Yield Curves in the Post-LIBOR World

On the Yield to Maturity Curve Implied by the Heath-Jarrow-Morton-fitted Forward Market Model

Master's thesis in mathematics

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DEPARTMENT OF MATHEMATICAL SCIENCES

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Abstract

The transition from the forward-looking LIBOR to backward-looking rates such as SOFR presents a modelling challenge, as the widely used LIBOR Market Model (LMM) is designed for forward-looking rates. To address this, the generalised Forward Market Model (FMM) has been proposed, capable of describing both forward- and backward-looking rates.

In this thesis, we study the properties of the FMM, focusing on the differentiability of the yield to maturity curve implied by the Heath-Jarrow-Morton-fitted FMM. We investigate the non-differentiability of the yield to maturity curve, and show that it is heavily influenced by the choice of zero-coupon bond interpolation method.

Keywords: Interest rates, backward-looking rates, LIBOR, generalised forward market model, yield curve.

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Victor Brun, Amsterdam, September 2023

List of Acronyms

Below is the list of acronyms that have been used throughout this thesis listed in alphabetical order.

CDOR	Canadian dollar offered rate
EURIBOR	Euro interbank offered rate
FMM	Forward market model
HJM	Heath-Jarrow-Morton
IBOR	Interbank offered rate
IRS	Interest rate swap
LIBOR	London interbank offered rate
LMM	LIBOR market model
ODE	Ordinary differential equation
PFS	Payer interest rate swap
RFS	Receiver interest rate swap
RNM	Risk-neutral measure
SDE	Stochastic differential equation
SOFR	Secured overnight financing rate
SONIA	Sterling overnight index average
TIBOR	Tokyo interbank offered rate
ZCB	Zero-coupon bond

Nomenclature

Below is the nomenclature of indices, sets, parameters, and variables that have been used throughout this thesis.

Measures and expectation

P	Real world measure.
Q	Risk-neutral measure.
Q^T	T-forward measure.
$\mathbb{E}^X$	Expectation in the measure X.

Interest rates

$r(t)$	Instantaneous risk-free interest rate.
$f(t,T)$	Continuously compounding forward rate.
$F_k(t)$	Forward-looking forward rate.
$R_k(t)$	Backward-looking forward rate.

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Introduction

Interest rates are central to the financial landscape, impacting everyone from families fixing their mortgage rates to Wall Street traders analysing price movements in sophisticated interest rate derivatives. Despite their apparent differences, these contracts share a commonality—the type of interest rate used.

The most frequently employed type of interest rates are Interbank Offered Rates (IBORs), of which the most well-known is the London Interbank Offered Rate (LIBOR), but there are various regional counterparts like EURIBOR, TIBOR, and CDOR. For simplicity, we will collectively refer to them as LIBOR or LIBOR rates, given their similar characteristics and functions.

LIBOR rates are simply compounding forward-looking rates specifying interest rates between banks and are utilised across a wide range of financial products, from everyday home mortgages to intricate interest rate derivatives. Due to their widespread usage, great efforts have been made to model them, with the most well-known model called the LIBOR Market Model (LMM). However, historically, these rates were determined by a panel of banks providing suggested rates. This resulted in significant concerns being raised regarding the potential for rate manipulations after the 2008 financial crisis, prompting the need for a more robust benchmark rate.[1]

In response to the vulnerability of LIBOR rates, financial authorities and market participants have acknowledged the necessity for a comprehensive solution. Consequently, LIBOR is being phased out, with the last U.S. dollar LIBOR published on the 30th of June.[2] In its place, various overnight rates have emerged. These rates are based on actual transactions and therefore resolve the issue that motivated them.

However, the problem with overnight rates is that they, as the name suggests, have an accrual period of only one night. To turn these rates into term rates like LIBOR (e.g., rates with different length tenors such as 1 month, 3 months, 6 months, etc.), a compounded setting-in-arrears rate is used. This rate is constructed from the overnight rates and is backward-looking in nature, meaning that it continues to evolve until maturity, i.e., it both settles and pays at maturity. Just as for the IBORs, these setting-in-arrears rates have regional variants; SOFR has been selected to replace the LIBOR in the US, while the UK has opted for SONIA as the replacement rate.

[1] https://archive.nytimes.com/www.nytimes.com/interactive/2012/07/10/business/dealbook/behind-the-libor-scandal.html?_r=0

[2] https://www.sec.gov/oiea/investor-alerts-and-bulletins/what-you-need-know-about-end-libor-investor-bulletin

From a modelling perspective, the transition from LIBOR to alternative rates like SOFR is, therefore, a shift from forward-looking to backward-looking rates. Because the LMM is only capable of modelling forward-looking rates, this transition has significant implications for financial institutions, as many of them rely on the LMM for pricing and hedging products based on interest rates. Therefore, a new model, capable of describing both forward- and backward-looking rates, has been proposed—the generalised Forward Market Model (FMM).

The generalised Forward Market Model, proposed in [9], enjoys many of the convenient properties of the LMM, such as martingale dynamics under the appropriate measure and the possibility for log-normal dynamics. But it also provides some additional features, such as having convenient dynamics under the risk-neutral measure. Moreover, because of its ability to model rates that continue to evolve until maturity, it is possible to fit a Heath-Jarrow-Morton model to each accrual period, which is done in [10].

In this thesis, we build upon the work done in [10] by studying the properties of the FMM. Specifically, we study the differentiability of the time to maturity curve implied by the Heath-Jarrow-Morton-fitted FMM.

2

Preliminaries

2.1 Stochastic calculus and related topics

In this section, we establish the mathematical foundation that will be utilised throughout this thesis. It is important to note that this section does not serve as an introductory explanation of the concepts involved. Instead, it can be utilised as a refresher and a reference for the definitions and theorems that will be employed later on.

For a gentle introduction to the concepts covered in this chapter within the context of financial mathematics, we recommend [12]. For a more technical and mathematically rigorous discussion, we highly recommend [8] and [7].

2.1.1 Functional analysis

Definition 2.1.1 (Variation)**.** Let $g : \mathbb{R} \to \mathbb{R}$. We define the *variation* of g on the interval $[a, b]$ as

$$V_g([a,b]) = \sup \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|,$$

where the supremum is taken over the set of all partitions such that

$$a = t_0^n < t_1^n < \cdots < t_n^n = b.$$

It is often convenient to consider the variation over an interval starting at 0, e.g. $[0, t]$. To simplify notation, we use the shorthand notation $V_g([0,t]) = V_g(t)$.

Definition 2.1.2 (Finite and bounded variation)**.** A function g is said to be of *finite variation* if $V_g(t) < \infty$ for all t. The function is moreover said to be of *bounded variation* if $\sup_t V_g(t) < \infty$, i.e., $V_g(t) < C$ where C is a t-independent constant.

Definition 2.1.3 (Quadratic variation)**.** Let $g : \mathbb{R} \to \mathbb{R}$. We define the *quadratic variation* of g on the interval $[0, t]$ as the limit (when it exists)

$$[g](t) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} (g(t_i^n) - g(t_{i-1}^n))^2,$$

where $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ and $\delta_n = \max_{1 \le i \le n}(t_i^n - t_{i-1}^n)$.

Theorem 2.1.1. *If $g : \mathbb{R} \to \mathbb{R}$ is continuous and of finite variation, it has zero quadratic variation.*

Proof. We will bound $[g](t)$ from above and then show the bound to be zero as a result of the properties imposed on g. By definition of quadratic variation it follows that

$$\begin{aligned}[g](t) &= \lim_{\delta_n \to 0} \sum_{i=i}^{n} \Big(g(t_i^n) - g(t_{i-1}^n) \Big)^2 \\ &\leq \lim_{\delta_n \to 0} \max_i |g(t_i^n) - g(t_{i-1}^n)| \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)| \\ &\leq \lim_{\delta_n \to 0} \max_i |g(t_i^n) - g(t_{i-1}^n)| V_g(t) = 0,\end{aligned}$$

where the last equality follows from $V_g(t) < \infty$ and continuity of g on $[0, t]$, as the latter implies that $\lim_{\delta_n \to 0} |g(t_i^n) - g(t_{i+1}^n)| = 0$. □

Definition 2.1.4 (Covariation)**.** Let $g, h : \mathbb{R} \to \mathbb{R}$. We define the *covariation* of g and h on the interval $[0, t]$ as the limit (when it exists)

$$[g, h](t) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} \Big(g(t_i^n) - g(t_{i-1}^n) \Big) \Big(h(t_i^n) - h(t_{i-1}^n) \Big),$$

where $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ and $\delta_n = \max_{1 \leq i \leq n} (t_i^n - t_{i-1}^n)$.

Theorem 2.1.2. *If g is continuous and h is of finite variation on $[0, t]$, then their covariation is zero, i.e., $[g, h](t) = 0$.*

The proof of theorem 2.1.2 is almost identical to that of theorem 2.1.1, with the only addition being that the function to move outside the sum must be the one which is continuous.

2.1.2 Measure theoretic probability theory

When dealing with uncountably large sample spaces, as we will throughout this thesis, it is not possible to assign a probability to each element while ensuring that the probability of the entire sample space is one. To address this technical issue, we rely on σ-fields.

Definition 2.1.5 (σ-field)**.** Let $\mathcal{F}$ be a field consisting of subsets of the sample space Ω. $\mathcal{F}$ is then called a *σ-field* if it satisfies:

1. $\varnothing, \Omega \in \mathcal{F}$,
2. $A \in \mathcal{F} \implies A^C \in \mathcal{F}$,
3. $A_1, A_2, \ldots, A_n, \dots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Definition 2.1.6 (Filtration)**.** A *filtration* $\{\mathcal{F}_t\}_{t \geq 0}$ is an increasing family of sub-σ-fields of the σ-field $\mathcal{F}$, i.e.

$$\{\varnothing\} \subseteq \mathcal{F}_{t_0} \subseteq \mathcal{F}_{t_1} \subseteq \cdots \mathcal{F}_{t_n} \subseteq \cdots \subseteq \mathcal{F}, \quad 0 \leq t_{n-1} \leq t_n.$$

The concept of a filtration will play a crucial role later on in this thesis by representing information. For example, we will often let all information available in some market up until, and including, time t be represented by the filtration $\mathcal{F}_t$.

Definition 2.1.7 (Measurable)**.** Any subset of Ω that belongs to the σ-field $\mathcal{F}$ is called *$\mathcal{F}$-measurable*.

Definition 2.1.8 (Probability measure)**.** A probability measure P on the measurable space $(\Omega, \mathcal{F})$ is a non-negative set function on the σ-field $\mathcal{F}$ such that

1. $P(\Omega) = 1$,
2. if $A \in \mathcal{F}$, then $P(A^C) = 1 - P(A)$,
3. and if $A_1, A_2, \dots, A_n, \dots \in \mathcal{F}$ are mutually exclusive, then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Definition 2.1.9 (Equivalent probability measures)**.** Let P and Q be two probability measures on a measurable space $(\Omega, \mathcal{F})$. We say that P and Q are *equivalent*, denoted $P \sim Q$, if they are zero on the same sets, i.e.,

$$P(A) = 0 \Longleftrightarrow Q(A) = 0 \quad \forall A \in \mathcal{F}.$$

Definition 2.1.10 (Absolutely continuous probability measures)**.** Let P and Q be two probability measures on a measurable space $(\Omega, \mathcal{F})$. We say that Q is *absolutely continuous* with respect to P, denoted $Q \ll P$, if

$$P(A) = 0 \Longrightarrow Q(A) = 0 \quad \forall A \in \mathcal{F}.$$

Note that if $Q \ll P$ and $P \ll Q$ the measures are equivalent.

The following is a central measure theoretic result and its proof relies on several concepts that are beyond scope of this thesis. Therefore, we will state the theorem in a probability theoretic formulation and leave it without proof.

Theorem 2.1.3 (Radon-Nikodym theorem)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space with two probability measures P and Q such that $Q \ll P$. Then there exists a P-almost surely unique random variable Λ such that $\Lambda \geq 0$, $\mathbb{E}^P \Lambda = 1$, and*

$$Q(A) = \mathbb{E}^P[\Lambda 1_A] = \int_A \Lambda dP, \quad \forall A \in \mathcal{F}, \tag{2.1}$$

where 1_A denotes the indicator function on A. Conversely, if there exists a random variable Λ with the above-stated properties, and Q is defined in accordance with (2.1)*, then it is a probability measure and $Q \ll P$.*

Definition 2.1.11 (Radon-Nikodym derivative)**.** We define the *Radon-Nikodym derivative* as the random variable Λ from the Radon-Nikodym theorem. Moreover, it is often denoted as the ratio of the differentials of the measures,

$$\Lambda = \frac{dQ}{dP}.$$

Theorem 2.1.4 (Abstract Bayes' theorem)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space with two probability measures P and Q, and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. If $Q \ll P$ with $\Lambda = dQ/dP$ and $\mathbb{E}^Q|X| < \infty$, then $\mathbb{E}^P|\Lambda X| < \infty$ and*

$$\mathbb{E}^Q[X \mid \mathcal{G}] = \frac{\mathbb{E}^P[\Lambda X \mid \mathcal{G}]}{\mathbb{E}^P[\Lambda \mid \mathcal{G}]}, \quad Q\text{-}a.s. \tag{2.2}$$

Proof. That $\mathbb{E}^Q|X| < \infty$ implies $\mathbb{E}^P|\Lambda X| < \infty$ follows directly from the definition of expectation and the fact that Λ is a Radon-Nikodym derivative.

Now, by the definition of conditional expectation provided by [8, page 44] we have, for any bounded $\mathcal{G}$-measurable random variable ξ, that

$$\mathbb{E}[\xi X] = \mathbb{E}[\xi \mathbb{E}[X \mid \mathcal{G}]]. \tag{2.3}$$

Then, by multiplying (2.2) with ξ and taking the Q-expectation of the right-hand side we get

$$\begin{aligned}
\mathbb{E}^Q\left[\frac{\mathbb{E}^P[\Lambda X \mid \mathcal{G}]}{\mathbb{E}^P[\Lambda \mid \mathcal{G}]}\xi\right] &= \mathbb{E}^P\left[\Lambda \frac{\mathbb{E}^P[\Lambda X \mid \mathcal{G}]}{\mathbb{E}^P[\Lambda \mid \mathcal{G}]}\xi\right] && (\Lambda \text{ is RN. derivative}) \\
&= \mathbb{E}^P\left[\mathbb{E}^P\left[\Lambda \frac{\mathbb{E}^P[\Lambda X \mid \mathcal{G}]}{\mathbb{E}^P[\Lambda \mid \mathcal{G}]} \mid \mathcal{G}\right]\xi\right] && (\text{use (2.3)}) \\
&= \mathbb{E}^P\left[\mathbb{E}^P[\Lambda \mid \mathcal{G}]\frac{\mathbb{E}^P[\Lambda X \mid \mathcal{G}]}{\mathbb{E}^P[\Lambda \mid \mathcal{G}]}\xi\right] && (\text{fraction } \mathcal{G}\text{-measurable}) \\
&= \mathbb{E}^P[\mathbb{E}^P[\Lambda X \mid \mathcal{G}]\xi] \\
&= \mathbb{E}^Q[X\xi]. && ((2.3), \text{ RN. derivative}) \\
&= \mathbb{E}^Q\left[\mathbb{E}^Q[X \mid \mathcal{G}]\xi\right]. && (\text{tower rule})
\end{aligned}$$

To finalise the proof we notice that by letting $\xi = 1_A$ for any $A \in \mathcal{F}$ the equality in expectation becomes equality in probability, i.e.

$$\int_A \frac{\mathbb{E}^P[\Lambda X \mid \mathcal{G}]}{\mathbb{E}^P[\Lambda \mid \mathcal{G}]} dQ = \int_A \mathbb{E}^Q[X \mid \mathcal{G}] dQ, \quad \forall A \in \mathcal{F}.$$

This means that above integrands are equal on every set, A, with non-zero Q-measure, or in other word; they are Q-almost surely equal. □

2.1.3 Stochastic processes

A stochastic process is an ordered set of random variables $\{X_n\}_{n=1}^{\infty}$, which can be uncountably large. This notation will be used when it is essential to emphasise the ordering. However, as this thesis primarily focuses on continuous stochastic processes, the notation $\{X(t)\}_{t\geq 0}$ will be used more frequently. Finally, when it is clear from the context, we will omit the set notation and denote the process as $X(t)$.

Definition 2.1.12 (Adapted process)**.** A stochastic process $\{X(t)\}_{t\geq 0}$ is said to be *adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for $t \geq 0$.

Definition 2.1.13 (Predictable process)**.** A stochastic process $\{Y_n\}_{n=1}^{\infty}$ is said to be *predictable with respect to the filtration* $\{\mathcal{F}_n\}_{n=0}^{\infty}$ if Y_n is $\mathcal{F}_{n-1}$-measurable.

Definition 2.1.14. A stochastic process $\{X(t)\}_{t\in[0,T]}$, adapted to some filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is said to be in

- S_T if there exists a grid $0 = t_0 < \cdots < t_n = T$, and random variables $\xi_0, \xi_1, \ldots, \xi_{n-1}$, with ξ_i being $\mathcal{F}_{t_i}$-measurable, and $E[\xi_i^2] < \infty$ for $i = 0, 1, \ldots, n-1$, such that

$$X(t) = \xi_0 1_{\{0\}}(t) + \sum_{i=1}^{n} \xi_{i-1} 1_{(t_{i-1},t_i]}(t) \quad \text{for } t \in [0, T].$$

- E_T if

$$E\left[\int_0^T X(t)^2 dt\right] < \infty.$$

- P_T if

$$P\left(\int_0^\infty X(t)^2 dt < \infty\right) = 1.$$

Definition 2.1.15 (Markov property)**.** Let $\mathcal{F}_t$ denote the σ-field generated by the stochastic process $X(t)$ up until time t. We say that $X(t)$ possesses the *Markov property* if for any t and s, the conditional distribution of $X(t+s)$ given $\mathcal{F}_t$ is the same as the conditional distribution of $X(t+s)$ given $X(t)$, i.e.,

$$P(X(t+s) \in A \mid \mathcal{F}_t) = P\left(X(t+s) \in A \mid X(t)\right) \quad \text{a.s.},$$

where A is an $\mathcal{F}_{t+s}$-measurable set.

Definition 2.1.16 (Brownian motion)**.** We call the stochastic process $W(t)$ a *Brownian motion* if it, for any $0 \le s \le t$, satisfies the following properties:

1. (Independent increments) $W(t) - W(s)$ is independent of the past, i.e., it is independent of the σ-algebra $\mathcal{F}^W(s)$ generated by $W(s)$.

2. (Normal increments) $W(t) - W(s) \sim \mathcal{N}(0, t-s)$.

3. (Continuity of paths) $W(t)$ is a continuous function of t.

From this definition of Brownian motion, it follows easily that it is a Markov process.

Theorem 2.1.5. *Let $W(t)$ be a Brownian motion, then $[W](t) = t$.*

Proof. Define

$$[W]_n(t) = \sum_{i=1}^{n} \left(W(t_i^n) - W(t_{i-1}^n)\right)^2.$$

Then, our objective is to prove that $[W]_n(t) \to t$ almost surely. To do this we will first show L^2-convergence. Let

$$\theta_i = \left(W(t_i^n) - W(t_{i-1}^n)\right)^2 - \left(t_i^n - t_{i-1}^n\right).$$

From the fact that the increments of Brownian motions are independent, it follows that θ_i and θ_j are independent for $i \neq j$. Moreover, the time increment term in θ_i will cancel the mean of the square Brownian increments, making θ_i have zero mean.

Next we observe that

$$[W]_n(t) - t = \sum_{i=1}^{n} \theta_i. \tag{2.4}$$

By relying on the result that $\mathbb{E}Z^4 = 3\sigma^4$ for $Z \sim \mathcal{N}(0, \sigma^2)$ and the properties of θ_i derived above, it is possible to bound the second moment of (2.4) from above:

$$\begin{aligned}
\mathbb{E}\left[([W]_n(t) - t)^2\right] &= \sum_{i=1}^{n} \mathbb{E}\left[\theta_i^2\right] + 2\sum_{i\neq j} \mathbb{E}[\theta_i \theta_j] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[\left(\left(W(t_i^n) - W(t_{i-1}^n)\right)^2 - \left(t_i^n - t_{i-1}^n\right)\right)^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[\left(W(t_i^n) - W(t_{i-1}^n)\right)^4\right] - 2\sum_{i=1}^{n}(t_i^n - t_{i-1}^n)^2 + \sum_{i=1}^{n}(t_i^n - t_{i-1}^n)^2 \\
&= 2\sum_{i=1}^{n}(t_i^n - t_{i-1}^n)^2 \\
&\leq 2\max_i |t_i^n - t_{i-1}^n| \sum_{i=1}^{n} |t_i^n - t_{i-1}^n| \\
&= 2t \max_i |t_i^n - t_{i-1}^n| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

The third equality utilises that θ_i has zero mean and that θ_i and θ_j are independent for $i \neq j$. The fourth equality takes advantage of the above result for the forth moment of Z, and finally the last equality notices that the sum is a telescope sum.

With L^2-convergence shown, we move on to convergence in probability. For this we consider some partition of $[0, t]$, such that it converges to 0 faster than n^{-2}. Moreover, let ϵ_n be a series decreasing to zero. We can then specify our partition by

$$\max_i |t_i^n - t_{i-1}^n| = \frac{\epsilon_n}{n^2}.$$

Markov's inequality yields that

$$P\left\{([W]_n(t) - t)^2 \geq 2\epsilon_n\right\} \leq \frac{\mathbb{E}\left[([W]_n(t) - t)^2\right]}{2\epsilon_n} \leq \frac{2t \max_i |t_i^n - t_{i-1}^n|}{2\epsilon_n} = \frac{t}{n^2}.$$

Notice that the sum over all probability is finite, i.e. $\sum_{n=1}^{\infty} tn^{-2} < \infty$. Therefore, by applying the Borel-Cantelli lemma we have

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{([W]_n(t) - t)^2 \geq 2\epsilon_n\right\}\right) = 0.$$

In other words, the probability of the squared difference between the partial quadratic variation and t being larger than $2\epsilon_n$ is zero. Moreover, since $\epsilon_n \to 0$, this must entail that the probability of $\{\lim_n [W]_n(t) \neq t\}$ is zero, which means that the quadratic variation of a Brownian motion is equal to t, almost surely. □

Definition 2.1.17 (Martingale)**.** A stochastic process $M(t)$, adapted to the filtration $\mathcal{F}_t$, with $\mathbb{E}|M(t)| < \infty$ for $t \geq 0$ is said to be a *martingale* if

$$\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$$

for $0 \leq s \leq t$.

Theorem 2.1.6 (Lévy's characterisation theorem)**.** *The stochastic process $W(t)$ is a Brownian motion if and only if*

1. $W(0) = 0$,
2. $W(t)$ *is a continuous martingale,*
3. *and* $[W, W](t) = t$.

Proof. See [8, proof of theorem 7.36]. □

Note that it is possible to relax the martingale property to a local martingale in Lévy's characterisation theorem. However, we will stay with the above formulation as it facilitates our use of the theorem. Moreover, if we have several Brownian motions, e.g. $W_1(t)$ and $W_2(t)$, we say that they are independent if $[W_1, W_2](t) = 0$ and correlated if $[W_1, W_2](t) = \rho_{12}t$, where $\rho_{12} \in [-1, 1]$.

Definition 2.1.18 (Radon-Nikodym derivative process)**.** Consider the filtered measurable space $(\Omega, \mathcal{F}_t, \mathcal{F})$ with two probability measures Q and P such that $Q \ll P$. Moreover, let Λ be the related Radon-Nikodym derivative, i.e., $\Lambda = dQ/dP$. We define the *Radon-Nikodym derivative process* as

$$\Lambda(t) = \mathbb{E}^P[\Lambda \mid \mathcal{F}_t].$$

The following result follows directly from the properties of conditional expectation since $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $s \leq t$.

