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K-Homology and Analytic Surgery of Finite Groups

Master's thesis in Engineering Mathematics and Computational Sciences

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Department of Mathematical Sciences
CHALMERS UNIVERSITY OF TECHNOLOGY
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Abstract

In this master's thesis, the K-homology of classifying spaces of finite groups and analytical surgery of finite groups are studied. The first part of the thesis introduces the theory necessary to formulate the main results. The main results, i.e. the calculation of the K-homology of the classifying space of a finite group and a generalization to certain infinite groups, are found in Chapter 5. In addition to the presentation of theoretical tools, examples are discussed to aid the reader's intuition. The intended audience is graduate students with basic understanding of topology and functional analysis.

Keywords: K-homology, analytic surgery, classifying spaces, finite groups, K-theory.

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Jack Ekenstam, Gothenburg, June 2020

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1
Introduction

In the study of topology, one aims to investigate the qualitative properties of space such as connectedness or compactness. Over the years, it has become clear that algebraic methods constitute an important set of tools to understand certain such properties. For example, algebraic methods allow for a short proof of the fact that if an open set $U \subseteq \mathbb{R}^n$ is homeomorphic to an open set $V \subseteq \mathbb{R}^m$, then $n = m$ (Theorem 2.26 of [7]). The proof relies on the machinery of homology theories, which will play an important role in this thesis as well.

The fundamental idea of homology theories - and the dual notion of cohomology theories - is to assign a collection of groups to a space X in a controlled way. If any of the homology groups differ between two spaces X and Y, it follows that X is not homeomorphic to Y. This idea can be quite fruitful since it usually is much easier to determine if any of the homology groups differ, rather than directly showing that two spaces are not homeomorphic. On the other hand, it is possible that two distinct spaces share the same homology groups. In other words, the process of mapping a space to the collection of its homology groups discards some information. In brief, homology groups are topological invariants of spaces.

The main focus of this thesis will be K-homology and the related cohomology theory, K-theory. What makes these theories especially attractive is that they extend naturally to C*-algebras, which can be thought of as generalized (or potentially noncommutative) spaces.

In particular, for a finite group G, we will study the assembly map $\mu : K_i(BG) \to K_i(C^*(G))$, for $i = 0, 1$. $K_i(BG)$ is the K-homology of the classifying space of a group G and $K_i(C^*(G))$ is the K-theory of the group C*-algebra of G. More will be said about the domain and codomain of μ throughout the text. For now, we will remark that μ fits into an exact sequence,

$$\begin{array}{ccccc} K_0(BG) & \xrightarrow{\mu} & K_0(C^*(G)) & \longrightarrow & \mathcal{S}_1(G) \\ \uparrow & & & & \downarrow \\ \mathcal{S}_0(G) & \longleftarrow & K_1(C^*(G)) & \xleftarrow[\mu]{} & K_1(BG) \end{array} \tag{1.1}$$

where $\mathcal{S}_*(G)$ can be seen as an explanation of why μ is not an isomorphism. For example, if $\mu : K_0(BG) \to K_0(C^*(G))$ is not injective, the difference of two elements with the same image can be used to define an element in $\mathcal{S}_0(G)$ that explain why these distinct elements map to same element under μ.

The sequence (1.1) is called the analytic surgery exact sequence. The name derives from surgery theory of geometric topology, where a similar exact sequence play an important role in classifying manifolds of dimension five or higher.

1.1 Structure of the Thesis

Chapter 2 begins by introducing CW-complexes and principal bundles with an aim towards defining classifying spaces of groups. A few results are mentioned but the proofs are either short or omitted. The main focus of the chapter is conceptualising the relevant space as the methods of the main results will be of algebraic nature. The chapter is concluded by a discussion of amalgamated free products of groups. Such products allows for the results of the thesis to be generalized to certain infinite groups, e.g. $SL(2,\mathbb{Z})$.

In Chapter 3, formalism and fundamental tools of homological algebra and category theory - such as exact sequences and some functors - are discussed. These are essential to the understanding of the following chapters. The abundance of technical results is increased.

Chapter 4 gives a brief introduction to the K-theory of C*-algebras. It must be stressed that entire books can be written about introductory K-theory. In this thesis, the aim has been to include discussion only of results that are fundamental to K-theory or play a vital role in the main results. A more detailed account of K-theory can be found in [15], for example.

The first part of Chapter 5 introduces K-homology following N. Higson and J. Roe [8]. In particular, the universal coefficient theorem (Theorem 5.4.1), which allows K-homology to be calculated in terms of K-theory, is presented. Section 5.5 contain the first main result of the thesis along with some examples.

Theorem 1.1.1 (K-Homology of Classifying Spaces of Finite Groups) *Let G be a finite group. Denote by $\mathcal{P}(G)$ the set of primes dividing $|G|$ and by $r(p) = |\operatorname{conj}_p(G)|$ the number of conjugacy classes consisting of elements whose order are p^d for some positive integer d. Then there are isomorphisms*

$$K_0(BG) \cong \mathbb{Z} \quad \textit{and} \quad K_1(BG) \cong \prod_{p\in\mathcal{P}(G)} (\mathbb{Z}/p^\infty)^{r(p)}$$

Here $\mathbb{Z}/p^\infty$ denotes the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of classes of rational numbers that are equivalent to $\frac{a}{b}$ with a and b coprime and $p|b$.

In section 5.6, we present the following theorem, which allows us to generalizes Theorem 1.1.1 to certain infinite groups.

Theorem 1.1.2 *Let $\varphi : K \to G$ and $\psi : K \to H$ be injective group homomorphisms and assume that φ is split by $\theta : G \to K$. Then*

$$K_i(B(G *_K H)) \cong (K_i(BG)/K_i(BK)) \oplus K_i(BH)$$

Assuming G, H, and K are finite, we may then apply Theorem 1.1.1 to $B(G*_K H)$. We find $K_0(B(G *_K H)) \cong \mathbb{Z}$ and

$$K_1(B(G *_K H)) \cong \prod_{p\in\mathcal{P}(G)} (\mathbb{Z}/p^\infty)^{r_G(p)-r_K(p)} \oplus \prod_{p\in\mathcal{P}(H)} (\mathbb{Z}/p^\infty)^{r_H(p)},$$

where $r_G(p) = |\operatorname{conj}_p(G)|$, $r_K(p) = |\operatorname{conj}_p(K)|$, and $r_H(p) = |\operatorname{conj}_p(H)|$.

In Chapter 6, we outline how these calculations could potentially be used to calculate all the terms of the analytic surgery exact sequence and give partial proofs.

1.2 Prerequisites

In order to follow the thesis it is recommended to be familiar with the basics of topology and functional analysis. An introductory course in each of the subjects should serve as sufficient preparation. Additional knowledge in areas such as algebraic topology, homological algebra, and K-theory will likely allow for a deeper understanding of the results.

2

Topological Preliminaries

This chapter aims to introduce classifying spaces, the terminology necessary to formulate the main results, and a few results that will be useful later on. For most of the chapter, the main emphasis is placed on examples, that are meant to provide the reader with intuition for the spaces that are studied. The bulk of the content is at the level of an introductory course in algebraic topology and is covered in more detail in Allen Hatcher's classical book [7] on the subject.

2.1 CW-Complexes

A CW-complex X is a topological space that is constructed in steps, starting with a set of points and inductively attaching closed disks, also called cells, of increasing dimension. On page 5 of [7], the process is described in the following three steps.

- Start with a discrete set of points. These points are referred to as 0-cells of X and the set is denoted X^0.
- X^n is obtained from X^{n-1} from attaching maps $\varphi_\alpha : D^n_\alpha \to X^{n-1}$ by identifying $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Formally, X^n is the quotient of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ by the equivalence relation $\sim$. It is required that φ_α is a homeomorphism from the interior of D^n_α to its image.
- $X = \bigcup_{n\geq 0} X^n$ is given the weak topology, induced by the maps φ_α. That is, $U \subseteq X$ is open if and only if its preimage is open in each D^n_α.

If $X = X^n$ for some $n \geq 0$, then the dimension of X is defined to be the least such n. If no such n exists, X is said to be infinite dimensional. X^n is called the n-skeleton of X.

Definition 2.1.1 (Finite Type) A CW-complex X is said to be finite if it can be constructed using finitely many cells. X is said to be of finite type if every n-skeleton X^n is finite.

Proposition 2.1.2 *A finite CW-complex X is compact.*

Proof. X is a quotient of a finite disjoint union of compact sets. □

From our construction of a model of the classifying spaces of a finite group in Section 2.4.2, it will be evident that they are of finite type. The above result then allows us to focus on K-homology of compact spaces, which simplifies some of the technical aspects.

2.2 Homotopy Type

In algebraic topology, homotopy of maps is studied in both homotopy theory and homology. We say that two continuous maps $f, g : X \to Y$ are homotopic if there is a continuous function (in the product topology) $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. In this case, we write $f \simeq g$.

Definition 2.2.1 (Homotopy Type) Let X and Y be topological spaces. X and Y are said to be homotopy equivalent or of the same homotopy type if there are functions $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq \mathrm{Id}_Y$ and $gf \simeq \mathrm{Id}_X$. The map g is called a homotopy inverse of f and we write $X \simeq Y$. If a space has the homotopy type of a point, it is said to be contractible.

