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Superconformal Higher Spin Theory in Three Dimensions

Master's Thesis in Physics and Astronomy

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MASTER'S THESIS

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Abstract

This thesis aims at constructing a superconformal higher spin theory in three spacetime dimensions. Such theories are of great interest, for instance, since they are believed to appear in the tensionless limit of string theory. They can also directly be related to theories of quantum gravity expressed as string and M-theories on AdS spaces, via the AdS/CFT correspondence. The construction of the superconformal higher spin theory employed in this thesis relies heavily on both the Cartan formalism of supergravity and gauge theory. After these have been introduced, both the conformal and superconformal algebras are constructed. These are then quantized in a way that enables a convenient extension to their respective higher spin version. The corresponding superconformal higher spin theory can either be formulated as a Chern-Simons gauge theory, or as a higher spin theory expressed in the unfolded formalism. By utilizing this, the field equations of the theory are derived using the equation of motion from respective approach, the zero field strength equation and the unfolded equation. By studying the consistent spin 2 truncation of the theory, the first steps towards a deeper understanding of the relation between the two approaches are taken. Finally, it is discussed how the constructed superconformal higher spin theory might also lead to new insights regarding the AdS/CFT correspondence and how this, in turn, can be used to obtain results in string and M-theory.

KEYWORDS: Conformal field theory, Chern-Simons theory, Higher spin theory, AdS/CFT, String theory, M-theory

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Contents

Chapter 1

Introduction

One of the most fundamental goals of theoretical physics is to unify gravity and quantum mechanics. The most promising candidate is string theory, in which the smallest constituents of the Universe are described as vibrating, one-dimensional strings. On sufficiently large length scales the strings appear as point particles, whose properties such as mass and charge (after compactification of the extra dimensions) are determined by the vibrational state of the string. String theories incorporating both bosonic and fermionic states are known as superstring theories, and consistency requires them to have ten spacetime dimensions. This means that six of them must be compactified on a very small compact manifold. In 1995, Witten showed that all known superstring theories could be unified into so-called M-theory, which instead describes the Universe's smallest constituents as two or five-dimensional branes living in eleven spacetime dimensions [1, 2].

The interaction between strings can be described by generalizing the perturbative formulation of quantum field theory, which is conveniently described using Feynman diagrams. Unlike for quantum field theory there is, however, no known non-perturbative formulation of string theory. This motivates why Maldacena's conjecture of the so-called AdS/CFT duality completely revolutionized the field in 1997 [3]. It provides a correspondence between the anti de-Sitter spaces that are used in the string and M-theoretical descriptions of quantum gravity, and the conformal quantum field theories living on their boundaries. Since quantum field theories can be formulated non-perturbatively, this may enable exact formulations of string and M-theory. The AdS/CFT correspondence is to this day one of the most active fields of research in theoretical physics [4]. The correspondence most relevant for this thesis is the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, which can be used to relate M-theory on the space $\mathrm{AdS}_4 \times S^7$, where S^7 denotes the seven compactified dimensions, to a three-dimensional conformal field theory. Since M-theory is a supersymmetric theory incorporating both bosons and fermions, this must be a superconformal field theory.

An intriguing property of this correspondence is that if we apply Neumann boundary conditions to the spin $s \geq 1$ fields in the AdS_4 space, the corresponding CFT_3 can be formulated as a Chern-Simons gauge theory [5]. This was first observed for the spin 1 case in [6] and then generalized to the spin 2 case in [7, 8]. The possibility of formulating the conformal field theories as Chern-Simons gauge theories enables us to express them in terms of an action that is invariant under some local symmetry transformation. From the action one can readily derive the Chern-Simons equation of motion, known as the zero field strength equation.

An important feature of all string theories is that they contain an infinite mass-spectrum of ever increasing spin. This property leads us to study the so-called higher spin theories based on the superconformal algebra. In their original formulation the superconformal field theories only include massless fields of spin 2 and lower, but by coupling them to a spin 0 and a spin $\frac{1}{2}$ field it can be shown that as soon as the spin $\frac{5}{2}$ fermion and the spin 3 boson are included, massless fields of *all* half-integer and integer spins will inevitably appear. Since this resembles the infinite tower of spins appearing in the tensionless (massless) limit of string theory, there are reasons to believe that one can gain a deeper understanding of string theory by studying superconformal higher spin theories [9].

Higher spin theory is an independent field of research which had been studied in several other contexts before the connection to string theory had been considered, *e.g.*, in supergravity in the seventies and eighties[1] [9]. A disadvantage of higher spin theory is that it is very difficult to find an exact formulation of the equation of motion. The only method that has managed to provide exactly formulated examples of interacting higher spin theories is the so-called unfolded formulation [10], developed by Vasiliev in [11–13].

With this background we understand how the equations of motion can be derived for a three-dimensional superconformal field theory living on the boundary of an AdS_4 space on which a higher spin theory with Neumann boundary conditions has been implemented. While the unfolded equation yields the spin 0 and $\frac{1}{2}$ equations of motion, the spin $s \geq 1$ equations can be derived from the Chern-Simons equation of motion. This indicates that the zero field strength and unfolded equations must be compatible (and thus also integrable). The exact relation between the two equations of motion is, however, still unknown, and understanding this might be one of the keys to gaining a deeper understanding of the AdS/CFT correspondence.

The purpose of this thesis is to construct a superconformal higher spin theory in three dimensions. To derive the field equations we will both utilize that the theory can be expressed as a Chern-Simons gauge theory, but also implement the unfolded formulation of higher spin theory. We are thereby able to compare the results stemming from the two approaches. However, to achieve this we first have to quantize the three-dimensional superconformal algebra in a way that enables a generalization to the corresponding higher spin algebra. We can then in detail study the equation of motion of respective approach, *i.e.* the field strength equation and the unfolded equation, for fields up to spin 2 in both the purely bosonic and the supersymmetric case. Since the spin 2 algebra is closed this is a consistent truncation.

In the bosonic case we follow the work of [14] and show that the field strength and unfolded equations yield equivalent results. In the supersymmetric case we solve the field strength equation, and by decomposing the unfolded equation into its irreducible representations we manage to take the first steps towards a deeper understanding of the exact relation between the two approaches in a superconformal higher spin context. The long-term motivation for this endeavor is to, in detail, be able to study the relation between three-dimensional superconformal field theories and M-theory on $\mathrm{AdS}_4 \times S^7$ that is proposed by the AdS/CFT correspondence.

[1]In fact, the low-energy limit of M-theory turns out to precisely correspond to eleven-dimensional supergravity, although that was obviously not known back then.

1.1 Outline of Thesis

The first four chapters of the thesis are introductory ones. These provide the concepts necessary for the reader to be able to follow the explicit construction and investigations of the superconformal higher spin theory that follows in the subsequent chapters. The content of respective chapter looks as follows.

Chapter 2 *Cartan Formalism of (Super)Gravity*
Introduces the Cartan formalism of gravity, which is necessary since the metric formulation of gravity can never include spinors. Also the extension to supergravity is discussed, where *contorsion* will be the key concept.

Chapter 3 *Gauge Theory*
Gives an introduction to both abelian and non-abelian gauge theories, with particular emphasis on Chern-Simons theory.

Chapter 4 *Conformal and Superconformal Symmetries*
The concepts of conformal and superconformal symmetries are defined, and the corresponding algebras are constructed explicitly from the symmetry transformations in three dimensions.

Chapter 5 *Higher Spin Theory and the AdS/CFT Duality*
Provides a general introduction to both higher spin theory, especially in the unfolded formulation, and the AdS/CFT correspondence. Although it presents some essential relations, in particular the unfolded equation, it is also a motivational chapter giving the detailed explanation to why superconformal higher spin theories are of interest.

Chapter 6 *Quantizing the Superconformal Algebra*
Presents an explicit representation of the generators of the superconformal algebra in three dimensions. This representation is constructed in a way that can easily be generalized to the full higher spin algebra. Also two possible ways of quantizing the algebra are given, the operator formulation and the star product formulation.

Chapter 7 *The Zero Field Strength Equation*
The zero field strength equation, stemming from Chern-Simons theory, is solved for the spin 2 restrictions of both the purely bosonic and the supersymmetric conformal field theory.

Chapter 8 *The Unfolded Equation*
The explicit unfolding of both the bosonic and the supersymmetric theory is carried out, and the results are continuously compared to those obtained by solving the zero field strength equation.

Chapter 9 *Conclusions and Future Directions*
Summarizes the obtained results and provides an outlook to future research.

Chapter 2

Cartan Formalism of (Super)Gravity

When Einstein in 1916 formulated the general theory of relativity he used the metric tensor to describe the geometric and causal structure of spacetime. The metric is uniquely defined at each point in spacetime given the coordinates x^μ in some coordinate system. For our purposes it will however prove convenient to use an alternative formulation of general relativity, the Cartan formalism. In the Cartan formalism, the set of basis vectors used in the tangent space of each point in spacetime is *not* derived from a specific coordinate system. Instead it is chosen as an arbitrary local Minkowski basis. This seemingly subtle change of perspective will turn out to be crucial since it will not only help us to formulate gravity as a gauge theory but also enable us to introduce spinors, which describe fermionic degrees of freedom. The Cartan formalism is consequently essential when we want to supersymmetrically extend general relativity to supergravity, which includes the fermionic gravitino field.

To understand the Cartan formalism, a certain amount of differential geometry is needed. This chapter will, thus, begin with a brief introduction to the subject, with focus on important concepts such as differential forms and exterior algebras. Then, the metric formulation of general relativity and some important quantities therein are briefly reviewed, before the transition to the Cartan formalism of general relativity is carried through. Finally we discuss how general relativity can be extended to a supersymmetric theory of gravity.

2.1 Differential Geometry

Since general relativity is a geometric theory where spacetime in general is curved, the concept of smooth manifolds is essential. These are infinitely differentiable manifolds for which all local regions resemble Euclidean (or Minkowski) space. Our first goal will thus be to understand how these can be described.

2.1.1 The Tangent and Cotangent Bases

A good starting point is to, given some local coordinates x^μ, identify a set of basis vectors on the manifold. Given some function $f = f(x^\mu)$ on the manifold one can form the directional derivatives $\partial_\mu f := \frac{\partial f}{\partial x^\mu}$. Thus it is natural to use the partial derivatives $\{\partial_\mu\}$ as a set of basis vectors. Note that this set, in general, differs for the different points of the manifold, since the partial derivatives always lie in the tangent space of the specific point of the manifold. For this reason, this basis is often referred to as the tangent basis. On the tangent space we form vectors as $V = V^\mu \partial_\mu$, although we often simply denote them as V^μ to refer to their components.

It will prove convenient to also introduce a basis for the dual of the tangent space, *i.e.*, the cotangent space. This dual basis will consist of one-forms which is a type of differential forms, an important concept that will be presented in more detail in the subsequent section. A one-form ω is a linear functional $\omega : T_p \to \mathbb{R}$, where T_p is the tangent space at point p. For now we only need to consider the gradient one-form of a function f. Its action on a vector $\frac{\mathrm{d}}{\mathrm{d}\lambda}$ is merely the directional derivative

$$\mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}f}{\mathrm{d}\lambda}. \tag{2.1}$$

In particular we note that

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \tag{2.2}$$

Hence, just like for the tangent basis, the coordinate system naturally induces a basis $\{\mathrm{d}x^\mu\}$ of gradient one-forms. This is the cotangent basis, which can be used to expand an arbitrary one-form ω into components as $\omega = \omega_\mu \mathrm{d}x^\mu$.

The tangent and cotangent bases can be used to expand all tensors on the curved manifold into components. A general tensor T having m contravariant and n covariant components can be expanded as

$$T = T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_m} \otimes \mathrm{d}x^{\nu_1} \otimes \dots \otimes \mathrm{d}x^{\nu_n}. \tag{2.3}$$

Once again it is worth pointing out that we often will use the phrase "tensor" to refer to its components $T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}$.

2.1.2 Differential Forms

The appeal of differential forms is that they provide a coordinate independent approach to defining integrands over manifolds of arbitrary dimension. The set of all differential k-forms on a manifold is always a vector space, with k being a non-negative integer. This implies that differential forms satisfy the basic operations of addition and multiplication by scalars induced by the vector space structure, but there are also other operations defined on differential forms. The two we will consider here are the wedge product and the exterior derivative. The former is even used in the construction of the differential forms of order $k \geq 2$.

An arbitrary differential form ω of order k can be constructed as

$$\omega = v_1 \wedge \ldots \wedge v_k \tag{2.4}$$

using k differential one-forms v_i and the wedge product. Recalling from the previous section that the k differential forms v_i can be related to k covariant vectors v_μ, via the relation $v_i = (v_\mu \mathrm{d}x^\mu)_i$, we realize that the geometrical interpretation of ω is the parallelepiped spanned by the covariant vectors v_μ. An important property of the wedge product of two one-forms is that it is anticommuting, meaning $v_1 \wedge v_2 = -v_2 \wedge v_1$. This gives the orientation of the parallelepiped.

The anticommutativity of the wedge product of one-forms, together with the construction of an arbitrary differential form in equation (2.4), directly tells us that two differential forms ω_1 and ω_2 of order p and q must satisfy the relation

$$\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1. \tag{2.5}$$

Equation (2.4) also implies that this object must be a differential $(p+q)$-form. The algebra having the wedge product as its multiplication operation is known as the exterior algebra, or the Grassmann algebra.

From equation (2.4) and the anticommutativity of the wedge product of one-forms we also realize that the differential forms $\mathrm{d}x^\mu$, equipped with the wedge product as the multiplication operator, form a basis for the totally antisymmetric covariant tensors. This means that we can extract the antisymmetric part of an arbitrary covariant tensor T as

$$T = \frac{1}{n!} T_{[\mu_1 \ldots \mu_n]} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}. \tag{2.6}$$

This basis will often be more convenient to use than the one in equation (2.3) using the tensor product $\mathrm{d}x^{\mu_1} \otimes \ldots \otimes \mathrm{d}x^{\mu_n}$.

Another important operation on differential forms is the exterior derivative. For our purposes we can define it as

$$\mathrm{d} := \mathrm{d}x^\mu \partial_\mu. \tag{2.7}$$

Note that this operation yields an $(n+1)$-form when acting on an n-form. In accordance with our observation in equation (2.6) we realize that the exterior derivative is the antisymmetric part of the partial derivative. The product rule for the exterior derivative reads

$$\mathrm{d}(\omega_1 \wedge \omega_2) = (\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \tag{2.8}$$

where ω_1 is a differential p-form and ω_2 is a differential form of arbitrary order.

The purpose of introducing the exterior derivative becomes obvious when studying the transformation of the partial derivative of a tensor A_ν under the coordinate transformation $x^\mu \to x'^\mu$. By using the product rule we find that it transforms according to

$$\partial_\mu A_\nu \to \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial A_\sigma}{\partial x^\rho} + A_\sigma \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}. \tag{2.9}$$

In order for the LHS to transform as a tensor, the second term must vanish. Since the partial derivatives commute, this term is symmetric in the μ and ν indices. If we instead study the transformation law for the exterior derivative $\mathrm{d}A$ of the one-form $A = A_\nu \mathrm{d}x^\nu$, the second term really vanishes. This due to our observation above that the exterior derivative is the antisymmetrized partial derivative, and the antisymmetric part of a symmetric expression is zero. Hence, the exterior derivative $\mathrm{d}A$ is a proper antisymmetric tensor.

Another consequence of the partial derivatives commuting and the exterior derivative being an antisymmetrized partial derivative is that the operator $\mathrm{d}^2 = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \partial_\mu \partial_\nu$ vanishes identically. Thus, if the exterior derivative is applied more than once, the result is always zero. Although they will not be used in this thesis it is worth pointing out that the exterior derivative enables metric-independent generalizations of Stokes', Gauss's, and Green's theorems to higher-dimensional manifolds.

2.2 The Metric Formulation of General Relativity

The setting of the metric formulation of general relativity is a four-dimensional Lorentzian manifold, *i.e.* a smooth manifold that has a metric of signature $(1, 3)$, with local coordinates x^μ. As we learned in section 2.1.1, these coordinates induce a tangent basis $\{\partial_\mu\}$ and a dual cotangent basis $\{\mathrm{d}x^\mu\}$. However, since both the tangent spaces T_p and the cotangent spaces T_p^* differ for the different points p of the manifold, both these bases are local. This, for instance, causes problems when we want to parallel transport a vector along the manifold. To resolve these we introduce an *affine connection*, which is used to connect the tangent spaces of different points on the manifold.

Hand in hand with the affine connection is often the covariant derivative introduced. Its importance is, however, self-evident. There must, for instance, be a way to generalize the equation for conservation of energy in flat spacetime, *i.e.* $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu}$ being the stress–energy tensor, to curved manifolds. As illustrated in equation (2.9) the partial derivative does *not* transform as a tensor, so we need to introduce a derivative that does. By the definition of the affine connection $\Gamma^\rho{}_{\mu\nu}$ we can form the covariant derivatives

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho \tag{2.10}$$

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho{}_{\mu\nu} V_\rho \tag{2.11}$$

acting on a contravariant and covariant vector, respectively. For higher-rank tensors, an additional term of the same form as the connection terms above is created for each free index. The covariant derivative of a tensor measures the rate of change of the tensor relative to parallel transport, in a certain direction.

Since the affine connection is defined in order for the covariant derivatives in equations (2.10) and (2.11) to transform like tensors under a coordinate transformation, we can use this to determine how the affine connection itself must transform. A straightforward calculation gives that it transforms as

$$\Gamma^\rho{}_{\mu\nu} \to \frac{\partial x'^\rho}{\partial x^\lambda} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\eta}{\partial x'^\nu} \Gamma^\lambda{}_{\sigma\eta} + \frac{\partial x'^\rho}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu}. \tag{2.12}$$

Note the second term which tells us that the affine connection is *not* a proper tensor.

Counting its components we realize that the affine connection introduced new degrees of freedom, meaning it is not uniquely determined by the metric. To remove these extra degrees of freedom we will introduce two constraints: metric compatibility and the torsion-free condition. The metric compatibility condition reads

$$D_\mu g_{\nu\rho} = 0 \tag{2.13}$$

and implies that the inner product of two vectors being parallel transported around the same path is preserved (meaning the lengths of the vectors and the angle between them are both preserved). It also implies the relation $D_\mu g^{\nu\rho} = 0$ for the inverse metric, and makes the covariant derivative commute with raising and lowering the indices, meaning

$$g_{\mu\nu} D_\rho V^\nu = D_\rho (g_{\mu\nu} V^\nu) = D_\rho V_\mu \tag{2.14}$$

for all vector fields V^μ.

To introduce the torsion-free connection we first need to define the torsion tensor. We do this by noting from equation (2.12) that the difference of any two affine connections $\Gamma^\rho{}_{\mu\nu}$ and $\tilde{\Gamma}^\rho{}_{\mu\nu}$ will transform as a proper tensor, since the second term will be identical in the two affine connections and thereby cancelling out. We also note that the affine connection with the two lower indices permuted, *i.e.* $\Gamma^\rho{}_{\nu\mu}$, will transform precisely as $\Gamma^\rho{}_{\mu\nu}$ since the partial derivatives in the second term commute. For every affine connection $\Gamma^\rho{}_{\mu\nu}$ we can thus construct a proper tensor

$$T_{\mu\nu}{}^\rho = 2\Gamma^\rho{}_{[\mu\nu]} \tag{2.15}$$

known as the torsion tensor. The torsion-free condition simply requires the torsion tensor to vanish. The affine connection then obtained is known as a *Levi-Civita connection*, and it is consequently symmetric in its lower two indices. The geometric interpretation of the torsion-free condition is that it imposes all parallelograms formed by parallel transport of two infinitesimal displacement vectors to be closed. Consistent theories of gravitation can actually be constructed without imposing the torsion-free condition, one example being the theory of supergravity that will be studied closer in section 2.5.

Having introduced the metric compatibility and torsion-free conditions we claim to have removed all the extra degrees of freedom introduced by the affine connection. To prove this it is enough to show that the Levi-Civita connection is uniquely determined by the metric. By permuting the indices in the metric compatibility condition (2.13) we find that

$$\begin{aligned} D_\rho g_{\mu\nu} - D_\mu g_{\nu\rho} - D_\nu g_{\rho\mu} = {} & \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} \\ & + 2\big(\Gamma^\sigma{}_{(\mu\nu)} g_{\sigma\rho} + \Gamma^\lambda{}_{[\mu\rho]} g_{\lambda\nu} + \Gamma^\lambda{}_{[\nu\rho]} g_{\lambda\mu}\big) = 0, \end{aligned} \tag{2.16}$$

where the torsion-free condition imposes the last two terms to vanish. By then multiplying by $g^{\lambda\rho}$ and solving for the Levi-Civita connection, we find it to read

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\big), \tag{2.17}$$

meaning it is uniquely fixed by the metric.

Finally we also want to introduce the important concept of *curvature*. It is obvious that the curvature is intimately related to the connection describing the manifold, but a local description of the curvature at each point of the manifold would be to prefer. This is provided by the *Riemann curvature tensor*. Loosely speaking, the curvature is supposed to measure how far from being flat the manifold is. A characteristic property of flat space is that parallel transport around a closed loop leaves a vector unchanged. We have already stated that the covariant derivative of a tensor measures the rate of change of the tensor relative to parallel transport, in a certain direction. A good measure of curvature should thus be the commutator of two covariant derivatives (since this, at least in the absence of torsion, corresponds to a closed parallelogram). A straightforward calculation using the definitions above yields

$$[D_\mu, D_\nu]V^\rho = \left(\partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}\right)V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho, \tag{2.18}$$

where the torsion-free condition has not yet been applied. This expression proves our claim above that the torsion-free condition imposes all parallelograms formed by parallel transport of two infinitesimal displacement vectors to be closed (since an infinitesimal parallelogram always is flat and thus has $\Gamma^\rho{}_{\mu\nu} \equiv 0$, so the entire expression in parentheses vanishes). It even shows that the torsion can be directly interpreted as the extent to which such infinitesimal parallelograms fail to close.

In general relativity we *impose* the torsion-free condition and realize that the expression in parentheses then provides a local description of curvature. Consequently, we define the Riemann curvature tensor to read

$$R^\rho{}_{\sigma\mu\nu} = 2(\partial_{[\mu}\Gamma^\rho{}_{\nu]\sigma} + \Gamma^\rho{}_{[\mu|\lambda|}\Gamma^\lambda{}_{\nu]\sigma}). \tag{2.19}$$

It is easily shown that the Riemann tensor in it its fully covariant form $R_{\lambda\sigma\mu\nu} = g_{\lambda\rho}R^\rho{}_{\sigma\mu\nu}$, is antisymmetric in both its first and second *pairs* of indices, meaning

$$R_{\lambda\sigma\mu\nu} = -R_{\sigma\lambda\mu\nu} = -R_{\lambda\sigma\nu\mu} = R_{\sigma\lambda\nu\mu} \tag{2.20}$$

and symmetric under the interchange of these pairs, meaning

$$R_{\lambda\sigma\mu\nu} = R_{\mu\nu\lambda\sigma}. \tag{2.21}$$

From the Riemann curvature tensor we can also construct the Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{2.22}$$

and the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{2.23}$$

These are important objects appearing in the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.24}$$

where Λ is the cosmological constant and G the gravitational constant. These equations describe gravity as the curvature of spacetime caused by matter and energy. They can be constructed from the so-called Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\,\mathrm{d}^d x, + \mathcal{S}_{\text{matter}} \tag{2.25}$$

where $g = \det(g_{\mu\nu})$ and the variation of the second term yields the RHS of the Einstein field equations including the stress-energy tensor $T_{\mu\nu}$.

2.3 The Cartan Formulation of General Relativity

In the metric formulation of general relativity we used the partial derivatives $\{\partial_\mu\}$ as the basis for the tangent spaces, and the metric $g_{\mu\nu}$ to describe the curvature of spacetime. Both of these are, at each point in spacetime, dependent on the coordinate system we choose. If we are to construct a quantum field theory of gravity we need to be able to separate the physical degrees of freedom from the ones that are artefacts of the coordinate system. This is very difficult to do using the metric tensor as our field, but standard procedure if we instead introduce gauge fields.

To achieve this we, for each point of the manifold, introduce an arbitrary orthonormal basis $\{e_a\}$ in the tangent space. Since the tangent space is Minkowskian we by "orthonormal" mean a basis having a Minkowski inner product satisfying $\eta(e_a, e_b) = \eta_{ab}$. To indicate that the indices labelling this basis are not related to a specific coordinate system, they will be denoted by Latin letters and be referred to as "flat", whereas the indices labelling the coordinate basis will be denoted by Greek letters and be referred to as "curved". The invertible matrices ${e_a}^\mu$ relating the basis $\{e_a\}$ to the coordinate system-induced basis $\{\partial_\mu\}$, thus satisfying

$$e_a = {e_a}^\mu \partial_\mu, \tag{2.26}$$

are known as the *frame fields* or *vielbeins*. Also the inverse matrices ${e_\mu}^a$ satisfying ${e_\mu}^a {e_b}^\mu = \delta^a_b$, are often referred to as the vielbeins.

Note that we for an arbitrary vector $V = V^\mu \partial_\mu = V^a e_a$ can use equation (2.26) to relate its components in the curved coordinate basis to the ones in the flat Minkowski basis by

$$V^\mu = {e_a}^\mu V^a. \tag{2.27}$$

Then acting with the inverse vielbein ${e_\mu}^b$ we find the expected relation

$$V^a = {e_\mu}^a V^\mu. \tag{2.28}$$

The same reasoning can be applied to *any* tensor, thereby expressing it in either the curved or the flat basis. For instance, we can write the metric of the spacetime manifold as

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}. \tag{2.29}$$

Due to this relation, the vielbeins are sometimes referred to as the "square root" of the metric.

It is also worth pointing out that the vielbein ${e_\mu}^a$ has $\frac{1}{2}D(D-1)$ additional degrees of freedom compared to $g_{\mu\nu}$, which is restricted to be symmetric (D being the dimension of spacetime). These extra degrees of freedom are, however, non-physical and correspond to local Lorentz transformations. In section 7.1.1 we will understand how they conveniently can be removed via a gauge choice.

Since also our new coordinate system-independent basis $\{e_a\}$ is a local basis spanning the tangent space of a specific point, we once again need to introduce a covariant derivative and a connection to be able to relate the tangent vectors of different tangent spaces.

The connection acting on tensors with flat indices is known as the spin connection. The covariant derivatives of vectors with flat indices thus take the forms

$$\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \\ D_\mu V_a &= \partial_\mu V_a - \omega_\mu{}^b{}_a V_b, \end{aligned} \tag{2.30}$$

where $\omega_\mu{}^a{}_b$ is the spin connection. Note in particular that the covariant derivative of the flat metric η_{ab} reads

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac}. \tag{2.31}$$

Since this metric is just Minkowskian throughout the entire manifold, both the covariant and the partial derivatives of it must vanish. This tells us that the spin connection must be antisymmetric in its two flat indices, meaning $\omega_{\mu(ab)} = 0$.

It is now natural to require the covariant derivatives of a vector $V = V^\mu \partial_\mu = V^a e_a$ to be equal whether its components are expressed in the flat or the curved basis. By imposing this we can express the Levi-Civita connection in terms of the vielbein and spin connection as

$$\Gamma^\rho{}_{\mu\nu} = e_a{}^\rho \big(\partial_\mu e_\nu{}^a + e_\nu{}^b \omega_\mu{}^a{}_b \big). \tag{2.32}$$

Multiplication by the inverse vielbein $e_\rho{}^a$ yields that the vanishing of the covariant derivatives of the vielbeins, *i.e.*

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \Gamma^\rho{}_{\mu\nu} e_\rho{}^a = 0, \tag{2.33}$$

is one possible solution to this equation. This is known as the "vielbein postulate". For the derivation of equation (2.32) we refer to [15], where it is claimed that this implies the vielbein postulate as its unique solution. This has however been questioned (see, *e.g.*, [16] for a rigorous mathematical discussion), so for our purposes we can really regard equation (2.33) as a postulate (the same is done also in, *e.g.*, [17]).

This far, the Cartan formulation of general relativity may seem almost completely analogous to the metric formulation. Important differences will, however, appear when we apply our knowledge of differential forms from section 2.1.2. We do this inspired by the antisymmetries of the torsion and Riemann tensors. The first step is to introduce a coordinate system-independent basis $\{e^a\}$ also in the cotangent spaces T_p^* of each point on the manifold. The transition matrix to the curved basis $\{\mathrm{d}x^\mu\}$, used for the cotangent spaces in the metric formulation, is obviously given by the inverse vielbeins $e_\mu{}^a$, meaning

$$e^a = e_\mu{}^a \mathrm{d}x^\mu. \tag{2.34}$$

By using this basis, an arbitrary tensor $A_{\mu_1 \ldots \mu_n}{}^{a_1 \ldots a_n}$ antisymmetric in its curved indices can be written as a tensor valued n-form

$$A^{a_1 \ldots a_n} = \frac{1}{n!} A_{\mu_1 \ldots \mu_n}{}^{a_1 \ldots a_n} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}. \tag{2.35}$$

Of particular interest is the torsion form

$$T^a = \frac{1}{2} T^a{}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{2.36}$$

and the curvature form

$$R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.37}$$

Similarly to how we expressed the torsion and Riemann tensors in terms of the Levi-Civita connection, in equations (2.15) and (2.19), it should be possible to express the torsion and curvature forms in terms of the spin connection. For this purpose we construct the spin connection 1-form

$$\omega^a{}_b = \omega_\mu{}^a{}_b \mathrm{d}x^\mu. \tag{2.38}$$

By inserting the expression for the Levi-Civita connection from equation (2.32) into equations (2.36) and (2.37) for the torsion and curvature forms, we find the expressions

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \tag{2.39}$$
$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{2.40}$$

which are known as the Maurer-Cartan structure equations.

We also want to be able to apply some sort of exterior derivative to the tensor valued forms. But since derivatives on curved manifolds should take the curvature into account, we need to extend the concept to an exterior *covariant* derivative. Since this derivative should possess the same antisymmetric nature as the ordinary exterior derivative, the Levi-Civita connection cannot appear because of its symmetry. The spin connection form ω_{ab} is, however, completely antisymmetric and should be included. The exterior covariant derivative of a contravariant and a covariant vector valued form thus read

$$\begin{aligned} \mathrm{D}V^a &= \mathrm{d}V^a + \omega^a{}_b \wedge V^b \\ \mathrm{D}V_a &= \mathrm{d}V_a + \omega_a{}^b \wedge V_b, \end{aligned} \tag{2.41}$$

respectively. The extension to higher-dimensional tensor valued forms is obvious, with an additional spin term for each free index.

In particular, we note that the exterior covariant derivative of the vielbein one-form yields the torsion form, *i.e.*, $T^a = \mathrm{D}e^a$. The torsion-free condition thus implies the relation

$$T^a = \mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \tag{2.42}$$

in the Cartan formalism. By massaging this a bit, we find the relation

$$\omega_{abc} = e_{[a}{}^\mu e_{b]}{}^\nu \partial_\mu e_{\nu c} - e_{[a}{}^\mu e_{c]}{}^\nu \partial_\mu e_{\nu b} - e_{[b}{}^\mu e_{c]}{}^\nu \partial_\mu e_{\nu a} \tag{2.43}$$

expressing the spin connection in terms of the vielbein.

2.4 Some Curvature-Related Tensors and Identities

We will now introduce some tensors and identities that are related to the curvature and will prove to be of great importance in the subsequent chapters. In section 2.2 we defined the Riemann curvature tensor to satisfy the relation

$$[D_\mu, D_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma, \tag{2.44}$$

where V^ρ is an arbitrary contravariant vector and the torsion-free condition has been assumed. This relation is known as the *Ricci identity*.

The Riemann tensor satisfies the so-called Bianchi identity

$$D_{[\mu}R_{\nu\rho]}{}^{\sigma}{}_{\lambda} = 0, \tag{2.45}$$

which will be derived in its most general form in section 3.3. Upon contraction of the ρ and λ indices, which in accordance with the metric compatibility condition (2.13) can be done by simply multiplying by the metric $g^{\rho\lambda}$, this yields

$$D_\mu R_\nu{}^\sigma - D_\nu R_\mu{}^\sigma + D_\rho R_{\mu\nu}{}^{\sigma\rho} = 0, \tag{2.46}$$

where we have used the symmetry properties (2.20) and (2.21) of the Riemann tensor and definition (2.22) of the Ricci tensor. By also contracting the ν and σ indices we find that

$$D_\mu R = 2D_\nu R_\mu{}^\nu, \tag{2.47}$$

where R is the Ricci scalar. This form of the Bianchi identity will prove useful later on.

Note that there must be a decomposition of the Riemann tensor into the Ricci tensor and Ricci scalar in the form

$$R_{\mu\nu}{}^{\rho\sigma} = x\delta^{[\rho}_{[\mu}R_{\nu]}{}^{\sigma]} + y\,\delta^{\rho\sigma}_{\mu\nu}R, \tag{2.48}$$

for some constants x and y. By contracting the ν and σ indices this yields

$$R_\mu{}^\rho = \frac{x}{4}\big(R_\mu{}^\rho + \delta^\rho_\mu R\big) + y\delta^\rho_\mu R, \tag{2.49}$$

which implies that $x = 4$ and $y = -1$, and thus

$$R_{\mu\nu}{}^{\rho\sigma} = 4\delta^{[\rho}_{[\mu}R_{\nu]}{}^{\sigma]} - \delta^{\rho\sigma}_{\mu\nu}R, \tag{2.50}$$

which is on the desired form.

There is also another, similar decomposition of the Riemann tensor reading

$$R_{\mu\nu}{}^{\rho\sigma} = W_{\mu\nu}{}^{\rho\sigma} + 4\delta^{[\rho}_{[\mu}S_{\nu]}{}^{\sigma]}, \tag{2.51}$$

where the *Weyl tensor* $W_{\mu\nu\rho\sigma}$ is defined as the completely traceless part of the Riemann tensor, and the *Schouten tensor* $S_{\mu\nu}$ is used to construct the trace part. The Weyl tensor has the same symmetry properties (2.20) and (2.21) as the Riemann tensor, but in addition also satisfies

$$W_{\mu\nu}{}^{\mu}{}_{\sigma} = 0, \tag{2.52}$$

which together with the other symmetry properties makes it traceless in *all* pairs of indices. The Weyl tensor is the only non-zero curvature component in the absence of matter, and it thus describes, *e.g.*, how gravitational waves propagate in vacuum. Another important property is that it is invariant under conformal transformations of the metric (a concept that will be introduced in chapter 4). This can be used to show that the vanishing of the Weyl tensor is a necessary and sufficient condition for spacetime to be conformally flat in $D \geq 4$ dimensions. In $D = 3$, the Weyl tensor vanishes identically.

This means that there is no curvature external to matter sources in three dimensions. By comparing equations (2.50) and (2.51) we then realize that the Schouten tensor must read

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \tag{2.53}$$

in three dimensions.

Also in three dimensions it is convenient to introduce a tensor whose vanishing implies that spacetime is conformally flat, like the Weyl tensor does in $D \geq 4$ dimensions. This tensor is known as the *Cotton tensor* and reads

$$C_{\mu\nu} = \varepsilon_\mu{}^{\rho\sigma} D_\rho S_{\sigma\nu}, \tag{2.54}$$

where $\varepsilon_\mu{}^{\rho\sigma}$ is the Levi-Civita tensor that is related to the Levi-Civita symbol $\epsilon_\mu{}^{\rho\sigma}$ (which is really a tensor density) via $\varepsilon_\mu{}^{\rho\sigma} = \sqrt{|g|}\epsilon_\mu{}^{\rho\sigma}$, with g denoting the determinant of the metric. Note that $C_{\mu\nu}$ is of third order in the derivatives of the metric[1]. We can easily show that the Cotton tensor must be symmetric by noting that

$$\varepsilon_\lambda{}^{\mu\nu} C_{\mu\nu} = -|g| \left(2\delta^{\rho\sigma}_{\nu\lambda} D_\rho S_\sigma{}^\nu - \frac{1}{2} D_\lambda R \right) = -|g| \left(D_\rho R_\lambda{}^\rho - \frac{1}{2} D_\lambda R \right) = 0, \tag{2.55}$$

where we in the last step have used the Bianchi identity (2.47). This can be used to rewrite the Cotton tensor as

$$C^{\mu\nu} = \varepsilon^{\rho\sigma(\mu} D_\rho S_\sigma{}^{\nu)} = \varepsilon^{\rho\sigma(\mu} D_\rho R_\sigma{}^{\nu)}, \tag{2.56}$$

where we have inserted the Schouten tensor from equation (2.53).

