Classifying Strictly Tessellating Polytopes

Connecting analysis, geometry, and algebra through strictly tessellating polytopes and investigating their unique properties.

MAX BLOM

Department of Mathematical Sciences
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2021

Master's thesis 2021

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Cover: A seal in a "banana pose", implying that it is feeling safe. The picture is not related to the subject.

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Abstract

This thesis consists of a paper and additional results. The paper shows a connection between the geometry of polytopal domains in Euclidean space and the eigenfunctions of the Dirichlet Laplacian. The necessary and sufficient geometric properties of a polytopal domain are shown for the first eigenfunction to extend to a real analytic function on the whole space. Furthermore, alcoves are essential for the proof of the main theorem. Additionally, the paper discusses how the results relate to crystallographic restrictions and lattices. Strictly tessellating polytopes are defined and used in connection to the main theorem. The paper concludes by formulating a conjecture akin to Fuglede's, replacing tessellation by translaton with strict tessellation. In addition to the paper, results on the geometric properties of strictly tessellating polytopes are presented, and bounds on the number of strictly tessellating polytopes up to equivalence are shown.

Acknowledgements

I would like to thank Magnus Goffeng and Julie Rowlett for providing guidance and insightful knowledge in many aspects. I would also like to thank Simon Jacobsson for providing useful comments.

Max Blom, Gothenburg, May 2021

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Introduction

Geometric analysis connects two major fields of mathematics. The Laplace eigenvalue equation is a typical example, where we study how the solutions to the partial differential equation relate to its geometric setting. The simplest form of the problem is in one dimension, where the only geometric property of a domain is its length, and we can read it off of the eigenvalues. In higher dimensions this becomes more difficult for two main reasons. First, one cannot in general compute the eigenvalues analytically. Second, the domains are much more complex and have additional properties such as boundary smoothness that cannot always be read off of the eigenvalues.

Here we focus on the Laplace eigenvalue equation for domains in Euclidean space. We assume Dirichlet boundary condition, which requires the solutions to vanish on the boundary of the domain. Moreover, we specify to polytopes. In one dimension, a polytope is a segment, and in two dimensions it is a bounded polygonal domain. There is a natural generalization to higher dimensions.

We show that real analytic functions satisfying the Laplace eigenvalue equation with Dirichlet boundary conditions are anti-symmetric with respect to the boundary of the polytopal domain, expanding on a result by Lamé [26]. Furthermore, we use that a real analytic function vanishing on an open, nonempty subset of a hyperplane vanishes on the whole hyperplane. This is used to show that, geometrically speaking, the eigenfunctions of interest repeat themselves throughout space, forming a tessellation with specific properties discussed in chapter 2. We say that a polytope tessellating space with these properties is *strictly tessellating*. Strict tessellation is of interest as it is shown to be a necessary and sufficient condition on a polytopal domain for the first eigenfunction for the Dirichlet Laplacian to extend to a real analytic function on the whole space.

The paper expands on two previous results. In 2008, McCartin proved [30] a theorem in two dimensions that showed on which polygons the Laplace operator with Dirichlet boundary conditions has an L^2-basis of trigonometric eigenfunctions. Going back to 1980, Bérard [2] showed that *alcoves* have a complete set of trigonometric eigenfunctions for the Dirichlet Laplacian. An alcove is a polytope that is formed from a root system when you "cut it up" with the planes of symmetry. A root system is a collection of vectors satisfying certain geometric properties. The details on this are given in chapter 2. We show that an alcove is strictly tessellating, and use this to prove a generalization of McCartin's theorem in $\mathbb{R}^n$.

We generalize the theorem by McCartin, that classifies polygonal domains, to polytopes. The theorem shows equivalence between a domain being a strictly tessellating polytopes, the first eigenfunction of the Dirichlet Laplacian extending to a real analytic function on the whole space, and a domain being congruent to an alcove. We further show that this that the eigenfunctions are trigonometric if the first eigenfunction extends to a real analytic function.

This thesis is basing itself on a paper written by Julie Rowlett, Max Blom, Henrik Nordell, Oliver Thim, and Jack Vahnberg. The paper constitutes chapter 2. The thesis builds on the paper by adding additional results for strictly tessellating polytopes in chapter 3. It is shown that the faces of a strictly tessellating polytope are also strictly tessellating polytopes themselves. Furthermore, it is shown that a strictly tessellating polytope of dimension n may have no more than 2^n vertices. A necessary condition for the solid angles of a strictly tessellating polytope is then stated. Finally, an upper and a lower bound are given for the number of strictly tessellating polytopes in $\mathbb{R}^n$ up to equivalence.

2

Main Paper

2.1 Introduction.

In *The Grammar of Ornament,* published in 1856, Owen Jones wrote [20]:

> Whenever any style of ornament commands universal admiration, it will always be found to be in accordance with the laws which regulate the distribution of forms in nature.

In the case of crystals, the laws that regulate their shape are dictated by the crystallographic groups.

2.1.1 Crystallographic groups.

A crystal or crystalline solid is a solid material whose constituents, such as atoms, molecules, or ions, are arranged in a highly ordered microscopic structure; for a two-dimensional example, see Figure 2.1. The crystal is often described in terms of its *symmetries,* those isometries of the ambient space under which the crystal remains unchanged. The three basic types of isometries of $\mathbb{R}^n$ are translations, rotations, and reflections. These form a group under composition. The patterns in Figure 2.2 have symmetry groups that are *plane crystallographic groups.* These are subgroups of the group of isometries of the plane that are topologically discrete and contain two linearly independent translations. Equivalently, a plane crystallographic group[1] is a co-compact subgroup of the group of isometries of the plane. A subgroup in this context is called co-compact if the quotient space $\mathbb{R}^2/\Gamma$ by the subgroup, Γ, is compact. The classification of these groups, up to equivalence, was achieved at the end of the 19th century by E. S. Fedorov [12, 13, 14, 15] and A. Schoenflies [32, 33, 34, 35]; for English references, see [15, 18, 36]. Two planar crystallographic groups are equivalent if they are isomorphic as abstract groups; equivalently if they are conjugate in the group of affine transformations of $\mathbb{R}^2$. In two dimensions, up to this notion of equivalence, there are seventeen crystallographic groups.

One can also consider crystals in three dimensions, and mathematically we may generalize all of these notions to $\mathbb{R}^n$. An n-dimensional crystallographic group is

[1] These are also known as wallpaper groups.

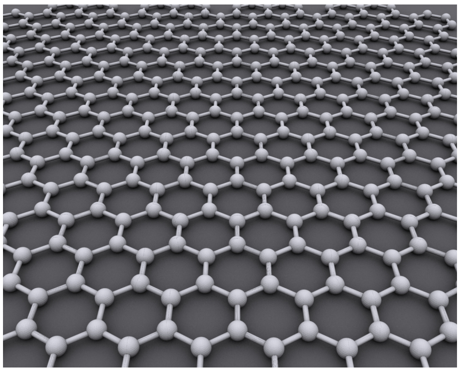

Figure 2.1: Graphene is an allotrope of carbon in the form of a two-dimensional, atomic-scale hexagonal lattice such that each point in the lattice corresponds to an atom. This image is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license at `commons.wikimedia.org/wiki/File:Graphen.jpg`.

a discrete group of isometries of $\mathbb{R}^n$ that is co-compact. Fedorov [12, 13, 14, 15] and Schoenflies [32, 33, 34, 35] proved that there are, up to equivalence, 219 crystallographic groups in $\mathbb{R}^3$. Two crystallographic groups in $\mathbb{R}^n$ are equivalent if they are conjugate in the group of affine transformations of $\mathbb{R}^n$; equivalently if they are isomorphic as abstract groups. In 1910, Bieberbach proved that for any n, there are only finitely many n-dimensional crystallographic groups up to equivalence [3, 4], thereby solving Hilbert's 18th problem; for an English reference, see [18, 36]. However, for general n, the precise number of crystallographic groups up to isometry in $\mathbb{R}^n$ is unknown. In four dimensions, it was not known until the 1970s that there are 4783 crystallographic groups up to isometry [7]. Can one obtain upper and lower bounds for the number of crystallographic groups up to isometry in $\mathbb{R}^n$ which depend on n? If so, does the lower bound tend to infinity, or is there a uniform upper bound? For higher dimensions, the classification is still in progress; a nonexhaustive list of recent results includes [8, 28, 31].

2.1.2 Strictly tessellating polytopes and our main result.

The constituents of a crystal create a perfectly regular pattern. Another way to create a perfectly regular pattern is by "strict tessellation." This is a notion specific to polytopes.

Definition 1. *The set of all* one-dimensional polytopes *is the set of all bounded open intervals*

$$\wp_1 := \{(a, b) : -\infty < a < b < \infty\}.$$

A domain *here is a connected, open set. Inductively, we define the set of polytopes $\wp_n$ in $\mathbb{R}^n$ for $n \geq 2$ to be the set of bounded domains $\Omega \subset \mathbb{R}^n$ such that*

$$\partial\Omega = \bigcup_{j=1}^{m} \overline{P_j}, \quad P_j \cong Q_j \in \wp_{n-1}.$$

Here, the boundary of Ω consists of the closures of $(n-1)$-dimensional polytopes, P_j. Each P_j is contained in an $(n-1)$-dimensional hyperplane, which is a set of the

Figure 2.2: Two Egyptian patterns whose symmetry groups are planar crystallographic groups. These patterns were documented by Owen Jones in 1856 [20, Egyptian No. 7 (plate 10), images 8 and 13]. These images were obtained from Wikimedia Commons and are in the public domain in their country of origin and other countries and areas where the copyright term is the author's life plus 100 years or fewer; this includes the United States `https://commons.wikimedia.org/wiki/Commons:Copyright_tags/Country-specific_tags#United_States_of_America`.

form

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{M} \cdot \boldsymbol{x} = b\},$$

for some fixed $\boldsymbol{M} \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$. *The meaning of* $P_j \cong Q_j$ *is that the hyperplane above is isometrically identified with* $\mathbb{R}^{n-1}$, *and with this identification* P_j *is isometrically identified with* Q_j. *Note that our definition of polytope makes no assumption of convexity; polytopes as defined here can be nonconvex.*

Next we introduce the notion of a strict tessellation. We are not aware of the term "strict tessellation" in the literature, but it might be known under a different name. An example of a strict tessellation of the plane is given in Figure 2.3; a tessellation of the plane which is not strict is given in Figure 2.4.