It follows from the definition that the identity map is homotopic to the constant map if and only if X is a contractible.

If two spaces are homeomorphic, they are of course also homotopy equivalent. While the converse is not true, homotopy equivalent spaces often share algebraic invariants, e.g. fundamental group and homology groups. This means that it suffice to find such invariants for a space that is homotopy equivalent to the space we are interested in.

2.3 Covering Spaces

Covering spaces arise as a kind of geometrical counterpart of the fundamental group and is also a useful tool for calculating the fundamental group of certain spaces. In particular, covering space techniques can be applied to calculate the fundamental group of the circle $\pi_1(S^1) \cong \mathbb{Z}$. This result implies several theorems, e.g. the 2-dimensional versions of Brouwer's fixed-point theorem (Theorem 1.9 of [7]) and the Borsuk-Ulam theorem (Theorem 1.10 of [7]).

A covering space $\widetilde{X}$ of a space X will be similar to X locally but may be quite different globally. This is encapsulated in the following definition and example.

Definition 2.3.1 (Covering Space, page 56 of [7]) A covering space, or simply a covering, of a topological space X is a topological space $\widetilde{X}$ together with a mapping $p : \widetilde{X} \to X$, enjoying the following property. For each $x \in X$ there is a neighbourhood $U \subseteq X$ of x such that $p^{-1}(U) \subseteq \widetilde{X}$ consists of a union of mutually disjoint open sets V_α and the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Such a neighbourhood U will be called evenly covered.

Example 2.3.2 (A Covering of S^1): Using the same notation as above, we may choose $X = S^1 \subseteq \mathbb{C}$, $\widetilde{X} = \mathbb{R}$, and $p(x) = \exp(2\pi i x)$. For a point $x_0 = \exp(2\pi i \theta_0) \in S^1$, let U be the quarter circle ranging from $\exp\left(2\pi i\theta_0 - \frac{\pi i}{4}\right)$ to $\exp\left(2\pi i\theta_0 + \frac{\pi i}{4}\right)$. The preimage of U is then the union of the disjoint sets $V_n = \{\theta_0 + n + x : -\frac{1}{8} < x < \frac{1}{8}\}$. It is straight forward to verify that $p|_{V_n}$ has inverse $\exp(2\pi i x) \mapsto n + x$. Since $p|_{V_n}$ and its inverse are continuous, $p|_{V_n}$ is a homeomorphism.

This example illustrates two relatively different spaces that nevertheless are locally homeomorphic.

Among the covering spaces of a given space X, there is a notion of universal cover which, in turn, covers every connected covering space if it exists. See Section 1.3 of [7] for further details.

Definition 2.3.3 (Universal Cover) A topological space X is said to be simply connected if it is path-connected and the fundamental group $\pi_1(X)$ is trivial. A covering $p : \widetilde{X} \to X$ is called a universal cover of X if $\widetilde{X}$ is simply connected.

Each connected CW-complex of finite type has a universal cover. This follows from the local contractibility of CW-complexes, proven as property (M) on page 230 of [17]. Moreover, this cover can be given a CW-complex structure (property (N), page 231 of [17]).

2.4 Principal Bundles

A generalization of covering spaces is fiber bundles. Specifically, we will consider principal bundles, which are fiber bundles with an associated group action.

Definition 2.4.1 (Fiber Bundle) A fiber bundle is given by (E, B, p, F), where E, B, and F are topological spaces and $p : E \to B$ is a continuous surjection such that each $x \in B$ is contained in a neighbourhood $U \subseteq B$, for which there is a homeomorphism $f : p^{-1}(U) \xrightarrow{\cong} U \times F$ such that the following diagram commutes.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{f} & U \times F \\ & \searrow^{p} & \downarrow \text{Proj} \\ & & U \end{array}$$

Heuristically, E is locally homeomorphic to $U \times F$.

Remark 2.4.2: It follows from the above diagram that the fiber over (i.e. preimage under p of) each point is homemorphic to F. We denote by F_x the fiber $p^{-1}(x)$ over $x \in B$.

Example 2.4.3 (Möbius Strip): Let E be a Möbius strip. Specifically, define E as the quotient of $[0, 1] \times [0, 1]$ by

$$(0, y) \sim (1, 1 - y) \text{ for all } y \in [0, 1].$$

Furthermore, define $p : E \to S^1 \subseteq \mathbb{C}$ by $p(x, y) = \exp(2\pi i x)$ and let $B = S^1$ and $F = [0, 1]$. To see that (E, B, p, F) defines a fiber bundle, we may - similarly to Example 2.3.2 - lift a quarter circle to a subspace homeomorphic to $(-\frac{1}{8}, \frac{1}{8}) \times [0, 1]$.

In order to introduce principal bundles we must first mention two properties of group actions.

Definition 2.4.4 (Transitive Action) Let G be a group acting on a space X. We say that G acts transitively on X if $\forall x, y \in X$, there is a $g \in G$ such that $gx = y$.

Example 2.4.5 (Action of $\mathbb{Z}$ on $\mathbb{R}$): We may define an action of $\mathbb{Z}$ on $\mathbb{R}$ by maps $x \mapsto x + n, \forall n \in \mathbb{Z}$. This group action is transitive on each subspace of the form $x + \mathbb{Z} := \{x + n : n \in \mathbb{Z}\},\ x \in \mathbb{R}$ but not on all of $\mathbb{R}$.

Definition 2.4.6 (Free Action) Let G be a group acting on a space X. The action of G will be called free if $gx \neq x$ for all $x \in X$ and $g \in G \setminus \{e\}$.

The action of Example 2.4.5 is free on all of $\mathbb{R}$.

Definition 2.4.7 (Principal Bundle) Let G denote a topological group. A principal G-bundle is a fiber bundle $p : E \to B$ together with a free action of G on E such that G takes F_x to itself for each $x \in B$. In addition, we require that G acts transitively on each fiber.

It is no coincidence that the action introduced in Example 2.4.5 is free and transitive on certain subspaces. Indeed, recall that $p : \mathbb{R} \to S^1,\ x \mapsto \exp(2\pi i x)$ is a covering space and note that, for each $z \in S^1$, the fiber $p^{-1}(z)$ is of the form $x + \mathbb{Z}$. Thus $p : \mathbb{R} \to S^1$ is a principal $\mathbb{Z}$-bundle. In fact, we have the following result.

Proposition 2.4.8 (Proposition 1.39 of [7]) *The universal cover $\widetilde{X}$ of X is a principal $\pi_1(X)$-bundle.*

The group action on $\widetilde{X}$ is by the group of homemorphisms $f : \widetilde{X} \to \widetilde{X}$ such that $p \circ f = p$. As a converse to the above proposition, we have

Proposition 2.4.9 (Proposition 1.40 of [7]) *If G acts on a path-connected and locally path-connected space Y such that each $y \in Y$ has a neighbourhood U with the property that $\{g(U)\}_{g \in G}$ is a collection of disjoint sets, then $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$. In particular, if Y is contractible, we find $G \cong \pi_1(Y/G)$.*

2.4.1 K(G,1)-Spaces

In this section, $K(G,1)$-spaces are introduced as a family of examples of principal G-bundles. Another reason to introduce $K(G,1)$-spaces, is that the classifying space of a discrete group G has the same homotopy type as $K(G,1)$-spaces. We will use this fact to construct spaces with the homotopy type of certain classifying spaces in section 2.5.

Definition 2.4.10 ($K(G,1)$-space) Given a group G, a path-connected space X is a $K(G,1)$-space if it has contractible universal cover and $\pi_1(X) = G$.

Remark 2.4.11: The above definition does not generally determine X up to homeomorphism. Therefore we may only talk about models of $K(G,1)$, as opposed to *the* $K(G,1)$*-space*. On the other hand, any two $K(G,1)$-spaces will have the same homotopy type. Hence, for (possibly generalized) homology theories $\mathcal{H}_*$, their value on the homotopy class of $K(G,1)$-spaces will be well-defined. Since the homotopy type is determined by G, this can be viewed as the study of homology of groups. In this thesis, the case when $\mathcal{H}_*$ is K-homology will be the main focus.

Example 2.4.12 (A Model of $K(\mathbb{Z},1)$): We have already encountered the covering $p : \mathbb{R} \to S^1$ numerous times. The fundamental group of S^1 is $\mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the homotopy class of loops traveling n laps around the center. $\mathbb{R}$ is contractible via $\mathbb{R} \times [0,1] \to \mathbb{R},\ (x,t) \mapsto tx$. It follows that S^1 is $K(\mathbb{Z},1)$-space.

Example 2.4.13 (A Model of $K(\mathbb{Z}/2,1)$): Considering, once again, S^1 we introduce an action by $\mathbb{Z}/2 := \mathbb{Z}/2\mathbb{Z}$ via the antipodal map $x \mapsto -x$. However, S^1 is not contractible. To amend this problem, we may embed S^1 as the equator in S^2 and

note that $S^1 \subseteq S^2$ is contractible. This idea generalises to embeddings of S^n into S^{n+1}.