Corollary 2.1.7. *The Radon-Nikodym derivative process is a P-martingale, i.e. for $0 \leq s \leq t$ it holds that*

$$\mathbb{E}^P[\Lambda(t) \mid \mathcal{F}_s] = \Lambda(s).$$

Lemma 2.1.8. *A process $M(t)$ is a Q-martingale if and only if $\Lambda(t)M(t)$ is a P-martingale.*

Proof. See [8, corollary 10.11]. □

2.1.4 Itô calculus

Definition 2.1.19 (Itô integral process)**.** Let X be an $\mathcal{F}_T$-measurable random variable in S_T and let $W(t)$ be a Brownian motion. Moreover, let $\xi_0, \xi_1, \ldots, \xi_{n-1}$ be random the variables used in the definition of S_T (see definition 2.1.14), i.e. ξ_i is

$\mathcal{F}_{t_i}$-measurable and $\mathbb{E}\xi_i^2 < \infty$. Then we define the *Itô integral process* $\{\int_0^t XdW\}_{t\geq 0}$ by

$$\int_0^0 XdW = 0,$$
$$\int_0^t XdW = \sum_{i=1}^{m} \xi_{i-1}(W(t_i) - W(t_{i-1})) + \xi_m(W(t) - W(t_m)),$$
$$\int_s^t XdW = \int_0^t XdW - \int_0^s XdW,$$

for $t \in (t_m, t_{m+1}]$, $m = 0, \ldots, n-1$ and $0 \leq s \leq t \leq T$.

Definition 2.1.20 (Itô process)**.** Let $\mu(t)$ and $\sigma(t)$ be two processes adapted to the filtration generated by the Brownian motion $W(t)$. Then, if $\int_0^T |\mu(s)|ds < \infty$ and $\sigma \in P_T$ we define an *Itô process* as a process that can be written on the form

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

Corollary 2.1.9 (Leibniz' stochastic product rule)**.** *Let $X(t)$ and $Y(t)$ be two Itô processes. Then the differential of their product is given by*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + d[X, Y](t),$$

where $[\cdot, \cdot]$ denotes the covariation.

Proof. See [8, page 113]. □

To be able to formally manipulate stochastic differentials we introduce the convention that, for two Itô processes $X(t)$ and $Y(t)$, the differential of the covariation is the product of the differentials, i.e. $d[X, Y](t) = dX(t)dY(t)$.

Theorem 2.1.10 (Itô's lemma)**.** *Let $X(t)$ be an Itô process, then for $f : \mathbb{R} \to \mathbb{R}$ in C^2 it holds that*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[X](s),$$

or equivalently, on differential form,

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d[X](t),$$

where $[\cdot]$ denotes the quadratic variation.

Proof. See [7, proof of theorem 3.3] □

Theorem 2.1.11 (Martingale representation theorem)**.** *Let $W(t)$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_t^W$ be the filtration generated by $W(t)$ up until time t. Moreover, let $X(t)$ be a martingale with respect to $\mathcal{F}_t^W$. Then there exists a predictable process $\Gamma(t)$ such that $\int_0^T \Gamma^2(s)ds < \infty$ with probability one, and it holds that*

$$X(t) = X(0) + \int_0^t \Gamma(s)dW(s).$$

2.1.5 Stochastic differential equations

Definition 2.1.21 (Stochastic differential equation)**.** Let $\mu : \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be two $\mathcal{F}_t$-adapted processes called the drift and diffusion coefficient, respectively. A *stochastic differential equation* (SDE) is then, on differential form, given by

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t),$$

for $t \in [0, T]$ and $X(0) = x_0$, where $W(t)$ is a Brownian motion. Or equivalently, on integrated form

$$X(t) = x_0 + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dW(s),$$

for $t \in [0, T]$. Additionally, the solution to this SDE is called a *diffusion Itô process.*

Later on in this thesis we will rely on the concept of SDEs. We will however use the notion in a somewhat extended manner, namely systems of SDEs where each individual equation is connected together by the same driving noise. This notion of multiple coupled SDEs are very closely related to that of systems of ODEs, with the addition of correlated Brownian motions.

Definition 2.1.22 (System of stochastic differential equations)**.** Let the drift coefficient $\boldsymbol{\mu} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and the diffusion coefficient $\boldsymbol{\sigma} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d\times n}$. A d-dimensional system of stochastic differential equations, driven by n Brownian motions, is then on differential form given by

$$d\boldsymbol{X}(t) = \boldsymbol{\mu}(\boldsymbol{X}(t), t)dt + \sigma(\boldsymbol{X}(t), t)d\boldsymbol{W}(t), \tag{2.5}$$

where $\boldsymbol{W}(t)$ is an n-dimensional vector Brownian motion with instantaneous correlation $dW_i(t)dW_j(t) = \rho_{ij}dt$. Alternatively, written without vector notation

$$dX_i(t) = \mu_i(\boldsymbol{X}(t), t)dt + \sum_{j=1}^{n} \sigma_{ij}(\boldsymbol{X}(t), t)dW_j(t).$$

The fact that above definition of systems of SDEs uses correlated Brownian motions makes it a powerful modelling tool, but it does have some draw backs. One such example is in the proof of Girsanov's theorem as we will see below. To solve this problem we embed the correlation within the drift coefficient making the driving noise vector easier to handle. To do this we simply utilise Cholesky decomposition of the correlation matrix, $\rho = CC^\intercal$. Then we can rewrite (2.5) as

$$d\boldsymbol{X}(t) = \boldsymbol{\mu}(\boldsymbol{X}(t), t)dt + \sigma(\boldsymbol{X}(t), t)Cd\widetilde{\boldsymbol{W}}(t),$$

where $\widetilde{\boldsymbol{W}}(t)$ is a vector Brownian motion with independent elements.

The existence and uniqueness of solutions to SDEs is not guaranteed. Instead, it is only assured under certain conditions, which we collect in the theorem below. An important remark is that these constraints will seldom be discussed in this thesis. Instead, unless otherwise stated, they are assumed to hold.

Theorem 2.1.12 (Existence and uniqueness of solution to SDE)**.** *Let $\boldsymbol{X}(t)$ satisfy* (2.5) *and define the matrix norm as $||\sigma||^2 = \sum_{i,j} \sigma_{ij}^2$. If the following three conditions are satisfied:*

1. *Coefficients are locally Lipschitz in $\boldsymbol{x}$ uniformly in t, i.e. for every T and N there exists a constant $K(T, N)$, such that for all $||\boldsymbol{x}||, ||\boldsymbol{y}|| \leq N$ and all $0 \leq t \leq T$ it holds that*
$$||\boldsymbol{\mu}(\boldsymbol{x}, t) - \boldsymbol{\mu}(\boldsymbol{y}, t)|| + ||\sigma(\boldsymbol{x}, t) - \sigma(\boldsymbol{y}, t)|| < K||\boldsymbol{x} - \boldsymbol{y}||.$$
2. *Coefficients satisfy the linear growth condition, i.e.*
$$||\boldsymbol{\mu}(\boldsymbol{x}, t)||^2 + ||\sigma(\boldsymbol{x}, t)||^2 \leq K^2(1 + ||\boldsymbol{x}||^2).$$
3. *$\boldsymbol{X}(0)$ is independent of $\{\boldsymbol{W}(t)\}_{0 \leq t \leq T}$, and $\mathbb{E}\left[X^2(0)\right] < \infty$.*

Then there exists a unique strong solution $\boldsymbol{X}(t)$ to (2.5) *with continuous paths, and*
$$\mathbb{E}\left[\sup_{0 \leq t \leq T} X^2(t)\right] < C\left(1 + \mathbb{E}\left[X^2(0)\right]\right),$$
where C is a constant only dependent on K and T.

Proof. See [7, proof of theorem 2.5 and theorem 2.9]. □

Definition 2.1.23 (Log-normal dynamics)**.** We say that $\boldsymbol{X}(t)$ evolves under *log-normal dynamics* if its differential is on the form
$$d\boldsymbol{X}(t) = (\boldsymbol{\mu}(t)dt + \sigma(t)d\boldsymbol{W}(t))\boldsymbol{X}(t)^\intercal, \tag{2.6}$$
where $\boldsymbol{\mu}(t)$ and $\sigma(t)$ are $\mathcal{F}_t$-adapted processes, $\sigma(t)$ is almost surely not zero, and $\boldsymbol{W}(t)$ is a vector Brownian motion with correlation given by $dW_i(t)dW_j(t) = \rho_{ij}dt$.

Notice that, for a one-dimensional system of SDEs, driven by a single Brownian motion, the solution to (2.6) is given by
$$X(t) = X(0) \exp\left\{\int_0^t \left(\mu(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_0^t \sigma(u)dW(u)\right\}.$$
If the drift and diffusion coefficient would be constants, this expression would coincide with that of a geometric Brownian motion. Thus, the solution to an SDE describing log-normal dynamics is sometimes called a generalised geometric Brownian motion.

Theorem 2.1.13 (Girsanov's theorem)**.** *Consider the probability space $(\Omega, \mathcal{F}, P)$ and the d-dimensional SDE*
$$d\boldsymbol{X}(t) = \boldsymbol{\mu}(\boldsymbol{X}(t), t)dt + \sigma(\boldsymbol{X}(t), t)d\boldsymbol{W}(t), \tag{2.7}$$
where $\boldsymbol{W}(t)$ is an n-dimensional P-Brownian motion vector with independent elements. Define $\boldsymbol{\Theta}(t)$ by
$$\sigma(\boldsymbol{X}(t), t)\boldsymbol{\Theta}(t) = \widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t), t) - \boldsymbol{\mu}(\boldsymbol{X}(t), t), \tag{2.8}$$

where $\widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t),t)$ *is the drift for* (2.11)*. Moreover, let*

$$\Lambda(t) = \exp\left\{-\int_0^t \boldsymbol{\Theta}(s)\cdot d\boldsymbol{W}(t) - \frac{1}{2}\int_0^t ||\boldsymbol{\Theta}^2(s)||ds\right\}, \tag{2.9}$$

for $t \in [0,T]$*. As* $\Lambda(t)$ *constitutes a Radon-Nikodym derivative process, define the measure* $Q(A) = \mathbb{E}^P[1_A\Lambda(T)]$ *for all* $A \in \mathcal{F}$*, and let*

$$\boldsymbol{B}(t) = \boldsymbol{W}(t) - \int_0^t \boldsymbol{\Theta}(s)ds. \tag{2.10}$$

Then, $\boldsymbol{X}(t)$ *satisfy the following SDE under* Q*:*

$$d\boldsymbol{X}(t) = \widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t),t)dt + \sigma(\boldsymbol{X}(t),t)d\boldsymbol{B}(t), \tag{2.11}$$

where $\boldsymbol{B}(t)$ *is an* n*-dimensional Brownian motion under* Q *with independent elements.*

Proof. Firstly we will verify that (2.8) indeed yields (2.11). Secondly we will confirm that $\Lambda(t)$ is a Radon-Nikodym derivative process, and lastly we will show that $\boldsymbol{B}(t)$ is a vector Brownian motion under Q.

Using the (2.10) on differential form in (2.11) yields:

$$\begin{aligned}
&\widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t),t)dt + \sigma(\boldsymbol{X}(t),t)d\boldsymbol{B}(t)\\
=&\widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t),t)dt + \sigma(\boldsymbol{X}(t),t)d\boldsymbol{W}(t) - \sigma(\boldsymbol{X}(t),t)\boldsymbol{\Theta}(t)dt\\
=&\widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t),t)dt + \sigma(\boldsymbol{X}(t),t)d\boldsymbol{W}(t) - \widetilde{\boldsymbol{\mu}}(\boldsymbol{X}(t),t)dt + \boldsymbol{\mu}(\boldsymbol{X}(t),t)dt\\
=&d\boldsymbol{X}(t),
\end{aligned}$$

where we in the last equality recognise the expression as (2.7).

Next we note that $\Lambda(t)$, by construction, is non-negative. Moreover, we have that $\Lambda(0) = 1$, and because, by corollary 2.1.7, Radon-Nikodym derivative processes are martingales it holds that $\Lambda(T) = \Lambda(0) = 1$.

To show that $\boldsymbol{B}(t)$ is a Brownian motion under Q we will employ Lévy's characterisation theorem (see theorem 2.1.6). That $\boldsymbol{B}(0) = 0$ follows directly from (2.10) since $\boldsymbol{W}(0) = 0$. Continuity of $\boldsymbol{B}(t)$ is ensured by the fact that we define the process as the sum of two continuous functions. The martingale property will require a bit more work.

From lemma 2.1.8 we know that $\boldsymbol{B}(t)$ is a Q-martingale if and only if $\Lambda(t)\boldsymbol{B}(t)$ is a P-martingale. To utilise this result we will employ the martingale representation theorem (see theorem 2.1.11) from which we know that the drift term of a martingale on differential form must be zero. Therefore, let us look at the differential of $\Lambda(t)\boldsymbol{B}(t)$. To avoid clutter we drop the explicit dependence notation:

$$\begin{aligned}
d\left(\Lambda\boldsymbol{B}\right) &= \Lambda d\boldsymbol{B} + \boldsymbol{B}d\Lambda + d\Lambda d\boldsymbol{B}\\
&= \Lambda(d\boldsymbol{W} - \boldsymbol{\Theta}dt) + \boldsymbol{B}\Lambda\boldsymbol{\Theta}d\boldsymbol{W} + \Lambda\boldsymbol{\Theta}d\boldsymbol{W}(d\boldsymbol{W} - \boldsymbol{\Theta}dt)\\
&= \Lambda(1 + \boldsymbol{B}\boldsymbol{\Theta})d\boldsymbol{W},
\end{aligned}$$

where we in the last step have used that $dW_i(t)dW_j(t) = 0$ for $i \neq j$, and

$$dW(t)dt = 0, \quad dtdt = 0,$$

which follow from theorem 2.1.2 for differentials and theorem 2.1.5. From this we conclude that $\boldsymbol{B}(t)$ is a martingale.

Lastly we need to show that $dB_i(t)dB_j(t) = 0$ for $i \neq j$ and $dB_i(t)dB_i(t) = dt$. But this follows directly from (2.10) and the fact that $\boldsymbol{W}(t)$ is a Brownian motion. We have therefore shown everything we intended to. □

2.2 Financial mathematics

In this section, we will introduce and discuss a selection of central concepts in mathematical finance. These concepts are essential for understanding the contents of this thesis and are closely related to the theory discussed in section 2.1. It's important to note that the contents of this section are not exhaustive; rather, they serve as a collection of ideas necessary to comprehend this thesis without requiring extensive knowledge of mathematical finance.

We begin by introducing the risk-free rate process, which forms the foundation for the concept of discounting. Building upon this, we proceed to describe traded assets and portfolios. Next, we put forward a simple model of asset value processes, and define portfolios and arbitrage. Thereafter, we connect the risk-free rate process with our asset model to introduce the notion of a risk-neutral measure and discuss how it is closely related to arbitrage. Additionally, we put forward the concept of numéraires and generalise the ideas behind the risk-neutral measure to introduce numéraire-associated measures. Next we introduce zero-coupon bonds, and demonstrate the numéraire-associated measure toolkit. Specifically, we define the T-forward measure, which will play a prominent role in this thesis. Lastly, we introduce different types of interest rates that will be frequently encountered throughout this thesis.

Throughout this section, if nothing else is mentioned, the filtration $\mathcal{F}_t$ represents all available information in the market at time t.

2.2.1 The risk-free rate and discounting

One of the most fundamental concepts in mathematical finance is that of an instantaneous risk-free interest rate process. This process, often stochastic, specifies the time evolution of an interest rate to which you may borrow or lend money over an infinitesimally short time period. Because of the instantaneous nature of this rate it can be seen as default free.

Definition 2.2.1 (Instantaneous risk-free interest rate process)**.** The *instantaneous risk-free interest rate process*, $r(t)$, is the time-t rate at which you can borrow or lend money, over an infinitesimally short period of time, in the money market.

The risk-free interest rate is often used to construct something called the *money market account* (MMA), or simply *bank account*. Assume that you at time $t = 0$ find yourself with one unit currency to spare and decide to lend it to some bank. You place the money in an account and in return the bank pay you interest, given by the instantaneous risk-free interest rate process. If you place the payments you receive by the bank back into the interest carrying account, the amount of money in the account will compound. Moreover, because the risk-free interest payments are done continuously, since the lending period is infinitesimally short, the money in the account at time t will be given by $\exp \int_0^t r(u)du$.

Definition 2.2.2 (Money market account process)**.** A bank account which lends its money to the money market at the risk-free rate $r(t)$ is called a *money market account*, and its time-t value is given by the process

$$M(t) = e^{\int_0^t r(u)du}.$$

Because the interest collected by a money market account is assumed to be risk-free it means that given some cash, you can either invest it in a money market account and be guaranteed a return, or place the cash in some other investment without the same guarantee. Subsequently, the money market account becomes a natural reference level for all other investments, as it is of interest to compare them with the risk-free one. The simplest example of such a comparison is when you have one unit currency of cash stuffed under your mattress. The nominal value of this investment will of course stay constant with time. Its value relative to our reference level will, however, not be constant. To check the value of your money under your mattress against our reference level, we denominate it by the money market account, i.e. at time t we have that our one unit currency of cash has the value of $1/M(t)$ units of cash in a money market account. Assuming a non-negative interest rate process, $M(t)$ will be increasing, and therefore we have that the relative value of our mattress investment decreases over time. In other words; it would have been a better investment to place the cash in a money market account.

The above reasoning alludes to the fact that the value of cash is not constant over time. In a market with a positive instantaneous risk-free interest rate, one unit of currency today has a higher value than the same amount tomorrow.

The notion of comparing investments against the money market account easily generalises to investments other than those under your mattress. In fact, to denominate the value of an investment by the MMA is of such importance within mathematical finance that the reciprocal of the MMA is defined as its own process.

Definition 2.2.3 (Stochastic discount process)**.** The *stochastic discount process* is defined as the reciprocal of the money market account, and is given by

$$D(t) = \frac{1}{M(t)} = e^{-\int_0^t r(u)du}.$$

2.2.2 Asset modelling, portfolios and arbitrage

With the foundational processes established, we now delve into how to model assets, portfolios and arbitrage. Let $S(t)$ represent the value of a non-dividend-paying stock at time t. Drawing from our experience, we anticipate the value of this stock to follow a certain trend, perhaps the average return of all stocks on a particular exchange. Additionally, we expect it to exhibit random fluctuations. These two characteristics of a stock's value process make a diffusion process (see definition 2.1.21) a natural choice for modelling it. The differential of the stock's value process becomes

$$dS(t) = \mu(S(t), t)dt + \sigma(S(t), t)dW^P(t),$$

where $W^P(t)$ is a Brownian motion under the real world measure, P. The drift coefficient, $\mu(S(t), t)$, determines the trend, while the diffusion coefficient, $\sigma(S(t), t)$, specifies the magnitude of the random movements.

The drift and diffusion coefficients are commonly selected such that the diffusion process becomes a geometric Brownian motion. In such a case the differential is given by

$$dS(t) = S(t)\left(\alpha dt + \sigma dW^P(t)\right),$$

with α and σ being constants. This assumption allows for an exact strong solution to the SDE:

$$S(t) = S(0)\exp\left\{\sigma W^P(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\},$$

which forms the foundation of the well-known Black-Scholes-Merton formula.

Throughout this thesis, we adopt log-normal dynamics, a generalisation of the aforementioned dynamics. We assume that α and σ are $\mathcal{F}_t$-adapted random processes, which applied to the stock's value process leads to the following dynamics:

$$dS(t) = S(t)\left(\alpha(t)dt + \sigma(t)dW^P(t)\right). \tag{2.12}$$

We call this model the log-normal asset model as it may describe any non-dividend-paying asset.

Definition 2.2.4 (Log-normal asset model)**.** Let $V_i(t)$ be the value process of some non-dividend-paying asset. Then, the *log-normal asset model* assumes it evolves according to

$$dV_i(t) = V_i(t)\left(\alpha_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_j^P(t)\right),$$

where $W_j^P(t)$ represents Brownian motions under the real-world measure such that $dW_i^P(t)dW_j^P(t) = \rho_{ij}dt$, and $\alpha_i(t)$ and $\sigma_{ij}(t)$ are $\mathcal{F}_t$-adapted processes.

This model is an extension of (2.12), where we introduce several, possibly correlated, driving noises. This provides greater flexibility to the model. As the number of the Brownian motions increases the model becomes capable of incorporating a wider range of variations. A model with more driving noises possesses greater degrees of freedom and can thus accommodate more diverse shapes. For a comprehensive discussion of this topic as it pertains to interest rate modelling, see [1, chapter 12].

Thus far, we have only considered a single asset. However, in practice, you are often in possession of several assets at the same time, forming a portfolio. The value process of this portfolio is called a portfolio process, and its definition arises naturally as it is just a linear combination of single value processes.

Definition 2.2.5 (Portfolio process)**.** Let $(V_1(t), V_2(t), \ldots, V_M(t))^\intercal$ be a vector of value processes for some type of assets. Moreover, let $(\Delta_1(t), \Delta_2(t), \ldots, \Delta_M(t))^\intercal$ be the number of each asset in the portfolio at time t. Then, the time-t value of the portfolio is called a *portfolio process* and is given by

$$X(t) = \sum_{i=1}^{M}\Delta_i(t)V_i(t).$$

Next, we will introduce the concept of arbitrage. Arbitrage refers to a portfolio process that requires no initial investment and performs as good, or better, than the money market account, with no risk involved.

Definition 2.2.6 (Arbitrage)**.** An *arbitrage* is a portfolio value process, $X(t)$, satisfying $X(0) = 0$ and, for some time $T > 0$,

$$P\{X(T) \geq 0\} = 1, \quad P\{X(T) > 0\} > 0.$$

2.2.3 The risk-neutral measure and risk-neutral pricing

The log-normal asset model introduced in the previous section is convenient, but it could be immensely improved by removing the drift term. As a result of the martingale representation theorem (see theorem 2.1.11), the value process would, in such a case, become a martingale. This would allow us to utilise the powerful machinery of martingale theory.

Consider a single value process, $V(t)$, following the log-normal asset model driven by a one-dimensional Brownian motion. To arrive at our desired model with the martingale property, we begin by taking the differential of the discounted value process. By applying Leibniz' stochastic product rule and Itô's lemma, we get:

$$\begin{aligned} d(D(t)V(t)) &= dD(t)V(t) + D(t)dV(t) + dD(t)dV(t) \\ &= D(t)V(t)\left([\alpha(t) - r(t)]dt + \sigma(t)dW^P(t)\right). \end{aligned} \tag{2.13}$$

It becomes evident that if the drift were equal to the instantaneous risk-free interest rate, we would have a martingale. However, it is seldom the case that $\alpha(t) = r(t)$. But luckily for us, we are equipped with Girsanov's theorem (see theorem 2.1.13). From this theorem, we know that

$$W^Q(t) = W^P(t) + \int_0^t \Theta(s)ds$$

is a Brownian motion under the measure defined by

$$Q(A) = \mathbb{E}^P\left[1_A \exp\left\{-\int_0^T \Theta(s)dW(s) - \frac{1}{2}\int_0^T \Theta^2(s)ds\right\}\right], \quad \forall A \in \mathcal{F}.$$

Therefore, by defining

$$\Theta(t) = \frac{\alpha(t) - r(t)}{\sigma(t)},$$

we are able to rewrite (2.13) as

$$d(D(t)V(t)) = D(t)V(t)\sigma(t)dW^Q(t),$$

which is our sought after martingale dynamics.

The measure Q defined above is called the *risk-neutral measure*, and by defining it in the proper way, it can be utilised to yield martingale dynamics for a discounted value process. The toy example used to illustrate the reasoning behind the construction of the measure is, however, a simplification of the context in which the model is normally used. In the toy example we considered a universe consisting of only one single asset, but in practice, there are often several. The same reasoning as above is still employed, but Girsanov's theorem becomes multidimensional, and the existence of the multidimensional equivalent of $\Theta(t)$ is not as straightforward. In the definition and theorem below, we collect the formal definition of the risk-neutral measure and provide the conditions under which it exists.

Definition 2.2.7 (Risk-neutral measure)**.** The probability measure Q is said to be *risk-neutral* if the discounted prices of all non-dividend-paying assets in a market, $D(t)V_i(t)$, are martingales under Q.

Theorem 2.2.1 (Existence of the risk-neutral measure)**.** *The risk-neutral measure, Q, exists if and only if there exists a $\mathcal{F}_t$-adapted process $\Theta_j(t)$ such that*

$$\alpha_i(t) - r(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t) \tag{2.14}$$

are solvable. (2.14) *are called the market price of risk equations.*

Proof. By definition of the risk-neutral measure, Q, we want every discounted non-dividend-paying asset to be a martingale under it. Assuming that they follow the log-normal asset model (see definition 2.2.4), their discounted value processes will be martingale under Q, according to the martingale representation theorem (see theorem 2.1.11), if the following holds:

$$\begin{aligned} d(D(t)V_i(t)) &= D(t)V_i(t)\left([\alpha_i(t) - r(t)]dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j^P(t)\right) \\ &= D(t)V_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)dW_j^Q(t), \end{aligned} \tag{2.15}$$

where $W_j^Q(t)$ is a Q-Brownian motion.