2.5 The Extension to Supergravity

When we formulated the theory of general relativity above, we imposed two constraints: metric compatibility and the torsion-free condition. In the Cartan formalism, the former was substituted by the vielbein postulate. In supergravity, we can still assume the metric/vielbein postulate to hold, but the torsion-free condition will be broken due to the fermionic gravitational fields that are introduced, the so-called gravitino fields [18]. There are still (less strict) constraints that can be imposed on the torsion, but for our purposes it is sufficient to work with a general non-zero torsion tensor.

Working in the metric formulation, a non-zero torsion tensor implies that the affine connection can no longer be written on the form of equation (2.17), since we reached that expression by applying the torsion-free condition to equation (2.16). We will denote the uniquely determined, torsion-free Levi-Civita connection of equation (2.17) by $\bar{\Gamma}^\lambda{}_{\mu\nu}$. We then define the *contorsion tensor* $K^\lambda{}_{\mu\nu}$ as the difference between the metric-compatible affine connection $\Gamma^\lambda{}_{\mu\nu}$ and the Levi-Civita connection $\bar{\Gamma}^\lambda{}_{\mu\nu}$, meaning

$$\Gamma^\lambda{}_{\mu\nu} = \bar{\Gamma}^\lambda{}_{\mu\nu} + K_{\mu\nu}{}^\lambda. \tag{2.57}$$

[1] This in contrast to the Weyl tensor which is of second order in the derivatives.

With this affine connection, equation (2.16) instead implies

$$\begin{aligned}\Gamma^{\sigma}{}_{(\mu\nu)} &= \frac{1}{2}g^{\sigma\rho}\big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\big) - g^{\sigma\rho}\big(g_{\lambda\nu}\Gamma^{\lambda}{}_{[\mu\rho]} + g_{\lambda\mu}\Gamma^{\lambda}{}_{[\nu\rho]}\big)\\ &= \bar{\Gamma}^{\sigma}{}_{(\mu\nu)} - \frac{1}{2}g^{\sigma\rho}\big(g_{\lambda\nu}T_{\mu\rho}{}^{\lambda} + g_{\lambda\mu}T_{\nu\rho}{}^{\lambda}\big).\end{aligned} \tag{2.58}$$

From equation (2.57) we then find the contorsion tensor to read

$$K_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}{}_{[\mu\nu]} + \Gamma^{\lambda}{}_{(\mu\nu)} - \bar{\Gamma}^{\lambda}{}_{(\mu\nu)} = \frac{1}{2}\big(T_{\mu\nu}{}^{\lambda} - T_{\nu}{}^{\lambda}{}_{\mu} + T^{\lambda}{}_{\mu\nu}\big), \tag{2.59}$$

which expresses it completely in terms of the torsion tensor. Note the antisymmetry in the ν and λ indices.

In the Cartan formalism we instead define the contorsion tensor as the difference between the metric-compatible and torsion-free spin connections, meaning

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + \tilde{K}_\mu{}^{ab}. \tag{2.60}$$

Here $\omega_\mu{}^{ab}(e)$ is the torsion-free spin connection from equation (2.43) and the tilde on the contorsion tensor denotes that it is not necessarily equivalent to the one introduced in the metric formulation; it will show that a sign differs. There are now two possible ways of formulating the supergravity version of the vielbein postulate (2.33). We will work with the convention

$$D_\mu e_\nu{}^a = K_{\mu\nu}{}^a. \tag{2.61}$$

This is reasonable since the LHS can be seen to evaluate to

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b(e)e_\nu{}^b + \tilde{K}_\mu{}^a{}_b e_\nu{}^b - \bar{\Gamma}^{\rho}{}_{\mu\nu}e_\rho{}^a, \tag{2.62}$$

where only $\tilde{K}_\mu{}^a{}_\nu$ remains on the RHS if we assume the bosonic vielbein postulate (2.33) to hold true. By then imposing equation (2.61) and using the antisymmetry of $K_{\mu\nu}{}^a$ observed from equation (2.59), we see that a sign differs for the two contorsion tensors we have introduced. This sign is merely a consequence of the differing index structure of the spin and affine connections when they act in the covariant derivative.

Note that one instead could have chosen to include the contorsion tensor on the RHS of equation (2.61) in the affine connection appearing in the covariant derivative on the LHS, thereby interchanging $\bar{\Gamma}^{\rho}{}_{\mu\nu}$ for $\Gamma^{\rho}{}_{\mu\nu}$ in equation (2.62), and then imposing $D_\mu e_\nu{}^a = 0$ as the vielbein postulate. This may seem like a more convenient choice, and is also in the same form as the bosonic postulate. However, in the Chern-Simons formulation we will employ in chapter 7, there are non-zero derivatives of the frame fields naturally appearing in a way that indicates that the formulation of the vielbein postulate from equation (2.61) is implicitly understood. It is consequently more convenient to use.

Chapter 3

Gauge Theory

The concept of symmetry is of great importance in physics. For instance, Noether's theorem states that each continuous symmetry corresponds to a conservation law. Of particular interest are the gauge theories, which are field theories in which the Lagrangian is invariant under a continuous group of local transformations. One of the most successful gauge theories is the Standard model which is based on the gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, where the symmetries corresponding to each subgroup describe the strong, weak and electromagnetic interaction, respectively. This is an example of a quantized gauge theory, where the quanta of each gauge field is known as a gauge boson that mediates interactions. For our purposes it is for the moment sufficient to consider the classical continuous gauge theories. By doing this we will introduce the most important concepts from gauge theory.

In this chapter we first give an introduction to Lie algebras, the mathematical foundation of gauge theory. We then define two types of gauge theories, abelian and non-abelian ones, and introduce a particularly important instance of the latter known as Chern-Simons theory. Finally we show how general relativity in three dimensions can be expressed as a Chern-Simons theory.

3.1 Brief Introduction to Lie Algebras

Motivated by Noether's theorem we want to find a way to describe continuous symmetries. This is done by continuous groups, and of particular interest are the Lie groups since they themselves are smooth manifolds. The exact definition reads:

Definition 3.1: *A group G is a Lie group if it is a smooth finite-dimensional manifold such that the group multiplication*

$$(g_1, g_2) \in G \times G \to g_1 g_2 \in G \tag{3.1}$$

and the group inversion

$$g_1 \in G \to g_1^{-1} \in G \tag{3.2}$$

are smooth maps for all group elements g_1 and g_2.

A Lie group can be characterized by its generators. These are transformations around the unit element that can produce any group element when they are combined and multiplied by arbitrary parameters. We let the generator T_i correspond to the group operation U_i specified by a parameter ϕ (*e.g.* rotations or Lorentz transformations)[1]. Since each group operation $U_i(\phi)$ can be divided into infinitesimally small parts, meaning

$$U_i(\phi) = \lim_{N\to\infty} \left(1 + \frac{i\phi T_i}{N}\right)^N, \tag{3.3}$$

it can be represented as

$$U_i(\phi) = e^{i\phi T_i}. \tag{3.4}$$

For a small parameter ϵ_i, this group operation can be expanded as

$$U_i(\epsilon_i) = 1 + i\epsilon_i T_i - \frac{1}{2}\epsilon_i^2 T_i^2 + \mathcal{O}(\epsilon_i^3). \tag{3.5}$$

Since closure is one of the group axioms, the element

$$U_j^{-1}U_i^{-1}U_jU_i = 1 + \epsilon_i\epsilon_j[T_i, T_j] + \mathcal{O}(\epsilon^3) \tag{3.6}$$

must belong to the group. Hence we can conclude that it must be possible to write the commutator of two generators as

$$[T_i, T_j] = i f_{ij}{}^k T_k, \tag{3.7}$$

where the coefficients $f_{ij}{}^k$ are known as the structure constants of the group. Since they are representation independent, the structure constants are often used to characterize a Lie group.

In equation (3.6) we defined the symbol $[.,.]$ to denote commutation, meaning we wrote $XY - YX = [X, Y]$. By defining this operation we have actually indicated how a Lie algebra $\mathfrak{g}$ can be constructed from a Lie group G. The definition of a Lie algebra reads:

Definition 3.2: *A Lie algebra $\mathfrak{g}$ is a vector space together with a bilinear mapping $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ known as the Lie bracket, such that all elements $x, y, z \in \mathfrak{g}$ satisfy*

$$[x, y] = -[y, x] \tag{3.8}$$

and the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{3.9}$$

Hence, the infinitesimal generators of a Lie group form a Lie algebra with the Lie bracket given by the commutator of these infinitesimal generators. For future purposes, we also give the definition of an abelian group:

Definition 3.3: *An abelian group is a group G together with an operator $\cdot$ for which all elements $g_1, g_2 \in G$ satisfy the commutative property*

$$g_1 \cdot g_2 = g_2 \cdot g_1. \tag{3.10}$$

By studying equation (3.8) in the definition of a Lie algebra, we realize that a Lie algebra is abelian if and only if the Lie bracket of any two elements in it is zero, *i.e.*, if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. As will soon be apparent, there are some significant differences between gauge theories having an abelian symmetry group and those having a non-abelian one.

[1] The group operations are often represented as matrices.

3.2 Abelian Gauge Theories

To introduce the most important concepts in non-abelian gauge theory, we will study the free Dirac Lagrangian

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi, \tag{3.11}$$

where $\psi(x)$ is a spinor field and $\bar{\psi} = \psi^\dagger\gamma^0$. The results obtained will be completely general. This Lagrangian is obviously invariant under global phase transformations

$$\psi(x) \to e^{i\theta}\psi(x), \tag{3.12}$$

with θ being a constant. Phase transitions like this generate the group of unitary 1×1 matrices, known as U(1). This is obviously an abelian group, since $e^{i\theta_1}e^{i\theta_2} = e^{i\theta_2}e^{i\theta_1}$.

Since we know from quantum mechanics that the phase of the spinor field ψ should never be an observable, we do not only expect the complete Lagrangian to be invariant under global phase transformations, but also *local* ones. Hence we want to construct a Lagrangian that is invariant under

$$\psi(x) \to e^{i\theta(x)}\psi(x). \tag{3.13}$$

The mass term in the Dirac Lagrangian $\mathcal{L}_D$ is already invariant under such transformations, but the first term causes problem since the partial derivative transforms as

$$\partial_\mu\psi(x) \to e^{i\theta(x)}\big(\partial_\mu\psi(x) + i\psi(x)\partial_\mu\theta(x)\big) \tag{3.14}$$

under $\psi(x) \to e^{i\theta(x)}\psi(x)$. In order for this term to be invariant, we need to construct a covariant derivative D_μ that transforms like the field itself, *i.e.*,

$$D_\mu\psi(x) \to e^{i\theta(x)}D_\mu\psi(x). \tag{3.15}$$

To achieve this we define the gauge potential $A_\mu(x)$ as the field that makes

$$D_\mu\psi(x) \coloneqq (\partial_\mu + iA_\mu(x))\psi(x) \tag{3.16}$$

transform covariantly. Comparison to equation (3.14) gives that the gauge potential must transform as

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\theta(x). \tag{3.17}$$

Note that when we use our new covariant derivative (3.15) in the Lagrangian, a new interaction term coupling the fermion to the gauge potential A_μ will appear. Hence, our extension of phase transformations from a global to a local symmetry not only necessitates the introduction of a new field, this field also carries physical meaning previously not present in the theory. Such fields are referred to as *gauge fields*.

To construct the complete interacting Dirac Lagrangian, *i.e.* the Lagrangian of quantum electrodynamics (QED), we should also include a purely kinetic term. Such a term must both be gauge invariant and independent of the spinor field. Since repeated application of covariant derivatives to a covariant field always will yield a covariant result, we are guided to study the quantity

$$[D_\mu, D_\nu]\psi(x) = i(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi(x) = iF_{\mu\nu}\psi(x), \tag{3.18}$$

where we at the last equality have defined the field strength tensor $F_{\mu\nu}$. Since the LHS is covariant and ψ transforms covariantly, the field strength tensor must be gauge invariant. As discussed in section 2.2 when introducing the Riemann tensor, the commutator of two covariant derivatives gives a measure for the curvature. Hence, $F_{\mu\nu}$ is sometimes referred to as the curvature tensor. Since the field strength tensor is itself gauge invariant we can write down the full QED Lagrangian as

$$\mathcal{L}_{\mathrm{QED}} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.19}$$

where the coefficient for the last term has been chosen to give it the customary normalization of a kinetic term. The field strength tensor itself being gauge invariant is a unique property of abelian gauge theories. For non-abelian gauge theories this will not be the case, and the kinetic term will take another form.

3.3 Non-Abelian Gauge Theories

Having discussed how a Lagrangian invariant under global $U(1)$ transformations can be extended to be invariant under local ones, the question arises how the corresponding procedure can be accomplished for non-abelian symmetry groups. To explore this we recall from section 3.1 that an arbitrary Lie group element can be written as $g = \exp(\theta^a T_a)$, with T_a being the generators of the group where we for future convenience have absorbed a factor of i. Our starting point will thus be an arbitrary Lagrangian that is invariant under the global transformations

$$\psi^i(x) \to \exp(\theta^a T_a)\psi^i(x), \tag{3.20}$$

where $\psi^i(x)$ is a vector of the fundamental fields of the theory. We now want to extend this Lagrangian to be invariant under the set of *local* gauge transformations

$$\psi^i(x) \to \psi'^i(x) = \exp(\theta^a(x)T_a)\psi^i(x). \tag{3.21}$$

The partial derivatives now transform as

$$\partial_\mu\psi^i(x) \to \exp\left(\theta^a(x)T_a\right)\left(\partial_\mu + T_a\partial_\mu\theta^a(x)\right)\psi^i(x), \tag{3.22}$$

so the second term, that breaks the invariance, is Lie algebra-valued. Hence, the covariant derivative is now introduced as

$$D_\mu\psi^i(x) \coloneqq (\partial_\mu + A^a_\mu T_a)\psi^i(x), \tag{3.23}$$

where we henceforth will use the more compact notation $A_\mu = A^a_\mu T_a$ for the gauge potential. By studying equation (3.22) we realize that in order for $D_\mu\psi^i(x)$ to be covariant, the gauge potential term must transform according to

$$\begin{aligned} A_\mu\psi^i(x) \to &- \partial_\mu\psi'^i(x) + (D_\mu\psi^i(x))' = -g(\partial_\mu\psi^i(x)) - (\partial_\mu g)\psi^i(x) + g(D_\mu\psi^i(x)) \\ &= \left(gA_\mu g^{-1} - (\partial_\mu g)g^{-1}\right)\psi'^i(x), \end{aligned} \tag{3.24}$$

where $g = \exp(\theta^a(x)T_a)$. A simple rewriting then gives the transformation rule

$$A_\mu \to gD_\mu g^{-1}, \tag{3.25}$$

for the gauge potential.

Using this construction of the covariant derivative we can now extend our globally gauge invariant Lagrangian to a locally gauge invariant one, thereby including interactions with the gauge fields. But analogously to what we did for the abelian gauge theories, we also want to include a kinetic term containing solely the gauge fields. For this purpose we once again need to construct a field strength tensor. Using equation (3.23) we this time find the field strength tensor to read

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.26}$$

By now using that repeated application of covariant derivatives on the fields $\psi^i(x)$ must yield a covariant quantity, we can find out how the field strength tensor transforms. We find that

$$F_{\mu\nu}\psi^i(x) \to F'_{\mu\nu} g\psi^i(x) = gF_{\mu\nu}\psi^i(x), \tag{3.27}$$

meaning the field strength tensor must transform as

$$F_{\mu\nu} \to gF_{\mu\nu}g^{-1}. \tag{3.28}$$

Hence, it is no longer a gauge invariant quantity and we can conclude that this is only the case for abelian gauge theories.

In the abelian gauge theory studied in the previous section we found the kinetic term to be proportional to $F_{\mu\nu}F^{\mu\nu}$. Neither this quantity is gauge invariant for non-abelian theories, since it transforms as

$$F_{\mu\nu}F^{\mu\nu} \to gF^{\mu\nu}F_{\mu\nu}g^{-1}. \tag{3.29}$$

To find a gauge invariant quantity reminiscent of the kinetic term in the abelian case is, however, easy. By applying the trace to both sides and using its cyclicity we find that the quantity $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant.

It is now interesting to investigate whether this term still can be interpreted as the kinetic energy. Since the field strength tensor is a Lie algebra-valued quantity we can expand it in terms of the generators as $F_{\mu\nu} = F^a_{\mu\nu}T_a$. Then

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = \mathrm{Tr}(F^a_{\mu\nu}T_aF^{\mu\nu b}T_b) = F^a_{\mu\nu}F^{\mu\nu b}\mathrm{Tr}(T_aT_b), \tag{3.30}$$

from which it is obvious that this term, after proper normalization, can be regarded as the kinetic energy if

$$\mathrm{Tr}(T_aT_b) \sim \delta_{ab}. \tag{3.31}$$

This is the case for compact groups. Examples of such include all $\mathrm{SU}(n)$ groups, which are studied in Yang-Mills theory. A customary normalization is to include a factor of $\frac{1}{2}$ on the RHS of equation (3.31), and consequently the kinetic term $-\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ in the Lagrangian.

Finally, we derive an important relation known as the Bianchi identity that the field strength tensor must always satisfy. We do this by writing

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0, \tag{3.32}$$

where the LHS vanishes identically after expansion of all terms. Then noting that

$$[D_\mu, [D_\nu, D_\rho]]\phi = D_\mu(F_{\nu\rho}\phi) - F_{\nu\rho}D_\mu\phi = (D_\mu F_{\nu\rho})\phi \tag{3.33}$$

for all fields ϕ, we can rewrite equation (3.32) as

$$D_{[\mu}F_{\nu\rho]} = 0. \tag{3.34}$$

This is known as the *Bianchi identity.*

3.3.1 In the Language of Differential Forms

It will prove useful to utilize our knowledge in differential forms to compactify the notation. First we form a Lie algebra-valued gauge potential one-form

$$A = \mathrm{d}x^\mu A^a_\mu T_a. \tag{3.35}$$

Studying equation (3.26) we then form the field strength two-form

$$F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \left(\partial_{[\mu}A^a_{\nu]} + \frac{1}{2}f_{bc}{}^a A^b_\mu A^c_\nu\right) T_a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}A + A \wedge A. \tag{3.36}$$

Finally, inspired by equation (3.23), we also construct the exterior covariant derivative $D = \mathrm{d} + A$, where A is understood to act together with a wedge product. We can then rewrite the Bianchi identity from equation (3.34) as

$$DF = 0, \tag{3.37}$$

since we when expanding this three-form into components directly obtain the LHS of the Bianchi identity (3.34).

3.4 Chern-Simons Theory

The formulation of the field strength as a two-form F enables us to write down Lagrangians that are easily integrated, thus suitable for an action formulation. Since we saw in section 3.3 that $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ was a gauge invariant quantity appearing in the kinetic terms of non-abelian gauge theories, it is obvious that also the Lagrangian term

$$\mathcal{L}_\mathrm{C} = \mathrm{Tr}(F \wedge F) \tag{3.38}$$

must be gauge invariant. In general, $\mathrm{Tr}(F^n)$ is referred to as the n-th Chern form, so this is known as the second Chern form. Note that it is to be integrated over four spacetime dimensions.

The Chern forms are particular in the sense that all gauge potentials A are stationary points of them. Consequently, they cannot directly be used to find the equations of motion of the theory. To prove this we note that the field strength two-form F, under a variation δA of the gauge potential, varies as

$$\delta F = \mathrm{d}(\delta A) + (\delta A) \wedge A + A \wedge \delta A = D(\delta A). \tag{3.39}$$

Using the additivity and the cyclicity of the trace we then find that the second Chern form $\mathcal{L}_\mathrm{C}$ varies as

$$\delta\mathcal{L}_\mathrm{C} = 2\mathrm{Tr}\big[D(\delta A) \wedge F\big]. \tag{3.40}$$

But here we notice that we can write

$$D(\delta A) \wedge F = D\big((\delta A) \wedge F\big) + (\delta A) \wedge DF. \tag{3.41}$$

The first term is a boundary term that will vanish identically when we integrate the Lagrangian to form the action, whereas the second term vanishes according to the Bianchi identity $DF = 0$. The action constructed by integrating the second Chern form in equation (3.39) can consequently *not* directly be used to find the equations of motion.

To construct a more convenient Chern-Simons action, we want to use the property that the Chern forms are exact differential forms. This means that they can be written on the form $\mathrm{d}\Omega$, for some differential form Ω known as the Chern-Simons form. In particular, the second Chern form $\mathrm{Tr}(F \wedge F)$ has

$$\Omega = \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right). \tag{3.42}$$

To show this we introduce a parameter s and define $F_s = s\mathrm{d}A + s^2 A \wedge A$. We can then write the second Chern form as

$$\begin{aligned}
\mathrm{Tr}(F \wedge F) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Tr}(F_s \wedge F_s)\mathrm{d}s = 2\int_0^1 \mathrm{Tr}\left(\frac{\mathrm{d}F_s}{\mathrm{d}s} \wedge F_s\right)\mathrm{d}s \\
&= 2\mathrm{d}\left(\int_0^1 \mathrm{Tr}(A \wedge F_s)\mathrm{d}s\right) = 2\mathrm{d}\left(\int_0^1 \mathrm{Tr}\left(sA \wedge \mathrm{d}A + s^2 A \wedge A \wedge A\right)\mathrm{d}s\right) \\
&= \mathrm{d}\mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge a\right),
\end{aligned} \tag{3.43}$$

which agrees with equation (3.42)[2]. By then applying Stokes' theorem we find that the three-dimensional Chern-Simons action can be written as

$$\mathcal{S}_{\mathrm{CS}} = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right), \tag{3.44}$$

where M is a three-dimensional manifold and k is a coupling constant. Notice that this action does *not* contain obviously gauge invariant quantities. As argued in [19], it can be shown that if the gauge group and the manifold M both are compact, the terms appearing in addition to the integral itself after a gauge transformation are a vanishing boundary term and a term of the form $2\pi n$ (for an appropriate constant k), with $n \in \mathbb{Z}$ being the so-called winding number. Hence, it is now the quantity $\exp(i\mathcal{S}_{\mathrm{CS}})$ that is gauge invariant, rather than the action itself. This makes the Chern-Simons action appropriate for the path integral formulation of several theories. For our purposes, the subtle change caused by the winding term can, however, be ignored, meaning we can still regard the Chern-Simons action as gauge invariant.

From the Chern-Simons action (3.44) we can easily find the corresponding equations of motion. Under a variation δA of the gauge potential it changes by

$$\delta\mathcal{S}_{\mathrm{CS}} = \frac{k}{4\pi}\int_M \mathrm{Tr}\big[(\delta A) \wedge \mathrm{d}A + A \wedge \mathrm{d}(\delta A) + 2(\delta A) \wedge A \wedge A\big]. \tag{3.45}$$

[2] The procedure is completely analogous for higher order Chern forms.

By using that the product rule (2.8) for exterior derivatives implies

$$A \wedge \mathrm{d}(\delta A) = (\delta A) \wedge \mathrm{d}A - \mathrm{d}(A \wedge \delta A), \tag{3.46}$$

where the second term is a boundary term that will vanish when it is integrated, we end up with

$$\delta \mathcal{S}_{\mathrm{CS}} = \frac{k}{2\pi} \int_M \mathrm{Tr}\big[(\delta A) \wedge (\mathrm{d}A + A \wedge A)\big]. \tag{3.47}$$

The principle of stationary action then gives us the equation of motion

$$F = \mathrm{d}A + A \wedge A = 0, \tag{3.48}$$

known as the zero field strength equation. As we noted earlier, the field strength measures the curvature. This means that the gauge potential can be regarded as a connection, and the vanishing of the field strength imposes this connection to be flat. Equation (3.48) is, thus, sometimes referred to as the flatness condition.

3.5 Gravity as a Chern Simons Theory

It can now be shown that general relativity in three dimensions can be written as a Chern Simons gauge theory. To prove this we need to show that the Einstein-Hilbert action (2.25) can be written on the form of the Chern-Simons action (3.44) in the absence of matter. Since the main focus of this thesis are conformal theories, and not general relativity, we will keep this discussion to the point. In the Cartan formalism, the Einstein-Hilbert action in three dimensions reads

$$\mathcal{S}_{\mathrm{EH}} = -\frac{1}{8\pi G} \int \left[e^a \wedge \left(\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abd}\omega^b \wedge \omega^c - \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right) \right]. \tag{3.49}$$

This is easily proven by using equation (2.40) for the curvature one-form R^a and expanding the differential forms into its components.

Next we will need a "gauge/gravity dictionary" identifying the tensors appearing in general relativity as gauge quantities introduced in this chapter. As we noted in the end of section 3.4, the gauge potential can be regarded as a connection. This means that the Levi-Civita connection $\Gamma^\rho{}_{\mu\nu}$ can conveniently be identified as the matrix valued gauge potential $(A_\mu)^\rho{}_\nu$. Insertion into equation (2.19) then yields that the Riemann tensor can be written as

$$R^\rho{}_{\sigma\mu\nu} = \big(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\big)^\rho{}_\sigma = (F_{\mu\nu})^\rho{}_\sigma, \tag{3.50}$$

meaning we can identify it as the matrix valued field strength tensor from equation (3.26). This is a very expected result, since we have already identified both the Riemann tensor and the field strength tensor as the commutator of two covariant derivatives, which measures the curvature.

Insertion of these results into equation (2.32) expressing the Levi-Civita connection in terms of the vielbeins and spin connection, gives

$$(A_\mu)^\rho{}_\nu = e^\rho{}_a \partial_\mu e^a{}_\nu + e^\rho{}_a \omega_\mu{}^a{}_b e^b{}_\nu, \tag{3.51}$$

which we note is a gauge transformation of the spin connection written on the matrix valued form $(\omega_\mu)^a{}_b$ by the gauge field $e^b{}_\nu$. This indicates that the frame fields and spin connections can be used as the gauge fields in a further formulation of general relativity as a gauge theory. The gauge group whose generators these gauge fields will correspond to is simply the isometry group of the background manifold.

Before being able to express the Einstein-Hilbert action as a Chern-Simons action, we also have to understand the role of the cosmological constant Λ. It is a scalar curvature measuring the vacuum energy density. The value of the cosmological constant determines the geometry of (empty) spacetime, which we know is a Lorentzian manifold. Flat space has $\Lambda = 0$ and can be described by the Minkowski metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{k=1}^{n} \mathrm{d}x_k^2 \tag{3.52}$$

in $n+1$ dimensions. Spaces of negative scalar curvature $\Lambda < 0$ are known as *Anti de-Sitter* (AdS) spaces. In n dimensions they can be regarded as embeddings of the Lorentzian manifold in the space $\mathbb{R}^{n-1,2}$, with the metric given by

$$\mathrm{d}s^2 = \sum_{k=1}^{n-1} \mathrm{d}x_k^2 - \mathrm{d}t_1^2 - \mathrm{d}t_2^2, \tag{3.53}$$

that are parametrized as

$$\sum_{k=1}^{n-1} x_k^2, -t_1^2 - t_2^2 = -\ell^2, \tag{3.54}$$

where ℓ is a non-zero constant known as the radius of curvature. AdS-spaces are of great interest in theoretical physics, in particular for their role in the AdS/CFT correspondence that will be discussed in chapter 5. Spaces of positive scalar curvature $\Lambda > 0$ are instead known as *de Sitter* (dS) spaces. These can be defined as submanifolds of Minkowski spaces of one higher dimensions. For the Minkowski space $\mathbb{R}^{n,1}$ with the metric given by equation (3.52), the de Sitter spaces are the submanifolds parametrized by

$$\sum_{k=1}^{n} x_k^2 - t^2 = \ell^2, \tag{3.55}$$

where ℓ is again the radius of curvature.

We can now formulate the Einstein-Hilbert action (3.49) as a Chern-Simons action for all possible values of the cosmological constant. For $\Lambda = 0$ we recall from special relativity that the isometry group of Minkowski spacetime is the Poincaré group. The Poincaré algebra is generated by the generators of translations and Lorentz transformations, which in three dimensions can be written as P_a and M_a, respectively. The algebra reads

$$\begin{aligned} [M_a, M_b] &= \epsilon_{ab}{}^c M_c \\ [M_a, P_b] &= \epsilon_{ab}{}^c P_c \\ [P_a, P_b] &= 0, \end{aligned} \tag{3.56}$$

where the trace relations

$$\begin{aligned}\operatorname{Tr}(P_a M_b) &= \frac{1}{2}\eta_{ab}\\ \operatorname{Tr}(M_a M_b) = \operatorname{Tr}(P_a P_b) &= 0.\end{aligned} \tag{3.57}$$

can be assumed to hold [19]. Since we want to use this isometry group as the gauge group of the Chern-Simons action, the gauge potential one-form will read

$$A = e^a P_a + \omega^a M_a. \tag{3.58}$$

By just inserting this into the Chern-Simons action (3.44) it is easily verified that it (up to vanishing boundary terms) recreates the Einstein-Hilbert action (3.49) with $\Lambda = 0$, assuming $k = -\frac{1}{4G}$.

To formulate the Einstein-Hilbert action as a Chern-Simons action for non-zero cosmological constants, we first need to determine the isometry groups of the AdS and dS-spaces. By studying the parametrizations given in equations (3.54) and (3.55), and recalling that the Lorentz group in $n+1$ dimensions is $\mathrm{SO}(n,1)$, we realize that the isometry groups of the $(n+1)$-dimensional AdS and dS-spaces in must be $\mathrm{SO}(n,2)$ and $\mathrm{SO}(n+1,1)$, respectively.

In AdS spaces, where $\Lambda = -\frac{1}{\ell^2} < 0$, the Einstein-Hilbert action can due to the isomorphism $\mathrm{so}(2,2) \cong \mathrm{sl}(2,\mathbb{R}) \times \mathrm{sl}(2,\mathbb{R})$ be constructed as the linear combination of two Chern-Simons actions having $\mathrm{SL}(2,\mathbb{R})$ as their gauge groups. We write the action as

$$\mathcal{S} = \mathcal{S}_{\mathrm{CS}}(A^+) - \mathcal{S}_{\mathrm{CS}}(A^-), \tag{3.59}$$

where the gauge connections are given by

$$A^\pm = \left(\omega^a \pm \frac{e^a}{\ell}\right) T_a, \tag{3.60}$$

with T_a being the generators of $\mathrm{sl}(2,\mathbb{R})$ satisfying $[T_a, T_b] = \epsilon_{ab}{}^c T_c$ and $\operatorname{Tr}(T_a T_b) = \frac{1}{2}\eta_{ab}$. Once again it is just a matter of inserting this into the Chern-Simons action (3.44) to prove that the correct Einstein-Hilbert action is recreated, this time with $k = -\frac{\ell}{4G}$.

In dS-spaces, where $\Lambda = \frac{1}{\ell^2} > 0$, we instead use the isomorphism $\mathrm{so}(3,1) \cong \mathrm{sl}(2,\mathbb{C})$. Upon complexification, $\mathrm{SL}(2,\mathbb{C})$ can be written as two copies of itself, which means we can once again create the Einstein-Hilbert action as a linear combination of two Chern-Simons actions, this time with $\mathrm{SL}(2,\mathbb{C})$ as their gauge groups. We once again write the action on the form of equation (3.59), but this time with

$$A^\pm = \left(\omega^a \pm i\frac{e^a}{\ell}\right) T_a, \tag{3.61}$$

with T_a being the generators of $\mathrm{SL}(2,\mathbb{C})$. In this case we find that the correct Einstein-Hilbert action is obtained for $k = -\frac{i\ell}{4G}$.

We have thus shown that the Einstein-Hilbert action (3.49) in three dimensions *always* can be formulated as a Chern-Simons action (3.44) and have thus formulated general relativity as a gauge theory, which was what we set out to do. There is, however, one subtlety that remains to be resolved. None of the isometry groups we used as gauge groups when constructing the Chern-Simons actions are compact. Recall how we in equations (3.30) and (3.31) found that the Lagrangian term $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ could be regarded as the kinetic energy only if the gauge group was compact. For non-compact gauge groups we will instead obtain terms of differing signs, meaning there will be terms of negative energy. In the quantum theory this would lead to non-unitarity due to negative norm states propagating locally. However, since general relativity in three dimensions has no local degrees of freedom there are no such states that can actually propagate, which resolves the issue.

Although this section has provided a good example of how theories of gravitation can be reformulated as Chern-Simons gauge theories, the main interest of this thesis is *not* general relativity but conformal and superconformal field theories. In the following chapter we introduce the symmetries upon which these theories are constructed.

Chapter 4

Conformal and Superconformal Symmetries

The aim of this thesis is to construct a superconformal higher spin theory. Superconformal field theories are by definition invariant under superconformal transformations, which is the supersymmetric extension of conformal transformations.

In this chapter we start from the definition of a conformal transformation and use it to construct the conformal algebra, also giving an explicit representation of the generators. We then discuss how this algebra can be supersymmetrically extended by also including fermionic generators, thereby constructing the superconformal algebra.

4.1 The Conformal Algebra

The conformal group is an extension of the Poincaré group, which is the symmetry group of special relativity requiring it to be invariant under spacetime translations and Lorentz transformations (in turn consisting of rotations in space and Lorentz boosts). The generator of translations has four degrees of freedom (in four-dimensional spacetime) and is denoted P_μ, whereas the Lorentz generators are denoted $M_{\mu\nu}$ and have six degrees of freedom due to the antisymmetry requirement. The Poincaré algebra takes the form

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[P_\mu, M_{\nu\rho}] &= i(\eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}),
\end{aligned} \tag{4.1}$$

with the sign convention $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. This can easily be derived by studying which infinitesimal coordinate transformation that leave the Minkowski metric unchanged.

A conformal transformation is defined as a coordinate transformation $x \to x'$ which leaves the Minkowski metric invariant up to a rescaling $\Lambda(x)$, *i.e.* such that

$$\eta_{\mu\nu}(x) \to \eta'_{\mu\nu}(x') = \Lambda(x)\eta_{\mu\nu}(x). \tag{4.2}$$

The name "conformal" refers to the fact that such transformations preserve the angle between any two vectors. Note that the special case $\Lambda(x) = 1$ yields the Poincaré group, whence it is already here clear that it must be a subgroup of the conformal group.

To investigate what conformal transformations there are, in addition to translations and Lorentz transformations, we study an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x). \tag{4.3}$$

To first order in ξ, the metric then transforms as

$$\begin{aligned}\eta_{\mu\nu} \to \eta'_{\mu\nu} &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} = (\delta^\rho_\mu - \partial_\mu\xi^\rho + \mathcal{O}(\xi^2))(\delta^\sigma_\nu - \partial_\nu\xi^\sigma + \mathcal{O}(\xi^2))\eta_{\rho\sigma} \\ &= \eta_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) + \mathcal{O}(\xi^2).\end{aligned} \tag{4.4}$$

In order for this to be a conformal transformation there must, in accordance with equation (4.2), be a function $f(x)$ such that

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = f(x)\eta_{\mu\nu}. \tag{4.5}$$

By computing the trace of both sides we find that

$$f(x) = \frac{2}{D}\partial_\mu\xi^\mu, \tag{4.6}$$

where D is the dimension of spacetime. By first applying another partial derivative to equation (4.5), then permuting the indices and constructing a linear combination we obtain

$$2\partial_\mu\partial_\nu\xi_\rho = \eta_{\mu\rho}\partial_\nu f(x) + \eta_{\nu\rho}\partial_\mu f(x) - \eta_{\mu\nu}\partial_\rho f(x). \tag{4.7}$$

Contraction with $\eta^{\mu\nu}$ then yields

$$2\partial^2\xi_\mu = (2-D)\partial_\mu f(x), \tag{4.8}$$

where $\partial^2 = \partial_\rho\partial^\rho$. By now applying the partial derivative ∂_ν to this equation (notice that this gives a RHS that is symmetric in the μ and ν indices) and ∂^2 to equation (4.5), we see that

$$(2-D)\partial_\mu\partial_\nu f(x) = \eta_{\mu\nu}\partial^2 f(x). \tag{4.9}$$

This, in turn, yields

$$(D-1)\partial^2 f(x) = 0. \tag{4.10}$$

By using equations (4.5)– (4.10) we can now easily derive the explicit forms of all possible conformal transformations in D dimensions.

