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Classical ghosts

A literature study of the classical aspects of the Batalin-Vilkovisky formalism

Master's thesis in Physics

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Abstract

Gauge symmetry appears in many physical theories and is often handled by just imposing gauge choices, i.e. removing the gauge symmetry as it consists of unphysical degrees of freedom. This approach may fail when symmetries do not close on-shell and obscures underlying structure of the theory. The Batalin-Vilkovisky formalism describes the connected structure of (higher) gauge symmetries, their corresponding (higher) Noether identities and equations of motion, by introducing a plethora of extra fields called ghosts, antifields and antighosts to represent these.

This literature study starts with the introduction of graded structures, from graded vector spaces to Q-manifolds and their duals L_∞-algebras, whereafter the physical phenomena and problems that motivates the BV formalism are explained. These phenomena and the formalism itself are then formulated in terms of the introduced mathematical framework. The extra fields are regarded as elements or sections of a symplectic Q-manifold. Central for the BV formalism is an action S that through the symplectic structure induces a differential which in turn gives the equations of motion, gauge transformation etc. when it acts upon the corresponding field. In the L_∞-dual description the gauge symmetries etc. are formulated in terms of higher products that generalise the Lie bracket, while equivalence between physical theories corresponds to quasi-isomorphisms. After some minor examples and calculations the thesis ends by revisiting a BV formulation of 10D supersymmetric Yang-Mills, with the help of bosonic spinors and weight partition functions.

Keywords: gauge symmetry, ghosts, antifields, L_∞-algebras, supersymmetry.

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Leo Andersson, Gothenburg, July 2026

Table of contents

1. Outline and background

This thesis assumes some prior knowledge. The reader should be acquainted with field theory, gauge transformations and Lie algebras. Also concepts from differential geometry such as fibre bundles and (co)homology are necessary. I have realised that I have acquaintances and friends, who are interested in this thesis but lack some of the prior knowledge. As a kind gesture I give a very brief and flawed introduction in Appendix A to some concepts from differential geometry that are needed. The description of the formalism tends to be quite mathematical which can lead to the discussion of subjects that some readers may find far-flung from physics. At the same time the goal is not to describe the underlying mathematics itself but the physics. Therefore proofs and (often trivial) technical details that mathematicians demand while more physics inclined people do not care about, e.g. smoothness, are often omitted.

The thesis is roughly divided into four parts. The first one contains the mathematical framework that will be used and the second explains much of the relevant physical problems and phenomena. The third part combines the first two, explaining how the introduced mathematical framework describes the physics. The fourth one deals with some chosen applications of the BV formalism, in particular super symmetry. Several sources has been used for writing this work but the ones I found most useful, and often cited most when writing this thesis are the following. The short "An introduction to the Batalin-Vilkovisky formalism" by D. Fiorenza [1] to give a short and good introduction to the mathematical structures, the much comprehensive "L_∞-algebras of classical field theories and the Batalin–Vilkovisky formalism" by B. Jurco, L. Raspollini, C. Saemann, and M. Wolf [2] to continue explaining the mathematical structures without demanding too much of the reader. To give an understanding for the physics and motivate the formalism, I found none better than "Lectures on the antifield-BRST formalism for gauge theories" by M. Henneaux [3]. Finally for the last part regarding super symmetry I made use of "Pure spinors in classical and quantum supergravity" by M. Cederwall [4].

The concept of gauge theory began already with a paper by Weyl in 1918 and became more formalised through the work of Yang and Mills 1953 [5], [6]. In a gauge theory there is a redundancy in the description of the physical world, or more precise the fields of the theory may be changed locally, in a specific way, without changing the physics described by the theory. These changes or transformations are referred to as gauge symmetries or gauge transformations. With the papers of W Ehrenberg and R E Siday 1949 and later Y. Aharonov and D. Bohm 1959, evidence was laid out that gauge theories are more than just a mathematical trick or manipulation but really have physical meaning that is not easily described otherwise [7], [8]. Today (and long before as well) gauge theory is a fundamental part of physics, appearing in general relativity, quantum field theories such as the Standard Model as well as in most attempts to go beyond these.

Gauge theories pose some difficulties. For example, when quantising gauge theories using the generating functional

$$Z \propto \int D\phi \, e^{-iS[\phi]},$$

the integral measure $D\phi$ may not be invariant under the gauge symmetry and thus break it for the theory [9]. Furthermore the gauge invariance of the action implies degeneracy of its (functional) Hessian, thus hindering conventional approaches to calculate the integral (i.e. Feynman diagrams) [10].

This is a problem that can be solved by introducing the Faddeev-Popov ghost [9], [11]. The ghost is a field[1] which takes on the role of the redundancy in description the that is associated to the gauge symmetry [3]. Hence it is mathematically constructed to, in the end, not have any effect at all upon the physical world. A more formal way of introducing ghosts is offered by the BRST method (from Becchi, Rouet, Stora and Tyutin) [10]. In some cases even further generalisation is needed, such as when the gauge symmetries themselves have redundancies, hence the need for ghosts for the ghosts. The Batalin-Vilkovisky formalism (in older literature known as the BRST antifield formalism), named after I.A. Batalin and G.A. Vilkovisky who extended the BRST framework, is suitable for vastly generalised settings and may, for example, handle even an infinite tower of ghosts [2], [12].

This thesis handles classical and not quantum theories. The BV formalism is namely excellent for investigating gauge symmetries, not only for quantisation purposes. It easily handles situations when symmetries do not close on-shell and is applicable for much more general cases.

[1]Sometimes the word ghost instead refers to the corresponding particle in quantum settings.

2. The mathematical frameworks

The reason for this chapter is to build up much of the mathematical framework that describes, is used for or is otherwise related to the Batalin-Vilkovisky formalism.

2.1 Graded structures

2.1.1 graded vector spaces

A $\mathbb{Z}_2$ graded vector space, or super vector space, is a vector space which is decomposable as $V = V_0 \oplus V_1$, where V_0 is called the even and V_1 the odd part [1]. The (super) dimension of this space is the tuple $(\dim V_0|\dim V_1)$. Vectors belonging to one of these components are called homogeneous and can be assigned a grade $|\cdot|$,

$$|\cdot| : v \mapsto |v| = i, \quad \text{for } v \in V_i. \tag{2.1}$$

The structure that surrounds vector spaces can be adjusted for super vector spaces so that the grading is respected, often in a way which feels natural when compared to how odd and even numbers behave. For example, morphisms in the category of super vector spaces are linear transformations that preserve the grading, i.e. $f : V \to W$ such that

$$f(V_i) \subseteq W_i \tag{2.2a}$$

$$(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1 \tag{2.2b}$$

$$(V \otimes W)_1 = V_1 \otimes W_0 \oplus V_0 \otimes W_1. \tag{2.2c}$$

The parity inversion of a super vector space V is denoted ΠV and is defined by interchanging the grades of V: $(\Pi V)_0 = V_1$, $(\Pi V)_1 = V_0$ [1]. As every vector space V can be regarded as a super vector space W, by letting $W_0 = V$ and $W_1 = \emptyset$, this operator makes it straightforward to construct more complex super vector spaces. For example,

$$\mathbb{R}^{(n|m)} := \mathbb{R}^n \oplus \Pi\mathbb{R}^m = \mathbb{R}^n \oplus \bigoplus_{k=1}^{m} \Pi\mathbb{R} \tag{2.3}$$

is a real vector space of dimension $(n|m)$.

2.1.2 algebras

A generalisation of these $\mathbb{Z}_2$-graded vector spaces are the so-called $\mathbb{Z}$-graded vector spaces which are decomposable as $V = \bigoplus_{k\in\mathbb{Z}} V_k$, where k is the grade of the subspaces. Similarly as for $\mathbb{Z}_2$-graded vector spaces, we can speak here of homogeneous elements with a corresponding grade operator $|\cdot| : v \mapsto |v| = k$, for $v \in V_k$ [2], [10].

After reading what follows, the reader may note that much of the properties and extra structures that applies or can be added to $\mathbb{Z}$-graded vector spaces also applies to $\mathbb{Z}_2$-graded vector spaces. This is by identifying the $\mathbb{Z}$-graded V as a $\mathbb{Z}_2$-graded space, with $\bigoplus_{k\in\mathbb{Z}} V_{2k}$ as the even part and $\bigoplus_{k\in\mathbb{Z}} V_{2k+1}$, and taking the grade of homogeneous elements mod 2. In the remainder of the thesis the word graded alone will refer to a more general $\mathbb{Z}$-grading.

For $l \in \mathbb{Z}$ and $V = \bigoplus_{k\in\mathbb{Z}} V_k$ the degree l shifted space $V[l]$ is defined as [2], [10]

$$V[l] := \bigoplus_{k\in\mathbb{Z}} V[l]_k, \qquad V[l]_k = V_{k+l}. \tag{2.4}$$

Thus the degree of the vectors will be lowered by l. Often the functions on the $\mathbb{Z}$ graded spaces are of higher interest than the elements themselves. Their building blocks are the coordinate functions belonging to the dual space V^* which is also a graded vector space, by the decomposition

$$V^* = \bigoplus_{k\in\mathbb{Z}} V^*_{-k}, \qquad V^*_{-k} = \left\{ w \in V^* \;\middle|\; w|_{V_l} = 0 \quad \text{for } l \neq k \right\} = (V_k)^*. \tag{2.5}$$

The choice of a minus sign is because the wish of making the grade a conserved quantity (here in the mathematical sense). With this choice, dual vectors of degree $-k$ sends vectors of degree k to real numbers of degree $-k+k=0$. It follows that for $V[l]$ the degree of the coordinate functions will be raised by l.

The grading of objects can be defined for things other than vector spaces, but in a similar way. A (commutative) graded algebra, for example, is a unital associative algebra decomposed as $A = \bigoplus_k A_k$, where elements in A_k are said to be homogeneous of degree k [2][13]. The multiplication of the algebra is associative, respects the grading, $|ab| = |a| + |b|$, and is graded commutative, $ab = -^{ab}ba$, where a, b are homogeneous elements of degree $|a|$ and $|b|$ respectively and $-^{ab}$ is short for $(-1)^{|a||b|}$. This notation will be used throughout the thesis, with some additions when needed. The property that a sign is picked up whenever two elements pass each other is a key property for graded structures and many properties from the corresponding ungraded versions are often straightforwardly generalised by this modification. Note that every graded vector space generates a graded algebra where the multiplication is defined as the tensor product modulo the ideal generated by $v \otimes w -^{1+vw} w \otimes v$. This product gives rise to the so-called n^{th} symmetric product of a graded vector space V, defined as $V^{\otimes n}$ modulo switches

$$v_1 \otimes \cdots v_i \otimes \cdots v_j \cdots v_n \mapsto -^{v_i v_j + (v_i+v_j)\sum_{i<k<j} v_k} v_1 \otimes \cdots v_j \otimes \cdots v_i \cdots v_n, \tag{2.6}$$

where $v_l \in V$ and $1 \leq i < j \leq n$. It is denoted by $\vee^n V$ and for even vectors it reduces to the ordinary symmetric product, using the same notation,

$$\vee^n V = V^{\otimes n} / \{ v_1 \otimes \cdots v_n - v_{\sigma(1)} \otimes \cdots v_{\sigma(n)} \mid v_i \in V,\ \sigma \in S_n \}. \tag{2.7}$$

For odd vectors it reduces to the exterior product [1].

$$\vee^2(\mathbb{R}[1]) \simeq (\wedge^2 \mathbb{R})[2]. \tag{2.8}$$

The notion and notation of a symmetric product $\vee$ may also be used when the multiplication, which is graded commutative, is not defined in terms of tensor products.

2.1.3 graded manifolds

Let $|M|$ be a topological manifold and $\mathscr{S}_M$ a sheaf of rings on it. If all stalks of the latter have unique maximal ideals, the pair $(|M|, \mathscr{S}_M)$ is called a local ring. Assume that every point in $|M|$ has a neighbourhood U such that there is an isomorphism (of locally ringed spaces)

$$\left(U, \mathscr{S}_M\Big|_U\right) \simeq \left(V, \vee^\bullet \mathscr{V}_V^* \otimes \mathcal{C}_V^\infty\right), \tag{2.9}$$

for V an open subset of $\mathbb{R}^{\dim |M|}$ with sheaf of smooth functions $\mathcal{C}_V^\infty$ and where $\mathscr{V}_V$ is a locally free $\mathbb{Z}$-graded sheaf of $\mathcal{C}_V^\infty$ modules on V. Then this local ring $M = (|M|, \mathscr{S}_M)$ is a $\mathbb{Z}$-graded manifold with body $M^\circ := (|M|, \mathcal{C}_M^\infty)$, with $\mathcal{C}_M^\infty$ the sheaf of smooth functions on $|M|$ (which is a subsheaf of $\mathscr{S}_M$) [2]. Alternatively it should locally look like

$$\left(\mathbb{R}^{\dim M^0}, \vee^\bullet W \otimes \mathcal{C}^\infty(\mathbb{R}^{\dim |M|})\right), \tag{2.10}$$

for some $\mathbb{Z}$-graded vector space W. Global functions on $\mathbb{Z}$-graded manifolds are denoted by $\mathcal{C}^\infty(M) := \Gamma(|M|, \mathscr{S}_M)$. Vector fields on these manifolds are as usual defined as derivations $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, though graded so that the Leibniz rule is modified to

$$v(fg) = v(f)g -^{vf} fv(g), \tag{2.11}$$

for a homogeneous vector field v and functions $f, g \in \mathcal{C}^\infty(M)$ [10].

Note that the above definition of a graded manifold identifies a manifold M with the pair of its body and its algebra of functions, i.e. the local version of $\mathcal{C}^\infty(M)$. To clarify, let $M = T[1]\mathbb{R}$ be defined such that the shifting only acts on the tangent space, i.e. $M \simeq \mathbb{R}[1] \oplus \mathbb{R}$. This is a graded manifold with body rather $M^\circ = \mathbb{R}$, though this is identified with $M^\circ = (\mathbb{R}, \mathcal{C}^\infty(\mathbb{R}))$. A further example of a $\mathbb{Z}$ graded manifold would be $T[1]\mathbb{R} \oplus T^*[-3]\mathbb{R}$. The manifold is isomorphic to $\mathbb{R} \oplus \mathbb{R}[1] \oplus \mathbb{R}[-3]$ and its coordinate functions are locally of the form

$$f(x, y, z) = f_0(x) + f_1(x)y + f_2(x)z + f_3(x)yz. \tag{2.12}$$

Here $x \in \mathbb{R}$ is the even and $y \in (\mathbb{R}[1])^* = \mathbb{R}[-1]$ and $z \in (\mathbb{R}[-3]^* = \mathbb{R}[3]$ the odd coordinates and f_i is any $\mathcal{C}^\infty(\mathbb{R})$ function. With this notation $\mathcal{C}^\infty(T[1]M) \simeq \Omega^\bullet(M)$.