From (2.15) we can see that we want the drift to be embedded in the new Brownian motion, rendering the expression a martingale under Q. By Girsanov's theorem (see theorem 2.1.13) we have that the new Brownian motion satisfies

$$dW_j^Q(t) = dW_j^P(t) - \Theta_j(t)dt.$$

Using this relation in (2.15) and matching the dt-terms, we arrive at the market price of risk equations

$$\alpha_i(t) - r(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\Theta_j(t),$$

which in turn determine $\Theta_j(t)$. Therefore, in order for Girsanov's theorem to hold, and for us to be able to construct Q there must exist a $\Theta_j(t)$ satisfying this system of equations. If it does not, Q does not exist. □

There are several important things to note in the above definitions and arguments. The first one is that it is possible that a set of non-dividend-paying assets, constituting a market or universe, makes the market price of risk equation non-solvable. In such a case, the model admits arbitrage. This connection between the market price of risk equations and arbitrage is considered to be one of the most fundamental theorems in mathematical finance, and is conveniently named the first fundamental theorem of asset pricing.

Theorem 2.2.2 (First fundamental theorem of asset pricing)**.** *If there exists a risk-neutral measure for a given market model, then the model is arbitrage-free.*

Proof. Let $X(t)$ be a portfolio value process with zero initial capital, i.e. $X(0) = 0$. Then, because the portfolio value process is just a linear combination of traded assets, the discounted value process is a martingale under the risk-neutral measure Q, i.e.

$$\mathbb{E}^Q[D(T)X(T) \mid \mathcal{F}_t] = D(t)X(t),$$

especially $\mathbb{E}^Q[D(T)X(T)] = 0$. Now assume that X(T) satisfies

$$P\{X(T) < 0\} = 0.$$

Because Q is equivalent to P the same must hold under the risk-neutral measure, i.e. $Q\{X(T) < 0\} = 0$. By reminding ourselves that $D(t)$ is non-negative for all t we can conclude that $Q\{X(T) > 0\} = 0$, because otherwise $\mathbb{E}^Q[D(T)X(T)] = 0$ would not hold. This means that the portfolio has a zero probability of having a positive return, and it is therefore not an arbitrage. □

The second important thing to note is that in the case of a model admitting arbitrage, it may be possible to select a subset of assets that make the market price of risk equations solvable. This means that a global model encompassing all assets may not be feasible, but several local models, each including a small subset of assets, may be possible. Although it is not possible to directly compare the results between the models, as each one will have its own risk-neutral measure, they are still useful for comparing behaviours within the model, which is often what is done in practice.

The third important remark to note is that there exists a more general formulation of the first fundamental theorem of asset pricing. In discrete time, the existence of a risk-neutral measure is equivalent to having an arbitrage-free model. Because we are working in continuous time, the formulation mentioned does, however, only state that the existence of a risk-neutral measure implies an arbitrage-free model. In the continuous case, it is possible to maintain the equivalence, but at the expense of introducing the concept of *no free lunch with vanishing risk*. This broader result is beyond the scope of discussion within this thesis. For readers interested in exploring this topic further, a comprehensive discussion can be found in [6] or in the original article [5]. Additionally, for a counterexample where the absence of arbitrage does not imply the existence of a risk-neutral measure, see [2].

The last detail worth pointing out is that the arguments leading up to the risk-neutral measure are all set in the real world, with the real-world measure. This is often not the case. Instead, when modelling assets of this kind, you assume the existence of a risk-neutral measure, and then, if necessary, calibrate the model against real-world data. If the aim of the model is only to compare behaviours of different assets within the model, a direct change to the real-world measure is not necessary. This motivates the following definition, which provides a tool to start the modelling directly in the risk-neutral measure. It becomes very useful in the coming context of numéraires.

Definition 2.2.8 (Traded asset)**.** A *traded asset* driven by d Brownian motions is an $\mathcal{F}_t$-adapted process with a differential given by

$$dV_i(t) = V_i(t)\left(r(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j^Q(t)\right),$$

where $W_j^Q(t)$ represents Brownian motions under the risk-neutral measure such that $dW_i^Q(t)dW_j^Q(t) = \rho_{ij}dt$, $r(t)$ is the instantaneous risk-free interest rate, and $\sigma_{ij}(t)$ is an $\mathcal{F}_t$-adapted processes.

This definition makes it evident that all discounted traded assets are martingales under the risk-neutral measure.

The introduction of the risk-neutral measure was motivated by the extensive and convenient mathematical toolkit available within martingale theory. An example of this is the ability to price any traded asset at any time t as the best approximation of the payoff at time t, i.e., the conditional expectation given all information available up until that time. Let $V(T)$ represent the value process of a traded asset at time T, such as the payoff of an option. Thanks to the martingale property of the risk-neutral measure, we can calculate the time-t value of this traded asset as follows:

$$D(t)V(t) = \mathbb{E}^Q\left[D(T)V(T) \mid \mathcal{F}_t\right] \iff V(t) = \mathbb{E}^Q\left[\exp\left(-\int_t^T r(s)ds\right)V(T) \mid \mathcal{F}_t\right],$$

where we have utilised the fact that $D(t)$ is $\mathcal{F}_t$-measurable. This formula for $V(t)$ is known as the risk-neutral pricing formula, and is summarised below proposition.

Proposition 2.2.3 (Risk-neutral pricing formula)**.** *Let $V(T)$ be a $\mathcal{F}_T$-measurable random variable representing some value at time T. Its time-t value is then given by*

$$D(t)V(t) = \mathbb{E}^Q[D(T)V(T) \mid \mathcal{F}_t].$$

We have now discussed the importance of the existence of the risk-neutral measure, but just like many mathematical concepts, uniqueness is equally crucial. Uniqueness of the risk-neutral measure is closely related to the concept of hedging, but as this is not addressed in this thesis, we will omit discussing the measure's uniqueness. The interested reader is referred to [12, section 5.4.4].

2.2.4 Numéraires and their associated measures

Up until now, we have only compared investments against the money market account. In this section, we expand this notion and begin comparing investments against traded assets.

Consider a non-zero, non-dividend-paying traded asset with a time-t price described by the $\mathcal{F}_t$-adapted process $N(t)$. Now, imagine you hold a contract that guarantees a payment at time T, represented by the $\mathcal{F}_T$-measurable random variable ξ. We can express the value of this payment at time T relative to the aforementioned asset as $\xi/N(T)$. In other words, we can quantify the value of our payment in terms of equivalent units of another asset. This is the essence of using $N(t)$ as a numéraire.

The most commonly used numéraire is the money market account, denoted as $M(t)$. By comparing the payment ξ to the value of the money market account at the payment time, $M(T)$, we can easily determine which investment is more favourable. Specifically, if $\xi/M(T) > 1$, it is more profitable to invest in the contract with the fixed payment. It is worth noting that the reciprocal of the MMA corresponds to the discount process, which explains why it is frequently chosen as a numéraire.

Definition 2.2.9 (Numéraire)**.** A *numéraire* is a strictly positive non-dividend-paying traded asset.

In section 2.2.3, we explored how the money market account is associated with the risk-neutral measure. This concept can be extended to arbitrary numéraires, leading us to the following definition, which is a generalisation of the ideas behind the risk-neutral measure (see definition 2.2.7).

Definition 2.2.10 (Numéraire-associated measure)**.** We define the probability measure Q^N, associated with the numéraire $N(t)$, as the measure that satisfies the following conditions:

1. Q^N and Q are equivalent measures, and
2. the value process of any traded asset $V(t)$, denominated by the numéraire $N(t)$, is a martingale under Q^N.

It is important to note that the above definition is formulated in terms of the risk-neutral measure Q. Therefore, the existence of numéraire-associated measures relies on the existence of Q. This connection ensures that the numéraire measures do not admit arbitrage, as they are constructed based on a measure whose existence implies an arbitrage-free model. However, this relationship is not immediately apparent, and we will therefore delve into the details in section 2.2.6.

Because the definition of a numéraire-associated measure is a generalisation of the risk-neutral measure, we get a slightly more general way of pricing traded assets. Let $V(t)$ be a traded asset with some payoff at time T, then its price at time t is given by

$$V(t) = \mathbb{E}^N \left[\frac{N(T)}{N(t)} V(T) \mid \mathcal{F}_t \right].$$

This formula will prove very useful in section 2.2.7.

2.2.5 Change of numéraire inside expectation

As will be proven throughout this thesis; changing numéraire, and thereby also the related measure, can simplify calculations immensely. In this section we derive the general formulas which makes these changes a game of plug-and-play.

Considering two numéraires, $N_1(t)$, and $N_2(t)$, where $\mathbb{E}^1$ and $\mathbb{E}^2$ denote expected value in the respective related measures. The goal of this section is to deduce the factor, Z, which facilitates the following change of measure inside conditional expectation:

$$\mathbb{E}^1[Z\xi \mid \mathcal{F}_t] = \mathbb{E}^2[\xi \mid \mathcal{F}_t], \tag{2.16}$$

for any random variable ξ.

By definition of a numéraire (see definition 2.2.9), $N_1(t)$ and $N_2(t)$ are two $\mathcal{F}_t$-adapted processes which each describe a strictly positive traded asset. We want to construct a Radon-Nikodym derivative which can be used to define the measure Q^{N_2} via Q^{N_1}. Let

$$Z_{1,2}(T) = \frac{N_1(0)N_2(T)}{N_2(0)N_1(T)}, \tag{2.17}$$

and notice that it indeed is a Radon-Nikodym derivative since $Z_{1,2}(T) > 0$ and

$$\mathbb{E}^1[Z_{1,2}(T)] = \frac{N_1(0)}{N_2(0)}\mathbb{E}^1\left[\frac{N_2(T)}{N_1(T)} \mid \mathcal{F}(0)\right] = \frac{N_1(0)}{N_2(0)}\frac{N_2(0)}{N_1(0)} = 1,$$

where the second to last equality follows from $N_2(T)/N_1(T)$ being a martingale under Q^{N_1}, by definition. Using this we can now, via the Radon-Nikodym theorem (see theorem 2.1.3), define the measure associated with N_2 as

$$Q^{N_2}(A) = \int_A Z_{1,2}(T)dQ^{N_1} = \frac{N_1(0)}{N_2(0)}\int_A \frac{N_2(T)}{N_1(T)}dQ^{N_1}, \quad \forall A \in \mathcal{F}.$$

In (2.16) we are however dealing with a change of measure inside conditional expectation, which means abstract Bayes' theorem (see theorem 2.1.4) needs to be utilised, and for that we need the Radon-Nikodym derivative process. Because of the martingale property it easily computes to

$$Z_{1,2}(t) = \mathbb{E}^1[Z_{1,2}(T) \mid \mathcal{F}_t] = \frac{N_1(0)N_2(t)}{N_2(0)N_1(t)}. \tag{2.18}$$

Now, let ξ be an $\mathcal{F}_T$-measurable random variable representing some payoff at a later time T. Then, by denominating the payoff by N_1, define the $\mathcal{F}_T$-measurable variable

$$X = \frac{\xi}{N_2(T)}.$$

Directly applying abstract Bayes' theorem yields

$$\begin{aligned}\mathbb{E}^2[X \mid \mathcal{F}_t] &= \frac{1}{Z_{1,2}(t)}\mathbb{E}^1[Z_{1,2}(T)X \mid \mathcal{F}_t] \\ &= \frac{N_1(t)}{N_2(t)}\mathbb{E}^1\left[\frac{\xi}{N_1(T)} \mid \mathcal{F}_t\right].\end{aligned}$$

Or equivalently

$$N_1(t)\mathbb{E}^1\left[\frac{\xi}{N_1(T)} \mid \mathcal{F}_t\right] = N_2(t)\mathbb{E}^2\left[\frac{\xi}{N_2(T)} \mid \mathcal{F}_t\right] \tag{2.19}$$

This result can in turn be utilised to answer the question posed by 2.16. For the arbitrary $\mathcal{F}_t$-measurable random variable Y we can write

$$\begin{aligned}\mathbb{E}^2\left[Y \mid \mathcal{F}_t\right] &= \mathbb{E}^2\left[\frac{N_2(t)}{N_2(T)}\left(Y\frac{N_2(T)}{N_2(t)}\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^1\left[\frac{N_1(t)}{N_1(T)}\left(Y\frac{N_2(T)}{N_2(t)}\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^1\left[\frac{Z_{1,2}(T)}{Z_{1,2}(t)}Y \mid \mathcal{F}_t\right].\end{aligned}$$

This result is stated in below corollary.

Corollary 2.2.4. *Let Y be an arbitrary $\mathcal{F}_t$-measurable random variable, $N_1(t)$ and $N_2(t)$ be numéraires with related measure Q^{N_1} and Q^{N_2}. Furthermore, let the Radon-Nikodym derivative process enabling the change of measure from Q^{N_1} to Q^{N_2} be denoted by $Z_{1,2}(t)$. Then the following relation holds:*

$$\mathbb{E}^2\left[Y \mid \mathcal{F}_t\right] = \mathbb{E}^1\left[\frac{N_1(t)N_2(T)}{N_2(t)N_1(T)}Y \mid \mathcal{F}_t\right] = \mathbb{E}^1\left[\frac{Z_{1,2}(T)}{Z_{1,2}(t)}Y \mid \mathcal{F}_t\right].$$

2.2.6 Numéraire's effect on Brownian motion

In this section, our goal is to derive two formulas used to construct the new Brownian motion and associated measure after changing the numéraire. Let $N_1(t)$ and $N_2(t)$ be the numéraires discussed in section 2.2.5, and let $Z_{1,2}$ be the Radon-Nikodym process defined in the same section. When the risk-neutral measure exists, the numéraire-related measures Q^{N_1} and Q^{N_2} also exist. By utilising Girsanov's theorem (see theorem 2.1.13), we know that if $W(t)$ is a Q^{N_1}-Brownian motion, the equation

$$d\widetilde{W}(t) = -\Theta(t)dt + dW(t)$$

represents a Q^{N_2}-Brownian motion. However, the specific form of $\Theta(t)$ is not immediately deduced from this line of reasoning. Therefore, the objective of this section is to determine this form and explain why it always exists given the existence of Q, as stated at the end of section 2.2.4. To accomplish this, we will derive two expressions for the Brownian motion under the new measure.

Deriving these expressions begins with the observation that the Radon-Nikodym process used to transition from Q^{N_1} to Q^{N_2}, as given by (2.18), must be a martingale according to Girsanov's theorem. The martingale representation theorem states that this is equivalent to its differential being driftless, i.e.,

$$dZ_{1,2}(t) = \frac{N_1(0)}{N_2(0)}d\left(\frac{N_2(t)}{N_1(t)}\right) = 0dt + Z_{1,2}(t)\Theta(t)dW(t).$$

Expanding the above differential using the Leibniz' stochastic product rule and Itô's lemma yields

$$\begin{aligned}dZ(t) &= \frac{N_1(0)}{N_2(0)}\left(dN_1(t)N_2^{-1}(t) + N_1(t)d\left(N_2^{-1}(t)\right) + dN_1(t)d\left(N_2^{-1}(t)\right)\right) \\ &= Z_{1,2}(t)(\sigma_2(t)dW_2 - \sigma_1(t)dW_1).\end{aligned}$$

In the last step, we utilise the fact that the numéraires are traded assets, and their differentials are given by

$$\begin{aligned} dN_1(t) &= N_1(t)((\cdots)dt + \sigma_1(t)dW_1(t)), \\ dN_2(t) &= N_2(t)((\cdots)dt + \sigma_2(t)dW_2(t)), \end{aligned} \tag{2.20}$$

where $W_1(t)$ and $W_2(t)$ are Brownian motions under Q^{N_1}. Note that the dt-terms above are not explicitly stated as they have no effect on the computation.

The next step is to notice that $\widetilde{W}(t)Z_{1,2}(t)$ must be a martingale under Q^{N^1}. To arrive at this we utilise corollary 2.2.4 and the fact that $\widetilde{W}(t)$ is a Brownian motion under Q^{N_2} in the following computation;

$$\begin{aligned} \mathbb{E}^1[\widetilde{W}(t)Z_{1,2}(t) \mid \mathcal{F}_s] &= Z_{1,2}(s)\mathbb{E}^1\left[\widetilde{W}(t)\frac{Z_{1,2}(t)}{Z_{1,2}(s)} \mid \mathcal{F}_s\right] \\ &= Z_{1,2}(s)\mathbb{E}^2[\widetilde{W}(t) \mid \mathcal{F}_s] = Z_{1,2}(s)\widetilde{W}(t). \end{aligned}$$

We will utilise the martingale property as it is equivalent with a driftless differential. Dropping the explicit time dependence to avoid clutter and, again, using Leibniz stochastic product rule and Itô's lemma we get

$$\begin{aligned} d\left(\widetilde{W}Z_{1,2}\right) &= Z_{1,2}d\widetilde{W} + dZ_{1,2}\widetilde{W} + dZ_{1,2}d\widetilde{W} \\ &= Z_{1,2}\left(-\Theta dt + dW\right) + \widetilde{W}Z_{1,2}\left(\sigma_2 dW_2 - \sigma_1 dW_1\right) \\ &+ Z_{1,2}(\sigma_2\mathrm{Corr}[dW_2, dW] - \sigma_1\mathrm{Corr}[dW_1, dW])dt. \end{aligned}$$

Setting the dt-term to zero to enforce the martingale property we finally arrive at

$$\Theta(t) = \sigma_2(t)\mathrm{Corr}[dW_2, dW](t) - \sigma_1(t)\mathrm{Corr}[dW_1, dW](t). \tag{2.21}$$

Notice that the expression will only be undefined if either the volatilities $\sigma_1(t)$ and $\sigma_2(t)$ or the Q^{N_1}-Brownian motions $dW(t)$, $dW_1(t)$, and $dW_2(t)$ are undefined. While we can assume the existence of the volatilities, the existence of Brownian motions may not be guaranteed. However, let's consider the case when $N_1(t)$ is the money market account. In this case, $Q^{N_1} = Q$, and the Brownian motions $W(t)$, $W_1(t)$, and $W_2(t)$ are Q-Brownian motions. We know that Q-Brownian motions exist when the market price of risk equations are solvable. Therefore, Q^{N_2} exists for any $N_2(t)$ as long as the market price of risk equations are solvable. By the same reasoning as in the proof of the first fundamental theorem of asset pricing (see theorem 2.2.2) this measure will not admit arbitrage.

We collect (2.21) in the below corollary.

Corollary 2.2.5. *Let $N_1(t)$ and $N_2(t)$ be numéraires with differentials on the same form as* (2.20) *and with related measure Q^{N_1} and Q^{N_2}. Moreover, let $W(t)$ be an arbitrary Q^{N_1}-Brownian motion, then*

$$d\widetilde{W}(t) = -\Theta(t)dt + dW(t)$$

is a Brownian motion under Q^{N_2} where

$$\Theta(t) = \sigma_2(t)\mathrm{Corr}[dW_2, dW](t) - \sigma_1(t)\mathrm{Corr}[dW_1, dW](t).$$

The above formulation requires knowledge about the actual processes that define the numéraires, which can sometimes be difficult to obtain. However, it is possible to reformulate the expression in terms of the log-returns between the numéraires and their relationship with the Brownian motion $W(t)$. This result is presented in the following corollary, and it can be shown to be equivalent to the previous formulation by applying Itô's lemma.

Corollary 2.2.6. *Let $N_1(t)$ and $N_2(t)$ be numéraires with associated measure Q^{N_1} and Q^{N_2}. Moreover, let $W(t)$ be an arbitrary Q^{N_1}-Brownian motion, then*

$$d\widetilde{W}(t) = -\Theta(t)dt + dW(t)$$

is a Brownian motion under Q^{N_2} where

$$\Theta(t) = \frac{d\left[W, \ln(N_2/N_1)\right]}{dt}(t).$$

2.2.7 Zero-coupon bond and the T-forward measure

Above we have introduced concepts which are at the epicentre of mathematical finance, but we have yet to discuss one of the ideas this thesis revolves around; the zero-coupon bond.

Definition 2.2.11 (Zero-coupon bond)**.** A T-maturity *zero-coupon bond* (ZCB) is a contract, and traded asset, that guarantees the holder a payment of one unit currency at time T, with no intermediate payments. Its time-t price is denoted by $P(t, T)$.

Proposition 2.2.7 (Risk-neutral zero-coupon bond price)**.** *The risk-neutral zero-coupon time $t \leq T$ price is given by*

$$P(t, T) = \mathbb{E}^Q\left[e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right].$$

Proof. As the contract price is a martingale under the risk-neutral measure, Q, it follows directly from the risk-neutral pricing formula (theorem 2.2.3) that

$$D(t)P(t, T) = \mathbb{E}^Q[D(T) \mid \mathcal{F}_t] \iff P(t, T) = \mathbb{E}^Q\left[e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right].$$

□

An important remark about the zero-coupon bond price is that t usually represents the observation time. This means that, even though conditional expectation is defined as a random variable, the price of a ZCB is deterministic. However, would you observe $P(t, T)$ at some time $s < t$, the price would be stochastic.

To illustrate the usefulness of this contract we consider two examples below.

Assume that you hold a contract guaranteeing a fixed payment of c at time T. As discussed in section 2.2.1, the time-t value of this contract is given by the discounted payment, i.e., $cD(T)/D(t)$. However, in the case of a stochastic risk-free rate, this discounted payment becomes a random variable. To address this, we aim to obtain a non-stochastic value that reflects the time-t value of the payment. The best approximation for this random variable is its expected value, resulting in

$$\mathbb{E}^Q\left[c\frac{D(T)}{D(t)} \mid \mathcal{F}_t\right] = c\mathbb{E}^Q\left[\frac{D(T)}{D(t)} \mid \mathcal{F}_t\right] = cP(t,T). \tag{2.22}$$

A subtle but important remark is that conditional expectation generally results in random variables, but this quantity is indeed deterministic because we are observing it from time t.

From (2.22), it becomes evident that the zero-coupon bond price serves as the deterministic equivalent of the stochastic discount process. To further establish this equivalence, we observe that for a deterministic risk-free rate $r(t)$, they are indeed equal:

$$P(t,T) = \mathbb{E}^Q\left[e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right] = e^{-\int_t^T r(u)du} = \frac{D(T)}{D(t)},$$

where $D(T)$ and $D(t)$ are deterministic due to the deterministic $r(t)$.

Now, let's consider another contract guaranteeing a payment $H(S(T))$ at the same time T, where $S(t)$ is an underlying traded asset, and $H(x)$ some payoff function. Unlike the previous payment, this payment is not predetermined but rather random, and it is fixed at the payment date. Similar to the previous case, the time-t value of this payment is given by $H(S(t))D(T)/D(t)$. However, a new challenge arises: we cannot employ the expectation in the same manner as in (2.22) since the payment itself is random. This means that to compute the expected value, we would need knowledge about the joint distribution of $H(S(t))$ and $D(T)$. Or do we?

Let $V(t)$ denote the time-t value of the aforementioned contract. By applying the risk-neutral pricing formula (see proposition 2.2.3), we have that

$$\begin{aligned} V(t) &= \mathbb{E}^Q\left[H(S(t))\frac{D(T)}{D(t)} \mid \mathcal{F}_t\right] \\ &= P(t,T)\mathbb{E}^Q\left[H(S(t))\frac{M(t)P(T,T)}{M(T)P(t,T)} \mid \mathcal{F}_t\right], \end{aligned}$$

where we have used $M(t) = 1/D(t)$, $P(T,T) = 1$, and the fact that $P(t,T)$ is $\mathcal{F}_t$-measurable. Notice that $M(t)$ serves as the numéraire for the risk-neutral measure, and $P(t,T)$ is a valid numéraire. By reminding us about the corollary for change of measure inside conditional expectation (see corollary 2.2.4) we recognise the factor within the expectation as the Radon-Nikodym derivative process used to change from Q to the measure associated with $P(t,T)$ as numéraire. This measure is called the T-forward measure denoted by Q^T. Consequently, the pricing formula simplifies to

$$V(t) = P(t,T)\mathbb{E}^T[H(S(t)) \mid \mathcal{F}_t].$$

The above example illustrates the usefulness of the T-forward measure, as it enables us to compute the value of a contract without requiring knowledge of the joint distribution of the random payout and the discount process. We formally define this measure as follows:

Definition 2.2.12 (T-forward measure)**.** Let Q be the risk-neutral measure, and T be the fixed maturity of the zero-coupon bond with time t price $P(t,T)$. Then, we define the T-forward measure, Q^T, as

$$Q^T(A) = \frac{1}{P(0,T)} \int_A D(T)dQ \quad \forall A \in \mathcal{F}.$$

2.2.8 Compounding rates

As the subject of study in this thesis is interest rates, it is essential to understand the different types of interest rates. In this section, we will briefly introduce three distinct types of interest rates that will be utilised throughout this thesis.