For spacetime dimensions $D \geq 3$, which will be the case throughout this thesis, equations (4.9) and (4.10) imply that $\partial_\mu\partial_\nu f(x) = 0$. This means that $f(x)$ is at most linear in the coordinates and, thus, can be written as

$$f(x) = A + B_\mu x^\mu \tag{4.11}$$

for some constants A and B_μ. Insertion of this expression into equation (4.7) shows that $\partial_\mu\partial_\nu\xi^\rho$ must be constant. Consequently, ξ_μ is at most quadratic in the coordinates and can be written on the form

$$\xi_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho, \tag{4.12}$$

where the symmetry condition $c_{\mu\nu\rho} = c_{\mu\rho\nu}$ must hold. Since the constraints (4.5)– (4.7) should hold for all coordinates x, we can insert our expression for ξ_μ order by order. Since only derivatives of ξ_μ appear in the constraints, the constant term $\xi_\mu = a_\mu$ is free of constraints. This corresponds to the infinitesimal translations $x^\mu \to x^\mu + a^\mu$ recognized from the Poincaré group.

For $\xi_\mu = b_{\mu\nu}x^\nu$, constraint (4.5) gives

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D} b_\rho{}^\rho \eta_{\mu\nu}, \tag{4.13}$$

which implies that $b_{\mu\nu}$ can be written as the sum of a pure trace and an antisymmetric part $m_{\mu\nu}$, *i.e.*, on the form

$$b_{\mu\nu} = \beta\eta_{\mu\nu} + m_{\mu\nu}. \tag{4.14}$$

The trace part corresponds to an infinitesimal rescaling (or "dilation")

$$x^\mu \to x^\mu + \beta\eta^{\mu\nu}x_\nu \equiv \alpha x^\mu, \tag{4.15}$$

whereas the antisymmetric part corresponds to the infinitesimal Lorentz transformations $x^\mu \to m^{\mu\nu}x_\nu$.

By finally inserting the quadratic term $x_\mu = c_{\mu\nu\rho}x^\nu x^\rho$ into constraint (4.7) we find that

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu, \tag{4.16}$$

where we have defined $b_\mu \equiv \frac{1}{D}c^\sigma{}_{\sigma\mu}$. The corresponding infinitesimal transformation is thus

$$x^\mu \to x^\mu + 2(b_\nu x^\nu)x^\mu - (x^\nu x_\nu)b^\mu \tag{4.17}$$

and is referred to as the special conformal transformation (SCT).

The generalizations to *finite* conformal transformations $x^\mu \to x'^\mu$ from the infinitesimal ones, read

$$\begin{aligned} \text{(translations):} &\qquad x'^\mu = x^\mu + a^\mu \\ \text{(dilations):} &\qquad x'^\mu = \alpha x^\mu \\ \text{(Lorentz transformations):} &\qquad x'^\mu = M^\mu{}_\nu x^\nu \\ \text{(SCT):} &\qquad x'^\mu = \frac{x^\mu - (x^\nu x_\nu)b^\mu}{1 - 2(b_\mu x^\mu) + (b^\mu b_\mu)(x^\nu x_\nu)}. \end{aligned} \tag{4.18}$$

These generalizations are obvious for the translations, dilations, and Lorentz transformations, but perhaps a bit less intuitive for the special conformal transformations. It is, however, trivial to verify that the infinitesimal version of this transformation really is the one in equation (4.17), and that it really is a conformal transformation (*i.e.*, that it can be written on the form of equation (4.2)) with a scale factor given by

$$\Lambda(x) = \big(1 - 2(b_\mu x^\mu) + (b^\mu b_\mu)(x^\nu x_\nu)\big)^2. \tag{4.19}$$

To summarize we have now found that conformal symmetry, in addition to the Poincaré symmetries of translations and Lorentz transformations, also contains dilations and special conformal transformations. In four-dimensional spacetime, the generators of these symmetry transformations introduce five additional degrees of freedom. We next want to find a representation of these generators and thereby be able to construct the conformal algebra.

4.1.1 Representations of the Conformal Algebra

Since we have already found the infinitesimal forms of the four types of conformal transformations, we can directly write down a representation of the generators. This representation reads

$$\begin{aligned} P_\mu &= i\partial_\mu \\ D &= -ix^\mu\partial_\mu \\ M_{\mu\nu} &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ K_\mu &= i(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu) \end{aligned} \tag{4.20}$$

for the generators of translations, dilations, Lorentz transformations, and special conformal transformations, respectively. Note that we, compared to when we in equation (3.4) constructed group transformations from the generators, have absorbed an additional i into the generators. The overall sign of respective generator can be chosen arbitrarily, and have been chosen to coincide with the conventions of [5].

Using this representation it is trivial to construct the conformal algebra, *i.e.*, the commutation relations of the generators. Straightforward calculations give the following non-zero commutation relations,

$$\begin{aligned} [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= -iK_\mu \\ [P_\mu, K_\nu] &= 2i(-\eta_{\mu\nu}D + M_{\mu\nu}) \\ [K_\mu, M_{\nu\rho}] &= i(\eta_{\mu\rho}K_\nu - \eta_{\mu\nu}K_\rho) \\ [P_\mu, M_{\nu\rho}] &= i(\eta_{\mu\rho}P_\nu - \eta_{\mu\nu}P_\rho) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}). \end{aligned} \tag{4.21}$$

These commutation relations define the conformal algebra. Note, in particular, that the Poincaré algebra (4.1) is recreated.

By studying the commutation relations a bit further one can prove that the conformal algebra in $D > 2$ dimensions must be isomorphic to $\mathrm{so}(D, 2)$ in Minkowski space (or $\mathrm{so}(D+1, 1)$ in Euclidean space). To do this we define the antisymmetric generators J_{ij} according to

$$\begin{aligned} J_{-1,D} &= D & J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\ J_{\mu\nu} &= M_{\mu\nu} & J_{D,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{4.22}$$

where $i, j \in \{-1, 0, \ldots, D\}$ and $\mu, \nu \in \{0, 1, \ldots, D-1\}$. It is then straightforward to verify that these generators satisfy the commutation relation

$$[J_{ij}, J_{kl}] = i(\eta_{il}J_{jk} + \eta_{jk}J_{il} - \eta_{ik}J_{jl} - \eta_{jl}J_{ik}), \tag{4.23}$$

where $\eta_{ij} = \mathrm{diag}(-1, -1, 1, \ldots, 1)$ in Minkowskian spacetime (and $\eta_{ij} = \mathrm{diag}(-1, 1, \ldots, 1)$ in Euclidean spacetime). This is precisely the $\mathrm{so}(D, 2)$ algebra. For our purposes, the generators J_{ij} would nevertheless be quite inconvenient to use, since we often want to keep the different degrees of freedom apart.

The conformal field theory we are to construct (starting in chapter 6) will be three-dimensional. Since the Lorentz generators $M_{\mu\nu}$ are antisymmetric we can in three dimensions use the Levi-Civita symbol to rewrite them as

$$M_\mu = \frac{1}{2}\epsilon_\mu{}^{\nu\rho} M_{\nu\rho}. \tag{4.24}$$

The inverted relation is found by acting with $\epsilon_{\sigma\lambda}{}^{\mu}$ on both sides and reads $M_{\mu\nu} = -\epsilon_{\mu\nu}{}^{\rho} M_\rho$ (the signs here are purely conventional). By using these two relations we can easily express the conformal algebra (4.21) in terms of the new Lorentz generator M_μ. For instance, we find that

$$[M_\mu, P_\nu] = \frac{1}{2}\epsilon_\mu{}^{\rho\sigma}[M_{\rho\sigma}, P_\nu] = \frac{i}{2}\epsilon_\mu{}^{\rho\sigma}(\eta_{\nu\rho}P_\sigma - \eta_{\nu\sigma}P_\rho) = i\epsilon_{\mu\nu}{}^{\sigma} P_\sigma. \tag{4.25}$$

Working analogously for the other commutation relations we find the conformal algebra to read

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[P_\mu, K_\nu] &= -2i(\eta_{\mu\nu}D + \epsilon_{\mu\nu}{}^{\rho} M_\rho) \\
[M_\mu, P_\nu] &= i\epsilon_{\mu\nu}{}^{\rho} P_\rho \\
[M_\mu, K_\nu] &= i\epsilon_{\mu\nu}{}^{\rho} K_\rho \\
[M_\mu, M_\nu] &= i\epsilon_{\mu\nu}{}^{\rho} M_\rho.
\end{aligned} \tag{4.26}$$

This is the representation of the three-dimensional conformal algebra that will henceforth be used in this thesis.

In particular we note that all commutators including the generator of dilations, D, and another generator X_μ can be written on the form

$$[D, X_\mu] = iwX_\mu. \tag{4.27}$$

We say that the algebra is graded by D and refer to the constant w as the weight of respective generator X_μ with respect to this grading. The weights of the different generators are summarized in table 4.1.

Weight	**Generator**
1	P_μ
0	M_μ, D
-1	K_μ

Table 4.1: The weights of the conformal generators with respect to the grading in D.

4.2 The Superconformal Algebra

The fact that all generators of the conformal group are of integer weight with respect to the grading in the dilation generator D reflects the fact that they all correspond to bosonic degrees of freedom. If we also want to include fermionic degrees of freedom, thereby enabling the group to describe a supersymmetric theory, we need to introduce generators of half-integer weight. We define the fermionic generator $Q^{\alpha I}$ to have weight $w = 1/2$ and $S_\alpha{}^I$ to have weight $w = -1/2$. Here, α is a spinor index indicating that we are now working with fermions, whereas I is an internal vector index related to the so-called R-symmetry in a way we will soon understand.

We now want to construct the superconformal algebra. By studying the grading in table 4.1 we realize that the commutator of the two fermionic generators $Q^{\alpha I}$ and $Q^{\beta J}$ must be proportional to the generator of translations. But since our construction of a Lie algebra from section 3.1 only applies for bosonic generators, and *not* for the Grassmann generators describing fermions, we first need to extend the concept of a Lie algebra to a Lie superalgebra. Its definition is very similar to definition 3.2 of a Lie algebra, but the Lie bracket is exchanged for the Lie superbracket. It reads:

Definition 4.1: *A Lie superalgebra $\mathfrak{g}$ is a vector space together with a bilinear mapping $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ known as the Lie superbracket, such that all elements $x, y, z \in \mathfrak{g}$ satisfy super skew-symmetry*

$$[x, y] = -(-1)^{|x||y|}[y, x] \tag{4.28}$$

and the super Jacobi identity

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0, \tag{4.29}$$

where $|x|$ denotes the Grassmann parity of x which is 0 (even) if x is bosonic and 1 (odd) if x is fermionic.

The Lie superbracket can be realized by the so-called supercommutator

$$[x, y\} = xy - (-1)^{|x||y|}yx. \tag{4.30}$$

Note that this becomes an anticommutator if both x and y are fermionic, and otherwise an ordinary commutator. Following our discussion above we can, thus, conclude that one of the commutation relations of the superconformal algebra, at least up to a multiplicative factor, must be of the form

$$\{Q^{\alpha I}, Q^{\beta J}\} = -2\delta^{IJ}(\gamma^\mu)^{\alpha\beta}P_\mu, \tag{4.31}$$

where γ^μ are the gamma matrices. A completely analogous argument for the fermionic operator $S_\alpha{}^I$ yields the relation

$$\{S_\alpha{}^I,\, S_\beta{}^J\} = -2\delta^{IJ}(\gamma^\mu)_{\alpha\beta}K_\mu. \tag{4.32}$$

The fermionic generator $Q^{\alpha I}$ is often referred to as the "generator of supersymmetry". Together with P_μ and $M_{\mu\nu}$ it forms the super-Poincaré algebra, which is a subalgebra of the superconformal algebra. In the same way as the generator K_μ of special conformal transformations naturally appeared when we extended the Poincaré algebra to the conformal algebra, the generator $S_\alpha{}^I$ appears when we want to combine this Q-supersymmetry with conformal symmetry. $S_\alpha{}^I$ is, thus, referred to as the generator of special conformal supersymmetry transformations [18].

Sticking to the analogy with the conformal algebra we realize that there should also be some symmetry relating the Q- and S-supersymmetries, just like P_μ and K_ν were related via M_μ and D in the algebra (4.26). Such symmetries are known as R-symmetries. The generator of this symmetry must have weight $w = 0$ and carry two vector indices, and we choose to denote it as T^{IJ}. Studying table 4.1 we see that also M_μ and D should appear in the commutator $\{Q^{\alpha I}, S^J_\beta\}$ since also they are of weight zero. It should consequently be of the form

$$\{Q^{\alpha I}, S^J_\beta\} = -2\delta^{IJ}\delta^\alpha_\beta D - 2\delta^{IJ}(\gamma^\mu)_\beta{}^\alpha M_\mu + 4\delta^\alpha_\beta T^{IJ}, \tag{4.33}$$

where we, once again, point out that the specific coefficients depend on conventions. By closer studying the Lie superalgebra constructed this way, one can show that the R-symmetry generators T^{IJ}, in three dimensions, must be the generators of SO(N) where N is the number of supersymmetries in the theory (*i.e.*, the number of values the vector indices I and J can take) [20]. Hence, T^{IJ} must satisfy the commutation relation (4.23) of the SO(N) generators. With a normalization that will prove to be convenient for the theory we will construct in chapter 6, we can thus write

$$[T_{IJ}, T^{KL}] = -2i\delta^{[K}_{[I}T_{J]}{}^{L]}. \tag{4.34}$$

Furthermore, since the generators $Q^{\alpha I}$ and $S_\alpha{}^I$ are the only generators of weight $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, we realize that their commutation relations with T^{IJ} must be of the forms

$$[T^{IJ}, Q^\alpha{}_K] = -i\delta^{IJ}_{KL}Q^{\alpha L}, \qquad [T^{IJ}, S_{\alpha K}] = -i\delta^{IJ}_{KL}S_\alpha{}^L. \tag{4.35}$$

The generators of the superconformal algebra are summarized in table 4.2, where also their weights with respect to the grading in D are given.

Weight	**Generator**
1	P_μ
1/2	$Q^{\alpha I}$
0	D, M_μ, T^{IJ}
−1/2	S^I_α
−1	K_μ

Table 4.2: The weights of the superconformal generators with respect to the grading in D.

Above we have derived all commutation relations including only the generators that did not appear in the conformal algebra, *i.e.*, $Q^{\alpha I}$, $S_\alpha{}^I$ and T^{IJ}. By construction we also know that

$$[D, Q^{\alpha I}] = \frac{1}{2}Q^{\alpha I}, \qquad [D, S^I_\alpha] = -\frac{1}{2}S^I_\alpha. \tag{4.36}$$

However, studying the grading in table 4.2 we realize that there must be four more non-zero commutators consisting of one "new" and one "old" generator in the algebra. We could in principle write down the form of these four commutators already, but since the coefficients will *not* be uniquely determined until we have chosen a specific representation of the generators, we leave it be until chapter 6. There we will construct the entire superconformal algebra working with a representation of the generators that is particularly suitable for higher spin theory.

It should be mentioned that there actually is an additional R-symmetry, corresponding to global U(1) transformations of the fermionic generators [21]. However, as we will only construct the superconformal algebra explicitly for the spin 2 case, the gauge fields corresponding to this symmetry will vanish. This is for the moment not at all obvious, but after having introduced both higher spin theory and our explicit construction of the superconformal field theory it will become very clear.

Chapter 5

Higher Spin Theory and the AdS/CFT Duality

Two of the main reasons why superconformal field theories are of interest in string and M-theory are their role in the AdS/CFT correspondence and the possibility to construct higher spin versions of them that have clear resemblances with essential features of string and M-theory. In this chapter we will explain these motivations for studying superconformal field theories in more detail.

The first part of the chapter motivates why higher spin theory in general is of interest, and why we want to apply it to the superconformal algebra. We then discuss how the higher spin algebras can be constructed and provide the so-called unfolded formalism of higher spin theory, which enables us to write down an explicit equation of motion known as the unfolded equation. In the second part of the chapter we introduce the AdS/CFT duality and discuss its relevance for the theory we are to construct.

5.1 Higher Spin Theory

Although all elementary particles contained in the Standard Model have spin 1 or lower and the (hypothetical) graviton and its supersymmetric partner the gravitino have spin 2 and 3/2 respectively, theories describing states of even higher spins have, for a long time, been studied in theoretical physics. The motivation for these studies have, however, varied in time. Already in 1939, Fierz and Pauli studied the free field equations for massive fields of arbitrary spin [22]. Their motivation was simply that there are higher spin representations of the Poincaré group, and therefore it was natural to search for field theories with particles carrying these representations (since Lorentz invariance is required, all elementary particles must be irreducible representations of the Poincaré group in Minkowski space).

Along with the discovery of supergravity in the mid-1970s, the search for consistent interactions of massless higher spin fields was intensified, *e.g.*, since one hoped that inclusion of such interactions would yield a better quantum behavior of the theory [9]. A big breakthrough came in 1978 when Flato and Fronsdal in [23] moved the setting from

Minkowski to Anti-de Sitter space. They there showed that two Dirac singletons in AdS_4 give rise to an infinite tower of massless higher spin representations (we will come back to the Dirac singleton representation in section 8.1). One year later, Fronsdal suggested that a theory of interacting singletons would yield interactions between massless higher spin fields [24]. In 1987, Fradkin and Vasiliev proved that gravitational interaction terms of massless higher spin fields exist but involve inverse powers of the cosmological constant [25]. This explained why the studies in the flat space limit $\Lambda \to 0$ had not been successful. These observations made in the 1970s and 1980s are the motivation for many applications of higher spin theory yet today, with the superconformal field theory we are about to construct being one of them. This will become clear in chapter 8.

In the late 1980s, also string and M-theorists were drawn to higher spin gauge theories. All string theories contain an infinite mass-spectrum of ever higher spins. This resembles the infinite tower of spins that inevitably appears in higher spin theories as soon as one state with spin greater than two is included (we will see explicitly how this infinite tower appears in section 6.3), with the difference that these states are massless. For this reason, the efforts of formulating string theories as higher spin gauge theories have been focused on the tensionless (and thereby massless) limits of the string theories. A successful example of such a reformulation was recently found in [26]. The motivation for M-theorists to study higher spin theories emerged with the discovery of the supermembrane in 1987, and in particular the observation that its local fermionic symmetries requires the eleven-dimensional supergravity equations of motion to be satisfied [27]. Theories studying supermembranes in $\text{AdS}_4 \times S^7$ backgrounds, whose spectra was proposed to contain the massless higher spin states created by two AdS_4 singletons, were launched the following year [28]. The singletons proposed in these theories were $N = 8$ singletons, meaning they obeyed 8 supersymmetries, and had propagating degrees of freedom only on the boundary of the AdS_4 space. In accordance with the AdS/CFT correspondence they can then be described by a three-dimensional conformal field theory. In section 8.1 we will see explicitly how such singletons appear.

Even more attention has been directed towards higher spin theories since Maldacena's discovery of the AdS/CFT correspondence in 1997. This conjectured duality relates theories in the bulk of AdS spaces, such as string and M-theories, to the conformal field theories on its boundaries [3]. The duality will be described in more detail in section 5.2, but the usefulness of higher spin theories when trying to understand it is already apparent. Since the precise formulation of string and M-theories in AdS spacetimes are yet not in place, it is mainly a weaker form of the duality that has been investigated; one including gauged supergravities in the AdS bulk. These are believed to describe the low energy limit of string/M-theory in AdS backgrounds [9]. Since both these theories and the conformal field theories living on the boundary can be formulated as higher spin gauge theories, this enables a convenient method of investigating the claimed correspondence between the two.

The AdS/CFT duality most relevant for this thesis is the $\text{AdS}_4/\text{CFT}_3$ duality, which enables a correspondence between M-theories in $\text{AdS}_4 \times S^7$ backgrounds and three-dimensional conformal field theories. As discussed above, the former can be described as a higher spin theory with a spectrum created by two $N = 8$ singletons. A complete higher spin formulation of the three-dimensional conformal field theories have, however, still not been found for $N > 1$. If this can be achieved, the $\text{AdS}_4/\text{CFT}_3$ duality could be studied in detail, and we would hopefully gain new insights about M-theory in $\mathrm{AdS}_4 \times S^7$ backgrounds, and thereby also about M-theory itself.

The explicit procedure of generalizing a theory to include higher spins varies a lot for different theories. In section 3.5 we briefly discussed how a three-dimensional AdS theory of gravity could be described as a Chern-Simons gauge theory with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ as its gauge group. For such a theory, the extension to higher spins is (at least in principle) almost trivial. To couple this theory of spin 2 gravity to all spins up to some integer n we simply have to extend the gauge group to $\mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R})$. Higher spin theories of this kind have proven to have a lot of interesting geometrical features, especially with regard to singularity resolution. They have for instance been used to resolve the Big Bang singularity in [29], and to construct higher spin black holes in [30].

For conformal theories, the extension to higher spins requires more sophisticated methods. One possibility is to employ a metric-like formulation where symmetric higher spin fields $\phi_{\mu_1 \ldots \mu_s}$ of spin s are introduced analogously to how the metric is introduced in the metric formulation of general relativity. This is often referred to as the Fronsdal formulation. Another possibility is a frame-like formulation, where the higher spin fields $\phi_{\mu_1 \ldots \mu_s}$ are replaced by frame-like fields $e_\mu{}^{a_1 \cdots a_{s-1}}$ and spin-like connections $\omega_\mu{}^{a_1 \cdots a_{s-1}}{}_b$. This is completely analogous to how we reformulated (super)gravity in the Cartan formalism, to be able to form spinors. Since we want to construct a supersymmetric higher spin theory we once again choose the frame-like formulation. To find the equations of motion we will use a method called unfolding, developed by Vasiliev in [11–13].

5.1.1 The Higher Spin Algebra and Unfolded Formalism

Unfolding is a general approach for reformulating systems of differential equations to first order form. It is especially valuable in applications to interacting higher spin systems, since it provides the only known examples of exact formulations of such systems [10]. The solutions to the equations of motion describing a field theory are, in general, zero-forms (functions) of the local coordinates of the manifold. By introducing the gauge potential 1-form, which makes it easy to ensure the necessary invariance under local gauge transformations, we unfortunately interweave the dynamical degrees of freedom with the ones that are gauge artefacts. In the unfolded formulation, one introduces a master gauge field and a master scalar field in an *extended* spacetime, which in addition to the usual (commuting) spacetime coordinates also has non-commutative (Grassmann even) spinorial coordinates [9]. By doing this, the physical degrees of freedom can be extracted in a clever way.

For the purposes of this thesis, only a fraction of the vast subject of unfolding will be of interest. First, a brief understanding of the three-dimensional conformal higher spin algebras in the unfolded formalism will be needed[1]. Then, the so-called unfolded equation (of scalar fields) can be formulated.

In section 4.1.1 we proved that the conformal algebra in D dimensions is $\mathrm{so}(D,2)$. In $D=3$ we can, due to the isomorphism $\mathrm{so}(3,2) \cong \mathrm{sp}(4,\mathbb{R})$, easily construct the conformal algebra using the two-dimensional quantum harmonic oscillator as our starting point. We

[1]There is a very similar formulation for the higher spin algebras of AdS_4, see, *e.g.*, [31]

know that this can be constructed from two pairs of creation and annihilation operators a_1, $a_1^\dagger$, a_2, and $a_2^\dagger$. In higher spin theory we gather these into a quartet $\hat{Y}^A$ (where $A = 1, 2, 3, 4$) of operators satisfying the canonical commutation relation

$$[\hat{Y}^A, \hat{Y}^B] = 2iC^{AB}, \tag{5.1}$$

where C^{AB} is the antisymmetric invariant tensor of $\mathrm{sp}(4, \mathbb{R})$ (often referred to as the charge conjugation matrix). The higher spin algebra is then defined as the algebra of all even functions $f(Y)$ in $\hat{Y}^A$, which is an associative algebra. Since it is often more convenient to work with ordinary commuting variable Y^A than the operators $\hat{Y}^A$ we want to introduce an associative product multiplying these. A convenient choice is the (Moyal) star product $\star$ defined by

$$(f \star g)(Y) = f(Y) \exp\left(i\overleftarrow{\partial}_A C^{AB} \vec{\partial}_B\right) g(Y), \tag{5.2}$$

where $\overleftarrow{\partial}$ denotes a partial derivative acting to the left.

One of the strengths of the unfolded formalism is that it encapsulates the entire infinite tower of spins into simple equations. To achieve this, some quantities must be gathered into so-called master forms: differential forms summing up the contributions from all spins. In the unfolded equation for scalar fields, two such master forms appear: the master gauge connection (which is a one-form) and the zero-form master field. The former reads

$$A = \sum_{n=1}^{\infty} (-i)^n A_n = \sum_{n=1}^{\infty} (-i)^n A_n^{a_1 \ldots a_n} T_{a_1 \ldots a_n} \tag{5.3}$$

and is a just a generalization to higher spin theory of the gauge potential one-form introduced in equation (3.35), although with a different normalization.

To form the zero-form master field we need a convenient basis to express the higher spin fields in. Due to the Lie algebra isomorphism $\mathfrak{so}(3, 2) \cong \mathfrak{sp}(4, \mathbb{R})$, the conformal algebra in three dimensions can be constructed by means of two commuting $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{Sp}(2, \mathbb{R})$ spinors q^α and p_α, since they together span a real four-dimensional spinor. The $\mathrm{SL}(2, \mathbb{R})$ spinors q^α and p_α will be studied in more detail in the next chapter, where they will also be given an interpretation, but for now we merely need to use one of them to expand the master field into its higher spin components. We write

$$\Phi(x) = \sum_{n=0}^{\infty} \phi^{\alpha_1 \ldots \alpha_n}(x) p_{\alpha_1} \ldots p_{\alpha_n}, \tag{5.4}$$

where the term with $n = 0$ corresponds to the scalar field $\phi(x)$.

With these concepts in place we can formulate the (sourceless) unfolded equation. It reads

$$\mathcal{D}\Phi(x) = 0, \tag{5.5}$$

where $\mathcal{D} = \mathrm{d} + A$ is the covariant derivative containing the master gauge potential and $\Phi(x)$ is the master field.

Motivation for the unfolded equation can be found in the Bargmann–Wigner equations, which describe free particles of arbitrary spin. For a free particle of spin s they are a set of $2s$ coupled linear partial differential equations, each resembling the Dirac equation, which read

$$(\gamma^\mu)^{\alpha_r \alpha_r'} D_\mu \psi_{\alpha_1 \ldots \alpha_r' \ldots \alpha_{2s}}(x) = 0 \tag{5.6}$$

for $r = 1, \ldots, 2s$. Here $D_\mu = \partial_\mu + \Omega_\mu$ (with Ω_μ being the connection) is the covariant derivative and $\psi_{\alpha_1 \ldots \alpha_r' \ldots \alpha_{2s}}(x)$ is the wave function. Massless fields in four dimensions can be described as higher-spin Weyl tensors $C_{\alpha_1 \ldots \alpha_{2s}}$ and $C_{\dot{\alpha}_1 \ldots \dot{\alpha}_{2s}}$ [32]. Such fields obey the Bargmann–Wigner equations in the form

$$\epsilon^{\beta\gamma} \partial_{\beta\dot{\beta}} C_{\gamma\alpha_2 \ldots \alpha_{2s}} = 0, \qquad \epsilon^{\dot{\beta}\dot{\gamma}} \partial_{\beta\dot{\beta}} C_{\dot{\gamma}\dot{\alpha}_2 \ldots \dot{\alpha}_{2s}} = 0, \tag{5.7}$$

where $\epsilon^{\beta\gamma}$ is the invariant metric of the Lorentz algebra $\mathrm{so}(3,1)$. By identifying the higher spin fields of our master field expansion (5.4) as a composition of such higher-spin Weyl tensors, we realize that the step from the Bargmann–Wigner equations (5.6) and (5.7) to the unfolded equation (5.5) is not very far.

5.2 The AdS/CFT Duality

The AdS/CFT duality was first proposed by Juan Maldacena in 1997 [3]. It gives a relation between the AdS spaces used in theories of quantum gravity and the conformal field theories used in, *e.g.*, Yang-Mills and Chern-Simons theories to describe elementary particles. It is often described as a holographic duality, since it claims that the boundary of the "bulk" AdS space can be regarded as the spacetime for the dual conformal field theory. This implies that the conformal field theory that is dual to a theory in d-dimensional AdS-space must be $(d-1)$-dimensional, meaning we have an $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence. The AdS/CFT duality provides the most successful realization of the conjectured holographic principle which, *e.g.*, could resolve the black hole information paradox [33].

The perhaps greatest appeal of the AdS/CFT duality is that it enables non-perturbative formulations of string and M-theories. The interactions of strings (or membranes) is usually expressed in perturbation series generalizing the Feynman diagrams of quantum field theory. However, since there is also a non-perturbative formulation of quantum field theory, the AdS/CFT duality opens the door for non-perturbative formulations of string and M-theories on AdS-spaces. One of the most interesting AdS/CFT correspondences is the one between type IIB string theory compactified on $\mathrm{AdS}_5 \times S^5$ (meaning the gravitational theory effectively lives in five-dimensional space-time, whereas the remaining five dimensions are compact) and $N = 4$ supersymmetric Yang-Mills theory, which is a four-dimensional CFT. Another interesting correspondence is the one between M-theory compactified on $\mathrm{AdS}_4 \times S^7$ and superconformal field theories that can either have $N = 6$ (ABJM theories) or $N = 8$ (BLG theories) depending on which kind of M-theory one is studying. Since these superconformal field theories will be three-dimensional, this correspondence will be to one of most relevance for this thesis.

Already in the beginning of this chapter we discussed the observation first made in the 1980s, that infinite towers of massless states with increasing spin appear in both higher spin theory and the tensionless limits of AdS compactifications of string and M-theories. This leads us to suspect that there might also be a correspondence between higher spin gauge theories on AdS-spaces and conformal field theories. In 2002, Klebanov and Polyakov conjectured such a correspondence for CFTs with $\mathrm{O}(N)$ symmetries [34]. This was further evidenced by Giombi and Yin when they in 2009 verified the correspondence for the higher spin theory in AdS_4 (expressed in the unfolded formalism) [35]. Hopefully, the higher spin correspondences can help to gain a better understanding of the string/M-theoretical versions of the AdS/CFT correspondences.

The AdS/CFT is a strong-weak duality in the sense that strongly interacting fields of the quantum field theory correspond to weakly interacting fields in the gravitational theory. Since weak fields are easier to treat mathematically, using *e.g.* perturbation theory, the AdS/CFT has found applications in several fields aiming at describing strongly coupled quantum systems. Examples include the study of quark-gluon plasmas in QCD and the phase transitions of superconductors and superfluids in condensed matter physics [36, 37].

Chapter 6

Quantizing the Superconformal Algebra

Any theory aiming at describing the most fundamental constituents of the Universe must be quantized. In analogy with this, we should before constructing the superconformal higher spin theory we are aiming at, first quantize the superconformal algebra.

We will begin this chapter by giving two equivalent ways of quantizing the (purely bosonic) conformal algebra, the star product formulation and the operator formulation. In this process we will also introduce a new representation of the generators of the conformal group, one that can easily be generalized to also include the corresponding higher spin generators. We then extend both methods supersymmetrically and use them to derive the superconformal algebra in its entirety. Finally we discuss how the higher spin algebras based on the conformal and superconformal algebras can be constructed.

6.1 The Bosonic Part of the Algebra

In the previous chapter we identified the Lie algebra isomorphism $\mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$, where $\mathrm{SO}(3,2)$ is the conformal group in three dimensions. This indicates that we can construct the algebra using two commuting $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$ spinors q^α and p_α, since they together span a real four-dimensional spinor. The variables q^α and p_α can be interpreted as the phase-space variables, which classically can be regarded as the generalized coordinates and conjugate momenta, respectively. This construction was first used in [38]. We now want to construct the explicit star product for this representation.

The invariant tensor appearing in equation (5.2) defining the star product can now be taken to be the invariant tensor $\epsilon^{\alpha\beta}$ of $\mathrm{SL}(2,\mathbb{R})$ (see appendix A for conventions). Note, however, that there is a factor of $\frac{1}{2}$ differing equation (5.1) from the ordinary canonical commutation relation of the position and momentum operators[1]. The star product in

[1]This assumes that we let the doublet $\hat{Y}^A$ consist of the operators $\hat{q}^\alpha$ and $\hat{p}_\alpha$ and *not*, *e.g.*, $\hat{q}^\alpha$ and $\hat{p}^\alpha$. Then we would also catch an additional minus sign, since the index of $\hat{p}_\gamma$ in the canonical commutation relation $[\hat{q}^\alpha, \hat{p}_\gamma] = i\delta^\alpha_\gamma$ can be raised with $\epsilon^{\beta\gamma}$, resulting in $[\hat{q}^\alpha, \hat{p}^\beta] = -i\epsilon^{\alpha\beta}$. In the end this is only a matter of convention.

our construction of the conformal algebra will consequently read

$$\begin{aligned}(f \star g)(q,p) &= f(q,p) \exp\left[\frac{i}{2}\begin{pmatrix}\overleftarrow{\partial}_q & \overleftarrow{\partial}_p\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{\partial}_q\\ \overrightarrow{\partial}_p\end{pmatrix}\right] g(q,p)\\ &= f(q,p)\exp\left[\frac{i}{2}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)\right] g(q,p).\end{aligned} \tag{6.1}$$

In explicit calculations it is often convenient to perform a Taylor expansion in this expression.

To construct the (bosonic) conformal algebra we will consider star commutators of the form $[f,g]_\star = f\star g - g\star f$ with f and g being generators. Before the extension to the corresponding higher spin algebra, the conformal algebra only contains spin 2 generators which are of second order in the phase-space variables q^α and p_α, something we will soon see explicitly. This means that the star product can be written as

$$(f\star g)^{\text{spin-2}} = f\left[1 + \frac{i}{2}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q\right) - \frac{1}{4}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q\right)^2\right] g, \tag{6.2}$$

which gives the star commutator

$$[f,g]_\star^{\text{spin-2}} = i\left(\frac{\partial f}{\partial q^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q^\alpha}\frac{\partial f}{\partial p_\alpha}\right) = i\{f,g\}_{\text{PB}}, \tag{6.3}$$

where the zeroth and second order terms vanish since q_α and p^α are commuting variables (of which f and g are functions)[2]. Note that the result, up to a factor of i, is precisely the Poisson bracket, which thus can be regarded as the spin 2 limit of the star commutator.

Let us also compute the star commutator of the phase-space variables. We find that

$$[q^\alpha, p_\beta]_\star = \frac{i}{2}\left[q^\alpha(\overleftarrow{\partial}_q\overrightarrow{\partial}_p)p_\beta + p_\beta(\overleftarrow{\partial}_p\overrightarrow{\partial}_q)q^\alpha\right] = i\delta^\alpha_\beta, \tag{6.4}$$

whereas $[q^\alpha, q^\beta] = [p_\alpha, p_\beta] = 0$. Note that these imply the Poisson bracket relations of the canonical coordinates well-known from the Hamiltonian formalism of classical mechanics.

6.1.1 An Explicit Representation of the Generators

We now want to find a representation of the generators, in terms of the phase-space variables q^α and p_α, that recreates the conformal algebra (4.26). To be able to construct vector representations of the generators from spinors, we will use the three-dimensional gamma matrices. Our representation follows the conventions of [39] and reads

$$\begin{aligned}P^a &= -\tfrac{1}{2}(\gamma^a)_{\alpha\beta}q^\alpha q^\beta\\ M^a &= -\tfrac{1}{2}(\gamma^a)_\alpha{}^\beta q^\alpha p_\beta\\ D &= -\tfrac{1}{2}q^\alpha p_\alpha\\ K^a &= -\tfrac{1}{2}(\gamma^a)^{\alpha\beta}p_\alpha p_\beta.\end{aligned} \tag{6.5}$$

[2] It is actually easy to realize that *all* terms of even order in the Taylor expansion of a star commutator must cancel out, since they appear pairwise with opposite signs.

It must now be verified that these generators really satisfy the conformal algebra, provided that we use the star commutator in the construction.