2.2 Q-manifolds

A vector field Q is said to be homological if it is of degree 1 and is a differential, $Q^2 = 0$ (equivalently the graded Lie bracket $[Q, Q] = 2Q^2 = 0$) [10]. The corresponding manifold is called a Q-manifold or differentially graded manifold and an algebra with such a differential (though not necessarily a vector field) is called a differentially graded algebra [2]. An obvious example of a Q-manifold is $T[1]M$, for some zero-degree/regular manifold M, with differential

$$Q = \xi^\mu \frac{\partial}{\partial x^\mu}, \tag{2.13}$$

where x^μ is the even coordinates on M and ξ^μ is the odd fibre coordinates [2]. Remembering the above correspondence $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$ and identifying $\mathrm{d}x^\mu = \xi^\mu$ this becomes the exterior differential.

Even more structure can be added to graded manifolds. A (graded) symplectic structure of degree k is a two-form $\omega \in \Omega^2(M)$ of degree k that is non-degenerate ($v \lrcorner \omega = 0 \longrightarrow v = 0$ for vector field v). If there is a homological vector field Q that is symplectic, i.e. the Lie derivative $\mathcal{L}_Q\omega = 0$, we have a symplectic Q-manifold of degree k. Here the generalised graded Lie derivative acts on a differential form as [2]

$$\mathcal{L}_Q\omega = Q \lrcorner \mathrm{d}\omega -^Q \mathrm{d}(Q \lrcorner \omega). \tag{2.14}$$

Given such a manifold, it is now possible to define a graded Poisson bracket. To every function $f \in \mathcal{C}^\infty(M)$ there is an associated vector field V_f of degree $|f| - k$ defined by [2]

$$V_f \lrcorner \omega = \mathrm{d}f. \tag{2.15}$$

The graded Poisson bracket is defined as

$$\{f, g\} := V_f \lrcorner V_g \lrcorner \omega \tag{2.16}$$

and fulfills

- $\{f, g\} = V_f[g]$,
- $\{f, g\} = -^{(f-k)(g-k)}\{g, f\}$,
- graded Jacobi
$$\{f, \{g, h\}\} -^{(f-k)(g+h-2k)+f-k} \{g, \{h, f\}\} -^{(h-k)(f+g-2k)+h-k} \{h, \{f, g\}\} = 0,$$
- and graded Leibniz $\{f, gh\} = \{f, g\}h -^{(f-k)g} \{f, h\}$.

The properties of ω implies that, locally, there are so-called Darboux coordinates x^i, ξ_i such that [2], [10]

$$\omega = \mathrm{d}\xi_i \wedge \mathrm{d}x^i. \tag{2.17}$$

In these coordinates[1]

$$\begin{aligned} V_f \lrcorner \omega &= \left(V_f^i \frac{\partial}{\partial x^i} + V_{fi} \frac{\partial}{\partial \xi_i} \right) \lrcorner \left(\mathrm{d}\xi_i \wedge \mathrm{d}x^i \right) = -^{1+\xi_i x^i} V_f^i \mathrm{d}\xi_i + V_{fi} \mathrm{d}x^i \\ &= \mathrm{d}f = \mathrm{d}x^i \frac{\overrightarrow{\partial}}{\partial x^i} f + \mathrm{d}\xi_i \frac{\overrightarrow{\partial}}{\partial \xi_i} f = f \frac{\overleftarrow{\partial}}{\partial x^i} \mathrm{d}x^i + f \frac{\overleftarrow{\partial}}{\partial \xi_i} \mathrm{d}\xi_i, \end{aligned} \tag{2.18}$$

[1]Note that there is not an extra implicit Einstein summation in the exponent of the minus sign. There is only one summation over the index i, which include all occurrences of it.

hence

$$V_f = -^{1+x^i\xi_i} f \frac{\overleftarrow{\partial}}{\partial \xi_i} \frac{\overrightarrow{\partial}}{\partial x^i} + f \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial \xi_i} . \tag{2.19}$$

The right and left derivatives introduced above are defined as

$$\frac{\overrightarrow{\partial}}{\partial y} f = \frac{\partial}{\partial y} f, \qquad f \frac{\overleftarrow{\partial}}{\partial y} = -^{y(1+f)} \frac{\partial}{\partial y} f. \tag{2.20}$$

It follows that the graded Poisson bracket can be expressed in terms of these coordinates as

$$\{f, g\} = V_f[g] = f \left(\frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial \xi_i} -^{1+x^i\xi_i} \frac{\overleftarrow{\partial}}{\partial \xi_i} \frac{\overrightarrow{\partial}}{\partial x^i} \right) g. \tag{2.21}$$

Let us assume that $Q \lrcorner \omega = \mathrm{d}S$ or equivalently that $Q = \{S, \cdot\}$ for some degree $k+1$ function S and d the exterior differential. Then by Jacobi $2Q^2 = \{\{S, S\}, \cdot\}$ and hence $\{S, S\}$ must be (locally) constant for Q to be a differential. The only constant that fits with the grade $|\{S, S\}| = 2 + k$ is 0 [2].

2.3 L_∞

2.3.1 Chevalley-Eilenberg

The usual Lie bracket is a map $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$. With the symmetric product the bracket may instead be defined as a map $\vee^2(\mathfrak{g}[1]) \to \Pi\mathfrak{g}[1]$ (or $\vee^2(\Pi\mathfrak{g}) \to \Pi\mathfrak{g}$) [1]. The dual $[\cdot, \cdot]^*$ is a map $\mathfrak{g}^*[-1] \to \vee^2(\mathfrak{g}^*[-1])$, that can be extended to degree 1 derivative of the so-called regular functions on $\mathfrak{g}$

$$Q : \mathcal{C}^\infty(\mathfrak{g}[1]) \to \mathcal{C}^\infty(\mathfrak{g}[1]) \tag{2.22}$$

using the Leibniz rule $Q(ab) = Q(a)b -^a aQ(b)$ [1][10]. This is then a Q-manifold with body a point, though it remains to show that Q is a differential. Going back to the equivalent structure of wedge products, the corresponding operator

$$\mathrm{d}_{\mathrm{CE}} : \wedge^\bullet \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^* \tag{2.23}$$

goes under the name Chevalley-Eilenberg differential. If T_a is the Lie algebra basis and ψ^a the dual basis, the derivative must (locally) be on the form

$$\mathrm{d}_{\mathrm{CE}} = \frac{1}{2} F^i_{jk} \psi^j \wedge \psi^k \frac{\partial}{\partial \psi^i}$$

as it raises the degree by one[2]. From the equality

$$\left\langle \mathrm{d}_{\mathrm{CE}}(\psi^i \wedge \psi^j) \middle| T_a \wedge T_b \wedge T_c \right\rangle = \left\langle \psi^i \wedge \psi^j \middle| [\cdot, \cdot](T_a \wedge T_b \wedge T_c) \right\rangle$$

[2]In this case degree rather refers to the number of components.

it may be deduced both that the Lie bracket should be extended to act as

$$[\cdot,\cdot] : T_a \wedge T_b \wedge T_c \mapsto [T_a, T_b] \wedge T_c + [T_b, T_c] \wedge T_a + [T_c, T_a] \wedge T_b$$

and that $F^i_{jk} = f^i_{jk}$. Now a simple calculations shows that the Jacobi identity is equivalent to the statement that d_{CE} is a differential [1].

2.3.2 L_∞-algebras

The above Chevalley-Eilenberg algebra may be generalised by dropping the assumption that the differential is of grade 1, call it δ. Still identifying $\wedge^\bullet \mathfrak{g}$ with $\mathcal{C}^\infty(\mathfrak{g}[1])$, let δ_n be the projection of δ on $\vee^n(\mathfrak{g}^*[-1])$ when it acts upon $\mathfrak{g}^*[-1]$ and define $[\cdot,\cdots,\cdot]_n$ as the dual map $[\cdot,\cdots,\cdot]_n : \vee^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$. From the equation

$$0 = \delta^2 = \underbrace{\delta_1^2}_{\text{deg }1} + \underbrace{\delta_1\delta_2 + \delta_2\delta_1}_{\text{deg }3} + \underbrace{\delta_1\delta_3 + \delta_3\delta_1 + \delta_2^2}_{\text{deg }4} + \cdots \tag{2.24}$$

follows that their duals

$$[\cdot]_1 = \delta_1^* : \mathfrak{g} \to \mathfrak{g} \tag{2.25a}$$
$$[\cdot,\cdot]_2 = \delta_2^* : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g} \tag{2.25b}$$
$$[\cdot,\cdot,\cdot]_3 = \delta_3^* : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g} \tag{2.25c}$$
$$\cdots$$

fulfill

$$[[f]_1]_1 = 0 \tag{2.26a}$$
$$[[f,g]_2]_1 = [[f]_1, g]_2 + [f, [g]_1]_2 \tag{2.26b}$$
$$\begin{aligned} &[[f,g]_2, h]_2 + [[g,h]_2, f]_2 + [[h,f]_2, g]_2 \\ =&[[f,g,h]_3]_1 - [[f]_1, g, h]_3 - [f, [g]_1, h]_3 - [f, g, [h]_1]_3. \end{aligned} \tag{2.26c}$$

This is interpreted as $[\cdot]_1$ being a differential that acts as a derivative on the bracket $[\cdot,\cdot]_2$ which in turn fulfills the Jacobi identity, modulo some homotopy given by $[\cdot,\cdot,\cdot]_3$ [1]. The above construction can be formalised as follows.

An L_∞ algebra is a graded vector space $L = \bigoplus_{n\in\mathbb{Z}} L_n$ equipped with higher products, that are multilinear graded antisymmetric maps $l_n : \underbrace{L \times \cdots \times L}_{n} \to L$ of degree $2-n$, for $m \geq 1$, satisfying the higher or homotopy Jacobi identities

$$\sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathrm{Sh}(n,m-n)} \chi(\sigma; \upharpoonleft_1, \cdots \upharpoonleft_m)(-1)^n l_{n+1}(l_{m-n}(\upharpoonleft_{\sigma(1)}, \cdots \upharpoonleft_{\sigma(m-n)}), \upharpoonleft_{\sigma(m-n+1)}, \cdots \upharpoonleft_{\sigma(m)}) = 0. \tag{2.27}$$

Here $\upharpoonleft_i \in L$ is the graded Koszul sign χ defined by

$$\upharpoonleft_1 \wedge \cdots \wedge \upharpoonleft_m = \chi(\sigma; \upharpoonleft_1, \cdots \upharpoonleft_m) \upharpoonleft_{\sigma(1)} \wedge \cdots \wedge \upharpoonleft_{\sigma(m)} \tag{2.28}$$

and $\mathrm{Sh}(n, m-n)$ is the set of shuffles[3] – permutations of the list $(1, 2, \cdots m)$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(m-n)$ and $\sigma(m-n+1) < \cdots < \sigma(m)$ [2], [10].

[3]Depending on wether you regard σ itself or the permutation of the $\upharpoonleft_i$'s, this is either a shuffle or an unshuffle.

The connection between L_∞-algebras and Q-manifolds is as follows. Let a Q-manifold consist of the differential Q and the manifold $M = V$ for some graded vector space V, so that the body of the manifold is just a point. Q may be decomposed as

$$Q = \sum_{n=1}^{\infty} Q_n, \qquad Q_n : V^*[-1] \to \vee^n(V^*[-1]). \tag{2.29}$$

The duals of each of these maps are D_n of grade -1,

$$D_n : \vee^n(V[1]) \to V[1], \qquad \langle Q_n v^* | w \rangle = \langle v^* | D_n w \rangle\,, \quad \forall v^* \in V^*[-1],\ \forall w \in \vee^n(V[1]). \tag{2.30}$$

Then $L = V[-1]$ is an L_∞-algebra with higher products [2], [10]

$$l_n = -^{\frac{1}{2}n(n-1)+1} s^{-1} \circ D_n \circ s^{\otimes n}, \tag{2.31}$$

where s is the so-called suspension symbol, which is the trivial isomorphism $s : V \to V[1]$ and

$$s^{\otimes n} : v_1 \wedge \cdots \wedge v_n \mapsto -^{\sum_{i=1}^{n-1}(n-i)v_i} sv_1 \vee \cdots \vee sv_n. \tag{2.32}$$

The (action of the) differential Q may be expressed directly in terms of the higher products l_i. Let $\psi^\alpha \in \mathcal{C}^\infty(L[1])$ be the coordinates of the Q-manifold and contract them to $\Psi \in \mathcal{C}^\infty(L[1]) \otimes L$. Then [2]

$$Q\Psi = -\sum_{n \geq 1} \frac{1}{n!} l_n(\Psi, \cdots \Psi). \tag{2.33}$$

Note that some signs appear when the higher products act on the coordinates,

$$\begin{aligned} l_n(\Psi, \cdots \Psi) &= l_n(\psi^{\alpha_1} \upharpoonleft_{\alpha_1}, \cdots \psi^{\alpha_n} \upharpoonleft_{\alpha_n}) \\ &= -^{\left(n \sum_{i=1}^{n} \psi^{\alpha_i} + \sum_{i=2}^{n} \psi^{\alpha_i} \sum_{l=1}^{n-1} \upharpoonleft^{\alpha_l}\right)} \psi^{\alpha_1} \cdots \psi^{\alpha_n} l_n(\upharpoonleft_{\alpha_1}, \cdots \upharpoonleft_{\alpha_n}), \end{aligned} \tag{2.34}$$

where the grading that appears implicitly in the exponent of -1 is the Q-manifold grading for ψ_i and the L_∞ grading for $\upharpoonleft_l$.

Skipping the details regarding signs, a proof for Formula (2.33) follows from

$$\begin{aligned} Q_n \Psi = Q\psi^\alpha \upharpoonleft_\alpha &= \psi^{\beta_1} \cdots \psi^{\beta_n} \left\langle s\upharpoonleft_{\beta_1} \vee \cdots \vee s\upharpoonleft_{\beta_n} \middle| Q_n \psi^\alpha \right\rangle \upharpoonleft_\alpha \\ &= \psi^{\beta_1} \cdots \psi^{\beta_n} \left\langle D_n \left(s\upharpoonleft_{\beta_1} \vee \cdots \vee s\upharpoonleft_{\beta_n} \right) \middle| \psi^\alpha \right\rangle \upharpoonleft_\alpha \\ &= \psi^{\beta_1} \cdots \psi^{\beta_n} \left\langle s\, l_n \left(\upharpoonleft_{\beta_1}, \cdots, \upharpoonleft_{\beta_n} \right) \middle| \psi^\alpha \right\rangle \upharpoonleft_\alpha \\ &= \psi^{\beta_1} \cdots \psi^{\beta_n} \left\langle s\upharpoonleft_\gamma l^{\gamma}_{n\ \beta_1, \cdots, \beta_n} \middle| \psi^\alpha \right\rangle \upharpoonleft_\alpha \\ &= \psi^{\beta_1} \cdots \psi^{\beta_n} l^{\alpha}_{n\ \alpha_1, \cdots, \beta_n} \upharpoonleft_\alpha \\ &= \psi^{\beta_1} \cdots \psi^{\beta_n} l_n \left(\upharpoonleft_{\beta_1}, \cdots, \upharpoonleft_{\beta_n} \right) \\ &= l_n(\Psi, \cdots, \Psi). \end{aligned} \tag{2.35}$$

2.3.3 L_∞-algebroids

If the above construction of an L_∞-algebra is carried out in the general case, when the body M° of a manifold M is not necessarily a point, the result is an L_∞-algebroid [2]. A special case is the Lie algebroid, which is a vector bundle $M \to M^\circ$, for some ordinary manifold M°, equipped with an antisymmetric bracket acting on sections, $[\cdot,\cdot] : \Gamma(M, M^\circ) \times \Gamma(M, M^\circ) \to \Gamma(M, M^\circ)$, satisfying the Jacobi identity. Moreover, it has a bundle map $\rho : M \to TM^\circ$ called anchor map, satisfying the Leibniz identity

[10]

$$[v, f \cdot w] = f \cdot [v, w] + \rho(v)(f) \cdot w, \tag{2.36}$$

for $v, w \in \Gamma(M, M^\circ)$, $f \in \mathcal{C}^\infty(M^\circ)$. The Lie algebroid is thus effectively a vector field that behaves like a Lie algebra and lives free in the fibre of the bundle, though the action of it needs to be anchored to the base manifold.