Formally, consider the space $\mathbb{R}^\infty$ of sequences of real numbers where all but finitely many elements are zero as well as the subspace

$$S^\infty = \{x \in \mathbb{R}^\infty : \|x\|^2 := \Sigma_{n\geq 0} x_n^2 = 1\}.$$

Moreover, consider the homotopy of maps

$$f_t : \mathbb{R}^\infty \to \mathbb{R}^\infty \setminus \{0\},\ (1-t)(x_1, x_2, x_3, \dots) + t(0, x_1, x_2, \dots)$$

and $f_t/\|f_t\|$, giving a homotopy in S^∞ between $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$ and the identity map. Finally,

$$g_t(x_1, x_2, x_3, \dots) = (1-t)(0, x_1, x_2, \dots) + t(1, 0, 0, \dots)$$

is a homotopy connecting the identity to a constant map, proving that S^∞ is contractible.

Similarly to the S^1-case, $\mathbb{Z}/2$ acts on S^∞ via the antipodal map and $\mathbb{RP}^\infty := S^\infty/(\mathbb{Z}/2)$. By Proposition 2.4.9, $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$ and it follows that $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2, 1)$-space.

Remark 2.4.14: To generalizes the above example, S^∞ can be considered a subspace of $\mathbb{C}^\infty$. Then, we may consider the action $(z_1, z_2, \dots) \mapsto (e^{2\pi i/n} z_1, e^{2\pi i/n} z_2, \dots)$ of $\mathbb{Z}/n$ for any $n \geq 1$. By similar arguments as above, this defines a $K(\mathbb{Z}/n, 1)$-space.

The models of $K(\mathbb{Z}/n, 1)$ are infinite dimensional. This is in fact a necessity, according to the following result.

Proposition 2.4.15 (Proposition 2.45 of [7]) *Let G be a group. If G has a torsion subgroup, then any $K(G, 1)$-space is infinite-dimensional.*

2.4.2 Classifying Spaces

As a motivation for studying classifying spaces we will mention that they are universal among principal bundles and are therefore widely studied in homotopy theory. The universal property of the classifying space BG of a group G is that given a principal G-bundle $p : Y \to X$, there exists a map $\varphi : X \to BG$ such that Y is isomorphic to the pullback of $EG \to BG$ along φ, where EG is the universal cover of BG. In other words, there is map $Y \to EG$, which makes the following diagram commute.

$$\begin{array}{ccc} Y & \dashrightarrow & EG \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & BG \end{array}$$

It is important to note that the vertical arrows exist in the homotopy category of CW-complexes, i.e. the category with CW-complexes as objects and homotopy classes of continuous maps as morphisms. Hence $EG \to BG$ is only defined up to homotopy. However, our main purpose of introducing classifying spaces is that for discrete groups, they are $K(G, 1)$-spaces. Moreover, there is a general method of

constructing $EG \to BG$. This method will be referred to as the nerve construction and is outlined in the following paragraph. We will denote by BG the specific model of the classifying space obtained from the nerve construction.

Fix an arbitrary group G. Let EG be the CW-complex with one n-simplex $[g_0, \ldots, g_n]$ for each element in G^{n+1}. This n-simplex is attached so that it has faces $[g_0, \ldots, \hat{g_i}, \ldots, g_n]$, where the "hat" denotes that the g_i-vertex is deleted. To see that EG is contractible, consider the homotopy that takes $x \in [g_0, \ldots, g_n]$ to $[e]$ via the line segment in $[e, g_0, \ldots, g_n]$, where $e \in G$ is the identity. G acts on EG by left multiplication and the quotient $BG := EG/G$ is a $K(G, 1)$-space by the same argument as in Example 2.4.13.

Example 2.4.16 (Nerve Construction for $\mathbb{Z}/2$): Let us study the low-dimensional cells of $B\mathbb{Z}/2$. We will consider $\mathbb{Z}/2 = \{0, 1\}$ with addition modulo 2. Firstly, one notes that the 0-cells are all identified via the group action.

The 1-cells of $E\mathbb{Z}/2$ are $[0, 0]$, $[1, 1]$, $[0, 1]$, and $[1, 0]$. A model for the 1-skeleton of $E\mathbb{Z}/2$ can be obtained by considering the square with the two top vertices labeled by 0 and the bottom two labeled by 1. The edges are labeled by as [starting point, end point] (We may choose a clockwise orientation but this is arbitrary). Topologically, this is the same as a circle and the group action identifies $[0, 0] \sim [1, 1]$ and $[0, 1] \sim [1, 0]$, which is the same as the antipodal action of $\mathbb{Z}/2$ on the circle.

Now, for the 2-cells, we can extend the model. Consider again the square described above and add two points, labelled 0 and 1, in the middle of the square and connect each these points by an edge to each of the vertices of the square. Add faces in the triangles spanned by two of the new edges and one edge of the original triangle. These faces are labelled by the 2-cells $[x, y, z]$, $x, y, z \in \{0, 1\}$ and make up a geometrical object homeomorphic to a sphere. The group action is equivalent to the antipodal action on a sphere.

It can be noted that $EG \to BG$, obtained from the nerve construction, may be quite large and not necessarily the most efficient model for a general $K(G, 1)$. For example, $\mathbb{R} \to S^1$ are very well known spaces while $E\mathbb{Z}$ will have infinitely many cells in every positive dimension. However, as mentioned above, if G has any torsion subgroup, any $K(G, 1)$-space is infinite dimensional. Therefore, it may be reasonable to use BG as a $K(G, 1)$-model for finite groups. Furthermore, as we can see from Example 2.4.16, $B\mathbb{Z}/2$ turns out to be the same as the model described in Example 2.4.13, at least in dimensions 0 through 2.

The strength of the nerve construction is that it is functorial, i.e. group homomorphisms $\varphi : G \to H$ induce maps $B\varphi : BG \to BH$ and $E\varphi : EG \to EH$ by taking $[g_0, \ldots, g_n]$ to $[\varphi(g_0), \ldots, \varphi(g_n)]$. In particular, the projection maps $G \times H \to G$ and $G \times H \to H$ induce an isomorphism $B(G \times H) \cong BG \times BH$.

In [10], a slight generalization of BG is considered. Given a G-space X, one may form the Borel construction $EG \times_G X := (EG \times X)/G$. The action on $EG \times X$ is the diagonal action. To understand the motivation of this object, one may note the action on $EG \times X$ may have better properties than the action on X. For example, even if the action on X is not free, the action on $EG \times X$ is. Moreover, if EG is contractible, then X and $EG \times X$ are homotopy equivalent. However, in this thesis, we shall focus exclusively on the case when $X = \{\mathrm{pt}\}$ is a point and G acts trivially.

Then $EG \times_G X = (EG \times \{\text{pt}\})/G \cong EG/G = BG$.

2.5 Amalgamated Free Products of Groups

Amalgamated free products is a slight generalization of the notion of free products, which is often encountered when studying the fundamental group of wedge sums of spaces. Other than providing tools to generalizes the Theorem 5.5.6, the theory also exposes some connections between algebra and topology that is of independent interest.

Definition 2.5.1 (Amalgamated Free Product of Groups) Let G, H, and K denote groups. Furthermore, let $\varphi : K \to G$ and $\psi : K \to G$ denote injective group homomorphisms. By $G *_K H$, we mean the quotient of $G * H$ by the normal subgroup generated by elements of the form $\psi(k)\varphi(k)^{-1}$, for $k \in K$. A group of the form $G *_K H$ is called an amalgam.

The main reason to study amalgamated free products in the context of this thesis is that they interact nicely with classifying spaces, which will be useful in Chapter 5. Namely, it follows from van Kampen's theorem that

$$\pi_1(BG \vee_{BK} BH) \cong \pi_1(BG) *_{\pi_1(BK)} \pi_1(BH) \cong G *_K H,$$

where $BG \vee_{BK} BH$ denotes the quotient of the disjoint union $BG \coprod BH$ obtained by identifying the subspaces $BK \subseteq BG$ with $BK \subseteq BH$. Here, BK is identified with subspaces of BG and BH via $B\varphi$ and $B\psi$, respectively.

We may also construct a contractible universal cover of $BG \vee_{BK} BH$. Firstly, we note that BK, under the above identification, lifts to several copies of EK contained in EG (and similarly for EH). We index the copies of EK in EG by $\alpha \in I$. Now, for each $\alpha \in I$, attach a copy of EH by identifying some $EK \subseteq EH$ with $EK_\alpha \subseteq EG$. The next step is to reverse the roles of EG and EH, i.e. attaching copies of EG analogously. Continuing this process indefinitely, one obtains the universal cover of $BG \vee_{BK} BH$. The resulting space is a covering space with the map induced by the maps $EG \to BG$ and $EH \to BH$. It is contractible since each copy of EG and EH retracts to the EK they are attached to. For a more precise account of this construction, the reader is referred to Section 1.B of [7].

Now, by noting that since the spaces BG and BH are path-connected, so is $BG \vee_{BK} BH$, we have proven the following lemma.

