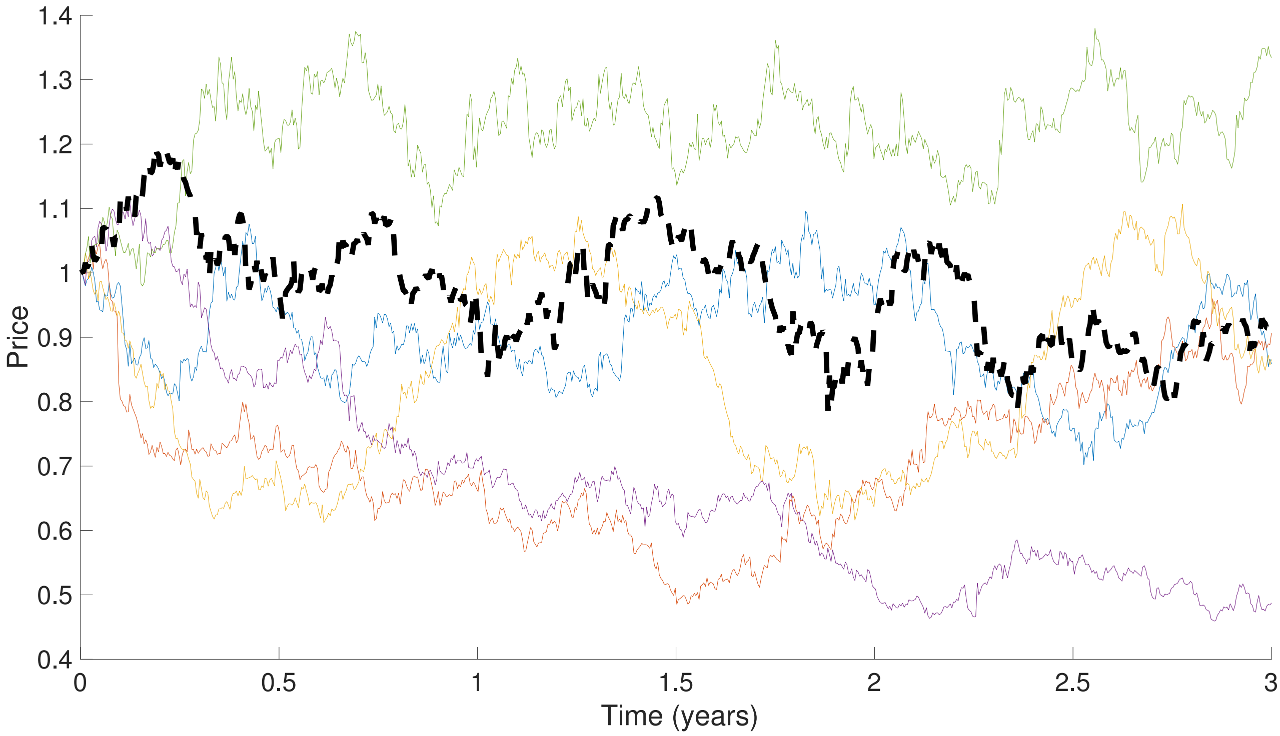

Investigation of Portfolio Strategies

using NIG-GARCH and CIR

Master's thesis in Engineering Mathematics and Computational Science

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CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2016

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Cover: 5 different simulations of the General Motors stock compared to the data.

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Abstract

This thesis creates a model for simulating stocks and interest rates to compare portfolio strategies. The two portfolio strategies used in the thesis are CPPI and OBPI. CPPI (constant proportion portfolio insurance) is a dynamic strategy that changes the amount in the risky asset and the safe asset at every timestep. OBPI (option based portfolio insurance) is a static strategy that invest an amount in the stock and the put option. It is found out that OBPI performs better in a decreasing market and that CPPI performs better in an increasing market.

The model used in this thesis can be seen as an extended Black-Scholes model. The stock will be modelled as a NIG (normal inverse Gaussian) with GARCH as stochastic volatility. The interest rate is modelled by a CIR (Cox, Ingersoll and Ross) model. There are some problems with this model, but it is better than the Black-Scholes model.

Keywords: CPPI, OBPI, NIG-GARCH, NIG, GARCH, CIR.

Acknowledgements

First of all I want to thank Patrik Albin for coming up with the idea for this thesis, and helping me complete this work. I also want to thank all my friends and family for supporting me during these 5 years.

Alexander Bore, Gothenburg, December 2016

Contents

List of Figures

List of Tables

1

Introduction

THIS thesis creates portfolio simulations, based on simulating a stock model and a model for the interest rate. In this chapter there is background to the problem and the goal of this thesis. In the next chapter the theory used in this thesis is presented. In Chapter 3 the model calibration and the numerical implementation of the model is described. Chapter 4 presents the results of the simulations. In Chapter 5 the results are discussed.

1.1 Background

When the Black-Scholes model was introduced it opened up a new area of financial mathematics. The model was a good start to be able to simulate a stock model, but there are of course some problems with the early model. As financial mathematics developed new models were created. This thesis will use a stock model based on the NIG distribution, instead of the normal distribution as in the Black-Scholes case. In the Black-Scholes case there is constant stochastic volatility, which is something that will be changed to a time varying stochastic volatility model in this thesis using GARCH. The Black-Scholes assumes a constant interest rate, but in this thesis it will be assumed that the interest rate follows a CIR process.

1.2 Aim

The aim of this thesis is to create a model to be able to compare the portfolio strategies. The aim can be summed up into these four points:

- Create a model for the interest rate using CIR process.
- Create a model for the stochastic volatility using GARCH process.
- Create a model for the stock using NIG distribution.
- Implement the models and compare the portfolio strategies.

2

Theory

THIS chapter contains the theory that is used in the thesis. First some basic theory and notation is introduced. Secondly the two portfolio strategies CPPI and OBPI are introduced. Thirdly the interest rate is introduced as a CIR process and GARCH is introduced as the stochastic volatility. In the end of the chapter the stock price is introduced as a type of NIG process. The parameters that are used to simulate the processes in the figures are presented in Chapter 4.

2.1 Basics

In this section definitions of some statistics and theory that will be used in the thesis are presented.

The stock model used in this thesis are defined by

$$S_t = S_0 e^{X_t} \tag{2.1}$$

or with the SDE

$$dS_t = S_t dX_t, \tag{2.2}$$

where the stochastic variable X_t is a Lévy process. More specifically X_t is in this thesis a NIG-GARCH process, that will be introduced later in this chapter. Equation (2.2) is also called a stochastic exponential.

Definition 2.1. *A stochastic process $X = (X_t)_{t \geq 0}$ is a Lévy process if it satisfies:*

- $X_0 = 0$ *almost surely.*
- *Independence of increments: for any* $0 \leq t_1 < ... < t_n < \infty$, $X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent*
- *Stationary increments: for any $s < t$: $X_t - X_s$ is equal in distribution to X_{t-s}.*
- *Continuity in probability: for any $\epsilon > 0$ and $t \geq 0$ it holds that* $\lim_{h \to 0} P(|X_{t+h} - X_t)| > \epsilon) = 0$.

Instead of working directly with the stock the logarithmic return is used or simply the log return.

Definition 2.2. *A log return X_t of a stock S_t in a discrete time interval t_i for $i = 1...n$ is defined by*

$$X_{t_i} = \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right),$$

with $X_{t_1} = 0$.

Looking at the log return instead at directly on the stock it is easier to compare the model with a specific probability distribution. Two important features working with log returns for a stock are the skewness and the kurtosis.

Definition 2.3. *The skewness of a random variable X with mean μ is its third standardized moment and defined by*

$$\text{Skew}(X) = \frac{\mathbb{E}[(X-\mu)^3]}{(\mathbb{E}[(X-\mu)^2])^{3/2}}.$$

The skewness is a measure of asymmetry around the mean. A distribution is symmetric if the skewness is 0. The normal distribution have 0 skewness. Log returns of a stock often are often negatively skewed.

Definition 2.4. *The kurtosis of a random variable X with mean μ is its fourth central moment and defined by*

$$\text{Kurt}(X) = \frac{\mathbb{E}[(X-\mu)^4]}{(\mathbb{E}[(X-\mu)^2])^2}.$$

Kurtosis is a measure of tail heaviness. Distributions with kurtosis 3 are said to be mesokurtic. The normal distribution have kurtosis 3. If the kurtosis of a distribution is less than 3 it is said to be platykurtic, which means the tails are thin or heavy. If the kurtosis of a distribution is more than 3 it is called leptokurtic, which means the tails are fat or light. Log returns of a stock is often leptokurtic.

2.2 Portfolio strategies

Portfolio strategies are used to maximize the growth in the portfolio, but at the same time minimize the risk. The basics of a portfolio strategy is that there is a safe asset and a risky asset which are invested in. The risky asset is in this thesis are the stock model that was introduced in Equation (2.1). The safe asset is a zero-coupon bond, that is defined by

$$B(t,T) = \mathbb{E}[K \exp\Big(-\int_t^T R(s)ds\Big)|\mathcal{F}(t)],$$

where K is the face value, $R(s)$ is a CIR process which will be introduced later in this chapter and $\mathcal{F}(t)$ is a filtration. For more information about the filtration see Shreve [12]. In this thesis we are working with 2 different portfolio strategies; the CPPI and the OBPI.

2.2.1 CPPI

The constant proportion portfolio insurance abbreviated as CPPI was first introduced in 1986 by Perold [10] and evolved later by Black and Perold in 1992 [5]. CPPI is a dynamic portfolio strategy, meaning it changes the amount invested in the risky asset and the safe asset at every time step. The strategy has a value V_t and also has a floor F_t which is the minimum value of the portfolio. There is a cushion C_t which is the excess over the floor, i.e. $C_t = V_t - F_t$. The exposure E_t is the investment in the risky asset, $E_t = \min\{mC_t, V_t\}$, where m is a multiplier which describes the risk. The change of value for the portfolio will described by

$$dV_t = E_t \frac{dS_t}{S_t} + (V_t - E_t)\frac{dB_t}{B_t}.$$

In Figure 2.1 the dynamics of a CPPI portfolio can be seen. The falling market to the left is a BMW stock together with a 12-month LIBOR interest rate and the raising market to the right is a Nasdaq index together with 12-month LIBOR. It can be seen in the picture that in a falling market the value of the CPPI never falls down under the floor and in a rising market the value of the portfolio grows with the stock.

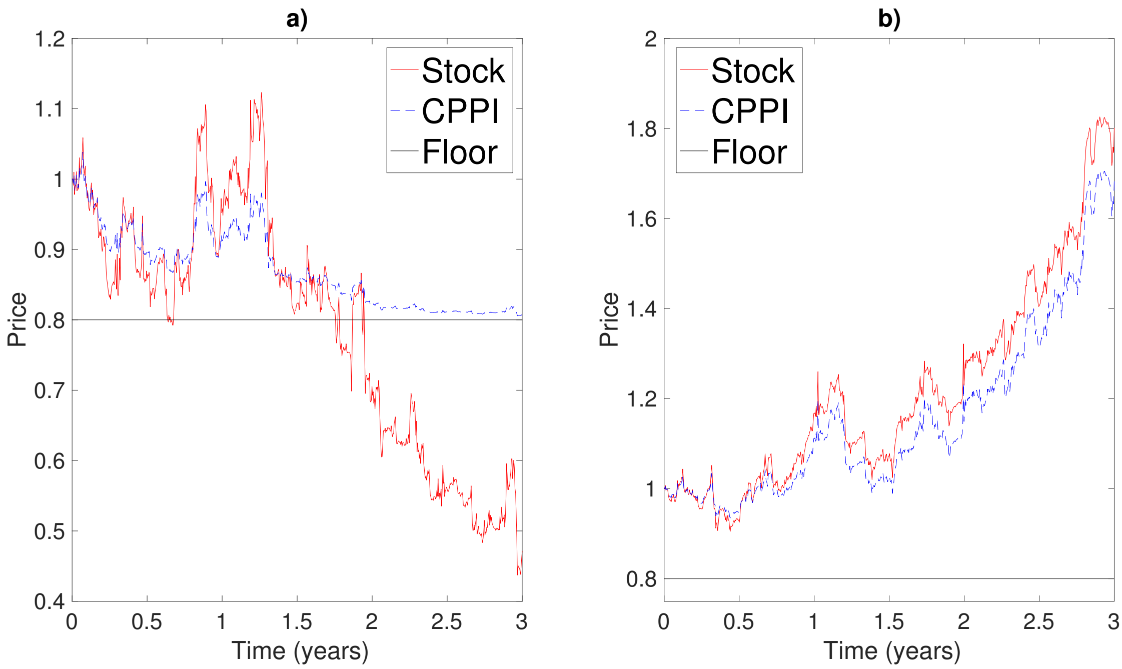

Figure 2.1: *Values for CPPI in a falling market a) and a rising market b). The falling market is based on a simulation using the BMW stock together with the 12-month LIBOR interest rate. Note how the stock is falling below the floor, but that the CPPI stays above the floor. The rising market is based on a simulation using the Nasdaq index togheter with 12-month LIBOR.*

