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# Pricing American Put options using the Binomial Model and the Finite Difference Method

MVEX03

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# A comparison between two numerical methods for solving the free boundary problem of pricing American put options

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# Abstract

In this thesis we have compared two numerical methods for pricing an American put option, the binomial model and a finite difference scheme. The two methods have been compared with regard to accuracy and computational speed. Thus, the best model is the one with the highest computational speed, which at the same time generates accurate results. We compare their exercise boundaries, see how they are affected by the amount of iterations and grid size, and analyze the theory behind the two methods. Since option pricing depends on market parameters, the methods have been compared multiple times, for different market parameters to obtain a complete comparison for multiple aspects. The thesis also includes the derivation of the binomial model algorithm as well as the theorem and a sketch of the proof for the boundary value problem of the partial differential equation used to price American put options.

# Acknowledgements

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Filip Johansson, Gothenburg, June 2021

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# List of Symbols

$S$ - Stock price
$S(t)$ - Stock price at time $t$
$h_S$ - Number of shares in the stock
$h_B$ - Numbers of shares in the risk-free asset
$K$ - Strike price
$\Pi_Y$ - Price of European option
$\hat{\Pi}_Y$ - Price of American option
$Y(t)$ - Pay-off for the option
$\sigma$ - volatility or instantaneous volatility
$\mathcal{F}$ - $\sigma$-algebra filtration
$T$ - Maturity
$r$ - Risk-free rate
$e^r$ - Continuously compounded rate
$V(h_B(t), h_S(t))$ - Portfolio Value
$\tau$ - Stopping time
$\mathcal{T}$ - Family of stopping times
$\mathcal{A}$ - Family of self-replicating and predictable portfolio strategies
$g(S(t))$ - Pay-off function
$\alpha$ - Instantanous mean of log return
$p$ - Physical (or real world) probability
$h$ - Step length
$v(t, x)$ - Price function for the Black-Scholes formula

# 1
# Introduction

## 1.1 Options

An option is a financial derivative that is based on a underlying asset, such as a stock, $S(t)$. The option constitutes of a contract between two entities, a buyer and a seller, which gives the buyer the opportunity to buy the stock for a specific price called strike price $K$, if the option is a call option. If the option is a put option, the buyer of the option is provided with the possibility to sell the stock at a specific price, strike price $K$. If this happen, the option is said to be exercised. It is clear that the buyer of the option holds a long position of the option (hoping that it will increase in value), while the seller holds a short position on the option (hoping that it will decrease in value). The difference between the stock price and the strike price is called pay-off, denoted $Y(t)$. Since options are based on one or multiple underlying asset, the options behavior are highly correlated with the behavior of the underlying asset but they are not the same. To get an option, the buyer pays a premium, denoted $\hat{\Pi}$ for American options in this thesis and $\Pi$ for European options. In this respect options are not free but they come with the advantage that they may be less risky than stocks in some situations and offer strategic alternatives when investing. Examples of this is that they are stable against big gap openings on the stock market, since options do not shut down when the market closes, which is the case for stocks. For the same reason, owning an option for the potential of a huge increase or decrease in value is a less risky strategy than owning the stock, where the option could be sold or exercised if it is in the money. But if the underlying asset should make a big move in the opposite of the desired direction, the investor will only have paid the price of the premium, but not suffered from the movement of the underlying asset. Another typical usage of call and put option is to use them for hedging, a type of insurance to offset losses. If an investor is long a stock then having a put option on the same stock will be an insurance that the loss won't be greater than the premium of the put option. The pay-off $Y(t)$ is also called the intrinsic value of the option and it is defined differently depending on the type of option.

$$\begin{aligned} \textbf{Call option:} \quad & Y(t) = (S(t) - K)_+ \\ \textbf{Put option:} \quad & Y(t) = (K - S(t))_+ \end{aligned} \tag{1.1}$$

The plus sign indicates that the parenthesis is always equal to or greater than zero, an option does not have a negative pay-off. For a call option the theoretical gain

is unlimited while for a put option it is limited by the value of the strike price $K$. When $Y(t) > 0$ the option is said to be in the money, hence $S(t) > K$ for call options and $S(t) < K$ for put options. When this occurs it is advantageous for the investor to either sell or exercise the option, but can also chose to do nothing and just keep the option.
The decision whether to sell or to exercise the option depends on which alternative is more profitable, if $\hat{\Pi}_Y(t) > Y(t)$ the profit of selling the option is greater than for exercising it, hence selling the option is the obvious choice. Most of the times, it is more profitable to sell an option instead of exercising it.

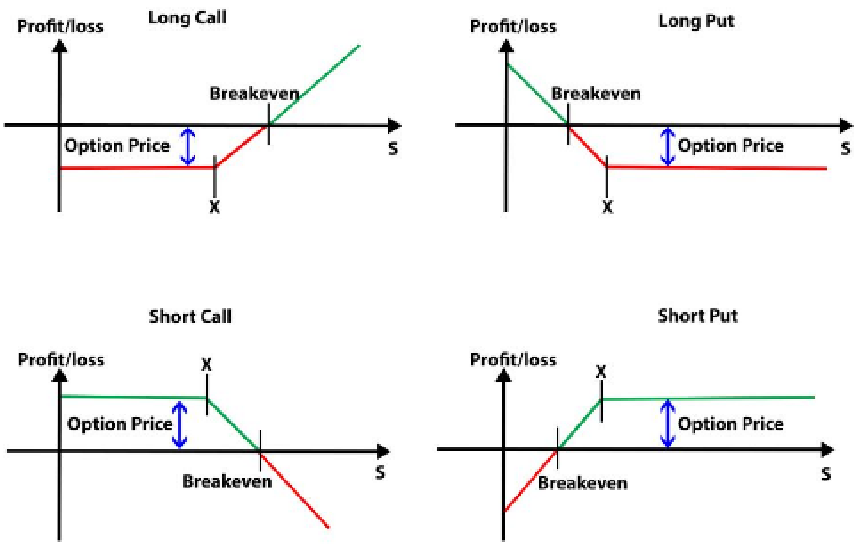



Figure 1.1: Pay-off for long and short positions on a call and put option [1].

### 1.1.1 American options

There are many types of options; European, American, Asian, Barrier options etc. But the focal point of this thesis is the American put option. There will be some reference to European options and therefore the difference should be explained. A European option, both call and put, can only be exercised at a specific time, called maturity, denoted $T$. Maturity is the expiration date of the option and is specified beforehand. After maturity, the option can no longer be exercised, which is the case for all types of options. Since the time in which the derivative can be exercised is known for the European derivative, the price depends only on the pay-off at this time, $T$. This gives rise to the fact that a closed form solution exists, the famous Black-Scholes formula, which is a solution of the partial differential equation modeling the dynamics of the financial market that have derivative investment instruments. In contrast to European options, American options can be exercised at any time, both prior to and including maturity. This extra benefit comes with the cost of a higher premium. The fact that the option can be exercised at any time prior to maturity, gives rise to a free-boundary problem, for which there exist no closed form solution. A free-boundary problem is a partial differential equation with both unknown function and unknown domain. For the American option, the unknown function is the pricing function, which as stated above, is solved by Black-Scholes formula for European options and the unknown domain is the time instant in which to exercise the option. For an American call option that does not pay dividend, it is never optimal to exercise prior to maturity, since the premium price is always greater than the pay-off; $\hat{\Pi}_Y(t) > Y(t)$. Thus, it is advantageous to sell the option instead of exercising it, since it generates more money. Hence, Black-Scholes formula can be used to price them as well. But for American call options that pay dividend and American put options this is not the case, hence Black-Scholes formula can not be used and we are instead dependent on using numerical methods for solving the problem. Specifically, the premium $\hat{\Pi}_Y(t)$ at all times $t$ needs to be calculated using numerical methods. Most options traded on the markets are American, hence giving a fair correct price and avoiding arbitrage opportunities clearly has important practical implications.

## 1.2 Highlander Downtown North AB

Highlander Downtown North AB is a financial start-up company, which is in the process of developing a system solution for global management of institutional assets that handles today's requirements for real-time updated data, using Kafka confluent. Confluent Platform is a full-scale event streaming platform that enables you to easily access, store, and manage data as continuous, real-time streams. Today at the capital market, relational database are used, which are not suited for real-time updated data. Financial derivatives are part of their product catalog. It is of crucial importance that the financial derivatives are correctly calculated, preferably with a high computational speed as well. Therefore, the coding performed in this thesis should offer a reliable and fast computational method for Highlander.

## 1.3 Problem formulation

Since many traded stocks and commodity options in today's markets are American, a fast and accurate numerical method is of crucial importance. For numerical pricing of options there are multiple suggested methods, such as the binomial method, Monte Carlo simulations, finite difference scheme and finite element methods. The standard method for American put options is the binomial model but other methods such as finite difference scheme and finite element method can also be used, Monte Carlo simulations are better fitted when the option has multiple sources of uncertainty or complicated features. Typically this would be the case for Asian or lookback options. But in the case of American put options, Monte Carlo simulations are typically too slow to be competitive. This thesis investigates the pricing of American put options, priced on a single underlying asset, thus Monte Carlo simulations will not be treated but instead the thesis aims to compare the binomial model with a finite difference scheme. The finite element method is not treated. Specifically, this thesis answers the following questions, as well as derive the proof of each method:

- Derive the binomial model algorithm and implement the binomial model.
- Derive the free boundary problem for the pricing PDE of American put options.
- Implement the finite difference scheme for the American put option, how should the grid be configured to obtain optimal results? and what are the optimal values for the parameters in this method?
- Compare the results of the two methods with respect to computational speed and investigate other advantages and disadvantages for the two methods.

Both methods will be implemented first in Python and afterwards optimized in Cython to achive maximal performance and then compared the results of the two methods with regards to computational speed.

### 1.3.1 Code-optimization

Optimization of code is an important aspect of the thesis, since it is of great importance for Highlander to have a fast calculating algorithm. One crucial aspect to have in mind is that the thesis focuses on measuring the performance of two different methods. Hence, keeping the coding implementation difference between the methods to a minimum is important. When optimizing the code, smaller things like multiplying a variable with itself instead of writing as an exponential function can save valuable time when iterating over many steps.
The coding will initially be performed in Python, largely with the help of standard libraries. To increase the performance for Highlander the code will be translated to Cython, which is a combination of Python and C/C++. Cython is mostly written in Python language with the addition of C/C++ syntaxes which result in an astonishing performance increase with regard to speed, up to $\sim 100$ times faster. Python is a interpreted language, meaning that it directly executes code without beforehand being compiled to a machine language program. Cython on the other hand is a compiled language, where the compiler does not run the program, but instead it translates the code into machine code. Compiler language is considerably faster than interpreted languages. Cython uses annotated Python code that is compiled to C code and then produces extensions that are used in Python but with faster computations than the original Python code.
To summarize, the Cython compilers will convert the Python code into C/C++ code, the Cython compiler is a set-up file which produces extensions that are used in Python but with faster computations than the original Python code, and a final python file is then needed to run the Cython file. Python variables are declared to have C data types - hence saving an incredible amount of time during large amounts of iterations. Thus the advantage of Cython is that it combines the best of two worlds; the easy to implement and all built in function advantage of Python with the advantageous speed of C/C++.

# 2
# Theory

## 2.1 Market assumptions

**In this thesis the following basic assumptions are made:**

- The market is arbitrage free - meaning that there is no sure way of making money without bearing any risk. Thus there exist at least one probability measure called martingale measure Q (or risk neutral measure) which is a probability measure such that the discounted price of assets in the market are martingales.
- The stocks that are being analyzed do not pay dividend.
- There is no bid/ask spread.
- There are no transaction costs and trades occur instantaneously.
- An investor can trade any fraction of shares.
- Both the volatility $\sigma$ and the risk-free rate $r$ are constant from $t = 0$ to maturity, $T$.

The Black Scholes equation is used to price the European derivative and is the time continuum version of the binomial model.

### 2.1.1 Black Scholes model

The Black Scholes methods was derived in 1973 by Fischer Black, Robert Merton, and Myron Scholes and is still today, widely used when it comes to option pricing [9]. The formula makes the assumption that the volatility $\sigma$ and the rate $r$ are constant, an assumption which received criticism [10]. The main idea from the Black-Scholes equation is that a perfect hedging strategy exists for the option by buying the underlying asset and thereby eliminating risk [8].

$$\frac{\partial \Pi_Y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi_Y}{\partial S^2} + rS\frac{\partial \Pi_Y}{\partial S} - r\Pi_Y = 0 \tag{2.1}$$

The components that decide the price of an option are the difference between the underlying asset price and the strike price – the pay-off $Y(t)$, the risk-free rate $r$, the time to maturity $T$, and the volatility $\sigma$. The risk-free rate is the return on a risk-free investment, such as a bond. The volatility measures the dispersion of returns for the asset. It can be measured in multiple ways but the most popular way is to define it as the standard deviation of returns for the assets, or standard deviation of log-returns. Since the volatility describes how much the stock varies in price for a specific time region, it is often an indication of risk, higher volatility implies a higher risk. When applied to options, a higher volatility means a higher price, since there is a bigger swing in the market and thus the potential of the option is greater. Equation 2.1 also does not include the expected return of the stock $\alpha$, which stems from the risk-neutral probability measure, or martingale measure Q, which will be explained below. But first we define the Brownian motion $W_t$, which is a stochastic process that has the following characteristics:

- $W_0 = 0$
- $W_t$ - has almost surely continuous paths
- $W_t$ - has independent increments
- $W_t - W_s \sim \mathcal{N}(0, t-s)$ - Normally distributed with zero mean and variance $t-s$, for $(0 \leq s \leq t)$

In the theory, the stock price $\{S(t)\}_{t\geq 0}$ is a positive time continuous stochastic process called a geometric Brownian motion (exponential Brownian motion) given by:

$$S(t) = S_0 e^{\alpha t + \sigma W(t)} \tag{2.2}$$

where $\sigma$ represents the instantaneous volatility of the stock and $\alpha$ is the mean of log-returns of the stock. Next, by Girsanov's theorem, a Brownian motion can be defined in the Girsanov probability measure with parameter $Q$:

$$W^Q(t) = W(t) + Qt \tag{2.3}$$

Using this, equation 2.2 can be written as:

$$S(t) = S_0 e^{(\alpha - Q\sigma)t + \sigma W^Q(t)} \tag{2.4}$$

By the fact that martingales have constant expectations, the discounted expected stock value is also a constant in the risk-neutral probability measure:

$$\mathbb{E}[S(t)] = S_0 e^{rt} \tag{2.5}$$

which holds if and only if the parameter in equation 2.3, Q = $\theta$, where:

$$\theta = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2} \tag{2.6}$$

In other words, the expected return of the stock is the same as the return on the risk-free asset in the martingale measure $Q$. Thus, leading to the definition of the martingale probability Q:

$$\mathbb{P}_Q(A) = \int_A p_Q(x)dx, \qquad p_Q(x) = \frac{1}{\sqrt{2\pi T}} e^{\frac{-(x+qT)^2}{2T}}, \qquad q = \frac{\alpha - r}{\sigma} + \frac{\sigma}{2} \tag{2.7}$$

For $\sigma > 0$, $T > 0$, $r \in \mathbb{R}$, $\alpha \in \mathbb{R}$. Rearranging equation 2.6 gives the following expression for the instantaneous mean of log-return $\alpha$ in the martingale measure Q:

$$\alpha = \theta\sigma + r - \frac{\sigma^2}{2} \tag{2.8}$$

substituted in equation 2.2 and we obtain the following expression for the stock price in the risk neutral measure:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W^Q(t)} \tag{2.9}$$

which describes the stock price in the risk-neutral probability measure, and notably without any dependence on the expected return of the stock $\alpha$. In conclusion, the Black-Scholes formula is widely used but it is commonly understood that it has some drawbacks for it to work, such as constant volatility and the fact that stocks are log-normally distributed (they tend to be fat-tailed distributed). The incorrect assumption that the stocks are log-normally instead of fat-tailed distributed means that the Black-Scholes models tends to underestimate extreme moves in the market. In practice, this means that buying or selling deep in the money or far out of the money options, are more expensive than what the Black-Scholes model says, giving rise to what's called a volatility smile. A volatility smile says that the further a stock is from being at the money, the more the option is overpriced.

