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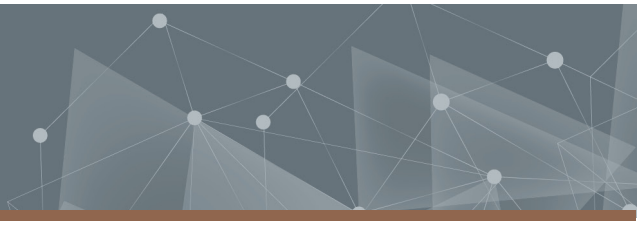

The mathematics of hearing the shape of a drum and filling Kac's holes

Master's thesis in Engineering mathematics and computational science

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Abstract

In 1966 Mark Kac asked whether it is possible to hear the shape of a drum, that is, if the set of eigenvalues of the Laplace operator on a smoothly bounded domain in the Euclidean plane determine the domain up to isometries. In fact, Kac credited his colleague, Professor Bers, for this picturesque language. The article not only explored this question, but also impressed upon readers the multitude of connections between this problem and various parts of mathematics and physics. Kac was forthcoming about his arguments being mostly heuristics justified by physics and as such not fully rigorous. However, it turns out that all of his arguments can be made rigorous using more recent techniques like microlocal analysis. Here we show how to make all of Kac's arguments rigorous. This includes demonstrating that if a sequence of convex polygons in the plane, with or without convex polygonal holes, converges in the sense of Hausdorff convergence to a convex smoothly bounded domain, then the first three coefficients in the asymptotic expansion of the heat trace of the polygons converge to those of the smoothly bounded domain. This appears to be the first time this has ever been done, showing that all of Kac's heuristics were in fact correct. Finally, we give a survey of Kac's legacy, the wealth of mathematics that Kac helped to inspire.

Keywords: Smoothly bounded domain, Spectral invariants, Heat kernel, Heat trace, Laplace operator, Eigenvalues, Mark Kac.

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Gustav Mårdby, Gothenburg, June 2023

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1
Introduction

This thesis is based on the article *Can one hear the shape of a drum?* written by Mark Kac in 1966 [64]. In the article Kac asks whether it is possible to determine the shape of a two-dimensional domain in the Euclidean plane if one is given its spectrum, i.e. the set of all eigenvalues of the Laplace operator. In other words, is the entire domain a spectral invariant? While one knew that the answer was no in 16 dimensions already in 1964 [75], the problem in two dimensions remained unsolved until 1992 when Carolyn Gordon, David Webb, and Scott Wolpert showed that the answer is no by constructing two domains which are different but have the same spectrum [49].

While the problem stated by Kac and its answer are interesting in their own right, what is arguably just as interesting are the branches of mathematics that were inspired and developed because of his article. One goal of this thesis is to give a tour of this development. Another goal of the thesis is to examine Kac's article in great detail. In particular, we want to explain the physics and statistical mechanics in the article, both ideas and calculations, and how they can be used to draw conclusions about what one can hear of a drum, i.e. what properties of the domain that are spectral invariants. For example, we will see that both the area and the perimeter are spectral invariants (under suitable assumptions), and we want to understand how Kac deduces this.

One part of Kac's article which is particularly interesting is when he tries to prove, using the short time asymptotic expansion of the heat trace, that one can "hear" the number of holes in the domain. In particular, Kac approximates a smooth domain by polygons (which have singular points, or "corners"), and the question is whether the expansion of the heat trace of the polygons converges to the expansion of the heat trace of the domain as one would expect. It turns out that Kac's "proof" is not complete, and has to this day never been completed. One main goal of this thesis is therefore to try to actually complete this. We know what the answer to the limit should be, but will Kac's method give us the right answer? One reason that this has not been done before is that it is not very well understood what happens when one approximates something smooth by something singular. Indeed, there are quite a few open questions regarding the behavior of spectral invariants, like the heat trace, under geometric deformations of domains. Therefore it is highly desirable to try to answer this question.

The structure of the thesis is as follows. In §2 we present the concepts and re-

sults that will be needed throughout the thesis. Many of these things should be already familiar to anyone who has taken a course in partial differential equations or Fourier analysis. Exceptions may include Hausdorff convergence, properties of the heat kernel and the heat trace, and Weyl's law. In §3 we study Kac's article. This includes summarizing his ideas and explaining his calculations. Then, we make all of his work rigorous in §4 using modern techniques. Finally, in §5 we give a comprehensive survey of the mathematics inspired by Kac's article, including when they actually solved the problem stated by Kac.

2

Prerequisites

Here we state the definitions and results that will be needed later.

2.1 Analysis

Definition 2.1.1. A *domain* $\Omega \subset \mathbb{R}^n$ is a connected open subset of $\mathbb{R}^n$.

From now on, we will always assume that Ω is a domain in $\mathbb{R}^n$, where n is some unspecified positive integer, unless otherwise stated. We denote the boundary of Ω by $\partial\Omega$.

Definition 2.1.2. For a domain Ω, $L^2(\Omega)$ is defined as the set of equivalence classes consisting of Lebesgue measurable functions $f : \Omega \to \mathbb{C}$ such that the Lebesgue integral $\int_\Omega |f(x)|^2 dx < \infty$, where two functions f, g are equivalent if

$$\{x \in \Omega : f(x) \neq g(x)\}$$

has Lebesgue measure 0. We write $f \in L^2(\Omega)$ to indicate that there exists an equivalence class in $L^2(\Omega)$ containing f. This makes $L^2(\Omega)$ a Hilbert space with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)}dx,$$

where again f and g are representatives from equivalence classes of $L^2(\Omega)$. From measure theory it follows that this inner product is well defined [40, p. 51].

We say that a function $f \in L^2(\Omega)$ is *normalized* if $\int_\Omega |f(x)|^2 dx = 1$.

Definition 2.1.3. Let k be a positive integer. If $f \in L^2(\Omega)$ and in addition the weak partial derivatives of f of order up to k exist and are in $L^2(\Omega)$, then we write $f \in H^k(\Omega)$. The inner product of $H^k(\Omega)$ is given by

$$\langle f, g \rangle_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \left\langle \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}, \frac{\partial^{|\alpha|} g}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n} \right\rangle_{L^2(\Omega)},$$

where

$$\alpha = (\alpha_1, \dots, \alpha_n),\ |\alpha| = \alpha_1 + \cdots + \alpha_n,\ \alpha_i \in \mathbb{N}.$$

Definition 2.1.4. For a function $f : \Omega \to \mathbb{C}$, we write $f \in \mathscr{C}^\infty(\Omega)$ if f is infinitely differentiable, that is, if f is N times differentiable for every $N \in \mathbb{N}$. Moreover, we write $f \in \mathscr{C}_c^\infty(\Omega)$ if $f \in \mathscr{C}^\infty(\Omega)$ and the support of f is compactly contained in Ω.

Note that if Ω is bounded, then $\mathscr{C}_c^\infty(\Omega) \subset H^1(\Omega)$ since f having compact support implies that f and any finite number of its partial derivatives attain a maximum and a minimum, from which it follows that they are L^2 integrable over Ω. The space $H_0^1(\Omega)$ is obtained by taking the closure of $\mathscr{C}_c^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm.

Definition 2.1.5. The *Laplace operator* is the partial differential operator given by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Definition 2.1.6. A function $f : \Omega \to \mathbb{C}$ is said to satisfy *Dirichlet boundary conditions* if $f = 0$ on $\partial\Omega$. Similarly, f satisfies *Neumann boundary conditions* if

$$\frac{\partial f}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\frac{\partial f}{\partial n}$ is the normal derivative of f pointing outwards from Ω.

When we consider partial differential equations containing the Laplace operator Δ, we will always (except when we're working with Weyl's law, see §2.6) have Dirichlet boundary conditions, meaning that our function f will be zero on the boundary. As a result, the domain of Δ on Ω will be $H_0^1(\Omega) \cap H^2(\Omega)$. Moreover, we will always assume that Ω has either smooth or piecewise smooth boundary.

Definition 2.1.7 ([41])**.** A one-dimensional *regular Sturm-Liouville problem* (SLP) on an interval $[a, b]$ is an equation of the form

$$\begin{cases} L(f) + \lambda w f = 0 \text{ on } [a, b], \\ B_1(f) = B_2(f) = 0. \end{cases} \tag{2.1}$$

Here, $L(f) = (rf')' + pf$ is a differential operator, where r, r', p are given real and continuous functions on $[a, b]$ satisfying $r > 0$ and

$$r(b)(\overline{g(b)}f'(b) - \overline{g'(b)}f(b)) = r(a)(\overline{g(a)}f'(a) - \overline{g'(a)}f(a))$$

for all $f, g \in L^2(a, b)$, w is a positive and continuous function on $[a, b]$, λ is some constant, and B_1 and B_2 are boundary conditions of the form

$$\begin{aligned} B_1(f) &= \alpha_1 f(a) + \alpha_1' f'(a) + \beta_1 f(b) + \beta_1' f'(b) = 0, \\ B_2(f) &= \alpha_2 f(a) + \alpha_2' f'(a) + \beta_2 f(b) + \beta_2' f'(b) = 0, \end{aligned}$$

where $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \beta_1, \beta_1', \beta_2, \beta_2'$ are constants. If (2.1) has a non-zero solution f, then λ is called an *eigenvalue* with corresponding *eigenfunction* f. The set of all eigenvalues is called the *spectrum* of the operator L.

We will see higher dimensional analogues of SLPs. However, in the one-dimensional case we will only be interested the case where $w = 1$. One reason that SLPs are important is because of the so called *spectral theorem*. We will need the following special case of this theorem.

Theorem 2.1.8 (Spectral theorem, [41])**.** *Given the SLP* (2.1) *with* $w = 1$*, there exists an orthonormal basis* $\{\phi_n\}_{n=1}^{\infty}$ *of* $L^2(a, b)$ *consisting of eigenfunctions. The eigenvalues* λ_n *of* ϕ_n *are real and satisfy* $\lim_{n\to\infty} \lambda_n = \infty$*. Moreover, if* $f \in \mathscr{C}^2(a, b)$ *and* f *satisfies the boundary conditions* $B_1(f) = B_2(f) = 0$*, then* $\sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ *converges uniformly to* f*.*

Definition 2.1.9. Given the SLP (2.1), we let $N(\lambda)$ denote the number of eigenvalues up to λ, i.e.

$$N(\lambda) = |\{n \in \mathbb{N} : \lambda_n \leq \lambda\}|. \tag{2.2}$$

Here we count multiplicity, so that an eigenvalue with, say, m linearly independent eigenfunctions is counted m times.

Definition 2.1.10. Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are *asymptotically equal to each other as* $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

We write this as $f(x) \sim g(x)$, $x \to \infty$. Functions being asymptotically equal as $x \to 0$ is defined analogously.

Now we state a standard result from calculus which we will use a lot, namely how to find all solutions to the ordinary differential equation $y'' + ay' + by = 0$. A proof can be found in most calculus textbooks, for example [88, p. 389].

Theorem 2.1.11. *Consider the equation*

$$y''(x) + ay'(x) + by(x) = 0, \ x \in \mathbb{R}, \tag{2.3}$$

where $a, b \in \mathbb{C}$ *are given constants. Let* r_1 *and* r_2 *be the (complex) roots to the characteristic polynomial* $p(r) = r^2 + ar + b$*. If* $r_1 \neq r_2$*, then* $\{e^{r_1 x}, e^{r_2 x}\}$ *is a basis of the solutions to* (2.3)*, meaning that all solutions to* (2.3) *are of the form* $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$*, where* $C_1, C_2 \in \mathbb{C}$*. If* $r_1 = r_2$*, then a basis is given by* $\{e^{r_1 x}, x e^{r_1 x}\}$*.*

2.2 Geometry

Definition 2.2.1. For a domain $\Omega \subset \mathbb{R}^n$, we let $|\Omega|$ denote the n-dimensional volume of Ω and $|\partial\Omega|$ the $(n-1)$-dimensional volume of the boundary of Ω. In particular, if $n = 2$ then $|\Omega|$ and $|\partial\Omega|$ denote the area and the perimeter of Ω, respectively.

In §4.4, we will need the notion of domains in the plane converging to another domain. To define this, we need a way of measuring the distance between two sets.

We will use Hausdorff distance (see [89, ch. 4-5]).

Definition 2.2.2. In $\mathbb{R}^2$, an *isometry* is an affine linear mapping from $\mathbb{R}^2$ to itself which corresponds to a composition of rotations, translations, and reflections. If Θ and Ω are two sets in $\mathbb{R}^2$, then the *Hausdorff distance* between Θ and Ω is defined as

$$D(\Theta, \Omega) = \inf_{T,S \text{ isometries}} d_H(T(\Theta), S(\Omega)),$$

where

$$d_H(\Theta, \Omega) = \max\left\{\sup_{x\in\Theta}\inf_{y\in\Omega}||x-y||,\ \sup_{x\in\Omega}\inf_{y\in\Theta}||x-y||\right\}.$$

Given a sequence of sets $\{\Omega_k\}_{k=1}^{\infty}$ in $\mathbb{R}^2$, we say that Ω_k *converges* to some set $\Omega \subset \mathbb{R}^2$, written $\Omega_k \to \Omega$ as $k \to \infty$, if $D(\Omega_k, \Omega) \to 0$ as $k \to \infty$.

For any $x \in \mathbb{R}^2$, $\inf_{y\in\Omega}||x-y||$ is often called the *distance* between x and Ω. The following two propositions give some justification of the above definition.

Proposition 2.2.3. *Given two non-empty sets $\Theta, \Omega \subset \mathbb{R}^2$, we have $d_H(\Theta, \Omega) = 0$ if and only if $\overline{\Theta} = \overline{\Omega}$.*

Proof. Assume first that $\overline{\Theta} = \overline{\Omega}$. We show that $\sup_{x\in\Theta}\inf_{y\in\Omega}||x-y|| = 0$ (the other quantity is shown to be zero in the same way). For any $x \in \Theta$, we have $x \in \overline{\Omega}$, which means that there for any $\epsilon > 0$ exists a $y \in \Omega$ such that $||x-y|| < \epsilon$. This shows that $\inf_{y\in\Omega}||x-y|| < \epsilon$, and so $\inf_{y\in\Omega}||x-y|| = 0$. Since this is true for any $x \in \Theta$, it follows that $\sup_{x\in\Theta}\inf_{y\in\Omega}||x-y|| = 0$.

Now assume that $d_H(\Theta, \Omega) = 0$. We want to show that $\overline{\Theta} \subset \overline{\Omega}$ (the proof that $\overline{\Omega} \subset \overline{\Theta}$ is similar). Let $x \in \overline{\Theta}$. Then, for any $\epsilon > 0$, we can choose $x' \in \Theta$ such that $||x-x'|| < \epsilon/2$. Moreover, we have by assumption that $\inf_{y\in\Omega}||x'-y|| = 0$, which gives that there exists a $y \in \Omega$ such that $||x'-y|| < \epsilon/2$. Then $||x-y|| \leq ||x-x'|| + ||x'-y|| < \epsilon$. Thus, we have $x \in \overline{\Omega}$, which completes the proof. □

Proposition 2.2.4. *Given two closed non-empty sets $\Theta, \Omega \subset \mathbb{R}^2$ and $\epsilon > 0$, we have $d_H(\Theta, \Omega) < \epsilon$ if and only if*

$$\begin{aligned}\Omega &\subset \bigcup_{x\in\Theta} B_\epsilon(x),\\ \Theta &\subset \bigcup_{x\in\Omega} B_\epsilon(x),\end{aligned} \tag{2.4}$$

where

$$B_\epsilon(x) = \{y \in \mathbb{R}^2 : ||x-y|| < \epsilon\}.$$

Proof. If $d_H(\Theta, \Omega) < \epsilon$, then for any $x \in \Omega$, we have $\inf_{y\in\Theta}||x-y|| < \epsilon$. This gives that there exists a $y_x \in \Theta$ such that $||x-y_x|| < \epsilon$, which shows that $x \in B_\epsilon(y_x)$. Analogously, for any $y \in \Theta$ there is an $x_y \in \Omega$ such that $y \in B_\epsilon(x_y)$.

Conversely, if (2.4) holds, then for any $x \in \Omega$ there is a $y_x \in \Theta$ such that $||x-y|| < \epsilon$, which implies that $\inf_{y\in\Theta}||x-y|| < \epsilon$. Since this holds for any $x \in \Omega$ and Θ, Ω are

closed, it follows that $\sup_{x\in\Omega}\inf_{y\in\Theta}||x-y|| < \epsilon$. Similarly $\sup_{x\in\Theta}\inf_{y\in\Omega}||x-y|| < \epsilon$, and thus $d_H(\Theta,\Omega) < \epsilon$. □

In §4 we will use the *Euler characteristic* (sometimes called the *Euler number*) of a domain in the plane.

Definition 2.2.5. For a compact set Ω consisting of finitely many polygons, the *Euler characteristic* of Ω is

$$\chi(\Omega) = V - E + F,$$

where V is the number of vertices in Ω, E is the number of edges in Ω, and F is the number of polygons in Ω.

For a smoothly bounded domain Ω, one computes the Euler characteristic using *triangulations*. Intuitively, a triangulation of Ω is a collection of "triangles" which need not have straight edges. Their union should be Ω and the intersection of two triangles is either empty, a common edge, or a common vertex. See [90, Definition 13.4.1] for the precise definition of a triangulation. The Euler characterstic of Ω is then given by the Euler characterstic of the given triangulation. By [90, Corollary 13.4.6], $\chi(\Omega)$ is independent of the choice of triangulation.

Definition 2.2.6. A domain $\Omega \subset \mathbb{R}^2$ has $r \geq 0$ *holes* if r is the smallest integer such that there exist a non-empty simply connected domain $\widetilde{\Omega}$ and r non-empty simply connected, disjoint domains $\Theta_1, \ldots, \Theta_r \subset \widetilde{\Omega}$ such that

$$\Omega = \widetilde{\Omega} \setminus \bigcup_{i=1}^{r} \overline{\Theta_i}.$$

The domains $\Theta_1, \ldots, \Theta_r$ are then called the *holes* of Ω.

Theorem 2.2.7. *If Ω has r holes, then $\chi(\Omega) = 1 - r$.*

Proof. This follows from [56, Thm. 2.44]. □

2.3 The heat kernel and the heat trace

Here we summarize the definitions and results needed for different objects connected to the heat equation, including the heat kernel and the heat trace. More details on these topics and the proofs of the statements below can be found in [1].

Consider the heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = 0, \ x \in \Omega, \ t > 0, \\ u(\cdot,t) \in H_0^1(\Omega) \cap H^2(\Omega), \ t > 0, \\ u(x,0) = f(x) \in \mathscr{C}_c^0(\overline{\Omega}). \end{cases} \tag{2.5}$$

We have the following result.

Theorem 2.3.1. *There exists a unique map $e^{-t\Delta} : \mathscr{C}^0(\overline{\Omega}) \to \mathscr{C}^\infty(\overline{\Omega} \times \mathbb{R}^+)$, called heat operator, which satisfies*

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) e^{-t\Delta}(f)(x,t) = 0, \ x \in \Omega, \ t > 0, \\ e^{-t\Delta}(f)(x,0) = f(x) \in \mathscr{C}^0_c(\overline{\Omega}). \end{cases}$$

Moreover, there exists a map $h : \Omega \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$ such that h is the Schwartz kernel of $e^{-t\Delta}$, meaning that

$$e^{-t\Delta}(f)(x,t) = \int_\Omega h(x,y,t) f(y) dy, \ x \in \Omega, \ t > 0.$$

The Schwartz kernel exists for many different partial differential equations. For the heat equation (2.5), the Schwartz kernel is also called the *heat kernel*.

Another way to describe the heat kernel h is that it is the *fundamental solution* to (2.5). In particular, h has the following properties.

Proposition 2.3.2. *The heat kernel h satisfies*

$$\begin{gathered} h(x,y,t) = h(y,x,t), \ x,y \in \Omega, \ t > 0, \\ \left(\frac{\partial}{\partial t} - \Delta\right) h(x,y,t) = 0 \text{ for } \Delta \text{ acting in either } x \text{ or } y, \ t > 0, \\ h(x,y,0) = \delta_0(x-y), \ x,y \in \Omega, \end{gathered}$$

where δ_0 is the Dirac delta distribution. The last property means that $\langle h(x,y,0), f(y)\rangle_{L^2(\Omega)} = f(x)$, or, more precisely, $\lim_{t\to 0}\langle h(x,y,t), f(y)\rangle_{L^2(\Omega)} = f(x)$.

