs, because the electron current is conserved. We examine the second term within the anapole tensor, $-q^\mu \not{q}$, which is contracted with the electron current. Expanding the momentum transfer $q_\mu = (k' - k)_\mu$, this contraction yields:

$$q^\mu [\bar{u}(k') \gamma_\mu u(k)] = \bar{u}(k') \not{q} u(k) = \bar{u}(k') (\not{k}' - \not{k}) u(k).$$

By applying the free-particle Dirac equation for the on-shell initial and final state electrons, $\not{k} u(k) = m_e u(k)$ and $\bar{u}(k') \not{k}' = m_e \bar{u}(k')$, the expression evaluates identically to zero:

$$\bar{u}(k') (m_e - m_e) u(k) = 0.$$

Physically, this guarantees that the longitudinal component of the virtual photon cannot couple to the conserved electron current. Consequently, the $q^\mu \not{q}$ term vanishes completely from the amplitude. We are left solely with the $q^2 \gamma^\mu$ term:

$$i\mathcal{M} = -ie f_A \frac{1}{q^2} [\bar{u}(p')(q^2 \gamma^\mu) \gamma^5 u(p)] [\bar{u}(k') \gamma_\mu u(k)].$$

The q^2 factor generated by the anapole tensor in the numerator perfectly cancels the $1/q^2$ pole from the photon propagator in the denominator. Dividing out the imaginary unit, we arrive at the final, compact form of the relativistic scattering amplitude,

$$\mathcal{M} = -e f_A [\bar{u}(p') \gamma^\mu \gamma^5 u(p)] [\bar{u}(k') \gamma_\mu u(k)]. \quad (4.1)$$

Physical interpretation

In standard quantum electrodynamics, the force between two charged particles is mediated by a massless photon, whose propagator goes as $1/q^2$. If dark matter coupled to ordinary matter in that way, the cross-section would diverge as $q \rightarrow 0$, and direct detection experiments would have been overwhelmed by collisions decades ago.

The anapole moment avoids this. As demonstrated in Equation (4.1), the loop generates an explicit factor of q^2 in the numerator of the vertex, which cancels the pole of the propagator exactly and leaves a constant interaction strength at low energy. The consequence is clearest in coordinate space, where the two momentum-space behaviours Fourier transform to

$$\frac{1}{|\mathbf{q}|^2} \longrightarrow \frac{1}{4\pi r'} \quad (4.2)$$

$$\text{constant} \longrightarrow \delta^{(3)}(\mathbf{r}). \quad (4.3)$$

The first is the Coulomb potential, which falls off slowly enough to have infinite range. The second vanishes everywhere except at the origin. Despite communicating through a massless photon, the dark matter therefore behaves exactly as though the mediator were super-heavy, and it is this zero-range *contact interaction* that suppresses the cross-section at low energy.

4.2 Non-relativistic effective field theory (NR-EFT)

As derived in Equation (4.1), the relativistic scattering amplitude is expressed in terms of four-component Dirac spinors. However, the kinematics relevant for direct detection experiments are non-relativistic. Dark matter particles from the galactic halo have typical velocities on the order of $v \sim 10^{-3}c$, and the bound target electrons in atomic orbitals also reside in non-relativistic momentum states.

Therefore, to properly evaluate the scattering rate and connect it to atomic response functions, we need to map our relativistic amplitude onto a Non-Relativistic Effective Field Theory (NR-EFT) framework. Following the formalism introduced by Fitzpatrick et al. [25] and later adapted for electron scattering by Catena et al. [1] we can replace the Lorentz-invariant Dirac structures with a basis of Galilean-invariant operators.

4.2.1 The Galilean invariant basis

The standard NR-EFT framework is canonically constructed by considering the elastic scattering process between a dark matter particle χ and a free target particle. In this idealised elastic limit, the kinematics can be fully described by four independent, three-dimensional vectors:

1. The momentum transfer: $\mathbf{q} = \mathbf{p} - \mathbf{p}' = \mathbf{k}' - \mathbf{k}$
2. The transverse relative velocity: $\mathbf{v}^\perp = \mathbf{v} - \mathbf{v}_e - \frac{\mathbf{q}}{2\mu_{\chi e}}$
3. The dark matter spin operator: $\mathbf{S}_\chi$
4. The electron spin operator: $\mathbf{S}_e$

where $\mathbf{v} = \mathbf{p}/M_1$ and $\mathbf{v}_e = \mathbf{k}/m_e$ are the initial velocities of the dark matter and the target electron, and $\mu_{\chi e} = (M_1 m_e)/(M_1 + m_e)$ is the reduced mass of the two-particle system.

In a strictly elastic collision, energy conservation enforces the orthogonality condition $\mathbf{v}^\perp \cdot \mathbf{q} = 0$. However, for sub-GeV dark matter interacting with liquid xenon detectors, the target electrons are bound in atomic orbitals. The scattering is therefore an *inelastic* ionisation process, requiring an energy transfer ΔE to overcome the electron's binding energy E_B and provide its final recoil kinetic energy E_R . Consequently, the strict elastic orthogonality condition is broken, and $\mathbf{v}^\perp \cdot \mathbf{q} = \Delta E - q^2/2m_e \neq 0$, as evaluated in Section 4.3.2.

Despite this inelasticity, as demonstrated by Catena et al. [1] the standard basis of Hermitian, Galilean-invariant quantum mechanical operators $\mathcal{O}_i$ constructed from these vectors remains the valid framework for evaluating the amplitude, provided the altered energy conservation is accounted for in the final phase-space integration.

By taking all possible products of the four fundamental vectors defined above, one can construct a complete mathematical basis of Hermitian, Galilean-invariant quantum operators. A generic spin-1/2 dark matter interaction can yield up to 14 independent operators, the full definitions of which are detailed in the foundational NR-EFT literature [25, 1].

However, the Majorana anapole interaction is constrained. Because the interaction proceeds exclusively via the contraction of the axial-vector dark matter current $(\bar{\chi}\gamma^\mu\gamma^5\chi)$ with the vector electron current $(\bar{e}\gamma_\mu e)$, the vast majority of these operators identically vanish. The surviving momentum and velocity dependencies of the anapole moment map onto just

two,

$$O_8 = \mathbf{S}_\chi \cdot \mathbf{v}^\perp, \quad (4.4)$$

$$O_9 = i\mathbf{S}_\chi \cdot \left(\mathbf{S}_e \times \frac{\mathbf{q}}{m_e} \right). \quad (4.5)$$

Here O_8 couples the dark matter spin to the transverse relative velocity defined above, and O_9 couples it to the electron spin.

To prove this mapping and find the exact effective coupling constants c_8 and c_9 that weight these operators, the next step is to reduce the relativistic Dirac spinors from our derived scattering amplitude into non-relativistic, two-component Pauli spinors.

4.2.2 Direct non-relativistic expansion of the amplitude

The canonical approach in Effective Field Theory (e.g., Del Nobile et al. [34]) is to apply the non-relativistic (NR) expansion directly to the fermion spinors before squaring. This approach explicitly preserves the momentum and spin dependencies of the quantum operators.

We begin with the relativistic amplitude derived from the anapole vertex,

$$\mathcal{M} = -e f_A [\bar{u}(p')\gamma^\mu\gamma^5 u(p)] [\bar{u}(k')\gamma_\mu u(k)]. \quad (4.6)$$

In the non-relativistic limit ($v \ll c$), we evaluate the four-component Dirac spinors in the standard Dirac basis, where the gamma matrices are defined as

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (4.7)$$

To leading order in momentum, the free-particle spinors decompose into two-component Pauli spinors ξ ,

$$u(p) \approx \sqrt{2m} \begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \xi \end{pmatrix}, \quad \bar{u}(p') = u^\dagger(p')\gamma^0 \approx \sqrt{2m} \left(\xi'^\dagger \quad -\xi'^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}'}{2m} \right). \quad (4.8)$$

Expanding the Dark Matter Axial-Vector Current

We systematically evaluate the temporal and spatial components of the dark matter axial-vector current, $J_{\text{DM}}^\mu = \bar{u}(p')\gamma^\mu\gamma^5 u(p)$. For the temporal component ($\mu = 0$),

$$\begin{aligned} J_{\text{DM}}^0 &= u^\dagger(p')\gamma^5 u(p) \\ &\approx 2M_1 \begin{pmatrix} \xi'^\dagger & \xi'^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}'}{2M_1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2M_1} \xi \end{pmatrix} \\ &= 2M_1 \left(\xi'^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}'}{2M_1} \xi + \xi'^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2M_1} \xi \right) = \xi'^\dagger [\boldsymbol{\sigma} \cdot (\mathbf{p}' + \mathbf{p})] \xi. \end{aligned} \quad (4.9)$$

Defining the standard EFT parameter for the sum of initial and outgoing velocities as $\mathbf{v}_{\text{sum},\chi} = \frac{\mathbf{p} + \mathbf{p}'}{2M_1}$, and substituting the spin operator $\mathbf{S}_\chi = \boldsymbol{\sigma}/2$, the temporal current simplifies to

$$J_{\text{DM}}^0 = 2\mathbf{S}_\chi \cdot (2M_1 \mathbf{v}_{\text{sum},\chi}) = 4M_1 \mathbf{S}_\chi \cdot \mathbf{v}_{\text{sum},\chi}. \quad (4.10)$$

For the spatial component ($\mu = i$),

$$\begin{aligned} \mathbf{J}_{\text{DM}} &= \bar{u}(p') \boldsymbol{\gamma} \gamma^5 u(p) = u^\dagger(p') \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} u(p) \\ &\approx 2M_1 \begin{pmatrix} \xi'^\dagger & \xi'^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}'}{2M_1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2M_1} \xi \end{pmatrix} \\ &= 2M_1 \left(\xi'^\dagger \boldsymbol{\sigma} \xi + \mathcal{O}(p^2/M_1^2) \right) \approx 4M_1 \mathbf{S}_\chi. \end{aligned} \quad (4.11)$$

Expanding the Electron Vector Current

We apply the same formalism to the electron vector current, $J_e^\mu = \bar{u}(k') \gamma^\mu u(k)$. The temporal component trivially evaluates to the unit matrix,

$$J_e^0 = u^\dagger(k') u(k) \approx 2m_e \xi'^\dagger \mathbb{1}_e \xi = 2m_e. \quad (4.12)$$

The spatial component requires evaluating the off-diagonal Dirac matrices,

$$\begin{aligned} \mathbf{J}_e &= \bar{u}(k') \boldsymbol{\gamma} u(k) = u^\dagger(k') \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} u(k) \\ &\approx 2m_e \left(\xi'^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{k}'}{2m_e} \boldsymbol{\sigma} \xi + \xi'^\dagger \boldsymbol{\sigma} \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{2m_e} \xi \right) \\ &= \xi'^\dagger [(\boldsymbol{\sigma} \cdot \mathbf{k}') \boldsymbol{\sigma} + \boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{k})] \xi. \end{aligned} \quad (4.13)$$

We reduce the bracket using the Pauli identity $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$. Letting $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ be the momentum transfer,

$$\begin{aligned} (\boldsymbol{\sigma} \cdot \mathbf{k}') \boldsymbol{\sigma} + \boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{k}) &= \mathbf{k}' + \mathbf{k} + i \boldsymbol{\sigma} \times (\mathbf{k}' - \mathbf{k}) \\ &= 2m_e \mathbf{v}_{\text{sum},e} + i \boldsymbol{\sigma} \times \mathbf{q}. \end{aligned} \quad (4.14)$$

Substituting $\boldsymbol{\sigma} = 2\mathbf{S}_e$, the electron spatial current is

$$\mathbf{J}_e = 2m_e \mathbf{v}_{\text{sum},e} + 2i(\mathbf{S}_e \times \mathbf{q}). \quad (4.15)$$

Contracting the Currents and EFT Matching

Contracting these currents via the Minkowski metric ($\mathcal{M} = -e f_A [J_{\text{DM}}^0 J_e^0 - \mathbf{J}_{\text{DM}} \cdot \mathbf{J}_e]$), the amplitude becomes

$$\begin{aligned} \mathcal{M} &= -e f_A \left[(4M_1 \mathbf{S}_\chi \cdot \mathbf{v}_{\text{sum},\chi}) (2m_e) - (4M_1 \mathbf{S}_\chi) \cdot (2m_e \mathbf{v}_{\text{sum},e} + 2i \mathbf{S}_e \times \mathbf{q}) \right] \\ &= -8M_1 m_e (e f_A) \left[\mathbf{S}_\chi \cdot \mathbf{v}_{\text{sum},\chi} - \mathbf{S}_\chi \cdot \mathbf{v}_{\text{sum},e} - \frac{i}{m_e} \mathbf{S}_\chi \cdot (\mathbf{S}_e \times \mathbf{q}) \right]. \end{aligned} \quad (4.16)$$

To match this to the standard NR-EFT framework, we group the velocities into the Galilean-invariant transverse velocity. By standard definition, $\mathbf{v}^\perp \equiv \mathbf{v}_{\text{sum},\chi} - \mathbf{v}_{\text{sum},e}$. Using the vector cross-product identity $\mathbf{S}_\chi \cdot (\mathbf{S}_e \times \mathbf{q}) = -\mathbf{S}_\chi \cdot (\mathbf{q} \times \mathbf{S}_e)$, the amplitude factorises as follows:

$$\boxed{\mathcal{M} = -8M_1 m_e (e f_A) \left[\mathbf{S}_\chi \cdot \mathbf{v}^\perp - i \mathbf{S}_\chi \cdot \left(\mathbf{S}_e \times \frac{\mathbf{q}}{m_e} \right) \right]}. \quad (4.17)$$

The two terms in the bracket are precisely O_8 and O_9 , which is the mapping anticipated above. The same pair, with the same relative weight, follows from squaring the relativistic amplitude and evaluating the resulting traces, carried out independently in Appendix B. No other operator appears, because the anapole fixes the Lorentz structure of the interaction to an axial dark matter current contracted with a vector electron current, and that contraction produces only these two.

4.3 Scattering off bound atomic states

In the preceding sections, the non-relativistic effective operators were derived under the assumption that the target electron was a free plane wave characterised by a definite momentum $\mathbf{k}$. However, in noble gas time-projection chambers utilising liquid xenon or argon, the electrons are bound within atomic orbitals.

Following the theoretical framework established by Catena et al. [1] we must transition from free-particle kinematics to bound-state quantum mechanics. In this framework, the free scattering amplitude $M(\mathbf{q}, \mathbf{v}_{el}^\perp)$ cannot be evaluated statically because the bound electron carries a distribution of momenta, and no longer a definite $\mathbf{k}$. Writing $\mathbf{v}_e = \mathbf{k}/m_e$ out explicitly, the transverse velocity of Section 4.2.1 reads

$$\mathbf{v}_{el}^\perp = \mathbf{v} - \frac{\mathbf{q}}{2\mu_{\chi e}} - \frac{\mathbf{k}}{m_e}, \quad (4.18)$$

which is the same quantity as $\mathbf{v}^\perp$ above. Following Catena et al. [1], the subscript is carried from here on as a reminder that $\mathbf{k}$ is no longer a definite quantity.