We will here verify two of the six non-zero commutation relations. First we note that

$$\begin{aligned}
[P^a, K^b]_\star &= \frac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)^{\gamma\delta}\left[q^\alpha q^\beta e^{\frac{i}{2}\left(\overleftarrow{\partial}_q\vec{\partial}_p-\overleftarrow{\partial}_p\vec{\partial}_q\right)}p_\gamma p_\delta - p_\gamma p_\delta e^{\frac{i}{2}\left(\overleftarrow{\partial}_q\vec{\partial}_p-\overleftarrow{\partial}_p\vec{\partial}_q\right)}q^\alpha q^\beta\right]\\
&= i(\gamma^a\gamma^b)_\alpha{}^\beta q^\alpha p_\beta\\
&= i\left(\epsilon^{ab}{}_c(\gamma^c)_\alpha{}^\beta q^\alpha p_\beta + \eta^{ab}q^\alpha p_\alpha\right)\\
&= -2i\epsilon^{ab}{}_c M^c - 2i\eta^{ab}D,
\end{aligned} \tag{6.6}$$

and then

$$\begin{aligned}
[D, P^a]_\star &= \frac{1}{4}(\gamma^a)_{\alpha\beta}\left[q^\alpha p_\alpha e^{\frac{i}{2}\left(\overleftarrow{\partial}_q\vec{\partial}_p-\overleftarrow{\partial}_p\vec{\partial}_q\right)}q^\alpha q^\beta - q^\alpha q^\beta e^{\frac{i}{2}\left(\overleftarrow{\partial}_q\vec{\partial}_p-\overleftarrow{\partial}_p\vec{\partial}_q\right)}q^\alpha p_\alpha\right]\\
&= -\frac{i}{2}(\gamma^a)_{\alpha\beta}q^\alpha q^\beta\\
&= iP^a.
\end{aligned} \tag{6.7}$$

Note that both results agree with the conformal algebra (4.26). Since the generators are only of second order in the phase-space variables and all terms of even orders in an expansion of a star commutator must vanish, what we have computed is (modulo a factor of i) just the Poisson brackets. Working completely analogously one easily verifies that also the rest of the conformal algebra is satisfied by the generators in equation (6.5).

6.1.2 The Equivalent Operator Formulation

Inspired by the quantum mechanical operator formulation of the canonical commutation relations, one may wonder if an analogous formulation is possible also here. Then we can abandon the star products. This would be especially tempting when we are to extend the theory to include higher spins, and thereby will obtain more non-vanishing terms in the star commutator expansions. In the operator formulation, the phase-space variable no longer commute, but are instead regarded as operators satisfying the canonical commutation relation

$$[q^\alpha, p_\beta] = i\delta^\alpha_\beta. \tag{6.8}$$

In a quantum mechanical formulation, each observable should correspond to a Hermitian operator having real eigenvalues. Since $[q^\alpha, q^\beta] = [p_\alpha, p_\beta] = 0$ still holds, the generators P^a and K^a from equation (6.5) are obviously already Hermitian.

To check if M^a is Hermitian we first need to clarify how the spinorial indices should be raised and lowered. The indices of p_α and q^α are by definition raised and lowered from the left with the antisymmetric $\epsilon^{\alpha\beta}$. Since the first spinor index of the gamma matrices $(\gamma^a)_\alpha{}^\beta$ is raised and lowered from the left but the second one from the right, this implies

$$q^\alpha(\gamma^a)_\alpha{}^\beta = \epsilon^{\alpha\gamma}q_\gamma(\gamma^a)_\alpha{}^\beta = -q_\gamma\epsilon^{\gamma\alpha}(\gamma^a)_\alpha{}^\beta = -q_\alpha(\gamma^a)^{\alpha\beta}, \tag{6.9}$$

whereas

$$(\gamma^a)_\alpha{}^\beta p_\beta = (\gamma^a)_\alpha{}^\beta\epsilon_{\beta\gamma}p^\gamma = (\gamma^a)_{\alpha\beta}p^\beta. \tag{6.10}$$

Hence we only pick up a sign when we flip the first spinorial index of a gamma matrix with the index of either q^α or p_α.

We can now easily identify M^a as Hermitian, since

$$(M^a)^\dagger = -\frac{1}{2}(\gamma^a)_\beta{}^\alpha p^\beta q_\alpha = -\frac{1}{2}(\gamma^a)_\alpha{}^\beta p_\beta q^\alpha = -\frac{1}{2}(\gamma^a)_\alpha{}^\beta q^\alpha p_\beta = M^a, \tag{6.11}$$

where we at the third equality used equation (6.8) and that the three-dimensional gamma matrices are traceless; see appendix A for an explicit representation. The generator D from equation (6.5) is, however, obviously not Hermitian. This can easily be adjusted by rewriting it as

$$D = -\frac{1}{4}(q^\alpha p_\alpha + p_\alpha q^\alpha), \tag{6.12}$$

which for the commuting phase-space variables is equivalent to equation (6.5). The process of constructing Hermitian generators is often referred to as a Weyl-ordering of the operators q^α and p_α.

We now want to verify that the four Hermitian generators, *i.e.* P^a, M^a and K^a from equation (6.5) and D from equation (6.11), satisfy the conformal algebra (4.26) provided that q^α and p_α are regarded as operators satisfying the canonical commutation relation (6.8). Let us verify the same two commutation relations as we did in equations (6.6) and (6.7) for the star product construction. We find that

$$\begin{aligned}
[P^a, K^b] &= \frac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)^{\gamma\delta}[q^\alpha q^\beta, p_\gamma p_\delta] \\
&= \frac{i}{2}(\gamma^a\gamma^b)_\alpha{}^\beta\big(q^\alpha p_\beta + p_\beta q^\alpha\big) \\
&= \frac{i}{2}\big[2\epsilon^{ab}{}_c(\gamma^c)_\alpha{}^\beta q^\alpha p_\beta + \eta^{ab}(q^\alpha p_\alpha + p_\alpha q^\alpha)\big] \\
&= -2i\epsilon^{ab}{}_c M^c - 2i\eta^{ab} D
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
[D, P^a] &= \frac{1}{8}(\gamma^a)_{\alpha\beta}\big([q^\gamma p_\gamma, q^\alpha q^\beta] + [p_\gamma q^\gamma, q^\alpha q^\beta]\big) \\
&= -\frac{i}{2}(\gamma^a)_{\alpha\beta} q^\alpha q^\beta \\
&= iP^a,
\end{aligned} \tag{6.14}$$

which both agree with the conformal algebra. The calculations for the remaining commutators are completely analogous, and show that the operator formulation constructed above obeys the conformal algebra.

6.2 The Supersymmetric Extension

In order to describe a supersymmetric theory we must also include the fermionic generators, *i.e.*, extending the conformal algebra to the superconformal algebra introduced in section 4.2. We will first conduct this extension in the operator formulation, before illustrating how the analogous procedure can be carried through via the introduction of an extended star product.

Since the fermionic generators, as we saw in section 4.2, are supposed to be anticommuting operators[3] we cannot construct them from the commuting (Grassmann even) operators q^α and p_α. Consequently, an anticommuting (Grassmann odd) object must be introduced in their construction. We will use the vectors λ^I to write the fermionic generators as

$$Q^{\alpha I} = q^\alpha \lambda^I \tag{6.15}$$

$$S_\alpha{}^I = p_\alpha \lambda^I. \tag{6.16}$$

We choose the normalization of these vector such that they obey the Clifford algebra

$$\{\lambda^I, \lambda^J\} = 2\delta^{IJ}. \tag{6.17}$$

By using the vectors λ^I we should also be able to construct the bosonic SO(N) R-symmetry generators T^{IJ} that was discussed in section 4.2.

Since we require all generators to be Hermitian, T^{IJ} must be of the form

$$T^{IJ} = \frac{i}{8}\big(\lambda^I\lambda^J - \lambda^J\lambda^I\big). \tag{6.18}$$

These generators commute as

$$\begin{aligned}
[T_{IJ}, T^{KL}] &= -\frac{1}{16}\Big[\lambda_{[I}\lambda_{J]}, \lambda^{[K}\lambda^{L]}\Big] = -\frac{1}{8}\Big(\lambda_{[I}\delta^{[K}_{J]}\lambda^{L]} - \delta^{[K}_{[I}\lambda_{J]}\lambda^{L]} + \lambda^{[K}\lambda_{[I}\delta^{L]}_{J]} - \lambda^{[K}\delta^{L]}_{[I}\lambda_{J]}\Big)\\
&= \frac{1}{4}\Big(\delta^{[K}_{[I}\lambda_{J]}\lambda^{L]} + \lambda^{[K}\delta^{L]}_{[I}\lambda_{J]}\Big) = \frac{1}{4}\Big(\delta^{[K}_{[I}\big(\lambda_{J]}\lambda^{L]} - \lambda^{L]}\lambda_{J]}\big)\Big)\\
&= -2i\delta^{[K}_{[I}T_{J]}{}^{L]},
\end{aligned} \tag{6.19}$$

which is precisely the so(N) algebra (4.34), as we expected. At the second equality we used the relation

$$[\lambda^I\lambda^J, \lambda^K] = \lambda^I\{\lambda^J, \lambda^K\} - \{\lambda^I, \lambda^K\}\lambda^J = 2(\delta^{JK}\lambda^I - \delta^{IK}\lambda^J), \tag{6.20}$$

from which also the commutation relations with the two fermionic generators follow directly as

$$\begin{aligned}
[T_{IJ}, Q^{\alpha K}] &= -i\delta^{KL}_{IJ}Q^\alpha{}_L\\
[T_{IJ}, S_\alpha{}^K] &= -i\delta^{KL}_{IJ}S_{\alpha L}.
\end{aligned} \tag{6.21}$$

We should also prove that these constructions of the fermionic generators satisfy the commutation relations (4.31)– (4.33). By acting with $(\gamma_a)^{\alpha\beta}$ on both sides of the definition of P^a in equation (6.5), we directly find that

$$q^\alpha q^\beta = -(\gamma_a)^{\alpha\beta}P^a \tag{6.22}$$

and thus

$$\{Q^{\alpha I}, Q^{\beta J}\} = q^\alpha q^\beta\{\lambda^I, \lambda^J\} = -2\delta^{IJ}(\gamma_a)^{\alpha\beta}P^a, \tag{6.23}$$

[3] In the sense that their Lie superbracket should become an anticommutator, *not* in the sense that they anticommute to zero.

which agrees with equation (4.31). A completely analogous calculation starting from the generator K^a yields equation (4.32), *i.e.*,

$$\{S_\alpha{}^I, S_\beta{}^J\} = -2\delta^{IJ}(\gamma_a)_{\alpha\beta}K^a. \tag{6.24}$$

To compute the commutator $\{Q^{\alpha I}, S_\beta{}^J\}$ we first note that the definitions of the (Hermitian) generators M^a in equation (6.5) and D in (6.12) imply that we can write the Hermitian operator $q^\alpha p_\beta + p_\beta q^\alpha$ as the linear combination

$$q^\alpha p_\beta + p_\beta q^\alpha = x\delta^\alpha_\beta D + y(\gamma_a)^\alpha{}_\beta M^a \tag{6.25}$$

for some constants x and y. Upon multiplication of both sides by δ^β_α we see that $x = -2$. Instead multiplying by $(\gamma^a)_\alpha{}^\beta$ we see that also $y = -2$. Consequently,

$$\begin{aligned}\{Q^{\alpha I}, S_\beta{}^J\} &= \{q^\alpha\lambda^I, p_\beta\lambda^J\} = q^\alpha p_\beta\{\lambda^I, \lambda^J\} - [q^\alpha, p_\beta]\lambda^J\lambda^I = 2\delta^{IJ}q^\alpha p_\beta - i\delta^\alpha_\beta\lambda^J\lambda^I \\ &= \delta^{IJ}\big(q^\alpha p_\beta + p_\beta q^\alpha\big) + \frac{i}{2}\delta^\alpha_\beta\big(\lambda^I\lambda^J - \lambda^J\lambda^I\big) \\ &= -2\delta^{IJ}\delta^\alpha_\beta D - 2\delta^{IJ}(\gamma^a)_\beta{}^\alpha M_a + 4\delta^\alpha_\beta T^{IJ},\end{aligned} \tag{6.26}$$

which agrees with equation (4.33). At the second line we used that the number terms that appear when we construct Hermitian objects from the two terms, *i.e.* when we Weyl order them, cancel each other out.

We have now derived all commutation relations that contain two of the new generators $Q^{\alpha I}$, $S_\alpha{}^I$ and T^{IJ}. There are also some non-zero commutation relations including one new generator and one of the conformal ones. These are completely trivial to derive in the operator formulation. One of them is derived as follows:

$$[K^a, Q^{\alpha I}] = -\frac{1}{2}(\gamma^a)^{\beta\gamma}[p_\beta p_\gamma, q^\alpha\lambda^I] = \frac{i}{2}(\gamma^a)^{\beta\gamma}\big(\delta^\alpha_\gamma p_\beta + \delta^\alpha_\beta p_\gamma\big)\lambda^I = i(\gamma^a)^{\alpha\beta}S_\beta{}^I, \tag{6.27}$$

and the other five in the same manner.

With these commutation relation in place, we have derived the entire superconformal algebra. It reads

$$\begin{aligned}[][M^a,\, M^b] &= i\epsilon^{ab}{}_c M^c \\ [M^a,\, P^b] &= i\epsilon^{ab}{}_c P^c \\ [M^a,\, K^b] &= i\epsilon^{ab}{}_c K^c \\ [P^a,\, K^b] &= -2i\epsilon^{ab}{}_c M^c - 2i\eta^{ab}D \\ [D,\, P^a] &= iP^a \\ [D,\, K^a] &= -iK^a \\ [D,\, Q^{\alpha I}] &= \tfrac{i}{2}Q^{\alpha I} \\ [D,\, S_\alpha{}^I] &= -\tfrac{i}{2}S_\alpha{}^I \\ [P^a,\, S_\beta{}^I] &= -i(\gamma^a)_{\beta\alpha}Q^{\alpha I} \\ [K^a,\, Q^{\alpha I}] &= i(\gamma^a)^{\alpha\beta}S_\beta{}^I \\ [M^a,\, Q^{\alpha I}] &= \tfrac{i}{2}(\gamma^a)_\beta{}^\alpha Q^{\beta I}\end{aligned} \tag{6.28}$$

$$
\begin{aligned}
[M^a,\, S_\alpha{}^I] &= -\tfrac{i}{2}(\gamma^a)_\alpha{}^\beta S_\beta{}^I \\
[T^{IJ},\, Q^\alpha{}_K] &= -i\delta^{IJ}_{KL} Q^{\alpha L} \\
[T^{IJ},\, S_{\alpha K}] &= -i\delta^{IJ}_{KL} S_\alpha{}^L \\
\{Q^{\alpha I},\, Q^{\beta J}\} &= -2\delta^{IJ}(\gamma^\mu)^{\alpha\beta} P_\mu \\
\{S_\alpha{}^I,\, S_\beta{}^J\} &= -2\delta^{IJ}(\gamma^\mu)_{\alpha\beta} K_\mu \\
\{Q^{\alpha I},\, S_\beta{}^J\} &= -2\delta^{IJ}\delta^\alpha_\beta D - 2\delta^{IJ}(\gamma_a)_\beta{}^\alpha M^a + 4\delta^\alpha_\beta T^{IJ} \\
[T_{IJ},\, T^{KL}] &= -2i\delta^{[K}_{[I} T_{J]}{}^{L]}
\end{aligned}
$$

and we will in the following chapters use it to construct a supersymmetric conformal field theory in three dimensions.

6.2.1 The Extended Star Product

Exactly like for the conformal algebra, the superconformal algebra can also be constructed using an extended (super)star product. In the star product formulation, the phase-space variables q^α and p_α were regarded as Grassmann even quantities commuting to zero. The vectors λ^I were, however, introduced as Grassmann odd quantities, meaning their Lie superbracket is an anticommutator. In the extended star product formulation they must, consequently, anticommute to zero, meaning

$$\lambda^I \lambda^J = -\lambda^J \lambda^I. \tag{6.29}$$

Such quantities are often simply referred to as “Grassmann numbers”.

In the previous section we saw that, in an operator formulation, λ^I can be taken to satisfy the Clifford algebra $\{\lambda^I, \lambda^J\} = 2\delta^{IJ}$. This indicates that their invariant tensor can simply be chosen as the Kronecker delta, which can also be regarded as a consequence of the vectors λ^I generating the whole so(N) algebra themselves. The generalization to an extended (super)star product must, consequently, read

$$(f \star g)(q,p,\lambda) = f(q,p,\lambda) \exp\left[\frac{i}{2}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q\right) + \overleftarrow{\partial}_\lambda \overrightarrow{\partial}_\lambda\right] g(q,p,\lambda). \tag{6.30}$$

To verify that also this construction satisfies the superconformal algebra (6.28) is now a trivial matter. The only difference to the purely bosonic case is that, since we are now studying a superalgebra, we have to extend the concept of the star commutator to the star supercommutator satisfying

$$[x,y\}_\star = x \star y - (-1)^{|x||y|} y \star x. \tag{6.31}$$

We will not give the calculations of all 18 non-zero star supercommutators of the generators, but settle with two to illustrate the process.

First we compute the star anticommutator

$$
\begin{aligned}
\{Q^{\alpha I}, S_\beta{}^J\}_\star &= q^\alpha \lambda^I e^{\frac{i}{2}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q\right) + \overleftarrow{\partial}_\lambda \overrightarrow{\partial}_\lambda} p_\beta \lambda^J + p_\beta \lambda^J e^{\frac{i}{2}\left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q\right) + \overleftarrow{\partial}_\lambda \overrightarrow{\partial}_\lambda} q^\alpha \lambda^I \\
&= q^\alpha p_\beta \{\lambda^I, \lambda^J\} + \frac{i}{2}\delta^\alpha_\beta [\lambda^I, \lambda^J] + \delta^{IJ}\left(q^\alpha p_\beta + p_\beta q^\alpha\right) + \frac{i}{2}\delta^\alpha_\beta \delta^{IJ}(1-1) \\
&= -2\delta^{IJ}\delta^\alpha_\beta D - 2\delta^{IJ}(\gamma^a)_\beta{}^\alpha M_a + 4\delta^\alpha_\beta T^{IJ},
\end{aligned} \tag{6.32}
$$

where we in the last step used equations (6.25) and (6.29). It is easy to realize that all terms of even order in the Taylor expansion must cancel out, also for the star anticommutator. We also compute the star commutator

$$\begin{aligned}[T^{IJ}, Q^{\alpha K}]_\star &= \frac{i}{4}\Big(\lambda^I\lambda^J e^{\bar{\partial}_\lambda\vec{\partial}_\lambda} q^\alpha\lambda^K - q^\alpha\lambda^K e^{\bar{\partial}_\lambda\vec{\partial}_\lambda}\lambda^I\lambda^J\Big) \\ &= \frac{i}{2} q^\alpha\big(\delta^{JK}\lambda^I - \delta^{IK}\lambda^J\big) \\ &= -i\delta^{IJ}_{KL}Q^{\alpha L}.\end{aligned} \tag{6.33}$$

Here there is a small subtlety involved at the second equality. One has to recall that the product rule for the Grassmann derivative ∂_θ of two Grassmann numbers θ_j and θ_k reads

$$\frac{\partial}{\partial\theta_i}(\theta_j\theta_k) = \frac{\partial\theta_j}{\partial\theta_i}\theta_k - \frac{\partial\theta_k}{\partial\theta_i}\theta_j, \tag{6.34}$$

since the Grassmann derivative must be anticommuted with θ_j to reach the second term, thereby picking up a sign. Note that both star supercommutators above agree with the superconformal algebra (6.28). It can easily be shown that the same holds for the entire superstar product construction. The reason we tenaciously keep formulating two equivalent constructions of the superconformal algebra is simply that the suitability of respective construction varies greatly depending on what one wants to compute.

6.3 Including Higher Spins

Our formulation of the bosonic generators in terms of the $\mathrm{SL}(2,\mathbb{R})$ spinors q^α and p_α is constructed in a way that is easily generalized to include higher spins. It is merely a question of including more spinors in the generators. If we let n_q and n_p denote the number of q^α and p_α spinors in the generators, the spin s generators must contain $n_q+n_p = 2(s-1)$ spinors in total. If we also let $c \leq \min(n_q, n_p)$ denote the number of contracted spinor pairs $q \cdot p \equiv q^\alpha p_\alpha$, the generators can be written on the form

$$G(n_q, n_p, c)^{\alpha_1\ldots\alpha_{2N}} = \big(-\tfrac{1}{2}\big)^{\frac{1}{2}(n_q+n_p)} q^{(\alpha_1}\ldots q^{\alpha_{n_q-c}}p^{\alpha_{n_q-c+1}}\ldots p^{\alpha_{2N})}(q\cdot p)^c, \tag{6.35}$$

where we have let $2N = n_q+n_p-2c$ (which makes it redundant to explicitly write out the value of c on the LHS). Note that due to the antisymmetry of the $\mathrm{SL}(2,\mathbb{R})$ metric ϵ_{ab}, all these generators must be irreducible representations (irreps) of $\mathrm{sl}(2,\mathbb{R})$ and thus totally traceless, *i.e.*, traceless in all pairs of indices. Also note that these generators contain all polynomials of even degree in q^α and p_α, which agrees with how we defined the higher spin algebra in section 5.1.1, and that they are *not* Weyl ordered by construction but that this easily can be implemented.

These generators can then easily be written on tensorial form carrying spacetime indices by combining the $2N$ spinors into N pairs contracted with one gamma matrix each. This results in

$$\begin{aligned}G(n_q, n_p)^{a_1\ldots a_N} =&(-1)^{\left\lfloor\frac{n_p-c}{2}\right\rfloor}\big(-\tfrac{1}{2}\big)^{\frac{1}{2}(n_q+n_p)} \\ &(\gamma^{a_1})_{\alpha_1\alpha_2}\cdots(\gamma^{a_N})_{\alpha_{2N-1}\alpha_{2N}}q^{(\alpha_1}\ldots q^{\alpha_{n_q-c}}p^{\alpha_{n_q-c+1}}\ldots p^{\alpha_{2N})}(q\cdot p)^c,\end{aligned} \tag{6.36}$$

where the new signs come from raising the indices on the $\left\lfloor \frac{n_p - c}{2} \right\rfloor$ p_{α_n} spinors that are contracted with the first index of a gamma matrix, in accordance with equation (6.9). Since the generators in equation (6.35) were irreducible representations of $\mathrm{sl}(2,\mathbb{R})$ and, thus, symmetric and traceless in the spinorial indices $\alpha_1, \ldots, \alpha_{2N}$, also the tensorial generators must be irreducible representations of $\mathrm{sl}(2,\mathbb{R})$. We say that the (flat) spacetime indices $a_1, \ldots, a_N$ are in the symmetrized traceless representation.

It is easily verified that equation (6.36) reproduces the generators from equation (6.5) in the spin 2 case, when $n_q + n_p = 2$. For spin 3, equation (6.36) instead yields the generators

$$\begin{aligned}
G^{ab}(4,0) &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}q^\alpha q^\beta q^\gamma q^\delta, & G(2,2) &= \tfrac{1}{4}(q\cdot p)^2, \\
G^{ab}(3,1) &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}q^{(\alpha} q^\beta q^\gamma p^{\delta)}, & G^{ab}(1,3) &= -\tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}q^{(\alpha} p^\beta p^\gamma p^{\delta)}, \\
G^{a}(3,1) &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}q^\alpha q^\beta (q\cdot p), & G^{a}(1,3) &= -\tfrac{1}{4}(\gamma^a)_{\alpha\beta}p^\alpha p^\beta (q\cdot p), \\
G^{ab}(2,2) &= -\tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}q^{(\alpha} q^\beta p^\gamma p^{\delta)}, & G^{ab}(0,4) &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}p^\alpha p^\beta p^\gamma p^\delta. \\
G^{a}(2,2) &= \tfrac{1}{4}(\gamma^a)_{\alpha\beta}q^\alpha p^\beta (q\cdot p), & &
\end{aligned} \tag{6.37}$$

To construct the spin 3 algebra we need to compute the star commutators of these generators, so let us first investigate how the star product of two generators of arbitrary spins look. We introduce the notation $G(2n)$, where $2n = n_p + n_q$, to denote an arbitrary generator of spin $s = n + 1$. By studying the star product given in equation (6.1) we see that the total number of spinors q^α and p_α will be decreased by two in the first non-vanishing term in the Taylor expansion of the star commutator $[G(2n_1), G(2n_2)]_\star$, by six in the second non-vanishing term, and so on[4]. Hence we can write

$$[G(2n_1), G(2n_2)]_\star = G\big(2(n_1+n_2-1)\big) + G\big(2(n_1+n_2-3)\big) + \ldots + G(n_0), \tag{6.38}$$

which continues down to the generator $G(n_0)$ corresponding to the lowest possible spin. This is either the spin 2 generator $G(2)$, if both n_1 and n_2 are either even or odd, or the spin 3 generator $G(4)$ if one of n_1 and n_2 is even and the other one odd. The spin 1 generator $G(0)$ would correspond to a central charge and is not included in the theory. Recall that the first term of the star product correspond to the Poisson bracket (modulo a factor of i).

We now note that the star commutator of two spin 2 generators $G(2)$ yields another spin 2 generator, which is consistent with the conformal algebra (4.26) being closed. If we instead want to construct the spin 2-spin 3 algebra we need to compute all commutation relations of one spin 2 generator $G(2)$ and one spin 3 generator $G(4)$. In accordance with equation (6.38), the result must always be another spin 3 generator $G(4)$. Since this is the only term appearing on the RHS of equation (6.38), deriving the spin 2-spin 3 algebra is merely a question of calculating the Poisson brackets of the spin 2 and spin 3 generators, found in equation (6.5) and (6.37), respectively. This can easily be done, and the resulting algebra can be found in [5].

We also want to construct the pure spin 3 algebra. Equation (6.38) then states that the commutation relations we need to compute are of the form

$$[G(4), G(4)]_\star = G(6) + G(2), \tag{6.39}$$

[4] Recall that all terms of even order in the Taylor expansion of the star commutator must vanish.

where the spin 4 generator $G(6)$ appears on the RHS. Hence it is impossible to construct a theory that includes spin 3 fields, without also including the spin 4 fields. The spin 4 generators can, however, easily be constructed from equation (6.36). But when we then want to construct the spin 3-spin 4 algebra we find that

$$[G(4), G(6)]_\star = G(8) + G(4), \tag{6.40}$$

meaning we must also include the spin 5 generators $G(8)$. In this way it continues until the entire infinite tower of integer spins has been included. We have thus illustrated the most essential property of all (bosonic) higher spin theories; as soon as the spin 3 field is included, fields of *all* integer spins must be included.

So far we have only constructed the higher spin algebra based on the conformal algebra. To construct the complete superconformal higher spin algebra is a bit more intricate and will not be done in detail, although it is conceptually easy. The reason for this is that the fermionic generators $Q_{\alpha I} = q^\alpha \lambda^I$ and $S_\alpha{}^I = p_\alpha \lambda^I$ contain one $\mathrm{SL}(2,\mathbb{R})$ spinor and one vector λ^I each, where $I = 1, \ldots, N$ with N being the number of supersymmetries of the theory. Since λ^I satisfies the Clifford algebra (6.17) they can be represented as the $\mathrm{SO}(N)$ gamma matrices. This tells us that the set $\{\lambda^I, \ldots, \lambda^{I_1 \ldots I_N}\}$ forms a basis, which in the matrix representation is the basis of all $2N \times 2N$ complex matrices.

Taking the fermionic spin 3 generators as an example we can, *e.g.*, construct generators of the form $Q^{a\alpha I} \sim P^a q^\alpha \lambda^I$, $Q^{aIJ} \sim P^a \lambda^{IJ}$ and $Q^{\alpha IJK} \sim q^\alpha \lambda^{IJK}$ and so on, where P^a is just the generator from equation (6.5). By studying the extended star product from equation (6.30) we realize that also the star supercommutator of these generators (which is now a star anticommutator) must satisfy equation (6.38), *i.e.*, the exact same relation as the bosonic higher spin generators satisfied. As we will soon understand, a fermionic spin s generator will correspond to a gauge field of spin $s - \frac{1}{2}$; for instance, the gauge field corresponding to the spin 2 generator $S_\alpha{}^I$ will be the spin $\frac{3}{2}$ gravitino field $\psi^\alpha{}_I$. Completely analogous to the bosonic case, equation (6.38) thus implies that as soon as the fermionic spin $\frac{5}{2}$ field is included in the theory, fields of *all* half-integer spins up to infinity must also be included in order to close the algebra.

In the following two chapters we will study two possible ways of deriving the field equations of the superconformal field theory. We will, however, restrict these calculations to only include fields of spins up to 2, meaning we will not actually need the explicit representations of neither the bosonic nor the fermionic higher spin generators. Note that this spin 2 truncation is consistent, since we have shown that the spin 2 algebra is closed.

Chapter 7

The Zero Field Strength Equation

Having constructed the superconformal higher spin algebra there are, at least, two methods for deriving the field equations of the corresponding theory. Either we can use the unfolded formalism known from section 5.1.1, or we can use that it can be formulated as a Chern-Simons gauge theory and apply the methods from section 3.4. In section 3.5 we illustrated how this can be done in general relativity by expressing the Einstein-Hilbert action as a Chern-Simons action. The explicit procedure for conformal gravity in three dimensions was first conducted by Horne and Witten in [40], and then extended to supergravity by Fradkin and Linetsky in [41]. The explicit construction of the Chern-Simons action will not be of interest to us. It suffices to know that it is feasible, so we can apply the equation of motion for Chern-Simons gauge theories.

The Chern-Simons equation of motion was in equation (3.48) derived to be the flatness condition $F = 0$, also known as the zero field strength equation. In this chapter we will solve this equation explicitly for the spin 2 fields, first for the purely bosonic theory and then for the supersymmetric theory.

7.1 The Setting

The zero-field strength equation reads

$$F = \mathrm{d}A + A \wedge A = 0, \tag{7.1}$$

where the gauge potential one-form is Lie-algebra valued and, in the spin 2 case, can be expanded in terms of the generators as $A = \mathrm{d}x^\mu A_\mu^a T_a$. We can thus write

$$\mathrm{d}(A_a T^a) + \frac{1}{2}\{A_a T^a, A_b T^b\} = 0. \tag{7.2}$$

When at least one of the generators is bosonic, the anticommutator evaluates to

$$\{A_a T^a, A_b T^b\} = A_a \wedge A_b [T^a, T^b], \tag{7.3}$$

where the sign in the second term of the commutator comes from changing the order of the one-forms A_a and A_b. For two fermionic generators, the anticommutator evaluates to

$$\{A_a T^a, A_b T^b\} = -A_a \wedge A_b \{T^a, T^b\}, \tag{7.4}$$

since both the fermionic generators and the one-forms are Grassmann odd. This can be summarized in the form

$$\mathrm{d}(A_a T^a) + \frac{(-1)^{|a||b|}}{2} A_a \wedge A_b [T^a, T^b\} = 0 \tag{7.5}$$

of the zero field strength equation, where $|\cdot|$ denotes the Grassmann parity that was introduced in definition 4.1.

Since we later intend to relate the obtained results to those emerging from the unfolded formulation of the theory, it is convenient to use consistent conventions in the two approaches[1]. We will (almost) follow the convention of the unfolded formalism, with the gauge potential given in equation (5.3). The gauge potential we will use to solve the spin 2 equation is thus $A = (-i)A_1$ with

$$A_1 = e_a P^a + \omega_a M^a + bD + f_a K^a + a_{IJ} T^{IJ} + i(\chi_{\alpha I} Q^{\alpha I} + \psi^{\alpha}{}_I S_{\alpha}{}^I), \tag{7.6}$$

where the factor of i in front of the fermionic generators makes these terms Hermitian. That the frame field and spin connection can be used as gauge fields was something we observed already in section 3.5. Similarly we introduce the gauge fields b and f_a corresponding to the generators of scalings and special conformal transformations. Since T^{IJ} generates SO(N), which is non-abelian for $N > 2$, the gauge fields a_{IJ} will usually be non-abelian. The gauge fields $\chi_{\alpha I}$ and $\psi^{\alpha}{}_I$ correspond to the fermionic generators and the latter will be given an interpretation as the gravitino field.

With these definitions in place, deriving the field equations is merely a matter of applying the flatness condition (7.5) to the superconformal algebra (6.28). It is then convenient to project out the terms proportional to respective generator, thereby obtaining seven equations in total. Before conducting this procedure it is convenient to contemplate the gauge freedoms of the theory. It will prove possible to fix a gauge that simplifies the calculations considerably.

7.1.1 Fixing a Gauge

For clarity, the gauge fixing procedure will be illustrated in the purely bosonic case. The procedure for the supersymmetric theory is completely analogous, and we will work in the same gauge in both cases. To study the gauge freedom of the system it is convenient to introduce the gauge parameter

$$\Lambda = \Lambda_a^{(P)} P^a + \Lambda_a^{(M)} M^a + \Lambda^{(D)} D + \Lambda_a^{(K)} K^a, \tag{7.7}$$

which is a Lie-algebra valued zero-form generating the gauge transformations. The Chern-Simons action (3.44) is then invariant under gauge transformations of the form

$$\delta A = \mathrm{d}\Lambda + [A, \Lambda]. \tag{7.8}$$

[1]It is important to point out that the exact relation between the two approaches is still unknown, meaning they are *not* necessarily equivalent. This motivates why we want to compare the results stemming from them.

By inserting the commutation relations of the conformal algebra and projecting out the equation corresponding to each of the four generators, this yields

$$\begin{aligned}\delta e_\mu{}^a &= D_\mu \Lambda^a_{(P)} - \epsilon_\mu{}^{ab}\Lambda_b^{(M)} - e_\mu{}^a \Lambda^{(D)} + b_\mu \Lambda^a_{(P)},\\ \delta\omega_\mu{}^a &= D_\mu \Lambda^a_{(M)} + 2\epsilon_\mu{}^{ab}\Lambda_b^{(K)} - 2\epsilon^{abc} f_{\mu b}\Lambda_c^{(P)},\\ \delta b_\mu &= D_\mu \Lambda^{(b)} - 2 e_\mu{}^a \Lambda_a^{(K)} + 2 f_\mu{}^a \Lambda_a^{(P)},\\ \delta f_\mu{}^a &= D_\mu \Lambda^a_{(K)} - b_\mu \Lambda^a_{(K)} + \epsilon^{abc} f_{\mu b}\Lambda_c^{(M)} + f_\mu{}^a \Lambda^{(D)}.\end{aligned} \tag{7.9}$$

Here, D_μ is the Lorentz covariant derivative, first defined in equation (2.41), containing the spin 2 connection $\omega_\mu{}^a$, meaning

$$De^a = \mathrm{d}e^a + \omega^a{}_c \wedge e^c = \mathrm{d}e^a + \frac{1}{2}\epsilon^a{}_{bc}\epsilon^{bde}\omega_{de} \wedge e^c = \mathrm{d}e^a + \epsilon^a{}_{bc}\omega^b \wedge e^c. \tag{7.10}$$

Studying the scaling transformations δb_μ above we realize that this is an equation that can be solved for the symmetry parameter $\Lambda_a^{(K)}$ assuming the vielbeins $e_\mu{}^a$ are invertible, which we declared them to be in section 2.3. This means that we can use the special conformal transformations to fix a gauge where $b_\mu = 0$, which is a very convenient choice. This should then, together with the $\Lambda_a^{(K)}$ solving $\delta b_\mu = 0$, be inserted into the expressions for $\delta e_\mu{}^a$, $\delta\omega_\mu{}^a$ and $\delta f_\mu{}^a$, resulting in new expressions for these gauge transformations.

7.2 The Bosonic Case

Let us first solve the zero field strength equation in the purely bosonic case, *i.e.*, when only the conformal algebra is considered. By first implementing our gauge choice $b = 0$ and then inserting the commutator relations of the conformal algebra into equation (7.5) we find the equations

$$\begin{aligned} (P_a):&\quad De^a = 0\\ (M_a):&\quad \mathrm{d}\omega^a + \tfrac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c - 2\epsilon^a{}_{bc} e^b \wedge f^c = 0\\ (D):&\quad e^a \wedge f_a = 0\\ (K_a):&\quad Df^a = 0,\end{aligned} \tag{7.11}$$

for the projection of respective generator.

The (P_a) equation is precisely the torsion-free condition (2.42), which can be used to express the spin connection in terms of the frame fields.