Lemma 2.5.2 *$BG \vee_{BK} BH$ is a $K(G *_K H, 1)$-space and therefore has the same homotopy type as $B(G *_K H)$.*

To motivate the above construction, we now present a few examples of infinite groups which can be represented as an amalgamated free product of finite groups.

Example 2.5.3: Consider the group of affine transformations of the real line. That is, the group of transformations of the form $x \mapsto \lambda x + \mu$ for $\lambda, \mu \in \mathbb{R}$. Identifying $\mathbb{R}$ with the line $y = 1$ in $\mathbb{R}^2$, we may represent the affine transformations by matrices of the following form.

$$\begin{bmatrix} \lambda & \mu \\ 0 & 1 \end{bmatrix}$$

Moreover, for $\lambda = \pm 1$ and $\mu \in \mathbb{Z}$, we obtain the infinite dihedral group that will be denoted D_∞. We will show that $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$.

Firstly, note that D_∞ is generated by reflections about two distinct points, e.g. $x = 0$ and $x = 1/2$. Indeed, these reflections can be written $A(x) = -x$ and $B(x) = B((x - 1/2) + 1/2) = (1/2 - x) + 1/2 = -x + 1$ or with matrix notation

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Now,

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

and we note that $\begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & \pm n \\ 0 & 1 \end{bmatrix}$ so that A and B generate D_∞.

We define a $\psi : \mathbb{Z}/2 * \mathbb{Z}/2 \to D_\infty$ by taking the generator of the left $\mathbb{Z}/2$-factor a to A and the generator of the right $\mathbb{Z}/2$-factor b to B. ψ is well-defined since both A and B have order 2. Moreover, ψ is surjective since A and B generate D_∞. Assume $\psi(x) = \mathrm{Id}$ for some arbitrary word $x = a^\delta (ba)^k b^\varepsilon$, where $\delta, \varepsilon \in \{0, 1\}$. Then $\psi(x) = A^\delta (BA)^k B^\varepsilon$. Using that

$$(BA)^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

a straight forward calculation verifies that the unique solution to the equation $A^\delta (BA)^k B^\varepsilon = \mathrm{Id}$ is $k = \delta = \varepsilon = 0$. This proves injectivity of ψ.

Example 2.5.4: Let $SL(2, \mathbb{Z})$ denote the group of 2x2-matrices over $\mathbb{Z}$ with determinant equal to 1 with matrix multiplication. To see that $SL(2, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$, one shows that $SL(2, \mathbb{Z})$ is generated by the following elements.

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

It is then possible to define a map $\varphi : \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4 \to SL(2, \mathbb{Z})$ by sending a generator s of $\mathbb{Z}/4$ to S and a generator t of $\mathbb{Z}/6$ to T. This map is well-defined since $S^2 = -\mathrm{Id}$ and $T^3 = -\mathrm{Id}$. In order to match the multiplicative notation of $SL(2, \mathbb{Z})$, we consider $\mathbb{Z}/n$ as roots of unity.

The following standard argument shows that S and T generate $SL(2, \mathbb{Z})$. Firstly, take $X_b \in SL(2, \mathbb{Z})$

$$X_b = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $c = 0$. Since $\det(X_b) = 1$, we must have $a = d = \pm 1$. Note that

$$(ST)^{\pm n} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{\pm n} = \begin{bmatrix} 1 & \pm n \\ 0 & 1 \end{bmatrix}.$$

Hence, by letting $n = |b|$, we have $X = \pm(ST)^{\pm n}$. For a general element of $SL(2,\mathbb{Z})$, we will reduce to the above case. Consider

$$Y = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z}).$$

Without loss of generality, we may assume that $|\gamma| \geq |\alpha|$. If not, we replace Y by SY. By the Euclidean algorithm, there are integers p, q such that $\gamma = p\alpha + q$ and $0 \leq q < \alpha$. Thus

$$(ST)^{-p}Y = \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma - p\alpha & \delta - p\beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ q & \delta' \end{bmatrix}.$$

Now, take $S(ST)^{-p}Y$ and apply the same algorithm again. In each step, the absolute value of left off-diagonal element is strictly decreased and since we started with a (finite) integer, we must reach zero after, say, m steps. Therefore, we have

$$\left(\prod_{i=0}^{m-1} S(ST)^{-p_{m-i}}\right) Y = X_k$$

for some integer k, whence we conclude

$$Y = \left(\prod_{i=1}^{m} (ST)^{p_i} S\right) X_k = \pm \left(\prod_{i=1}^{m} (ST)^{p_i} S\right) (ST)^k.$$

To prove injectivity of φ, we must show that $\varphi(x) = \mathrm{Id}$ implies that x is the empty word. An arbitrary element of $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$ can be written $x = \pm s^{m_1} t^{n_1} \dots s^{m_k} t^{n_k}$. By noting that $s^2 = -1$ and $t^3 = -1$, we may assume $m_i < 2$ and $n_i < 3$. In addition, we may assume that all exponents are positive except for m_1 and n_k. However, we may multiply $\varphi(x) = \mathrm{Id}$ by S^{1-m_1} from the left and T^{2-n_k} from the right. Then we must show that if $\pm S^{1-m_1} T^{2-n_k} = \varphi(s^{1-m_1} x t^{2-n_k}) = \varphi(s t^{n_1} \dots s^{m_k} t^2)$, then x is the empty word. Proving injectivity therefore amounts to showing that Id, S, T, T^2, ST, ST^2, or their additive inverses do not appear in any non-trivial way as a product of $ST = \varphi(st)$ and $ST^2 = \varphi(st^2)$. To this end, note that any product of the factors

$$ST = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad ST^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

has non-negative entries. This follows from the fact that each entry is non-decreasing under left-multiplication by ST and ST^2. Indeed,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}.$$

It can be seen that any twofold product of ST and ST^2 has an entry strictly greater than 1. Thus Id, S, T, T^2 cannot be written as products of ST and ST^2. Furthermore, ST and ST^2 have unique representations as products of ST and ST^2.

Remark 2.5.5: By considering the quotient by the subgroup generated by $-\mathrm{Id}$ in $SL(2,\mathbb{Z})$, we find a third example of an amalgam of finite groups.

$$\mathrm{PSL}(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\langle -\mathrm{Id}\rangle \cong (\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4)/\langle -\mathrm{Id}\rangle \cong \mathbb{Z}/3 * \mathbb{Z}/2.$$

It turns out that the theory of amalgams is closely related to the theory of groups acting of trees, i.e. connected graphs with no circuits. This connection is thoroughly studied in [14]. The machinery presented in the book allows for a shorter proof that $SL(2,\mathbb{Z}) \cong \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$ and provides conditions for when a group cannot be written as an amalgam.

Theorem 2.5.6 (Theorem 15 of [14]) *Assume that G is a countable group. The set of fix points of the action by G on any tree is nonempty if and only if the following conditions hold.*

(a) *G is not an amalgam*
(b) *no quotient of G is isomorphic to $\mathbb{Z}$*
(c) *G is finitely generated*

Hypothetically, it is possible to use the above theorem to determine whether a group is an amalgam. If G is a countable group that acts freely on some tree, is finitely generated, and has no quotient isomorphic to $\mathbb{Z}$, then it follows that G is an amalgam. However, it may be simpler to utilize the theorem if you are interested in proving that a certain group is not an amalgam.

3

Category Theory and Homological Algebra

Initially, homology arose from the study of topology. It has since grown to become its own discipline with applications in areas seemingly divorced from topology such as the classification of regular rings (Chapter 4 of [16]) and in several parts of algebraic geometry. In the following chapter, a few homological tools - that will be useful in this thesis - are presented.

3.1 Exact Sequences

One of the most important computational tools of homological algebra is the notion of exact sequences. The notion extends to more general contexts but we shall only need to concern ourselves with exact sequences of modules.

Definition 3.1.1 (Exact Sequence of Modules) Let R be a ring and $M_i,\ i \in \mathbb{Z}$ be (left) R-modules. Furthermore, let $m_i : M_i \to M_{i+1}$ be R-linear maps. Then

$$\ldots \xrightarrow{m_{i-2}} M_{i-1} \xrightarrow{m_{i-1}} M_i \xrightarrow{m_i} M_{i+1} \xrightarrow{m_{i+1}} \ldots$$

is an exact sequence if $\ker(m_i) = \mathrm{im}(m_{i-1}),\ \forall i \in \mathbb{Z}$. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

will be called a short exact sequence.

It is useful to note that certain information can be extracted directly from some exact sequences. For example, if $\varphi : A \to B$ fits into an exact sequence of the form $0 \to A \xrightarrow{\varphi} B$, then Definition 3.1.1 tells us that $\ker \varphi = \mathrm{im}(0 \to A) = 0$. It follows that φ is injective. Similarly, if $A \xrightarrow{\psi} B \to 0$ is an exact sequence, then $\mathrm{im}(\psi) = \ker(B \to 0) = B$ so that ψ is surjective. These two observations can be combined to deduce that if $0 \to A \xrightarrow{\theta} B \to 0$ is exact, then $\theta : A \xrightarrow{\cong} B$ is an isomorphism.