For the sake of simplicity, let us assume that the Lie algebroid is of the form $\mathfrak{g} \times M^\circ \to M^\circ$. Similar as there are Lie groups that are differentiated to Lie algebras, there are Lie groupoids that are differentiated to Lie algebroids [2]. They are categories

$$G \ltimes M^\circ \rightrightarrows M^\circ, \tag{2.37}$$

for some Lie group G corresponding to $\mathfrak{g}$, with morphisms (g, x) that are maps $x \mapsto gx$, $x \in M^\circ$, $g \in G$. Here the action of g upon x is the unlinearised version of the anchor map ρ. The groupoid is a mathematically much more pleasant alternative to the orbit space M°/G [2]. For example, the latter alternative is not always a smooth manifold.

2.3.4 L_∞ morphisms

A morphism between two L_∞-algebras (L, l_i) and (M, m_i) is defined as a set of multilinear graded symmetric maps $\{\phi_i\}_{i\in\mathbb{Z}^+}$ of degree $1 - i$

$$\phi_i : \underbrace{L \times \cdots L}_{i} \to L' \tag{2.38}$$

fulfilling

$$\begin{aligned}
&\sum_{j=1}^{i} \sum_{\sigma\in\mathrm{Sh}(j,i-j)} -^{i-j}\chi(\sigma; \upharpoonright_1, \cdots \upharpoonright_i)\phi_{i-j+1}(l_j(\upharpoonright_{\sigma(1)}, \cdots \upharpoonright_{\sigma(j)}), \upharpoonright_{\sigma(j+1)}, \cdots \upharpoonright_{\sigma(i)}) \\
=&\sum_{j=1}^{i} \frac{1}{j!} \sum_{k_1+\cdots k_j=i} \sum_{\sigma\in\mathrm{Sh}(k_1,\cdots k_j)} \chi(\sigma; \upharpoonright_1, \cdots \upharpoonright_i)\zeta(\sigma; \upharpoonright_1, \cdots \upharpoonright_i)\times \\
&\qquad \times m_j \left(\phi_{k_1}(\upharpoonright_{\sigma(1)}, \cdots \upharpoonright_{\sigma(k_1)}), \cdots \phi_{k_j}(\upharpoonright_{\sigma(i-k_j+1)}, \cdots \upharpoonright_{\sigma(i)})\right),
\end{aligned} \tag{2.39}$$

$$\zeta(\sigma; \upharpoonright_1, \cdots \upharpoonright_i) = -^{\left(\sum_{1\le m\le n\le j} k_n k_m + \sum_{m=1}^{j-1} k_m(j-m) + \sum_{m=2}^{j}(1-k_m)\sum_{k=1}^{k_1+\cdot+k_{m-1}} l_{\sigma(k)}\right)}. \tag{2.40}$$

With this definition, an L_∞-morphism is an isomorphism if and only if ϕ_1 is an isomorphism [2]. In the Q-manifold picture, an L_∞-morphism between L_1 and L_2 corresponds to a Q-morphism, that is a map $\Phi : M_1 \to M_2$ such that [14]

$$Q_1 \circ \Phi^* = \Phi^* \circ Q_2. \tag{2.41}$$

A weaker equivalence set of relations that better captures the physics are the L_∞-quasi-isomorphisms, which are morphisms between L_∞-algebras $(L, l_i) \to (M, m_i)$ such that $\phi_1 : H^\bullet_{l_1}(L) \to H^\bullet_{m_1}(M)$ is an isomorphism. The notion of quasi-isomorphism is carried over to the corresponding Q-manifolds.

The symplectic structure of a Q-manifold corresponds to an inner product on the L_∞-algebra, with the cyclic property that [2]

$$\langle \upharpoonright_{n+1} | l_n(\upharpoonright_1, \cdots \upharpoonright_n) \rangle = -^{n(1+\upharpoonright_1+\upharpoonright_{n+1})+\upharpoonright_{n+1}\sum_{i=1}^n \upharpoonright_i} \langle \upharpoonright_1 | l_n(\upharpoonright_2, \cdots \upharpoonright_n, \upharpoonright_{n+1}) \rangle \,. \tag{2.42}$$

In the context of this thesis these inner products are typically an integration together with a trace. Given L_∞-algebras L and L', each equipped with such a cyclic product of its own, we will restrict the morphisms $\phi_i : L \to L'$ to those that preserve the inner product, [2]

$$\sum_{j=1}^{i-1} \Big\langle \phi_j(\upharpoonright_1, \cdots \upharpoonright_j) \Big| \phi_{i-j}(\upharpoonright_{j+1}, \cdots \upharpoonright_i) \Big\rangle_{L'} = \delta^i_2 \langle \upharpoonright_1 | \upharpoonright_2 \rangle_L \,. \tag{2.43}$$

2.4 The homological perturbation lemma

This is a quite useful lemma that towards the end of the thesis will be applied to show equivalence between different formulations of theories.

Let (L, b) and (M, b) be two chain/cochain complexes such that

- there are quasi-isomorphisms $i : (L, b) \mapsto (M, b)$ (imbedding) and $p : (M, b) \mapsto (L, b)$ (projection),
- there is a homotopy h between 1 and ip:

$$ip = 1 + hb + bh. \tag{2.44}$$

All the above setup defines a homotopy equivalence and is compactly notated as

$$(L, b) \underset{p}{\overset{i}{\rightleftarrows}} (M, b) \circlearrowright h. \tag{2.45}$$

b is of course two (in general) different operators – b and the projected version of it.

Now consider a small perturbation δ. Here small means that $\exists (1-\delta h)^{-1}$ and perturbation means that δ is a map on M of the same degree as b and that $(b+\delta)^2 = 0$. The homological perturbation lemma states that the following will be a homotopy equivalence as well [15].

$$(L, b_1) \underset{p_1}{\overset{i_1}{\rightleftarrows}} (M, b+\delta) \circlearrowleft h_1 \tag{2.46}$$

where

$$i_1 = i + hAi \tag{2.47a}$$
$$p_1 = p + pAh \tag{2.47b}$$
$$h_1 = h + hAh \tag{2.47c}$$
$$b_1 = b + pAi \tag{2.47d}$$
$$A = (1-\delta h)^{-1}\delta. \tag{2.47e}$$

3. The physical theory

3.1 General gauge theory

As well known, a physical field ϕ can be viewed as a fibre bundle $E \to M$ locally isomorphic to $M \times F$, where M is the base space (the space-time) and the fibre F is the possible local values of the field. Given multiple fields we may add them (by Whitney sums) into a single fibre bundle E with local coordinates x^μ, ϕ^i. The new fibre F of fields can be extended to the so-called k$^{\text{th}}$ jet space F^k, with coordinates $\phi^i, \phi^i_\mu, \phi^i_{\mu\nu}, \cdots \phi^i_{\mu_1\mu_2\cdots\mu_k}$ [16]. If this in turn is combined with M we have the k$^{\text{th}}$ jet bundle $J^k(E) \simeq_{\text{locally}} M \times F^k$. A section $s : M \to J^k(E)$ becomes identifiable with not just the fields but also their space-time derivatives, up to order k, as we can view it as a map $x^\mu \mapsto \big(x^\mu, \phi^i(x), \partial_\mu \phi^i(x), \cdots \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_k}\phi^i(x)\big)$ [16]. To avoid dealing with questions regarding if a chosen jet bundle covers enough orders of derivatives we introduce the infinite jet-bundle $J^\infty(E)$ to cover them all [17]. Functions on these jet bundles are called local functions.

An action $S[\phi^i] = \int \mathrm{dvol}\ \mathscr{L}(x, \phi^i(x), \partial_\mu\phi^i(x), \cdots)$ is as known a functional of the fields ϕ^i. More precisely it is a local functional, that is of the form $\int \mathrm{dvol}\ \mathscr{L} \circ s$, for some section $s : M \to J^k(E)$ and a local function $\mathscr{L} : J^k(E) \to \mathbb{R}$ [16]. The field equations corresponding to the action arise from setting its functional derivative, with respect to the fields, to zero,

$$0 = \frac{\delta S}{\delta \phi^i}(x) := \frac{\partial \mathscr{L}}{\partial \phi^i}(x) - \partial_\mu \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i}(x) + \partial_\nu\partial_\mu \frac{\partial \mathscr{L}}{\partial \partial_\nu \partial_\mu \phi^i}(x) - \cdots . \tag{3.1}$$

As by definition, the study of local functionals can be reduced to the study of local functions. These local functions are in turn generated by a finite number of coordinate functions and this motivates why seemingly infinite-dimensional problems or expressions involving functionals may formally be treated as finite-dimensional. In what follows, the words function and functional may be used interchangeably and locality is assumed.

As known, a gauge transformation (also known as gauge/local symmetry) of the fields is the local action of a group upon the fields such that the action is invariant. It should also be known to the reader that the study of gauge symmetries can (nearly) be reduced to the study of infinitesimal gauge symmetries, related to the Lie algebra of the gauge group in question. Now gauge transformations (which henceforth will refer the infinitesimal gauge transformations rather than the finite) are a subset of transformations of the form $\phi^i \mapsto \phi^i + \delta_c \phi^i$, where

$$\delta_c \phi^i = \left(r^i_\alpha + r^{i\mu}_\alpha \partial_\mu + r^{i\mu\nu}_\alpha \partial_\mu\partial_\nu + \cdots\right) c^\alpha(x), \tag{3.2}$$

$c^\alpha(x)$ is the arbitrarily chosen gauge parameter and the r's are functions on the jet bundle [3]. By defining

$$R^i_\alpha(x,y) := r^i_\alpha(x)\delta(x-y) + r^{i\mu}_\alpha(x)\partial_\mu\delta(x-y) + \cdots \tag{3.3}$$

the transformation is

$$\delta_c\phi^i(x) = \int \mathrm{dvol}\; R^i_\alpha(x,y)c^\alpha(y) \tag{3.4}$$

which in turn can be compactified even more into

$$\delta_c\phi^i = R^i_\alpha c^\alpha \tag{3.5}$$

using de Witt's notation, where every index also contains a coordinate dependence and where the Einstein summation/notation includes not just an implicit sum over indices but also a corresponding integral [18]. Note that this notation fits well with the mentioned fact that local functionals can be identified with local functions.

To be a gauge transformation it must also leave the action invariant, that is

$$0 = \delta_c S = \frac{\delta S}{\delta\phi^i}\delta_c\phi^i = \frac{\delta S}{\delta\phi^i}R^i_\alpha c^\alpha \quad \forall c^\alpha \tag{3.6}$$

and from this follows the Noether identities

$$\frac{\delta S}{\delta\phi^i}R^i_\alpha = 0 \tag{3.7}$$

that are relations among the equations of motion [19]. Taking good old 4 D Yang-Mills as an example, the action is

$$S = \frac{1}{2}\int \mathrm{tr}\, F \wedge *F = \frac{1}{4}\int \mathrm{d}^4x\;\; \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} \tag{3.8}$$

and the gauge symmetry takes the form

$$\delta_c A_\mu(x) = D_\mu\Lambda(x) = -\int \mathrm{d}y\; D_\mu(y)\delta(x-y)\Lambda(y), \tag{3.9}$$

for some scalar Λ. D_μ is as usual the covariant derivative and the field strength is $F = \mathrm{d}A + A\wedge A$ [9]. We can identify $R_\mu(x,y) = -D_\mu(y)\delta(x-y)$ and the corresponding Noether identity is

$$0 = R_\nu\frac{\delta S}{\delta A_\nu} = -\int D_\nu(y)\delta(x-y)D_\mu F^{\mu\nu}(y) = D_\nu D_\mu F^{\mu\nu}(x). \tag{3.10}$$

If a source term is added to the action,

$$S = \frac{1}{2}\int \mathrm{tr}\, F\wedge *F + A\wedge *J = \frac{1}{4}\int \mathrm{d}^4x\;\; \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{d}^4x\;\; \mathrm{tr}\, A_\mu J^\mu, \tag{3.11}$$

the gauge symmetry is the same. The corresponding Noether identity reduces to

$$0 = D_\nu(D_\mu F^{\mu\nu} - J^\nu) = -D_\nu J^\nu, \tag{3.12}$$

which is the continuity equation.

Given some gauge transformations, new ones can be constructed by combining these with each other and/or local functions. These new transformations will of course not bring any new knowledge, hence the need for some kind of prime factorisation of gauge symmetries. A solution to this factorisation problem is a generating set, which is a set of gauge transformations $\delta_c\phi^i = R^i_\alpha c^\alpha$ such that all gauge transformations (that we consider) can be written in the general form

$$\delta\phi^i = R^i_\alpha \Lambda^\alpha + M^{[ij]}\frac{\delta S}{\delta\phi^j}, \tag{3.13}$$

where Λ^α and $M^{[ij]}$ are functions and $M^{[ij]}$ is antisymmetric in indices i, j [3]. The latter part, $M^{[ij]}\delta S/\delta\phi^j$, is called a trivial gauge transformation and these are the gauge transformations that vanish on-shell, $M^{[ij]}\delta S/\delta\phi^j \approx 0$ using Dirac's notation. In general, all smooth functions f that vanish on-shell can be written as

$$f(\phi) = k^i\frac{\delta S}{\delta\phi^i}, \tag{3.14}$$

for some function k^i.

Note that the bracket of two elements R^i_α and R^i_β in the generating set is a new gauge transformation of the form

$$R^j_\alpha\frac{\delta R^i_\beta}{\delta\phi^j} - R^j_\beta\frac{\delta R^i_\alpha}{\delta\phi^j} = R^i_\gamma C^\gamma_{\alpha\beta} + M^{[ij]}_{\alpha\beta}\frac{\delta S}{\delta\phi^j}, \tag{3.15}$$

according to the definition of generating sets. The generating sets are only Lie algebras if $C^\gamma_{\alpha\beta}$ is constant and if it closes off-shell, $M^{[ij]}_{\alpha\beta} \equiv 0$ [3].