Studying the curvature form $R^a{}_b$ from equation (2.40), we realize that it written in the one-index form $R^a = \frac{1}{2}\epsilon^a{}_{bc}R^{bc}$ precisely contains the first two terms in the (M_a) equation above. By extracting the components of the differential forms we can thus write the (M_a) equation as

$$R_{\mu\nu}{}^a = 4\epsilon^a{}_{bc} e_{[\mu}{}^b f_{\nu]}{}^c. \tag{7.12}$$

Multiplication by $\epsilon^{\rho\mu}{}_a$ yields the Ricci tensor $R_\nu{}^\rho$ on the LHS, since

$$\epsilon^{\rho\mu}{}_a R_{\mu\nu}{}^a = \frac{1}{2}\epsilon^{\rho\mu}{}_a \epsilon^a{}_{bc} R_{\mu\nu}{}^{bc} = -\delta^{\rho\mu}_{bc} R_{\mu\nu}{}^{bc} = -R_{\mu\nu}{}^{\rho\mu} = R_\nu{}^\rho, \tag{7.13}$$

which means that equation (7.12) implies

$$R_\nu{}^\rho = -8\delta^{\rho\mu}_{bc} e_{[\mu}{}^b f_{\nu]}{}^c = -8\delta^{[\rho}_{[\mu} f_{\nu]}{}^{\mu]} = 2\big(f_\nu{}^\rho + \delta^\rho_\nu f_\mu{}^\mu\big). \tag{7.14}$$

By computing the trace of both sides we find that

$$f_\mu{}^\mu = \frac{1}{8}R, \tag{7.15}$$

where R is the Ricci scalar. Insertion of this into equation (7.14) then yields

$$f_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right), \tag{7.16}$$

after some renaming of the indices and usage of the metric to lower the upper index. Note that this, up to a factor of $\frac{1}{2}$, is precisely the Schouten tensor $S_{\mu\nu}$ introduced in equation (2.53).

Expanding the differential forms, the (D) equation simply states that $f_{[\mu\nu]} = 0$, *i.e.*, that $f_{\mu\nu}$ is symmetric. Since the (M_a) equation implied $f_{\mu\nu} = \frac{1}{2}S_{\mu\nu}$ which we already know is a symmetric tensor (since both the Ricci tensor and the metric are symmetric), the (D) equation is merely an identity providing no new physical information.

The (K_a) equation can be written as

$$D_{[\mu} f_{\nu]}{}^a = 0. \tag{7.17}$$

By using the vielbein $e_\rho{}^a$ to lower the upper index, which the vielbein postulate (2.33) allows us to, and implementing equation (7.16) this can be rewritten as

$$\varepsilon_\sigma{}^{\mu\nu} D_\mu S_{\nu\rho} = 0. \tag{7.18}$$

The quantity on the LHS is precisely the Cotton tensor $C_{\sigma\rho}$, introduced in equation (2.54). The vanishing of the Cotton tensor implies that spacetime is conformally flat, which is an expected result since our theory has not been coupled to matter.

While the (P_a) and (M_a) equations related the different gauge fields to each other and the (D) equation was merely an identity, the Cotton equation $C_{\mu\nu} = 0$ is the field equation for free three-dimensional conformal gravity. The seven equations projected from $F = 0$ in the supersymmetric case will play similar roles. But here, also a corresponding field equation for the fermionic fields, the so-called Cottino equation, and a field equation for the non-abelian gauge field a_{IJ} must appear. Both the (super)Cotton and Cottino equations will prove to couple the bosonic gauge fields to the fermionic ones.

7.3 The Supersymmetric Case

By inserting the superconformal algebra (6.28) into the zero field strength equation (7.5) we find the following seven equations:

$$\begin{aligned}
(P_a): &\quad \mathrm{d}e^a + \epsilon^a{}_{bc}\omega^b \wedge e^c + i(\gamma^a)^{\alpha\beta}\chi_{\alpha I} \wedge \chi_\beta{}^I = 0 \\
(Q^{\alpha I}): &\quad \mathrm{d}\chi_{\alpha I} - a_{IJ} \wedge \chi_\alpha{}^J - (\gamma_a)_{\alpha\beta}e^a \wedge \psi^\beta{}_I + \tfrac{1}{2}(\gamma_a)_\alpha{}^\beta\omega^a \wedge \chi_{\beta I} = 0 \\
(M_a): &\quad \mathrm{d}\omega^a + \tfrac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c - 2\epsilon^a{}_{bc}e^b \wedge f^c + 2i(\gamma^a)_\beta{}^\alpha\chi_{\alpha I} \wedge \psi^{\beta I} = 0 \\
(D): &\quad -e_a \wedge f^a + i\chi_{\alpha I} \wedge \psi^{\alpha I} = 0 \\
(T^{IJ}): &\quad \mathrm{d}a_{IJ} + a_{[I|K|} \wedge a^K{}_{J]} + 4i\chi_{\alpha[I} \wedge \psi^\alpha{}_{J]} = 0 \\
(S_\alpha{}^I): &\quad \mathrm{d}\psi^\alpha{}_I - a_{IJ} \wedge \psi^{\alpha J} - \tfrac{1}{2}(\gamma_a)_\beta{}^\alpha\omega^a \wedge \psi^\beta{}_I + (\gamma_a)^{\alpha\beta}f^a \wedge \chi_{\beta I} = 0 \\
(K_a): &\quad \mathrm{d}f^a + \epsilon^a{}_{bc}\omega^b \wedge f^c - i(\gamma^a)_{\alpha\beta}\psi^\alpha{}_I \wedge \psi^{\beta I} = 0.
\end{aligned} \tag{7.19}$$

After having introduced the shorthand notation

$$\bar{\chi}_\mu{}^I\gamma^a\chi_{\nu I} \equiv \chi_\mu{}^{\alpha I}(\gamma^a)_\alpha{}^\beta\chi_{\nu\beta I}, \tag{7.20}$$

which will be used frequently throughout this thesis, the (P_a) equation can be written as

$$\partial_{[\mu}e_{\nu]}{}^a + \epsilon^a{}_{b[\nu}\omega_{\mu]}{}^b - i\bar{\chi}_{[\mu}{}^I\gamma^a\chi_{\nu]}{}^I = 0. \tag{7.21}$$

By multiplying with $\epsilon_\rho{}^{\mu\nu}$ and solving for the spin connection we find that

$$\omega_a{}^\rho = e_a{}^\rho\omega - \epsilon^{\rho\mu\nu}\partial_\mu e_\nu{}^a + i\epsilon_\rho{}^{\mu\nu}\bar{\chi}_{\mu I}\gamma_a\chi_\nu{}^I, \tag{7.22}$$

where the trace $\omega \equiv \omega_\mu{}^\mu$ can easily be solved for after multiplication by $e^a{}_\rho$. This yields

$$\omega = \frac{1}{2}\left(\epsilon^{a\mu\nu}\partial_\mu e_{\nu a} + i\epsilon^{\mu\nu\rho}\bar{\chi}_{\mu I}\gamma_\rho\chi_\nu{}^I\right), \tag{7.23}$$

which after insertion into equation (7.22) gives the spin connection

$$\omega_a{}^\rho = -\epsilon^{\rho\mu\nu}\partial_\mu e_{\nu a} + \frac{1}{2}e_a{}^\rho\left(\epsilon_b{}^{\mu\nu}\partial_\mu e_\nu{}^b\right) + i\epsilon^{\rho\mu\nu}\bar{\chi}_{\mu I}\gamma_a\chi_\nu{}^I + \frac{i}{2}e_a{}^\rho\left(\epsilon^{\mu\nu\sigma}\bar{\chi}_{\mu I}\gamma_\nu\chi_\sigma{}^I\right). \tag{7.24}$$

To show that the first two terms after multiplication by $\epsilon_{b\rho c}$ recreates the torsion-free connection $\omega_{abc}(e)$ from equation (2.43) is merely a question of multiplying Levi-Civita symbols. In section 2.5 we defined the contorsion as the difference between the complete (supersymmetric) spin connection and the torsion-free (purely bosonic) one. Hence we write

$$\omega_{a\rho} = \omega_{a\rho}(e) + K_{a\rho}(\chi), \tag{7.25}$$

where the contorsion is given by

$$K_{a\rho}(\chi) = i\left(\epsilon_\rho{}^{\mu\nu}\bar{\chi}_{\mu I}\gamma_a\chi_\nu{}^I + \frac{1}{2}e_{a\rho}\left(\epsilon^{\mu\nu\sigma}\bar{\chi}_{\mu I}\gamma_\nu\chi_\sigma{}^I\right)\right). \tag{7.26}$$

Before setting out to solve the $(Q^{\alpha I})$ equation we will introduce the relevant covariant derivative. Since the superconformal algebra contains a non-abelian generator, its gauge field a_{IJ} must be included. The full covariant derivative thus reads

$$\hat{D} = \mathrm{d} + \omega + a, \tag{7.27}$$

where ω is the supersymmetric spin connection from equation (7.25), including the contorsion. To determine how this derivative acts on fields carrying spinor indices we recall that the generators of the Lorentz group in the spinor representation[2] can be written as $S_{ab} = \frac{1}{4}\gamma_{ab}$. Consequently,

$$\begin{aligned}\hat{D}\chi_{\alpha I} &= \mathrm{d}\chi_{\alpha I} + \omega^{ab}(S_{ab})_\alpha{}^\beta \wedge \chi_{\beta I} + a_{JI} \wedge \chi_\alpha{}^J \\ &= \mathrm{d}\chi_{\alpha I} - \tfrac{1}{4}(\gamma_{ab})_\alpha{}^\beta \epsilon^{abc}\omega_c \wedge \chi_{\beta I} - a_{IJ} \wedge \chi_\alpha{}^J \\ &= \mathrm{d}\chi_{\alpha I} + \tfrac{1}{2}(\gamma^a)_\alpha{}^\beta \omega_a \wedge \chi_{\beta I} - a_{IJ} \wedge \chi_\alpha{}^J.\end{aligned} \tag{7.28}$$

With this in mind, the $(Q^{\alpha I})$ equation can be written as

$$\hat{D}_{[\mu}\chi_{\nu]\alpha I} = (\gamma_a)_{\alpha\beta} e_{[\mu}{}^a \psi_{\nu]}{}^\beta{}_I, \tag{7.29}$$

which implies

$$\epsilon_\rho{}^{\mu\nu}\hat{D}_\mu \chi_{\nu\alpha I} = \psi_{\rho\alpha I} - (\gamma_\rho)_\alpha{}^\gamma (\gamma^\nu)_{\gamma\beta}\psi_\nu{}^\beta{}_I, \tag{7.30}$$

where we have used the gamma relation (A.7). Upon multiplication of both sides by $(\gamma^\rho)_\delta{}^\alpha$ we find the γ-trace

$$(\gamma^\rho)_\delta{}^\alpha \psi_{\rho\alpha I} = -\frac{1}{2}\epsilon_\rho{}^{\mu\nu}(\gamma^\rho)_\delta{}^\alpha \hat{D}_\mu \chi_{\nu\alpha I}. \tag{7.31}$$

Insertion into equation (7.30) then yields

$$\epsilon_\rho{}^{\mu\nu}\hat{D}_\mu\chi_{\nu\alpha I} = \frac{1}{2}\epsilon_\sigma{}^{\mu\nu}(\gamma_\rho\gamma^\sigma)_\alpha{}^\beta \hat{D}_\mu \chi_{\nu\beta I} + \psi_{\rho\alpha I}. \tag{7.32}$$

By once again using the gamma relation (A.7) and solving for $\psi_{\rho\alpha I}$ we end up at

$$\psi_{\rho\alpha I} = \frac{1}{2}\big(\epsilon_\rho{}^{\mu\nu}\hat{D}_\mu\chi_{\nu\alpha I} + (\gamma^\mu)_\alpha{}^\beta \hat{D}_\mu \chi_{\rho\beta I} - (\gamma^\nu)_\alpha{}^\beta \hat{D}_\rho \chi_{\nu\beta I}\big), \tag{7.33}$$

which expresses the gravitino field ψ completely in terms of the gauge field χ corresponding to the Q-supersymmetry.

Precisely as in the bosonic case we note that the first two terms in the (M_a) equation constitute the curvature form R^a written in one-index form. It thus implies

$$\epsilon^{\rho\mu\nu}R_{\mu\nu}{}^a - 4\epsilon^{\rho\mu\nu}\epsilon^a{}_{bc}e_\mu{}^b f_\nu{}^c + 2i\epsilon^{\rho\mu\nu}\bar{\psi}_\mu\gamma^a\chi_\nu = 0. \tag{7.34}$$

Solving for the gauge field $f^{a\rho}$ this yields

$$f^{a\rho} = \frac{1}{4}\epsilon^{\rho\mu\nu}R_{\mu\nu}{}^a + e^{\rho a} f + i\epsilon^{\rho\mu\nu}\bar{\psi}_{\mu I}\gamma^a\chi_\nu{}^I. \tag{7.35}$$

By computing the trace of both sides we obtain

$$f = -\frac{1}{8}\epsilon_a{}^{\mu\nu}R_{\mu\nu}{}^a + \frac{i}{2}\epsilon^{\mu\nu\lambda}\bar{\psi}_{\mu I}\gamma_\nu\chi_\lambda{}^I, \tag{7.36}$$

[2]It is easy to verify that these generators obey the Lorentz algebra (4.23).

which inserted into equation (7.35) gives the gauge field

$$f^{a\rho} = \frac{1}{8}\left(2\epsilon^{\rho\mu\nu}R_{\mu\nu}{}^{a} - e^{\rho a}\epsilon_b{}^{\mu\nu}R_{\mu\nu}{}^{b}\right) + i\left(\epsilon^{\rho\mu\nu}\bar{\psi}_{\mu I}\gamma^a\chi_\nu{}^I + \frac{1}{2}e^{\rho a}\left(\epsilon^{\mu\nu\lambda}\bar{\psi}_{\mu I}\gamma_\nu\chi_\lambda{}^I\right)\right). \quad (7.37)$$

It will prove convenient to decompose the dual Riemann tensor $R_{\mu\nu}{}^{a}$ appearing here into its (bosonic) torsion-free and supersymmetric parts. By using equation (7.25) we can write the curvature form as

$$\begin{aligned} R^a &= \mathrm{d}\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b\wedge\omega^c = \mathrm{d}\big(\omega^a(e) + K^a\big) + \frac{1}{2}\big(\omega^b(e)+K^b\big)\wedge\big(\omega^c(e)+K^c\big) \\ &= \left(\mathrm{d}\omega^a(e) + \frac{1}{2}\epsilon^a{}_{bc}\omega^b(e)\wedge\omega^c(e)\right) + \big(\mathrm{d}K^a + \epsilon^a{}_{bc}\omega^b(e)\wedge K^c\big) + \frac{1}{2}\epsilon^a{}_{bc}K^b\wedge K^c \\ &= R^a(e) + D(e)K^a + \frac{1}{2}\epsilon^a{}_{bc}K^b\wedge K^c, \end{aligned} \quad (7.38)$$

where we have let $R^a(e)$ denote the purely bosonic curvature form and $D(e) = \mathrm{d} + \omega(e)$ is the purely bosonic covariant derivative. Then we can write

$$R_{\mu\nu}{}^{a} = R_{\mu\nu}{}^{a}(e) + 2D_{[\mu}(e)K_{\nu]}{}^{a} + \epsilon^a{}_{bc}K_\mu{}^bK_\nu{}^c, \quad (7.39)$$

which is the dual Riemann tensor that should be inserted into equation (7.37) for the gauge field $f^{a\rho}$. Note that all quantities appearing in the expression for $f^{a\rho}$ – *i.e.* the bosonic spin connection, the contorsion and the gravitino gauge field – have already been expressed in terms of the frame field and the gauge field $\chi_{\mu\alpha I}$ corresponding to the Q-supersymmetry. So what we have found is really a solution for $f^{a\rho}$.

The (D) equation simply yields

$$e_{[\mu}{}^{a}f_{\nu]a} = i\chi_{[\mu|\alpha I|}\psi_{\nu]}{}^{\alpha I}, \quad (7.40)$$

or equivalently

$$f_{[\mu\nu]} = -i\bar{\psi}_{[\mu}{}^{I}\chi_{\nu]I}. \quad (7.41)$$

Since we have already found the solutions for the gauge fields $f_{\mu\nu}$ and $\psi_{\mu\alpha I}$, in equations (7.37) and (7.33) respectively, this equation must precisely as in the bosonic case be an identity. To verify that this identity is really satisfied by our obtained solutions provides an excellent check of our previous calculations. Since this calculation is quite lengthy we will only perform the first steps here, and then leave the rest for appendix B.

Insertion of the dual Riemann tensor $R_{\mu\nu}{}^{a}$ into equation (7.37) yields

$$\begin{aligned} f^{\sigma\rho} =& \frac{1}{2}\Big(\frac{1}{2}\epsilon^{\rho\mu\nu}R_{\mu\nu}{}^{\sigma}(e) - \frac{1}{4}g^{\rho\sigma}\epsilon_b{}^{\mu\nu}R_{\mu\nu}{}^{b}(e) + \epsilon^{\rho\mu\nu}D_{[\mu}(e)K_{\nu]}{}^{\sigma} + \frac{1}{2}\epsilon^{\rho\mu\nu}\epsilon^\sigma{}_{bc}K_\mu{}^bK_\nu{}^c \\ &+ 2i\epsilon^{\rho\mu\nu}\bar{\psi}_\mu{}^I\gamma^\sigma\chi_{\nu I} + ig^{\rho\sigma}\epsilon^{\mu\nu\lambda}\bar{\psi}_\mu{}^I\gamma_\nu\chi_{\lambda I}\Big), \end{aligned} \quad (7.42)$$

where we have also multiplied by $e_a{}^\sigma$. The first two terms constitute the purely bosonic part which we, in accordance with equation (7.16), expect to equal $\frac{1}{2}S^{\sigma\rho}$ with $S^{\sigma\rho}$ being the bosonic Schouten tensor. This can easily be proven by noting that the relation $R_{\mu\nu}{}^{\sigma}(e) = \frac{1}{2}\epsilon^\sigma{}_{\lambda\eta}R_{\mu\nu}{}^{\lambda\eta}(e)$ implies

$$\epsilon^{\rho\mu\nu}R_{\mu\nu}{}^{\sigma}(e) = 2\epsilon^{\rho\mu\nu}\epsilon^\sigma{}_{\lambda\eta}\delta^\lambda_\mu S^\eta_\nu(e) = 2\big(S^{\sigma\rho}(e) - g^{\rho\sigma}S(e)\big), \quad (7.43)$$

where we have used equation (2.51) relating the Riemann tensor to the Schouten tensor. By computing the trace of both sides we obtain

$$S(e) = -\frac{1}{4}\epsilon_\sigma{}^{\mu\nu}R_{\mu\nu}{}^\sigma(e). \tag{7.44}$$

These results imply that the purely bosonic part of $f^{\sigma\rho}$ from equation (7.42) can be written as

$$\frac{1}{2}\left(\frac{1}{2}\epsilon^{\rho\mu\nu}R_{\mu\nu}{}^\sigma(e) - \frac{1}{4}g^{\rho\sigma}\epsilon_b{}^{\mu\nu}R_{\mu\nu}{}^b(e)\right) = \frac{1}{2}S^{\sigma\rho}(e), \tag{7.45}$$

just like we expected. Since we know that both the Schouten tensor and the metric are symmetric, the antisymmetric part of $f^{\sigma\rho}$ can be written as

$$f^{[\sigma\rho]} = \epsilon^{\mu\nu[\rho}\left(\frac{1}{2}D_\mu(e)K_\nu{}^{\sigma]} + \frac{1}{4}\epsilon^{\sigma]}{}_{bc}K_\mu{}^bK_\nu{}^c + i\bar{\psi}_{\mu I}\gamma^{\sigma]}\chi_\nu{}^I\right). \tag{7.46}$$

In appendix B we prove that this expression for $f^{[\sigma\rho]}$ satisfies the identity (7.41).

The (T^{IJ}) equation implies

$$i\chi_{[\mu}{}^{\alpha[I}\psi_{\nu]\alpha}{}^{J]} = -\frac{1}{4}\left(\partial_{[\mu}a_{\nu]}{}^{IJ} + a_{[\mu}{}^{[I|K|}a_{\nu]K}{}^{J]}\right). \tag{7.47}$$

Since a_{IJ} is a non-abelian gauge field it has an associated field strength tensor $F^{IJ}_{\mu\nu}$ on the form of equation (3.26), reading[3]

$$F^{IJ}_{\mu\nu} = 2\left(\partial_{[\mu}a_{\nu]}{}^{IJ} + a_{[\mu}{}^{[I|K|}a_{\nu]K}{}^{J]}\right). \tag{7.48}$$

Then we can simply write

$$i\chi_{[\mu}{}^{\alpha[I}\psi_{\nu]\alpha}{}^{J]} = -\frac{1}{8}F^{IJ}_{\mu\nu}, \tag{7.49}$$

which is the field equation for the non-abelian gauge field.

The $(S_\alpha{}^I)$ equation can be written as

$$\hat{D}_{[\mu}\psi_{\nu]}{}^\alpha{}_I + (\gamma_a)^{\alpha\beta}f_{[\mu}{}^a\chi_{\nu]\beta I} = 0. \tag{7.50}$$

This is the field equation for the spin $\frac{3}{2}$ gravitino field. Since it is the fermionic analogue to the spin 2 Cotton equation it is known as the *Cottino equation*. From the solution (7.33) for the gravitino field, we can conclude that the Cottino equation contains derivatives up to second order on the gauge field $\chi_{\mu\alpha I}$.

The (K_a) equation reads

$$\tilde{D}_{[\mu}f_{\nu]}{}^a - i(\gamma^a)_{\alpha\beta}\psi_{[\mu}{}^{\alpha I}\psi^\beta_{\nu]I} = 0, \tag{7.51}$$

where $\tilde{D} \equiv \mathrm{d} + \omega(e) + K$ is the full supersymmetric covariant derivative. Note that when the first two terms in this derivative act on the bosonic part of $f_\nu{}^a$ they recreate the LHS of the bosonic (K_a) equation (7.17), which upon multiplication by $\varepsilon_\sigma{}^{\mu\nu}e_\rho{}^a$ resulted

[3] *Cf.*, *e.g*, the gluon field strength tensor in QCD.

in the bosonic Cotton tensor $C_{\sigma\rho}(e)$. Since we yet have not coupled the field theory to matter, it must still be conformally flat. This implies that the LHS of equation (7.51), upon multiplication by $\varepsilon_\sigma{}^{\mu\nu} e_\rho{}^a$, must yield the supersymmetric Cotton tensor $C_{\sigma\rho}$. The (K_a) equation, thus, once again results in the Cotton equation $C_{\sigma\rho} = 0$, which is the field equation for the spin 2 gravitational field.

To summarize, the zero field strength equation resulted in three equations that could be solved to express the gauge fields $\omega_\mu{}^a$, $\psi_{\mu\alpha I}$ and $f_\mu{}^a$ in terms of a_{IJ} (appearing only in the covariant derivatives), $e_\mu{}^a$ and $\chi_{\mu\alpha I}$, one identity that these solutions satisfy and three field equations describing the dynamics of the theory.

Chapter 8

The Unfolded Equation

In the previous chapter we derived the field equations of our superconformal field theory by solving the zero field strength equation stemming from Chern-Simons theory. In this chapter we will instead employ the unfolded formulation of higher spin theory to describe the theory. The exact relation between these two approaches is still unknown. An understanding of this might be one of the keys in the process of utilizing the theory to gain new insights in string and M-theory via the AdS/CFT correspondence.

The initial part of this chapter will be devoted to establishing the setup of the unfolded formalism. In particular, we will give the explicit form of the unfolded equation in the context of superconformal field theories. Then we will perform the explicit unfolding of the bosonic spin 2 theory. By continuously comparing the obtained results with those from section 7.2 stemming from the zero field strength equation we will illustrate that the two approaches yield equivalent result in this case, something that was first observed in [14]. Finally, we will explicitly unfold the spin 2 truncation of the full supersymmetric theory.

8.1 The Unfolded Setup

From equation (5.4) we know that the zero-form master field appearing in the unfolded equation (5.5) can be expanded in terms of the $\mathrm{SL}(2,\mathbb{R})$ spinors p_α as

$$\Phi(x) = \sum_{n=0}^{\infty} \phi^{\alpha_1 \dots \alpha_n}(x) p_{\alpha_1} \dots p_{\alpha_n}. \tag{8.1}$$

In chapter 6 we gave two possible interpretations of p_α, either as the phase-space variable of conjugate momentum or as the momentum operator. We also expressed the generator of special conformal transformations as $K_a = -\frac{1}{2}(\gamma_a)^{\alpha\beta} p_\alpha p_\beta$. This indicates that $\Phi(x)$ can instead be expanded in terms of products of the generators K_a. However, since we will employ the operator formulation of the algebra, $\Phi(x)$ must act on a state. We let the operators p_α take the role of creation operators and introduce a vacuum $|0\rangle$ that is excited by them. The other $\mathrm{SL}(2,\mathbb{R})$ operator q^α must then correspond to the operators annihilating the vacuum, which we consequently denote by $|0\rangle_q$[1].

Following this analysis we can write the bosonic part of $\Phi(x)$, *i.e.* the terms containing an even number of p_α operators, as

$$\tilde{\Phi}(x)\,|0\rangle_q = \sum_{n=0}^{\infty}(-i)^n \phi^{a_1\ldots a_n}(x) K_{a_1\ldots a_n}\,|0\rangle_q\,, \tag{8.2}$$

where $K_{a_1\ldots a_n} \equiv K_{a_1}\ldots K_{a_n}$ and we have employed a different normalization. Similarly, we can write the fermionic part of $\Phi(x)$, *i.e.* the terms containing an odd number of p_α operators, as

$$\varphi(x)\,|0\rangle_q = \sum_{n=0}(-i)^n \varphi^{\alpha a_1\ldots a_n}(x) K_{a_1\ldots a_n} p_\alpha\,|0\rangle_q\,. \tag{8.3}$$

Note that the fields on the RHS will always carry a spinor index α, indicating that they are fermionic.

Although this decomposition of the master field $\Phi(x)$ into its bosonic and fermionic parts is conceptually correct, we need to understand it in more detail to be able to solve the unfolded equation. In particular the role played by the vacuum state will prove to be of great importance. We have already introduced a vacuum state $|0\rangle_q$ and argued that it should be excited by the p_α operators and annihilated by the q^α operators. However, when we in section 6.2 constructed the superconformal algebra we also had to introduce the generators $T^{IJ} = \frac{i}{4}\lambda^{[I}\lambda^{J]}$ of the $\mathrm{SO}(N)$ R-symmetry, with N being the number of supersymmetries of the theory. Also this symmetry must have associated creation and annihilation operators which we will denote $a_i^\dagger$ and a^i, respectively, where $i = 1,\ldots,\frac{N}{2}$. Assuming[2] $N = 8$, the complete decomposition of $\Phi(x)$ must thus be of the form

$$\begin{aligned}\Phi(x)\,|0\rangle_{q,a} = \Big[&\big(\phi + \phi^{ij}a_i^\dagger a_j^\dagger + \tilde{\phi}a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger\big) + \big(\varphi^{\alpha i}a_i^\dagger + \varphi^{\alpha ijk}a_i^\dagger a_j^\dagger a_k^\dagger\big)p_\alpha \\ &-i\big(\phi^a + \phi^{aij}a_i^\dagger a_j^\dagger + \tilde{\phi}^a a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger\big)K_a - i\big(\varphi^{\alpha a i}a_i^\dagger + \varphi^{\alpha a ijk}a_i^\dagger a_j^\dagger a_k^\dagger\big)K_a p_\alpha \\ &+\ldots\Big]\,|0\rangle_{q,a}\,,\end{aligned} \tag{8.4}$$

where we have introduced a vacuum state $|0\rangle_{q,a}$ that is annihilated by both q^α and a^i and the spacetime dependence of the fields has not been written out explicitly.

It is now convenient to collect the bosonic and the fermionic states that are annihilated by the operator q^α into one vector each. We name them $|S\rangle_A$ and $|C\rangle_{\dot{A}}$, respectively, and assuming that $|0\rangle_{q,a}$ is bosonic they must read

$$|S\rangle_A \equiv \begin{pmatrix} |0\rangle_{q,a} \\ a_i^\dagger a_j^\dagger\,|0\rangle_{q,a} \\ a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger\,|0\rangle_{q,a}\end{pmatrix}, \qquad |C\rangle_{\dot{A}} \equiv \begin{pmatrix} a_i^\dagger\,|0\rangle_{q,a} \\ a_i^\dagger a_j^\dagger a_k^\dagger\,|0\rangle_{q,a}\end{pmatrix}. \tag{8.5}$$

[1] To be completely rigorous we should *not* regard this vacuum state as a Fock vacuum but as an eigenstate with eigenvalue zero, meaning $|0\rangle_q \equiv |q=0\rangle$, a description well-known from ordinary quantum mechanics.

[2] Such theories are known as BLG theories and were, in the context of M-theory, introduced independently by Bagger and Lambert in [42, 43] and by Gustavsson in [44].

Although these vectors are only annihilated by q^α, and *not* by a^i, we will refer to them as the $|S\rangle_A$ and $|C\rangle_{\dot{A}}$-vacua. Since the indices labelling these vacua are in different representations of SO(8) (the so-called *chiral* irreducible representations $\mathbf{8_s}$ and $\mathbf{8_c}$ associated with bosonic and fermionic fields, respectively) we have put a distinguishing dot on one of them. Note, however, that $|S\rangle_A$ and $|C\rangle_{\dot{A}}$ both contain 8 states. We can now rewrite the expansion of $\Phi(x)$ from equation (8.4) as

$$\Phi(x)\,|0\rangle_q = \sum_{n=0}^{\infty}(-i)^n\Big(\phi^{Aa_1\dots a_n}(x)K_{a_1\dots a_n}\,|S\rangle_A + \varphi^{\dot{A}\alpha a_1\dots a_n}(x)K_{a_1\dots a_n}p_\alpha\,|C\rangle_{\dot{A}}\Big), \tag{8.6}$$

which we note resembles the expansions from equations (8.2) and (8.3) a lot. This is the expansion of the master field $\Phi(x)$ we will use in the unfolded equation.

In section 6.3 we identified the higher spin generators $K_{a_1\dots a_n}$ as totally traceless and symmetric in all pairs of the indices $a_1,\dots,a_n$ (meaning they are irreducible representations of $\mathrm{sl}(2,\mathbb{R})$). The same must consequently hold both for the bosonic fields $\phi^{Aa_1\dots a_n}$ and the fermionic fields $\varphi^{\dot{A}\alpha a_1\dots a_n}$. Note, however, that also the operators $K_{a_1\dots a_n}p_\alpha$ must be irreducible representations. This implies that also the gamma traces of the fermionic fields must vanish, meaning

$$(\gamma_{a_1})_{\beta\alpha}\varphi^{\dot{A}\alpha a_1\dots a_n} = 0. \tag{8.7}$$

These are all properties that will be frequently used when the explicit unfolding is to be performed.

The possibility of representing the three-dimensional conformal group SO(3, 2) as the direct sum of one fermionic and one bosonic representation was first discovered by Dirac in 1963 [45], and is referred to as the *singleton* representation. In the context of field theory, this corresponds to decomposing the massless master field $\Phi(x)\,|0\rangle$ into the direct sum of a spinorial master field (the second term in equation (8.6)) and a scalar master field (the first term). Cleverly enough, the former is known as the *Rac*-singleton $Rac^A\,|S\rangle_A$ and the latter as the *Di*-singleton $Di^{\dot{A}}\,|C\rangle_{\dot{A}}$, so the decomposition can be written as $\Phi(x)\,|0\rangle = Di \oplus Rac$.

8.1.1 Explicit Construction of the Singleton Representation

In the previous section we decomposed the master field $\Phi(x)\,|0\rangle_{q,a}$ into the Dirac singleton representation by describing the SO(N) R-symmetry in terms of abstract annihilation and creation operators a^i and $a_i^\dagger$, where $i = 1,\dots,\frac{N}{2}$. In this section we will illustrate how this is done explicitly in terms of the operators λ^I that we in section 6.2 used to construct the SO(N) generators T^{IJ}. In this process we will understand how both λ^I and T^{IJ} act on the two singleton vacua $|S\rangle_A$ and $|C\rangle_{\dot{A}}$. This will be absolutely necessary to know when we in section 8.3 want to solve the supersymmetric unfolded equation.

Since the operators λ^I in accordance with equation (6.17) satisfy the Clifford algebra, they can be represented as the 16×16 dimensional gamma matrices of SO(8), which we will denote Γ^I (we keep studying BLG theories, having $N = 8$). A convenient way to find a representation of these is to construct direct products of the three-dimensional gamma matrices listed in appendix A. Assuming we want Γ^I to be real and off-diagonal matrices we can use the construction

$$\begin{aligned} \Gamma^1 &= \gamma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, & \Gamma^5 &= \gamma^0 \otimes \mathbb{1} \otimes \gamma^1 \otimes \gamma^0, \\ \Gamma^2 &= \gamma^0 \otimes \gamma^0 \otimes \gamma^0 \otimes \gamma^0, & \Gamma^6 &= \gamma^0 \otimes \mathbb{1} \otimes \gamma^2 \otimes \gamma^0, \\ \Gamma^3 &= \gamma^0 \otimes \gamma^1 \otimes \gamma^0 \otimes \mathbb{1}, & \Gamma^7 &= \gamma^0 \otimes \gamma^0 \otimes \mathbb{1} \otimes \gamma^1, \\ \Gamma^4 &= \gamma^0 \otimes \gamma^2 \otimes \gamma^0 \otimes \mathbb{1}, & \Gamma^8 &= \gamma^0 \otimes \gamma^0 \otimes \mathbb{1} \otimes \gamma^2. \end{aligned} \tag{8.8}$$

By studying the representations of γ^μ we see that $\Gamma^1 = \left(\begin{smallmatrix} 0 & \mathbb{1}_8 \\ \mathbb{1}_8 & 0 \end{smallmatrix}\right)$. The remaining Γ^I have γ^0 as the leftmost gamma matrix in their direct products. This implies that all Γ^I can be written in the form

$$\Gamma^I = \begin{pmatrix} 0 & \sigma^I \\ \bar{\sigma}^I & 0 \end{pmatrix}, \tag{8.9}$$

where $\sigma^I = (\mathbb{1}, \sigma^{\tilde{I}})$ and $\bar{\sigma}^I = (\mathbb{1}, -\sigma^{\tilde{I}})$ for some 8×8 matrices $\sigma^{\tilde{I}}$, $\tilde{I} = 2, \ldots, 8$.

Knowing that the operators λ^I can be represented as gamma matrices Γ^I, the method for constructing the creation and annihilation operators $a_i^\dagger$ and a^i is well-known. We simply let

$$a_i^\dagger = \frac{1}{2}\big(\lambda_{2i-1} - i\lambda_{2i}\big), \qquad a^i = \frac{1}{2}\big(\lambda_{2i-1} + i\lambda_{2i}\big), \tag{8.10}$$

where $i = 1, \ldots, \frac{N}{2}$. It is trivial to verify that these operators satisfy the commutation relations of creation and annihilation operators, *i.e.*,

$$\begin{aligned} \{a_i^\dagger, a^j\} &= \delta_i^j \\ \{a_i^\dagger, a_j^\dagger\} = \{a^i, a^j\} &= 0. \end{aligned} \tag{8.11}$$

By inverting the relations in equation (8.10) we find that the operators λ^I can be written as

$$\lambda_{2i-1} = a^i + a_i^\dagger, \qquad \lambda_{2i} = i(a_i^\dagger - a^i), \tag{8.12}$$

for $i = 1, \ldots, \frac{N}{2}$. When working with this representation we should modify the extended star product from equation (6.30) to contain the creation and annihilation operators instead of λ.

In the previous section we decomposed $|0\rangle_q$ – *i.e.*, the set of states created by powers of the creation operators $a_i^\dagger$ acting on the vacuum state $|0\rangle_{q,a}$ – into a direct sum of a bosonic vacuum $|S\rangle_A$ and a fermionic vacuum $|C\rangle_{\dot{A}}$. This means it can be written on the form[3]

$$|0\rangle_q = \begin{pmatrix} |S\rangle_A \\ |C\rangle_{\dot{A}} \end{pmatrix}. \tag{8.13}$$

By acting with the operators from equation (8.12) on this state, one easily verifies that

$$\lambda^I \begin{pmatrix} |S\rangle_A \\ |C\rangle_{\dot{A}} \end{pmatrix} = \Gamma^I \begin{pmatrix} |S\rangle_A \\ |C\rangle_{\dot{A}} \end{pmatrix}, \tag{8.14}$$

[3]Note that the notation in equation (8.6) thus is a little bit sloppy. Instead of writing $|S\rangle_A$ and $|C\rangle_{\dot{A}}$ on the RHS we should really have written the vectors $\left(\begin{smallmatrix} |S\rangle_A \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ |C\rangle_{\dot{A}} \end{smallmatrix}\right)$, respectively. This subtlety is henceforth implicitly understood.

with Γ^I being the gamma matrix representations from equation (8.8) (at least up to uninteresting multiplicative factors). By using that the matrices Γ^I can be written on the form in equation (8.9) we can write

$$\lambda^I \begin{pmatrix} |S\rangle_A \\ |C\rangle_{\dot{A}} \end{pmatrix} = \begin{pmatrix} 0 & (\sigma^I)_A{}^{\dot{A}} \\ (\bar{\sigma}^I)_{\dot{A}}{}^A & 0 \end{pmatrix} \begin{pmatrix} |S\rangle_A \\ |C\rangle_{\dot{A}} \end{pmatrix} = \begin{pmatrix} (\sigma^I)_A{}^{\dot{A}} |C\rangle_{\dot{A}} \\ (\bar{\sigma}^I)_{\dot{A}}{}^A |S\rangle_A \end{pmatrix}, \tag{8.15}$$

which tells us how the operators λ^I act on the different vacua. This way of representing the gamma matrices is generally referred to as the Weyl (chiral) representation. There, the two subrepresentations are often referred to as the S (positive chirality) and C (negative chirality) representations, which explains our naming of the vacua.

