CHALMERS
UNIVERSITY OF TECHNOLOGY

The Borsuk-Ulam Theorem in Synthetic Stone Duality

Master's Thesis in Computer Science and Engineering

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CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2025
www.chalmers.se

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Master's in Computer Science - algorithms, languages and logic
Date: June 03, 2025

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Abstract

The recently introduced *synthetic Stone duality* [1] is an extension of homotopy type theory (HoTT), which is conjecturally modeled in the higher topos of *light condensed anima* as recently introduced by Clausen and Scholze [2]. As it turns out, synthetic Stone duality turns out to be an appropriate setting in which to develop a restricted form of point-set topology. We survey the development of point-set topology within synthetic Stone duality, recovering a working theory of second countable compact Hausdorff spaces and, in particular, a working interval whose topology is the usual metric topology. Using the interval, we are able to define topological paths and loops, as well as the topological fundamental group, in the standard way.

We then turn to the theory of *higher modalities* in HoTT to relate topological spaces, as developed within synthetic Stone duality, to their homotopy types. After recalling the basics of the theory of higher modalities we are able to define the *shape* modality as the nullification of the interval. The shape modality, which is conjectured to more generally convert (topological) finite CW complexes to their homotopy types, converts the topological circle $\mathbb{S}^1$ into its higher inductive counterpart S^1. Using shape, we are able to pass back and forth between reasoning about the topological and homotopical circle as convenient. As an application, we use the shape modality to provide a characterization of $\mathbb{R}$, long known as folklore, in terms of the modal unit $\eta : \mathbb{S}^1 \to S^1$ for the topological circle. Similarly using the shape modality, we are able to show that topological paths in $\mathbb{S}^1$ lift through the standard covering map $\mathbb{R} \twoheadrightarrow \mathbb{S}^1$ while bypassing the standard covering space theory, and prove the two-dimensional Borsuk-Ulam theorem as an application.

Keywords

Homotopy Type Theory, Condensed Mathematics

Acknowledgments

Thanks to David for being my sounding board throughout the whole process of writing this. And special thanks to my supervisor Felix, who was willing to take me on and patiently guided me through the process of learning, even when I felt out of my depth (which was almost all the time). Our many in-person discussions are at the root of essentially every idea in this thesis.

Gothenburg, June 2025
Armaan Rashid

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Chapter 1
Introduction

The present work is a new proof of an old theorem: the *Borsuk-Ulam* theorem in topology, which states that any continuous function from the n-sphere $\mathbb{S}^n$ to $\mathbb{R}^n$ must map two antipodal points to the same point in $\mathbb{R}^n$. This is a very concrete and intuitive geometric statement in low dimensions. Recall that $\mathbb{S}^n$ is the sphere that lives in $n + 1$ dimensions, so $\mathbb{S}^1$ is precisely a circle on the 2D plane and $\mathbb{S}^2$ is what would be called a sphere in everyday parlance. The Borsuk-Ulam theorem states that no matter how one continuously "squishes" a circle onto a line, or a sphere onto a plane, two opposing points must be "squished" onto the same point.

The Borsuk-Ulam theorem is, in truth, a theorem that bridges many areas of mathematics, given its many different proofs – some of which are heavily combinatorial, indicating a profound connection with discrete mathematics. However, this thesis is less about the actual theorem, and more about exhibiting the power of some new mathematical technology.

Part of that technology is *homotopy type theory* (HoTT) [3] a recently introduced new logical foundation for mathematics, based on the dependent type theory [4] of Per Martin-Löf. The key idea of HoTT is to take *type* as the basic notion of mathematical object instead of *set*, where a type can be equipped with higher homotopical structure very concisely. For the reader with a background in set-theoretic mathematics, we provide an introduction to "naïve homotopy type theory" in the next chapter.

The other part is the recently introduced *condensed mathematics* [2] of Dustin Clausen and Peter Scholze, a completely new formalism for doing geometry and topology whose goal is to combine algebraic and analytic geometry under one framework. To oversimplify things, the basic object of geometry in modern math is ordinarily a set X with "extra structure" – e.g. the structure of a metric space, affine scheme or smooth manifold, where this structure can take the form of a specified set of subsets of X and/or functions on X, depending on the context. Arguably the most basic such structure is that of a *topology*, a distinguished set of subsets which indirectly specify the relative proximity of points in X; a set with a topology, or *topological space*, is what Clausen and Scholze aim to replace with their *light condensed sets.*

Clausen and Scholze have developed not just the formalism (see [2]) but also reproven some classic theorems (see [5]) like the Hirzebruch-Riemann-Roch theorem within it, using "formal nonsense" (i.e. category theory) made available to them in the framework. In some sense this thesis continues the work of reproving old theorems from a condensed point of view – but there is a type-theoretic twist.

Light condensed sets are defined, belying their name, not as sets but as special functors, *sheaves* on the category of *Stone spaces*, the dual spaces of Boolean algebras. These functors themselves assemble into a category, the category of light condensed sets **LCond**; as a category of sheaves **LCond** is in fact a *topos*. We do not rigorously define this notion, originally due to Grothendieck, but just informally think of toposes as categories which are amenable to some very abstract kind of geometry. There is a deep relationship between toposes – or rather their modern variant ∞-toposes – and the aforementioned homotopy type theory (HoTT): any ∞-topos interprets HoTT semantically, as shown by Shulman [6]. With that in mind, one is able to either work directly with a topos like **LCond**, sometimes referred to as working *externally*, or in the topos' *internal* type theory. For **LCond** it is conjectured that the recently introduced *synthetic Stone duality* [1] is this internal type theory. Indeed, current research by Coquand, Höfer and Sattler [7] builds a conjectured constructive model of synthetic Stone duality in **LCond** or, rather, a higher topos variant of **LCond**.[1] Since we assume most readers are unfamiliar with how a type theory can be the "internal language" of a topos, a vast topic, we briefly digress to explain this notion by means of an analogy.

As the "internal type theory" of light condensed sets, we mean that synthetic Stone duality is a formal language that natively expresses the properties of the category **LCond**. We will go into a bit more detail in Chapter 2, but for now we consider the parallel here with Euclidean geometry, which can be done in two ways. Assuming the usual axioms, including the parallel postulate, and working with points and lines directly, was seen as orginary geometry for millennia. In light of the introduction of Cartesian coordinates, however, this geometry is sometimes dubbed "synthetic" geometry, as opposed to the same geometry which can be done *with* coordinates in $\mathbb{R}^2$, "analytic" geometry. $\mathbb{R}^2$ might be seen as a *model* that interprets Euclidean geometry. By this we mean exactly that there is a dictionary translating from Euclidean geometry into $\mathbb{R}^2$: one goes from synthetic to analytic essentially by fixing an origin. Working in synthetic Stone duality, which is just homotopy type theory extended by a few axioms, is like working with (synthetic) Euclidean geometry: whatever we prove in this setting is (conjectured to be) translatable back into the category **LCond** of light condensed sets.

Given that **LCond** was introduced as an environment in which to do geometry, it is not entirely surprising that its type-theoretic counterpart, synthetic Stone duality, turns out to be especially attractive as an environment in which to work with geometry and topology. As it turns out, we can use its axioms to redevelop a new kind of point-set topology, and stage an interaction between this topology and the homotopical notions that are "builtin" to HoTT. By topology we mean the conventional setup: a bare set of points equipped with some set of "open sets" that describe the relative proximity of points. The ability to stage the interaction between topology and homotopy so cleanly, using an advanced HoTT tool known as a *higher modality*, is the key interaction we explore while proving the two-dimensional Borsuk-Ulam theorem.

[1] To be precise since we usually model HoTT in higher toposes, while **LCond** is a 1-topos, the type theory is conjectured to be modeled by the ∞-topos of condensed anima.

This thesis is in large part expository, detailing some of the recently developed tools for doing geometry and topology in HoTT and proving the Borsuk-Ulam theorem in two dimensions as a demonstration of the power of these tools. The combination of several ingredients that allows us to give the proof. First of all we are able, via the axioms of synthetic Stone duality, to develop a reasonable notion of topology for an important class of spaces: second countable compact Hausdorff spaces. In particular we can develop a well-behaved interval with the expected topology, which allows us to define the topological circle $\mathbb{S}^1$ in the expected way, as the set of points in $\mathbb{R}^2$ with unit norm. This permits the standard development of the theory of topological paths and loops in a space. The standard proof of Borsuk-Ulam from [8], whose outline we still broadly follow, would normally invoke the theory of covering spaces. However in HoTT, by using higher modalities, we are able to use a 'synthetic' version of covering space theory as originally outlined in [9]. Both the ordinary topology that becomes possible with synthetic Stone duality, and the higher modalities (as well as higher inductive types) available in HoTT, make the proof possible.

Chapter 2 offers a rapid introduction to homotopy type theory, with a particular emphasis on some of its more advanced features (like propositional truncation) that we use frequently. We also offer a very brief recap of some aspects of classical point-set topology, with an eye to the parts we will redevelop from scratch in Chapter 4. Finally we describe condensed mathematics in a nontechnical way – in some sense, one of the main perks of working in synthetic Stone duality is to avoid the topos-theoretic technicalities of condensed maths. The reader is encouraged to only take what they need from this chapter at a first pass, potentially coming back to it as a reference.

Chapter 3 offers preliminaries on Boolean algebras and their dual spaces, called spectra or *Stone spaces* in honor of Marshall Stone, the originator of Stone duality. Again this may be skipped by the reader who is familiar, the reader keeping in mind that we restrict our attention to *countably presented* Boolean algebras throughout this thesis.

Chapter 4 is the exposition of synthetic Stone duality as featured in [1]. All results here are naturally due to the cited paper; in many cases we give heavily expanded proofs, emphasizing the development of the topology. Not all of this chapter is used in the subsequent chapters, but we offer a more complete exposition in order to thoroughly characterize Boolean algebras and their duals, Stone spaces, since these both lie at the root of the topology that emerges. However, these results are not necessary to understand the development of the synthetic topology, particularly of (second countable) compact Hausdorff spaces, which occupies the last part of this chapter. A striking feature that emerges is how much of the topology of compact Hausdorff spaces that we characterize in this chapter emerges from algebraic properties of Boolean algebras.

Chapter 5 actually uses the theory developed in the previous chapter to characterize a familiar topological space: the interval $\mathbb{I} = [0, 1]$, from which we can define the topological spheres $\mathbb{S}^n$. We are able to recover many of the familiar topological properties, often with different methods.

Chapter 6 takes a break from synthetic Stone duality to spend some time on the theory of modalities in HoTT, where we give the different equivalent characterizations of *higher modality* as developed in [10]. We introduce the crucial *shape modality* using the interval defined in the previous chapter, which is conjectured to take a well-behaved class of topological spaces to their homotopy types as higher inductive types. This does hold in the case of the topological circle $\mathbb{S}^1$, whose shape is homotopical S^1; we use this to give a new characterization of the real numbers.

In Chapter 7 we relate the material developed in the previous chapters, introducing the classic topological notions of path, loop and fundamental group, since we are able to develop a well-behaved interval. Using the shape modality, we are able to develop a synthetic variant of the traditional covering space theory for topological spaces, showing that we can lift topological paths through the traditional map $\mathbb{R} \twoheadrightarrow \mathbb{S}^1$.

Finally, in Chapter 8 we are able to prove the two-dimensional Borsuk-Ulam theorem relatively easily using the machinery developed in the prior chapters.

Chapter 2
Context & Related Work

In this chapter we attempt to provide some mathematical context for synthetic Stone duality. The relevant parts of this chapter will largely depend on the background and expertise of the reader. Fundamentally the only "real" prerequisite to understand the exposition to come is expertise in homotopy type theory, whose advanced features (higher inductive types, modality) are used in a crucial way. But this would be dishonest: though we will develop point-set topology from scratch in synthetic Stone duality, the construction will likely appear bewildering for the reader who has not had prior exposure to ordinary point-set topology. We also assume a minimal background in category theory (in particular familiarity with limits and colimits) in this thesis.

2.1. Naïve Homotopy Type Theory

We very quickly recall the basics of homotopy type theory (HoTT), Martin-Löf dependent type theory (MLTT) [4] enriched with the univalence axiom which we will state momentarily. The canonical reference for HoTT [11] and its more modern variant [3] are rather encyclopedic. A shorter introduction is given by [12] which is written for people with a strong mathematical background but no type theory exposure.

In type theory *type* (denoted with capital letters $A, B, C\ldots$) is taken as the basic notion instead of *set*. Like sets in set theory, types have elements (denoted with lowercase letters $a, b, c\ldots$). In set theory, though it is often forgotten in practice, the elements are always sets. In type theory however there is total conceptual separation between types and their *elements* (also called *points* or *terms*). In particular types cannot be elements of other types, and elements cannot themselves be types. We use $:$ instead of $\in$ to denote membership of elements in types, as in the formula $x : A$. We note here also that type theory is its own deductive language. There are technically four valid judgments; we have already mentioned the most common one, the aforementioned element-in-type formula. This is again unlike set theory, whose formulas are written in first-order logic. Just like first-order logic type theory has an equality built in, referred to as *judgmental* equality and denoted by $\doteq$. Two types or two elements of the same type can be compared for judgmental equality but never two elements of the same type nor an element with a type.

Types are allowed to depend on other types and, crucially, on elements of other types. So given an element $x : A$ one can form the type $B(x)$ where the function-style notation indicates that the syntactic formulation of the type $B(x)$ uses the element x somewhere. Because of the possibility of dependence, we place every judgment in a context to make sure all types are always well-formed.

> **Definition 2.1: Context**
>
> A *context* is a list of elements in types, defined inductively as follows.
>
> The empty list is a context. Given a context Γ, if we have a type B depending *only* on elements in Γ then $\Gamma, y : B$ is a context.

With contexts defined, we can state precisely when we know two types A and B are *not* dependent on each other: whenever, in the *same* context Γ, A and B are both well-formed. When a type depends on an element of another type we use context to formally write

$$\Gamma, x : A \vdash B(x) \text{ type}$$

where $B(x)$ type is an instance of another judgment asserting that $B(x)$ is well-formed. In practice we will usually omit the context Γ and not explicitly state the well-formedness of types via this judgment, under the assumption that types are well-formed given the elements that have been mentioned in prior discussion.

When a type $B(x)$ depends on an element of A, i.e. $x : A$, to be well-formed, we call B a *type family over A*. A type family can also be considered as a function into the *universe* of types.

> **Definition 2.2: Universes**
>
> Every type A has an encoding in a *universe type* $\hat{A} : \mathcal{U}$ and with every universe we have a *universal type family* $\mathcal{T}$ over the universe which recovers the original types from their encodings. That is, if $\hat{A} : \mathcal{U}$ encodes the type A then $\mathcal{T}\hat{A} \doteq A$.
>
> Almost always we abuse notation and write $A : \mathcal{U}$ for the encoding of a type A. With this abuse of notation in hand we can consider a type family over A as just a function $B : A \to \mathcal{U}$ into the universe of types.

Notice that it is not the type A itself that is an element of the universe but rather its encoding, so the above definition does not technically contradict the dictum that types are not elements; in practice though we often conflate types and their encodings and consider types elements of the universe. Since $\mathcal{U}$ is itself a type it also lives in another universe type, but we neglect this detail as well since the entire present work does not ever need to go up in universe level.

Recall that in HoTT types are built *inductively* via introduction rules that specify how to build elements, induction rules specifying how to build dependent functions (defined below) out of a type, and computation rules ensuring that the induction rules behave consistently. (The induction rule for dependent function types themselves works a bit differently.) We rarely work with these rules explicitly when dealing with types so omit a formal treatment; we refer the reader to Chapters 1-5 of [3] for a comprehensive introduction. We recall the basic *type formers* which allow us to build types out of each other.

Definition 2.3: Dependent Function

Say B is a type family over A. Then there is a *dependent function* type

$$\prod_{x:A} B(x)$$

If $\Gamma, x : A \vdash b : B(x)$ then via the introduction rule we can form the function $\Gamma \vdash \lambda x.b : \prod_{x:A} B(x)$. Similarly given a function in context $\Gamma \vdash f : \prod_{x:A} B(x)$ we can always call the function on a term x in context $\Gamma, x : A \vdash f(x) \prod_{x:A} B(x)$. In type-theory parlance this is the *elimination rule* for functions, what one might naturally refer to as "function application."

When A and B do not depend on each other $\prod_{x:A} B$ is written just $A \to B$, recovering the ordinary notion of function.

Definition 2.4: Dependent Sum

Say B is a type family over A. Then there is a *dependent sum* (sometimes *dependent pair*) type

$$\sum_{x:A} B(x)$$

with essentially the same introduction rule and induction rule. Given $a : A, b : B(a)$ we have a term $(a, b) : \sum_{x:A} B(x)$. And to give a dependent function

$$\prod_{z:\sum_{x:A} B(x)} P(z)$$

where P is some type depending on z, it suffices to give a dependent function

$$\prod_{x:A} \prod_{y:B(x)} P((x, y))$$

Where $B : A \to \mathcal{U}$ is a type family over A recall that the type $\sum_{x:A} B(x)$ is oftentimes referred to as the *total space* of the type family. Finally, we discuss the one type former that behaves as usual, i.e. does not have a "dependent" generalization.

Definition 2.5: Coproducts

Given A, B there is the *coproduct type* $A + B$ with introduction rule as follows. Given $a : A$ we have $\mathrm{inl}(a) : A + B$ and similarly given $b : B$ we have $\mathrm{inr}(b) : A + B$. Furthermore the induction rule states that to give a function $A + B \to C$ it suffices to give $f : A \to C$ and $g : B \to C$.

We assume that the universe of types is closed under all of the aforementioned type formers. Finally we recall the essential example of a dependent type, which allows us to express equality in type theory.

Example 2.5.1: Identity Type

Say $x, y : A$. Then we assert that the type

$$x \underset{A}{=} y$$

exists and is the *identity type* of x and y; almost always we drop the subscript $\underset{A}{=}$ under the $=$ sign when the type is clear from context. It comes equipped with the introduction rule $\mathrm{refl}_x : x = x$ ("reflexivity") and the induction rule which states that, in order to define a function

$$f : \prod_{x,y:A} \prod_{p:x=y} P(x, y, p)$$

it suffices to define

$$f_{\mathrm{refl}} : \prod_{x:A} P(x, x, \mathrm{refl}_x)$$

This induction rule is often referred to as *path induction*. On that note, the identity type is sometimes referred to as the *path space* of a type.

We further assume the universe is closed under identity types. We assume familiarity with some of the basic types that the above type formers make available to us, like the type of homotopies between functions, equivalences between types and the fiber types of non-dependent functions; further we assume that $\emptyset$, $\mathbb{1}$, the finite types $\mathbb{F}$ and $\mathbb{N}$ exist with their usual induction principles. Again we refer to [3] for a clear treatment. Finally we recall the fundamental notion of *truncation level* and, related to it, the higher inductive *n-truncation*.

Definition 2.6: Truncation Level

Let $\mathbb{T} \simeq \mathbb{N}$ be the natural numbers reindexed starting at −2. Let $k : \mathbb{T}$. We define what it means for a type A to be k-truncated by induction on k

$$\begin{aligned}(-2)\text{-trunc }(A) &:= \sum_{a:A} \prod_{x:A} a = x \\ (k+1)\text{-trunc }(A) &:= \prod_{x,y:A} k\text{-trunc }(x = y)\end{aligned}$$

If a type is -2 truncated we say it is *contractible*.

If a type is -1 truncated we call it a *proposition*, since any two elements (if they exist) can be identified. That is, if a proposition is inhabited, it is contractible.

If a type is 0 truncated we call it a *set*: if two elements are equal they are equal in just one way.

If a type is k truncated it is immediately $k+1$ truncated. When a type is k truncated we often say it is a *k-type*.

Recall that every type has an identity type, so the 'generic' identity type over any two elements $\prod_{x,y:A} x = y$ can be arbitrarily iterated – for example given two points $x, y : A$ we can form the type $x = y$ and then the generic identity type over any two points in this type, i.e. $\prod_{p,q} p = q$. The truncation level k of a type A, when it exists, expresses that the iteration of the identity types of A in this manner $k + 2$ times leads to a contractible type. This is a bit finicky to think about for the general identity type quantified over any two elements; if we focus on just one element $x : A$ and iterate the *loop space* (i.e. the paths from x to itself) as follows it is a bit more concrete.

$$\begin{aligned}
\Omega(X, x) &:= x = x \\
\Omega^2(X, x) &:= \mathrm{refl}_x = \mathrm{refl}_x \\
\Omega^3(X, x) &:= \mathrm{refl}_{\mathrm{refl}_x} = \mathrm{refl}_{\mathrm{refl}_x} \\
&\vdots
\end{aligned}$$

Letting $\Omega^0(X) := X$ for convenience, if X is a −2-type then tautologically $\Omega^0(X) \doteq X$ itself is contractible, if X is a −1-type then $\Omega^1(X, x) \doteq x = x$ is contractible, and so on. We interpret the presence of nontrivial (i.e. not contractible) structure in the higher identity types of a type as higher homotopical structure. In general when a type has a distinguished point $x : X$ we write $\Omega(X)$ for its loop space, omitting explicit mention of the point x.

Given a type, there is a universal way to force it to be k-truncated.

Definition 2.7: The n-Truncation

The n truncation is a higher inductive type that receives a type A and "cuts off" all of the path structure above truncation level n.

We explicitly give the introduction rules for the -1 truncation or *propositional truncation* $\| - \|_{-1}$.

$$\begin{aligned}
| - |_{-1} &: A \to \|A\|_{-1} \\
\alpha &: \prod_{x,y:\ \|A\|_{-1}} x = y
\end{aligned}$$

and *set truncation* $\| - \|_0$

$$\begin{aligned}
| - |_0 &: A \to \|A\|_0 \\
\alpha &: \prod_{x,y:\ \|A\|_0} \prod_{p,q:x=y} p = q
\end{aligned}$$

We reserve the general $\|A\|$ (without a subscript) to mean $\|A\|_{-1}$, propositional truncation. When there is a term in the propositional truncation of a type, $a : \|A\|$, we say that A is *merely* inhabited.

The map $| - |_n : A \to \|A\|_n$ is often called the *unit* of the n-truncation.[2]

> From these first two examples of truncation the overall pattern is hopefully clear: we force a type A to be n truncated by just asserting all elements of the $n + 1$st iterated path space to be equal via an introduction rule.

The n-truncation is a fundamental operation that is maybe best understood through its universal property. That is, if A is any type and B is n truncated, then the below diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle |-|_n}\big\downarrow & \overset{\exists!}{\nearrow} & \\ \|A\|_n & & \end{array}$$

always admits a unique lift. In other words a function from a type A into an n-truncated type B always factors through $\|A\|_n$, so in some sense $\|A\|_n$ is the "minimal" way to truncate the type A. Furthermore the existence of the unique lift shows that when mapping into a type B of truncation level n from $\|A\|_n$ it suffices to give a map from A itself. We use this, especially for propositional truncation, constantly in the following, often without comment. n-truncation allows us to make an important homotopical definition.

> **Definition 2.8: Fundamental Group and Homotopy Group**
>
> The nth homotopy group of a pointed type (X, x) is the set truncation of the nth iterated loop space as defined above, i.e.
>
> $$\pi_n(X) := \|\Omega^n(X)\|_0$$
>
> $\pi_1(X)$ is called the *fundamental group*.

Observe that the name "group" is justified as the type of paths from an element to itself automatically has a group structure via path concatenation and inversion.

The n-truncation also allows us to discuss a property of types dual to the truncation level.

> **Definition 2.9: n-Connectedness**
>
> Given a type A we say it is n-connected when $\|A\|_n$ is contractible. When A is 0-connected, so its set truncation is contractible, we just say it is connected.

Where n-truncatedness expressed the property that a type had no higher path structure "above" level n (where the identity type of a type is "higher" than the original type), connectedness expresses that a type has no nontrivial structure *below* level n.

With truncation level in hand, we can discuss subtypes and logic. Subtypes of sets, i.e. subsets, are used extensively in this thesis.

---

[2]Not to be confused with the unit type.

Definition 2.10: Subtype

Given a family of propositions $P : A \to \mathcal{U}$ over a type A, the type

$$\sum_{x:A} P(x)$$

is a *subtype* of A, where elements (x, p) in the type are just an element $x : A$ along with the witness $p : P(x)$ that it satisfies a proposition, analogous to the usual notion of subset in set theory. We force the type family P to be a family of propositions so that we can "ignore" the witness p.

When A itself is a set we say a subtype is a *subset*. In this case we do use subset notation $B \subset A$ to denote the embedding $\mathrm{pr}_1 : \sum_{x:A} B(x) \hookrightarrow A$.

Using the propositional truncation we can also give the equivalent of the traditional (intuitionistic) logic connectives, recalling that in general the logic of HoTT is not classical, so one cannot apply the law of excluded middle in general.

Definition 2.11: Type-theoretic Logic

We use the dependent function to define a $\forall$ quantifier as a type

$$\forall x : A.P(x) := \prod_{x:A} P(x)$$

We use the non-dependent product to define "conjunction" of types

$$A \wedge B := A \times B$$

We use the function type to define implication: here the notation already coincides, so $A \to B$ means A implies B. We define $\bot := \emptyset$ and the "negation" of a type to be $\neg A := A \to \bot$.

We use the truncated dependent sum to define an $\exists$ quantifier as a type

$$\exists_{x:A}.P(x) := \|\sum_{x:A} P(x)\|$$

We use the truncated coproduct to define disjunction

$$A \vee B := \|A + B\|$$

In what comes we will sometimes switch back and forth between the "logic" notation and the native type-theoretic notation depending on our emphasis: when we want to emphasize the actual terms in the type we prefer the type theoretic notation.

Finally, we recall the fundamental univalence axiom and one of its consequences, function extensionality.

Definition 2.12: Univalence Axiom

Let A, B be types, and consider their corresponding elements in the universe $A, B : \mathcal{U}$. As elements of $\mathcal{U}$ there is an identity type $A =_{\mathcal{U}} B$. By path induction we obtain a map

$$\text{univalence} : A = B \to A \simeq B$$

by taking $\text{refl}_A \mapsto \text{id}_A$ the latter of which is clearly an equivalence.

The univalence axiom asserts that the map univalence is an equivalence, so

$$A =_{\mathcal{U}} B \simeq A \simeq B$$

It is not obvious to prove, but univalence implies function extensionality, which also comes in a surprising form in HoTT.