Definition 2. *A polytope* $\Omega \in \wp_n$ strictly tessellates $\mathbb{R}^n$ *if*

1. $\mathbb{R}^n = \bigcup_{j \in \mathbb{Z}} \overline{\Omega_j}$, *such that each* Ω_j *is isometric to* Ω, *and* $\Omega_j \cap \Omega_k = \emptyset$ *for any* $j \neq k$.

2. *Let* m *be the number of boundary faces of* Ω, *and let* $\{\mathcal{H}_{j,i}\}_{i=1}^m$ *be the corresponding* m *hyperplanes containing the* m *boundary faces of* Ω_j. *Then* $\mathcal{H}_{j,i} \cap \Omega_k = \emptyset$ *for all* $1 \leq i \leq m$, *and for all* j *and* $k \in \mathbb{Z}$ *(including* $k = j$*). Note that this immediately implies that the polytope is convex.*

3. *For each* $k \neq j$, *for some* $N \in \mathbb{N}$, $\Omega_k = R_N \circ \cdots \circ R_1(\Omega_j)$. *Here,* R_1 *is reflection across one of the boundary faces of* Ω_j. *For* $I \geq 2$, R_I *is reflection across a boundary face of* $R_{I-1} \circ \cdots \circ R_1(\Omega_j)$.

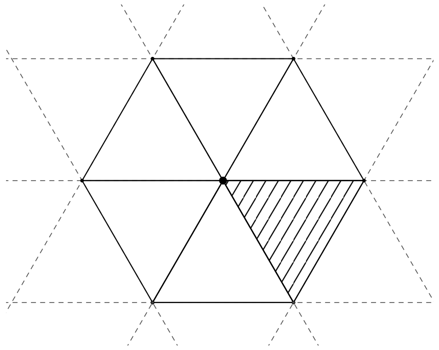

Figure 2.3: Equilateral triangles are shown here to strictly tessellate the plane.

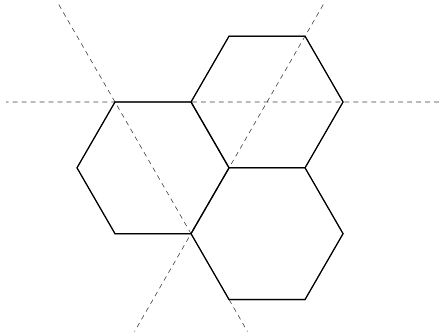

Figure 2.4: Although it is well known that regular hexagons tessellate the plane by reflection, the tessellation is not strict, because the lines that contain the edges of the hexagon cut through the interior of the reflected copies.

For any real numbers $a < b$,

$$\mathbb{R} = \bigcup_{j \in \mathbb{Z}} \overline{\Omega_j}, \quad \text{where } \Omega_j := (j(b-a)+a, j(b-a)+b)\,.$$

In this case, the boundary faces are points, $\{j(b-a)+a, j(b-a)+b\}_{j\in\mathbb{Z}}$, and therefore the hyperplanes that contain these faces are simply the points themselves. This shows that conditions (1) and (2) above are satisfied. Moreover, for any $k \neq j$, for example $k = j+\ell$, if $\ell > 0$, then Ω_k is obtained by reflecting across the boundary faces $(j+i)(b-a)+b$ for $i = 0, \ldots, \ell-1$. If $\ell < 0$, then Ω_k is obtained by reflecting across the boundary faces $(j+i)(b-a)+a$ for $i = 0, \ldots, \ell-1$. Consequently, every element of $\wp_1$ strictly tessellates $\mathbb{R}$.

In 2008, McCartin proved a remarkable classification theorem [30], connecting geometry and analysis. Recall the Laplacian on $\mathbb{R}^n$ is the partial differential operator

$$\Delta := -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

The Laplace eigenvalue problem for a domain $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary condition is to find all functions $u : \overline{\Omega} \to \mathbb{C}$ that are not identically zero and satisfy

$$\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega, \text{ for some constant } \lambda, \quad \text{and } \; u|_{\partial\Omega} = 0.$$

This is a difficult problem, because in general it is impossible to compute the numbers λ. However, using the tools of functional analysis [9] one can prove that these eigenvalues are discrete and positive and therefore can be ordered, counting multiplicity, as

$$0 < \lambda_1 < \lambda_2 \leq \cdots \uparrow \infty.$$

Since we define all domains here to be connected, the first eigenvalue is simple, and its corresponding eigenfunction is uniquely defined, up to multiplication by scalars.

In this way we may speak of the first eigenfunction that has eigenvalue λ_1. In one dimension, by Definition 1, a polytope is a bounded open interval (a, b) for some real numbers $a < b$. The Laplace eigenvalue equation with the Dirichlet boundary condition on such a polytope is to find all functions u defined on $[a, b]$ such that there exists $\lambda \in \mathbb{C}$ with

$$-u''(x) = \lambda u(x), \quad a < x < b, \quad u(a) = u(b) = 0.$$

This is a classical ordinary differential equation (see [6]), and all solutions to this equation are precisely (up to multiplication by constants) given by

$$u_k(x) = \sin\left(\frac{x-a}{b-a} k\pi\right), \quad \lambda_k = \frac{k^2\pi^2}{(b-a)^2}, \quad k \in \mathbb{N}.$$

These eigenfunctions are trigonometric functions. We can also define trigonometric functions on $\mathbb{R}^n$.

Definition 3. *An eigenfunction $u : \mathbb{R}^n \to \mathbb{C}$ for the Laplacian is* trigonometric *if it can be expressed as a finite sum of trigonometric functions*

$$u(\boldsymbol{x}) = \sum_{j=1}^{m} a_j \sin(\boldsymbol{L}_j \cdot \boldsymbol{x}) + b_j \cos(\boldsymbol{M}_j \cdot \boldsymbol{x}).$$

Here, $a_j, b_j, \in \mathbb{C}$ and $\boldsymbol{L}_j, \boldsymbol{M}_j \in \mathbb{R}^n$ satisfy $||\boldsymbol{L}_j||^2 = ||\boldsymbol{M}_j||^2 = \lambda$ for all $j = 1, \ldots, m$, where λ is the eigenvalue corresponding to u.

Remark 1. *Since*

$$\cos(t) = \sin(t + \pi/2), \quad \textit{for all } t \in \mathbb{R},$$

it is equivalent to define a trigonometric eigenfunction to be a function of the form

$$u(\boldsymbol{x}) = \sum_{j=1}^{m} a_j \sin(\boldsymbol{L}_j \cdot \boldsymbol{x} + \phi_j).$$

Here, $a_j \in \mathbb{C}$, $\boldsymbol{L}_j \in \mathbb{R}^n$, $\phi_j \in \left\{0, \frac{\pi}{2}\right\}$, and $||\boldsymbol{L}_j||$ are the same for all $j = 1, \ldots, m$. We note that some authors refer to these functions as "quasi-periodic."

In general, it is impossible to compute the eigenfunctions of an arbitrary polygonal domain. Nonetheless, McCartin proved the following classification theorem which shows the equivalence of the analytic property, having trigonometric eigenfunctions, with the geometric property, strictly tessellating.

Theorem 1 (McCartin [30])**.** *Assume that Ω is a polygonal domain in the plane (a two-dimensional polytope). Then the following are equivalent:*

1. *Ω has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue problem with the Dirichlet boundary condition.*

2. *Ω strictly tessellates the plane.*

3. *Ω is one of the following: a rectangle, an isosceles right triangle, an equilateral triangle, or a hemi-equilateral triangle, also known as 30-60-90 triangle because its interior angles have degree measures 30, 60, and 90.*

We note that if any of the above three conditions are satisfied, it follows immediately that Ω is convex.

Remark 2. *The Laplace eigenfunctions for a rectangular domain with vertices at the points $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) with the Dirichlet boundary condition can be computed using separation of variables, which reduces the problem to two one-dimensional problems. The resulting eigenfunctions are indexed by $m,n \in \mathbb{N}$. For Cartesian coordinates $\boldsymbol{x} = (x,y) \in \mathbb{R}^2$, the eigenfunctions are*

$$u_{m,n}(x,y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

Using trigonometric identities, we have

$$u_{m,n}(x,y) = \frac{1}{2}\left[\cos\left(\begin{bmatrix}\frac{m\pi}{a}\\ -\frac{n\pi}{b}\end{bmatrix}\cdot\boldsymbol{x}\right) - \cos\left(\begin{bmatrix}\frac{m\pi}{a}\\ \frac{n\pi}{b}\end{bmatrix}\cdot\boldsymbol{x}\right)\right].$$

Consequently, these are trigonometric eigenfunctions.

Our main result is a generalization to all dimensions.

Theorem 2. *Assume that Ω is a polytope in $\mathbb{R}^n$. Then the following are equivalent:*

1. *The first eigenfunction for the Laplace eigenvalue equation with the Dirichlet boundary condition extends to a real analytic function on $\mathbb{R}^n$.*

2. *Ω strictly tessellates $\mathbb{R}^n$.*

3. *Ω is congruent to a fundamental domain of a crystallographic Coxeter group as defined in Bourbaki [5, VI.25, Proposition 9, p. 180], and is also known as an alcove [2, p. 179]; see also Section 3.*

The three equivalent statements in Theorem 2 are respectively analytic, geometric, and algebraic. These statements and how they were proved are depicted in Figure 2.5. Our work therefore reveals an intimate connection between analysis, geometry, and algebra. Moreover, combining our theorem with Bérard's proposition, see [2, Proposition 9, p. 181] or Proposition 2 below, we obtain the following rather remarkable result.