We also want to know how the SO(N) generators $T^{IJ} \equiv \frac{i}{4}\lambda^{[I}\lambda^{J]}$ look in this representation. By definition they can be represented as the matrices

$$\frac{i}{4}\Gamma^{[I}\Gamma^{J]} = \frac{i}{4}\begin{pmatrix} \sigma^{[I}\bar{\sigma}^{J]} & 0 \\ 0 & \bar{\sigma}^{[I}\sigma^{J]} \end{pmatrix} = \frac{i}{4}\begin{pmatrix} \sigma^{IJ} & 0 \\ 0 & \bar{\sigma}^{IJ} \end{pmatrix}, \tag{8.16}$$

where the last equality is customary notation. When T^{IJ} acts on the vacuum we obtain

$$T^{IJ} \begin{pmatrix} |S\rangle_A \\ |C\rangle_{\dot{A}} \end{pmatrix} = \frac{i}{4}\begin{pmatrix} (\sigma^{IJ})_A{}^B |S\rangle_B \\ (\bar{\sigma}^{IJ})_{\dot{A}}{}^{\dot{B}} |C\rangle_{\dot{B}} \end{pmatrix}, \tag{8.17}$$

which will be a useful fact when we in section 8.3 set out to solve the supersymmetric unfolded equation.

8.2 Unfolding the Bosonic Equation

To illustrate the explicit unfolding procedure we will begin by solving the significantly less complicated bosonic unfolded equation, truncated at the spin 2 level. Since we in section 6.3 showed that the spin 2 algebra is closed, this is a consistent truncation. In addition to showing how the field equations can be extracted, we will continuously compare the obtained results with those stemming from the bosonic zero field strength equation, which was studied in section 7.2. By doing so we will be able to show that the two approaches yield completely equivalent results in this case.

The source-free, bosonic unfolded equation reads

$$\mathcal{D}\Phi(x)\,|0\rangle_q = 0, \tag{8.18}$$

where the master field $\Phi(x)$ can be expanded as in equation (8.2), *i.e.*,

$$\Phi(x)\,|0\rangle_q = \sum_{n=0}^{\infty}(-i)^n \phi^{a_1\ldots a_n}(x) K_{a_1\ldots a_n}\,|0\rangle_q\,. \tag{8.19}$$

The exterior covariant derivative is given by $\mathcal{D} = \mathrm{d} + A$ with

$$A = \sum_{n=1}^{\infty}(-i)^n A_n, \tag{8.20}$$

where the m-th term contains the spin $s = m + 1$ generators

$$A_m = e^{a_1 \dots a_m} P_{a_1 \dots a_m} + \omega^{a_1 \dots a_m} M_{a_1 \dots a_m} + f^{a_1 \dots a_m} K_{a_1 \dots a_m}, \tag{8.21}$$

and we have implemented the gauge choice $b = 0$ that was discussed in section 7.1.1.

The LHS of the unfolded equation (8.18) thus contains terms of all spins s and all levels n of the generators $K_{a_1 \dots a_n}$ up to infinity. We will truncate it at spin $s = 2$ and level $n = 2$. Due to the former we can set $A = (-i)A_1$ and due to the latter we can truncate the expansion (8.19) of the master field after the $n = 3$ term[4]. Hence we can write the LHS of the unfolded equation as

$$\mathcal{D}\Phi(x) \left|0\right\rangle_q = \left[\left(\mathrm{d} - iA_1 \right) \sum_{n=0}^{3} (-i)^n \phi^{a_1 \dots a_n} K_{a_1 \dots a_n} \right] \left|0\right\rangle_q + \mathcal{O}(n = 3, s = 3). \tag{8.22}$$

To explicitly unfold this expression into respective level $n = 0, 1, 2$, we first need to investigate how the different generators act on the vacuum. We find that

$$P_a \left|0\right\rangle_q = -\frac{1}{2} (\gamma_a)_{\alpha\beta} q^\alpha q^\beta \left|0\right\rangle_q = 0, \tag{8.23}$$

$$M_a \left|0\right\rangle_q = -\frac{1}{2} (\gamma_a)_\alpha{}^\beta q^\alpha p_\beta \left|0\right\rangle_q = -\frac{1}{2} (\gamma_a)_\alpha{}^\beta [q^\alpha, p_\beta] \left|0\right\rangle_q = -\frac{i}{2} (\gamma_a)_\alpha{}^\alpha \left|0\right\rangle_q = 0, \tag{8.24}$$

$$D \left|0\right\rangle_q = -\frac{1}{4} (q^\alpha p_\alpha + p_\alpha q^\alpha) \left|0\right\rangle_q = -\frac{i}{4} \delta^\alpha_\alpha \left|0\right\rangle_q = -\frac{i}{2} \left|0\right\rangle_q, \tag{8.25}$$

meaning that the vacuum is translationally and rotationally invariant but not scale invariant, which is a reasonable result.

To derive the level $n = 0, 1$ and 2 equations is now simply a matter of inserting the gauge potential A_1 into equation (8.22) and using the commutation relations of the conformal algebra (4.26) to eliminate the generators appearing in addition to $K_{a_1 \dots a_n}$. At level $n = 0$ we get the contribution $\mathrm{d}\phi(x) \left|0\right\rangle_q$ from the exterior derivative, but also

$$- e^b \phi^a P_b K_a \left|0\right\rangle_q = 2ie^b \phi^a (\eta_{ab} D + \epsilon_{ab}{}^c M_c) \left|0\right\rangle_q = e^a \phi_a \left|0\right\rangle_q \tag{8.26}$$

from the term $-A_1 \phi^a K_a$ in the expansion. By writing out the form-index explicitly, the level $n = 0$ unfolded equation can thus be written as

$$\left(\partial_\mu \phi + \phi_\mu + \mathcal{O}(s = 3) \right) \left|0\right\rangle_q = 0, \tag{8.27}$$

where $\mathcal{O}(s = 3)$ denotes all terms of spin 3 and higher.

At level $n = 1$ we have, in addition to the derivative $(-i)(\mathrm{d}\phi^a)K_a \left|0\right\rangle_q$, contributions from the first three potential terms in equation (8.22). The first one yields the contribution $(-i) f^a \phi K_a \left|0\right\rangle_q$, the second one yields

$$(-i)^2 \omega^b \phi^a M_b K_a \left|0\right\rangle_q = -i\epsilon_{ab}{}^c \omega^a \phi^b K_c \left|0\right\rangle_q, \tag{8.28}$$

[4]Recall that we want to keep terms of order K_{ab} in the generators *after* having applied the covariant derivative, and the generators of A_1 appearing in this derivative may lower this level at most one step.

and the third one yields

$$\begin{aligned}(-i)^3\phi^{ab}e^cP_cK_aK_b\left|0\right\rangle_q &= i\phi^{ab}e^c\big(K_aP_c - 2i(\epsilon_{ca}{}^dM^d + \eta_{ac}D)\big)K_b\left|0\right\rangle_q \\ &= 2\phi^{ab}e^c\big(K_a\eta_{bc}D + i\epsilon_{ac}{}^d\epsilon_{db}{}^eK_e - \frac{i}{2}\eta_{ac}K_b - i\eta_{ac}K_b\big)\left|0\right\rangle_q \\ &= -i\big(\phi^{ab}e_bK_a + 3\phi^{ab}e_aK_b + 2\phi^{ab}e_bK_a - 2\phi_a{}^ae^bK_b\big)\left|0\right\rangle_q, \end{aligned} \tag{8.29}$$

where we have used equation (8.25). By now using our observation that the flat indices are in symmetrized traceless representations, which implies that $\phi^{ab} = \phi^{ba}$ and $\phi_a{}^a = 0$, this can be written as

$$i\phi^{ab}e^cP_cK_aK_b\left|0\right\rangle_q = -6i\phi^{ab}e_bK_a\left|0\right\rangle_q. \tag{8.30}$$

By adding up all contributions, also writing out the form-index, we find the level $n = 1$ equation to read

$$\big(D_\mu\phi^a + f_\mu{}^a\phi + 6\phi_\mu{}^a + \mathcal{O}(s=3)\big)K_a\left|0\right\rangle_q = 0. \tag{8.31}$$

Here we have combined the partial derivative and the contribution from equation (8.28) into the covariant derivative $D = d + \omega$ containing only the spin 2 connection, used extensively previously in this thesis (see, *e.g*, equation (7.10)).

Also at level $n = 2$, there are three potential terms contributing. The first contribution reads $(-i)^2f^bK_b\phi^aK_a\left|0\right\rangle_q = -f^a\phi^bK_{ab}\left|0\right\rangle_q$ and the second one reads

$$(-i)^3\omega^cM_c\phi^{ab}K_{ab}\left|0\right\rangle_q = i\omega^c\phi^{ab}(K_aM_c + i\epsilon_{ca}{}^dK_d)K_b\left|0\right\rangle_q = 2\epsilon_{ac}{}^d\omega^c\phi^{ab}K_{bd}\left|0\right\rangle_q, \tag{8.32}$$

which we note is the spin connection part of the derivative term $(-i)^2(D\phi^{ab})K_{ab}\left|0\right\rangle_q$. Finally, from the term $(-i)^4A_1\phi^{abc}K_{abc}\left|0\right\rangle_q$ we get the contribution

$$\begin{aligned}\phi^{abc}e^dP_dK_{abc}\left|0\right\rangle_q &= \phi^{abc}e^d(K_aP_d - 2i\epsilon_{da}{}^eM_e - 2i\eta_{ad}D)K_bK_c\left|0\right\rangle_q \\ &= \phi^{abc}e^d\big(K_aK_bP_d - 2iK_a\epsilon_{db}{}^eM_e - 2iK_a\eta_{bd}D - 2i\epsilon_{da}{}^eK_bM_e \\ &\quad + 2\epsilon_{da}{}^e\epsilon_{eb}{}^fK_f - 2i\eta_{ad}K_bD - 2\eta_{ad}K_b\big)K_c\left|0\right\rangle_q \\ &= -\phi^{ab}{}_c\big(e^cK_aK_b + 4\delta^{ce}_{db}e^dK_aK_e + 3e_bK_aK^c + 4\delta^{ce}_{da}e^dK_bK_e \\ &\quad + 2e_bK_aK^c + 5e_aK_bK^c\big)\left|0\right\rangle_q \\ &= -15\phi^{abc}e_cK_{ab}\left|0\right\rangle_q, \end{aligned} \tag{8.33}$$

where we again have used that the flat indices are in the symmetrized traceless representation. Adding up the different contributions we find the level $n = 2$ unfolded equation to read

$$\big(D_\mu\phi^{ab} + f_\mu{}^a\phi^b + 15\phi_\mu{}^{ab} + \mathcal{O}(s=3)\big)K_{ab}\left|0\right\rangle_q = 0. \tag{8.34}$$

Here we need to be careful, we *cannot* just set the expression in parentheses to zero. While the scalar fields $\phi^{a_1\cdots a_n}$ are symmetric and traceless in the flat indices, this is *not* the case for the term $f_\mu{}^a\phi^b$. Hence, if we want to eliminate the generator K_{ab} from the equation (which imposes the a and b indices to be in the symmetrized traceless representation), we must only include the symmetric and traceless part of $f_\mu{}^a\phi^b$, which reads $f_\mu{}^{(a}\phi^{b)} - \frac{1}{3}\eta^{ab}f_{\mu c}\phi^c$.

We have now derived the unfolded equations up to spin 2 at level $n = 0$, 1 and 2. For clarity, we repeat our results:

$$n = 0: \quad \partial_\mu \phi + \phi_\mu + \mathcal{O}(s=3) = 0, \tag{8.35}$$

$$n = 1: \quad D_\mu \phi^a + f_\mu{}^a \phi + 6\phi_\mu{}^a + \mathcal{O}(s=3) = 0, \tag{8.36}$$

$$n = 2: \quad D_\mu \phi^{ab} + f_\mu{}^{(a}\phi^{b)} - \frac{1}{3}\eta^{ab} f_{\mu c}\phi^c + 15\phi_\mu{}^{ab} + \mathcal{O}(s=3) = 0. \tag{8.37}$$

To solve this set of equations we will decompose each of them into its constituent irreducible representations.

8.2.1 The Decomposition into Irreducible Representations

Since the $n = 0$ unfolded equation only carries one index, it is already irreducible. The level $n \geq 1$ equations can, however, be split further into three irreducible parts: a trace, an antisymmetric part and a traceless symmetric part[5]. We will refer to these as the n^-, n^0 and n^+ equations, respectively. To reach the n^- equations we need, for instance, to multiply the original equations by the frame field $e_a{}^\mu$. Note that we thanks to the vielbein postulate (2.33) can let $e_a{}^\mu$ pass the covariant derivative without any correctional term.

For $n = 1$ we then find the irreducible equations

$$n = 1^-: \quad D_\mu \phi^\mu + f_\mu{}^\mu \phi = 0, \tag{8.38}$$

$$n = 1^0: \quad \epsilon^{\mu\nu a}(D_\mu \phi_\nu + f_{\mu\nu}\phi) = 0, \tag{8.39}$$

$$n = 1^+: \quad D_{(\mu}\phi_{\nu)} + f_{(\mu\nu)}\phi + 6\phi_{\mu\nu} = 0, \tag{8.40}$$

where we have omitted the higher spin terms. By using that the level $n = 0$ equation (8.35) implies $\phi_\mu = -\partial_\mu \phi$, these can be rewritten as

$$n = 1^-: \quad -\Box\phi + f_\mu{}^\mu \phi = 0, \tag{8.41}$$

$$n = 1^0: \quad \epsilon^{\mu\nu a} f_{\mu\nu}\phi = 0, \tag{8.42}$$

$$n = 1^+: \quad -D_{(\mu}\partial_{\nu)}\phi + f_{(\mu\nu)}\phi + 6\phi_{\mu\nu} = 0. \tag{8.43}$$

The $n = 1^-$ is the Klein-Gordon equation in curved spacetime. When we in a moment will have found the solution for the gauge field $f_{\mu\nu}$ (from the $n = 2^-$ equation), we will see how it looks explicitly. The $n = 1^0$ equation states that $f_{[\mu\nu]} = 0$, *i.e.*, that the gauge field $f_{\mu\nu}$ is symmetric. This agrees with what we found from the zero field strength equation in section 7.2. The $n = 1^+$ can be solved for the field $\phi_{\mu\nu}$, yielding

$$\phi_{\mu\nu} = \frac{1}{6}\big(D_{(\mu}\partial_{\nu)}\phi - f_{(\mu\nu)}\phi\big), \tag{8.44}$$

which can be used to solve the $n = 2$ equations.

[5]This decomposition should be implemented for the curved and one of the flat indices; we already know that the flat indices are in the symmetrized traceless representation in these equations.

Similarly, we find the irreducible representations of the $n = 2$ equation (8.37) to read

$$n = 2^- : \quad D_\mu \phi^{\mu a} + \frac{1}{2} f_\mu{}^\mu \phi^a + \frac{1}{6} f^{ab} \phi_b = 0, \tag{8.45}$$

$$n = 2^0 : \quad \epsilon^{\mu\nu(a} \left(D_\mu \phi_\nu{}^{b)} + \frac{1}{2} f_\mu{}^{b)} \phi_\nu \right) = 0, \tag{8.46}$$

$$n = 2^+ : \quad D_{(\mu} \phi_{\nu\rho)} + f_{(\mu\nu} \phi_{\rho)} + 15 \phi_{\mu\nu\rho} = 0, \tag{8.47}$$

where we have repeatedly used that the $n = 1^0$ equation implied $f_{\mu\nu}$ to be symmetric. We will now solve the $n = 2^-$ equation. By inserting the expression (8.44) for $\phi_{\mu\nu}$ stemming from the $n = 1^+$ equation, it can be written as

$$\Box \partial_\mu \phi - \frac{1}{3} D_\mu \Box \phi - (D_\nu f_\mu{}^\nu) \phi - 2 f_\mu{}^\nu \partial_\nu \phi + \frac{1}{3} (D_\mu f_\nu{}^\nu) \phi - \frac{8}{3} f_\nu{}^\nu \partial_\mu \phi = 0. \tag{8.48}$$

The first term can be rewritten as

$$\Box \partial_\mu \phi = D_\mu \Box \phi + [\Box, D_\mu] \phi = D_\mu \Box \phi + R^\nu{}_{\mu\nu}{}^\rho D_\rho \phi = D_\mu \Box \phi + R_\mu{}^\nu \partial_\nu \phi, \tag{8.49}$$

where we have used the Ricci identity (2.44) and that the Riemann curvature tensor acting on the scalar field ϕ yields zero. We can then use the Klein-Gordon equation (8.38) to write

$$D_\mu \Box \phi = D_\mu (f_\nu{}^\nu \phi) = (D_\mu f_\nu{}^\nu) \phi + f_\nu{}^\nu D_\mu \phi. \tag{8.50}$$

Insertion of these results into equation (8.48) yields

$$(R_{\mu\nu} - 2 f_{\mu\nu} - 2 f_\rho{}^\rho g_{\mu\nu}) \partial^\nu \phi - (D_\nu f_\mu{}^\nu - D_\mu f_\nu{}^\nu) \phi = 0. \tag{8.51}$$

The solution to this equation is simply

$$f_{\mu\nu} = \frac{1}{2} S_{\mu\nu} = \frac{1}{2} \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right), \tag{8.52}$$

since both expressions in parentheses then vanish. The first one vanishes directly upon insertion and the second one after implementation of the relation in equation (2.47), stemming from the Bianchi identity. Note that this is the exact same solution for $f_{\mu\nu}$ as the one we found in equation (7.16), by solving the zero field strength equation.

By inserting the solution for $f_{\mu\nu}$ into the Klein-Gordon equation (8.41) we find it to read

$$\Box \phi - \frac{1}{8} R \phi = 0, \tag{8.53}$$

which is the correct form of the Klein-Gordon equation in curved spacetime [14]. This indicates that the unfolded formulation we used when constructing the scalar master field was really correct. We also want to insert the solution for $f_{\mu\nu}$ into the $n = 2^0$ equation (8.46). By first using the result $\phi_\mu = -\partial_\mu \phi$ from the $n = 0$ equation and the solution (8.44) for $\phi_{\mu\nu}$ from the $n = 1^+$ equation, we can write the $n = 2^0$ equation as

$$\epsilon^{\mu\nu(a} \big[D_\mu D_\nu \partial^{b)} \phi - (D_\mu f_\nu{}^{b)}) \phi - 2 f_\mu{}^{b)} \partial_\nu \phi \big] = 0. \tag{8.54}$$

We now note that the Ricci identity and equation (2.50) for the Riemann tensor, can be used to write

$$\epsilon^{\mu\nu(\rho}D_\mu D_\nu \partial^{\sigma)}\phi = \frac{1}{2}\epsilon^{\mu\nu(\rho}R_{\mu\nu}{}^{\sigma)\lambda}\partial_\lambda\phi = \epsilon^{\nu\lambda(\rho}R_\nu{}^{\sigma)}\partial_\lambda\phi, \tag{8.55}$$

which can be applied to the first term above. We then find the $n = 2^0$ equation to read

$$\epsilon^{\mu\nu(a}\big[R_\mu{}^{b)}\partial_\nu\phi - (D_\mu f_\nu{}^{b)})\phi - 2f_\mu{}^{b)}\partial_\nu\phi\big] = 0. \tag{8.56}$$

By finally inserting the solution (8.52) for $f_{\mu\nu}$ this yields

$$\epsilon^{\mu\nu(a}D_\mu R_\nu{}^{b)}\phi = 0. \tag{8.57}$$

Note that the LHS is precisely $C^{ab}\phi$, with the Cotton tensor C^{ab} written on the form of equation (2.56). Hence we have once again obtained the Cotton equation, just like we did by solving the zero field strength equation.

To summarize we have now showed that the zero field strength equation and the unfolded equation yield equivalent results for the purely bosonic theory, the solution $f_{\mu\nu} = \frac{1}{2}S_{\mu\nu}$ for one of the gauge fields and the Cotton equation as the field equation. The Klein-Gordon equation we found from the unfolded $n = 1^-$ equation only contains information on the scalar field that is introduced together with the unfolded formalism. Since it neither contains dynamics for the higher spin fields nor couples them to the scalar field, it carries no physical information on the higher spin theory itself.

8.3 Unfolding the Supersymmetric Equation

To solve the supersymmetric unfolded equation, we will use the singleton expansion constructed in equation (8.6) as our master field $\Phi(x)\left|0\right\rangle_q$. The spin 2 gauge potential appearing in the covariant derivative $\mathcal{D} = \mathrm{d} + A$ is now

$$A_1 = e_a P^a + \omega_a M^a + f_a K^a + a_{IJ}T^{IJ} + i(\chi_{\alpha I}Q^{\alpha I} + \psi^\alpha{}_I S_\alpha{}^I), \tag{8.58}$$

which is equation (7.6) with the gauge choice $b = 0$ implemented. By once again truncating after level $n = 2$ and spin $s = 2$[6], we find the contributing terms on the LHS of the unfolded equation to read

$$\begin{aligned}\mathcal{D}\Phi(x)\left|0\right\rangle_q =& (\mathrm{d} - iA_1)\Big[\big(\phi^A - i\phi^{Aa}K_a + (-i)^2\phi^{Aab}K_{ab} + (-i)^3\phi^{Aabc}K_{abc}\big)\left|S\right\rangle_A \\ &+ \big(\varphi^{\dot{A}\alpha} - i\varphi^{\dot{A}\alpha a}K_a + (-i)^2\varphi^{\dot{A}\alpha ab}K_{ab}\big)p_\alpha\left|C\right\rangle_{\dot{A}}\Big] + \mathcal{O}\big(n = \tfrac{5}{2}, s = 3\big).\end{aligned} \tag{8.59}$$

The unfolding procedure will be analogous to the bosonic case. We will first implement the commutation relations of the superconformal algebra to split the unfolded equation into its different levels n, and then further decompose these into their irreducible representations. Note that the terms appearing in the bosonic equations (8.35)-(8.37) must once again appear[7]. However, there will also be supersymmetric contributions, stemming

[6] Recall that this is consistent since we in section 6.3 showed that the spin 2 algebra is closed.

[7] But now with an additional index A on the scalar fields $\phi^{a_1\cdots a_n}$ and acting on the bosonic $\left|S\right\rangle_A$ vacuum.

both from the new terms in the gauge potential (8.58) and from the new spinorial singleton fields $\varphi^{\dot{A}\alpha a_1\dots a_n}$. It will also appear two new equations, the level $n=\frac{1}{2}$ and $n=\frac{3}{2}$ equations yielding the fermionic field equations.

At level $n=0$ there are two supersymmetric contributions. From the term $(-i)A_1\phi^A\left|S\right\rangle_A$ we get the contribution

$$(-i)a_{IJ}T^{IJ}\phi^A\left|S\right\rangle_A=\frac{1}{4}a_{IJ}(\sigma^{IJ})_A{}^B\phi^A\left|S\right\rangle_B\,, \tag{8.60}$$

where we have used our observation from equation (8.17) how the generators T^{IJ} act on the vacuum states. Note that this is the non-abelian part of the covariant derivative $D\phi^A\left|S\right\rangle_A$ written in the spinor representation, where $D=\mathrm{d}+\omega+a$ as introduced in equation (7.27). Furthermore, the term $(-i)A_1\varphi^{\dot{A}\alpha}p_\alpha\left|C\right\rangle_{\dot{A}}$ yields the contribution

$$\begin{aligned}\chi_{\beta I}Q^{\beta I}\varphi^{\dot{A}\alpha}p_\alpha\left|C\right\rangle_{\dot{A}}&=\chi_{\beta I}\lambda^I q^\beta\varphi^{\dot{A}\alpha}p_\alpha\left|C\right\rangle_{\dot{A}}=-i\chi_{\alpha I}\varphi^{\dot{A}\alpha}\lambda^I\left|C\right\rangle_{\dot{A}}\\&=-i\chi_{\alpha I}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\dot{A}\alpha}\left|S\right\rangle_A\,,\end{aligned} \tag{8.61}$$

where we have used that λ^I and $\varphi^{\dot{A}\alpha}$ are both Grassmann odd, meaning they anti-commute, and we in the last step used equation (8.14) telling us how λ^I acts on the vacuum states. By combining these contributions with the purely bosonic ones from equation (8.35), we find the $n=0$ unfolded equation to read

$$(D_\mu\phi^A+\phi_\mu{}^A-i\chi_{\mu\alpha I}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\dot{A}\alpha})\left|S\right\rangle_A=0, \tag{8.62}$$

where we have written out the form-index explicitly and used that the spin connection term in the covariant derivative does not exist for $n=0$.

To derive the $n=\frac{1}{2}$ equation, it is first convenient to derive how the bosonic generators act on the state $p_\alpha\left|0\right\rangle_q$. We find that

$$P_a p_\alpha\left|0\right\rangle_q=-\frac{1}{2}(\gamma_a)_{\beta\gamma}q^\beta q^\gamma p_\alpha\left|0\right\rangle_q=-\frac{1}{2}(\gamma_a)_{\beta\gamma}q^\beta(p_\alpha q^\gamma+i\delta^\gamma_\alpha)\left|0\right\rangle_q=0, \tag{8.63}$$

$$M_a p_\alpha\left|0\right\rangle_q=-\frac{1}{2}(\gamma_a)_\beta{}^\gamma q^\beta p_\gamma p_\alpha\left|0\right\rangle_q=-\frac{i}{2}(\gamma_a)_\beta{}^\gamma(\delta^\beta_\alpha p_\gamma+\delta^\beta_\gamma p_\alpha)\left|0\right\rangle_q=-\frac{i}{2}(\gamma_a)_\alpha{}^\gamma p_\gamma\left|0\right\rangle_q, \tag{8.64}$$

$$Dp_\alpha\left|0\right\rangle_q=-\frac{1}{4}(q^\beta p_\beta+p_\beta q^\beta)p_\alpha\left|0\right\rangle_q=-\frac{1}{2}(i+p_\beta q^\beta)p_\alpha\left|0\right\rangle_q=-ip_\alpha\left|0\right\rangle_q\,, \tag{8.65}$$

which will be used repeatedly below. Studying equation (8.59) we realize that there, in addition to the exterior derivative $\mathrm{d}\varphi^{\dot{A}\alpha}\left|C\right\rangle_{\dot{A}}$, are four more terms contributing to the unfolded $n=\frac{1}{2}$ equation. The term $(-i)A_1\varphi^{\dot{A}\alpha}p_\alpha\left|C\right\rangle_{\dot{A}}$ yields the contribution

$$-i\big(\omega_a M^a+a_{IJ}T^{IJ}\big)\varphi^{\dot{A}\alpha}p_\alpha\left|C\right\rangle_{\dot{A}}=\left(-\frac{1}{2}(\gamma^a)_\beta{}^\alpha\omega_a\varphi^{\dot{A}\beta}+\frac{1}{4}a_{IJ}(\bar{\sigma}^{IJ})_{\dot{B}}{}^{\dot{A}}\varphi^{\dot{B}\alpha}\right)p_\alpha\left|C\right\rangle_{\dot{A}}\,, \tag{8.66}$$

which we note is precisely the spin connection and non-abelian parts of the covariant derivative $(D\varphi^{\dot{A}\alpha})p_\alpha\left|C\right\rangle_{\dot{A}}$, see equation (7.28). There are also contributions from the terms

$$\psi^\alpha{}_I S_\alpha{}^I\phi^A\left|S\right\rangle_A=\psi^\alpha{}_I\phi^A\lambda^I p_\alpha\left|S\right\rangle_A=\psi^\alpha{}_I(\sigma^I)_A{}^{\dot{A}}\phi^A p_\alpha\left|C\right\rangle_{\dot{A}} \tag{8.67}$$

and

$$(-i)\chi_{\alpha I}Q^{\alpha I}\phi^{Aa}K_a\,|S\rangle_A{=}-\chi_{\alpha I}(\gamma_a)^{\alpha\beta}S_\beta{}^I\phi^{Aa}\,|S\rangle_A{=}-\chi_{\alpha I}(\gamma_a)^{\alpha\beta}(\sigma^I)_A{}^{\dot A}\phi^{Aa}p_\beta\,|C\rangle_{\dot A}\,, \quad (8.68)$$

where we have used that ϕ^A and ϕ^{Aa} by construction are Grassmann even (bosonic). The final contribution to the $n=\frac{1}{2}$ equation reads

$$\begin{aligned} -\varphi^{\dot a\alpha a}e^bP_bK_ap_\alpha\,|C\rangle_{\dot A} &= 2i\varphi^{\dot A\alpha a}e^b(\epsilon_{ba}{}^cM_c+\eta_{ab}D)p_\alpha\,|C\rangle_{\dot A} \\ &= \varphi^{\dot A\alpha a}e^b\big(\epsilon_{ba}{}^c(\gamma_c)_\alpha{}^\beta p_\beta+2\eta_{ab}p_\alpha\big)\,|C\rangle_{\dot A} \\ &= \varphi^{\dot A\alpha a}e^b\big((\gamma_b\gamma_a)_\alpha{}^\beta p_\beta+\eta_{ab}p_\alpha\big)\,|C\rangle_{\dot A}\,. \end{aligned} \quad (8.69)$$

Here, we note that the first term can be written on the form

$$\varphi^{\dot A\alpha a}e^b(\gamma_b\gamma_a)_\alpha{}^\beta p_\beta\,|C\rangle_{\dot A} = \varphi^{\dot A\alpha a}e^b\big((\gamma_a)_\alpha{}^\gamma(\gamma_b)_\gamma{}^\beta+2\eta_{ab}\delta_\alpha^\beta\big)p_\beta\,|C\rangle_{\dot A}\,, \quad (8.70)$$

where the first term is the gamma trace we from equation (8.7) know must vanish. The total contribution from this term is thus

$$-\varphi^{\dot a\alpha a}e^bP_bK_ap_\alpha\,|C\rangle_{\dot A} = 3\varphi^{\dot A\alpha b}e_bp_\alpha\,|C\rangle_{\dot A}\,. \quad (8.71)$$

By collecting the terms we then find the unfolded $n=\frac{1}{2}$ equation to read

$$(D_\mu\varphi^{\dot A\alpha}+3\varphi_\mu{}^{\dot A\alpha}+\psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot A}\phi^A-\chi_{\mu\beta I}(\sigma^I)_A{}^{\dot A}(\gamma_a)^{\beta\alpha}\phi^{Aa})p_\alpha\,|C\rangle_{\dot A}=0, \quad (8.72)$$

with the form-index written out explicitly.

The procedure for deriving the level $n=1,\frac{3}{2}$ and 2 equations is completely analogous. Since the calculations of some of the terms appearing in these equations are a bit lengthy, and to avoid repetition, we refer to appendix C for these derivations. There we show that the first five levels of the source-free unfolded equations can be written as

$$\begin{aligned} n=0:&\quad (D_\mu\phi^A+\phi_\mu{}^A-i\chi_{\mu\alpha I}(\bar\sigma^I)_{\dot A}{}^A\varphi^{\dot A\alpha})\,|S\rangle_A=0, && (8.73)\\ n=\tfrac{1}{2}:&\quad (D_\mu\varphi^{\dot A\alpha}+3\varphi_\mu{}^{\dot A\alpha}+\psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot A}\phi^A-\chi_{\mu\beta I}(\gamma_a)^{\beta\alpha}(\sigma^I)_A{}^{\dot A}\phi^{Aa})p_\alpha\,|C\rangle_{\dot A}=0, && (8.74)\\ n=1:&\quad (D_\mu\phi^{Aa}+f_\mu{}^a\phi^A+6\phi_\mu{}^{Aa}+i\psi_\mu{}^\beta{}_I(\gamma^a)_{\beta\alpha}(\bar\sigma^I)_{\dot A}{}^A\varphi^{\dot A\alpha} \\ &\quad -3i\chi_{\mu\alpha I}(\bar\sigma^I)_{\dot A}{}^A\varphi^{\dot A\alpha a})K_a\,|S\rangle_A=0, && (8.75)\\ n=\tfrac{3}{2}:&\quad \big(D_\mu\varphi^{\dot A\alpha a}+10\varphi_\mu{}^{\dot A\alpha a}+f_\mu{}^a\varphi^{\dot A\alpha}+\psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot A}\phi^{Aa} \\ &\quad -2\chi_{\mu\beta I}(\gamma_b)^{\beta\alpha}(\sigma^I)_A{}^{\dot A}\phi^{Aab}\big)K_ap_\alpha\,|C\rangle_{\dot A}=0, && (8.76)\\ n=2:&\quad \big(D_\mu\phi^{Aab}+f_\mu{}^a\phi^{bA}+15\phi_\mu{}^{Aab}+i\psi_\mu{}^\beta{}_I(\gamma^b)_{\beta\alpha}(\bar\sigma^I)_{\dot A}{}^A\varphi^{\dot A\alpha a} \\ &\quad -5i\chi_{\mu\alpha I}(\bar\sigma^I)_{\dot A}{}^A\varphi^{\dot A\alpha ab}\big)K_{ab}\,|S\rangle_A=0, && (8.77) \end{aligned}$$

where all terms of spin 3 and higher have been omitted. The next step will be to split these equations into their irreducible representations.

8.3.1 The Decomposition into Irreducible Representations

The unfolded $n=0$ equation is already irreducible and can be solved for the field $\phi_\mu{}^A$, yielding

$$\phi_\mu{}^A = -D_\mu\phi^A + i\chi_{\mu\alpha I}(\bar\sigma^I)_{\dot A}{}^A\varphi^{\dot A\alpha}. \tag{8.78}$$

The $n=\frac{1}{2}$ equation contains two irreducible representations. The $n=\frac{1}{2}^-$ equation is found by computing the gamma trace of the equation, and by then subtracting this from the original equation we obtain the $n=\frac{1}{2}^+$ equation. Thus we find the equations

$$n=\tfrac{1}{2}^-:\quad (\gamma^\mu)_{\gamma\alpha}\big(D_\mu\varphi^{\dot A\alpha} + \psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot A}\phi^A - \chi_{\mu\beta I}(\gamma_a)^{\beta\alpha}(\sigma^I)_A{}^{\dot A}\phi^{Aa}\big) = 0, \tag{8.79}$$

$$n=\tfrac{1}{2}^+:\quad D_\mu\varphi^{\dot A\alpha} + 3\varphi_\mu{}^{\dot A\alpha} + \psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot A}\phi^A - \chi_{\mu\beta I}(\gamma_a)^{\beta\alpha}(\sigma^I)_A{}^{\dot A}\phi^{Aa} - \text{trace} = 0, \tag{8.80}$$

where we in the $n=\frac{1}{2}^-$ equation have used that the field $\varphi_\mu{}^{\dot A\alpha}$, in accordance with equation (8.7), is γ-traceless. By studying the first term we identify the $n=\frac{1}{2}^-$ equation as the Dirac equation in curved spacetime. Note, in particular, that it contains terms that couple the matter fields to gravity.

The term written as "trace" in the $n=\frac{1}{2}^+$ equation simply denotes the gamma trace from the $n=\frac{1}{2}^-$ equation. This means that it must always vanish on shell, *i.e.*, when the Dirac equation is satisfied. Assuming this is the case, the $n=\frac{1}{2}^+$ equation can be solved for the field $\varphi_\mu{}^{\dot A\alpha}$, yielding

$$\varphi_\mu{}^{\dot A\alpha} = -\frac{1}{3}\big(D_\mu\varphi^{\dot A\alpha} + \psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot A}\phi^A - \chi_{\mu\beta I}(\gamma_a)^{\beta\alpha}(\sigma^I)_A{}^{\dot A}\phi^{Aa}\big), \tag{8.81}$$

which can then be inserted into the higher level equations.