We say that two exact sequences

$$\ldots \xrightarrow{m_{i-1}} M_i \xrightarrow{m_i} M_{i+1} \xrightarrow{m_{i+1}} \ldots \text{ and } \ldots \xrightarrow{n_{i-1}} N_i \xrightarrow{n_i} N_{i+1} \xrightarrow{n_{i+1}} \ldots$$

are isomorphic if there are isomorphims $\varphi_i : M_i \to N_i$ for each $i \in \mathbb{Z}$ that are compatible with the maps m_i and n_i, i.e. $\varphi_{i+1} \circ m_i = n_i \circ \varphi_i$.

We now turn to an example from the category of abelian groups with group homomorphisms Ab.

Example 3.1.2 ($\mathbb{Z}/n$): For $n \in \mathbb{Z}$ the quotient map $p : \mathbb{Z} \to \mathbb{Z}/n$ is a surjective map. In terms of exact sequences, this means that $\mathbb{Z} \xrightarrow{p} \mathbb{Z}/n \to 0$ is exact. If we want to extend this exact sequence to the left, we need an abelian group G and a map $G \to \mathbb{Z}$ with image $\ker(p) = n\mathbb{Z}$. For example, we can take $G = \mathbb{Z}$ and the map $x \to nx$. Then, since multiplication by n is injective,

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/n \to 0$$

is a short exact sequence.

Note that even though the first and the second term of the short exact sequence of Example 3.1.2 are isomorphic, $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0$ is not exact. This demonstrates the easily overlooked fact that the mappings are a crucial component of exact sequences. Nevertheless, mappings are sometimes omitted from the notation. For example, when it is clear from the context what the maps must be or it suffices to know that there exists an isomorphism.

3.2 Functors

In category theory, functors play the role of mappings between categories. However, one should note that a functor is not necessarily a map. This is because categories are not generally sets, e.g. Ab.

Definition 3.2.1 (Functor) Let $\mathcal{C}$ and $\mathcal{D}$ denote categories. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is a rule that associates an object $F(C)$ in $\mathcal{D}$ to each object C in $\mathcal{C}$, and morphism $F(f) : F(C_1) \to F(C_2)$ in $\mathcal{D}$ to every morphism $f : C_1 \to C_2$ in $\mathcal{C}$ such that $F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)}$ and $F(f \circ g) = F(f) \circ F(g)$. A contravariant functor instead takes morphisms $g : C_1 \to C_2$ to $F(g) : F(C_2) \to F(C_1)$ and satisfies $F(f \circ g) = F(g) \circ F(f)$.

A property of functors which will be important in the discussion of derived functors is left-exactness. We will only define left-exactness and some variants for contravariant functors but analogous definitions are possible for covariant functors.

Definition 3.2.2 (Exact Functor) Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. F is said to be half-exact if

$$0 \to A \to B \to C \to 0 \text{ is exact } \implies F(C) \to F(B) \to F(A) \text{ is exact,}$$

left-exact if

$$0 \to A \to B \to C \to 0 \text{ is exact } \implies 0 \to F(C) \to F(B) \to F(A) \text{ is exact,}$$

and right-exact if

$$0 \to A \to B \to C \to 0 \text{ is exact } \implies F(C) \to F(B) \to F(A) \to 0 \text{ is exact.}$$

F is called exact if it is both left-exact and right-exact.

In order to study functors it is sometimes useful to consider maps between functors. Formally, such maps are called a natural transformation.

Definition 3.2.3 (Natural Transformation) Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\eta : F \to G$ associates to each object $C \in \mathcal{C}$ a morphism $\eta_C : F(C) \to G(C)$ such that, for each morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

Three functors that will be used extensively in this thesis are $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$, $\varinjlim$, and $\mathrm{Ext}^1_{\mathbb{Z}}(\cdot, \mathbb{Z})$. The first two will be introduced below and $\mathrm{Ext}^1_{\mathbb{Z}}(\cdot, \mathbb{Z})$ will be presented at a later stage, following a brief introduction to derived functors.

3.2.1 The Hom Functor

Given an abelian group $M \in \mathrm{Obj}(\mathsf{Ab})$, the set of homomorphisms $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, with addition given by $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$, is an abelian group. Furthermore, a group homomorphism $\theta : M \to N$ induces a map $\theta^* : \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ by precomposition, i.e. $\theta^*(\psi) = \psi \circ \theta$. Hence, $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z}) : \mathsf{Ab} \to \mathsf{Ab}$ is a contravariant functor.

It is useful to note that $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(M \oplus N, \mathbb{Z})$. Using this observation together with the fact that every finitely generated abelian group is a direct sum of cyclic groups, it suffices to calculate $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$ in order to know the value of $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ for any finitely generated abelian group M.

Proposition 3.2.4 (Proposition 15, page 58 of [14]) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ *and* $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \cong 0,\ \forall n \geq 1$.

Proof. A homomorphism $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is determined by where it sends the generator $1 \in \mathbb{Z}$. Thus $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z},\ \varphi \mapsto \varphi(1)$ is an isomorphism and it follows that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$.

For any $\psi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$, we have

$$n\psi([1]) = \psi([n]) = \psi([0]) = 0 \implies \psi([1]) = 0 \implies \psi = 0,$$

so that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \cong 0$. □

Proposition 3.2.5 $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ *is left-exact.*

Proof. Fix an exact sequence of abelian groups $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. $\mathrm{Hom}_{\mathbb{Z}}$ takes the sequence to

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z}) \xrightarrow{g^*} \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Z}) \xrightarrow{f^*} \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}).$$

We will prove that this sequence is exact at $\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Z})$.

Exactness at $\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$: Take $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$. If $\varphi \neq 0$, there is some $c \in C$ such that $\varphi(c) \neq 0$. Since g is surjective, there is an element $b \in B$ such that $g(b) = c$. Hence $\varphi \circ g(b) = \varphi(c) \neq 0$. Thus $\psi \mapsto \psi \circ g$ is injective.

$\underline{\text{im } g* \supseteq \ker f^*}$: Take $\psi \in \ker f^*$. We shall construct a map $\theta : C \to \mathbb{Z}$ such that $\psi = \theta \circ g$. By the surjectivity of g; for any $c \in C$, there is a $b_c \in B$ with $g(b_c) = c$. Let $\theta(c) := \psi(b_c)$. We claim that this is well-defined. Indeed, for $b_c, b'_c \in B$ with $g(b_c) = g(b'_c) = c$, it follows that $g(b_c - b'_c) = 0$. That is, $b_c - b'_c \in \ker g = \text{im } f$ so that $b_c - b'_c = f(a)$ for some $a \in A$ and consequently

$$\psi(b_c) = \psi(b'_c - b'_c + b_c) = \psi(b'_c) + \psi(f(a)) = \psi(b'_c)$$

$\underline{\text{im } g* \subseteq \ker f^*}$: for any $\varphi \in \text{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$, $f^*(g^*(\varphi)) = \varphi \circ (g \circ f) = 0$. □

3.2.2 The Colimit Functor

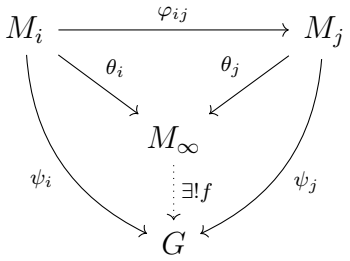
Colimits make precise the idea that $\mathbb{Q}$ is the limit of $\left\{\frac{1}{n}\mathbb{Z}\right\}_{n\geq 1}$. In other words, it is a way of making sense of limits involving algebraic structures. In this thesis, it will suffice to consider colimits in Ab. Furthermore, the indexing set will always be the positive integers. Such colimits are also called direct limits. In order to define direct limits we will need the notion of a direct system.

Definition 3.2.6 (Direct system) A direct system of abelian groups $\mathcal{M} = \{M_i; \varphi_{ij}\}$ consists of an abelian group M_i for each $i \geq 1$ and group homomorphisms $\varphi_{ij} : M_i \to M_j$ for all $i \leq j$. These maps are subject to the following conditions. For all $i \leq j \leq k$,

$$\varphi_{ii} = \text{Id} \quad \text{and} \quad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}.$$

For a group G, a homomorphism $\psi : \mathcal{M} \to G$ is defined as a system of homomorphisms $\psi_i : M_i \to G$, $\forall i \geq 1$, such that $\psi_i = \psi_j \circ \varphi_{ij}$ for $i \leq j$.

Definition 3.2.7 (Direct Limit) A homomorphism $\theta : \mathcal{M} \to M_\infty$, where M_∞ is a group, is said to have the universal property for homomorphisms from $\mathcal{M}$ to groups if for any homomorphism $\psi : \mathcal{M} \to G$, there is a unique map $f : M_\infty \to G$ such that $\psi_i = f \circ \theta_i$ for all $i \geq 1$. Then M_∞ is called the colimit of $\mathcal{M}$ and is denoted by $\underrightarrow{\text{colim}}_{i\geq 1} M_i$. In other words, there is a commutative diagram



While this definition - like other universal properties - characterizes colimits up to isomorphism, it may not provide an overwhelming amount of intuition for such an object. In order to get a better understanding of colimits, we will now construct a model for the colimit of a general direct system of abelian groups and then consider a specific example.