Proposition 2.3.3. *If $\Omega = \mathbb{R}^n$, then the heat kernel has the explicit expression*

$$h_{\mathbb{R}^n}(x,y,t) = \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t}, \ x,y \in \mathbb{R}^n, \ t > 0.$$

Proof. The reader can easily check that the first two conditions in Proposition 2.3.2 hold. To show that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} f(y) dy = f(x),$$

we follow [1, Page 149-150 and exercise 7]. Fix $x \in \mathbb{R}^n$, let $\epsilon > 0$, and choose $\delta > 0$ so that $|f(y) - f(x)| < \epsilon$ whenever $||y - x|| < \delta$. It is a straightforward computation to show that

$$\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} dx = 1,$$

which gives that

$$\left| \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} f(y) dy - f(x) \right| = \left| \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} (f(y) - f(x)) dy \right|$$
$$\leq \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} |f(y) - f(x)| dy$$

$$\begin{aligned} &= \int_{B_\delta(x)} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} |f(y) - f(x)| dy \\ &+ \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} |f(y) - f(x)| dy. \end{aligned} \tag{2.6}$$

Here, $B_\delta(x)$ is the open ball centered at x with radius r, i.e.

$$B_\delta(x) = \{y \in \mathbb{R}^n : ||y - x|| < \delta\}.$$

Since $|f(y) - f(x)| < \epsilon$ for all $y \in B_\delta(x)$, the first term in (2.6) can be estimated as

$$\int_{B_\delta(x)} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} |f(y) - f(x)| dy < \epsilon \int_{B_\delta(x)} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} dy \leq \epsilon.$$

For the second term in (2.6), recall that f has compact support (in particular, f is bounded) and write

$$C = \frac{2 \max_{x \in \mathbb{R}^n} |f(x)|}{(4\pi)^{n/2}}.$$

Then

$$\int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} |f(y) - f(x)| dy \leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(x)} e^{-||x-y||^2/4t} dy,$$

which, after change of variables, can be written as

$$C \int_{\mathbb{R}^n \setminus B_{\delta/\sqrt{t}}(0)} e^{-||y||^2/4} dy.$$

This goes to 0 as $t \to 0$. In particular, the second term in (2.6) will be less than ϵ for t sufficiently close to 0. Then

$$\left| \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-||x-y||^2/4t} f(y) dy - f(x) \right| < 2\epsilon,$$

which completes the proof. □

If $\Omega = \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$, then it can similarly be shown, using the method of images [43], that the heat kernel with Dirichlet boundary conditions at $x_n = 0$ is

$$h_{\mathbb{R}^n_+}(x, y, t) = \frac{e^{-||x-y||^2/4t} - e^{-||x+y||^2/4t}}{(4\pi t)^{n/2}}, \ x, y \in \mathbb{R}^n. \tag{2.7}$$

For an arbitrary domain $\Omega \subset \mathbb{R}^n$ it is usually not possible to write out h explicitly. However, we have the following.

Proposition 2.3.4. *If Ω is bounded, then*

$$h(x, y, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \overline{\phi_n(y)}, \tag{2.8}$$

where ϕ_n are normalized eigenfunctions with corresponding eigenvalues λ_n, so that

$$\begin{cases} \Delta \phi_n(x) = \lambda_n \phi_n(x), \ x \in \Omega, \\ \phi_n \in H^1_0(\Omega) \cap H^2(\Omega). \end{cases}$$

Definition 2.3.5. The *heat trace* is defined as

$$\int_\Omega h(x,x,t)dx = \sum_{n=1}^{\infty} e^{-\lambda_n t} \int_\Omega |\phi_n(x)|^2 dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

Integrating termwise is justified by the monotone convergence theorem (see [40, p. 51]).

2.4 The wave equation in one dimension

Suppose we have a string of length l which is held fixed in its two endpoints $x = 0$ and $x = l$. We give it some initial position $f(x)$ and some initial velocity $g(x)$. Let $F(x,t)$ denote the height of the string at a point x and at time t relative to the endpoints. Then $F(x,t)$ satisfies the one-dimensional wave equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2} F(x,t) = c^2 \Delta F(x,t), \ 0 < x < l, \ t > 0, \\ F(0,t) = F(l,t) = 0, \ t > 0, \\ F(x,0) = f(x), \ 0 < x < l, \\ F_t(x,0) = g(x), \ 0 < x < l. \end{cases} \tag{2.9}$$

Here, $c > 0$ is a given constant, and it is common to choose units so that $c^2 = 1$. However, Kac instead chooses $c^2 = 1/2$ because that makes one of his formulas from physics simpler (see [64, p. 5]). Right now we will use $c^2 = 1$, but later in §3 when we describe Kac's work in more detail, we will explain the changes that occur due to his choice of $c^2 = 1/2$.

We solve (2.9) by separating variables. That is, we look for solutions of the form $X(x)T(t)$ for some unknown functions X and T. Inserting this into (2.9) gives

$$X(x)T''(t) = X''(x)T(t),$$

or

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{2.10}$$

Since the left hand side in (2.10) is independent of x and the right hand side is independent of t, both sides must be constant. Thus, we have

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

for some $-\lambda \in \mathbb{C}$. The equation for X combined with the boundary conditions in (2.9) then gives

$$\begin{cases} X''(x) + \lambda X(x) = 0, \ 0 < x < l, \\ X(0) = X(l) = 0. \end{cases} \tag{2.11}$$

This is a regular SLP (see Definition 2.1.7). By the spectral theorem for SLPs (see Theorem 2.1.8), there exists an orthogonal basis for $L^2(0,l)$ consisting of eigenfunctions X_n with corresponding eigenvalues λ_n which together satisfy (2.11). Thus, we

can indeed obtain the complete solution to (2.9) by first solving (2.11). Now, if $\lambda = 0$, then the equation becomes $X''(x) = 0$, whose general solution is $X(x) = Ax + B$. However, the boundary conditions immediately give that $A = B = 0$, so $X(x) = 0$ is the only solution in this case.

If $\lambda < 0$, then the characteristic polynomial $p(r) = r^2 + \lambda$ has the roots $r = \pm\sqrt{|\lambda|}$, and so by Theorem 2.1.11 the general solution becomes

$$X(x) = Ae^{x\sqrt{|\lambda|}} + Be^{-x\sqrt{|\lambda|}},$$

for some constants $A, B \in \mathbb{C}$. If we set $A = (A' + B')/2$ and $B = (A' - B')/2$, then we can rewrite the solutions in the equivalent way

$$X(x) = A' \cosh\Big(x\sqrt{|\lambda|}\Big) + B' \sinh\Big(x\sqrt{|\lambda|}\Big).$$

The boundary condition $X(0) = 0$ now gives that $A' = 0$, so $X(x) = B' \sinh\Big(x\sqrt{|\lambda|}\Big)$. The other boundary condition then gives $X(l) = B' \sinh\Big(l\sqrt{|\lambda|}\Big) = 0$, which can only hold if $B' = 0$. We conclude that $X(x) = 0$ is the only solution when $\lambda < 0$.

Finally, if $\lambda > 0$, then the characteristic polynomial $p(r) = r^2 + \lambda$ instead has the roots $r = \pm i\sqrt{\lambda}$, which gives that the general solution becomes

$$X(x) = Ae^{ix\sqrt{\lambda}} + Be^{-ix\sqrt{\lambda}},$$

or, equivalently,

$$X(x) = A' \cos\Big(x\sqrt{\lambda}\Big) + B' \sin\Big(x\sqrt{\lambda}\Big),\ A', B' \in \mathbb{C}.$$

The first boundary condition gives $X(0) = A' = 0$, so that $X(x) = B' \sin\Big(x\sqrt{\lambda}\Big)$. From $X(l) = 0$, we then get

$$B' \sin\Big(l\sqrt{\lambda}\Big) = 0.$$

Since we are seeking non-zero solutions, we assume $B' \neq 0$ and solve $\sin\Big(l\sqrt{\lambda}\Big) = 0$. The solutions are given by $l\sqrt{\lambda} = \pi n$, $n \in \mathbb{Z}$. Thus, we get the eigenvalues

$$\lambda_n = \frac{\pi^2 n^2}{l^2}$$

with corresponding eigenfunctions

$$X_n(x) = B'_n \sin\Big(\frac{\pi n x}{l}\Big),\ 0 < x < l,\ n \in \mathbb{Z}.$$

However, note that $\lambda_n = \lambda_{-n}$ and that X_n and X_{-n} are scalar multiples of each other. Moreover, $X_0(x) = 0$ and is therefore not an eigenfunction. Thus, we only need $n \geq 1$.

We now insert the values of λ_n into the equation for T to obtain

$$T_n''(t) + \frac{\pi^2 n^2}{l^2} T_n(t) = 0,\ t > 0.$$

By Theorem 2.1.11, the solutions are given by

$$T_n(t) = a_n \cos\Big(\frac{\pi n t}{l}\Big) + b_n \sin\Big(\frac{\pi n t}{l}\Big),\ t > 0,\ n \geq 1,\ a_n, b_n \in \mathbb{C}.$$

We have therefore obtained infinitely many solutions to (2.9), namely

$$\begin{aligned} F_n(x,t) = X_n(x)T_n(t) &= B_n' \sin\Big(\frac{\pi n x}{l}\Big)\left(a_n \cos\Big(\frac{\pi n t}{l}\Big) + b_n \sin\Big(\frac{\pi n t}{l}\Big)\right) \\ &= \sin\Big(\frac{\pi n x}{l}\Big)\left(A_n'' \cos\Big(\frac{\pi n t}{l}\Big) + B_n'' \sin\Big(\frac{\pi n t}{l}\Big)\right),\ t > 0,\ n \geq 1,\ A_n'', B_n'' \in \mathbb{C}, \end{aligned}$$

where $A_n'' = a_n B_n'$ and $B_n'' = b_n B_n'$. Finally, by the spectral theorem for SLPs (see Theorem 2.1.8), it follows that $\{X_n\}_{n\geq 1}$ is an orthogonal basis of $L^2(0,l)$. Moreover, the full solution to (2.9) is

$$F(x,t) = \sum_{n=1}^{\infty} \sin\Big(\frac{\pi n x}{l}\Big)\left(A_n'' \cos\Big(\frac{\pi n t}{l}\Big) + B_n'' \sin\Big(\frac{\pi n t}{l}\Big)\right),$$

where

$$\begin{aligned} A_n'' &= \frac{\int_0^l f(x)\sin\Big(\frac{\pi n x}{l}\Big)dx}{\int_0^l \sin^2(\frac{\pi n x}{l})dx} = \frac{2}{l}\int_0^l f(x)\sin\Big(\frac{\pi n x}{l}\Big)dx, \\ B_n'' &= \frac{\int_0^l g(x)\frac{\pi n}{l}\sin\Big(\frac{\pi n x}{l}\Big)dx}{\int_0^l \frac{\pi^2 n^2}{l^2}\sin^2(\frac{\pi n x}{l})dx} = \frac{2}{\pi n}\int_0^l g(x)\sin\Big(\frac{\pi n x}{l}\Big)dx \end{aligned}$$

by the initial conditions.

2.5 The wave equation in two dimensions

Now let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain in the Euclidean plane which is held fixed along its boundary $\partial\Omega$. If we give Ω some initial position $f(x)$ and velocity $g(x)$, then the displacement $F(x,t)$ of Ω perpendicular to the plane at $x \in \Omega$ and at time t satisfies the two-dimensional wave equation given by

$$\begin{cases} \frac{\partial^2}{\partial t^2}F(x,t) = \Delta F(x,t),\ x \in \Omega,\ t > 0, \\ F(x,t) = 0 \text{ on } \partial\Omega, \\ F(x,0) = f(x),\ x \in \Omega, \\ F_t(x,0) = g(x),\ x \in \Omega. \end{cases} \tag{2.12}$$

Analogous to the one-dimensional case, we can solve (2.12) by separating variables. We then insert $X(x)T(t)$ into (2.12) to obtain

$$\frac{T''(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda \in \mathbb{C}.$$

Thus, $X(x)$ satisfies

$$\begin{cases} \Delta X(x) + \lambda X(x) = 0 \text{ in } \Omega, \\ X(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.13}$$

By [70, p. 83], there exists an orthogonal basis for $L^2(\Omega)$ consisting of eigenfunctions X_n with corresponding positive eigenvalues λ_n which together satisfy (2.13). We may without loss of generality assume that the eigenfunctions X_n are normalized. The equation for T,

$$T_n''(t) + \lambda_n T_n(t) = 0,$$

is the same as in the one-dimensional case and therefore has the same solution:

$$T_n(t) = a_n \cos\left(t\sqrt{\lambda_n}\right) + b_n \sin\left(t\sqrt{\lambda_n}\right),\ t > 0,\ n \geq 1,\ a_n, b_n \in \mathbb{C}.$$

Hence, as before, the full solution to (2.12) becomes

$$F(x,t) = \sum_{n=1}^{\infty} X_n(x) \left(a_n \cos\left(t\sqrt{\lambda_n}\right) + b_n \sin\left(t\sqrt{\lambda_n}\right) \right)$$

with

$$\begin{aligned} a_n &= \iint_\Omega f(x)\overline{X_n(x)}dx, \\ b_n &= \frac{1}{\sqrt{\lambda_n}} \iint_\Omega g(x)\overline{X_n(x)}dx. \end{aligned}$$

2.6 Weyl's law

Here we show how Weyl proved what is now known as *Weyl's law*. In two dimensions, Weyl's law says that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{4\pi}, \tag{2.14}$$

where $N(\lambda)$ is the number of eigenvalues up to λ (see Definition 2.1.9). The first things we need in order to prove this are the so called variational principles from PDE theory, which say the following.

Theorem 2.6.1 ([6])**.** *Let Ω be a domain with piecewise smooth boundary. If λ_k denotes the kth eigenvalue of the Laplace operator with corresponding eigenfunction f_k, then*

$$\lambda_k = \begin{cases} \inf_{\substack{f\in H \\ f\neq 0}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2} \text{ and is attained by } f_1, \text{ if } k = 1, \\ \inf_{\substack{f\in H \\ f\neq 0}} \left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2} : \langle f, f_j\rangle_{L^2(\Omega)} = 0,\ j = 1, \ldots, k-1 \right\} \\ \text{and is attained by } f_k, \text{ if } k \geq 2, \end{cases} \tag{2.15}$$

$$\lambda_k = \sup_{\substack{L\subset H \\ dim\ L = k-1}} \inf_{\substack{f\in L^\perp \\ f\neq 0}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2}, \tag{2.16}$$

$$\lambda_k = \inf_{\substack{L\subset H \\ dim\ L = k}} \sup_{\substack{f\in L \\ f\neq 0}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2}, \tag{2.17}$$

where the integrals are taken with the respect to the Lebesgue measure. Here, $H = H_0^1(\Omega)$ for Dirichlet boundary conditions, and $H = H^1(\Omega)$ for Neumann boundary conditions.

Remark 2.6.2. The expression

$$\frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2}$$

is called the *Rayleigh quotient* of f.

Another key ingredient that we need for the proof of (2.14) is to cover Ω by rectangles and consider the eigenvalues on those. This will be useful because on rectangles the eigenvalues and eigenfunctions can be computed analytically. Indeed, suppose $\Omega = [0, l_1] \times [0, l_2]$ for some $l_1, l_2 > 0$ and consider the eigenvalue problem with Dirichlet boundary conditions:

$$\begin{cases} \Delta U(x,y) + \lambda U(x,y) = 0 \text{ in } [0,l_1] \times [0,l_2], \\ U(x,y) = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.18}$$

To solve this, as usual we separate variables and look for solutions of the form $X(x)Y(y)$. Inserting this gives

$$X''(x)Y(y) + X(x)Y''(y) + \lambda X(x)Y(y) = 0$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \lambda.$$

Each side depends on different variables, so they are constant, say $-\mu$. Thus, we have reduced the problem to the one-dimensional eigenvalue problems

$$\begin{cases} X''(x) + \mu X(x) = 0, \ 0 < x < l_1, \\ X(0) = X(l_1) = 0, \end{cases}$$

$$\begin{cases} Y''(y) + (\lambda - \mu) Y(y) = 0, \ 0 < y < l_2, \\ Y(0) = Y(l_2) = 0, \end{cases}$$

which can be solved as in §2.4. The eigenvalues for X then become $\mu_n = \pi^2 n^2 / l_1^2$ with eigenfunctions $X_n(x) = A_n \sin\left(\frac{\pi n x}{l_1}\right)$, $n \geq 1$. Similarly for Y, the eigenvalues become $\omega_m = \pi^2 m^2 / l_2^2$ with eigenfunctions $Y_m(y) = B_m \sin\left(\frac{\pi m y}{l_2}\right)$, $m \geq 1$. By the spectral theorem, $\{\sin\left(\frac{\pi n x}{l_1}\right)\}_{n \geq 1}$ and $\{\sin\left(\frac{\pi m y}{l_2}\right)\}_{m \geq 1}$ are orthogonal bases of $L^2(0, l_1)$ and $L^2(0, l_2)$, respectively (see Theorem 2.1.8).

Now, note that if μ is an eigenvalue of X and ω is an eigenvalue of Y, then $\mu + \omega$ is an eigenvalue of U since

$$\Delta U(x,y) = X''(x)Y(y) + X(x)Y''(y) = \mu X(x)Y(y) + \omega X(x)Y(y) = (\mu + \omega)U(x,y).$$

This shows that all eigenvalues of (2.18) are given by $\lambda_{n,m} = \pi^2 n^2/l_1^2 + \pi^2 m^2/l_2^2$, and the eigenfunctions are $U_{n,m}(x,y) = X_n(x)Y_m(y) = C_{n,m}\sin\left(\frac{\pi n x}{l_1}\right)\sin\left(\frac{\pi m y}{l_2}\right)$. Indeed, by [70, p. 83] $\{U_{n,m}\}_{n,m\geq 1}$ is an orthogonal basis of $L^2([0,l_1]\times[0,l_2])$.

Now assume that (2.18) has Neumann boundary conditions instead. Then we solve the equation in the same way. However, after separating variables, the equation for X becomes

$$\begin{cases} X''(x) + \mu X(x) = 0, \ 0 < x < l_1, \\ X'(0) = X'(l_1) = 0. \end{cases}$$

This is different from the equations that we've seen earlier, but we proceed in the same way and consider different cases for μ. If $\mu = 0$, the equation becomes $X''(x) = 0$ and has the general solution $X(x) = Ax + B$. This time the boundary conditions only imply that $A = 0$, so all non-zero constant functions $X(x) = B$ are eigenfunctions corresponding to $\mu = 0$.

If instead $\mu < 0$, then by Theorem 2.1.11 the general solution becomes $X(x) = A\cosh\left(x\sqrt{|\mu|}\right) + B\sinh\left(x\sqrt{|\mu|}\right)$ with $X'(x) = \sqrt{|\mu|}\left(A\sinh\left(x\sqrt{|\mu|}\right) + B\cosh\left(x\sqrt{|\mu|}\right)\right)$. The boundary condition $X'(0) = 0$ gives $B = 0$, and then $X'(l_1) = 0$ gives $A = 0$, so that $X(x) = 0$ is the only solution.

Finally, if $\mu > 0$, then we (again by Theorem 2.1.11) get $X(x) = A\cos\left(x\sqrt{\mu}\right) + B\sin\left(x\sqrt{\mu}\right)$ and $X'(x) = \sqrt{\mu}\left(-A\sin\left(x\sqrt{\mu}\right) + B\cos\left(x\sqrt{\mu}\right)\right)$, from which $X'(0) = 0$ implies that $B = 0$, so $X(x) = A\cos\left(x\sqrt{\mu}\right)$. Then, from $X'(l_1) = 0$ we obtain the eigenvalues $\mu_n = \pi^2 n^2/l_1^2$ with corresponding eigenfunctions

$$X_n(x) = A_n \cos\left(\frac{\pi n x}{l_1}\right), \ n \geq 1.$$

Carrying out the same calculation for $Y(y)$ yields the eigenvalues $\omega_m = \pi^2 m^2/l_2^2$ and eigenfunctions

$$Y_m(y) = B_m \cos\left(\frac{\pi m y}{l_2}\right), \ m \geq 1.$$

Thus, arguing as above with the Dirichlet boundary conditions, we conclude that the eigenvalues for Neumann boundary conditions are the same as for the Dirichlet boundary conditions, except that for Neumann zero is also an eigenvalue. However, the eigenfunctions now become $U_{n,m}(x,y) = C_{n,m}\cos\left(\frac{\pi n x}{l_1}\right)\cos\left(\frac{\pi m y}{l_2}\right)$, which constitute a basis for $L^2([0,l_1]\times[0,l_2])$ [70, p. 83].

Now that we know the eigenvalues and eigenfunctions on rectangles, we can prove that Weyl's law holds on them.