Sometimes the gauge transformations have gauge transformations themselves. Concretely, this means that there exists some non-trivial (i.e. on-shell non-vanishing) Ξ^α such that

$$\Xi^\alpha R^i_\alpha = M^{[ij]}\frac{\delta S}{\delta\phi^j}. \tag{3.16}$$

In that case the generating set is reducible, otherwise it is called irreducible [3]. A common example is when the field ϕ^i is some differential form of de Rham degree $p > 1$ but the action depends only on the field through $\mathrm{d}\phi^i$. A gauge transformation would then be $\delta_c\phi^i = \mathrm{d}c$ for some differential form c of degree $p-1$ while $\delta_k c = \mathrm{d}k$ with k a $p-2$ form is a gauge transformation of the gauge transformation. If $p-2 > 0$ we could even add a gauge transformation of the gauge transformation of the gauge transformation. In general a theory is called k-stage reducible if

$$\begin{cases} \delta_{c_1}\phi^i = R^i_{1\,\alpha_1}c_1^{\alpha_1}, \quad \delta_{c_1}S = \dfrac{\delta S}{\delta\phi^i}\delta_{c_1}\phi^i = \dfrac{\delta S}{\delta\phi^i}R^i_{1\,\alpha_1}c_1^{\alpha_1} = 0 \\ \delta_{c_n}c_{n-1}^{\alpha_{n-1}} = R^{\alpha_{n-1}}_{n\,\alpha_n}c_n^{\alpha_n}, \quad \delta_{c_n}c_{n-2}^{\alpha_{n-2}} = R^{\alpha_{n-2}}_{n-1\,\alpha_{n-1}}R^{\alpha_{n-1}}_{n\,\alpha_n}c_n^{\alpha_n} = \dfrac{\delta S}{\delta\phi^i}N^{i\alpha_{n-2}}_{n\,\alpha_n}c_n^{\alpha_n} \quad 1 < n < k \end{cases}, \tag{3.17}$$

for some functionals $R^{\alpha_{n-1}}_{n\,\alpha_n}$ and $N^{i\alpha_{n-2}}_{n\,\alpha_n}$ [19]. The equal signs furthest to the right are the criterias that fields two stages below should not be affected by the gauge transformations, modulo trivial transformations. These will be called higher Noether identities because of their similarities to regular Noether identities. Note that the lower set of equations incorporates the first if $\phi^i = c_0^{\alpha_0}$ etc. Also the $R^{\alpha_{n-1}}_{n\,\alpha_n}$'s are taken to be generating sets.

3.2 Towards the Batalin-Vilkovisky formalism

Let F_0 denote the space of all possible fields $\phi^i(x)$ and let Σ denote the submanifold which is the restriction to those that obey the equations of motion. In classical physics we are interested in observables, that are (local) functions of the fields belonging to $\mathcal{C}^\infty(F_0)$. Of course much of this larger class of functions is irrelevant and the interest is in $\mathcal{C}^\infty(\Sigma) = \mathcal{C}^\infty(F_0)/N$, where N is the set of functions that vanish on Σ. If the theory has gauge symmetries with generating set G, this should be restricted even further to $\mathcal{C}^\infty(\Sigma/G)$ [3]. As mentioned earlier, a quotient like this poses some problems such as lack of smoothness and this is further complicated if the symmetry does not close off-shell [2], [3]. The goal from here is to find something called a resolution of $\mathcal{C}^\infty(\Sigma/G)$, that is, a differential operator Q such that $H^0_Q = \mathcal{C}^\infty(\Sigma/G)$. In what follows, all functional derivatives are without respect to the boundary data (that is, all boundary terms in the functional derivatives are omitted).

Let the gauge symmetries be ordered by a number k, called ghost degree, such that the gauge symmetries of degree k are acted upon by gauge symmetries of degree $k+1$ and the gauge symmetries of degree 1 are those that act upon the fields. Introduce fields c_k that are coordinates on $\{\mathfrak{g}_k\}[k]$ (i.e. $c_k \in (\{\mathfrak{g}_k\}[k])^*$), where $\{\mathfrak{g}_k\}$ is a generating set of $\mathfrak{g}_k$, which is the Lie algebra of the gauge symmetries of degree k. For simplicity, we will just write $\mathfrak{g}_k[k]$ instead. As the c_k's are just gauge parameters (or rather the dual of degree-shifted gauge parameters) promoted to be fields, they should not affect the physics at all. They bear the fitting, but also somewhat confusing, name ghosts (for $k > 1$ they may also be referred to as higher ghosts).

Now let the total space of fields be $\mathscr{F} = T^*[-1]\,(\cdots \mathfrak{g}_2[2] \oplus \mathfrak{g}_1[1] \oplus F_0)$. The cotangent fibre coordinates for $T^*[-1](\mathfrak{g}_k[k]) \simeq \mathfrak{g}_k^*[-k-1]$ are denoted by c_k^+ and go under the name of (higher) antighosts. The cotangent coordinates for $T^*[-1]F_0$, ϕ^+, are called the antifields [3], [10]. The physical fields[1] and the ghost fields will be collectively referred to as just the fields, while the physical and ghost antifields will be referred to as antifields. Together these two sets constitutes the Batalin-Vilkovisky (BV) fields.

[1]A theory may have many physical fields ϕ^i, or just a single. Here it does not matter – the first case is equally just a single field with many components.

The Koszul-Tate differential δ is defined by

$$\delta\phi^i = 0 \tag{3.18a}$$
$$\delta\phi_i^+ = \frac{\delta S}{\delta\phi^i} \tag{3.18b}$$
$$\delta c^+_{1\,\alpha_1} = R^i_{1\,\alpha_1}\phi^+_i \tag{3.18c}$$
$$\delta c^+_{2\,\alpha_2} = R^{\alpha_1}_{2\,\alpha_2} c^+_{1\,\alpha_1} + \frac{1}{2}N^{[ij]}_{2\,\alpha_2}\phi^+_i\phi^+_j \tag{3.18d}$$
$$\dots$$

This construction is such that all the would-be non-exact closed expressions in terms of ϕ_i^+ are made exact by the introduction of $c^+_{1\alpha_1}$ [3]. Similarly, the introduction of $c^+_{k\,\alpha_k}$ ensures that the would-be non-exact closed combinations of preceding antifields[2] are made exact. Hence all other cohomology[3], $H^k(\delta)$, $k \neq 0$, vanishes and

$$H^0_\delta = \frac{\ker\delta_0}{\mathrm{im}\ \delta_{-1}} = \frac{\mathcal{C}^\infty(F_0)}{N} = \mathcal{C}^\infty(\Sigma). \tag{3.19}$$

The reason for the ghosts c_k comes with the next differential, the Chevalley–Eilenberg operator γ [10]. It is defined as

$$\gamma\phi^i = R^i_{1\,\alpha_1}c_1^{\alpha_1} \tag{3.20a}$$
$$\gamma c_1^{\alpha_1} = \frac{1}{2}f^{\alpha_1}{}_{\beta_1\gamma_1}c_1^{\beta_1}c_1^{\gamma_1} + R^{\alpha_1}_{2\,\alpha_2}c_2^{\alpha_2} \tag{3.20b}$$
$$\gamma c_2^{\alpha_2} = \cdots,$$

where $f^{\alpha_1}{}_{\beta_1\gamma_1}$ is the structure function for $\mathfrak{g}_1$. It follows that the zeroth cohomology for this operator is all gauge invariant functions.

Both differentials δ and γ can be extended trivially to all BV fields by setting them to zero where they were not defined before. The sum of these two differentials is not necessarily a differential in itself but can be completed by, possibly infinitely many, other terms to become a differential. There are certain demands regarding the minimum of antighosts they include[4] and they will not destroy the desired structure. The zeroth cohomology of this differential,

$$Q = \delta + \gamma + \text{more}, \tag{3.21}$$

thus constitutes all combinations of physical field configurations that are both gauge invariant and on-shell, i.e. all observables [18]. In fact there is even the stronger

[2]The expressions also contain ϕ^i, through the $R^{\dots}_{\dots}$ and $N^{\dots}_{\dots}$ functions.

[3]Some literature give the ghost fields c_k a grading equaling the corresponding ghost grade, called the pure ghost grade while the antighosts are given a grading called antighost grade equaling minus their ghost grade. From this they define ghost grade = pure ghost grade − antighost grade. There the above cohomology is referred to as homology, as it is viewed with respect to the antighost grade.

[4]Or rather a minimum of their antighost grade, see previous footnote.

statement that [3]

$$H_Q^k = \begin{cases} 0, & k < 0 \\ H_\gamma^k(\Sigma), & k \geq 0 \end{cases} . \tag{3.22}$$

4. The mathematical formulation

In this chapter the observations made in Chapter 3 are formulated using the language of Chapter 2.

4.1 The Batalin-Vilkovisky formalism

A (classical) Batalin-Vilkovisky complex is a symplectic Q-manifold of degree -1 such that the homological vector field Q is homological with respect to the symplectic structure ω,

$$Q \lrcorner \omega = \delta S, \tag{4.1}$$

for some degree zero functional S, where δ now denotes the exterior derivative on the manifold. There is also a requirement on the cohomology that $H^k(Q) = 0, \forall k < 0$ [10], [20]. To make use of this complex to describe the structures investigated in Chapter 3, som further requirements are added. To begin with, the manifold should be chosen as

$$\mathscr{F} := T^*[-1](\cdots \ltimes \mathfrak{g}_2[2] \ltimes \mathfrak{g}_1[1] \ltimes F_0), \tag{4.2}$$

where F_0 is the physical field(s), $\mathfrak{g}_k[k]$ the (higher) ghosts, $T^*[-1]F_0$ the physical antifields, and $T^*[-1](\mathfrak{g}_k[k])$ the (higher) antighosts. The coordinates for the fields (i.e. the fields themselves) are denoted Φ^A and the coordinates for the antifields are denoted Φ^+_A. For simplicity it will be assumed that these coordinates are global. Furthermore, the action of Q upon a component of Φ^A should return the same result as when the corresponding Lie algebra acts upon Φ^A, modulo terms including Φ^+_A. There is no need to also specify the action upon the antifields Φ^+_A as this follows from the already stated requirement through the symplectic structure.

The zero degree functional $S \in \mathcal{C}^\infty(\mathscr{F})$ in Equation (4.1) will be called the (BV) action and is required to fulfill [20]

$$S\Big|_{F_0} = S_0, \tag{4.3}$$

where S_0 is the usual action for the relevant theory. Equation (4.1) is equivalent to

$$Q = \{S, \cdot\}, \tag{4.4}$$

where $\{\cdot, \cdot\}$ is the graded Poisson bracket induced by the symplectic structure ω, which in under these circumstances goes under the name antibracket[1]. The requirement that Q is a differential is then equivalent to the so-called classical master

[1] In this case where it is of grade 1, i.e. the symplectic structure is of grade -1, it is also called a Gerstenhaber bracket [13].

equation

$$\{S, S\} = 0. \tag{4.5}$$

Let Φ^A and Φ^+_A be Darboux coordinates such that

$$\omega = \delta\Phi^+_A \wedge \delta\Phi^A. \tag{4.6}$$

Using Equation (2.21), the homological vector field is

$$Q = \{S, \cdot\} = \frac{S\overleftarrow{\delta}}{\delta\Phi^A}\frac{\overrightarrow{\delta}}{\delta\Phi^+_A} - \frac{S\overleftarrow{\delta}}{\delta\Phi^+_A}\frac{\overrightarrow{\delta}}{\delta\Phi^A} \tag{4.7}$$

and the master equation is

$$\{S, S\} = 2\,\frac{S\overleftarrow{\delta}}{\delta\Phi^A}\frac{\overrightarrow{\delta}S}{\delta\Phi^+_A} = 0. \tag{4.8}$$

Using Equation (4.7) it is easy to see that the action of Q on the fields Φ^A implies that the (generalised) Noether identites show up when Q acts on Φ^+_A, as in Equations (3.18).

The role of S in the Batalin-Vilkovisky formalism reminds much of the role that the Hamiltonian H plays in Hamiltonian mechanics. The latter generates time evolution[2] through the regular Poisson bracket,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \{\cdot, H\}_{\mathrm{PB}}. \tag{4.9}$$

The Hamiltonian itself should not evolve under time,

$$\{H, H\}_{\mathrm{PB}} = 0. \tag{4.10}$$

The BV action S uses a graded Poisson bracket to generate generalised equations of motion: regular equations of motion when acting upon the physical antifields, gauge transformations when acting upon the physical fields etc. This is summarised in Table 4.1 below. Similarly as in the Hamiltonian case, the action should be invariant under all transformations it generates, thus the classical master equation

$$\{S, S\} = 0. \tag{4.11}$$

When using the BV formalism in practice, a subgoal may be to formulate a BV action S given knowledge of the gauge symmetries. It can be shown that for boundary conditions on S that captures much of what is required of the differential Q that was constructed in Equation (3.21), there exists an unique S that solves the master equation $\{S, S\} = 0$ (modulo canonical transforms using the antibracket) [21].

[2]Implicit time evolution $\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial}{\partial t}$ to be exact.

phenomena		higher gauge	higher tr.'s	gauge	tr.'s	(observables) fields	EoM	equations of motion		Noether identities	
field	$\cdots \xleftarrow[Q]{}$	c_2	$\xleftarrow[Q]{}$	c_1	$\xleftarrow[Q]{}$	ϕ^i	$\xleftarrow[Q]{}$	ϕ_i^+	$\xleftarrow[Q]{}$	c_1^+	$\cdots$
ghost degree		2		1		0		-1		-2	

Table 4.1: The phenomenological names of the field content (in parenthesis the relevant cohomology) and how the differential Q acts on these. Note that the arrows are in the direction of the differential, i.e. towards higher ghost grades, but opposite the direction fields act on each other (gauge fields transforms physical fields etc.).

4.2 Further properties

Now remember the L_∞-algebras from Chapter 2. The BV differential Q is equivalent to a set of generalised brackets l_i for the corresponding L_∞-algebra. As far as I understand, the Q-manifold is rather dual to an L_∞-algebroid where the body is the spacetime. L_∞-algebras are the special case when there is no space-time. For the purpose of just investigating the gauge structure this is enough. The first generalised bracket, the differential l_1, corresponds to the linearised action of S (on the BV fields), thus including the linearised equations of motion [2]. The second bracket l_2 captures the ordinary bracket of transformations that would be closed, but is rendered open when taking the higher homotopies $l_3, \cdots$ into account. According to Equation (2.33) they constitute the total differential,

$$Q\Psi = -\sum_{n\geq 1} \frac{1}{n!} l_n(\Psi, \cdots, \Psi), \tag{4.12}$$

which clearly is a generalisation of the Maurer-Cartan equation [2], [9]

$$\mathrm{d}\Psi + \frac{1}{2}[\Psi, \Psi] = 0. \tag{4.13}$$

The action corresponding to the above differential is [2], [22]

$$S = \sum_{n\geq 1} \frac{1}{(n+1)!} \langle \Psi | l_n(\Psi, \cdots \Psi) \rangle \,. \tag{4.14}$$

The higher brackets $l_2, l_3, \cdots$ thus represent all (generalised) interaction terms. For example in [23] these products have been used to describe n^{th} order string interactions.