We begin by examining the continuously compounding forward interest rate, denoted as $f(t,T)$. This rate captures the rate at which a zero-coupon bond continuously compounds between times t and T.

Definition 2.2.13 (Continuously compounding forward interest rate)**.** For $0 \leq t \leq T$, let $P(t,T)$ be the price at time t of the zero-coupon bond paying 1 at time T. Then, the *continuously compounding forward interest rate* $f(t,T)$ is defined as

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T).$$

Moving on, we consider the simply compounding forward interest rate, denoted as $F(t,T,T+\delta)$. This rate represents the rate at which a zero-coupon bond simply compounds between times T and $T+\delta$, where δ denotes the time difference between the maturity and expiry dates. The computation of this rate involves linearly interpolating the prices of zero-coupon bonds at times t with maturities T and $T+\delta$. Note that $\lim_{\delta \to 0} F(t,T,T+\delta) = f(t,T)$.

Definition 2.2.14 (Simply compounding forward interest rate)**.** For $\delta > 0$ and $0 \leq t \leq T$, let $P(t,T)$ and $P(t,T+\delta)$ be the prices at time t of the zero-coupon bonds that pay 1 at time T and $T+\delta$, respectively. The *simply compounding forward interest rate* $F(t,T,T+\delta)$, with expiry date T and maturity date $T+\delta$, is defined as

$$1 + \delta F(t,T,T+\delta) = \frac{P(t,T)}{P(t,T+\delta)}.$$

Lastly, we define the yield to maturity curve.

Definition 2.2.15 (Yield to maturity curve)**.** The *yield to maturity curve*, denoted as $y(t,T)$, represents the average continuously compounding interest rate over a given period $[t,T]$ and is given by

$$y(t,T) = \frac{1}{T-t} \int_t^T f(t,u)du.$$

The yield to maturity curve becomes the yield curve when $T = t+s$, i.e. $R_t(s) = y(t,t+s)$.

2.3 Financial instruments

Definition 2.3.1 (Interest rate swap)**.** Consider the notional amount N, and let $\{T_\alpha, T_{\alpha+1}, \ldots, T_\beta\}$ be a set of dates, τ_j be the year fraction between T_j and T_{j-1}, and $F_j(t)$ be the simply compounding forward interest rate with maturity T_j and expiry T_{j-1} (floating leg). Lastly, let K be some fixed interest rate (fixed leg). The *interest rate swap* (IRS) is then defined as the contract which exchanges payments between the fixed and floating leg. This means that at every instant $T_j \in \{T_\alpha, T_{\alpha+1}, \ldots, T_\beta\}$ the payment of $N\tau_j K$ is exchanged for the payment $N\tau_j F_j(T_{j-1})$. The times at which the floating rate is settled for the upcoming payment are called reset dates and those are $T_\alpha, \ldots, T_{\beta-1}$.

When the fixed leg is paid and the floating leg is received, we call it a *payer interest rate swap* (PFS), whilst in the case when the fixed leg is received and the floating leg is paid we call it a *receiver interest rate swap* (RFS).

The PFS discounted payoff is given by

$$\sum_{i=\alpha+1}^{\beta} \frac{D(t)}{D(T_i)} N\tau_i (F_i(T_{i-1}) - K).$$

The RFS discounted payoff is given by

$$\sum_{i=\alpha+1}^{\beta} \frac{D(t)}{D(T_i)} N\tau_i (K - F_i(T_{i-1})).$$

Proposition 2.3.1 (Risk-neutral interest rate swap price)**.** *The time t risk-neutral payer interest rate swap value is given by*

$$\begin{aligned}\mathbf{PFS}(t, N, K) &= N \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(F_i(t) - K) \\ &= -NP(t, T_\alpha) + NP(t, T_\beta) + N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i).\end{aligned}$$

and the corresponding value for the receiver interest rate swap is given by

$$\begin{aligned}\mathbf{RFS}(t, N, K) &= N \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(K - F_i(t)) \\ &= NP(t, T_\alpha) - NP(t, T_\beta) - N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i),\end{aligned}$$

Proof. The idea of our proof is to use the risk-neutral formula (see proposition 2.2.3) and then utilise corollary 2.2.4 to change from the risk-neutral measure, Q, to the T_i-forward measure Q^{T_i}. This change will allow us to utilise the martingale property of the simply compounding forward interest rate under this measure to achieve an expression for the price free from expectation operators. By the risk-neutral pricing formula the time-t price, $V(t)$, of a payer interest rate swap payoff is given by

$$\begin{aligned} D(t)V(t) &= \mathbb{E}_Q\left[N \sum_{i=\alpha+1}^{\beta} \tau_i D(T_i)(K - F_i(T_{i-1})) \mid \mathcal{F}(t)\right] \\ &= N \sum_{i=\alpha+1}^{\beta} \tau_i \mathbb{E}_Q\left[D(T_i)(K - F_i(T_{i-1})) \mid \mathcal{F}(t)\right]. \end{aligned} \tag{2.23}$$

Now, by dividing by the left-hand side discount factor and multiplying by $P(t, T_i)/P(t, T_i)$ inside the expectation, each expectation part of (2.23) becomes

$$\begin{aligned} &\frac{1}{D(t)}\mathbb{E}_Q\left[D(T_i)(K - F_i(T_{i-1})) \mid \mathcal{F}(t)\right] \\ &= \frac{1}{D(t)}\mathbb{E}_Q\left[D(T_i)\frac{P(t,T_i)}{P(t,T_i)}(K - F_i(T_{i-1})) \mid \mathcal{F}(t)\right] \\ &= \frac{1}{D(t)P(t,T_i)}\mathbb{E}_Q\left[D(T_i)P(T_i,T_i)P(t,T_i)(K - F_i(T_{i-1})) \mid \mathcal{F}(t)\right], \end{aligned}$$

where the introduction of $P(T_i, T_i)$ is allowed as it equals one. In our expression we recognise

$$\frac{D(T_i)P(T_i,T_i)}{D(t)P(t,T_i)} = \frac{M(t)P(T_i,T_i)}{M(T_i)P(t,T_i)}$$

as the ratio between numéraires in corollary 2.2.4 used to transition between the risk-neutral measure and the T_i-forward measure. It thus follows that

$$\begin{aligned} V(t) &= N \sum_{i=\alpha+1}^{\beta} \tau_i \mathbb{E}_{T_i}[P(t,T_i)(K - F_i(T_{i-1})) \mid \mathcal{F}(t)] \\ &= N \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)(K - F_i(t)) \\ &= \mathbf{PFS}(t, N, K). \end{aligned}$$

The second equality follows from the facts that $P(t, T_i)$ is $\mathcal{F}_t$-measurable, and $F_i(\cdot)$ is a martingale under the T_i-forward measure. $\mathbf{RFS}(t, N, K)$ is achieved mutatis mutandis. ☐

Definition 2.3.2 (Forward swap rate)**.** The time-t *forward swap rate* is the fixed rate $S_{\alpha,\beta}(t) = K$ for a receiver interest rate swap, for which the discounted value of the price at first payment is fair, i.e. the K such that

$$\begin{aligned} &\frac{D(T_\alpha)}{D(t)} \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)(K - F_i(t)) = 0 \\ &\implies S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}. \end{aligned}$$

The swap rate allows us to find an alternative expression for the time t discounted value of interest rate swap payoffs. For a receiver IRS we have that the discounted value is given by

$$\frac{D(T_\alpha)}{D(t)}(S_{\alpha,\beta}(T_\alpha) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i). \tag{2.24}$$

Definition 2.3.3 (Interest rate caplet)**.** We define an *interest rate caplet* as a payer interest rate swap consisting of one payment, where the payment is executed only if it has a positive value. Using the same notation as above the caplet payoff is given by:

$$N\tau_i(F_i(T_{i-1}) - K)^+,$$

where $\cdot^+ = \max(0, \cdot)$. This is payoff form is referred to as a call option.

Definition 2.3.4 (Interest rate cap)**.** We define an *interest rate cap* as a payer interest rate swap where each exchange payment is executed only if it has a positive value. Using the same notation as above the cap discounted payoff is given by:

$$\sum_{i=\alpha+1}^{\beta} \frac{D(T_i)}{D(t)} N\tau_i(F_i(T_{i-1}) - K)^+.$$

2.4 The Heath-Jarrow-Morton framework

In this section we briefly introduce the Heath-Jarrow-Morton framework. This will be needed in section 4.2 where the framework is extended and later used to derive the full zero-coupon bond curve for the Forward Market Model.

2.4.1 Construction of the framework and its no-arbitrage condition

The objective of the HJM framework is to model the instantaneous forward rate (see definition 2.2.13) as a diffusion process, where its dynamics under the physical measure P are given by

$$df(t,T) = \mu(t,T)dt + \boldsymbol{\sigma}(t,T)^\intercal d\boldsymbol{W}^P(t), \tag{2.25}$$

where $\boldsymbol{W}^P(t)$ is a d-dimensional Brownian motion with independent elements under the physical measure. To simplify notation, we will implicitly assume that the coefficients may depend on the rate itself.

Up until now, we have only encountered one particular model, namely the log-normal asset model (see definition 2.2.4). The no-arbitrage condition for assets following the log-normal asset model has been discussed in section 2.2.3, and it relies on selecting a drift such that the discounted asset value is a martingale under the risk-neutral measure. However, this approach does not apply in the HJM case. The reason for this is that $f(t,T)$ is not a traded asset. But we do know how it relates to one very common traded asset.

The zero-coupon bond is a traded asset, and as it is related to the instantaneous forward rate by

$$P(t,T) = \exp\left\{-\int_t^T f(t,u)du\right\}, \tag{2.26}$$

we aim to utilise it to derive the no-arbitrage condition for the HJM framework. More specifically we will derive the dynamics of (2.26) under the physical measure, and then use them to define the risk-neutral measure.

We begin by assuming f is nice enough. It is then possible to apply Leibniz' integral rule, yielding the differential of the exponent:

$$\begin{aligned} d\left(-\int_t^T f(t,u)du\right) &= f(t,t)dt - \int_t^T df(t,u)du \\ &= r(t)dt - \int_t^T \alpha(t,u)dtdu - \int_t^T \boldsymbol{\sigma}(t,u)^\intercal d\boldsymbol{W}^P(t)du. \end{aligned}$$

Moreover, assuming that sufficient conditions are satisfied by the drift and diffusion coefficient, we may apply Fubini's stochastic theorem to change the order of integration. This results in:

$$\begin{aligned} d\left(-\int_t^T f(t,u)du\right) &= r(t)dt - \int_t^T \alpha(t,u)dudt - \int_t^T \boldsymbol{\sigma}(t,u)^\intercal dud\boldsymbol{W}^P(t) \\ &= r(t)dt - \alpha^*(t,T)dt - \boldsymbol{\sigma}^*(t,T)^\intercal d\boldsymbol{W}^P(t), \end{aligned} \tag{2.27}$$

where the shorthand notation

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)du, \quad \boldsymbol{\sigma}^*(t,T) = \int_t^T \boldsymbol{\sigma}(t,T)^\intercal du,$$

is used.

Via Itô's lemma and (2.27) the dynamics for (2.26) now become:

$$dP(t,T) = P(t,T)\left[\left(r(t) - \alpha^*(t,T) + \frac{1}{2}\boldsymbol{\sigma}^*(t,T)^\intercal\boldsymbol{\sigma}^*(t,T)\right)dt - \boldsymbol{\sigma}^*(t,T)d\boldsymbol{W}^P(t)\right].$$

And subsequently, the discounted ZCB dynamics become:

$$d(D(t)P(t,T)) = D(t)P(t,T)\left[\left(-\alpha^*(t,T) + \frac{1}{2}\boldsymbol{\sigma}^*(t,T)^\intercal\boldsymbol{\sigma}^*(t,T)\right)dt - \boldsymbol{\sigma}^*(t,T)^\intercal d\boldsymbol{W}^P(t)\right].$$

From Girsanov's theorem, we know that if there exists a unique process $\boldsymbol{\Theta}(t)$, such that

$$\left(-\alpha^*(t,T) + \frac{1}{2}\boldsymbol{\sigma}^*(t,T)^\intercal\boldsymbol{\sigma}^*(t,T)\right)dt - \boldsymbol{\sigma}^*(t,T)^\intercal d\boldsymbol{W}^P(t) \tag{2.28}$$

$$= -\boldsymbol{\sigma}^*(t,T)^\intercal\left(\boldsymbol{\Theta}(t,T)dt + d\boldsymbol{W}^P(t)\right), \tag{2.29}$$

then

$$\boldsymbol{W}^Q(t) = \int_0^t \boldsymbol{\Theta}(s)ds + \boldsymbol{W}^P(t),$$

will be a Brownian motion under the measure $Q(A) = \mathbb{E}^P[1_A\Lambda(T)]$, where $\Lambda(T)$ is defined by (2.9). Furthermore, the discounted zero-coupon bond will under this measure be a martingale, meaning that it is a risk-neutral measure.

From (2.28) one can easily derive the condition that $\boldsymbol{\Theta}(t)$ must satisfy for the risk-neutral measure to exist, which by the first fundamental theorem of asset pricing also becomes the condition for an arbitrage-free model. By differentiating this condition with respect to T, we arrive at below theorem.

Theorem 2.4.1 (Heath-Jarrow-Morton no-arbitrage condition)**.** *A model in the HJM framework, i.e. on the form given by* (2.25)*, does not admit arbitrage if there exists a process* $\boldsymbol{\Theta}(t)$ *which uniquely and for all* $0 \leq t \leq T$ *satisfy*

$$\boldsymbol{\sigma}(t,T)^\intercal\boldsymbol{\Theta}(t) = \alpha(t,T)^\intercal - \boldsymbol{\sigma}^*(t,T)^\intercal\boldsymbol{\sigma}(t,T).$$

2.4.2 Risk-neutral dynamics

In the previous section we established the condition needed for a model in the HJM framework to not admit arbitrage. In this section we will assume that it holds and derive the dynamics of the instantaneous forward rate under the risk-neutral measure.

Since theorem 2.4.1 is assumed to be satisfied, we know that an SDE on the following form exists:

$$df(t,T) = \mu^f(t,T)dt + \boldsymbol{\sigma}^f(t,T)^\intercal d\boldsymbol{W}^Q(t),$$

where $\boldsymbol{W}^Q(t)$ is a d-dimensional Brownian motion under the risk-neutral measure. Here the superscript is used to emphasise that the coefficients relate to the instantaneous forward rate, as we will later make a similar ansatz for the ZCB dynamics. Moreover, to simplify notation, the coefficients may implicitly be dependent on $f(t,T)$.

To derive the risk-neutral dynamics for the instantaneous forward rate, or more specifically show that the dynamics are fully determined by the diffusion $\boldsymbol{\sigma}(t,T)$, we will rely on the zero-coupon bond. Because the zero-coupon is a traded asset, we assume the following risk-neutral dynamics:

$$dP(t,T) = P(t,T)\left(r(t)dt + \boldsymbol{\sigma}^P(t,T)^\intercal d\boldsymbol{W}^Q(t)\right), \tag{2.30}$$

where $r(t)$ represents the risk-free rate. Next, we introduce the auxiliary process

$$Y(t) = \ln P(t,T) = -\int_t^T f(t,u)du.$$

By assuming f is nice enough, it is possible to apply Leibniz' integral rule, yielding the differential of this process:

$$\begin{aligned} dY(t) &= f(t,t)dt - \int_t^T df(t,u)du \\ &= r(t)dt - \int_t^T \mu^f(t,u)dtdu - \int_t^T \boldsymbol{\sigma}^f(t,u)^\intercal \boldsymbol{W}^Q(t)du. \end{aligned}$$

Moreover, assuming sufficient conditions are satisfied by the drift and diffusion coefficient we may apply Fubini's stochastic theorem to change the order of integration. This results in

$$dY(t) = \left(r(t) - \int_t^T \mu^f(t,u)du\right)dt - \left(\int_t^T \boldsymbol{\sigma}^f(t,u)du\right)^\intercal d\boldsymbol{W}^Q(t). \tag{2.31}$$

Since $Y(t)$ is the logarithm of the ZCB, we must be able to arrive at the same dynamics via Itô's lemma applied to (2.30):

$$dY(t) = \left(r(t) - \frac{1}{2}\boldsymbol{\sigma}^P(t,T)^\intercal \boldsymbol{\sigma}^P(t,T)\right)dt + \boldsymbol{\sigma}^P(t,T)^\intercal d\boldsymbol{W}^Q(t). \tag{2.32}$$

By matching the drift and diffusion terms in (2.31) with the ones in (2.32) and differentiating with respect to T, we obtain the drift term

$$\mu^f(t,T) = \boldsymbol{\sigma}^f(t,T)^\intercal \int_t^T \boldsymbol{\sigma}^f(t,u)du.$$

We summarise this in below definition, which makes it obvious that the dynamics are fully determined by the diffusion coefficient.

Definition 2.4.1. The risk-neutral dynamics of the instantaneous forward rate, $f(t,T)$, in the *HJM framework*, is given by:

$$df(t,T) = \left(\boldsymbol{\sigma}^f(t,T)^\intercal \int_t^T \boldsymbol{\sigma}^f(t,u)du \right) dt + \boldsymbol{\sigma}^f(t,T) d\boldsymbol{W}^Q(t),$$

where $\boldsymbol{W}^Q(t)$ is a d-dimensional vector of independent Q-Brownian motions.

In general, the process $f(t,T)$ does not adhere to the Markov property. However, by selecting the correct volatility ansatz, it is possible to make it a Markov process. We will briefly discuss one possible ansatz in the context of the extended HJM framework in section 4.2. For a more detailed discussion, we refer to [4] and [1, chapter 12].

3

The LIBOR Market Model

Interest rate derivatives with LIBOR rates as underlying have been prevalent in the markets for decades. Consequently, the LIBOR Market Model (LMM) has been the preferred model for studying many of these derivatives. The model postulates log-normal dynamics for LIBOR rates, leading to convenient pricing formulas for caps and caplets.

In this chapter, we offer a brief overview of the LIBOR Market Model (LMM) and provide motivation for its use. We begin by presenting the necessary theory required to develop the model. Next, we demonstrate its utility by valuing caplets and caps, after which we explore the zero-coupon bond curve and the interpolations required to construct it. Finally, we consider the yield to maturity curve implied by the LMM.

The contents of this chapter are heavily inspired by [12] and [3], the latter of which we refer to for a more nuanced discussion about the LMM and all its intricacies. Moreover, [1] also contains some illuminating discussions about the LMM.

3.1 Theory

In order to introduce the LIBOR Market Model, we require some notation. Let us assume that we are on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$, where Q is the risk-neutral measure. Additionally, let $\mathcal{T} = \{T_0, T_1, \ldots, T_N\}$ represent a set of dates, called the *time grid*, and let $F_k(t)$ denote the simple compounding forward rate at time t with an expiry date of T_{k-1} and a maturity date of T_k (see definition 2.2.14). Consequently, the set of expiry dates becomes $\mathcal{E} = \{T_0, T_1, \ldots T_{N-1}\}$, and the set of maturity dates becomes $\mathcal{M} = \{T_1, T_2, \ldots, T_N\}$. Finally, we define τ_k as the tenor for rate $F_k(t)$, representing the time from expiry to maturity: $\tau_k = T_k - T_{k-1}$.

By rewriting the definition for the simple compounding rate, we have that

$$F_k(t)P(t, T_k) = \frac{P(t, T_{k-1}) - P(t, T_k)}{\tau_k}. \tag{3.1}$$

The right-hand side of (3.1) is a linear combination of the traded assets $P(t, T_{k-1})$ and $P(t, T_k)$. Therefore, since linear combinations of traded assets are also traded assets themselves, the quantity $F_k(t)P(t, T_k)$ represents a traded asset.

Now, let us select $P(\cdot, T_k)$ as the numéraire, then the related measure, Q^{T_k}, is the T_k-forward measure (see definition 2.2.12). Our quantity denoted in this numéraire is, by definition, a martingale under the T_k-forward measure. Therefore, we have that

$$\mathbb{E}^{T_k}\left[\frac{F_k(t)P(t,T_k)}{P(t,T_k)} \mid \mathcal{F}_s\right] = \frac{F_k(s)P(s,T_k)}{P(s,T_k)} = F_k(s), \quad s \leq t.$$

In other words, the simple compounding rate $F_k(t)$ is a martingale under the T_k-forward measure. By the martingale representation theorem, it conveniently follows that there exists an $\mathcal{F}_t$-adapted process $\Gamma_k(t)$ such that

$$dF_k(t) = \Gamma_k(t)dW_k^{T_k}(t),$$

where $\boldsymbol{W}^{T_k}(t) = (W_1^{T_k}(t), W_2^{T_k}(t), \ldots, W_N^{T_k}(t))^T$ is a vector Brownian motion under the T_k-forward measure with instantaneous correlation $\rho_{ij}dt = dW_i^{T_k}(t)dW_j^{T_k}(t)$. Moreover, to enforce log-normal dynamics we set $\Gamma_k(t) = \sigma_k(t)F_k(t)$. Together, this provides the following definition of the LIBOR Market Model.

Definition 3.1.1 (LIBOR Market Model)**.** Let $\{T_0, T_1, \ldots, T_N\}$ be a set of expiry and maturity dates, and let $F_k(t)$ be the simply compounding rate with expiry date T_{k-1} and maturity date T_k. Then the LIBOR market model assumes that the rate's dynamics under the T_k-forward measure follows the log-normal martingale dynamics

$$dF_k(t) = \sigma_k(t)F_k(t)dW_k^{T_k}(t), \tag{3.2}$$

where $\sigma_k(t)$ is a deterministic function, and $W_k^{T_k}(t)$ is a Brownian motion under the T_k-forward measure with instantaneous correlation

$$dW_i^{T_k}(t)dW_j^{T_k}(t) = \rho_{ij}dt.$$

However, when dealing with these rates in practice, the above dynamics are not sufficient. When comparing simulations of multiple rates or performing computations involving multiple rates, it is necessary to express them in the same measure. Therefore, our objective is to fix the measure to a specific T_j-forward measure and utilise the tools developed in section 2.2.6 to express every rate's dynamics in this measure.

There are three cases to consider. The first case has already been considered, namely when $j = k$. The other two cases are for $j > k$ and $j < k$, respectively.

In the second case; begin by fixing $j > k$. Next, notice that Q^{T_k}, for which we have the dynamics, is the measure associated to $P(t, T_k)$ as numéraire, while $Q^{T_{k+1}}$ is the measure associated with $P(t, T_{k+1})$ as numéraire. We can therefore utilise corollary 2.2.6 to yield the T_{k+1}-forward dynamics:

$$dF_k(t) = -\sigma_k(t)F_k(t)\frac{\rho_{k,k+1}\tau_{k+1}\sigma_{k+1}(t)F_{k+1}(t)}{1+\tau_{k+1}F_{k+1}(t)}dt + \sigma_k(t)F_k(t)dW_k^{T_{k+1}}(t). \tag{3.3}$$

To arrive at the T_j-forward dynamics, take (3.3) and add one to the maturity index and derive the new drift. Repeat this procedure until the maturity index is equal to j

We employ the same method in the third case, but with a slight modification. Instead of adding one to the maturity index, we subtract one. This yields a similar formula, with the important distinction that the time at which we express the forward rate, t, should not exceed T_j. The rationale behind this is that T_j represents the maturity of the underlying ZCB, which serves as the numéraire for the measure. As the numéraire is not defined beyond T_j, neither is the associated measure.

The results are collected in the theorem below.