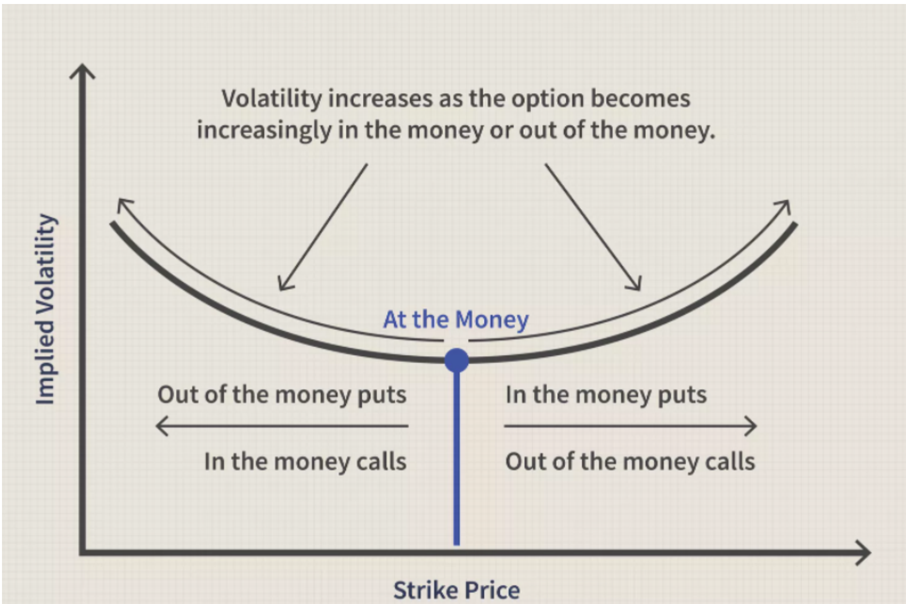



Figure 2.1: Illustration of a volatility smile. If the market behaved exactly as the Black-Scholes theory suggests, then this convex function would instead be a straight horizontal line. [3]

## The Greeks

The Greeks are used in option pricing to measure sensitivity, and are defined as the partial derivatives on the different parameters that decided the price in the Black Scholes model model:

$$\Delta = \frac{\partial V}{\partial S} \tag{2.10}$$

$$\Gamma = \frac{\partial^2 V}{\partial S^2} \tag{2.11}$$

$$\nu = \frac{\partial V}{\partial \sigma} \tag{2.12}$$

$$\Theta = -\frac{\partial V}{\partial \tau} \tag{2.13}$$

$$\rho = \frac{\partial V}{\partial r} \tag{2.14}$$

The Greeks are important tools for an investor to better understand risk and potential reward of an option position [13]. In this thesis, some aspects of delta will be investigated and implemented using the binomial model. Delta describes the expected price change of the option with respect to the change of the underlying asset and ranges differently depending on the type of option; $\Delta \in [0, 1]$ for call options and $\Delta \in [-1, 0]$ for put options. For put options, $\Delta = -1$ means that an increase of 1 dollar in stock price generates a 1 dollar decrease in put price. The typical value for delta at the money is 0.5 and the more in the money the put option is the closer delta is to -1, reversely the more out of the money the put option is, the closer delta will be to 0.

## 2.2 The American put option

Pricing of American put options is a free-boundary problem, where the boundary itself is also a part of the solution, for free-boundary problems no closed form solution exist. The boundary for the American put option is the optimal exercise curve and the solution is the price at this unknown exercise time. Thus, as previously stated, the American put option can not be calculated using the Black-Scholes model.
In the case of American put options there exist a critical exercise boundary, and if the stock price falls below this boundary, then it is optimal to exercise the option. Thus, for American put options, one could divide the option in to two regions; the early exercise region and the hold region. Moreover, by the no arbitrage assumption, we have that the return from a portfolio in the risk-free probability, should not exceed the return in the risk-free asset, thus the Black-Scholes equation becomes the inequality:

$$\frac{\partial \hat{\Pi}_Y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{\Pi}_Y}{\partial S^2} + rS\frac{\partial \hat{\Pi}_Y}{\partial S} - r\hat{\Pi}_Y \leq 0 \tag{2.15}$$

The assumption that the market is arbitrage-free implies; $\hat{\Pi}_Y(t) \geq Y(t)$, since otherwise the option would allow for arbitrage by exercising immediately and guarantee a profit at no risk. Denote the stock price $S^*(t)$ for the value in which it is advantageous for the holder of the option to early exercise. Hence, $S^*(t)$ represents the stock price in the exercise region, $0 \leq S(t) \leq S^*(t)$ and it is optimal to exercise since, $\hat{\Pi}_Y(t) = Y(t) = (K - S(t))_+$, thus the buyer takes full advantage of the option. In the region $S(t) > S^*(t)$, the option should be held since, $\hat{\Pi}_Y(t) > Y(t)$, in this region it is similar to European option since Black-Scholes equality holds for this region. Conclusion is that the option can be early exercised when the present value of the intrinsic value $Y(t)$ is greater the remaining option value of holding to maturity $T$. This typically happens when the option is very deep in the money and interest rates are high. The problem of the optimal exercise time is called stopping time, and is the first time the stock enters the exercise region. The stopping time is denoted $\tau$ and is a random variable for a given filtration $\mathcal{F}_t$ taking values in $(0, \infty)$ and satisfies:

$$\tau \leq t \in \mathcal{F}_t \tag{2.16}$$

Denoting $\mathcal{T}_{t,T}$ the set of all stopping times for $S(t)$ with values in interval $(t, T)$, the American option is given in the following way in the martingale measure Q, where $v(t, S(t)) = \hat{\Pi}_Y(t)$ denotes the price process of the option:

$$v(t, S(t)) \colon = \max_{\tau \in \mathcal{T}_{t,T}} \mathbf{E}^Q[e^{-r(\tau - t)}(K - S(\tau))_+ | \mathcal{F}_t] \tag{2.17}$$

This is a probabilistic representation of the price and as the case with the European option there is a close connection to a deterministic, partial differential equation based representation of the price, which is represented by the system of inequalities, including equation (2.15):

$$\begin{aligned}
&\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv \leq 0 \\
&v(t, S(t)) \geq (K - S(t))_+ \\
&\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv\right)(Y(t) - v(t, S(t))) = 0
\end{aligned} \tag{2.18}$$

Equation 2.17 and 2.18 are derived and proved in section 3.3 and are then used for implementation of the finite difference scheme, section 3.3.1.

# 3

# Methods

## 3.1 Introduction to the Binomial model

The binomial model is the most used method for finding an arbitrage free price of the American put option. It works by the construction of a tree that represents different stock prices from the initial time $t = 0$ to maturity time $T$. The model builds on the initial price of the stock and then the assumption that in each time step the stock price is either increasing or decreasing in value, giving rise to the tree structure, figure 3.1. The amount of iterations that is performed in the calculation is equal to the number of time steps. The valuation of the option price using the binomial tree is done iteratively, starting at maturity and then calculating the price of the option backwards in time. At maturity the price of the option is the same as for the European option, which can be calculated using the final step in the binomial tree for the stock price.

$$\hat{\Pi}_Y(T) = (K - S(T))_+ \tag{3.1}$$

Before describing the recurrence formula used for calculating the price in previous time steps some necessary descriptions are needed. The description is short and for a deeper understanding see [5].
Firstly, $p$ represents the physical (or real world) probability, where $p$ is the probability that the stock price increases and the probability $(1 - p)$ is the probability that the stock price decreases. For maximal convergence speed this probability is set to $p = 0.5$. Thus, a specific node in the binomial tree has the following possibilities for the next time step:

$$S(t) = \begin{cases} S(t-1)e^u & \text{– Stock price increases with probability } p \\ S(t-1)e^d & \text{– Stock price decreases with probability } \textit{1 - p} \end{cases} \tag{3.2}$$

The two quantities $u$ and $d$ represent the fact that the stock price is increasing or decreasing and are calculated using the physical probability $p$, the instantaneous mean of log return $\alpha$ and the instantaneous volatility $\sigma$. The instantaneous mean of log return is the logarithm of the expected future return of the stock price and the instantaneous volatility is the expected volatility for the future of the stock. Where $d < r < u$, thus the binomial model does not admit arbitrage and $h$ is the step length of the time instances in the binomial model.

$$u = \alpha h + \sigma \sqrt{\frac{1-p}{p}} \sqrt{h} \tag{3.3}$$

$$d = \alpha h - \sigma\sqrt{\frac{1-p}{p}}\sqrt{h} \tag{3.4}$$

These properties are used in the following ways for the reccurance formula:

$$q_u = \frac{e^r - e^d}{e^u - e^d} \in (0,1), \qquad q_d = 1 - q_u \tag{3.5}$$

Equation 3.1, 3.2, 3.3, 3.4 and 3.5 together gives the following recurrence formula which is used to calculate the binomial price recursively, proved below.

$$\hat{\Pi}_Y(t) = \max[(K - S(t))_+, e^{-r}(q_u\hat{\Pi}^u_Y(t+1) + q_d\hat{\Pi}^d_Y(t+1))] \tag{3.6}$$

The price at maturity calculated by equation 3.1, is the last column in the price matrix. As the number of iterations for the binomial model increases it tends to the Black Scholes formula. Thus, a larger number of iterations generates a more precise option price, but it comes at the expense of computational time. The precision is decided based on the number of decimals which do not change when the amount of iterations change. The Binomial model is the time discrete approximation of the Black Scholes model. Through the derivation of this method one obtains the tools to give a fair arbitrage free price of the American option and also an optimal exercise strategy of the option.

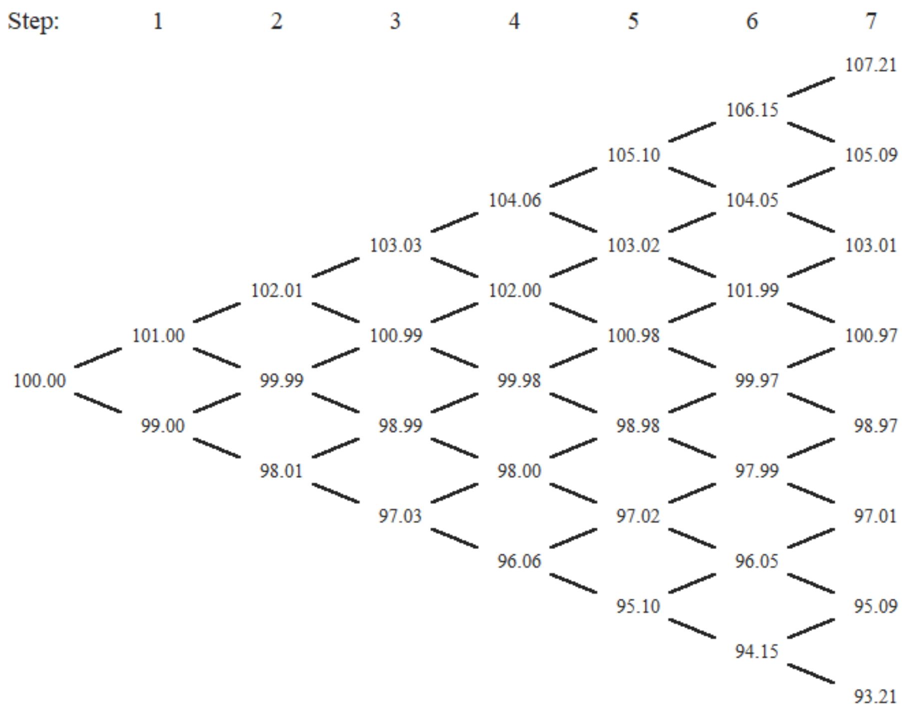



Figure 3.1: Illustration of the a Binomial tree, where the numbers at the nodes represent the stock price for different time instances [4].

### Delta using the Binomial model:

From the theory section we learned that using the Black Scholes model, it is possible to measure sensitivity of the option price, using the Greeks. In this thesis, different relationship for delta will be investigated, hence a definition of delta using the Binomial model will be necessary.

$$\Delta(t) = \frac{\partial \hat{\Pi}_Y(t)}{\partial S(t)} \tag{3.7}$$

Translating delta from the time continuum case using Black Scholes to the time discrete binomial method we obtain the following equation to calculate.

$$\Delta(i,j) = \frac{\hat{\Pi}_Y(i,j+1) - \hat{\Pi}_Y(i+1,j+1)}{S(i,j+1) - S(i+1,j+1)} \tag{3.8}$$

## 3.2 Derivation of the Binomial model algorithm

In the coming sections the binomial model algorithm will be described and proved along with the necessary definitions and proofs along the way.