Corollary 1. *Assume that Ω is a polytope in $\mathbb{R}^n$. If the first eigenfunction for the Laplace eigenvalue equation with the Dirichlet boundary condition extends to a real analytic function on $\mathbb{R}^n$, then it is a trigonometric eigenfunction. Moreover, in that case,* all *the eigenfunctions of Ω are trigonometric.*

Remark 3. *Every trigonometric eigenfunction satisfies the first condition of Theorem 2. However, there are many functions that satisfy this condition but are* not *trigonometric. Examples include the eigenfunctions for a disk in $\mathbb{R}^2$ that are products of Bessel functions and trigonometric functions. There is no contradiction with Corollary 1 because a disk is not a polygonal domain.*

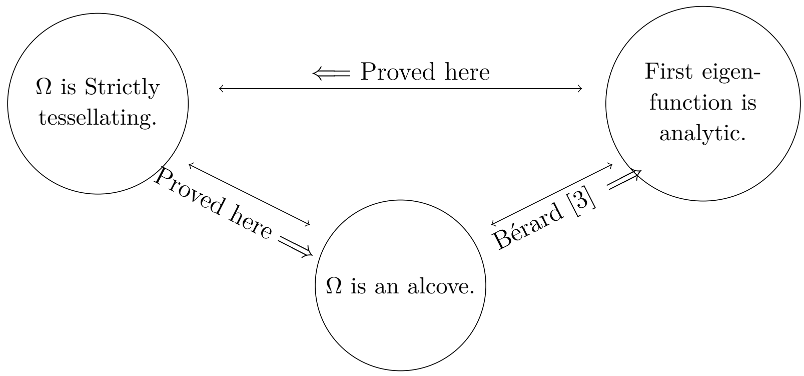

Figure 2.5: This diagram shows the three statements of Theorem 2 and how they were proved.

2.1.3 Organization.

In Section 2.2, we prove that if the first eigenfunction of a polytope satisfies the hypotheses of Theorem 2, then the polytope strictly tessellates $\mathbb{R}^n$. We prove this by generalizing classical results of Lamé [26]. In Section 2.3, we introduce the notions of root systems and alcoves and prove that all polytopes that strictly tessellate $\mathbb{R}^n$ are alcoves. We then recall the result of Bérard [2]: all alcoves have a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition. These results together complete the proofs of Theorem 2 and Corollary 1. In Section 2.4 we discuss connections to the Fuglede and Goldbach conjectures. We make our own conjecture and conclude with a purely geometric conjecture which is equivalent to the strong Goldbach conjecture.

2.2 The first eigenfunction and strict tessellation.

There is no known method to explicitly compute the eigenfunctions and eigenvalues for an arbitrary polytope. However, using the tools of functional analysis, one can prove general facts about them. We summarize briefly here. Here a domain refers to an open, connected set. For the Dirichlet boundary condition for the Laplace eigenvalue equation on a bounded domain, $\Omega \subset \mathbb{R}^n$, the eigenvalues form a discrete positive set which accumulates only at infinity [9]. We can therefore order the eigenvalues as they increase and counting multiplicity by repeating an eigenvalue according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 \cdots \uparrow \infty.$$

We may correspondingly order the eigenfunctions. Since we define all domains here to be connected, the first eigenvalue is simple, and its corresponding eigenfunction is uniquely defined, up to multiplication by scalars. In this way, we may speak of the "first" eigenfunction, which is the eigenfunction whose eigenvalue is equal to λ_1. The eigenfunctions form an orthogonal basis of the Hilbert space $\mathcal{L}^2(\Omega)$. We shall require the following well-known fact about the first eigenfunction. The proof of this theorem can be found in the classical PDE textbook of Evans [10, §6.5].

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then the first eigenfunction of the Laplace eigenvalue equation with the Dirichlet boundary condition does not vanish anywhere inside Ω.*

The following result is originally due to Lamé [26] in two dimensions and restricted to trigonometric eigenfunctions. Here, we immediately obtain the following generalization to $\mathbb{R}^n$ for all n as well as to real analytic functions by applying the identity theorem for real analytic functions; see [25].

Lemma 1 (Vanishing planes)**.** *Let u be a real analytic function on $\mathbb{R}^n$. Assume that u vanishes on an open, nonempty subset of a hyperplane*

$$\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{M} \cdot \mathbf{x} = b\}.$$

Then u vanishes on all of $\mathcal{P}$.

We will also generalize Lamé's fundamental theorem, which was originally proved in two dimensions and for trigonometric functions, to n dimensions and real analytic eigenfunctions.

Theorem 4 (Lamé's Fundamental Theorem)**.** *Assume that u is a real analytic function on $\mathbb{R}^n$ that satisfies the Laplace eigenvalue equation with the Dirichlet boundary condition on a polytope $\Omega \in \wp_n$. Then u is anti-symmetric with respect to all $(n-1)$-dimensional hyperplanes on which u vanishes.*

Proof. Let λ be the eigenvalue corresponding to u, so that on Ω we have

$$\Delta u(\mathbf{x}) = \lambda u(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega.$$

Then, since u is real analytic, Δu is also real analytic on Ω. The function

$$\Delta u - \lambda u$$

is real analytic and vanishes on Ω which is an open subset of $\mathbb{R}^n$. Consequently, by Lemma 1 this function vanishes on all of $\mathbb{R}^n$, and therefore u satisfies the same Laplace eigenvalue equation on all of $\mathbb{R}^n$.

Now, let H be an $(n-1)$-dimensional hyperplane on which u vanishes. Let $\mathbf{v} \in \mathbb{R}^n$ be a normal vector to H of length one, such that $\mathbf{v}$ points away from the interior of Ω. Let

$$u(r, \mathbf{z}) := u(\mathbf{z} + r\mathbf{v}), \quad \text{for } \mathbf{z} \in H \text{ and } r \in \mathbb{R}.$$

The hyperplane H splits $\mathbb{R}^n$ into the disjoint union

$$\mathbb{R}^n \setminus H = \mathcal{R}_+ \cup \mathcal{R}_-, \quad \mathbb{R}^n = \mathcal{R}_+ \cup H \cup \mathcal{R}_-,$$

such that $\mathbf{v}$ points from $\mathcal{R}_+$ to $\mathcal{R}_-$.

We now define the function

$$\widetilde{u}(r,\mathbf{z}) := \begin{cases} u(r,\mathbf{z}), & (r,\mathbf{z}) \in \overline{\mathcal{R}_+}; \\ -u(-r,\mathbf{z}), & (r,\mathbf{z}) \in \mathcal{R}_-. \end{cases}$$

With this definition, $\widetilde{u}$ is anti-symmetric with respect to H. By the definition of u, there is an open, connected, nonempty subset $O \subset \mathbb{R}^n$ that contains an open, connected, nonempty subset of $\partial\Omega \subset H$, and such that

$$(\Delta - \lambda)(u - \widetilde{u}) = 0 \text{ on } O \setminus H, \quad u - \widetilde{u} = 0 \text{ on } O \cap \overline{\mathcal{R}_+},$$

and the normal derivatives

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\partial \widetilde{u}}{\partial \mathbf{v}} \text{ on } O \cap H.$$

Consequently, by standard uniqueness theory of partial differential equations [9, 10], $u = \widetilde{u}$ on O. It therefore follows that $\widetilde{u}$ is also real analytic on O. By the identity theorem for real analytic functions [25, Chapter 2], we obtain that $u = \widetilde{u}$ on $\mathbb{R}^n$. We therefore obtain that u, like $\widetilde{u}$, is anti-symmetric with respect to H. □

We are now poised to prove the first implication in Theorem 2.

Proposition 1. *Assume that Ω is a polytope in $\mathbb{R}^n$, and the first eigenfunction satisfies the first condition of Theorem 2. Then Ω strictly tessellates $\mathbb{R}^n$.*

Proof. Let Ω be a polytope in $\mathbb{R}^n$ as in the statement of the proposition. If $n = 1$, then Ω is a segment and may be written as (a, b) for some real numbers $a < b$. We have computed the eigenfunctions explicitly in this case. They are

$$u_k(x) = \sin\left(\frac{x-a}{b-a} k\pi\right).$$

The first eigenfunction in particular satisfies the hypotheses of Theorem 2, and we have also shown that all one-dimensional polytopes strictly tessellate $\mathbb{R}^1$. Hence the proposition is proved in one dimension. So let us assume that $n \geq 2$. By Lemma 1, for an affine hyperplane $\mathcal{P}$ that contains a boundary face of Ω, all eigenfunctions of Ω vanish on $\mathcal{P}$. Since the first eigenfunction never vanishes in the interior of Ω by Theorem 3, it follows that all of the hyperplanes that contain the boundary faces of Ω have empty intersection with the interior of Ω. In simpler terms, this means that the polytope Ω is convex.

Since the first eigenfunction, u_1, of Ω satisfies $(\Delta - \lambda_1)u_1 = 0$ on Ω which is an open, connected, nonempty subset of $\mathbb{R}^n$, and u_1 is real analytic, this equation is satisfied on all of $\mathbb{R}^n$. Consider a reflection of Ω across one of its boundary faces. By Theorem 4, u_1 is odd with respect to this reflection and therefore satisfies the Dirichlet boundary condition as well as the Laplace eigenvalue equation on the reflected copy of Ω. Consequently, by standard uniqueness theory [9, 10], the first eigenfunction on the reflected copy of Ω is equal to a scalar multiple of u_1. Moreover, since the first eigenfunction does not vanish inside the domain, we obtain that u_1 does not vanish in the reflected copy of Ω. We repeat this argument to cover $\mathbb{R}^n$ with copies of Ω obtained by repeated reflections across boundary faces. Since u_1 does not vanish inside any of the reflected copies of Ω, by Lemma 1 and Definition 2 the tessellation must be strict. $\square$

2.3 Root systems, alcoves, and strictly tessellating polytopes.

In 1980, Pierre Bérard showed that a certain type of bounded domain in $\mathbb{R}^n$, known as an *alcove,* always has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition. To define alcoves, we must first define root systems. The concept of a root system was originally introduced by Wilhelm Killing in 1888 [21, 22]. His motivation was to classify all simple Lie algebras over the field of complex numbers. In this section, we will see how our analytic problem, the study of the Laplace eigenvalue equation, is connected to these abstract algebraic concepts from Lie theory and representation theory.