Definition 2.13: Function Extensionality and Weak Function Extensionality

In HoTT as a consequence of univalence we have *function extensionality*

$$f \sim g \simeq f = g$$

where $f, g : A \to B$. This is equivalent to *weak function extensionality* which asserts that truncation level commutes with the dependent function type, that is

$$\prod_{x:A} \text{is-contr}(B(x)) \simeq \text{is-contr}\left(\prod_{x:A} B(x)\right)$$

which immediately generalizes to any truncation level $k : \mathbb{T}$ by induction.

Both function extensionality and the univalence axiom are used extensively in the following, often without comment.

Finally, we close out this rapid-fire introduction by introducing the canonical example of a higher inductive type, the homotopical circle S^1.

Definition 2.14: The Homotopical Circle

The homotopical circle S^1 is a type with the following constructors

$$\begin{aligned} &\text{base} : S^1 \\ &\text{loop} : \text{base} = \text{base} \end{aligned}$$

The induction rule for the circle states that to give a function $f : \prod_{x:S^1} P(x)$ it suffices to give a point $u : P(\text{base})$ and a witness that u is invariant to transporting over loop, i.e.

$$\alpha : \text{tr}_P(\text{loop}, u) = u$$

This circle lacks any geometric information we associate with a circle; it is more accurately seen as a type representing the *homotopy type* of a circle. It is not trivial to prove, but the fundamental group of the circle in fact turns out to be $\mathbb{Z}$ as expected; that is

$$\|\text{base} = \text{base}\|_0 \simeq \mathbb{Z}$$

In fact the untruncated loop space base = base is equivalent to $\mathbb{Z}$, expressing the fact that the circle has no higher homotopy groups.

The homotopical circle is a prominent character later on in this thesis; in fact we will be able to relate it to the topological circle $\mathbb{S}^1$ (i.e. the conventional set of points in $\mathbb{R}^2$ at distance 1 from the origin) to show that it does actually represent the homotopy type of the circle we are familiar with.

2.2. Topology and the Borsuk-Ulam Theorem

The goal of the present thesis is to prove the two-dimensional Borsuk-Ulam theorem, a classic result in point-set topology, in synthetic Stone duality. Chapters 3 and 4 will develop a version of point-set topology from scratch, but we briefly recall the basics to motivate the constructions made there. This section is written "in set theory", even though we know from the previous section that we can define and discuss sets and subsets in type theory, to highlight the fact that the traditional presentation of point-set topology (e.g. [13]) relies in a crucial way on the nonconstructive *axiom of choice* (the "C" in ZFC). Throughout this brief section we highlight these nonconstructive uses in order to observe the obstacles synthetic Stone duality, which only admits a weak form of nonconstructivity, faces on the way to develop a working theory of point-set topology. The following is quite far from a comprehensive account, for which the standard is still [13].

Given a set X, a topology on X is simply a distinguished set of subsets of X.

Definition 2.15: Topology on a Set

Given a set X a *topology* on X is a subset $\mathcal{T} \subseteq \mathcal{P}(X)$ of the powerset obeying the following axioms

1. $\emptyset \in \mathcal{T}$
2. $X \in \mathcal{T}$
3. $\mathcal{F} \subseteq \mathcal{T} \Rightarrow \bigcup \mathcal{F} \in \mathcal{T}$ for any family of sets $\mathcal{F}$ in $\mathcal{T}$
4. $\mathcal{J} \subseteq \mathcal{T} \Rightarrow \bigcap \mathcal{F} \in \mathcal{T}$ for any *finite* family of sets $\mathcal{J}$ in $\mathcal{T}$

A set X equipped with a topology $\mathcal{T}$ is called a *topological space*. Almost always the topology $\mathcal{T}$ is left implicit since it is clear from context and we just refer to X when we mean $X, \mathcal{T}$. The elements of $\mathcal{T}$ are called *open sets*. The complement of an open set is a *closed set*.

The "surprise" in the definition of a topology $\mathcal{T}$ – to be closed under only finite intersection but all unions – is indeed what gives the subject its distinctive feel. Since (arbitrary) union and intersection of sets distribute over each other, a topology also forms a *complete Heyting algebra* or a *frame* and much of topology can be developed by relegating the points of X to the background and privileging the set of subsets $\mathcal{T}$, which is also often called a *locale* viewed in the opposite category. The locale viewpoint is explored deeply in some modern accounts such as [14].

Historically, topology arose in large part from the desire to make calculus, known in its modern incarnation as analysis, more rigorous. Indeed, certain properties of $\mathbb{R}$ and functions on $\mathbb{R}$ were known to be fundamental, but difficult to isolate. One such key property was the *continuity* of functions on $\mathbb{R}$, a premathematically intuitive notion that stubbornly resists formalization. A key contribution of topology is to make this notion rigorous by giving a precise definition.

Definition 2.16: Continuous Function

A continuous function between topological spaces $f : X \to Y$ is an ordinary set-function such that, for every open set $U \subseteq Y$ the preimage $f^{-1}(U)$ is open in X.

In other words, a function f is continuous when it induces a frame homomorphism $f^{-1} : \mathcal{T}(Y) \to \mathcal{T}(X)$.

The Borsuk-Ulam theorem in general states that any continuous function

$$f : \mathbb{S}^n \to \mathbb{R}^n$$

must fail to be injective on antipodal points of the sphere, so there exists $x : \mathbb{S}^n$ such that $f(x) = f(-x)$. (Here we use the notation $\mathbb{S}^n$ to mean the standard n-dimensional sphere, to differentiate from the higher inductive types S^n in HoTT.) The one-dimensional theorem $f : \mathbb{S}^1 \to \mathbb{R}$ falls out immediately as a consequence of the intermediate value theorem, which has been proved in synthetic Stone duality [1]. In two dimensions, the theorem is still geometrically intuitive, though not easy to prove. The conventional proof uses covering space theory, but our proof will use the advanced HoTT tool of modality [10] available in HoTT.

Here we recall two fundamental properties that a topological space can have.

Definition 2.17: Connectedness

A topological space X is *connected* whenever there do not exist two disjoint nonempty open subsets $U, V \subseteq X$ whose union is X. When U, V do exist they are necessarily closed and open or *clopen* and we refer to the pair as a *separation* of X.

Equivalently, X is connected when every function $f : X \to 2$ is constant, where 2 is the set of two elements with the discrete topology.

When a space is not connected, so admits a separation we say it is *disconnected.* When for any two points $x, y \in X$ there is a separation U, V of X so that $x \in U$ and $y \in V$ we say X is *totally disconnected.*

When we discuss connectedness in our development of synthetic topology, we will opt for the latter of the two equivalent definitions given above. Finally we have the all-important notion of compactness.

Definition 2.18: Compactness

A space X is compact when every open cover $\bigcup_{i \in I} U_i$ of X admits a finite subcover. In other words, every open cover is "really" just a finite open cover since all but finitely many sets from the cover can be removed.

Compactness is especially well-behaved when a topological space's open sets are very well-separated by open sets. There is a hierarchy of *separation axioms* for general topological spaces which we partially recap here.

Definition 2.19: Separation Axioms

There is a hierarchy of increasingly strong *separation axioms* for a general topological space X.

We say X is T_1 when for any $x, y \in X$ there is U, V open such that $x \in U, y \notin U$ and vice versa for V. This turns out to be equivalent to having all singleton sets, so all finite sets, be closed in X.

We say X is T_2 or *Hausdorff* when for any $x, y \in X$ there is U, V open **and disjoint** so that $x \in U, y \notin U$ and vice versa for V.

We say X is T_3 or *regular* when it is T_1 and for any $x \in X$ and C closed in X, there is U, V open and disjoint so that $x \in U$ and $C \subseteq V$. This implies $x \notin V$ and $C \cap U = \emptyset$.

We say X is T_4 or *normal* when it is T_1 and for any C, D closed in X, there is U, V open **and disjoint** so that $C \subseteq U$ and $D \subseteq V$.

When a space is both compact and Hausdorff, it is particularly well-behaved, and we pay special attention to them in the development of synthetic Stone duality. In fact a great deal of the exposition in that chapter is devoted to showing that the compact Hausdorff spaces in synthetic Stone duality – whose definition has apparently nothing to do with compactness or Hausdorffness as defined above – behave as expected.

Indeed, the synthetic definition begins with the following theorem from classical topology which we state without proof.

Theorem 2.20: Every Compact Hausdorff Space Is a Quotient of Stone Space

A *Stone space* is a totally disconnected compact Hausdorff space. Every compact Hausdorff space X admits a quotient map from a Stone space.

The classical theory of Stone duality reveals that these arise naturally as the "dual spaces" of Boolean algebras. As it turns out, a Stone space is always a cofiltered limit of finite sets with the discrete topology or a *profinite set*, the limit being taken in the category **Top** of topological spaces and continuous functions. Synthetic Stone duality naturally begins with the characterization of Stone spaces, using that to then generalize to characterizing a synthetic kind of compact Hausdorff space.

Finally, we note that one crucial difference between ordinary point-set topology and the topology we will see in synthetic Stone duality is that on a given set X there is in fact an entire complete lattice of different topologies it could have. On the other hand, open and closed sets will have a very different definition than Definition 2.15. This will force the spaces we discuss to come equipped with one *particular* topology, and so we will often refer to *the* topology on a set in that case.

2.3. Condensed Mathematics

Synthetic Stone duality, the type theory we introduce in §4, is conjectured to be the internal type theory of the recently introduced topos of *light condensed sets* [2]. Dustin Clausen and Peter Scholze, the progenitors of the theory, argue condensed sets should replace topological spaces in many cases, as part of the larger program of condensed mathematics. Clausen and Scholze were motivated to introduce the condensed formalism in order to have a new way of working with algebraic structures that come equipped with a topology (e.g. topological vector spaces and topological groups) since in general such objects can have degenerate properties making their categories ill-behaved. Unlike the simple axioms for a topology however, condensed sets require considerable machinery from category theory. For readers familiar with category theory and Grothendieck topologies, a light condensed set is a sheaf on the category of light profinite sets $\mathbf{ProFin}^{\text{light}}$ equipped with the coherent topology, where a (light) profinite set is a cofiltered limit of a (countable) diagram of finite sets.

By working with the internal type theory in [1] we can avoid much of the categorical machinery involved in condensed mahts, but passing to a slightly different perspective. Recall that in classical topology a *Stone space* is a totally disconnected compact Hausdorff space. Every Stone space is in fact a profinite set (see Lemma 4.12.4 below for a "synthetic" proof of this). But Stone spaces are, by the classical Stone representation theorem, the dual spaces of Boolean algebras, so there is an isomorphism of categories

$$\mathbf{Stone} \cong \mathbf{Boole}^{\text{op}}$$

We are interested in the restriction of this to the "light" version of each category: second countable Stone spaces and countably presented Boolean algebras respectively, which all induce equivalent categories of sheaves. In view of the above equivalence, the type theory of light condensed sets, hereafter referred to as *synthetic Stone duality*, takes Boolean algebras as a starting point as opposed to the profinite sets Clausen and Scholze spotlight, as is detailed further in the next chapter. Most importantly, synthetic Stone duality allows for the redevelopment of a large part of ordinary point-set topology in HoTT, which allows us to reprove some of the classic theorems of the subject in this new context, like the Borsuk-Ulam theorem we highlight in the present work.

2.4. Notation

We fix notation briefly; throughout this thesis since most of the types we deal with are sets we use a lot of set-theoretic shorthand. When $D : A \to \mathcal{U}$ is a family of propositions over a set A we write $D \subseteq A$. We will often conflate using D to refer to the total space $\sum_{a:A} D(a)$ and the embedding $\mathrm{pr}_1 : \sum_{a:A} D(a) \hookrightarrow A$. Given a subset $D \subseteq A$ and a term $x : A$ we write $x : D$ to indicate that $D(x)$ is inhabited. We use the notation of intersection and union as follows

$$\bigcap_{i:I} D_i :\equiv \sum_{a:A} \prod_{i:I} D_i(a)$$
$$\bigcup_{i:I} D_i :\equiv \sum_{a:A} \exists_{i:I} D_i(a)$$

A subset $D \subseteq A$ is decidable when there is a function in the type

$$\prod_{a:A} D(a) + \neg D(a)$$

which is the same thing as a map $A \to 2$. The singleton $\{a\}$, where $a : A$, naturally denotes the (contractible) subset

$$\sum_{x:A} a = x$$

Sometimes, when a dependent sum type is heavily nested, we swap the sum notation

$$\sum_{x:A} \sum_{y:B(x)} C(x, y)$$

for "named tuple" notation as follows

$$x : A, y : B(x), C(x, y)$$

to make nested types easier to read.

Chapter 3
Boolean Algebras and Spectra

To prepare for an exposition of the *synthetic Stone duality* presented by Cherubini et al. in [1], we very briefly review some basics of Boolean algebras and their spectra; this chapter is a heavily expanded version of the first section of the aforementioned paper. For the reader familiar with Boolean algebras and especially the classical Stone duality between totally disconnected compact Hausdorff topological spaces, i.e. *Stone spaces*, and Boolean algebras, this chapter can be easily skipped.

3.1. Boolean Algebras

We rapidly recall the basics of Boolean algebras, which are distributive lattices with a bijective complementation function satisfying De Morgan's laws. There are many fantastic references on the subject, though [15] is the most pertinent to our discussion of Stone duality.

Definition 3.1: Boolean Algebra

A Boolean algebra B is a set equipped with the structure of a bounded distributive lattice, along with a distinguished bijection.

For B to be a bounded distributive lattice means that it is equipped with associative and commutative operations $\wedge$ (conjunction) and $\vee$ (disjunction) obeying the *absorption* identities

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

Absorption allows a natural partial order structure to emerge by setting $a \leq b \leftrightarrow a \vee b = b \leftrightarrow a \wedge b = a$. As it turns out $\wedge$ and $\vee$ define the infimum and supremum of any two elements with respect to this order, and conversely any finite poset with finite suprema and infima form a lattice. Distributivity means that $\wedge$ and $\vee$ distribute over each other in the natural way

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

and boundedness just means that there are least and greatest elements $0, 1$ respectively, sometimes also denoted $\bot, \top$.

Furthermore a Boolean algebra comes with a bijection $\neg : B \simeq B$ that interacts with $\wedge$ and $\vee$ via *De Morgan's laws*:

$$\neg(a \vee b) = \neg a \wedge \neg b$$
$$\neg(a \wedge b) = \neg a \vee \neg b$$

For the reader familiar with classical propositional logic, the perhaps arcane-seeming algebraic equations in the above definition have a very intuitive logical meaning, interpreting $\wedge$ as "and", $\vee$ as "or" and $\neg$ as "not". Indeed we expand on this via an example.

Example 3.1.1: Boolean Algebras on Finitely Many Atoms

Take a finite set $p_1 \ldots p_n$; the logically inclined might like to interpret these as the atoms in a propositional language. The (free) Boolean algebra generated by these elements has 2^n elements and every element corresponds precisely to an equivalence class of formulae in the logic, where the equivalence relation is given by interderivability in one's chosen proof theory. Formulae with implication get interpreted precisely as one might expect

$$p \to q := \neg p \vee q$$

in the algebra. Since $\neg$ in a Boolean algebra is bijective, one has that $\neg\neg p = p$ or, in other words, the logic modeled is classical.

In view of the connection with logic, an element of the form $\neg b$ may be referred to as a *negated* element or the *negation* of b.

As usual for an algebraic structure there we often want to study a Boolean algebra in terms of a smaller subset of *generators*, some subset $G \subseteq B$ such that every element in B is some finite combination of operations on a subset of G. In the example just given the finite set of atoms are generators: every element is expressible as some combination of the atoms $p_1 \ldots p_n$ under the operations, observing that $p_1 \vee \neg p_1 = 1$ and dually $p_1 \wedge \neg p_1 = 0$. (One can always trivially take B itself to be the set of generators, though this does not help to characterize the algebra more simply.)

De Morgan's laws allow any element of a Boolean algebra to be rewritten in *conjunctive normal form*, i.e. a disjunction of conjunctions of its generators.

Lemma 3.1.2:

(Conjunctive Normal Form). Given an element $b : B$ in some Boolean algebra that is a finite combination of generators (g_i) under meet, join, and negation, b can be written as the finite disjunction of a conjunction of (possibly negated) generators. That is,

$$b = \bigvee_{j \in J} g_{j_1} \wedge \ldots \wedge g_{j_n}$$

where J is finite, and any g_{j_i} may be negated.

Proof. De Morgan's laws and distributivity essentially spell out the algorithm.

1. Shift any outer negations inwards repeatedly using de Morgan's laws, as in the following example

$$\neg(\varphi \vee \psi) = \neg\varphi \wedge \neg\psi$$
$$\neg(\varphi \wedge \psi) = \neg\varphi \vee \neg\psi$$

where φ, ψ themselves may be nested elements involving further generators, for which the procedure can be reapplied recursively. If φ, ψ are both generators, this step is finished. While executing this step, if φ or ψ are themselves negated, reduce the double negated $\neg\neg\varphi$ to φ.

2. Distribute $\wedge$ over $\vee$ recursively: if an element is of the form

$$\varphi \wedge (\psi \vee \chi) \text{ or } (\psi \vee \chi) \wedge \varphi$$

distribute $\wedge$ over $\vee$; applied to the left example above

$$(\varphi \wedge \psi) \vee (\varphi \vee \chi)$$

and apply recursively to φ, ψ, χ until all are generators or negated generators.

This algorithm terminates because formulae are written from finitely many generators.

We briefly note that the dual form is available.

Lemma 3.1.3:

(Disjunctive Normal Form). Given an element $b : B$ in some Boolean algebra that is a finite combination of generators (g_i) under meet, join, and negation, b can be written as the finite conjunction of a disjunction of (possibly negated) generators. That is,

$$b = \bigwedge_{j \in J} g_{j_1} \vee \ldots \vee g_{j_n}$$

where J is finite, and any g_{j_i} may be negated.

Proof. Apply the algorithm described for CNF using the *other* distributive law.

The ability to write terms in conjunctive normal form for Boolean algebras is surprisingly crucial to many of the proofs in synthetic Stone duality, especially those of a topological nature.

The prior example is actually typical for finite Boolean algebras: *all* finite Boolean algebras are of this form.

The following is the fundamental example of a Boolean algebra for the rest of this thesis.

Example 3.1.4: Free Countable Boolean Algebra

The *free Boolean algebra* on countably many generators $(g_n)_{n:\mathbb{N}}$, or "letters", denoted $2[\mathbb{N}]$, is the Boolean algebra generated by taking arbitrary finite meets and joins of (g_n) as well as the "negated letters" $(\neg g_n)_{n:\mathbb{N}}$. These "words" are quotiented out by the defining identities of a Boolean algebra, with a top and bottom element $0, 1$ adjoined.

The above example restricts to the finitely generated case of Example 3.1.1 by choosing cofinitely many (g_n) to be among the relations.

For those readers less familiar with lattice theory, one can translate everything to ring theory since Boolean algebras are in bijection with *Boolean rings*, commutative rings that are idempotent, but we do not go further into this characterization ([15] does take this approach, for the interested reader). We offer a final important example given by maps.

Example 3.1.5: $\mathbf{Hom}(-, \mathbb{2})$

Given a type X, the maps $X \to \mathbb{2}$ have a natural Boolean algebra structure computed pointwise. For example the join $f \vee g : X \to \mathbb{2}$ is given by

$$x \mapsto f(x) \vee g(x)$$

and similarly for the other operations.

The function space $X \to \mathbb{2}$ corresponds naturally to the decidable subtypes of X by using the decidable equality in $\mathbb{2}$; here we are making the simple observation that decidable subtypes are endowed with a Boolean algebra structure.

Like any algebraic structure defined by an equational theory, one can quotient a Boolean algebra B by a congruence, an equivalence relation $\sim$ that respects the operations. For Boolean algebras this explicitly means, assuming $\sim$ is the equivalence relation:

$$\begin{gathered} \text{if } x \sim y \text{ and } a \sim b \text{ then} \\ x \wedge a \sim y \wedge b \\ x \vee a \sim y \vee b \\ \neg x \sim \neg y \end{gathered}$$

An equivalence relation θ that has the above properties is a congruence and the equivalence classes enjoy a Boolean algebra structure with the join and meet defined representative-wise. One easily observes that, similar to how for groups a congruence is determined by the equivalence class of the unit, congruence classes of a Boolean algebra are in fact determined by the equivalence class at 0, motivating the definition below.

Definition 3.2: Ideal (Boolean Algebra)

An *ideal* in a Boolean algebra is the equivalence class of 0 in some congruence θ on B. Explicitly an ideal is a subset $I \subseteq B$ such that

$$\begin{aligned} &(1)\ 0 : I \\ &(2)\ a, b : I \to a \vee b : I \\ &(3)\ a : I \to b : B \to a \wedge b : I \end{aligned}$$

Observe the crucial fact that in the third condition b is *any* element in B, reminiscent of ideals in rings.

Condition (3) is equivalent to demanding that I be downward closed in the partial order on B, which we recall from Definition 3.1 is induced by the absorption axioms.

The *ideal generated by a subset* $S \subseteq B$, denoted (S), is the smallest ideal containing the set S, i.e. the intersection of all ideals containing S. Concretely this is just obtained by closing S under the above operations. When S is the image of some other set, i.e. $S = (x_j)_{j:J}$ we abuse notation and write $(x_j)_{j:J}$ for the corresponding ideal that is generated, when it is clear from context.

A *filter* is defined in precisely the order-dual manner.

Luckily, the ideal generated by a subset, a notion we use constantly, admits a satisfyingly explicit description; we recall the below theorem, originally shown by Stone.

Lemma 3.2.1:

Given a subset $S \subseteq B$ the ideal (S) generated by S is precisely the closure of S under finite joins, and further closed downwards. Explicitly, $p \in (S)$ iff for some finite subset $E \subseteq S$ one has

$$p \leq \bigvee E$$

Proof. See [16] Theorem 11.

In light of the above definition we will almost never refer to a Boolean algebra B quotiented by a congruence but, as is standard, rather to a Boolean algebra quotiented by an ideal, i.e. B/I or $B/(x_j)$ as appropriate. Having clarified the notion of ideal, we can finally make our fundamental definition.

Definition 3.3: Countably Presented Boolean Algebra

A Boolean algebra B is *countably presented* when

$$\| B \simeq 2[\mathbb{N}]/(f_j)_{j \in J} \|$$

and $J \subseteq \mathbb{N}$ is a decidable subset.

In other words, a countably presented Boolean algebra is the free Boolean algebra $2[\mathbb{N}]$ quotiented by a *countably generated* ideal, which is a stronger condition than just taking any quotient of $2[\mathbb{N}]$. We define Boole_ω to be exactly the subuniverse of countable Boolean algebras, with a natural category structure given by Boolean morphisms. Observe that depending on the type X, Example 3.1.5 above, while certainly being a Boolean algebra, may not be countably presented.

Now we can define the *spectrum* of a Boolean algebra which works somewhat like the *dual space* of a vector space; these spectra are more famously known as Stone spaces, since they come equipped with a natural topology. Exploring this topology occupies a large part of the next chapter.

In classical mathematics Stone spaces are generated by the ultrafilters[3] on a Boolean algebra. These are classified by Boolean morphisms into $\mathbb{2}$, hence the following definition.

Hereafter we suppress the adjective "countable" / "countably presented" except for emphasis.

Definition 3.4: Spectra and Stone Spaces

The *spectrum* of a Boolean algebra B is

$$\mathrm{Sp}(B) := \mathrm{Hom}_{\mathrm{Boole}_\omega}(B, \mathbb{2})$$

all the Boolean algebra morphisms (i.e. preserving meet, join, complement, 0, 1) from B into the Boolean algebra $\mathbb{2}$. A space that is equivalent to $\mathrm{Sp}(B)$ for some B is called a *Stone space*.

We highlight several important examples. First, we consider the spectrum of the free countable Boolean algebra.

Example 3.4.1: Cantor Space

Consider $f : \mathrm{Hom}_{\mathrm{Boole}_\omega}(\mathbb{2}[\mathbb{N}], \mathbb{2})$. Any such morphism is completely determined by its action on the generators $(g_n)_{n:\mathbb{N}}$ and this action is representable as a binary digit; naturally then a morphism is uniquely represented as a binary sequence, so

$$\mathrm{Sp}(\mathbb{2}[\mathbb{N}]) = \mathbb{2}^{\mathbb{N}}$$

applying univalence.

The elements of Cantor space $\mathbf{0} : \mathbb{2}^{\mathbb{N}}$, $\mathbf{1} : \mathbb{2}^{\mathbb{N}}$ denote the constant zero, one sequences respectively in what follows.

Next we highlight an equally important subspace of the above.

Example 3.4.2: Truncated Sequences

Consider the free Boolean algebra on a countable set of generators, $\mathbb{2}[\mathbb{N}]$ generated by $(g_n)_{n:\mathbb{N}}$ quotiented by the ideal

$$(g_m \wedge g_n)_{m \neq n}$$

We denote this B_∞; intuitively this is just 2[ℕ] with the "bottom half" removed.

An essential example of a Stone space is

$$\mathbb{N}_\infty := \mathrm{Sp}(B_\infty)$$

We claim

[3] Or prime ideals, or prime filters, or maximal ideals. In a Boolean algebra maximal and prime coincide for both filters and ideals, and prime filters and ideals are in bijection with each other by complementation.

$$\mathbb{N}_\infty \simeq \Sigma_{\alpha:2^{\mathbb{N}}} \prod_{i,j:\mathbb{N}} i \neq j \to \alpha_i \alpha_j = 0$$

That is, N_∞ is equivalent to the set of binary sequences that attain 1 at most once.

To see this, consider a Boolean morphism $f : B_\infty \to 2$. It is determined by its action on generators $g : \mathbb{N} \to B_\infty$. Define $\alpha := f \circ g$. It remains to show that for any f, this induced sequence hits 1 at most once; explicitly suppose $i \neq j$.

We want $f(g_i) = 0 \vee f(g_j) = 0$. Equality in $\mathbb{2}$ is decidable, so we can assume $f(g_i) = 1 \wedge f(g_j) = 1$ and derive a contradiction. Then $f(g_i) \wedge f(g_j) = 1 \wedge 1 = 1$ but by the relation and since f is a homomorphism, $f(g_i) \wedge f(g_j) = f(g_i \wedge g_j) = 0$, so $0 \underset{\mathbb{2}}{=} 1$, a contradiction.

Truncated sequences will come into play when we come to characterize the nonconstructivity available to us in synthetic Stone duality. We have a well-behaved relationship between the above two Stone spaces.

Lemma 3.4.3:

There is a natural map $\kappa : \mathbb{2}^{\mathbb{N}} \to \mathbb{N}_\infty$ that retracts the inclusion $\iota : \mathbb{N}_\infty \to \mathbb{2}^{\mathbb{N}}$, i.e. $\kappa \circ \iota = \mathrm{id}_{\mathbb{N}_\infty}$.