**The following basic notations is used through the rest of the chapter:**

- Early exercise - must describe the price $\hat{\Pi}_Y$ at all times $t$, where $t = \{0..., N\}$, and $N$ is the time at maturity.
- $V(h_S(t),\, h_B(t))$ denotes the value of a portfolio at time $t$, $h_S(t)$ are the number of stocks in the portfolio and $h_B(t)$ are the number of shares in the risk-free asset.
- Standard American derivatives: $Y(t) = g(S(t))$, $t = 0, 1, .., N$, where $g$ denotes the pay-off function of the derivative, for the American put: $(K - S(t))_+$.
- When to exercise an American option depends only on the information up to that time period, i.e an adapted process.
- $\mathrm{V}_\tau$ and $\mathrm{Y}_\tau$ denotes the portfolio value and the pay-off with respect to the exercise strategy $\tau$, which is a random variable.

### 3.2.1 Arbitage free pricing:

**Definition of exercise strategy, also called stopping time**

$$\begin{aligned} &\tau : \Omega \to \{0, 1, ..., N\} \\ s.t \qquad \{\tau = t\} \in \mathcal{F}_t, \qquad &t = \{0, 1, ..., N\} \end{aligned} \tag{3.9}$$

An assumption is that there is an underlying probability space $\Omega$, where $\Omega$ has a finite number of elements and hence all the expectations of $\tau$ exist and are finite. The exercise strategy for the American derivative (stopping time), $\tau(\omega)$, represents

the time steps from 0 to $N$, which are the different time steps in the Binomial model, under the filtration $\mathcal{F}_t$. Where $\omega$ denotes a probability event on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Equation 3.9 specifies that the decision to exercise the option at time $t$ only depends on $\mathcal{F}_t$, the information available at that time instant. The exercise strategy is part of the collection of subsets that is used to model the information available at time $t$.
Denote $\mathcal{T}_0$ the family of all exercise strategies, i.e all nodes in the Binomial tree.

**Pay-off definition for the American put option derivative**

$$Y(t) = (K - S(t))_+, \qquad t = \{0, ..., N\} \tag{3.10}$$

*The pay-off is a random variable, where the stopping time $\tau \in \mathcal{T}_0$*

$$(Y_\tau)(\omega) = (Y)_{\tau(\omega)}(\omega), \qquad \omega \in \Omega \tag{3.11}$$

This is the pay-off relative to the exercise strategy, or time instant $\tau$. For a martingale measure Q, the stock price is the discounted expectation of this stock price, i.e the stock price is recalculated to its current value. We call a stopping time optimal, $\tau_0$, for the pay-off $Y(t)$ in the Martingale measure Q if:

$$\mathbf{E}^Q[Y^*_{\tau_0}] = \max_{\tau \in \mathcal{T}_0} \mathbf{E}^Q[Y^*_\tau] \tag{3.12}$$

Where $\mathbf{E}^Q[Y^*_\tau]$ is the risk-neutral price for an early exercise. For American options it is in general not possible to decide upon a replicating strategy, due to the early exercise option. This is since for the option to be a replicating strategy $Y(t) = V(h_S(t)\,,\, h_B(t))$ for all time instances $t$. The portfolio value $V(h_S(t), h_B(t))$ is a Q-martingale but the discounted pay-off $Y^*_\tau$ is only an adapted process and not a Q-martingale [2].

But by the Arbitage free property, a lower and upper bound can be decided for the price of the derivative, $\hat{\Pi}_Y(t)$. The initial price of the option-process is denoted $\hat{\Pi}_Y(0)$, $\mathcal{A}$ denotes the family of self-replicating and predictable strategies. Therefore, $\mathcal{A}^+_Y$ represents the super-replicating family of portfolios and $\mathcal{A}^-_Y$ represents the sub-replicating family of portfolios for a general American option, which are denoted.

$$\mathcal{A}^+_Y = \{(h_S(t),\, h_B(t)) \in \mathcal{A} |\, V(h_S(t),\, h_B(t)) \geq Y(t), \quad t = \{0, .., N\} \tag{3.13}$$

$$\mathcal{A}^-_Y = \{(h_S(t),\, h_B(t)) \in \mathcal{A} |\, \exists\, \tau \in \mathcal{T}_0 \;\; s.t \;\; Y_\tau \leq V_\tau(h_S,\, h_B)\} \tag{3.14}$$

To avoid arbitrage opportunities, the initial price $\hat{\Pi}_Y(0)$ has to be less than or equal to the initial value of the portfolio $V(h_S(0),\, h_B(0))$ for any number of shares; $(h_S(0),\, h_B(0) \in A^+_Y)$.

$$\hat{\Pi}_Y(0) \leq \inf_{(h_S(t),\, h_B(t)) \in \mathcal{A}^+_Y} V(h_S(0),\, h_B(t)) \tag{3.15}$$

analogously, to avoid arbitrage when taking a short position of the option, the initial price must be greater than or equal to the initial portfolio value.

$$\hat{\Pi}_Y(0) \geq \sup_{(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^-} V(h_S(0),\, h_B(0)) \tag{3.16}$$

equation 3.15 and 3.16 combined gives the following for the initial price of the derivative for an arbitrage free market with a martingale measure Q:

$$\sup_{(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^-} V(h_S(0),\, h_B(t)) \leq \;\hat{\Pi}_Y(0)\; \leq \inf_{(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^+} V(h_S(0), h_B(t)) \tag{3.17}$$

this defines an interval for arbitrage free prices of the initial option price.

**The risk-neutral pay-off value, for an optimal exercise strategy belongs to this same interval:**

$$\sup_{(h_S(0),\, h_B(0)) \in \mathcal{A}_Y^-} V(h_S(0),\, h_B(0)) \leq \max_{\tau \in \mathcal{T}_0} \mathbf{E}^Q[Y_\tau^*] \leq \inf_{(h_S(0),\, h_B(0)) \in \mathcal{A}_Y^+} V(h_S(0), h_B(0)) \tag{3.18}$$

For proof, the Optional sampling theorem will be needed and it is defined by:

$$\mathbf{E}[M_\tau] = M_0 \tag{3.19}$$

The Optional sampling theorem says that under certain conditions the expected value of the portfolio at later time is the same as the initial expected value.

**Proof equation (3.18)**

For each $(h_S,\, h_B) \in \mathcal{A}_Y^+$ there exist a process $\tau_0 \in \mathcal{T}_0$ such that the value of the portfolio process is less than or equal to the pay-off process of the derivative: $V(h_S(\tau_0),\, h_B(\tau_0)) \leq Y_{\tau_0}$.
Moreover, the discounted portfolio value: $V^*(h_S(t),\, h_B(t))$ is a Q-martingale and therefore by the Optional sampling theorem, we have for $(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^-$:

$$\begin{aligned} &V(h_S(0),\, h_B(0)) = V^*(h_S(0),\, h_B(0)) = \mathbf{E}^Q[V^*(h_S(\tau_0),\, h_B(\tau_0))] \leq \\ &\mathbf{E}^Q[Y^*(\tau_0)] \leq \operatorname*{Sup}_{\tau \in \mathcal{T}_0} \mathbf{E}^Q[Y_\tau^*] \end{aligned} \tag{3.20}$$

And for $(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^+$, again by the optional sampling theorem, we have for each process $\tau \in \mathcal{T}_0$

$$V(h_S(0),\, h_B(0)) = \mathbf{E}^Q[V^*(h_S(\tau),\, h_B(\tau))] \geq \mathbf{E}^Q[Y_\tau^*] \tag{3.21}$$

Which concludes the proof of equation (3.18)
□

The proof of equation (3.18) demonstrates that it is possible to generate a replicating and hedging strategy for the American option. To give a unique price of the American derivative, not just the initial price, we assume that the market is arbitrage free and to derive the proof we shall need the following theorem first.

### 3.2.2 Doob's decomposition theorem and the Snell envelope:

Every adapted process can be decomposed in a unique way into the sum $X = M + A$, where $M$ is a martingale such that $M_0 = X_0$ and $A$ is an adapted process with $A_0 = 0$. Moreover, $X$ is a super-martingale if and only if A is a decreasing process, for proof [2].
Another tool that is needed for deciding upon a hedging and the optimal exercise strategy for a arbitrage free market is the Snell envelope of the process $Y(t)$ that defines the American derivative. The Snell envelope is the smallest super-martingale which dominates the process $Y(t) \Leftrightarrow Y(t) \geq \mathbf{E}[Y(t)]$.

**The American derivative is defined recursively in the binomial model**

$$\hat{\Pi}_Y(t) = \begin{cases} Y(N) & t = N \\ \max\{Y(t),\, e^{-r}[q_u \hat{\Pi}^u_{put}(t+1) + q_d \hat{\Pi}^d_{put}(t+1)]\} & t = \{0\,,\,\ldots\,,\,N-1\} \end{cases} \tag{3.22}$$

and the discounted value of the derivative:

$$\hat{\Pi}^*_Y(t) = \begin{cases} Y^*(N) & t = N \\ \max\{Y^*(t),\, e^{-r}[q_u \hat{\Pi}^*_u(t+1) + q_d \hat{\Pi}^*_d(t+1)]\} & t = \{0\,,\,\ldots\,,\,N-1\} \end{cases} \tag{3.23}$$

In this case: $\hat{\Pi}_Y(t)$ dominates the pay-off $Y(t)$, i.e $\hat{\Pi}_Y(t)$ is always equal to or greater than $Y(t)$. $\hat{\Pi}^u_Y(t)$ represent the price of the derivative when the price is moving up in the next node of the binomial tree and $\hat{\Pi}^d_Y(t)$ represent the price is moving down in the next node of the binomial tree.

**Proof of the Snell envelope:**
$\hat{\Pi}^*_Y(t)$ is an adapted and non-negative process. For each time instant t, the price of the derivative fulfills:

$$\hat{\Pi}^*_Y(t) \geq \mathbf{E}^Q[\hat{\Pi}^*_Y(t+1) \,|\, \mathcal{F}_t] \tag{3.24}$$

Hence that $\hat{\Pi}^*_Y$ is a Q-super-martingale. Moreover, $\hat{\Pi}^*_Y(t)$ is the smallest Q-super-martingale that dominates the pay-off process $Y^*(t)$. In fact if $Z$ is a super-martingale such that $Z(t) \geq Y^*(t)$, then we have:

$$\hat{\Pi}^*_Y(N) = Y^*(N) \leq Z(N) \tag{3.25}$$

The statement follows by induction, assuming that $\hat{\Pi}^*_Y(t) \leq Z(t)$, we obtain:

$$\hat{\Pi}^*(t-1) = \max\{Y^*(t-1),\, \mathbf{E}[\hat{\Pi}^*_Y(t-1)\,|\,\mathcal{F}_{t-1}]\} \leq \\ \max\{Y^*(t-1),\, \mathbf{E}[Z(t)\,|\mathcal{F}_{t-1}] \leq \max\{Y^*(t-1)\,,\, Z(t-1)\} = Z(t-1) \tag{3.26}$$

And the proof of the Snell envelope is complete.
□

### 3.2.3 Theorem and proof of the binomial model algorithm:

Assuming there exits a unique martingale measure Q, then there exist a strategy:

$$(h_S(t),\, h_B(t)) \in \mathcal{A}^+_Y \cap \mathcal{A}^-_Y \tag{3.27}$$

Such that:

a) $V(h_S(t),\, h_B(t)) \geq Y(t), \qquad t \in \{0, ..., N\}$
b) There exist a stopping time, $\tau_0 \in \mathcal{T}_0$ such that the process
$V(h_S(\tau_0),\, h_B(\tau_0)) = Y(\tau_0)$

Moreover,

$$\mathbf{E}^Q[Y^*(\tau_0)] = V(h_S(0), h_B(0)) = \max_{\tau \in \mathcal{T}_0} \mathbf{E}^Q[Y^*(\tau)] \tag{3.28}$$

Where $Y^*(\tau)$ is the discounted pay-off process and this value defines the initial arbitrage free price $\hat{\Pi}_Y(0)$ of the process and $\tau_0$ is equal to an optimal stopping time.

**Proof:**
Firstly, construct the Snell envelope for the discounted pay-off process, $Y^*$. Then use Doob's decomposition theorem to separate out the martingale part of the price process $\hat{\Pi}^*_Y(t)$ to be able to decide upon a position strategy of the portfolio, i.e $(h_S(t),\, h_B(t)) \in \mathcal{A}^+_Y \cap \mathcal{A}^-_Y$.
The proof is concluded by stating that:

$$\hat{\Pi}^*_Y(0) = V^*(h_S(0),\, h_B(0)) = V(h_S(0),\, h_B(0)) \tag{3.29}$$

Introduce the price process of the American option; $\hat{\Pi}_Y(t) = \hat{\Pi}^*_Y(t)B(t)$, where $B(t)$, denotes the risk-free asset, a bond at time $t$. Which is defined recursively and at maturity $N$ the derivative has the value:

$$\hat{\Pi}_Y(N) = Y(N) \tag{3.30}$$

i.e the price is equal to the pay-off of the derivative and at an earlier times it is defined as:

- $Y(N-1)$ – if the option is exercised
- $\frac{1}{1+r}\mathbf{E}^Q[\hat{\Pi}_Y(N)\,|\,\mathcal{F}_{N-1}]$

hence leading to the following definition:

$$\hat{\Pi}_Y(N-1) = \max\{Y(N-1),\, \frac{1}{1+r}\mathbf{E}^Q[\,\hat{\Pi}_Y(N)\,|\,F_{N-1}\,]\} \tag{3.31}$$

where

$$\frac{1}{1+r}\mathbf{E}^Q[\,\hat{\Pi}_Y(N)\,|\,F_{N-1}\,] = \frac{1}{1+r}[q_u\hat{\Pi}_Y^u + q_d\hat{\Pi}_Y^d] \tag{3.32}$$

$\frac{1}{1+r}$ is used for calculating the discounted value in the time discrete case, in the time continuous case the definition is: $e^{-rt}$. Now by repeating the argument for equation 3.31 backwards in time, we get for $\hat{\Pi}_Y^*(t) = \frac{\hat{\Pi}_Y(t)}{B(t)}$, equation 3.23. Thus, we have constructed the Snell envelope, the price is decreasing on average, meaning that it is a super-martingale. This implies that as time goes by, the advantage of an early exercise diminishes.

Next, we use Doob's decomposition theorem and prove that there exist a $(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^+ \cap \mathcal{A}_Y^-$. Since $\hat{\Pi}_Y^*$ is a Q-super-martingale, then by Doob's decomposition:

$$\hat{\Pi}_Y^* = M + A \tag{3.33}$$

where M is a Q-martingale such that at time zero, it is equal to the initial price of the option; $\hat{\Pi}_Y^* = M_0$ and A is a predictable and decreasing process such that at time zero, $A_0 = 0$.