2.2.2 OBPI

The option based portfolio insurance or OBPI was first introduced by Leland in 1976 [9]. This strategy is a strategy based on options, hence its name. The strategy is static, meaning an amount is invested in the risky asset and an amount is invested in the safe asset at the start and this amount stays the same over time. OBPI can be seen as a generalized CPPI [4]. The value of an OBPI is

$$V_t = S_t + p(t, S_t, K),$$

where $p(t, S_t, K)$ is the European put price. The fair price of an option is calculated by assuming the the risk neutral measure from Shreve [12, p. 218]. The formula for calculating the fair price of an option is

$$\Pi_Y(t) = \tilde{\mathbb{E}}[\exp\Big(\int_t^T -R(s)ds\Big) g(S_T)|\mathcal{F}(t)],$$

where $g(x)$ is the payoff function. The payoff function for a European put is $g(x) = \max(K - S_T, 0) = (K - S_T)^+$ at time of maturity T and strike price K. This leads to that the price of a European put is

$$\begin{aligned} p(t, S_t, K) &= \exp\Big(\int_t^T -R(s)ds\Big)\, \tilde{\mathbb{E}}[(K - S_T)^+|\mathcal{F}(t)] \\ &= \exp\Big(\int_t^T -R(s)ds\Big) \int_{-\infty}^{d} (K - S_t \exp(x)) f(x) dx, \end{aligned}$$

where $f(x)$ is the pdf of the random variable x and $d = log(K/S_t)$. For the Black-Scholes model the put can be expressed in an explicit formula, but in the case of NIG distribution the integral must be calculated. Figure 2.2 presents the put price with strike price $K = 1.25$ and time of maturity $T = 5$ years for a stock based on the Ford stock and the interest rate based on the 12-month LIBOR rate.

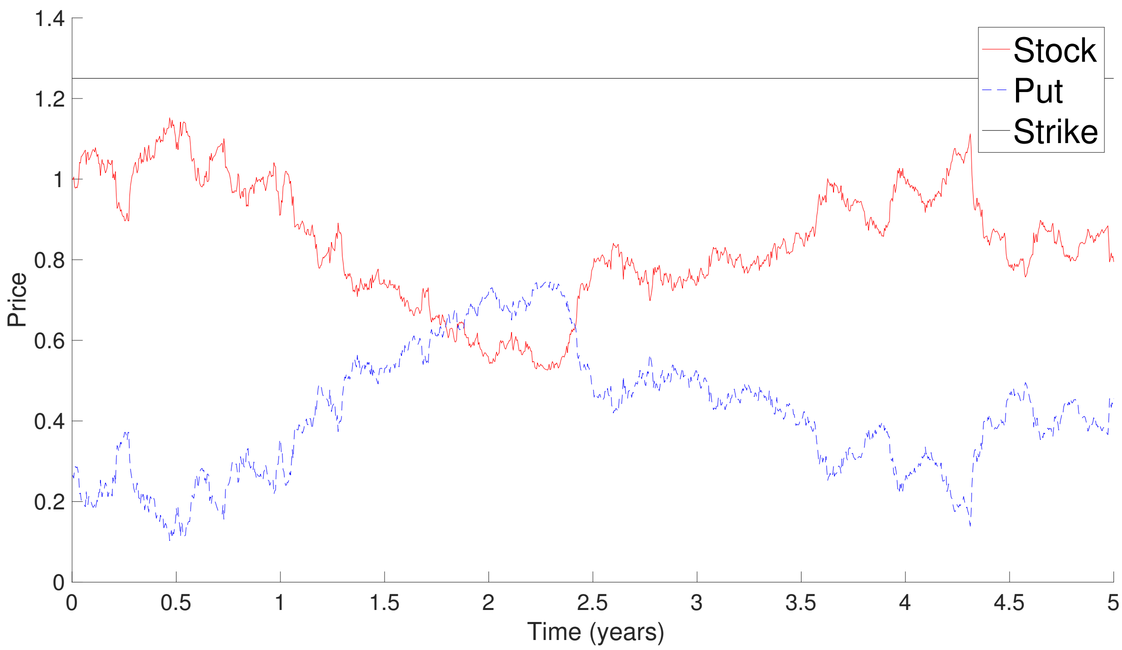

Figure 2.2: *Put price with strike price $K = 1.25$ and time of maturity $T = 5$ years for a stock based on Ford and 12-month LIBOR.*

In the OBPI strategy q shares are invested in the stock and q shares are invested in the put with time of maturity T and strike price K. The value of the OBPI portfolio is expressed by

$$V_t = q(S_t + p(t, S_t, K)).$$

The OBPI strategy also has a floor at time of maturity T. Using options on the stock it is guaranteed that the value of the portfolio is at least $V_T = qK$, independent of the stock price S_T. The dynamics of a OBPI portfolio are presented in Figure 2.3. The falling market is based on a simulation using the BMW stock and 12-month LIBOR. The rising market is based on a simulation using the Nasdaq index and 12-month LIBOR. The OBPI strategy used in the figure is extended from what is presented here, and will be explained in the next chapter. Note that in comparison the floor for OBPI is higher than for CPPI. This depends of course on how the values in both strategies are chosen. Note also that in a rising market the OBPI strategy grows slower than the CPPI strategy, because the OBPI strategy increases only after the stock price is higher than the strike price K.

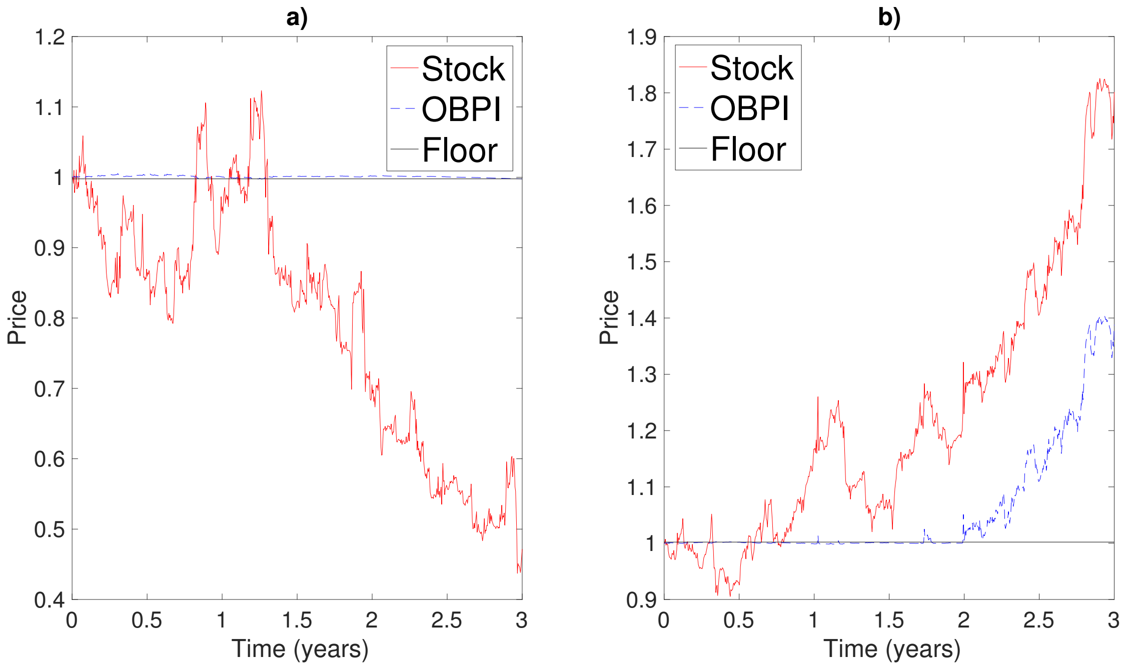

Figure 2.3: *Values for OBPI in a falling market a) and a rising market b). The falling market is based on a simulation using the BMW stock together with the 12-month LIBOR interest rate. Note how the stock is falling below the floor, but that the OBPI stays above the floor. The rising market is based on a simulation using the Nasdaq index together with 12-month LIBOR. Note how the OBPI strategy does not increase before the stock grows over* $K = 1.25$.

2.3 CIR process

The interest in this project is modeled by a CIR process. The CIR process was first introduced by Feller but later changed name by Cox, Ingersoll and Ross in 1985[7]. The CIR process is expressed by

$$dR_t = \kappa(\theta - R_t)dt + \sigma\sqrt{R_t}dW_t, \tag{2.3}$$

where κ, $\theta > 0$, $\sigma^2 < 2\kappa\theta$ and W_t is a standard Wiener process. The parameter κ determines the rate of adjustment. The parameter θ is the long term mean and σ

is the volatility. The 2 conditions on the parameters means that $R_t > 0$, i.e. the interest rate will always be positive. Having a positive interest rate means, of course, that we are guaranteed to increase an investment in the bond with a CIR interest rate at time of maturity.

The transition density function for a CIR process, at time s given the current time t, is expressed by

$$f(R_s, s|R_t, t) = ce^{-u-v}\Big(\frac{u}{v}\Big)^{q/2} I_q(2\sqrt{uv}),$$

where

$$\begin{aligned} c &= \frac{2\kappa}{\sigma^2(1-e^{-\kappa(s-t)})}, \\ u &= cR_t e^{\kappa(s-t)}, \\ v &= crR_s, \\ q &= \frac{2\kappa\theta}{\sigma^2} - 1 \end{aligned}$$

and $I_q(.)$ is the modified Bessel function of first kind order q.

The bond price of a zero-coupon bond for a CIR process is calculated by

$$B(t,T) = f(t, R_t) = \exp(-R_t C(t,T) - A(t,T)), \tag{2.4}$$

where

$$C(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\kappa\sinh(\gamma(T-t))},$$

$$A(t,T) = -\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{\gamma\exp(\frac{1}{2}\kappa(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\kappa\sinh(\gamma(T-t))}\right)$$

and

$$\gamma = \frac{1}{2}\sqrt{\kappa^2 + 2\sigma^2}.$$

This zero-coupon bond guarantees that the bond price at time of maturity is $B(T,T) = 1$. For the interested reader there is a more extensive calculation of the bond price in Appendix A.

Figure 2.4 presents a CIR process based on 12-month LIBOR and its corresponding bond price. Note that the bond price is decreasing at some points, but at the time of maturity the we have the guaranteed value at $B(T,T) = 1$.

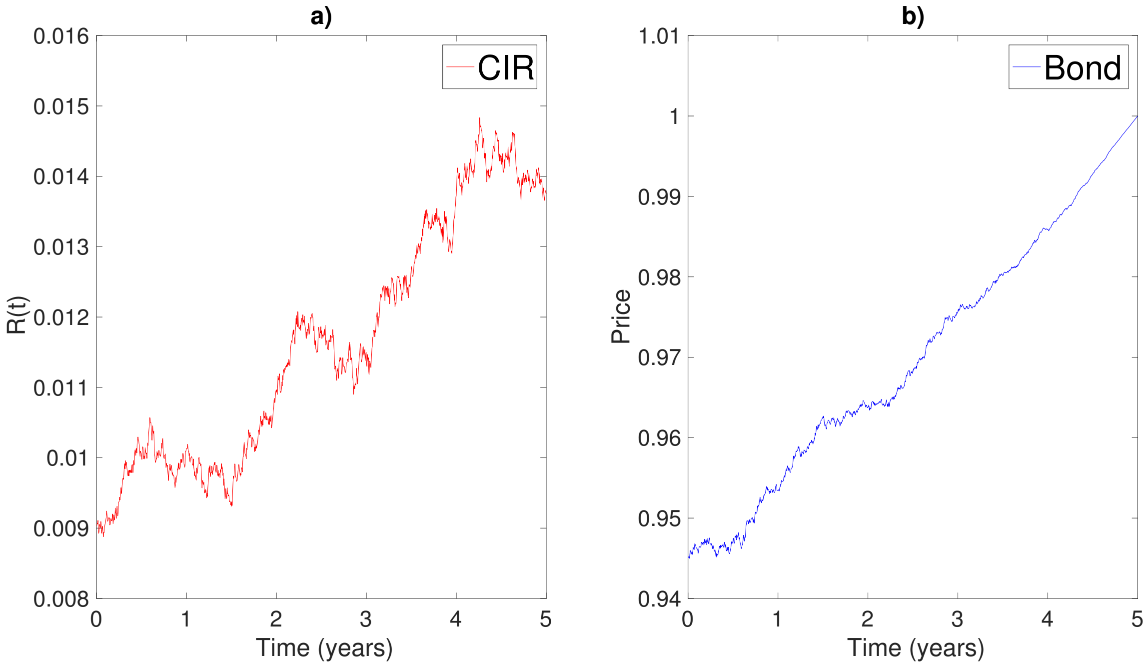

Figure 2.4: *a) shows a CIR process based on the 12-month LIBOR rate and b) shows the bond price for the CIR process.*

2.4 GARCH

To model the stochastic volatility this thesis uses GARCH. Generalized autoregressive conditional heteroscedasticity or GARCH was first introduced by Bollerslev in 1986 [6]. If X_t is the log return and σ_t is the stochastic volatility, the GARCH(p,q) model will be expressed by

$$X_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 \sum_{j=1}^{q} b_j \sigma_{t-j}^2,$$

where ϵ_t are iid random variables with mean 0 and variance 1, and $a_i \geq 0$, $b_j \geq 0$ and $\sum_{i=1}^{\max(p,q)} (a_i + b_i) < 1$.