Proposition 2.6.3. *Let $I = (a_1,b_1)\times(a_2,b_2) \subset \mathbb{R}^2$ be a rectangle. The both the Dirichlet and Neumann eigenvalues on I satisfy*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{|I|}{4\pi}.$$

Proof. Write $l_i = b_i - a_i,\ i = 1, 2$. We saw that the Dirichlet eigenvalues are

$$\frac{n^2\pi^2}{l_1^2} + \frac{m^2\pi^2}{l_2^2}$$

with eigenfunctions

$$\sin\left(\frac{n\pi x}{l_1}\right)\sin\left(\frac{m\pi x}{l_2}\right),\ n, m \geq 1.$$

The Neumann eigenvalues are the same as the Dirichlet eigenvalues except that for Neumann we also include $n = 0$ and $m = 0$. The eigenfunctions for Neumann are

$$\cos\left(\frac{n\pi x}{l_1}\right)\cos\left(\frac{m\pi x}{l_2}\right),\ n, m \geq 0.$$

By possibly translating I, we may assume that $a_i = 0$ and $b_i = l_i$.

Now, for any $R > 0$, note that $N(R^2)$ is the number of lattice points $p = (n\pi/l_1, m\pi/l_2)$ such that $|p|^2 \leq R^2$. This is asymptotically equal to $l_1 l_2 R^2/4\pi$. To see why, note that in the rectangle $r := [0, n\pi/l_1] \times [0, m\pi/l_2]$ there are exactly nm such lattice points. Since $|r| = nm\pi^2/l_1 l_2$, it follows that the number of lattice points equals the area multiplied by the factor $l_1 l_2/\pi^2$. As the lattice points are evenly spaced in the plane, this relation holds for any shape, up to asymptotic equality. So, in the case of $n^2\pi^2/l_1^2 + m^2\pi^2/l_2^2 \leq R^2$, the corresponding shape is a quarter circle with radius R, which has area $\pi R^2/4$. Hence the number of lattice points in it becomes asymptotically equal to $\pi R^2/4 \cdot l_1 l_2/\pi^2 = l_1 l_2 R^2/4\pi$, as claimed.

Thus, we get

$$N(R^2) \sim \frac{l_1 l_2 R^2}{4\pi}$$

or

$$N(\lambda) \sim \frac{l_1 l_2 \lambda}{4\pi},\ \lambda \to \infty.$$

The statement now follows by noting that $|I| = l_1 l_2$. □

We can now show that Weyl's law holds for an arbitrary domain Ω by approximating it with rectangles and using the Dirichlet and Neumann eigenvalues on the approximating rectangles together with the variational principles, following Weyl's original proof [113]. This is known as Dirichlet-Neumann bracketing.

Theorem 2.6.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the Dirichlet eigenvalues of the Laplace operator on Ω. If the boundary $\partial\Omega$ is piecewise smooth, then*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{4\pi}.$$

Proof. Let $\epsilon > 0$ and choose an inner covering $\Omega_I = \bigcup_{i=1}^{N} \Omega_{I,i}$ and an outer covering $\Omega_O = \bigcup_{j=1}^{M} \Omega_{O,j}$ consisting of finitely many disjoint rectangles (the rectangles may only overlap on their boundaries) such that $|\Omega_0 \backslash \Omega_1| < \epsilon$. We've already shown Weyl's law for each rectangle (see Proposition 2.6.3) and we want to show it for Ω.

Let $\lambda_k(\Omega_\cdot)$ and $\mu_k(\Omega_\cdot)$ denote the kth Dirichlet and Neumann eigenvalue on $\Omega_\cdot$, respectively. Take the unions

$$\bigcup_{i,k}\{\lambda_k(\Omega_{I,i})\},\ \bigcup_{j,k}\{\lambda_k(\Omega_{O,j})\},\ \bigcup_{i,k}\{\mu_k(\Omega_{I,i})\},\ \bigcup_{j,k}\{\lambda_k(\Omega_{O,j})\}$$

of the respective eigenvalues over the rectangles and reorder them into increasing sequences. Denote the results by $\{\tilde{\lambda}_k(\Omega_I)\}$, $\{\tilde{\lambda}_k(\Omega_O)\}$, $\{\tilde{\mu}_k(\Omega_I)\}$, and $\{\tilde{\mu}_k(\Omega_O)\}$, respectively.

To proceed in the proof, we prove the following lemma.

Lemma 2.6.5. *With the notations above, we have for every k*

$$\tilde{\mu}_k(\Omega_O) \overset{(1)}{\leq} \mu_k(\Omega_O) \overset{(2)}{\leq} \lambda_k(\Omega_O) \overset{(3)}{\leq} \lambda_k(\Omega) \overset{(4)}{\leq} \lambda_k(\Omega_I) \overset{(5)}{\leq} \tilde{\lambda}_k(\Omega_I).$$

Proof of lemma. There are five inequalities to prove, (1) through (5). Let's start with (3). Since $\Omega \subset \Omega_O$, we can extend functions $f \in H_0^1(\Omega)$ to equal 0 on $\Omega_O \backslash \Omega$. Then we have $f \in H_0^1(\Omega_O)$, and so $H_0^1(\Omega) \subset H_0^1(\Omega_O)$. Now (3) follows immediately from the variational principle (2.17) because taking the infimum over a larger set yields a smaller result. Inequality (4) follows in the same way as $\Omega_I \subset \Omega$. In fact, the same argument works for (5) because for each rectangle we can extend its H_0^1 functions to be zero outside it, and then it will be included in Ω_I. Similarly, (2) follows from the fact that for Neumann boundary conditions (2.17) minimizes over H^1, which is larger than what we minimize over for Dirichlet boundary conditions, namely H_0^1.

Thus, it remains to show (1). Recall that for Neumann boundary conditions on a rectangle, 0 is an eigenvalue with corresponding eigenfunctions being non-zero constant functions. Thus, if we let $R_1, \ldots, R_M$ denote the rectangles of Ω_O, then each R_i has 0 as an eigenvalue. This means that $\tilde{\mu}_1(\Omega_O) = \cdots = \tilde{\mu}_M(\Omega_O) = 0$. Since $\mu_i(\Omega_O) \geq 0$ for every i, it follows that $\tilde{\mu}_i(\Omega_O) \leq \mu_i(\Omega_O)$ for $i = 1, \ldots, M$. The next eigenvalue, $\tilde{\mu}_{M+1}(\Omega_O)$, will then be the smallest positive eigenvalue coming from all the rectangles $R_1, \ldots, R_M$. After re-labeling, we may assume that $\tilde{\mu}_{M+1}(\Omega_O)$ comes from rectangle R_1.

The next step is to use the variational principle (2.16). We have

$$\mu_{M+1}(\Omega_O) = \sup_{\substack{L \subset H^1(\Omega_O) \\ \dim L = M}} \inf_{\substack{f \in L^\perp \\ f \neq 0}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2}. \tag{2.19}$$

Now let $\tilde{L}$ be the M-dimensional subspace of $H^1(\Omega_O)$ spanned by the characteristic functions on the rectangles, so that a basis of $\tilde{L}$ is given by $\{f_1, \ldots, f_M\}$ where

$$f_i(x) = \begin{cases} 1, & x \in R_i, \\ 0, & x \notin R_i. \end{cases}$$

To show that $\tilde{\mu}_{M+1}(\Omega_O) \leq \mu_{M+1}(\Omega_O)$, by (2.19) it's enough to show that

$$\tilde{\mu}_{M+1}(\Omega_O) \leq \inf_{\substack{f \in \tilde{L}^\perp \\ f \neq 0}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2}.$$

Note that the eigenfunction u corresponding to $\tilde{\mu}_{M+1}(\Omega_O)$ is in $\tilde{L}^\perp$. Indeed, this is immediate for $R_2, \ldots, R_M$ since we extend u to be 0 on those. For R_1, recall the variational principle (2.15) which says that u is orthogonal to the eigenfunctions corresponding to $\tilde{\mu}_1, \ldots, \tilde{\mu}_M$. But these eigenfunctions are constant functions, so we obtain $\int_{R_1} u = 0$ and therefore u is also orthogonal to the characteristic functions on R_1, hence $u \in \tilde{L}^\perp$.

In fact, for any $f \in \tilde{L}^\perp$ we have $\int_{R_i} f = 0$ for every i by definition of $\tilde{L}$. In particular, f is orthogonal to the eigenfunctions corresponding to $\tilde{\mu}_1(\Omega_O), \ldots, \tilde{\mu}_M(\Omega_O)$. This gives, by (2.15), that for every i

$$\tilde{\mu}^{(i)} \leq \frac{\int_{R_i} |\nabla f|^2}{\int_{R_i} |f|^2},$$

where $\tilde{\mu}^{(i)}$ is the first positive Neumann eigenvalue on R_i. Then

$$\int_{R_i} |\nabla f|^2 \geq \tilde{\mu}^{(i)} \int_{R_i} |f|^2 \geq \tilde{\mu}_{M+1}(\Omega_O) \int_{R_i} |f|^2,$$

and so

$$\frac{\int_{\Omega_O} |\nabla f|^2}{\int_{\Omega_O} |f|^2} = \frac{\sum_{i=1}^M \int_{R_i} |\nabla f|^2}{\sum_{i=1}^M \int_{R_i} |f|^2} \geq \frac{\tilde{\mu}_{M+1}(\Omega_O) \sum_{i=1}^M \int_{R_i} |f|^2}{\sum_{i=1}^M \int_{R_i} |f|^2} = \tilde{\mu}_{M+1}(\Omega_O).$$

Since this holds for all $f \in \tilde{L}^\perp$, it also holds for the infimum:

$$\tilde{\mu}_{M+1}(\Omega_O) \leq \inf_{\substack{f \in \tilde{L}^\perp \\ f \neq 0}} \frac{\int_{\Omega_O} |\nabla f|^2}{\int_{\Omega_O} |f|^2}.$$

This completes the proof that $\tilde{\mu}_{M+1}(\Omega_O) \leq \mu_{M+1}(\Omega_O)$. Inequality (1) now follows by induction on k. □

Continuing with the proof of the theorem, let $N_N(\lambda, \Omega_\cdot)$ and $N_D(\lambda, \Omega_\cdot)$ denote the number of Neumann and Dirichlet eigenvalues less than or equal to λ on $\Omega_\cdot$, respectively. By Lemma 2.6.5, we have

$$\frac{\sum_{i=1}^N N_D(\lambda, \Omega_{I,i})}{\lambda} \leq \frac{N_D(\lambda, \Omega_I)}{\lambda} \leq \frac{N_D(\lambda, \Omega)}{\lambda} \leq \frac{N_D(\lambda, \Omega_O)}{\lambda} \leq \frac{N_N(\lambda, \Omega_O)}{\lambda} \leq \frac{\sum_{j=1}^M N_N(\lambda, \Omega_{O,j})}{\lambda}$$

because having larger eigenvalues means that there are fewer eigenvalues up to some fixed number. By Weyl's law on rectangles (see Proposition 2.6.3), we have

$$\lim_{\lambda \to \infty} \frac{\sum_{i=1}^N N_D(\lambda, \Omega_{I,i})}{\lambda} = \frac{\sum_{i=1}^N |\Omega_{I,i}|}{4\pi} = \frac{|\Omega_I|}{4\pi}$$

and similarly

$$\lim_{\lambda\to\infty} \frac{\sum_{j=1}^{M} N_N(\lambda, \Omega_{O,j})}{\lambda} = \frac{|\Omega_O|}{4\pi}.$$

Thus, we obtain

$$\frac{|\Omega_I|}{4\pi} \leq \liminf_{\lambda\to\infty} \frac{N_D(\lambda,\Omega)}{\lambda} \leq \limsup_{\lambda\to\infty} \frac{N_D(\lambda,\Omega)}{\lambda} \leq \frac{|\Omega_O|}{4\pi}.$$

Letting $\epsilon \to 0$ gives

$$\lim_{\lambda\to\infty} \frac{N_D(\lambda,\Omega)}{\lambda} = \frac{|\Omega|}{4\pi},$$

as desired. □

A consequence of Weyl's law is that the area of Ω is determined by the eigenvalues of the Laplace operator. More generally, any quantity that is determined by its spectrum, i.e. the set of eigenvalues, is a *spectral invariant*. In Kac's article Kac is interested to know to what extent the geometry of a planar domain is determined by its spectrum. In the next section, we will examine how Kac studies this using heuristic arguments from physics.

3

The mathematics and physics in Kac's paper

Before we begin studying Kac's article, we explain its title, *Can one hear the shape of a drum?*. Suppose the domain $\Omega \subset \mathbb{R}^2$ represents a membrane which can be identified with the head of a drum. If it is set in motion by some initial position and velocity, then, as we stated in §2.5, its position function satisfies the two-dimensional wave equation (2.12). Then, from a physical point of view, the eigenvalues of the corresponding eigenvalue problem (2.13) represent the *frequencies* that Ω can produce. Therefore, one might say that a quantity may be "heard" if it is determined by the eigenvalues. In other words, the quantity can be heard precisely if it is a spectral invariant [64, p. 1-2].

In §2.6 we showed how Weyl proved that the area of Ω is a spectral invariant. Kac is aware of this result and wants to find other quantities of Ω that are spectral invariants. Is perhaps the entire set Ω a spectral invariant (up to isometries)? In other words, can one hear the shape of a drum? Studying spectral invariants is interesting for several reasons, one of which being its connections to physics. Indeed, one of Kac's goals in his article with studying spectral invariants is to "impress upon you the multitude of connections between our problem and various parts of mathematics and physics" [64, p. 3].

To explore these connections between mathematics and physics, we now consider a system of M particles in a three-dimensional volume Ω such that the system is in equilibrium with temperature T. We assume that the particles comprise an ideal gas. The Schrödinger equation for such an ideal gas is a separable equation given by

$$\begin{cases} \frac{\hbar^2}{2m}\Delta\phi(x) = -E\phi(x) \text{ in } \Omega, \\ \phi(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

Above, $\hbar = h/2\pi$ with h being Planck's constant, m is the mass of each particle, and $E > 0$ is some constant. Rearranging this equation is equivalent to the Laplace eigenvalue equation

$$\begin{cases} \frac{1}{2}\Delta\phi(x) + \lambda\phi(x) = 0 \text{ in } \Omega, \\ \phi(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{3.2}$$

with $\lambda = mE/\hbar^2$. In particular, Kac chooses to include the factor $c^2 = 1/2$ in his eigenvalue equation. One could just as easily get the more standard eigenvalue

equation

$$\begin{cases} \Delta\phi(x) + \Lambda\phi(x) = 0 \text{ in } \Omega, \\ \phi(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{3.3}$$

by setting $\Lambda = 2\lambda$.

If (λ_n, ϕ_n) denotes the nth normalized eigenpair of (3.2), then we know from physics (more specifically, Boltzmann statistics, see e.g. [96, ch. 6] or [57, ch. 6]) that the probability of finding specified particles at $x_1, \ldots, x_M$ within $dx_1, \ldots, dx_M$ is

$$\prod_{k=1}^{M} \frac{\sum_{n=1}^{\infty} e^{-\lambda_n \tau} \phi_n^2(x_k)}{\sum_{n=1}^{\infty} e^{-\lambda_n \tau}} dx_k, \quad \tau = \frac{\hbar^2}{mkT}. \tag{3.4}$$

If the particles were instead moving uniformly in Ω, then the probability of finding specified particles would be

$$\frac{dx_1 \ldots dx_M}{|\Omega|^M}. \tag{3.5}$$

By the correspondence principle in quantum mechanics, the quantum perspective (3.4) and classical perspective (3.5) will approach each other as time goes to zero [105, p. 160-161]. This motivates the study of the short time asymptotic behavior of the heat trace

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}, \ t \to 0,$$

which will (hopefully) reveal geometric properties of Ω. At this point, Kac explains that one of the goals of the article is to obtain the first three terms in the short time asymptotic expansion of the heat trace in the case where Ω is a smoothly bounded convex domain in the plane, namely

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{|\partial\Omega|}{4\sqrt{2\pi t}} + \frac{1}{6}, \quad t \to 0. \tag{3.6}$$

This would show that in addition to the area, the perimeter is also a spectral invariant. The significance of 1/6 will be seen later.

3.1 The t^{-1}-term

To obtain the first term in the heat trace, Kac considers the diffusion equation, which in this case is identical to the heat equation, given by

$$\begin{cases} \frac{\partial}{\partial t} P_\Omega(x, y, t) = \frac{1}{2} \Delta P_\Omega(x, y, t), \ x \in \Omega, \\ P_\Omega(x, y, t) = 0 \text{ for } x \in \partial\Omega, \\ P_\Omega(x, y, 0) = \delta_0(x - y), \ x \in \Omega, \end{cases} \tag{3.7}$$

for some fixed $y \in \Omega$. The Laplace operator Δ is taken with respect to x. Kac explains the equation as letting a collection of particles be concentrated at y at $t = 0$ and then letting them diffuse inside Ω. The boundary condition $P_\Omega = 0$ on $\partial\Omega$ is

interpreted as the particles being "eaten" when they reach the boundary.

Mathematically, Kac is trying to find the heat kernel, i.e. the Schwartz kernel of the fundamental solution of the heat equation (see §2.3). From Proposition 2.3.4, it follows that

$$P_\Omega(x,y,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y), \tag{3.8}$$

where (λ_n, ϕ_n) are the normalized eigenpairs of (3.2). Kac argues that for small values of t the boundary $\partial\Omega$ should not have had much time to eat the particles coming from y. Thus, we should have $P_\Omega(x,y,t) \sim P_0(x,y,t)$, $t \to 0$, where P_0 is the solution to the same problem but without the boundary:

$$\begin{cases} \frac{\partial}{\partial t}P_0(x,y,t) = \frac{1}{2}\Delta P_0(x,y,t) \text{ in } \mathbb{R}^2, \\ P_0(x,y,0) = \delta_0(x-y). \end{cases}$$

Because of the factor of $1/2$ in the equation, the heat kernel on $\mathbb{R}^2$, known as the free heat kernel, will now be

$$P_0(x,y,t) = \frac{1}{2\pi t}e^{-||x-y||^2/2t} = \frac{1}{2\pi t}e^{-((x_1-y_1)^2+(x_2-y_2)^2)/2t},\ x=(x_1,x_2),\ y=(y_1,y_2).$$

To see why, we need to check that the conditions in Proposition 2.3.2 are satisfied. Clearly $P_0(x,y,t) = P_0(y,x,t)$. Moreover, since

$$\begin{aligned} \frac{\partial}{\partial t}P_0(x,y,t) &= \frac{e^{-||x-y||/2t}}{2\pi t^2}\left(\frac{||x-y||^2}{2t} - 1\right), \\ \frac{\partial^2}{\partial x_1^2}P_0(x,y,t) &= \frac{e^{-||x-y||/2t}}{2\pi t^2}\left(\frac{(x_1-y_1)^2}{t} - 1\right), \\ \frac{\partial^2}{\partial x_2^2}P_0(x,y,t) &= \frac{e^{-||x-y||/2t}}{2\pi t^2}\left(\frac{(x_2-y_2)^2}{t} - 1\right), \end{aligned}$$

we see that

$$\frac{\partial}{\partial t}P_0(x,y,t) = \frac{1}{2}\Delta P_0(x,y,t)$$

holds when Δ is taken with respect to x. By symmetry it then also holds with respect to y. Finally, for a function $f \in \mathscr{C}_c^0(\overline{\Omega})$, we need to show that

$$\lim_{t\to 0}\int_{\mathbb{R}^2}\frac{1}{2\pi t}e^{-||x-y||^2/2t}f(y)dy = f(x).$$

However, the proof of this is exactly the same as the proof of Proposition 2.3.3 and is therefore left as an exercise to the reader.

The relation $P_\Omega(x,y,t) \sim P_0(x,y,t)$, $t \to 0$, would now imply that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(x)\psi_n(y) \sim \frac{1}{2\pi t}e^{-||x-y||^2/2t},\ \ t \to 0.$$

Taking the trace, i.e. letting $x = y$ and integrating over Ω, then gives

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t}, \quad t \to 0, \tag{3.9}$$

which is the first term in the asymptotic expansion. Kac then attempts to make this argument rigorous by inscribing a square Q inside Ω and considering the corresponding solution P_Q with Q as its domain instead. Since $Q \subset \Omega$, Kac argues that the boundary of Q should eat more of the particles than the boundary of Ω, and therefore $P_Q(x, y, t) \leq P_\Omega(x, y, t) \leq P_0(x, y, t)$, which ultimately leads to (3.9). We will make the argument rigorous in a more elementary way using Weyl's law (see Theorem 2.6.4) in §4.1.