Since the market is complete, i.e each derivative is replicable, there exist a portfolio $(h_S(t),\, h_B(t)) \in \mathcal{A}$ which replicates the European derivative with pay-off $M(N)$ at maturity, in the sense that the value of the portfolio is equal to the pay-off, $V^*(h_S(N),\, h_B(N)) = M(N)$. Moreover, since both M and $V^*(h_S,\, h_B)$ are Q-martingales with the same value at maturity, they are equal and:

$$V^*(h_S(t),\, h_B(t)) = \mathbf{E}^Q[V^*(h_S(N),\, h_B(N))\,|\,\mathcal{F}_t] = \mathbf{E}^Q[M_N|\mathcal{F}_t] = M_t \tag{3.34}$$

therefore $(h_S(t),\, h_B(t)) \in \mathcal{A}_X^+$, since $A_n \leq 0$. Moreover since $A_0 = 0$, the price of the derivative is equal to the value which is a the initial Q-martingale.

$$V(h_S(0)\,,\, h_B(0)) = M_0 = \hat{\Pi}_Y(0) \tag{3.35}$$

To confirm that $(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^-$, we set:

$$\tau_0(\omega) = \min\{\,t\,|\,\hat{\Pi}_t^*(\omega) = Y_t^*(\omega)\}\,, \qquad \omega \in \Omega \tag{3.36}$$

since,

$$\begin{aligned}\{\tau_0 = t\} = \{\hat{\Pi}_Y^*(0) > Y^*(0)\} \cap \cdots \cap \{\hat{\Pi}_Y^*(t-1) \\ > Y^*(t-1)\} \cap \{\hat{\Pi}_Y^*(t) > Y^*(t)\} \in \mathcal{F}_t\end{aligned} \tag{3.37}$$

for all times $t$, $\tau_0$ is a stopping time. Moreover $\tau_0$ is the first time instant in which it is advantageous to exercise the option, which is to say:

$$Y^*(t) \geq \mathbf{E}^Q[\hat{\Pi}_Y^*(t+1) \,|\, \mathcal{F}_t] \tag{3.38}$$

From Doob's decomposition theorem we have that:

$$M_n = \hat{\Pi}_Y^*(t) + \sum_{k=0}^{t-1} \Big(\hat{\Pi}_Y^*(k) - \mathbf{E}^Q[\,\hat{\Pi}_Y^*(k+1) \,|\, \mathcal{F}_k]\Big) \tag{3.39}$$

And from this it is obtained that the martingale-process is equal to the price at the optimal exercise time:

$$M_{\tau_0} = \hat{\Pi}_Y^*(\tau_0) \tag{3.40}$$

since,

$$\hat{\Pi}^*_Y(k) = \mathbf{E}^Q[\,\hat{\Pi}^*_Y(k+1) \,|\, F_k] \quad \text{on} \quad \{k \leq \tau_0\} \tag{3.41}$$

then by equation 3.34 we have that

$$V^*(h_S(\tau_0)),\, h_B(\tau_0)) = M_{\tau_0} = \hat{\Pi}_Y^*(\tau_0) = Y^*(\tau_0) \tag{3.42}$$

thus, $(h_S(t), h_B(t)) \in \mathcal{A}_Y^-$.

**Conclusion:**
Lastly, we want to confirm that $\tau_0$ is an optimal stopping time. From the fact that $(h_S(t),\, h_B(t)) \in \mathcal{A}_Y^+ \cap \mathcal{A}_Y^-$, by equation 3.18 we obtain:

$$V(h_S(0),\, h_B(0)) = \max_{\tau \in \mathcal{T}_0} \mathbf{E}^Q[Y^*(\tau)] \tag{3.43}$$

And by the Optional Sampling theorem:

$$V(h_S(0),\, h_B(0)) = \mathbf{E}^Q[Y^*(\tau_0)] \tag{3.44}$$

□

The derived algorithm can be used for the following applications; by using 3.35, the initial option price can be calculated using the recurrence formula. An optimal stopping time defined by equation 3.36, where we have:

$$\mathbf{E}^Q[Y(\tau_0)] = \max_{\tau \in \mathcal{T}_0} \mathbf{E}^Q[Y(\tau)] \tag{3.45}$$

we also can obtain a hedging strategy for the derivative, such that $V(h_S(t),\, h_B(t)) \geq Y(t)$ for every $t$.

## 3.3 Theorem and proof of the free-boundary problem

From the theory in section 2.2, we have that the stock price $S(t)$ in the risk-neutral measure Q is a geometric Brownian motion:

$$dS(t) = rS(t)dt + \sigma S(t)d(W)^Q \tag{3.46}$$

Integrating, we obtain equation 2.9:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W^Q(t)}$$

We have the following definition, theorem and proof of the free boundary problem for the American put option.

**Definition of stopping time for the free-boundary problem**
Assume S(t) = x and let $\mathcal{F}_u^{(t)}$ for $t \leq u \leq T$ denote the $\sigma$-algebra generated by the stock process S(v), v $\in$ [t,u]. $\mathcal{T}_{t,T}$ denotes the set of stopping times in the filtration $\mathcal{F}_u^{(t)}$ for $t \leq u \leq T$ taking values in [t,T]. Thus, $\{\tau \leq t\} \in \mathcal{F}_u^{(t)}$ for every $u \in [t,T]$; a stopping time in the family of stopping times $\mathcal{T}_{t,T}$ makes the decision to stop at u $\in$ [t,T] based only on the path of $S(t)$ from $t$ to $u$. Thus, the following definition for the price of the American put option, where the price function is denoted as; $\hat{\Pi}_Y(t) = v(t, S(t))$

$$v(t,x) = \max_{\tau \in \mathcal{T}_{t,T}} \mathbf{E}^Q[e^{-r(\tau-t)}(K - S(\tau))|S(t) = x] \tag{3.47}$$

As stated in section 2.2, the American put option satisfies the system of linear inequalities:

$$v(t,x) \geq (K-x)_+ \quad \text{for } \ t \in [0,T], \ \ x \geq 0 \tag{3.48}$$

$$\begin{gathered} rv(t,x) - \frac{\partial v}{\partial t} - rx\frac{\partial v}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{x^2} \geq 0 \\ \text{for all } \ t \in [0,T], \ \ x \ \geq 0 \text{ and for each } \ t \in [0,T] \end{gathered} \tag{3.49}$$

Equality holds in either equation 3.48 or 3.49. As stated in section 2.2, the American put option should be exercised as the stock price falls below a certain level, denoted $S^*(T-t)$, the notation is obvious because of its dependence until maturity. The stock level in which its advantageous to exercise the option, $S^*(t)$, is unknown, but can be determined numerically by the presented methods (section 3.2 and section 3.3.1). Thus, creating the two regions; a hold region $\mathcal{H}$ - where it is advantageous to hold on to the option and early exercise region $\mathcal{S}$ - where it is advantageous to exercise the option. Defined in the following way:

$$\begin{aligned} \mathcal{H} &= \{(t,x); v(t,x) > (K-x)_+\} \quad \text{Hold region} \\ \mathcal{S} &= \{(t,x); v(t,x) = (K-x)_+\} \quad \text{Stopping region} \end{aligned} \tag{3.50}$$

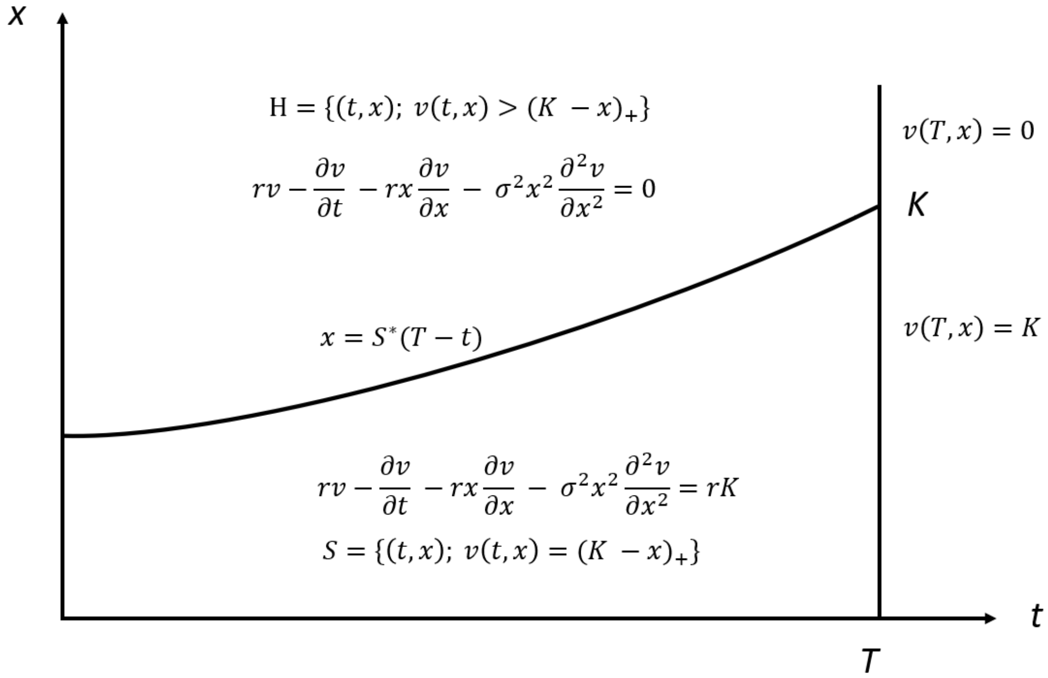



Figure 3.2: Illustration of the exercise boundary and the two regions; holding region and stopping region, as the options approaches maturity, the price goes to the strike price.

The line $S^*(T-t)$ is included in the stopping region $\mathcal{S}$. In the region $0 \leq x \leq S^*(T-t)$, we have $v(t,x) = K - x$, thus; the left hand derivative on the curve $x = S^*(T-t)$ is $\frac{\partial v}{\partial x} = -1$. At $x = S^*(T-t)$, $\frac{\partial v}{\partial x}$ is continuous and hence, the so called smooth pasting condition for the option is fulfilled:

$$\frac{\partial v}{\partial x^-} = \frac{\partial v}{\partial x^+} \qquad \text{for } S^*(T-t),\ 0 \leq t < T \tag{3.51}$$

However, at $t = T$, $L(T-T) = L(0) = K$ and $v(T,x) = (K-x)_+$, thus, the smooth pasting condition does not hold:

$$\begin{aligned} \frac{\partial v}{\partial x^-} &= -1 \\ \frac{\partial v}{\partial x^+} &= 0 \end{aligned} \tag{3.52}$$

which is clear from figure 3.2, note that this derivative is delta from the greeks in section 2.1.1. Equation 3.51 and 3.52 give the following equations:

$$\begin{aligned} rv(t,x) - \frac{\partial v}{\partial t} - rx\frac{\partial v}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{x^2} &= 0, \qquad x \geq S^*(T-t) \\ v(t,x) = K - x&, \qquad 0 \leq x \leq S^*(T-t) \end{aligned} \tag{3.53}$$

Equation 3.51, the terminal condition in equation 3.52 and equation 3.53 together with the asymptotic condition: $\underset{x\to\infty}{lim}\, v(t,x) = 0$ are used to determine the function

$v(t,x)$ by setting up a finite difference scheme (section 3.3.1) one can solve $v(t,x)$ and $S^*(T-t)$.

We want to show that $v(t,x)$ defined by equation 3.47 satisfies the smooth-pasting condition, such that $v_x^{'}$ is continuous at the curve that separates the two regions, $x = S^*(T-t)$ and everywhere else. But first, we show that the stopping process is a martingale and the price process is a super-martingale.

**The stopping process is a martingale:**

Let $S(u)$ be the stock price starting at $S(t) = x$ and $t \le u \le T$, then the stopping time is defined as

$$\tau^* = \min\{u \in [t,T];\ (u, S(u)) \in \mathcal{S}\} \tag{3.54}$$

The price process $e^{-ru}v(u, S(u))$ is a super-martingale under the risk-neutral probability $\mathbf{P}^Q$ and the stopping process $e^{-r(u\wedge\tau^*)}v(u, S(u\wedge\tau^*))$ is a martingale, where $u \wedge \tau^* = \min(u, \tau^*)$. If $(u, S(u))$ never enters the stopping region, $\mathcal{S}$ for any $u \in [t,T]$, hence always in the hold region, is interpreted as: $\tau^* \to \infty$. Where $u \wedge \tau^* = \min(u, \tau^*)$

Note that equation 3.54, is almost identical to equation 3.36, which is the stopping time for the binomial model algorithm.

**Proof:**

By continuity of $v(t,x)$ we can apply *Itô's* formula and use the following relation:

$$dS(u)dS(u) = du$$

$$dS(u) = rS(u)v_x(u, S(u))du + \sigma S(u)v_x(u, S(u))dW^Q(u)$$

$$\begin{aligned}\implies d[e^{-ru}v(u,S(u))] = e^{-ru}[-ru(u,S(u))du + v_u^{'}(u,S(u))du + v_x^{'}(u,S(u))dS(u) + \\ \frac{1}{2}v_{xx}^{''}(u,S(u))dS(u)dS(u))] = e^{-ru}[-rv(u,S(u)) + v_u^{'}(u,S(u)) + rS(u)v_x^{'}(u,S(u)) + \\ \frac{1}{2}\sigma^2 S^2(u)v_{xx}^{''}(u,S(u))]du + e^{-ru}\sigma S(u)v_x(u,S(u))dW^Q(u)\end{aligned} \tag{3.55}$$

comparing equation 3.55 to figure 3.2, we see that the $du$ term in equation 3.55 is: $e^{-ru}rK\mathbf{1}_{\{S(u)<S^*(T-t)\}}$. As this is non-positive, $e^{-ru}v(u,S(u))$ is a super-martingale in the risk-neutral probability $\mathbf{P}^Q$. Starting at $u = t$, by the fact that it is optimal to exercise as soon the stock price enters the stopping region $\implies S(u) > S^*(T-t)$, thus the $du$ term is zero until $\tau^*$, hence the stopping process $e^{-r(u\wedge\tau^*)}v(u\wedge\tau^*, S(u \wedge \tau^*))$, for $t \le u \le T$, is a martingale.