Theorem 3.1.1 (T_j-forward dynamics)**.** *Let $F_k(t)$ be the simply compounding rate with expiry date T_{k-1} and maturity date T_k. The dynamics of $F_k(t)$ under the T_j-forward measure is then, in the LMM, given by:*

$$dF_k(t) = \begin{cases} -\sigma_k(t)F_k(t)\sum_{i=k+1}^{j} \frac{\rho_{ki}\tau_i\sigma_i(t)F_i(t)}{1+\tau_i F_i(t)}dt + \sigma_k(t)F_k(t)dW_k^{T_j}(t), & j > k,\ t \leq T_{k-1}, \\ \sigma_k(t)F_k(t)dW_k^{T_k}(t), & j = k,\ t \leq T_{k-1}, \\ \sigma_k(t)F_k(t)\sum_{i=j+1}^{k} \frac{\rho_{ki}\tau_i\sigma_i(t)F_i(t)}{1+\tau_i F_i(t)}dt + \sigma_k(t)F_k(t)dW_k^{T_j}(t), & j < k,\ t \leq T_j. \end{cases}$$

A subtle, but important, detail that can be observed when carrying out the derivation of (3.3) is that the instantaneous correlations, ρ_{ij}, are measure independent, i.e. we need only specify one correlation matrix, and it will be the same in all measures. This follows directly from Girsanov's theorem.

Consider the Q^{T_i}-Brownian motions $W_i^{T_i}(t)$ and the Q^{T_j}-Brownian motion $W_j^{T_j}(t)$ for $i \neq j$. We then know that the instantaneous correlation can be written as

$$\begin{aligned} dW_i^{T_i}(t)dW_j^{T_j}(t) &= \left([...]dt + dW_i^{T_k}(t)\right)\left([...]dt + dW_i^{T_k}(t)\right) \\ &= dW_i^{T_k}(t)dW_j^{T_k}(t) \\ &= \rho_{ij}dt, \end{aligned}$$

where the last equality follows from the facts that $dW(t)dt = d[W,\cdot](t) = 0$, for any Brownian motion $W(t)$, and $d[\cdot,\cdot](t) = 0$ (see theorem 2.1.2 and theorem 2.1.1).

3.1.1 Valuation of LIBOR caplets and caps

Although the focus of this thesis is not the valuation of interest rate derivatives, we want to show the utility of the LMM in this setting, as it is one of its motivating factors.

With the dynamics of our model defined we are able to price interest rate caplets on simply compounding forward rates. From the definition of a caplet (see definition 2.3.3) we have that its payoff at T_{k-1} is given by

$$\tau_i(F_k(T_{k-1}) - K)^+. \tag{3.4}$$

This payoff is strikingly similar to that of a call option with maturity T_{k-1} on the "stock" F_k struck at K. Therefore, by assuming that F_k adheres to the LMM, and thereby also following log-normal dynamics, its price has a closed form expression given by the Black-Scholes-Merton formula. For a proof of the Black-Scholes-Merton formula see [12, section 5.2.5].

Proposition 3.1.2 (Black caplet formula)**.** *Let $F_k(t)$ be the forward rate with expiry date T_{k-1} and maturity date T_k and assume that it follows the log-normal dynamics in the LIBOR Market Model (see definition 3.1.1). Moreover, consider the T_{k-1} caplet which has payoff*

$$\tau_k(F_k(T_{k-1}) - K)^+,$$

where $\tau_k = T_k - T_{k-1}$*, and* K *is the strike rate of the caplet. Then, the time-*t *Black price for this caplet is given by*

$$\mathbf{Caplet}^{\text{Black}}(t, k, K) = P(t, T_k)(F_k(T_{k-1})\Phi(d_-) - K\Phi(d_+)),$$

where $P(t, T_k)$ *is the time-*t *price of the ZCB with maturity* T_k*,*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz$$

is the cumulative function for the normal distribution, and

$$d_\pm = \frac{1}{\sqrt{\int_0^{T_{k-1}} \sigma_k^2(t)dt}} \left(\log \frac{F_k(0)}{K} \pm \frac{1}{2} \int_0^{T_{k-1}} \sigma_k^2(t)dt \right).$$

Black's caplet formula is easily extended to a caps (see definition 2.3.4) formula. Notice that a cap's payoff is the sum of caplet payoffs with a common strike. Therefore, as a consequence of the linearity of the expectation used by the risk-neutral pricing formula in the proof of the Black-Scholes-Merton formula, the time-t price of a cap with first payment at $T_{\alpha+1}$ and last payment T_β is the sum of time-t prices for the T_α-, $T_{\alpha+1}$-, ..., $T_{\beta-1}$-caplets with a common strike, i.e.

$$\mathbf{Cap}^{\text{Black}}(t, \alpha, \beta, K) = \sum_{i=\alpha+1}^{\beta} \mathbf{Caplet}^{\text{Black}}(t, i, K).$$

3.2 Completing the zero-coupon bond curve

Thus far, our primary focus has been on the forward rate $F_k(t)$ defined in terms of the zero-coupon bond price. However, in this section, we shift our attention to the ZCB itself, as it serves as one of the fundamental building blocks in derivatives pricing.

It is possible to rewrite the relationship between the forward rate and the zero-coupon bond, as given in (3.1), in terms of the forward rates, resulting in

$$P(T_i, T_j) = \prod_{k=i}^{j} \frac{1}{1 + \tau_k F_k(T_i)}, \tag{3.5}$$

where T_i and T_j belong to the time grid $\mathcal{T}$, and $T_i \leq T_j$. This representation highlights that the ZCB can be viewed as the inverse product of the compounded rate of changes dictated by the simply compounding forward rates.

However, there is a caveat with (3.5): it requires both the observation time and the maturity to align with the time grid. Since many derivative pricing calculations require $P(t, T)$ for arbitrary t and T, additional considerations are needed.

Consider the case when $T_{i-1} < t < T_i$. Because t is between two time grid points we would like to compound the ZCB using only a portion of the forward rate that settles at T_{i-1} and matures at T_i. The problem therefore becomes to valuate the forward rate inside its accrual period. Because the LMM does not provide the tools necessary for doing this, an interpolation of $P(t,T)$ in $T \leq T_i$ must be constructed. This interpolation is referred to as the front-stub interpolation and is denoted as $P(t, T_{\eta(t)})$, where

$$\eta(t) = \min\{j : T_j \geq t,\, T_j \in \mathcal{T}\}.$$

Similarly, when the maturity lies between two time grid points yet another interpolation must be constructed. This interpolation is called back-stub interpolation, and is denoted as

$$P(t, T_{\eta(T)-1}, T) = \frac{P(t,T)}{P(t, T_{\eta(T)-1})}.$$

Notice that this expression ought to be interpreted as the discount factor between $T_{\eta(T)-1}$ and T viewed from t. Figure 3.1 visualises how both stubs relates to the whole ZCB curve.

With these interpolations specified it is possible to use the Schlögl ZCB representation, specified in [11], to construct the zero-coupon bond price for arbitrary t and T:

$$P(t,T) = P(t, T_{\eta(t)}) \left(\prod_{j=\eta(t)+1}^{\eta(T)-1} \frac{1}{1 + \tau_j F_j(t)} \right) P(t, T_{\eta(T)-1}, T). \tag{3.6}$$

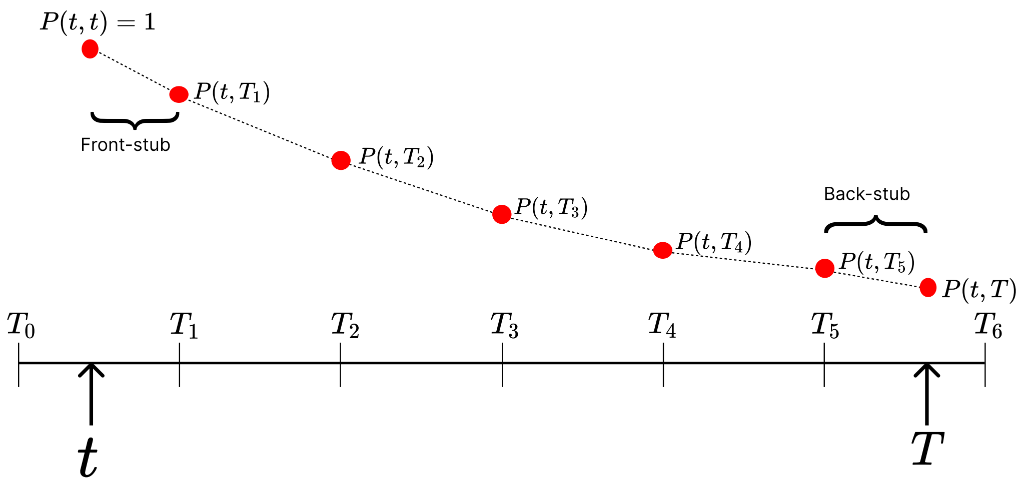

Figure 3.1: Visualisation of (3.6). The front- and back-stub factors can be observed to represent the portions of the ZCB curve that do not align with the time grid.

As mentioned, the LMM does not provide the necessary tools to specify these interpolations, which is one of the significant drawbacks of the model. Since the model does not imply the entire structure of the zero-coupon bond curve, a choice must be made, introducing the potential for undesired behaviour such as arbitrage.

The remainder of this section will discuss some possible choices for interpolations.

3.2.1 Linear back-stub interpolation

The simplest possible interpolation is naturally a linear one matching the values provided by the LMM at the time grid points. Assume that $0 \leq t \leq T_{k-1} \leq T \leq$

T_k, where $k = \eta(T)$, and observe the following limits between which we wish to interpolate:

$$\lim_{T \to T_{k-1}} P(t, T_{k-1}, T) = 1, \tag{3.7}$$

$$\lim_{T \to T_k} P(t, T_{k-1}, T) = (1 + \tau_k F_k(t))^{-1}. \tag{3.8}$$

The interpolation, in terms of T, between these two points is given by:

$$\begin{aligned} P(t, T_{k-1}, T) &= \frac{T_k - T}{T_k - T_{k-1}} P(t, T_{k-1}, T_{k-1}) + \frac{T - T_{k-1}}{T_k - T_{k-1}} P(t, T_{k-1}, T_k) \\ &= \frac{T_k - T}{T_k - T_{k-1}} + \frac{T - T_{k-1}}{T_k - T_{k-1}} (1 + \tau_k F_k(t))^{-1}. \end{aligned}$$

We collect this result in the following definition.

Definition 3.2.1 (Linear back-stub interpolation). Assume that $0 \leq t \leq T_{k-1} \leq T \leq T_k$, where $k = \eta(T)$. Then, the linear back-stub interpolation is given by:

$$\begin{aligned} P(t, T_{k-1}, T) &= \frac{T_k - T}{T_k - T_{k-1}} + \frac{T - T_{k-1}}{T_k - T_{k-1}} P(t, T_{k-1}, T_k) \\ &= \frac{T_k - T}{T_k - T_{k-1}} + \frac{T - T_{k-1}}{T_k - T_{k-1}} (1 + \tau_k F_k(t))^{-1}. \end{aligned} \tag{3.9}$$

This linear interpolation is simple in nature, but it possesses one fatal flaw, it admits arbitrage. To show this we observe the following necessary conditions:

$$P(0, T_{k-1}, T) = \mathbb{E}^{T_{k-1}}[P(t, T_{k-1}, T)], \tag{3.10}$$

$$P(0, T_{k-1}, T_k) = \mathbb{E}^{T_{k-1}}[P(t, T_{k-1}, T_k)]. \tag{3.11}$$

The reasoning behind these conditions is that they relate the initial ZCB curves, $P(0, T_{k-1}, T)$, given by the market, to the ones produced by the interpolation. Therefore, if the interpolation enforces something that is not guaranteed by the market, arbitrage can occur.

Moreover, notice that (3.11) is a necessary no-arbitrage condition for ZCBs maturing on the time grid. In the setting of the LMM this condition is always satisfied because the model builds on the assumption (actually included in the definition in this thesis) that the zero-coupon bond is a traded asset.

By taking expectation of (3.9), we get that condition (3.10) imposes

$$P(0, T_{k-1}, T) = \frac{T_k - T}{T_k - T_{k-1}} + \frac{T - T_{k-1}}{T_k - T_{k-1}} P(0, T_{k-1}, T_k)$$

on the initial ZCB curves $P(0, T)$ and $P(0, T_{k-1})$. These are given by the market, and there is no guarantee they hold true, and thus the model may admit arbitrage.

3.2.2 Arbitrage-free back-stub interpolation

As seen in the previous section, the no-arbitrage conditions impose a relation between the initial ZCB curves provided by the market. By having this in mind, it is possible to select an interpolation that is guaranteed to satisfy these constraints.

Let $\alpha(t,T)$ be a deterministic function of observation time and maturity such that $\alpha(t,T_{k-1}) = 0$ and $\alpha(t,T_k) = 1$, and set

$$P(t,T_{k-1},T) = P(0,T_{k-1},T) + \alpha(t,T)\left(P(t,T_{k-1},T_k) - P(0,T_{k-1},T_k)\right). \quad (3.12)$$

Because the choice of $\alpha(t,T)$ at the time grid points, this expression will line up with (3.7) and (3.8). It will also satisfy (3.10), by construction, as long as (3.11) is respected.

Following the discussion in [1, section 15.1.2], the function $\alpha(t,T)$ can be specified almost arbitrarily. However, to avoid unwanted consequences, e.g. unrealistic ZCB dynamics, one possible choice is via the HJM framework.

Taking the differential of (3.12) yields:

$$\begin{aligned} dP(t,T_{k-1},T) &= [P(t,T_{k-1},T_k) - P(0,T_{k-1},T_k)]d\alpha(t,T) + \alpha(t,T)dP(t,T_{k-1},T_k) \\ &= (\cdots)dt + \alpha(t,T)dP(t,T_{k-1},T_k), \end{aligned}$$

where we have realised that since $\alpha(t,T)$ is deterministic, its differential will only contain a dt-term. Next, by employing the same ZCB dynamics ansatz as in the HJM framework, (2.30), and using Leibniz' stochastic product rule and Itô's lemma, we arrive at

$$\begin{aligned} dP(t,T_{k-1},T) &= P(t,T_{k-1},T)\left[(\cdots)dt + \left(\boldsymbol{\sigma}_P(t,T_{k-1}) - \boldsymbol{\sigma}_P(t,T)\right)^{\intercal} d\boldsymbol{W}^Q(t)\right], \\ dP(t,T_{k-1},T_k) &= P(t,T_{k-1},T_k)\left[(\cdots)dt + \left(\boldsymbol{\sigma}_P(t,T_{k-1}) - \boldsymbol{\sigma}_P(t,T_k)\right)^{\intercal} d\boldsymbol{W}^Q(t)\right]. \end{aligned}$$

Combining the dynamics derived from our interpolation ansatz with the dynamics used in the HJM framework, we arrive at an equality between two SDEs:

$$\begin{aligned} &(\cdots)dt + \alpha(t,T)P(t,T_{k-1},T_k)(\boldsymbol{\sigma}_P(t,T_{k-1}) - \boldsymbol{\sigma}_P(t,T_k))^{\intercal} d\boldsymbol{W}^Q(t) \\ &\quad = (\cdots)dt + P(t,T_{k-1},T)(\boldsymbol{\sigma}_P(t,T_{k-1}) - \boldsymbol{\sigma}_P(t,T))^{\intercal} d\boldsymbol{W}^Q(t). \end{aligned}$$

To determine $\alpha(t,T)$ it suffices to match the diffusion terms. Via Lévy's characterisation theorem this leads to

$$\alpha(t,T)P(t,T_{k-1},T_k)||\boldsymbol{\sigma}(t,T_{k-1}) - \boldsymbol{\sigma}(t,T_k)||_2 = P(t,T_{k-1},T)||\boldsymbol{\sigma}(t,T_{k-1}) - \boldsymbol{\sigma}(t,T)||_2,$$

or equivalently

$$\alpha(t,T) = P(t,T_{k-1},T_k)\frac{||\boldsymbol{\sigma}(t,T_{k-1}) - \boldsymbol{\sigma}(t,T)||_2}{||\boldsymbol{\sigma}(t,T_{k-1}) - \boldsymbol{\sigma}(t,T_k)||_2}.$$

Because $\alpha(t,T)$ is deterministic, the right-hand side must be as well. This can be achieved by freezing the stochastic factors at $t = 0$, which yields the following approximation:

$$\alpha(t,T) \approx P(0,T_{k-1},T_k)\frac{||\boldsymbol{\sigma}(t,T_{k-1}) - \boldsymbol{\sigma}(t,T)||_2}{||\boldsymbol{\sigma}(t,T_{k-1}) - \boldsymbol{\sigma}(t,T_k)||_2}.$$

Notice that the volatilities are assumed to be deterministic. But if they were to be stochastic, freezing them as well is a viable option to arrive at an approximate deterministic expression for $\alpha(t, T)$.

The problem has hereby been changed from interpolating bond prices to interpolating bond volatilities. Because the former, as shown, has the challenge of tampering with the model in such a way that arbitrage may be emitted, the latter is preferred. Reason being that there are almost no constraints on volatilities to ensure an arbitrage-free model.

We collect this result in the following definition.

Definition 3.2.2 (Arbitrage-free back-stub interpolation)**.** Assume that $0 \leq t \leq T_{k-1} < T \leq T_k$, where $k = \eta(T)$. Then, one possible arbitrage-free back-stub interpolation is given by

$$P(t, T_{k-1}, T) = P(0, T_{k-1}, T) + \alpha(t, T) \left(P(t, T_{k-1}, T_k) - P(0, T_{k-1}, T_k) \right).$$

where

$$\alpha(t, T) = P(0, T_{k-1}, T_k) \frac{||\boldsymbol{\sigma}(t, T_{k-1}) - \boldsymbol{\sigma}(t, T)||_2}{||\boldsymbol{\sigma}(t, T_{k-1}) - \boldsymbol{\sigma}(t, T_k)||_2}.$$

3.2.3 Simple front-stub interpolation

Assume $k = \eta(t) = \eta(T)$, and $T_{k-1} \leq t < T \leq T_k$. Then, the front-stub is given by $P(t, T)$. Comparing this setting to that of the back-stub, we notice that the limit corresponding to (3.7), $P(t, t) = 1$, is given. However, the limit corresponding to (3.8), $P(t, T_k)$, is not provided. Therefore, performing a linear interpolation in the same fashion as for the back-stub is not possible.

The second simplest interpolation, after a linear one, is based on a piece-wise constant instantaneous forward rate. Assume that the instantaneous forward rate, $f(t, u) = C$, is constant for $u \in [t, T_{k+1}]$. Then we obtain the following relations:

$$P(t, T_k, T_{k+1}) = (1 + \tau_{k+1} F_{k+1}(t))^{-1} = \exp\left\{ -\int_{T_k}^{T_{k+1}} f(t, u) du \right\} = \exp\left\{ -\tau_{k+1} C \right\}.$$

From this it follows that $C = -\tau_{k+1}^{-1} \ln P(t, T_k, T_{k+1})$. Thus

$$\begin{aligned} P(t, T) &= \exp\left\{ -\int_t^T f(t, u) du \right\} \\ &= \exp\left\{ -(T - t) C \right\} \\ &= P(t, T_k, T_{k+1})^{\frac{T-t}{\tau_{k+1}}} \\ &= (1 + \tau_{k+1} F_{k+1}(t))^{-\frac{T-t}{\tau_{k+1}}}. \end{aligned}$$

Therefore, by simulating the LIBOR rate with maturity T_{k+1} up until t, we are able to interpolate the front-stub $P(t, T)$, for $k = \eta(t) = \eta(T)$. We collect this result in the following definition.

Definition 3.2.3 (Log-linear front-stub interpolation)**.** Assume that $0 \leq T_{k-1} \leq t < T \leq T_k$, where $k = \eta(t) = \eta(T)$. Then, the log-linear front-stub interpolation of the time-t price the of zero-coupon bond with maturity T is given by:

$$\begin{aligned} P(t,T) &= P(t,T_k,T_{k+1})^{\frac{T-t}{\tau_{k+1}}} \\ &= (1+\tau_{k+1}F_{k+1}(t))^{-\frac{T-t}{\tau_{k+1}}} . \end{aligned} \tag{3.13}$$

Much like the linear back-stub interpolation, this interpolation is flawed. Firstly it admits arbitrage. To show this we use the same reasoning that is behind (3.10); if the interpolation imposes some relation on the initial ZCB curves that is not guaranteed by the market, the interpolation admits arbitrage.

Consider front-stubs in the first accrual period, i.e. let $T_0 = 0 \leq t < T \leq T_1$. Moreover, introduce a new time $0 \leq s \leq t$. Then, following the arguments of (3.10)

$$P(0,t,T) = \mathbb{E}^t[P(s,t,T)],$$

imposes a necessary condition on the initial ZCB curves for the interpolation to not admit arbitrage. However, we have that

$$\mathbb{E}^t[P(s,t,T)] = \mathbb{E}^t\left[P(t,T_1,T_2)^{\frac{T-s}{\tau_2}-\frac{t-s}{\tau_2}}\right] = \mathbb{E}^t\left[P(s,T_1,T_2)^{\frac{T-t}{\tau_2}}\right] \neq P(0,t,T),$$

in general.

The second flaw with this interpolation is the constraint it imposes on the volatility. Via Itô's lemma and (2.30) we have the following dynamics:

$$\begin{aligned} d\ln P(t,T) &= (\cdots)dt + \boldsymbol{\sigma}_P(t,T)^\intercal d\boldsymbol{W}^Q(t), \\ d\ln P(t,T_k,T_{k+1}) &= (\cdots)dt + (\boldsymbol{\sigma}_P(t,T_{k+1}) - \boldsymbol{\sigma}_P(t,T_k))^\intercal\, d\boldsymbol{W}(t), \end{aligned}$$

Moreover, from (3.13) we have that these dynamics are related via

$$\begin{aligned} d\ln P(t,T) =& d\left(\frac{T-t}{\tau_{k+1}}\ln P(t,T_k,T_{k+1})\right) \\ =& \frac{T-t}{\tau_{k+1}} d\ln P(t,T_k,T_{k+1}) + \ln P(t,T_k,T_{k+1}) d\left(\frac{T-t}{\tau_{k+1}}\right) \\ &+ d\left(\frac{T-t}{\tau_{k+1}}\right) d\ln P(t,T_k,T_{k+1}), \end{aligned}$$

where the last term is zero. Thus, by matching the diffusion coefficients we get that the interpolation imposes

$$\boldsymbol{\sigma}_P(t,T) = \frac{T-t}{\tau_{k+1}}(\boldsymbol{\sigma}_P(t,T_{k+1}) - \boldsymbol{\sigma}_P(t,T_k)),$$

which is not ensured to be respected by the market at $t = 0$.

It is possible to construct more advanced front-stub interpolations which are better suited for pricing applications. Such interpolations will however not be discussed in this thesis as they require some more preliminaries, but most importantly this simple interpolation suffices for our intents and purposes. The interested reader is referred to [1].

3.3 Implied yield to maturity curve

When equipped with interpolations for both the front- and back-stub, (3.6) constitutes a complete ZCB curve, defined for all $0 \leq t \leq T \leq T_N$. We may therefore turn our focus to the yield to maturity curve, $y(t,T)$, implied by this ZCB curve. From definition 2.2.15 and definition 2.2.14 it follows directly that

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du = -\frac{\log P(t,T)}{T-t}. \tag{3.14}$$

There are many interesting properties of the yield to maturity curve worth studying. Our focus will be on the non-differentiability in maturity at the time grid points.

3.3.1 Non-differentiability

In this section we will provide the conditions necessary for a yield to maturity curve, constructed from the log-linear front-stub interpolation (see definition 3.2.3) and the linear back-stub interpolation (see definition 3.2.1), to be differentiable in maturity. We will furthermore show that these conditions are in general not met.

When differentiating (3.14) there are two different cases to consider.

Case 1 ($T_{k-1} \leq t < T \leq T_k$, $k = \eta(t) = \eta(T)$). In this we will only have a front-stub (see figure 3.1). The ZCB, given by (3.6), in the yield to maturity expression will therefore reduce to only the front-stub $P(t,T)$. This yields the following derivative:

$$\frac{\partial y(t,T)}{\partial T} = \frac{\ln P(t,T)}{(T-t)^2} - \frac{1}{T-t}\frac{\partial}{\partial T}\ln P(t,T). \tag{3.15}$$

As this case only occurs when approaching T_k from the left, we need only consider the left limit of the derivative in (3.15). It follows directly from definition 3.2.3 and is given by:

$$\lim_{T \nearrow T_k} \frac{\partial}{\partial T}\ln P(t,T) = \frac{\ln P(t,T_k,T_{k+1})}{\tau_{k+1}}. \tag{3.16}$$

Together this yields that

$$\lim_{T \nearrow T_k} \frac{\partial y(t,T)}{\partial T} = (T_k - t)^{-2}\ln P(t,T_k) - (T_k - t)^{-1}\frac{\ln P(t,T_k,T_{k+1})}{\tau_{k+1}} = 0, \tag{3.17}$$

where the second equality comes from substituting $P(t,T_k)$ in the first term by the front-stub interpolation.