The field content of a BV theory is not restricted to be on the form presented in Equation (4.2). Add, for example, a and b to the set of fields, fulfilling

$$Qa = b, \quad Qb = 0. \tag{4.15}$$

Using the symplectic structure to derive how the action S change, and from this how Q is altered, it follows that their antifields fulfill [2]

$$Qa^+ = 0, \quad Qb^+ = -{}^{a}a^+. \tag{4.16}$$

None of these four BV fields couples to the rest, thus not affecting the physics at all. At the same time the cohomology is neither altered as the fields come in trivial pairs – the one that is killed by the differential is an exact expression of the other. General equivalence though lie rather in the linear part l_1 of the differential than the whole of Q.

As l_1 gives the linearised transformations etc. it describes the free theory. While the cohomology of Q was

$$H_Q^\bullet = \frac{\text{invariant BV fields}}{\text{generalised on-shell conditions}}, \quad \text{e.g.} \quad H_Q^0 = \frac{\text{invariant fields}}{\text{on-shell conditions}} = \text{observables} \tag{4.17}$$

the cohomology for l_1 is the other way around,

$$H_{l_1}^\bullet = \frac{\text{BV fields on-shell}}{\text{transformations}}, \quad \text{e.g.} \quad H_{l_1}^0 = \frac{\text{fields on-shell}}{\text{gauge transformation}} = \text{physical fields.} \tag{4.18}$$

Thus it equals the independent field content of the free theory. The action of l_1 is shown in Table 4.2.

phenomena			higher gauge		gauge		(physical) fields		equations of motion		Noether identities	
field	$\cdots$	$\xrightarrow{l_1}$	c_2	$\xrightarrow[\text{higher tr.'s}]{l_1}$	c_1	$\xrightarrow[\text{tr.'s}]{l_1}$	ϕ^i	$\xrightarrow[\text{EoM}]{l_1}$	ϕ_i^+	$\xrightarrow{l_1}$	c_1^+	$\cdots$
L_∞ degree			-1		0		1		2		3	

Table 4.2: The phenomenological names of the field content (in parenthesis also the corresponding cohomology) and how the differential l_1 acts on these, though linearised. Here the direction of the arrows, i.e. the action of l_1, is the same as the fields act upon each other.

It can be shown that a quasi-isomorphism between two L_∞-algebras[3] is the same as semi-classical equivalence between the corresponding classical physical theories [24]. The latter statement means that their quantum theories have the same tree-level scattering amplitudes, i.e. the same S-matrices, modulo similarity transformations. This covers equivalences between theories obtained by integrating in and out fields, where the integrating out (or in) fields in turn corresponds to homotopy transfers. In [24] it is conjectured that this covering is in fact an equivalence. Also so-called Seiberg-Witten maps, which are another way of stating equivalence between physical theories, corresponds to quasi-isomorphisms [25].

[3]Which are equipped with metrics, hence corresponding Q-manifolds are symplectic.

5. Examples

Here follows three examples to highlight and better explain some aspects of the BV formalism. The first shows what the higher products l_n looks like in a less complex setting – a scalar field theory without gauge symmetry. Then follows a pair of examples in the settings of Yang-Mills theory. The first of these shows how Yang-Mills may be described through the BV formalism. The other shows the equivalence between two different formulations of Yang-Mills, both through homotopy transfer and integrating out fields. For simplicity it is assumed that there is no boundary.

5.1 L_∞-algebra for scalar field theory

Consider the scalar theory described by the action

$$S_0 = -\frac{1}{2!}\varphi\Delta\varphi + \frac{a}{3!}\varphi^3 + \frac{b}{4!}\varphi^4, \tag{5.1}$$

where the space-time integral for the stated Lagrangian density is implicit and $\Delta = -\partial_\mu\partial^\mu$. As there is no gauge symmetry the only relevant fields for a BV theory are φ and φ^+. From the equations of motion and lack of gauge symmetry it follows that

$$Q = \left(-\Delta\varphi + \frac{a}{2!}\varphi^2 + \frac{b}{3!}\varphi^3\right)\delta_{\varphi^+} + 0\delta_\varphi. \tag{5.2}$$

The differential Q may be decomposed according to the number of term it returns,

$$Q_1\psi^\alpha = -\Delta\psi^\beta\delta^\varphi_\beta\delta^\alpha_{\varphi^+} \tag{5.3a}$$
$$Q_2\psi^\alpha = \frac{a}{2!}\psi^\beta\psi^\gamma\delta^\varphi_\beta\delta^\varphi_\gamma\delta^\alpha_{\varphi^+} \tag{5.3b}$$
$$Q_3\psi^\alpha = \frac{b}{3!}\psi^\beta\psi^\gamma\psi^\delta\delta^\varphi_\beta\delta^\varphi_\gamma\delta^\varphi_\delta\delta^\alpha_{\varphi^+}. \tag{5.3c}$$

Here all BV fields have been merged into ψ^α, whose components are $\psi^\varphi = \varphi$ and $\psi^{\varphi^+} = \varphi^+$.

Introducing the L_∞-basis $\upharpoonleft_\varphi \in L_1$, $\upharpoonleft_{\varphi^+} \in L_2$ (so that $\varphi \in (L_1[1])^*$, $\varphi^+ \in (L_2[1])^*$) , we may perform a contraction

$$\Psi = \psi^\alpha\upharpoonleft_\alpha = \psi^\varphi\upharpoonleft_\varphi + \psi^{\varphi^+}\upharpoonleft_{\varphi^+} \in \mathcal{C}^\infty(L[1]) \otimes L. \tag{5.4}$$

According to Equation (2.33)

$$Q_1\Psi = -l_1(\Psi) = -^{1+\psi^\alpha}\psi^\alpha l_1(\Gamma_\alpha) \tag{5.5a}$$

$$Q_2\Psi = -\frac{1}{2!}l_2(\Psi,\Psi) = \frac{-^{1+\Gamma_\alpha\psi^\beta}}{2!}\psi^\alpha\psi^\beta l_2(\Gamma_\alpha,\Gamma_\beta) \tag{5.5b}$$

$$Q_3\Psi = -\frac{1}{3!}l_3(\Psi,\Psi,\Psi) = \frac{-^{1+\Gamma_\alpha\psi^\beta+(\Gamma_\alpha+\Gamma_\beta)\psi^\gamma}}{3!}\psi^\alpha\psi^\beta\psi^\gamma l_3(\Gamma_\alpha,\Gamma_\beta,\Gamma_\gamma). \tag{5.5c}$$

Comparing Equations (5.5) with Equations (5.3) it follows that the higher products are

$$l_1\Gamma_\alpha = \delta^\varphi_\alpha\delta^\beta_{\varphi^+}\Delta^\dagger\Gamma_\beta \tag{5.6a}$$

$$l_2(\Gamma_\alpha,\Gamma_\beta) = -a\,\delta^\varphi_\alpha\delta^\varphi_\beta\delta^\gamma_{\varphi^+}\Gamma_\gamma \tag{5.6b}$$

$$l_3(\Gamma_\alpha,\Gamma_\beta,\Gamma_\gamma) = -b\,\delta^\varphi_\alpha\delta^\varphi_\beta\delta^\varphi_\gamma\delta^\delta_{\varphi^+}\Gamma_\delta, \tag{5.6c}$$

$\Delta^\dagger$ is the dual of Δ. These products are clearly antisymmetric, as the right hand sides look symmetric and Γ_φ is odd. They also fulfill the higher homotopies required by Equation (2.27).

Using Equation (4.14), a corresponding BV action is

$$S = -\frac{1}{2!}\langle\Psi|l_1(\Psi)\rangle - \frac{1}{3!}\langle\Psi|l_2(\Psi,\Psi)\rangle - \frac{1}{3!}\langle\Psi|l_3(\Psi,\Psi,\Psi)\rangle, \tag{5.7}$$

which reduces to the ordinary action S_0 as was expected.

5.2 Yang-Mills formulated in BV

As stated above, the Yang-Mills action takes the well-known form

$$S_0 = \int\frac{1}{2}\operatorname{tr}F\wedge *F = \begin{bmatrix}\text{let integrals and trace}\\ \text{be implicit henceforth}\end{bmatrix} = \frac{1}{2}\mathrm{F}*\mathrm{F}, \tag{5.8}$$

where A is the connection and $F = \mathrm{d}A + A\wedge A$. For formulating Yang-Mills using the Batalin-Vilkovisky formalism the relevant BV fields should be A (physical field), c (ghost), A^+ (physical antifield), and c^+ (antighost), with ghost numbers 0, 1, -1, and -2 respectively. The action of Q should approximately (i.e. modulo the extra terms in the sum $\delta+\gamma+\text{more}$) fulfill [10], [19]

$$\underset{\text{the gauge transformation}}{QA = \delta_c A := Dc}\,, \qquad \underset{\text{the adjoint action of the gauge group}}{Qc = -\frac{1}{2}[c,c]}\,, \quad \underset{\text{EoM}}{QA^+ = D*F}, \quad \underset{\text{Noether identity}}{Qc^+ = -DA^+}. \tag{5.9}$$

The action is $S = S_0 + \cdots$, where S_0 contributes to Q by

$$\{S_0,\cdot\} = S_0\left(\frac{\overleftarrow{\delta}}{\delta\Phi^A}\frac{\overrightarrow{\delta}}{\delta\Phi^+_A} - \frac{\overleftarrow{\delta}}{\delta\Phi^+_A}\frac{\overrightarrow{\delta}}{\delta\Phi^A}\right) = D*F\frac{\delta}{\delta A^+}, \tag{5.10}$$

where Φ^A and Φ^+_A are the fields and antifields respectively. Adding the term

$$-DcA^+ = -(\mathrm{d}c + [A, c])^\alpha A^+_\alpha \tag{5.11}$$

to the action gives

$$\begin{aligned} &\mathrm{d}c^\alpha \delta_{A^\alpha} - \mathrm{d}A^+_\alpha \delta_{c^+_\alpha} + f^\gamma_{\alpha\beta} c^\alpha A^+_\gamma \delta_{A^+_\beta} + f^\gamma_{\alpha\beta} A^\alpha A^+_\gamma \delta_{c^+_\beta} + f^\gamma_{\alpha\beta} A^\alpha c^\beta \delta_{A^\gamma} \\ = &\, Dc\delta_A - DA^+\delta_{c^+} - [c, A^+]\delta_{A^+}, \end{aligned} \tag{5.12}$$

which seems to be on the right track. Note that some Lie algebra index contractions are implicit.

If the total action is

$$S = \frac{1}{2} F * F - \mathrm{D}cA^+ + \frac{1}{2}[c, c]c^+, \tag{5.13}$$

it acts as

$$Q = -\frac{1}{2}[c, c]\delta_c + \mathrm{D}c\delta_A + (\mathrm{D} * F - [c, A^+])\delta_{A^+} + (-\mathrm{D}A^+ - [c, c^+])\delta_{c^+} \tag{5.14}$$

and fulfills the master equation

$$0 = \frac{1}{2}\{S, S\} = S \,\frac{\overleftarrow{\delta}}{\delta\Phi^A}\, \frac{\overrightarrow{\delta}}{\delta\Phi^+_A}\, S = D * FDc - [c, A^+]Dc - \frac{1}{2}DA^+[c, c] - \frac{1}{2}[c, c^+][c, c]. \tag{5.15}$$

The last term disappears by the Jacobi identity, the two in the middle combine to one exact term and one that also disappears by Jacobi. The first one disappears by similar means. For further details in this calculation the reader is referred to Appendix B.

5.3 Homological perturbation for Yang-Mills

The above BV formulation of Yang-Mills may be reformulated as follows.

Let two complexes be denoted as

$$\begin{array}{ccc} \begin{array}{cccc} c & A & B^+ & \\ & & \nearrow_{\sigma} & \\ & B & A^+ & c^+ \end{array} & \begin{array}{c} \xrightarrow{p} \\ \\ \\ \xleftarrow[i]{} \end{array} & \begin{array}{cccc} c & A & & \\ & & & \\ & & A^+ & c^+ \end{array} \end{array} . \tag{5.16}$$

One complex is left of the p and i arrows, including BV fields c, A, A^+, B, B^+ and c^+, and one is right of them and includes BV fields c, A, A^+ and c^+. The p is a projection from left to right and i is an inclusion the other way around. The complexes are equipped with one differential each, the right one is just the restricted version of the left one. The left differential is $D = \sigma = \sigma|_B$ (i.e. non-zero only at B) and the right restricted is $D' = 0$. Where the differentials act non-trivially is marked by an arrow giving the corresponding action (in this case only between B and B^+). As B and B^+ make a trivial pair the maps p and i are quasi-isomorphisms. There is also a homotopy $h = -\sigma^{-1} = -\sigma^{-1}|_{B^+}$ between ip and 1, $ip = 1 + \sigma h + h\sigma$.

Add a signed (and restricted) de Rham differential $\delta = \pm\mathrm{d}$ to the left complex,

$$\begin{tikzcd} c \arrow[r, "\mathrm{d}"] & A \arrow[r, "\mathrm{d}"] & B^+ & \\ & B \arrow[ur, "\sigma"] \arrow[r, "\mathrm{d}"] & A^+ \arrow[r, "-\mathrm{d}"] & c^+ \end{tikzcd} \quad . \tag{5.17}$$

Then the homological perturbation lemma implies that

$$\begin{tikzcd} c \arrow[r, "\mathrm{d}"] & A \arrow[r, "\mathrm{d}"] & B^+ & \\ & B \arrow[ur, "\sigma"] \arrow[r, "\mathrm{d}"] & A^+ \arrow[r, "-\mathrm{d}"] & c^+ \end{tikzcd} \quad \overset{p_1}{\longrightarrow} \quad \begin{tikzcd} c \arrow[r, "\mathrm{d}"] & A \arrow[dr, "-\mathrm{d}\sigma^{-1}\mathrm{d}"] & & \\ & & A^+ \arrow[r, "-\mathrm{d}"] & c^+ \end{tikzcd} \quad \underset{i_1}{\longleftarrow} \tag{5.18}$$

are quasi-isomorphic with homotopy h_1, $i_1 p_1 = 1 + (\delta + \sigma)h_1 + h_1(\sigma + \delta)$, where

$$\begin{aligned}
&\alpha = (1 - \delta h)^{-1}\delta = (1 + \delta\sigma^{-1})^{-1}\delta = \sum_{n\in\mathbb{N}} (-\delta\sigma^{-1})^n \delta \\
&i_1 = i + h\alpha i = (1 + \sigma^{-1}\delta)^{-1} i \\
&p_1 = p + p\alpha h = p(1 + \delta\sigma^{-1})^{-1} \\
&h_1 = h + h\alpha h = -\sigma^{-1}(1 + \delta\sigma^{-1})^{-1} \\
&D'_1 = D' + p\alpha i = p(1 + \delta\sigma^{-1})^{-1}\delta i = \sum_{n\in\mathbb{N}} p(-\delta\sigma^{-1})^n \delta i \text{ is the right complex differential.}
\end{aligned} \tag{5.19}$$

Let an operator be of the form p[product of n δ's and m σ^{-1}'s]i. If this operator acts non-trivially upon a product of fields, the operator must have at least one δ for every σ^{-1}, to produce the B^+'s needed for avoiding zero. The from σ^{-1} produced B's will be killed by p, hence the need of yet one more δ for every σ^{-1}. The new right complex differential is thus simplified as $D'_1 = p(\delta - \delta\sigma^{-1}\delta)i = p(\delta - \mathrm{d}\sigma^{-1}\mathrm{d})i$.