□

Now, to show that $v(t,x)$ is in fact the only equation that fulfills the smooth pasting condition, such that it is continuous. Fix $t$ with $0 \le t \le T$, then the super-martingale property $e^{-ru}v(t,S(t))$ and the optional sampling theorem (equation 3.19) implies:

$$e^{-r(t\wedge\tau)}v(t\wedge\tau, S(t\wedge\tau)) \geq \mathbf{E}^Q[e^{-r(T\wedge\tau)}v(T\wedge\tau, S(T\wedge\tau))|\mathcal{F}_t]$$

where $r \wedge \tau$ is defined as the minimum, $min(t, \tau)$. For $\tau \in \mathcal{T}_{t,T}$ (family of exercise strategies), we have $t \wedge \tau = t$, but for $\tau \wedge T = \tau$ if $\tau < \infty$ and $T$, if $\tau = \infty$. Hence, for $\tau \in \mathcal{T}_{t,T}$:

$$\begin{aligned} e^{-rt}v(t, S(t)) \geq \mathbf{E}^Q[e^{-r\tau}v(\tau, S(\tau))\mathbf{1}_{\{\tau<\infty\}} + e^{-rt}\mathbf{E}^Q[e^{-rT}v(T, S(T))\mathbf{1}_{\{\tau=\infty\}}|\mathcal{F}_t] \\ \geq \mathbf{E}^Q[e^{-r\tau}v(\tau, S(\tau))|\mathcal{F}_t] \end{aligned} \tag{3.56}$$

Where $e^{-r\tau} = 0$ for $\tau = \infty$ and using equation 3.48, and $(K - S(t))_+ \geq K - S(t)$ gives:

$$\begin{aligned} \mathbf{E}^Q[e^{-r\tau}v(\tau, S(\tau))|\mathcal{F}_t] \geq \mathbf{E}^Q[e^{-r\tau}(K - S(\tau))|\mathcal{F}_t] \\ \implies e^{-rt}v(t, S(t)) \geq \mathbf{E}^Q[e^{-r\tau}(K - S(\tau))|\mathcal{F}_t] \end{aligned} \tag{3.57}$$

Since $S(t)$ is a Brownian motion, the last part of equation 3.57 is a function of $t$ and $S(t)$, by denoting $S(t) = x$, we can replace the filtration and since this holds for any stopping time $\tau \in \mathcal{T}_{t,T}$, we get:

$$e^{-rt}v(t, x) \geq \max_{\tau\in\mathcal{T}_{t,T}} \mathbf{E}^Q[e^{-r(\tau-t}(K - S(\tau))|S(t) = x] \tag{3.58}$$

Using that the stopping process is a martingale, where $\tau^*$ is defined by equation 3.54 and $v(\tau^*, S(\tau^*)) = K - S(\tau^*)$ for $\tau^* < \infty$. By replacing $\tau$ by $\tau^*$ in equation 3.56, the first inequality becomes an equality:

$$\begin{aligned} e^{-rt}v(t, S(t)) \geq \mathbf{E}^Q[e^{-r\tau}v(\tau, S(\tau))\mathbf{1}_{\tau<\infty} + e^{-rt} \geq \mathbf{E}^Q[e^{-rT}v(T, S(T))\mathbf{1}_{\tau=\infty}|\mathcal{F}_t] \\ = \mathbf{E}^Q[e^{-r\tau^*}v(\tau^*, S(\tau^*))] = \mathbf{E}^Q[e^{-r\tau^*}(K - S(\tau^*)] \end{aligned} \tag{3.59}$$

For the case when $\tau^* = \infty$, then the process is in the holding region; $(T, S(T)) \in \mathcal{H}$. Thus, the second inequality in 3.56 becomes an equality, hence for all $\tau^* \geq 0$:

$$v(t, x) = \mathbf{E}^Q[e^{-r(\tau^*-t)}(K - S(\tau^*))|S(t) = x] \tag{3.60}$$

□

### 3.3.1 Finite Difference Scheme

A finite difference scheme is a numerical method for solving differential equations. This is achieved through a discrete grid and approximating the derivative part of the PDE using finite differences. For American put options the grid consists of all the time instances from the initial time, $t = 0$ to maturity, $T$, and the stock prices are set to a minimum value ($S(t) = 0$) and a highest value $S_{max}$. The theoretical limit for the maximal stock price is infinity, but that is both unlikely and impractical, hence a highest unlikely to occur value is used as $S_{max}$ in the grid. The option value at the different grid points is then solved by a system of linear equations consisting of finite differences from the closest nodes in the grid. Specifically the Euler backward method is used in the case for American put. The transition from PDE to a finite difference is called discretization and this gives rise to two kinds of errors compared to an exact analytical solution. One error occurs due to the fact that computers only handles a certain amount of decimals when performing calculations, this error is very small. The other error is called discretization error and occurs because of the discretization itself, error when a continuous function is represented by a finite number of evaluations [11]. The first error is unavoidable and unchangeable when performing this method but the discretization error can be minimized by increasing the number of grid points, but it comes at the expense of computational cost [11].

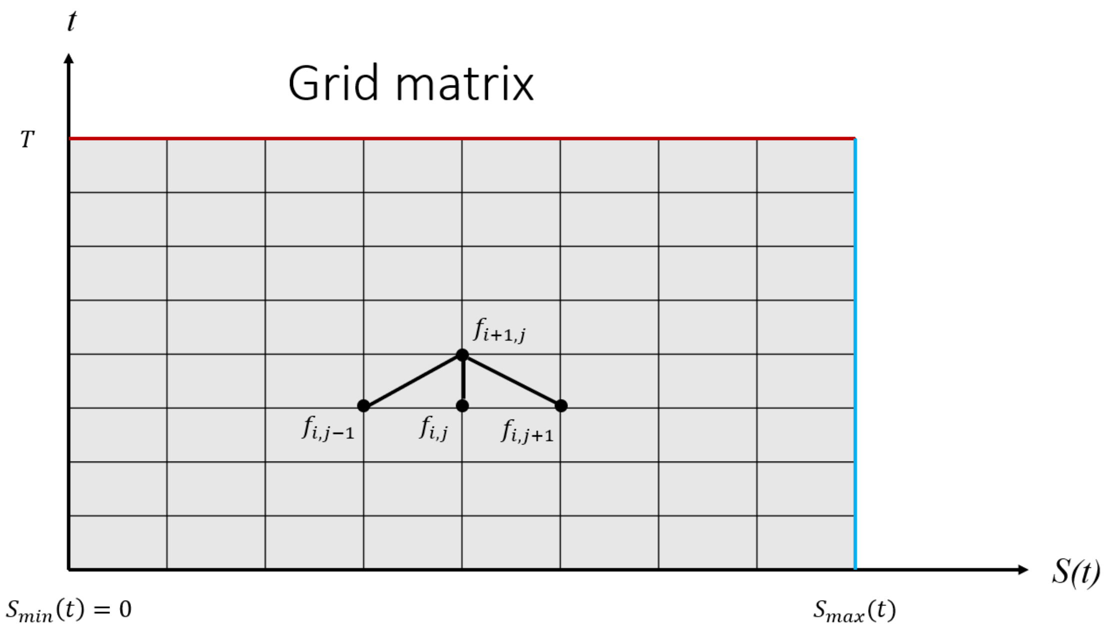



Figure 3.3: Illustration of the grid for finite difference scheme that replaces the PDE space for the implicit method. The red line illustrates maturity $T$, the blue line is the maximal allowed stock value $S_{max}$. $\Pi_{i,j}$ is the option price at node $(i, j)$, where $i$ represents the time instances and $j$ represents the stock prices.

The difference scheme has the following boundary condition at $S(t) = 0$ for $N$ number of time steps:

$$\hat{\Pi}(i,0) = Ke^{T-i\Delta t}, \quad i = 0,1...,N \tag{3.61}$$

and initial condition at $t = 0$ for $M$ number of stock values:

$$\begin{aligned} \hat{\Pi}(0,j) &= K - j\Delta S \\ K - j\Delta S \geq 0, &\quad j = 0,1...,M \end{aligned} \tag{3.62}$$

$\Delta t$ and $\Delta S$ are defined in equation 3.66 and 3.67. There are three types of approaches when applying a finite difference scheme, the explicit method, implicit method and Crank–Nicolson method. The explicit method is the easiest to implement but can generate unstable results and negative option price, both the implicit method and Crank–Nicolson method are reliable with Crank-Nicolson being the most accurate while the implicit is less computationally heavy. Since this thesis focus on computational speed comparison, the implicit method will be implemented using projected successive over realaxation algorithm, (PSOR).

### 3.3.2 Difference Approximation

Discretization of the derivatives in the Black Scholes PDE is performed using Taylor expansion:

$$f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f^{(2)}(x_0)}{2!}h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n + R_n(x) \tag{3.63}$$

$R_n(x)$ is the remaining term, denoting the difference between the Taylor polynomial of degree $n$ and the original function [11], $h$ is small and $f \in C^{(n+1)}$. For first order Taylor expansion:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + R_1(x) \tag{3.64}$$

$$\implies f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{R_1(x)}{h} \tag{3.65}$$

the grid points in the finite difference scheme are defined by in the following way.

**Time instances grid points:**

$$t_i = ih, \quad i = 0,1,...,N, \quad h = \frac{T}{N} = \Delta t \tag{3.66}$$

**Stock prices grid points:**

$$\begin{gathered} S_j = S_{min} + jk, \qquad \{S_{min} = 0 \implies S_j = j\Delta S\} \qquad j = 0,1,...,M+1, \\ k = \frac{S_{max} - S_{min}}{M+1} = \Delta S \end{gathered} \tag{3.67}$$

### 3.3.3 Implicit method:

The implicit method finds a solution by solving an equation involving both the current state of the system and a later time, while the explicit method only calculates the later times from the current time [12]. The implicit method leads to a sequence of complementary problems, which are solved iteratively using PSOR algorithm, in other words a system of linear equations is used to solve the option value for one time step before solving for the previous time step, figure 3.3, thus iterating backwards in time.

Approximation of equation 2.1 with Euler backwards gives the following finite difference equations of first and second order derivative of the option price $\hat{\Pi}_{i,j} = \hat{\Pi}(t_i, S_i,)$ with respect to the stock price and time:

$$\frac{\partial \hat{\Pi}}{\partial t} = \frac{\hat{\Pi}_{i+1,j} - \hat{\Pi}_{i,j}}{\Delta t} + \mathcal{O}(\Delta t) \tag{3.68}$$

$$\frac{\partial \hat{\Pi}}{\partial S} = \frac{\hat{\Pi}_{i,j+1} - \hat{\Pi}_{i,j-1}}{2\Delta S} + \mathcal{O}(\Delta t) \tag{3.69}$$

$$\frac{\partial^2 \hat{\Pi}}{\partial S^2} = \frac{\hat{\Pi}_{i,j+1} - 2\hat{\Pi}_{i,j} + \hat{\Pi}_{i,j-1}}{\Delta S^2} + \mathcal{O}(\Delta S^2) \tag{3.70}$$

applying this discretizations for Black Scholes equation 2.1 and we obtain:

$$\frac{\hat{\Pi}_{i+1,j} - \hat{\Pi}_{i,j}}{\Delta t} + \frac{1}{2}\sigma^2 (j\Delta S)^2 \frac{\hat{\Pi}_{i,j+1} - 2\hat{\Pi}_{i,j} + \hat{\Pi}_{i,j-1}}{\Delta S^2} + r(j\Delta S)\frac{\hat{\Pi}_{i,j+1} - \hat{\Pi}_{i,j-1}}{2\Delta S} = r\hat{\Pi}_{i,j} \tag{3.71}$$

which can be reformulated as:

$$\begin{cases} \alpha_j = 1 + r\Delta t + (\sigma j)^2 \Delta t \\ \beta_j = -\frac{1}{2}(\sigma j)^2 \Delta t - rj\Delta t \\ \gamma_j = -\frac{1}{2}(\sigma j)^2 \Delta t + \frac{1}{2} rj\Delta t \end{cases} \tag{3.72}$$

$$\implies \alpha_j \hat{\Pi}_{i,j} + \beta_j \hat{\Pi}_{i,j+1} + \gamma_j \hat{\Pi}_{i,j-1} = \hat{\Pi}_{i+1,j} \tag{3.73}$$

### 3.3.4 PSOR - algorithm

When solving the implicit method, the projected successive over relaxation algorithm is used. The SOR algorithm is an iterative algorithm used to solve a system of linear equations. It is a version of Gauss–Seidel method [14], but modified to speed up convergence by adding a relaxation factor $\omega \in (0, 2)$.
The algorithm works by approximating new $x_j^{(k+1)}$ using the old $x_j^{(k)}$, and has the following formula:

$$x_j^{(k+1)} = \max\left[K - S_j,\ (1-\omega)x_j^{(k)} + \frac{\omega}{a_{jj}}\left(b_j - \sum_{i<j} a_{ji} x_i^{(k+1)} - \sum_{i>j} a_{ji} x_i^{(k)}\right)\right. \tag{3.74}$$

$$j = 1, ..., M$$

iteration proceeds until some predetermined convergence level;

$$|x_j^{(k+1)} - x_j^{(k)}| < \epsilon \tag{3.75}$$

where $x_j$ is an option price in the grid matrix, for a specific time instant and a specific stock price, equation 3.74 corresponds to the discretization of equation 3.47. In this thesis the tolerance level $\epsilon$ takes values $10^{-4}$, $10^{-6}$, $10^{-8}$. $a_{jj}$ is the tridiagonal matrix consisting of the coefficients from equation 3.72:

$$a_{jj} = \begin{bmatrix} \alpha & \beta & & & \\ \gamma & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \beta \\ & & & \gamma & \alpha \end{bmatrix}$$

$b_j$ is the column vector that contains all the option values in the grid matrix $Q$ for one specific time instance, which is then iterated forward in time to obtain full matrix $Q$, containing all the option prices and the initial condition is obtained from equation 3.62. After fulfilling the tolerance level, $b_j$ is updated and applies the newly approximated option prices for the next iteration:

$$b_j = \begin{bmatrix} Q(i,2) \\ \vdots \\ \vdots \\ Q(i,M) \end{bmatrix}$$

Generating the grid matrix containing the option prices calculated using the finite difference scheme by the implicit method, illustrated by the following pseudocode:

**Algorithm 1** PSOR

```
for t = 1, 2, ..., N do
    x = np.copy(GridMatrix[t 1:M])
    n = len(x)
    xold = 100 * x

    while np.linalg.norm(xold-x) > tol do
        xold = np.copy(x)
        for i in range(n) do
            z=((d[i]-TridiagonalMatrix[i,i-1]*x[i-1]-TridiagonalMatrix[i,i+1]*x[i+1])
/(TridiagonalMatrix[i,i]))
            x[i]=max(omega*z+(1-omega)*xold[i],StrikePrice - (i+1)*ds)
        end for
    end while
    GridMatrix[t+1,1:-1] = x
end for
```

# 4

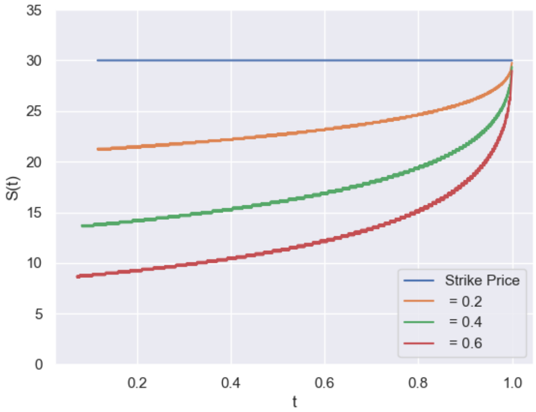
# Results

The results from the Binomial model are presented first followed by the results from the finite difference method. The results are then compared in different tables and figures for different inputs. Besides the code, the performance is also hardware dependent and the results presented comes from a laptop with AMD Ryzen™ 7 4700U processor with 16 GB DDR4 RAM.