Case 2 ($t \leq T_{k-1} < T < T_k$, $k = \eta(T)$). In this case we have all three parts of (4.17). The front-stub and middle factor(s) do however have fixed maturities, meaning that their T-derivatives become zero. This results in the following:

$$\begin{aligned}\frac{\partial y(t,T)}{\partial T} =&(T-t)^{-2}\ln P(t,T)\\&-(T-t)^{-1}\frac{\partial}{\partial T}\left(\ln P(t,T_{\eta(t)})-\sum_{i=\eta(t)+1}^{\eta(T)-1}\ln(1+\tau_i R_i(t))+\ln P(t,T_{\eta(T)-1},T)\right)\\=&\frac{\ln P(t,T_{\eta(t)})}{(T-t)^2}-\sum_{i=\eta(t)+1}^{\eta(T)-1}\frac{\ln(1+\tau_i R_i(t))}{(T-t)^2}+\frac{\ln P(t,T_{\eta(T)-1},T)}{(T-t)^2}\\&-\frac{1}{T-t}\frac{\partial}{\partial T}\ln P(t,T_{\eta(T)-1},T).\end{aligned} \tag{3.18}$$

Notice that above expression for the yield to maturity derivative is undefined for T on the time grid points. This follows because at these times, the number of factors in the Schlögl ZCB representation changes, leaving the derivative of the sum undefined. As we will only consider the limits as T approaches these points, this poses no issue for us.

Using definition 3.2.1, the left limit of the derivative in (3.18) becomes

$$\lim_{T\nearrow T_k}\frac{\partial}{\partial T}\ln P(t,T_{k-1},T)=\frac{1}{\tau_k}-\frac{1}{\tau_k P(t,T_{k-1},T_k)}. \tag{3.19}$$

To compute the right limit of the derivative in (3.18) we note that approaching T_k from the right means approaching it from the interval $(T_k,T_{k+1}]$. Therefore, the limit will be that of the derivative of $\ln P(t,T_k,T)$. Again, using definition 3.2.1 yields

$$\lim_{T\searrow T_k}\frac{\partial}{\partial T}\ln P(t,T_k,T)=\frac{P(t,T_k,T_{k+1})}{\tau_{k+1}}-\frac{1}{\tau_{k+1}}. \tag{3.20}$$

By expanding the first term in (3.18) using the Schlögl ZCB representation, (3.6), and combining it with (3.19) we arrive at the following left limit of (3.18):

$$\begin{aligned}\lim_{T\nearrow T_k}\frac{\partial y(t,T)}{\partial T}=&\frac{\ln P(t,T_{\eta(t)})}{(T_k-t)^2}-\sum_{j=\eta(t)+1}^{k}\frac{\ln(1+\tau_j F_j(t))}{(T_k-t)^2}\\&-\frac{1}{\tau_k(T_k-t)}+\frac{1}{\tau_k(T_k-t)P(t,T_{k-1},T_k)}.\end{aligned} \tag{3.21}$$

Next, by the same procedure but using (3.20) instead of (3.19) we arrive at the following right limit of (3.18):

$$\begin{aligned}\lim_{T\searrow T_k}\frac{\partial y(t,T)}{\partial T}=&\frac{\ln P(t,T_{\eta(t)})}{(T_k-t)^2}-\sum_{j=\eta(t)+1}^{k}\frac{\ln(1+\tau_j F_j(t))}{(T_k-t)^2}\\&-\frac{P(t,T_k,T_{k+1})}{\tau_{k+1}(T_k-t)}+\frac{1}{\tau_{k+1}(T_k-t)}.\end{aligned} \tag{3.22}$$

With these limits established we can finally analyse the yield to maturity curve's differentiability. It turns out that the right limit does not coincide with any of the left limits. To convince ourselves of this, consider the differences between the limits.

Beginning with, (3.17) and (3.22), when $T_{k-1} \leq t < T \leq T_k$ and $k = \eta(t) = \eta(T)$:

$$\begin{aligned}\lim_{T \nearrow T_k} \frac{\partial y(t,T)}{\partial T} - \lim_{T \searrow T_k} \frac{\partial y(t,T)}{\partial T} &= -\frac{\ln P(t,T_k)}{(T_k - t)^2} + \frac{P(t,T_k,T_{k+1})}{\tau_{k+1}(T_k - t)} - \frac{1}{\tau_{k+1}(T_k - t)} \\ &= -\frac{\ln P(t,T_k,T_{k+1})}{\tau_{k+1}(T_k - t)} + \frac{P(t,T_k,T_{k+1})}{\tau_{k+1}(T_k - t)} - \frac{1}{\tau_{k+1}(T_k - t)}.\end{aligned}$$

This difference is not guaranteed to be zero, but equating with zero and solving for $P(t,T_k,T_{k+1})$ would give the conditions on the zero-coupon bonds necessary for the yield to maturity derivative to be defined at T_k. However, the only real positive solution to the resulting equation is for $P(t,T_k,T_{k+1}) = 1$, i.e. $P(t,T_k) = P(t,T_{k+1})$. As this generally does not hold, the chosen front- and back-stub interpolations will produce a discontinuity at T_k in the T-derivative of the yield to maturity curve.

Next, in the case when $t \leq T_{k-1} < T < T_k$ and $k = \eta(T)$, the limit difference becomes the difference between (3.21) and (3.22). This works out to be:

$$\begin{aligned}\lim_{T \nearrow T_k} \frac{\partial y(t,T)}{\partial T} - \lim_{T \searrow T_k} \frac{\partial y(t,T)}{\partial T} =& \frac{P(t,T_k,T_{k+1})}{\tau_{k+1}(T_k - t)} + \frac{1}{\tau_k(T_k - t)P(t,T_{k-1},T_k)} \\ &- \frac{1}{\tau_{k+1}(T_k - t)} - \frac{1}{\tau_k(T_k - t)}.\end{aligned}$$

Equating this with zero and using that $P(t,T_i,T_j) = P(t,T_j)/P(t,T_i)$ for $T_i \leq T_j$ yields

$$\tau_{k+1}(P(t,T_{k-1}) - P(t,T_k)) = \tau_k(P(t,T_k) - P(t,T_{k+1})),$$

and is the condition necessary for the T-derivative of the yield to maturity curve to exist at T_k, given the chosen interpolations. This can be seen to not hold in general by taking the differential of the relation and using (2.30).

We have hereby shown that the log-linear front-stub and linear back-stub interpolations produce a yield to maturity curve which is non-differentiable at the time grid points $T_1, T_2, \ldots, T_N$.

4

The Generalised Forward Market Model

The Generalised Forward Market Model (FMM) is an extension of the LIBOR Market Model presented in [9]. The FMM retains many of the analytical features of the LMM, such as the martingale property under the related T-forward measure, while also providing additional benefits. Notably, the model offers a closed analytical form for the forward rates under the risk-neutral measure, eliminating the need for an approximate measure like the spot-LIBOR measure in derivative valuation.

This chapter will commence by introducing the concepts needed to define the FMM, such as the extended zero-coupon bond price and backward-looking forward rate. Subsequently, the chapter will introduce the model and its peculiarities. Following that, we will recapitulate the results of [10], which extends the FMM to fit a Markovian Heath-Jarrow-Morton model (HJM-fitted FMM). Lastly, we will analyse the T-differentiability of the yield to maturity curve implied by the HJM-fitted FMM.

4.1 Theory

4.1.1 The extended zero-coupon bond and related measure

The foundation of the generalised Forward Market Model rests on the concept of a zero-coupon price, which continues to be defined even after the bond's maturity. We call this type of bond an extended zero-coupon bond. Proposition 2.2.7 provides us with the time t price of a zero-coupon bond, which represents the expected value of the accumulated risk-free interest rate over the period from t to the bond's maturity, conditioned on the information available at time t. Mathematically, we express it as

$$P(t,T) = \mathbb{E}^{Q}\left[e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right],$$

where Q denotes the risk-neutral measure.

In our previous discussion of this quantity, we were content with stating that it is undefined for times beyond the bond's maturity, i.e., $t > T$. However, there is no inherent requirement for us to define this behaviour at maturity. By allowing time to surpass the maturity in the above expression, we observe that we are left with something $\mathcal{F}_t$-measurable and closely linked to the money market account. In mathematical terms, for $t > T$, we have that

$$P(t,T) = \mathbb{E}^Q \left[e^{\int_T^t r(u)du} \mid \mathcal{F}_t \right] = e^{\int_T^t r(u)du} = \frac{M(t)}{M(T)}.$$

This leads us to the definition of the extended zero-coupon bond price.

Definition 4.1.1 (Extended zero-coupon bond price)**.** The time-t price of an *extended zero-coupon bond* with maturity T is defined as:

$$P(t,T) = \begin{cases} \mathbb{E}^Q \left[e^{-\int_t^T r(u)du} \mid \mathcal{F}_t \right] & t \leq T, \\ \frac{M(t)}{M(T)} & t > T. \end{cases}$$

In the remainder of this chapter, if not explicitly stated otherwise, $P(t,T)$ will denote the time-t price of the extended zero-coupon bond.

Similarly to the standard ZCB, the extended ZCB is a traded asset. This follows directly from the fact that for $t \leq T$, it is equal to the ZCB, and for $t > T$ its, differential is given by

$$dP(t,T) = \frac{r(t)}{M(T)} dt,$$

which has drift equal to that of the instantaneous risk-free interest rate. Therefore, by definition (see definition 2.2.8) it is a traded asset. Consequently, the extended ZCB is a traded asset for all t.

Since $P(t,T)$ is a traded asset that is strictly positive and does not pay dividends, it also serves as a numéraire (see definition 2.2.9). We refer to the measure associated with this numéraire as the extended T-forward measure, which is defined below.

Definition 4.1.2 (Extended T-forward measure)**.** Let $V(t)$ be a traded asset, and $P(t,T)$ be the time-t price of an extended zero-coupon bond. The *extended T-forward measure* is then defined as the measure Q^T such that $Q^T \sim Q$, and, for $s \leq t$,

$$\mathbb{E}^T \left[\frac{V(t)}{P(t,T)} \mid \mathcal{F}_s \right] = \frac{V(s)}{P(s,T)}.$$

Similar to other measures associated with numéraires, the existence of Q^T is guaranteed in an arbitrage-free market model, as extensively discussed in section 2.2.6. However, it is worth noting that this measure is a hybrid measure since the numéraire is the ordinary ZCB before T and switches to the money market account after maturity.

4.1.2 Backward- and forward-looking forward rates

In chapter 3, we have primarily dealt with forward rates, which are rates that settles at a specified expiry and are paid at a later maturity. In this section, we generalise the concept of forward rates by introducing two types: one that settles at the expiry and another that settles at the maturity.

Let's consider a time grid $\mathcal{T} = \{0 = T_0, T_1, \ldots, T_N\}$ consisting of expiry and maturity dates, where τ_k represents the year fraction for the interval $[T_{k-1}, T_k)$. Using the risk-free rate $r(t)$, we can construct a rate with expiry T_{k-1} and maturity T_k, denoted as $R(T_{k-1}, T_k)$, that accumulates interest linearly over its accrual period $[T_{k-1}, T_k)$, similar to the definition of simply compounding forward rates (see definition 2.2.14). This rate is given by

$$R(T_{k-1}, T_k) = \frac{1}{\tau_k}\left[e^{\int_{T_{k-1}}^{T_k} r(u)du} - 1\right] = \frac{1}{\tau_k}\left[\frac{M(T_k)}{M(T_{k-1})} - 1\right] = \frac{1}{\tau_k}\left[P(T_k, T_{k-1}) - 1\right]. \tag{4.1}$$

The notable aspect of (4.1) is that it aligns with the framework of extended zero-coupon bonds. As demonstrated above, we can express the rate in terms of the time-T_k price of the extended ZCB with maturity T_{k-1}. This implies that $R(T_{k-1}, T_k)$ continues to be a random variable until the maturity of the rate.

This property of the rate continuing to be a random variable until the end of the accrual period is called backward-looking or settle-in-arrears. We will primarily use the former name, motivated by the fact that the rate waits until the end of the period and then "looks back", taking into account the realised evolution of $r(t)$ over the accrual period before determining its value. In contrast, the forward rate we are accustomed to, as modelled by LMM, is referred to as forward-looking. This name derives from the fact that it determines its value at the beginning of the accrual period through a prediction of the future cumulative evolution of $r(t)$ (see proposition 2.2.7).

It is possible to define a simply compounding rate that settles at the beginning of the accrual period, i.e. a forward rate, as the best approximation of the backward-looking rate at expiry. Mathematically, the forward-looking rate with an expiry at T_{k-1} and maturity at T_k, denoted as $F(T_{k-1}, T_k)$, is defined as

$$F(T_{k-1}, T_k) = \mathbb{E}^{T_k}\left[R(T_{k-1}, T_k) \mid \mathcal{F}_{T_{k-1}}\right]. \tag{4.2}$$

As the measures we are working with are equivalent, the above definition can be made in any of the discussed measures. However, we adopt the definition above using the extended T_k-forward measure as it provides a convenient expression for the time-t approximation of both $R(T_{k-1}, T_k)$ and $F(T_{k-1}, T_k)$, as will be demonstrated below.

Note that this definition also coincides with the time-T_{k-1} fair price of an interest rate swap on the backward-looking rate over the single period $(T_{k-1}, T_k]$.

However, both of these rates are random variables, meaning they are unknown until T_{k-1} and T_k, respectively, at which point they are realised. However, the model we intend to construct later relies on random processes describing the rates. To construct random processes from these random variables, we utilise the best time-t approximation of the backward- and forward-looking rates.

Definition 4.1.3 (Backward-looking forward rate)**.** Let $R(T_{k-1}, T_k)$ be the simply compounding backward-looking rate given by (4.1). We then define the *backward-looking forward rate* as the best time-t approximation of this rate, i.e.

$$R_k(t) = \mathbb{E}^{T_k}\left[R(T_{k-1}, T_k) \mid \mathcal{F}_t\right].$$

Definition 4.1.4 (Forward-looking forward rate)**.** Let $F(T_{k-1}, T_k)$ be the simply compounding forward-looking rate given by (4.2). We then define the *forward-looking forward rate* as the best time-t approximation this rate, i.e.

$$F_k(t) = \mathbb{E}^{T_k}\left[F(T_{k-1}, T_k) \mid \mathcal{F}_t\right].$$

Again, notice that this approximation construction coincides with the time-t fair price of single period interest rate swaps on the two rates, respectively.

Both the backward- and forward-looking forward rates have explicit formulas that can be derived by changing to the risk-neutral measure. Starting with the backward-looking rate, we have that

$$\begin{aligned}
R_k(t) &= \mathbb{E}^{T_k}[R(T_{k-1}, T_k) \mid \mathcal{F}_t] \\
&= \frac{1}{D(t)P(t, T_k)}\mathbb{E}^Q[D(T_k)P(T_k, T_k)R(T_{k-1}, T_k) \mid \mathcal{F}_t] \\
&= \frac{1}{\tau_k P(t, T_k)}\mathbb{E}^Q\left[\left(e^{\int_{T_{k-1}}^{T_k} r(u)du} - 1\right) e^{-\int_t^{T_k} r(u)du} \mid \mathcal{F}_t\right] \\
&= \frac{1}{\tau_k P(t, T_k)}\mathbb{E}^Q\left[e^{-\int_t^{T_{k-1}} r(u)du}\right] - \frac{1}{\tau_k} \\
&= \frac{1}{\tau_k}\left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right).
\end{aligned} \tag{4.3}$$

This expression for $R_k(t)$ reveals that it is a martingale under Q^{T_k}, that $R_k(T_k) = R(T_{k-1}, T_k)$, and that the rate is constant for $t > T_k$. The latter is due to the cancellation of the time-varying money market account in the definition of the extended ZCB.

Regarding the forward-looking rate, it can be rewritten as

$$\begin{aligned}
F_k(t) &= \mathbb{E}^{T_k}[F(T_{k-1}, T_k) \mid \mathcal{F}_t] \\
&= \mathbb{E}^{T_k}\left[\mathbb{E}^{T_k}[R(T_{k-1}, T_k) \mid \mathcal{F}_{T_{k-1}}] \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{T_k}[R(T_{k-1}, T_k) \mid \mathcal{F}_t] = R_k(t).
\end{aligned}$$

This clearly demonstrates that by modelling $R_k(t)$, we also obtain a model for $F_k(t)$ "for free". Furthermore, the computation above shows that $R_k(T_{k-1}) = F(T_{k-1}, T_k)$.

4.1.3 The model

In Section 4.1.2, we derived several important characteristics of both the backward- and forward-looking (forward) rates that our model should incorporate. These characteristics are:

1. $R_k(t)$ is a martingale under the extended T_k-forward measure.
2. $R_k(T_{k-1}) = F(T_{k-1}, T_k)$.
3. $R_k(T_k) = R(T_{k-1}, T_k)$.
4. $R_k(t)$ is constant for $t > T_k$.

Characteristic 1 allows us to conveniently select our model by specifying the differential of $R_k(t)$ under the extended T_k-forward measure as a martingale. To satisfy characteristic 4 is simply a matter of multiplying the diffusion term by a function that is zero for $t > T_k$. This function is chosen, in accordance with [9], to be linearly decreasing from one to zero over the accrual period. It's important to note that this choice can be made differently, and the current choice affects the differentiability of the yield to maturity curve, as seen in section 4.4.1. The remaining two characteristics follow directly from the definition of the rates.

Based on these considerations, we specify the following dynamics for the backward-looking forward rate.

Definition 4.1.5 (Generalised Forward Market Model)**.** The differential of the backward-looking forward rate is assumed to be given by

$$dR_k(t) = \sigma_k(t)\gamma_k(t)dW_k^{T_k}(t), \tag{4.4}$$

where $\sigma_k(t)$ is an $\mathcal{F}_t$-adapted process, $W_k^{T_k}(t)$ is a Brownian motion under T_k-forward measure such that $dW_i^{T_k}(t)dW_j^{T_k}(t) = \rho_{ij}dt$, and

$$\gamma_k(t) = \min\left\{\frac{(T_k - t)^+}{T_k - T_{k-1}}, 1\right\}. \tag{4.5}$$

Comparing this model with the LMM presented in definition 3.1.1, we can see that this model does not enforce log-normal dynamics. In fact, the FMM dynamics are more general than the presented LMM dynamics. To enforce log-normal dynamics resembling those of the LMM is however only a matter of selecting $\sigma_k(t) = \widetilde{\sigma}_k(t)R_k(t)$, where $\widetilde{\sigma}_k(t)$ is a deterministic function.

Another significant advantage of the FMM is that one can derive explicit dynamics under the risk-neutral measure. This arises from the fact that the quotient between the numéraires, i.e., the money market account and the extended zero-coupon bond, has an explicit formula, unlike in the LMM setting. We employ the tools developed in section 2.2 to derive the FMM dynamics under the risk-neutral measure in the theorem below.

Theorem 4.1.1 (FMM risk-neutral dynamics)**.** *The risk-neutral dynamics of the backward-looking forward rate is given by*

$$dR_k(t) = \sigma_k(t)\gamma_k(t)\sum_{i=\eta(t)}^{k}\rho_{ik}\frac{\tau_i\sigma_i(t)\gamma_i(t)}{1+\tau_i R_i(t)}dt + \sigma_k(t)\gamma_k(t)dW_k^Q(t), \tag{4.6}$$

where $W_k^Q(t)$ is a Q-Brownian motion such that $dW_i^Q(t)dW_j^Q(t) = \rho_{ij}dt$, and

$$\eta(t) = \min\{j : T_j \geq t,\, T_j \in \mathcal{T}\}.$$

Proof. We will simply employ the result in corollary 2.2.6 to directly arrive at the above-stated differential. According to this result, we want to replace the Brownian motion in (4.4) with

$$dW_k^{T_k}(t) = d\left[W_k^{T_k}, \log(M/P(\cdot, T_k))\right](t) + dW_k^Q(t),$$

where $dW_k^Q(t)$ is a Brownian motion under the risk-neutral measure.

Let us first simplify the logarithm of the numéraire quotient

$$\ln \frac{M(t)}{P(t,T_k)} = \ln \frac{P(t,0)}{P(t,T_k)} = \ln \prod_{i=1}^{k} \frac{P(t,T_{i-1})}{P(t,T_i)} = \sum_{i=1}^{k} \ln\left(1+\tau_i R_i(t)\right),$$

where the first equality utilises the relation between the extended ZCB and the money market account. The second and third equality arise from it being a telescoping product and the definition of the extended ZCB, respectively.

This means that the dynamics specified by (4.4) becomes, under the risk-neutral measure,

$$\begin{aligned} dR_k(t) &= \sigma_k(t)\gamma_k(t) dW_k^{T_k}(t) \\ &= \sigma_k(t)\gamma_k(t)\left(d\left[W_k^{T_k}, \ln(M/P(\cdot,T_k))\right](t) + dW_k^Q(t)\right) \\ &= \sigma_k(t)\gamma_k(t)\left(dW_k^{T_k}(t)\sum_{i=1}^{k} d\ln\left(1+\tau_i R_i(t)\right) + dW_k^Q(t)\right) \\ &= \sigma_k(t)\gamma_k(t)\left(\sum_{i=1}^{k} \frac{\tau_i}{1+\tau_i R_i(t)} dW_k^{T_k}(t) dR_i(t) + dW_k^Q(t)\right) \\ &= \sigma_k(t)\gamma_k(t)\sum_{i=1}^{k} \rho_{ik}\frac{\tau_i\sigma_i(t)\gamma_i(t)}{1+\tau_i R_i(t)} dt + \sigma_k(t)\gamma_k(t) dW_k^Q. \end{aligned}$$

The final step is to recognise that $\gamma_i(t) = 0$ for $i \leq \eta(t)$, which yields the sought after dynamics. □

By replicating the steps in the above proof, it is possible to derive the FMM dynamics under, for example, the T_j-forward measure or the spot-LIBOR measure, which can be useful for valuing derivatives. However, the valuation of derivatives is not the focus of this thesis, and interested readers are referred to [9].

The dynamics defined by (4.6) rely on one driving noise per forward rate. However, to combine the FMM with an extended version of the HJM framework later in this chapter, each forward rate must be driven by a vector Brownian motion with the identity matrix as the correlation matrix. Fortunately, it is possible to rewrite the above dynamics in such a form.

Let $\boldsymbol{C}_i$ be the i:th row of the N-dimensional matrix C, such that $\rho_{ij} = \boldsymbol{C}_i\boldsymbol{C}_j^\intercal$. The existence of C follows from Cholesky decomposition and the fact that correlation matrices are positive semi-definite. Moreover, let $\widetilde{\boldsymbol{W}}(t) = (\widetilde{W}_1(t), \dots, \widetilde{W}_N(t))$, where the correlation matrix is given by the identity matrix. Then, we can write

$$(W_1^Q(t), \dots, W_N^Q(t)) = C\widetilde{\boldsymbol{W}}(t),$$

where $W_1^Q(t), \dots, W_N^Q(t)$ are the Brownian motions from theorem 4.1.1. By introducing the N-dimensional vector process

$$\boldsymbol{\sigma}_k^R(t) = \sigma_k(t)\boldsymbol{C}_k$$

if follows that

$$\boldsymbol{\sigma}_i^R(t)^\intercal \boldsymbol{\sigma}_j^R(t) = \sigma_i(t)\boldsymbol{C}_i^\intercal\boldsymbol{C}_j\sigma_j(t) = \rho_{ij}\sigma_i(t)\sigma_j(t). \tag{4.7}$$

Hence, (4.6) can be reformulated as

$$dR_k(t) = \gamma_k(t)\boldsymbol{\sigma}_k^R(t) \sum_{i=\eta(t)}^{k} \boldsymbol{\sigma}_i^R(t) \frac{\tau_i \gamma_i(t)}{1+\tau_i R_i(t)} dt + \gamma_k(t)\boldsymbol{\sigma}_k^R(t) d\widetilde{\boldsymbol{W}}(t). \tag{4.8}$$

4.2 The extended HJM framework

In section 2.4 we introduced the well known HJM framework for modelling the instantaneous forward rate. In this section we intend to extend this framework to be applicable to the extended zero-coupon bond.