Volatility clustering is a phenomena where small changes tends to be followed by small changes and large changes tends to be followed by large changes. This means that if the stochastic volatility is small today, it is likely that the stochastic volatility will be small tomorrow as well, and the same for large volatilities. Since GARCH is based on previous observations it exhibits volatility clustering.

In Figure 2.5 a stock based on the Dow Jones index and its corresponding GARCH(1,1) process is presented. Note how large changes in the stock are followed by high values of the stochastic volatility. GARCH(1,1) is often considered a reasonable model [13], and is used in this thesis.

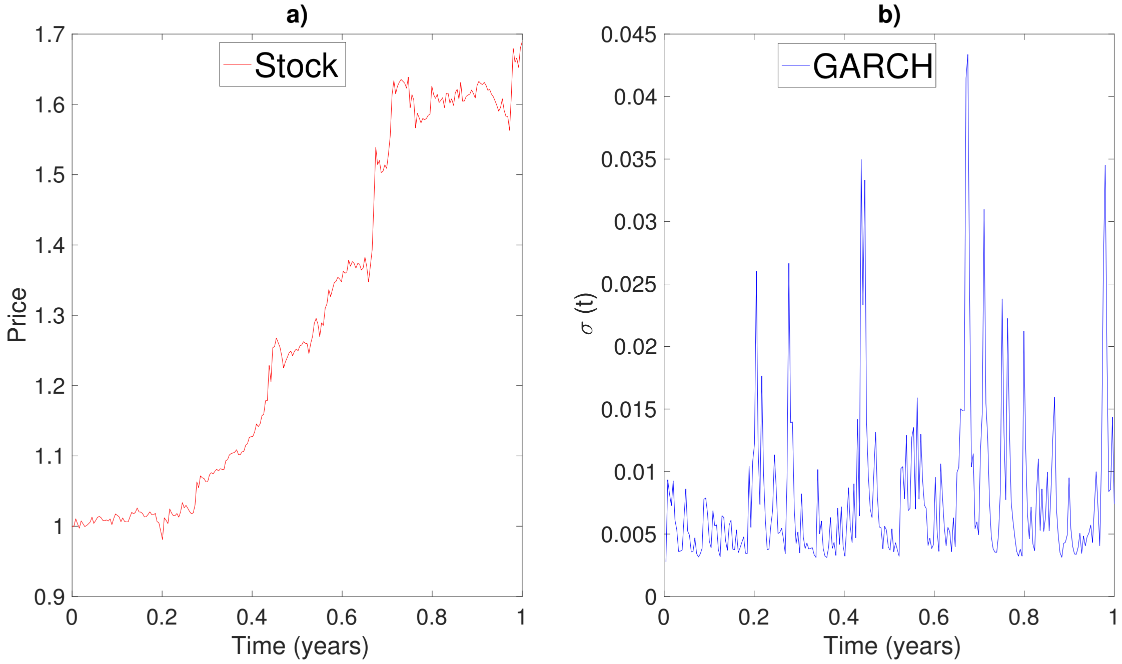

Figure 2.5: *Stock in a) based on the Dow Jones index with its corresponding GARCH(1,1) in b).*

2.5 NIG process

The NIG process will be used to model the stock price in this thesis. The NIG or normal inverse Gaussian process is based on the normal distribution and IG (inverse Gaussian) distribution. The normal, IG and NIG processes are all three Lévy processes [11], which was defined in Definition 2.1. The NIG distribution was introduced in 1995 by Barndorff-Nielsen [3]. In Appendix B a goodness of fit for the NIG distribution are presented, which motivates why NIG is a good choice to model the stock.

2.5.1 Normal distribution

The normal (or Gaussian) distribution is one of the most well-known and important distributions. The normal distribution has 2 parameters, $\mathrm{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is the mean and $\sigma^2 > 0$ is the variance. The probability density function for a normal distribution is given by

$$f_{\mathrm{N}}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right).$$

In the Black-Scholes model, which was the first model in this area, the normal distribution was used. There are some problems with the normal distribution, which makes us want to use a better model. As mentioned before the normal distribution have skewness 0 and kurtosis 3. This means that the normal distribution are symmetric, when usually log return of a stock are negatively skewed and the tails are heavier than the log returns are. Due to the problems with the normal distribution there is need for a better model. In this thesis the NIG distribution, which is introduced below after the IG distribution, is chosen to be the model.

2.5.2 IG distribution

The IG distribution has probability density function

$$f_{\text{IG}}(x|\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x}} \exp\left(-\frac{\lambda}{2\mu^2 x}(x-\mu)^2 \right),$$

where $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter.

2.5.3 NIG distribution

The NIG distribution is based on both the normal distribution and the IG distribution. The NIG distribution has 4 parameters α, β, μ and δ. The probability density function for NIG is

$$f_{\text{NIG}}(x|\alpha,\beta,\mu,\delta) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2-\beta^2} - \beta(x-\mu) \right) q\Big(\frac{x-\mu}{\delta}\Big)^{-1} K_1\left(\delta\alpha q\Big(\frac{x-\mu}{\delta}\Big) \right),$$

where $q(x) = \sqrt{1+x^2}$ and $K_1(.)$ is the modified Bessel function of third order and index 1. The parameters α and β are shape parameters, where α is steepness and β is asymmetry, and it holds that $0 \leq |\beta| \leq \alpha$. The parameter μ is the location parameter, where $\mu \in \mathbb{R}$ and δ is the scale parameter, where $\delta > 0$. Figure 2.6 shows how the different parameters change the pdf. Note how these different parameters makes the NIG distribution more versatile than the normal distribution and how the asymmetry and shape can be changed by changing the parameters in NIG.

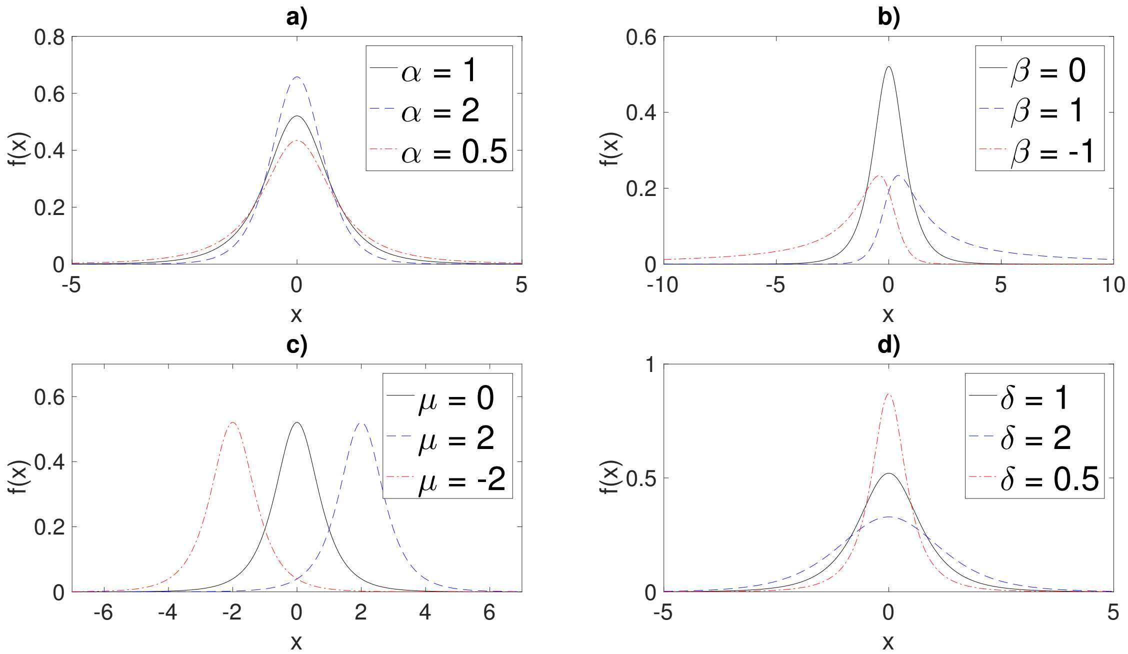

Figure 2.6: *Probability density function for NIG. The standard variables used are $\alpha = 1$, $\beta = 0$, $\mu = 0$ and $\delta = 1$. In a) α is changed, in b) β is changed, in c) μ is changed and in d) δ is changed.*

The NIG distribution have no direct way to generate a random variable. To generate a NIG random variable both normal distribution and IG distribution are used. According to Schoutens a NIG(α, β, μ, δ) random variable can be generated by

$$X_t = \mu + \beta\delta^2 \mathrm{IG} + \delta \mathrm{N}, \tag{2.5}$$

where IG is an IG(μ,λ) random variable with $\mu = (\delta\sqrt{\alpha^2 - \beta^2})^{-1}$ and $\lambda = 1$ and N is a N(0,σ^2) with $\sigma^2 = \mathrm{IG}$.

To have a more convenient way of working with the NIG distribution the invariant parameters $\bar{\alpha} = \alpha\delta$ and $\bar{\beta} = \beta\delta$ are introduced. The probability density function for the invariant parameters is

$$\bar{f}_{\mathrm{NIG}}(x|\bar{\alpha}, \bar{\beta}, \mu, \delta) = \frac{\bar{\alpha}}{\pi\delta} \exp\left(\sqrt{\bar{\alpha}^2 - \bar{\beta}^2} - \bar{\beta}\frac{(x-\mu)}{\delta}\right) q\Big(\frac{x-\mu}{\delta}\Big)^{-1} K_1\left(\bar{\alpha} q\Big(\frac{x-\mu}{\delta}\Big)\right).$$

One good thing about this parametrization is that NIG($\bar{\alpha}, \bar{\beta}, \mu, \delta$) is a location-scale family, i.e.

$$X \sim \mathrm{NIG}(\bar{\alpha}, \bar{\beta}, \mu, \delta) \Leftrightarrow \frac{X-\mu}{\delta} \sim \mathrm{NIG}(\bar{\alpha}, \bar{\beta}, 0, 1).$$

This feature will be very useful later when a time dependent δ is used and when calculating the put price.

For simpler notation the variable $\bar{\rho} = \bar{\beta}/\bar{\alpha} = \beta/\alpha$ is defined. With this parametrisation the formula for the mean is

$$\mathbb{E}(X) = \mu + \frac{\bar{\rho}\delta}{\sqrt{1-\bar{\rho}^2}},$$

the variance is

$$\mathrm{Var}(X) = \frac{\delta^2}{\bar{\alpha}(\sqrt{1-\bar{\rho}^2})^{3/2}},$$

the skewness is

$$\mathrm{Skew}(X) = 3\frac{\bar{\rho}}{\sqrt[4]{1-\bar{\rho}^2}\sqrt{\bar{\alpha}}}$$

and the kurtosis is

$$\mathrm{Kurt}(X) = 3\Big(1 + \frac{4\bar{\rho}^2+1}{\bar{\alpha}\sqrt{1-\bar{\rho}^2}}\Big).$$

Note here that both the skewness and kurtosis can change values depending on the parameters that are used.