3.2 The $t^{-1/2}$-term

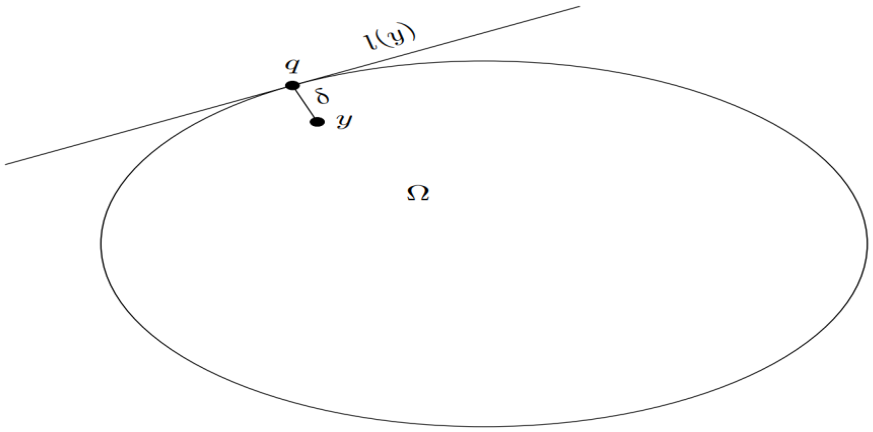

Figure 3.1: The boundary is approximated by a line.

For the second term in (3.6), Kac assumes that Ω is convex and that y is close to the boundary $\partial\Omega$, so that $\partial\Omega$ can be approximated by the line $l(y)$ which is tangent and passes through the point q on $\partial\Omega$ which is the closest to y (see Figure 3.1). The corresponding equation is

$$\begin{cases} \frac{\partial}{\partial t} P_{l(y)}(x, y, t) = \frac{1}{2} \Delta P_{l(y)}(x, y, t) \text{ in } \Omega, \\ P_{l(y)}(x, y, t) = 0 \text{ on } l(y), \\ P_{l(y)}(x, y, 0) = \delta_0(x - y) \text{ in } \Omega, \end{cases}$$

which again means that we're seeking the heat kernel. By the method of images [43], it is given by

$$P_{l(y)}(x, y, t) = \frac{e^{-||x-y||^2/2t} - e^{-||x-\tilde{y}||^2/2t}}{2\pi t},$$

where $\tilde{y}$ is the point obtained when y is reflected across $l(y)$. In particular,

$$P_{l(y)}(y, y, t) = \frac{1 - e^{-2\delta^2/t}}{2\pi t}, \ \delta = ||q - y||.$$

For small t, Kac argues that P_Ω and $P_{l(y)}$ should be so close that in fact

$$\iint_\Omega P_\Omega(y,y,t)d\rho \sim \iint_\Omega P_{l(y)}(y,y,t)dy,\ t\to 0.$$

This would imply that

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-\lambda_n t} &= \iint_\Omega P_\Omega(y,y,t)dy \sim \iint_\Omega P_{l(y)}(y,y,t)dy = \iint_\Omega \frac{1-e^{-2\delta^2/t}}{2\pi t}dy = \\ &= \frac{|\Omega|}{2\pi t} - \frac{1}{2\pi t}\iint_\Omega e^{-2\delta^2/t}dy.\end{aligned} \tag{3.10}$$

To approximate $\iint_\Omega e^{-2\delta^2/t}dy$, define $\partial\Omega(\delta)$ as the curve inside Ω with distance δ and the same curvature as $\partial\Omega$. Assuming $\tilde{\delta}$ is chosen small enough so that $\partial\Omega(\tilde{\delta})$ is well defined, we approximate

$$\iint_\Omega e^{-2\delta^2/t}dy \approx \int_0^{\tilde{\delta}} e^{-2\delta^2/t}|\partial\Omega(\delta)|d\delta \approx \int_0^\infty e^{-2x^2}|\partial\Omega|\sqrt{t}dx = \frac{|\partial\Omega|\sqrt{2\pi t}}{4}.$$

Inserting this into (3.10) gives

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{|\partial\Omega|}{4\sqrt{2\pi t}},\ t\to 0, \tag{3.11}$$

which gives the second term in the asymptotic expansion. Kac tries to make these arguments more rigorous by inscribing a rectangle R inside Ω and considering the corresponding solution P_R for that domain. With P_R as a lower bound and $P_{l(y)}$ as an upper bound for P_Ω, he again obtains (3.11). We will show that the argument can be made rigorous by using a locality principle that holds up to the boundary. We may call this *Kac's principle of feeling the boundary.*

3.3 The t^0-term

3.3.1 Preparations

Before Kac obtains the next term in the asymptotic expansion of the heat trace, he considers the corresponding diffusion equation when $\Omega = S(\theta)$ is an infinite sector with an angle $\theta \in (\pi/2, \pi)$ (see Figure 3.2). With polar coordinates $(r,\phi),(R,\alpha) \in S(\theta)$ and Δ acting with respect to (r,ϕ), the equation becomes

$$\begin{cases} \frac{\partial}{\partial t}P_{S(\theta)}(r,\phi,R,\alpha,t) = \frac{1}{2}\Delta P_{S(\theta)}(r,\phi,R,\alpha,t) \text{ in } S(\theta), \\ P_{S(\theta)}(r,\phi,R,\alpha,t) = 0 \text{ on } \partial S(\theta), \\ P_{S(\theta)}(r,\phi,R,\alpha,t) = \delta_0(r-R) \text{ in } S(\theta). \end{cases}$$

To obtain the heat kernel, Kac uses a formula by Carslaw and Sommerfeld, based on the method of images [21], given by

$$P_{S(\theta)}(r,\phi,R,\alpha,t) = v(r,\phi,R,\alpha,t) - v(r,\phi,R,-\alpha,t), \tag{3.12}$$

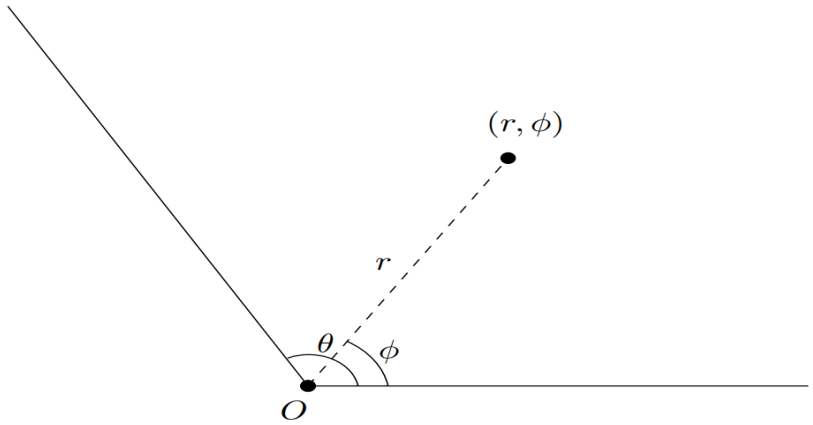

Figure 3.2: An infinite sector with angle $\theta \in (\pi/2, \pi)$.

where

$$\begin{aligned} v(r,\phi,R,\alpha,t) &= \frac{1}{2\pi t} \sum_{\phi-\alpha-\pi<2k\theta<\phi-\alpha+\pi} e^{-(r^2-2rR\cos(\phi-\alpha-2k\theta)+R^2)/2t} \\ &- \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-(r^2+R^2)/2t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-rR\cosh(s)/t}}{\cosh\left(\pi s/\theta + i\pi(\phi-\alpha)/\theta\right) - \cos(\pi^2/\theta)} ds. \end{aligned} \tag{3.13}$$

If we now let $(r, \phi) = (R, \alpha)$, then it follows that for $0 < \phi < \theta - \pi/2$

$$\begin{aligned} P_{S(\theta)}(r,\phi,r,\phi,t) &= \frac{1}{2\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/t}}{2\pi t} \\ &- \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)} ds \\ &+ \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)} ds. \end{aligned}$$

For $\theta - \pi/2 < \phi < \pi/2$, we get

$$\begin{aligned} P_{S(\theta)}(r,\phi,r,\phi,t) &= \frac{1}{2\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/t}}{2\pi t} - \frac{e^{-r^2(1-\cos(2(\theta-\phi))/t}}{2\pi t} \\ &- \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)} ds \\ &+ \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)} ds, \end{aligned}$$

and finally, for $\pi/2 < \phi < \theta$,

$$\begin{aligned} P_{S(\theta)}(r,\phi,r,\phi,t) &= \frac{1}{2\pi t} - \frac{e^{-r^2(1-\cos(2(\theta-\phi))/t}}{2\pi t} \\ &- \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)} ds \\ &+ \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/t}}{4\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)} ds. \end{aligned}$$

We will show in §4.2 how to derive these expressions of the heat kernel. However, there we will use $c^2 = 1$ (see (3.2)), which means that there will be some factors of 2 which are different from the expressions here. More precisely, we will see that if $c^2 = 1$, then for $0 < \phi < \theta - \pi/2$

$$\begin{aligned} P_{S(\theta)}(r,\phi,r,\phi,t) = \frac{1}{4\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/2t}}{4\pi t} \\ - \sin\left(\frac{\pi^2}{\theta}\right)\frac{e^{-r^2/2t}}{8\pi\theta t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)}ds \\ + \sin\left(\frac{\pi^2}{\theta}\right)\frac{e^{-r^2/2t}}{8\pi\theta t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)}ds, \end{aligned}$$

for $\theta - \pi/2 < \phi < \pi/2$,

$$\begin{aligned} P_{S(\theta)}(r,\phi,r,\phi,t) = \frac{1}{4\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/2t}}{4\pi t} - \frac{e^{-r^2(1-\cos(2(\theta-\phi))/2t}}{4\pi t} \\ - \sin\left(\frac{\pi^2}{\theta}\right)\frac{e^{-r^2/2t}}{8\pi\theta t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)}ds \\ + \sin\left(\frac{\pi^2}{\theta}\right)\frac{e^{-r^2/2t}}{8\pi\theta t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)}ds, \end{aligned}$$

and for $\pi/2 < \phi < \theta$,

$$\begin{aligned} P_{S(\theta)}(r,\phi,r,\phi,t) = \frac{1}{4\pi t} - \frac{e^{-r^2(1-\cos(2(\theta-\phi))/2t}}{4\pi t} \\ - \sin\left(\frac{\pi^2}{\theta}\right)\frac{e^{-r^2/2t}}{8\pi\theta t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)}ds \\ + \sin\left(\frac{\pi^2}{\theta}\right)\frac{e^{-r^2/2t}}{8\pi\theta t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)}ds. \end{aligned}$$

3.3.2 When Ω is a polygon

Now let Ω be a convex polygon with edges $E_1, \ldots, E_N$ and vertices $V_1, \ldots, V_N$ of obtuse angles $\theta_1, \ldots, \theta_N$ with V_i connecting E_i and E_{i+1}. Kac's idea is that particles diffusing from a point $y \in \Omega$ should either see the boundary $\partial\Omega$ as a straight line or an infinite sector with y close to a vertex V_i. He argues that this makes it reasonable to approximate the boundary as the nearest sector, and therefore he approximates P_Ω as $P_{S(\theta_i)}$.

Now, to obtain the heat trace around V_i, we take the heat kernel $P_{S(\theta_i)}(r,\phi,R,\alpha,t)$, set $(r,\phi) = (R,\alpha)$, and integrate over $S(\theta_i)$. As stated by Kac, the second integral in $P_{S(\theta_i)}(r,\phi,r,\phi,t)$ vanishes when one integrates it over $S(\theta_i)$. Kac doesn't include the proof of this, but it is included in [2]. We follow their calculation (but we use $c^2 = 1/2$, see (3.2)) and note that when we integrate over the angle $d\phi$, we get something proportional to

$$\int_0^{\theta_i}\frac{d\phi}{\cosh(\pi s/\theta_i + 2\pi i\phi/\theta_i) - \cos(\pi^2/\theta_i)}.$$

After the substitution $\theta = \pi s/\theta_i + 2\pi i\phi/\theta_i$, this turns into

$$\frac{\theta_i}{2\pi i}\int_{\pi s/\theta_i}^{\pi s/\theta_i + 2\pi i} \frac{d\theta}{\cosh(\theta) - \cos(\pi^2/\theta_i)},$$

which equals

$$-\frac{2}{\sqrt{1-\cos^2(\pi^2/\theta_i)}}\arctan\left(-\frac{(1+\cos(\pi^2/\theta_i))\tanh(\theta/2)}{\sqrt{1-\cos^2(\pi^2/\theta_i)}}\right)\Bigg|_{\pi s/\theta_i}^{\pi s/\theta_i + 2\pi i}.$$

One can check that this has the same value at the two endpoints, so indeed the integral becomes 0, just like Kac claims.

For the first integral in $P_{S(\theta_i)}(r,\phi,r,\phi,t)$, we get, when integrating over S_{θ_i},

$$\begin{aligned}\int_0^{\theta_i}\int_0^{\infty} -\sin\left(\frac{\pi^2}{\theta_i}\right)\frac{e^{-r^2/t}}{4\pi\theta_i t}\int_{-\infty}^{\infty}\frac{re^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i)}dsdrd\phi \\ = -\frac{1}{4\pi t}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_0^{\infty}\int_{-\infty}^{\infty}\frac{re^{-r^2(1+\cosh(s))/t}}{\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i)}dsdr.\end{aligned}$$

After interchanging integrals according to Fubini's theorem, we obtain

$$-\frac{1}{4\pi t}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i)}\int_0^{\infty} re^{-r^2(1+\cosh(s))/t}drds,$$

which simplifies to

$$-\frac{1}{8\pi}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{ds}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}.$$

This is the contribution of just one sector in Ω. To obtain the total contribution, we sum over all sectors:

$$-\frac{1}{8\pi}\sum_{i=1}^{N}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{ds}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}.$$

One of the terms in $P_{S(\theta_i)}(r,\phi,r,\phi,t)$ which we have not yet integrated over is $1/2\pi t$. Of course it will give $|\Omega|/2\pi t$. The remaining number of terms in $P_{S(\theta_i)}(r,\phi,r,\phi,t)$ depends on ϕ according to §3.3.1. In particular, integrating the last terms over $S(\theta_i)$ restricted to $\theta_i - \pi/2 \le \phi \le \pi/2$ gives

$$\begin{aligned}-\frac{1}{2\pi t}\int_{\theta_i-\pi/2}^{\pi/2}\int_0^{\infty}\left(e^{-r^2(1-\cos(2\phi))/t}+e^{-r^2(1-\cos(2(\theta_i-\phi)))/t}\right)rdrd\phi \\ = -\frac{1}{8\pi}\int_{\theta_i-\pi/2}^{\pi/2}\left(\frac{1}{\sin^2(\phi)}+\frac{1}{\sin^2(\theta_i-\phi)}\right)d\phi = -\frac{1}{4\pi}\cot\left(\theta_i-\frac{\pi}{2}\right).\end{aligned}$$

Summing over all sectors, this turns into

$$-\frac{1}{4\pi}\sum_{i=1}^{N}\cot\left(\theta_i-\frac{\pi}{2}\right).$$

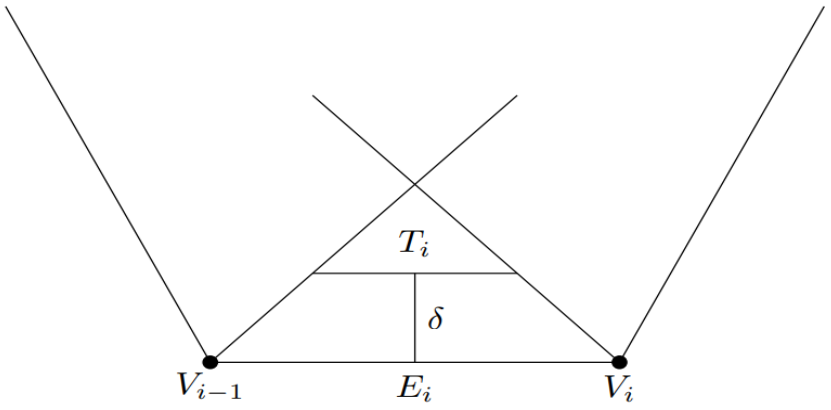

Figure 3.3: The region between two vertices in the polygon where the angles are small make up a triangle.

Now consider the region in the polygon Ω where $0 < \phi < \theta_{i-1} - \pi/2$ and $\pi/2 < \phi < \theta_i$ (here we identify θ_0 with θ_N). Note that the region is a triangle T_i with base E_i and base angles $\theta_{i-1} - \pi/2$ and $\theta_i - \pi/2$. Next, draw a line segment in T_i parallel to E_i with distance δ to E_i (see Figure 3.3). Assuming $\theta_{i-1} = \theta_i$, the length of this line segment becomes, by similar triangles,

$$|E_i| - 2\delta \cot\left(\theta_i - \frac{\pi}{2}\right).$$

Moreover, it is apparent in T_i that $\delta = r\sin(\phi)$, which can be rewritten as $2\delta^2 = r^2(1 - \cos(2\phi))$. Thus, when integrating the heat kernel over this region, we get, approximately,

$$\begin{aligned} &-\frac{1}{2\pi t}\int_0^\infty \left(|E_i| - 2\delta\cot\left(\theta_i - \frac{\pi}{2}\right)\right) e^{-2\delta^2/t} d\delta \\ &= -\frac{1}{2\pi t}\left(\frac{|E_i|}{2}\sqrt{\frac{\pi t}{2}} - \frac{t}{2}\cot\left(\theta_i - \frac{\pi}{2}\right)\right) = -\frac{|E_i|}{4\sqrt{2\pi t}} + \frac{1}{4\pi}\cot\left(\theta_i - \frac{\pi}{2}\right). \end{aligned}$$

Since $\sum_{i=1}^N |E_i| = |\partial\Omega|$, the total contribution becomes

$$-\frac{|\partial\Omega|}{4\sqrt{2\pi t}} + \frac{1}{4\pi}\sum_{i=1}^N \cot\left(\theta_i - \frac{\pi}{2}\right).$$

Finally, adding everything up, we get that the heat trace for our polygon is

$$\begin{aligned} &\sum_{n=1}^\infty e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{|\partial\Omega|}{4\sqrt{2\pi t}} + \frac{1}{4\pi}\sum_{i=1}^N \cot\left(\theta_i - \frac{\pi}{2}\right) - \frac{1}{4\pi}\sum_{i=1}^N \cot\left(\theta_i - \frac{\pi}{2}\right) \\ &- \frac{1}{8\pi}\sum_{i=1}^N \sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^\infty \frac{ds}{(1+\cosh(s))(\cosh(\pi s/\theta_i) - \cos(\pi^2/\theta_i))}. \end{aligned} \tag{3.14}$$

In particular, the sums containing $\cot\left(\theta_i - \frac{\pi}{2}\right)$ cancel. Now let $N \to \infty$ in a way that makes every $\theta_i \to \pi$. Then we should have $\sum_{i=1}^N \sin\frac{\pi^2}{\theta_i} \to -2\pi$ since, heuristically, if

all angles are equal, then $\theta_i = \pi(N-2)/N$ and therefore

$$\sum_{i=1}^{N} \sin \frac{\pi^2}{\theta_i} = N \sin\Big(\frac{\pi N}{N-2}\Big) \xrightarrow[N\to\infty]{} -2\pi$$

by a straightforward calculation from elementary calculus. Hence, the last term in (3.14) should approach

$$\frac{2\pi}{8\pi} \int_{-\infty}^{\infty} \frac{ds}{(1+\cosh(s))^2} = \frac{1}{6}, \tag{3.15}$$

which would imply that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{|\partial\Omega|}{4\sqrt{2\pi t}} + \frac{1}{6}, \;\; t \to 0.$$