Proof. The map is given by

$$\alpha \mapsto n \mapsto \begin{cases} \alpha_n & \text{if} \quad \bigvee_{i<n} \alpha_i = 0 \\ 0 & \text{else} \end{cases}$$

that retains the first 1 in $\alpha : \mathbb{2}^{\mathbb{N}}$ if it exists and discards all further.

Binary sequences will play a key role in the theory we develop.

Related to the above example, we can rephrase a basic fact about binary sequences in terms of spectra.

Lemma 3.4.4:

Consider $\alpha : 2^{\mathbb{N}}$, a binary sequence.

$$\alpha = \mathbf{0} \leftrightarrow \mathrm{Sp}\left(\mathbb{2}/(\alpha_n)_{n:\mathbb{N}}\right)$$

Proof. The spectrum of the trivial one-element Boolean algebra is empty (since any morphism $\mathbb{1} \to \mathbb{2}$ maps 0 and 1 to the same element), and $\mathbb{2}/(\alpha_n)_{n:\mathbb{N}}$ is trivial if and only if for some k, $\alpha_k = 1$.

The above lemma also has a quasi-generalization: the spectrum of any Boolean algebra B is inhabited iff the algebra is nontrivial.

Lemma 3.4.5:

For any $B : \mathrm{Boole}_\omega$, $\mathrm{Sp}(B)$ is empty iff B is trivial, i.e. $0 \underset{B}{=} 1$.

Proof. Suppose we have $\neg\ \mathrm{Sp}(B)$. Then since Sp is an embedding we know B is (equivalent to) the trivial Boolean algebra. In the other direction, if B is trivial, there is no map into $\mathbb{2}$ preserving 0 and 1.

Because we do not have double negation stability in general in HoTT, it is surprising that we can show the dual of the above lemma: that $\mathrm{Sp}(B) \leftrightarrow \neg\left(0 =_B 1\right)$.

In §2.3 we described the relationship between condensed mathematics and Boolean algebras: condensed sets are sheaves on the category of *light profinite sets*, countable limits of finite sets. In classical mathematics, a profinite set is the "same thing" as a Stone space, i.e. the spectrum of a Boolean algebra, a result we recover in the next chapter, hence the development of the basic theory of Boolean algebras and Stone spaces above.

In classical mathematics one proves Stone duality as a theorem about Boolean algebras. We are going to assert Stone duality as an *axiom* in homotopy type theory and then show everything turns out as expected – in particular that Stone spaces are exactly profinite sets – and gain the ability to do topology with Stone spaces as a sort of "basis" for a broader class of topological spaces.

Chapter 4
Synthetic Stone Duality

Synthetic Stone duality is homotopy type theory enlarged with four axioms. It is the conjectured internal type theory of light condensed sets, and it is in this setting that we eventually provide a proof of the Borsuk-Ulam theorem. Dustin Clausen and Peter Scholze originally introduced condensed sets as drop-in replacement for general topological spaces, at least for central applications in number theory and algebraic topology. It is perhaps somewhat ironic then that working in the internal type theory of condensed sets allows one to redevelop a theory of point-set topology, albeit a theory that looks quite different from the classical one.

Our primary goal in this chapter is to develop exactly this theory, such that we can smoothly apply it in an ordinary topological proof of the Borsuk-Ulam theorem in two dimensions. The topology that emerges from synthetic Stone duality differs in many key aspects from ordinary point-set topology, however, so we detail the theory from scratch. Throughout we are guided by trying to synthetically recapture the familiar notions from point-set topology.

All of the results in this chapter appear in [1] originally. We occasionally omit proofs, but more frequently we offer more explicit proofs than the original.

4.1. Axioms

With the preliminaries about Boolean algebras and their spectra defined, we detail the four axioms of synthetic Stone duality.

Axiom 4.1: Stone Duality

For $B : \mathrm{Boole}_\omega$, the evaluation map

$$\begin{aligned} \mathrm{ev} : B &\to 2^{\mathrm{Sp}(B)} \\ b &\mapsto (f \mapsto f(b)) \end{aligned}$$

is an equivalence.

Using the axiom, we can show that being a Stone space is in fact a proposition.

Lemma 4.1.1:

For any $A : \mathcal{U}$, $\mathrm{Stone}(A)$ is a proposition.

Proof. We give a sketch. The nontrivial part is to show $\mathrm{Sp} : \mathrm{Boole}_\omega \hookrightarrow \mathrm{Stone}$ is an embedding, which is possible by Axiom 4.1. But then its fibers are propositions, and

$$\mathrm{fib}_{\mathrm{Sp}}(A) \doteq \sum_{B:\mathrm{Boole}_\omega} A = \mathrm{Sp}(B)$$

which is equivalent to $\mathrm{Stone}(A) \doteq \sum_{B:\mathrm{Boole}_\omega} A \simeq \mathrm{Sp}(B)$ by univalence.

Recall that 2^S gives just the decidable subsets of S for any set S, and enjoys a natural Boolean algebra structure, so the equivalence in Stone duality makes sense. The axiom is “forcing” the classical Stone duality to hold between elements of the Boolean algebra B and the decidable subsets of the Stone space $\mathrm{Sp}(B)$. In classical Stone duality, these decidable subsets are in fact the clopen, i.e. closed and open, subsets of a topological space. When we define a suitable notion of open and closed set we will see that these decidable subsets are precisely the clopen subsets.

We can extend Axiom 1 to Stone spaces themselves.

Lemma 4.1.2:

If S : Stone then

$$S \simeq \mathrm{Sp}(2^S)$$

Proof. Recall that, where $S = \mathrm{Sp}(B)$ we have $B \simeq 2^S$ by Stone duality under the equivalence $\mathrm{ev} := b \mapsto (f \mapsto f(b))$. Then $- \circ \mathrm{ev} : \mathrm{Sp}(2^S) \to S$ is an equivalence, recalling that $S = \mathrm{Sp}(B)$; it is easy to verify that for any $f : \mathrm{Sp}(2^S)$ that $f \circ \mathrm{ev} : B \to 2$ is in fact a Boolean morphism so that $f \circ \mathrm{ev} : S$. We can give its inverse explicitly in fact:

$$\begin{aligned} \gamma : S &\to \mathrm{Sp}(2^S) \\ s &\mapsto (\alpha \mapsto \alpha(s)) \end{aligned}$$

and show at least that this is a left inverse, which suffices. For any $s : S$ then we claim

$$(\alpha \mapsto \alpha(s)) \circ \mathrm{ev} = s$$

which we just show pointwise. Assuming $b : B$, then

$$(\alpha \mapsto \alpha(s))(\mathrm{ev}(b)) = \mathrm{ev}(b)(s) = s(b)$$

which suffices.

Observe that unlike in Axiom 4.1, the above is just an equivalence of sets, not an equivalence of Boolean algebras since S : Stone is in general not a Boolean algebra.

Stone duality is the axiom we use the most throughout the following, so we remark upon the general technique for its use here. Given a map $D : S \to 2$ for any S we fix the convention $s : S$ is in D, as a subset of S, if and only if $D(s) = 0$; indeed this is the proposition we use to define a decidable subset D from such a map.

In view of this, suppose that $S = \mathrm{Sp}(B)$ for some $B : \mathrm{Boole}_\omega$. (Henceforth when we refer to S : Stone and invoke $B : \mathrm{Boole}_\omega$ without comment, we of course mean the underlying algebra for which $S = \mathrm{Sp}(B)$.) Then by Stone duality $D : 2^S$ induces a unique element $d : B$ by a map $2^S \to B$, which is sent back to D under the evaluation map. That is, $D = s \mapsto s(d)$, so as a family of propositions over S, D is the subset given by $s(d) = 0$. We often "pass back and forth" between decidable subsets $D : 2^S$ and the unique corresponding elements $d : B$ silently. Here we remark that because of the convention we choose, the interaction with intersections and unions is contravariant: intersections of decidable subsets become joins of elements. (Of course one might also choose $D(s) = 1$ to be the defining proposition for being in a decidable subset.)

Lemma 4.1.3:

Suppose that $(D_n)_{n:\mathbb{N}}$ is some countable family of decidable subsets of S : Stone, with $(d_n)_{n:\mathbb{N}}$ the corresponding elements of the underlying Boolean algebra B. Then if

$$\bigcap_n D_n = \emptyset$$

then for some finite set $J \subseteq \mathbb{N}$

$$\bigvee_{j:J} d_j = 1$$

Proof. Consider $S/(d_n) := \mathrm{Sp}\big(B/(d_n)_{n:\mathbb{N}}\big)$ which has a natural inclusion $S/(d_n) \hookrightarrow S$. The maps $s : S/(d_n)$ are those maps $s : S$ such that, for all n, $s(d_n) = 0$, so in particular $S/(d_n) \subseteq \bigcap_n D_n$ and we know then that $S/(d_n) = \emptyset$ by assumption. But then by Lemma 3.4.5 the Boolean algebra $B/(d_n)$ is trivial, so in particular the ideal $(d_n)_{n:\mathbb{N}}$ includes 1. By Lemma 3.2.1 this means

$$1 \leq \bigvee_{j:J} d_j$$

where $J \subseteq \mathbb{N}$ finite, but of course this is an equality since 1 is the greatest element.

The above lemma, though apparently easy to show, is fundamental; the ability to pass from a countably infinite collection to a finite collection is the heart of topological compactness, and will be at the center of characterizing the topology on Stone spaces in the next section.

Now we introduce the next axiom, which is essential to characterizing synthetically the fact that Stone $\simeq \mathrm{Boole}_\omega^{\mathrm{op}}$.

Axiom 4.2: Mono and Epi

Given $B, C : \mathrm{Boole}_\omega$, one has

$$B \hookrightarrow C \leftrightarrow \mathrm{Sp}(C) \twoheadrightarrow \mathrm{Sp}(B)$$

This is a necessary property for the categories (represented as subuniverses) Stone and Boole_ω to be anti-equivalent – monomorphisms need map to epimorphisms and vice versa.

The next axiom is more surprising, but emerges from a categorical property in the topos of light condensed sets.

Axiom 4.3: Local Choice

Let $B : \mathrm{Boole}_\omega$ and $S := \mathrm{Sp}(B)$. If there is an inhabited family of propositions

$$p : \prod_{s:S} \|P(s)\| \text{ where } P : S \to \mathcal{U}$$

then there is merely another Stone space of which S is a quotient with a function into the untruncated term. Formally, there is a term

$$\|C : \mathrm{Boole}_\omega, q : \mathrm{Sp}(C) \twoheadrightarrow \mathrm{Sp}(B), \prod_{t:\,\mathrm{Sp}(C)} P(q(t))\|$$

The essential point of this is that if we are trying to prove a proposition, we can use the axiom to get access to an “actual term” in $P(s)$ for any s by surjectivity of q.

We recall the fundamental equivalence for a type $A : \mathcal{U}$

$$\left(\sum_{X:\mathcal{U}} X \to A\right) \simeq A \to \mathcal{U}$$
$$(X, f) \mapsto \mathrm{fib}_f$$

When f is surjective, that information is retained on the other side of the equivalence since we merely have a fiber, giving us the extended

$$\left(\sum_{X:\mathcal{U}} X \twoheadrightarrow A\right) \simeq \sum_{P:A\to\mathcal{U}} \prod_{x:A} \|P(x)\|$$
$$(X, f) \mapsto \mathrm{fib}_f$$

where above $P : A \to \mathcal{U}$ is any type family over A, not necessarily just a family of propositions. So the axiom can be rephrased in terms of the other side of the equivalence. Assume there is a surjection $E \twoheadrightarrow F$ and a map $f : S \to F$ where S : Stone. The left hand side of the equivalence then gives T : Stone and the data of the two dashed maps below making the square commute

$$\begin{array}{ccc} T & \dashrightarrow & E \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ S & \xrightarrow{\;f\;} & F \end{array}$$

The final axiom is separate from Stone concerns, and just expresses that the topos has a weak form of nonconstructivity, dependent choice.

Axiom 4.4: Dependent Choice

For a descending sequence of surjections E_n such that we have

$$\prod_{n:\mathbb{N}} E_{n+1} \twoheadrightarrow E_n$$

the sequential limit $\lim_k E_k$ exists and $\pi_0 : \lim_k E_k \to E_0$ is a surjection.

This is a categorical way of presenting "ordinary" dependent choice. In particular we have countable choice and so have an equivalence between Cauchy and Dedekind reals, which we use in constructing the interval. Using the above axioms, it is possible to very precisely characterize the nonconstructivity of synthetic Stone duality.

Theorem 4.5: No Weak Limited Principle of Omniscience

It is not the case that a binary sequence being constantly zero can be decided in general, that is:

$$\neg \prod_{\alpha:2^{\mathbb{N}}} \alpha = \mathbf{0} \vee \neg(\alpha = \mathbf{0})$$

Theorem 4.6: Lesser Limited Principle of Omniscience

For a binary sequence that hits 1 at most once, either the even or odd elements are all 0.

$$\prod_{\alpha:\mathbb{N}_\infty} \left(\prod_{k:\mathbb{N}} \alpha_{2k} = 0\right) \vee \left(\prod_{k:\mathbb{N}} \alpha_{2k+1} = 0\right)$$

The proofs of both of the above theorems are available in [1]. Using them, one can in fact show that the theory is strong enough to prove a natural proposition, that for any binary sequence $\alpha : 2^{\mathbb{N}}$ one has the following, sometimes called *Markov's principle.*

Lemma 4.6.1:

For $\alpha : 2^{\mathbb{N}}$ one has:

$$\neg \prod_{n:\mathbb{N}} \alpha_n = 0 \to \sum_{n:\mathbb{N}} \alpha_n = 1$$

Proof. See [1].

4.2. Synthetic Point-Set Topology

With the axioms of synthetic Stone duality in hand, we begin the development of point-set topology in this setting.

Following prior work of Escardó [17], Synthetic Stone Duality admits a synthetic approach to point-set topology with an essential differences: our open and closed subsets are only closed under *countable* union and intersection respectively. We define openness and closedness in terms of binary sequences.

Definition 4.7: Open and Closed Proposition

A proposition P is open whenever there is merely a sequence so that P holds if and only if the sequence attains 1 somewhere. That is,

$$\text{Open}(P) := \exists_{\alpha:2^{\mathbb{N}}}(P \leftrightarrow \exists_{k:\mathbb{N}}\alpha_k = 1)$$

Dually, a proposition is closed whenever there is merely a sequence so that P holds if and only if the sequence is everywhere 0, or

$$\text{Closed}(P) := \exists_{\alpha:2^{\mathbb{N}}}(P \leftrightarrow \forall_{k:\mathbb{N}}\alpha_k = 0)$$

From here it is simple to define open and closed sets.

Definition 4.8: Open and Closed Sets

Given a set X and subset U, we say U is *open* (resp. *closed*) whenever $U(x)$ is an open (resp. closed) proposition for all $x : X$. A closed and open or *clopen* set is also called decidable.

We will soon justify the terminology choice stated above by showing that clopen sets are precisely the decidable subsets; we show further that we can apply proof by contradiction for closed and open propositions. Throughout many of the following proofs where our goal is a proposition, we silently use the universal property of propositional truncation.

First, we give an important example of a closed proposition, returning to $\mathbb{N}_\infty$ from Example 3.4.2.

Example 4.8.1: $\mathbb{N}_\infty$ Closed

Recall that $\mathbb{N}_\infty$ is the spectrum of the Boolean algebra B_∞, which is equivalent to the set of binary sequences which attain 1 at most once. Formally $\mathbb{N}_\infty$ can be defined as a proposition on binary sequences $\iota : 2^{\mathbb{N}}$

$$\mathbb{N}_\infty(\iota) \leftrightarrow \forall_{i,j} i \neq j \to \alpha_i \alpha_j = 0$$

Given $\iota : 2^{\mathbb{N}}$, the proposition $\mathbb{N}_\infty(\iota)$ is closed. Consider $U \subseteq \mathbb{N} \times \mathbb{N}$ where $U(i,j) := i \neq j$, and let $s : \mathbb{N} \to U$ be any standard bijection. Define $\sigma_{m,n} := \iota_m \iota_n$. By definition, $\mathbb{N}_\infty(\iota) \leftrightarrow \prod_{m,n:U} \sigma_{m,n} = 0$. Observe that $\sigma \circ s$ is a binary sequence. It follows that $\mathbb{N}_\infty(\iota) \leftrightarrow \prod_{n:\mathbb{N}} (\sigma \circ s)_n = 0$ since s is surjective.

The above example is essential later to prove the finite closure properties for open, closed sets. For now we show that closed, open sets behave as expected under complementation.

Lemma 4.8.2:

If P : Closed then $\neg P$: Open and if P : Open then $\neg P$: Closed.

Proof. We only show the negation of a closed proposition is open, since the other direction is immediate. Say P : Closed. A proposition being open or closed is itself a proposition, so we apply the universal property of propositional truncation to obtain a sequence $\alpha : 2^{\mathbb{N}}$ such that $P \leftrightarrow \alpha = \mathbf{0}$. Then $\neg P \leftrightarrow \neg(\alpha = \mathbf{0})$, which by Markov's principle (Lemma 4.6.1) means that $\neg P \leftrightarrow \exists_{n:\mathbb{N}} \alpha_n = 1$ (the right to left direction being immediate).

Lemma 4.8.3:

If P : Closed or P : Open then $\neg\neg P \to P$.

Proof. For an open proposition P let α represent openness, with $\alpha : 2^{\mathbb{N}}$. Then $\neg\exists_n \alpha_n = 1 \to \alpha = \mathbf{0}$ immediately by case splitting on equality in 2, so $\neg P$ is closed. Similarly if P closed with α representing closedness, we have $\neg\alpha = \mathbf{0} \to \exists_n \alpha_n = 1$ by Markov's principle so $\neg P$ open. Then for P either open or closed, $\neg\neg P$ open or closed respectively with the same representing sequence α, so $P \leftrightarrow \neg\neg P$.

In particular the above two sets state that a subset U is open in a set X if and only if $\neg U$ is closed and vice versa. We observe that for a decidable subset D it is instant to show the proposition $D(x)$ is open and closed: if $D(x)$ use the representing sequence $\mathbf{0}$ else the representing sequence $\mathbf{1}$ to show it is closed, or vice versa to show it is open.

Corollary 4.8.4:

If P : Open and P : Closed then $P + \neg P$.

Proof. Let α, β represent openness, closedness of P respectively. Using the natural map $2^{\mathbb{N}} \to \mathbb{N}_\infty$, assume $\alpha, \beta : \mathbb{N}_\infty$. Since $\sum_{n:\mathbb{N}} \alpha_n = 1$ is a proposition for $\alpha : \mathbb{N}_\infty$, we have

$$\sum_{n:\mathbb{N}} \alpha_n = 1 \leftrightarrow P \leftrightarrow \beta = \mathbf{0}$$

Intersperse α, β on odd, even indices respectively to form $\chi : 2^{\mathbb{N}}$. The above equivalence tells us further that $\chi : \mathbb{N}_\infty$ so apply LLPO. If $\forall_k \chi_{2k} = 0$ then $\beta = \mathbf{0}$, so P. Otherwise $\neg \sum_{n:\mathbb{N}} \alpha_n = 1$ so $\neg P$.

Up to countability open and closed propositions satisfy the axioms of topology in the usual way (which necessarily implies the sets do), where disjunction of propositions takes the role of union, and conjunction the role of intersection.

Lemma 4.8.5:

Given $\prod_{n:\mathbb{N}} P_n$, where P_n is open (resp. closed) for all n, then $\bigvee_n P_n$ open (resp. $\bigwedge_n P_n$ closed).

Proof. We show the proof that if P_n open for all n, then so is $\bigvee_n P_n$. Let α^n be the sequence representing openness for P_n. Simply intersperse all the sequences by defining

$$\begin{aligned} \alpha : \mathbb{N} \times \mathbb{N} &\to 2 \\ m, n &\mapsto \alpha^m_n \end{aligned}$$

Consider any standard bijection $s : \mathbb{N} \simeq \mathbb{N} \times \mathbb{N}$. Let $\beta : \mathbb{N} \to 2$ be given by $\alpha \circ s$. By definition of α we have $\bigvee_n P_n \leftrightarrow \exists_{i,j:\mathbb{N}\times\mathbb{N}} \alpha_i^j = 1$ and because s is a surjection, we have $\exists_{i,j:\mathbb{N}\times\mathbb{N}} \alpha_i^j = 1 \leftrightarrow \exists_{n:\mathbb{N}} \beta_n = 1$. ■

The above result is the conventional property of topological spaces, except open, closed sets are usually closed under *arbitrary* union, intersection respectively. Similarly we have that open, closed sets are closed under finite intersection, union respectively, though the proof is more complex.

Lemma 4.8.6:

If $(P_n)_{n \leq k}$ is some finite set of open (resp. closed) propositions, then $\bigwedge_{n \leq k} P_n$ is open (resp. $\bigvee_{n \leq k} P_n$ is closed).

Proof. We just show that a finite disjunction of closed propositions is closed, as the other proof is similar. It suffices to show that if P, Q closed then so is $P \vee Q$. Let α, β be the sequences representing closedness of P, Q respectively; assume that $\alpha, \beta : \mathbb{N}_\infty$.

By definition, we have

$$P \vee Q \leftrightarrow \alpha = \mathbf{0} \vee \beta = \mathbf{0}$$

Intersperse α, β to form the sequence $\iota : 2^{\mathbb{N}}$ with α, β on odd, even indices respectively. Then

$$\alpha = \mathbf{0} \vee \beta = \mathbf{0} \leftrightarrow \mathbb{N}_\infty(\iota)$$

since, if they are both nonzero ι contains at least two ones. Since $\mathbb{N}_\infty(\iota)$ is closed so is $P \vee Q$.

Closed and open propositions also enjoy a subtle "stability" with respect to implication. Given that C, U are closed, open propositions respectively, an implication in either direction has the topological property of the codomain. More precisely:

Lemma 4.8.7:

Let C : Closed and U : Open. Then $C \to U$: Open and $U \to C$: Closed.

Proof. Because closed and open propositions are both $\neg\neg$-stable we can show in the standard way that $\neg C \vee U \leftrightarrow C \to U$ and similar for $\neg U \vee C \leftrightarrow U \to C$. But $C \vee \neg U$ is closed, and $C \vee \neg U$ is open.

Finally, we observe that any function definable in HoTT is automatically continuous with respect to the topology.

Theorem 4.9: Continuity

Let $f : A \to X$ be a map. The preimage of an open (resp. closed) set $U \subseteq X$ is open (resp. closed).

Proof. Since $U : X \to \mathrm{Prop}_{\mathcal{U}}$ is an open/closed family it follows $U \circ f : A \to \mathrm{Prop}_{\mathcal{U}}$ is still an open/closed family, precisely on the preimage of U.

Hereafter we frequently use the fact that any function is continuous without comment.

4.3. Overtly Discrete Types

In the previous section we demonstrated that open and closed propositions behave as expected under negation: the negation of an open proposition is closed and vice versa, so that open and closed sets are complements of each other as expected. As it turns out, there is a deeper way to understand the duality between open and closed propositions. Here we discuss the subuniverse of *overtly discrete* or ODisc types, which correspond topologically to (countable) discrete spaces but also characterize open propositions. Furthermore, we use ODisc to better understand the category Boole_ω.

Definition 4.10: Overtly Discrete Types

A type X is overtly discrete whenever it is merely a sequential *colimit* of finite sets:

$$\mathrm{ODisc}(X) := \| \sum_{F : \mathbb{N} \to \mathbb{F}} X \simeq \mathrm{colim}\ F \|$$

where F is a functor from $\mathbb{N}$ considered as a category with its natural order to $\mathbb{F}$, the subuniverse of finite types.

By definition, all finite and countable sets are ODisc. We give a precise characterization of ODisc types below to better understand Boole_ω.

Recall the classical notation for the category of general filtered colimits of finite sets as $\mathrm{Ind}(\mathbf{FinSet})$, and dually the category of cofiltered limits of finite sets as $\mathrm{Pro}(\mathbf{FinSet})$; in the "light" setting it suffices to just consider sequential colimits and limits respectively, and as usual we assume everything is light when we refer to these categories. As it turns out, the open and closed propositions internalize a quasi-duality between these two categories.

The canonical work on sequential colimits [18] in HoTT immediately gives, by the closure properties of sequential colimits, the following properties which we give without full proofs.

Corollary 4.10.1:

1. If X : ODisc, $Y : X \to \mathrm{ODisc}$ then $\sum_{x:X} Y(x)$: ODisc.
2. If F_n : ODisc for all n, colim F_n : ODisc
3. If X : ODisc then $\|X\|$: ODisc.

Proof. See [18]. This holds for all sequential colimits, which commute with $\sum$, colim and propositional truncation.

Corollary 4.10.2:

A map $f : X \to Y$ where X, Y : ODisc is the sequential colimit of maps between finite sets.

Proof. Same as above.

The above properties follow from general facts about sequential colimits. Crucially, we can use this to characterize open propositions.

Theorem 4.11: Open Propositions Are ODisc

$\text{Open}(P) \leftrightarrow \text{ODisc}(P)$

Proof. Assume $P : \text{Open}$ and take the representing sequence $\alpha : \mathbb{N}_\infty$ (by Lemma 3.4.3). Build a sequence $F : \mathbb{N} \to \mathbb{F}$ of finite sets by induction

$$0 \mapsto \mathbb{F}_{\alpha(0)}$$
$$S(n) \mapsto F \mapsto \mathbb{F}_{\alpha(S(n))} + F$$

where $\mathbb{F}_k$ is just the finite type with k elements, and implicit maps $f_n : F_n \to F_{n+1}$ given by inr, so we abuse notation and treat F as a functor $\mathbb{N} \to \mathbb{F}$. By definition, $\text{colim } F \leftrightarrow \exists_n \; \|F_n\| \leftrightarrow \exists_n \alpha_n = 1 \leftrightarrow P$. It is not hard to see that since $\alpha : \mathbb{N}_\infty$ that $\text{colim } F$ is a proposition, so we have an equivalence.

The proof in the other direction is similar. Assuming $\text{ODisc}(P)$, so we have some implicit functor $F : \mathbb{N} \to \mathbb{F}$ such that $P = \text{colim } F$, consider the sequence given by

$$n \mapsto \|F_n\|$$

and since $\|F_n\|$ is decidable we are implicitly using Boolean reflection on it to obtain a binary sequence. Then $P \leftrightarrow \exists_n \alpha_n = 1$.

Corollary 4.11.1:

Open propositions are closed under $\sum$.