The derivation and motivation behind the drift condition that ensures the arbitrage-free nature of the HJM framework relied on the fact that the ordinary ZCB is, by definition, a traded asset. As discussed, our extended ZCB is also a traded asset. However, due to the fact that it is defined for times after its maturity, the dynamics change.

Consider the definition of the extended ZCB together with the definition of the continuously compounding forward rate. This yields the following relation:

$$P(t,T) = \exp\left\{\int_T^t r(u)du\right\} = \exp\left\{\int_T^t f(t,u)du\right\}, \quad t > T.$$

For our continuously compounding forward rate to be consistent with this relation it must hold that $r(u) = f(u,u) = f(t,u)$ for $u \in [T,t]$. This implies that the value of $f(t,u)$ must be set at u, leading to the following dynamics ansatz:

$$df(t,u) = 1_{\{t \le u\}} \left[\mu^f(t,u)dt + \boldsymbol{\sigma}^f(t,u)^\intercal d\boldsymbol{W}^Q(t)\right], \tag{4.9}$$

where $\mu^f(t,T)$ is a 1-dimensional $\mathcal{F}_T$-adapted process with possible $f(t,u)$ dependence being implicit, $\boldsymbol{\sigma}^f(t,T)$ is an N-dimensional $\mathcal{F}_T$-adapted process with possible $f(t,u)$ dependence being implicit, and $\boldsymbol{W}^Q(t)$ is an N-dimensional Q-Brownian motion, with the identity matrix as correlation matrix. The latter means that the correlation structure for the driving noise must be embedded into the volatility vector. The method for doing this relies on Cholesky decomposition of the correlation matrix and is discussed at the end of section 4.1.3.

Next, let's consider the dynamics of the extended zero-coupon bond. Since it evolves as the money market account after maturity, its dynamics must take the form

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + 1_{\{t \le T\}}\boldsymbol{\sigma}^P(t,T)^\intercal d\boldsymbol{W}^Q(t), \tag{4.10}$$

where $\boldsymbol{\sigma}^P(t,T)$ is an N-dimensional $\mathcal{F}_T$-adapted process with possible $P(t,T)$ dependence being implicit, and $\boldsymbol{W}^Q(t)$ is the same Brownian motion as in (4.9). The indicator function ensures that the volatility is zero for $t > T$, which means that the extended zero-coupon bond becomes deterministic after maturity, in accordance with its definition.

By utilising Leibniz' integral rule, Fubini's theorem, and Itô's lemma in the same way as in section 2.4 we arrive at the following relations:

$$\mu^f(t,T) = 1_{\{t\leq T\}}\boldsymbol{\sigma}^f(t,T)^\intercal \int_t^T \boldsymbol{\sigma}^f(t,u)du, \tag{4.11}$$

$$\boldsymbol{\sigma}^P(t,T) = \int_t^T 1_{\{t\leq u\}}\boldsymbol{\sigma}^f(t,u)du, \tag{4.12}$$

which allow us to define the arbitrage-free risk-neutral dynamics of the instantaneous forward rate.

Definition 4.2.1. The arbitrage-free risk-neutral dynamics of the instantaneous forward rate, $f(t,T)$, in the *extended HJM framework*, are given by:

$$df(t,T) = 1_{\{t\leq T\}}\left[\left(\boldsymbol{\sigma}^f(t,T)^\intercal \int_t^T \boldsymbol{\sigma}^f(t,u)du\right)dt + \boldsymbol{\sigma}^f(t,T)^\intercal d\boldsymbol{W}^Q(t)\right],$$

where $\mu^f(t,T)$ is a 1-dimensional $\mathcal{F}_T$-adapted, $\boldsymbol{\sigma}^f(t,T)$ is an N-dimensional $\mathcal{F}_T$-adapted process, and $\boldsymbol{W}^Q(t)$ is an N-dimensional Q-Brownian motion, with the identity matrix as correlation matrix.

Akin to the ordinary HJM framework, the process $f(t,T)$ becomes Markov by selecting the correct volatility ansatz. One such possible choice is considered in below theorem.

Theorem 4.2.1. *Let the instantaneous forward rate volatility have the following separable structure:*

$$\begin{aligned}\boldsymbol{\sigma}^f(t,T) &= \sum_{k=1}^N \boldsymbol{\zeta}_k(t)g_k(T)1_{\{T\in(T_{k-1},T_k]\}}\\ &= \boldsymbol{\zeta}_{\eta(T)}(t)g_{\eta(T)}(T),\end{aligned}$$

where N number of elements in the time set grid, $\boldsymbol{\zeta}_k(t)$ is an N-dimensional $\mathcal{F}_t$-adapted process, and $g_k(T)$ is a deterministic scalar function. Then,

$$f(t,T) = \begin{cases} f(0,T) + \boldsymbol{g}(T)^\intercal\boldsymbol{X}(t) + \boldsymbol{g}(T)^\intercal Y(t)\boldsymbol{G}(t,T), & t<T,\\ r(T) = f(0,T) + \boldsymbol{g}(T)^\intercal \boldsymbol{X}(T), & t\geq T,\end{cases} \tag{4.13}$$

where

$$\begin{aligned} d\boldsymbol{X}(t) &= Y(t)\boldsymbol{g}(t)dt + \zeta(t)d\boldsymbol{W}^Q(t), & \boldsymbol{X}(0)=0,\\ dY(t) &= \zeta(t)^\intercal\zeta(t)dt, & Y(0)=0,\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{g}(T) &= (g_1(T),\ldots,g_N(T))^\intercal,\\ \boldsymbol{G}(t,T) &= (G_1(t,T),\ldots,G_N(t,T))^\intercal,\\ \zeta(t) &= (\boldsymbol{\zeta}_1(t)^\intercal,\ldots,\boldsymbol{\zeta}_N(t)^\intercal),\end{aligned}$$

where $G_k(t,T) = \int_t^T g_k(u)du$. Moreover, $f(t,T)$ is Markov in $\boldsymbol{X}(t)$ and $Y(t)$.

The proof of this theorem can be found in [10, appendix A]. It mainly contains straight forward computations to show that the integrated forward rate dynamics indeed satisfy (4.13). However, when (4.13) is established, it can be directly seen from the formulation of theorem 4.2.1 that $f(t,T)$ is Markov in $\boldsymbol{X}(t)$ and $Y(t)$. The state variables $\boldsymbol{X}(t)$ and $Y(t)$ form a Markov process as neither of them are dependent on anything before t. Moreover, because $\boldsymbol{g}$, and therefore also $\boldsymbol{G}$, is deterministic, $f(t,T)$ becomes Markov in $\boldsymbol{X}(t)$ and $Y(t)$.

4.3 Completing the extended zero-coupon bond curve

Section 3.2 explains why the LMM only provides the zero-coupon bond price for t and T on the time gird. Similarly, in the current context, we face a similar situation. However, instead of relying on an arbitrary interpolation method, we can employ an alternative approach—the extended HJM framework.

In this section, we will first align the extended HJM dynamics for the instantaneous forward rate with the dynamics postulated by the FMM, resulting in a model for the instantaneous forward rate called the HJM-fitted FMM. This model is defined for all positive t and T, and encompasses the entire extended ZCB curve, thereby providing us with the back- and front-stub formulas without any assumptions other than those made in the FMM and the extended HJM framework.

4.3.1 Fitting the model to the extended HJM framework

The idea behind utilising the extended HJM framework to yield back- and front-stub interpolations hinges on the fact that the forward rate we are modelling is defined for all times, especially inside the accrual periods. It allows for the possibility of connecting the evolution of the simple forward rate, dictated by the FMM, in the accrual period with the evolution of the instantaneous forward rate, dictated by the HJM framework. Thereby creating a HJM model, defined for all times and maturities, which implies the FMM dynamics. This is also the reason why this type of interpolation is not possible in the setting of the LMM, because there the modelled forward rate becomes undefined for times after expiry, and there is thus no evolution of the simple forward rate to be matched with the instantaneous forward rate inside the accrual period.

Following procedures similar to those below, it is however possible to define a HJM model for the instantaneous forward rate which match, and thereby also implies, the LMM. But again, this model will only be defined up until expiry, and it therefore provides no interpolations.

To connect the FMM with the extended HJM framework we take the differential of the definition of $R_k(t)$ and combine it with (4.10). This yields the risk-neutral dynamics of the backward-looking forward rate in the extended HJM framework:

$$
\begin{aligned}
dR_k(t) = &\left(R_k(t) + \frac{1}{\tau_k}\right) \left(\int_t^{T_k} 1_{\{t \le u\}} \boldsymbol{\sigma}^f(t,u)du\right)^{\intercal} \left(\int_t^{T_k} 1_{\{t \le u\}} \boldsymbol{\sigma}^f(t,u)du\right) dt \\
&+ \tau_k P(t, T_{k-1}) P(t, T_k) \left(\int_t^{T_{k-1}} 1_{\{t \le u\}} \boldsymbol{\sigma}^f(t,u)du\right)^{\intercal} \left(\int_t^{T_k} 1_{\{t \le u\}} \boldsymbol{\sigma}^f(t,u)du\right) dt \\
&+ \left(R_k(t) + \frac{1}{\tau_k}\right) \left(\int_{T_{k-1}}^{T_k} 1_{\{t \le u\}} \boldsymbol{\sigma}^f(t,u)du\right)^{T} d\boldsymbol{W}^Q(t).
\end{aligned}
\tag{4.14}
$$

To make this extended HJM model align with the FMM it suffices to match the diffusion terms in (4.14) and (4.8), i.e.

$$
\begin{aligned}
\gamma_k(t)\boldsymbol{\sigma}_k^R(t) &= \left(R_k(t) + \frac{1}{\tau_k}\right) \left(\int_{T_{k-1}}^{T_k} 1_{\{t \le u\}} \boldsymbol{\sigma}_k^f(t,u)du\right)^{\intercal} \\
&= \left(R_k(t) + \frac{1}{\tau_k}\right) \left(\int_{T_{k-1} \vee t}^{T_k \vee t} \boldsymbol{\sigma}_k^f(t,u)du\right)^{\intercal}.
\end{aligned}
\tag{4.15}
$$

Moreover, to make the underlying HJM model Markovian we utilise theorem 4.2.1 by assuming the following separable structure for the volatility:

$$
\begin{aligned}
\boldsymbol{\sigma}^f(t,T) &= \sum_{i=1}^{N} \boldsymbol{\zeta}_i(t) g_i(T) 1_{T \in (T_{k-1}, T_k]} \\
&= \boldsymbol{\zeta}_{\eta(T)}(t) g_{\eta(T)}(T),
\end{aligned}
$$

where $\boldsymbol{\zeta}_i(t)$ is an N-dimensional $\mathcal{F}_t$-adaptable process and $g_i(T)$ is a deterministic function, for each $i = 1, \dots, N$.

Now, by introducing

$$
G_i(t,T) = \int_t^T g_i(u)du,
$$

the integral over the instantaneous forward rate volatility can be written as

$$
\int_{T_{k-1} \vee t}^{T_k \vee t} \boldsymbol{\sigma}^f(t,T) = \boldsymbol{\zeta}_k(t) \int_{T_{k-1} \vee t}^{T_k \vee t} g_k(u) = \boldsymbol{\zeta}_k(t) G_k(T_{k-1} \vee t, T_k \vee t).
$$

Substituting this into (4.15) yields

$$
\gamma_k(t)\boldsymbol{\sigma}_k^R(t) = \left(R_k(t) + \frac{1}{\tau_k}\right) G_k(T_{k-1} \vee t, T_k \vee t) \boldsymbol{\zeta}_k(t).
$$

To simplify our expression we, without loss of generality, assume that $G_k(T_{k-1}, T_k) = 1$. Now, by matching the deterministic and stochastic factors we conclude that

$$
\begin{aligned}
\gamma_k(t) &= G_k(T_{k-1} \vee t, T_k \vee t), \\
\boldsymbol{\sigma}_k^R(t) &= \left(R_k(t) + \frac{1}{\tau_k}\right) \boldsymbol{\zeta}_k(t).
\end{aligned}
$$

By solving for the components in the separable volatility structure we get

$$\begin{aligned} g_k(t) &= -\frac{d}{dt}\gamma_k(t), \quad t \in (T_{k-1}, T_k), \\ \boldsymbol{\zeta}_k(t) &= \frac{1}{R_k(t) + \frac{1}{\tau_k}}\boldsymbol{\sigma}_k^R(t). \end{aligned} \tag{4.16}$$

This entails that, by selecting the instantaneous forward rate volatility in accordance with (4.16), the implied forward rate dynamics will coincide with the dynamics defined by the FMM. Combining this with theorem 4.2.1 results in below proposition.

Proposition 4.3.1 (HJM-fitted FMM)**.** *Let $f(t,T)$ satisfy theorem 4.2.1 and choose the volatility in accordance with* (4.16)*. Then, the integrated forward dynamics will imply definition 4.1.5 and are given by*

$$f(t,T) = f(0,T) - \sum_{k=1}^{N} X_k(t)\frac{d}{dT}\gamma_k(T) + \sum_{k,h=1}^{N} Y_{k,h}(t)[\gamma_k(T) - \gamma_k(t)]\frac{d}{dT}\gamma_k(T),$$

where

$$\begin{aligned} dX_k(t) &= -\sum_{h=1}^{N} Y_{k,h}(t)d\gamma_k(t) + \frac{1}{R_k(t) + \frac{1}{\tau_k}}\boldsymbol{\sigma}_k^R(t)^\intercal d\boldsymbol{W}^Q(t), \\ dY_{k,h}(t) &= \frac{1}{\left(R_k(t) + \frac{1}{\tau_k}\right)\left(R_h(t) + \frac{1}{\tau_h}\right)}\boldsymbol{\sigma}_k^R(t)^\intercal \boldsymbol{\sigma}_h^R(t)dt. \end{aligned}$$

4.3.2 Implied extended zero-coupon bond curve

After the work we thus far have done, the integrated extended zero-coupon bond dynamics follow easily. By combining definition 2.2.13 and (4.13) we directly get the result in the below proposition.

Proposition 4.3.2. *Using the same notation as in theorem 4.2.1, the extended zero-coupon bond price is, for any non-negative t and T, given by*

$$P(t,T) = \begin{cases} P(0,t,T)\exp\left\{-\boldsymbol{G}(t,T)^\intercal \boldsymbol{X}(t) - \frac{1}{2}\boldsymbol{G}(t,T)^\intercal Y(t)\boldsymbol{G}(t,T)\right\}, & t < T, \\ \exp\left\{\int_t^T r(u)du\right\}, & t \geq T, \end{cases}$$

where $P(s,t,T) = P(s,T)/P(s,t)$ is the forward discount factor between t and T observed at time s.

By the same reasoning as for the (4.13), the above zero-coupon bond price process is Markov in $\boldsymbol{X}(t)$ and $Y(t)$.

The formulation of the extended ZCB curve presented in proposition 4.3.2 is however cumbersome to use in a practical setting where we rely on simulations. The reason being that in order to compute $P(t,T)$ for every $0 < t < T \leq T_N$, where T_N is the last element in the time grid $\mathcal{T}$, one need to simulate

- $Y_{k,h}(t)$ for each $k, h = 1, \dots, N$, up until time T_h,

- and, $X_k(t)$ for each $k = 1, \dots, N$ up until time T_k.

The number of state variable to simulate thus grows quadratically with the number of forward rates, R_k. We therefore seek for a more efficient way of computing $P(t,T)$.

To find a more computationally efficient expression for the extended ZCB we remind us of the Schlögl ZCB representation, (3.6), described in section 3.2. Because $R_k(t)$ is defined much in the same way as the forward-looking forward rate we are concerned with in chapter 3, this representation is just as valid for the extended zero-coupon bond. Using $R_k(t)$ it becomes:

$$P(t,T) = P(t,T_{\eta(t)}) \left(\prod_{j=\eta(t)+1}^{\eta(T)-1} \frac{1}{1+\tau_j R_j(t)} \right) P(t,T_{\eta(T)-1},T). \tag{4.17}$$

Since the central factor(s) can be computed directly from the FMM-simulated forward rates, we only need to deploy proposition 4.3.2 in the stubs. This will make the number of state variables grow linearly as the dependence of $X_k(t)$ on the off-diagonals, $Y_{h,k}(t)$, $h \neq k$, will be eliminated when $k = \eta(t)$.

This more efficient way of computing the extended ZCB curve does however come at a cost, the Markov property. As will be seen in section 4.3.4 the front-stub interpolation becomes path-dependent.

4.3.3 The back-stub interpolation

Proposition 4.3.3 (Back-stub interpolation)**.** *Let $t \leq T_{k-1} < T \leq T_k$. The back-stub formula for the forward extended ZCB, $P(t,T_{\eta(T)-1},T) = P(t,T_{k-1},T)$, is then given by*

$$\begin{aligned} P(t,T_{k-1},T) =& P(0,T_{k-1},T)(1+\tau_k R_k(t))^{-G_k(T_{k-1},T)} P(0,T_{k-1},T_k)^{-G_k(T_{k-1},T)} \\ & \cdot \exp\left\{ \frac{1}{2} G_k(T_{k-1},T) G_k(T,T_k) Y_{k,k}(t) \right\}. \end{aligned} \tag{4.18}$$

Proof. Let $t \leq T_{k-1}$ where $k = \eta(T)$, then we are sure of there being a back-stub.

Using proposition 4.3.2 and the additivity of $\boldsymbol{G}(t,T)$ in its arguments, we can write

$$\begin{aligned}
P(t,T_{k-1},T) =& \frac{P(t,T)}{P(t,T_{k-1})} \\
=& P(0,T_{k-1},T_k)\exp\Big\{-\boldsymbol{G}(T_{k-1},T)^\intercal X(t) - \frac{1}{2}\boldsymbol{G}(t,T)^\intercal Y(t)\boldsymbol{G}(t,T) \\
&+ \frac{1}{2}\boldsymbol{G}(t,T_{k-1})^\intercal Y(t)\boldsymbol{G}(t,T_{k-1})\Big\} \\
=& P(0,T_{k-1},T_k)\exp\Big\{-\boldsymbol{G}(T_{k-1},T_k)^\intercal \boldsymbol{X}(t) \\
&- \frac{1}{2}(\boldsymbol{G}(t,T_{k-1}) + \boldsymbol{G}(T_{k-1},T))^\intercal Y(t)(\boldsymbol{G}(t,T_{k-1}) + \boldsymbol{G}(T_{k-1},T)) \\
&+ \frac{1}{2}\boldsymbol{G}(T_{k-1},T)^\intercal Y(t)\boldsymbol{G}(t,T_{k-1})\Big\} \\
=& P(0,T_{k-1},T_k)\exp\Big\{-\boldsymbol{G}(T_{k-1},T)^\intercal[\boldsymbol{X}(t) + Y(t)\boldsymbol{G}(t,T_{k-1})] \\
&- \frac{1}{2}\boldsymbol{G}(T_{k-1},T)^\intercal Y(t)\boldsymbol{G}(T_{k-1},T)\Big\},
\end{aligned}$$

where we have used that $\boldsymbol{G}(t,T_{k-1})^\intercal Y(t)\boldsymbol{G}_(T_{k-1},T) = \boldsymbol{G}(T_{k-1},T)^\intercal Y(t)\boldsymbol{G}(t,T_{k-1})$ stemming from the fact that $Y(t)$ is symmetrical. We can simplify this expression by realising that the all components, except at index $k = \eta(T)$, of $\boldsymbol{G}(T_{k-1},T)$ are zero because $T_{k-1} \leq T \leq T_k$. Therefore, we can drop the vector notation yielding

$$P(t,T_{k-1},T) = P(0,T_{k-1},T)\exp\left\{-Z_k(t)G_k(T_{k-1},T) - \frac{1}{2}Y_{k,k}(t)G_k^2(t,T_{k-1})\right\}, \tag{4.19}$$

with the auxiliary process $Z_k(t)$ defined as

$$Z_k(t) = X_k(t) + \sum_{i=\eta(t)}^{k-1} Y_{k,i}(t)G_i(t,T_{k-1}).$$

The last step is finding an alternative formulation for $Z_k(t)$. By letting $T = T_k$ we get

$$P(t,T_{k-1},T_k) = \frac{1}{1+\tau_k R_k(t)} = P(0,T_{k-1},T_k)\exp\left\{-Z_k(t) - \frac{1}{2}Y_{k,k}(t)\right\},$$

where we have used our assumption that $G(T_{k-1},T_k) = 1$. Solving for the auxiliary process yields

$$Z_k(t) = \ln(1+\tau_k R_k(t)) + \ln P(0,T_{k-1},T_k) - \frac{1}{2}Y_{k,k}(t).$$

Substituting this into (4.19) results in the sought after back-stub formula. □

4.3.4 The front-stub interpolation

For us to be able to establish an explicit expression for the front-stub we first need to do some preparatory leg work. Theorem 4.2.1 tells us how the instantaneous forward rate at time t differs from the initial curve $f(0,T)$. However, to derive the front-stub interpolation we are interested in the more general case: how $f(t,T)$ differs from the same quantity but evaluated at a time different from 0.

Let $0 \leq S \leq t \leq T$, then by utilising (4.13) we get the following difference:

$$\begin{aligned} f(t,T) - f(S,T) =& \boldsymbol{g}(T)^\intercal[\boldsymbol{X}(t) - \boldsymbol{X}(S)] + \boldsymbol{g}(T)^\intercal[Y(t)\boldsymbol{G}(t,T) - Y(S)\boldsymbol{G}(S,T)] \\ =& \boldsymbol{g}(T)^\intercal[\boldsymbol{X}(t) - \boldsymbol{X}(S)] + \boldsymbol{g}(T)^\intercal[Y(t)\boldsymbol{G}(t,T) - Y(S)\boldsymbol{G}(t,T)] \\ &+ \boldsymbol{g}(T)^\intercal[Y(S)\boldsymbol{G}(t,T) - Y(S)\boldsymbol{G}(S,T)] \\ =& \boldsymbol{g}(T)^\intercal[\boldsymbol{X}(t) - \boldsymbol{X}(S) - Y(S)\boldsymbol{G}(S,t)] \\ &+ \boldsymbol{g}(T)^\intercal[Y(t) - Y(S)]\boldsymbol{G}(t,T) \\ =& \boldsymbol{g}(T)^\intercal\boldsymbol{X}^S(t) + \boldsymbol{g}(T)^\intercal Y^S(t)\boldsymbol{G}(t,T), \end{aligned} \tag{4.20}$$

where we have used the additivity of $\boldsymbol{G}(t,T)$ and introduced

$$\begin{aligned} \boldsymbol{X}^S(t) &= \boldsymbol{X}(t) - \boldsymbol{X}(S) - Y(S)\boldsymbol{G}(S,t), \\ Y^S(t) &= Y(t) - Y(S). \end{aligned}$$

Now, when comparing (4.20) to (4.13) we notice that they have the same form, only introducing an S-dependence in the former case. This means that we can express the time-t price of an extended zero-coupon bond in terms of the time-S curve instead of the time-0 curve as:

$$P^S(t,T) = P(S,t,T)\exp\left\{-\boldsymbol{G}(t,T)^\intercal\boldsymbol{X}^S(t) - \frac{1}{2}\boldsymbol{G}(t,T)^\intercal Y^S(t)\boldsymbol{G}(t,T)\right\},$$

for $S \leq t \leq T$.

To tie this to the front-stub interpolation we consider the case when $\eta(t) = \eta(T) = k$ and $S = T_{k-1}$, i.e. the case when we are ensured to have a front-stub. Then, by simple substitution we get the following instantaneous forward rate and extended ZCB formulas:

$$\begin{aligned} f(t,T) &= f(T_{k-1},T) + g_k(T)x_k(t) + g_k(T)y_k(t)G_k(t,T), \\ P(t,T) &= P(T_{k-1},t,T)\exp\left\{-G_k(t,T)x_k(t) - \frac{1}{2}G_k^2(t,T)y_k(t)\right\}, \end{aligned} \tag{4.21}$$

where we have introduced

$$\begin{aligned} x_k(t) &= X_k^{T_{k-1}}(t) = X_k(t) - X_k(T_{k-1}) - G_k(T_{k-1},t)Y_{k,k}(T_{k-1}), \\ y_k(t) &= Y_k^{T_{k-1}}(t) = Y_{k,k}(t) - Y_{k,k}(T_{k-1}), \end{aligned}$$

for $T_{k-1} \leq t \leq T_k$. This means that the scalar processes $x_k(t)$ and $y_k(t)$ are only defined locally on the intervals between the time grid points $[T_{k-1}, T_k]$. Moreover, their dynamics are given by

$$\begin{aligned} dx_k(t) &= g_k(t)y_k(t)dt + \boldsymbol{\zeta}_k(t)^\intercal d\boldsymbol{W}^Q(t) \\ &= g_k(t)y_k(t)dt + \frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}}dW_k(t), \\ dy_k(t) &= \boldsymbol{\zeta}_k(t)^\intercal\boldsymbol{\zeta}_k(t)dt \\ &= \left(\frac{\sigma_k(t)}{R_k(t) + \frac{1}{\tau_k}}\right)^2 dt, \end{aligned}$$

with $x_k(T_{k-1}) = y_k(T_{k-1}) = 0$. Notice that we in these scalar processes have, via (4.7), reverted to the scalar volatility used in the definition of the FMM (see definition 3.1.1).