Before decomposing the higher level equations into irreducible representations, we will make an important observation. The solutions we have found above for $\phi_\mu{}^A$ and $\varphi_\mu{}^{\dot A\alpha}$ carry a curved index, whereas the $n=1$ equation contains these fields with a flat index. Hence we want to act on the $n=1$ equation with a vielbein $e_{\nu a}$. But then we need to recall that the vielbein postulate $D_\mu e_\nu{}^a = 0$ only holds in the bosonic case. In equation (2.61) we saw that the vielbein postulate of supergravity reads

$$D_\mu e_\nu{}^a = K_{\mu\nu}{}^a = \epsilon^a{}_{bc}K_\mu{}^b e_\nu{}^c, \tag{8.82}$$

where $K_\mu{}^b$ is the contorsion tensor. Since

$$e_{\nu a}D_\mu\phi^{Aa} = D_\mu\phi_\nu{}^A - (D_\mu e_{\nu a})\phi^{Aa}, \tag{8.83}$$

the contorsion term will appear as a compensating term in all higher level equations. The contorsion tensor of our theory was derived in equation (7.26). Insertion of this into equation (8.82) yields

$$\begin{aligned}
D_\mu e_{\nu a} &= i\epsilon_{abc}\Big(\epsilon^{b\rho\sigma}\bar\chi_\rho\gamma_\mu\chi_\sigma + \frac{1}{2}e_\mu{}^b\epsilon^{\rho\sigma\lambda}\bar\chi_\rho\gamma_\sigma\chi_\lambda\Big)e_\nu{}^c\\
&= i\Big(2\delta^{\rho\sigma}_{ac}\bar\chi_\rho\gamma_\mu\chi_\sigma e_\nu{}^c + \frac{1}{2}\epsilon_{a\mu\nu}(\epsilon^{\rho\sigma\lambda}\chi_\rho\gamma_\sigma\chi_\lambda)\Big)\\
&= i\big(2\bar\chi_a\gamma_\mu\chi_\nu - \bar\chi_\mu\gamma_\nu\chi_a + \bar\chi_\mu\gamma_a\chi_\nu + \bar\chi_\nu\gamma_\mu\chi_a\big)\\
&= i\big(\bar\chi_a\gamma_\mu\chi_\nu - \bar\chi_\mu\gamma_\nu\chi_a + \bar\chi_\mu\gamma_a\chi_\nu\big),
\end{aligned} \tag{8.84}$$

where we repeatedly have used that

$$\bar{\chi}_\nu \gamma_\mu \chi_a = \chi_\nu{}^\alpha (\gamma_\mu)_\alpha{}^\beta \chi_{a\beta} = -\chi_{\nu\alpha}(\gamma_\mu)_\beta{}^\alpha \chi_a{}^\beta = -\bar{\chi}_a \gamma_\mu \chi_\nu \tag{8.85}$$

and we have suppressed the vector index I labelling all χ-fields. Equation (8.84) can now be used to very conveniently derive the contorsion terms that should be included in the irreducible representations of the level $n = 1, \frac{3}{2}$ and 2 unfolded equations.

By using what we just learned we can write the $n = 1$ equation (8.75) as

$$\begin{aligned} &D_\mu \phi_\nu{}^A + f_{\mu\nu}\phi^A + 6\phi_{\mu\nu}{}^A + i\psi_\mu{}^\beta{}_I(\gamma_\nu)_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi^{\dot{A}\alpha} - 3i\chi_{\mu\alpha I}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi_\nu{}^{\dot{A}\alpha} \\ &- i\big(\bar{\chi}_{aI}\gamma_\mu \chi_\nu{}^I - \bar{\chi}_{\mu I}\gamma_\nu \chi_a{}^I + \bar{\chi}_{\mu I}\gamma_a \chi_\nu{}^I\big) = 0. \end{aligned} \tag{8.86}$$

The decomposition into irreducible representation then reads

$$\begin{aligned} n = 1^- :\ & D_\mu \phi^{\mu A} + f_\mu{}^\mu \phi^A + i\psi_\mu{}^\beta{}_I(\gamma^\mu)_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi^{\dot{A}\alpha} - 3i\chi_{\mu\alpha I}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi^{\mu\dot{A}\alpha} \\ & - 2i\bar{\chi}_{aI}\gamma^\mu \chi_\mu{}^I \phi^{Aa} = 0, \end{aligned} \tag{8.87}$$

$$\begin{aligned} n = 1^0 :\ & \epsilon^{\mu\nu a}\big[D_\mu \phi_\nu{}^A + f_{\mu\nu}\phi^A + i\psi_\mu{}^\beta{}_I(\gamma_\nu)_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi^{\dot{A}\alpha} - 3i\chi_{\mu\alpha I}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi_\nu{}^{\dot{A}\alpha} \\ & - i\bar{\chi}_{\mu I}\gamma_b \chi_\nu{}^I \phi^{Ab}\big] = 0, \end{aligned} \tag{8.88}$$

$$\begin{aligned} n = 1^+ :\ & D_{(\mu}\phi_{\nu)}{}^A + f_{(\mu\nu)}\phi^A + 6\phi_{\mu\nu}{}^A + i\psi_{(\mu}{}^\beta{}_{|I}(\gamma_{\nu)})_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A \varphi^{\dot{A}\alpha} \\ & - 3i\chi_{(\mu|\alpha I}(\bar{\sigma}^I)_{\dot{A}|}{}^A \varphi_{\nu)}{}^{\dot{A}\alpha} - 2i\bar{\chi}_{aI}\gamma_{(\mu}\chi_{\nu)}{}^I \phi^{Aa} = 0. \end{aligned} \tag{8.89}$$

By inserting the solution (8.78) for $\phi_\mu{}^A$ we see that the first term in the $n = 1^-$ equation reads $-\Box\phi^A$ so this is once again the Klein-Gordon equation, although this time with additional supersymmetric corrections. After insertion of the solution for $\phi_\mu{}^A$ into the $n = 1^0$ equation, the first term can be rewritten as the field strength tensor for the non-abelian gauge field. This is, thus, the field equation for the gauge field a_{IJ}. Hence, if we insert the expression for the field strength tensor we found in equation (7.49) by solving the zero field strength equation, this should become an identity. In the end of this chapter we will prove this, thereby verifying that the unfolded level $n \leq 1$ equations we have derived are consistent with the results we derived in section 7.3 from the zero field strength equation. The $n = 1^+$ equation can directly be solved to give the field $\phi_{\mu\nu}{}^A$.

When we want to decompose the $n = \frac{3}{2}$ equation (8.76) into irreducible representations, we once again need to be a bit careful. If we want to set the expression in parenthesis to zero, we need to explicitly implement the gamma tracelessness that the operator $K_a p_\alpha$ implies. This is in complete analogy with how we explicitly had to impose symmetricity and tracelessness in the a and b-indices when eliminating K_{ab} from the $n = 2$ equation in section 8.2. By subtracting the gamma trace we find the $n = \frac{3}{2}$ equation to read

$$\begin{aligned} &D_\mu \varphi^{\dot{A}\alpha a} + 10\varphi_\mu{}^{\dot{A}\alpha a} + f_\mu{}^a \varphi^{\dot{A}\alpha} + \psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot{A}}\phi^{Aa} - 2\chi_{\mu\beta I}(\gamma_b)^{\beta\alpha}(\sigma^I)_A{}^{\dot{A}}\phi^{Aab} \\ &- \frac{1}{3}(\gamma^a)^\alpha{}_\beta(\gamma_b)^\beta{}_\gamma\big(f_\mu{}^b\varphi^{\dot{A}\gamma} + \psi_\mu{}^\gamma{}_I(\sigma^I)_A{}^{\dot{A}}\phi^{Ab}\big) = 0. \end{aligned} \tag{8.90}$$

The gamma traces of the first and second terms have vanished due to equation (8.7), *i.e.*, the gamma tracelessness of $\varphi^{\dot{A}\alpha a}$. The gamma trace of the last term has vanished since

$$(\gamma_b\gamma_c)^{\beta\delta}\phi^{Abc} = (\eta_{bc} + \gamma_{bc})^{\beta\delta}\phi^{Abc} = 0, \tag{8.91}$$

where the first term equals zero due to the tracelessness and the second one due to the symmetry of ϕ^{Abc} in the flat indices.

This equation can now be decomposed into three irreducible representations. We will first act on the entire equation with the vielbein $e_\nu{}^a$ to change a vector index to a curved one, meaning we will once again need to include the contorsion terms from equation (8.84). The irreducible representations can then be obtained in exactly the same way as for the integer level equations, *i.e.*, as the trace, the antisymmetric part and the traceless symmetric part in the curved indices. Thus we find the equations

$$\begin{aligned} n = \tfrac{3}{2}^- :\ & D_\mu \varphi^{\mu\dot{A}\alpha} + f_\mu{}^\mu \varphi^{\dot{A}\alpha} + \psi_\mu{}^\alpha{}_I (\sigma^I)_A{}^{\dot{A}} \phi^{\mu A} - 2\chi_{\mu\beta I}(\sigma^I)_A{}^{\dot{A}}(\gamma_b)^{\beta\alpha}\phi^{\mu A b} \\ & - 2i\bar{\chi}_{aI}\gamma^\mu \chi_\mu{}^I \varphi^{\dot{A}\alpha a} - \frac{1}{3}(\gamma^\mu\gamma_b)^\alpha{}_\beta \big(f_\mu{}^b \varphi^{\dot{A}\beta} + \psi_\mu{}^\beta{}_I (\sigma^I)_A{}^{\dot{A}}\phi^{Ab}\big) = 0, \qquad (8.92) \\ n = \tfrac{3}{2}^0 :\ & \epsilon^{\mu\nu a}\Big[D_\mu \varphi_\nu{}^{\dot{A}\alpha} + f_{\mu\nu}\varphi^{\dot{A}\alpha} + \psi_\mu{}^\alpha{}_I(\sigma^I)_A{}^{\dot{A}}\phi_\nu{}^A - 2\chi_{\mu\beta I}(\sigma^I)_A{}^{\dot{A}}(\gamma_b)^{\beta\alpha}\phi_\nu{}^{bA} \\ & - i\bar{\chi}_{\mu I}\gamma_b \chi_\nu{}^I \varphi^{\dot{A}\alpha b} - \frac{1}{3}(\gamma_\nu\gamma_b)^\alpha{}_\beta \big(f_\mu{}^b\varphi^{\dot{A}\beta} + \psi_\mu{}^\beta{}_I(\sigma^I)_A{}^{\dot{A}}\phi^{Ab}\big)\Big] = 0, \qquad (8.93) \\ n = \tfrac{3}{2}^+ :\ & D_{(\mu}\varphi_{\nu)}{}^{\dot{A}\alpha} + 10\varphi_{\mu\nu}{}^{\dot{A}\alpha} + f_{(\mu\nu)}\varphi^{\dot{A}\alpha} + \psi_{(\mu}{}^\alpha{}_{|I}(\sigma^I)_{A|}{}^{\dot{A}}\phi_{\nu)}{}^A \\ & - 2\chi_{(\mu|\beta I}(\gamma_b)^{\beta\alpha}(\sigma^I)_{A|}{}^{\dot{A}}\phi_{\nu)}{}^{Ab} - 2i\bar{\chi}_{aI}\gamma_{(\mu}\chi_{\nu)}{}^I\varphi^{\dot{A}\alpha a} \\ & - \frac{1}{3}(\gamma_{(\nu})^\alpha{}_{|\beta}(\gamma_b)^\beta{}_{\gamma|}\big[f_{\mu)}{}^b\varphi^{\dot{A}\gamma} + \psi_{\mu)}{}^\gamma{}_I(\sigma^I)_A{}^{\dot{A}}\phi^{Ab}\big] = 0. \qquad (8.94) \end{aligned}$$

Since the $n = \frac{3}{2}^-$ equation contains both gauge fields $f_{\mu\nu}$ and $\psi_{\mu\alpha I}$ it cannot yet be used to find a solution expressed only in terms of the frame fields $e_\mu{}^a$ and $\chi_{\mu\alpha I}$ for any of them. This problem will, however, soon be resolved as also the $n = 2^-$ equation will contain these two gauge fields. The $n = \frac{3}{2}^0$ equation must be the spin $\frac{3}{2}$ field equation, *i.e.*, the Cottino equation. It will not be of interest to write it out on a more explicit form. However, by inserting the solution (8.81) for $\varphi_\nu{}^{\dot{A}\alpha}$ we note that it, exactly as the form of the Cottino equation we found in equation (7.50) by solving $F = 0$, contains derivatives up to second order. The $\frac{3}{2}^+$ equation can easily be solved to give the field $\varphi_{\mu\nu}{}^{\dot{A}\alpha}$.

To decompose the $n = 2$ equation (8.77) into its irreducible parts, we first have to recall to explicitly make the equation traceless and symmetric in the two flat indices when we eliminate the generator K_{ab}. Then we also need to include the contorsion terms from equation (8.84). After having done so, we can decompose the equation into the usual irreducible representations and end up with the equations

$$\begin{aligned} n = 2^- :\ & D_\mu\phi^{\mu A a} + \frac{1}{2}f_\mu{}^\mu\phi^{Aa} + \frac{1}{2}f^{ba}\phi^A{}_b - \frac{1}{3}f^{ab}\phi^A{}_b + i\psi_\mu{}^\beta{}_I(\gamma^a)_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\mu\dot{A}\alpha} \\ & - 5i\chi_{\mu\alpha I}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\mu\dot{A}\alpha a} - 2i\bar{\chi}_{bI}\gamma_\mu\chi^{\mu I}\phi^{Aab} = 0, \qquad (8.95) \\ n = 2^0 :\ & \epsilon^{\mu\nu(a}\big[D_\mu\phi_\nu{}^{b)A} + \frac{1}{2}f_\mu{}^{b)}\phi_\nu{}^A + \frac{1}{2}f_{\mu\nu}\phi^{b)A} + i(\gamma^{b)})_{\beta\alpha}\psi_\mu{}^\beta{}_I(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi_\nu{}^{\dot{A}\alpha} \\ & - 5i\chi_{\mu\alpha I}(\bar{\sigma}^{|I})_{\dot{A}}{}^A\varphi_\nu{}^{\dot{A}\alpha|b)} - i\bar{\chi}_{\mu I}\gamma_c\chi_\nu{}^{|I}\phi^{A|b)c}\big] = 0, \qquad (8.96) \\ n = 2^+ :\ & D_{(\mu}\phi_{\nu\rho)}{}^A + f_{(\mu\nu}\phi_{\rho)}{}^A - \frac{1}{3}g_{(\mu\nu}f_{\rho)a}\phi^{Aa} + 15\phi_{\mu\nu\rho}{}^A + i\psi_{(\mu}{}^\beta{}_{|I|}(\gamma_\rho)_{|\beta\alpha}(\bar{\sigma}^I)_{\dot{A}|}{}^A\varphi_{\nu)}{}^{\dot{A}\alpha} \\ & - 5i\chi_{(\mu|\alpha I}(\bar{\sigma}^I)_{\dot{A}|}{}^A\varphi_{\nu\rho)}{}^{\dot{A}\alpha} - 2i\big(\bar{\chi}^{aI}\gamma_{(\mu}\chi_{\nu)I}e_\rho{}^b + \bar{\chi}^{bI}\gamma_{(\mu}\chi_{\rho)I}e_\nu{}^a\big)\phi^A{}_{ab} = 0. \qquad (8.97) \end{aligned}$$

Note that the $n = 2^-$ equation contains both $f_{\mu\nu}$ and $\psi_{\mu\alpha I}$. Together with the $n = \frac{3}{2}^-$ equation it can thus be used to solve for these fields in terms of the frame fields $e_\mu{}^a$ and the gauge fields $\chi_{\mu\alpha I}$. In analogy with what we found in the bosonic case in equation (8.57), the $n = 2^0$ equation must be the spin 2 field equation, *i.e.*, the Cotton equation. By inserting the solution of the $n = 1^+$ equation for the field $\phi_\nu{}^{bA}$, we see that our Cotton equation is of third order in the derivatives, which agrees with what we observed in section 2.4. The $n = 2^+$ equation is easily solved for the field $\phi_{\mu\nu\rho}{}^A$.

We have now proposed a set of supersymmetric unfolded equations (8.73)–(8.77) and divided them into their constitutional irreducible representations. In the bosonic case we could at this stage prove that these equations were equivalent to those stemming from the zero field strength equation. We did this by showing that both the solution for the gauge field $f_{\mu\nu}$, which was $f_{\mu\nu} = \frac{1}{2}S_{\mu\nu}$, and the field equation, which was the Cotton equation $C_{\mu\nu} = 0$, that emerged were equivalent to those we found from solving $F = 0$. In the supersymmetric case this would correspond to first proving that the $n = \frac{3}{2}^-$ and $n = 2^-$ unfolded equations yield the same solutions for the gauge fields $f_{\mu\nu}$ and $\psi_{\mu\alpha I}$ as $F = 0$ did, *i.e.*, the solutions found in equations (7.42) and (7.33), respectively. Then one would have to verify that the $n = \frac{3}{2}^0$ equation is really the correct Cottino equation and can be rewritten on the form of equation (7.50) derived from $F = 0$, and that the $n = 2^0$ equation is the same supersymmetric Cotton equation as the $F = 0$ equation (7.51).

Since this procedure would be very complicated in the supersymmetric case, we will instead use another (weaker) approach to check the consistency of the unfolded and zero field strength equations. By implementing our solutions for the gauge fields $f_{\mu\nu}$ and $\psi_{\mu\alpha I}$ from $F = 0$, we intend to verify that the level $n = 1^0$ unfolded equation (8.88) is really the same field equation for the non-abelian gauge field as equation (7.49) stemming from $F = 0$. Hence, if we assume this field equation to hold true we should obtain an identity. In the process of proving this identity we will not only need to use results from the zero field strength equation, but also from all level $n \leq 1$ unfolded equations. By proving the identity we will thus have checked the consistency with $F = 0$ for *all* level $n \leq 1$ unfolded equations derived above.

8.3.2 Proof of the $n = 1^0$ Identity

The $n = 1^0$ unfolded equation (8.88) can be written as

$$\begin{aligned} &D_{[\mu}\phi_{\nu]}{}^A + f_{[\mu\nu]}\phi^A + i\psi_{[\mu}{}^\beta{}_{|I|}(\gamma_{\nu]})_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\dot{A}\alpha} - 3i\chi_{[\mu|\alpha I}(\bar{\sigma}^I)_{\dot{A}|}{}^A\varphi_{\nu]}{}^{\dot{A}\alpha} \\ &- i\bar{\chi}_{[\mu}{}^I\gamma_{|a|}\chi_{\nu]I}\phi^{Aa} = 0. \end{aligned} \tag{8.98}$$

Insertion of the solutions for $\phi_\mu{}^A$ and $\varphi_\mu{}^{\dot{A}\alpha}$ that we derived in equations (8.78) and (8.81) from the $n = 0$ and $n = \frac{1}{2}^+$ equations, respectively, yields

$$\begin{aligned} &- D_{[\mu}D_{\nu]}\phi^A + i(\bar{\sigma}^I)_{\dot{A}}{}^A(D_{[\mu}\chi_{\nu]\alpha I})\varphi^{\dot{A}\alpha} + f_{[\mu\nu]}\phi^A + i\psi_{[\mu}{}^\beta{}_{|I|}(\gamma_{\nu]})_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\dot{A}\alpha} \\ &- i(\bar{\sigma}^I\sigma^J)^A{}_B\big(\bar{\chi}_{[\mu|I|}\psi_{\nu]J}\phi^B + \chi_{[\mu|\alpha I}(\gamma_{a|})^{\alpha\beta}\chi_{\nu]\beta J}\phi^{Ba}\big) - i\bar{\chi}_{[\mu}{}^I\gamma_{|a|}\chi_{\nu]I}\phi^{Aa} = 0, \end{aligned} \tag{8.99}$$

where the product rule has been used to cancel one of the terms emerging from the first term of equation (8.98) with one emerging from the fourth term. We now need to recall what the matrices σ^I represent. We introduced them in equation (8.9) to construct the Weyl representation of the SO(8) gamma matrices Γ^I. Since these satisfy the relation

$$\Gamma^I\Gamma^J = \Gamma^{IJ} + \delta^{IJ}, \tag{8.100}$$

we can by studying the representation of Γ^{IJ} derived in equation (8.16) conclude that

$$(\bar{\sigma}^I\sigma^J)^A{}_B = \delta^{IJ}\delta^A_B + (\bar{\sigma}^{IJ})^A{}_B, \tag{8.101}$$

which should be inserted on the second line of equation (8.99). By also implementing the condition

$$f_{[\mu\nu]} = -i\bar{\psi}_{[\mu}{}^I\chi_{\nu]I} = i\bar{\chi}_{[\mu}{}^I\psi_{\nu]I}, \tag{8.102}$$

which was derived from the (D) projection of $F = 0$ in equation (7.41), we find two more terms that cancel out. Also note that since the middlemost term on the second line of equation (8.99) is antisymmetric in μ and ν but symmetric in α and β, it must be symmetric in the I and J indices. This means that the $\bar{\sigma}^{IJ}$-term emerging after application of equation (8.101) will vanish for this term. We are thus left with

$$\begin{aligned}
&- D_{[\mu}D_{\nu]}\phi^A + i(\bar{\sigma}^I)_{\dot{A}}{}^A\big(D_{[\mu}\chi_{\nu]\alpha I}\big)\varphi^{\dot{A}\alpha} + i\psi_{[\mu}{}^\beta{}_{|I|}(\gamma_{\nu]})_{\beta\alpha}(\bar{\sigma}^I)_{\dot{A}}{}^A\varphi^{\dot{A}\alpha} \\
&- i\bar{\chi}_{[\mu|I|}\psi_{\nu]J}(\bar{\sigma}^{IJ})^A{}_B\phi^B + i\bar{\chi}_{[\mu}{}^I\gamma_{|a|}\chi_{\nu]I}\phi^{Aa} - i\bar{\chi}_{[\mu}{}^I\gamma_{|a|}\chi_{\nu]I}\phi^{Aa} = 0,
\end{aligned} \tag{8.103}$$

where we note that also the last two terms cancel out. Since the last term is the contorsion term, this illustrates the importance of remembering to include these in the supersymmetric unfolded equations.

Next we will use that we in equation (7.49) found the (T^{IJ}) projection of the zero field strength equation to imply

$$i\bar{\chi}_{[\mu}{}^{[I}\psi_{\nu]}{}^{J]} = -\frac{1}{8}F^{IJ}_{\mu\nu}. \tag{8.104}$$

as the non-abelian field equation. Furthermore, definition (3.26) of the non-abelian field strength tensor, in the (conjugate) spinor representation, implies that

$$D_{[\mu}D_{\nu]}\phi^A = \frac{1}{8}F^{IJ}_{\mu\nu}(\bar{\sigma}_{IJ})^A{}_B\phi^B. \tag{8.105}$$

This gives that the leftmost term on the first line and the leftmost term on the second line of equation (8.103) will cancel each other out. After this observation we are left with merely

$$i(\bar{\sigma}^I)_{\dot{A}}{}^A\big[(D_{[\mu}\chi_{\nu]\alpha I}) + \psi_{[\mu}{}^\beta{}_I(\gamma_{\nu]})_{\beta\alpha}\big]\varphi^{\dot{A}\alpha} = 0. \tag{8.106}$$

To prove that this hold true we will insert our solution (7.33) from $F = 0$ for the gravitino field. After application of the gamma relation $\gamma^\mu\gamma^\nu = \epsilon^{\mu\nu\rho}\gamma_\rho + g^{\mu\nu}$, the equation evaluates to

$$\frac{i}{2}(\bar{\sigma}^I)_{\dot{A}}{}^A\epsilon^{\rho\sigma}{}_{[\mu}\big[(\gamma_{\nu]})_\alpha{}^\beta D_\rho\chi_{\sigma\beta I} - (\gamma_{|\rho})_{\alpha|}{}^\beta D_{\nu]}\chi_{\sigma\beta I} + (\gamma_{|\rho})_\alpha{}^\beta D_{\sigma|}\chi_{\nu]}{}^\beta{}_I\big]\varphi^{\dot{A}\alpha} = 0. \tag{8.107}$$

Here we note that the LHS can be obtained via cycling in the four curved indices according to

$$6\epsilon^{\rho\sigma}{}_{[\mu}\gamma_\nu D_\rho\chi_{\sigma]} = \epsilon^{\rho\sigma}{}_{[\mu}\big(\gamma_{\nu]}D_\rho\chi_\sigma - \gamma_{|\rho|}D_{\nu]}\chi_\sigma + \gamma_{|\rho}D_{\sigma|}\chi_{\nu]}\big), \tag{8.108}$$

where the other indices have been omitted for clarity. But since we are working in three dimensions, all antisymmetrizations over more than three spacetime indices must vanish identically. Consequently, the LHS of equation (8.107) vanishes identically.

This proves that the $n = 1^0$ unfolded equation is really satisfied by our solutions stemming from $F = 0$. Note that if we would not have assumed the non-abelian field equation (8.104) stemming from $F = 0$ to hold true, this field equation would be precisely what we would have been left with instead of an identity. We have thereby shown that our level $n \leq 1$ unfolded equations are consistent with the zero field strength equation, as desired.

Chapter 9

Conclusions and Future Directions

In this thesis we have realized the superconformal algebra in terms of two $\mathrm{SL}(2,\mathbb{R})$ spinors q^α and p_α and a Grassmann odd vector λ^I, a construction that conveniently can be generalized to the infinite dimensional higher spin algebra. We gave two possible ways of quantizing this algebra, either utilizing the operator formulation or the (super)star product.

After having introduced the unfolded formulation of higher spin theory we set out to solve both the zero field strength equation, stemming from Chern-Simons gauge theory, and the unfolded equation. This was performed for fields up to spin 2 in both the purely bosonic case (following the work of [14]) and the supersymmetric case. In the bosonic case we showed that the two approaches yielded both the same solution for the gauge field $f_{\mu\nu}$ and the same field equation, the Cotton equation. Also for the supersymmetric extension we managed to solve the zero field strength equation, resulting in explicit solutions for the gauge field $f_{\mu\nu}$ and the gravitino field, as well as three field equations; the Cotton equation, the Cottino equation and a Chern-Simons equation for the non-abelian gauge field. We then decomposed the unfolded equation into its irreducible representations and gave an interpretation to each of them. In particular, we showed that the $n = 1^0$ equation is consistent with our solutions to the zero field strength equation.

A natural continuation of this work would be to explicitly solve the given $n = \frac{3}{2}^-$ and $n = 2^-$ unfolded equations, and verifying that the solutions for the gauge field $f_{\mu\nu}$ and the gravitino field agree with those solving the zero field strength equation. It would then be interesting to verify that the unfolded spin $\frac{3}{2}$ and spin 2 field equations really can be rewritten on the form of the Cottino and Cotton equations obtained from the zero field strength equation. If this can be achieved, it has been shown that the zero field strength and unfolded equations yield compatible results, as in the bosonic case.

In the case of $N = 8$, the obtained (BLG) theory for spins $s \leq 2$ should be coupled to three-dimensional superconformal gravity. This coupled theory was first constructed in [46], where also its relation to M-theory was discussed. To be able to show that the obtained results agree with this theory would be very intriguing and a first step towards possible applications in M-theory.

An interesting extension would be to study the zero field strength and unfolded equations for fields of spins higher than two. Using our construction of the algebra this would conceptually be very easy, it is merely a question of including the higher spin generators defined in section 6.3 and truncating the unfolded equation at higher spins than two. Hopefully this process would also uncover a possible action formulation of the higher spin theory, with the unfolded equation as its field equation. Such a formulation may be the key to a detailed understanding of the relation between the zero field strength and unfolded equations.

There are also several intriguing continuations related to the AdS/CFT correspondence. It has been shown that higher spin theories in AdS_4 can be constructed from complex $\mathrm{SL}(2,\mathbb{C})$ variables which are able to relate to the $\mathrm{SL}(2,\mathbb{R})$ variables q^α and p_α that we used to construct the three-dimensional conformal higher spin theory [31]. Since this enables very explicit studies of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence in a higher spin context, it is of great interest. A detailed understanding of this correspondence may result in a deeper understanding of the AdS/CFT correspondence in general.

Recall also the earlier discussed observation that an infinite tower of massless higher spin states is constructed by the product of two Dirac singletons in AdS_4, which in addition only have propagating degrees of freedom on the boundary. Since the theory living on the boundary is a three-dimensional conformal field theory, there are hopes that theories similar to the one we have constructed can actually be used to prove the AdS/CFT correspondence.

A long-term goal of constructing the superconformal higher spin theory in three dimensions is also to gain new insights in M-theory on $\mathrm{AdS}_4 \times S^7$, via the AdS/CFT correspondence. Perhaps it may even be the first step towards a complete, non-perturbative formulation of M-theory.

Appendix A

Conventions

We here present the conventions that have been adopted in the thesis. Spinorial indices are denoted by the first part of the Greek alphabet, *i.e.* by $\alpha, \beta, \gamma, \delta$. For the curved spacetime indices we use the remaining part of the Greek alphabet, while we for the flat (tangent) space indices use the Latin lowercase letters $a, b, \ldots$. The uppercase Latin letters I, J, K denote the internal SO(N) R-symmetry indices.

A.1 The Levi-Civita Symbol

The Levi-Civita symbol is a totally antisymmetric tensor density which, in three dimensions, is defined as

$$\epsilon^{\mu\nu\rho} = \begin{cases} 1 & \text{if } (\mu\nu\rho) \text{ is an even permutation of } (012), \\ -1 & \text{if } (\mu\nu\rho) \text{ is an odd permutation of } (012), \\ 0 & \text{if two indices are equal.} \end{cases} \tag{A.1}$$

The generalization to other dimensions is obvious. The indices are raised and lowered using the metric $g_{\mu\nu}$, meaning $\epsilon_{\mu\nu\rho} = g_{\mu\sigma} g_{\nu\eta} g_{\rho\tau} \epsilon^{\sigma\eta\tau}$. Two Levi-Civita symbols can be contracted as

$$\begin{aligned} \epsilon^{\mu\nu\rho}\epsilon_{\sigma\lambda\kappa} &= -6\delta^{\mu\nu\rho}_{\sigma\lambda\kappa} \\ \epsilon^{\mu\nu\rho}\epsilon_{\rho\sigma\lambda} &= -2\delta^{\mu\nu}_{\sigma\lambda} \\ \epsilon^{\mu\nu\rho}\epsilon_{\nu\rho\sigma} &= -2\delta^{\mu}_{\sigma} \\ \epsilon^{\mu\nu\rho}\epsilon_{\mu\nu\rho} &= -6, \end{aligned} \tag{A.2}$$

where the generalized Kronecker delta is defined as

$$\delta^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p} = \begin{cases} \frac{1}{p!} & \text{if } (\nu_1 \ldots \nu_p) \text{ is an even permutation of } (\mu_1 \ldots \mu_p), \\ -\frac{1}{p!} & \text{if } (\nu_1 \ldots \nu_p) \text{ is an odd permutation of } (\mu_1 \ldots \mu_p), \\ 0 & \text{otherwise.} \end{cases} \tag{A.3}$$

Furthermore, we let $\varepsilon^{\mu\nu\rho} = \sqrt{|g|}\epsilon^{\mu\nu\rho}$, with g being the determinant of the metric, denote the Levi-Civita tensor which is really a proper tensor and *not* a tensor density.

A.2 The Three-Dimensional Gamma Matrices

The three-dimensional gamma matrices constitute a basis for the real, traceless 2×2 matrices, and can be constructed from the Pauli matrices σ^i as

$$\begin{aligned}
(\gamma^0)_\alpha{}^\beta &= (i\sigma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
(\gamma^1)_\alpha{}^\beta &= (\sigma^1)_\alpha{}^\beta \;\, = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
(\gamma^2)_\alpha{}^\beta &= (\sigma^3)_\alpha{}^\beta \;\, = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned} \tag{A.4}$$

It is easily checked that they satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. To obtain the full basis for the real 2×2 matrices we also need to include the identity matrix $\mathbb{1} = \gamma^0\gamma^1\gamma^2$.

The spinorial indices α and β are raised and lowered using the two-dimensional Levi-Civita symbol $\epsilon^{\alpha\beta}$ and its inverse $\epsilon_{\alpha\beta}$. From definition (A.1) of the Levi-Civita symbol we note that we must have $\epsilon^{\alpha\beta} = \gamma^0$ and consequently $\epsilon_{\alpha\beta} = -\gamma^0$, so that $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = (\mathbb{1})_\alpha{}^\beta$. By using this we find that

$$(\gamma^0)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (\gamma^1)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\gamma^2)_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A.5}$$

which we note are all symmetric. By also including the antisymmetric $\epsilon_{\alpha\beta}$ we can once again form the full basis of real 2×2 matrices.

We also define the antisymmetrized products of gamma matrices as

$$\begin{aligned}
\gamma^{\mu\nu} &= \gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - g^{\mu\nu} \\
\gamma^{\mu\nu\rho} &= \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \epsilon^{\mu\nu\rho}.
\end{aligned} \tag{A.6}$$

These matrices satisfy some important identities, including

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 3 \\
\gamma^{\mu\nu}\gamma_\nu &= 2\gamma^\mu \\
\gamma^{\mu\nu\rho}\gamma_\rho &= \gamma^{\mu\nu} \\
\gamma^{\mu\nu\rho}\gamma_{\nu\rho} &= -2\gamma^\mu \\
\gamma^\mu\gamma^\nu\gamma^\rho &= \gamma^{\mu\nu\rho} + 2\gamma^{[\mu}g^{\nu]\rho} + g^{\mu\nu}\gamma^\rho,
\end{aligned} \tag{A.7}$$

which have all been used repeatedly in this thesis.

A.3 The Superconformal Algebra

We define the generators of the spin 2 conformal algebra using the two bosonic $SL(2,\mathbb{R})$ spinors q^α and p_α. Classically they can be regarded as the phase-space variables satisfying the Poisson bracket relation $\{q^\alpha,\, p_\beta\}_{\mathrm{PB}} = \delta^\alpha_\beta$. In the quantized case they instead correspond to the position and momentum operators satisfying the canonical commutation relation $[q^\alpha,\, p_\beta] = i\delta^\alpha_\beta$. The conformal generators can then be represented as

$$\begin{aligned} P^a &= -\tfrac{1}{2}(\gamma^a)_{\alpha\beta}q^\alpha q^\beta \\ M^a &= -\tfrac{1}{2}(\gamma^a)_\alpha{}^\beta q^\alpha p_\beta \\ D &= -\tfrac{1}{4}(q^\alpha p_\alpha + p_\alpha q^\alpha) \\ K^a &= -\tfrac{1}{2}(\gamma^a)^{\alpha\beta}p_\alpha p_\beta. \end{aligned} \tag{A.8}$$

To extend the algebra supersymmetrically we also need to include the generators

$$\begin{aligned} Q^{\alpha I} &= q^\alpha \lambda^I \\ S_\alpha{}^I &= p_\alpha \lambda^I \\ T^{IJ} &= \tfrac{i}{8}(\lambda^I\lambda^J - \lambda^J\lambda^I), \end{aligned} \tag{A.9}$$

where λ^I is a Grassmann odd vector satisfying the Clifford algebra $\{\lambda^I,\, \lambda^J\} = 2\delta^{IJ}$. $Q^{\alpha I}$ and $S_\alpha{}^I$ are the fermionic generators, whereas T^{IJ} are the generators of the $SO(N)$ R-symmetry “rotating” the N supersymmetries of the theory into each other.