Fix $\mathcal{M} = \{M_i; \varphi_{ij}\}$ and consider the disjoint union $\coprod_{i\geq 1} M_i$. Define an equivalence relation $\sim$ on $\coprod_{i\geq 1} M_i$ by the following. For $m_i \in M_i$ and $m_j \in M_j$,

$$m_i \sim m_j \iff \exists k \geq i, j \text{ such that } \varphi_{ik}(m_i) = \varphi_{jk}(m_j).$$

That is, m_i is equivalent to m_j if their image coincide for some index k.

The set $\coprod_{i\geq 1} M_i/\sim$ possesses a natural abelian group structure. Indeed, the classes of $m_i \in M_i$ and $m_j \in M_j$, with $i \leq j$, can be added by noting that $m_i \sim \varphi_{ij}(m_i)$. Hence one obtains well defined addition by

$$[m_i] + [m_j] = [\varphi_{ij}(m_i) + m_j],$$

where the addition on the right hand side is inherited from the addition on M_j. It can be verified that the map $M_j \to \coprod_{i\geq 1} M_i/\sim,\ x \to [x]$ satisfies the universal property for colimits.

Example 3.2.8 ($\mathbb{Z}/p^\infty$): The following direct system will play an important role in Chapter 5. Let p be a prime and consider

$$\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \xrightarrow{p} \dots,$$

where the maps are given by multiplication by p. Denote the colimit of the system by $\mathbb{Z}/p^\infty$.

We shall prove that

$$\mathbb{Z}/p^\infty \cong \mathbb{Z}[1/p]/\mathbb{Z} = \{m/p^n \in \mathbb{Q} :\ p \nmid m,\ n \geq 1\},$$

i.e. that $\mathbb{Z}/p^\infty$ isomorphic to the quotient of additive group of the ring $\mathbb{Z}[1/p]$ by $\mathbb{Z}$. To this end, let

$$\psi : \mathbb{Z}[1/p]/\mathbb{Z} \to \mathbb{Z}/p^\infty = \coprod_{n\geq 1} (\mathbb{Z}/p^n)/\sim \text{ be defined by } m/p^n \mapsto [(m,n)].$$

If $\psi(m/p^n) = 0$, then m must be 0 and hence ψ is injective. Moreover, for any $(m,n) \in \coprod_{n\in\mathbb{Z}}(\mathbb{Z}/p^n)$, we may uniquely write $0 \leq m = p^k l < p^n$, where $p \nmid l$ and $0 \leq k < n$. Now, $l/p^{n-k} \in \mathbb{Z}[1/p]/\mathbb{Z}$ and $\psi(l/p^{n-k}) = [(l, n-k)] = [p^k l, n] = [(m,n)]$.

Lastly, we shall prove that $\varinjlim$ commutes with direct products and that it is exact. The first fact can be derived from the universal property.

Proposition 3.2.9 *Let $\mathcal{M} = \{M_i; \varphi_{ij}\}$ and $\mathcal{N} = \{N_i; \varphi'_{ij}\}$ be direct systems of abelian groups. Denote by M_∞ and N_∞ the colimits of $\mathcal{M}$ and $\mathcal{N}$, respectively. The maps associated to these colimits are denoted*

$$\theta : \mathcal{M} \to M_\infty \text{ and } \theta' : \mathcal{N} \to N_\infty.$$

Consider the direct system of abelian groups

$$\mathcal{M} \oplus \mathcal{N} := \{M_i \oplus N_i; \Phi_{ij} = (\varphi_{ij}, \varphi'_{ij})\}.$$

The map $\Theta = (\theta, \theta') : \mathcal{M} \oplus \mathcal{N} \to M_\infty \oplus N_\infty$ has the universal property of maps from $\mathcal{M} \oplus \mathcal{N}$ to sets. Hence $M_\infty \oplus N_\infty$ is the colimit of $\mathcal{M} \oplus \mathcal{N}$.

Proof. Fix a set X and a map $\Psi : \mathcal{M} \oplus \mathcal{N} \to X$. Since Ψ factors as maps $\psi : \mathcal{M} \to X$ and $\psi' : \mathcal{N} \to X$, the universal property provides maps $\alpha : M_\infty \to X$ and $\beta : N_\infty \to X$ with $\psi_i = \alpha \circ \theta_i$ and similarly for β. Now,

$$\begin{aligned}(\alpha + \beta) \circ \Theta_i(m_i, n_i) &= (\alpha + \beta)(\theta_i(m_i), \theta'_i(n_i)) = \alpha(\theta_i(m_i)) + \beta(\theta'_i(n_i)) \\ &= \psi_i(m_i) + \psi'_i(n_i) = \Psi_i(m_i, n_i)\end{aligned}$$

and if $F : M_\infty \oplus N_\infty \to X$ also satisfies $\Psi_i = F \circ \Theta_i$, then, for $i \leq j$, we have

$$\begin{aligned} F([m_i],[n_j]) &= F(\theta_i(m_i), \theta'_j(n_j)) = F(\theta_j(\varphi_{ij}(m_i)), \theta'_j(n_j)) \\ &= F \circ \Theta_j(\varphi_{ij}(m_i), n_j) = \Psi_j(\varphi_{ij}(m_i), n_j) \\ &= (\alpha + \beta) \circ \Theta_i(\varphi_{ij}(m_i), n_j) \\ &= (\alpha + \beta)([\varphi_{ij}(m_i)], [n_j]) \\ &= (\alpha + \beta)([m_i], [n_j]). \end{aligned}$$

We conclude that $\Theta : \mathcal{M} \oplus \mathcal{N} \to M_\infty \oplus N_\infty$ has the universal property for maps from $\mathcal{M} \oplus \mathcal{N}$ to sets. □

Proposition 3.2.10 $\varinjlim$ *is an exact functor.*

Proof. Assume $0 \to \{A_i; \alpha_{ij}\} \xrightarrow{f} \{B_i; \beta_{ij}\} \xrightarrow{g} \{C_i; \gamma_{ij}\} \to 0$ is an exact sequence of direct systems of abelian groups, i.e. $0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$ is exact for each i. We will use the realization as a quotient of the coproduct to show that $\varinjlim$ is exact. That is, we are proving that

$$0 \to \coprod A_i/\sim \xrightarrow{f_*} \coprod B_i/\sim \xrightarrow{g_*} \coprod C_i/\sim \to 0$$

is exact.

Exactness at $\coprod A_i/\sim$: Assume $f_*([a_i]) = [0]$. Pick representatives $a_i \in A_i$ and $0 \in B_j$ of the equivalence classes. Without loss of generality, we may assume $i \leq j$. Now,

$$\begin{aligned} f_i(a_i) \sim 0 \in B_j &\implies 0 = \beta_{ij} \circ f_i(a_i) = f_j(\alpha_{ij}(a_i)) = 0 \implies \alpha_{ij}(a_i) = 0 \\ &\implies [a_i] = [\alpha_{ij}(a_i)] = 0. \end{aligned}$$

For the second implication, injectivity of f_j was used.

Exactness at $\coprod B_i/\sim$: $\ker g_* \supseteq \operatorname{im} f_*$ is clear since

$$g_j \circ (\beta_{ij} \circ f_i(a_i)) = g_j \circ (f_j \circ \alpha_{ij}(a_i)) = 0.$$

To see that $\ker g_* \subseteq \operatorname{im} f_*$, take $[b_i] \in \coprod B_i/\sim$ such that $g_*([b_i]) = 0$. Then there is a j such that $0 = \gamma_{ij} \circ g_i(b_i) = g_j \circ \beta_{ij}(b_i)$. It follows that $\beta_{ij}(b_i) \in \operatorname{im} f_j$, i.e. $[b_i] = [\beta_{ij}(b_i)] \in \operatorname{im} f_*$.

Exactness at $\coprod C_i/\sim$: Pick a representative $c_i \in C_i$ for any class in $\coprod C_i/\sim$. By surjectivity of g_i, there is a b_i such that $[c_i] = [g_i(b_i)] = g_*([b_i])$. □

3.3 (Co)homology Theories

Having introduced some basic notions of homological algebra, we are now ready to define homology and cohomology theories. There are several examples of such theories, such as simplicial or singular (co)homology, but one can also study (co)homology theories axiomatically. The latter is the approach we will introduce presently. In Chapters 4 and 5, we will introduce K-theory - a generalized cohomology theory - as well as K-homology - a generalized homology theory.

Definition 3.3.1 (Homology Theory) Let $(H_n)_{n\in\mathbb{Z}}$ be a collection of covariant functors from the category of pairs (X, A), where X is a topological space and A is a subspace of X, to Ab. Associated to these groups, there is a natural transformation $\partial : H_n(X, A) \to H_{n-1}(A)$, where $H_{n-1}(A) := H_{n-1}(A, \emptyset)$. The functors $(H_n)_{n\in\mathbb{Z}}$ together with ∂ is called a homology theory if the following axioms are satisfied.