To evaluate the transition amplitude between an initial atomic state $|1\rangle$ and an ionised final state $|2\rangle$, we perform a Taylor expansion of the free amplitude around the zero-momentum limit ($\mathbf{k} = 0$),

$$M(\mathbf{q}, \mathbf{v}_{el}^\perp) \approx M\Big|_{\mathbf{k}=0} + \left(\frac{\mathbf{k}}{m_e}\right) \cdot m_e \nabla_k M\Big|_{\mathbf{k}=0}. \quad (4.19)$$

When this expanded amplitude is substituted into the squared transition matrix element, the physical transition probability separates into three distinct mathematical terms,

$$|M_{1\rightarrow 2}|^2 = |M|^2 |f_{1\rightarrow 2}|^2 + 2m_e \text{Re}\left[M f_{1\rightarrow 2} \nabla_k M^* \cdot \mathbf{f}_{1\rightarrow 2}^*\right] + m_e^2 \left|\nabla_k M \cdot \mathbf{f}_{1\rightarrow 2}\right|^2. \quad (4.20)$$

Here, the atomic structure is entirely captured by two form factors evaluated in coordinate space. The scalar form factor $f_{1\rightarrow 2}(\mathbf{q})$ measures the standard density overlap, while the vectorial form factor $\mathbf{f}_{1\rightarrow 2}(\mathbf{q})$ promotes the electron momentum to the quantum mechanical derivative operator ∇ ,

$$f_{1\rightarrow 2}(\mathbf{q}) = \int d^3x \psi_2^*(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}} \psi_1(\mathbf{x}), \quad (4.21)$$

$$\mathbf{f}_{1\rightarrow 2}(\mathbf{q}) = \int d^3x \psi_2^*(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}} \left(\frac{i\nabla}{m_e}\right) \psi_1(\mathbf{x}). \quad (4.22)$$

4.3.1 The four atomic response functions

By evaluating the three expansion terms in the squared transition matrix element over all occupied initial and final electron states, Catena et al. [1] show that any general dark

matter-electron interaction can be characterised by four atomic response functions because the last term splits, according to whether $\nabla_k M$ contracts with $\mathbf{f}_{1 \rightarrow 2}$ freely or along $\mathbf{q}$. In terms of the two form factors above,

$$W_i^{n\ell}(k', \mathbf{q}) \equiv V \frac{4k'^3}{(2\pi)^3} \sum_{m=-\ell}^{\ell} \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \begin{cases} |f_{1 \rightarrow 2}(\mathbf{q})|^2, & i = 1, \\ \frac{\mathbf{q}}{m_e} \cdot f_{1 \rightarrow 2}(\mathbf{q}) \mathbf{f}_{1 \rightarrow 2}^*(\mathbf{q}), & i = 2, \\ |f_{1 \rightarrow 2}(\mathbf{q})|^2, & i = 3, \\ \left| \frac{\mathbf{q}}{m_e} \cdot \mathbf{f}_{1 \rightarrow 2}(\mathbf{q}) \right|^2, & i = 4, \end{cases} \quad (4.23)$$

where V is the normalisation volume, k' the asymptotic momentum of the ionised electron, and the sums run over the magnetic quantum number of the occupied orbital together with the angular quantum numbers of the final state. The first response is the ionisation form factor already standard in the sub-GeV literature, and the only one that survives for an interaction with no velocity or spin structure. The other three arise only from a general treatment of the interaction, and were introduced there.

None of the four carries an electron spin index. The spin sum is taken at the level of the amplitude, before the spatial integrals, so any dependence on $\mathbf{S}_e$ is carried by the kinematic coefficients of Section 4.3.2 rather than by the atomic responses themselves.

4.3.2 The dark matter response for the anapole moment

To compute the exact differential scattering rate for the scotogenic Majorana dark matter, we must quantitatively evaluate the dark matter response functions, $R_i^{n\ell}$, which act as the kinematic coefficients for the corresponding atomic response functions $W_i^{n\ell}$. The total squared matrix element for atomic ionisation is factorised as

$$|M_{ion}^{n\ell}|^2 = \sum_{i=1}^4 R_i^{n\ell} \left(v_{el}^\perp, \frac{q}{m_e} \right) W_i^{n\ell}(k', q). \quad (4.24)$$

By matching our derived non-relativistic amplitude (Equation (4.17)) to the EFT operator basis ($M = c_8 \mathcal{O}_8 + c_9 \mathcal{O}_9$), we identify the specific coupling constants generated by our one-loop anapole form factor f_A ,

$$c_8 = -8M_1 m_e e f_A, \quad c_9 = +8M_1 m_e e f_A. \quad (4.25)$$

Note that the squares of these couplings are identical: $c_8^2 = c_9^2 = 64M_1^2 m_e^2 e^2 f_A^2$.

By substituting c_8 and c_9 into the general expressions for $R_i^{n\ell}$ given by Catena et al. [1], which carry all fifteen couplings, and setting the dark matter spin to $j_\chi = 1/2$, the four kinematic coefficients evaluate to

$$R_1^{n\ell} = \frac{1}{4} c_8^2 (v_{el}^\perp)^2 + \frac{1}{8} c_8^2 \left(\frac{q}{m_e} \right)^2, \quad (4.26)$$

$$R_2^{n\ell} = -\frac{1}{2} c_8^2 \left(\mathbf{v}_{el}^\perp \cdot \frac{\mathbf{q}}{m_e} \right) \left(\frac{m_e}{q} \right)^2, \quad (4.27)$$

$$R_3^{n\ell} = \frac{1}{4} c_8^2, \quad (4.28)$$

$$R_4^{n\ell} = 0. \quad (4.29)$$

Both the squared transverse electron velocity $(v_{el}^\perp)^2$ and its projection onto the momentum transfer $(\mathbf{v}_{el}^\perp \cdot \mathbf{q})$ are evaluated at $\mathbf{k} = 0$. This does not undo the point that $\mathbf{k}$ is now described by a distribution: the expansion of Equation (4.19) has already carried that dependence into the form factors, leaving the amplitude itself to be taken at $\mathbf{k} = 0$. Energy conservation then sets the two combinations that survive,

$$(v_{el}^\perp)^2 = v^2 + \frac{q^2}{4\mu_{\chi e}^2} \frac{M_1 - m_e}{M_1 + m_e} - \frac{\Delta E_{n\ell}}{\mu_{\chi e}}, \quad (4.30)$$

$$\mathbf{v}_{el}^\perp \cdot \mathbf{q} = \Delta E_{n\ell} - \frac{q^2}{2m_e}, \quad (4.31)$$

where $\Delta E_{n\ell} = E_B^{n\ell} + E_R$ is the total energy deposited to overcome the atomic binding energy $E_B^{n\ell}$ and generate a free electron recoil E_R .

Chapter 5

Phenomenology and Results

EVALUATING the viability of the scotogenic anapole model requires translating the formal quantum field theory derivations into physical observables. This chapter bridges the microscopic matrix elements of the dark matter-electron interaction to the macroscopic event rates expected in direct detection experiments. The non-relativistic effective theory of Chapter 4 has already separated the dark matter physics from the atomic physics. What remains is to convolve that result with the halo velocity distribution and with the response of a dual-phase Time Projection Chamber (TPC), which records not a continuous energy spectrum but a scintillation pulse produced by ionised electrons.

5.1 Global constraints on the scotogenic parameter space

Before scanning the parameter space, we must impose the theoretical and experimental boundaries restricting the free parameters of the scotogenic model. The scotogenic framework couples the dark matter phenomenology directly to the generation of active neutrino masses, collider kinematics, and early-universe cosmology. Consequently, the scalar mediator mass m_η , the dark matter mass M_1 , and the Yukawa coupling y cannot be treated as entirely free parameters.

5.1.1 Collider kinematics and the LEP bound

The scalar mediator η is an $SU(2)_L$ electroweak doublet, meaning it contains both neutral (η_R, η_I) and charged ($\eta^\pm$) components. Because these scalars couple directly to the electroweak gauge bosons, their masses are constrained by precision collider data.

During the operation of the Large Electron-Positron (LEP) collider, precision measurements of the Z -boson decay width forbade the existence of any new charged particles with a mass below $m_Z/2 \approx 45.6$ GeV, as this would open a new $Z \rightarrow \eta^+ \eta^-$ decay channel. LEP-II expanded on this, searching directly for pair-produced charged scalars in the form of charged Higgs bosons. A charged scalar is pair produced at an e^+e^- collider through the photon and the Z , so the production rate is determined by its electric charge and weak isospin and not the specific model it belongs to. Those searches therefore bound $\eta^\pm$ as well. They set $m_{\eta^\pm} \gtrsim 70 - 90$ GeV [32, 27].

The anapole loop of Chapter 3 runs on the charged component, so this bound constrains it directly. A large splitting between the charged and neutral components would also

show in electroweak precision data. An approximate symmetry of the Higgs sector, called custodial symmetry, fixes the ratio $\rho = m_W^2/(m_Z^2 \cos^2 \theta_W)$ to unity at tree level, and splitting the members of a new doublet breaks it. Holding ρ at its measured value requires $\Delta m \lesssim O(10)$ GeV [35, 36], placing the neutral components of the scalar mediator η at the same scale as the charged one. That is important because it is the neutral component, not the charged one, that mediates the annihilation $N_1 N_1 \rightarrow \nu \bar{\nu}$ setting the relic density in Section 5.1.4. Both scalar masses are written m_η from here on,

$$m_\eta \gtrsim 90 \text{ GeV}. \quad (5.1)$$

5.1.2 Charged scalar searches at the LHC

At the LHC the charged scalar would be pair produced through the Drell–Yan process $pp \rightarrow \gamma^*/Z^* \rightarrow \eta^+ \eta^-$, and under the τ -philic assumption each would decay through $\eta^\pm \rightarrow \tau^\pm N_1$. Since N_1 escapes the detector, the event appears as two τ -leptons and missing transverse momentum.

That signature is the one targeted by searches for direct stau production. Such a search bounds the rate for a topology, namely a pair-produced charged scalar decaying to a τ and an invisible particle, and the stau is only the state used to label it. What must match is the production rate, and for a scalar produced this way that is fixed by the electric charge and the weak isospin alone. The charged component of the inert doublet carries $(T_3, Q) = (+1/2, +1)$. Supersymmetry supplies two staus, the partners of the left- and right-handed τ , and the left-handed one carries $(-1/2, -1)$. The two sets of quantum numbers therefore differ only by an overall sign. The photon term in the cross-section goes as Q^2 and the Z term as the square of the Z coupling, so neither sees that sign, while the photon– Z interference is linear in both and the two sign flips cancel against each other. The cross-sections are identical. The right-handed stau is an $SU(2)$ singlet, with $T_3 = 0$ and a different Z coupling, so $\eta^\pm$ is the analogue of $\tilde{\tau}_L$, and not of the right-handed $\tilde{\tau}_R$.

ATLAS quote the two cases separately [37]. Their combined result, 120 to 390 GeV, assumes stau-left and stau-right to be degenerate and therefore twice the production. The result for stau-left pair production alone is the one that applies here, and excludes

$$155 \text{ GeV} \lesssim m_{\eta^\pm} \lesssim 310 \text{ GeV}. \quad (5.2)$$

The limit is quoted within a simplified model, a benchmark with the production mode and decay chain fixed and only the masses left free, and its assumptions hold here: the neutralino is taken massless, which matches $M_1 \ll m_{\eta^\pm}$, and the branching ratio is taken to be unity, which holds under the strict τ -philic hierarchy. The excluded interval lies well above the LEP bound of the previous section, and well above the benchmark used throughout this chapter, so it does not further restrict the mediator masses studied here.

5.1.3 Lepton flavour violation and the MEG bound

As established in Section 2.2, the Yukawa couplings that generate the active neutrino masses also drive lepton flavour violating (LFV) decays, most notably $\mu \rightarrow e \gamma$, whose branching ratio the MEG experiment constrains to $\text{BR}(\mu \rightarrow e \gamma) < 1.5 \times 10^{-13}$ [31]. We now make that constraint quantitative.

In the scotogenic model, the theoretical prediction scales as [29, 30],

$$\text{BR}(\mu \rightarrow e \gamma) \approx \frac{3\alpha}{64\pi G_F^2 m_\eta^4} \left| \sum_k y_{\mu k}^* y_{ek} F_2 \left(\frac{M_k^2}{m_\eta^2} \right) \right|^2, \quad (5.3)$$

where α is the fine-structure constant, G_F the Fermi constant, and $F_2(x)$ a dimensionless loop function of order unity. Requiring an $\mathcal{O}(1)$ Yukawa coupling to satisfy the thermal relic density typically overproduces LFV decays unless the masses of the loop particles reside at the multi-TeV scale.

The τ -philic hierarchy adopted throughout this work ($y_{\tau 1} \gg y_{\mu 1}, y_{e 1}$) strongly suppresses the $k = 1$ term in the sum above. This hierarchy is not an arbitrary choice: scanning the viable parameter space of the same model, Ávila et al. [22] find that the cLFV constraints drive the dark matter Yukawa couplings to obey $|y_{e 1}| \ll |y_{\mu 1}| < |y_{\tau 1}|$, the same ordering assumed here. The Yukawa coupling that sets the anapole scattering rate is therefore not itself constrained by MEG. The bound applies instead to the heavier singlets $N_{2,3}$, which carry the flavour mixing demanded by oscillation data and are free to reside at the multi-TeV scale. The direct detection analysis that follows is in this sense decoupled from the LFV sector.

5.1.4 Thermal freeze-out and the relic density target

In the standard cosmological model, dark matter is produced thermally in the early universe. At high temperatures, the Majorana dark matter particles N_1 were in thermal equilibrium with the Standard Model plasma, continuously annihilating into Standard Model leptons through exchange of the scalar η . For the sub-GeV masses studied here annihilation into charged leptons is not possible: the τ -philic hierarchy leaves only the τ available, and N_1 is far too light to produce a pair of τ -leptons. The only available annihilation channel is $N_1 N_1 \rightarrow \nu \bar{\nu}$ through the neutral component. As the universe expanded and cooled below the dark matter mass ($T \ll M_1$), the annihilation rate dropped below the Hubble expansion rate, causing the dark matter comoving number density to freeze out.

Precision measurements of the cosmic microwave background by the Planck satellite constrain the present-day cold dark matter relic density to [4],

$$\Omega_{DM} h^2 \approx 0.120. \quad (5.4)$$

Meeting this abundance fixes the thermally averaged annihilation cross-section at freeze-out. The canonical value of $3 \times 10^{-26} \text{ cm}^3/\text{s}$ is derived for s -wave annihilation and cannot be used here, because the s -wave channel of this model is helicity-suppressed to zero [38].

We therefore adopt the relic condition derived for this model directly. Kubo, Ma and Suematsu [29] solve the Boltzmann equation for pure p -wave annihilation of the scotogenic singlet and obtain, for $\Omega_d h^2 = 0.12$ and in GeV units,

$$M_1 = 5.86 \times 10^{-8} x_f^{-1/2} e^{x_f}, \quad (5.5)$$

$$b = 2.44 \times 10^{-11} x_f^2, \quad (5.6)$$

where b is the p -wave coefficient of $\langle \sigma v \rangle = b \langle v^2 \rangle$. The first relation fixes the freeze-out parameter x_f for each dark matter mass, giving $x_f \approx 15.7$ at $M_1 = 100 \text{ MeV}$ and $x_f \approx 18.1$ at $M_1 = 1 \text{ GeV}$. The corresponding thermally averaged cross-section at freeze-out, $\langle \sigma v \rangle = 6b/x_f \approx (2.7 - 3.1) \times 10^{-26} \text{ cm}^3/\text{s}$ across the sub-GeV range, is close to the canonical benchmark but is here obtained from a p -wave calculation. In the heavy-mediator limit these relations give $\langle \sigma v \rangle \propto y^4 M_1^2 / m_\eta^4$, so that raising the mediator mass suppresses the annihilation rate steeply and forces a larger coupling in compensation.

The calculation above includes only $N_1 N_1$ annihilation, and leaves out coannihilation with η . Coannihilation is exponentially suppressed by the gap between the two masses [39],

and with $m_\eta \gtrsim 90$ GeV against a sub-GeV N_1 that gap is a factor of ninety or more, so it can safely be neglected here.

By requiring the model to satisfy the Planck relic density, the Yukawa coupling y is no longer a free parameter. For any given coordinate in the (M_1, m_η) parameter plane, the universe dictates a concrete coupling y_{relic} . This constrains the theoretical parameter space, providing a framework to evaluate direct detection bounds.

5.2 The differential event rate

To evaluate the observable event rate, we must isolate the fundamental particle and atomic physics from the astrophysical dark matter distribution. This requires defining the kinematic boundaries of the interaction, computing the differential cross-section at a fixed velocity, and integrating the result over the local dark matter flux. Two further steps then turn that rate into something a detector reports: the continuous recoil spectrum is mapped onto a discrete number of ionised electrons, and those onto the scintillation signal actually recorded.