The front-stub formula is now only a matter of computing $P(T_{k-1}, t, T)$ using the back-stub formula and plugging it into (4.21). We collect this result as below proposition.

Proposition 4.3.4 (Front-stub interpolation)**.** *Let $T_{k-1} \leq t < T \leq T_k$. The front-stub formula for the ZCB $P(t,T)$ is then given by*

$$\begin{aligned} P(t,T) =& P(0,t,T)P(0,T_{k-1},T_k)^{-G_k(t,T)}(1+\tau_k R_k(T_{k-1}))^{-G_k(t,T)} \exp\Big\{ - G_k(t,T)x_k(t) \\ & - \frac{1}{2}G_k^2(t,T)y_k(t) + \frac{1}{2}Y_{k,k}(T_{k-1})[G_k(T,T_k)G_k(T_{k-1},T) - G_k(t,T_k)G_k(T_{k-1},t)]\Big\}. \end{aligned} \tag{4.22}$$

It can be directly observed that the front-stub interpolation is not Markov. When $T_{k-1} < t$, (4.22) depends on the forward rate $R_k(T_{k-1})$ and the state variable $Y_{k,k}(T_{k-1})$, thus making it path-dependent.

4.4 Implied yield to maturity curve

As already discussed, one of the advantages provided by the FMM-fitted HJM is that it directly implies the value of a ZCB for any maturity and at any time. Due to the relation between the ZCB value and the yield to maturity curve, the FMM-fitted HJM also provides an expression for the yield to maturity curve for any $(t,T) \in \mathbb{R}^2_{\geq 0}$. From definition 2.2.15 and definition 2.2.13 it follows directly that

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du = -\frac{\log P(t,T)}{T-t}. \tag{4.23}$$

Apart from providing a closed formula for the yield to maturity curve, the model also provides a way to simulate the yield to maturity curve. By using the simulation scheme for $P(t,T)$ presented in [10] the yield to maturity curve can efficiently be computed via (4.23) and (3.1). In figure 4.1 and 4.2 two different realisations of such simulated yield to maturity curves are presented. Note that these curves are sample paths where the underlying rates have been simulated with log-normal dynamics under the risk-neutral measure with diag(1) as instantaneous correlation matrix, and instantaneous volatility given by $\sigma_k(t) = 0.5$. Therefore, the characteristics of these presented curves can be very much influenced by the randomness in the underlying rates.

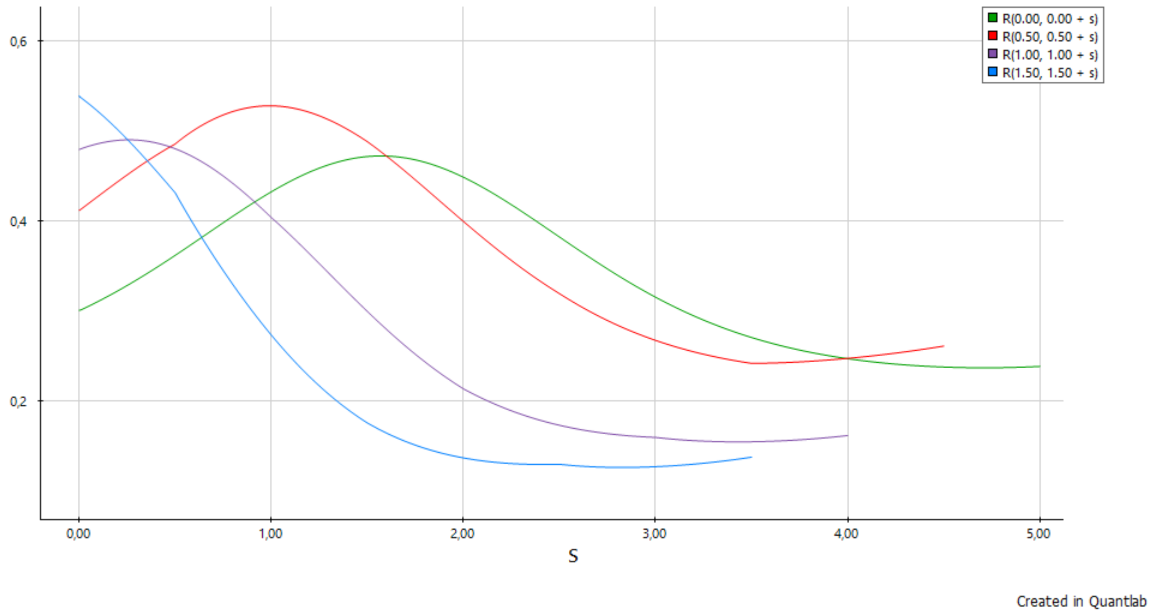

Figure 4.1: Yield curves at different observation times $R_t(s)$, $t \in \{0, 1/2, 1, 3/2\}$. The extended zero-coupon bonds used to compute the curves have been simulated according to the scheme presented in [10], which relies on the Schlögl ZCB representation, (4.17). Moreover, the underlying forward rates $R_k(t)$ have been simulated under the log-normal formulation of theorem 4.1.1 with uncorrelated driving noise and instantaneous volatility $\sigma_k(t) = 0.5$.

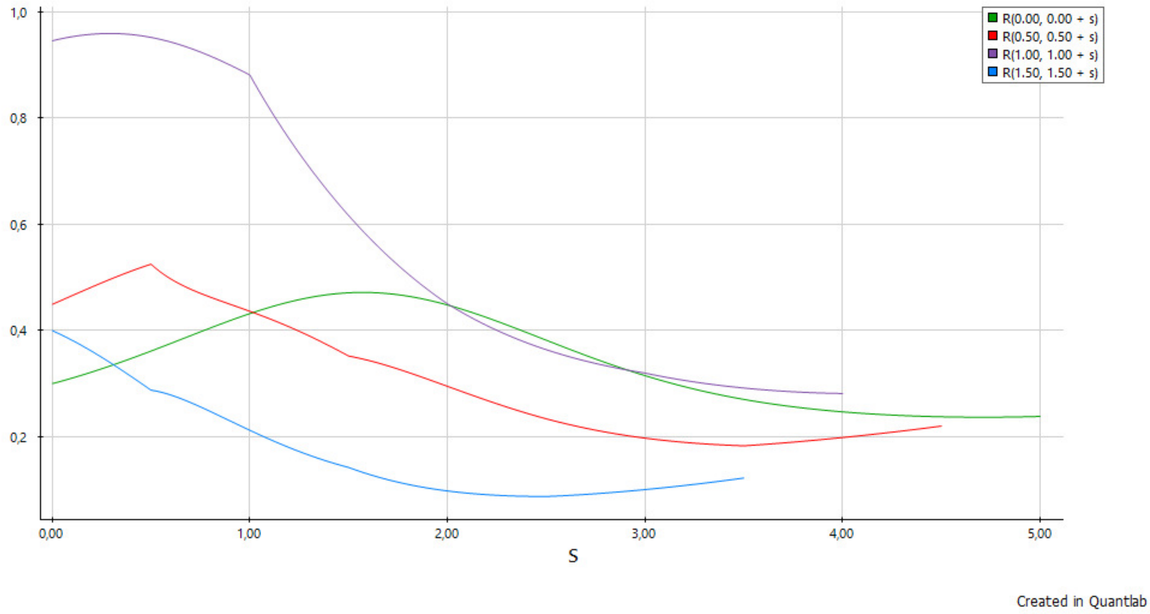

Figure 4.2: Yield curves at different observation times $R_t(s)$, $t \in \{0, 1/2, 1, 3/2\}$. The extended zero-coupon bonds used to compute the curves have been simulated according to the scheme presented in [10], which relies on the Schlögl ZCB representation, (4.17). Moreover, the underlying forward rates $R_k(t)$ have been simulated under the log-normal formulation of theorem 4.1.1 with uncorrelated driving noise and instantaneous volatility $\sigma_k(t) = 0.5$.

With the randomness' influence on the characteristics of the curves in mind, there is still one specific behaviour which stands out, namely that the curves in figure 4.2 seem to have points, coinciding with the time grid points $\{0, T_1, T_2, \ldots, T_N\}$, where they are non-differentiable.

4.4.1 Non-differentiability

In this section we will provide the conditions necessary for the yield to maturity curve, constructed from the Schlögl ZCB representation and the front- and back-stub interpolations given by the HJM-fitted FMM, to be differentiable in maturity. We will subsequently show that these conditions are not guaranteed to hold.

To investigate the property we differentiate the yield to maturity curve and derive its left and right limits as T approaches T_k.

When differentiating (4.23) there are two different cases to consider.

Case 1 ($T_{k-1} \leq t < T \leq T_k$, $k = \eta(t) = \eta(T)$). In this case we will only have a front-stub (see figure 3.1). The ZCB, given by (4.17), in the yield to maturity expression will therefore only reduce to the front-stub $P(t, T)$. This yields the following derivative:

$$\frac{\partial y(t,T)}{\partial T} = \frac{\ln P(t,T)}{(T-t)^2} - \frac{1}{T-t}\frac{\partial}{\partial T} \ln P(t,T), \tag{4.24}$$

where the derivative of the logarithm of the front-stub follows directly from (4.22), and is given by

$$\begin{aligned}\frac{\partial \ln P(t,T)}{\partial T} &= -f(0,T) + [\ln P(0,T_{k-1},T_k) + \ln(1+\tau_k R_k(T_{k-1})) + x_k(t)] \frac{d\gamma_k(T)}{dT} \\ &+\frac{1}{2} Y_{k,k}(T_{k-1})[G_k(T_{k-1},T) - G_k(T,T_k)] \frac{d\gamma_k(T)}{dT} + G_k(t,T) y_k(t) \frac{d\gamma_k(T)}{dT}.\end{aligned} \tag{4.25}$$

Since the front-stub will only be present when approaching T_k from the left, only the left limit needs to be considered. By recognising that

$$\lim_{T\nearrow T_k} \frac{d\gamma_k(T)}{dT} = -\frac{1}{T_k - T_{k-1}} = -\frac{1}{\tau_k},$$

the left limit of (4.25) simplifies to

$$\begin{aligned}\lim_{T\nearrow T_k} \frac{\partial \ln P(t,T)}{\partial T} &= -f(0,T_k) - \frac{1}{\tau_k}[\ln P(0,T_{k-1},T_k) + \ln(1+\tau_k R_k(T_{k-1})) + x_k(t)] \\ &-\frac{1}{\tau_k}\left[\frac{1}{2} Y_{k,k}(T_{k-1}) + \gamma_k(t) y_k(t)\right].\end{aligned}$$

Now, by combining the above limit with (4.24) and performing some simplifications, we arrive at one of our sough after results. The left limit of the T-derivative of the yield to maturity curve, in the case when $T_{k-1} \le t < T \le T_k$ and $k = \eta(t) = \eta(T)$, is given by:

$$\lim_{T\nearrow T_k} \frac{\partial y(t,T)}{\partial T} = \frac{\ln P(0,t,T_k)}{(T_k - t)^2} + \frac{f(0,T_k)}{T_k - t} + \frac{y_k(t)}{2\tau_k^2} + \frac{Y_{k,k}(T_{k-1})}{2\tau_k}. \tag{4.26}$$

Case 2 ($t \le T_{k-1} < T < T_k$, $k = \eta(T)$). In this case we have all three parts of (4.17). The front-stub and middle factor(s) do however have fixed maturities, meaning that their T-derivatives become zero. This results in the following:

$$\begin{aligned}\frac{\partial y(t,T)}{\partial T} =&(T-t)^{-2} \ln P(t,T) \\ &- (T-t)^{-1} \frac{\partial}{\partial T}\left(\ln P(t,T_{\eta(t)}) - \sum_{i=\eta(t)+1}^{\eta(T)-1} \ln(1+\tau_i R_i(t)) + \ln P(t,T_{\eta(T)-1},T)\right) \\ =&\frac{\ln P(t,T_{\eta(t)})}{(T-t)^2} - \sum_{i=\eta(t)+1}^{\eta(T)-1} \frac{\ln(1+\tau_i R_i(t))}{(T-t)^2} + \frac{\ln P(t,T_{\eta(T)-1},T)}{(T-t)^2} \\ &- \frac{1}{T-t}\frac{\partial}{\partial T} \ln P(t,T_{\eta(T)-1},T),\end{aligned} \tag{4.27}$$

where the derivative in the last term follows directly from (4.18), and is given by

$$\begin{aligned}\frac{\partial \ln P(t,T_{k-1},T)}{\partial T} &= -f(0,T) + [\ln(1+\tau_k R_k(t)) + \ln P(0,T_{k-1},T_k)] \frac{d\gamma_k(T)}{dT} \\ &+\frac{1}{2} Y_{k,k}(t)[G_k(T_{k-1},T) - G_k(T,T_k)] \frac{d\gamma_k(T)}{dT}.\end{aligned} \tag{4.28}$$

Moreover, notice that above expression for the yield to maturity derivative is undefined for T on the time grid points. This follows because at these times, the number of factors in the Schlögl ZCB representation changes, leaving the derivative of the sum undefined. As we will only consider the limits as T approaches these points, this poses no issue for us.

The left limit of (4.28) simplifies

$$\lim_{T\nearrow T_k} \frac{\partial \ln P(t,T_{k-1},T)}{\partial T} = -f(0,T_k) - \frac{1}{\tau_k}\left[\ln(1+\tau_k R_k(t)) + \ln P(0,T_{k-1},T_k) + \frac{1}{2}Y_{k,k}(t)\right].$$

To compute the right limit we note that approaching T_k from the right means approaching it from the interval $(T_k, T_{k+1}]$. Therefore, the limit will be that of the derivative of $\ln P(t,T_k,T)$. With this in mind, and by recognising that

$$\lim_{T\searrow T_k} \frac{d\gamma_{k+1}(T)}{dT} = -\frac{1}{T_{k+1}-T_k} = -\frac{1}{\tau_{k+1}},$$

the right limit of (4.28) becomes

$$\begin{aligned}\lim_{T\searrow T_k} \frac{\partial \ln P(t,T_k,T)}{\partial T} =& -f(0,T_k) - \frac{1}{\tau_{k+1}}\left[\ln(1+\tau_{k+1}R_{k+1}(t)) + \ln P(0,T_k,T_{k+1})\right] \\ &+ \frac{1}{2\tau_{k+1}}Y_{k+1,k+1}(t).\end{aligned}$$

Now, by combining (4.27) with above limits and performing some simplifications, we get the two remaining sought after results. The left and right limits of the T-derivative of the yield to maturity curve, in the case when $t \le T_{k-1} < T < T_k$ and $k = \eta(T)$, are given by:

$$\begin{aligned}\lim_{T\nearrow T_k} \frac{\partial y(t,T)}{\partial T} =& \frac{\ln P(t,T_{\eta(t)})}{(T_k-t)^2} - \sum_{j=\eta(t)+1}^{k} \frac{\ln(1+\tau_j R_j(t))}{(T_k-t)^2} + \frac{f(0,T_k)}{T_k-t} \\ &+ \frac{1}{\tau_k(T_k-t)}\left[\ln(1+\tau_k R_k(t)) + \ln P(0,T_{k-1},T_k) + \frac{1}{2}Y_{k,k}(t)\right],\end{aligned} \tag{4.29}$$

$$\begin{aligned}\lim_{T\searrow T_k} \frac{\partial y(t,T)}{\partial T} =& \frac{\ln P(t,T_{\eta(t)})}{(T_k-t)^2} - \sum_{j=\eta(t)+1}^{k} \frac{\ln(1+\tau_j R_j(t))}{(T_k-t)^2} + \frac{f(0,T_k)}{T_k-t} \\ &+ \frac{1}{\tau_{k+1}(T_k-t)}\left[\ln(1+\tau_{k+1} R_{k+1}(t)) + \ln P(0,T_k,T_{k+1}) - \frac{1}{2}Y_{k+1,k+1}(t)\right].\end{aligned} \tag{4.30}$$

To tie the derivative limits back to the characteristics seen in figure 4.2 we begin by observing that when $T_{k-1} \le t < T \le T_k$ and $k = \eta(t) = \eta(T)$, the difference between (4.26) and (4.30) becomes:

$$\begin{aligned}\lim_{T\nearrow T_k} \frac{\partial y(t,T)}{\partial T} - \lim_{T\searrow T_k} \frac{\partial y(t,T)}{\partial T} =& \frac{1}{T_k-t}\left[\frac{\ln(1+\tau_k R_k(t))}{\tau_k} - \frac{\ln(1+\tau_{k+1}R_{k+1}(t))}{\tau_{k+1}}\right] \\ &+ \frac{1}{T_k-t}\left[\frac{\ln P(0,T_{k-1},T_k)}{\tau_k} - \frac{\ln P(0,T_k,T_{k+1}))}{\tau_{k+1}}\right] \\ &+ \frac{1}{T_k-t}\left[\frac{Y_{k+1,k+1}(t)}{2\tau_{k+1}} + \frac{Y_{k,k}(T_{k-1})}{2\tau_k} + \frac{x_k(t)}{\tau_k}\right] + \frac{y_k(t)}{\tau_k^2}.\end{aligned}$$

By equating this with zero, using that $P(t,T_i,T_j) = (1+\tau_j R_j(t))^{-1}$, and by rearranging we get that

$$\frac{1}{\tau_{k+1}} \ln \frac{P(t,T_k,T_{k+1})}{P(0,T_k,T_{k+1})} + \frac{1}{\tau_k} \ln \frac{P(t,T_{k-1},T_k)}{P(0,T_{k-1},T_k)} = \left[\frac{Y_{k+1,k+1}(t)}{2\tau_{k+1}} + \frac{Y_{k,k}(T_{k-1})}{2\tau_k} + \frac{x_k(t)}{\tau_k} \right] + (T_k - t)\frac{y_k(t)}{\tau_k^2}. \tag{4.31}$$

If this relation holds true, the derivative will be continuous in this case. However, in figure 4.2 simulations of the curve $R_t(s) = y(t, t+s)$ clearly suggests, due to its sharp corners, that the yield to maturity curve is non-differentiable in T_1, and therefore, that (4.31) does not hold in general.

In the second case, when $t \leq T_{k-1} < T < T_k$ and $k = \eta(T)$, the corresponding difference is between (4.29) and (4.30). This is given by

$$\lim_{T \nearrow T_k} \frac{\partial y(t,T)}{\partial T} - \lim_{T \searrow T_k} \frac{\partial y(t,T)}{\partial T} = \frac{1}{T_k - t} \left[\frac{\ln(1+\tau_k R_k(t))}{\tau_k} - \frac{\ln(1+\tau_{k+1} R_{k+1}(t))}{\tau_{k+1}} \right] + \frac{1}{T_k - t} \left[\frac{\ln P(0,T_{k-1},T_k)}{\tau_k} - \frac{\ln P(0,T_k,T_{k+1})}{\tau_{k+1}} \right] + \frac{1}{2(T_k - t)} \left[\frac{Y_{k,k}(t)}{\tau_k} + \frac{Y_{k+1,k+1}(t)}{\tau_{k+1}} \right],$$

and mutatis mutandis we get the following constraint for differentiability at the time grid points:

$$\frac{1}{\tau_{k+1}} \ln \frac{P(t,T_k,T_{k+1})}{P(0,T_k,T_{k+1})} + \frac{1}{\tau_k} \ln \frac{P(t,T_{k-1},T_k)}{P(0,T_{k-1},T_k)} = \frac{1}{2} \left[\frac{Y_{k,k}(t)}{\tau_k} + \frac{Y_{k+1,k+1}(t)}{\tau_{k+1}} \right]. \tag{4.32}$$

Again, via the same argument as for (4.31) we conclude from figure 4.2 that (4.32) is not satisfied in general.

We have hereby used the simulations, that made us question whether the yield to maturity curve had a continuous derivative, to show that it indeed does not. Furthermore, (4.31) and (4.32) show that the non-differentiability of the curve at the time grid points is something that is closely tied to the interpolation chosen for the front- and back-stub.

5

Discussion and future research

5.1 Discussion

The LIBOR has, to varying success, been in use since the 1980s in everything from sophisticated financial derivatives to bank loans. During recent years, there has, however, been a growing concern about the manipulation of the LIBOR. This has resulted in the decision to phase out the LIBOR, with the last US-dollar LIBOR published on 30th June 2023. The replacing rates are overnight rates, which must be converted into term rates. To do this, a compounded setting-in-arrears method is used, leaving the replacing term rates backward-looking, i.e., both settling and paying at maturity. Because the term rates modelled by the LIBOR Market Model are undefined after expiry, a new model is required to describe the replacing rates.

In this thesis, we have considered the LIBOR Market Model and one model capable of replacing it, namely the generalised Forward Market Model, first proposed in [9]. As for the LIBOR Market Model, we introduced the theory necessary to define it. Moreover, we stated its dynamics under the T_j-forward measure and thereafter demonstrated how the ansatz of log-normal dynamics for the LIBOR results in a Black-Scholes-Merton-like formula for valuing caplets. The last topic concerning the LIBOR Market Model is the zero-coupon bond curve and the yield to maturity curve it implies. We developed three different interpolations, of which the log-linear front-stub interpolation and the linear back-stub interpolation was were to derive conditions necessary for a continuous T-derivative of the yield to maturity curve. These conditions were then shown to not be respected by our model in the general case.

Armed with the theory behind the LIBOR Market Model, we extend it and introduce the generalised Forward Market Model. Particularly, we define the extended zero-coupon bond, which allows us to construct both forward- and backward-looking rates. The former exactly coincides with the rate modelled by the LIBOR Market Model, and the latter possesses all the desired properties of the rates replacing the LIBOR. Then, following the approach of [10], we match an HJM model to the FMM. Since the extended zero-coupon bond is defined after its maturity, the HJM-fitted Forward Market Model enables us to directly derive the dynamics for the entire extended zero-coupon curve. However, the expression for this curve is computationally cumbersome, leading us to utilise the Schlögl ZCB representation presented in [11]. This drastically improves the computational efficiency of simulating the ZCB curve as the number of state variables needed to simulate goes from increasing quadratically to increasing linearly with the number of forward rates.

Finaly we, similarly to the chapter about the LIBOR Market Model, focus on the yield to maturity curve. Akin to the analysis of this curve in the LMM setting, we derive conditions necessary for the T-derivative of the curve to be differentiable at the time grid points. We note that these conditions are closely related to the fitted HJM model and use simulations to show that they are generally not met. We do however recognise that these simulation results could be made more rigorous, instead of relying solely on the observable characteristics of the curves in figure 4.2. As the left and right limits of $\frac{\partial y(t,T)}{\partial T}$ can be computed explicitly, this could be done by ensuring that their differences are non-zero at the time grid points.

5.2 Future work

This thesis focuses on only a small part of the yield to maturity curve implied by the Heath-Jarrow-Morton-fitted Forward Market Model, and there are many questions left unanswered.

Firstly, we have seen that the non-differentiability of the yield to maturity curve is influenced by the choice of front- and back-stub interpolations. This begs the question; is it possible to choose interpolations such that differentiability is guaranteed?

Next, because yield curves observed in the market are generally not differentiable in maturity it is essential to investigate whether the property of non-differentiability of the yield to maturity curve holds practical significance. Can real-world scenarios be constructed where this property poses an actual issue?

Furthermore, while we have considered the differentiability of the yield to maturity curve implied by the HJM-fitted FMM directly, it is of interest to examine its behaviour under expectation. We have preliminary results utilising simulations to suggest that the curve is non-differentiable under expectation as well. Can this be proven analytically, and does the front- and back-stub interpolation play a role in this case?

Lastly, in [1], they use the yield to maturity curve to showcase the possible shapes that the yield curve can take. Do these shapes coincide with the ones that can be produced by the HJM-fitted Forward Market Model?

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