Definition 4. *A* root system *in $\mathbb{R}^n$ is a finite set R of vectors that satisfy:*

1. *0 is not in R.*

2. *The vectors in R span $\mathbb{R}^n$.*

3. *For $\boldsymbol{v} \in R$, the only scalar multiples of $\boldsymbol{v}$ which also belong to R are $\pm\boldsymbol{v}$.*

4. *R is closed with respect to reflection across any hyperplane whose normal is an element of R, that is,*

$$\boldsymbol{v} - 2\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{||\boldsymbol{u}||^2}\boldsymbol{u} \in R, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in R;$$

5. *If $\boldsymbol{u}; \boldsymbol{v} \in R$, then the projection of $\boldsymbol{u}$ onto the line through $\boldsymbol{v}$ is an integer or half-integer multiple of $\boldsymbol{v}$. The mathematical formulation of this is that*

$$2\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{||\boldsymbol{v}||^2} \in \mathbb{Z}, \quad \textit{for all } \boldsymbol{u}, \boldsymbol{v} \in R.$$

The elements of a root system are often referred to as roots. *Four root systems in $\mathbb{R}^2$ are shown in Figure 2.6.*

Remark 4. *There are different variations of Definition 4 of a root system depending on the context. Sometimes only conditions 1–4 are used to define a root system. When the additional assumption 5 is included, then the root system is said to be* crystallographic. *In other contexts, condition 3 is omitted, and one would call a root system that satisfies condition 3* reduced.

We will need the dual root system to define the eigenvalues of the polytope that will be naturally associated with the root system.

Definition 5. *Let R be a root system. Then for $\boldsymbol{v} \in R$ the* coroot $\boldsymbol{v}^\vee$ *is defined to be*

$$\boldsymbol{v}^\vee = \frac{2}{||\boldsymbol{v}||^2}\boldsymbol{v}.$$

The set of coroots $R^\vee := \{\boldsymbol{v}^\vee\}_{v\in R}$. This is called the dual root system, *and may also be called the inverse root system. It is a straightforward exercise requiring only the definitions to prove that the dual root system is itself a root system.*

We associate a Weyl group to a root system. These Weyl groups are subgroups of the orthogonal group $O(n)$.

Definition 6. *For any root system $R \subset \mathbb{R}^n$ we associate a subgroup of the orthogonal group $O(n)$ known as its* Weyl group. *This is the subgroup $W < O(n)$ generated by the set of reflections by hyperplanes whose normal vectors are elements of R. For $\boldsymbol{v} \in R$ reflection across the hyperplane with normal vector equal to $\boldsymbol{v}$ is explicitly*

$$\sigma_{\boldsymbol{v}} : \mathbb{R}^n \to \mathbb{R}^n, \quad \sigma_{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{x} - 2\frac{(\boldsymbol{v}\cdot\boldsymbol{x})}{||\boldsymbol{v}||^2}\boldsymbol{v}.$$

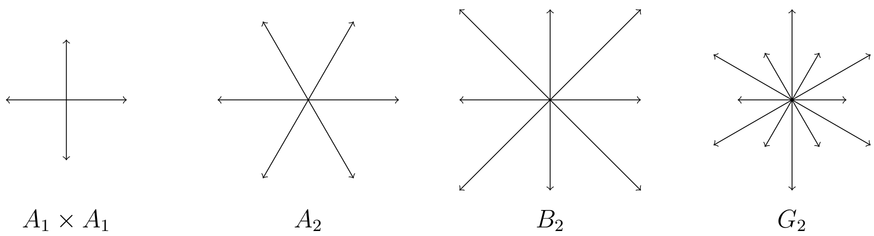

Figure 2.6: Here are four root systems in $\mathbb{R}^2$. Below each root system is the name of its Weyl group. The name of the Weyl group may also be used as the name of the root system.

By the definition of a root system, the associated Weyl group is finite. To explain what was proved in [2] by Bérard, we require the notion of Weyl chamber.

Definition 7. *For a root system $R \subset \mathbb{R}^n$ for each $\boldsymbol{v} \in R$, let $H_{\boldsymbol{v}}$ denote the hyperplane that contains the origin and whose normal vector is $\boldsymbol{v}$. In particular,*

$$H_{\boldsymbol{v}} := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}\cdot\boldsymbol{v} = 0\}.$$

Let $\mathcal{H} = \{H_{\boldsymbol{v}}\}_{\boldsymbol{v}\in R}$. Then $\mathbb{R}^n \setminus (\cup_{H\in\mathcal{H}} H)$ is disconnected, and each connected open component is known as a Weyl chamber. *A Weyl chamber of the Weyl group A_2 is shown in Figure 2.7.*

Definition 8. *Let R be a root system. Denote by $H_{\boldsymbol{v}}$ the hyperplane in $\mathbb{R}^n$ that contains the origin and whose normal vector is equal to $\boldsymbol{v}$ for $\boldsymbol{v} \in R$. Let*

$$H_{\boldsymbol{v},k} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{v}\cdot\boldsymbol{x} = k\},$$

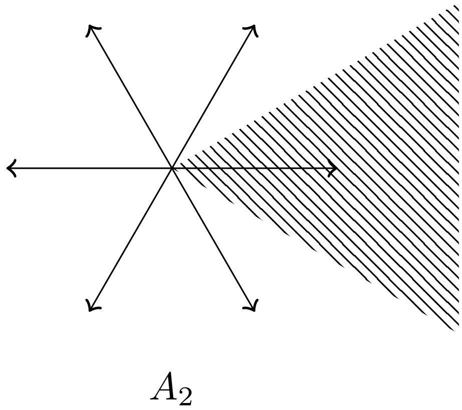

Figure 2.7: Extending the shaded area to infinity shows a Weyl chamber of the Weyl group A_2.

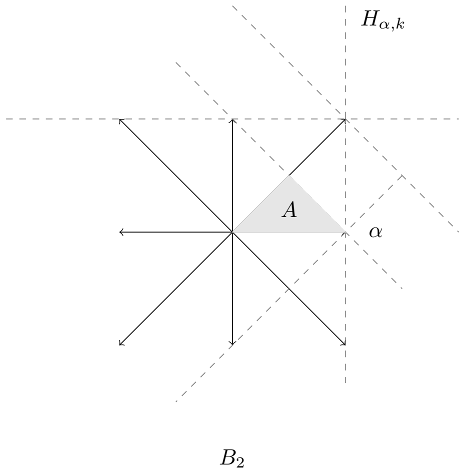

Figure 2.8: This shows an alcove, A, corresponding to the root system with Weyl group B_2. For $\alpha \in B_2$, the hyperplanes $H_{\alpha,k}$ for $k \in \mathbb{Z}$ are the parallel hyperplanes which have normal vector equal to α. Note that A is an isosceles right triangle.

for $k \in \mathbb{Z}$. Then $H_{\boldsymbol{v},0} = H_{\boldsymbol{v}}$. For $k \neq 0$, the hyperplane $H_{\boldsymbol{v},k}$ is parallel to $H_{\boldsymbol{v}}$. We define an alcove *to be a connected component of*

$$\mathbb{R}^n \setminus \left\{ \bigcup_{\boldsymbol{v} \in R, k \in \mathbb{Z}} H_{\boldsymbol{v},k} \right\}.$$

We note that the definition of an alcove immediately implies that it is a polytope in $\mathbb{R}^n$. An example of an alcove is shown in Figure 2.8.

Proposition 2. *[2, Proposition 9, p. 181] Let $\Omega \subset \mathbb{R}^n$ be an alcove. Then Ω has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition.*

For readers who understand French and read [2], you may notice that the statement of Proposition 2 is not the English translation of [2, Proposition 9, p. 181]. Bérard proved a stronger result; he specified the eigenvalues and corresponding eigenfunctions. To understand what Bérard proved, let R be a root system. Let $C(R)$ denote a Weyl chamber, and let $D(R)$ denote an alcove that is contained in the Weyl chamber $C(R)$. Consider the dual root system $R^\vee$. The vertices of the closures of the alcoves associated to $R^\vee$ create a lattice. Let us denote this lattice by Γ. The dual lattice is

$$\Gamma^* := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \boldsymbol{\gamma} \in \mathbb{Z}, \quad \forall \boldsymbol{\gamma} \in \Gamma\}.$$

Bérard referred to the points contained in this dual lattice as "the group of weights of R" ("le groupe des poids de R") [2]. He proved that the eigenvalues for the alcove

$D(R)$ are given by

$$\{4\pi^2||\mathbf{q}||^2 : \mathbf{q} \in \Gamma^* \cap C(R)\}.$$

The multiplicity of the eigenvalue $\lambda = 4\pi^2||\mathbf{q}||^2$ is equal to the number of vectors $\mathbf{q} \in \Gamma^* \cap C(R)$ that satisfy $\lambda = 4\pi^2||\mathbf{q}||^2$. The eigenfunctions are certain linear combinations of $e^{2\pi i \mathbf{x}\cdot w(\mathbf{q})}$, where $w(\mathbf{q})$ is in the affine Weyl group of R. The affine Weyl group of R is the semi-direct product of the Weyl group and the lattice Γ. Combining our Proposition 1 with Bérard's Proposition 2, we obtain the following corollary which states that every alcove is a strictly tessellating polytope.

Corollary 2. *Let $\Omega \subset \mathbb{R}^n$ be an alcove. Then Ω is a polytope that strictly tessellates $\mathbb{R}^n$.*

In the following proposition we prove the converse: every strictly tessellating polytope is an alcove of a root system.

Proposition 3. *Let $\Omega \subset \mathbb{R}^n$ be a polytope that strictly tessellates $\mathbb{R}^n$. Then Ω is an alcove.*

Proof. We will build a root system, R, using the fact that Ω strictly tessellates space. The tessellation defines hyperplanes in $\mathbb{R}^n$ that contain the boundary faces of the copies of Ω in the tessellation. Assume that Ω has m boundary faces. By the definition of strict tessellation, there is a discrete set of vectors

$$\{v_{j,k}\}_{j\in\mathbb{Z},1\leq k\leq m},$$

where $v_{j,k}$ is a unit normal vector to the hyperplane containing the kth boundary face of Ω_j. We first define $\mathcal{R}$ to be the set that contains each distinct $v_{j,k}$ together with its opposite $-v_{j,k}$. Since Ω is a bounded, connected, open set with boundary consisting of flat faces, the set of vectors $\mathcal{R}$ defined in this way spans $\mathbb{R}^n$. To see this, we observe that if this were not the case, then Ω would be contained in a k-dimensional hyperplane in $\mathbb{R}^n$ and thus would not an open set in $\mathbb{R}^n$. By definition, we note that $0 \notin \mathcal{R}$. By Definition 2 the set of vectors $\mathcal{R}$ is finite.