5.2.1 Kinematic integration limits

For an incoming dark matter particle N_1 of mass M_1 to deposit a total energy $\Delta E_{n\ell} = E_R + E_B^{n\ell}$ into an atomic shell via a momentum transfer q , energy and momentum conservation fix how fast it must be travelling. Writing the deposit in terms of the incoming momentum $\mathbf{p} = M_1 \mathbf{v}$ and the transfer $\mathbf{q}$ gives $\Delta E_{n\ell} = (2\mathbf{p} \cdot \mathbf{q} - q^2)/2M_1$, so that $v \cos \theta = \Delta E_{n\ell}/q + q/2M_1$ with θ the angle between the two. Since $\cos \theta \leq 1$, the smallest velocity able to supply the deposit is

$$v_{min}(q, \Delta E_{n\ell}) = \frac{\Delta E_{n\ell}}{q} + \frac{q}{2M_1}. \quad (5.7)$$

The maximum possible velocity of a dark matter particle scattering in the laboratory frame, v_{max} , is determined by the Standard Halo Model (SHM), a Maxwell-Boltzmann distribution whose width is set by the local circular speed v_0 , the orbital speed of the Galactic rotation curve at the Sun. Particles exceeding the galactic escape velocity v_{esc} are not gravitationally bound to the Milky Way. Boosting into the rest frame of the Earth (which moves with velocity $v_\oplus$), the upper kinematic cutoff is

$$v_{max} = v_{esc} + v_\oplus. \quad (5.8)$$

Standard astrophysical parameters [1] are assumed to be $v_0 \approx 220$ km/s, $v_{esc} \approx 544$ km/s and $v_\oplus \approx 244$ km/s.

Because the required minimum velocity v_{min} cannot exceed the absolute galactic maximum velocity v_{max} , the allowed momentum transfer q is bounded. Setting $v_{min} = v_{max}$ and solving the quadratic equation for q defines the integration limits,

$$q_{min,max} = M_1 v_{max} \mp \sqrt{M_1^2 v_{max}^2 - 2M_1 \Delta E_{n\ell}}. \quad (5.9)$$

If the term inside the square root is negative ($M_1 v_{max}^2 < 2\Delta E_{n\ell}$), the dark matter particle possesses insufficient kinetic energy to ionise the shell, and the integration evaluates to zero.

5.2.2 The differential ionisation cross-section

Following the NR-EFT framework for dark matter-electron scattering established by Catena et al. [1], we define the differential ionisation cross-section for a specific atomic shell (n, ℓ) . The cross-section evaluates the transition probability for an incoming dark matter particle with a fixed velocity v . By integrating the squared matrix element over the kinematically allowed momentum transfer q , the differential cross-section with respect to the logarithmic electron recoil energy E_R is given by [1],

$$\frac{d\sigma_{ion}^{n\ell}}{d \ln E_R} = \frac{1}{128\pi M_1^2 m_e^2 v^2} \int_{q_{min}}^{q_{max}} dq q |M_{ion}^{n\ell}|^2. \quad (5.10)$$

Substituting the evaluated anapole matrix element and extracting the loop-induced coupling coefficient $c_8^2 = 64M_1^2 m_e^2 e^2 f_A^2$, the mass terms cancel against the denominator. The differential cross-section for scotogenic anapole dark matter reduces to

$$\boxed{\frac{d\sigma_{ion}^{n\ell}}{d \ln E_R} = \frac{e^2 f_A^2}{2\pi v^2} \int_{q_{min}}^{q_{max}} dq q \left[R_1^{n\ell} W_1^{n\ell}(k', q) + R_2^{n\ell} W_2^{n\ell}(k', q) + R_3^{n\ell} W_3^{n\ell}(k', q) \right]}, \quad (5.11)$$

where f_A is the anapole form factor. Following the exact decomposition formalism of [1], the terms $R_i^{n\ell}$ represent the kinematic response functions associated with the NR-EFT operator $\mathcal{O}_8$, and $W_i^{n\ell}(k', q)$ are the atomic response functions evaluated at the outgoing electron momentum $k' = \sqrt{2m_e E_R}$. The atomic response functions for xenon used throughout this work were tabulated with the DarkART package [40], the C++ successor to the code released with [1], on the logarithmic grid it constructs internally: one hundred points spanning $k' \in [0.1, 100]$ keV and $q \in [1, 1000]$ keV. The spacing is logarithmic because the responses fall off as power laws in both variables, and because for sub-GeV recoils the momentum transfer sits near the lower end of the tabulated range.

5.2.3 Astrophysical velocity integrals and the ionisation rate

While the cross-section defines the scattering probability for a fixed velocity, the local dark matter halo consists of a continuous velocity distribution $f(\mathbf{v})$. The differential interaction rate per unit detector mass is obtained by integrating the cross-section over the local dark matter flux [1],

$$\frac{dR_{ion}^{n\ell}}{d \ln E_R} = \frac{\rho_{DM}}{M_1 m_T} \int d^3v v f(\mathbf{v}) \Theta(v - v_{min}) \frac{d\sigma_{ion}^{n\ell}}{d \ln E_R}, \quad (5.12)$$

where ρ_{DM} is the local dark matter mass density, m_T is the mass of a single target atom, and the Heaviside step function $\Theta(v - v_{min})$ enforces the absolute kinematic cutoff.

In standard spin-independent dark matter models, the matrix element is independent of the dark matter velocity, allowing the convolution to factor entirely into a single mean inverse speed integral, $\eta(v_{min})$. However, the NR-EFT operator $\mathcal{O}_8$ introduces a direct dependence on $(v_{el}^\perp)^2$, hidden within the $R_1^{n\ell}$ response function.

It is essential to note that, for the inelastic scattering considered here, $\mathbf{v}_{el}^\perp$ is *not* orthogonal to the momentum transfer: as noted in Section 4.3, energy conservation gives $\mathbf{v}_{el}^\perp \cdot \mathbf{q} =$

$\Delta E_{n\ell} - q^2/2m_e \neq 0$. Decomposing $\mathbf{v}_{el}^\perp$ into components orthogonal to and along $\mathbf{q}$, and using $\mathbf{v} \cdot \hat{\mathbf{q}} = v_{min}$, its square separates into

$$(v_{el}^\perp)^2 = \underbrace{(v^2 - v_{min}^2)}_{\text{orthogonal to } \mathbf{q}} + \underbrace{\left(\frac{\Delta E_{n\ell}}{q} - \frac{q}{2m_e}\right)^2}_{\text{along } \mathbf{q}}. \quad (5.13)$$

Only the first term carries a dependence on the halo velocity. The second is fixed by the kinematics of the recoil and therefore factors out of the velocity integral.

Consequently, the velocity integration splits into two astrophysical functions. We define the standard integral $\eta(v_{min})$ and the higher-order velocity moment $h(v_{min})$ in the Earth frame (the halo integral $\eta(v_{min})$, always written with its argument, is not to be confused with the inert scalar η of the model),

$$\eta(v_{min}) = \int_{v>v_{min}} d^3v \frac{f(\mathbf{v})}{v} \quad (5.14)$$

$$h(v_{min}) = \int_{v>v_{min}} d^3v v f(\mathbf{v}). \quad (5.15)$$

Applying Fubini's theorem to swap the order of integration between q and v , the velocity-dependent piece evaluates analytically, while the longitudinal piece simply multiplies $\eta(v_{min})$,

$$\int_{v>v_{min}} d^3v \frac{f(\mathbf{v})}{v} (v_{el}^\perp)^2 = \left[h(v_{min}) - v_{min}^2 \eta(v_{min}) \right] + \left(\frac{\Delta E_{n\ell}}{q} - \frac{q}{2m_e} \right)^2 \eta(v_{min}). \quad (5.16)$$

By separating the terms in the anapole response functions, the continuous differential ionisation rate becomes a single numerical integral over the momentum transfer q . Converting from logarithmic to linear recoil energy ($d \ln E_R = dE_R/E_R$), the total rate summed over all occupied atomic orbitals is

$$\boxed{\frac{dR_{ion}}{dE_R} = \frac{1}{E_R} \frac{\rho_{DM}}{M_1 m_T} \frac{e^2 f_A^2}{2\pi} \sum_{n,\ell} \int_{q_{min}}^{q_{max}} dq q \left[\tilde{R}_1^{n\ell} W_1^{n\ell} + R_2^{n\ell} W_2^{n\ell} \eta(v_{min}) + R_3^{n\ell} W_3^{n\ell} \eta(v_{min}) \right]}, \quad (5.17)$$

where $\tilde{R}_1^{n\ell}$ absorbs the higher-order velocity integrations,

$$\tilde{R}_1^{n\ell} = \frac{1}{4} \left[h(v_{min}) - v_{min}^2 \eta(v_{min}) + \left(\frac{\Delta E_{n\ell}}{q} - \frac{q}{2m_e} \right)^2 \eta(v_{min}) \right] + \frac{1}{8} \left(\frac{q}{m_e} \right)^2 \eta(v_{min}). \quad (5.18)$$

This formulation captures both the velocity dispersion of the Milky Way halo and the longitudinal component of $\mathbf{v}_{el}^\perp$ generated by the inelasticity of the process, providing the theoretical input required for numerical evaluation. The longitudinal term is not negligible: dropping it, as a strictly elastic treatment would, shifts the limit by about 6%.

5.2.4 Discrete signal modelling for dual-phase TPCs

The rate derived in Section 5.2.3 is differential in the recoil energy E_R , but a dual-phase time-projection chamber does not measure energy directly. The route to the measured

signal runs through the discrete number of ionised electrons n_e that reach the gas phase, and the first step is the distribution $P(n_e | E_R)$, which follows the model established for sub-GeV scattering [12, 15] in three stages.

First, the recoiling electron creates $n^{(1)} = \lfloor E_R/W \rfloor$ additional quanta [15], either electrons or scintillation photons, where $W \approx 13.8$ eV is the average energy per quantum in liquid xenon. This W carries no index and is not the atomic response function $W_i^{n\ell}$ of Section 4.3.1. If an inner shell is ionised, such as $4d$ or $4p$, the subsequent de-excitation photoionises neighbouring atoms and adds $n^{(2)}$ secondary quanta, so that $N_Q = n^{(1)} + n^{(2)}$.

Second, the primary electron either recombines, with probability f_R , or escapes. At these recoil energies recombination is negligible ($f_R \approx 0$), so the primary electron always contributes exactly one observed electron, $n'_e = 1$.

Third, each of the N_Q additional quanta is observed as an extracted electron with probability $f_e \approx 0.83$ [15]. These are independent, so the number of secondary electrons n''_e is binomially distributed over N_Q trials.

Because the total number of observed electrons is $n_e = n'_e + n''_e = 1 + n''_e$, finding exactly n_e electrons requires exactly $n_e - 1$ secondary electrons to be produced from the N_Q available quanta. Therefore, the probability distribution is binomial and evaluated at $n_e - 1$ [1],

$$P(n_e | E_R) = \binom{N_Q}{n_e - 1} f_e^{n_e - 1} (1 - f_e)^{N_Q - (n_e - 1)}, \quad (5.19)$$

for $1 \leq n_e \leq N_Q + 1$, and evaluates to zero otherwise.

The observable event rate, representing the expected number of events per unit target mass and time yielding exactly n_e electrons, is obtained by integrating the continuous theoretical spectrum over this binomial probability distribution [1],

$$\frac{dR_{ion}}{dn_e} = \int dE_R P(n_e | E_R) \frac{dR_{ion}}{dE_R}. \quad (5.20)$$

5.2.5 The photomultiplier signal (S2)

The rate dR_{ion}/dn_e gives the spectrum in electrons reaching the gas phase, but the S2-only analyses measure the secondary proportional scintillation instead, quantified as a number of photoelectrons (PE). Because photon emission in the gas phase and the quantum efficiency of the photomultipliers are both statistical, a fixed n_e does not always give the same S2. The recorded PEs fluctuate about a mean set by the secondary-scintillation gain g_2 , which is captured by a Gaussian smearing [16, 15],

$$P(S2 | n_e) = \text{Gauss}(S2 | \mu = n_e g_2, \sigma = \sqrt{n_e} \sigma_{S2}). \quad (5.21)$$

The width scales as the square root of the electron number, weighted by an empirical factor σ_{S2} . XENON1T is characterised by $g_2 = 33$ PE per electron and $\sigma_{S2} = 7$ [16].

The final, continuous differential rate with respect to the observable S2 signal is obtained by summing the probability distribution over all possible integer values of n_e , multiplied by a detector-specific cut efficiency $\epsilon(S2)$ [1],

$$\boxed{\frac{dR_{ion}}{dS2} = \epsilon(S2) \sum_{n_e=1}^{\infty} P(S2 | n_e) \frac{dR_{ion}}{dn_e}}. \quad (5.22)$$

Integrating this over the S2 bins defined by the collaboration, $S2 \in [150, 200)$ PE and the three above it, gives the predicted number of events. Poisson statistics are then applied independently to each bin, and comparison with the observed counts at the quoted exposure sets the 90% confidence upper limits.

XENON1T report the cut efficiency $\epsilon(S2)$ in two pieces: a set of S2-dependent selection efficiencies plotted in Figure 2 of [16], and a flat 93% covering the selections not plotted there. Their product runs from $\epsilon \approx 0.009$ at $S2 = 150$ PE to $\epsilon \approx 0.027$ at 350 PE. We gather the efficiency in the same manner as Catena et al. [1]: the seven curves in that figure are read directly out of its vector graphics, calibrated against the printed axis ticks, and with the radius efficiency squared as they specify. The product then multiplies the raw fiducial exposure. The complete set of experimental and astrophysical inputs entering the limit is collected in Table 5.1.

Table 5.1: Experimental and astrophysical inputs used to derive the exclusion limits. The XENON1T quantities follow the re-analysis prescription of Catena et al. [1], and the quanta model follows Essig et al. [15].

Quantity	Symbol	Value	Source
Local DM density	ρ_{DM}	$0.4 \text{ GeV}/\text{cm}^3$	[1]
Circular velocity	v_0	220 km/s	[1]
Galactic escape velocity	v_{esc}	544 km/s	[1]
Earth velocity	$v_{\oplus}$	244 km/s	[1]
Fiducial exposure	—	80 755 kg days	[1, 16]
S2 bins [PE]	—	150–350, four bins of width 50	[1]
Observed events	N_{obs}	8, 7, 2, 1	[1]
Selection efficiency	$\epsilon(S2)$	S2-dependent, $\sim 10^{-2}$	[16]
Secondary gain	g_2	33 PE/electron	[16]
S2 width factor	σ_{S2}	7	[16]
Energy per quantum	W	13.8 eV	[15]
Electron fraction	f_e	0.833	[15]
Recombination fraction	f_R	0	[15]
Secondary quanta	$n^{(2)}$	(0, 0, 4, 6, 3) for (5p, 5s, 4d, 4p, 4s)	[15]

5.3 Results and parameter space exclusion

We are now ready to evaluate the viability of the sub-GeV scotogenic model.

To study the kinematic behaviour of the model before applying cosmological constraints, we define a theoretical benchmark: $M_1 = 100 \text{ MeV}$, $m_\eta = 100 \text{ GeV}$, and a dark Yukawa coupling $y = 1.0$. Consistent with the lepton flavour violation constraints discussed in Section 5.1.3, a τ -philic flavour structure ($m_\ell = m_\tau$) is assumed.

5.3.1 Differential cross-section and event rates

The primary physical quantity influencing the detector response is the differential ionisation cross-section, $d\sigma_{\text{ion}}/d \ln E_R$.