The non-zero commutation relations of these seven generators are

$$\begin{aligned} [M^a,\, M^b] &= i\epsilon^{ab}{}_c M^c \\ [M^a,\, P^b] &= i\epsilon^{ab}{}_c P^c \\ [M^a,\, K^b] &= i\epsilon^{ab}{}_c K^c \\ [P^a,\, K^b] &= -2i\epsilon^{ab}{}_c M^c - 2i\eta^{ab}D \\ [D,\, P^a] &= iP^a \\ [D,\, K^a] &= -iK^a \\ [D,\, Q^{\alpha I}] &= \tfrac{i}{2}Q^{\alpha I} \\ [D,\, S_\alpha{}^I] &= -\tfrac{i}{2}S_\alpha{}^I \\ [P^a,\, S_\beta{}^I] &= -i(\gamma^a)_{\beta\alpha}Q^{\alpha I} \\ [K^a,\, Q^{\beta I}] &= i(\gamma^a)^{\beta\alpha}S_\alpha{}^I \\ [M^a,\, Q^{\beta I}] &= \tfrac{i}{2}Q^{\alpha I}(\gamma^a)_\alpha{}^\beta \\ [M^a,\, S_\beta{}^I] &= -\tfrac{i}{2}(\gamma^a)_\beta{}^\alpha S_\alpha{}^I \\ [T^{IJ},\, Q^\alpha{}_K] &= -i\delta^{IJ}_{KL}Q^{\alpha L} \\ [T^{IJ},\, S_{\alpha K}] &= -i\delta^{IJ}_{KL}S_\alpha{}^L \\ \{Q^{\alpha I},\, Q^{\beta J}\} &= -2\delta^{IJ}(\gamma^a)^{\alpha\beta}P_a \\ \{S_\alpha{}^I,\, S_\beta{}^J\} &= -2\delta^{IJ}(\gamma^a)_{\alpha\beta}K_a \\ \{Q^{\alpha I},\, S_\beta{}^J\} &= -2\delta^{IJ}\delta^\alpha_\beta D - 2\delta^{IJ}(\gamma_a)_\beta{}^\alpha M^a + 4\delta^\alpha_\beta T^{IJ} \\ [T_{IJ},\, T^{KL}] &= -2i\delta^{[K}_{[I}T_{J]}{}^{L]}. \end{aligned} \tag{A.10}$$

We can here notice an important feature of the superconformal algebra; it is graded by the generator D in the sense that all generators X of the algebra satisfy the relation

$$[D,\, X\} = iwX, \tag{A.11}$$

for some weight w. The weights of the generators are summarized in table A.1.

Weight	**Generator**
1	P^a
1/2	$Q^{\alpha I}$
0	D, M^a, T^{IJ}
−1/2	$S_\alpha{}^I$
−1	K^a

Table A.1: The weights of the superconformal generators with respect to the grading in D.

Appendix B

Proof of the (D)-identity

Here we set out to show that the identity

$$f^{[\sigma\rho]} + i\bar{\psi}^{[\sigma|I|}\chi^{\rho]}{}_I = 0, \tag{B.1}$$

originating from the (D)-projection of $F = 0$, really is satisfied by the expression we found for $f^{\sigma\rho}$ and $\psi_{\mu\alpha I}$. In equation (7.46) we found that

$$f^{[\sigma\rho]} = \epsilon^{\mu\nu[\rho}\left(\frac{1}{2}D_\mu(e)K_\nu{}^{\sigma]} + \frac{1}{4}\epsilon^{\sigma]}{}_{bc}K_\mu{}^b K_\nu{}^c + i\bar{\psi}_\mu\gamma^{\sigma]}\chi_\nu\right), \tag{B.2}$$

where the contorsion, in accordance with equation (7.26), is given by

$$K_{a\rho} = i\left(\epsilon_\rho{}^{\mu\nu}\bar{\chi}_{\mu I}\gamma_a\chi_\nu{}^I + \frac{1}{2}e_{\rho a}\epsilon^{\mu\nu\sigma}\bar{\chi}_{\mu I}\gamma_\nu\chi_\sigma{}^I\right). \tag{B.3}$$

The gauge field $\psi_{\mu\alpha I}$ was in equation (7.33) found to read

$$\psi_{\mu\alpha I} = \frac{1}{2}\left(\epsilon_\mu{}^{\nu\rho}\hat{D}_\nu\chi_{\rho\alpha I} - (\gamma^\nu)_\alpha{}^\beta\hat{D}_\mu\chi_{\nu\beta I} + (\gamma^\nu)_\alpha{}^\beta\hat{D}_\nu\chi_{\mu\beta I}\right). \tag{B.4}$$

Let us first comment on the different derivatives appearing in these expressions. $D(e) = \mathrm{d} + \omega(e)$ is the purely bosonic covariant derivative, $\hat{D}(e) = \mathrm{d} + \omega(e) + a$ also includes the non-abelian gauge field a but not the contorsion, whereas $\hat{D} = \mathrm{d} + \omega + a$ is the full covariant derivative. Consequently $\hat{D} = \hat{D}(e) + K$, which can be applied to the expression for $\psi_{\mu\alpha I}$ above. The fact that $f^{[\sigma\rho]}$ in equation (B.2) contains the purely bosonic derivative may seem like a problem. However, as it only acts on $K \sim \chi^2$ we can exchange it for $\hat{D}(e)$, since the additional non-abelian terms appearing in $\hat{D}(e)K_\nu{}^{\sigma]}$ vanish when the internal I-indices of the χ-fields are contracted as in equation (B.3). Since the only derivative appearing after these rewritings is $\hat{D}(e)$ we will for convenience simply refer to it as D.

From our expression for f, K, and ψ above it is obvious that equation (B.1) only contains two kinds of terms: $\chi D\chi$-terms and χ^4-terms. We will begin by showing that the former of these cancel out.

B.1 Cancellation of the $\chi D\chi$-terms

After multiplication of both sides by $(-2i)$, the $\chi D\chi$-terms on the LHS of equation (B.1) read

$$\begin{aligned}\mathrm{LHS}(\chi D\chi) =&\epsilon^{\mu\nu[\rho}\Big(- iD_\mu K_\nu{}^{\sigma]} + \epsilon_\mu{}^{|\eta\tau}(D_\eta\bar{\chi}_\tau)^{\alpha|}(\gamma^{\sigma]}\chi_\nu)_\alpha + (D_\mu\bar{\chi}_\tau)\gamma^{|\tau|}\gamma^{\sigma]}\chi_\nu \\ &+ (\gamma^{|\tau}D_\tau\chi_\mu)^{\alpha|}(\gamma^{\sigma]}\chi_\nu)_\alpha\Big) \\ &+ \epsilon^{\mu\nu[\sigma}(D_\mu\bar{\chi}_\nu)\chi^{\rho]} - (\gamma^\mu D^{[\sigma}\chi_\mu)^{|\alpha|}\chi^{\rho]}{}_\alpha + (\gamma^\mu D_\mu\chi^{[\sigma})^{|\alpha|}\chi^{\rho]}{}_\alpha, \end{aligned} \tag{B.5}$$

where all χ-fields also carry an internal index I which has not been written out. The sign of the third term is explained by the relation

$$-(\gamma^\tau)^{\alpha\beta}(D_\mu\chi_{\tau\beta})(\gamma^\sigma)_\alpha{}^\gamma\chi_{\nu\gamma} = -(D_\mu\chi_{\tau\beta})(\gamma^\tau\gamma^\sigma)^{\beta\gamma}\chi_{\nu\gamma} = (D_\mu\chi_\tau{}^\beta)(\gamma^\tau\gamma^\sigma)_\beta{}^\gamma\chi_{\nu\gamma}. \tag{B.6}$$

Let us now rewrite these seven terms a bit:

$$\begin{aligned}\mathrm{I} &= -i\epsilon^{\mu\nu[\rho}D_\mu K_\nu{}^{\sigma]} = \epsilon^{\mu\nu[\rho}\left(D_\mu\epsilon^{\sigma]\tau\eta}(\bar{\chi}_\tau\gamma_\nu\chi_\eta) + \frac{1}{2}D_\mu(\delta^\sigma_\nu\epsilon^{abc}\bar{\chi}_a\gamma_b\chi_c)\right) \\ &= \epsilon^{\mu\nu[\rho}\epsilon^{\sigma]\tau\eta}\big((D_\mu\bar{\chi}_\tau)\gamma_\nu\chi_\eta + \bar{\chi}_\tau\gamma_\nu(D_\mu\chi_\eta)\big) + \frac{1}{2}\epsilon^{\mu\sigma\rho}\epsilon^{abc}\big((D_\mu\bar{\chi}_a)\gamma_b\chi_c + \bar{\chi}_a\gamma_b(D_\mu\chi_c)\big) \\ \mathrm{II} &= \epsilon^{\mu\nu[\rho}\epsilon_\mu{}^{|\eta\tau|}(D_\eta\bar{\chi}_\tau)^\alpha(\gamma^{\sigma]}\chi_\nu)_\alpha = -(D_\nu\bar{\chi}^{[\rho})(\gamma^{\sigma]}\chi^\nu) + (D^{[\rho}\bar{\chi}_\nu)(\gamma^{\sigma]}\chi^\nu) \\ \mathrm{III} &= \epsilon^{\mu\nu[\rho}(D_\mu\bar{\chi}^\tau)\gamma_\tau\gamma^{\sigma]}\chi_\nu = \epsilon^{\mu\nu[\rho}(D_\mu\bar{\chi}^\tau)(\epsilon_\tau{}^{\sigma]\eta}\gamma_\eta + \delta^{\sigma]}_\tau)\chi_\nu \\ &= \epsilon^{\mu\nu[\rho}\epsilon_\tau{}^{\sigma]\eta}(D_\mu\bar{\chi}^\tau)\gamma_\eta\chi_\nu + \epsilon^{\mu\nu[\rho}(D_\mu\bar{\chi}^{\sigma]})\chi_\nu \\ \mathrm{IV} &= \epsilon^{\mu\nu[\rho}(\gamma^\tau D_\tau\chi_\mu)^\alpha(\gamma^{\sigma]}\chi_\nu)_\alpha = \epsilon^{\mu\nu[\rho}(\gamma^\tau)^{\alpha\beta}(D_\tau\chi_{\mu\beta})(\gamma^{\sigma]})_\alpha{}^\gamma\chi_{\nu\gamma} \\ &= \epsilon^{\mu\nu[\rho}(D_\tau\bar{\chi}_\mu)\epsilon^{\sigma]\tau\eta}\gamma_\eta\chi_\nu - \epsilon^{\mu\nu[\rho}(D^{\sigma]}\bar{\chi}_\mu)\chi_\nu \\ \mathrm{V} &= \epsilon^{\mu\nu[\sigma}(D_\mu\bar{\chi}_\nu)\chi^{\rho]} = -\epsilon^{\mu\nu[\rho}(D_\mu\bar{\chi}_\nu)\chi^{\sigma]} \\ \mathrm{VI} &= -(\gamma^\mu D^{[\sigma}\chi_\mu)^{|\alpha|}\chi^{\rho]}{}_\alpha = -(\gamma^\mu)^{\alpha\beta}(D^{[\sigma}\chi_{\mu\beta})\chi^{\rho]}{}_\alpha = -(D^{[\rho}\bar{\chi}_\mu)\gamma^{|\mu|}\chi^{\sigma]} \\ \mathrm{VII} &= (\gamma^\mu D_\mu\chi^{[\sigma})^{|\alpha|}\chi^{\rho]}{}_\alpha = (\gamma^\mu)^{\alpha\beta}(D_\mu\chi^{[\sigma}{}_\beta)\chi^{\rho]}{}_\alpha = (D_\mu\bar{\chi}^{[\rho})\gamma^{|\mu|}\chi^{\sigma]}. \end{aligned} \tag{B.7}$$

Note that the covariant derivatives in all expressions but the first one always act on the conjugate field $\bar{\chi}$. Hence it is desirable to rewrite also the first expression on this form. By using that

$$\bar{\chi}_\tau\gamma_\nu(D_\mu\chi_\eta) = -\chi_{\tau\alpha}(\gamma_\nu)^{\alpha\beta}(D_\mu\chi_{\eta\eta}) = (D_\mu\chi_{\eta\beta})(\gamma_\nu)^{\beta\alpha}\chi_{\tau\alpha} = -(D_\mu\bar{\chi}_\eta)\gamma_\nu\chi_\tau \tag{B.8}$$

we find that

$$\mathrm{I} = \epsilon^{\mu\nu[\rho}D_\mu K_\nu{}^{\sigma]} = 2\epsilon^{\mu\nu[\rho}\epsilon^{\sigma]\tau\eta}(D_\mu\bar{\chi}_\tau)\gamma_\nu\chi_\eta + \epsilon^{\mu\sigma\rho}\epsilon^{abc}(D_\mu\bar{\chi}_a)\gamma_b\chi_c. \tag{B.9}$$

In the list above we can identify two kinds of terms, those with a γ-matrix and those without. These kinds of terms must consequently cancel out separately.

We begin by showing that the terms without a γ-matrix cancel out. Such terms appear in the expressions denoted by II, III and IV. Collecting them they read

$$\epsilon^{\mu\nu[\rho}\left[(D_\mu\bar{\chi}^{\sigma]})\chi_\nu - (D^{\sigma]}\bar{\chi}_\mu)\chi_\nu - (D_\mu\bar{\chi}_\nu)\chi^{\sigma]}\right], \tag{B.10}$$

which can be written as

$$\epsilon^{\mu\nu[\rho}\left[(D_\mu\bar{\chi}^{\sigma]})\chi_\nu - (D^{\sigma]}\bar{\chi}_\mu)\chi_\nu - (D_\mu\bar{\chi}_\nu)\chi^{\sigma]}\right] = -6\epsilon_{\mu\nu}{}^{[\rho}(D^\sigma\bar{\chi}^\mu)\chi^{\nu]}. \tag{B.11}$$

Note that as we write out the antisymmetrization in the μ and ν-indices explicitly, half of the terms will vanish as an index will be repeated in the Levi-Civita symbol, thereby only leaving the three terms on the LHS. Since the RHS is antisymmetrized over four spacetime indices and we are working in three dimensions it must vanish identically.

Let us now move on to the terms that include a γ-matrix. These appear in all expressions but the fifth one in our list above. To show that they vanish we first have to compute the products of Levi-Civita symbols appearing in some of the expressions. We find that

$$\begin{aligned}
-\epsilon^{\mu\nu[\rho}\epsilon^{\sigma]\tau\eta}(D_\mu\bar{\chi}_\tau)\gamma_\eta\chi_\nu &= -(D^{[\rho}\bar{\chi}_\nu)\gamma^{\sigma]}\chi^\nu + (D_\mu\bar{\chi}^{[\rho})\gamma^{|\mu|}\chi^{\sigma]} - (D_\mu\bar{\chi}^\mu)\gamma^{[\rho}\chi^{\sigma]} + (D^{[\rho}\bar{\chi}^{\sigma]})\gamma^\nu\chi_\nu \\
\epsilon^{\mu\nu[\rho}(D_\tau\bar{\chi}_\mu)\epsilon^{\sigma]\tau\eta}\gamma_\eta\chi_\nu &= (D_\nu\bar{\chi}^{[\rho})\gamma^{\sigma]}\chi^\nu - (D^{[\rho}\bar{\chi}_\mu)\gamma^{|\mu|}\chi^{\sigma]} + (D_\mu\bar{\chi}^\mu)\gamma^{[\rho}\chi^{\sigma]} + (D^{[\rho}\bar{\chi}^{\sigma]})\gamma^\nu\chi_\nu \\
2\epsilon^{\mu\nu[\rho}\epsilon^{\sigma]\tau\eta}(D_\mu\bar{\chi}_\tau)\gamma_\nu\chi_\eta &= 2\big((D^{[\rho}\bar{\chi}_\nu)\gamma^{|\nu|}\chi^{\sigma]} - (D_\mu\bar{\chi}^{[\rho})\gamma^{\sigma]}\chi^\mu - (D_\mu\bar{\chi}^\mu)\gamma^{[\rho}\chi^{\sigma]} - (D^{[\rho}\bar{\chi}^{\sigma]})\gamma^\nu\chi_\nu\big) \\
\epsilon^{\mu\sigma\rho}\epsilon^{abc}(D_\mu\bar{\chi}_a)\gamma_b\chi_c &= 2\big((D_\mu\bar{\chi}^\mu)\gamma^{[\rho}\chi^{\sigma]} - (D_\mu\bar{\chi}^{[\rho})\gamma^{|\mu|}\chi^{\sigma]} + (D_\mu\bar{\chi}^{[\rho})\gamma^{\sigma]}\chi^\mu\big).
\end{aligned} \tag{B.12}$$

After these rewritings we simply add all the γ-terms from equation (B.7) together. We identify six different kinds of terms and find directly that they all vanish identically.

B.2 Cancellation of the χ^4-terms

It remains to show that also the χ^4-terms on the LHS of equation (B.1) vanish identically. To do this it is first convenient to note that it can be rewritten as

$$f^{[\sigma\rho]} - i\bar{\chi}^{[\sigma}\psi^{\rho]} = 0, \tag{B.13}$$

where both χ and ψ carry an internal index I which has not been written out. We note that there are seven χ^4-terms which, after multiplication by a factor of $(-4i)$, read

$$\begin{aligned}
\mathrm{I} &= -i\epsilon^{\mu\nu[\rho}\epsilon^{\sigma]}{}_{bc}K_\mu{}^bK_\nu{}^c = 2iK^{[\rho\sigma]}K + 2iK_\mu{}^{[\rho}K^{\sigma]\mu} \\
\mathrm{II} &= \epsilon^{\mu\nu[\rho}\bar{\chi}_\mu\gamma^{\sigma]}\epsilon_\nu{}^{\eta\tau}\gamma^aK_{\eta a}\chi_\tau = K^\mu{}_a\bar{\chi}_\mu(\gamma^{[\sigma}\gamma^{|a|})\chi^{\rho]} - K^{[\rho}{}_a\bar{\chi}_\nu(\gamma^{\sigma]}\gamma^a)\chi^\nu \\
&= -\epsilon^{a\tau[\rho}K^{|\mu|}{}_a\bar{\chi}_\mu\gamma_\tau\chi^{\sigma]} - K^{\mu[\rho}\bar{\chi}_\mu\chi^{\sigma]} - K^{[\rho\sigma]}\bar{\chi}^\nu\chi_\nu \\
\mathrm{III} &= -\epsilon^{\mu\nu[\rho}\bar{\chi}_\mu(\gamma^{\sigma]}\gamma^\eta\gamma^a)K_{\nu a}\chi_\eta = \epsilon^{\mu\nu[\rho}K_{\mu a}\bar{\chi}_\nu\big(\epsilon^{\sigma]\eta a} + \gamma^{\sigma]}g^{\eta a} - g^{\sigma]a}\gamma^\eta + g^{\sigma]\eta}\gamma^a\big)\chi_\eta \\
&= -K^{[\rho|\nu|}\bar{\chi}^{\sigma]}\chi_\nu + K^{\eta[\rho}\bar{\chi}_\eta\chi^{\sigma]} + K^{[\rho\sigma]}\bar{\chi}^\nu\chi_\nu \\
&\quad + \epsilon^{\mu\nu[\rho}\big(K_\mu{}^{\sigma]}\bar{\chi}_\eta\gamma^\eta\chi_\nu - K_{\mu a}\bar{\chi}^{\sigma]}\gamma^a\chi_\nu - K_\mu{}^{|\eta|}\bar{\chi}_\eta\gamma^{\sigma]}\chi_\nu\big) \\
\mathrm{IV} &= \epsilon^{\mu\nu[\rho}K_{\eta a}\bar{\chi}_\mu(\gamma^{\sigma]}\gamma^\eta\gamma^a)\chi_\nu = \epsilon^{\mu\nu[\rho}K_{\eta a}\bar{\chi}_\mu\big(\epsilon^{\sigma]\eta a} + \gamma^{\sigma]}g^{\eta a} - g^{\sigma]a}\gamma^\eta + g^{\sigma]\eta}\gamma^a\big)\chi_\nu \\
&= \epsilon^{\mu\nu[\rho}\big(- K_\eta{}^{\sigma]}\bar{\chi}_\mu\gamma^\eta\chi_\nu + K^{\sigma]}{}_a\bar{\chi}_\mu\gamma^a\chi_\nu + K\bar{\chi}_\mu\gamma^{\sigma]}\chi_\nu\big) \\
\mathrm{V} &= -\epsilon^{\mu\nu[\rho}K_{\mu a}\bar{\chi}^{\sigma]}\gamma^a\chi_\nu \\
\mathrm{VI} &= K^{[\rho}{}_a\bar{\chi}^{\sigma]}(\gamma^\mu\gamma^a)\chi_\mu = \epsilon^{\mu a\eta}K^{[\rho}{}_a\bar{\chi}^{\sigma]}\gamma_\eta\chi_\mu + K^{[\rho|\mu|}\bar{\chi}^{\sigma]}\chi_\mu \\
\mathrm{VII} &= K_{\mu a}\bar{\chi}^{[\rho}(\gamma^{|\mu}\gamma^{a|})\chi^{\sigma]} = \epsilon^{\mu a\eta}K_{\mu a}\bar{\chi}^{[\rho}\gamma_\eta\chi^{\sigma]}.
\end{aligned} \tag{B.14}$$

Here we have repeatedly used that $\bar{\chi}^{[\rho}\chi^{\sigma]} = 0$ and in term I we used that

$$\begin{aligned} K^{[\rho}{}_{\mu}K^{|\mu|\sigma]} &= K^{\mu[\sigma}K^{\rho]}{}_{\mu} = -K^{\mu[\rho}K^{\sigma]}{}_{\mu}, \\ K^{[\rho}{}_{\mu}K^{\sigma]\mu} &= 0, \\ K^{\mu[\rho}K_{\mu}{}^{\sigma]} &= 0, \end{aligned} \tag{B.15}$$

where the latter two equalities follow from $[K^{\mu\nu}, K^{\rho\sigma}] = 0$. K simply denotes the trace, *i.e.* $K = K_{\mu}{}^{\mu}$.

To check that these seven terms cancel out it is useful to use equation (B.3) for the contorsion to derive an expression for $\bar{\chi}_{\mu}\gamma_{\nu}\chi_{\rho}$. By multiplying equation (B.3) by $e_{\tau}{}^{a}$ and $e^{\rho a}$ one finds that

$$K_{\tau\rho} = i\left(\epsilon_{\rho}{}^{\mu\nu}\bar{\chi}_{\mu}\gamma_{\tau}\chi_{\nu} + \frac{1}{2}e_{\tau\rho}\epsilon^{\mu\nu\sigma}\bar{\chi}_{\mu}\gamma_{\nu}\chi_{\sigma}\right) \tag{B.16}$$

and

$$K = \frac{i}{2}\epsilon^{\mu\nu\rho}\bar{\chi}_{\mu}\gamma_{\nu}\chi_{\rho}, \tag{B.17}$$

respectively. Consequently,

$$K_{\tau\rho} = i\epsilon_{\rho}{}^{\mu\nu}\bar{\chi}_{\mu}\gamma_{\tau}\chi_{\nu} + e_{\tau\rho}K. \tag{B.18}$$

By now multiplying this by $\epsilon_{\sigma\eta}{}^{\rho}$ and using that $\bar{\chi}_{\mu}\gamma_{\tau}\chi_{\nu} = -\bar{\chi}_{\nu}\gamma_{\tau}\chi_{\mu}$ we, after some renaming of the indices, find that

$$\bar{\chi}_{\mu}\gamma_{\nu}\chi_{\rho} = \frac{i}{2}\big(\epsilon_{\mu\rho}{}^{\sigma}K_{\nu\sigma} + \epsilon_{\mu\nu\rho}K\big). \tag{B.19}$$

This result can now be applied to several of the terms in the list above. After having multiplied all Levi-Civita symbols together and using the relations in equation (B.15) we end up at

$$\begin{aligned} \mathrm{I} &= 2iK_{\mu}{}^{[\rho}K^{\sigma]\mu} + 2iK^{[\rho\sigma]}K \\ \mathrm{II} &= -\tfrac{i}{2}K^{[\rho\sigma]}K - K^{\mu[\rho}\bar{\chi}^{\sigma]}\chi_{\mu} - K^{[\rho\sigma]}\bar{\chi}^{\mu}\chi_{\mu} \\ \mathrm{III} &= -K^{[\rho|\mu|}\bar{\chi}^{\sigma]}\chi_{\mu} + K^{\mu[\rho}\bar{\chi}^{\sigma]}\chi_{\mu} + K^{[\rho\sigma]}\bar{\chi}^{\mu}\chi_{\mu} - \tfrac{i}{2}K_{a}{}^{[\rho}K^{\sigma]a} - \tfrac{i}{2}K^{[\rho\sigma]}K \\ \mathrm{IV} &= -iK_{a}{}^{[\rho}K^{\sigma]a} - iK^{[\rho\sigma]}K \\ \mathrm{V} &= \tfrac{i}{2}K_{a}{}^{[\rho}K^{\sigma]a} + \tfrac{i}{2}K^{[\rho\sigma]}K \\ \mathrm{VI} &= -\tfrac{i}{2}K^{[\rho\sigma]}K + K^{[\rho|\mu|}\bar{\chi}^{\sigma]}\chi_{\mu} \\ \mathrm{VII} &= -iK_{\mu}{}^{[\rho}K^{\sigma]\mu}. \end{aligned} \tag{B.20}$$

We identify two types of terms, $K\bar{\chi}\chi$ and KK-terms, that must cancel separately.

By collecting all $K\bar{\chi}\chi$-terms we find them to yield

$$(-1+1)K^{[\rho\sigma]}\bar{\chi}^{\mu}\chi_{\mu} + (-1+1)K^{\mu[\rho}\bar{\chi}^{\sigma]}\chi_{\mu} + (-1+1)K^{[\rho|\mu|}\bar{\chi}^{\sigma]}\chi_{\mu} = 0. \tag{B.21}$$

Similarly, the KK-terms yield

$$i\big(2 - \tfrac{1}{2} - 1 + \tfrac{1}{2} - 1\big)K_{\mu}{}^{[\rho}K^{\sigma]\mu} + i\big(2 - \tfrac{1}{2} - \tfrac{1}{2} - 1 + \tfrac{1}{2} - \tfrac{1}{2}\big)K^{[\rho\sigma]}K = 0. \tag{B.22}$$

This proves that also the χ^4-terms on the LHS of equation (B.1) vanish. Consequently we have now proved that equation (B.1), which is the (D)-projection of the zero field strength equation, really is an identity for our solutions for the gauge fields f and ψ.

Appendix C

Derivation of the Supersymmetric Unfolded Equations

We here set out to derive the supersymmetric contributions to the unfolded equations of level $n = 1$, $n = \frac{3}{2}$ and $n = 2$. This is just a question of applying the commutation relations of the superconformal algebra (6.28) to equation (8.59), which is the unfolded equation truncated after level $n = 2$ and spin $s = 2$, and identify the terms of the different levels. Note that the bosonic terms of the $n = 1$ and $n = 2$ equations were derived already in section 8.2.

At level $n = 1$ we first find the supersymmetric contribution

$$(-i)^2 a_{IJ} T^{IJ} \phi^{Aa} \left|S\right\rangle_A = -\frac{i}{4} a_{ij} (\sigma^{IJ})_B{}^A \phi^{Ba} \left|S\right\rangle_A , \tag{C.1}$$

where we have used equation (8.17). This is the non-abelian part of the term including the covariant derivative acting on the field ϕ^{Aa}. It appears in the exact same way at all levels. We also have the supersymmetric contributions

$$\psi_I^\beta S_\beta^I \varphi^{\dot{A}\alpha} p_\alpha \left|C\right\rangle_{\dot{A}} = -\psi_I^\beta p_\beta p_\alpha \varphi^{\dot{A}\alpha} \lambda^I \left|C\right\rangle_{\dot{A}} = \psi_I^\beta (\bar{\sigma}^I)_{\dot{A}}{}^A (\gamma^a)_{\beta\alpha} \varphi^{\dot{A}\alpha} K_a \left|S\right\rangle_A , \tag{C.2}$$

where we have used that $K^a = -\frac{1}{2}(\gamma^a)^{\alpha\beta} p_\alpha p_\beta$ implies $p_\alpha p_\beta = -(\gamma^a)_{\alpha\beta} K_a$, and

$$\begin{aligned}
-i\chi_{\beta I} Q^{\beta I} \varphi^{\dot{A}\alpha a} K_a p_\alpha \left|C\right\rangle_{\dot{A}} &= i\varphi^{\dot{A}\alpha a} \chi_{\beta I} \lambda^I (-K_a q^\beta p_\alpha + i(\gamma_a)^{\beta\gamma} p_\gamma p_\alpha) \left|C\right\rangle_{\dot{A}} \\
&= \varphi^{\dot{A}\alpha a} (\bar{\sigma}^I)_{\dot{A}}{}^A (\chi_{\alpha I} K_a - \chi_{\beta I} (\gamma_a)^{\beta\gamma} p_\gamma p_\alpha) \left|S\right\rangle_A \\
&= \varphi^{\dot{A}\alpha a} (\bar{\sigma}^I)_{\dot{A}}{}^A \big(\chi_{\alpha I} K_a + \chi_{\beta I} (\gamma_a)^{\beta\gamma} (\gamma^b)_{\gamma\alpha} K_b\big) \left|S\right\rangle_A \\
&= \varphi^{\dot{A}\alpha a} (\bar{\sigma}^I)_{\dot{A}}{}^A \big(\chi_{\alpha I} K_a + \chi_{\beta I} (-\gamma^b \gamma_a + 2\eta_a^b)^\beta{}_\alpha K_b\big) \left|S\right\rangle_A \\
&= -3\chi_{\alpha I} (\bar{\sigma}^I)_{\dot{A}}{}^A \varphi^{\dot{A}\alpha a} K_a \left|S\right\rangle_A ,
\end{aligned} \tag{C.3}$$

where we have used equation (8.7) for the vanishing of the gamma trace. Thus we end up with equation (8.75) as our unfolded $n = 1$ equation.

At level $n = \frac{3}{2}$ there is no bosonic analogue, so we have to derive all terms from the beginning. First there is the trivial contribution $-if^a \varphi^{\dot{A}\alpha} K_a p_\alpha \left|C\right\rangle_{\dot{A}}$. The term

$(-i)^2 A_1 \varphi^{\dot{A}\alpha a} K_a p_\alpha \left|C\right\rangle_{\dot{A}}$ then yields both the non-abelian part of the term including the covariant derivative $D\varphi^{\dot{A}\alpha a}$, reading $-\frac{i}{4} a_{IJ} (\bar{\sigma}^{IJ})_{\dot{B}}{}^{\dot{A}} \varphi^{\dot{B}\alpha a} K_a p_\alpha \left|C\right\rangle_{\dot{A}}$, and the spin connection part

$$\begin{aligned}
-\varphi^{\dot{A}\alpha a} \omega^b M_b K_a p_\alpha \left|C\right\rangle_{\dot{A}} &= -\varphi^{\dot{A}\alpha a} (K_a \omega^b M_b + i \omega^b \epsilon_{ba}{}^c K_c) p_\alpha \left|C\right\rangle_{\dot{A}} \\
&= i \left(\frac{1}{2} \omega^b (\gamma_b)_\alpha{}^\beta \varphi^{\dot{A}\alpha a} - \epsilon^a{}_{bc} \omega^b \varphi^{\dot{A}\beta c} \right) K_a p_\beta \left|C\right\rangle_{\dot{A}} . \qquad \text{(C.4)}
\end{aligned}$$

Furthermore, we also find the contribution

$$-i\psi_I^\alpha S_\alpha^I \phi^{Aa} K_a \left|S\right\rangle_A = -\, i\psi_I^\alpha \phi^{Aa} \lambda^I K_a p_\alpha \left|S\right\rangle_A = -i\psi_I^\alpha (\sigma^I)_A{}^{\dot{A}} \phi^{Aa} K_a p_\alpha \left|C\right\rangle_{\dot{A}} , \qquad \text{(C.5)}$$

as well as

$$\begin{aligned}
-\chi_{\beta I} Q^{\beta I} \phi^{Aab} K_{ab} \left|S\right\rangle_A &= -\phi^{Aab} \chi_{\beta I} \big(K_a Q^{\beta I} - i (\gamma_a)^{\beta\gamma} S_{\gamma I} \big) K_b \left|S\right\rangle_A \\
&= i \phi^{Aab} \chi_{\beta I} \big(K_a (\gamma_b)^{\beta\gamma} + K_b (\gamma_a)^{\beta\gamma} \big) S_{\gamma I} \left|S\right\rangle_A \\
&= 2i \phi^{Aab} \chi_{\beta I} (\sigma^I)_A{}^{\dot{A}} (\gamma_b)^{\beta\gamma} K_a p_\gamma \left|C\right\rangle_{\dot{A}} , \qquad \text{(C.6)}
\end{aligned}$$

and

$$\begin{aligned}
i\varphi^{\dot{A}\alpha ab} e^c P_c K_{ab} p_\alpha \left|C\right\rangle_{\dot{A}} &= i\varphi^{\dot{A}\alpha ab} \big[K_a e^c P_c - 2ie^c (\epsilon_{ca}{}^d M_d + \eta_{ac} D) \big] K_b p_\alpha \left|C\right\rangle_{\dot{A}} \\
&= 2\varphi^{\dot{A}\alpha ab} e^c \big[K_a (\epsilon_{cb}{}^d M_d + \eta_{bc} D) + K_b (\epsilon_{ca}{}^d M_d + \eta_{ac} D) + i\epsilon_{ca}{}^d \epsilon_{db}{}^e K_e \\
&\quad - i\eta_{ac} K_b \big] p_\alpha \left|C\right\rangle_{\dot{A}} \\
&= -\, 2i\varphi^{\dot{A}\alpha ab} e^c \big[(\epsilon_{cb}{}^d (\gamma_d)_\alpha{}^\beta p_\beta + 2\eta_{bc} p_\alpha) K_a + (\eta_{bc} K_a - \eta_{ab} K_c) p_\alpha \\
&\quad + \eta_{ac} K_b p_\alpha \big] \left|C\right\rangle_{\dot{A}} \\
&= -\, 2i\varphi^{\dot{A}\alpha ab} e^c \big[(-\gamma_b \gamma_c + \eta_{bc})_\alpha{}^\beta p_\beta + 4\eta_{bc} p_\alpha \big] K_a \left|C\right\rangle_{\dot{A}} \\
&= -\, 10i\varphi^{\dot{A}\alpha ab} e_b K_a p_\alpha \left|C\right\rangle_{\dot{A}} , \qquad \text{(C.7)}
\end{aligned}$$

where we repeatedly have used that the indices of the singleton fields are in the symmetrized traceless representations. By adding these six contributions together we obtain the unfolded $n = \frac{3}{2}$ equation (8.76).

At level $n = 2$ we once again find the term including the non-abelian part of the covariant derivative, *i.e.*, $\frac{1}{4} a_{IJ} (\sigma^{IJ})_B{}^A \phi^{Bab} \left|S\right\rangle_A$. In addition to this, we find the contributions

$$\begin{aligned}
-i\psi_I^\beta S_\beta^I \varphi^{\dot{A}\alpha a} K_a p_\alpha \left|C\right\rangle_{\dot{A}} &= -i\varphi^{\dot{A}\alpha a} \psi_I^\beta \lambda^I K_a p_\beta p_\alpha \left|C\right\rangle_{\dot{A}} \\
&= i\varphi^{\dot{A}\alpha a} \psi_I^\beta (\bar{\sigma}^I)_{\dot{A}}{}^A (\gamma^b)_{\alpha\beta} K_{ab} \left|S\right\rangle_A \qquad \text{(C.8)}
\end{aligned}$$

and

$$\begin{aligned}
-\chi_{\beta I} Q^{\beta I} \varphi^{\dot{A}\alpha ab} K_a K_b p_\alpha \left|C\right\rangle_{\dot{A}} &= -\varphi^{\dot{A}\alpha ab} \chi_{\beta I} (K_a Q^{\beta I} - i(\gamma_a)^{\beta\gamma} S_\gamma^I) K_b p_\alpha \left|C\right\rangle_{\dot{A}} \\
&= -\varphi^{\dot{A}\alpha ab} \chi_{\beta I} (K_{ab} \lambda^I q^\beta - iK_a (\gamma_b)^{\beta\gamma} \lambda^I p_\gamma - iK_b (\gamma_a)^{\beta\gamma} \lambda^I p_\gamma) p_\alpha \left|C\right\rangle_{\dot{A}} \\
&= -i\varphi^{\dot{A}\alpha ab} (\chi_{\alpha I} \lambda^I K_{ab} + 2\chi^{\beta I} \lambda_I K_a (\gamma_b)_\beta{}^\gamma (\gamma^c)_{\gamma\alpha} K_c) \left|C\right\rangle_{\dot{A}} \\
&= -5i\varphi^{\dot{A}\alpha ab} \chi_{\alpha I} (\bar{\sigma}^I)_{\dot{A}}{}^A K_{ab} \left|S\right\rangle_A , \qquad \text{(C.9)}
\end{aligned}$$

which give us equation (8.77) as the unfolded $n = 2$ equation.

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