1. Homotopy: If $f, g : (X, A) \to (Y, B)$ and $f \simeq g$, then $f_* = g_*$. That is, homotopic maps induce the same maps on homology.
2. Excision: Let U be an open subset of X. If the closure of U is contained in the interior of A, then $H(X \setminus U, A \setminus U) \cong H(X, A)$ via the map induced by the inclusion.
3. Additivity: If $X = \coprod_{\alpha\in I} X_i$, where I is an arbitrary indexing set, then the inclusions $i_\alpha : X_\alpha \to X$ induce an isomorphism $\oplus_{\alpha\in I} H_n(X_i) \to H_n(X)$.
4. Exactness: The inclusions $i : A \to X$ and $j : X \to (X, A)$ induce a long exact sequence
$$\ldots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \ldots$$
5. Dimension: for all $n \neq 0$, $H_n(\{\mathrm{pt}\}) = 0$.

We may similarly define a cohomology theory. The difference is that the functors will be contravariant, all maps will change direction, and the additivity axiom instead provides an isomorphism $H^n(X) \to \prod_{\alpha\in I} H^n(X_i)$.

Remark 3.3.2: If the dimension axiom is omitted, we obtain a generalized (co)-homology theory.

Example 3.3.3: Consider a chain complex C_* of abelian groups. That is, a sequence
$$\ldots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots,$$
where $d_{n-1} \circ d_n = 0$. Define the homology of C_* by $H_n(C_*) := \ker(d_n)/\operatorname{im}(d_{n+1})$. Many homology theories are defined by first specifying a functor from some subcategory of Top [1] to the category of chain complexes of abelian groups and then using the above construction (See Chapter 2 of [7]).

Similarly, for a cochain complex C^* of abelian groups
$$\ldots \xleftarrow{d^{n+1}} C^{n+1} \xleftarrow{d^n} C^n \xleftarrow{d^{n-1}} C^{n-1} \xleftarrow{d^{n-2}} \ldots,$$
the cohomology of C^*, i.e. $H^n(C^*) := \ker(d^n)/\operatorname{im}(d^{n-1})$, is used to define cohomology theories.

The reason homology is indexed using subscript and cohomology is indexed using superscript derives from the fact that homology is covariant while cohomology is contravariant.

3.4 Derived Functors

Introducing derived functors in full generality requires us to first introduce some notions which have not been mentioned above. However, for our present purpose it

[1] Top is the category of topological spaces with continuous functions as morphisms.

will suffice to consider the derived functor of $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$, i.e. $\mathrm{Ext}^1_{\mathbb{Z}}(\cdot, \mathbb{Z})$, which we will introduce after a few general remarks.

Let F be a left-exact contravariant functor taking a short exact sequence

$$\xi : 0 \to A \to B \to C \to 0$$

to the exact sequence

$$F(\xi) : 0 \to F(C) \to F(B) \to F(A).$$

Since we would like to use the machinery of exact sequences, it would be useful to extend $F(\xi)$ to the right such that the sequence remains exact. The derived functor of F achieves exactly this.

3.4.1 The Ext Functor

Definition 3.4.1 ($\mathrm{Ext}^*_{\mathbb{Z}}(\cdot, \mathbb{Z})$) For $M \in \mathrm{Obj}(\mathsf{Ab})$, take an exact sequence

$$\cdots \to R_1 \to R_0 \to M \to 0,$$

where R_i are free abelian groups. Such a sequence will be referred to as a free resolution of M. Consider the cochain complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(R_0, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R_1, \mathbb{Z}) \to \dots.$$

$\mathrm{Ext}^*_{\mathbb{Z}}(G, \mathbb{Z})$ is defined as the cohomology of this cochain complex.

Remark 3.4.2: It can be shown that a free resolution always exists and that $\mathrm{Ext}^*_{\mathbb{Z}}$ does not depend on the choice of free resolution. The proofs of these statements can be found in Chapter 2 of [16].

The first thing to note about this definition is that by the left-exactness of $\mathrm{Hom}_{\mathbb{Z}}$ (Proposition 3.2.5), the sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R_0, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R_1, \mathbb{Z}) \to \dots$$

is exact. It follows that the kernel of $\mathrm{Hom}_{\mathbb{Z}}(R_0, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R_1, \mathbb{Z})$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and consequently $\mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z})$, the cohomology at position 0 of

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(R_0, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R_1, \mathbb{Z}) \to \dots,$$

is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

For a short exact sequence of abelian groups $\xi : 0 \to A \to B \to C \to 0$, the horseshoe lemma (Lemma 2.2.8 of [16]) implies that we can chose free resolutions

such that

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_1^A & \longrightarrow & R_1^B & \longrightarrow & R_1^C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_0^A & \longrightarrow & R_0^B & \longrightarrow & R_0^C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative and all rows except the lowest are split exact. Exchanging the bottom row with zeros and applying $\mathrm{Hom}_{\mathbb{Z}}$, we obtain

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R_1^C, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R_1^B, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R_1^A, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R_0^C, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R_0^B, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(R_0^A, \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The snake lemma (see Lemma 1.3.2 of [16]) now implies that the following sequence is exact.

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \\ \to \mathrm{Ext}^1_{\mathbb{Z}}(C, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(B, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \to \dots$$

As with $\mathrm{Hom}_{\mathbb{Z}}$, we may note that $\mathrm{Ext}^*_{\mathbb{Z}}$ commutes with direct sums and thereby restrict our attention to cyclic groups. Moreover, only $\mathrm{Ext}^1_{\mathbb{Z}}$ will be needed. In fact, for any $A, B \in \mathrm{Obj}(\mathsf{Ab})$, $\mathrm{Ext}^i_{\mathbb{Z}}(A, B)$ is always zero for $i \geq 2$ (Lemma 3.3.1 of [16]).

Proposition 3.4.3 *If $A, B \in \mathrm{Obj}(\mathsf{Ab})$, then*

$$\mathrm{Ext}^*_{\mathbb{Z}}(A \oplus B, \mathbb{Z}) \cong \mathrm{Ext}^*_{\mathbb{Z}}(A, \mathbb{Z}) \oplus \mathrm{Ext}^*_{\mathbb{Z}}(B, \mathbb{Z}).$$

Proof. Take free resolutions R_i^A and R_i^B of A and B, respectivly. Then $R_i^A \oplus R_i^B$ is a free resolution of $A \oplus B$ and the cohomology of

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(R_0^A \oplus R_0^B, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(R_1^A \oplus R_1^B, \mathbb{Z}) \to \dots.$$

is $\mathrm{Ext}^*_{\mathbb{Z}}(A, Z) \oplus \mathrm{Ext}^*_{\mathbb{Z}}(B, \mathbb{Z})$ since $\mathrm{Hom}_{\mathbb{Z}}$ commutes with direct sums. □

Proposition 3.4.4 $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong 0$ *and* $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n, \ \forall n \geq 1.$

Proof. Consider the free resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ of $\mathbb{Z}$, i.e. $R_0 = \mathbb{Z}$ and $R_i = 0,\ i \geq 1$. The cohomology of

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(R_0, \mathbb{Z}) \to 0$$

at $i = 1$ is $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$.

For an integer $n \geq 1$, we may consider the free resolution of $\mathbb{Z}/n$ given by $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$. Using the left-exactness of $\mathrm{Hom}_{\mathbb{Z}}$ (Proposition 3.2.5), we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$$
$$\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \ldots.$$

By Proposition 3.2.4 and the preceding result this exact sequence can be written

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \to 0.$$

Hence $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$ has a single generator of order n, i.e. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n$. □

3.5 Pro-groups

Often, some information is lost in the process of taking a colimit of a direct system. To make use of this information, it is sometimes possible to define maps before passing to the colimit. It is then useful to know how these maps behave after the colimit is taken. For example, this idea is used in the proof of the Cocompletion Theorem (Theorem 5.5.1), which plays a major role in the main results of this thesis. The following exposition is based on section 1 of [10].

Definition 3.5.1 (Inverse System) An inverse system of abelian groups $\mathcal{M} = \{M_i; \psi_{ij}\}$, indexed by the nonnegative integers, consists of an abelian group M_i for each $i \geq 0$ and group homomorphisms $\psi_{ij} : M_i \to M_j$ for all $i \geq j$ such that, for all $i \geq j \geq k$,

$$\varphi_{ii} = \mathrm{Id} \text{ and } \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}.$$

The category of pro-groups has inverse systems of abelian groups as objects. By pro-homorphisms $\theta : \{M_i; \psi_{ij}\} \to \{N_i; \psi'_{ij}\}$, we mean a set of group homomorphisms $\theta_i : M_i \to N_i$ for each $i \geq 0$ such that $\theta_j \circ \psi_{ij} = \psi'_{ij} \circ \theta_i$. Kernels and cokernels are defined as the pro-groups given by $\ker \theta_i$ and $\mathrm{coker}\, \theta_i$. θ will be called an isomorphism if each θ_i is an isomorphism.

$\{M_i; \psi_{ij}\}$ is called pro-trivial if $\forall j \geq 0$, there is an $i \geq j$ such that $\psi_{ij} = 0$. If $\ker \theta$ and $\mathrm{coker}\, \theta$ are both pro-trivial, then θ is said to be a pro-isomorphism.