Proof. Since ODisc is closed under $\sum$.

Using this, we can show that subspace topology is well-behaved.

Corollary 4.11.2:

If $W \subseteq V$ and $V \subseteq T$, both open in their respective supersets, then $W \subseteq T$ is open.

Proof. $V \subseteq T$ means that for all $t : T$, $V(t)$ is an open proposition. If $W \subseteq V$ is open then $W : \sum_{t:T} V(t) \to \text{Prop}_{\mathcal{U}}$ and $W(t, v)$ open for any $t : T, v : V(t)$. The total space of W over V

$$\sum_{t:T} \sum_{v:V(t)} W(t, v)$$

gives a natural way to define W as a subset of T that respects transitivity: taking the total space over just $V(t)$ for a fixed t

$$t \mapsto \sum_{v:V(t)} W(t, v)$$

Since open propositions are closed under $\sum$ by Corollary 4.11.1, this is a family of open propositions over T, i.e. an open subset.

Now we are prepared to explicitly characterize general ODisc types, not just propositions. Before doing so, we give a very explicit characterization of the colimit of finite sets.

Lemma 4.11.3:

Given $(F_n)_{n:\mathbb{N}}$, a sequence of finite sets, and $\prod_{n:\mathbb{N}} f_n : F_n \to F_{n+1}$ the colimit colim F is an open quotient of $\sum_{n:\mathbb{N}} F_n$.

Proof. We give a sketch. The open relation on $\sum_{n:\mathbb{N}} F_n$ is given by relating $(m, x) \sim (n, y)$ if they are eventually mapped to the same element. Explicitly this is the proposition that

$$\exists_{k:\mathbb{N}} f_k \circ \ldots \circ f_m(x) = f_k \circ \ldots \circ f_n(y)$$

Since $\mathbb{N}$: ODisc and equality in finite types is decidable, thus open and ODisc. Since ODisc types are $\sum$-closed, this is the propositional truncation of an ODisc type, which is still ODisc, thus open. The construction of the equivalence from the set quotient to the colimit is a lengthy but standard calculation.

With the above technicality in hand, we can explicitly characterize ODisc types.

Theorem 4.12: ODisc Types Are Open Quotients of Countable Sets

Any ODisc type is a countable set, quotiented by an open equivalence relation, i.e. the equivalence classes are open in the original set.

Proof. Suppose X : ODisc. Let $F : \mathbb{F}^{\mathbb{N}}$ be a functor, with $f_n : F_n \to F_{n+1}$; we have $X \simeq$ colim F. Applying Lemma 4.11.3 we immediately have that colim $F \simeq \sum_{n:\mathbb{N}} F_n / \sim$ where $\sim$ is open, and $\sum_{n:\mathbb{N}} F_n$ is clearly countable as a countable union of finite sets.

Now suppose that $X \simeq \mathbb{N}/R$ where R open, that is, $R(x, y)$ is an open proposition for all $x, y : \mathbb{N}$. We need to build a series of finite sets so $X \simeq$ colim F; a natural candidate is to 'accumulate' finite pieces. That is, define

$$F_n \mapsto [n]/R_n$$

where $[n]$ is the image of the natural inclusion $\mathbb{F}_n \hookrightarrow \mathbb{N}$. R_n is just the restriction of R to $[n]$.

Using dependent choice on the open propositions $R(x, y)$ and any bijection $s : \mathbb{N} \simeq \mathbb{N} \times \mathbb{N}$, we obtain a matrix of sequences $\alpha : \mathbb{N} \to 2^{\mathbb{N}}$ where

$$R(x, y) \leftrightarrow \exists_n \alpha_{s(x,y)}(n) = 1$$

using the openness of $R(x, y)$ for all x, y.

It is easy to see that an open quotient of a finite set is still a finite set, so this is well-defined; the maps in the sequence are the natural embeddings given by $f_n : F_n \hookrightarrow F_{n+1}$ composed with the quotient. By definition $N/R \simeq \text{colim } F$ so $X \simeq \text{colim } F$. ■

This lengthy characterization of ODisc can be applied immediately to characterize Boole_ω.

Lemma 4.12.1:

Given $B : \text{Boole}_\omega$, there is a sequence of finite Boolean algebras B_n such that $B = \text{colim } B_n$. In particular, $B : \text{ODisc}$.

Proof. We apply "accumulation" again. Given a finite prefix of the countable set of relations $(f_i)_{i=1}^n$, obtain $\left(g_{j_k}\right)_{k=1}^m \subseteq (f_j)$, the finite set of generators in $f_1 ... f_n$, and including $g_1 ... g_n$. Let $B_n := \left(g_{j_k}\right)_{i=1}^m / (f_k)_{k=1}^n$. Then $B = \text{colim } B_n$.

This lemma goes the other way around so $\text{Boole}_\omega \hookrightarrow \text{ODisc}$ as exactly the subcategory of Boolean algebras with Boolean morphisms.

Lemma 4.12.2:

Suppose $\text{ODisc}(B)$, where B is a Boolean algebra. Then $B : \text{Boole}_\omega$, i.e. B is countably presented.

Proof. Suppose B is the sequential colimit of finite sets F_n. Any ODisc type is a quotient of a countable set by an open equivalence relation, so B has open equality and there is a surjection $s : \mathbb{N} \twoheadrightarrow B$. Then $\text{im}(s)$ is the countable set of B-generators. Since the equivalence relation is open, we can consider the sequence $\alpha_{i,j}$ which is 1 somewhere if and only if $g_i \sim g_j$. Choose any standard bijection $p : \mathbb{N} \simeq \mathbb{N}^3$. Define a sequence of relations via the map

$$\begin{aligned} f &: \mathbb{N} \to 2^{\mathbb{N}} \\ n &\mapsto \begin{cases} 0 \text{ if } \alpha_{i,j}^k = 0 \\ g_i \wedge \neg g_j \text{ if } \alpha_{i,j}^k = 1 \end{cases} \text{where } (i, j, k) = p(n) \end{aligned}$$

Since p is in particular a surjection, then $g_i \sim g_j$ will be included in the set of relations if and only if its representing sequence attains 1 at some point. Then $B = (g_n)/(f_n)$.

This characterization has a vital consequence.

Corollary 4.12.3:

If $B : \text{Boole}_\omega$, then $x =_B y$ is open for any $x, y : B$. By Stone duality, it follows $x =_{2^S} y$ is open for any $S : \text{Stone}$.

Proof. Since B : Boole_ω implies B : ODisc it is the quotient of a countable set by an open equivalence relation, so it follows that equality in B is open. The second claim is immediate.

4.4. Stone Spaces

It is not too difficult to see that Stone spaces are exactly the cofiltered (or just sequential) limits of countable diagrams of finite sets also known as *light profinite sets* in the terminology of [2]; we denote the category of light profinite sets as $\text{Pro}(\mathbf{FinSet})$ (as usual omitting the light qualifier). After establishing this, just as in the previous section we used the category $\text{Ind}(\mathbf{FinSet})$ to help characterize Boole_ω, we will use $\text{Pro}(\mathbf{FinSet})$ to help characterize Stone.

Lemma 4.12.4:

Suppose that S : Stone, so $S = \text{Sp}(B)$ for some B : Boole_ω. Then

$$\exists_{D:\mathbb{N}\to\ \text{FinStone}} S = \lim_n D_n$$

where FinStone $\hookrightarrow$ Stone is the subcategory of finite Stone spaces and D is a functor.

Proof. By Stone duality and epi-mono exchange we have an antiequivalence Stone $\simeq$ $\text{Boole}_\omega^{\text{op}}$, so limits are carried to colimits. We know that $S = \text{Sp}(B)$ and $B = \text{colim}_n B_n$ for some sequence of finite Boolean algebras B_n, so by the anti-equivalence

$$\lim \text{Sp}(B_n) = \text{Sp}(\text{colim } B_n) = \text{Sp}(B)$$

Defining $S_n \coloneqq \text{Sp}(B_n)$ gives the result, since the spectrum of a finite Boolean algebra is finite.

The following special case of the above is mostly technical, but it is important to keep in mind.

Corollary 4.12.5:

Stone spaces are stable under finite limits, in particular the Cartesian product $S \times T$ of two Stone spaces is Stone.

Proof. Using Lemma 4.12.4 any finite limit of Stone spaces can be turned into a sequential limit of finite sets, which is again Stone.

Thus Stone spaces are exactly the light profinite sets spotlighted in Clausen and Scholze's formulation of condensed mathematics [2]. In the same way Open and ODisc types are related, we have a relation between Closed and Stone.

Lemma 4.12.6:

If P is a closed proposition then P : Stone.

Proof. To show P is Stone it suffices to show it is equivalent to the spectrum of some Boolean algebra. The algebra in question is given by quotienting 2 by the sequence α representing closedness, since

$$\alpha = \mathbf{0} \leftrightarrow \mathrm{Sp}\Big(2/(\alpha_n)_{n:\mathbb{N}}\Big)$$

So $P \leftrightarrow \mathrm{Sp}\Big(2/(\alpha_n)_{n:\mathbb{N}}\Big)$.

Lemma 4.12.7:

If S : Stone, then $\neg\neg S \leftrightarrow \|S\|$.

Proof. If $\neg\neg S$ where $S = \mathrm{Sp}(B)$, since $\neg\ \mathrm{Sp}(B) \leftrightarrow \left(0 =_B 1\right)$ by Lemma 3.4.5 it follows we have $\neg\left(0 =_B 1\right)$. This gives an embedding $\mathrm{id} : 2 \hookrightarrow B$, but then $\mathrm{Sp}(B) \twoheadrightarrow \mathrm{Sp}(2)$ by axiom Axiom 4.2. $\mathrm{Sp}(2)$ is inhabited by $\mathrm{id} : 2 \to 2$, so we merely have a fiber $\|\mathrm{Sp}(B)\|$ by surjectivity.

The other direction is immediate by the universal property of propositional truncation.

Corollary 4.12.8:

If S : Stone, then $\|S\|$: Closed and $\|S\|$: Stone.

Proof. As shown above, $\|S\| \leftrightarrow \neg\neg S \leftrightarrow \neg\left(0 =_B 1\right)$ where the RHS is the negation of an open proposition, hence closed. The final statement is an application of Lemma 4.12.6.

We finally have, using the above, the dual of Lemma 3.4.5, where we showed that $\neg\ \mathrm{Sp}(B) \leftrightarrow 0 =_B 1$, which we announced at the end of §3.

Corollary 4.12.9:

Given $S = \mathrm{Sp}(B)$ we have $\|S\| \leftrightarrow 0 \neq_B 1$.

Proof. From Lemma 3.4.5 and applying negation to both sides we know that

$$\neg\neg S \leftrightarrow \neg\left(0 =_B 1\right)$$

and applying the equivalence $\neg\neg S \leftrightarrow \|S\|$ we have $\|S\| \leftrightarrow \neg\left(0 =_B 1\right)$.

Since closed propositions are Stone it is an immediate corollary that Stone is closed under propositional truncation.

Complementing the characterization of identity types in ODisc types as open propositions, we have the following result.

Lemma 4.12.10:

If S : Stone and $s, t : S$ then $s = t$: Closed.

Proof. If $s = t$ then assuming $S = \mathrm{Sp}(B)$ it follows by function extensionality $s \sim t$, as Boolean morphisms $B \to 2$. (Recall by weak function extensionality $s \sim t \doteq$

$\prod_{b:B} s(b) = t(b)$ is a product of propositions, so is a proposition.) Since 2 has decidable and therefore closed equality, and B is countably presented, the action of s : $\mathrm{Sp}(B)$ is determined by the action on countably many generators. It follows that $s \sim t \simeq s = t$ is a countable conjunction of closed propositions.

Corollary 4.12.11:

Any S : Stone is T_1, that is, any two distinct $s, t : S$ have open sets U, V separating them; more precisely, so that $s : U$ and $t : \neg U$, and $s : \neg V$ and $t : V$.

Proof. It suffices to show that any singleton set (and thus any finite point set) is closed; then, for any $s, t : S$, the complements of $\{s\}, \{t\}$ will be precisely the separating opens. But this is immediate by the above lemma.

We will soon show much stronger separation properties for Stone spaces and the compact Hausdorff spaces we introduce in the next section.

To recap our duality between Stone and ODisc, we have shown that Stone = Closed and ODisc = Open *within* $\mathrm{Prop}_{\mathcal{U}}$. Furthermore, identity types in Stone are closed propositions, and identity types in ODisc are open propositions, which also implies that Stone and ODisc are closed under identity types. Finally there is a strict embedding $\mathrm{Boole}_\omega \hookrightarrow$ ODisc where we showed ODisc is explicitly characterized as the category of countable sets quotiented by open equivalence relations.

Since we have axiomatized that Stone $\simeq \mathrm{Boole}_\omega^{\mathrm{op}}$ and Boole_ω sits within the larger ODisc context, it naturally begs the question if Stone also lives naturally within a larger category. This will turn out to be the category CHaus of (second-countable) compact Hausdorff spaces, which we define and characterize in the next section.

To close out this section we begin building towards the synthetic version of the result, in classical point-set topology, that Stone spaces are totally disconnected compact Hausdorff spaces (equivalently, compact Hausdorff spaces admitting a basis of clopen sets).

We begin by characterizing the closed sets in Stone spaces.

Theorem 4.13: Stone Topology

For $A \subseteq S$ with S : Stone all the following are equivalent:

1. A is the countable intersection of decidable subsets $D_n \subseteq S$.
2. A is the image of a map, in fact an embedding, of some T : Stone.
3. A is closed.

Proof. The first condition is a reformulation of the fact that Stone spaces have a "clopen" (in our terms, decidable) basis. The second condition helpfully shows that closed subsets of Stone spaces are Stone spaces. There are two closed circuits in this proof: we first show (i) $\leftrightarrow$ (ii) and then use that to show (i) $\leftrightarrow$ (iii). Throughout we assume $S = \mathrm{Sp}(B)$.

(i) → (ii) Suppose $A = \bigcap_{n:\mathbb{N}} D_n$. By decidability each D_n corresponds uniquely to $d_n : B$, where we have $s : D_n$ if and only if $s(d_n) = 0$.

Consider the algebra $B/(d_n)_{n:\mathbb{N}}$ and its spectrum. There is a natural embedding $T := \mathrm{Sp}\big(B/(d_n)_{n:\mathbb{N}}\big) \hookrightarrow S$ just given by the identity map. Then for any $t : S$, we have $t : T$ if and only if it is 0 on every d_n, i.e. iff $t : \bigcap_{n:\mathbb{N}} D_n = A$, so T is the Stone space embedding onto A.

(ii) → (i) Suppose $q : T \to S$ is a map of Stone spaces whose image is A; let $p : B \to C$ be the underlying map of Boolean algebras. Factor p by the first isomorphism theorem

$$B \xrightarrow{\ \pi\ } B/\ker(p) \longrightarrow \mathrm{im}(p) \xhookrightarrow{\ \iota\ } C$$

where the middle map is an isomorphism. By the anti-equivalence induced by Sp we have corresponding maps $T \twoheadrightarrow \mathrm{Sp}(B/\ker(p)) \hookrightarrow S$, in particular that $\mathrm{Sp}(B/\ker(p)) \simeq A$ since in Boole_ω one has $B/\ker(p) \simeq \mathrm{im}(p)$.

As a subset of B, $\ker(p)$: ODisc since ODisc is closed under $\sum$, and $\ker(p)$ is a family of open (thus ODisc) propositions by Corollary 4.12.3. Then $B/\ker(p)$: ODisc by Theorem 4.12 and therefore $B/\ker(p) : \mathrm{Boole}_\omega$ by Lemma 4.12.2. We just want to show that $\mathrm{Sp}(B/\ker(p)) \simeq A$, can be written as a conjunction of decidable subsets, which we define as

$$D_n(s) : s(a_n) = 0$$

where $a : \mathbb{N} \to \ker(p)$ is a map into the countably many generators of $\ker(p)$. Each D_n is obviously decidable, and any $s : S = \mathrm{Sp}(B)$ is in $\mathrm{Sp}(B/\ker(p))$ iff $s(a_n) = 0$ for all n.

Now we do the other circuit assuming the above two steps.

(i) → (iii) Given $A = \bigcap_{n:\mathbb{N}} D_n$, for any $x : S$ define a sequence naturally by $\alpha_x(n) := D_n(x)$ where the right hand side can be converted into a binary value precisely by decidability. Then naturally $\bigcap_n D_n(x) \leftrightarrow A(x) \leftrightarrow \alpha_x = \mathbf{0}$.

(iii) → (i) Observe the map $\alpha \mapsto \alpha = \mathbf{0}$ is a surjection from $2^{\mathbb{N}} \twoheadrightarrow \mathrm{Closed}$. If $A \subseteq S$ closed we can apply local choice and obtain

$$\begin{array}{ccc} T & \xrightarrow{\ \beta\ } & 2^{\mathbb{N}} \\ {\scriptstyle e}\big\downarrow & & \big\downarrow \\ S & \xrightarrow{\ A\ } & \mathrm{Closed} \end{array}$$

the maps e, β in the commuting diagram above. Define $B \subseteq T$ by $B(t) := \beta_t = \mathbf{0}$ which is closed by definition. By commutation $A(e(t)) = \beta_t = \mathbf{0}$ so $A(e(t)) = B(t)$; since e is surjective, it follows A is the image of B. By (ii) -> (i) above we have (i).

Several corollaries fall out of this theorem immediately. Firstly we are able to explicitly characterize Stone types, as we were able to do for ODisc types.

Lemma 4.13.1:

Recall $2^{\mathbb{N}} = \mathrm{Sp}(2[\mathbb{N}])$. Every S : Stone is a closed subset of $2^{\mathbb{N}}$.

Proof. Since $S = \mathrm{Sp}(B)$ and $B = 2[\mathbb{N}]/(r_n)$ since it is countably presented, there is an embedding $S \hookrightarrow 2^{\mathbb{N}}$ given by just taking the identity. But the image of a Stone space in another is closed.

In other words, all Stone spaces are closed subsets of Cantor space: this is one place where our "light" hypothesis is absolutely essential, since this would definitely not be true for Stone spaces of arbitrary cardinality. We can also use the topological characterization to derive an intuitive result about closed propositions, namely that they are stable under $\sum$-types.

Corollary 4.13.2:

If $A \subseteq S$ closed, then $\exists_{x:S} A(x)$ closed. Then, dually, it follows that for an open $U \subseteq S$ one has $\prod_{x:X} U(x)$ open.

Proof. We know $\Sigma_{x:S} A(x)$: Stone for a closed subset A by Theorem 4.13 because A admits an embedding from a Stone space. Then the truncation is closed and Stone by Corollary 4.12.8. Now $\neg\exists_{x:S} A(x) = \forall_{x:S} \neg A(x)$ is open and every open U arises as the complement of some closed A.

Corollary 4.13.3:

If P : Closed and $Q : P \to$ Closed then

$$\sum_{p:P} Q(p) : \text{Closed}$$

Proof. By Corollary 4.13.2 one has $\exists_{p:P} Q(p)$ closed. By the universal property of propositional truncation $\exists_{p:P} Q(p) \leftrightarrow \sum_{p:P} Q(p)$.

This yields that closedness is transitive, i.e. if $A \subseteq B$ closed and $B \subseteq C$ closed, then $A \subseteq C$ is also closed in C.

We start demonstrating some of the expected topological properties of Stone spaces. First of all, it seems totally natural to expect a limit of finite sets – which we have shown all Stone spaces arise as – to be compact in the open cover sense, and we can in fact recover this using the theorem on Stone topology, since we can characterize closed sets as arising from the clopen basis. Of course in our setting, compactness means that any *countable* open cover has a finite subcover. We give the equivalent phrasing in terms of closed covers.

Theorem 4.14: Stone Spaces Are Compact

Given a sequence of closed subsets $C_n \subseteq S$ of a Stone space S such that $\bigcap_n C_n = \emptyset$, there is some $k : \mathbb{N}$ so that

$$\bigcap_{n \leq k} C_n = \emptyset$$

Proof. Fix any standard bijection $s : \mathbb{N} \simeq \mathbb{N} \times \mathbb{N}$. Then we can turn every C_n into a countable intersection $C_n = \bigcap_{k:\mathbb{N}} D_n^k$ of decidable subsets, and use the bijection to reindex the D_n^k over all n, k over just n, so that $\bigcap_n C_n = \bigcap_n D_n$. Applying Lemma 4.1.3 one has that some finite subcollection $\left(d_j\right)_{j:J}$ of the corresponding elements join to 1; passing back through the duality it follows that for any $s : S$

$$1 = s(1) = s\left(\bigvee_{j:J} d_j\right) = \bigvee s\left(d_j\right)$$

so some D_j does not contain s for all $s : S$, hence

$$\bigcap_{j:J} D_j = \emptyset$$

where J finite. Closing J downward gives the empty intersection as stated in the theorem.

Next, we show that Stone spaces admit a strong kind of normality, where any two closed sets are separable not just by open, but clopen sets. This further implies Stone spaces are regular and Hausdorff. The proof technique is the same once again, relying on Lemma 4.1.3 in an essential way.

Lemma 4.14.1:

Let $F, G \subseteq S$ be disjoint nonempty closed subspaces of S : Stone. Then there is a decidable $D \subseteq S$ such that $F \subseteq D$ and $G \subseteq \neg D$.

Proof. By Theorem 4.13 we know

$$F = \bigcap_{n:\mathbb{N}} F_n$$
$$G = \bigcap_{n:\mathbb{N}} G_n$$

with F_n, G_n decidable. Let H_n be the sequence of sets interspersing F_n, G_n; then $\bigcap_{n:\mathbb{N}} H_n = \emptyset$ by disjointness of F, G. This is in particular an intersection of closed sets, so we can pass to an intersection of finitely many of them, which is still empty, using Theorem 4.14. We observe then that for finite sets I, J we have

$$\left(\bigcap_{i:I} F_i\right) \cap \left(\bigcap_{j:J} G_j\right) = \emptyset$$

observing that I, J must both be nonempty, since otherwise the intersection of just finitely many F_i or G_j would not be the empty set.

Since decidable subsets are closed under finite intersection/union we observe that

$$D = \neg \bigcap_{i:I} F_i$$

suffices: it is clearly disjoint from $F \subseteq \bigcap_{i:I} F_i$ while $G \subseteq \bigcap_{j:J} G_j \subseteq D$ by definition.

This strong form of normality yields Hausdorff separation for Stone space.

Corollary 4.14.2:

Say S : Stone. If $s, t : S$ distinct, then they are separable by disjoint open neighborhoods; i.e. S is Hausdorff.

Proof. Immediate since S is T_1 by Corollary 4.12.11.

Finally, we show that Stone spaces are totally disconnected. We define the notion of connected component first.

Definition 4.15: Connected Component

Suppose X is a set and $x : X$. Then the *connected component* of x, Q_x, is given by

$$Q_x := \bigcap D_x$$

where D_x is the set of all decidable subsets containing X.

We say a space is *connected* whenever for any $x : X$, $Q_x = X$.

Recalling that decidable subsets are exactly the clopen subsets, this is the standard definition. We show that *connectedness* of a space can be interpreted in terms of maps, as in the standard setting.

Lemma 4.15.1:

A pointed set X is connected iff all maps $f : X \to 2$ are constant, or in other words, the only decidable subsets are X and $\emptyset$.

Proof. From right to left suppose $f : X \to 2$ is constant for all f. But then this means the only decidable subsets of X are either $\emptyset$ or X itself; then trivially the connected component is X.

Suppose X is connected. Then for all $x : X$ one has $Q_x = X$. Let D_X be the set of all nonempty decidable subsets of X (which is well-defined since X is pointed) and D_x be the set of decidable subsets containing a given point $x : X$. Then we observe

$$\begin{aligned}\bigcap D_X &= \bigcap_{x:X} \bigcap D_x \\ &= \bigcap_{x:X} Q_x \\ &= X\end{aligned}$$

so X is the only nonempty decidable subset, corresponding to the map $\mathrm{const}_0 : X \to 2$.

Our definition of connected component is given in terms of possibly arbitary set intersection, which might pose problems in our countable version of topology. We would like the intersection used in the above definition to be countable so that at the very least we can conclude that the connected component is closed; the following lemma handles this technical issue.

Lemma 4.15.2:

Let S : Stone and $(d_n) : B^{\mathbb{N}}$ be the generators of the underlying Boolean algebra, and $(D_n) : (2^S)^{\mathbb{N}}$ the corresponding decidable subsets. For any $x : S$ let

$$D_n^x := \begin{cases} D_n & \text{if } x(d_n) = 0 \\ S & \text{if } x(d_n) \neq 0 \end{cases}$$

Then $Q_x = \bigcap_{n:\mathbb{N}} D_n^x$.

Proof. Throughout $x : S$ is fixed. It suffices to show $Q_x \supseteq \bigcap_{n:\mathbb{N}} D_n^x$; the other inclusion is by definition. Suppose $y : \bigcap_{n:\mathbb{N}} D_n^x$.

Suppose $D : 2^S$ so that $x : D$. We want to show $y : D$. The corresponding $d : B$ can be written in CNF as

$$d = \bigwedge_{i:I} \bigvee_{j:J_i} d_{ij}$$

where each d_{ij} is a generator of B. Since $x : D$ it follows $d(x) = 0$, and this is true iff there is some i so that for all $j : J_i$, $x(d_{ij}) = 0$. Then since each d_{ij} is a generator and $y : \bigcap_{n:\mathbb{N}} D_n^x$, it follows that $y(d_{ij}) = 0$ for all $j : J_i$ also, so $y : D$.

Finally we can characterize the essential property of Stone spaces that, as we will see in the next section, characterizes them among the larger class of compact Hausdorff spaces that we will introduce shortly.

Corollary 4.15.3:

For S : Stone and $x : S$, $Q_x = \{x\}$.

Proof. By Lemma 4.15.2 $Q_x = \bigcap_{n:\mathbb{N}} D_n^x$. Recall that D_n^x is defined for each n by the action of x on the generator d_n.

If $y : Q_x$ then it agrees with x on every generator of B, the underlying Boolean algebra. But $x, y : S$ are completely determined by their action on generators, so $x = y$.

We pause to note how essential Stone duality was to the proofs of compactness, normality and total disconnectedness above. The synthetic definition of compact Hausdorff spaces allows us to easily generalize these to a more general class of space.

4.5. Compact Hausdorff Spaces

Now we attempt to extend as much as possible our results on the topology of Stone spaces to the all-important class of compact Hausdorff spaces, a class which includes many fundamental topological spaces like the sphere $\mathbb{S}^n$ or the interval $\mathbb{I}$. Our synthetic definition has apparently nothing to do with either being compact or Hausdorff; we take the vantage point instead that compact Hausdorff spaces arise as quotients of Stone spaces (which is true in the classical setting). One of our primary goals in this section will be to show that these spaces are, in fact, compact and Hausdorff.