2.5.4 NIG-GARCH distribution

The final model will be presented in this section. This model is a NIG process with stochastic volatility, which was first introduced by Bandorff-Nielsen in 1997 [2]. Work from Andersson in 2001 [1] gives the NIGSV (NIG - stochastic volatility) model, which is $\mathrm{NIG}(\alpha, \beta, \mu_t, \delta_t)$, where δ_t is the stochastic volatility. The model that is used in this thesis is a model from Jensen and Lunde in 2001 [8] which uses GARCH as stochastic volatility. This model assumes a NIG process with the invariant parameters, i.e $X(t) \sim \mathrm{NIG}(\bar{\alpha}, \bar{\beta}, \mu, \delta)$. A time dependent δ_t is introduced and the process is $X_t \sim \mathrm{NIG}(\bar{\alpha}, \bar{\beta}, \mu, \delta_t)$. Introducing the parameter $\gamma = \sqrt{\alpha^2-\beta^2}$ (and $\bar{\gamma} = \delta\sqrt{\alpha^2-\beta^2} = \sqrt{\bar{\alpha}^2-\bar{\beta}^2}$) it follows that

$$\mathbb{E}(X_t) = \mu + \delta_t\frac{\bar{\beta}}{\bar{\gamma}}$$

and

$$\mathrm{Var}(X_t) = \delta_t^2\frac{\bar{\alpha}^2}{\bar{\gamma}^3}.$$

Since the invariant parameters $\bar{\alpha}$ and $\bar{\beta}$ are used the model can be rewritten in a more convienient way as

$$X_t = \mu + \frac{\sqrt{\bar{\gamma}}\bar{\beta}}{\bar{\alpha}}\delta_t + \delta_t\eta_t,$$

where $\eta_t \sim \mathrm{NIG}(\bar{\alpha}, \bar{\beta}, -\frac{\sqrt{\bar{\gamma}}\bar{\beta}}{\bar{\alpha}}, \frac{\bar{\gamma}^{3/2}}{\bar{\alpha}})$. This expression has to be rewritten so that GARCH can be used. The process η_t has mean

$$\mathbb{E}(\eta_t) = \frac{\bar{\gamma}^{3/2}\bar{\beta}}{\bar{\alpha}\bar{\gamma}} - \frac{\sqrt{\bar{\gamma}}\bar{\beta}}{\bar{\alpha}} = 0$$

and variance

$$\mathrm{Var}(\eta_t) = \Big(\frac{\bar{\gamma}^{3/2}}{\bar{\alpha}}\Big)^2\frac{\bar{\alpha}^2}{\bar{\gamma}^3} = 1,$$

and because of this the model can be changed to $\mathrm{NIG}(\bar{\alpha}, \bar{\beta}, \mu, \delta_t\frac{\bar{\gamma}^{3/2}}{\bar{\alpha}})$. The mean for this is

$$\mathbb{E}(X_t) = \mu + \delta_t\frac{\sqrt{\bar{\gamma}}\bar{\beta}}{\bar{\alpha}}$$

and the variance is

$$\mathrm{Var}(X_t) = \delta_t^2.$$

The time dependent parameter δ_t is in fact a GARCH process and in this specific case a GARCH(1,1). The GARCH(1,1) is defined as

$$\sigma_t^2 = a_0 + a_1\sigma_{t-1} + b_1\epsilon_{t-1}, \tag{2.6}$$

where $\epsilon_{t-1} = X_{t-1} - \mathbb{E}(X_{t-1}) = X_{t-1} - \mu - \sigma_{t-1}\frac{\sqrt{\bar{\gamma}}\bar{\beta}}{\bar{\alpha}}$, so that ϵ_t fulfills the GARCH conditions with mean 0 and variance 1. Other conditions that also have to be fulfilled is that $a_0 > 0$ and $a_1 + b_1 < 1$. In conclusion the stock model used in this thesis is

$$X_t \sim \mathrm{NIG}(\bar{\alpha}, \bar{\beta}, \mu, \sigma_t\frac{\bar{\gamma}^{3/2}}{\bar{\alpha}}),$$

where σ_t is a GARCH(1,1) process as in Equation (2.6).

3
Methods

THIS chapter contains the information about the parameter estimation, or calibration of the model and the numerical implementation of the model. The model is calibrated using the maximum likelihood method. In this chapter it is assumed that a year consists of 250 trading days, i.e, $dt = 1/250$. It is also assumed that there is a discrete time interval $(t_i)_{i=1}^n$ and that n is 250, 750 or 1250, so that $T = ndt$ is 1, 3 or 5 years. Also note that lowercase x_{t_i} and r_{t_i} is used to represent the data and capital X_{t_i} and R_{t_i} is used to represent the simulated random variable.

3.1 Model calibration

Model calibration is the name for parameter estimation. To calibrate the model the maximum log-likelihood estimation is used. For NIG-GARCH this will be

$$\begin{aligned}\ln L(\alpha, \beta, \mu, \delta, a_0, a_1, b_1 | x_{t_1}, ...x_{t_n}) &= \sum_{i=1}^{n} \ln \bar{f}_{\text{NIG}}(x_{t_i} | \bar{\alpha}, \bar{\beta}, \mu, \delta_{t_i}) \\ &= \sum_{i=1}^{n} \ln \bar{f}_{\text{NIG}}\Big(x_{t_i} | \bar{\alpha}, \bar{\beta}, \mu, \sigma_{t_i} \frac{\bar{\gamma}^{3/2}}{\bar{\alpha}}\Big),\end{aligned}$$

where $\bar{\gamma} = \sqrt{\bar{\alpha}^2 - \bar{\beta}^2} = \delta\sqrt{\alpha^2 - \beta^2}$ and σ_{t_i} is a GARCH(1,1) process. This means that σ_{t_i} is modeled by

$$\sigma_{t_{i+1}}^2 = a_0 + a_i \sigma_{t_i}^2 + b_1 \epsilon_{t_i}^2 \text{ for } i = 1, ..., n-1,$$

where $\epsilon_{t_i} = x_{t_i} - \sigma_{t_i} \frac{\sqrt{\bar{\gamma}}\bar{\beta}}{\bar{\alpha}} - \mu$ and $\sigma_{t_1}^2$ is chosen as the variance from the first 20 observations [1]. This is solved numerically in MATLAB. The starting values are chosen by method of moments. The parameters from the calibration will be used to generate random variables to simulate the stock price.

For the CIR process the log-likelihood function is

$$\begin{aligned}&\ln L(\kappa, \theta, \sigma | r_{t_1}, ..., r_{t_n}) \\ &= \sum_{i=1}^{n-1} \ln f_{\text{CIR}}(r_{t_{i+1}} | \kappa, \theta, \sigma, r_{t_i}) \\ &= (n-1) \ln c + \sum_{i=1}^{n-1} \Bigg(- u_{t_i} - v_{t_{i+1}} \quad + \frac{1}{2} q \ln \left(\frac{v_{t_{i+1}}}{u_{t_i}} \right) + \ln \Big(I_q (2\sqrt{u_{t_i} v_{t_{i+1}}}) \Big) \Bigg),\end{aligned}$$

where $c = \frac{2\kappa}{\sigma^2(1-e^{-\kappa dt})}$, $u_{t_i} = cr_{t_i} e^{\kappa dt}$ for $i = 1...n-1$, $v_{t_{i+1}} = cr_{t_{i+1}}$ for $i = 1, .., n-1$ and $q = \frac{2\kappa\theta}{\sigma^2} - 1$. This is solved numerically in MATLAB and the values will be used to simulate the interest rate with a CIR process.

3.2 Model simulation

For the portfolio simulations the formulas from previous chapter are discretized. It is assumed that the stocks, the bonds and the puts are available in the quantity that is needed, for example it is possible to invest in a fraction of a stock. It is assumed that the initial value for the portfolio is $V_{t_1} = 1$ and that the initial value for the stock $S_{t_1} = 1$, and that the stock is modeled as discrete version of Equation (2.1), i.e,

$$S_{t_i} = S_{t_{i-1}} \exp(X_{t_i}) \text{ for } i = 2, ..., n,$$

where X_{t_i} is a NIG-GARCH random variable generated by using Equation (2.5).

The CIR process is simulated by using a discrete version of Equation (2.4) and is expressd by

$$R_{t_{i+1}} = R_{t_i} + dR_{t_i} = R_{t_i} + \kappa(\theta - R_{t_i})dt + \sigma\sqrt{R_{t_i}}\sqrt{dt} \cdot \epsilon_i \text{ for } i = 1, ..., n-1,$$

where ϵ_i are standard normal variables and $R_{t_1} = r_0$ is given from the data used. The bond is modeled by a discrete version of Equation (2.4), which is

$$B(t_i, t_n) = f(t_i, R_{t_i}) = \exp\Big(- R_{t_i} C(t_i, t_n) - A(t_i, t_n)\Big) \text{ for } i = 1, ..., n,$$

with

$$C(t_i, t_n) = \frac{\sinh(\gamma(t_n - t_i))}{\gamma\cosh(\gamma(t_n - t_i)) + \frac{1}{2}\kappa\sinh(\gamma(t_n - t_i))},$$

$$A(t_i, t_n) = -\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{\gamma\exp(\frac{1}{2}\kappa(t_n - t_i))}{\gamma\cosh(\gamma(t_n - t_i)) + \frac{1}{2}\kappa\sinh(\gamma(t_n - t_i))}\right)$$

and

$$\gamma = \frac{1}{2}\sqrt{\kappa^2 + 2\sigma^2}.$$

The European put is modeled by

$$p(t_i, S_{t_i}, K) = \exp\Big(\int_{t_i}^{t_n} -R(s)ds\Big)\int_{-\infty}^{d}(K - S_{t_i}\exp(x))f\Big(x|\bar{\alpha}, \bar{\beta}, \tau\mu, \tau\sigma_t\frac{\bar{\gamma}^{3/2}}{\bar{\alpha}}\Big)dx,$$

where $d = \log(K/S_{t_i})$ and $\tau = t_n - t_i$, and the integrals are solved numerically using MATLAB.

To be able to do comparisons between CPPI and OBPI the two strategies will have the same amount invested in the risky asset. This means that E_{t_1} from CPPI and q from OBPI will have the same value, i.e., $E_{t_1} = q$. The both strategies will also have the same initial amount, $V_{t_1}^{\text{OBPI}} = V_{t_1}^{\text{CPPI}} = 1$. The rest of the amount for the OBPI is filled up with the bond, making the value for the OBPI to be

$$V_{t_i}^{\text{OBPI}} = q(S_{t_i} + p(t_i, S_{t_i}, K)) + hB_{t_i},$$

where h is the amount invested in the bond, and is calculated by

$$h = \frac{V_{t_1}^{\text{OBPI}} - q(S_{t_1} + p(t_i, S_{t_1}, K))}{B_{t_1}}.$$

The strike price for the bond is chosen to be $K = 1.25$. The amount invested in the risky asset and the put is $q = 0.7$. The floor for this type of OBPI will be $F_{t_n} = qK + h$, which changes in each simulation, depending on the put and bond, but will be approximately $F_{t_n} \approx 0.7 \cdot 1.25 + .12 \approx 0.99$. For CPPI the value of the portfolio is calculated by

$$V_{t_{i+1}}^{\text{CPPI}} = V_{t_i}^{\text{CPPI}} + dV_{t_i},$$

where

$$\begin{aligned} dV_{t_i} &= E_{t_i} \frac{dS_{t_i}}{S_{t_i}} + (V_{t_i}^{\text{CPPI}} - E_{t_i}) \frac{dB_{t_i}}{B_{t_i}} \\ &= E_{t_i} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} + (V_{t_i}^{\text{CPPI}} - E_{t_i}) \frac{B_{t_{i+1}} - B_{t_i}}{B_{t_i}}, \end{aligned}$$

where $E_{t_i} = \min(mC_{t_i}, V_{t_i}^{\text{CPPI}})$ and with $m = 3.5$. The cushion is chosen as $C_{t_i} = V_{t_i}^{\text{CPPI}} - F_{t_i}$ with the floor chosen as $F_{t_i} = F = 0.8$.

4
Results

In this chapter the results for the calibrations are presented, using the methods described in the previous chapter. The portfolio simulations, that were described in the previous chapter, have been made by Monte Carlo simulation with $N = 50000$ number of simulations. The simulations have used LIBOR (London Interbank Offered Rate) as interest rate. The stocks have been simulated using the financial indices Dow Jones, Nasdaq, Nikkei 225 and S&P 500, and the stocks from BMW, Ford, General Motors and Volkswagen.

4.1 Calibration results

In Table 4.1 the calibration results for the 3 month, 6 month and 12 month LIBOR interest rates are presented. Note that the results are quite similar for each year.