If we instead had $c^2 = 1$ (see (3.2)), we would have gotten

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{6}, \;\; t \to 0.$$

3.3.3 If Ω has holes

The last thing that Kac mentions in his article is what happens with the heat trace when Ω is a polygon with $r \geq 1$ holes (see Definition 2.2.6) such that each hole is also a polygon. Then, points close to an N-sided hole Θ with angles $\theta_1, \ldots, \theta_N$ may view the boundary as a sector, so that the heat kernel can be computed as before. However, this time the angles θ_i will be between π and 2π. Using the formula for the heat kernel given by (3.12) and (3.13) then gives that for $0 < \phi < \pi/2$

$$\begin{aligned} P_{S(\theta_i)}(r,\phi,r,\phi,t) = {} & \frac{1}{2\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/t}}{2\pi t} \\ & - \sin\left(\frac{\pi^2}{\theta_i}\right) \frac{e^{-r^2/t}}{4\pi\theta_i t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i) - \cos(\pi^2/\theta_i)} ds \\ & + \sin\left(\frac{\pi^2}{\theta_i}\right) \frac{e^{-r^2/t}}{4\pi\theta_i t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i + 2\pi i\phi/\theta_i) - \cos(\pi^2/\theta_i)} ds, \end{aligned}$$

for $\pi/2 < \phi < \theta - \pi/2$

$$\begin{aligned} P_{S(\theta_i)}(r,\phi,r,\phi,t) = {} & \frac{1}{2\pi t} \\ & - \sin\left(\frac{\pi^2}{\theta_i}\right) \frac{e^{-r^2/t}}{4\pi\theta_i t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i) - \cos(\pi^2/\theta_i)} ds \\ & + \sin\left(\frac{\pi^2}{\theta_i}\right) \frac{e^{-r^2/t}}{4\pi\theta_i t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i + 2\pi i\phi/\theta_i) - \cos(\pi^2/\theta_i)} ds, \end{aligned}$$

and for $\theta - \pi/2 < \phi < \theta$

$$\begin{aligned} P_{S(\theta_i)}(r,\phi,r,\phi,t) = \frac{1}{2\pi t} - \frac{e^{-r^2(1-\cos(2(\theta_i-\phi))/t}}{2\pi t} \\ - \sin\left(\frac{\pi^2}{\theta_i}\right)\frac{e^{-r^2/t}}{4\pi\theta_i t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i)}ds \\ + \sin\left(\frac{\pi^2}{\theta_i}\right)\frac{e^{-r^2/t}}{4\pi\theta_i t}\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/t}}{\cosh(\pi s/\theta_i+2\pi i\phi/\theta_i)-\cos(\pi^2/\theta_i)}ds. \end{aligned}$$

Thus, when we compute the heat trace by integrating $P_{S(\theta_i)}(r,\phi,r,\phi,t)$, just like in §3.3.2 the constant term becomes

$$-\frac{1}{8\pi}\sum_{i=1}^{N}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{ds}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}.$$

However, this time, if we let $N\to\infty$ so that each $\theta_i\to\pi$, then since $\theta_i>\pi$ we now have $\sin(\pi^2/\theta_i)>0$, and so the constant should instead approach

$$-\frac{2\pi}{8\pi}\int_{-\infty}^{\infty}\frac{ds}{(1+\cosh(s))^2} = -\frac{1}{6}.$$

It therefore seems like every polygonal hole in Ω gives a negative contribution to the constant term which converges to $-1/6$ as the hole converges to a smoothly bounded domain. Thus, if Ω has r holes, then the constant term should approach $(1-r)/6$. This makes it reasonable to suspect that for any smoothly bounded domain Ω with r holes, we have

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}\sim\frac{|\Omega|}{2\pi t}-\frac{|\partial\Omega|}{4\sqrt{2\pi t}}+\frac{1-r}{6},\ t\to 0,$$

or, if $c^2=1$ (see (3.2)),

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}\sim\frac{|\Omega|}{4\pi t}-\frac{|\partial\Omega|}{8\sqrt{\pi t}}+\frac{1-r}{6},\ t\to 0.$$

Since $\chi(\Omega)=1-r$ (see Theorem 2.2.7), this would imply that one can hear the Euler characteristic of Ω .

4

Filling Kac's holes

Many of Kac's arguments are not rigorous. Starting with the diffusion equation (3.7) itself, he doesn't explain the meaning of the Dirac delta distribution δ_0, but only gives a vague physical interpretation that all particles are concentrated at one point y at $t = 0$, somehow indicating that for $t = 0$ the function maybe takes an infinite value at y and equals 0 everywhere else. Obviously this doesn't make sense from a mathematical point of view (see Proposition 2.3.2 for the actual meaning of δ_0). Then, when he obtains the first term in the asymptotic expansion of the heat trace, he assumes that $P_\Omega(x, y, t) \sim P_0(x, y, t)$, $t \to 0$, without proving it. This relation is now known as the *principle of not feeling the boundary*. A rigorous proof can be found in [107]. Kac only gives a physical explanation, in this case that for small time the boundary should not have had much time to eat any of the particles. As we briefly mentioned at the end of §3.1, he does spend some time trying to explain how to rigorously prove that $P_\Omega(x, y, t) \leq P_0(x, y, t)$, but here there are also a lot of details missing.

When Kac obtains the second term in the asymptotic expansion of the heat trace, he uses that

$$\iint_\Omega P_\Omega(y, y, t)dy \sim \iint_\Omega P_{l(y)}(y, y, t)dy, \ t \to 0.$$

Of course, while it may seem reasonable that the boundary almost looks like a line for a small time if y is close to the boundary, the asymptotic equality must be proved. It is also not clear what "close" means in this setting. Again he tries to make some of the arguments rigorous using inscribed rectangles, but as before there are a lot of details and rigorous steps missing.

Kac's vague use of "close" reoccurs many times, especially when he works with the sectors and polygons in §3.3.2, where he argues that the boundary can be approximated by an infinite sector when a point is close to one of the vertices. After making such approximations, he also obtains a lot of integrals of which he makes more unjustified approximations. For example, on several occasions he replaces the upper limit of an integral by infinity in order to make it easier to compute. It is no longer clear why the asymptotic equalities should hold after such steps, and Kac never addresses this.

Finally, at then end of §3.3.2 where we described how Kac lets the number of sides N of his polygons go to infinity, he also assumes that every $\theta_i \to \pi$ and simply interchanges the limit with the integral in (3.14). That gives him an integral (3.15) which

is easy to compute, allowing him to obtain the third in the asymptotic expansion of the heat trace. However, as the reader obviously knows, interchanging limits with integrals is not at all a trivial endeavor. He also does not address in what sense the polygons converge, and why the convergence implies that the angles $\theta_i \to \pi$. These make up for yet even more holes in Kac's article which remain to be fixed.

The goal of this section is to make all of Kac's arguments rigorous. In particular, for the asymptotic expansion of the heat trace on Ω, we will use Weyl's law to obtain the first term if Ω is any bounded domain. Then, we will use Kac's principle of feeling the boundary to obtain the second term if Ω is a polygonal domain, and we will use the microlocal construction of the heat kernel [78] to obtain the third term if Ω is a curvilinear polygonal domain. We will then show that if a sequence of polygons converges to a smooth domain Ω, then the heat trace coefficients of the polygons converge to those of Ω. This will once and for all show that Kac's heuristic arguments from physics actually give the correct results. However, in our diffusion and eigenvalue equation we will use $c^2 = 1$ (see (2.9) and (3.2)), making our eigenvalues twice as large as those of Kac. This will again change some terms by a factor of 2 compared to Kac's, and we will be careful to explain where these changes occur.

4.1 The t^{-1}-term

We can obtain the first term in the heat trace rigorously using Weyl's law (see Theorem 2.6.4). Note that if we insert λ_n into (2.14), we get $\lambda_n \sim \frac{4\pi n}{|\Omega|}$, $n \to \infty$. This implies that we, for any given $\epsilon > 0$, can choose an $N \in \mathbb{N}$ such that

$$\frac{4\pi n}{|\Omega|}(1-\epsilon) \leq \lambda_n \leq \frac{4\pi n}{|\Omega|}(1+\epsilon) \tag{4.1}$$

for $n \geq N$. Then

$$\sum_{n=N}^{\infty} e^{-4\pi n t(1+\epsilon)/|\Omega|} \leq \sum_{n=N}^{\infty} e^{-\lambda_n t} \leq \sum_{n=N}^{\infty} e^{-4\pi n t(1-\epsilon)/|\Omega|} \tag{4.2}$$

for $t > 0$. Summing the geometric series, we obtain

$$\sum_{n=N}^{\infty} e^{-4\pi n t(1\pm\epsilon)/|\Omega|} = \frac{e^{-4\pi t(N-1)(1\pm\epsilon)/|\Omega|}}{e^{4\pi t(1\pm\epsilon)/|\Omega|}-1}.$$

We have thus shown that

$$\sum_{n=1}^{N-1} e^{-\lambda_n t} + \frac{e^{-4\pi t(N-1)(1+\epsilon)/|\Omega|}}{e^{4\pi t(1+\epsilon)/|\Omega|}-1} \leq \sum_{n=1}^{\infty} e^{-\lambda_n t} \leq \sum_{n=1}^{N-1} e^{-\lambda_n t} + \frac{e^{-4\pi t(N-1)(1-\epsilon)/|\Omega|}}{e^{4\pi t(1-\epsilon)/|\Omega|}-1}.$$

In particular,

$$\begin{aligned}\liminf_{t\to 0}\left(t\sum_{n=1}^{N-1} e^{-\lambda_n t} + t\frac{e^{-4\pi t(N-1)(1+\epsilon)/|\Omega|}}{e^{4\pi t(1+\epsilon)/|\Omega|}-1}\right) &\leq \liminf_{t\to 0} t\sum_{n=1}^{\infty} e^{-\lambda_n t} \\ \leq \limsup_{t\to 0} t\sum_{n=1}^{\infty} e^{-\lambda_n t} &\leq \limsup_{t\to 0}\left(t\sum_{n=1}^{N-1} e^{-\lambda_n t} + t\frac{e^{-4\pi t(N-1)(1-\epsilon)/|\Omega|}}{e^{4\pi t(1-\epsilon)/|\Omega|}-1}\right),\end{aligned}$$

and since

$$\lim_{t\to 0} t\frac{e^{-4\pi t(N-1)(1\pm\epsilon)/|\Omega|}}{e^{4\pi t(1\pm\epsilon)/|\Omega|}-1} = \frac{|\Omega|}{4\pi(1\pm\epsilon)},$$

we get

$$\frac{|\Omega|}{4\pi(1+\epsilon)} \leq \liminf_{t\to 0} t\sum_{n=1}^{\infty} e^{-\lambda_n t} \leq \limsup_{t\to 0} t\sum_{n=1}^{\infty} e^{-\lambda_n t} \leq \frac{|\Omega|}{4\pi(1-\epsilon)}.$$

By letting $\epsilon \to 0$, it follows that

$$\lim_{t\to 0} t\sum_{n=1}^{\infty} e^{-\lambda_n t}$$

exists and equals $|\Omega|/4\pi$, which shows that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{4\pi t}, \ \ t\to 0.$$

If we, like Kac, would have chosen $c^2 = 1/2$, then by Weyl's law we would instead have $\lambda_n \sim \frac{2\pi n}{|\Omega|}$, $n\to\infty$, from which the same calculation as above would have given

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t}, \ \ t\to 0.$$

4.2 Kac's principle of feeling the boundary and the $t^{-1/2}$-term

Here we show how to use *Kac's principle of feeling the boundary* to make Kac's heuristics rigorous for the $t^{-1/2}$ term. Recall from (2.7) that the heat kernel over the half space $\mathbb{R}^2_+ = \{(x_1,x_2)\in\mathbb{R}^2 : x_2 \geq 0\}$ with Dirichlet boundary conditions at $x_2 = 0$ is given by

$$h_{\mathbb{R}^2_+}(x,y,t) = \frac{e^{-||x-y||^2/4t} - e^{-||x+y||^2/4t}}{4\pi t}, \ \ x,y\in\mathbb{R}^2_+.$$

Moreover, for a function $f:(0,\infty)\to\mathbb{R}$, we write $f(t) = \mathcal{O}(t^\infty)$, $t\to 0$, to indicate that f decays faster than any polynomial as $t\to 0$, i.e. for all $N\in\mathbb{N}$,

$$\lim_{t\to 0}\frac{f(t)}{t^N} = 0.$$

With this notation, Kac's principle of feeling the boundary says the following.

Theorem 4.2.1 ([108], [77])**.** *Let $\Omega\subset\mathbb{R}^2$ be a polygonal domain. Let h_Ω denote the heat kernel for the Laplacian on Ω with Dirichlet boundary conditions. Then, for $S = S_\theta$, an infinite sector of opening angle θ and with heat kernel $h_{S(\theta)}$ for Dirichlet boundary conditions, and for any corner of Ω with opening angle θ, there is a neighborhood U_θ such that*

$$|h_\Omega(x,y,t) - h_{S(\theta)}(x,y,t)| = \mathcal{O}(t^\infty) \text{ uniformly in } (x,y)\in U_\theta\times U_\theta.$$

Moreover, for any open set $U_e \in \Omega$ with positive distance to all corners of Ω, we have

$$|h_\Omega(x,y,t) - h_{\mathbb{R}^2_+}(x,y,t)| = \mathcal{O}(t^\infty) \text{ uniformly in } (x,y) \in U_e \times U_e.$$

This shows that to obtain the asymptotic expansion of the heat trace in the polygon Ω, it's enough to replace the actual heat kernel h_Ω with either $h_{S(\theta)}$ if we're close to a corner with angle θ, or with $h_{\mathbb{R}^2_+}$ otherwise. This is indeed what Kac does, so in essence he is correct, but he does not provide a full rigorous proof.

To make the notion of "close" precise, suppose Ω has N vertices $V_1, \dots, V_N$ with angles $\theta_1, \dots, \theta_N$. For each $i = 1, \dots, N$, choose U_{θ_i} according to Theorem 4.2.1, and choose $R_i > 0$ such that the sector $S(R_i, \theta_i)$ with radius R_i and angle θ_i is contained in U_{θ_i}. Then each $S(R_i, \theta_i)$ is contained in the infinite sector $S(\theta_i)$. Assuming, without loss of generality, that the sectors $S(R_i, \theta_i)$ are disjoint, the heat trace becomes

$$\iint_\Omega h_\Omega(x,x,t)dx \sim \sum_{i=1}^N \iint_{S(R_i,\theta_i)} h_{S(\theta_i)}(x,x,t)dx + \iint_{\Omega \setminus \bigcup_{i=1}^N S(R_i,\theta_i)} h_{\mathbb{R}^2_+}(x,x,t)dx, \ t \to 0.$$

To compute $\iint_{S(R_i,\theta_i)} h_{S(\theta_i)}(x,x,t)dx$, we do as in §3.3.2. Since $S(R_i,\theta_i)$ has finite radius R_i, we now get

$$\iint_{S(R_i,\theta_i)} h_{S(\theta_i)}(x,x,t)dx = \int_0^{\theta_i}\int_0^{R_i} h_{S(\theta_i)}(r,\phi,r,\phi,t) r dr d\phi.$$

Thus, while the second integral in $h_{S(\theta_i)}$ (see (4.14)) still becomes 0 as in §3.3.2, the first integral becomes

$$\begin{aligned}
&\int_0^{\theta_i}\int_0^{R_i} -\sin\left(\frac{\pi^2}{\theta_i}\right)\frac{e^{-r^2/2t}}{8\pi\theta_i t}\int_{-\infty}^{\infty}\frac{re^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i)}dsdrd\phi \\
&= -\frac{1}{8\pi}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{1-e^{-R_i^2(1+\cosh(s))/2t}}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}ds.
\end{aligned}$$

However, it is clear that

$$\int_{-\infty}^{\infty}\frac{e^{-R_i^2(1+\cosh(s))/2t}}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}ds = \mathcal{O}(t^\infty), \ t \to 0,$$

which gives that

$$\begin{aligned}
&\int_0^{\theta_i}\int_0^{R_i} -\sin\left(\frac{\pi^2}{\theta_i}\right)\frac{e^{-r^2/2t}}{8\pi\theta_i t}\int_{-\infty}^{\infty}\frac{re^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i)}dsdrd\phi \\
&\sim -\frac{1}{8\pi}\sum_{i=1}^N\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{1}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}ds, \ t \to 0.
\end{aligned}$$

By §3.3.2, the remaining contribution from the sectors is

$$\begin{aligned}
&\frac{1}{4\pi t}\sum_{i=1}^{N}\Bigg[\int_0^{\theta_i-\pi/2}\int_0^{R_i}\Big(1-e^{-r^2(1-\cos(2\phi))/2t}\Big)\,r dr d\phi\\
&+\int_{\theta_i-\pi/2}^{\pi/2}\int_0^{R_i}\Big(1-e^{-r^2(1-\cos(2\phi))/2t}-e^{-r^2(1-\cos(2(\theta_i-\phi)))/2t}\Big)\,r dr d\phi\\
&+\int_{\pi/2}^{\theta_i}\int_0^{R_i}\Big(1-e^{-r^2(1-\cos(2(\theta_i-\phi)))/2t}\Big)\,r dr d\phi\Bigg]\\
&=\frac{1}{4\pi t}\sum_{i=1}^{N}|S(R_i,\theta_i)|-\frac{1}{4\pi t}\sum_{i=1}^{N}\int_0^{R_i}re^{-r^2/2t}\Bigg(\int_0^{\theta_i-\pi/2}e^{r^2\cos(2\phi)/2t}d\phi\\
&+\int_{\theta_i-\pi/2}^{\pi/2}\Big(e^{r^2\cos(2\phi)/2t}+e^{r^2\cos(2(\theta_i-\phi))/2t}\Big)\,d\phi+\int_{\pi/2}^{\theta_i}e^{r^2\cos(2(\theta_i-\phi))/2t}d\phi\Bigg)dr
\end{aligned}$$

To simplify this, we use the following lemma.

Lemma 4.2.2 ([2], Lemma 7)**.** *For any $\pi/2<\theta_i<\pi$,*

$$\begin{aligned}
&\int_0^{\theta_i-\pi/2}e^{r^2\cos(2\phi)/2t}d\phi+\int_{\theta_i-\pi/2}^{\pi/2}\Big(e^{r^2\cos(2\phi)/2t}+e^{r^2\cos(2(\theta_i-\phi))/2t}\Big)\,d\phi+\int_{\pi/2}^{\theta_i}e^{r^2\cos(2(\theta_i-\phi))/2t}d\phi\\
&=\pi I_0(r^2/2t),
\end{aligned}$$

where I_0 is the modified Bessel function of order 0:

$$I_n(x)=\Big(\frac{x}{2}\Big)^n\sum_{k=0}^{\infty}\frac{(\frac{1}{4}x^2)^k}{k!\Gamma(n+k+1)},\ x\geq 0,\ n\geq 0.$$

Thus, the above contribution from the sectors becomes

$$\begin{aligned}
&\frac{1}{4\pi t}\sum_{i=1}^{N}|S(R_i,\theta_i)|-\frac{1}{4t}\sum_{i=1}^{N}\int_0^{R_i}re^{-r^2/2t}I_0(r^2/2t)dr\\
=&\frac{1}{4\pi t}\sum_{i=1}^{N}|S(R_i,\theta_i)|-\frac{1}{4}\sum_{i=1}^{N}\int_0^{R_i^2/2t}e^{-u}I_0(u)du.
\end{aligned}$$

Since

$$\frac{d}{du}\Big(ue^{-u}\big(I_0(u)+I_1(u)\big)\Big)=e^{-u}I_0(u)$$

(see [112]), we get

$$\frac{1}{4\pi t}\sum_{i=1}^{N}|S(R_i,\theta_i)|-\frac{1}{8t}\sum_{i=1}^{N}R_i^2e^{-R_i^2/2t}\big(I_0(R_i^2/2t)+I_1(R_i^2/2t)\big).$$

Now we use that

$$\begin{aligned}
I_0(x)&\sim\frac{e^x}{\sqrt{2\pi x}},\ x\to\infty,\\
I_1(x)&\sim\frac{e^x}{\sqrt{2\pi x}},\ x\to\infty
\end{aligned}$$

(see [112]). In particular,

$$I_0(R_i^2/2t) \sim \frac{1}{R_i}\sqrt{\frac{t}{\pi}}e^{R_i^2/2t},\ t \to 0,$$
$$I_1(R_i^2/2t) \sim \frac{1}{R_i}\sqrt{\frac{t}{\pi}}e^{R_i^2/2t},\ t \to 0.$$

This gives that the total contribution from the sectors becomes asymptotically equal to

$$\frac{1}{4\pi t}\sum_{i=1}^{N}|S(R_i,\theta_i)| - \frac{1}{4\sqrt{\pi t}}\sum_{i=1}^{N}R_i$$
$$-\frac{1}{8\pi}\sum_{i=1}^{N}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{1}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}ds,\ t\to 0.$$

Next, let E_i be an open set containing the part of $\partial\Omega$ between V_{i-1} and V_i (here

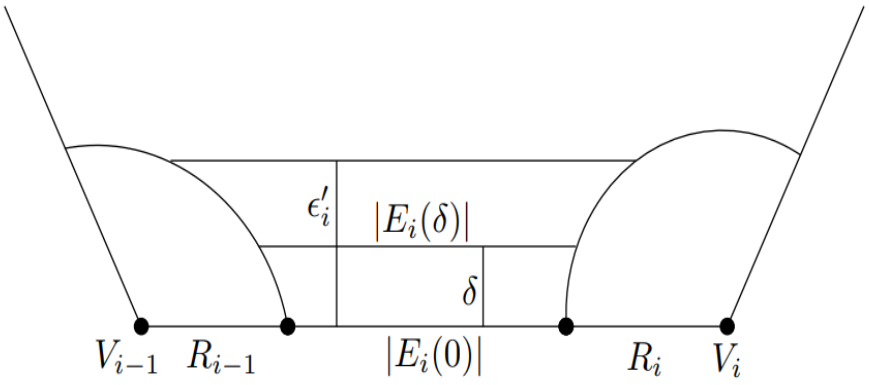

Figure 4.1: The polygon is split up into sectors which are close to the vertices, and regions close to the boundary but away from the vertices.