Definition 4.16: Compact Hausdorff Type

X is *compact Hausdorff*, written $X : \text{CHaus}$, whenever its equality types are closed propositions and it merely admits a surjection $q : S \twoheadrightarrow X$ where $S : \text{Stone}$.

The definition implies that whenever $X : \text{CHaus}$ that X is a set. As we will ratify later, CHaus corresponds to *second countable* compact Hausdorff spaces (i.e. those having a countable basis of opens) in the classical setting since our topologies are only closed under countable unions; as usual throughout these notes we will omit this ‘countable’ qualifier most of the time.

Since we are taking Stone as a primitive notion of space and defining CHaus in terms of that, we start to characterize the topology on CHaus spaces by looking at Stone spaces.

The closed equality condition is a necessary, though not sufficient, condition to ensure Hausdorffness, as we will demonstrate explicitly later on. Explicitly since $\sum_{y:X} x = y$ is the HoTT encoding of the singleton set $\{x\}$ we have that, since $x = y$ is a closed proposition, singletons are closed. That is, CHaus spaces are T_1.

By definition of CHaus we have an embedding $\text{Stone} \hookrightarrow \text{CHaus}$ using Lemma 4.12.10. Since CHaus spaces are defined as closed quotients of Stone, CHaus spaces are also closed quotients of Cantor space in view of Lemma 4.13.1. We ratify some properties of CHaus spaces being a quotient. In particular we want to show we can characterize closed sets similarly to how we characterized closed sets in Stone space.

We note that many proofs in this section use the same technique: when proving a proposition one can use the surjection $S \twoheadrightarrow X$ in order to pull back to the Stone context where the result has already been proved.

Lemma 4.16.1:

Suppose X : CHaus. $B \subseteq X$ is closed iff there is a closed subset of some Stone space $A \subseteq S$ (or alternatively, just a Stone space) so that $A \twoheadrightarrow B$.

Proof. Suppose $B \subseteq X$ closed. Recall that there is S : Stone with $q : S \twoheadrightarrow X$. Then $q^{-1}(B)$ is closed in S; by surjectivity one has that $q(q^{-1}(B)) = B$.

Now suppose we have the surjection $q : A \twoheadrightarrow B$ where $A \subseteq S$ closed with S : Stone; in particular A : Stone. Then $\mathrm{im}(q) = B$ but $\mathrm{im}(q) := \sum_{b:B} \|\mathrm{fib}_q(b)\|$ is closed, since $\mathrm{fib}_q(b) \doteq \sum_{a:A} q(a) = b \subseteq A$ is a closed subset by Lemma 4.12.10. Then since identity types in X are closed, it is Stone, so the truncation is Stone and closed.

Corollary 4.16.2:

If $C(x)$ is closed, then $\exists x : C(x)$ is closed. Dually, if $U(x)$ open so is $\prod_{x:X} U(x)$.

Proof. We know by Lemma 4.16.1 that C is surjected on by a Stone space S, so $\exists x C(x) \leftrightarrow \|S\|$ which is Closed and Stone by Corollary 4.12.8. The proof of the second statement is similar to that in Corollary 4.13.2.

Corollary 4.16.3:

If $B \subseteq X$ where X : CHaus and B is a closed subset then B : CHaus.

Proof. By Lemma 4.16.1 B admits a surjection from a Stone space, and B inherits equality from X so its identity types are still closed.

Using the minimal characterization of CHaus provided so far, though we definitely do not have Stone duality $2^S \simeq B$ where $S = \mathrm{Sp}(B)$ for compact Hausdorff spaces in general, we *can* show that the Boolean algebra 2^X is countably presented.

Lemma 4.16.4:

Suppose X : CHaus. Then the Boolean algebra 2^X (maps $X \to 2$ with operations given pointwise) is countably presented.

Proof. Another "pulling back to Stone" style proof. Let $q : S \twoheadrightarrow X$ where S : Stone. There is a natural injection $- \circ q : 2^X \hookrightarrow 2^S$, we

Take $\alpha : 2^S$. Then $\alpha : 2^X$ (abusing notation) if and only if it is constant on the fibers of q. together, i.e.

$$\forall_{x,y:S} q(x) = q(y) \to \alpha(x) = \alpha(y)$$

Observe that $q(x) = q(y)$: Closed and $\alpha(x) = \alpha(y)$: Open so the whole implication is open. Then $2^X \subseteq 2^S$ is an open subalgebra and since 2^S is countably presented by Stone duality, it follows from Theorem 4.12 that 2^X : ODisc; since 2^X is a Boolean algebra and by Lemma 4.12.2 2^X is countably presented.

The simple observations above are enough for compactness.

Theorem 4.17: Compact Hausdorff Space Is Compact

Consider $(C_n)_{n:\mathbb{N}}$, a sequence of closed subsets of X : CHaus. Then

$$\bigcap_{n:\mathbb{N}} C_n = \emptyset \to \exists_{k:\mathbb{N}} \bigcap_{n \leq k} C_k = \emptyset$$

Proof. Take $C_n \subseteq X$ closed for all n. We can pull back through the surjection $q : S \twoheadrightarrow X$ (where S : Stone) so denote $A_n := q^{-1}(C_n)$ which are closed subsets of a Stone space. Then

$$\begin{aligned}\bigcap_{n:\mathbb{N}} A_n &= \bigcap_{n:\mathbb{N}} q^{-1}(C_n) \\ &= q^{-1}\left(\bigcap_{n:\mathbb{N}} C_n\right) = \emptyset\end{aligned}$$

But we already know then by Theorem 4.14 one can find a finite subcollection $I \subseteq \mathbb{N}$ so that $\bigcap_{i:I} A_i = \emptyset$. Then $\bigcap_{i:I} q(A_i) = \bigcap_{i:I} C_i$ since q is a surjective map and $A_i = q^{-1}(C_i)$. Since preimage commutes with finite intersection we have

$$\bigcap_{i:I} q^{-1}(C_i) = q^{-1}\left(\bigcap_{i:I} C_i\right) = \emptyset$$

and so q on the right hand side is $\bigcap_i A_i$ which is empty.

Corollary 4.17.1:

A closed subset $B \subseteq X$ of a compact Hausdorff space X is compact.

Proof. Combining Corollary 4.16.3 and Theorem 4.17.

We highlight here a particular special case of compactness which helps to give a feel for its “finite” character.

Example 4.17.2: Compactness on Subset Chains

Suppose that C_n is a descending chain of closed subsets in a compact Hausdorff space. If each C_n is nonempty, then $\bigcap_{n:\mathbb{N}} C_n \neq \emptyset$.

In other words, if the intersection $\bigcap_{n:\mathbb{N}} C_n$ is empty then one of the C_n itself must be empty, “terminating” the chain at some finite stage by the descending chain condition. We prove this.

Proof. Suppose that $\forall_n \ \|C_n\|$ is merely inhabited, which is a product of closed propositions (by Corollary 4.16.2) so closed; as is $\|\bigcap_{n:\mathbb{N}} C_n\|$ by the same corollary. It suffices then to proceed with a proof by contradiction.

Suppose the intersection $\bigcap_{n:\mathbb{N}} C_n$ is in fact empty. Then by compactness there is some $k : \mathbb{N}$ so that $\bigcap_{n \leq k} C_n = \emptyset$. But $\bigcap_{n \leq k} C_n = C_k$ by the descending chain condition, so $C_{k'} = \emptyset$ for all $k' \geq k$, contradicting our assumption.

Since we have established we are working with (countably) compact spaces indeed, let us also establish normality which will imply the Hausdorff property.

Lemma 4.17.3:

If $A, B \subseteq X$ are closed disjoint sets in a compact Hausdorff space X, there are disjoint open sets $U, V \subseteq X$ separating them.

Proof. Pull A, B back to closed sets in S via the surjection $q : S \twoheadrightarrow X$, say A', B', which are still disjoint in S. Separate A', B' by decidable $D, \neg D$ via Lemma 4.14.1. Then $q(D), q(\neg D)$ closed by Lemma 4.16.1, and naturally $A \subseteq \neg q(D)$ and $B \subseteq \neg q(\neg D)$ which are both open, which suffices.

We observe that $q(D) \cap q(\neg D)$ might have been inhabited, but even so their complements are still disjoint.

Corollary 4.17.4:

For any two distinct points $x, y : X$ where X : CHaus there are disjoint opens U, V so that $x : U$ and $y : V$.

Proof. The singleton sets $\{x\}, \{y\}$ are closed by definition of CHaus, so just apply Lemma 4.17.3.

From hereon we can be confident that our compact Hausdorff spaces do behave like the second countable compact Hausdorff spaces we are familiar with.

In the last part of the proof of Corollary 4.17.4, recall we needed to carefully interchange the map q with an intersection of closed subsets. In fact, when the closed subsets are descending chains, continuous maps are well-behaved in general. The following lemma is mostly a technicality, but allows us to more deeply characterize the topology on CHaus spaces.

Lemma 4.17.5:

Suppose $f : X \to Y$ with X, Y : CHaus. Suppose G_n is a descending chain of closed subsets of X. Then f commutes with intersection

$$f\left(\bigcap_{n:\mathbb{N}} G_n\right) = \bigcap_{n:\mathbb{N}} f(G_n)$$

Proof. The lemma as stated is using a lot of nested "set theory shorthand" as described in §2.4, which can be confusing to type-theoretically translate all at once, so to be clear, we need to show

$$\|\mathrm{fib}_{f|_{\cap_n G_n}}(y)\| \leftrightarrow \prod_{n:\mathbb{N}} \|\mathrm{fib}_{f|_{G_n}}(y)\|$$

for all y.

Left to right, we can use the universal property of propositional truncation and assume we actually have $x : \bigcap_n G_n$ so $f(x) = y$, and give this as the fiber for each n.

Right to left we first observe that we can use proof by contradiction. Let $F := \mathrm{fib}_f(y) \doteq f^{-1}(y)$ which is closed because CHaus spaces are T_1. The left hand side is just $\|F \cap \bigcap_{n:\mathbb{N}} G_n\|$ the truncation of a closed set, so closed by Corollary 4.16.2. Thus, assume the RHS which can be phrased as $\|F \cap G_n\|$ for all n, and assume $\bigcap_{n:\mathbb{N}} F \cap G_n$ is empty. $F \cap G_n$ clearly forms a descending chain of subsets, so by Example 4.17.2 it follows that there is some n for which $F \cap G_n$ is empty, a contradiction.

Using our hard earned lemma, we can generalize the other statement in Theorem 4.13, that closed sets are the intersections of decidables, to a compact Hausdorff space.

Corollary 4.17.6:

Take X : CHaus with $p : S \twoheadrightarrow X$. Then any closed $A \subseteq X$ is the countable intersection of the image of decidable subsets D_n:

$$A = \bigcap_{n:\mathbb{N}} p(D_n)$$

Proof. A is surjected on by $p^{-1}(A)$, a closed set in S, which by Theorem 4.13 is the countable intersection of decidable subsets, say D_n. Apply Lemma 4.17.5 to obtain the final equality below

$$A = p(p^{-1}(A)) = p\left(\bigcap_n D_n\right) = \bigcap_n p(D_n)$$

At this point, the topology on compact Hausdorff spaces is mostly characterized; the interested reader can indeed skip to the main contents of this thesis in the following chapters. For the remainder of this section, however, we show the perhaps unenlightening but very technically important results that both CHaus and Stone are subuniverses closed under $\sum$-types, and further that we have a converse of Corollary 4.15.3 – that totally disconnected CHaus spaces are Stone.

Surprisingly, even though we have that CHaus is a more general class of spaces than Stone, it is easier to show CHaus is $\sum$-stable.

Theorem 4.18: Compact Hausdorff Space Is $\sum$-Stable

If X : CHaus and $Y : X \to \mathrm{CHaus}$ then

$$\sum_{x:X} Y(x) : \mathrm{CHaus}$$

Proof. The identity types are closed in $\sum_{x:X} Y(x)$ as closed propositions are closed under $\sum$ by Corollary 4.13.3. Then we just need to exhibit a surjection from Stone space. To do this we essentially rely on the characterization of Stone space from Lemma 4.13.1.

Let $q : S \twoheadrightarrow X$ be the surjection X has from some S : Stone.

Then $Y \circ q : S \to$ CHaus. Since each $Y(x)$: Stone we also have for every $x : S$ that there is merely some S_x : Stone surjecting onto $Y(q(x))$: CHaus. S_x is equivalent by Lemma 4.13.1 to some closed subset $C_x \subseteq 2^{\mathbb{N}}$ of Cantor space, so we have

$$\prod_{x:S} \|C_x \twoheadrightarrow Y(q(x))\|$$

and we can apply local choice, obtaining T : Stone, $p : T \twoheadrightarrow S$ and $\prod_{t:T} C_t \twoheadrightarrow Y(r(t))$ where $r : T \twoheadrightarrow X$ is given by $r := q \circ p$. We can convert this into a surjection $\sum_{t:T} C_t \twoheadrightarrow \sum_{t:T} Y(r(t))$ and since r is surjective we have a map $\sum_{t:T} Y(r(t)) \twoheadrightarrow \sum_{x:X} Y(x)$, so there is a surjection

$$\sum_{t:T} C_t \twoheadrightarrow \sum_{x:X} Y(x)$$

It remains to show that $\sum_{t:T} C_t$: Stone. T is itself a closed subset of $2^{\mathbb{N}}$ as is each C_t, thus $\sum_{t:T} C_t \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is a closed subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Stone spaces are stable under finite products by Corollary 4.12.5, and by Theorem 4.13 closed subsets of Stone spaces are Stone.

Notice the subtlety in the above proof: we needed to pass to the "universal" characterization of Stone spaces as seen in Lemma 4.13.1 because we cannot assume in general that Stone types are stable under $\sum$ yet. This universal characterization essentially depends on the fact that our Stone spaces are "countably generated," and does not hold for Stone spaces of arbitrary cardinality. Showing that Stone spaces are stable under dependent sum is the crowning achievement of our section; first we characterize Stone types as exactly the totally disconnected CHaus types. To do this we spend more time interacting with and characterizing connectedness in CHaus types. First we handle the usual technicality to make sure we are only ever dealing with countable intersections.

Corollary 4.18.1:

The connected component Q_x of some $x : X$ where X : CHaus is a countable intersection

$$\bigcap_{n:\mathbb{N}} D_n$$

of decidable subsets.

Proof. Decidable subsets 2^X are countably presented by Lemma 4.16.4 so one has $d : \mathbb{N} \twoheadrightarrow 2^X$. The decidable subsets containing x, say D_x, are a subset $D_x \subseteq 2^X$, so one has $E : \mathbb{N} \twoheadrightarrow D_x$ by defining, similar to Lemma 4.15.2

$$E_n := \begin{cases} D_n & \text{if } d_n(x) = 0 \\ X & \text{if } d_n(x) \neq 0 \end{cases}$$

and clearly $Q_x = \bigcap_{n:\mathbb{N}} E_n$.

This actually allows us to show that connected components in CHaus have a special topological property related to *local connectedness*. Local connectedness in a topological space states that every open set contains a connected subset, which is equivalent to the connected components being open. This is a similar but less strong statement showing that every open set around a connected component contains a clopen neighborhood.

Lemma 4.18.2:

Suppose X : CHaus and $x : X$. Let $U \subseteq X$ open so that $Q_x \subseteq U$. Then there is decidable E so that

$$Q_x \subseteq E \subseteq U$$

Proof. Suppose $x : X$ and $Q_x \subseteq U$. Then by Corollary 4.18.1 one has $Q_x = \bigcap_{n:\mathbb{N}} D_n \subseteq U$ where D_n decidable. This lemma is a direct consequence of compactness.

Let $C = \neg U$. It follows then

$$Q_x \cap C = \bigcap_{n:\mathbb{N}} (D_n \cap C) = \emptyset$$

which by compactness means that there is finite k so that

$$\bigcap_{n \leq k} (D_n \cap C) = \emptyset$$

Let $E := \bigcap_{n \leq k} D_n$. $Q_x \subseteq E$ by definition, and E is decidable as the finite intersection of decidable sets; since the intersection with C is finite then $E \subseteq U$.

The above lemma allows us to show that our definition of connectedness lines up with intuition.

Lemma 4.18.3:

Let X : CHaus, $x : X$. Any map $f : Q_x \to 2$ is a constant map.

Proof. Let A, B be the preimages of $0, 1 : 2$ respectively. Then since $\{0\}, \{1\}$ are decidable subsets 2, it follows A, B are each open and closed by continuity, and disjoint by definition.

Given $x : X$, either $x : A$ or $x : B$ is nonempty; assume $x : A$ without loss of generality. Then by Lemma 4.18.2, we have some E decidable so that

$$A \sqcup B = Q_x \subseteq E \subseteq U$$

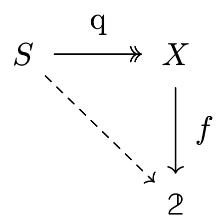
in particular that $B \subseteq U$. Since $B \cap U = \emptyset$ then $B = \emptyset$, so $\mathrm{im}(f) = \{0\}$.

Finally, we show that we can use proof by contradiction to show connectedness, since a function $f : X \to 2$ being constant is an open proposition on CHaus spaces. This is a highly useful characterization for the applications to actual topological spaces in §4.

Lemma 4.18.4:

A function $f : X \to 2$ being constant, where X : CHaus, is an open proposition.

Proof. Since being an open proposition is itself a proposition, we obtain X : CHaus with a surjection $q : S \twoheadrightarrow X$. Let $f : X \to 2$ as in the following diagram



inducing a map $f \circ q : S \to 2$, i.e. a decidable subset $D : 2^S$. Since q is surjective, f is constant if and only if $D = \emptyset \vee D = S$, which is true iff (by Stone duality) the corresponding element $d : B$ is 0 or 1. But $d = 0 \vee d = 1$: Open since Boolean algebras have open equality by Corollary 4.12.3.

All of that having been established, we can finally fully characterize the embedding Stone $\hookrightarrow$ CHaus.

Lemma 4.18.5:

Suppose X : CHaus, and for all $x : X$, $Q_x = \{x\}$. Then X : Stone.

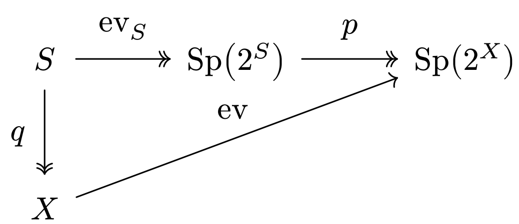
Proof. Assume X : CHaus so that $Q_x = \{x\}$ for all x. We show that the evaluation map $\mathrm{ev} : X \to \mathrm{Sp}(2^X)$ is an equivalence, for which it suffices to show injectivity and surjectivity as a map of sets.

Total disconnectedness is precisely the statement of injectivity: suppose $\mathrm{ev}(x) = \mathrm{ev}(y)$. This just means that for every $f : 2^X$, $f(x) = f(y)$. Then x, y are in exactly the same decidable subsets, so $x, y : Q_x$ and since every connected component is a singleton, $x = y$.

Next recall we have a surjection $q : S \twoheadrightarrow X$. This induces a natural embedding $- \circ q : 2^X \hookrightarrow 2^S$, from which we obtain the surjection (via Axiom 4.2 and since Lemma 4.16.4) $p : \mathrm{Sp}(2^S) \twoheadrightarrow \mathrm{Sp}(2^X)$ defined by the functorial action of Sp on $- \circ q$. Explicitly, p is the map

$$\varphi \mapsto (f \mapsto \varphi(f \circ q))$$

where $\varphi : \mathrm{Sp}(2^S)$ and $f : 2^X$. We have the diagram



Recall that ev_S was shown to be an equivalence in Lemma 4.1.2, so it is in particular surjective. It is immediate if the diagram commutes that $\mathrm{ev} : X \to \mathrm{Sp}(2^X)$ is surjective. To see this, we need to show that $p \circ \mathrm{ev}_S = \mathrm{ev} \circ q$. This is effectively by definition of p but we unfold the calculation.

Computing the LHS pointwise on $s : S$

$$\begin{aligned}(p \circ \mathrm{ev}_S)(s) &= p(\mathrm{ev}_S(s)) \\ &= (f \mapsto \mathrm{ev}_S(s)(f \circ q)) = (f \mapsto f(q(s)))\end{aligned}$$

which is just $\mathrm{ev} \circ q$.

Really, the meaningful part of the above proof is the proof of *injectivity* for the map $\mathrm{ev} : X \to 2^X$ which is related directly to total disconnectedness; the surjectivity is effectively just unfolding the surjection $S \twoheadrightarrow X$ and the anti-equivalence ensured by Axiom 4.1 and Axiom 4.2. We close out this chapter by giving an important application of the above result, namely to demonstrate the fact that Stone spaces are not just stable under product but under the dependent product.

Theorem 4.19: Stone Spaces Are $\sum$-Stable

Let S : Stone, $T : S \to$ Stone. Then $\sum_{s:S} T(s)$: Stone.

Proof. Since Stone $\hookrightarrow$ CHaus, we have $\sum_{s:S} T(s)$: CHaus by Theorem 4.18. Now by it suffices to show that the connected components are singletons by Lemma 4.18.5. Fix $(s, t) : \sum_{s:S} T(s)$. Suppose that $(u, v) : Q_{s,t}$.

We can convert decidable subsets in S into decidable subsets on $\sum_{s:S} T(s)$ via the function

$$- \circ \pi_1 : (S \to 2) \to \left(\sum_{s:S} T(s) \to 2\right)$$

Consider any $f : 2^S$. We have that $f(s) = f(\pi_1(s,t)) = f(\pi_1(u,v)) = f(u)$ where the middle equality follows since $(s,t), (u,v)$ are in the same connected component and thus take the same values on all functions into 2. But since for every $f : S \to 2$, $f(u) = f(s)$, it follows that u, s are in the same connected component of S, which is totally disconnected. So $u = s$.

Now we have established that, abusing notation slightly, $Q_{s,t} \subseteq \{s\} \times T(s)$. Suppose now $a, b : Q_{s,t}$; we know $a = (s, u)$ and $b = (s, v)$. Applying the same argument again for the type $T(s)$, let $f : T(s) \to 2$, and promote it to a function from the total space into 2; then we know $f(u) = f(v)$, so u, v are in the same connected component of $T(s)$: Stone and $u = v$.

Chapter 5
The Interval

In this chapter we define and prove basic properties about some classic topological spaces – the closed interval $\mathbb{I} = [0, 1]$, the topological n-Sphere $\mathbb{S}^n$, and real projective space $\mathbb{RP}^n$. We show that the topology induced by synthetic Stone duality on these spaces is in fact the "usual topology." Crucially the well-behaved interval available in synthetic Stone duality, developed in this chapter, makes it possible for us to define the ordinary topological versions of paths, loops and homotopy groups. The next chapter does exactly this and relates these to HoTT's builtin homotopical notions.

5.1. The Interval via Cantor Space

We note that most of this section already appears in [1]; our only original contribution is a proof of connectedness that bypasses cohomology.

Since we have countable choice, the Cauchy and Dedekind reals are equivalent (so we refer to "the" reals) and we can use either to define $\mathbb{I}$ (cf. Cor 11.4.3 in [11]); we assume $\mathbb{I}$ is defined by Cauchy reals. In particular because we have countable choice we observe that we can make the ordinary construction of Cauchy reals as equivalence classes of rational Cauchy sequences; we use this implicitly in some proofs. We recall that we have a partial order on reals in HoTT that is not a total (linear) order: it is not decidable for general and distinct $x, y : \mathbb{R}$ if $x < y$ or $x > y$. HoTT reals satisfy a weaker form of linearity.

Definition 5.1: Weak Linearity

A strict partial order $<$ is *weakly linear* when, for $x < y$, we have for all z that $x < z \vee z < y$.

HoTT reals are weakly linear cf. Thm 11.2.8 in [11] Since the definition is a bit cryptic we unfold the use of the above order.

Example 5.1.1: Equality Up to ε

Given $x, y : \mathbb{R}$ we can use weak linearity to ascertain equality up to ε where $\varepsilon > 0$. That is, because $x - \varepsilon < x + \varepsilon$ by weak linearity we have either that $x - \varepsilon < y \vee x + \varepsilon < y$.

In this section we show that $\mathbb{I}$: CHaus and that we will in fact be able to obtain a stronger order on $\mathbb{I}$ than we usually have in HoTT. Furthermore we will be able to show that its topology is in fact as expected: every open set is the countable (in fact, finite by compactness) union of open intervals.

We begin by establishing a relation on finite binary sequences that we eventually extend to Cantor space; the interval will turn out to be equivalent to Cantor space quotiented by this relation. Consider the family of functions parameterized by $n : \mathbb{N}$, $\text{cs}_n : 2^n \to \mathbb{I}$ defined by $\text{cs}_n(\alpha) := \sum_{i=0}^{n-1} \frac{\alpha_i}{2^{i+1}}$ with $\text{cs} : 2^{\mathbb{N}} \to \mathbb{I}$ given by just taking the infinite sum. We can use this family to characterize $\mathbb{I}$ as the quotient of $2^{\mathbb{N}}$ via the following equivalence relation(s). In the limit we obtain the function

$$\begin{aligned} \text{cs} : 2^{\mathbb{N}} &\to \mathbb{I} \\ \alpha &\mapsto \sum_{i=0} \frac{\alpha_i}{2^{i+1}} \end{aligned}$$

Definition 5.2: n-Closeness

Given $\alpha, \beta : 2^n$ define

$$\alpha \underset{n}{\sim} \beta \leftrightarrow |\text{cs}_n(\alpha) - \text{cs}_n(\beta)| \leq \frac{1}{2^n}$$

i.e. we relate α, β finite sequences if and only if their difference is no greater than the place value of the final digit.

Given $\alpha, \beta : 2^{\mathbb{N}}$ we say they are n-close whenever, letting α^n, β^n be the finite sequences consisting of their first n digits, we have that $\alpha^n \underset{n}{\sim} \beta^n$.

The idea between two finite sequences being n-close is that when we come to infinite sequences $\alpha, \beta : 2^{\mathbb{N}}$ we can identify two sequences under cs iff they are n-close everywhere: either they agree on all digits, or differ on an infinite tail, identifying

...01111...
...10000...

where the prefixes agree digitwise. Of course this corresponds to the fact that for any $k : \mathbb{N}$ one has $2^{-k} = \sum_{n=k+1}^{\infty} 2^{-n}$.