Table 4.1: *Calibration results for different interest rates.*

	κ	θ	σ
LIBOR 3-month 1 year	0.3716	0.0204	0.0117
LIBOR 3-month 3 years	0.0004	3.0222	0.0089
LIBOR 3-month 5 years	0.0004	1.3968	0.0076
LIBOR 6-month 1 year	0.0594	0.1296	0.0132
LIBOR 6-month 3 years	0.0004	4.5747	0.0099
LIBOR 6-month 5 years	0.0009	0.8251	0.0084
LIBOR 12-month 1 year	0.1492	0.0595	0.0155
LIBOR 12-month 3 years	0.0009	2.5157	0.0129
LIBOR 12-month 5 years	0.0011	0.8651	0.0107

Figure 4.1 shows the result of 3 different simulations for the 12-month LIBOR. In the 1 year simulation in a) it shows that the simulation is similar to the data. Also in the 3 years simulation in b) the simulation and the data are quite similar. In the 5 year simulation in c) it shows that the simulation and the data are quite different. The result for simulating 3-month LIBOR rate and 6-month LIBOR rate looks similar to the 12 month LIBOR, since the parameters from Table 4.1 are similar.

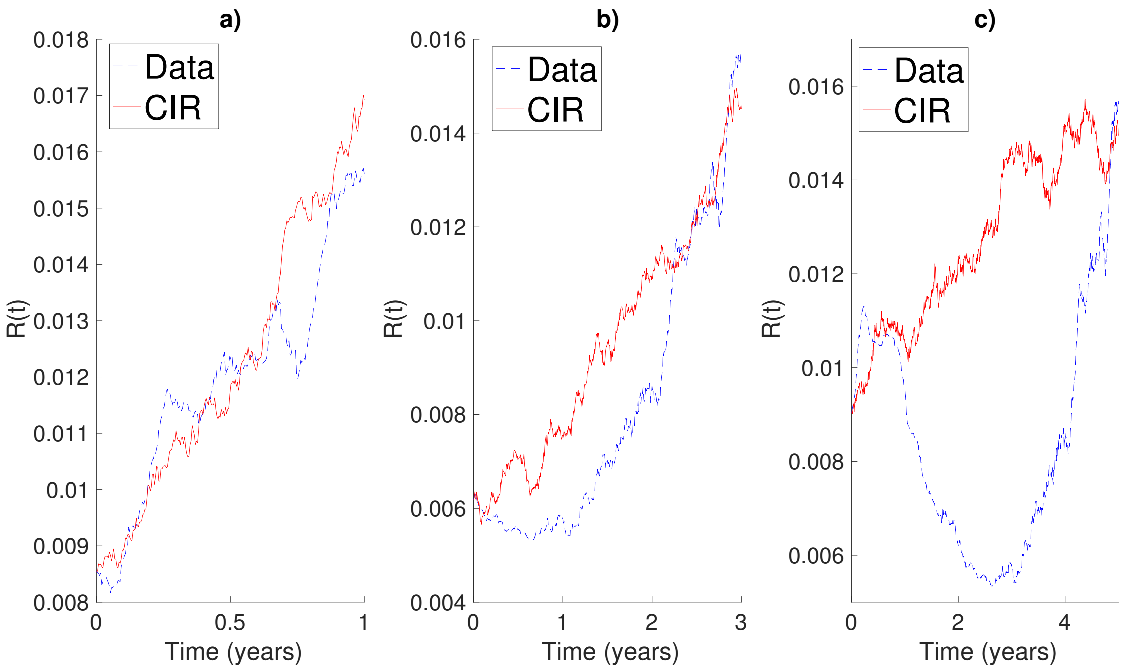

Figure 4.1: *Simulation results for 12 month LIBOR compared to the data, a) 1 year, b) 3 years and c) 5 years.*

The calibration results for the stocks are presented in Table 4.2. Since the distribution changes in time it is not possible to plot a pdf and compare it to the data. By looking at the variable β, we can conclude whether the skewness is positive or negative, but we cannot know its magnitude. The variable μ tells us that the distribution is located around 0 for all stocks. The variables a_0, a_1 and b_1 shows that b_1 will be most important when calculating the stochastic volatility.

Table 4.2: *Calibration results for stocks.*

	α	β	μ	δ	a_0	a_1	b_1
BMW 1 year	52.1318	-2.2490	0.0000	0.0496	0.0000	0.0470	0.8572
BMW 3 years	161.3547	-1.1124	0.0000	0.0524	0.0000	0.0601	0.9207
BMW 5 years	81.6365	-3.0610	0.0015	0.0782	0.0000	0.0288	0.9649
Dow Jones 1 year	161.7483	7.6339	0.0000	0.0160	0.0000	0.1659	0.7250
Dow Jones 3 years	173.3850	12.8098	0.0000	0.0167	0.0000	0.1811	0.7203
Dow Jones 5 years	173.0489	4.5408	0.0005	0.0184	0.0000	0.1577	0.7676
Ford 1 year	61.8963	-3.7252	0.0000	0.0185	0.0001	0.1893	0.4928
Ford 3 years	55.3888	-1.7224	0.0000	0.0115	0.0001	0.4361	0.2987
Ford 5 years	64.9243	-0.1999	0.0001	0.0176	0.0001	0.2784	0.3860
General Motors 1 year	82.8415	-0.9244	0.0000	0.0280	0.0001	0.1031	0.6069
General Motors 3 years	64.5421	-0.3543	0.0000	0.0180	0.0001	0.2286	0.2507
General Motors 5 years	64.2914	0.4841	0.0001	0.0209	0.0002	0.1345	0.2745
Nasdaq 1 year	118.6849	8.3449	0.0000	0.0251	0.0000	0.0800	0.8718
Nasdaq 3 years	118.8003	-1.4298	0.0012	0.0180	0.0000	0.1952	0.7192
Nasdaq 5 years	136.2092	-2.2369	0.0014	0.0224	0.0000	0.1308	0.7962
Nikkei 225 1 year	81.9400	-2.1287	-0.0000	0.0422	0.0000	0.0743	0.8929
Nikkei 225 3 years	114.7528	0.6292	0.0002	0.0448	0.0000	0.0825	0.8952
Nikkei 225 5 years	136.9897	3.4473	0.0000	0.0540	0.0000	0.0703	0.9069
S&P500 1 year	135.1133	11.4463	0.0000	0.0144	0.0000	0.1430	0.7491
S&P500 3 years	169.6456	16.5561	0.0000	0.0161	0.0000	0.1674	0.7320
S&P500 5 years	155.7278	5.2378	0.0006	0.0174	0.0000	0.1522	0.7693
Volkswagen 1 year	50.6959	1.7989	0.0001	0.0522	0.0000	0.1697	0.7980
Volkswagen 3 years	61.0874	0.0357	0.0000	0.0643	0.0000	0.0746	0.9221
Volkswagen 5 years	69.7610	-0.0036	0.0008	0.0669	0.0000	0.0594	0.9297

Figure 4.2 shows the results of the Kernel densities for 3 different simulations with the Volkswagen stock compared to the data. The Kernel density can be thought of as the empirical pdf. In a), which is a 1 year simulation, it shows some problems. First of all the data are not as smooth as for 3 or 5 years of data. Secondly, in the area at the beginning of the tails the simulation have more observations than the data and the simulation does not have the same peak as the data. The 3 year simulation looks a bit better, and the 5 year simulation looks even better.

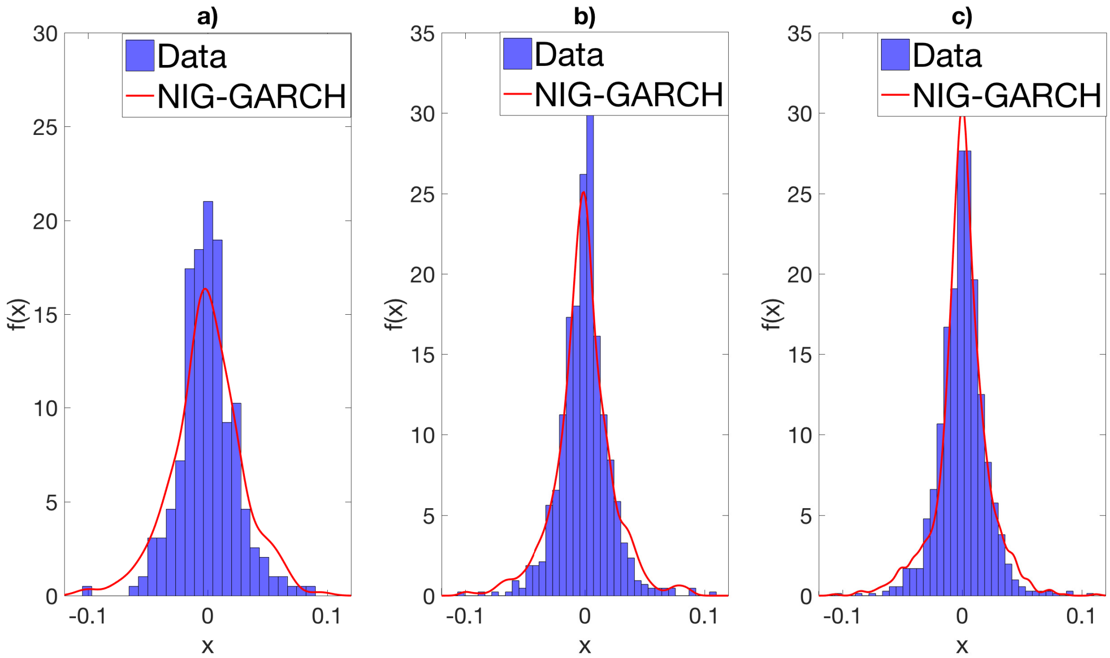

Figure 4.2: *Kernel density for Volkswagen stock simulation compared to the data, for a) 1 year, b) 3 years and c) 5 years.*

4.2 Portfolio results

In Table 4.3-4.5 the results for the portolio simulations are presented. The 3 tables give similar answers for the different interest rates. In a decreasing market the OBPI strategy performs better. In a slightly increasing market the both strategies performs similarly, sometimes OBPI is better and sometimes CPPI is better. In a significant increasing stock market the CPPI strategy performs better.

Table 4.3: *Results for portfolio simulations with LIBOR 3-month. CPPI and OBPI show the values of both portfolio strategies. S ($\bar{\sigma}$) is the value of the stock and the mean volatility in parenthesis.*

Value	Initial	1 year	3 years	5 years
BMW				
CPPI	1	0.8920	0.9268	1.2226
OBPI	1	1.0127	1.0574	1.3227
S ($\bar{\sigma}$)	1	0.7794 (0.0189)	0.8349 (0.0154)	1.3653 (0.0155)
Dow Jones				
CPPI	1	1.1254	1.9205	2.7595
OBPI	1	1.0191	1.5209	2.1349
S ($\bar{\sigma}$)	1	1.1555 (0.0071)	1.9925 (0.0070)	2.8703 (0.0070)
Ford				
CPPI	1	0.9029	0.9107	1.1177
OBPI	1	1.0031	1.0225	1.1671
S ($\bar{\sigma}$)	1	0.8173 (0.0149)	0.8118 (0.0161)	1.1761 (0.0159)
General Motors				
CPPI	1	0.9874	1.0239	1.4919
OBPI	1	1.0178	1.0784	1.4425
S ($\bar{\sigma}$)	1	0.9761 (0.0146)	1.0298 (0.0164)	1.7079 (0.0174)
Nasdaq				
CPPI	1	1.2707	2.1626	4.1098
OBPI	1	1.0902	1.7109	3.1296
S ($\bar{\sigma}$)	1	1.3186 (0.0085)	2.2595 (0.0088)	4.2871 (0.0085)
Nikkei225				
CPPI	1	0.9345	1.3385	2.8230
OBPI	1	1.0087	1.2329	2.3134
S ($\bar{\sigma}$)	1	0.8836 (0.0128)	1.4495 (0.0114)	3.1098 (0.0116)
S&P500				
CPPI	1	1.2045	2.2776	3.5928
OBPI	1	1.0446	1.7726	2.7324
S ($\bar{\sigma}$)	1	1.2429 (0.0070)	2.3542 (0.0068)	3.7324 (0.0700)
Volkswagen				
CPPI	1	1.3629	1.1666	2.9301
OBPI	1	1.2470	1.2324	2.6590
S ($\bar{\sigma}$)	1	1.4796 (0.0206)	1.2754 (0.0172)	3.5457 (0.0165)

Table 4.4: *Results for portfolio simulations with LIBOR 6-month. CPPI and OBPI show the values of both portfolio strategies. S ($\bar{\sigma}$) is the value of the stock and the mean volatility in parenthesis.*