$V_0 := V_N$) which doesn't intersect $S(R_i,\theta_i)$, so that it has some positive distance to V_{i-1} and V_i. By translating the heat kernel $h_{\mathbb{R}^2_+}$, we can write its contribution to the heat trace at E_i as

$$\iint_{E_i} h_{\mathbb{R}^2_+}(x,x,t)dx = \int_0^{\epsilon_i'}\frac{1-e^{-\delta^2/t}}{4\pi t}|E_i(\delta)|d\delta$$

for some $\epsilon_i' > 0$, where $|E_i(\delta)|$ is the length of the line segment which is parallel and has distance δ to the edge connecting V_{i-1} and V_i (see Figure 4.1). If $|E_i(0)| := \lim_{\delta\to 0}|E_i(\delta)|$, then we let $\epsilon_i > 0$ be arbitrary and choose $\epsilon_i' > 0$ so that

$$\Big||E_i(\delta)| - |E_i(0)|\Big| < \epsilon_i$$

for $0 < \delta < \epsilon_i'$. This is possible because the endpoints of $E_i(\delta)$ are determined by $S(R_{i-1},\theta_{i-1})$ and $S(R_i,\theta_i)$, which have smooth boundary. Then

$$(|E_i(0)|-\epsilon_i)\int_0^{\epsilon_i'}\frac{1-e^{-\delta^2/t}}{4\pi t}d\delta \le \int_0^{\epsilon_i'}\frac{1-e^{-\delta^2/t}}{4\pi t}|E_i(\delta)|d\delta \le (|E_i(0)|+\epsilon_i)\int_0^{\epsilon_i'}\frac{1-e^{-\delta^2/t}}{4\pi t}d\delta.$$

Since

$$(|E_i(0)|\pm\epsilon_i)\int_0^{\epsilon_i'}\frac{1-e^{-\delta^2/t}}{4\pi t}d\delta=\frac{\epsilon_i'(|E_i(0)|\pm\epsilon_i)}{4\pi t}-\frac{|E_i(0)|\pm\epsilon_i}{8\sqrt{\pi t}}+\frac{|E_i(0)|\pm\epsilon_i}{8\sqrt{\pi t}}\operatorname{erfc}\left(\frac{\epsilon_i'}{\sqrt{t}}\right),$$

where

$$\operatorname{erfc}\left(\frac{\epsilon_i'}{\sqrt{t}}\right)=\frac{2}{\sqrt{\pi}}\int_{\epsilon_i'/t}^{\infty}e^{-x^2}dx=\mathcal{O}(t^\infty),\ t\to 0,$$

we get

$$\begin{aligned}\frac{\epsilon_i'(|E_i(0)|-\epsilon_i)}{4\pi t}-\frac{|E_i(0)|-\epsilon_i}{8\sqrt{\pi t}}+\mathcal{O}(t^\infty)&\le\int_0^{\epsilon_i'}\frac{1-e^{-\delta^2/t}}{4\pi t}|E_i(\delta)|d\delta\\&\le\frac{\epsilon_i'(|E_i(0)|+\epsilon_i)}{4\pi t}-\frac{|E_i(0)|+\epsilon_i}{8\sqrt{\pi t}}+\mathcal{O}(t^\infty).\end{aligned}$$

By choosing ϵ_i arbitrarily small, it follows that the contribution to the heat trace at E_i becomes asymptotically equal to

$$\frac{\epsilon_i'|E_i(0)|}{4\pi t}-\frac{|E_i(0)|}{8\sqrt{\pi t}},\ t\to 0.$$

The total contribution from $E_1,\dots,E_N$ is then

$$\sum_{i=1}^{N}\left(\frac{\epsilon_i'|E_i(0)|}{4\pi t}-\frac{|E_i(0)|}{8\sqrt{\pi t}}\right),\ t\to 0.$$

For the remaining part of Ω, which we denote by U, we have

$$\iint_U h_{\mathbb{R}^2_+}(x,x,t)=\frac{|U|}{4\pi t}-\frac{1}{4\pi t}\iint_U e^{-||x||^2/t}dx.$$

By construction of E_i, it is clear that every point in U has distance at least

$$\min\{\min_{1\le i\le N}\epsilon_i',\min_{1\le i\le N}R_i\}>0$$

to $\partial\Omega$, from which it immediately follows that

$$\iint_U e^{-||x||^2/t}dx=\mathcal{O}(t^\infty),\ t\to 0.$$

Thus, the contribution from U, up to asymptotic equality, is simply

$$\frac{|U|}{4\pi t}.$$

Adding everything up, we get for our polygonal domain Ω that

$$\begin{aligned}\sum_{n=1}^{\infty}e^{-\lambda_n t}\sim&\frac{1}{4\pi t}\sum_{i=1}^{N}|S(R_i,\theta_i)|-\frac{1}{4\sqrt{\pi t}}\sum_{i=1}^{N}R_i\\&-\frac{1}{8\pi}\sum_{i=1}^{N}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{1}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}ds\\&+\sum_{i=1}^{N}\left(\frac{\epsilon_i'|E_i(0)|}{4\pi t}-\frac{|E_i(0)|}{8\sqrt{\pi t}}\right)+\frac{|U|}{4\pi t},\ t\to 0.\end{aligned}$$

Since

$$\sum_{i=1}^{\infty}(|S(R_i,\theta_i)|+\epsilon_i'|E_i(0)|)+|U|=|\Omega|,$$
$$\sum_{i=1}^{N}(2R_i+|E_i(0)|)=|\partial\Omega|,$$

this simplifies to

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}\sim\frac{|\Omega|}{4\pi t}-\frac{|\partial\Omega|}{8\sqrt{\pi t}}$$
$$-\frac{1}{8\pi}\sum_{i=1}^{N}\sin\left(\frac{\pi^2}{\theta_i}\right)\int_{-\infty}^{\infty}\frac{1}{(1+\cosh(s))(\cosh(\pi s/\theta_i)-\cos(\pi^2/\theta_i))}ds,\ t\to 0.$$

In particular, we have

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}\sim\frac{|\Omega|}{4\pi t}-\frac{|\partial\Omega|}{8\sqrt{\pi t}}+a_0(\Omega),$$

for some $a_0(\Omega)$ independent of t. We have therefore obtained the first two terms in the asymptotic heat trace expansion. If we would had used $c^2=1/2$ (see (3.2)) like Kac, we would have obtained

$$\sum_{n=1}^{\infty}e^{-\lambda_n t}\sim\frac{|\Omega|}{2\pi t}-\frac{|\partial\Omega|}{4\sqrt{2\pi t}}+a_0(\Omega),$$

partly because the heat kernel over the sector $S(\theta)$ would change according to §3.3.1, and partly because the heat kernel over the half space would then be

$$h_{\mathbb{R}^2_+}(x,y,t)=\frac{e^{-||x-y||^2/2t}-e^{-||x+y||^2/2t}}{2\pi t},\ x,y\in\mathbb{R}^2_+.$$

4.2.1 Derivation of the heat kernel on infinite sectors

To see how to obtain $h_{S(\theta)}$, we follow the derivation in [2, §6]. They refer to Carslaw [21] and note that the heat kernel can be written as

$$\tilde{h}_{S(\theta)}(r,\phi,R,\alpha,t)=\frac{e^{-(r^2+R^2)/4t}}{8\pi\theta t}\int_A e^{rR\cos(z-\phi)/2t}\frac{e^{i\pi z/\theta}}{e^{i\pi z/\theta}-e^{i\pi\alpha/\theta}}dz.$$

Here, A is a contour in $\mathbb{C}$ which can either be taken as A_ϕ, defined as the union of

$A_{\phi_1}=\{z\in\mathbb{C}:\phi-\pi<\mathrm{Re}(z)<\phi+\pi,\ \mathrm{Im}(z)>0\}$ going from $\phi+\pi+i\infty$ to $\phi-\pi+i\infty$,
$A_{\phi_2}=\{z\in\mathbb{C}:\phi-\pi<\mathrm{Re}(z)<\phi+\pi,\ \mathrm{Im}(z)<0\}$ going from $\phi-\pi-i\infty$ to $\phi+\pi-i\infty$,

or A can be taken as the union of

$$l_1=\{\phi-\pi+iy:y\in\mathbb{R}\}\text{ going from }\phi-\pi-i\infty\text{ to }\phi-\pi+i\infty, \tag{4.3}$$
$$l_2=\{\phi+\pi+iy:y\in\mathbb{R}\}\text{ going from }\phi+\pi+i\infty\text{ to }\phi+\pi-i\infty, \tag{4.4}$$
$$\text{small circles around the poles in } z\in(\phi-\pi,\phi+\pi). \tag{4.5}$$

The values of ϕ and α are assumed to be close enough to each other so that no pole lie on l_1 or l_2. Writing $z \in (\phi - \pi, \phi + \pi)$ in (4.5) makes sense because every pole is real (see (4.7)).

Now, to get the heat kernel for Dirichlet boundary conditions, one needs to use the *method of images* (see [43]). One then sets $w = z - \phi$ and obtains a direct term

$$\tilde{h}_{S(\theta)}(r, \phi, R, \alpha, t) = \frac{e^{-(r^2+R^2)/4t}}{8\pi\theta t} \int_{A_0} e^{rR\cos(w)/2t} \frac{1}{1 - e^{i\pi(\alpha-\phi-w)/\theta}} dw$$

and an indirect (also called reflected) term

$$\tilde{h}_{S(\theta)}(r, -\phi, R, \alpha, t) = \frac{e^{-(r^2+R^2)/4t}}{8\pi\theta t} \int_{A_0} e^{rR\cos(w)/2t} \frac{1}{1 - e^{i\pi(\alpha+\phi-w)/\theta}} dw.$$

The heat kernel for Dirichlet boundary conditions then becomes

$$\begin{aligned} &h_{S(\theta)}(r, \phi, R, \alpha, t) = \tilde{h}_{S(\theta)}(r, \phi, R, \alpha, t) - \tilde{h}_{S(\theta)}(r, -\phi, R, \alpha, t) \\ &= \frac{e^{-(r^2+R^2)/4t}}{8\pi\theta t} \int_{A_0} e^{rR\cos(w)/2t} \left(\frac{1}{1 - e^{i\pi(\alpha-\phi-w)/\theta}} - \frac{1}{1 - e^{i\pi(\alpha+\phi-w)/\theta}} \right) dw. \end{aligned} \tag{4.6}$$

To compute the integral, we need to find the contribution from the lines and from the circles around the poles.

For the circles, we find the residues at the poles between $-\pi$ and π. Note that

$$\begin{aligned} 1 - e^{i\pi(\alpha-\phi-z)/\theta} &= 0, \\ 1 - e^{i\pi(\alpha+\phi-w)/\theta} &= 0 \end{aligned}$$

when

$$\begin{aligned} z &= \alpha - \phi + 2k\theta, \ k \in \mathbb{Z}, \\ w &= \alpha + \phi + 2j\theta, \ j \in \mathbb{Z}. \end{aligned} \tag{4.7}$$

Since the poles are simple, the residues become

$$\begin{aligned} &\mathrm{Res}_{z=\alpha-\phi+2k\theta} \frac{e^{rR\cos(z)/2t}}{1 - e^{i\pi(\alpha-\phi-z)/\theta}} = \lim_{z\to\alpha-\phi+2k\theta} \frac{(z - (\alpha - \phi + 2k\theta))e^{rR\cos(z)/2t}}{1 - e^{i\pi(\alpha-\phi-z)/\theta}} \\ &= \lim_{u\to 0} \frac{u}{1 - e^{-i\pi u/\theta}} e^{rR\cos(u+\alpha-\phi+2k\theta)/2t} = \frac{\theta}{i\pi} e^{rR\cos(\alpha-\phi+2k\theta)/2t} \end{aligned}$$

and similarly

$$\mathrm{Res}_{w=\alpha+\phi+2j\theta} \frac{e^{rR\cos(w)/2t}}{1 - e^{i\pi(\alpha+\phi-w)/\theta}} = \frac{\theta}{i\pi} e^{rR\cos(\alpha+\phi+2j\theta)/2t}.$$

Thus, by the residue theorem the integral around these poles becomes

$$2\theta e^{rR\cos(\alpha-\phi+2k\theta)/2t}$$

and

$$2\theta e^{rR\cos(\alpha+\phi+2j\theta)/2t},$$

respectively. It remains to find for which values of k and j the poles are between $-\pi$ and π. By (4.7), this holds when

$$-\frac{\pi}{2\theta} < \frac{\alpha-\phi}{2\theta} + k < \frac{\pi}{2\theta},\ k \in \mathbb{Z}, \tag{4.8}$$

$$-\frac{\pi}{2\theta} < \frac{\alpha+\phi}{2\theta} + j < \frac{\pi}{2\theta},\ j \in \mathbb{Z}. \tag{4.9}$$

Since we are interested in the case where ϕ and α are close but different (afterwards we will set $\phi = \alpha$), for now we may assume that $\phi = 0$ and $0 < \alpha \leq \theta/4$. Then (4.8) becomes

$$\frac{-\pi-\alpha}{2\theta} < k < \frac{\pi-\alpha}{2\theta} \text{ and } 0 < \frac{\alpha}{2\theta} \leq \frac{1}{8},$$

from which it follows that the smallest and largest value of k are

$$k_{\min} = \left\lceil -\frac{\pi}{2\theta} \right\rceil, \tag{4.10}$$

$$k_{\max} = \begin{cases} \lfloor \frac{\pi}{2\theta} \rfloor \text{ if } \frac{\pi}{2\theta} \notin \mathbb{Z}, \\ \frac{\pi}{2\theta} - 1 \text{ if } \frac{\pi}{2\theta} \in \mathbb{Z}, \end{cases} \tag{4.11}$$

respectively. For j, we see that (4.9) is satisfied when

$$\frac{-\pi-\phi-\alpha}{2\theta} < j < \frac{\pi-\phi-\alpha}{2\theta},$$

or, equivalently,

$$j \in \left(\frac{-\pi-\phi-\alpha}{2\theta}, \frac{\pi-\phi-\alpha}{2\theta} \right) \cap \mathbb{Z} =: V_{\phi,\alpha}. \tag{4.12}$$

We conclude that the total contribution to the integral in the heat kernel given by (4.6) from all the poles is

$$\sum_{k_{\min} \leq k \leq \max} 2\theta e^{rR\cos(\alpha-\phi+2k\theta)/2t} - \sum_{j \in V_{\phi,\alpha}} 2\theta e^{rR\cos(\alpha+\phi+2j\theta)/2t}.$$

Next we consider the integral in (4.6) over the lines l_1 and l_2. For A_0 we have $\phi = 0$ and so, by (4.3) and (4.4), we parametrize l_1, l_2 as $l_1(s) = -\pi + is$ with s going from $-\infty$ to ∞ and $l_2(s) = \pi + is$ with s going from ∞ to $-\infty$.

Note that if $\theta = \pi/n$ for some $n \in \mathbb{N}$, then the integrand in (4.6) becomes 2π-periodic:

$$\begin{aligned} &e^{rR\cos(w+2\pi)/2t} \left(\frac{1}{1 - e^{in\pi(\alpha-\phi-w-2\pi)/\pi}} - \frac{1}{1 - e^{in\pi(\alpha+\phi-w-2\pi)/\pi}} \right) \\ &= e^{rR\cos(w)/2t} \left(\frac{1}{1 - e^{in\pi(\alpha-\phi-w)/\pi}} - \frac{1}{1 - e^{in\pi(\alpha+\phi-w)/\pi}} \right). \end{aligned}$$

Therefore the integrand takes the same values on l_1 and l_2. Since the lines have opposite orientations, it follows that they cancel out and their contribution to the integral is 0.

For a general θ, the first term in the integral (4.6) becomes

$$\int_{l_1\cup l_2} \frac{e^{rR\cos(w)/2t}}{1-e^{i\pi(\alpha-\phi-w)/\theta}}dw = \int_{-\infty}^{\infty} \frac{e^{rR\cos(-\pi+is)/2t}}{1-e^{i\pi(\alpha-\phi+\pi-is)/\theta}}ids - \int_{-\infty}^{\infty} \frac{e^{rR\cos(\pi+is)/2t}}{1-e^{i\pi(\alpha-\phi-\pi-is)/\theta}}ids$$
$$= i\int_{-\infty}^{\infty}\left(\frac{e^{-rR\cosh(s)/2t}}{1-e^{\pi s/\theta}e^{\pi i(\alpha-\phi+\pi)/\theta}} - \frac{e^{-rR\cosh(s)/2t}}{1-e^{\pi s/\theta}e^{\pi i(\alpha-\phi-\pi)/\theta}}\right)ds.$$

If we let $r=R$ and $\phi=\alpha$, this can be simplified to

$$-\sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta)-\cos(\pi^2/\theta)}ds.$$

For the second term in the integral in (4.6), we make a similar computation and obtain

$$\int_{-\infty}^{\infty} -i\left(\frac{e^{-rR\cosh(s)/2t}}{1-e^{\pi s/\theta}e^{i\pi(\alpha+\phi+\pi)/\theta}} - \frac{e^{-rR\cosh(s)/2t}}{1-e^{\pi s/\theta}e^{i\pi(\alpha+\phi-\pi)/\theta}}\right)ds,$$

which for $r=R$ and $\phi=\alpha$ becomes

$$\sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta+2\pi i\phi/\theta)-\cos(\pi^2/\theta)}ds.$$

Now, if we add up the contributions from the poles and lines, we get that the heat kernel for our sector $S(\theta)$ is

$$\begin{aligned}h_{S(\theta)}(r,\phi,R,\alpha,t) =& \frac{e^{-(r^2+R^2)/4t}}{8\pi t\theta}\Bigg(\sum_{k_{\min}\le k\le \max} 2\theta e^{rR\cos(\alpha-\phi+2k\theta)/2t} - \sum_{j\in V_{\phi,\alpha}} 2\theta e^{rR\cos(\alpha+\phi+2j\theta)/2t}\\ &-\sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\frac{e^{-rR\cosh(s)/2t}}{\cosh(\pi s/\theta+\pi i(\alpha-\phi)/\theta)-\cos(\pi^2/\theta)}ds\\ &+\sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\frac{e^{-rR\cosh(s)/2t}}{\cosh(\pi s/\theta+\pi i(\alpha+\phi)/\theta)-\cos(\pi^2/\theta)}ds\Bigg),\end{aligned}$$

where $k_{\min}, k_{\max}$, and $V_{\phi,\alpha}$ are given by (4.10), (4.11), and (4.12), respectively. In particular, for $r=R$ and $\phi=\alpha$,

$$\begin{aligned}h_{S(\theta)}(r,\phi,r,\phi,t) =& \frac{e^{-r^2/2t}}{8\pi t\theta}\Bigg(\sum_{k_{\min}\le k\le \max} 2\theta e^{r^2\cos(2k\theta)/2t} - \sum_{j\in V_{\phi,\phi}} 2\theta e^{r^2\cos(2\phi+2j\theta)/2t}\\ &-\sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta)-\cos(\pi^2/\theta)}ds\\ &+\sin\left(\frac{\pi^2}{\theta}\right)\int_{-\infty}^{\infty}\frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta+2\pi i\phi/\theta)-\cos(\pi^2/\theta)}ds\Bigg)\end{aligned} \tag{4.13}$$

However, note that since $\pi/2 < \theta < \pi$, we have

$$\frac{1}{2} < \frac{\pi}{2\theta} < 1,$$
$$-1 < -\frac{\pi}{2\theta} < -\frac{1}{2},$$

and therefore

$$k_{\min} = \left\lceil -\frac{\pi}{2\theta} \right\rceil = 0,$$
$$k_{\max} = \left\lfloor \frac{\pi}{2\theta} \right\rfloor = 0.$$

Thus, the sum over k only contains the term corresponding to $k = 0$, namely $2\theta e^{r^2/2t}$. For the sum over j, we have that $j \in V_{\phi,\phi}$ if and only if

$$\frac{-\pi - 2\phi}{2\theta} < j < \frac{\pi - 2\phi}{2\theta},$$

which can be rewritten as

$$-\frac{\pi}{2} - j\theta < \phi < \frac{\pi}{2} - j\theta,\ j \in \mathbb{Z}.$$

Now it is straightforward to see that only $j = 0$ works if $0 < \phi < \theta - \pi/2$, both $j = 0$ and $j = -1$ work if $\theta - \pi/2 < \phi < \pi/2$, and only $j = -1$ works if $\pi/2 < \phi < \theta$. This gives that for $0 < \phi < \theta - \pi/2$

$$\begin{aligned} h_{S(\theta)}(r,\phi,r,\phi,t) = {} & \frac{1}{4\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/2t}}{4\pi t} \\ & - \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/2t}}{8\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)} ds \\ & + \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/2t}}{8\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)} ds. \end{aligned} \tag{4.14}$$

For $\theta - \pi/2 < \phi < \pi/2$,

$$\begin{aligned} h_{S(\theta)}(r,\phi,r,\phi,t) = {} & \frac{1}{4\pi t} - \frac{e^{-r^2(1-\cos(2\phi))/2t}}{4\pi t} - \frac{e^{-r^2(1-\cos(2(\theta-\phi))/2t}}{4\pi t} \\ & - \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/2t}}{8\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)} ds \\ & + \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/2t}}{8\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)} ds, \end{aligned} \tag{4.15}$$

and for $\pi/2 < \phi < \theta$,

$$\begin{aligned} h_{S(\theta)}(r,\phi,r,\phi,t) = {} & \frac{1}{4\pi t} - \frac{e^{-r^2(1-\cos(2(\theta-\phi))/2t}}{4\pi t} \\ & - \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/2t}}{8\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta) - \cos(\pi^2/\theta)} ds \\ & + \sin\left(\frac{\pi^2}{\theta}\right) \frac{e^{-r^2/2t}}{8\pi\theta t} \int_{-\infty}^{\infty} \frac{e^{-r^2\cosh(s)/2t}}{\cosh(\pi s/\theta + 2\pi i\phi/\theta) - \cos(\pi^2/\theta)} ds. \end{aligned} \tag{4.16}$$

This is what Kac obtains, except for some factors of 2 differing from his expressions. Of course, this is because Kac has the factor $c^2 = 1/2$ in his eigenvalue equation (see (3.2)). One can easily justify these changes in the heat kernel in the same way as we did in §3.1.