We want to show that cs is surjective, i.e. that every $x : \mathbb{I}$ has a binary expansion, so we can eventually show that $\mathbb{I}$: CHaus. The construction of the binary expansion for any $x : \mathbb{I}$ is similar to the standard one, but in the absence of the law of excluded middle one does not have a linear order for $\mathbb{R}$.

Lemma 5.2.1:

Any two binary sequences $\alpha, \beta : 2^{\mathbb{N}}$ are equal under cs precisely when they are n-Close for all n, i.e.

$$\text{cs}(\alpha) = \text{cs}(\beta) \leftrightarrow \prod_{n:\mathbb{N}} \alpha \underset{n}{\sim} \beta$$

Proof. From left to right assume

$$\sum_{i=0}^{\infty} \frac{\alpha_i}{2^{i+1}} = \sum_{i=0}^{\infty} \frac{\beta_i}{2^{i+1}}$$

We induct on the index k of the sequences α, β. If $\alpha_k = \beta_k$ the above equality implies that that holds for all $k' \leq k$, otherwise the infinite sums would not be equal.

Else we have k so that $\alpha_k \neq \beta_k$. Assume without loss of generality $\alpha_k = 1$ and $\beta_k = 0$; we claim that for all $j > k$, $\alpha_j = 0$ and $\beta_j = 1$. Since $\mathrm{cs}_{k+1}(\alpha) - \mathrm{cs}_{k+1}(\beta) = \frac{1}{2^{k+1}}$ and $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$ it follows that

$$\sum_{i=k+1}^{\infty} \frac{\alpha_i - \beta_i}{2^{i+1}} = -\frac{1}{2^{k+1}}$$

but this is true if and only if $\alpha_i - \beta_i = -1$ for all $i > k$, proving the claim.

Now we show by induction that for all $i \geq k$, $\alpha \underset{i}{\sim} \beta$. For $i = k$ we actually have an equality $|\mathrm{cs}_k(\alpha) - \mathrm{cs}_k(\beta)| = \frac{1}{2^{k+1}}$. Now assume $i > k$ and $\mathrm{cs}_i(\alpha) - \mathrm{cs}_i(\beta) \leq \frac{1}{2^i}$. We know by the above claim that $\alpha_i - \beta_i = -1$ so

$$\begin{aligned}\mathrm{cs}_{i+1}(\alpha) - \mathrm{cs}_{i+1}(\beta) &= \mathrm{cs}_i(\alpha) - \mathrm{cs}_i(\beta) + \frac{\alpha_i - \beta_i}{2^{i+1}} \\ &\leq \frac{1}{2^i} - \frac{1}{2^{i+1}} = \frac{1}{2^{i+1}}\end{aligned}$$

so $\alpha \underset{i+1}{\sim} \beta$.

Now assume $\prod_{n:\mathbb{N}} \alpha \underset{n}{\sim} \beta$. It suffices to show that $\mathrm{cs}(\alpha) - \mathrm{cs}(\beta) = 0$. But for any n we have that $0 \leq |\mathrm{cs}(\alpha) - \mathrm{cs}(\beta)| \leq \frac{1}{2^n}$ so we have $|\mathrm{cs}(\alpha) - \mathrm{cs}(\beta)| = 0$.

To see that cs is surjective we explicitly construct the binary expansion of a point in $\mathbb{I}$. The problem is that in the absence of LEM, the usual construction – which decides at every step if $x > 2^{-n}$ for some $x : \mathbb{I}$ – does not immediately go through. In the presence of LLPO, however, we can get a notion of linear order that is good enough. We will deal with Cauchy reals and use a *slightly* adapted definition of nonnegativity originally given in [19].

Definition 5.3: Nonnegativity

Given a real number $r = [\{q_n\}]$ defined as an equivalence class of Cauchy sequences of rationals, we define $r \geq 0$ when for every $n : \mathbb{N}$ there exists $N_n : \mathbb{N}$ so that $\forall k \geq N_n$

$$q_k \geq -\frac{1}{2^n}$$

where q is some representing Cauchy sequence of r.

Nonpositivity is defined analogously by asserting the existence of a tail of the sequence that is $\leq \frac{1}{2^n}$.

Lemma 5.3.1:

For $x, y : \mathbb{I}$ we have $x \leq y \vee y \leq x$.

Proof. It suffices to show $x - y \leq 0 \vee y - x \leq 0$, and without loss of generality just $x \leq 0 + x \geq 0$. The idea is to first construct a binary sequence $\alpha : 2^{\mathbb{N}}$ whose even indices represent nonnegativity up to $-\frac{1}{2^n}$ and whose odd indices represent nonnegativity up to $\frac{1}{2^n}$. We just construct the even indices in the sequence as the construction of the odd indices is completely analogous.

Let $x = \{q_n\}$ where $\{q_n\}$ is some representing Cauchy sequence of rational numbers. To define α_{2n} obtain the least index N_n such that for all $i, j \geq N_n$, $|q_i - q_j| \leq \frac{1}{2^{n+1}}$. If $\alpha_{N_n} \leq -\frac{1}{2^n}$ where here we use the decidable order on $\mathbb{Q}$, then set $\alpha_{2n} = 1$ else $\alpha_{2n} = 0$. Morally, we are setting $\alpha_{2n} = 1$ whenever we know the sequence cannot be nonnegative because of the Cauchy condition.

Furthermore, by definition, if $\alpha_{2n} = 1$ for some n then for all $j \geq n$ $\alpha_{2j+1} = 0$, since for large enough k – that is $k \geq N_n$ – $q_k \leq 0 \leq \frac{1}{2^n}$.

Recall the map $f : 2^{\mathbb{N}} \to \mathbb{N}_\infty$ defined in Lemma 3.4.3 which retains only the first 1 in the input sequence. We can call LLPO on $f(\alpha)$ (by the universal property of propositional truncation) and we obtain WLOG that $\forall n : f(\alpha)_{2n} = 0$. This implies that the first 1 (if any) that occurs in α is for some odd index $2j + 1$, and by the above claim this means for all $n \geq j$ one has $\alpha_{2n} = 0$. But then for large enough n we have a tail of q such that

$$q_k \geq -\frac{1}{2^n}$$

where k is in the tail, so q is nonnegative. In the other case we similarly obtain nonpositivity.

We sometimes refer to the above as *dull linearity* of the order on $\mathbb{I}$.

Lemma 5.3.2:

Any $x : \mathbb{I}$ merely admits an expansion in terms of a sequence α so that

$$x = \sum_{i=0} \frac{\alpha_i}{2^{i+1}} = \mathrm{cs}(\alpha)$$

in particular implying cs is a surjection.

Proof. We give the traditional construction of binary expansion, which goes through exactly the same with dull linearity; we apply the universal property of propositional truncation to Lemma 5.3.1 implicitly to obtain either $x \geq \frac{1}{2}$ or $x \leq \frac{1}{2}$ at each step.

$$\alpha_0 := x \geq \frac{1}{2}$$

$$\alpha_{n+1} := x - \text{cs}_{n+1}(\alpha^n) \geq \frac{1}{2^{n+1}}$$

where in the second line α^n is the sequence constructed so far by induction. In both cases we are defining α_n to be 1 if $x \geq \frac{1}{2}$ is obtained by dull linearity, and 0 otherwise.

Corollary 5.3.3:

$\mathbb{I}$: CHaus. Further, $\mathbb{I}^n$: CHaus for any $n : \mathbb{N}$.

Proof. By the above lemma $\mathbb{I}$ admits a surjection cs from the Stone space $2^{\mathbb{N}} = \text{Sp}(2[\mathbb{N}])$. Then for any $x, y : \mathbb{I}$ we have merely that $\text{cs}(\alpha) = x$ and $\text{cs}(\beta) = y$ for some α, β but by Lemma 5.2.1 $\text{cs}(\alpha) = \text{cs}(\beta) \leftrightarrow \forall_{n:\mathbb{N}} \alpha \underset{n}{\sim} \beta$ which is a countable conjunction of decidable propositions, and in particular of closed propositions, so closed. The last statement is a consequence of Theorem 4.18.

Given the cs characterization of reals we have a different (more "computable") way of defining order on the interval. That is, we take a binary sequence in the fiber, and find n large enough so that the error is larger than $\frac{1}{2^n}$.

Definition 5.4: Order on the Interval

Given $x, y : \mathbb{I}$ we define the proposition $x < y$ by

$$x < y := \exists_n \text{cs}_n(\alpha) + \frac{1}{2^n} \underset{\mathbb{Q}}{<} \text{cs}_n(\beta)$$

where the order used in the definition is the order on the rationals, and implicitly we use the universal property of propositional truncation to obtain α, β so that cs $(\alpha) = x$ and cs $(\beta) = y$.

Of course we know it is possible that distinct α, α' might have the same image under cs, so we show that the above is well-defined regardless of the choice of α, β.

Lemma 5.4.1:

Suppose $\alpha, \alpha' : 2^{\mathbb{N}}$ such that $\text{cs}(\alpha) = \text{cs}(\alpha')$. Then

$$\exists_n \text{cs}_n(\alpha) + \frac{1}{2^n} \underset{\mathbb{Q}}{<} \text{cs}_n(\beta) \leftrightarrow \exists_n \text{cs}_n(\alpha') + \frac{1}{2^n} \underset{\mathbb{Q}}{<} \text{cs}_n(\beta)$$

Proof. We just give a sketch. Use the *explicit* characterization of when α, α' are unequal digitwise but agree under cs. That is, α, α' agree digitwise on some finite prefix up to k and then without loss of generality $(\alpha_i)_{i=k} = 0111...$ while $(\alpha'_i)_{i=k} = 1000...$ as shown in Lemma 5.2.1.

Then it is easy to show that as n increases the error between the sequences will eventually be small enough such that if $\alpha < \beta$ then $\alpha' < \beta$ and vice versa.

Definition 5.4 is a very explicit way of stating $\neg \prod_n \alpha \underset{n}{\sim} \beta$, so $x < y$: Open, as the negation of a closed proposition. Naturally $x < y \lor y < x \leftrightarrow x \neq y$ so $x \neq y$ is open as well. Now we look at seriously characterizing the order topology on $\mathbb{I}$.

Lemma 5.4.2:

Suppose $D \subseteq (2^{\mathbb{N}})$ is decidable. Then its image in $\mathbb{I}$, i.e. $\mathrm{cs}(D)$, is a finite union of closed intervals and therefore a closed set.

Proof. First we want to characterize what the decidable subsets of $2^{\mathbb{N}}$ are; morally these should be those binary sequences that agree on some finite subset. By Stone duality D corresponds precisely to d, some element of the free Boolean algebra $2[\mathbb{N}]$, recalling that $2^{\mathbb{N}} = \mathrm{Sp}(2[\mathbb{N}])$. By conjunctive normal form, we know

$$d = \bigwedge_{i \in I} \bigvee_{j \in J_i} d_{ij}$$

where each d_{ij} is among the generators of $2[\mathbb{N}]$. We just consider *basic* decidable subsets, i.e. $D : 2^{2^{\mathbb{N}}}$ so that corresponding d can be written as

$$\bigvee_{j \in J} d_j$$

for some finite set J; the union of such subsets is given on the Boolean side by the finite meet of disjunctions as above. To slightly simplify the arithmetic, suppose the first $|J|$ generators of $2^{[\mathbb{N}]}$ are precisely $\left(d_j\right)_{j \in J}$. Given $\alpha : \mathrm{Sp}(2[\mathbb{N}])$ we have $\alpha : D$ if and only if $\alpha\left(d_j\right) = 0$ for all j. Under the equivalence with Cantor space this gives $\alpha : 2^{\mathbb{N}}$ that is 0 on the first $|J|$ entries, so $\alpha : D$ if and only if $\alpha_j = 0$ for $j < |J|$.

Now consider the image of D under cs, precisely those $x : \mathbb{I}$ so that

$$x = \sum_{i=|J|}^{\infty} \frac{\alpha_i}{2^{i+1}}$$

where $\alpha_i : i \geq |J|$ might vary freely. But this is precisely the interval $\left[0, 2^{-|J|}\right]$ with the extremes being achieved by $\mathbf{0}$ and $\mathbf{1}$ shifted right by $|J|$ indices. In the general case where $\left(d_j\right)_{j \in J}$ are not the first $|J|$ generators of $2[\mathbb{N}]$, the interval is just shifted by some fixed amount.

Now we can use our earlier work characterizing the topology of compact Hausdorff spaces to describe the topology on the interval; in particular we recall that all closed (resp. open) sets arise as the image of countable intersections (resp. unions) of decidable subsets. In the following proof, we use closed, open set to mean closed and open in the sense of Definition 4.8 but refer to open and closed *intervals* in $\mathbb{I}$ in the sense of the usual order topology on $\mathbb{I}$.

Lemma 5.4.3:

Any open set $U \subseteq \mathbb{I}$ is a countable union of open intervals.

Proof. By Corollary 4.17.6 we know that every closed set in $\mathbb{I}$ is the countable intersection of the image of decidable subsets in $2^{\mathbb{N}}$, which we characterized as finite unions of closed intervals in Lemma 5.4.2. So every closed set is a countable intersection of finite unions of closed intervals, i.e. a countable intersection of closed intervals. Then a set U is open iff it is the complement of such a set, i.e., a countable union of open intervals.

We note that in the usual order topology on $\mathbb{I}$, open intervals include half opens of the form $[0, x), (y, 1]$ for $x > 0$ and $y < 1$.

Since the order and metric topologies coincide on $\mathbb{R}$, every $f : \mathbb{I} \to X$ to a type X is continuous in the normal metric sense. Now we work to prove the last basic topological fact about $\mathbb{I}$: that it is connected, which in SSD means precisely that any function $f : \mathbb{I} \to 2$ is constant.

Lemma 5.4.4:

$\mathbb{I}$ is connected; that is, any function $f : \mathbb{I} \to 2$ is constant.

Proof. By Lemma 4.18.4 we can proceed with a proof by contradiction. Say f is not constant. Then $D \subseteq 2^{\mathbb{N}}$ induced by $f \circ \mathrm{cs}$ is a proper nonempty decidable subset. Then by Lemma 5.4.2 $\mathrm{cs}(D)$ is a proper closed subset of $\mathbb{I}$, in particular a finite union of closed intervals. In particular since it is a proper subset it is not a countable union of open intervals, so by Lemma 5.4.3 it is not an open subset. But $\mathrm{cs}(D) = f^{-1}(f(\mathrm{cs}(D))) = f^{-1}(0)$ and since $\{0\}$ is open in 2, so f is not continuous.

Lemma 5.4.5:

Any function $f : \mathbb{I} \to \mathbb{Z}$ is constant.

Proof. Consider $x, y : \mathbb{I}$ and let $\ell := f(x), m := f(y)$. If $\ell = m$ we are done. Otherwise take the quotient of $\mathbb{Z}$ by the relation $\sim$ generated by

$$a \sim b \leftrightarrow (a = b = m) \vee (a = \ell \wedge b \neq m)$$

Then $2 = \mathbb{Z}/\sim$, and by connectedness, $\pi \circ f : \mathbb{I} \to 2$ is constant where $\pi : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/\sim$.

$(\pi \circ f)(x) = [m]$ since $f(x) = m$ and by constancy $(\pi \circ f)(y) = [m]$ and since $[m] = \{m\}$, we know $f(x) = f(y) = m$.

Finally we observe that everything just stated about the interval $\mathbb{I}$ immediately extends to any closed interval $[a, b] \subseteq \mathbb{R}$ via the standard homeomorphism $\mathbb{I} \to [a, b]$ given by $x \mapsto x(b - a) + a$. Indeed, we observe a version of the gluing lemma originally appearing as Lemma 1.2.2 in [20] for continuous functions on subintervals which agree on the overlap.

Lemma 5.4.6:

Let $f_1 : [a,b] \to X$ and $f_2 : [b,c] \to X$ be maps into a set X. If $f_1(b) = f_2(b)$ then there is a function $g : [a,c] \to X$ such that

$$g(x) = \begin{cases} f_1(x) \text{ if } x \le b \\ f_2(x) \text{ if } x \ge b \end{cases}$$

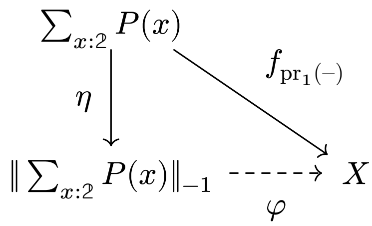
Proof. This is a special case of Lemma 1.2.2 in [20]. Let $\iota : [a,b] \cup [b,c] \to [a,c]$ be the obvious inclusion map.

By dull linearity we obtain for any $x : [a,c]$ that $x \le b \vee b \le x$ so the map is surjective. Define the type family $P : \mathbb{2} \to ([a,b] \to \mathrm{Prop}_{\mathcal{U}})$ by

$$P(1) := x \mapsto x \le b$$
$$P(2) := x \mapsto x \ge b$$

or in other words $P(1) := [a,b]$ and $P(2) := [b,c]$. Then $f_1 : P(1) \to X$ and $f_2 : P(2) \to X$, where $P(1) \cap P(2) = \{b\}$ and by assumption $f_1(b) = f_2(b)$.

We can thus apply the main theorem of [21], which states that since X is 0-truncated there is a factorization of the total map $f_{\mathrm{pr}_1(-)} : \sum_{x:\mathbb{2}} P(x) \to X$ through the propositional truncation



The truncation is obviously inhabited and thus contractible, implying that $f_{\mathrm{pr}_1(-)}$ is constant on the intersection $P(1) \cap P(2)$. Then by surjectivity of ι, given $y : \mathbb{I}$ we have $z : \| \sum_{x:\mathbb{2}} P(x) \|$ and can just define $g(y) := \varphi(z)$, where $g : \mathbb{I} \to X$ as desired.

The above proof gives the spirit of the gluing lemma, but we observe that there was nothing special about the interval in the above; we used it to be concrete, but we state the general version.

Lemma 5.4.7:

Let $(f_i : C_i \to X)_{i \in I}$ be a finite collection of maps into a set X, where $\bigcup_i C_i = Y$ is a finite closed cover of a set Y. Further suppose that $f_i = f_j$ on all pairwise intersections $C_i \cap C_j$. Then there is a function $g : Y \to X$ so that $g(y) = f_i(y)$ where $y : C_i$.

Proof. Reapply the above proof, creating the type family over the finite set I and applying the factorization theorem of [21].

Finally, with a well-behaved interval in hand, it is easy to define $\mathbb{S}^n$ as a closed subset of its $n+1$-fold product.

Definition 5.5: Topological n-Sphere

The *topological sphere* $\mathbb{S}^n$ (sometimes the "sphere with points") is the subset of the $n+1$-fold product of the closed interval $[-1, 1]$ given by

$$\mathbb{S}^n := \sum_{x:[-1,1]^{n+1}} \|x\| = 1$$

where $\| - \|$ is the Euclidean norm.

Immediately by the theory developed in the prior section we find that the n-sphere is compact Hausdorff.

Lemma 5.5.1:

$\mathbb{S}^n$: CHaus.

Proof. By Corollary 4.16.3 we just need to exhibit that $\mathbb{S}^n$ is a closed subset since $\mathbb{I}^{n+1}$: CHaus. Equality in $[-1, 1]$ is closed since $[-1, 1]$: CHaus, so equality of two points $x, y : \mathbb{S}^n$ is an equality in $[-1, 1]$ and a trivial equality of terms in propositions, which is closed.

We can also easily show that at least the topological circle is connected in the sense of Definition 4.15.

Lemma 5.5.2:

The topological circle $\mathbb{S}^1$ is connected.

Proof. Consider any function $f : \mathbb{S}^1 \to 2$, and recall the standard map $\pi : \mathbb{I} \twoheadrightarrow \mathbb{S}^1$ arising as the restriction of the covering map $\mathbb{R} \twoheadrightarrow \mathbb{S}^1$. Then since π is surjective it follows that f is constant if and only if $f \circ \pi$ is, but $f \circ \pi$ is constant by Lemma 5.4.4.

Chapter 6
The Shape Modality

In this chapter we introduce the very basics of the theory of *higher modalities* in HoTT, as originally developed in [10]. One particular modality – the *shape* modality – will come into play heavily in the proof of the 2D Borsuk-Ulam theorem.

Put simply, a modality is an operator $\bigcirc : \mathcal{U} \to \mathcal{U}$ on types, generalizing from modal logic where modalities are operators on propositions. A higher modality is a modality that somehow operates on a type and all its path spaces, in a way that we will make precise shortly. As it turns out a higher modality admits several equivalent characterizations that we give in the first section of this chapter, using a familiar notion – the n-truncation, which turns out to be an example of a modality – as a guiding example.

We then delve into those modalities that arise specifically as *nullifications* of a type family, of which the shape modality is a crucial example. As a closing application, we use the shape modality to give a different characterization of $\mathbb{R}$.

6.1. Modalities

We recall briefly the theory of *modalities* in HoTT, whose basic theory is significantly developed in [10] and furthered in [9], both of which this section pulls from.

Definition 6.1: Modal Operator

A *modal operator* is an operator on types

$$\bigcirc : \mathcal{U} \to \mathcal{U}$$

and a family of *units*

$$\eta_X : X \to \bigcirc X$$

taking any $X : \mathcal{U}$ to its image in the modal subuniverse. Often we drop the subscript and just write η when the type is clear.

Given a modal operator $\bigcirc$ we say a type X is $\bigcirc$-*modal* when η_X is an equivalence, and similarly we say a map $f : X \to Y$ is $\bigcirc$-modal when all its fiber types are $\bigcirc$-modal. $\mathcal{U}_\bigcirc$, defined as

$$\mathcal{U}_\bigcirc := \sum_{X:\mathcal{U}} \text{isModal}(X)$$

is the subuniverse of modal types where $\text{isModal}(X) := \text{is-equiv}(\eta_X)$.

We say a type X is $\bigcirc$-*connected* when $\bigcirc X$ is contractible, and say f is $\bigcirc$-connected when its fibers are.

When a modality $\bigcirc$ is clear from context we will just say a type/map is *modal*, but we do not overload "connected" in this way, which always means the set truncation is contractible.

Given a modal operator we consider it a (higher) modality when it satisfies one of three equivalent conditions which we tour in this section. A modality can also be characterized by either of two orthogonal factorization systems that it induces, which we only briefly remark upon. The proofs of their mutual equivalence are in [10] (and [9] for reflective factorization systems).

Firstly we give the condition from which higher modalities get their name.

Definition 6.2: Higher Modality

A *higher modality* is a modal operator that comes with

1. An induction rule for modal types

$$\mathrm{ind}_{\bigcirc} : \prod_{x:A} P(\eta(x)) \to \prod_{u:\bigcirc A} P(u)$$

   where $P : \bigcirc A \to U$ is any type family and $A : \mathcal{U}$ is any type in the universe, along with a "computation rule"

$$\mathrm{comp}_A(f, x) : \mathrm{ind}_{\bigcirc}(f, \eta(x)) = f(x)$$

   for every dependent function $f : \prod_{x:A} P(\eta(x))$.

2. A function for every modal type $\bigcirc A$ witnessing that the identity types are modal, that is

$$\prod_{x,y:\bigcirc A} \mathrm{isModal}(x = y)$$

The second datum in the above definition is why these are "higher" modalities: as soon as a type is modal, so are all of its path spaces. We begin our running example: propositional truncation is in fact a higher modality.

Example 6.2.1: Propositional Truncation as a Higher Modality

From the definition of higher modality, it is immediate that the propositional truncation as a higher inductive type is a higher modality.

The operator $\| - \| : \mathcal{U} \to \mathcal{U}$ is the modal operator with units $| - |_A : A \to \|A\|$. The induction and computation rule are given by its definition as a higher inductive type. Since for $x, y : \|A\|$, $x = y$ is a proposition (in fact contractible), and $\eta : P \to \|P\|$ is an equivalence for propositions P, the result follows.

Definition 6.2 gives the definition that is closest in spirit to the idea of a modality as an operator on types. Higher modalities are equivalently defined by a universal property determining maps out of them.

Definition 6.3: Uniquely Eliminating Modality

Given a modal operator $\bigcirc$, it is a *uniquely eliminating modality* when the map

$$- \circ \eta : \prod_{u : \bigcirc A} P(u) \to \prod_{x : A} P(\eta(x))$$

is an equivalence for any type A and family of modal types $P : \bigcirc A \to \mathcal{U}_{\bigcirc}$ over $\bigcirc A$.

We often informally refer to this equivalence as the *dependent universal property* for a modality.

Returning to our running example of propositional truncation, we show the dependent universal property of the propositional truncation which incarnates propositional truncation as a uniquely eliminating modality.

Example 6.3.1: Dependent Universal Property of Propositional Truncation

The dependent universal property of the propositional truncation, that is, the equivalence

$$- \circ |-| : \prod_{u : \|A\|} P(u) \simeq \prod_{x : A} P(|x|)$$

where $P : \|A\| \to \mathrm{Prop}_{\mathcal{U}}$, shows that propositional truncation is a uniquely eliminating modality. Recall that the inverse map is given precisely by the induction rule for propositional truncation, which for a family of propositions suffices to give the equivalence.

Inspired by this example we often do refer to the equivalence given by a uniquely eliminating modality as a dependent universal property for the modal operator $\bigcirc$. The next characterization of modality instead defines a modal operator to be a modality in terms of the subuniverse induced by the modal operator.

Definition 6.4: Σ-closed Reflective Subuniverse

Given a modal operator $\bigcirc$, the subuniverse $\mathcal{U}_{\bigcirc}$ defines a *Σ-closed reflective subuniverse* whenever

1. For any $B : \mathcal{U}$ such that $\mathrm{isModal}(B)$, the canonical map

$$(\bigcirc A \to B) \to (A \to B)$$
$$f \mapsto f \circ \eta$$

   is an equivalence, and
2. The subuniverse $\mathcal{U}_{\bigcirc}$ is closed under Σ-types.

We sometimes refer to the first condition on a subuniverse here as *reflectivity*.

We observe that condition 1 in this definition is almost just the statement that $\bigcirc$ is left adjoint to the inclusion $\mathcal{U}_\bigcirc \hookrightarrow \mathcal{U}$ though we did not state naturality, nor yet show that $\bigcirc$ is actually functorial.

Example 6.4.1: Propositions as Reflective Subuniverse

In the case of propositional truncation the reflective subuniverse is, naturally, $\mathrm{Prop}_\mathcal{U}$. The reflectivity is given by the non-dependent universal property of the propositional truncation

$$\|A\| \to P \simeq A \to P$$

for all propositions $P : \mathrm{Prop}_\mathcal{U}$, and it is easy to show that propositions are closed under Σ (in fact this holds for n-truncated types generally).