Let $-\sigma^{-1} = *$ the Hodge star. Assume that $D + \delta$ and D'_1 are the linearised differentials for two Q-manifolds,

$$D'_1 = Q'_1 = \mathrm{d}c\delta_A + \mathrm{d} * \mathrm{d}A\delta_{A^+} - \mathrm{d}A^+\delta_{c^+} \tag{5.20a}$$

$$D + \delta = Q_1 = \mathrm{d}c\delta_A + \mathrm{d}A\delta_{B^+} - *^{-1}B\delta_{B^+} + \mathrm{d}B\delta_{A^+} - \mathrm{d}A^+\delta_{c^+}. \tag{5.20b}$$

Two possible actions corresponding to these are

$$S_1' = \frac{1}{2} F * F - \mathrm{d}cA^+ \tag{5.21a}$$

$$S_1 = -\frac{1}{2} B *^{-1} B + \mathrm{d}AB - \mathrm{d}cA^+ \tag{5.21b}$$

and from the discussions above it follows that they describe the same physical system. Inspired by the last example, a non-linearised version of the action is conjectured to be

$$S = -\frac{1}{2} B *^{-1} B + FB - DcA^+ + \frac{1}{2}[c,c]c^+. \tag{5.22}$$

B is thought of an auxiliary field which plays the role of F (or more exactly $*^{-1}F$), but with this action it is invariant under the gauge transformation. The full action is [2]

$$S = -\frac{1}{2} B *^{-1} B + FB - DcA^+ - B^+[B,c] + \frac{1}{2}[c,c]c^+. \tag{5.23}$$

The cohomological vector field given by this is

$$\begin{aligned} Q = &-\frac{1}{2}[c,c]\delta_c + Dc\delta_A + [c,B]\delta_B + (DB - [c,A^+])\delta_{A^+} + (F - *^{-1}B - [c,B^+])\delta_{B^+} \\ &+ (-DA^+ - [c,c^+] + [B,B^+])\delta_{c^+}. \end{aligned} \tag{5.24}$$

The outline for proving equality between these nonlinearised versions using the L_∞-machinery is as follows.

- Split the differentials into n-terms differentials – parts corresponding to interactions of order $n+1$. For example

$$Q' = Q_1' + Q_2' + Q_3' \tag{5.25a}$$

$$Q_1' = \mathrm{d}c\delta_A + \mathrm{d} * \mathrm{d}A\delta_{A^+} - \mathrm{d}A^+\delta_{c^+} \tag{5.25b}$$

$$\begin{aligned} Q_2' = &-\frac{1}{2}[c,c]\delta_c + [A,c]\delta_A + \left([A,*\mathrm{d}A] + \frac{1}{2}\mathrm{d} * [A,A] - [c,A^+]\right)\delta_{A^+} \\ &- \left([A,A^+] + [c,c^+]\right)\delta_{c^+} \end{aligned} \tag{5.25c}$$

$$Q_3' = \frac{1}{2}[A,*[A,A]]\delta_{A^+} \quad . \tag{5.25d}$$

- Find higher products l_n and l_n' through which these $n+1$-interactions can be expressed.

$$l_1' \text{ᛚ}_\alpha = -\delta^c_\alpha \mathrm{d}\text{ᛚ}_A -^n \delta^A_\alpha \mathrm{d} * \mathrm{d}\text{ᛚ}_{A^+} + \delta^{A^+}_\alpha \mathrm{d}\text{ᛚ}_{c^+} \tag{5.26a}$$

$$l_2'(\text{ᛚ}_{\alpha\text{ᚠ}}, \text{ᛚ}_{\beta\text{ᚢ}}) = \delta^c_\alpha\delta^c_\beta f^{\text{ᚦ}}_{\text{ᚠᚢ}}\text{ᛚ}_{c\text{ᚦ}} + \delta^A_\alpha\delta^c_\beta f^{\text{ᚦ}}_{\text{ᚠᚢ}}\text{ᛚ}_{A\text{ᚦ}} - \delta^c_\alpha\delta^A_\beta f^{\text{ᚦ}}_{\text{ᚢᚠ}}\text{ᛚ}_{A\text{ᚦ}} -^{n-1} \delta^A_\alpha\delta^A_\beta f^{\text{ᚦ}}_{\text{ᚠᚢ}} + \cdots \tag{5.26b}$$

$$l_3'(\text{ᛚ}_{\alpha\text{ᚠ}}, \text{ᛚ}_{\beta\text{ᚢ}}, \text{ᛚ}_{\gamma\text{ᚦ}}) = \cdots \tag{5.26c}$$

Here greek letters denote fields while runes are used as Lie indices. n denotes the dimension of space-time.

- Find a quasi-isomorphism ϕ_1, ϕ_2, ϕ_3 from left to right complex. ϕ_1 should project out B and B^+ and therefore preserve the l_1 cohomology.

This approach requires the handling of quite many terms. Following [2], we instead find a Q-morphism between the Q-manifolds, as this is equivalent to an L_∞-morphism. Define a map Φ from the larger to the smaller complex,

$$\Phi^* c = c \qquad \Phi^* c^+ = c^+ \tag{5.27a}$$
$$\Phi^* A = A \qquad \Phi^* A^+ = A^+ \tag{5.27b}$$
$$\Phi^* B = *F \quad \Phi^* B^+ = 0\ . \tag{5.27c}$$

This map clearly satisfies Equation (2.41),

$$\Phi^* \circ Q = Q' \circ \Phi^*. \tag{5.28}$$

The orthodox way of proving equivalence between theories would be to integrate out fields. In this case it is complicated by the fact that both B and the corresponding antifield B^+ make appearance in the action. In the linearised case would be to solve the corresponding equations of motion

$$0 = \{S, B^+\} = - *^{-1} B + F \longleftrightarrow B = *F \tag{5.29}$$

and substitute this back into the action to obtain the effective one,

$$S_{1,\text{eff}} := S \Big|_{B=*F} = S_1'. \tag{5.30}$$

6. 10D super Yang-Mills

In the following the Batalin-Vilkovisky formalism is used to explore 10 dimensional super Yang-Mills theory. This chapter is not much more than a mere summary of what has been written in [4], with some calculations redone by own hand.

Let $Z^M = (x^m, \theta^\mu)$ and $M = (m, \mu)$ denote coordinates and indices for $(10|16)$ dimensional superspace, i.e. x is a space-time coordinate and θ a spinor one. With the help of a super vielbein the indices can be exchanged to Lorentz frame indices $A = (a, \alpha)$. This superspace also has a covariant derivative, whose fermionic part is $D_\alpha = \partial_\alpha - (\gamma^a\theta)_\alpha \partial_a$ and moreover, there is a torsion whose non-vanishing part is in the Lorentz frame $T^a{}_{\alpha\beta} = 2\gamma^a_{\alpha\beta}$ [4]. Here $(\cdot)$ indicates index contraction, $(\gamma^a\theta)_\alpha = \gamma^a_{\alpha\beta}\theta^\beta$.

For this to be a Yang-Mills theory there must of course be a connection (for some corresponding gauge group) $A = \mathrm{d}Z^M A_M$, which can be split into the two fields $A_m(x,\theta)$ and $A_\mu(x,\theta)$ [4]. Thanks to the torsion the field strength $F = \mathrm{d}A + A \wedge A$ will in components be expressed as

$$F_{\alpha\beta} = 2D_{(\alpha}A_{\beta)} + 2A_{(\alpha}A_{\beta)} + 2\gamma^a_{\alpha\beta}A_a. \tag{6.1}$$

It turns out that this field strength can be decomposed into

$$\text{a vector } F_a = \frac{1}{16}\gamma_a^{\alpha\beta}F_{\alpha\beta} \quad \text{and a self dual five form } F_{abcde} = \frac{1}{2\cdot 5!\cdot 16}\gamma^{\alpha\beta}_{abcde}F_{\alpha\beta}. \tag{6.2}$$

In ordinary Yang-Mills there is only one field, a connection with space-time indices, that matters. A convention to get into this situation is by simply demanding that $F_a = 0$ to make the spinor part of the connection dependent of the space-time part, though we will treat the dependence the other way around. Letting also the five-form part be zero corresponds to the on-shell condition.

The famous supersymmetry algebra has generators [4]

$$q_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\gamma^a\theta)_\alpha \frac{\partial}{\partial x^a}, \tag{6.3}$$

fulfilling the relations

$$\{q_\alpha, q_\beta\} = 2\gamma^a_{\alpha\beta}\frac{\partial}{\partial x^a}, \qquad \{D_\alpha, q_\beta\} = 0. \tag{6.4}$$

Let $Q = \lambda^\alpha D_\alpha$ for some spinor valued but bosonic λ^α, hence a ghost variable because of the inverse statistics[1]. Then

$$Q^2 = \frac{1}{2}\lambda^\alpha\lambda^\beta\{D_\alpha, D_\beta\} = -(\lambda\gamma^a\lambda)\partial_a.$$

Thus if λ^α is a pure spinor, defined here as fulfilling $(\lambda\gamma^a\lambda) = 0$, Q will be a differential. Now let λ have ghost number one and incorporate the connection A_α into the λ expansion in a ghost 1 field $\Psi(x, \theta, \lambda)$, according to

$$\Psi(x, \theta, \lambda) = C(x, \theta) + \lambda^\alpha A_\alpha(x, \theta) + \cdots . \tag{6.5}$$

At the second power of λ

$$\left(Q\Psi + \Psi^2\right)[\lambda^2] = \lambda^\alpha\lambda^\beta F_{\alpha\beta}, \tag{6.6}$$

hence the equations of motion are incorporated into the equation

$$Q\Psi + \Psi^2 = 0 \tag{6.7}$$

and hence again, the differential Q gives the linearised equations of motion

$$Q\Psi = 0. \tag{6.8}$$

To find the BV field content the zero mode cohomology is investigated, i.e. the cohomology for x independent functions, i.e. again the cohomology for the differential $Q_0 = \lambda^\alpha\partial_\alpha$. A useful tool for this is the partition function defined as

$$P_{\text{ꝁ}}(t) = \bigoplus_{n\in\mathbb{N}} R_n t^n, \tag{6.9}$$

R_n is the representations occuring at n^{th} power of the object α (respect taken to parity and e.g. eliminating ideals) [26].

The partition function for θ, the unconstrained spinor is

$$P_\theta(t) = \bigoplus_{i=1}^{\infty} \wedge^i(00001)t^i, \tag{6.10}$$

which follows directly from that θ has weight (00001) and is fermionic.

The representation for the (symmetrised) n^{th} power of an ordinary (but bosonic) spinor can be split to irreducible ones by contracting pairs with gamma matrices γ^a or the (antisymmetric) product of five of these, γ^5_{abcde} (contractions with other powers of gamma's yield zero) [26]. Now the the corresponding representations for pure spinors λ is obtained by quoting away the ideal (generated by) $(\lambda\gamma^a\lambda)$, hence the only surviving representation is where all, but one in the case of an odd power,

[1]The spinor of nature λ gives it a sign when commuting etc., in addition to the sign from the ghost grade. These eliminate each others and the overall parity is bosonic. This setup may be viewed as a double grading: one $\mathbb{Z}_2$-grading for the spinor properties and one $\mathbb{Z}$-grading giving the ghost number.

of the λ's are contracted into pairs through γ^5. This representation is $(0000n)^2$. The corresponding for λ is thus

$$P_\lambda(t) = \bigoplus_{i=1}^{\infty}(0000i)t^i. \tag{6.11}$$

With the use of LiE [27] it can be shown that

$$P_\lambda(t)P_\theta(t) = \Big((00000) \ominus (10000)t^2 \oplus (00010)t^3 \ominus (00001)t^5 \oplus (10000)t^6 \ominus (00000)t^8\Big)\,, \tag{6.12}$$

up to the order of t^{20}. Without giving proof, we assume that this factorisation holds. This is the partition function for the differential Q_0.

As the differential Q_0 is a scalar with odd parity, it only changes the representation of objects it acts upon with a sign, hence trivial pairs will annihilate each other in the partition function. As no other monomials except these can form annihilating pairs, the partition function will be the same as the one for the zero mode cohomology of the same complex [26]. Define

$$\mathbb{C}_{\lambda\theta} := \mathbb{C}(\theta) \otimes \mathbb{C}(\lambda)/\langle(\lambda\gamma^a\lambda)\rangle, \tag{6.13}$$

where $\langle(\lambda\gamma^a\lambda)\rangle$ in turn denotes the ideal spanned by $(\lambda\gamma^a\lambda)$. There is then a homotopy equivalence

$$(H^\bullet_{Q_0}(\mathbb{C}_{\lambda\theta}), 0) \underset{p}{\overset{i}{\rightleftarrows}} (\mathbb{C}_{\lambda\theta}, Q_0) \circlearrowleft h, \tag{6.14}$$

where p is the obvious projection (non-closed monomials are sent to zero) and the inclusion i sends equivalence classes to the unique monomials that represents them (except in the zero case, where the choice is the trivial one). h is chosen as $h = h^0h^1$,

$$h^0 : \begin{cases} Q_0\eta & \mapsto -[\eta] \\ \text{non-exact} \mapsto 0 \end{cases}, \qquad h^1 : \omega \mapsto \omega - [\omega]$$

where $[\omega]$ denotes a chosen representative for the corresponding equivalence class when quoting with exact forms (the detail of how this choice is done is omitted). Then for monomials the homotopy criteria (2.44) reduces to

$$\begin{aligned} \text{exact: } 0 &= Q_0\eta + Q_0hQ_0\eta = Q_0(\eta - [\eta]) \\ \text{closed, not exact: } [\omega] &= \omega + Q_0h\omega = \omega + Q_0h^0(\omega - [\omega]) \\ \text{non-closed: } 0 &= \omega + Q_0h\omega + hQ_0\omega = [\omega] + hQ_0\omega. \end{aligned} \tag{6.15}$$

The formal inverse of $1 - Q_1h$ is formally

$$\sum_{n\in\mathbb{N}}(Q_1h)^n. \tag{6.16}$$

[2] In the case of $n = 2p$ all representations are $(p, 0, 0, 0, 0), (p-1, 0, 0, 0, 2)\cdots(1, 0, 0, 0, 2p-2), (0, 0, 0, 0, 2p)$, ordered in decreasing number of contractions with γ^a (i.e. increasing number of contractions with γ^5). For $n = 2p+1$, just add 1 in the fifth place [26].