Since $\mathcal{R}$ is a finite set, and there are countably many hyperplanes defined by the tessellation, this means that for each $v \in \mathcal{R}$, there are countably infinitely many hyperplanes whose normal direction is $\pm v$. Fix some $v \in \mathcal{R}$, and by possibly moving the entire picture, assume that there is a hyperplane $H_{v,0}$ with normal direction $\pm v$ that contains the origin. Let the closest parallel hyperplane to $H_{v,0}$ in the direction of v be $H_{v,1}$. We repeat this process for each $v \in \mathcal{R}$ and then define

$$R := \left\{\mathbf{v} := \frac{v}{\mathrm{dist}(0, H_{v,1})}\right\}_{v\in\mathcal{R}}.$$

We therefore have

$$H_{v,1} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\cdot\mathbf{v} = 1.\}$$

Since the tessellation is unchanged by reflection in the direction of $\pm v$, the distance between adjacent parallel hyperplanes with normal vector $\pm v$ is equal to

$\mathrm{dist}(0, H_{v,1})$. Consequently, we may enumerate the parallel hyperplanes as

$$H_{v,j} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = j\}, \quad j \in \mathbb{Z}.$$

A schematic image is given in Figure 2.9. For ease of notation, let us define $H_{\mathbf{v},k} := H_{v,k}$.

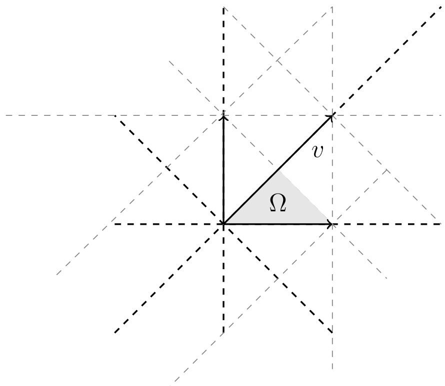

Figure 2.9: Given a polytope Ω, we construct the hyperplanes $H_{\mathbf{v},0}$, here in the thicker black dotted lines, and the normal vectors $\mathbf{v}$. The set $\{H_{\mathbf{v},k}\}$ includes the thinner gray dotted lines.

Let $\mathbf{w} \in R$. By possibly translating the entire picture, assume that there is a hyperplane in the tessellation with normal direction $\pm\mathbf{w}$ and that contains the origin, such that the origin is a vertex of a copy of Ω in the tessellation. Thus $H_{\mathbf{w},0}$ is a hyperplane in the tessellation. Consider the reflection with normal direction $\mathbf{v}$, denoted by $\sigma_{\mathbf{v}}$, that is,

$$\sigma_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} - 2\frac{\mathbf{x} \cdot \mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}.$$

Then $\sigma_{\mathbf{v}}(0) = 0$. Consequently, $\sigma_{\mathbf{v}}(H_{\mathbf{w},0})$ is another hyperplane in the strict tessellation which also contains the origin: thus it is $H_{\mathbf{u},0}$ for some $\mathbf{u} \in R$. Similarly, we also have $\sigma_{\mathbf{v}}(H_{\mathbf{w},1}) = H_{\mathbf{u},j}$ for some $j \in \mathbb{Z}$. Since $\sigma_{\mathbf{v}}$ preserves the scalar product, for $\mathbf{x} \in H_{\mathbf{w},1}$, by definition we have

$$\mathbf{x} \cdot \mathbf{w} = 1 \implies \sigma_{\mathbf{v}}(\mathbf{x}) \cdot \sigma_{\mathbf{v}}(\mathbf{w}) = 1.$$

Since $\sigma_{\mathbf{v}}$ sends $\mathbf{x}$ to a point in $H_{\mathbf{u},j}$ we also have

$$\sigma_{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{u} = j.$$

Since $\sigma_{\mathbf{v}}(H_{\mathbf{w},0}) = H_{\mathbf{u},0}$, we must have that $\sigma_{\mathbf{v}}(\mathbf{w}) = \alpha\mathbf{u}$ for some $\alpha \in \mathbb{R}$. Therefore, combining with the above, we obtain

$$1 = \mathbf{x} \cdot \mathbf{w} = \sigma_{\mathbf{v}}(\mathbf{x}) \cdot \sigma_{\mathbf{v}}(\mathbf{w}) = \alpha\sigma_{\mathbf{v}}(\mathbf{x}) \cdot \mathbf{u} = \alpha j \implies \alpha = \frac{1}{j}.$$

So we have proved that

$$\sigma_{\mathbf{v}}(\mathbf{w}) = \frac{1}{j}\mathbf{u}.$$

The vector $\mathbf{y_w} := \mathrm{dist}(0, H_{\mathbf{w},1})\frac{\mathbf{w}}{||\mathbf{w}||} = \frac{\mathbf{w}}{||\mathbf{w}||^2}$ is orthogonal to the hyperplanes $H_{\mathbf{w},0}$ and $H_{\mathbf{w},1}$ and connects the origin to the nearest point in $H_{\mathbf{w},1}$. When this vector is reflected by $\sigma_{\mathbf{v}}$, it will again start from the origin and have its endpoint lying on one of the parallel hyperplanes, by virtue of the strict tessellation. Let us define the vector $\mathbf{y_v}$ in the analogous way. We compute explicitly that

$$\sigma_{\mathbf{v}}(\mathbf{y_w}) = \mathbf{y_w} - 2\mathbf{y_w} \cdot \mathbf{v}\frac{\mathbf{v}}{||\mathbf{v}||^2} = \mathbf{y_w} - 2(\mathbf{y_w} \cdot \mathbf{v})\mathbf{y_v}.$$

On the other hand, since $\sigma_{\mathbf{v}}(\mathbf{w}) = \frac{1}{j}\mathbf{u}$, we compute that

$$\sigma_{\mathbf{v}}(\mathbf{y_w}) = \sigma_{\mathbf{v}}\left(\frac{\mathbf{w}}{||\mathbf{w}||^2}\right) = \frac{1}{||\mathbf{w}||^2}\sigma_{\mathbf{v}}(\mathbf{w}) = \frac{1}{||\mathbf{w}||^2}\frac{1}{j}\mathbf{u}.$$

Now, since $||\mathbf{u}||^2 = j^2||\mathbf{w}||^2$, we have $\sigma_{\mathbf{v}}(\mathbf{y_w}) = j\left(\frac{\mathbf{u}}{||\mathbf{w}||^2 j^2}\right) = j\mathbf{y_u}$. Combining these calculations, we obtain

$$\sigma_{\mathbf{v}}(\mathbf{y_w}) = \mathbf{y_w} - 2(\mathbf{y_w} \cdot \mathbf{v})\mathbf{y_v} = j\mathbf{y_u} \implies 2(\mathbf{y_w} \cdot \mathbf{v})\mathbf{y_v} = \mathbf{y_w} - j\mathbf{y_u}.$$

The vector $\mathbf{y_w}$ goes from the origin to $H_{\mathbf{w},1}$, while the vector $-j\mathbf{y_u}$ goes from the origin to $H_{\mathbf{u},-j}$. By vector addition and the strict tessellation, the sum $\mathbf{y_w} - j\mathbf{y_u}$ must go from the origin and end precisely at one of the parallel hyperplanes. Consequently, the vector

$$2(\mathbf{y_w} \cdot \mathbf{v})\mathbf{y_v}$$

must be an integer multiple of $\mathbf{y_v}$ because it goes from the origin in the direction of $\mathbf{y_v}$ and lands at one of the parallel hyperplanes $H_{\mathbf{v},k}$ for some $k \in \mathbb{Z}$. Therefore,

$$2(\mathbf{y_w} \cdot \mathbf{v}) = k \in \mathbb{Z}.$$

By the definitions of $\mathbf{y_w}$ and $\mathbf{v}$,

$$2(\mathbf{y_w} \cdot \mathbf{v}) = 2\frac{\mathbf{w} \cdot \mathbf{v}}{||\mathbf{w}||^2} = k \in \mathbb{Z}.$$

In a similar way, reversing the roles of $\mathbf{w}$ and $\mathbf{v}$, we also obtain

$$2\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||^2} \in \mathbb{Z}.$$

Since $\mathbf{w}, \mathbf{v} \in R$ were arbitrary, this shows the final condition needed for R to be a root system in Definition 4 is satisfied. We conclude that R is a root system and that Ω is one of its alcoves. $\square$

The proofs of Theorem 2 and Corollary 1 will now follow from Propositions 1 and 3 and Bérard's Proposition 2.

2.3.0.1 Proof of Theorem 2

By Proposition 1, if Ω is a polytope, and its first eigenfunction is real analytic on $\mathbb{R}^n$, then Ω strictly tessellates $\mathbb{R}^n$. By Proposition 3, if Ω is a polytope that strictly tessellates $\mathbb{R}^n$, then Ω is an alcove. By Bérard's Proposition 2, if Ω is an alcove, then all its eigenfunctions are trigonometric. We have therefore proved that the statements in Theorem 2 satisfy: $1 \implies 2 \implies 3 \implies 1$. $\square$

2.3.0.2 Proof of Corollary 1

If the first eigenfunction of a polytope in $\mathbb{R}^n$ satisfies the hypotheses of Theorem 2, then the polytope is an alcove. By Bérard's Proposition 2, all of the eigenfunctions of the polytope are trigonometric. $\square$

2.4 Concluding remarks and conjectures.

We have now answered the analysis question: When does a polytope in $\mathbb{R}^n$ have a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation? In geometric terms, the necessary and sufficient condition for a polytope to have a complete set of trigonometric eigenfunctions is that the polytope strictly tessellates $\mathbb{R}^n$. In algebraic terms, in the language of Bourbaki, the equivalent necessary and sufficient condition is that the polytope is congruent to a fundamental domain of a crystallographic Coxeter group [2, p. 179], [5, VI.25, Proposition 9, p. 180]. Returning to the analysis problem, it is interesting to note that it is enough to know that the first eigenfunction is real analytic and satisfies the Laplace eigenvalue equation on $\mathbb{R}^n$ to conclude that it is a trigonometric function and moreover, *all* the eigenfunctions are trigonometric. This is a remarkable fact. Moreover, the equivalence of analytic, geometric, and algebraic statements shows that these different areas of mathematics are intimately connected. The Fuglede conjecture similarly brings together different areas of mathematics in the study of a single question.

2.4.1 The Fuglede Conjecture.

To state the Fuglede conjecture, we introduce a few concepts.