4.3 Microlocal construction

Here we explain the main ideas of the microlocal construction of the heat kernel [78], which will be needed in order to make some of Kac's results rigorous. As stated in §4, the microlocal construction is done on curvilinear polygonal domains.

Definition 4.3.1 ([78], Def. 1.3)**.** A domain $\Omega \subset \mathbb{R}^2$ is a *curvilinear polygonal domain* if it has piecewise smooth boundary and each non-smooth point of $\partial\Omega$ is a vertex. Here, a point $p \in \partial\Omega$ is a *vertex* if there is a continuous function $\gamma : (-a, a) \to \partial\Omega$ with $a > 0$ and $\gamma(0) = p$ such that γ is smooth on $(-a, 0]$ and $[0, a)$, $||\gamma'(t)|| = 1$ for all $t \in (-a, a)$, and

$$\lim_{t\uparrow 0} \gamma'(t) = v_1, \ \lim_{t\downarrow 0} \gamma'(t) = v_2,$$

for some vectors v_1, v_2 with $-v_1 \neq v_2$.

The condition that $-v_1 \neq v_2$ implies that there will be an interior angle in Ω at p between 0 and 2π. The angle π is allowed. Note that the edges between the corners need not be straight. Thus, the notion of curvilinear polygonal domains is a generalization of polygons.

The main result in [78] that we will need is the following.

Theorem 4.3.2 ([78], Thm. 6.10)**.** *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygonal domain with edges $E_1, \ldots, E_N$ and vertices $V_1, \ldots, V_N$ of angles $\alpha_1, \ldots, \alpha_N$ with vertex V_i connecting edges E_i and E_{i+1} (here, $E_{N+1} = E_1$). Then the heat trace of the Laplace operator with Dirichlet boundary conditions* (3.3) *satisfies*

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} = \frac{a_{-1}(\Omega)}{t} + \frac{a_{-1/2}(\Omega)}{\sqrt{t}} + a_0(\Omega) + \mathcal{O}(\sqrt{t}\log(t)), \ t \to 0,$$

where

$$\begin{aligned} a_{-1}(\Omega) &= \frac{|\Omega|}{4\pi}, \\ a_{-1/2}(\Omega) &= -\frac{|\partial\Omega|}{8\sqrt{\pi t}}, \\ a_0(\Omega) &= \frac{\chi(\Omega)}{6} - \frac{N}{12} + \frac{1}{24\pi}\sum_{i=1}^{N} \alpha_i + \frac{\pi}{24}\sum_{i=1}^{N} \frac{1}{\alpha_i} \end{aligned}$$

with $\chi(\Omega)$ being the Euler characteristic of Ω (see Definition 2.2.5).

Remark 4.3.3. In [78] they prove Theorem 4.3.2 also for Neumann and Robin boundary conditions.

To prove Theorem 4.3.2, geometric microlocal analysis is used to construct the heat kernel on a so called *heat space* Ω_h^2. The heat space is obtained by applying a *blow up* on $\Omega \times \Omega \times [0, 1)$, which yields an asymptotic description of not only the heat trace but the heat kernel as $t \to 0$. Other tools needed to prove Theorem 4.3.2 include integral representations of the heat kernel on infinite sectors using Green's functions, functional calculus to prove that the heat kernel equals the inverse Laplace transform of the Green's function, and solving different model problems for the smooth parts of the boundary and for the vertices obtained after blowing up the domain [78, §1].

We can now use Theorem 4.3.2 to obtain the asymptotic expansion of the heat trace for a smoothly bounded domain.

Corollary 4.3.4. *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain with r holes (see Definition 2.2.6). Then the heat trace of the Laplace operator* (3.3) *satisfies*

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1-r}{6}, \ t \to 0.$$

In particular, if Ω is a smoothly bounded convex domain, then

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{6}, \ t \to 0.$$

Proof. Since $\partial\Omega$ is smooth, we have no angles, so $N = 0$. The result now follows immediately from Theorem 4.3.2 since $\chi(\Omega) = 1 - r$ (see Theorem 2.2.7), and that $r = 0$ if Ω is convex. □

4.4 Polygons converging to a smoothly bounded domain

We will now rigorously prove what Kac is trying to prove with the convex polygons converging to a smooth domain (see §3.3.2 and §3.3.3), namely that the heat trace coefficients of the polygons converge to those of the smooth domain. We start with the case where the polygons have no holes.

Theorem 4.4.1. *Let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of N_k-sided convex polygonal domains with interior angles $\alpha_{k,j}$, $k \geq 1$, $1 \leq j \leq N_k$. Assume that $\overline{\Omega_k} \to \overline{\Omega}$ as $k \to \infty$ (see Definition 2.2.2), with Ω being a non-empty smoothly bounded convex domain. Then, the heat trace coefficients of t^{-1}, $t^{-1/2}$, and t^0 for Ω_k converge to those of Ω.*

Proof. By Theorem 4.3.2, the heat trace coefficients of t^{-1} and $t^{-1/2}$ for Ω are $a_{-1}(\Omega) = |\Omega|/4\pi$ and $a_{-1/2}(\Omega) = -|\partial\Omega|/8\sqrt{\pi}$, respectively. Thus, for the convergence of the first two coefficients it's enough to show that $|\Omega_k| \to |\Omega|$ and

$|\partial\Omega_k| \to |\partial\Omega|$ as $k \to \infty$. Showing that $|\Omega_k| \to |\Omega|$ is relatively straightforward and doesn't even require the convexity assumption. To see this, note that if $|\Omega_k| \not\to |\Omega|$ then there exists and $\epsilon > 0$ and a subsequence $\{\Omega_{k_l}\}_{l=1}^{\infty}$ such that $\left||\Omega_{k_l}| - |\Omega|\right| > \epsilon$ for all l. Since the polygons are compact and simply connected, it follows that there is a $\delta > 0$ such that for all l there is an $x_0 \in \mathbb{R}^2$ so that either

$$B_\delta(x_0) = \{x \in \mathbb{R}^2 : ||x - x_0|| < \delta\} \subset \Omega_{k_l} \text{ and } B_\delta(x_0) \cap \Omega = \emptyset$$

or

$$B_\delta(x_0) \subset \Omega \text{ and } B_\delta(x_0) \cap \Omega_k = \emptyset.$$

In the former case, we get

$$d_H(\Omega_k, \Omega) \geq \sup_{x \in \Omega_{k_l}} \inf_{y \in \Omega} ||x - y|| \geq \inf_{y \in \Omega} ||x_0 - y|| \geq \delta,$$

and in the latter case we similarly get $d_H(\Omega_k, \Omega) \geq \inf_{y \in \Omega_{k_l}} ||x_0 - y|| \geq \delta$. This shows that $\Omega_k \not\to \Omega$.

The proof that $|\partial\Omega_k| \to |\partial\Omega|$ relies heavily on the convexity assumption and becomes a subtle issue without it. In fact, if we don't have convexity, then all we can guarantee is that

$$|\partial\Omega| \leq \liminf_{k \to \infty} |\partial\Omega_k|.$$

See [89, §4] for more details.

For the coefficient of t^0, we have (again by Theorem 4.3.2)

$$a_0(\Omega_k) = \frac{1}{6} - \frac{N_k}{12} + \frac{1}{24\pi} \sum_{j=1}^{N_k} \alpha_{k,j} + \frac{\pi}{24} \sum_{j=1}^{N_k} \frac{1}{\alpha_{k,j}}.$$

Since

$$\sum_{j=1}^{N_k} \alpha_{k,j} = \pi(N_k - 2), \tag{4.17}$$

this simplifies to

$$a_0(\Omega_k) = \frac{1}{12} - \frac{N_k}{24} + \frac{\pi}{24} \sum_{j=1}^{N_k} \frac{1}{\alpha_{k,j}}. \tag{4.18}$$

Next, we show that the angles approach π as $k \to \infty$.

Lemma 4.4.2. *Under the assumptions in Theorem 4.4.1, we have*

$$\lim_{\substack{k \to \infty \\ 1 \leq j \leq N_k}} \alpha_{k,j} = \pi.$$

Proof of lemma. By convexity, the angles $\alpha_{k,j}$ are between 0 and π. In particular the sequence of angles is bounded, so we know by Bolzano-Weierstrass' theorem that there is at least one convergent subsequence $\alpha_{k_l,j_l} \to \alpha \in [0, \pi]$. We want to show that $\alpha = \pi$. If $\alpha = 0$, then the polygons collapse and $|\Omega| = 0$, which contradicts that Ω is a domain. Now assume that $0 < \alpha < \pi$. After applying isometries of $\mathbb{R}^2$ and

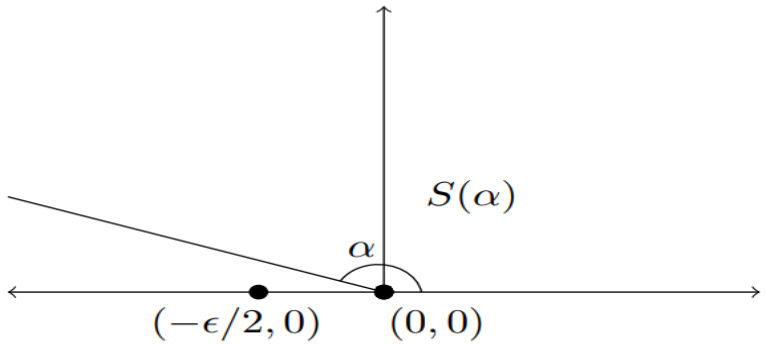

Figure 4.2: We apply isometries so that one vertex is at the origin, one edge is parallel to the x-axis, and one edge is in the upper half plane. If the opening angle $\alpha < \pi$, then points on the left x-axis will have a positive distance to $S(\alpha)$.

re-naming, we have a sequence of angles $\alpha_k \to \alpha$ at a vertex in Ω_k which we place at the origin $(0,0) \in \mathbb{R}^2$ such that one edge of Ω_k adjacent to the vertex is contained in $\{(x,0) : x \geq 0\}$, and the other edge is contained in $\{(r\cos(\alpha_k), r\sin(\alpha_k)) : r > 0\}$ (see Figure 4.2). Then, by convexity, $\overline{\Omega_k}$ is contained in the infinite sector

$$S(\alpha_k) := \{(r\cos(\phi), r\sin(\phi)) : r \geq 0,\ 0 \leq \phi \leq \alpha_k\}.$$

Moreover, we have $S(\alpha_k) \to S(\alpha)$ since $\alpha_k \to \alpha$. Indeed, this follows from the fact that for any $r \geq 0$,

$$\lim_{k\to\infty} r\sqrt{(\cos(\alpha_k) - \cos(\alpha))^2 + (\sin(\alpha_k) - \sin(\alpha))^2} = 0.$$

We claim that this implies that $\overline{\Omega} \subset S(\alpha)$. To show this, suppose $x \in \overline{\Omega} \backslash S(\alpha)$. Since $\overline{\Omega}$ and $S(\alpha)$ are closed, this would imply that

$$\inf_{y\in S(\alpha)} ||x - y|| = \epsilon,$$

for some $\epsilon > 0$. Now choose $k \geq 1$ such that $d_H(\overline{\Omega}, \overline{\Omega_k}) \leq \epsilon/3$ and $d_H(S(\alpha), S(\alpha_k)) \leq \epsilon/3$. Then there exists an $x_k \in \overline{\Omega_k}$ with $||x - x_k|| \leq \epsilon/3$. Moreover, for any $y_k \in S(\alpha_k)$ there is a $y \in S(\alpha)$ with $||y - y_k|| \leq \epsilon/3$. Thus,

$$||x_k - y_k|| \geq ||x - y|| - ||x_k - x|| - ||y - y_k|| \geq \epsilon/3,$$

which shows that $x_k \notin S(\alpha_k)$. But this contradicts that $\overline{\Omega_k} \subset S(\alpha_k)$, so indeed we must have $\overline{\Omega} \subset S(\alpha)$.

Next, we want to show that $(0,0) \in \partial\Omega$. If $(0,0) \notin \overline{\Omega}$, then by closure of $\overline{\Omega}$

$$\inf_{y\in\overline{\Omega}} ||x - y|| =: \delta > 0.$$

But then $d_H(\overline{\Omega_k}, \overline{\Omega}) \geq \delta > 0$, contradicting that $d_H(\overline{\Omega_k}, \overline{\Omega}) \to 0$. So, $(0,0) \in \overline{\Omega}$. If $(0,0) \in \Omega$ then, since Ω is a domain and hence open, there is an $\epsilon > 0$ such that

$B_\epsilon((0,0)) \subset \Omega$. Then we have $(-\epsilon/2, 0) \in \overline{\Omega}$. But $\overline{\Omega} \subset S(\alpha)$, so this implies that $(-\epsilon/2, 0) \in S(\alpha)$. This is not true since $\alpha < \pi$ (see Figure 4.2). Thus, we conclude that $(0,0) \notin \Omega$, hence $(0,0) \in \partial\Omega$.

Finally, we consider the tangent line to $\partial\Omega$ at $(0,0)$. Since Ω is smoothly bounded and convex, its tangent line at $(0,0)$ can be characterized as the unique line in the plane that intersects $\overline{\Omega}$ at $(0,0)$ and such that $\overline{\Omega}$ is entirely contained on one side of the line. Now, extend the edge of Ω that lies on the x-axis to the entire x-axis. Then it intersects $\overline{\Omega}$ at $(0,0)$, and Ω is entirely on one side of the line (above) since $\overline{\Omega} \subset S(\alpha)$. Thus, the x-axis is a tangent line to $\partial\Omega$ at $(0,0)$. However, we can do the same thing with the ray $\{(r\cos(\alpha), r\sin(\alpha)) : r \geq 0\}$ along the angle α. Thus, $\partial\Omega$ has two tangent lines at $(0,0)$, which contradicts the uniqueness. We therefore have a contradiction, and we conclude that α cannot be less than π. □

It now follows that we can write $\alpha_{k,j} = \pi(1 - f(k,j))$ for some function

$$f : \{(k,j) : k \in \mathbb{N},\ 1 \leq j \leq N_k\} \to (0,1)$$

such that

$$\lim_{\substack{k\to\infty \\ 1\leq j\leq N_k}} f(k,j) = 0.$$

Note that (4.17) implies

$$\sum_{j=1}^{N_k} f(k,j) = 2$$

for every $k \in \mathbb{N}$. Thus,

$$\begin{aligned}
&\frac{\pi}{24}\sum_{j=1}^{N_k}\frac{1}{\alpha_{k,j}} = \frac{1}{24}\sum_{j=1}^{N_k}\frac{1}{1-f(k,j)} = \frac{1}{24}\sum_{j=1}^{N_k}\sum_{m=0}^{\infty} f(k,j)^m = \frac{N_k}{24} + \frac{1}{24}\sum_{j=1}^{N_k}\sum_{m=1}^{\infty} f(k,j)^m \\
&= \frac{N_k}{24} + \frac{1}{24}\sum_{j=1}^{N_k} f(k,j)\sum_{m=0}^{\infty} f(k,j)^m = \frac{N_k}{24} + \frac{1}{12} + \frac{1}{24}\sum_{j=1}^{N_k}\sum_{m=2}^{\infty} f(k,j)^m \\
&= \frac{N_k}{24} + \frac{1}{12} + \frac{1}{24}\sum_{j=1}^{N_k}\frac{f(k,j)^2}{1-f(k,j)}.
\end{aligned}$$

Combining this with (4.18) gives

$$a_0(\Omega_k) = \frac{1}{6} + \frac{1}{24}\sum_{j=1}^{N_k}\frac{f(k,j)^2}{1-f(k,j)}.$$

It therefore remains to show that

$$\lim_{k\to\infty}\sum_{j=1}^{N_k}\frac{f(k,j)^2}{1-f(k,j)} = 0.$$

To do this, define for each $k \in \mathbb{N}$

$$\epsilon_k = \max_{1\leq j\leq N_k} f(k,j).$$

These tend to zero because all angles tend to π (see Lemma 4.4.2), so

$$0 \leq \sum_{j=1}^{N_k} \frac{f(k,j)^2}{1-f(k,j)} \leq \frac{\epsilon_k}{1-\epsilon_k} \sum_{j=1}^{N_k} f(k,j) = \frac{2\epsilon_k}{1-\epsilon_k}.$$

Since $2\epsilon_k/(1-\epsilon_k) \to 0$ as $\epsilon_k \to 0$, the result follows. □

Next we want to prove that the conclusion in Theorem 4.4.1 holds even if Ω_k have holes.

Theorem 4.4.3. *Fix $r \geq 0$ and let $\{\Omega_k\}_{k=1}^{\infty}$ be a sequence of domains such that each Ω_k has r holes $\Theta_{k,1}, \ldots, \Theta_{k,r}$, so that*

$$\Omega_k = \widetilde{\Omega}_k \setminus \bigcup_{i=1}^{r} \overline{\Theta_{k,i}}$$

according to Definition 2.2.6. Assume that $\widetilde{\Omega}_k$ and $\Theta_{k,i}$ are convex polygonal domains satisfying $\widetilde{\Omega}_k \to \Omega$ and $\Theta_{k,i} \to \Theta_i$ as $k \to \infty$. If, in addition,

$$\widetilde{\Omega}_k \setminus \bigcup_{i=1}^{r} \overline{\Theta_{k,i}} \to \Omega \setminus \bigcup_{i=1}^{r} \overline{\Theta_i},$$

then the heat trace coefficients of t^{-1}, $t^{-1/2}$, and t^0 for Ω_k converge to those of $\Omega \setminus \bigcup_{i=1}^{r} \overline{\Theta_i}$.