Every pair $Q_1 h$ raises the power of θ by one and as the spinor space is 16 dimensional the sum is finite and well-defined. The homological perturbation lemma is thus applicable here and we have the homotopy equivalence

$$(H^{\bullet}_{Q_0}(\mathbb{C}_{\lambda\theta}), D) \underset{p'}{\overset{i'}{\rightleftarrows}} (\mathbb{C}_{\lambda\theta}, Q) \circlearrowright h', \tag{6.17}$$

for some differential D and i', p', h' defined as in the lemma.

What this tell us is that the original theory can be projected onto the zero mode cohomology which describes the same physics and therefore the content of the latter must include all field content. Thanks to the properties of Q_0 the BV fields will all have integer ghost number and integer or half integer dimension (this is actually a major reason for why the cohomology of Q_0 and not, say Q_1, was investigated). Given the sign of representations ($\oplus$ or $\ominus$) the content of the cohomology can be found in the monomials seen in Table 6.1 below.

	0	1	2	3	4	5	6	7	8	θ
0	0					5	6			
1		2	3					8		
2				5	6					
3	3					8				
4		5	6							
5				8						
6	6									
7		8								
8										
λ										

Table 6.1: Monomials of λ and θ whose representation has at least the correct sign. The depth of the table indicates the exponent of λ in the monomial while the the exponent of θ is indicated by the width. All possible places where the cohomology whose overall exponent is 6 is indicated by the same symbol in the table.

Using LiE once more, we can investigate if these candidate monomials actually include the relevant representation (e.g. which sixth order monomial in λ and θ that actually include the representation (10000)). This reduces the cohomology table to Table 6.2 below.

For formulating a BV action some kind of integral measure is needed and it can be shown that the following is sufficient for our needs, [4]

$$\int \mathrm{d}[Z] f := \int \mathrm{d}^{10}x \int \mathrm{d}^{16}\theta \int e^{-t\{Q,\theta^\alpha \bar{\lambda}_\alpha\}} \Omega \wedge f. \tag{6.18}$$

Here $\bar{\lambda}_\alpha$ is a conjugate pure spinor introduced to fix degenerations in the pure spinor measure. In order to not alter the cohomology the differential has been changed to

$$Q = (\lambda D) + \mathrm{d}\bar{\lambda}_\alpha \frac{\partial}{\partial \bar{\lambda}_\alpha} \tag{6.19}$$

	0	1	2	3	4	5	6	7	8	θ
0	0									
1		2	3							
2				5	6					
3						8				
4										
5										
6										
7										
8										
λ										

Table 6.2: Monomials in λ and θ containing the required representations. The rest is as in Table 6.1.

($\mathrm{d}\bar{\lambda}_\alpha$ and $\bar{\lambda}_\alpha$ is thus a trivial pair). Further $\Omega \sim \lambda^{-3}(\mathrm{d}\lambda)^{11}$ is a holomorphic top form for the pure spinor space (which happens to be Calabi-Yau). The factor $e^{-t\{Q,\theta^\alpha\bar{\lambda}_\alpha\}} = e^{-t(\lambda\bar{\lambda})-t(\theta\mathrm{d}\bar{\lambda})\}}$ is for regularisation and is not dependent upon t.

As ψ includes all BV fields, it is self-conjugate with respect to the BV bracket and may be formulated as

$$\{A, B\} = \int A\,\frac{\overleftarrow{\delta}}{\delta\psi}\,\mathrm{d}[Z]\,\frac{\overrightarrow{\delta}}{\delta\psi}\,B \tag{6.20}$$

instead of using individual fields/antifields [4], [26]. As stated above the generalised equations of motion are

$$0 = \{S, \psi\} = \int S\,\frac{\overleftarrow{\delta}}{\delta\psi}\,\mathrm{d}[Z] = Q\psi + \psi^2 \tag{6.21}$$

and without knowledge of details, it seems from here reasonable that the action should be on the form

$$S \sim \int \mathrm{d}[Z]\left(\psi Q\psi + \psi^3\right). \tag{6.22}$$

The assumption behind this is that Q is self-conjugate, which is true as

$$AQB = -^{A}Q(AB) -^{A+1} QAB \tag{6.23}$$

and

$$Q(AB) = \frac{\partial}{\partial\theta^\alpha}\left(\lambda^\alpha AB\right) - \frac{\partial}{\partial x^a}\left(\lambda^\alpha(\gamma^a\theta)_\alpha AB\right) \tag{6.24}$$

is killed by the integral. Now the action

$$S = \int \mathrm{d}[Z]\,\mathrm{Tr}\left(\frac{1}{2}\psi Q\psi + \frac{1}{3}\psi^3\right), \tag{6.25}$$

fulfills the requirements that its corresponding differential is

$$Q = \{S, \cdot\} = \int S\,\frac{\overleftarrow{\delta}}{\delta\psi}\,\mathrm{d}[Z]\,\frac{\overrightarrow{\delta}}{\delta\psi} = \int \mathrm{d}[Z]\,\frac{\overleftarrow{\delta}}{\delta\psi}\,S\,\frac{\overrightarrow{\delta}}{\delta\psi} = \int \mathrm{d}[Z]\left(Q\psi + \psi^2\right)\frac{\overrightarrow{\delta}}{\delta\psi} \tag{6.26}$$

and master equation

$$\begin{aligned}\{S, S\} &= \int \mathrm{d}[Z] \left(Q\psi + \psi^2\right) \frac{\overrightarrow{\delta}}{\delta\psi} S = \int \mathrm{d}[Z] \left(Q\psi + \psi^2\right)^2 \\ &= \int \mathrm{d}[Z] \operatorname{Tr} \left(Q\psi Q\psi + \psi^2 Q\psi + Q\psi\psi^2 + \psi^4\right) = 0.\end{aligned}$$

The first term in the last step is zero by self-conjugacy (and second degree nilpotency) of Q and the last is zero because ψ is odd while the trace is cyclic. The middle two integrands are exact.

What has been reviewed in this thesis are some of the more basic and fundamental properties of the classical aspects of the Batalin-Vilkovisky formalism. There are still many further details and related topics which this thesis does not cover. For example much more can be said of the L_∞-algebras and the quantum side of the BV formalism was barely touched upon in the background, Chapter 1. Of course there is probably much more that yet has to be discovered in this area of mathematical physics. As the BV formalism is probably the most sophisticated method to deal with gauge symmetry (so far), which in turn is an essential part of modern, fundamental physics, it is clear that research in this field will at some point demand the use of the Batalin-Vilkovisky formalism.

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A. Preliminaries

The reasons for this appendix are that I found the material covered here is a subject of its own that is more common than ghost formalisms and that I myself was already acquainted with it before starting this project. An introduction to this material worthy to be in the main thesis would be too long and different from what I wish to focus on. At the same time I know that many (of the few) who will read this thesis lack these preliminaries. As I now write in the appendix I allow myself to explain short and less formal, with the goal of giving the reader an understanding of the subjects.

A.1 Manifolds and fibre bundles

We begin with manifolds. A manifold is something that locally looks like $\mathbb{R}^n$. Examples are the torus, the sphere and $\mathbb{R}^n$ itself. Earth is flat, just zoom in enough. To describe this mathematically, a manifold is a topological space M. Given a point p, there is a neighbourhood U in M, open set $V \in \mathbb{R}^n$ and a bijective map $\phi : U \to V$, both ϕ itself and its inverse ϕ^{-1} are continuous. This map gives the manifold a local description by $(x^1, x^2, \cdots x^n) \in \mathbb{R}^n$ called coordinates. One example is spherical coordinates for the sphere S^2. Often one assumes smoothness of the manifold, i.e. you may differentiate functions on the manifold.

In every point $p \in M$ there is a tangent space T_pM, that is an n dimensional plane that is tangent to M in p (this construction of course assume smoothness). See Figure A.1. Points in T_pM are called tangent vectors. A basis for this tangent plane is

$$\frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial x^2}, \quad \cdots \frac{\partial}{\partial x^n}. \tag{A.1}$$

It may look peculiar to denote vectors by derivatives but there are reasons for this. Basis vector $\partial/\partial x^k$ is the one that, at the given point p, points in the direction of coordinate x^k, i.e. generates translation in x^k direction.

These tangent spaces T_pM for different point p can be put together into the so-called tangent bundle TM, that is the original manifold M itself together with all its tangent planes. This is a $2n$ dimensional manifold where every point corresponds to one point in M and a position in the corresponding tangent space.

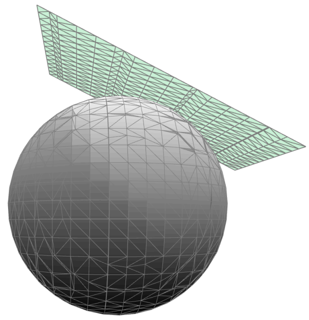
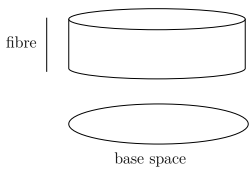
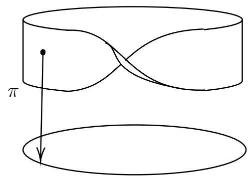


Figure A.1: A tangent plane to the sphere.

Tangent bundles are a subset of the more general class of fibre bundles. A fibre bundle is a manifold E that is constructed by two other manifolds, M and F, called the base space and the fibre respectively. E looks locally as the Cartesian product $M \times F$, in the same sense as a manifold looks locally like $\mathbb{R}^n$. The criteria is that given a point $E \ni u \simeq (p, f) \in M \times F$, there is a unique and globally (i.e. for the total manifold E) defined projection $\pi : u \mapsto p$ to the base space. If the fibre bundle is not just locally but globally the product $M \times F$ it is called trivial. Figure A.2 illustrates a pair of examples.



(a) A cylinder, that is the trivial fibre bundle which has the circle S^1 as base space and the interval $[0, 1]$ as fibre.



(b) The Möbius strip is a non-trivial example of a fibre bundle where the base space and the fibre are the same as for the cylinder. The projection π from strip to circle is still uniquely defined.

Figure A.2

A section is a (local or global) map $s : M \to E$ such that $\pi \circ s = \mathrm{id}$, i.e. $s : p \mapsto (p, z(p))$, for some function z. The set of sections is denoted by $\Gamma(E, M)$. Sections for TM are called vector fields. Vector fields $v \in \Gamma(TM, M)$ are equipped with a derivative action upon functions $f \in \mathcal{C}^\infty(M)$,

$$v : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad v : f \mapsto v(f) = v^\mu \frac{\partial f}{\partial x^\mu}. \tag{A.2}$$

A.2 Differential forms

Given two vectors v and u their wedge product or exterior product is defined as

$$v \wedge u = v \otimes u - u \otimes v, \tag{A.3}$$

with the defining property that $v \wedge u = -u \wedge v$. More vectors may be multiplied together,

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k, \tag{A.4}$$

which is well-defined without need of parentheses by associativity. The space of k number of vectors wedged together is denoted by $\wedge^k V$, if V is the space the vectors belong to. Sometimes sums of different degrees of these wedged together vectors are studied, hence this convenient bullet notation,

$$\wedge^\bullet V := \bigoplus_{k=0}^{\infty} \wedge^k V. \tag{A.5}$$

The sum is finite by antisymmetry of $\wedge$, if V is finite.

The dual of a vector space V denoted by V^* is defined as all linear functions from V to $\mathbb{R}$ (or $\mathbb{C}$ or, in general, the relevant scalar field). Given a basis e_μ for V the dual basis ϵ^μ is defined by

$$\langle \epsilon^\mu, e_\nu \rangle = \delta^\mu{}_\nu. \tag{A.6}$$

The cotangent space T^*M is defined as TM but with cotangent vectors, vectors dual to the tangent vectors, instead. The space of sections $\Gamma(T^*M, M)$ is denoted by $\Omega^1(M)$ and its elements are called one-forms. They may be represented as

$$\omega = \omega_\mu \mathrm{d}x^\mu, \tag{A.7}$$

for some function ω_μ, where $\mathrm{d}x^\mu$ is the dual basis to $\partial/\partial x^\mu$,

$$\left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \delta^\mu{}_\nu. \tag{A.8}$$

One-forms may be wedged together to p-forms,

$$\omega = \omega_{\mu_1\mu_2,\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \cdots \wedge \mathrm{d}x^{\mu_p}, \tag{A.9}$$

for some function $\omega_{\mu_1\mu_2,\cdots\mu_p}$. The set of these are denoted by $\Omega^p(M)$. Note that $\Omega^0(M) = \mathcal{C}^\infty(M)$. Vector fields may also act on these higher forms through the inner product $\lrcorner$ defined as

$$\begin{aligned}
&\left(v^\mu \frac{\partial}{\partial x^\mu}\right) \lrcorner \left(\omega_{\mu_1\mu_2,\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \cdots \wedge \mathrm{d}x^{\mu_p}\right) \\
&= v^\mu \omega_{\mu\mu_2,\cdots\mu_p} \mathrm{d}x^{\mu_2} \cdots \wedge \mathrm{d}x^{\mu_p} - v^\mu \omega_{\mu_1\mu,\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_3} \cdots \wedge \mathrm{d}x^{\mu_p} + \cdots \\
&\quad \cdots + (-1)^{p-1} v^\mu \omega_{\mu_1\mu_2,\cdots\mu} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \cdots \wedge \mathrm{d}x^{\mu_{p-1}}.
\end{aligned} \tag{A.10}$$

The exterior derivative, or de Rham operator, d is defined as

$$\mathrm{d} = \text{“ } \mathrm{d}x^\mu \wedge \frac{\partial}{\partial x^\mu} \text{ ”}, \quad \mathrm{d} : \Omega^\bullet(M) \to \Omega^\bullet(M), \tag{A.11}$$

$$\mathrm{d}\omega = \mathrm{d}\left(\omega_{\mu_1\mu_2,\cdots\mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \cdots \wedge \mathrm{d}x^{\mu_p}\right) = \frac{\partial}{\partial x^\mu} \omega_{\mu_1\mu_2,\cdots\mu_p} \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{A.12}$$

It is a differential as it is a derivative, i.e. obeys Leibniz rule

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta, \quad \omega \in \Omega^k(M), \tag{A.13}$$

and squares to zero, $\mathrm{d}^2 = 0$. The latter property follows by antisymmetry of the wedge product and Clairaut's rule. We have the nice properties that

$$v(f) = \langle \mathrm{d}f, v \rangle, \quad v \in TM, f \in \mathcal{C}^\infty(M) \tag{A.14}$$

and if $x^{ᚠ}(p)$, for fixed ᚠ, is the funtion that sends $p = (x^1, \cdots x^n)$ to the ᚠ$^{\text{th}}$ coordinate $x^{ᚠ}$,

$$\mathrm{d}x^{ᚠ}(p) = \mathrm{d}x^{ᚠ}. \tag{A.15}$$

The exterior derivative of a 0-form, that will say a function, is

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu, \tag{A.16}$$

which is the same as the gradient, if vector fields are translated to covector fields/one-forms. In $\mathbb{R}^3$ the exterior derivative of a one-form is

$$\begin{aligned} &\mathrm{d}\left(\omega_x \mathrm{d}x + \omega_y \mathrm{d}y + \omega_z \mathrm{d}z\right) \\ &= \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \end{aligned} \tag{A.17}$$

which is strongly reminiscent of the curl

$$\begin{aligned} &\nabla \times \left(V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} + V^z \frac{\partial}{\partial z}\right) \\ &= \left(\frac{\partial V^z}{\partial y} - \frac{\partial V^y}{\partial z}\right) \frac{\partial}{\partial x} + \left(\frac{\partial V^x}{\partial z} - \frac{\partial V^z}{\partial x}\right) \frac{\partial}{\partial y} + \left(\frac{\partial V^y}{\partial x} - \frac{\partial V^x}{\partial y}\right) \frac{\partial}{\partial z}. \end{aligned} \tag{A.18}$$

Similarly, the exterior derivative of a 3-form looks like the divergence. The property that $\nabla \times \nabla f = 0$ for all functions f is just a special case of $\mathrm{d}^2 = 0$.