The relevant differential cross-sections for the case of sub-GeV DM scattering off a xenon atom are illustrated in the top panel of Figure 5.1. They correspond to the outermost xenon electron shells (5p, 5s, 4d, 4p, 4s), which dominate the signal entirely. The full calculation additionally includes the 3d, 3p and 3s shells, whose binding energies (0.711, 0.958 and

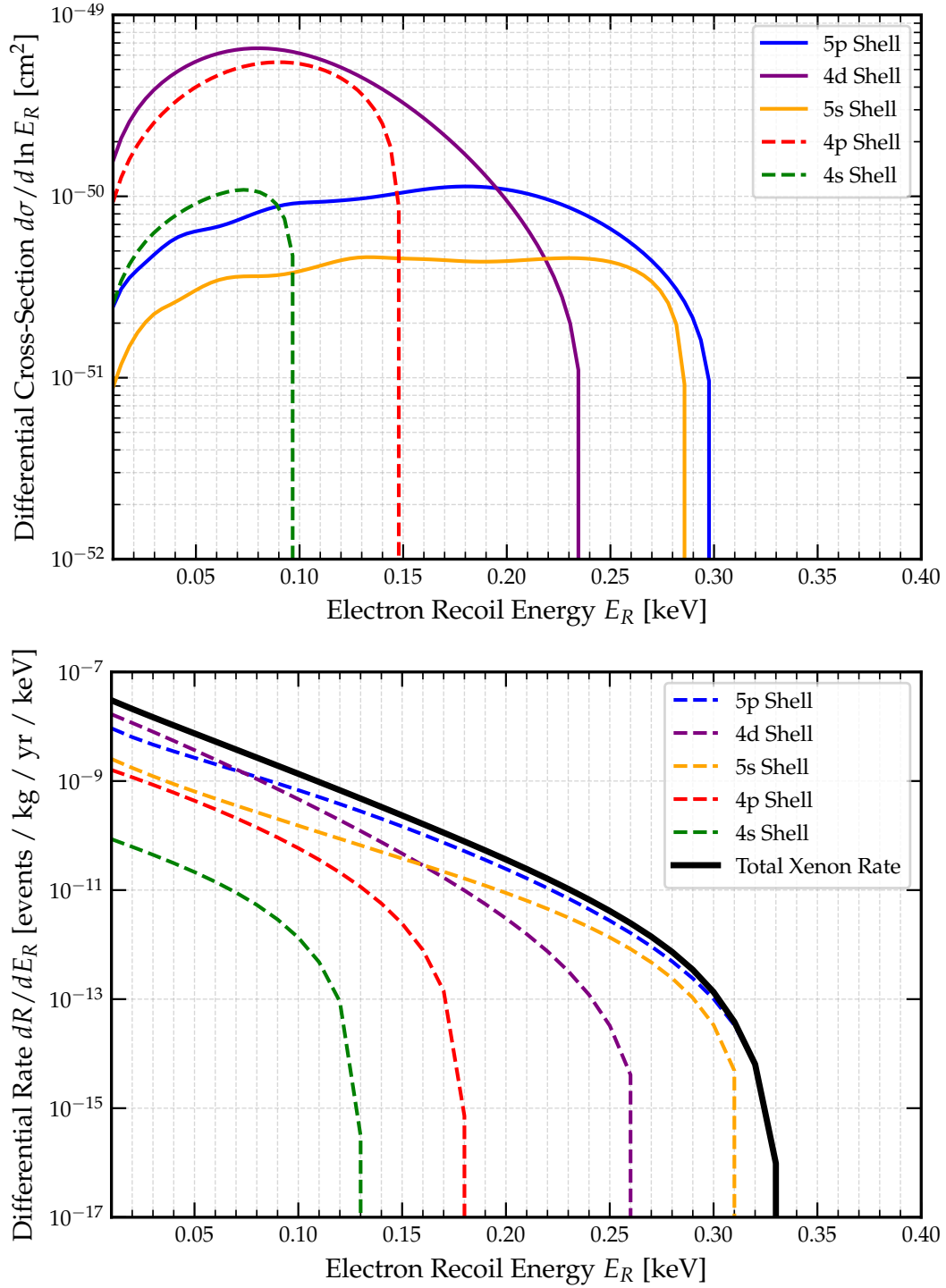


Figure 5.1: Top: The differential ionisation cross-section $d\sigma_{ion}/d\ln E_R$ for the benchmark point $M_1 = 100$ MeV, evaluated at the fixed relative velocity $v = 750$ km/s. The distinct step-like features correspond to the kinematic thresholds of the individual xenon atomic shells. Bottom: The corresponding differential event rate dR_{ion}/dE_R for a mediator mass $m_\eta = 100$ GeV, after convolution with the Standard Halo Model velocity distribution.

1.093 keV) place them within reach only for $M_1 \gtrsim 200$ MeV. Even at $M_1 = 1$ GeV they contribute below 1% of the total rate, and including them changes the exclusion limit by less than one per cent. The remaining shells are kinematically inaccessible: with $E_B = 4.84$ keV, the $2p$ shell would require $M_1 > 1.4$ GeV merely to open at the galactic escape velocity. All binding energies are taken from the Roothaan–Hartree–Fock tables distributed with DarkART [40], ensuring that the kinematics are consistent with the tabulated response functions. As the deposited recoil energy E_R increases, the total required energy transfer $\Delta E = E_R + E_B$ eventually exceeds the maximum kinetic energy of the incoming dark matter particles. When a specific shell demands more kinetic energy than is kinematically available, it becomes inaccessible, resulting in a sharp, step-like drop in the cross-section.

The bottom panel of Figure 5.1 displays the differential event rate dR_{ion}/dE_R . Convolution with the Standard Halo Model velocity distribution smooths the sharp atomic thresholds and imposes an absolute cutoff at the maximum kinetic energy available in the halo.

5.3.2 EFT-level exclusion limit on the anapole coupling

To translate the continuous energy spectrum into an experimental exclusion limit, the differential rate is mapped onto the discrete S2 photoelectron bins utilised in the XENON1T analysis, as derived in Section 5.2.5.

We first derive an intermediate upper limit at the effective field theory level by constraining the NR-EFT coupling constant, c_8 . The advantage is that it separates the atomic physics and the detector response from the loop mechanics of the underlying model. Because the total predicted event rate scales quadratically with this effective coupling ($N \propto c_8^2$), the upper limit at a 90% Confidence Level (C.L.) can be determined analytically. By computing the expected number of events N_{test} for a chosen benchmark coupling $c_{8,\text{test}}$, the maximum allowed coupling $c_{8,\text{max}}$ for a specific S2 bin is obtained by scaling against the 90% C.L. upper limit on signal events (N_{up}^{90}) for that bin,

$$c_{8,\text{max}} = c_{8,\text{test}} \sqrt{\frac{N_{\text{up}}^{90}}{N_{\text{test}}}}. \quad (5.23)$$

This scaling is evaluated across all S2 signal bins independently, and the most restrictive bin dictates the final phenomenological exclusion for a given dark matter mass. Once this $c_{8,\text{max}}$ curve is established, any combination of the model parameters generating the anapole moment can be evaluated against it.

The resulting 1D exclusion limit is presented in Figure 5.2. The solid curve denotes the maximum allowed coupling c_8 , with the shaded region above it excluded by the experimental data. The most stringent constraint reaches $c_8 \approx 6.0 \times 10^{-6}$ at $M_1 = 71$ MeV. Numerical values of the limit at representative masses are collected in Table 5.2, together with the equivalent bounds on the anapole moment g/Λ^2 (defined in Section 5.3.3) and on the Yukawa coupling.

Above the optimum the limit gradually weakens. While heavier particles possess sufficient kinetic energy to trigger the detector, the local dark matter number density ($n_\chi = \rho_{\text{DM}}/M_1$) decreases inversely with mass. This reduces the total particle flux passing through the target, resulting in a weaker constraint.

Conversely, as the mass decreases below the optimum the limit weakens exponentially, and the origin of that collapse is the *quantisation* of the ionisation signal. To register in

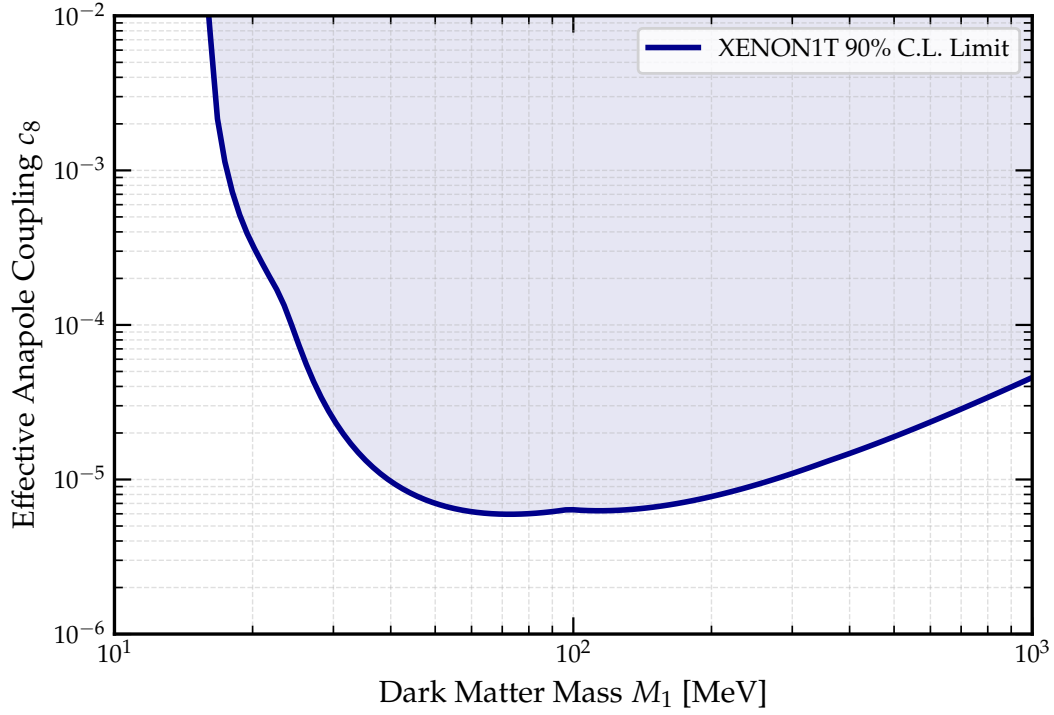


Figure 5.2: The 90% C.L. upper limit on the effective anapole coupling c_8 derived from the XENON1T S2-only data. The shaded region above the solid curve represents the parameter space excluded by the experiment. The limit weakens sharply below $M_1 \approx 30$ MeV since the ionisation signal is quantised, as explained below.

Table 5.2: The 90% C.L. upper limit on the effective anapole coupling c_8 , the equivalent limit on the anapole moment $g/\Lambda^2 = c_8/(8eM_1m_e)$, and the corresponding bound on the Yukawa coupling for $m_\eta = 100$ GeV. Couplings above these values are excluded.

M_1 [MeV]	c_8	g/Λ^2 [GeV $^{-2}$]	y ($m_\eta = 100$ GeV)
20	3.3×10^{-4}	13.4	5649
30	2.5×10^{-5}	0.667	1260
50	7.0×10^{-6}	0.113	520
100	6.4×10^{-6}	0.052	350
200	7.8×10^{-6}	0.031	273
300	1.1×10^{-5}	0.029	265
500	1.9×10^{-5}	0.031	270
700	2.9×10^{-5}	0.033	280
1000	4.6×10^{-5}	0.037	296

the analysis window an event must produce an S2 signal above 150 PE, which for $g_2 = 33$ requires roughly $n_e \gtrsim 5$ extracted electrons. The largest number of electrons a halo particle can possibly generate is set by the maximum energy it can deposit,

$$n_e^{\max} = 1 + \left\lfloor \frac{\frac{1}{2}M_1 v_{\max}^2 - E_B^{n\ell}}{W} \right\rfloor, \quad (5.24)$$

with $W = 13.8$ eV the average energy per quantum. Evaluated for the loosely bound $5p$ shell, this ceiling falls from $n_e^{\max} = 7$ at $M_1 = 30$ MeV to $n_e^{\max} = 5$ at 20 MeV, and to $n_e^{\max} = 2$ at 11 MeV. Once $n_e^{\max}$ drops below the $n_e \approx 5$ required by the threshold, only the far tail of the Gaussian S2 smearing can contribute: at $n_e = 2$ the mean signal of 66 PE lies more than 8σ below the threshold, and the detection probability becomes numerically zero. The scattering itself remains kinematically open throughout. At $M_1 = 30$ MeV a particle at $v_{\max} \approx 788$ km/s still carries ≈ 104 eV, far above the 12.4 eV binding energy, but the signal simply cannot be made large enough to be seen.

5.3.3 Validation against a published exclusion limit

The exclusion limits presented here rest on an independent implementation of the scattering rate, the detector response model and the binned Poisson statistics. It is therefore essential to verify the complete pipeline against an established result before applying it to the scotogenic model.

A direct benchmark is available. Catena et al. [1] apply the same XENON1T dataset and the same atomic response functions to a set of generic electromagnetic dark matter models, including a Majorana anapole interaction defined by

$$\mathcal{L}_{\text{anapole}} = \frac{g}{2\Lambda^2} \bar{\chi} \gamma^\mu \gamma^5 \chi \partial^\nu F_{\mu\nu}. \quad (5.25)$$

Because a Majorana bilinear acquires a factor of two from the second Wick contraction, the physical vertex coefficient of this Lagrangian is g/Λ^2 , which maps directly onto the form factor derived in Chapter 3. Their coupling is therefore identified with the anapole form factor of this work through

$$\frac{g}{\Lambda^2} \equiv |f_A| = \frac{c_8}{8eM_1 m_e}, \quad (5.26)$$

allowing us to compare our c_8 limit of Figure 5.2 directly to their results.

The comparison is shown in Figure 5.3. Across the full range of the published curve, $M_1 = 17 - 1000$ MeV, the limit derived here reproduces it to within 3%.

This is a good validation, because it tests the whole pipeline of this thesis at once: the anapole form factor of Chapter 3, the non-relativistic matching of Chapter 4, and the atomic responses, the detector model and the statistics of this chapter. None of the numerical inputs are shared. The response functions used in this work come from DarkART [40], whereas [1] computed theirs with DarkARC, the earlier Python code published with their preprint. Furthermore, the detector efficiency $\epsilon(S2)$ has been reconstructed independently from the XENON1T figure. Two tabulations of the same atomic integrals, produced by different codes, could very well explain the $\sim 3\%$ deviation of these two results.

Neither this work nor [1] attempts to reproduce the *official* XENON1T limit, which uses a mass-optimised signal region and subtracts the estimated background. For the official XENON1T results [16], in the $S2 \in [150, 200)$ PE bin the collaboration estimates 2.43

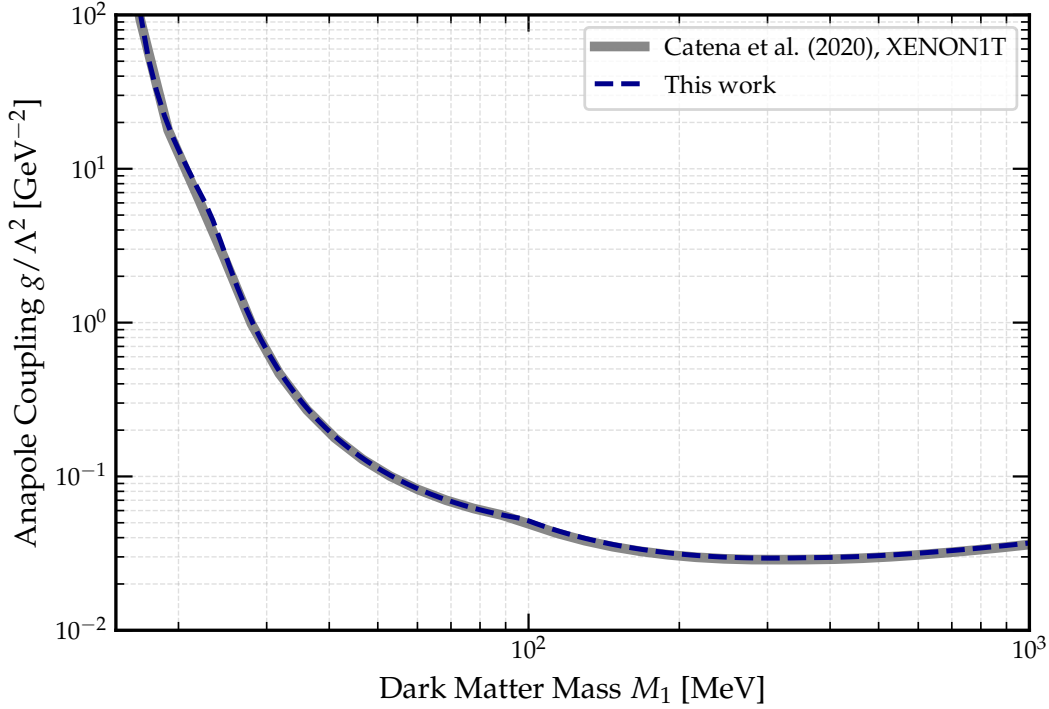


Figure 5.3: The 90% C.L. XENON1T exclusion limit on the anapole coupling g/Λ^2 derived in this work (dashed), compared with the published limit of Catena et al. [1] (solid grey). The reference curve was extracted from the vector graphics of their Figure 10 and calibrated against the printed axis tick marks. The two agree to within 3%.

background events among the 8 observed, so ignoring the background raises the 90% upper limit from 10.57 to 12.99 signal events, a factor 1.11 in c_8 . The limits presented here are therefore conservative.