Proof. The convergence of $a_{-1}(\Omega_k)$ and $a_{-1/2}(\Omega_k)$ is proved in the same way as in Theorem 4.4.1. For $a_0(\Omega_k)$, let $N_{k,0}$ and $N_{k,i}$ be the number of sides of $\widetilde{\Omega}_k$ and $\Theta_{k,i}$, respectively, let $\alpha_{k,0,j}$ be the interior angles of $\widetilde{\Omega}_k$, and let $\alpha_{k,i,j}$ be the exterior angles of $\Theta_{k,i}$. By Theorem 4.3.2, we have

$$a_0(\Omega_k) = \frac{\chi(\Omega)}{6} - \frac{1}{12} \sum_{i=0}^{r} N_{k,i} + \frac{1}{24\pi} \sum_{i=0}^{r} \sum_{j=1}^{N_{k,i}} \alpha_{k,i,j} + \frac{\pi}{24} \sum_{i=0}^{r} \sum_{j=1}^{N_{k,i}} \frac{1}{\alpha_{k,i,j}}.$$

Since $\chi(\Omega) = 1 - r$ (see Theorem 2.2.7) and

$$\sum_{j=1}^{N_{k,i}} \alpha_{k,i,j} = \begin{cases} \pi(N_{k,0} - 2), & \text{if } i = 0, \\ \pi(N_{k,i} + 2), & \text{if } i \geq 1, \end{cases} \tag{4.19}$$

we get

$$a_0(\Omega_k) = \frac{1-r}{12} - \frac{1}{24} \sum_{i=0}^{r} N_{k,i} + \frac{\pi}{24} \sum_{i=0}^{r} \sum_{j=1}^{N_{k,i}} \frac{1}{\alpha_{k,i,j}}.$$

Analogous to the proof of Lemma 4.4.2, the angles $\alpha_{k,0,j} \uparrow \pi$ and $\alpha_{k,i,j} \downarrow \pi$ for $i \geq 1$ as $k \to \infty$, so we write $\alpha_{k,0,j} = \pi(1 - f(k,0,j))$ and $\alpha_{k,i,j} = \pi(1 + f(k,i,j))$ for $i \geq 1$ with

$$f : \{(k,i,j) : k \in \mathbb{N},\ 0 \leq i \leq r,\ 1 \leq j \leq N_{k,i}\} \to (0,1).$$

We have, by (4.19),

$$\sum_{j=1}^{N_{k,i}} f(k,i,j) = 2,\ 0 \leq i \leq r.$$

Then, for $i = 0$ we get as before

$$\frac{\pi}{24}\sum_{j=1}^{N_{k,0}}\frac{1}{\alpha_{k,0,j}} = \frac{N_{k,0}}{24} + \frac{1}{12} + \frac{1}{24}\sum_{j=1}^{N_{k,0}}\frac{f(k,0,j)^2}{1-f(k,0,j)},$$

and for $i \geq 1$ we similarly get

$$\begin{aligned}
&\frac{\pi}{24}\sum_{j=1}^{N_{k,i}}\frac{1}{\alpha_{k,i,j}} = \frac{1}{24}\sum_{j=1}^{N_{k,i}}\frac{1}{1+f(k,i,j)} \\
&= \frac{1}{24}\sum_{j=1}^{N_{k,i}}\sum_{m=0}^{\infty}(-f(k,i,j))^m = \frac{N_{k,i}}{24} + \frac{1}{24}\sum_{j=1}^{N_{k,i}}\sum_{m=1}^{\infty}(-f(k,i,j))^m \\
&= \frac{N_{k,i}}{24} - \frac{1}{24}\sum_{j=1}^{N_{k,i}} f(k,i,j)\sum_{m=0}^{\infty}(-f(k,i,j))^m = \frac{N_{k,i}}{24} - \frac{1}{12} + \frac{1}{24}\sum_{j=1}^{N_{k,i}}\sum_{m=2}^{\infty}(-f(k,i,j))^m \\
&= \frac{N_{k,i}}{24} - \frac{1}{12} + \frac{1}{24}\sum_{j=1}^{N_{k,i}}\frac{f(k,i,j)^2}{1+f(k,i,j)}.
\end{aligned}$$

Thus,

$$a_0(\Omega_k) = \frac{1-r}{6} + \frac{1}{24}\sum_{j=1}^{N_{k,0}}\frac{f(k,0,j)^2}{1-f(k,0,j)} + \frac{1}{24}\sum_{i=1}^{r}\sum_{j=1}^{N_{k,i}}\frac{f(k,i,j)^2}{1+f(k,i,j)}$$

Now we let $k \to \infty$. Write

$$\epsilon_{i,k} = \max_{1\leq j\leq N_{k,i}} f(k,i,j),\ 0 \leq i \leq r$$

and note that

$$0 \leq \sum_{j=1}^{N_{k,0}}\frac{f(k,0,j)^2}{1-f(k,0,j)} \leq \frac{2\epsilon_{0,k}}{1-\epsilon_{0,k}}$$

and for $i \geq 1$

$$0 \leq \sum_{j=1}^{N_{k,i}}\frac{f(k,i,j)^2}{1+f(k,i,j)} \leq 2\epsilon_{i,k}.$$

Since all $\epsilon_{i,k} \to 0$, it follows that

$$a_0(\Omega_k) \to \frac{1-r}{6}.$$

□

The assumption that the holes in Ω_k are convex polygons is crucial. Indeed, it is not hard to construct non-convex domains $\{\Omega_k\}_{k=1}^{\infty}$ converging to a smooth domain Ω such that the perimeter of the Ω_k doesn't converge to the perimeter of Ω and the interior angles don't converge to π. For example, let Ω be the unit disc and for each k, let Ω_k be an enclosed 2^k-sided polygon such that every edge has length $1/k$ and every other vertex lies on the boundary $\partial\Omega$. Then it is clear that $\Omega_k \to \Omega$ in Hausdorff convergence since $||x-y|| \leq 1/k$ for every $x \in \Omega_k$ and $y \in \Omega$. However,

the perimeter of Ω_k is $2^k/k$, which goes to ∞ as $k \to \infty$, while the perimeter of Ω is π. This shows that $\Omega_k \to \Omega$ does not necessarily imply that $|\partial\Omega_k| \to |\partial\Omega|$. Moreover, every interior angle of Ω_k converges to 0. Figure 4.3 illustrates Ω_k for $k = 7$ and $k = 10$.

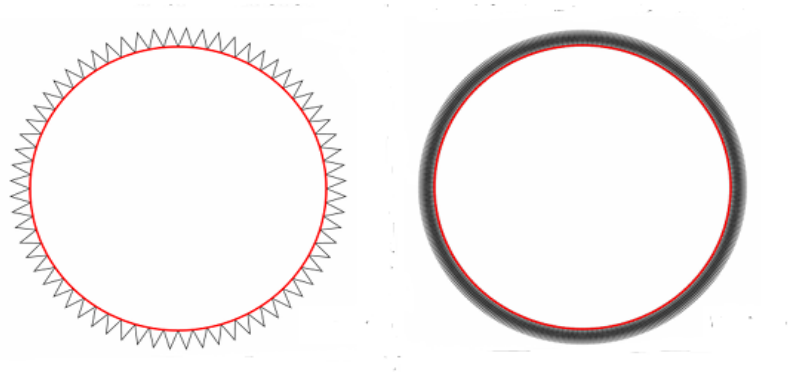

Figure 4.3: The polygons converge to the unit disc in the sense of Hausdorff, but the perimeter diverges. The left shows $k = 7$ and the right $k = 10$.

It is important to note that there are many different notions of convergence of sets, some more suitable than others depending on the situation. In this thesis we used Hausdorff convergence because it is relatively simple while also being good enough for our purposes here. Other examples of convergence of sets are Gromov-Hausdorff convergence [89, §5], pointed Gromov-Hausdorff convergence [63], and geometric convergence [3]. A natural continuation of this thesis would therefore be to ask under which notions of convergence the heat kernel and the heat trace coefficients converge to those of the limit set, and under what assumptions this holds in the non-convex case. Another possible continuation of the thesis would be to examine other types of domains Ω_k converging to something smooth, rather than just polygons. One could also try to find more terms in the heat trace expansion for a smoothly bounded domain, and see if they converge in the same way as the first three for convex polygonal domains. One could consider what would happen if we replaced Dirichlet boundary conditions for our problem with, say, Neumann or Robin boundary conditions. Finally, one could try to do what we have done in more general spaces like Riemannian manifolds.

5

Kac's legacy

The list of results given below as well as the open questions in this field of study are all part of Kac's legacy. Indeed, Kac's article inspired us and many more to study spectral invariants. The amount of mathematics that has been developed because of his article is truly spectacular, and it will most likely continue to develop for a long time. In this section, we try to summarize the mathematics inspired by Kac's article, starting from 1966 when his article was published and gradually working towards the time as this thesis is written.

The story began when Kac published the article *Can one hear the shape of a drum?* in 1966 [64]. It was partly inspired by Milnor's article who found two isospectral non-isometric 16-dimensional flat tori in 1964 [75]. A year after Kac's article was published, Kneser found two 12-dimensional flat tori which are isospectral but non-isometric [68]. In the same year, Grunbaum and Shaphard gave a survey of the study of convex polytopes in Euclidean space of two and three dimensions [53]. Kac's article also inspired Gerst to study nth power residues in 1970 [45]. Following that year, Berger, Gauduchon and Mazet wrote lecture notes about the spectrum of Riemann varieties [44]. Cheng became equally inspired in 1975 and studied eigenvalues and eigenfunctions of the Laplacian [25].

At this point in time, people also started using number theory to prove similar results. For example, in 1977 Kitaoka found (expressed in number theoretic language) two isospectral non-isometric 8-dimensional flat tori [67] and Perlis constructed algebraic number fields which are arithmetically equivalent but non-isometric [86]. In the same year Tammela studied Minkowski reduction regions for positive quadratic forms in seven variables [103], and Gangolli studied zeta functions of Selberg's type for compact space forms [42].

In 1978, Wolpert showed that flat tori are determined by their spectrum up to isometries, with the exception of some subvariety in the moduli space. Hence, in some sense almost all flat tori are determined by their spectrum [115]. In the same year Thurston wrote lecture notes about the geometry and topology of 3-manifolds [104]. In the next year Wolpert continued his work and studied the length spectra as moduli spaces for compact Riemann surfaces [116]. Ikeda then joined the journey in 1980 and gave examples of lens spaces which are isospectral but non-isometric [59]. Not long after in the same year he showed that three-dimensional spherical space forms are determined by their spectrum [60], and in 1983 he studied when two isospectral spherical space forms with non-cyclic fundamental groups must be

isometric [61]. 1983 was in fact a productive year in this subject as Scott also discussed the geometries that arise and the significance for the theory of 3-manifolds [98]. Moreover, Sunada studied trace formulas, Wiener integrals, and asymptotics for compact Riemannian manifolds [100], and Melrose examined the asymptotic expansion of the heat trace by perturbation arguments [73]. Not long after, Gordon and Wilson constructed continuous families of isospectral non-isometric manifolds in 1984 [51].

Before that, however, Venkov studied the spectral theory of automorphic functions [109], Brooks found another relationship between the fundamental groups and the spectrum of the Laplacian [12], and Hsia showed that the forms in a genus of positive ternary quadratic forms are not classified by the set of integers they represent [58], all in 1981. Urakawa also gave examples of bounded domains of dimension not less than four which are isospectral but non-isometric in 1982 [106].

A particularly important milestone for solving the problem proposed by Kac was achieved in 1985 when Sunada provided a new approach of proving that two isospectral Riemannian manifolds are non-isometric[1] [101]. This result was improved by Buser in 1986 as he proved that $\dim V_g > 0$ for a larger class of g-values [18]. At this point developments were made rapidly, and in 1987 Brooks constructed a manifold of negative curvature and two different potentials Φ_1 and Φ_2 such that $\Delta + \Phi_1$ and $\Delta + \Phi_2$ have the same spectrum [13]. In the same year, Brooks and Tse constructed surfaces which are isospectral but non-isometric [16], and DeTurck and Gordon constructed isospectral deformations of Riemannian metrics on manifolds on whose coverings certain nilpotent Lie groups act [35]. Finally, Protter described developments about whether one can hear the shape of a drum, and discussed the status of Kac's conjecture as well as similar conjectures for other operators [91]. Following that year, Buser visualized the combinatorial structure of Sunada's examples by looking at Cayley graphs [19], Brooks showed how to use Sunada's method to construct isospectral manifolds [14], and DeTurck studied which geometric properties are and aren't determined by their spectrum [34]. Then, Osgood, Phillips, and Sarnak studied the extremals of the determinants of Laplacians as functions of the metrics on surfaces [83]. In another article they studied compact sets of surfaces which are isospectral with respect to the Laplace operator [82], and in yet another article they showed that any isospectral family of two-dimensional Euclidean domains is compact in the C^∞ topology [85]. The work was still not quite done for that year as Durso also used the solution operator for the wave equation to prove several results in the inverse spectral theory of polygonal domains [37]. Furthermore, van den Berg and Srisatkunarajah studied the asymptotic expansion of the heat trace $e^{t\Delta}$ [108], and Widom showed that one of the inequalities used and proved in [83] follows quickly from theorems of G. Szegö on Toeplitz determinants [114].

Not surprisingly, 1989 was another strong year in the developments of this subject. First, Brooks, Perry and Yang studied isospectral sets of conformally equivalent

[1] Sunada may not have realized the importance of this result at the time as he simply described it as "a geometric analogue of a routine method in number theory".

metrics [15], and Gordon, DeTurck and Lee generalized the methods by Sunada by giving a general method for constructing isospectral manifolds [36]. Then, Gordon studied which geometric and topological properties of a Riemannian manifold are determined by their spectrum and constructed different manifolds with the same spectrum [47]. Bérard studied isospectral but non-isometric Riemann varieties [7], and Osgood, Phillips and Sarnak showed that any isospectral set of plane domains is compact in the C^∞ topology [84]. At the end of the year, Chang and DeTurck also proved that to determine whether two triangles in the Euclidean plane are congruent it's enough to know that they have their first N eigenvalues in common, where N depends on the first two eigenvalues of the triangles [23]. Just as much contribution was given in the following year as Schiemann found two isospectral non-isometric four-dimensional flat tori [93] and Sunada and Nishio established certain trace formulas [102]. Later, Branson, Gilkey, and Orsted found relations between linear and quadratic terms in the asymptotic expansion of the heat kernel, and used these to compute the heat variants modulo cubic terms [11]. Durso also showed that triangles are spectrally determined, and that for general polygons the Poisson relation holds [38]. In 1991, Shiota found two isospectral non-isometric flat tori [99], and Earnest and Nipp described and found inequivalent positive definite integral quaternary quadratic forms with identical theta series [39]. Finally, Chow showed that if g is any metric on a Riemann surface, then under Hamilton's Ricci flow g converges to a metric of constant curvature [26].

1992 was the year where Kac's problem was finally solved. First, Gordon, Webb and Wolpert used ideas from Sunada to construct a pair of isospectral, non-isometric domains in a hyperbolic setting and in the 2-sphere [49]. Later that year, they were in fact able to construct two isospectral manifolds in the Euclidean plane which are not isometric, thus showing that one cannot hear the shape of a drum [50]. The story doesn't end here, however, partly because one was still interested in the properties of domains that are spectral invariants, and partly because one can ask similar questions in more general spaces. For example, in the same year Conway introduced simple notation for orbifolds [28], and together with Sloane he found two isospectral non-isometric five- and six-dimensional flat tori [30]. Bérard also studied isospectral transplantation [8]. Conway then continued his work together with Buser, Doyle, and Semmler by giving more examples of isospectral non-congruent domains in the plane, and a "simple method of proof" [20]. Also, in 1994 Schiemann solved the problem about what the lowest dimension is where you can have two flat tori which are isospectral but non-isometric. The answer turned out to be four [94]. One year later, Chapman constructed several examples of isospectral but non-isometric domains [24], and in 1996 Gordon and Webb continued to describe the solution of Kac's problem whether you can hear the shape of a drum [48]. They still weren't done as in 1997 Conway proved that the genus of a quadratic form is determined by their spectrum if and only if the dimension is at most four [29]. Moreover, Kaplansky and Schiemann proved that there are at most 913 and at least 891 regular ternary quadratic forms [62]. Not long after Schiemann proved that ternary positive definite quadratic forms are determined by their theta series [95]. Zelditch continued the development in 1998 by describing some results on the inverse spectral problem

on a compact Riemannian manifold which is based on V.Guellemin's strategy of normal forms [117]. Later in 1999 Zelditch proved that bounded simply connected real analytic plane domains with the symmetry of an ellipse are determined by their spectrum [118].

At this point a new millennium had started, but the mathematicians were still eager to study properties of manifolds and domains that are and aren't spectrally determined. In 2000, Watanabe studied plane domains which are isospectrally determined by studying the trace of the heat kernel [111], and Hassell and Zelditch showed that any isophasal (i.e. having the same scattering-phase) family is compact in the C^∞ topology [55]. Later in 2005, Okada, Shudo, Tasaki, and Harayama revisited Kac's paper on whether one can hear the shape of a drum [80], and Perry discussed Kac's article and some generalizations [87]. Kurasov and Nowaczyk investigated the inverse spectral problem for the Laplace operator on a finite metric graph and showed that the problem has a unique solution for graphs with rationally independent edges and without vertices of valence 2 [69]. Gnutzmann, Smilansky and Sondergaard also showed in 2005 that the spectral ambiguity can be resolved by comparing the nodal sequences, i.e. the numbers of nodal domains of eigenfunctions arranged by increasing eigenvalues [46]. Band, Shapira, and Smilansky presented and discussed isospectral quantum graphs which are not isometric in 2006 [5]. Then, in 2007 Guillarmou and Guillopé computed the regularized determinant of the Dirichelt-to-Neumann map in terms of particular values of dynamical zeta functions using natural uniformization [54]. At the end of the year, Colin De Verdiére proved a semi-classical trace formula [27].

The development kept on going and in 2008 Kim joined the research subject and showed that on some compact surfaces the set of flat metrics which have the same Laplacian spectrum is compact in the C^∞ topology [66]. In 2009 Schurmann described relations between between quadratic forms, sphere packings, and coverings [97]. In the following year Colin De Verdiére, Guillemin and Jerison proved that the wave trace $h(t)$ has a finite limit on $(2\pi, 8)$ and is infinitely differentiable at $t = 2\pi$ from the right [33]. 2011 turned out to be yet another productive year as Hein and Cervino proved, using theory of invariants, that every pair in the family that Conway and Sloane constructed in [30] are non-isometric whenever the parameters are pairwise different [22]. Simultaneously, Oh proved that 8 of the remaining 22 ternary quadratic forms are regular [79], and Antunes and Freitas gave numerical evidence that you only need the first three eigenvalues to determine a triangle completely [4]. At the end of the year, Datchev and Hezari also gave a survey of positive inverse spectral and inverse resonant results for Laplacians on bounded domains, Laplace-Beltrami operators on compact manifolds, Shrödinger operators, Laplacians on exterior domains, and Laplacians on manifolds which are hyperbolic near infinity [32]. One year later Borrelli, Lazarus, and Thibert converted convex integration theory into an algorithm that produces isometric maps of flat tori, and provided an implementation of a convex integration process leading to images of an embedding of a flat torus [9]. Bruning and Fajman gave an explicit formula for the number of nodal domains of certain eigenfunctions on a flat torus [17], and Oren and Band studied generalized Laplacians on discrete graphs and used them to show

that isospectral objects have the same nodal count sequences [81]. Borrelli, Jabrane, Lazarus, and Thibert continued their work in the next year and gave explicit constructions of isometric embeddings of a square flat torus in the three-dimensional Euclidean space [10]. In that year Conway and Sloane also described relations between sphere packings, lattices and groups [31]. In yet the same year Grieser and Maronna reviewed and added a new twist to the story that lead to a curious elementary geometric problem about triangles, which they then solved [52].

Then, in 2015 Richter and Raum established Sturm bounds for degree g on Siegel modular forms modulo a prime p [92]. In 2016 Lu and Rowlett showed that one can hear the corners of a drum [72]. In 2017 Meyerson and McDonald proved that the heat content determines planar triangles [74], and in 2018 Aldana and Rowlett proved a variational formula which shows how the zeta-regularized determinant of the Laplacian varies with respect to the opening angle [2]. Some of these mathematicians continued making discoveries not long after, and in 2019 Lin gave an analytic method to detect the dislocation of lattices [71]. Nursultanov, Rowlett, and Sher also constructed the heat kernel on curvilinear polygonal domains with Dirichlet, Neumann, and Robin conditions [78], and later in the same year they showed that the corners of a planar domain are a spectral invariant under the same boundary conditions [77].

In 2020, Kim and Oh made a return and proved that there are exactly 49 primitive regular ternary triangular forms [65]. Two years later, Nilsson, Rowlett and Rydell studied isospectral and isometric flat tori from three different perspectives: analysis, geometry, and number theory [76]. Then, Verhoeven explored some tools used for the inverse spectral problem, and attempted to prove some inverse spectral results for quadrilaterals [110].

As stated in the beginning of the section, it is evident that the mathematics inspired by Kac's article has been developed by a substantial amount over the years. It will be interesting to see how much further it develop in the coming years.

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