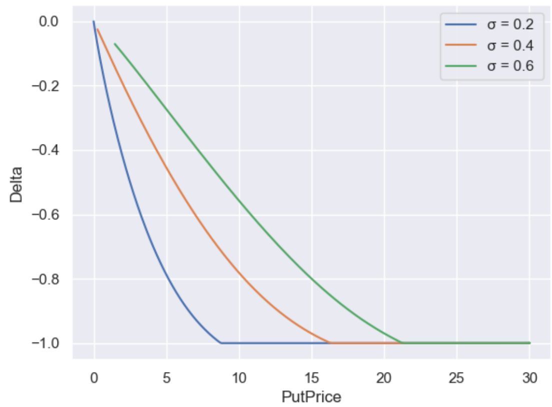
## 4.1 Binomial Model

An overview of how different volatilities $\sigma$ impact the exercise boundary, pay-off function, put price against delta and also the stock price against delta. From figure 4.1 (a) its clear that as the volatility of the underlying stock increases, the exercise boundary becomes lower. This stems from the fact that as the volatility increases, the put price also increases, which is clear from the second part of equation 3.22, that depends on $\sigma$ in equation 3.3 and equation 3.4. An increase in volatility also gives a greater discrepancy between the pay-off curve and the price curve figure 4.1 (d), for the same reason as the lower exercise boundary in figure 4.1 (a). Figure 4.1 (b) and (c) shows that delta decreases faster with lower volatility against the put price and it increases faster in relation to the stock price.

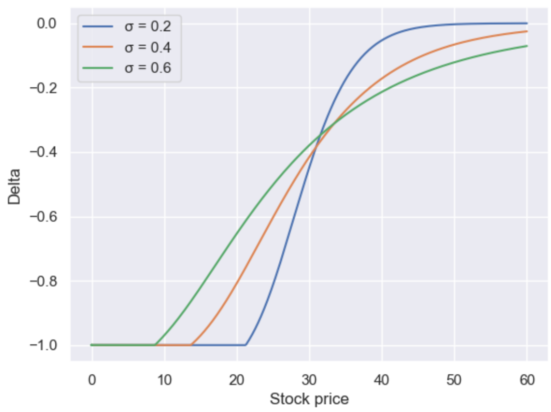
Delta measures the expected change in option price with respect to changes in the underlying stock, equation 3.8. Thus, an increase in volatility gives a higher value for delta (closer to zero for the put option), for a specific put price. The relationship between volatility and stock price can be described as follows: when the put option is in the money, delta increases with volatility for a specific stock price. But as the put option is out of the money, delta decreases as the volatility increases for a given stock price. In practice this means that an in the money put option is less in the money as volatility increases, i.e the option is more overpriced and an out of the money put option is less out of the money as volatility increases.



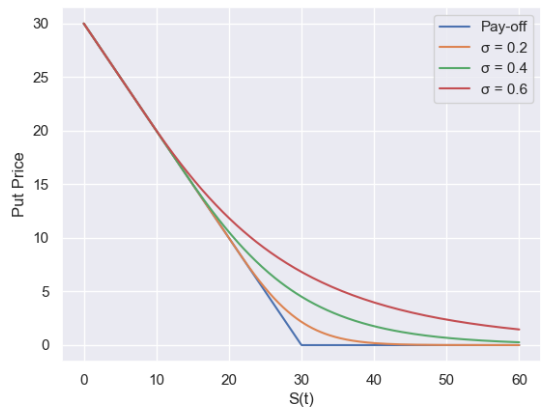

(a) Exercise boundary for different volatility $\sigma$, an increase in volatility results in a lower exercise bound.




(b) Put price against Delta for different volatilities.



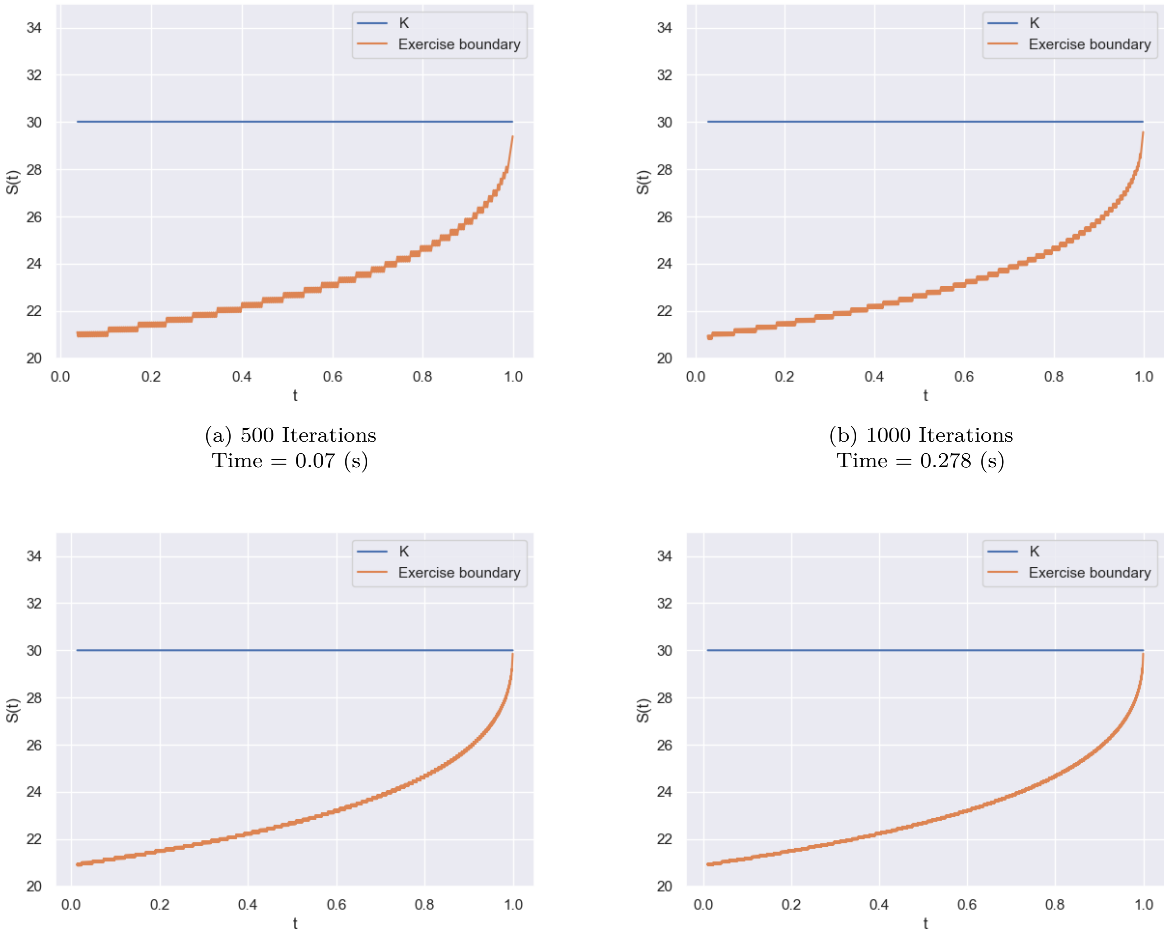

(c) Delta against Stock price for different volatilities




(d) The pay-off curve (intrinsic value) and the option value for different volatilities

Figure 4.1: Binomial model for different values of the volatility $\sigma$: 2000 iterations, $K = 30$, $\sigma = 0.2$, $r = 0.01$, $T = 1$, $\alpha = 0.01$

The area under the exercise curve represents the region for where it is advantageous to exercise the option. Figure 4.2, show that the smoothness and thereby the accuracy of the exercise curve depends on the number of iterations.




(a) 500 Iterations
Time = 0.07 (s)

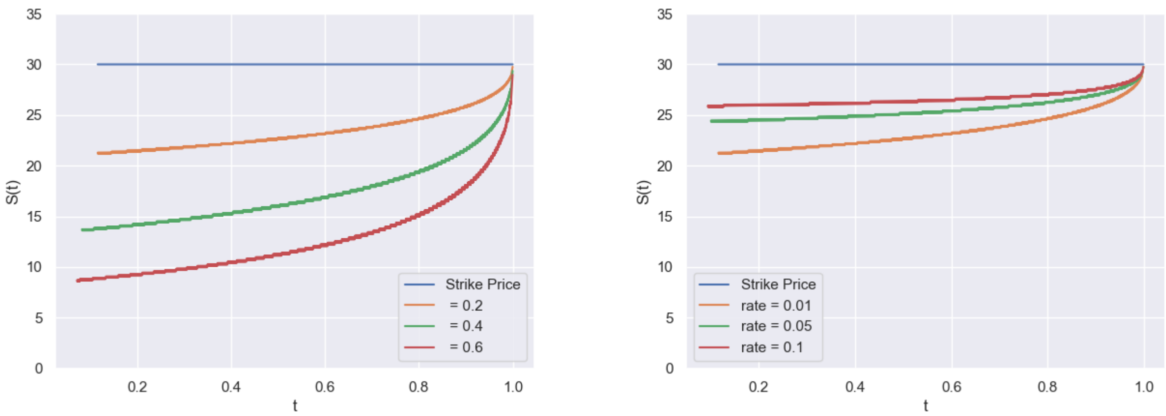
(b) 1000 Iterations
Time = 0.278 (s)

(c) 4000 iterations
Time = 4.80 (s)

(d) 8000 Iterations
Time = 21.80 (s)

Figure 4.2: Binomial model - exercise boundary figures for different amount of iterations. $K = 30$, $\sigma = 0.2$, $r = 0.01$, $\alpha = 0.1$, $T = 1$. The region under the orange curve is the exercise region, plots are generated in Python but times comes from Cython.

Comparison on the effect on the exercise boundary between the differences in volatility and differences in rate. The rate has the opposite effect compared to the volatility, when the rate increases the exercise boundary becomes lower. Hence in the case when the rate is high, it is more often profitable for an early exercise of the American put. This is because if the interest rate is high, exercising the option and investing the money in the risk-free asset is more profitable than selling the option at later times and getting the time value of the option.



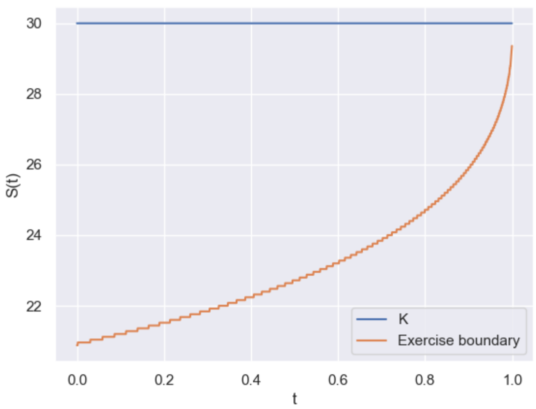

(a) Exercise boundary for different volatilties $\sigma$

(b) Exercise boundary for different rates, $r$

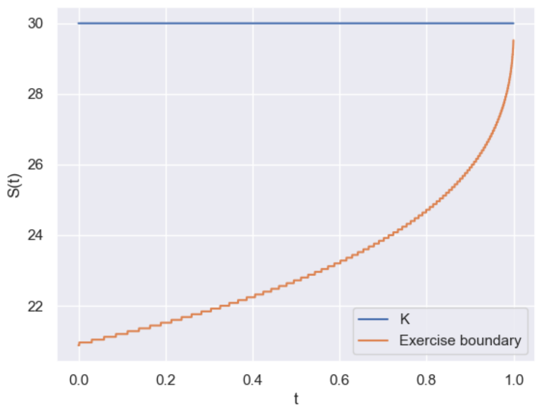
Figure 4.3: Binomial model for different values of the volatility $\sigma$ and different rate $r$, 2000 iterations, $K = 30$, $\sigma = 0.2$ for rate comparison and $r = 0.01$ for $\sigma$ comparison, $T = 1$, $\alpha = 0.01$

## 4.2 Finite Difference Scheme

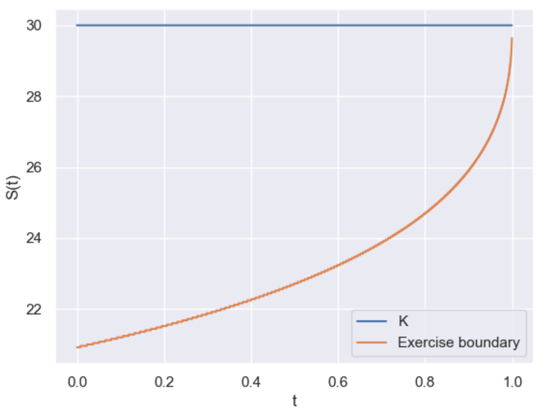
Initially the same graphs will be presented for the finite difference method, except for the graphs related to delta since it has not been computed using the finite difference model. As for the binomial model the preciseness of the method increases with the number of time steps. As can be seen in figure 4.4 the exercise boundaries are similar to the ones for the binomial model, where the smoothness of the curve increases as the grid size increases.