5.3.4 Global constraint landscape and the thermal relic tension

While the experimental limits on the effective coupling c_8 are robust, evaluating the viability of the scotogenic model requires mapping these constraints back onto the fundamental parameter space. The parameter that carries the mapping is the single Yukawa coupling $y \equiv y_{\tau 1}$ of Equation (2.3). Because η is an $SU(2)_L$ doublet containing both charged and neutral scalars, that one coupling controls two different things at once. It sets the strength of the loop involving $\eta^\pm$ and the τ , which generates the anapole moment studied here, and it connects N_1 to the neutral scalar and ν_τ , which governs the tree-level annihilation rate in the early universe. The central tension of the model follows from that shared dependence.

If N_1 was produced by standard thermal freeze-out, its annihilation cross-section is set by the relic condition of Section 5.1.4. Annihilation into the τ sector is kinematically closed for sub-GeV masses ($2M_1 < 2m_\tau$), so N_1 annihilates entirely into active neutrinos. Inverting the relic condition of Kubo et al. [29] gives the Yukawa coupling required to meet the thermal target,

$$y_{\text{req}} = \left(\frac{4\pi M_1^2 x_f \langle \sigma v \rangle_{\text{target}}}{r_1^2 (1 - 2r_1 + 2r_1^2)} \right)^{1/4}, \quad (5.27)$$

where x_f is the freeze-out parameter, evaluated at each mass as in Section 5.1.4, and

$r_1 = M_1^2/(M_1^2 + m_\eta^2)$. This required coupling must also satisfy the perturbative limit of quantum field theory ($y < \sqrt{4\pi} \approx 3.54$). If y_{req} exceeds this ceiling, the loop expansion no longer converges.

To show how these constraints combine, we map the full $(M_1, m_{\eta^\pm})$ plane in Figure 5.4. At every point the Yukawa coupling required to reproduce the observed relic density is obtained from the p -wave freeze-out condition of Section 5.1.4, and the point is retained only where that coupling remains perturbative.

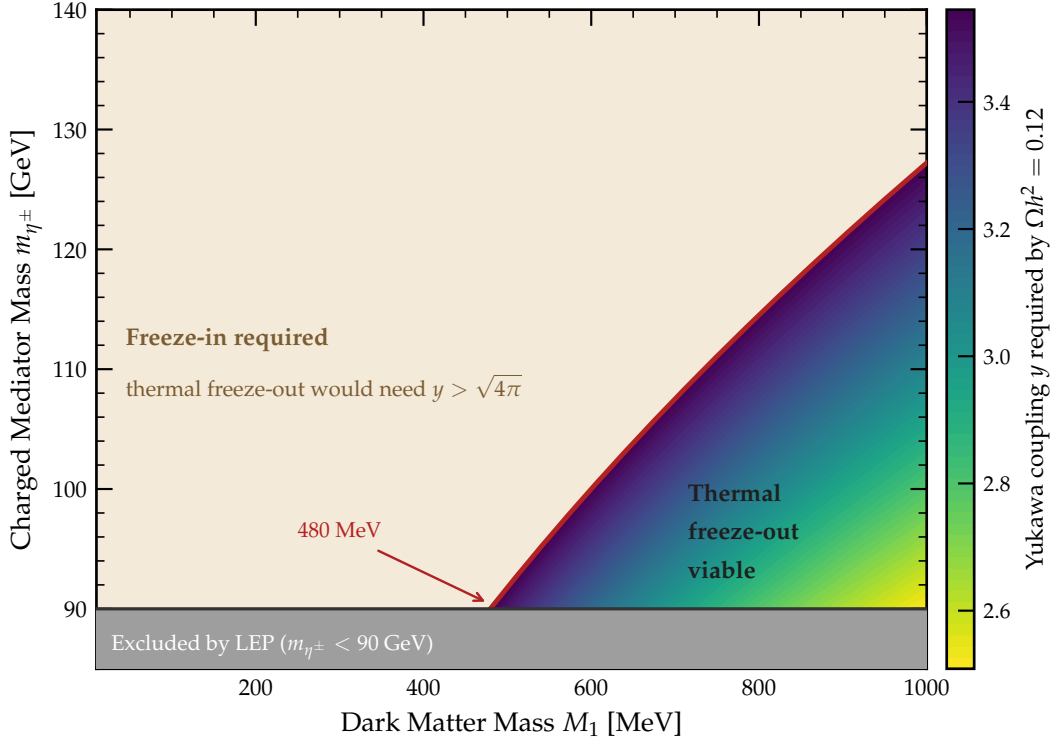


Figure 5.4: Parameter space of the sub-GeV scotogenic model. In the coloured wedge, thermal freeze-out reproduces $\Omega h^2 = 0.12$ with a perturbative Yukawa coupling, whose required value sets the shading. The wedge is bounded from below by the LEP constraint on the charged scalar (solid black) and from the left by the perturbativity condition $y < \sqrt{4\pi}$ (solid red). In the beige region, thermal freeze-out is ruled out, and dark matter must be produced non-thermally, as discussed in Section 5.3.6.

Two boundaries constrain the surviving region. The LEP-II searches for charged scalars bound $m_{\eta^\pm}$ from below. The relic condition restricts the viable parameter space from the left, as reproducing the observed abundance at low M_1 demands a coupling beyond the perturbative ceiling.

At the LEP floor of $m_\eta = 90$ GeV the sub-GeV window opens at $M_1 \approx 480$ MeV, and at the lower end of the LEP range quoted in Section 5.1.1, $m_\eta = 70$ GeV, dark matter masses down to roughly 280 MeV would meet the relic density target. At the $m_\eta = 100$ GeV benchmark, which is used later in this chapter, the floor rises to 600 MeV. Below these boundaries, and for any mediator heavier than about 127 GeV, the required Yukawa coupling is instead driven into the non-perturbative regime ($y_{\text{req}} > \sqrt{4\pi}$).

Thermal freeze-out is therefore *viable* for sub-GeV scotogenic dark matter, but only in a

narrow corner of the parameter space, namely a dark matter mass above roughly 480 MeV, a mediator confined between the LEP bound and roughly 127 GeV, and a Yukawa coupling ($y_{\text{req}} \approx 2.5 - 3.5$) that stays within 30% of the perturbative ceiling. Herms et al. [41] reach the opposite verdict for this model, because they require N_1 itself to generate the observed neutrino masses. However, the τ -philic structure adopted here removes that requirement and makes sub-GeV scotogenic dark matter perturbatively viable.

Direct detection is not able to probe the viable parameter space. The XENON1T limit only begins to exclude couplings around $y \approx 265$, which exceeds the thermal requirement by a factor of 78 at $M_1 = 600$ MeV, the lightest mass with a perturbative thermal solution at this benchmark, and by 106 at $M_1 = 1$ GeV. The experimentally excluded region and the cosmologically preferred region are separated by roughly two orders of magnitude in coupling, and therefore by eight in event rate. That separation does not depend on where the LEP bound is placed, since the excluded coupling and the thermal requirement both scale as m_η .

5.3.5 The thermal window over the full mass range

Looking at the parameter space above, there is no compelling reason to restrict the scope of the scotogenic model to a sub-GeV window. The perturbativity condition allows a much larger viable mass range above that cut-off: because the required coupling scales as $y_{\text{req}} \propto m_\eta / \sqrt{M_1}$ for $m_\eta \gg M_1$, the heaviest allowed mediator grows roughly as $\sqrt{M_1}$, reaching $m_{\eta^\pm} \approx 180$ GeV at $M_1 = 2$ GeV and $m_{\eta^\pm} \approx 380$ GeV at $M_1 = 10$ GeV. Repeating the construction of Figure 5.4 over five decades in mass, we obtain the complete thermal window of the model shown in Figure 5.5.

Along the LEP floor the required coupling falls from $y \approx 3.5$ at $M_1 = 0.5$ GeV, within a per cent of the perturbative ceiling, to $y \approx 1.2$ at 5 GeV and $y \approx 0.5$ near 100 GeV, where the model is at its most comfortable. Beyond that the window narrows again, because N_1 can only be the dark matter while it is lighter than the mediator, which follows from taking the fermion as the candidate, instead of the inert scalar. It closes altogether at $M_1 \approx 3.9$ TeV where that requirement meets the perturbativity bound. The scotogenic model is therefore considerably better suited to the GeV–TeV range than to the sub-GeV masses studied here, and the tension documented above should be read as a statement about light dark matter specifically, and not the whole scotogenic model itself. This is consistent with the emphasis of the recent review of Ávila et al. [22], which restricts its treatment of fermionic scotogenic dark matter to masses above 1 GeV. Lighter candidates are achievable only under particular circumstances, and this work quantifies what those circumstances are.

5.3.6 Viability through non-thermal production

The parameter scans from Section 5.3.4 show that standard thermal freeze-out survives only above $M_1 \approx 480$ MeV, and only for mediator masses close to the LEP bound. For the lower half of the sub-GeV range the required coupling is non-perturbative, and an alternative production mechanism becomes necessary. Coannihilation [39] cannot resolve this tension either, since it requires the dark matter and its partner to be degenerate to within a few per cent, as Section 5.1.4 noted. LEP, however, forces $m_\eta \gtrsim 90$ GeV against a sub-GeV N_1 , and since N_1 is a gauge singlet, every channel that could deplete it runs through the same heavy scalar.

In this non-perturbative regime the model must rely on a non-thermal cosmological history,

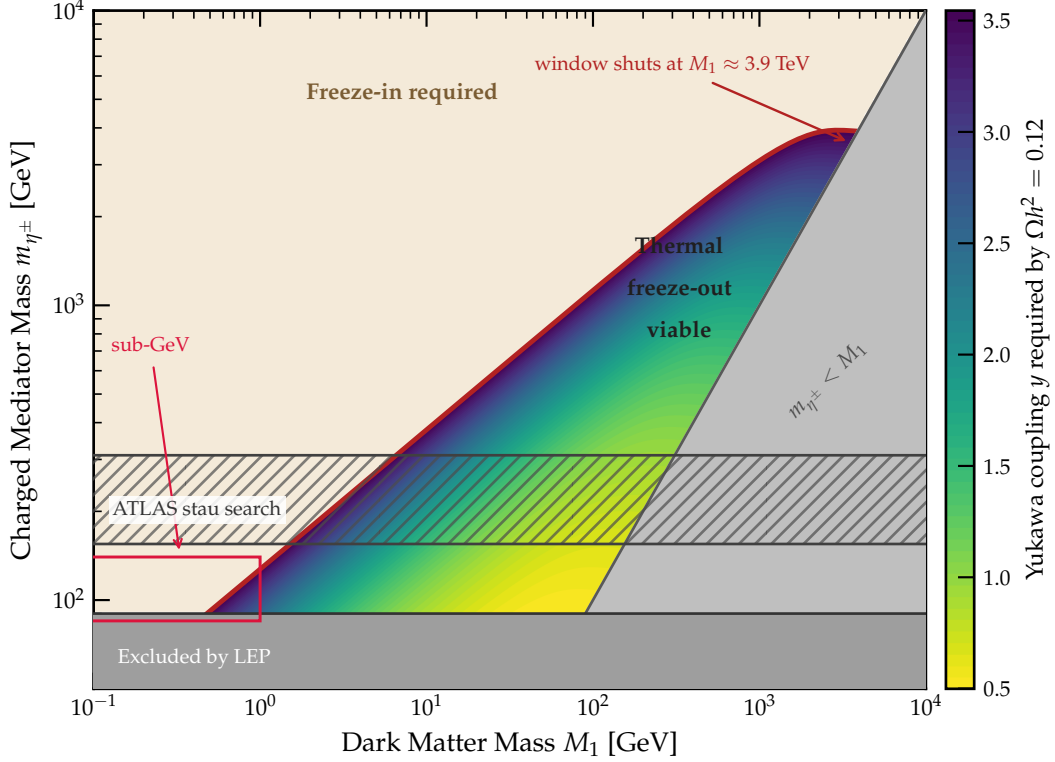


Figure 5.5: The thermal window of the model over the full mass range. As in Figure 5.4, the coloured region is where freeze-out reproduces $\Omega h^2 = 0.12$ with a perturbative coupling. Three boundaries constrain it: the LEP constraint (grey, below), perturbativity (solid red, above), and the requirement that N_1 be the lightest $\mathbb{Z}_2$ -odd state (grey, right). The red box marks the sub-GeV region of Figure 5.4, and the hatched band is the ATLAS stau-left exclusion of Section 5.1.2. Direct detection exclusion limits are not drawn: below 1 GeV the limit lies far above the perturbative ceiling, and above it a nuclear-recoil analysis would be needed.

such as the Feebly Interacting Massive Particle (FIMP) freeze-in mechanism [42]. Here N_1 never thermalises. Its abundance is instead built up slowly out of the thermal bath, through the rare decays of η particles that are themselves in equilibrium. Every member of the doublet contributes, since $\eta^\pm \rightarrow \tau^\pm N_1$ and $\eta^0 \rightarrow \nu_\tau N_1$ all run through the same Yukawa coupling and therefore proceed at the same rate.

The coupling this requires can be computed. Inverting the freeze-in relation of Hall et al. [42], summed over all four decaying states of the doublet, each of which the chiral Yukawa gives the width $\Gamma = y^2 m_\eta / 16\pi$ in the limit $m_\eta \gg M_1, m_\tau$, the observed abundance requires

$$y_{\text{FIMP}} \approx 1 \times 10^{-11} - 7 \times 10^{-11} \quad (5.28)$$

across $M_1 = 30 - 1000$ MeV at $m_\eta = 100$ GeV, which agrees with the order of magnitude usually quoted for freeze-in. Ávila et al. [22] obtain the same dependence on both masses for this mechanism, with normalisations differing by a factor of a few.

That coupling unfortunately closes the door on direct detection. The anapole rate scales as y^4 , so moving from the thermal branch to the freeze-in one suppresses it by roughly 45 orders of magnitude. Figure 5.6 places both mechanisms on the same axis as the XENON1T

exclusion limit of Section 5.3.2. Even on the thermal branch the predicted c_8 lies four orders of magnitude below the XENON1T bound, which is eight orders in event rate. The freeze-in prediction $c_8 \approx 8 \times 10^{-32}$ lies more than twenty-five orders below it. Reaching the thermal prediction at $M_1 = 1$ GeV would take on the order of 3×10^7 tonne-years, against the 200 tonne-years projected for DARWIN [43], which is a proposed next-generation xenon observatory.

The same small coupling that closes the direct detection door opens a different one. A particle that couples feebly also decays slowly, and the decay width quoted above gives the charged scalar a proper decay length

$$c\tau_\eta = \frac{16\pi\hbar c}{y^2 m_\eta}, \quad (5.29)$$

which is about 10^{-17} m for the $y \approx 3$ of the thermal branch but 20 – 650 km for the freeze-in coupling. In a collider detector the thermal mediator would decay almost instantly, while the freeze-in mediator crosses the entire apparatus and leaves a distinctive track behind it.

The scotogenic model is therefore not untestable in general, only in a direct-detection chamber. The coupling that suppresses the anapole rate by 45 orders of magnitude is the same one that gives the mediator a macroscopic lifetime, which is what makes it possible to probe the model at particle colliders instead of underground TPCs.