Value	Initial	1 year	3 years	5 years
BMW				
CPPI	1	0.8931	0.9305	1.2345
OBPI	1	1.0129	1.0577	1.3253
S ($\bar{\sigma}$)	1	0.7788 (0.0190)	0.8350 (0.0155)	1.3709 (0.0155)
Dow Jones				
CPPI	1	1.1261	1.9249	2.7574
OBPI	1	1.0190	1.5231	2.1313
S ($\bar{\sigma}$)	1	1.1556 (0.0071)	1.9963 (0.0070)	2.8663 (0.0070)
Ford				
CPPI	1	0.9032	0.9112	1.1283
OBPI	1	1.0028	1.0213	1.1678
S ($\bar{\sigma}$)	1	0.8162 (0.0149)	0.8093 (0.0161)	1.1841 (0.0159)
General Motors				
CPPI	1	0.9872	1.0276	1.4955
OBPI	1	1.0173	1.0784	1.4366
S ($\bar{\sigma}$)	1	0.9744 (0.0146)	1.0327 (0.0164)	1.6955 (0.0173)
Nasdaq				
CPPI	1	1.2704	2.1580	4.1145
OBPI	1	1.0896	1.7066	3.1305
S ($\bar{\sigma}$)	1	1.3178 (0.0085)	2.2539(0.0088)	4.2894(0.0085)
Nikkei225				
CPPI	1	0.9360	1.3448	2.8282
OBPI	1	1.0086	1.2343	2.3131
S ($\bar{\sigma}$)	1	0.8846(0.0128)	1.4541 (0.0114)	3.1101(0.0116)
S&P500				
CPPI	1	1.2055	2.2715	3.6036
OBPI	1	1.0447	1.7674	2.7383
S ($\bar{\sigma}$)	1	1.2424 (0.0070)	2.3473 (0.0068)	3.7383 (0.0070)
Volkswagen				
CPPI	1	1.3624	1.1670	2.9288
OBPI	1	1.2461	1.2306	2.6457
S ($\bar{\sigma}$)	1	1.4779 (0.0206)	1.2710 (0.0172)	3.5276 (0.0165)

Table 4.5: *Results for portfolio simulations with LIBOR 12-month. CPPI and OBPI show the values of both portfolio strategies. S ($\bar{\sigma}$) is the value of the stock and the mean volatility in parenthesis.*

Value	Initial	1 year	3 years	5 years
BMW				
CPPI	1	0.8936	0.9365	1.2494
OBPI	1	1.0124	1.0589	1.3264
S ($\bar{\sigma}$)	1	0.7780 (0.0190)	0.8405 (0.0154)	1.3722 (0.0155)
Dow Jones				
CPPI	1	1.1255	1.9241	2.7700
OBPI	1	1.0186	1.5211	2.1372
S ($\bar{\sigma}$)	1	1.1539 (0.0071)	1.9940 (0.0070)	2.8764 (0.0069)
Ford				
CPPI	1	0.9047	0.9139	1.1345
OBPI	1	1.0027	1.0201	1.1661
S ($\bar{\sigma}$)	1	0.8174 (0.0149)	0.8098 (0.0161)	1.1803 (0.0159)
General Motors				
CPPI	1	0.9886	1.0327	1.5092
OBPI	1	1.0172	1.0782	1.4395
S ($\bar{\sigma}$)	1	0.9754 (0.0146)	1.0333 (0.0164)	1.7046 (0.0173)
Nasdaq				
CPPI	1	1.2719	2.1581	4.1294
OBPI	1	1.0900	1.7054	3.1383
S ($\bar{\sigma}$)	1	1.3184 (0.0085)	2.2527 (0.0088)	4.3021 (0.0085)
Nikkei225				
CPPI	1	0.9371	1.3510	2.8436
OBPI	1	1.0085	1.2351	2.3170
S ($\bar{\sigma}$)	1	0.8841 (0.0128)	1.4568 (0.0114)	3.1177(0.0116)
S&P500				
CPPI	1	1.2056	2.2770	3.5976
OBPI	1	1.0440	1.7701	2.7310
S ($\bar{\sigma}$)	1	1.2418 (0.0070)	2.3520 (0.0068)	3.7242 (0.0070)
Volkswagen				
CPPI	1	1.3694	1.1776	2.9373
OBPI	1	1.2492	1.2330	2.6395
S ($\bar{\sigma}$)	1	1.4852 (0.0206)	1.2765 (0.0172)	3.5210 (0.0165)

5
Conclusion

In Table 4.1 the results for the calibration of the interest rates are presented. The calibration was done by numerical maximum likelihood method, so there is the possibility that a local minimum has been found and not the global minimum. The table shows that for each year the 3-month, 6-month and 12-month LIBOR rates are quite similar. The parameter θ, which is the long term mean, have a much higher value than the values are for the interest rate. This means that these parameters can only be used in a limited time interval. Looking at Figure 4.1 it shows that the result for 1 year looks good and 3 years looks quite good. The result for 5 years does not look so good. This means that the 1 year interest rates are a good fit by a CIR process and also the 3 years interest rates can be simulated as a CIR process. The 5 year interest rates are not a CIR process and should better be estimated by some other random process.

The calibration results for the stocks opens up concerns in some cases. It is know from Appendix B that the skewness is negative for all cases. The calibrations results in Table 4.2 shows that β is positive in many cases, meaning that the distribution will have positive skewness, when in fact the data is negatively skewed. Since there is time varying parameters there is no way to know the quantity of the skewness, but it is know that the skewness is positive, when in fact it should be negative. The calibration has been done numerically by maximum likelihood method, so there is the possibility that the variables found are not the true variables.

Looking at Figure 4.2 it shows the result for 3 different simulations. In a), which is a 1 year simulation, the results are not as good as wanted. The simulation does not get as high peak as the data and the simulation have more observations in the beginning of the tails than the data. One problem is that the data is not as smooth as for 5 years. The simulation for 1 year does not give as good fit as the 5 year simulation. A one year simulation is only 250 simulated random variables which also means that only a few 'bad' variables have much effect on the result. This is a problem that happens often with the 1 year simulations. In the 5 year simulation in c) the results are good. A 5 year simulation includes 1250 random variables, which makes it more robust than the 1 year simulation. The 3 years simulation is based on 750 random variables, and it is better than the 1 year simulation, but not as good as the 5 year simulation. For the area in the beginning of the tails, there are some more observations than in the data. This is a problem with the NIG-GARCH model that happens quite often due to GARCH. The GARCH model is based on previous observations, so an extreme variable leads to higher probability of more extreme observations, due to the fact that the distribution of the log returns changes over time based on the GARCH model. It can be concluded that the NIG-GARCH model is a good model, but there are still some problems with this model.

For the portfolio strategy simulations it is clear that the stock price that is most important for how the portfolio change. For a decreasing stock market the OBPI performs better than CPPI. For the values that are chosen in the simulations used in this thesis the floor for the OBPI is higher than the floor for CPPI, which explains the performance in a decreasing market. For a slightly increasing stock market we have that OBPI and CPPI performs very similarly. Which of the strategies that performs better in this case depends both on the time and the stochastic volatility. For a significant increasing stock market the CPPI performs better than the OBPI. The CPPI strategy will in this case invest more and more money in the stock, making the growth better than in the case of OBPI where the investment in the stock is fixed.

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A

Bond price for a CIR process

This appendix includes a calculation for a zero-coupon bond for a CIR process. The calculations are based on Shreve [12, p. 272-276]. The interest rate model is a CIR process which is expressed by

$$dR_t = \kappa(\theta - R_t)dt + \sigma\sqrt{R_t}dW_t,$$

where κ, $\theta > 0$, $\sigma^2 < 2\kappa\theta$ and W_t is a standard Wiener process. The discounted process D_t is expressed by

$$D_t = \exp(-\int_0^t R(s)ds)$$

and it follows from this that the differential of D_t is

$$dD_t = -R_t D_t dt.$$

A zero-coupon bond is a contract that promises to pay the face value K (here it is chosen that $K = 1$) at time of maturity T. At time T it is guaranteed to get the value K and at the time leading up to T the bond will have a value lower than K, depending on the value of interest rate R_t. Considering the risk-neutral pricing formula it is known that the bond should be a martingale under the risk-neutral measure, meaning that the price of the bond is

$$D_t B(t,T) = \tilde{\mathbb{E}}[D_T|\mathcal{F}(t)], \tag{A.1}$$

with $B(T,T) = 1$. This will give the price formula for a zero-coupon bond to be

$$B(t,T) = \tilde{\mathbb{E}}[\exp(\int_t^T R(s)ds)|\mathcal{F}(t)].$$

The interest rate R_t is given by a stochastic differential equation (SDE) and it is a Markov process, and it must also hold that

$$B(t,T) = f(t,R_t)$$

for some function $f(t,r)$. We want to find the partial differential equation (PDE) for the function $f(t,r)$, so we find a martingale, take its differential and set its dt term equal to zero. The martingale will be the function from equation (A.1), i.e

$D_tB(t,T) = D_tf(t,R_t)$, which have the differential

$$\begin{aligned} d(D_tf(t,R_t)) &= f(t,R_t)dD_t + D_tdf(t,R_t) \\ &= D_t\Big(-Rf(t,R_t)dt + f_t(t,R_t)dt + f_R(t,R_t)dR_t + \frac{1}{2}f_{RR}dR_tdR_t\Big) \\ &= D_t\Big(-R_tf(t,R_t) + f_t(t,R_t) + \kappa(\theta - R_t)f_R(t,R_t) \\ &\quad + \frac{1}{2}\sigma^2R_tf_{RR}(t,R_t)\Big) + D_t\sigma\sqrt{R_t}dW_t, \end{aligned}$$

which is calculated by using Ito's formula and Ito's product rule. Now use the dt term for calculating the bond price the PDE of the bond price is

$$f_t(t,r) + \kappa(\theta - r)f_r(t,r) + \frac{1}{2}\sigma^2rf_{rr}(t,r) = rf(t,r). \tag{A.2}$$

Now guess the solution to be

$$f(t,r) = \exp(-rC(t,T) - A(t,T))$$

and put this expression of $f(t,r)$ in Equation (A.2). This will be

$$\Big((-C'(t,T) + \kappa C(t,T) + \frac{1}{2}\sigma^2C^2(t,T) - 1)r - A'(t,T) - \kappa\theta C(t,T)\Big)f(t,r) = 0,$$

where it must hold that $C(T,T) = A(T,T) = 0$. Now it must be that the r term is zero and therefore there are the two ordinary differential equations (ODE)

$$C'(t,T) = \kappa C(t,T) + \frac{1}{2}\sigma^2C^2(t,T) - 1$$

and

$$A'(t,T) = -\kappa\theta C(t,T).$$

The solution of $A(t,T)$ and $C(t,T)$ is

$$C(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\kappa\sinh(\gamma(T-t))}, \tag{A.3}$$

and

$$A(T,t) = -\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{\gamma\exp(\frac{1}{2}\kappa(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\kappa\sinh(\gamma(T-t))}\right), \tag{A.4}$$

where $\gamma = \frac{1}{2}\sqrt{\kappa^2 + 2\sigma^2}$. The bond price will be

$$B(t,T) = \exp(-R_tC(t,T) - A(t,T)),$$

with $C(t,T)$ as in Equation (A.3), $A(t,T)$ as in Equation (A.4) and R_t is the interest rate from a CIR process.

B

NIG goodness of fit

This appendix contains information on the goodness of fit for the NIG distribution compared to the normal distribution. Note that no statistical test have been made and that the goodness of fit is done by looking at the values and graphs and compare the NIG distribution with the normal distribution. In Table B.1 are the calibrations results for a NIG process.