The electromagnetic potential may be expressed as the one-form $A = A_\mu \mathrm{d}x^\mu$. Its exterior derivative is

$$\mathrm{d}A = \frac{1}{2!}(\partial_\mu A_\nu - \partial_\nu A_\mu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2!} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = F, \tag{A.19}$$

$F_{\mu\nu}$ is the electromagnetic field strength. In the case of a non-Abelian gauge group (e.g. $SU(2)$) the corresponding field, which here also is denoted by A, will be the Lie algebra valued

$$A = A_\mu^\alpha T_\alpha \mathrm{d}x^\mu \in \Omega^1(M) \otimes \mathfrak{g}, \tag{A.20}$$

if T_α is a basis for the relevant gauge Lie algebra $\mathfrak{g}$. Now the field strength is $F = \mathrm{d}A + A \wedge A$ and there is a covariant derivative D,

$$D\omega = \mathrm{d}\omega + [A, \omega], \tag{A.21}$$

for $\omega \in \Omega^\bullet(M) \otimes \mathfrak{g}$ a Lie algebra valued form. The Lie bracket acts on a p-form ω and q-form η as

$$[\omega, \eta] := \omega \wedge \eta + (-1)^{pq} \eta \wedge \omega = \omega^\alpha \wedge \eta^\beta [T_\alpha, T_\beta]. \tag{A.22}$$

A.3 Chain complexes and integration

If $\mathrm{d}\omega = 0$ it is called closed. If $\omega = \mathrm{d}\eta$, for some form η it is called exact. There is often interest in which forms are closed, except for the exact one as they are trivially closed, hence a need for separation between the exact and the non-trivial closed but non-exact ones. The de Rham differential d may be viewed as a collection of maps $\mathrm{d}_p : \Omega^p(M) \to \Omega^{p+1}(M)$ in the chain complex

$$\cdots \xrightarrow{\mathrm{d}_{p-2}} \Omega^{p-1}(M) \xrightarrow{\mathrm{d}_{p-1}} \Omega^p(M) \xrightarrow{\mathrm{d}_p} \Omega^{p+1}(M) \xrightarrow{\mathrm{d}_{p+1}} \cdots . \tag{A.23}$$

From this the de Rham cohomology is defined as

$$H^p := \frac{\ker \mathrm{d}_p}{\mathrm{im}\ \mathrm{d}_{p-1}}. \tag{A.24}$$

For the case $p = 0$, d_{-1} is defined as the zero map. For other differentials that also act on a complex (in this case $\Omega^\bullet(M)$), that is graded ($\Omega^\bullet(M) = \bigoplus_k \Omega^k(M)$) and raises the grading by one ($\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$), a cohomology may be defined analogously. To separate different cohomologies from each other we write $H^\bullet_\mathrm{d}(M)$. For differentials ∂ that instead lowers the grading by one, a homology may be defined as

$$H_{\partial,p} := \frac{\ker \partial_p}{\mathrm{im}\ \partial_{p+1}}. \tag{A.25}$$

To give a slightly better feeling for cohomology, consider $\mathbb{R}^3$. As mentioned d_0 corresponds to the gradient ∇ and d_1 corresponds to the curl $\nabla\times$. The fact that every vector field with vanishing curl is the gradient of some scalar function, can be translated to the statement that every closed one-form is exact. Thus, the cohomology is zero. For $\mathbb{R}^3 \setminus \{0\}$, there are exceptions – vector fields whose curls are zero but are not the gradients of scalar fields. Adding a gradient of a scalar field to such an exception will technically give another exception even though the added term does not really contribute to something new. The cohomology views these two exceptions as the same.

Another example is in electromagnetism, where there is a gauge transformation

$$A \mapsto A + \mathrm{d}\Lambda, \tag{A.26}$$

for some function Λ. This does not alter the physics as what matters in the end (modulo some topological effects) is the field strength F which is invariant,

$$F = \mathrm{d}A \mapsto \mathrm{d}(A + \mathrm{d}\Lambda) = F. \tag{A.27}$$

The elements of $H^1_\mathrm{d}(M)$ are equivalence classes of fields A that only differs by a gauge transformation, i.e. have the same field strength F.

For ω a top form, i.e. of grade n if the manifold M is of this dimension, the integral of ω is defined as

$$\int_M \omega := \int_M \omega_{\mu_1,\cdots\mu_n}(x_1, \cdots x_n)\mathrm{d}x^{\mu_1}\mathrm{d}x^{\mu_2}\cdots\mathrm{d}x^{\mu_n} \tag{A.28}$$

(the actual definition is a sum of integrals similar to this one, but weighted, as just one coordinate chart is not enough in general). The famous generalisation of Stokes' theorem states that

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \tag{A.29}$$

where ∂M denotes the boundary of M (e.g. ∂M is the sphere if M is the ball). Exact forms are thus killed by integrals if there is no boundary, $\partial M = \emptyset$. Using Equation (A.13), Stokes' generalised formula may be recast into the form of a generalised partial integration formula

$$\int_M \mathrm{d}\omega \wedge \eta = \int_{\partial M} \mathrm{d}(\omega \wedge \eta) + (-1)^{1+p} \int_M \omega \wedge \mathrm{d}\eta, \quad \omega \in \Omega^p(M). \tag{A.30}$$

With a metric $g_{\mu\nu}$ it is possible to go from vectors to one-forms and vice versa (also known as going from contravariant to covariant vectors when using index notation). Furthermore, we can define an invariant volume form (i.e. non-vanishing top form)

$$\mathrm{dvol} = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n, \tag{A.31}$$

$g = \det g_{\mu\nu}$. There is also an operator $*$ called the Hodge star which sends p-forms to $(n-p)$-forms. The new $(n-p)$-form is roughly the same, except for a factor that depends on $g_{\mu\nu}$ and p. Using this, the curl $\nabla\times$ may better be expressed as $*\mathrm{d}$ and the Yang-Mills action can be written as [9]

$$S = -\int \mathrm{dvol}\, \frac{1}{8} F^{\mu\nu\alpha} F_{\mu\nu\alpha} = \int \mathrm{dvol} \frac{1}{4} \operatorname{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right) = \int \frac{1}{2} \operatorname{Tr} F \wedge *F, \tag{A.32}$$

with the normalisation $\operatorname{Tr}\left(T_\alpha T_\beta\right) = -\frac{1}{2}\delta_{\alpha\beta}$ of the Lie algebra basis and $[T_\alpha, T_\beta] = f^{\gamma}_{\alpha\beta} T_\gamma$.

Finally, there are a few more terms that may show up without explanation, such as sheaves and morphisms. The latter shows up many times in mathematics and means in general a map which preserve the relevant structure(s). The former I myself have poor experience of and I can certainly not explain it without taking this appendix to be further from reference free. As far as I understand, it suffices in this thesis to think of them as fibre bundles, which are special cases of sheaves.

B. Miscellaneous calculations

The action is $S = S_0 + \cdots$, where S_0 contributes to Q by

$$\{S_0, \cdot\} = S_0 \left(\frac{\overleftarrow{\delta}}{\delta \Phi^A} \frac{\overrightarrow{\delta}}{\delta \Phi^+_A} - \frac{\overleftarrow{\delta}}{\delta \Phi^+_A} \frac{\overrightarrow{\delta}}{\delta \Phi^A} \right) = -D * F \frac{\delta}{\delta A^+}, \tag{B.1}$$

where Φ^A and Φ^+_A are the (ghost)field and anti ghost/field. Adding the term

$$-DcA^+ = -(\mathrm{d}c + [A, c])^\alpha A^+_\alpha \tag{B.2}$$

to the action gives

$$\begin{aligned} &\mathrm{d}c^\alpha \delta_{A^\alpha} - \mathrm{d}A^+_\alpha \delta_{c^+_\alpha} + f^\gamma_{\alpha\beta} c^\alpha A^+_\gamma \delta_{A^+_\beta} + f^\gamma_{\alpha\beta} A^\alpha A^+_\gamma \delta_{c^+_\beta} + f^\gamma_{\alpha\beta} A^\alpha c^\beta \delta_{A^\gamma} \\ &= Dc\delta_A - DA^+ \delta_{c^+} - [c, A^+]\delta_{A^+}. \end{aligned} \tag{B.3}$$

Here some implicit Lie algebra index contractions were made and for objects that has a lower Lie algebra index, we defined

$$D\Phi^+_\beta := \mathrm{d}\Phi^+_\alpha + [A, \Phi^+]_\alpha, \quad [\Phi, \Phi^+]_\alpha := f^\gamma_{\alpha\beta} \Phi^\beta \Phi^+_\gamma, \tag{B.4}$$

where Φ (Φ^+) is any (anti)field. Some supporting results are that

$$D\Phi\Phi^+ = \mathrm{d}\Phi\Phi^+ + [A, \Phi]A^+ = \mathrm{d}\Phi\Phi^+ + f^\gamma_{\alpha\beta} A^\alpha \Phi^\beta \Phi^+_\gamma = -\Phi \mathrm{d}\Phi^+ - \Phi[A, \Phi^+] \equiv -\Phi D\Phi^+ \tag{B.5a}$$

$$\frac{\delta}{\delta \Phi^1}[\Phi^1, \Phi^2]\Phi^+ = \frac{\delta}{\delta \Phi^1}\left(f^\gamma_{\alpha\beta} \Phi^{1\alpha} \Phi^{2\beta} \Phi^+_\gamma \right) = f^\gamma_{\alpha\beta} \Phi^{2\beta} \Phi^+_\gamma \equiv [\Phi^2, \Phi^+], \tag{B.5b}$$

for fields Φ, Φ^1, Φ^2 and antifield Φ^+.

If the total action is

$$S = \frac{1}{2} F * F - \mathrm{D}cA^+ + \frac{1}{2}[c, c]c^+, \tag{B.6}$$

it acts as

$$Q = -\frac{1}{2}[c, c]\delta_c + \mathrm{D}c\delta_A + (-\mathrm{D} * F - [c, A^+])\delta_{A^+} + (-\mathrm{D}A^+ - [c, c^+])\delta_{c^+} \tag{B.7}$$

and the master equation is

$$0 = \frac{1}{2}\{S, S\} = S \frac{\overleftarrow{\delta}}{\delta \Phi^A} \frac{\overrightarrow{\delta}}{\delta \Phi^+_A} S = -D * FDc - [c, A^+]Dc - \frac{1}{2} DA^+[c, c] - \frac{1}{2}[c, c^+][c, c]. \tag{B.8}$$

The last term,

$$-\frac{1}{2}[c,c^+][c,c] = -\frac{1}{2}f^\gamma_{\alpha\beta}f^\alpha_{\delta\epsilon}c^\beta c^+_\gamma c^\delta c^\epsilon \tag{B.9}$$

$$= -\frac{1}{6}c^+_\gamma\left(f^\gamma_{\alpha\beta}f^\alpha_{\delta\epsilon}c^\beta c^\delta c^\epsilon + f^\gamma_{\alpha\epsilon}f^\alpha_{\beta\delta}c^\epsilon c^\delta c^\beta + f^\gamma_{\alpha\delta}f^\alpha_{\epsilon\beta}c^\delta c^\epsilon c^\beta\right) \tag{B.10}$$

$$= -\frac{1}{6}c^+_\gamma\left(f^\gamma_{\alpha\beta}f^\alpha_{\delta\epsilon} + f^\gamma_{\alpha\epsilon}f^\alpha_{\beta\delta} + f^\gamma_{\alpha\delta}f^\alpha_{\epsilon\beta}\right)c^\beta c^\delta c^\epsilon \tag{B.11}$$

is zero by the Jacobi identity. The two in the middle,

$$-[c,A^+]Dc = -f^\gamma_{\alpha\beta}c^\beta A^+_\gamma \mathrm{d}c^\alpha - f^\gamma_{\alpha\beta}f^\alpha_{\delta\epsilon}c^\beta A^+_\gamma A^\delta c^\epsilon \tag{B.12}$$

respectively

$$-\frac{1}{2}DA^+[c,c] = -\frac{1}{2}f^\gamma_{\alpha\beta}\mathrm{d}A^+_\gamma c^\alpha c^\beta - \frac{1}{2}f^\gamma_{\alpha\beta}f^\alpha_{\delta\epsilon}A^\beta A^+_\gamma c^\delta c^\epsilon \tag{B.13}$$

are together zero. Their first two terms are together exact. The two later are

$$\begin{aligned}&\frac{1}{2}A^+_\gamma A^\beta c^\delta c^\epsilon(2f^\gamma_{\alpha\delta}f^\alpha_{\beta\epsilon} - f^\gamma_{\alpha\beta}f^\alpha_{\delta\epsilon}) = \frac{1}{2}A^+_\gamma A^\beta c^\delta c^\epsilon(2f^\gamma_{\alpha\delta}f^\alpha_{\beta\epsilon} + f^\gamma_{\alpha\delta}f^\alpha_{\epsilon\beta} + f^\gamma_{\alpha\epsilon}f^\alpha_{\beta\delta})\\ &= \frac{1}{2}A^+_\gamma A^\beta c^\delta c^\epsilon(f^\gamma_{\alpha\delta}f^\alpha_{\beta\epsilon} + f^\gamma_{\alpha\epsilon}f^\alpha_{\beta\delta}) = 0\end{aligned} \tag{B.14}$$

as the field part is odd in δ, ϵ while the structure constant chunk is even. Using the identity [9]

$$D^2c = [F,c] \tag{B.15}$$

the first term in the equation is

$$-D * FDc = *FD^2c = *F[F,c], \tag{B.16}$$

which in turn is zero by similar reasoning as above [19].

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