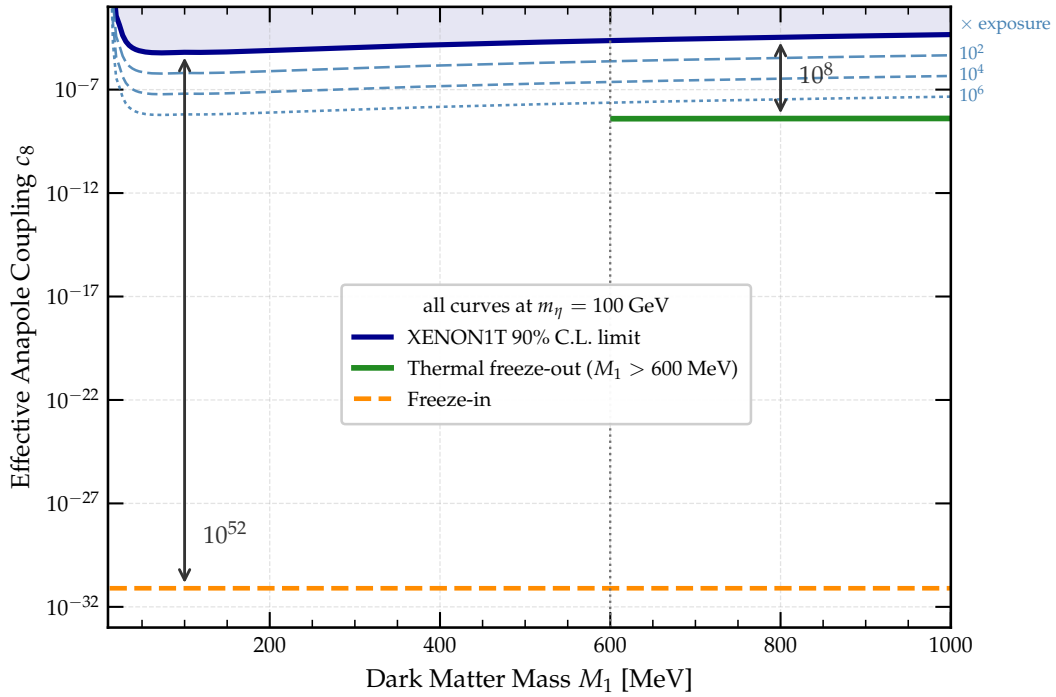


Figure 5.6: The 90% C.L. upper limit on c_8 from XENON1T (blue), compared with the coupling predicted by thermal freeze-out (green) and by freeze-in (orange), all at $m_\eta = 100$ GeV. The thermal curve is drawn only above $M_1 = 600$ MeV, below which the required coupling is non-perturbative (dotted line). Faint dashed curves give the reach of 10^2 , 10^4 and 10^6 times this exposure under background-free scaling. Arrows give the shortfall in event rate, which scales as c_8^2 .

The same conclusion has been reached once before for a related model. Fiaschi et al. [44]

study MeV-to-GeV right-handed neutrino dark matter in the SLIM model and also find the predicted cross-sections too small for direct detection, for the same reason found here: LEP only allows for heavy charged mediators, and the rate is therefore significantly suppressed.

Chapter 6

Conclusion and Outlook

DIRECT detection phenomenology of sub-GeV Majorana dark matter within the minimal scotogenic model is the main focus of this thesis. To avoid strict experimental bounds from charged lepton flavour violation, such as $\mu \rightarrow e\gamma$, we assumed a strictly τ -philic flavour structure. Since the dark matter candidate N_1 is a Majorana fermion, its vector charge and both its magnetic and electric dipole moments vanish identically under charge conjugation. This leaves the loop-induced anapole moment as its leading-order electromagnetic interaction with Standard Model particles.

Mapping this loop amplitude onto a non-relativistic effective field theory allowed us to derive the inelastic scattering cross-section against bound atomic electrons. We then convolved the continuous differential event rate with the Standard Halo Model kinematics and mapped it onto the discrete S2 photoelectron bins of the XENON1T analysis. This procedure yielded an EFT-level upper limit on the effective anapole coupling constant c_8 . At the same time, it revealed a steep loss of sensitivity for xenon-based detectors below $M_1 \approx 30$ MeV, which we traced to the quantisation of the ionisation signal: below this mass the maximum number of electrons a halo particle can ionise falls below the number required to exceed the 150 PE analysis threshold, even though the scattering itself remains kinematically allowed.

Our most important result is the evaluation of the scotogenic parameter space under global constraints. Under the τ -philic assumption the relevant physics is governed by a single entry of the Yukawa matrix, $y \equiv y_{\tau 1}$, which controls both the loop-induced anapole scattering rate and the tree-level p -wave annihilation into active neutrinos. Satisfying the Planck relic density target requires a specific coupling strength, but precision measurements from LEP bound the charged scalar mediator to $m_{\eta^\pm} \gtrsim 90$ GeV. Our parameter scans demonstrated that enforcing this heavy mediator mass severely suppresses the early universe annihilation rate.

Consequently, achieving the correct thermal relic density requires Yukawa couplings that violate the perturbative limit of quantum field theory ($y > \sqrt{4\pi}$) for $M_1 \lesssim 600$ MeV at $m_\eta = 100$ GeV, or $M_1 \lesssim 480$ MeV at the LEP floor. Thermal freeze-out therefore remains viable in the upper part of the sub-GeV range, provided the mediator sits close to its collider bound. The ATLAS search for direct stau production excludes charged scalars between 155 and 310 GeV once the single doublet scalar is matched to stau-left production alone, which lies above the mediator masses a sub-GeV candidate can have. The mediator is therefore confined from below by LEP and from above by perturbativity, to $m_{\eta^\pm} \approx 90 - 127$ GeV,

while the dark matter mass is bounded only from below, $M_1 \gtrsim 480$ MeV. Additionally, the heavy mediator strongly suppresses the anapole loop function. This shifts the XENON1T exclusion limit to unphysical couplings ($y \gtrsim 265$). Because this experimental limit lies far above the perturbativity bound, direct detection currently has no sensitivity to the physically viable thermal parameter space in this model.

Below that window, where thermal freeze-out would demand non-perturbative couplings, the model must rely on a non-thermal cosmological history. A mathematically consistent alternative is the Feebly Interacting Massive Particle (FIMP) freeze-in mechanism. In this scenario, dark matter is produced slowly in the early universe via rare decays from the thermal bath ($\eta^\pm \rightarrow \tau^\pm + N_1$ and $\eta^0 \rightarrow \nu_\tau + N_1$). To achieve the correct relic density without overclosing the universe, this mechanism requires microscopic Yukawa couplings on the order of $y \sim \mathcal{O}(10^{-11})$.

Adopting the FIMP framework resolves the perturbativity issue, but at a cost to the experimental outlook. Because the anapole scattering cross-section scales with y^4 , the transition to a freeze-in coupling ($y_{\text{FIMP}} \approx 1 \times 10^{-11} - 7 \times 10^{-11}$, computed from the freeze-in relic condition) suppresses the expected direct detection event rate by roughly 45 orders of magnitude. The dark matter signal therefore falls far below the sensitivity threshold of noble gas TPCs like XENONnT [45], LZ [46] or DARWIN [43]. Consequently, experimental verification of sub-GeV scotogenic dark matter cannot be achieved through direct detection. Instead, it must rely on collider searches probing the long-lived charged scalar $\eta^\pm$ at facilities like the High-Luminosity LHC [47] or future lepton colliders such as the FCC-ee [48]. On the theoretical side, a natural continuation is to extend the analysis from isolated atoms to semiconductor targets, for which the crystal response formalism of Griffin et al. [49] provides the appropriate framework, and to the coherent nuclear-recoil regime that governs masses above a GeV.

Appendix A

Algebraic Reductions for the Electromagnetic Vertex

A.1 Algebraic reduction of the photon-lepton numerator

To extract the physical anapole moment, the raw loop momenta within the finite numerator of Figure 3.6 must be algebraically reduced into the external dark matter mass M_1 and the momentum transfer $q = p' - p$.

The finite numerator is given by

$$\mathcal{N}_{c,fin}^\mu = \left[(\not{p}' - \not{Q}_c) \gamma^\mu (\not{p} - \not{Q}_c) + m_{\ell_\alpha}^2 \gamma^\mu \right] P_R. \quad (\text{A.1})$$

Substituting the shift vector $Q_c = xp' + yp$, the brackets expand into four distinct momentum structures,

$$\begin{aligned} \mathcal{N}_{c,fin}^\mu = & \left[(1-x)(1-y) \not{p}' \gamma^\mu \not{p} - x(1-x) \not{p}' \gamma^\mu \not{p}' \right. \\ & \left. - y(1-y) \not{p} \gamma^\mu \not{p} + xy \not{p} \gamma^\mu \not{p}' + m_{\ell_\alpha}^2 \gamma^\mu \right] P_R. \end{aligned} \quad (\text{A.2})$$

Evaluating these terms between the external Majorana spinors $\bar{u}(p')$ and $u(p)$, we systematically apply the Dirac equations of motion ($\not{p}u(p) = M_1 u(p)$ and $\bar{u}(p')\not{p}' = M_1 \bar{u}(p')$) [3], the chiral commutation relation ($\not{p}P_R = P_L \not{p}$), and the Clifford anti-commutator ($\not{A}\gamma^\mu \not{A} = 2A^\mu \not{A} - A^2 \gamma^\mu$).

Term 1:

$$\begin{aligned} \bar{u}(p') [\not{p}' \gamma^\mu \not{p} P_R] u(p) &= M_1 \bar{u}(p') \gamma^\mu \not{p} P_R u(p) \\ &= M_1 \bar{u}(p') \gamma^\mu P_L \not{p} u(p) \\ &= M_1^2 \bar{u}(p') \gamma^\mu P_L u(p) \end{aligned} \quad (\text{A.3})$$

Term 2:

$$\begin{aligned}\bar{u}(p') [\not{p}' \gamma^\mu \not{p}' P_R] u(p) &= \bar{u}(p') [2p'^\mu \not{p}' - M_1^2 \gamma^\mu] P_R u(p) \\ &= 2M_1 p'^\mu \bar{u}(p') P_R u(p) - M_1^2 \bar{u}(p') \gamma^\mu P_R u(p)\end{aligned}\quad (\text{A.4})$$

Term 3:

$$\begin{aligned}\bar{u}(p') [\not{p} \gamma^\mu \not{p} P_R] u(p) &= \bar{u}(p') [2p^\mu \not{p} - M_1^2 \gamma^\mu] P_R u(p) \\ &= 2p^\mu \bar{u}(p') P_L \not{p} u(p) - M_1^2 \bar{u}(p') \gamma^\mu P_R u(p) \\ &= 2M_1 p^\mu \bar{u}(p') P_L u(p) - M_1^2 \bar{u}(p') \gamma^\mu P_R u(p)\end{aligned}\quad (\text{A.5})$$

Term 4:

Using $q = p' - p$, the substitutions $\not{p}' = \not{p} + \not{q}$ and $\not{p} = \not{p}' - \not{q}$ are applied to allow the momentum operators to act directly upon their corresponding external spinors,

$$\begin{aligned}\bar{u}(p') [\not{p} \gamma^\mu \not{p}' P_R] u(p) &= \bar{u}(p') [(\not{p}' - \not{q}) \gamma^\mu (\not{p} + \not{q}) P_R] u(p) \\ &= \bar{u}(p') [(\not{p}' - \not{q}) \gamma^\mu P_L (\not{p} + \not{q})] u(p).\end{aligned}\quad (\text{A.6})$$

Expanding the brackets and applying the equations of motion yields

$$\begin{aligned}&= \bar{u}(p') \left[M_1^2 \gamma^\mu P_L + M_1 \gamma^\mu P_L \not{q} - M_1 \not{q} \gamma^\mu P_L \right. \\ &\quad \left. - 2q^\mu \not{q} P_R + q^2 \gamma^\mu P_R \right] u(p).\end{aligned}\quad (\text{A.7})$$

Final Reduced Numerator for the photon-lepton diagram

Summing the four reduced momentum terms alongside the lepton mass term, the completely evaluated finite numerator for this diagram, evaluated between the external Majorana spinors, takes the form

$$\begin{aligned}\bar{u}(p') \mathcal{N}_{c,fin}^\mu u(p) &= \bar{u}(p') \left\{ (1-x)(1-y) M_1^2 \gamma^\mu P_L \right. \\ &\quad - x(1-x) [2M_1 p'^\mu P_R - M_1^2 \gamma^\mu P_R] \\ &\quad - y(1-y) [2M_1 p^\mu P_L - M_1^2 \gamma^\mu P_R] \\ &\quad + xy [M_1^2 \gamma^\mu P_L + M_1 \gamma^\mu P_L \not{q} - M_1 \not{q} \gamma^\mu P_L - 2q^\mu \not{q} P_R + q^2 \gamma^\mu P_R] \\ &\quad \left. + m_{\ell_\alpha}^2 \gamma^\mu P_R \right\} u(p).\end{aligned}\quad (\text{A.8})$$

This expression is now entirely free of internal loop momenta and is prepared for combination with Figure 3.7 prior to the final application of the Gordon decomposition.

A.2 Algebraic reduction of the photon-scalar numerator

An identical reduction is now applied to the finite numerator of Figure 3.7. Because the external photon couples directly to the internal scalar loop instead of the fermion line, the chiral projector P_R is not separated by an internal γ^μ matrix.

The finite numerator is

$$\mathcal{N}_{d,fin}^\mu = (\not{p} - \not{Q}_d)P_R(2Q_d^\mu + q^\mu). \quad (\text{A.9})$$

Substituting the shift vector $Q_d = zp - xq$, the two bracketed factors evaluate to

$$\not{p} - \not{Q}_d = \not{p} - (z\not{p} - x\not{q}) = (1-z)\not{p} + x\not{q} \quad (\text{A.10})$$

$$2Q_d^\mu + q^\mu = 2(zp^\mu - xq^\mu) + q^\mu = 2zp^\mu + (1-2x)q^\mu. \quad (\text{A.11})$$

Multiplying these brackets together yields four distinct terms. Because the momentum vectors p^μ and q^μ commute with the Dirac matrices, they can be factored to the front of the spinor bilinear,

$$\begin{aligned} \bar{u}(p')\mathcal{N}_{d,fin}^\mu u(p) = \bar{u}(p') \Big[& \underbrace{2z(1-z)p^\mu \not{p} P_R}_{\text{Term 1}} + \underbrace{(1-z)(1-2x)q^\mu \not{p} P_R}_{\text{Term 2}} \\ & + \underbrace{2zx p^\mu \not{q} P_R}_{\text{Term 3}} + \underbrace{x(1-2x)q^\mu \not{q} P_R}_{\text{Term 4}} \Big] u(p). \end{aligned} \quad (\text{A.12})$$

We systematically apply the chiral commutation relation ($\not{p}P_R = P_L\not{p}$) and the Dirac equation of motion ($\not{p}u(p) = M_1u(p)$) to the terms containing $\not{p}$.

Term 1:

$$\begin{aligned} 2z(1-z)p^\mu \bar{u}(p') \not{p} P_R u(p) &= 2z(1-z)p^\mu \bar{u}(p') P_L \not{p} u(p) \\ &= 2z(1-z)M_1 p^\mu \bar{u}(p') P_L u(p) \end{aligned} \quad (\text{A.13})$$

Term 2:

$$\begin{aligned} (1-z)(1-2x)q^\mu \bar{u}(p') \not{p} P_R u(p) &= (1-z)(1-2x)q^\mu \bar{u}(p') P_L \not{p} u(p) \\ &= (1-z)(1-2x)M_1 q^\mu \bar{u}(p') P_L u(p) \end{aligned} \quad (\text{A.14})$$

Terms 3 and 4:

In the remaining terms, the $\not{q}$ matrix is situated between $\bar{u}(p')$ and the projector $P_R u(p)$. Because $\not{q}$ cannot be directly reduced by the equations of motion without further kinematic expansion, these terms remain unchanged for the time being.

Final Reduced Numerator for Figure 3.7

Summing these evaluated components, the fully reduced finite numerator for this diagram takes the form

$$\begin{aligned} \bar{u}(p') \mathcal{N}_{d,fin}^\mu u(p) = \bar{u}(p') \Big\{ & 2z(1-z)M_1 p^\mu P_L + (1-z)(1-2x)M_1 q^\mu P_L \\ & + 2zx p^\mu \not{q} P_R + x(1-2x)q^\mu \not{q} P_R \Big\} u(p). \end{aligned} \quad (\text{A.15})$$

A.3 The chiral Gordon decomposition

To extract the physical $\mathcal{O}(q^2)$ anapole moment from the reduced amplitudes, all instances of the bare incoming momentum (p^μ) must be eliminated in favour of the momentum transfer q^μ . Because the terms are bound by chiral projectors (P_R and P_L), the specific chiral Gordon identities must be derived [3].