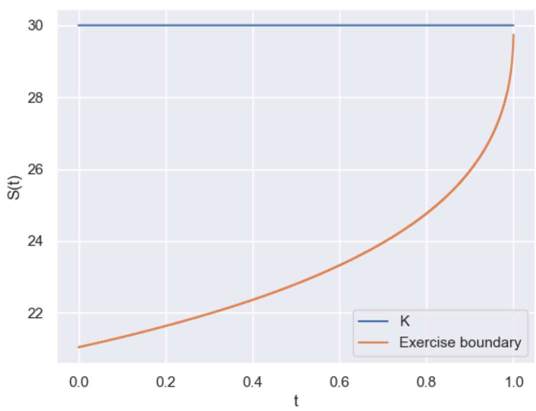

(a) Grid: [2000×1000]
Time = 38.8 (s)




(b) Grid: [4000×1000]
Time = 233.4 (s)

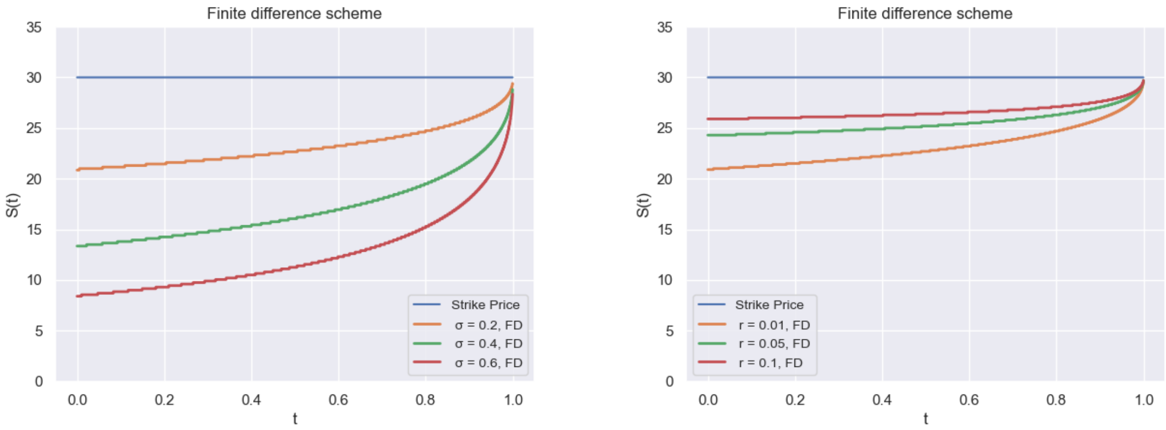



(c) Grid: [8000×2000]
Time = 273.4 (s)




(d) Grid: [10000×4000]
Time = 298.1 (s)

Figure 4.4: Finite difference scheme - exercise boundary figures for different grid size [t $\times$ $S(t)$], $K = 30$, $\sigma = 0.2$, $r = 0.01$, $\epsilon = 10^{-6}$, $\omega = 1.2$, $S_{max} = 150$, $T = 1$




(a) Exercise boundary for different volatilties $\sigma$

(b) Exercise boundary for different rates, $r$

Figure 4.5: Finite difference scheme for different values of the volatility $\sigma$ and different rate $r$, $[4000 \times 2000]$ , $K = 30$, $\sigma = 0.2$ for comparison of different rates, and $r = 0.01$ for $\sigma$ comparison, $T = 1$, $\epsilon = 10^{-6}$, $\omega = 1.2$, $S_{max} = 150$.

Table 4.1, 4.2, 4.3 and figure 4.7 show results for the finite difference method for different combinations of grid size, tolerance $\epsilon$ and relaxation factor $\omega$. Before a more optimized composition of this parameters is presented and compared to the binomial model in section 4.3. Calculations has only been performed in Cython and not in Python, since the purpose if to find optimized parameters, thus the language comparison is of no importance for this. The initial prices $\hat{\Pi}_Y(0)$ are interpolated using Cythons built in function, *interp1d*.

| **Iterations** | $\hat{\Pi}_Y(0)$ $\epsilon = 10^{-4}$ | Time (s) |
|---|---|---|
| $[500 \times 500]$ | 1.489240 | 1.18 |
| $[1000 \times 500]$ | 1.488203 | 2.35 |
| $[1500 \times 500]$ | 1.489006 | 3.58 |
| $[1000 \times 1000]$ | 1.487879 | 6.33 |
| $[2000 \times 1000]$ | 1.488490 | 9.75 |
| $[3000 \times 1000]$ | 1.488956 | 14.83 |
| $[2000 \times 2000]$ | 1.484364 | 30.16 |
| $[4000 \times 2000]$ | 1.485702 | 41.96 |
| $[6000 \times 2000]$ | 1.488663 | 64.57 |
| $[3000 \times 3000]$ | 1.479573 | 259.93 |
| $[6000 \times 3000]$ | 1.484696 | 474.22 |
| $[8000 \times 1500]$ | 1.489532 | 67.21 |
| $[8000 \times 2000]$ | 1.488198 | 66.29 |
| $[10000 \times 2000]$ | 1.487164 | 85.09 |

Table 4.1: Table for difference grid composition, $\hat{\Pi}_Y(0) = 1.489125$. $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $T = 1$, $\epsilon = 10^{-4}$, $\omega = 1.2$, $S_{max} = 200$

It is clear from the table 4.1 that the number of stock values should be approximately 30%, thus, approximately this composition for the comparison between the two methods.

| **Iterations** | $\hat{\Pi}_Y(0)$ $\epsilon = 10^{-4}$ | $\hat{\Pi}_Y(0)$ $\epsilon = 10^{-6}$ | $\hat{\Pi}_Y(0)$ $\epsilon = 10^{-8}$ | Time (s) $\epsilon = 10^{-4}$ | Time (s) $\epsilon = 10^{-6}$ | Time (s) $\epsilon = 10^{-8}$ |
|---|---|---|---|---|---|---|
| [1500 × 500] | 1.489006 | 1.487546 | 1.487551 | 3.58 | 6.66 | 8.57 |
| [2000 × 1000] | 1.488490 | 1.488567 | 1.488569 | 9.75 | 14.03 | 18.96 |
| [3000 × 1000] | 1.488956 | 1.488631 | 1.488629 | 14.83 | 24.36 | 30.00 |
| [6000 × 2000] | 1.488663 | 1.488925 | 1.488934 | 64.57 | 90.78 | 126.53 |
| [8000 × 1500] | 1.489532 | 1.488891 | 1.488886 | 67.21 | 98.77 | 129.47 |
| [8000 × 2000] | 1.488198 | 1.488948 | 1.488949 | 67.86 | 106.86 | 136.65 |
| [10000 × 2000] | 1.487164 | 1.488952 | 1.488958 | 87.97 | 133.72 | 177.04 |

Table 4.2: Table for different value of the tolerance $\epsilon$, $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\omega = 1.2$, $S_{max} = 200$

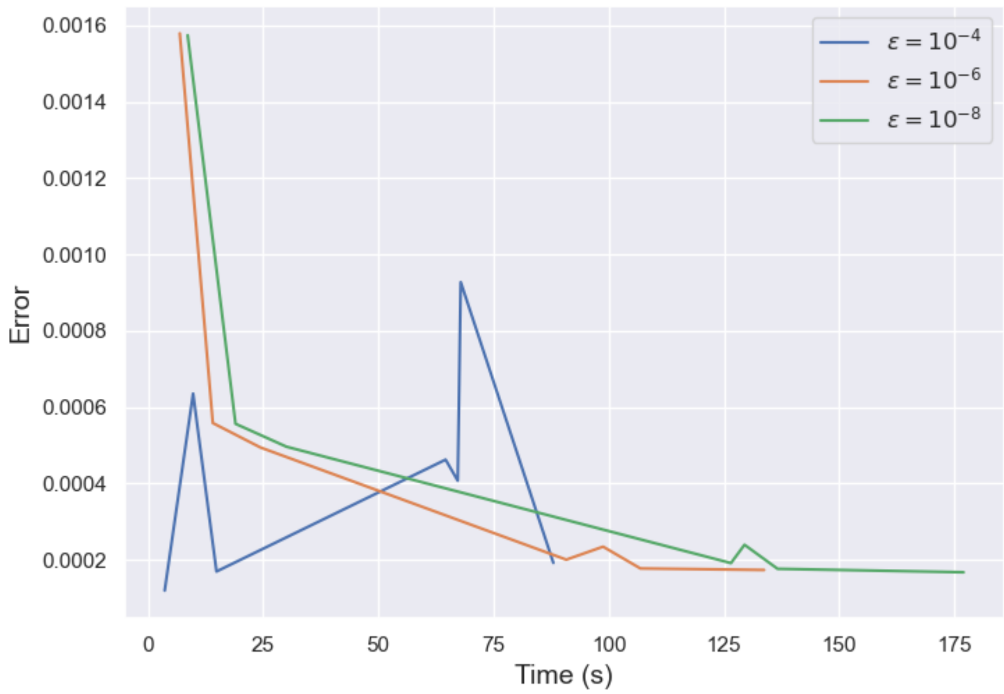



Figure 4.6: illustration of the different tolerance $\epsilon$, where the error is calculated with $\hat{\Pi}_Y^{Bin}(0) = 1.489125$, for 12 000 iterations using the binomial model as a reference point. Error $= |\hat{\Pi}_Y^{Bin}(0) - \hat{\Pi}_Y^{FD}(0)|$, $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\omega = 1.2$, $S_{max} = 200$

From table 4.2 and figure 4.6 one can see that for the finite difference method to converge on the third decimal, then the fastest grid composition is $[2000 \times 1000]$ and $\epsilon = 10^{-4}$. But for the method to converge on the forth decimal a grid composition of approximately $[8000 \times 2000]$ and $\epsilon = 10^{-6}$ is needed.

| **Iterations** | $\hat{\Pi}_Y(0)$ $(\omega = 1.2)$ | $\hat{\Pi}_Y(0)$ $(\omega = 1.5)$ | $\hat{\Pi}_Y(0)$ $(\omega = 1.7)$ | Time (s) $(\omega = 1.2)$ | Time (s) $(\omega = 1.5)$ | Time (s) $(\omega = 1.7)$ |
|---|---|---|---|---|---|---|
| $[1500 \times 500]$ | 1.489006 | 1.487144 | 1.485808 | 3.58 | 6.76 | 11.69 |
| $[2000 \times 1000]$ | 1.488490 | 1.491138 | 1.4895044 | 9.75 | 16.45 | 28.36 |
| $[3000 \times 1000]$ | 1.488956 | 1.490855 | 1.486655 | 14.83 | 25.93 | 41.47 |
| $[6000 \times 2000]$ | 1.488663 | 1.483421 | 1.492550 | 64.57 | 82.89 | 135.04 |
| $[8000 \times 1500]$ | 1.489532 | 1.499158 | 1.480144 | 67.21 | 99.61 | 152.14 |
| $[8000 \times 2000]$ | 1.488198 | 1.482195 | 1.489400 | 67.86 | 114.16 | 182.00 |
| $[10000 \times 2000]$ | 1.487164 | 1.480713 | 1.489752 | 87.97 | 140.67 | 225.4 |

Table 4.3: Table for different value of the relaxation factor $\omega$. Reference point - Binomial model 12000 iterations: $\hat{\Pi}_Y(0) = 1.489125$. $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\epsilon = 10^{-4}$, $S_{max} = 200$

An increase in relaxation factor does not increase the accuracy but it does affect the computational time negatively, hence setting the relaxation factor $\omega = 1.2$. Further investigation into this can be performed to get closer to an optimum of the PSOR algorithm but that is not the focus of this thesis and hence for a comparison with the binomial model, $\omega = 1.2$ is used.

## 4.3 Comparison

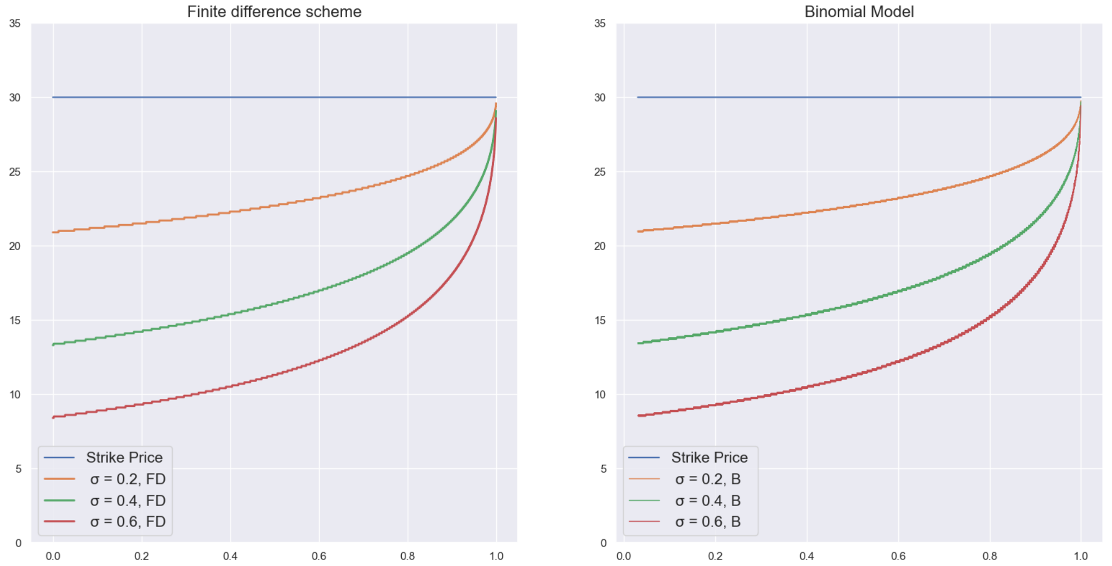



Figure 4.7: Comparison of the exercise curves for the finite difference scheme and the binomial model with different volatilities. Number of iterations = 5000, $[4500 \times 1500]$, $K = 30$, $r = 0.01$, $T = 1$, $S_max = 150$, $\epsilon = 10^{-6}$, $\omega = 1.2$

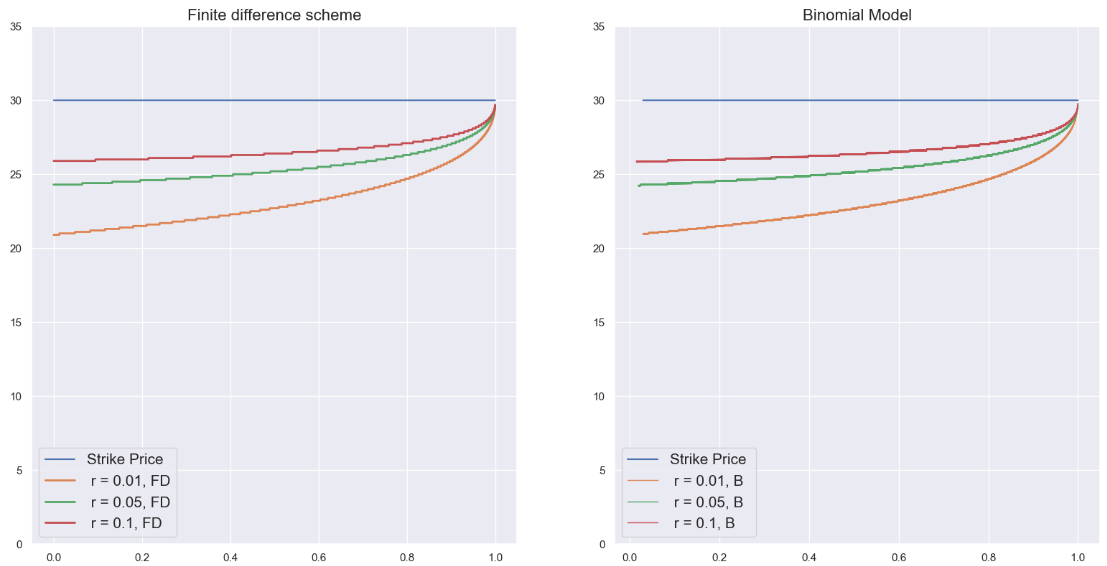



Figure 4.8: Comparison of the exercise curves for the finite difference scheme and the binomial model with different rates. Number of iterations = 5000, $[4500 \times 1500]$, $K = 30$, $\sigma = 0.2$ $T = 1$, $S_{max} = 150$, $\epsilon = 10^{-6}$, $\omega = 1.2$

From figure 4.7 and 4.8 its clear the the two methods generates approximately the same result, the finite difference method has a bit smoother exercise curve and its exercise curve tends to start a bit earlier but needs more iterations to reach the strike price at maturity, also seen in figure 4.2 and figure 4.4.