Definition 9. *A domain* $\Omega \subset \mathbb{R}^d$ *is said to be a* spectral set *if there exists* $\Lambda \subset \mathbb{R}^n$ *such that the functions*

$$\{e^{2\pi i \lambda \cdot \boldsymbol{x}}\}_{\lambda \in \Lambda}$$

are an orthogonal basis for $\mathcal{L}^2(\Omega)$. *The set* Λ *is then said to be* a spectrum of Ω, *and* (Ω, Λ) *is called* a spectral pair.

To relate these notions to our work here, we observe that if a domain Ω were to have all its eigenfunctions for the Laplace eigenvalue equation of the form $e^{2\pi i \lambda \cdot \mathbf{x}}$, then these functions would comprise an orthogonal basis for $\mathcal{L}^2(\Omega)$. Consequently, knowing that the eigenfunctions are precisely of this form implies that the domain is a spectral set. However, the converse is not true, in the sense that if Ω is a spectral set, then its eigenfunctions are not necessarily individual complex exponential functions. If Ω is a spectral set, then the eigenfunctions must be linear combinations of the $e^{2\pi i \lambda \cdot \mathbf{x}}$, since these are a basis for $\mathcal{L}^2(\Omega)$. However, the linear combinations could have countably infinitely many terms, so it is not clear what precise form the eigenfunctions will take.

Conjecture 1 (Fuglede [16])**.** *Every domain of* $\mathbb{R}^n$ *which has positive Lebesgue measure is a spectral set if and only if it tiles* $\mathbb{R}^n$ *by translation.*

Fuglede proved in 1974 that the conjecture holds if one assumes that the domain is the fundamental domain of a lattice [16]. Only several years later, in 2003, was further progress made by Iosevich, Katz, and Tao [19] who proved that the Fuglede conjecture is true if one restricts to convex planar domains. In the following year, Tao proved that the Fuglede conjecture is false in dimension 5 and higher [37]. In 2006, the works of Farkas, Kolounzakis, Matolcsi and Mora [11, 23, 24, 29] proved that the conjecture is also false for dimensions 3 and 4. In 2017, Greenfeld and Lev proved that Fuglede's conjecture is true if one restricts attention to domains that are convex polytopes, but only in $\mathbb{R}^3$ [17]. In 2019, Lev and Matolcsi proved that Fuglede's conjecture is true if one restricts attention to convex domains, in any dimension [27]. Interestingly, the Fuglede conjecture is still an open problem for arbitrary domains in dimensions one and two. Here we make the following conjecture which is related to yet independent from Fuglede's.

Conjecture 2. *Let Ω be a domain in $\mathbb{R}^n$. Then Ω has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition if and only if Ω is a polytope that strictly tessellates $\mathbb{R}^n$. Equivalently, Ω has a complete set of trigonometric eigenfunctions for the Laplace eigenvalue equation with the Dirichlet boundary condition if and only if Ω is an alcove.*

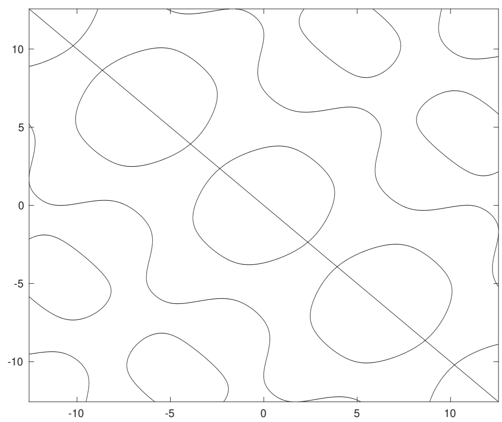

Figure 2.10: This figure shows the null set of the function $u(x, y) = \sin(x) + \sin(y) + \sin((x + y)/\sqrt{2})$ in a square-shaped region of $\mathbb{R}^2$. The null set includes the line $y = -x$ as well as the other curves in the region. Consequently, by uniqueness, this function is the first eigenfunction of the connected, open domains that are bounded by these curves, since it vanishes on the boundary but not on the interior and satisfies the Laplace eigenvalue equation. Hence the first eigenfunction satisfies the first condition of Theorem 2, but we do not obtain any further conclusions because the domain is not a polytope.

The difficulty in treating arbitrary domains is that we do not have a replacement for Lamé's results which are central to our proof. Moreover, it is possible to construct linear combinations of trigonometric functions which vanish on curved regions; an example is given in Figure 2.10. Consequently, we cannot immediately conclude that domains which have trigonometric eigenfunctions have flat boundary faces, and hence they are polytopes. A domain with a curved boundary could have a few trigonometric eigenfunctions. What is reasonable to expect, however, is that it does not have a *complete* set of trigonometric eigenfunctions.

2.4.2 The crystallographic restriction theorem and a geometric approach to the Goldbach conjecture.

The vertices of the strict tessellation given by a polytope that is an alcove are in fact the set of points in a full-rank lattice. We note that two different polytopes may give rise to the same lattice; for example, an isosceles right triangle and the square obtained by two copies of that triangle will produce the same lattice. For any discrete group of isometries of $\mathbb{R}^n$, an element g in such a group has finite order if there is an integer $k > 0$ such that g composed with itself k times is the identity. The minimal such k is the order of g. To state the crystallographic restriction theorem, we define a function which is like an extension of the Euler totient function. For an odd prime p and $r \geq 1$,

$$\psi(p^r) := \phi(p^r), \quad \phi(p^r) = p^r - p^{r-1}.$$

Here ϕ denotes the Euler totient function. The Euler totient function of a positive integer n counts the positive integers that are relatively prime to, and at most n. So, for example, for an odd prime p, the positive integers that are not relatively prime to p^r are p, $2p$, $3p$, $\ldots$,$p^{r-1}p = p^r$. There are p^{r-1} of these. All other positive integers are relatively prime to p^r, hence $\phi(p^r) = p^r - p^{r-1}$. The function ψ is further defined as follows:

$$\psi(1) = \psi(2) = 0, \; \psi(2^r) := \phi(2^r) \text{ for } r > 1,$$

and

$$\text{for } m = \prod_i p_i^{r_i}, \quad \psi(m) := \sum_i \psi(p_i^{r_i}).$$

Theorem 5 (Crystallographic Restriction I)**.** *For any discrete group G of isometries of $\mathbb{R}^n$, for $n \geq 2$ the set of orders of the elements G that have finite order is equal to*

$$\mathrm{Ord}_n = \{m \in \mathbb{N} : \psi(m) \leq n\}.$$

The crystallographic restriction theorem is connected to the mathematics of crystals when we reformulate the theorem in the context of lattices. A full-rank lattice is a set of points in $\mathbb{R}^n$ of the form

$$\Gamma = \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} = L\mathbf{x}, \quad L \in \mathrm{GL}(n, \mathbb{R}), \quad \mathbf{x} \in \mathbb{Z}^n\}. \tag{2.1}$$

Here $\mathrm{GL}(n, \mathbb{R})$ is the set of $n \times n$ invertible matrices with real entries, and $\mathbb{Z}^n$ are the elements of $\mathbb{R}^n$ whose entries are integers. We say that the matrix L generates the lattice Γ. The generating matrix L is not unique, because for any $M \in \mathrm{GL}(n, \mathbb{Z})$ the set of points in (2.1) is equal to

$$\{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} = LM\mathbf{x}, \quad \mathbf{x} \in \mathbb{Z}^n\}.$$

Here $\mathrm{GL}(n,\mathbb{Z})$ is the group of invertible $n \times n$ matrices whose entries are integers. Note that to be a group, this requires the determinant of all elements of $\mathrm{GL}(n,\mathbb{Z})$ to be equal to ± 1. Two matrices $L_1, L_2 \in \mathrm{GL}(n,\mathbb{R})$ generate the same lattice if and only if there is an $M \in \mathrm{GL}(n,\mathbb{Z})$ such that $L_1 = L_2 M$. For a matrix $M \in \mathrm{GL}(n,\mathbb{Z})$, we identify it with the isometry of $\mathbb{R}^n$ that maps $\mathbf{x} \in \mathbb{R}^n$ to $M\mathbf{x}$. The matrices in $\mathrm{GL}(n,\mathbb{Z})$ can therefore be identified with the group of symmetries of the crystal whose atoms lie on the points of the lattice. Hence, the order of M is equal to the smallest positive integer k such that M^k is the identity matrix. It turns out that the set of orders of the elements of any discrete group G of isometries of $\mathbb{R}^n$ that have finite order is equal to the set of orders of the elements of $\mathrm{GL}(n,\mathbb{Z})$. Consequently, the crystallographic restriction may be reformulated as follows.

Theorem 6 (Crystallographic Restriction II). *For any $n \geq 2$, the set of orders of the elements of $\mathrm{GL}(n,\mathbb{Z})$ is equal to*

$$\mathrm{Ord}_n = \{m \in \mathbb{N} : \psi(m) \leq n\}.$$

In [1], Bamberg, Cairns, and Kilminster proved that one may reformulate the strong Goldbach conjecture in terms of the orders of elements of $\mathrm{GL}(n,\mathbb{Z})$.

Conjecture 3 (Strong Goldbach). *Every even natural number greater than six can be written as the sum of two distinct odd primes.*

Theorem 7 (Theorem 3 of [1]). *The following statements are equivalent:*

1. *The strong Goldbach conjecture is true;*
2. *For each even $n \geq 6$ there is a matrix $M \in \mathrm{GL}(n,\mathbb{Z})$ which has order pq for distinct primes p and q, and there is no matrix in $\mathrm{GL}(k,\mathbb{Z})$ of order pq for any $k < n$.*

The Goldbach conjecture is an extremely difficult problem. Difficult, long-standing open problems have sometimes been solved by translating the problem into a different field of mathematics. The proof of Fermat's last theorem, also a statement in number theory, was achieved using newly-developed techniques in algebraic geometry [38, 39]. To approach the Goldbach conjecture geometrically, we ask

Question 1. Is there a geometric reason for the existence of a symmetry for full-rank lattices in $\mathbb{R}^n$, with $n \geq 6$ an even number, such that this symmetry is of order pq for two odd primes $p \neq q$ such that $p + q = n + 2$?