By evaluating the commutator of $\not{q}$ and $\gamma^\mu P_{R,L}$ between the external spinors, applying the chiral commutation relations ($\not{p}P_R = P_L\not{p}$), and enforcing the Dirac equations of motion ($\not{p}u(p) = M_1 u(p)$ and $\bar{u}(p')\not{p}' = M_1 \bar{u}(p')$), we find

$$\begin{aligned} \bar{u}(p') \not{q} \gamma^\mu P_R u(p) &= \bar{u}(p') (\not{p}' - \not{p}) \gamma^\mu P_R u(p) \\ &= \bar{u}(p') \left[M_1 \gamma^\mu P_R - (2p^\mu - \gamma^\mu \not{p}) P_R \right] u(p) \\ &= \bar{u}(p') \left[M_1 \gamma^\mu P_R - 2p^\mu P_R + M_1 \gamma^\mu P_L \right] u(p). \end{aligned} \quad (\text{A.16})$$

Because $P_R + P_L = 1$, the M_1 terms combine to yield $\bar{u}(p') [M_1 \gamma^\mu - 2p^\mu P_R] u(p)$. Rearranging this expression provides the direct substitution rules for the bare momenta,

$$\bar{u}(p') 2p^\mu P_R u(p) = \bar{u}(p') \left[M_1 \gamma^\mu - \not{q} \gamma^\mu P_R \right] u(p) \quad (\text{A.17})$$

$$\bar{u}(p') 2p^\mu P_L u(p) = \bar{u}(p') \left[M_1 \gamma^\mu - \not{q} \gamma^\mu P_L \right] u(p). \quad (\text{A.18})$$

A.3.1 Applying Gordon decomposition to the photon-lepton diagram

Returning to the evaluated finite numerator for Figure 3.6, the bare incoming (p^μ) and outgoing (p'^μ) momenta are present in Term 2 and Term 3,

$$\begin{aligned} \bar{u}(p') \mathcal{N}_{c,fin}^\mu u(p) = \bar{u}(p') \Big\{ & (1-x)(1-y)M_1^2 \gamma^\mu P_L \\ & - x(1-x) \left[2M_1 p'^\mu P_R - M_1^2 \gamma^\mu P_R \right] \\ & - y(1-y) \left[2M_1 p^\mu P_L - M_1^2 \gamma^\mu P_R \right] \\ & + xy \left[M_1^2 \gamma^\mu P_L + M_1 \gamma^\mu P_L \not{q} - M_1 \not{q} \gamma^\mu P_L - 2q^\mu \not{q} P_R + q^2 \gamma^\mu P_R \right] \\ & + m_{\ell_\alpha}^2 \gamma^\mu P_R \Big\} u(p). \end{aligned} \quad (\text{A.19})$$

The chiral Gordon identities are systematically applied. For the outgoing momentum in Term 2, utilising the relation $2p'^\mu P_R = M_1 \gamma^\mu + \gamma^\mu \not{q} P_R$ yields

$$\begin{aligned} 2M_1 p'^\mu P_R - M_1^2 \gamma^\mu P_R &= M_1 (M_1 \gamma^\mu + \gamma^\mu \not{q} P_R) - M_1^2 \gamma^\mu P_R \\ &= M_1^2 \gamma^\mu (1 - P_R) + M_1 \gamma^\mu \not{q} P_R \\ &= M_1^2 \gamma^\mu P_L + M_1 \gamma^\mu \not{q} P_R. \end{aligned} \quad (\text{A.20})$$

For the incoming momentum in Term 3, utilising $2p^\mu P_L = M_1 \gamma^\mu - \not{q} \gamma^\mu P_L$ gives

$$\begin{aligned} 2M_1 p^\mu P_L - M_1^2 \gamma^\mu P_R &= M_1 (M_1 \gamma^\mu - \not{q} \gamma^\mu P_L) - M_1^2 \gamma^\mu P_R \\ &= M_1^2 \gamma^\mu (1 - P_R) - M_1 \not{q} \gamma^\mu P_L \\ &= M_1^2 \gamma^\mu P_L - M_1 \not{q} \gamma^\mu P_L. \end{aligned} \quad (\text{A.21})$$

We observe that the structure $M_1^2 \gamma^\mu P_L$ can be factored out of both substitutions.

The Fully Decomposed photon-lepton numerator:

Substituting these evaluated components back into the main expression, the Gordon-decomposed finite numerator for Figure 3.6 is

$$\begin{aligned} \bar{u}(p') \mathcal{N}_{c,fin}^\mu u(p) &= \bar{u}(p') \left\{ (1-x)(1-y) M_1^2 \gamma^\mu P_L \right. \\ &\quad - x(1-x) [M_1^2 \gamma^\mu P_L + M_1 \gamma^\mu \not{q} P_R] \\ &\quad - y(1-y) [M_1^2 \gamma^\mu P_L - M_1 \not{q} \gamma^\mu P_L] \\ &\quad + xy [M_1^2 \gamma^\mu P_L + M_1 \gamma^\mu P_L \not{q} - M_1 \not{q} \gamma^\mu P_L - 2q^\mu \not{q} P_R + q^2 \gamma^\mu P_R] \\ &\quad \left. + m_{\ell_a}^2 \gamma^\mu P_R \right\} u(p). \end{aligned} \quad (\text{A.22})$$

A.3.2 Applying Gordon decomposition to the photon-scalar diagram

The finite numerator for Figure 3.7 before decomposition is given by

$$\begin{aligned} \bar{u}(p') \mathcal{N}_{d,fin}^\mu u(p) &= \bar{u}(p') \left\{ 2z(1-z) M_1 p^\mu P_L + (1-z)(1-2x) M_1 q^\mu P_L \right. \\ &\quad \left. + 2zx p^\mu \not{q} P_R + x(1-2x) q^\mu \not{q} P_R \right\} u(p). \end{aligned} \quad (\text{A.23})$$

The chiral Gordon identities are applied to the terms containing the bare momentum p^μ (Term 1 and Term 3).

Term 1 Substitution:

Using the derived identity for $2p^\mu P_L$,

$$\begin{aligned} z(1-z) M_1 (2p^\mu P_L) &= z(1-z) M_1 (M_1 \gamma^\mu - \not{q} \gamma^\mu P_L) \\ &= z(1-z) M_1^2 \gamma^\mu - z(1-z) M_1 \not{q} \gamma^\mu P_L. \end{aligned} \quad (\text{A.24})$$

Term 3 Substitution:

This term contains an internal $\not{q} P_R$ contraction. This inner structure is first reduced utilising the equations of motion,

$$\begin{aligned} \bar{u}(p') \not{q} P_R u(p) &= \bar{u}(p') (\not{p}' - \not{p}) P_R u(p) \\ &= M_1 \bar{u}(p') P_R u(p) - \bar{u}(p') P_L \not{p} u(p) \\ &= M_1 \bar{u}(p') (P_R - P_L) u(p). \end{aligned} \quad (\text{A.25})$$

Thus, Term 3 is rewritten as $zxM_1(2p^\mu P_R - 2p^\mu P_L)$. Both chiral Gordon identities are then substituted,

$$zxM_1 \left[\left(M_1 \gamma^\mu - \not{q} \gamma^\mu P_R \right) - \left(M_1 \gamma^\mu - \not{q} \gamma^\mu P_L \right) \right] = zxM_1 \left(-\not{q} \gamma^\mu P_R + \not{q} \gamma^\mu P_L \right). \quad (\text{A.26})$$

The Fully Decomposed photon-scalar numerator:

By collecting the expanded Term 1, the unchanged Term 2, the expanded Term 3, and the unchanged Term 4, the complete, Gordon-decomposed numerator for the photon-scalar diagram is assembled,

$$\begin{aligned} \bar{u}(p') \mathcal{N}_{d,fin}^\mu u(p) = \bar{u}(p') \Big\{ & z(1-z)M_1^2 \gamma^\mu - z(1-z)M_1 \not{q} \gamma^\mu P_L \\ & + (1-z)(1-2x)M_1 \not{q}^\mu P_L \\ & - zxM_1 \not{q} \gamma^\mu P_R + zxM_1 \not{q} \gamma^\mu P_L \\ & + x(1-2x)q^\mu \not{q} P_R \Big\} u(p). \end{aligned} \quad (\text{A.27})$$

Through this procedure, all instances of the bare momentum p^μ have been eliminated in favour of the momentum transfer q^μ , while preserving the P_R and P_L chiral projection structures.

A.4 Analytical reduction of the integral to the Kopp formalism

While the one-dimensional integral for the anapole form factor f_A can be evaluated via computer algebra systems such as Mathematica [33], the raw output is rather chaotic. Evaluating the integral over the Feynman parameter z yields the following mathematical expression,

$$\begin{aligned} \mathcal{I}_{\text{raw}} = & \frac{1}{12\sqrt{-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu)}} \times \\ & \left[(2 - 6\epsilon + 6\mu) \arctan \left(\frac{1 + \epsilon - \mu}{\sqrt{-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu)}} \right) \right. \\ & + (2 - 6\epsilon + 6\mu) \arctan \left(\frac{1 - \epsilon + \mu}{\sqrt{-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu)}} \right) \\ & \left. + 3\sqrt{-(-1 + \epsilon)^2 + 2(1 + \epsilon)\mu - \mu^2} \ln \left(\frac{\epsilon}{\mu} \right) \right]. \end{aligned} \quad (\text{A.28})$$

To demonstrate equivalence with the closed-form Effective Field Theory expression derived by Kopp et al. [26], this raw output must be analytically simplified. Expanding the argument of the square roots present in the denominator and the logarithms reveals that it corresponds exactly to the negative square of the standard kinematic discriminant R ,

$$-\epsilon^2 - (-1 + \mu)^2 + 2\epsilon(1 + \mu) = -\left[(\mu - 1 - \epsilon)^2 - 4\epsilon \right] \equiv -R^2. \quad (\text{A.29})$$

Because the physical parameter space dictates that $R^2 > 0$, the algorithmic evaluation has analytically continued the integration into the complex plane, making the square root imaginary: $\sqrt{-R^2} = iR$.

Substituting iR into the raw integral output, the imaginary unit can be extracted from the inverse tangent functions via the complex identity $\arctan(-iz) = -i \operatorname{arctanh}(z)$. Multiplying the global prefactor $\frac{1}{12iR}$ by the extracted $-i$ introduces an overall minus sign to the $\operatorname{arctanh}$ terms,

$$\begin{aligned} \mathcal{I} &= \frac{1}{12iR} \left[(2 - 6\epsilon + 6\mu)(-i) \left(\operatorname{arctanh} \frac{1 + \epsilon - \mu}{R} + \operatorname{arctanh} \frac{1 - \epsilon + \mu}{R} \right) + 3iR \ln \left(\frac{\epsilon}{\mu} \right) \right] \\ &= -\frac{2 - 6\epsilon + 6\mu}{12R} \left(\operatorname{arctanh} \frac{1 + \epsilon - \mu}{R} + \operatorname{arctanh} \frac{1 - \epsilon + \mu}{R} \right) + \frac{1}{4} \ln \left(\frac{\epsilon}{\mu} \right). \end{aligned} \quad (\text{A.30})$$

The inverse hyperbolic tangent addition formula, $\operatorname{arctanh}(x) + \operatorname{arctanh}(y) = \operatorname{arctanh} \left(\frac{x+y}{1+xy} \right)$, is now applied to combine the remaining terms. Defining $x = (1 + \epsilon - \mu)/R$ and $y = (1 - \epsilon + \mu)/R$, the numerator of the combined argument simplifies to $x + y = 2/R$.

The denominator expands as

$$1 + xy = 1 + \frac{1 - (\epsilon - \mu)^2}{R^2} = \frac{R^2 + 1 - \epsilon^2 + 2\epsilon\mu - \mu^2}{R^2}. \quad (\text{A.31})$$

Expanding $R^2 = \mu^2 + \epsilon^2 + 1 - 2\mu - 2\epsilon\mu - 2\epsilon$, the sum evaluates directly to

$$1 + xy = \frac{2 - 2\mu - 2\epsilon}{R^2} = \frac{2(1 - \mu - \epsilon)}{R^2}. \quad (\text{A.32})$$

Dividing the numerator by the denominator, the argument of the $\operatorname{arctanh}$ function simplifies to

$$\frac{x + y}{1 + xy} = \frac{2/R}{2(1 - \mu - \epsilon)/R^2} = \frac{R}{1 - \mu - \epsilon}. \quad (\text{A.33})$$

Thus, the sum of the two $\operatorname{arctanh}$ functions is equal to $\operatorname{arctanh} \left(\frac{R}{1 - \mu - \epsilon} \right)$. To match the sign convention in the literature, a minus sign is factored out of the argument denominator, utilising the antisymmetric property $\operatorname{arctanh}(-z) = -\operatorname{arctanh}(z)$,

$$\operatorname{arctanh} \left(\frac{R}{1 - \mu - \epsilon} \right) = -\operatorname{arctanh} \left(\frac{R}{\mu - 1 + \epsilon} \right). \quad (\text{A.34})$$

Substituting this simplified argument back into the simplified integral, the minus sign extracted from the $\operatorname{arctanh}$ cancels the leading minus sign from earlier. Simultaneously, the logarithm argument is inverted ($\ln(\epsilon/\mu) = -\ln(\mu/\epsilon)$),

$$\begin{aligned} \mathcal{I} &= \left(-\frac{2(1 - 3\epsilon + 3\mu)}{12R} \right) \left(-\operatorname{arctanh} \left(\frac{R}{\mu - 1 + \epsilon} \right) \right) - \frac{1}{4} \ln \left(\frac{\mu}{\epsilon} \right) \\ &= -\frac{1}{4} \ln \left(\frac{\mu}{\epsilon} \right) + \frac{1 - 3\epsilon + 3\mu}{6R} \operatorname{arctanh} \left(\frac{R}{\mu - 1 + \epsilon} \right). \end{aligned} \quad (\text{A.35})$$

Finally, factoring out a global $-1/6$ mathematically organises the terms into the exact structure required,

$$\mathcal{I} = -\frac{1}{6} \left[\frac{3}{2} \ln \left(\frac{\mu}{\epsilon} \right) - \frac{1 + 3\mu - 3\epsilon}{R} \operatorname{arctanh} \left(\frac{R}{\mu - 1 + \epsilon} \right) \right]. \quad (\text{A.36})$$

Restoring the global loop prefactor $\frac{e|y_{\alpha 1}|^2}{16\pi^2 M_1^2}$ from the full amplitude evaluation, the factor of $1/6$ extracted from the integration multiplies the denominator to $96\pi^2$, yielding the final effective vertex form factor,

$$f_A = -\frac{e|y_{\alpha 1}|^2}{96\pi^2 M_1^2} \left[\frac{3}{2} \ln\left(\frac{\mu}{\epsilon}\right) - \frac{1 + 3\mu - 3\epsilon}{\sqrt{(\mu - 1 - \epsilon)^2 - 4\epsilon}} \operatorname{arctanh}\left(\frac{\sqrt{(\mu - 1 - \epsilon)^2 - 4\epsilon}}{\mu - 1 + \epsilon}\right) \right]. \quad (\text{A.37})$$

This algebraic reduction proves that the raw momentum-space loop integration derived in this work is mathematically identical to the coordinate-space EFT derivation presented by Kopp et al.

Appendix B

The Relativistic Amplitude

B.1 Squaring the relativistic amplitude and trace evaluation

To determine the scattering cross-section, the matrix element $\mathcal{M}$ must be converted to the unpolarised squared amplitude $|\overline{\mathcal{M}}|^2$. By squaring the amplitude, summing over the final spin states, and averaging over the initial spin states, the Dirac spinors are mapped into evaluable momentum traces via Casimir's trick.

The fully contracted t -channel amplitude is given by

$$\mathcal{M} = -e f_A [\bar{u}(p') \gamma^\mu \gamma^5 u(p)] [\bar{u}(k') \gamma_\mu u(k)]. \quad (\text{B.1})$$

Computing $|\mathcal{M}|^2 = \mathcal{M} \mathcal{M}^\dagger$ requires the Hermitian conjugate of the amplitude. Taking the Dirac adjoint ($\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$) of the anapole vertex yields

$$(\gamma^\mu \gamma^5)^\dagger = \gamma^{5\dagger} \gamma^{\mu\dagger} = \gamma^5 \gamma^0 \gamma^\mu \gamma^0 = -\gamma^0 \gamma^\mu \gamma^5 \gamma^0. \quad (\text{B.2})$$

Since $(\gamma^0)^2 = \mathbb{I}$, the adjoint vertex reduces to $\gamma^\mu \gamma^5$. Therefore, the unpolarised squared amplitude factorises into the contraction of a dark matter tensor $L_{\text{DM}}^{\mu\nu}$ and an electron tensor $L_{e,\mu\nu}$,

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4} (e f_A)^2 L_{\text{DM}}^{\mu\nu} L_{e,\mu\nu}, \quad (\text{B.3})$$

where the prefactor of $1/4$ accounts for averaging over the two initial spin orientations of both the dark matter particle and the target electron.