Table B.1: *Calibration results for the NIG distribution for the stocks.*

	α	β	μ	δ
BMW 1 year	55.5664	-9.5762	0.0037	0.0255
BMW 3 years	55.2221	-7.3534	0.0023	0.0185
BMW 5 years	67.4266	-8.1323	0.0028	0.0204
Dow Jones 1 year	142.0513	-20.0863	0.0017	0.0095
Dow Jones 3 years	117.7326	-11.7186	0.0010	0.0078
Dow Jones 5 years	115.6097	-7.6197	0.0009	0.0077
Ford 1 year	63.4294	-10.2994	0.0019	0.0156
Ford 3 years	48.6697	-4.1206	0.0005	0.0117
Ford 5 years	58.5758	-2.1980	0.0006	0.0156
General Motors 1 year	84.4321	-6.5504	0.0016	0.0212
General Motors 3 years	58.8766	-3.9140	0.0010	0.0164
General Motors 5 years	61.5733	-1.8088	0.0008	0.0193
Nasdaq 1 year	83.0211	-14.9955	0.0020	0.0087
Nasdaq 3 years	85.4966	-20.3406	0.0026	0.0086
Nasdaq 5 years	103.5277	-24.9447	0.0030	0.0096
Nikkei 225 1 year	63.7231	-9.1142	0.0017	0.0139
Nikkei 225 3 years	82.3038	-12.5492	0.0022	0.0135
Nikkei 225 5 years	102.6659	-8.4403	0.0020	0.0177
S&P500 1 year	131.1383	-6.5769	0.0008	0.0090
S&P500 3 years	116.7815	-10.4292	0.0010	0.0074
S&P500 5 years	105.7119	-7.3283	0.0010	0.0074
Volkswagen 1 year	42.0533	5.5355	-0.0030	0.0244
Volkswagen 3 years	33.7683	-3.0903	0.0009	0.0159
Volkswagen 5 years	42.8087	-3.6565	0.0016	0.0168

Table B.2 contains the result for mean, variance, skewness and kurtosis, with the sample values in parenthesis. It is especially skewness and kurtosis that are of interest in this case. The normal distribution has skewness 0 and kurtosis 3. Comparing the NIG values and normal values to the sample values it is clear that NIG is a better fit than normal, for both skewness and kurtosis.

Table B.2: *Mean, variance, skewness and kurtosis for a NIG distribution, with its corresponding sample values in parenthesis.*

	Mean	Variance	Skewness	Kurtosis
BMW 1 year	-0.0007 (-0.0007)	0.0010 (0.0005)	-0.4372 (-0.8491)	5.4003 (7.5674)
BMW 3 years	-0.0002 (-0.0002)	0.0007 (0.0003)	-0.3970 (-0.6114)	6.1726 (7.3232)
BMW 5 years	0.0003 (0.0003)	0.0006 (0.0003)	-0.3097 (-0.5310)	5.3250 (6.4804)
Dow Jones 1 year	0.0003 (0.0003)	0.0001 (0.0001)	-0.3661(-0.4024)	5.4128(4.1728)
Dow Jones 3 years	0.0003 (0.0003)	0.0001 (0.0001)	-0.3133 (-0.2822)	6.4332 (4.9728)
Dow Jones 5 years	0.0004 (0.0004)	0.0001 (0.0001)	-0.2095 (-0.1227)	6.4260 (5.1586)
Ford 1 year	-0.0007 (-0.0007)	0.0005 (0.0003)	-0.4927 (-0.6565)	6.3932 (5.7259)
Ford 3 years	-0.0005 (-0.0005)	0.0005 (0.0003)	-0.3377 (-0.8679)	8.4552 (19.4473)
Ford 5 years	0.0000 (0.0000)	0.0005 (0.0003)	-0.1177 (-0.5935)	6.2973 (12.4187)
General Motors 1 year	-0.0000 (-0.0000)	0.0005 (0.0003)	-0.1742 (-0.2296)	4.7206 (3.8237)
General Motors 3 years	-0.0001 (-0.0001)	0.0006 (0.0003)	-0.2031 (-0.3349)	6.1658 (6.8195)
General Motors 5 years	0.0002 (0.0002)	0.0006 (0.0003)	-0.0809 (-0.0323)	5.5352 (5.8314)
Nasdaq 1 year	0.0004 (0.0004)	0.0002 (0.0001)	-0.6420 (-0.7083)	7.7605 (5.3678)
Nasdaq 3 years	0.0005 (0.0005)	0.0002 (0.0001)	-0.8454 (-1.0172)	8.1622 (15.9547)
Nasdaq 5 years	0.0006 (0.0006)	0.0002 (0.0001)	-0.7364 (-0.7785)	6.8371 (12.9921)
Nikkei 225 1 year	-0.0003(-0.0003)	0.0005 (0.0002)	-0.4575 (-0.6843)	6.6899 (6.5125)
Nikkei 225 3 years	0.0001(0.0001)	0.0004 (0.0002)	-0.4365 (-0.4756)	5.9860 (6.2913)
Nikkei 225 5 years	0.0005 (0.0005)	0.0004 (0.0002)	-0.1830 (-0.3082)	4.6970 (5.5281)
S&P500 1 year	0.0003 (0.0003)	0.0001 (0.0001)	-0.1386 (-0.3590)	5.5725 (4.4398)
S&P500 3 years	0.0003 (0.0003)	0.0001 (0.0001)	-0.2888(-0.3651)	6.5965(5.2995)
S&P500 5 years	0.0005 (0.0005)	0.0001 (0.0001)	-0.2357 (-0.1898)	6.9286(5.3162)
Volkswagen 1 year	0.0002 (0.0002)	0.0012 (0.0006)	0.3917 (0.1611)	6.1556 (4.9450)
Volkswagen 3 years	-0.0005 (-0.0005)	0.0010 (0.0005)	-0.3752 (-1.0288)	8.7922 (13.8383)
Volkswagen 5 years	0.0002 (0.0002)	0.0008 (0.0004)	-0.3023 (-0.8724)	7.2965 (12.9956)

Figure B.1-B.8 shows the histogram of the log return compared to the pdf of both NIG and normal distributions. By looking at the figures it is clear that the NIG distribution is a better fit than the normal distribution. Notice how the NIG distribution follows the peak much higher than the normal distribution. Even if it is hard to see it is noticeable that the NIG are negatively skewed. Also notice that the tails for the NIG distribution are lighter, than for the normal distribution.

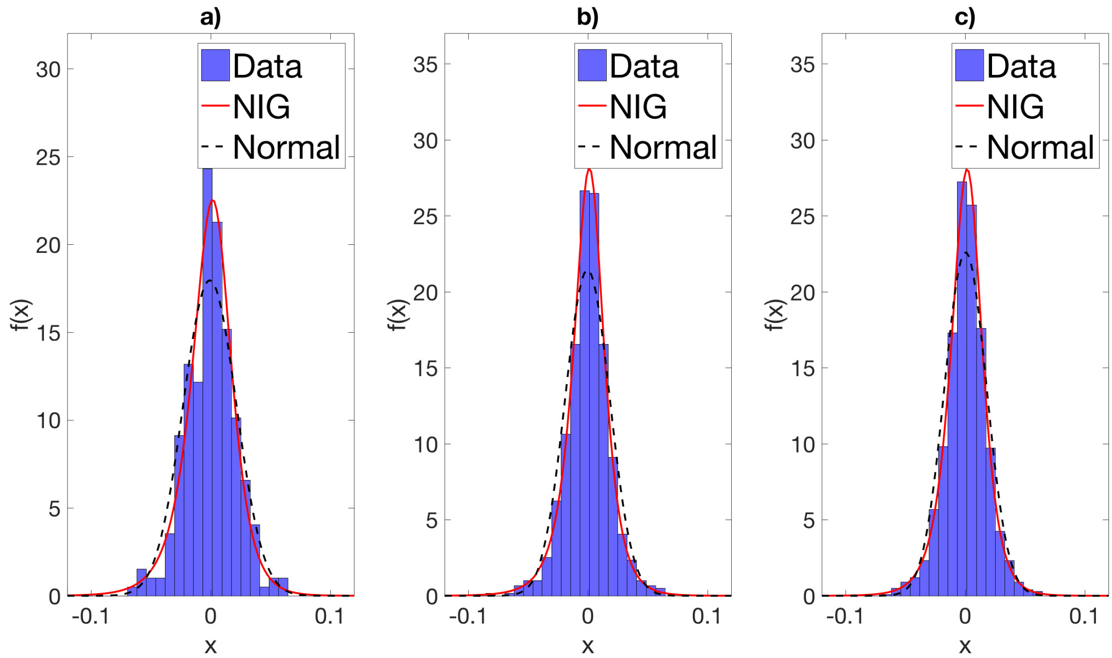

Figure B.1: *Histogram for the log returns for BMW with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

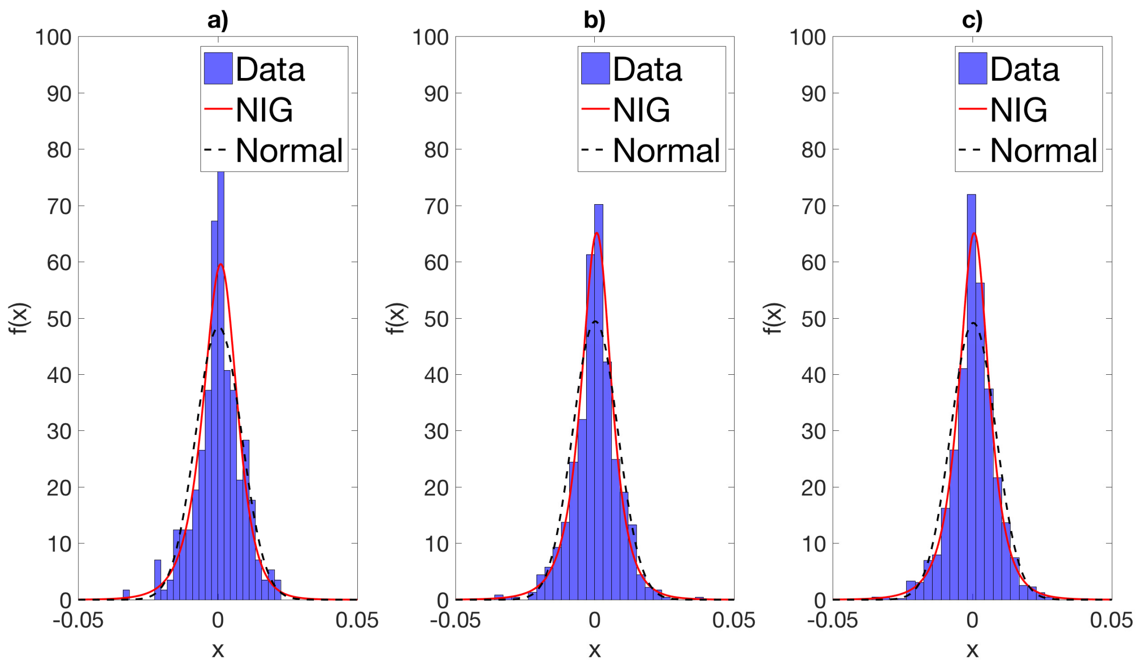

Figure B.2: *Histogram for the log returns for Dow Jones with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

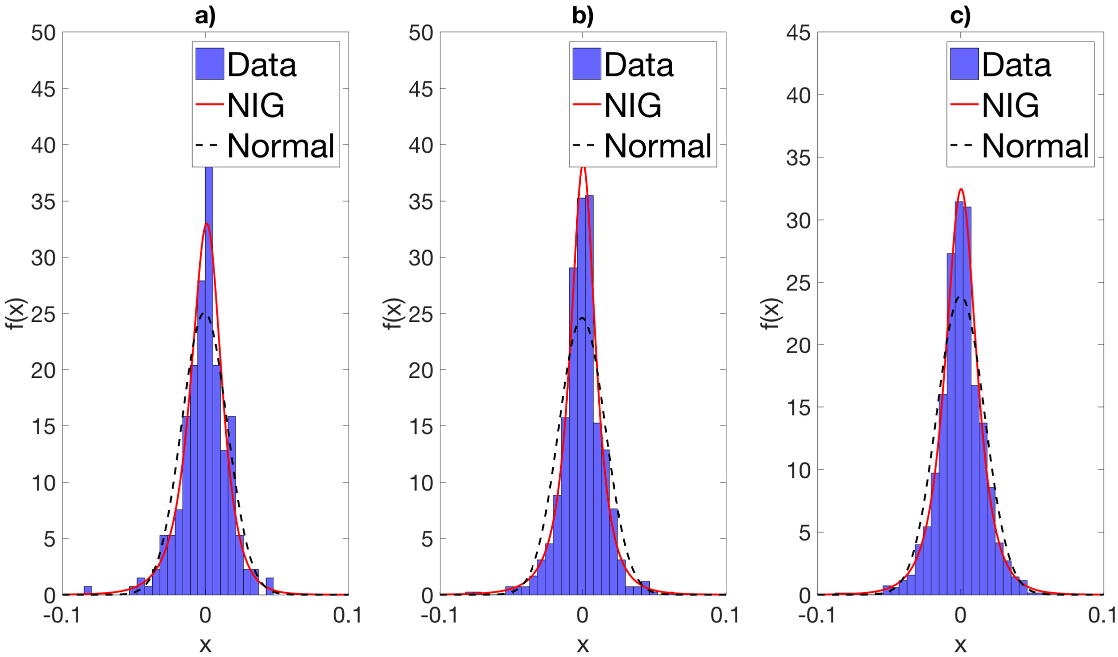

Figure B.3: *Histogram for the log returns for Ford with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