We observe that reflectivity of the subuniverse induces a canonical naturality square.

Lemma 6.4.2:

Given a modal operator $\bigcirc$ inducing a Σ-closed reflective subuniverse, the reflectivity induces for any map $f : A \to B$ a naturality square

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \eta_A}\big\downarrow & & \big\downarrow{\scriptstyle \eta_B} \\ \bigcirc A & \overset{\bigcirc f}{\dashrightarrow} & \bigcirc B \end{array}$$

Proof. Immediate by applying reflectivity to the map $\eta_B \circ f$.

All three of the above characterizations depend on the data of a modal operator. We state the equivalence explicitly without proof; the proof can be found in §1 of [10].

Theorem 6.5: Equivalent Characterizations of Modalities

Given a modal operator $\bigcirc$, the following are equivalent

1. $\bigcirc$ is a higher modality.
2. $\bigcirc$ is a uniquely eliminating modality.
3. The subuniverse $\mathcal{U}_\bigcirc$ is a Σ-closed reflective subuniverse.

When any of these are satisfied we just say $\bigcirc$ is a *modality*.

Hereafter when discussing a modality we freely use any of the properties explicitly described in Definition 6.2, Definition 6.3, and Definition 6.4. We show some basic properties.

Lemma 6.5.1:

The units $\eta_X : X \to \bigcirc X$ are $\bigcirc$-connected.

Proof. We want to show that the modal fibers of η_X are contractible. Using the dependent universal property we obtain a function

$$s : \prod_{z:\bigcirc X} \bigcirc \left(\mathrm{fib}_\eta(z)\right)$$

by defining $s(\eta(x)) :\equiv \eta\left(x, \mathrm{refl}_{\eta(x)}\right)$, giving the centers of contraction. Next we need a contracting homotopy for every $s(z)$ or in other words a term in the type

$$\prod_{z:\bigcirc X} \prod_{w:\bigcirc\left(\mathrm{fib}_\eta(z)\right)} w = s(z)$$

Applying the dependent universal property to $\bigcirc \left(\mathrm{fib}_\eta(z)\right)$ it suffices to give a term in the type

$$\prod_{z:\bigcirc X} \prod_{x:X} \prod_{p:\eta(x)=z} \eta(x, p) = s(z)$$

and by path induction on p it suffices to show for every $x : X$ that $s(\eta(x)) = \eta\left(x, \mathrm{refl}_{\eta(x)}\right)$ which holds by definition.

We return to our canonical example.

Example 6.5.2: -1-connected Maps

The $\| - \|$-connected maps (or -1-connected maps) are those maps $f : A \to B$ for which $\|\mathrm{fib}_f(b)\|$ is contractible, i.e. inhabited, for all $b : B$. These are precisely the surjective maps.

Lemma 6.5.1 implies, therefore, that the units for propositional truncation $|-|_A : A \to \|A\|$ are surjective.

Inspecting the proof of Lemma 6.5.1 we observe that it relied essentially on the dependent universal property for modal units as stated in Definition 6.3. $\bigcirc$-connected maps are in fact precisely the maps that satisfy a dependent universal property.

Lemma 6.5.3:

Given a modality $\bigcirc$ and a $\bigcirc$-connected map $f : A \to B$, as well as a family of modal types $P : B \to \mathcal{U}_\bigcirc$ over B, the map

$$- \circ f : \prod_{y:B} P(y) \to \prod_{x:A} P(f(x))$$

is an equivalence.

Proof. Observe that $\bigcirc$-connectedness means precisely that $\bigcirc \left(\mathrm{fib}_f(y)\right)$ is contractible, so there is an equivalence

$$\left(\prod_{y:B} \bigcirc \left(\mathrm{fib}_f(y)\right) \to P(y)\right) \simeq \left(\prod_{y:B} \mathbb{1} \to P(y)\right) \simeq \prod_{y:B} P(y)$$

Similarly for every $y : B$ reflectivity of the modality implies that there is an equivalence

$$(\bigcirc (\mathrm{fib}_f(y)) \to P(y)) \simeq (\mathrm{fib}_f(y) \to P(y))$$

which assembles into an equivalence of the dependent function types

$$\left(\prod_{y:B} \bigcirc (\mathrm{fib}_f(y)) \to P(y)\right) \simeq \left(\prod_{y:B} \mathrm{fib}_f(y) \to P(y)\right)$$

and the equivalence $\left(\prod_{y:B} \mathrm{fib}_f(y) \to P(y)\right) \simeq \prod_{x:A} P(f(x))$ is standard. These assemble into a commuting square

$$\begin{array}{ccc} \prod_{y:B} \bigcirc (\mathrm{fib}_f(y)) \to P(y) & \longrightarrow & \prod_{y:B} \mathrm{fib}_f(y) \to P(y) \\ \downarrow & & \downarrow \\ \prod_{y:B} P(y) & \xrightarrow{\quad - \circ f \quad} & \prod_{x:A} P(f(x)) \end{array}$$

and by 2-of-3 the bottom map is an equivalence.

In fact the converse of the above lemma is true (see [10] Lemma 1.36) and leads naturally to the notion of a stable orthogonal factorization system.

At this point we have stated that a modality is an operator $\bigcirc$ on types whose induced subuniverse is reflective and closed under Σ-types, and which comes equipped with a dependent universal property for defining functions out of $\bigcirc$-types. Furthermore, if a type is modal, so are all of its path spaces. Our final characterization of modalities is not based on modal operators at all, however.

Definition 6.6: Orthogonal Factorization System

Let $\mathrm{Func}_{\mathcal{U}} :\equiv \sum_{A,B:\mathcal{U}} A \to B$ be the type of all non-dependent functions between any two types in the universe; we will leave the types A, B implicit usually and just write $f : \mathrm{Func}_{\mathcal{U}}$. An *orthogonal factorization system* is

1. The data of two families of propositions

$$\mathcal{L}, \mathcal{R} : \mathrm{Func}_{\mathcal{U}} \to \mathrm{Prop}_{\mathcal{U}}$$

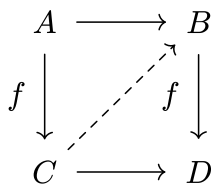
   over $\mathrm{Func}_{\mathcal{U}}$, such that all equivalences are in both $\mathcal{L}, \mathcal{R}$ and both classes are closed under composition of maps. Given a map $f : A \to B$ then if $\mathcal{L}(f)$ we say f is in the *left class* or is just a *left* map, and similarly if $\mathcal{R}(f)$ we say f is in the *right class* or is a *right* map.

2. A type family

$$\mathrm{im}_{\mathcal{L},\mathcal{R}} : \mathrm{Func}_{\mathcal{U}} \to \mathcal{U}$$

over all functions in the universe, corresponding to the type a map f factors through.

3. A witness for any $f : \mathrm{Func}_{\mathcal{U}}$ that the type of functions $g : \mathcal{L}$ such that $g : A \to \mathrm{im}_{\mathcal{L},\mathcal{R}}(f)$ and $h : \mathcal{R}$ so that $h : \mathrm{im}_{\mathcal{L},\mathcal{R}}(f) \to B$ equipped with a homotopy witnessing that $f \sim h \circ g$, is contractible.
4. The left maps lift against the right maps; that is, given a commutative square where $f : \mathcal{L}$, $g : \mathcal{R}$



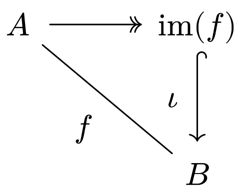
the type of diagonal fillers, i.e. maps $C \to B$ with homotopies witnessing commutation of the upper and lower triangle, is contractible.

In other words an orthogonal factorization system distinguishes a left and right class of maps so that any $f : A \to B$ factorizes uniquely as a left map followed by a right map, and so that the left and right maps lift against each other. Returning to our example of propositional truncation, we see that it induces a familiar factorization system.

Example 6.6.1: Embedding and Surjection

Recall that a $\| - \|$-truncated map, i.e. a map with fibers that are propositions, is an embedding. Further, in view of Example 6.5.2 a $\| - \|$-connected map is surjective. For the n-truncation in general, the n-connected maps form the left class and the n-truncated maps form the right class of an orthogonal factorization system. (See Chapter 7 in [11] for more.)

In the case of propositional truncation it follows that every map factors as a surjection (onto the image of f) followed by an embedding as below



This is an orthogonal factorization system induced by $\| - \|$.

The factorization system described in Example 6.6.1 induced by the modality $\| - \|$ is the *stable* factorization system. This factorization system induced by a modality $\bigcirc$ has $\bigcirc$-connected maps as the left class and $\bigcirc$-modal maps as the right class.

Theorem 6.7: Modalities Induce a Stable Factorization System

A modality $\bigcirc$ induces an orthogonal factorization system for which the left class is the $\bigcirc$-connected and the right class is the $\bigcirc$-modal maps. The left maps of this factorization system are stable under pullback.

Proof. Theorem 1.34 in [10].

The factorization system allows us to show definitively the suspicion raised by Lemma 6.5.3 – that if a map $f : A \to \bigcirc A$ is $\bigcirc$-connected, it actually is the modal unit.

Corollary 6.7.1:

If $f : A \to \bigcirc A$ is $\bigcirc$-connected, then $f = \eta_A$.

Proof. For a modal type $\bigcirc A$ the fiber type terminal projection $! : \bigcirc A \to \mathbb{1}$ is equivalent to $\bigcirc A$ so the terminal projection is modal. Then since f is $\bigcirc$-connected both

$$A \xrightarrow{f} \bigcirc A \xrightarrow{!} \mathbb{1}$$

and

$$A \xrightarrow{\eta_A} \bigcirc A \xrightarrow{!} \mathbb{1}$$

factorize $! : A \to \mathbb{1}$ as a modal map following a $\bigcirc$-connected map. By uniqueness of such a factorization it follows $f = \eta_A$.

A stable factorization system can be characterized abstractly as an orthogonal factorization system whose left maps are stable under pullback, and any such system in fact induces a modality.

We highlight another orthogonal factorization system, the *reflective* factorization system as featured in [9]. The left and right classes of a reflective factorization system have a very conrete description. The left class is the class of maps whose induced map on modal types is an equivalence.

Definition 6.8: $\bigcirc$-Equivalences

A map $f : A \to B$ is a $\bigcirc$-equivalence when $\bigcirc f : \bigcirc A \to \bigcirc B$ is an equivalence.

The proof is not completely obvious, but it is not difficult to show that $\bigcirc$-connected maps are $\bigcirc$-equivalences, as in Proposition 6.3 of [9].

The right class is the class of maps whose naturality squares are pullbacks.

Definition 6.9: $\bigcirc$-Étale Maps

Given a map $f : A \to B$ and a modality $\bigcirc$ we say f is $\bigcirc$-etale when

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \bigcirc A & \xrightarrow{\bigcirc f} & \bigcirc B \end{array}$$

is a pullback square.

Lemma 6.9.1:

Given a modality $\bigcirc$, the $\bigcirc$-equivalences and the $\bigcirc$-etale maps form the left and right classes, respectively, of an orthogonal factorization system.

Proof. Theorem 7.2 in [9].

As it turns out this factorization system satisfies additional properties [9, Prop 7.3, 7.4] which define the abstract notion of a *reflective* factorization system. We do not detail this notion here since we only need to know about the classes of maps, but like the stable factorization system, when a factorization system is reflective it actually induces a modality, specifically a Σ-closed reflective subuniverse. For our purposes we are particularly interested in the right class of $\bigcirc$-etale maps, which will be essential to developing the synthetic theory of covering spaces as detailed in Chapter 7.

6.2. Localization and Nullification

The modality primarily of interest to us, the shape modality, arises from *nullification* of certain types. Nullification is a special case of *localization*, the process of formally inverting a class of maps. To introduce this notion, fix an A-indexed family of maps $\mathcal{F} : \prod_{x:A} B(x) \to C(x)$.

Definition 6.10: $\mathcal{F}$ -Local Types

A type X is $\mathcal{F}$-local whenever for all $x : A$ the induced map

$$\begin{gathered} (C(x) \to X) \to (B(x) \to X) \\ f \mapsto f \circ \mathcal{F}_x \end{gathered}$$

is an equivalence. More concretely, any map $f : B(x) \to X$ admits a unique extension through $C(x)$

$$\begin{array}{ccc} B(x) & \xrightarrow{f} & X \\ {\scriptstyle \mathcal{F}_x}\downarrow & \overset{\exists!}{\nearrow} & \\ C(x) & & \end{array}$$

making the above diagram commute. When $C(x) = \mathbb{1}$ for all $x : A$, we say that X is $\mathcal{F}$-null.

Morally, an $\mathcal{F}$-local type X "sees $B(x)$ and $C(x)$ as equivalent" via maps from $B(x)$ and $C(x)$. In the case where $C(x) = \mathbb{1}$, one has that X sees $B(x)$ as contractible for all x, hence the terminology $\mathcal{F}$-null.

Using a complex higher inductive type, it is possible to create a modality that arises as the nullification of a specific type A, sending a type X to a modal type $\bigcirc X$ that is A -null. Thus $\bigcirc X$ "sees A as contractible" through maps into it, i.e. all maps $A \to \bigcirc X$ are constant.

Now, $\mathbb{1}$ is A-null for every A, so theoretically $\bigcirc X$ could be $\mathbb{1}$ for every X, making it trivial; however, it is possible to define the nullification modality in such a way that it sends X to $\mathbb{1}$ for as few types X as possible, while remaining a modality. The following is not a precise definition, but is precise enough for our purposes.

Definition 6.11: Nullification

Given a type family $\mathcal{F} : A \to \mathcal{U}$, the *nullification* of $\mathcal{F}$ is the smallest Σ-closed reflective subuniverse of $\mathcal{U}$, with the modality $\bigcirc$ acting as the reflector, such that $\bigcirc \mathcal{F}(x)$ is contractible for all x.

The precise definition of nullification from [10] is as a complex higher inductive type, and we redirect the interested reader to that article for the constructors and the proof that they induce a modality. Instead we focus on two key examples which are enough to understand the shape modality.

Example 6.11.1: n-Truncation

The n-truncation $\| - \|_n$, ordinarily defined as a higher inductive type in HoTT, is equivalent to the nullification of S^{n+1} where $S^0 = \mathbb{2}$. In particular propositional truncation $\| - \|_{-1}$ arising as $\mathbb{2}$ nullification means that all maps $\mathbb{2} \to \|A\|_{-1}$ are constant for any $A : \mathcal{U}$. Then naturally there can be at most one element of $\|A\|_{-1}$, so if it is inhabited it is contractible, hence a proposition.

More important for synthetic Stone duality is another modality arising as a nullification: the *shape* modality.

Definition 6.12: Shape Modality

The *shape* modality, written as $\int -$, is the nullification of $\mathbb{I}$.

A variant of shape, $\int_1 -$, is defined as the *meet* of $\int -$ and 1-truncation. Specifically, the Σ-closed reflective subuniverse corresponding to $\int_1 -$ is the largest such subuniverse in which all types are both $\int$-modal and 1-truncated.

Morally, shape nullifies the "points" in any type that admits nontrivial continuous functions from $\mathbb{I}$. On sets, which by definition have no higher path structure, but may admit a nontrivial topology, $\int X$ in some casesgives the homotopy type of X induced by the topology. (It is an open conjecture that shape on a topological finite CW complex yields the homotopy type as a higher inductive type.) Using the definition it is possible to show that $\int \mathbb{R} = \mathbb{1}$, for example; in fact the shape of any path-connected type is $\mathbb{1}$ by Lemma 5.2.4 in [1]. On the other hand $\int \mathbb{Z} = \mathbb{Z}$, since all functions $\mathbb{I} \to \mathbb{Z}$ are constant by Lemma 5.4.5. Crucially, by Prop 6.5.6 in [1], we have that

$$\int \mathbb{S}^1 = S^1$$

whose proof we elaborate below, a fact that will be essential in our proof of the 2D Borsuk-Ulam theorem.

We note that we interpret the variant $\int_1 -$ as the 1-truncated homotopy type (under the naive view of $\int -$ as yielding the homotopy type), but it has not been shown to be equivalent to $\| - \|_1 \circ \int$.

6.3. Shape and the Reals

As an application of the shape modality, we give a surprising new characterization of $\mathbb{R}$.

Definition 6.13: $\mathbb{R}$ as Covering Space

Consider the following pullback square arising from the modal unit $\eta : A \to \int_1 A$ for $\int_1$ applied to $\mathbb{S}^1$:

$$\begin{array}{ccc} \mathrm{fib}_\eta(\star) & \longrightarrow & \mathbb{1} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \star} \\ \mathbb{S}^1 & \xrightarrow{\ \eta\ } & \int_1 \mathbb{S}^1 = S^1 \end{array}$$

We describe the top-left type, which is judgmentally $\sum_{x:\mathbb{S}^1} \eta(x) = \star$, as $\mathbb{R}_{\mathcal{U}}$, or informally as "covering space" $\mathbb{R}$.

Morally, covering space $\mathbb{R}$ pairs an integer, hidden as an element of the identity type $\eta(x) \underset{S^1}{=} \star$, and $x : \mathbb{S}^1$ modeling the non-integral part of a real number. However, we demonstrate the equivalence in a considerably more abstract way. To do so, we briefly recall the theory of torsors for a group G, and how torsors are related to the *delooping* of types, a notion we define shortly. As it turns out, we will be able to show that $\mathbb{R}_{\mathcal{U}} \simeq \mathbb{R}$ by examining the delooping of $\mathbb{Z}$.

Definition 6.14: G-Torsor

Let G be a group so, in particular, a set. We consider G to be pointed at its unit. The type of *G-torsors* is given by the type of merely inhabited types which at each point are pointedly isomorphic to G, that is

$$\mathrm{T}G := \left(X : \mathcal{U}, p : \|X\|, \prod_{x:X} (X, x) \underset{\mathrm{pt}}{\simeq} G \right)$$

In particular this implies that X has a group structure with x as the unit for all $x : X$.

Example 6.14.1: The Trivial Torsor

Every group G is a G-torsor, with the family of equivalences given either by left of right multiplication which is always a bijection in a group. This is often referred to as the *trivial torsor* over G.

In this case we have an equivalence $\mathrm{T}G \simeq (\mathrm{T}G \simeq G)$ where every element $x : \mathrm{T}G$ corresponds to the (pointed) equivalence $\mathrm{T}G \simeq G$ given by sending x to the unit in G.

Torsors have a much more general definition for arbitrary types, in which case a type may not be a torsor over itself (those that are are called *homogeneous*), but we do not need this level of generality. In the context of groups a torsor T acquires a very concrete interpretation as being the group G that has "forgotten the unit", but for which any element $t : T$ might be chosen as the unit, giving the pointed equivalence $T, t \underset{\mathrm{pt}}{\simeq} G$ which is part of the data of a torsor. For groups (and in fact more generally) torsors are intimately related to their deloopings, that is, the types whose loop spaces are equivalent to the group G.

Definition 6.15: Delooping

Given a type A, the *delooping* of A, if it exists, is a pointed connected type $\mathrm{B}A$ whose loop space at the point is equivalent to the type. Explicitly

$$\mathrm{pt} \underset{\mathrm{B}A}{=} \mathrm{pt} \underset{\mathrm{pt}}{\simeq} A$$

where the equivalence is pointed at $\mathrm{refl}_{\mathrm{pt}}$. In the specific context of group theory, the delooping is sometimes referred to as the *classifying type* of a group.

The classic example of a delooping is given by the homotopical circle.

Example 6.15.1: The Circle Deloops $\mathbb{Z}$

For the homotopical circle S^1, since $\mathrm{base} = \mathrm{base} \simeq \mathbb{Z}$, it follows that the pointed type (S^1, base) is a delooping of $\mathbb{Z}$.

The loop space $\mathrm{base} = \mathrm{base}$ is in fact a $\mathbb{Z}$-torsor, with the pointed equivalences given by composing the standard equivalence $\mathrm{base} = \mathrm{base} \simeq \mathbb{Z}$ with addition by k on $\mathbb{Z}$ as necessary.

The general theory of torsors and delooping is developed in the context of HoTT in [22] where the following theorem is shown in, in fact, much greater generality.

Theorem 6.16: G-Torsors Deloop G

If the type of pointed G-torsors is contractible, then it is the unique delooping of G up to equivalence.

Proof. Theorem 2 in [22].

Wärn in fact shows this for a much more general notion of torsor over any type. The type of pointed G-torsors *is* in fact contractible, and as we know inhabited by the trivial torsor G. These facts are used implicitly throughout the proof of the below theorem.

Theorem 6.17: Covering Space ℝ Is ℝ

$\mathbb{R}_{\mathcal{U}} \simeq \mathbb{R}$.

Proof. Throughout the proof wherever we use $\mathbb{R}$ we mean Cauchy reals. We begin by elaborating the proof of Prop 6.5.6 in [1]. Consider the following commutative square

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{1} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{Z}} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow[\text{fib}_\pi]{} & B\mathbb{Z} \end{array}$$

Recall that $B\mathbb{Z} \simeq S^1$ is the type of $\mathbb{Z}$-torsors; the bottom map is well-defined, recalling that $\text{fib}_\pi([r])$ is a $\mathbb{Z}$-torsor under the action $r \mapsto k + r$ for any integer k, since $[r] \underset{\mathbb{R}/\mathbb{Z}}{=} [k + r]$. The pullback type of this square is just

$$\sum_{[r]:\mathbb{R}/\mathbb{Z}} \text{fib}_\pi([r]) \underset{B\mathbb{Z}}{=} \mathbb{Z}$$

By the equivalence $B\mathbb{Z} \simeq (B\mathbb{Z} \simeq \mathbb{Z})$ and univalence the above type is equivalent to

$$\sum_{[r]:\mathbb{R}/\mathbb{Z}} \text{fib}_\pi([r])$$

and this is equivalent to $\mathbb{R}$, so the above square is indeed a pullback square.

Recalling the standard equivalences $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$ and $B\mathbb{Z} \simeq S^1$, if we can show that fib_π is the modal unit η for $\int_1$ then by the universal property of the pullback, it is immediate that $\mathbb{R} \simeq \mathbb{R}_{\mathcal{U}}$.

By Corollary 6.7.1 it suffices to check that the map is $\int_1$-connected. Since $B\mathbb{Z}$ is connected, we can just check the fiber at $\mathbb{Z}$, is contractible; but as just shown the fiber at $\mathbb{Z}$ is $\mathbb{R}$ and $\int_1 \mathbb{R} = \mathbb{1}$.

With the new characterization of $\mathbb{R}$ established, we can demonstrate that identity types in $\mathbb{R}^n$ are closed propositions.

Corollary 6.17.1:

For any $x, y : \mathbb{R}^n$, $x = y$ is a closed proposition.

Proof. Let $(x, p), (y, q) : \mathbb{R}_{\mathcal{U}}$. The identity type is thus

$$\sum_{\alpha : x \underset{\mathbb{S}^1}{=} y} \text{tr}(\alpha, p) \underset{\eta(y)=\star}{=} q$$

Since $\mathbb{R} = \mathbb{R}_{\mathcal{U}}$ we know that this type must be a proposition. Since $\mathbb{S}^1$: CHaus and $\eta(y) = \star$ is merely equivalent to $\mathbb{Z}$, this is the dependent product of a closed and a decidable proposition. Since closed propositions are Σ-stable, this is a closed proposition.

Since equality in $\mathbb{R}^n$ is characterizable as the n-fold product of the above, it is the n-fold product of closed propositions, so closed.

We will at times silently pass to covering space $\mathbb{R}$ below in view of the above theorem. At this point, we are ready to relate the topological versions of paths, loops and homotopy, to the homotopical versions builtin to HoTT.

Chapter 7
Topology and Homotopy

In this section, we use the tools of synthetic Stone duality, particularly the development of a topologically well-behaved interval, to develop the elementary notions of algebraic topology for topological spaces. We relate this to synthetic homotopy theory via the shape modality. This section extends and elaborates the methods of §5.5 of [1] in which Cherubini et al. use the shape modality to prove Brouwer's fixed-point theorem.

7.1. Topological Paths and Loops

With the interval in hand, we can make analytic some of the traditional notions from point-set topology, particularly the traditional point-set notion of homotopy *just enough* to prepare for the Borsuk-Ulam theorem. Throughout the following we implicitly assume that a type X is 0-truncated, i.e. a set.

Definition 7.1: Topological Paths and Loops

A *topological path* in a type X is a function $p : \mathbb{I} \to X$. When X has a basepoint x and $p(0) = p(1) = x$ we say that p is a topological *loop*, a subtype of paths:

$$\mathrm{Loop}(X, x) :\equiv \sum_{p:\mathbb{I}\to X} p(0) = x \times p(1) = x$$

If the context is clear that we do not mean a path in the sense of the identity type, we just say p is a path; similarly we will often write $\mathrm{Loop}(X)$ when the basepoint is clear from context.

Definition 7.2: Topological Homotopy

A *topological homotopy* between two functions $f, g : A \to B$ is a function

$$H : A \times \mathbb{I} \to B$$

further equipped with the data witnessing that

$$H(-, 0) = f$$
$$H(-, 1) = g$$

If such a homotopy exists for two functions f, g we write $H : f \overset{\sim}{\sim} g$ to differentiate from the ordinary HoTT notion of homotopy.

A topological homotopy as in Definition 7.2 can be thought of as a family of continuous functions $H_t : A \to B$ parameterized by t that vary continuously with t. Since paths and loops are just functions, a homotopy between paths or loops could be defined as just an ordinary topological homotopy. However, it is natural to impose the condition that each H_t in such a homotopy is not just a continuous function but still a path or loop.

Definition 7.3: Path Homotopy and Loop Homotopy

Given $p, q : \mathbb{I} \to X$ as paths in X with the same endpoints, i.e. such that

$$p(0) = q(0) \wedge p(1) = q(1)$$

a *path homotopy* is a continuous function $H : \mathbb{I} \times \mathbb{I} \to X$ so that $H(-, t)$ is a path from $p(0)$ to $p(1)$ for all t. When p, q are loops based at the same point we say H is a *loop homotopy*.

In this setting we can easily show that these homotopies also admit left and right whiskering.

Lemma 7.3.1:

Let $H : f \sim g$ where $f, g : A \to B$. If $h : B \to C$ then there is a homotopy $h \blacksquare H : hf \sim hg$ (colloquially "left-whiskering" by h) and similarly if $k : D \to A$ then there exists the "right-whisker" $H \blacksquare k : fk \sim gk$.