The condition that there is no matrix in $\mathrm{GL}(k,\mathbb{Z})$ of order pq for any $k < n$ is equivalent to requiring $p + q = n + 2$. This follows from Theorem 6, which states that the orders of the elements of $\mathrm{GL}(k,\mathbb{Z})$ are equal to the set of nonnegative integers m with $\psi(m) \leq k$. In order to guarantee that

$$\psi(pq) = p + q - 2 > k \text{ for all } k < n, \text{ but } \psi(pq) \leq n,$$

we must have $\psi(pq) = p + q - 2 = n$.

Consequently, the symmetry of order pq would correspond to a matrix $M \in \mathrm{GL}(n, \mathbb{Z})$ that does not admit a diagonal decomposition into two matrices of smaller dimensions. Geometrically, this matrix would not arise as a product of symmetries of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ for any $k = 1, \ldots, n-1$. It would be a new symmetry occurring first in $\mathbb{R}^n$. Since [1] already realized the connection between the Goldbach conjecture and the crystallographic restriction theorem, this geometric approach would seem unlikely to lead to any new developments. Nonetheless, it is interesting that a famous number-theoretic conjecture can be equivalently phrased as a simple question about the orders of symmetries of full-rank lattices in $\mathbb{R}^n$.

3

Further Results

Theorem 2 in chapter 2 shows a connection between analysis, algebra, and geometry. We can see that the properties of strictly tessellating polytopes can be used in each field. It is a motivation to study further whether a polytope is strictly tessellating or not. In this chapter we prove geometric characteristics for strictly tessellating polytopes.

Knowing that there are four possible interior angles for strictly tessellating polygons, we show that the property that angles between 1-dimensional planes can only assume 4 distinct values is preserved in higher dimensions. In order to do this, we show that the k-dimensional faces inherit the property of being strictly tessellting. This is then used to show an upper bound on the number of vertices a strictly tessellating polytope can have. Moreover, we prove a result for the solid angles of strictly tessellating polytopes. Consequently, we show an upper and lower bound on the number of strictly tessellating polytopes $\mathbb{R}^n$ up to equivalence. To recap quickly; in chapter 2 it was shown that strictly tessellating polytopes have the property that their tessellation is obtained by reflection in the faces of the polytopes. Furthermore, the planes that contain the polytopes making up the boundary do not intersect the interior of the tessellation – i.e. the union of interiors of the tessellated copies of the polytope.

Terminology 1. *We introduce some terminology for polytopes. For a polytope P_n we have that*

1. *An* $(n-1)$-face *of P_n is an $(n-1)$-dimensional polytope F contained in the boundary of P_n which is maximal in the sense that if F' such that $F \subseteq F'$ is another $(n-1)$-dimensional polytope contained in the boundary, then $F' = F$.*

2. *For $k = 0, \dots, n-2$ we inductively define a* k-face *F of P_n as the k-dimensional polytope that is the intersection of two $(k+1)$-faces. Here F must also be in the boundary of P_n. Maximality is inherited from 1 as both intersecting polytopes are maximal.*

3. *The* extension *of a k-face F of P_n is the k-dimensional plane* Π *containing F.*

Note that the plane Π is unique since if two planes Π_1, Π_2 are extensions of F, then $F \subset \Pi_1 \cap \Pi_2$. Therefore, $\Pi_1 = \Pi_1 \cap \Pi_2 = \Pi_2$ since Π_1, Π_2 and F all have the same dimension.

This is useful terminology for strict tessellations, as the extension of each $(n-1)$-face of a strictly tessellating polytope must not intersect the interior of the tessellation. Furthermore, the $(k-1)$-faces of a polytope are intersections of k-faces, or in other words, they connect k-faces to each other. This gives a recursive relation between the k-faces for $k = 0, 1, \ldots, n-1$.

Remark 5. *If a k-face F has the extension of a $(k-1)$-face intersecting its interior, then there is an extension of another k-face $\hat{F} \neq F$ intersecting it. If we look at this for $k = n-1$, it would mean that the extension of $\hat{F}$ cuts through the strict tessellation, but this is not allowed by definition. This motivates a theorem regarding the $(n-1)$-faces that constitute the boundary of the polytope.*

3.1 A strictly tessellating polytope and its faces

Theorem 8. *Let $P_n \subset \mathbb{R}^n$ be an n-dimensional strictly tessellating polytope, then its boundary ∂P_n consists of strictly tessellating polytopes P_{n-1}^k of dimension $n-1$. In other words, $\partial P_n = \cup_{k \in K} P_{n-1}^k$, where K is the set of indices for its $(n-1)$-faces.*

The proof will follow from the ideas presented in remark 5. We will use the fact that the intersection of two k-dimensional planes in $\mathbb{R}^n$ is a $(k-1)$-dimensional plane, given that the two intersecting planes are not identical or have empty intersection. In the proof we work with a polytope of dimension n in $\mathbb{R}^n$.

Proof. Let $P_n \subset \mathbb{R}^n$ be a strictly tessellating polytope. Take a face $F \subset \partial P_n$ of P_n. F is a polytope since P_n is a polytope. The extension F_e of F will contain the $(n-1)$-faces of the tessellated copies of P_n, since the tessellation of P_n fills $\mathbb{R}^n$, and since F_e cannot intersect the interior of the tessellation by definition it must only intersect the exterior. The exterior is the union of boundaries of the tessellated copies of P_n. This must itself be a tessellation that we call τ, because if there is a gap there is no $(n-1)$-face present there, and therefore the tessellation of P_n has a gap, which would be a contradiction. We will go through the three conditions of definition 2 and show that τ is a strict tessellation.

We want to show that no extensions of $(n-2)$-faces in τ may intersect the interior of τ. Assume that there exists an $(n-2)$-face E with extension E_e that cuts through the interior of τ. Then there exists an $(n-1)$-face $\hat{F}$ – not equal to F – with extension $\hat{F}_e$ such that $E_e = F_e \cap \hat{F}_e$. However, $(n-1)$-face extensions may only intersect with the exterior of the tessellation of P_n, and thus it cannot intersect the interior of an $(n-1)$-face, but it does so since E_e does.

Now, we want to show that reflections around the $(n-2)$-faces preserve τ. Take the $(n-1)$-face F^0 and it's extension F_e^0. This time, let E and E_e be an $(n-2)$-face and its extension, both in F_e^0. These do not intersect the interior of τ. There exist $(n-1)$-faces F^k such that $E_e = \bigcap_{k \in K} \left(F_e^k \cap F_e \right)$, where F_e^k is the extension of F^k.

K is the set of indices for the $(n-1)$-faces satisfying this condition. This is simply a technical way of describing all the $(n-1)$-faces intersecting F_e in E_e.

By definition, reflections around F_e and F_e^k preserve the tessellation. Take N^k to be the normal of F_e^k. We claim that the set $\mathcal{N} := \left(N^k\right)_{k\in K} \cup \left(-N^k\right)_{k\in K}$ is *symmetrically* distributed in $\mathbb{R}^n$, i.e. that for every vector pair $n_1, n_2 \in \mathcal{N},\ n_1 \neq n_2$ there exists an $n_3 \in \mathcal{N},\ n_3 \neq -n_2$ such that $n_1 \parallel (n_2 + n_3)$. This follows from the fact that P_n has an associated root system, which coincides with our $\mathcal{N}$ by definition of the fundamental chamber. Thus we can always find a vector w such that $w \perp E_e$ and $w \parallel F_e^0$. By symmetry, if we pick the point in F_e^0 we can thus reflect it using this new vector, which will serve as the normal of E_e.

Finally, we see that τ consists of reflected copies of F, due to the reflection symmetry. If τ is symmetric in every $(n-1)$-face, then we can tile F_e with F by repeated reflection.

□

We have thus established that the $(n-1)$-faces of strictly tessellating polytopes are strictly tessellating. This greatly limits what a strictly tessellating polytope can look like. If used recursively, we see that also the $(n-2)$-faces are strictly tessellating as well, and so forth to dimension 0. Interestingly, this means that once we come down to dimension 2, it says that the 2-dimensional faces of any strictly tessellating polytope can only be one of the strictly tessellating polygons.

Corollary 3. *All k-faces of a strictly tessellating polytope P_n are strictly tessellating for $k = 1, 2, \ldots, n-1$.*

Corollary 4. *The angles between 1-faces (i.e. line segments) of P_n are either $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$.*

Proof. By corollary 3, the faces of dimension 2 are strictly tessellating. Thus by McCartin's results [30] we have that the angles between the 1-faces of these 2-faces are exactly those given in the statement. □

While practically unfeasible, we could try to build all the different strictly tessellating polytopes using the lower-dimensional building blocks. Still, this is interesting in itself and useful for our other results.

3.2 Bound on vertices

We present a simple bound on the maximum number of vertices in $\mathbb{R}^n$. This result can be deduced from the fact that a polytope with more than 2^n vertices will have at least one obtuse angle. This was conjectured by Erdös around 1950, and later proven in 1962 by L. Danzer and B. Grünbaum in [40]. We use this in conjunction with corollary 4 – implying that the angles between 1-faces can at most be $\frac{\pi}{2}$.

Proposition 4. *The maximum number of vertices in a strictly tessellating polytope in $\mathbb{R}^n$ is 2^n.*

Proof. By the results of L. Danzer and B. Grünbaum in [40] we have that a polytope of dimension n with more than 2^n vertices will at least have one obtuse angle between its 1-faces. By corollary 4 we cannot have angles larger than $\frac{\pi}{2}$, and thus a polytope with more than 2^n vertices cannot be strictly tessellating. Furthermore, we want to show that there exists an n-dimensional strictly tessellating polytope with 2^n. To do this, let $P_n = [0,1]^n$ be the unit n-cube. P_n trivially strictly tessellates $\mathbb{R}^n$ and has 2^n vertices. □

3.3 Necessary criteria for solid angles of a strictly tessellating polytope

To recap, a solid angle at a vertex v of a polytope P is the $(n-1)$-volume of the intersection between the polytope and an ε-sphere centered in v, i.e. $\angle v := \text{Area}\,(P \cap \partial B(v,\varepsilon)) \cdot \varepsilon^{-(n-1)}$ where $\varepsilon > 0$. Normally, one would take the limit when $\varepsilon \to 0$, but since we are working with polytopes, the *corner* can be seen as a frustrum, and thus a sufficiently small ε will yield the limit value. A *full* solid angle, being the surface area of an n-sphere, is

$$A_n := \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{3.1}$$

We see that in two dimensions we have $A_2 = 2\pi$, so the definition is consistent with "regular angles".