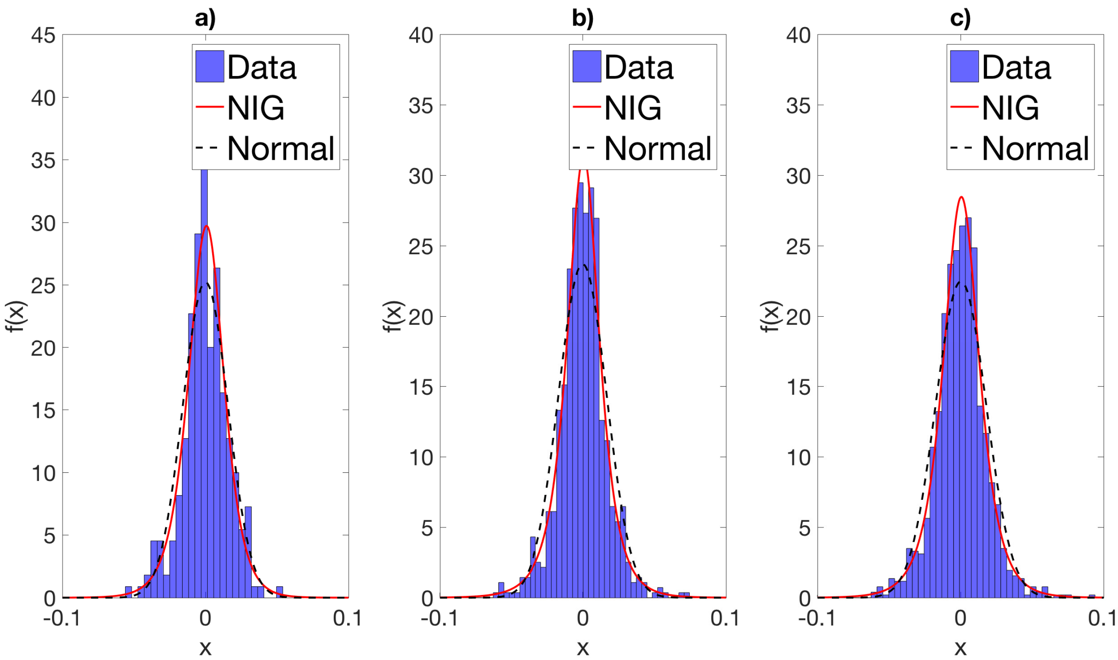

Figure B.4: *Histogram for the log returns for General Motors with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

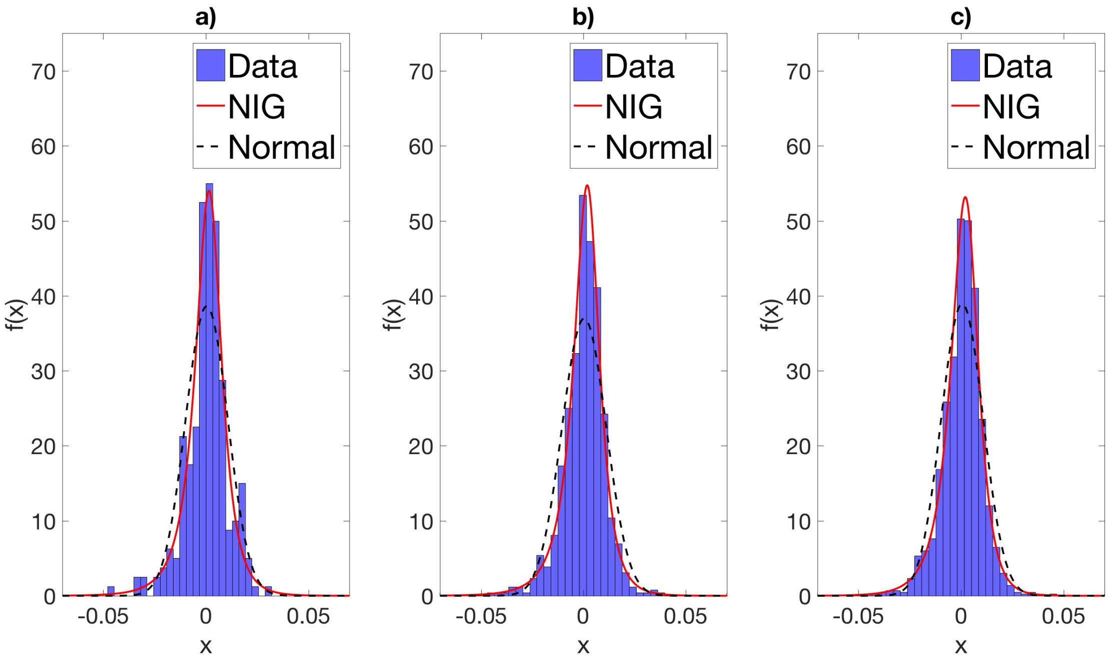

Figure B.5: *Histogram for the log returns for Nasdaq with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

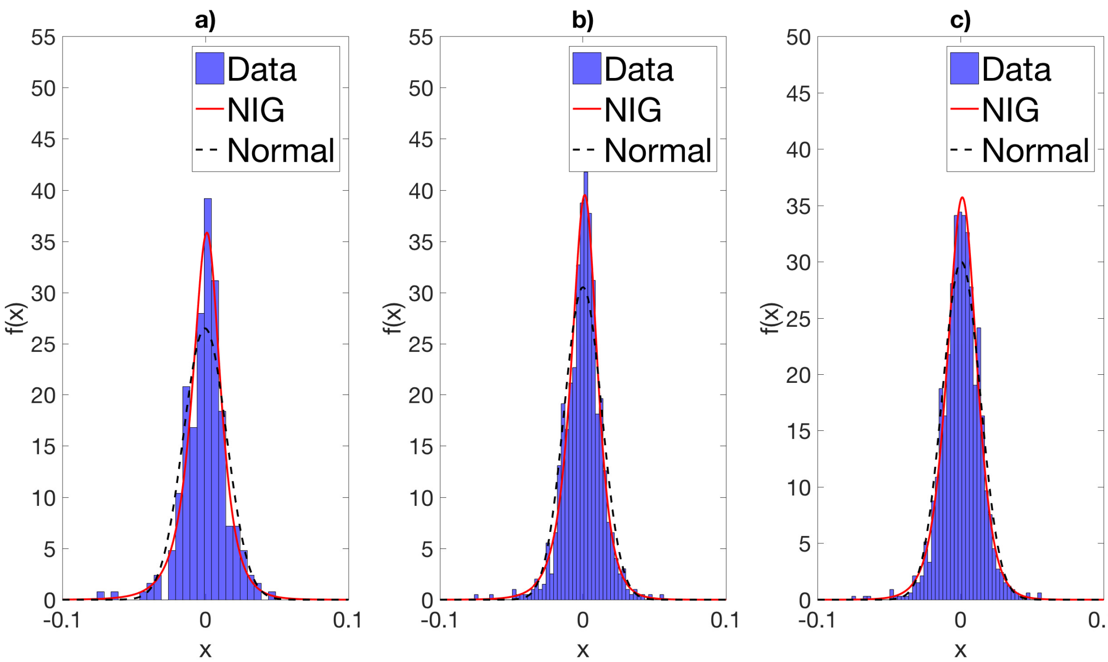

Figure B.6: *Histogram for the log returns for Nikkei225 with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

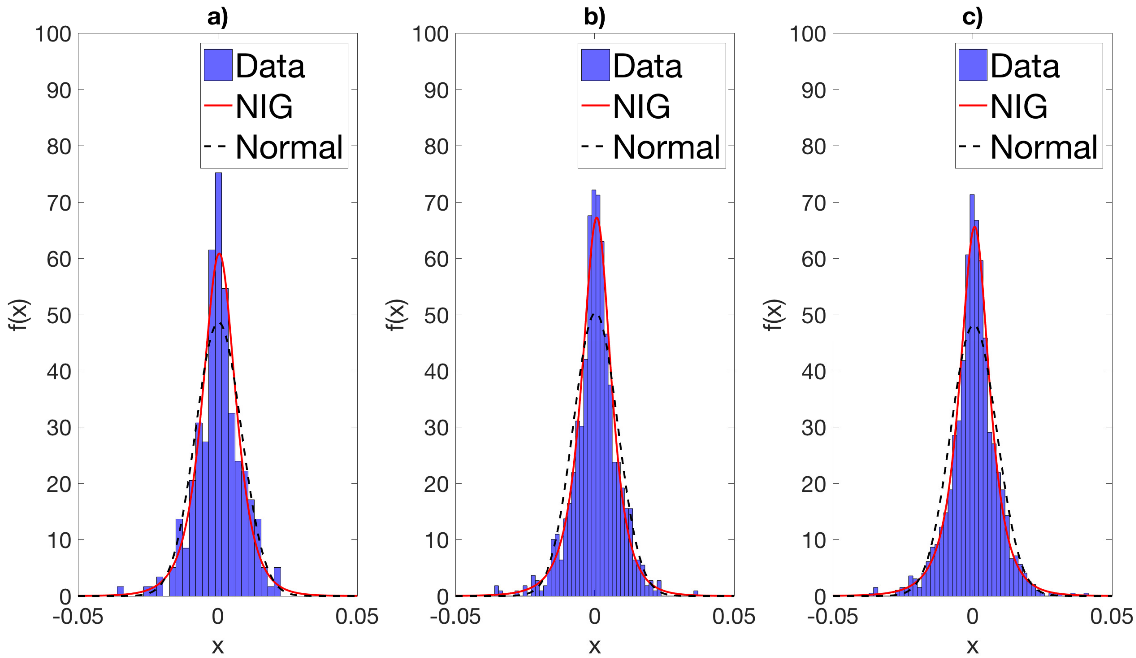

Figure B.7: *Histogram for the log returns for S&P 500 with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*

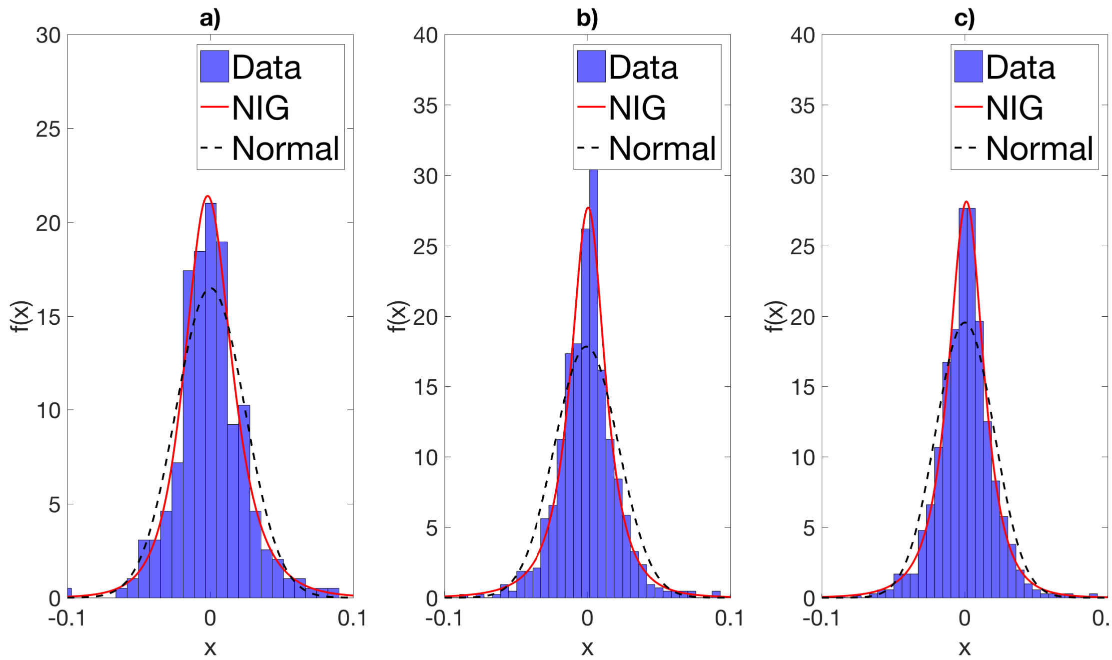

Figure B.8: *Histogram for the log returns for Volkswagen with its corresponding pdf for NIG and normal distribution for a) 1 year, b) 3 years and c) 5 years.*