Proof. To define "left-whiskering" $h \blacksquare H$, just post-compose by h, so $h \blacksquare H := h \circ H$. To see this is indeed a homotopy between the two functions hf, hg observe that for any $x : A$ one has $h(H(x, 0)) = h(f(x))$ and similarly $h(H(x, 1)) = h(g(x))$ since $H : f \sim g$.

The definition of right-whiskering is similar, except we have to remember that the domain is a product of types, so given $k : D \to A$ we define $H \blacksquare k := H \circ (k \times \mathrm{id})$. To verify this induces a homotopy $fk \sim gk$, observe that $H((k \times \mathrm{id})(x, 0)) = H(k(x), 0) = f(k(x))$ and similarly at $t = 1$ we have $H((k \times \mathrm{id})(x, 1)) = g(k(x))$ because H is a homotopy.

Lemma 7.3.2:

Given two paths $p : \mathbb{I} \to X$ and $q : \mathbb{I} \to X$ so that $p(1) = q(0)$, there exists a concatenated path $p \star q : \mathbb{I} \to X$ from $p(0)$ to $q(1)$ so that

$$(p \star q)(x) = \begin{cases} p(2x) \text{ if } x \leq \frac{1}{2} \\ q(2x - 1) \text{ if } x \geq \frac{1}{2} \end{cases}$$

Proof. Let $r_0 : \left[0, \frac{1}{2}\right] \to \mathbb{I}$ and $r_2 : \left[\frac{1}{2}, 1\right] \to \mathbb{I}$ be the standard homeomorphisms. Apply function gluing cf. Lemma 5.4.6 to $p \circ r_0 : \left[0, \frac{1}{2}\right] \to X$ and $q \circ r_1 : \left[\frac{1}{2}, 1\right] \to X$.

From here we can define the fundamental group in the traditional topological sense, introducing some new notation to not confuse with HoTT's builtin synthetic notions. First we make a simple observation.

Lemma 7.3.3:

Where X is n-truncated, the type of path homotopies between $p, q : \mathbb{I} \to X$ is n-truncated.

Proof. Since truncation level is stable under dependent sum, it suffices to check each component is n-truncated. The type of functions $\mathbb{I} \times \mathbb{I} \to X$ is n-truncated, as does the data showing that $H(t,-)$ is a loop based at x for all t, as the product of two paths $H(t, 0) = x \times H(t, 1) = x$ which are $n - 1$ truncated and applying weak function extensionality.

This means that in order to define the topological fundamental group in the standard way, we need to propositionally truncate in order to have a well-defined equivalence relation.

Definition 7.4: Topological Fundamental Group

Given a set X with a specified base point x, the *topological fundamental group* of X, denoted as $\dot{\pi}_1(X)$ is given by homotopy classes of loops at x, with the group operation given by path concatenation. A *homotopy class* of loops is a subset of $\mathrm{Loop}(X)$ where for any loops p, q in the set, there merely exists a loop homotopy $p \overset{\sim}{\leadsto} q$.

In view of Lemma 7.3.3, which implies that there can be a set's worth of loop homotopies between loops $p, q : \mathbb{I} \to X$ into a set X, the truncation to *mere* existence is nontrivial. At this point we also redefine the notion of fundamental group for sets in terms of shape.

For any pointed type (X, x) in HoTT, one traditionally defines the fundamental group to be the set-truncated loop space

$$\pi_1(X, x) := \|x = x\|_0$$

but for sets X, this is always trivial, since $x = x$ is contractible to refl_x when the underlying type is a set.

In view of this, we use the shape modality and overload this definition, giving a notion of fundamental group for topological spaces that can actually be nontrivial.

Definition 7.5: Fundamental Group

The *fundamental group* of a pointed set (X, x) is given by

$$\pi_1(X) := \|\eta(x) =_{\int X} \eta(x)\|_0$$

where η is the unit for the shape modality.

We close this section with two applications of our machinery.

First, we show that topological loops can be reformulated as maps from $\mathbb{S}^1$.

Lemma 7.5.1:

Let $p : \mathbb{I} \to X$ be a loop based at $x : X$. Then the diagram below

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{p} & X \\ {\scriptstyle \pi}\big\downarrow & \exists! \nearrow & \\ \mathbb{S}^1 & & \end{array}$$

admits a unique filler, where π is the standard covering map

$$x \mapsto \cos(2\pi x), \sin(2\pi x)$$

Proof. We apply function gluing. Cover the circle by the upper and lower semicircle including the endpoints $(1, 0), (-1, 0)$ in both; this is a finite closed cover C_1, C_2, corresponding to $y \geq 0, y \leq 0$, which are closed propositions as the negations of the open propositions $y < 0, y > 0$ respectively.

Then $\pi^{-1}(C_1) = \left[0, \frac{1}{2}\right]$ and similarly $\pi^{-1}(C_2) = \left[\frac{1}{2}, 1\right]$ and the fibers of π are contractible restricted to C_1, C_2. Then $f_i((x, y)) := p(f_i^{-1}((x, y)))$ is well-defined.

f_1, f_2 agree on the intersection, particularly at $(1, 0)$, precisely because p is a loop. The uniqueness is forced by commutativity.

We sometimes call a map $\mathbb{S}^1 \to \mathbb{S}^1$ induced by a topological loop $\mathbb{I} \to \mathbb{S}^1$ an *induced loop.*

Finally, in preparation for the Borsuk-Ulam theorem, we show a standard result about the topological sphere $\mathbb{S}^2$.

Lemma 7.5.2:

The imbedding of $\iota : \mathbb{I} \to \mathbb{S}^1 \hookrightarrow \mathbb{S}^2$ on the equator, i.e. where $\mathbb{S}^2$ intersects the xy-plane in $\mathbb{R}^3$, is topologically homotopic to a constant loop.

Proof. Recall the standard loop

$$\begin{aligned} \pi : \mathbb{I} &\to \mathbb{S}^1 \\ t &\mapsto \cos 2\pi t, \sin 2\pi t \end{aligned}$$

and the embedding $p := (x, y) \mapsto (x, y, 0)$ from $\mathbb{S}^1 \hookrightarrow \mathbb{S}^2$. We define $\iota : \mathbb{I} \to \mathbb{S}^2$ as just $p \circ \pi$. Geometrically this is a loop on the sphere in $\mathbb{R}^3$ lying on the xy plane. We give a homotopy from ι to $\text{const}_{0,0,1}$ trigonometrically, “squeezing” the circle to a point.

We define a function $H : [0, 2\pi] \times \left[0, \frac{\pi}{2}\right] \to \mathbb{S}^2$ for geometric clarity; composing with the standard homeomorphisms $[0, 2\pi] \simeq \mathbb{I}$, $\left[0, \frac{\pi}{2}\right] \simeq \mathbb{I}$ in each component gives the topological homotopy. Consider the map

$$\theta, \rho \mapsto \cos\theta \cos\rho, \sin\theta \cos\rho, \sin\rho$$

Morally, θ represents the angle within $\mathbb{S}^1$ viewed “on the xy-plane” and so $\cos\theta, \sin\theta$ is just a point on $\mathbb{S}^1$. But we scale by $\cos\rho$ in each component where ρ is an angle from the z-axis to the xy plane (in any direction). Geometrically this “squeezes” the components as ρ goes from 0 to $\frac{\pi}{2}$, and $\sin\rho$ denotes the z-component of the squeezed circle on the surface of $\mathbb{S}^2$. This is in fact a map into $\mathbb{S}^2$ since

$$\begin{aligned}\cos^2\theta\cos^2\rho + \sin^2\theta\cos^2\rho + \sin^2\rho &= \cos^2\rho(\cos^2\theta + \sin^2\theta) + \sin^2\rho \\ &= \cos^2\rho + \sin^2\rho = 1\end{aligned}$$

This is also a loop in θ since $\cos, \sin$ are 2π-periodic, and the homotopy is in fact between the claimed functions: $H(\theta, 0) = (\cos\theta, \sin\theta, 0)$ and $H(\theta, \frac{\pi}{2}) = (0, 0, \cos\frac{\pi}{2}) = (0, 0, 1)$.

It is easy to extend the above homotopy to a loop homotopy based at $(1, 0, 0)$ by instead parameterizing by an angle ρ between the xy-plane and the vertical line through $(1, 0, 0)$, similarly squeezing the imbedded circle towards a point, while being significantly messier algebraically.

7.2. Synthetic Covering Theory

In this section we briefly develop a synthetic version, using the shape modality, of the classical theory of covering spaces for topological spaces. We recall that a covering space of a (path connected and locally path connected) space is a local homeomorphism $p : E \twoheadrightarrow B$ where B admits an open covering such that, for all U in the cover, $p^{-1}(U)$ decomposes into the coproduct of "sheets" homeomorphic to U. These maps are an archetypal example of fibrations, since they admit path-lifting: a topological path in B lifts uniquely, having chosen a "sheet" in the preimage, to a path in E. Covering space theory is the essential ingredient in many classical topological proofs, including the proof of the 2D Borsuk-Ulam theorem.

Having developed the basics of the interval and topological paths and loops, it is natural to ask if it is possible to redevelop this theory in synthetic Stone duality. We take a different approach, however, and use the shape modality to bypass the theory of covering maps completely, using the class of étale maps defined in §6.1. With surprisingly little effort, we are able to recover topological path lifting for the standard covering map $\mathbb{R} \twoheadrightarrow \mathbb{S}^1$. A minor variant of this section originally appears as Section 8 of [9], with a slightly different shape modality (defined as the nullification of $\mathbb{R}$) in the context of a different variation of HoTT, Shulman's real-cohesive type theory.

Recall the canonical map

$$\begin{aligned}\pi : \mathbb{R} &\to \mathbb{S}^1 \\ x &\mapsto \cos 2\pi x, \sin 2\pi x\end{aligned}$$

that "wraps" the real line around the topological circle in a helix. In ordinary point-set topology this would be shown to be a covering map, a special kind of local homeomorphism. Using modality, we can obtain the path-lifting directly without having to characterize explicitly the topological properties of the map. We first give a preliminary definition.

The synthetic definition of covering space utilizes the étale notion from Definition 6.9.

Definition 7.6: Covering Space

A map $f : A \to B$ is a *covering map* whenever the fibers are sets and f is $\int_1$ -étale. When f is a covering map we often refer to its domain A as a *covering space.*

The above definition has ostensibly nothing to do with covering spaces, but we partially justify the name by exhibiting the essential feature of a covering space: generalized path-lifting. More inherited properties are detailed in Section 8 of [9].

Theorem 7.7: Map Lifting for Covering Spaces

Suppose $f : Y \to X$ is a covering map and $g : Z \to X$ is any map. Then if a filler for the below diagram

$$\begin{array}{ccc} & & \int_1 Y \\ & \overset{\exists}{\nearrow} & \downarrow \\ \int_1 Z & \longrightarrow & \int_1 X \end{array}$$

exists, there is a unique corresponding lift

$$\begin{array}{ccc} & & Y \\ & \overset{\exists!}{\nearrow} & \downarrow f \\ Z & \xrightarrow{g} & X \end{array}$$

for the underlying types.

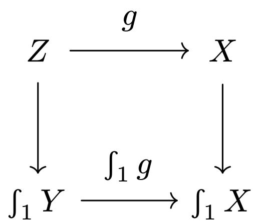
Proof. This falls out immediately from the definition of covering space; namely that f is $\int_1$ -etale, so we know the square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \int_1 Y & \xrightarrow{\int_1 f} & \int_1 X \end{array}$$

is a pullback square. There is also a commuting square induced by g

$$\begin{array}{ccc} Z & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ \int_1 Z & \xrightarrow{\int_1 g} & \int_1 X \end{array}$$

Now assuming we have a triangle lift $h : \int_1 Z \to \int_1 Y$ we obtain a map $h \circ \eta : Z \to \int_1 Y$ that, inheriting commutativity from both of the above squares, yields a commuting square

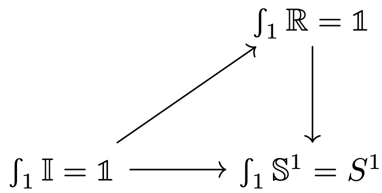


Then by the universal property of the pullback we obtain a unique map $k : Z \to Y$ so that, in particular, $g = f \circ k$.

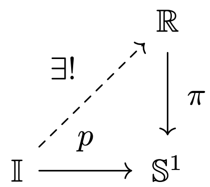
As an immediate application, we can show that paths lift for the standard covering map $\mathbb{R} \to \mathbb{S}^1$.

Corollary 7.7.1:

Given a topological path $p : \mathbb{I} \to \mathbb{S}^1$ and a choice of commutation for the diagram

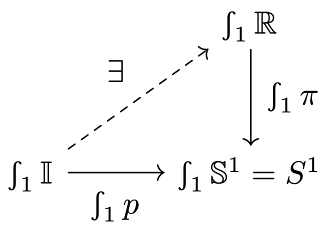


there is a unique corresponding lift



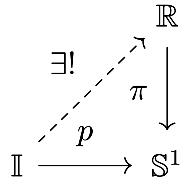
where π is the standard covering map.

Proof. In order to apply the previous theorem, we need to show that π is a covering map, which is immediate since we have *defined* covering space $\mathbb{R}$ as the appropriate pullback and shown its equivalence to $\mathbb{R}$ in Theorem 6.17. Furthermore we need that the triangle



has a lift. But $\int_1 \mathbb{R} = \int_1 \mathbb{I} = \mathbb{1}$ so the bottom and right maps are constant maps, and the lift is given by the constant map. Commutation is exhibited by any $p : \text{base} = \text{base}$.

There are thus precisely $\mathbb{Z}$ ways for this diagram to lift, inducing the corresponding unique lift



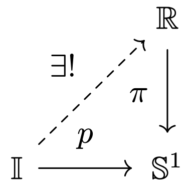
as desired.

Observe that in the above proof, a "choice of commutation" for the $\int_1$ triangle corresponds precisely to choosing a point in the fiber of a conventionally-defined covering map in set-theoretic topology.

With path-lifting in hand, we can define the winding number of a circle in a standard way.

Definition 7.8: Winding Number

Let $h : \mathbb{I} \to \mathbb{S}^1$ be a loop based at $\star$ and $\tilde{h} : \mathbb{I} \to \mathbb{R}$ a lift to $\mathbb{R}$ commuting with the ordinary covering map $\pi : \mathbb{R} \to \mathbb{S}^1$ as in



Since h is a loop based at $\star$, $h(0) = h(1)$ so $\tilde{h}(0), \tilde{h}(1) : \mathrm{fib}_\pi(\star)$ by commutation. (We abuse notation and describe elements of $\mathrm{fib}_\pi(\star) \doteq \sum_{x:\mathbb{R}} \pi(x) = \star$ just in terms of the first coordinate.) Since $\pi : \mathbb{R} \twoheadrightarrow \mathbb{S}^1$ is just the quotient map $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ up to isomorphism, we have

$$\alpha : \mathrm{fib}_\pi(\star) \simeq \mathbb{Z}$$

In this context we can define a map

$$\begin{aligned} w : \mathrm{Loop}(\mathbb{S}^1) &\to \mathbb{Z} \\ h &\mapsto \alpha\big(\tilde{h}(1)\big) - \alpha\big(\tilde{h}(0)\big) \end{aligned}$$

and refer to $w(h)$ as the *winding number* of the loop h.

It is immediate by definition that $w(\mathrm{const}_\star) = 0$, where $\star : \mathbb{S}^1$ is any point.

In the rest of this subsection, we want to show that the winding number is in fact a well-defined homotopy invariant. First we establish some basic notions.

In view of Lemma 7.5.1, a loop homotopy can be considered as a map $H : \mathbb{I} \times \mathbb{S}^1 \to \mathbb{S}^1$ after inverting the first two arguments, so that $t : \mathbb{I}$ is the first argument. We observe the following relationship between induced loops $\mathbb{S}^1 \to \mathbb{S}^1$ and basepoint preserving maps in the circle.

Lemma 7.8.1:

Suppose $p : \mathbb{S}^1 \to \mathbb{S}^1$ is an induced loop. Then the bottom map in the square below arising from applying the 1-shape

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{p} & \mathbb{S}^1 \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\int_1 p} & S^1 \end{array}$$

is a basepoint preserving map.

Proof. Let $b := (0, 1) : \mathbb{S}^1$. Assume without loss of generality that $\eta(b) = \text{base}$, where η is the modal unit, and p is induced from a loop based at b. Then $p(b) = b$. By commutation of the diagram

$$\eta(p(b)) = \textstyle\int_1 p(\eta(b))$$

so $\eta(b) = \int_1 p(\text{base})$, and $\eta(b) = \text{base}$ so $\text{base} = \int_1 p(\text{base})$.

Recalling the standard equivalence

$$\left(\sum_{f : S^1 \to S^1} f(\text{base}) = \text{base} \right) \simeq \mathbb{Z}$$

we can now define the degree of a loop. We write $S^1 \to_{\cdot} S^1$ to denote basepoint preserving maps.

Definition 7.9: Degree

Let $p : \mathbb{S}^1 \to \mathbb{S}^1$ be an induced loop and $p^\dagger : S^1 \to_{\cdot} S^1$ the unique induced basepoint-preserving map. The unique integer corresponding to $p^\dagger$ is the *degree* of p.

Now, using the same technique, we can show that degree and winding number are homotopy invariant.

Lemma 7.9.1:

Let $H : \mathbb{I} \times \mathbb{S}^1 \to \mathbb{S}^1$ be a loop homotopy between two topological loops $p, q : \text{Loop}(\mathbb{S}^1)$. Then p, q have the same degree.

Proof. We simply factor the map as in the following diagram

$$\mathbb{I} \xrightarrow{H} (\mathbb{S}^1 \to \mathbb{S}^1) \xrightarrow{\int_1} (S^1 \to_{\cdot} S^1) = \mathbb{Z}$$

implicitly currying H, where the last map is an equivalence and the arrow $\to_\cdot$ indicates a basepoint-preserving map. Then $\textstyle\int_1 \circ H$ must be constant.

Lemma 7.9.2:

Let $H : \mathbb{I} \times \mathbb{S}^1 \to \mathbb{S}^1$ be a loop homotopy between two topological loops $p, q :$ $\mathrm{Loop}(\mathbb{S}^1)$. Then p, q have the same winding number.

Proof. Recalling the map $w : \mathrm{Loop}(\mathbb{S}^1) \to \mathbb{Z}$ we factor over a loop homotopy

$$\mathbb{I} \xrightarrow{H} (\mathbb{S}^1 \to \mathbb{S}^1) \xrightarrow{w} \mathbb{Z}$$

currying H again. Constancy of $\mathbb{I} \to \mathbb{Z}$ functions implies that $w(p) = w(H(0)) =$ $w(H(1)) = w(q)$.

We phrase this in the terms needed for the proof of the Borsuk-Ulam theorem.

Corollary 7.9.3:

Two loops p, q that are homotopic share degree and winding number. In particular, two loops with differing degree, winding number cannot be homotopic.

Proof. Immediate by the prior two lemmas.

Chapter 8
The Borsuk-Ulam Theorem

As an application of the machinery developed in the previous chapters, we prove the two-dimensional Borsuk-Ulam theorem in the setting of synthetic Stone duality. The Borsuk-Ulam theorem is a statement about continuous maps from the topological n-sphere $\mathbb{S}^n$ into $\mathbb{R}^n$. The theorem states that all such maps are not injective: in particular, $f : \mathbb{S}^n \to \mathbb{R}^n$ fails to be injective on antipodes, so there merely exists some $x : \mathbb{S}^n$ so that $f(x) = f(-x)$. The rest of this chapter is devoted to demonstrating the theorem in dimension two, then in general based on a yet-to-be-completed cohomology calculation.

8.1. The 2D Borsuk-Ulam Theorem

The proof strategy for the two-dimensional Borsuk-Ulam theorem, taken from [8], is proof by contradiction. We want to assume that our given function $f : \mathbb{S}^2 \to \mathbb{R}^2$ takes distinct values on every antipodal pair and then derive a contradiction, i.e. we want to show

$$\neg \prod_{x:\mathbb{S}^2} f(x) \neq f(-x)$$

In order to show this is a valid strategy we need of course show that the proposition is $\neg\neg$-stable. We show it is closed.

Lemma 8.0.1:

The Borsuk-Ulam theorem in n dimensions, i.e.

$$\prod_{f:\mathbb{S}^n \to \mathbb{R}^n} \exists_{x:\mathbb{S}^n} f(x) = f(-x)$$

is a product of closed propositions.

Proof. Recall that $\mathbb{R}^n$ equality is closed by Corollary 6.17.1. For propositions, we have an embedding Closed $\to$ Stone $\hookrightarrow$ CHaus. Then

$$\sum_{x:\mathbb{S}^n} f(x) = f(-x) : \text{CHaus}$$

and admits a surjection $S \twoheadrightarrow \sum_{x:\mathbb{S}^n} f(x) = f(-x)$. But any surjective map induces an equivalence on the propositional truncations so we have $\|S\| \leftrightarrow \| \sum_{x:\mathbb{S}^n} f(x) = f(-x)\|$ and $\|S\|$ is closed.

With the above in hand we can show that the Hatcher proof strategy works in the synthetic Stone duality setting.

Lemma 8.0.2:

Showing that an "antipodally injective" function $f : \mathbb{S}^2 \to \mathbb{R}^2$ cannot exist is enough to show the Borsuk-Ulam theorem, or formally,

$$\neg\forall_{x:\mathbb{S}^2} f(x) \neq f(-x) \to \exists_{x:\mathbb{S}^2} f(x) = f(-x)$$

Proof. We first show

$$\forall_{x:\mathbb{S}^2} f(x) \neq f(-x) \leftrightarrow \neg\exists_{x:\mathbb{S}^2} f(x) = f(-x)$$

From left to right, since we want to show a proposition we can assume we have x so that $f(x) = f(-x)$ and give this to $g : \forall_{x:\mathbb{S}^2} f(x) \neq f(-x)$. From right to left, given arbitrary $x : \mathbb{S}^2$ and $p : f(x) = f(-x)$ we can call $h : \neg\exists_{x:\mathbb{S}^2} f(x) = f(-x)$ on $\eta(x, p)$. Given the equivalence of propositions, we further obtain

$$\neg\forall_{x:\mathbb{S}^2} f(x) \neq f(-x) \leftrightarrow \neg\neg\exists_{x:\mathbb{S}^2} f(x) = f(-x)$$

from left to right in particular, where our proof directly demonstrates the left hand side. Finally by the previous lemma the right hand side (without negations) is closed and therefore $\neg\neg$-stable, so we have $\exists_{x:\mathbb{S}^2} f(x) = f(-x)$.

Now we proceed to show that a given continuous function $f : S^2 \to \mathbb{R}^2$ cannot take different values on different antipodes.

Theorem 8.1: Borsuk-Ulam for $\mathbb{S}^2$

There is no $f : \mathbb{S}^2 \to \mathbb{R}^2$ so that $f(x) \neq f(-x)$ for all $x : \mathbb{S}^2$.

Proof. Suppose there is such an f. Then we can define

$$g(x) := \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

where $\| - \|$ is the Euclidean norm. By definition we observe that

$$g(x) = -g(-x)$$

This is well-defined by the assumption that $f(x) - f(-x) \neq 0$ as a map from $\mathbb{S}^2 \to \mathbb{S}^1$. We precompose g with the canonical embedding $\iota : \mathbb{S}^1 \hookrightarrow \mathbb{S}^2$ of $\mathbb{S}^1$ into the intersection of $\mathbb{S}^2$ with the xy-plane, and the canonical loop $\pi : \mathbb{I} \to \mathbb{S}^1$ given by $x \mapsto \cos 2\pi x, \sin 2\pi x$. Thus we obtain a topological loop $h : \mathbb{I} \to \mathbb{S}^1$

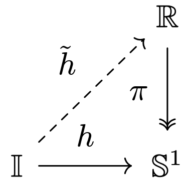
$$h := \mathbb{I} \xrightarrow{\pi} \mathbb{S}^1 \xhookrightarrow{\iota} \mathbb{S}^2 \xrightarrow{g} \mathbb{S}^1$$

By whiskering the topological homotopy $H : \iota \circ \pi \sim \mathrm{const}_{0,0,1}$ defined in Lemma 7.5.2 with g we obtain a homotopy $h \sim \mathrm{const}_{g(0,0,1)}$, so h is nullhomotopic. Now we show that h *cannot* be nullhomotopic.

Observe that ι has a symmetry like g by definition: $\iota(x) = -\iota(-x)$. Therefore for $s \in \left[0, \frac{1}{2}\right]$, since $\pi\left(s + \frac{1}{2}\right) = -\pi(s)$ (geometrically corresponding to going "halfway around" the circle), it follows that

$$\begin{aligned} h\left(s + \frac{1}{2}\right) &= g\left(\iota\left(\pi\left(s + \frac{1}{2}\right)\right)\right) \\ &= -g(\iota(\pi(s))) \\ &= -h(s) \end{aligned}$$

Now by using the path-lifting available for $\mathbb{S}^1$ by Corollary 7.7.1, we have a commuting triangle



where π is the canonical covering map. Applying the symmetry for h discussed above through the commutation, we obtain

$$\pi\left(\tilde{h}\left(s + \frac{1}{2}\right)\right) = h\left(s + \frac{1}{2}\right) = -h(s) = -\pi\big(\tilde{h}(s)\big)$$

Since for any $r \in \mathbb{R}$ we have $\pi(r) = -\pi(s)$ if and only if $s = r + k + \frac{1}{2}$ for an integer k, it follows that $\tilde{h}\left(s + \frac{1}{2}\right) = \tilde{h}(s) + k + \frac{1}{2}$ for some k that might depend continuously on s. However by Lemma Lemma 5.4.5 we know that all $\mathbb{I} \to \mathbb{Z}$ functions must be constant, so k is constant for all $s \in \left[0, \frac{1}{2}\right]$. Therefore setting $q := 2k + 1$ we have

$$\tilde{h}\left(s + \frac{1}{2}\right) = \tilde{h}(s) + \frac{q}{2}$$

where q is an odd integer. We can apply this identity twice to see that

$$\tilde{h}(1) = \tilde{h}\left(\frac{1}{2}\right) + \frac{q}{2} = \tilde{h}(0) + q$$

Finally we observe that $\tilde{h}(1) - \tilde{h}(0) = q$ and since q is odd $\tilde{h}(1) \neq \tilde{h}(0)$. Observe then that the winding number of h, $\alpha\big(\tilde{h}(1)\big) - \alpha\big(\tilde{h}(0)\big)$, where $\alpha : \mathrm{fib}_\pi(\star) \simeq \mathbb{Z}$, is nonzero. But since $h \sim \mathrm{const}_{g(0,0,1)}$ then $w(h) = 0$, a contradiction.

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