We compare both computational speed and accuracy, where the accuracy is measured using the binomial model with 12000 iterations as benchmark, since it converges to the Black-Scholes model in the limit.

| **Iterations** | | $\hat{\Pi}_Y^B(0)$ | $\hat{\Pi}_Y^{FD}(0)$ | Difference (B) | Difference (FD) |
|---|---|---|---|---|---|
| 500 | [1500 × 500] | 1.489240 | 1.487546 | 0.006725 | 0.001579 |
| 1000 | [2000 × 1000] | 1.489245 | 1.488567 | 0.00012 | 0.000558 |
| 2000 | [3000 × 1000] | 1.489301 | 1.488631 | 0.000176 | 0.000494 |
| 5000 | [6000 × 2000] | 1.489105 | 1.488925 | 0.00002 | 0.0002 |
| 8000 | [8000 × 1500] | 1.489138 | 1.488891 | 0.000013 | 0.000234 |
| 10 000 | [8000 × 2000] | 1.489130 | 1.488948 | 0.000005 | 0.000177 |
| 12 000 | [10 000 × 2000] | 1.489125 | 1.488952 | 0 | 0.000173 |

Table 4.4: Comparison between the binomial model and a finite difference scheme in Cython, the difference in value for both models is the absolute value measured against the Binomial model for 12 000 iterations. $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\epsilon = 10^{-4}$, $\omega = 1.2$, $S_{max} = 200$.

| **Iterations** | | $\hat{\Pi}_Y^B(0)$ Time (s) | $\hat{\Pi}_Y^{FD}(0)$ Time (s) | Difference (B) | Difference (FD) |
|---|---|---|---|---|---|
| 500 | [1500 × 500] | 0.008 | 6.66 | 9.202 | 2.55 |
| 1000 | [2000 × 1000] | 0.042 | 14.03 | 9.168 | -4.82 |
| 2000 | [3000 × 1000] | 0.14 | 24.36 | 9.07 | -15.15 |
| 5000 | [6000 × 2000] | 1.47 | 90.78 | 7.74 | -81.57 |
| 8000 | [8000 × 1500] | 2.48 | 98.77 | 6.73 | -89.56 |
| 10 000 | [8000 × 2000] | 5.63 | 106.86 | 3.58 | -97.65 |
| 12 000 | [10 000 × 2000] | 9.21 | 133.72 | 0 | -124.51 |

Table 4.5: Comparison between the binomial model and a finite difference scheme in Cython, the difference in time for both models is measured against the Binomial model for 12 000 iterations. Greater positive value - faster computational compared to the reference (9.21 s) and increased negative value means slower computational speed compared to the reference value. $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\epsilon = 10^{-6}$, $\omega = 1.2$, $S_{max} = 200$.

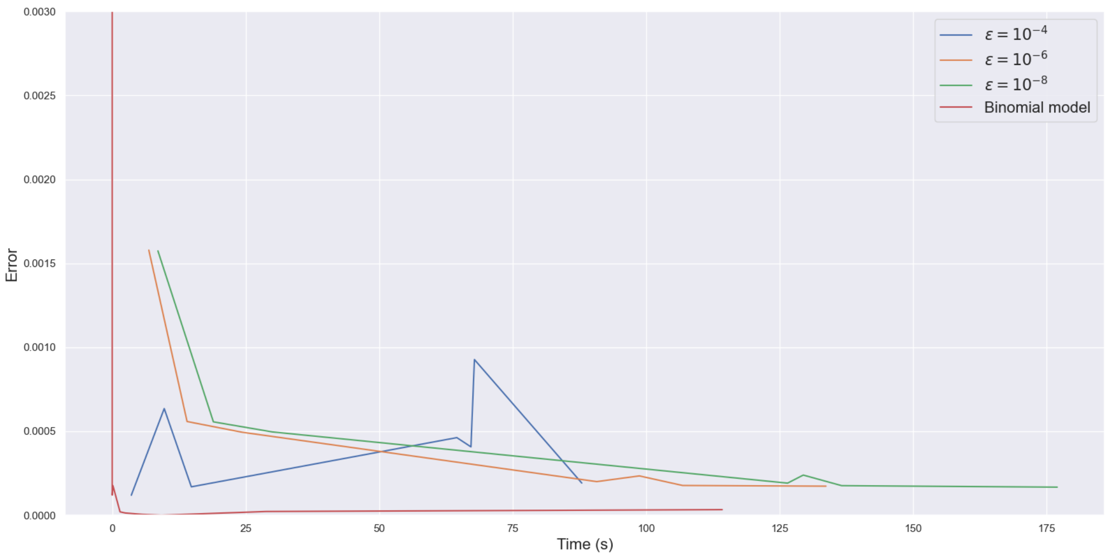



Figure 4.9: Comparison of computational speed, reference is the binomial model for 12000 iterations, $\sigma = 0.2$, $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\epsilon = 10^{-6}$, $\omega = 1.2$, $S_{max} = 200$.

By table 4.4, 4.5 and figure 4.9, the binomial model is superior. Note, that for $\epsilon = 10^{-4}$ the error does not seem to converge for the finite difference scheme. Hence, even if it is faster to use $\epsilon = 10^{-4}$, it is less reliable than $\epsilon = 10^{-6}$ and $\epsilon = 10^{-8}$, which behaves very similar.

| **No iterations** | 100 [100×100] | 1000 [1000×500] | 5000 [5000×1500] | 8000 [8000×2000] | 10000 [10 000×2000] |
|---|---|---|---|---|---|
| $\hat{\Pi}_Y^B(0)$ - Python (s) | 0.61 | 1.3 | 16.78 | 45.7 | 77.1 |
| $\hat{\Pi}_Y^B(0)$ - Cython (s) | 0.00033 | 0.0237 | 0.75 | 2.465 | 4.28 |
| $\hat{\Pi}_Y^{FD}(0)$ - Python (s) | 0.86 | 10.69 | 189.34 | 403.81 | 428.86 |
| $\hat{\Pi}_Y^{FD}(0)$ - Cython (s) | 0.1526 | 6.3754 | 126.12528 | 150.01 | 287.36 |

Table 4.6: Speed comparison for the two methods between Python and Cython, clearly the binomial method has been better optimized than the finite difference scheme, $T = 3$. $\sigma = 0.2$, $K = 10$, $S(0) = 11$, $r = 0.01$, $\alpha = 0.01$, $\omega = 1.2$, $\epsilon = 10^{-6}$, $S_{max} = 200$.

| **Iterations** | $\sigma = 0.1$ | $\sigma = 0.2$ | $\sigma = 0.4$ | $\sigma = 0.6$ |
|---|---|---|---|---|
| $\hat{\Pi}_Y^B(0)$- Initial value | 0.414423 | 1.489138 | 3.842134 | 6.200934 |
| $\hat{\Pi}_Y^{FD}(0)$ - Initial value | 0.413897 | 1.488631 | 3.841569 | 6.200183 |
| $\hat{\Pi}_Y^B(0)$ Time (s) | 1.98 | 2.14 | 2.05 | 2.1 |
| $\hat{\Pi}_Y^{FD}(0)$ Time (s) | 23.08 | 22.64 | 49.1 | 174.4 |

Table 4.7: Comparison of speed and calculation accuracy for differences in volatilities in Cython. $\hat{\Pi}_Y^B(0)$ - 8000 iterations, $\hat{\Pi}_Y^{FD}(0)$ - [3000 × 1000]. $K = 30$, $S(0) = 32$, $r = 0.01$, $\alpha = 0.01$, $T = 1$, $\epsilon = 10^{-6}$, $\omega = 1.2$, $S_{max} = 200$.

Grid configuration [3000 $\times$ 1000] is based on previous results for accuracy and speed for finite difference method, section 4.2. As the volatility increases the difference in price between the two methods decreases, but when studying the time for the two methods there is a substantial difference; the binomial model is unaffected by an increase in volatility, while the computational time for the finite difference scheme increases as the volatility increases.

# 5

# Summary & Conclusion

After an initial summary of the theory necessary to understand the different models, this thesis derives the binomial model algorithm and the boundary value problem, section 3.2 and 3.3. For the binomial model there are results for differences in delta and how it is affected by differences in volatility. For the finite difference scheme a justification for the choice of parameters in the projected successive over relaxation algorithm are presented, and then the exercise boundaries and performances of the two methods are compared. The structure of the theoretical exercise boundary from figure 3.2 is reproduced both for the binomial model and the finite difference scheme. By figure 4.7, 4.8 it is clear that the exercise boundaries are similar for the two methods, both when studying different volatilities and different rates, but the binomial method tends to converge to the strike price more reliably than the finite difference scheme, figure 4.2 compared to figure 4.4. An increase in volatility has great effect on the computational time for the finite difference scheme, but for the binomial model, an increase does not affect the computational speed, table 4.7. When comparing the two models performance in regard to computational speed and accuracy, its clear that the binomial model is the best, which can be seen in table 4.4, 4.5 and figure 4.9. Although, a problem that was highlighted initially has been hard to avoid; difference in performance due to to code implementation. One big aspect of this error is the conversion from Python to Cython code. As can be seen from table 4.6, the language conversion done using the binomial model is much better than it is for the finite difference model, where Cython is $\sim 60$ times faster than Python for the binomial model, but only $\sim 13$ times faster using the finite difference scheme.

## Further work

Firstly the implementation in Cython for the finite difference method should be further optimized. Also, a deeper study of the optimal relaxation factor $\omega$ and the optimal tolerance level $\epsilon$ would potentially benefit the finite difference model. Both parameters have an effect on the computational speed, this becomes a further optimization problem, as lower tolerance level does increase accuracy, table 4.2, but at the cost of computational time. We also see from figure 4.6, that a lower tolerance is needed for convergence in the finite difference scheme, but it does not seem necessary to have $\epsilon$ lower than $10^{-6}$. It is important to note that Black-Scholes model itself is also just an approximation, with many assumptions considered to be incorrect. As stated in section 2.1.1, the stock prices tend to be fat-tailed instead of log-normally distributed, this could potentially be overcome by applying a student-t distribution or the Weibull distribution for modeling the stock prices. Also, another flaw in the

Black-Scholes method is the assumed constant volatility, which potentially could be overcome by applying a GARCH-process to the volatility in Black-Scholes model [15]. The advantages of using the binomial model for Highlander is obvious, but there are some situations where the finite difference model could perhaps be advantageous. If the option being priced is evaluated on a stock that pays dividend, then the finite difference model could be better than the binomial model, also to make up for one of the drawbacks from the Black-Scholes model – the constant volatility, one can apply local volatility to the finite difference method, more easily than for the binomial model. In addition, the boundary condition used for the finite difference scheme is the Dirichlet boundary condition, which is necessary to accurately price barrier options. Thus, to implement on these types of options, it is necessary to use a finite difference scheme instead of the binomial model. Another possible implementation is a finite element method, and see how well it performs in comparison to the methods presented in this thesis.

# Bibliography


[1] Rekhit Pachanekar. Basics Of Options Trading Explained, 2019. https://blog.quantinsti.com/basics-options-trading/

[2] Andrea Pascucci, Wolfgang J. Runggaldier. Financial Mathematics - Theory and Problems for Multi-period Models, 2009

[3] Cory Mitchell Reviewed by Julius Mansa, Volatility Smile Definition and Uses, 2021. https://www.investopedia.com/terms/v/volatilitysmile.asp

[4] How Binomial Trees Work in Option Pricing. https://www.macroption.com/binomial-trees/

[5] Simone Calogero, Option pricing theory (Lecture notes), 2020, Chalmers University of Technology.

[6] Norbert Hillber, Oleg Reichmann, Christoph Schwab, Christoph Winter. Computational Methods for Quantitative Finance, 2013.

[7] F. Black, M. Scholes: The Pricing of Options and Corporate Liabilities. The Journal of Political Economy 81, 637–654 (1973) https://www.cs.princeton.edu/courses/archive/fall09/cos323/papers/black_scholes73.pdf

[8] Black-Scholes equation, Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_equation

[9] Adam Hayes, reviewd by Gordon Scott, Black-Scholes Model, 2021. https://www.investopedia.com/terms/b/blackscholes.asp

[10] Bradford Cornell: Warren Buffett, Black-Scholes and the Valuation of Long-dated Options (2009). https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1433622

[11] Finite Difference Method, Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Finite_difference_method

[12] Explicit and implicit methods, Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Explicit_and_implicit_methods

[13] Mary Hall, Using the "Greeks" to Understand Options, 2021. https://www.investopedia.com/trading/using-the-greeks-to-understand-options/

[14] Successive over-relaxation, Wikipedia, the free encyclopedia. https://en.wikipedia.org/wiki/Successive_over-relaxation

[15] H.Gong, A. Thavanewaran, J singh: A Black-Scholes model with GARCH volatility (2010) https://www.researchgate.net/publication/265442526_A_Black-Scholes_model_with_GARCH_volatility

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