Evaluating the dark matter tensor, the fermion completeness relation $\sum_s u(p, s) \bar{u}(p, s) = \not{p} + M_1$ is applied to convert the spinor brackets into a closed trace over Dirac matrices,

$$L_{\text{DM}}^{\mu\nu} = \text{Tr} [(\not{p}' + M_1) \gamma^\mu \gamma^5 (\not{p} + M_1) \gamma^\nu \gamma^5]. \quad (\text{B.4})$$

Utilising the anti-commutation relation $\{\gamma^\mu, \gamma^5\} = 0$, the first γ^5 is commuted through the momentum slash $\not{p}$, yielding $\gamma^5 \not{p} = -\not{p} \gamma^5$. Because γ^5 commutes with the scalar mass M_1 , the trace becomes

$$L_{\text{DM}}^{\mu\nu} = \text{Tr} [(\not{p}' + M_1) \gamma^\mu (-\not{p} + M_1) \gamma^\nu \gamma^5]. \quad (\text{B.5})$$

Commuting γ^5 past γ^ν introduces an additional minus sign, and applying $(\gamma^5)^2 = \mathbb{I}$ removes the axial components entirely,

$$L_{\text{DM}}^{\mu\nu} = \text{Tr} [(\not{p}' + M_1) \gamma^\mu (\not{p} - M_1) \gamma^\nu]. \quad (\text{B.6})$$

Since the trace of an odd number of gamma matrices vanishes, the mass-linear cross-terms evaluate to zero. The remaining terms are expanded using the standard trace identities $\text{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}$ and $\text{Tr}[\not{a} \gamma^\mu \not{b} \gamma^\nu] = 4(a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu}(a \cdot b))$,

$$L_{\text{DM}}^{\mu\nu} = 4(p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu}(p \cdot p')) - 4M_1^2 g^{\mu\nu}. \quad (\text{B.7})$$

Rearranging the terms, the dark matter tensor evaluates to

$$L_{\text{DM}}^{\mu\nu} = 4[p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu}(p \cdot p' + M_1^2)]. \quad (\text{B.8})$$

The electron tensor $L_{e,\mu\nu}$ is evaluated similarly, lacking the initial axial γ^5 terms. This yields the standard QED vector current tensor,

$$L_{e,\mu\nu} = \text{Tr}[(\not{k}' + m_e)\gamma_\mu(\not{k} + m_e)\gamma_\nu] = 4\left[k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu}(k \cdot k' - m_e^2)\right]. \quad (\text{B.9})$$

The two tensors $L_{\text{DM}}^{\mu\nu}$ and $L_{e,\mu\nu}$ are subsequently contracted. Expanding the algebraic product yields

$$\begin{aligned} L_{\text{DM}} \cdot L_e = 16 & \left[2(p' \cdot k)(p \cdot k') + 2(p' \cdot k')(p \cdot k) \right. \\ & - 2(p \cdot p')(k \cdot k' - m_e^2) - 2(k \cdot k')(p \cdot p' + M_1^2) \\ & \left. + 4(p \cdot p' + M_1^2)(k \cdot k' - m_e^2) \right]. \end{aligned} \quad (\text{B.10})$$

Expanding the terms involving metric contractions,

$$\begin{aligned} \text{Term 1: } & -2(p \cdot p')(k \cdot k') + 2m_e^2(p \cdot p') \\ \text{Term 2: } & -2(p \cdot p')(k \cdot k') - 2M_1^2(k \cdot k') \\ \text{Term 3: } & +4(p \cdot p')(k \cdot k') - 4m_e^2(p \cdot p') + 4M_1^2(k \cdot k') - 4M_1^2 m_e^2.. \end{aligned}$$

Summing these terms, the pure momentum-squared components perfectly cancel. The remaining mass-dependent terms simplify to

$$2M_1^2(k \cdot k') - 2m_e^2(p \cdot p') - 4M_1^2 m_e^2. \quad (\text{B.11})$$

Factoring out a common factor of 2 and substituting back into the full tensor contraction yields

$$L_{\text{DM}} \cdot L_e = 32 \left[(p' \cdot k')(p \cdot k) + (p' \cdot k)(p \cdot k') + M_1^2(k \cdot k') - m_e^2(p \cdot p') - 2M_1^2 m_e^2 \right]. \quad (\text{B.12})$$

Restoring the initial spin-averaging factor of 1/4 yields the unpolarised squared matrix element,

$$|\overline{\mathcal{M}}|^2 = 8(e f_A)^2 \left[(p' \cdot k')(p \cdot k) + (p' \cdot k)(p \cdot k') + M_1^2(k \cdot k') - m_e^2(p \cdot p') - 2M_1^2 m_e^2 \right]. \quad (\text{B.13})$$

To optimise this expression for direct detection kinematics, the initial and final state dot products are rewritten in terms of the four-momentum transfer $q = p - p' = k' - k$. Squaring the momentum transfer provides relations for the two dot products,

$$q^2 = (p - p')^2 = 2M_1^2 - 2(p \cdot p') \implies p \cdot p' = M_1^2 - \frac{q^2}{2} \quad (\text{B.14})$$

$$q^2 = (k' - k)^2 = 2m_e^2 - 2(k \cdot k') \implies k \cdot k' = m_e^2 - \frac{q^2}{2}. \quad (\text{B.15})$$

The mixed initial-final state dot products ($p \cdot k'$, $p' \cdot k$, and $p' \cdot k'$) are subsequently expressed entirely in terms of the initial state collision parameter ($p \cdot k$) and q^2 . Using $p' = p - q$ and $k' = k + q$,

$$p' \cdot k = (p - q) \cdot k = p \cdot k - q \cdot k \quad (\text{B.16})$$

$$p \cdot k' = p \cdot (k + q) = p \cdot k + p \cdot q \quad (\text{B.17})$$

$$p' \cdot k' = (p - q) \cdot (k + q) = p \cdot k + p \cdot q - q \cdot k - q^2. \quad (\text{B.18})$$

The q dot products are evaluated by contracting q with the definitions $q = p - p'$ and $q = k' - k$,

$$p \cdot q = p \cdot (p - p') = M_1^2 - \left(M_1^2 - \frac{q^2}{2} \right) = \frac{q^2}{2} \quad (\text{B.19})$$

$$q \cdot k = (k' - k) \cdot k = \left(m_e^2 - \frac{q^2}{2} \right) - m_e^2 = -\frac{q^2}{2}. \quad (\text{B.20})$$

Substituting these identities back, the mixed dot products simplify to

$$p' \cdot k = p \cdot k + \frac{q^2}{2} \quad (\text{B.21})$$

$$p \cdot k' = p \cdot k + \frac{q^2}{2} \quad (\text{B.22})$$

$$p' \cdot k' = p \cdot k + \frac{q^2}{2} - \left(-\frac{q^2}{2} \right) - q^2 = p \cdot k. \quad (\text{B.23})$$

Applying these q^2 -dependent relations to Equation (B.13), the mass terms evaluate to $-2M_1^2 m_e^2 - \frac{q^2}{2}(M_1^2 - m_e^2)$, while the cross-terms expand into polynomials of $(p \cdot k)$. The relativistic squared amplitude simplifies to

$$\overline{|\mathcal{M}|^2} = 8(e f_A)^2 \left[2(p \cdot k)^2 + q^2(p \cdot k) - \frac{q^2}{2}(M_1^2 - m_e^2) + \frac{q^4}{4} - 2M_1^2 m_e^2 \right]. \quad (\text{B.24})$$

To align this expression with direct detection phenomenology, the remaining unknown dot product ($p \cdot k$) is evaluated in the target rest frame (the lab frame). Assuming the target electron is initially at rest, its four-momentum is $k = (m_e, \mathbf{0})$. The contraction with the initial dark matter four-momentum $p = (E_1, \mathbf{p})$ isolates the dark matter energy,

$$p \cdot k = E_1 m_e. \quad (\text{B.25})$$

Inserting this into the amplitude,

$$\overline{|\mathcal{M}|^2} = 8(e f_A)^2 \left[2E_1^2 m_e^2 + q^2 E_1 m_e - \frac{q^2}{2}(M_1^2 - m_e^2) + \frac{q^4}{4} - 2M_1^2 m_e^2 \right]. \quad (\text{B.26})$$

The first and fifth terms can be combined by factoring out $2m_e^2$, yielding the standard relativistic energy-momentum relation, $E_1^2 - M_1^2 = |\mathbf{p}|^2$, where $\mathbf{p}$ is the three-momentum of the dark matter particle,

$$2E_1^2 m_e^2 - 2M_1^2 m_e^2 = 2m_e^2(E_1^2 - M_1^2) = 2m_e^2|\mathbf{p}|^2. \quad (\text{B.27})$$

Substituting this momentum relation back into the amplitude yields

$$\overline{|\mathcal{M}|^2} = 8(e f_A)^2 \left[2m_e^2 |\mathbf{p}|^2 + q^2 E_1 m_e - \frac{q^2}{2} (M_1^2 - m_e^2) + \frac{q^4}{4} \right]. \quad (\text{B.28})$$

Distributing the global factor of 8 clears the denominators, resulting in the lab-frame squared matrix element,

$$\overline{|\mathcal{M}|^2} = (e f_A)^2 \left[16m_e^2 |\mathbf{p}|^2 + 2q^2 (4E_1 m_e - 2M_1^2 + 2m_e^2) + 2q^4 \right]. \quad (\text{B.29})$$

This algebraic form facilitates the non-relativistic expansion. Because the dark matter velocity is $v = |\mathbf{p}|/M_1$, isolating the three-momentum explicitly exposes the leading-order velocity dependence, while the grouped q^2 terms systematically organise the momentum-dependent kinematics.

B.2 Matching of squared amplitude to NR-EFT

While the exact lab-frame squared amplitude derived in Equation (B.29) is valid at all energies, terrestrial direct detection experiments probe a strictly non-relativistic regime. Dark matter confined to the Milky Way halo travels at velocities of $v \sim 10^{-3}c$, and the momentum transfer to the target electrons is typically on the order of $\mathcal{O}(\text{keV})$. To extract the phenomenological operators required for evaluating the scattering rate, the non-relativistic (NR) limit must be applied to the matrix element.

Starting from the final result of the previous section,

$$\overline{|\mathcal{M}|^2} = (e f_A)^2 \left[16m_e^2 |\mathbf{p}|^2 + 2q^2 (4E_1 m_e - 2M_1^2 + 2m_e^2) + 2q^4 \right]. \quad (\text{B.30})$$

In the NR limit ($v \ll c$), the dark matter four-momentum $p^\mu = (E_1, \mathbf{p})$ and the four-momentum transfer $q^\mu = (q_0, \mathbf{q})$ simplify significantly. The following standard kinematic approximations are applied:

1. The dark matter kinetic energy is negligible compared to its rest mass, implying $E_1 \approx M_1$.
2. The dark matter three-momentum squared is strictly a function of its velocity: $|\mathbf{p}|^2 = M_1^2 v^2$.
3. For elastic scattering, the energy transfer $q_0 = |\mathbf{q}|^2 / 2\mu_{\chi e}$ is highly suppressed compared to the three-momentum transfer $|\mathbf{q}|$. Thus, the four-momentum transfer squared reduces to just its spatial component: $q^2 = q_0^2 - |\mathbf{q}|^2 \approx -|\mathbf{q}|^2$.

Substituting these limits into Equation (B.30), and dropping the internal $q^4 \approx |\mathbf{q}|^4$ term as it is highly suppressed by $\mathcal{O}(|\mathbf{q}|^4/m^4)$ compared to the mass-dominant terms, the amplitude becomes

$$\overline{|\mathcal{M}|^2} \approx (e f_A)^2 \left[16M_1^2 m_e^2 v^2 - 2|\mathbf{q}|^2 (4M_1 m_e - 2M_1^2 + 2m_e^2) \right]. \quad (\text{B.31})$$

To match this to the standard NR-EFT framework, the DM velocity squared v^2 must be expressed in terms of the Galilean-invariant transverse velocity, $\mathbf{v}^\perp$. By definition, $\mathbf{v}^\perp$ is the component of the initial relative velocity orthogonal to the momentum transfer,

$$\mathbf{v}^\perp = \mathbf{v} - \frac{\mathbf{q}}{2\mu_{\chi e}}, \quad (\text{B.32})$$

where $\mu_{\chi e} = \frac{M_1 m_e}{M_1 + m_e}$ is the reduced mass of the dark matter-electron system. The matching is performed for a free electron at rest, so the scattering is elastic and $\mathbf{v}^\perp \cdot \mathbf{q} = 0$. The bound-state corrections that break this orthogonality are taken up in Section 4.3 and do not affect the operator content matched here. Squaring the definition then yields a direct relation for v^2 ,

$$v^2 = (v^\perp)^2 + \frac{|\mathbf{q}|^2}{4\mu_{\chi e}^2}. \quad (\text{B.33})$$

Substituting this v^2 relation back into the leading term of the squared amplitude,

$$16M_1^2 m_e^2 v^2 = 16M_1^2 m_e^2 (v^\perp)^2 + 16M_1^2 m_e^2 \left(\frac{|\mathbf{q}|^2}{4\mu_{\chi e}^2} \right). \quad (\text{B.34})$$

Expanding the reduced mass in the denominator of the second term ($\mu_{\chi e}^2 = \frac{M_1^2 m_e^2}{(M_1 + m_e)^2}$), the product $M_1^2 m_e^2$ perfectly cancels,

$$16M_1^2 m_e^2 \left(\frac{(M_1 + m_e)^2 |\mathbf{q}|^2}{4M_1^2 m_e^2} \right) = 4(M_1 + m_e)^2 |\mathbf{q}|^2 = |\mathbf{q}|^2 (4M_1^2 + 8M_1 m_e + 4m_e^2). \quad (\text{B.35})$$

This substitution generates a new $|\mathbf{q}|^2$ term sourced directly from the velocity kinematics. This term is now grouped with the existing $|\mathbf{q}|^2$ block from the interaction tensor. Distributing the -2 prefactor into the original q^2 bracket allows the algebraic sum to be evaluated,

$$\begin{aligned} \text{From } v^2 : & \quad |\mathbf{q}|^2 (4M_1^2 + 8M_1 m_e + 4m_e^2) \\ \text{From } q^2 : & \quad -2|\mathbf{q}|^2 (4M_1 m_e - 2M_1^2 + 2m_e^2) = |\mathbf{q}|^2 (4M_1^2 - 8M_1 m_e - 4m_e^2). \end{aligned}$$

Summing these two components results in an exact cancellation of the mixed mass terms ($\pm 8M_1 m_e$) and the electron mass terms ($\pm 4m_e^2$). The total contribution simplifies to a pure dependence on the dark matter mass,

$$\text{Total } |\mathbf{q}|^2 \text{ contribution: } |\mathbf{q}|^2 (4M_1^2 + 4M_1^2) = 8M_1^2 |\mathbf{q}|^2. \quad (\text{B.36})$$

Inserting this evaluated sum back into the expression, the full non-relativistic squared matrix element reduces into two terms,

$$\overline{|\mathcal{M}|^2} \approx (e f_A)^2 [16M_1^2 m_e^2 (v^\perp)^2 + 8M_1^2 |\mathbf{q}|^2]. \quad (\text{B.37})$$

Factoring out the global kinematic prefactor $16M_1^2 m_e^2$, the final NR-EFT representation of the scattering amplitude is obtained,

$$\boxed{\overline{|\mathcal{M}|^2}_{NR} \approx 16M_1^2 m_e^2 (e f_A)^2 \left[(v^\perp)^2 + \frac{|\mathbf{q}|^2}{2m_e^2} \right]}. \quad (\text{B.38})$$

By mapping this expression to the standard NR-EFT operator basis defined by Fitzpatrick et al. [25], the two distinct physical interactions driving the scattering event are identified:

- The first term is proportional to the transverse velocity squared, $(v^\perp)^2$. This corresponds to operator $O_8 = \mathbf{S}_\chi \cdot \mathbf{v}^\perp$, representing the interaction between the dark matter spin and the classical moving charge of the electron.
- The second term is proportional to the momentum transfer squared over the electron mass squared, $|\mathbf{q}|^2/m_e^2$. This corresponds to operator $O_9 = i\mathbf{S}_\chi \cdot (\mathbf{S}_e \times \frac{\mathbf{q}}{m_e})$, representing the quantum mechanical interaction between the dark matter spin and the intrinsic magnetic dipole moment (spin) of the electron.

Having established a bridge between the relativistic quantum field theory and the non-relativistic effective operators, these two separated terms serve as the direct inputs for the evaluation of the differential event rate in the target material.

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