So far, only angles between 1-faces have been mentioned. However, the solid angles of a strictly tessellating polytope are also of importance. Since a strictly tessellating polytope is tessellated by reflection, if we take a vertex v in the tessellation the *corners* adjacent to v are the same. A corner is simply a solid angle of dimension n if our polytope is n-dimensional. If consider an n-ball centered in v, the corners must fill up the ball, or there will be a gap in the tessellation. Furthermore, the corners may not overlap, because there may not be overlaps in a tessellation. Thus, the solid angle at v will be on the form $\frac{A_n}{m}$, where $m \in \mathbb{N}$. We formulate this as a theorem.

Theorem 9. *The solid angles $\angle v$, $v \in \text{vert}(P_n)$ of a strictly tessellating polytope P_n of dimension n are of the form*

$$\angle v = \frac{A_n}{m} \tag{3.2}$$

where $m \in \mathbb{N}$.

Proof. If we take a vertex v in a strict tessellation of $P_n \subset \mathbb{R}^n$, we have that the adjacent corners, i.e. the solid angles taken in v, are the same. To show this we use the reflection symmetry in definition 1. Assume that the corners are not the same. Then we can find two adjacent corners that are not the same, i.e. they share an $(n-1)$-face. However, due to the reflection symmetry in the $(n-1)$-face they must be reflected copies of each other.

Therefore, all corners in v are the same. Since the number of corners is a natural number and the *full* solid angle is A_n, it follows that each angle must be of the form in 3.2 in order to not have any gaps or overlaps. $\square$

Using corollary 3, we get that theorem 9 holds for all k-faces as well.

Corollary 5. *All solid angles between k-faces are of the form $\frac{A_k}{m}$ for some $m \in \mathbb{N}$.*

3.4 Bound on number of strictly tessellation polytopes in $\mathbb{R}^n$

We denote the number of equivalence classes of strictly tessellating polytopes of dimension n by $\mathcal{P}(n)$. Two polytopes are said to be equivalent if they differ by a bijective transformation that preserves the angles between 1-faces. In other words, we can scale polytopes in one or more directions as long as the angles between 1-faces are unaffected, but the polytope is considered the same up equivalence. Same goes for certain basis changes. Naturally, translations, rotations, and reflections are allowed as well since they do not affect any angles.

We say that two polytopes $P_1, P_2 \subset \mathbb{R}^n$ are equivalent $P_1 \sim_{\text{ang}} P_2$ if there exists a bijective transformation T that preserves angles between 1-faces such that $P_1 = TP_2$ and $P_2 = T^{-1}P_1$. When we say "number of strictly tessellating polytopes", we refer to the number of equivalence classes.

Knowing that there are at most 2^n vertices in a strictly tessellating polytope in $\mathbb{R}^n$, and that angles between 1-faces can only take 4 distinct values, we can make some rough estimates on the maximum number of strictly tessellating polytopes in $\mathbb{R}^n$. An exact number of strictly tessellating polytopes is most likely difficult to obtain, as we would have to take into consideration all the angles not only for the polytopes themselves but also for all their k-faces.

Proposition 5. *The number of strictly tessellating polytopes of dimension n is bounded as*

$$\mathcal{P}(n) \leq 4^{16^n}. \tag{3.3}$$

Proof. We want to estimate the number by constructing an excessive amount of polytopes with the angles given by corollary 4. This requires that such a construction gives a unique polytope. We note that strictly tessellating polytopes are convex and thus are the convex hull of their vertices. Since a convex hull is unique, the 1-faces are also determined. We want to only consider the angles between 1-faces alone, and have the rest of the polytope be determined up to equivalence.

Claim: A convex polytope P is decided by the angles between its 1-faces up to equivalence.

Proof of claim: Consider a convex polytope P. We note that if we try to move one or more vertices there are two possibilities. First possibility is that the internal angles change, which is not allowed as we have fixed angles between the 1-faces. The second possibility is that the internal angles do not change, which means the polytope has been transformed – in particular its vertices – by a transformation that preserves angles between 1-faces. If we have another convex polytope P' with the same angles and can't transform P into P' without changing the angles, but then P and P' don't have the same angles, which is a contradiction. The transformation is bijective since we should be able to transform back to P from P', and that means the transformation back is just the inverse of the transformation from P to P'.

Given that we can construct a polytope up to equivalene by knowing the angles between the 1-faces, we get the number of ways to permute these by taking the number of angles – say a angles – and raise the number of possible angles to the power of a. The number of permutations of angles thus becomes 4^a. Not all of these permutations will give a strictly tessellating polytope, and most likely not even a polytope for the matter, but it suffices for an upper bound. Though, we cannot simply determine a, so we make an upper estimate on the number of angles we can pick.

If we have a maximum of e_j 1-faces in the polytope with j vertices, we can pick $\binom{e_j}{2}$ different pairs of 1-faces. These will have an angle between them, so for each we pick one of 4 angles from corollary 4. Finally, we sum over the possible number of vertices. The minimum number of vertices is $n+1$ in order to have a full polytope, and at most 2^n by proposition 4. The estimate as such becomes

$$\mathcal{P}(n) \leq \sum_{j=n+1}^{2^n} 4^{\binom{e_j}{2}}. \tag{3.4}$$

We do not know a simple way to obtain a maximum value for e, but we can estimate it from above, and thus also give an upper bound for the right hand side of 3.4. Since we can at most have 2^n vertices in $\mathbb{R}^n$, the maximum number of 1-faces is obtained by forming a 1-face between all pairs of vertices, thus $e \leq \binom{2^n}{2}$. From this we can bound 3.4 from above by substituting e for the upper bound. The exponent thus becomes $\binom{\binom{2^n}{2}}{2} = 2^{n-2} - 2^{2n-3} - 2^{3n-2} + 2^{4n-3}$ if calculated explicitly, and we thus get

$$\mathcal{P}(n) \leq \sum_{j=n+1}^{2^n} 4^{2^{n-2}-2^{2n-3}-2^{3n-2}+2^{4n-3}}. \tag{3.5}$$

The sum can be bounded from above by multiplication by setting the index to its maximum $j = 2^n$ and then multiplying with 2^n to get rid of the sum, giving us

$$\mathcal{P}(n) \leq 4^{2^{n-2}-2^{2n-3}-2^{3n-2}+2^{4n-3}+\frac{n}{2}} \leq 4^{16^n}. \tag{3.6}$$

□

This upper bound isn't very exiting because it grows very fast, however, it does show us that the number of strictly tessellating polytopes bounded by the dimension n. Now we want to show a lower bound for the number of strictly tessellating polytopes in $\mathbb{R}^n$. The simplest way to construct a new strictly tessellating polytope is to take the cartesian product between two or more strictly tessellating polytopes. Thus, in dimension n we have that the strictly tessellating polytopes are *at least* the ones that are products of lower-dimensional strictly tessellating polytopes matching up to n in sum.

Proposition 6. *The number of strictly tessellating polytopes of dimension n is bounded as*

$$\mathcal{P}(n) \geq \frac{1}{6} \left(\lfloor n/2 \rfloor + 1\right) \left(\lfloor n/2 \rfloor + 2\right) \left(\lfloor n/2 \rfloor + 3\right). \tag{3.7}$$

Proof. The cartesian product $P_1 \times P_2$ of two strictly tessellating polytopes P_1, P_2 is also a strictly tessellating polytope. We can see this quickly as P_1 and P_2 are orthogonal. This means that we can multiply strictly tessellating polytopes of lower dimensions to obtain one of higher dimension. The dimension of the new polytope is the sum of the dimensions of its factors. Since we know all strictly tessellating polytopes in $\mathbb{R}$ and $\mathbb{R}^2$, we will use these to construct n-dimensional polytopes. Therefore, we want to know how many different ways we can sum up to n with 1 and 2.

This is a simple problem, as we can write $n = 2k + 1$ if n is odd or $n = 2k$ if n is even. If n is odd, we always multiply with a line segment, and thus the number will be the same as for $n - 1$. Thus it suffices to only consider $n = 2k$. Now, if $n = 2k$ we have k twos that we can split into two ones each, i.e. write a 2 as $1 + 1$ therefore using two line segments instead one of the three 2-dimensional strictly tessellating polytopes. Thus, k ways of splitting it. We don't care about the order. Since the polytopes we multiply are all be orthogonal to each other, we can simply perform a basis change and the angles will be preserved.

For each of the k ways to construct the sum, each occurence of a two represents one of the three strictly tessellating polytopes that cannot be constructed from 1-

dimensional strictly tessellating polytopes – we exclude rectangles as they are the cartesian product of two line segements. We are thus interested in how many ways we can fill up k' slots where $0 \leq k' \leq k$ with 3 elements, ignoring order.

If we have k' twos, then the number of combinations of 2-dimensional strictly tessellating polytopes filling k' slots will be $c_{k'} := \left(\binom{3}{k}\right) = \binom{k'+2}{k'}$, since we allow repetition. This gives the number of combinations for k' twos. Since k' can range from 0 to k, we obtain the total number of combiations by summing over $c_{k'}$ when k' goes from 0 to k. Now, set $k = \lfloor n/2 \rfloor$, as we have the same number for $n = 2k$ and $n = 2k+1$, so we can also account for odd dimensions in our estimate. Alltogether this becomes

$$\mathcal{P}(n) \geq \sum_{k'=0}^{\lfloor n/2 \rfloor} \binom{k'+2}{k'} = \frac{1}{6} \left(\lfloor n/2 \rfloor + 1\right) \left(\lfloor n/2 \rfloor + 2\right) \left(\lfloor n/2 \rfloor + 3\right). \tag{3.8}$$

The equality is obatined by explicitly calculating the sum, which is straightforward since $\binom{k'+2}{k'} = \frac{(k'+2)(k'+1)}{2}$.

□

We have given a lower bound for the number of strictly tessellating polytopes of dimension n.

3.5 Concluding Remarks

The results shown in this chapter have been necessary, but no sufficient results have been shown. Due to the amount of requirements on the k-faces, solid angles, and the relationship between dimensions, it appears difficult to provide a *simple* sufficient requirement for a polytope to be strictly tessellating. We can use theorem 2 and compare to alcoves, but it would be tedious to do that by hand, or even with a computer in high enough dimensions. This is a suitable area for possible future work on strictly tesselating polytopes.

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