A sequence $\xi : \{M_i; \psi_{ij}\} \xrightarrow{\alpha} \{M'_i; \psi'_{ij}\} \xrightarrow{\beta} \{M''_i; \psi''_{ij}\}$ is called exact if, for each $i \geq 0$, the sequence $M_i \xrightarrow{\alpha_i} M'_i \xrightarrow{\beta_i} M''_i$ is exact.

The direct systems of interest in this thesis will arise as duals of inverse systems. Therefore, the following lemma will be useful

Lemma 3.5.2 (Lemma 1.2 of [10]) *Let*

$$0 \to \{M_i\} \to \{M_i'\} \to \{M_i''\} \to 0$$

be a exact sequence of pro-groups. Then there is an exact sequence.
$0 \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(M_i'', \mathbb{Z}) \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(M_i', \mathbb{Z}) \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(M_i, \mathbb{Z})$
$\to \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(M_i'', \mathbb{Z}) \to \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(M_i', \mathbb{Z}) \to \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(M_i, \mathbb{Z}) \to 0.$
Moreover, if $f : \{M_i\} \to \{N_i\}$ is a pro-isomorphism, then it induce isomorphisms

$$\varinjlim \mathrm{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z}) \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}}(M_i, \mathbb{Z})$$
$$\varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(N_i, \mathbb{Z}) \cong \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(M_i, \mathbb{Z})$$

Proof. First apply $\mathrm{Hom}_{\mathbb{Z}}$ and to obtain an exact sequence with six terms. Applying the exact functor $\varinjlim$ (Proposition 3.2.10) proves the first statement.

To see that f induces isomorphisms on the colimits, consider the following exact sequences

$$0 \longrightarrow \{\ker f_i\} \longrightarrow \{M_i\} \longrightarrow \{\mathrm{im}\, f_i\} \longrightarrow 0$$
$$0 \longrightarrow \{\mathrm{im}\, f_i\} \longrightarrow \{N_i\} \longrightarrow \{\mathrm{coker}\, f_i\} \longrightarrow 0.$$

Applying the first part of the lemma and noting that $\varinjlim \mathrm{Hom}_{\mathbb{Z}}(\ker f_i, \mathbb{Z}) = 0 = \varinjlim \mathrm{Hom}_{\mathbb{Z}}(\mathrm{coker}\, f_i, \mathbb{Z})$ since each element is identified with 0, we obtain isomorphisms

$$\varinjlim \mathrm{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z}) \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}}(\mathrm{im}\, f_i, \mathbb{Z}) \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}}(M_i, \mathbb{Z})$$
$$\varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(N_i, \mathbb{Z}) \cong \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{im}\, f_i, \mathbb{Z}) \cong \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(M_i, \mathbb{Z})$$

□

Example 3.5.3: As an application of the above lemma, we consider the quotient of the ring $\mathbb{Z}[x]$ by $\langle x^3 - 1\rangle$ and the ideal $I = \langle x - 1\rangle$ generated by $x - 1$. We obtain a pro-group from

$$I/I \leftarrow I/I^2 \leftarrow I/I^3 \leftarrow \dots$$

by consider the underlying additive groups. The maps take $x + I^{n+1}$ to $x + I^n$. As a free group, I is generated by $x - 1$ and $x^2 - 1$. If I^n was generated by $3^{n-1}(x-1)$ and $3^{n-1}(x^2-1)$, it would follow that $I/I^n \cong \mathbb{Z}/3^{n-1} \times \mathbb{Z}/3^{n-1}$ so that we could calculate the direct limit of $\mathrm{Hom}_{\mathbb{Z}}(I/I^n, \mathbb{Z})$ similarly to Example 3.2.8. This is not the case but we will see that $\{I/I^n\}$ is still pro-isomorphic to $\{\mathbb{Z}/3^{n-1} \times \mathbb{Z}/3^{n-1}\}$.

Note that $(x-1)^3 = x^3 - 3x^2 + 3x - 1 = -3x(x-1)$. Thus $I^3 = \langle (x-1)^3\rangle$ is generated by

$$\begin{aligned} -3x(x-1) &= 3(x^2 - x), \\ a := -3x(x-1) \cdot x &= 3(x^2 - 1), \text{ and} \\ b := -3x(x-1) \cdot x^2 &= 3(1 - x) \end{aligned}$$

as a group. Moreover, since $a + b = 3(x^2 - x)$, I^3 is in fact generated by a and b as a group. By induction, it follows that I^{2k+1} is generated by $3^k(x-1)$ and $3^k(x^2-1)$ so that $3^k I \subseteq I^m,\ \forall m \leq 2k+1$. Now, consider

$$\begin{array}{ccccccc} I/I & \longleftarrow & I/3I & \longleftarrow & I/3^2I & \longleftarrow & \dots \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ I/I & \longleftarrow & I/I^2 & \longleftarrow & I/I^3 & \longleftarrow & \dots. \end{array}$$

The vertical maps $x+3^k I \mapsto x+I^{k+1}$ are well-defined and surjective since $3^k I \subseteq I^{k+1}$. The kernel consists of the elements mapped to I^{k+1}, i.e. $\ker f_k = I^{k+1}/3^k I$. This is a pro-trivial pro-group. Indeed, the image of $I^{2k+1}/3^{2k}I$ in $I^{k+1}/3^k I$ is zero since $I^{2k+1} = 3^k I$. By Lemma 3.5.2,

$$\begin{aligned} \varinjlim \operatorname{Hom}_{\mathbb{Z}}(I/I^k, \mathbb{Z}) &\cong \varinjlim \operatorname{Hom}_{\mathbb{Z}}(I/3^k I, \mathbb{Z}) \\ &\cong \varinjlim \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/3^k \times \mathbb{Z}/3^k, \mathbb{Z}) \\ &\cong \mathbb{Z}/3^\infty \times \mathbb{Z}/3^\infty. \end{aligned}$$

4

K-Theory

There are several different definitions of K-theory; some algebraic and others topological. Historically, it was a great challenge to define algebraic K-theory for a general ring. Topological K-theory is easier to define and enjoys a certain periodicity, which simplifies its structure. In particular, the Bott periodicity theorem (Theorem 4.4.11) states that $K_n(X) \cong K_{n+2}(X)$, i.e. that there only are two distinct K-theory groups. In this chapter, we will focus on the K-theory of C*-algebras, which generalizes topological K-theory. In this setting, the important periodicity theorem still applies.

4.1 C*-algebras

Definition 4.1.1 (C*-algebra) Let $(A, \|\cdot\|)$ be a complete normed vector space over $\mathbb{C}$ with the following additional structure.

- Associative and distributive multiplication $A \times A \to A$ such that

$$\forall \lambda \in \mathbb{C},\ \forall a, b \in A,\ \lambda(ab) = (\lambda a)b = a(\lambda b) \text{ and } \|ab\| \leq \|a\| \cdot \|b\|.$$

- An involution $* : A \to A$ with the following properties. $\forall \lambda \in \mathbb{C},\ \forall a, b \in A$,

$$\begin{gathered}(a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*, \\ (\lambda a)^* = \overline{\lambda} a^*, \quad (a^*)^* = a, \\ \|a^* a\| = \|a\|^2\end{gathered}$$

Remark 4.1.2: A C*-algebra A does not necessarily contain a multiplicative unit. If it does, it is called unital. Then it follows that $1^* a = (a^* 1)^* = (a^*)^* = a$, so that $1^* = 1$ by uniqueness of unit. Moreover, $\|1\| = \|1^* 1\| = \|1\|^2$, i.e. $\|1\| = 1$ (unless $A = 0$).

Definition 4.1.3 (Homomorphism of C*-algebra) Let A and B be C*-algebra. A map $\varphi : A \to B$ is called a $*$-homomorphism if it is bounded, linear, multiplicative ($\varphi(ab) = \varphi(a)\varphi(b)$), and $\varphi(a^*) = \varphi(a)^*$. If $\varphi(1) = 1$, then φ is called unital.

The archetypal example of a C*-algebra is a norm-closed subalgebra of $\mathcal{B}(H) := \{T : H \to H \text{ linear and bounded}\}$, where H is a Hilbert space, and it is possible to faithfully represent any C*-algebra A as bounded linear operators on a Hilbert space H_A. Another important example is the following.

Example 4.1.4: Let X be a locally compact Hausdorff space and consider

$$C_0(X) = \{f : X \to \mathbb{C} \text{ continuous } \mid \forall \varepsilon > 0, \exists K \subseteq X \text{ compact, s.t. } \forall x \in X \setminus K,\ |f(x)| < \varepsilon\}.$$

Take the operations to be pointwise addition, pointwise multiplication, and pointwise scalar multiplication. Let the norm and involution be given by

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f^*(x) = \overline{f(x)}$$

then $C_0(X)$ is a C*-algebra. Note that if X is compact $C_0(X)$ is equal to

$$C(X) = \{f : X \to \mathbb{C} \text{ continuous }\}.$$

Remark 4.1.5: If $X = \{\text{pt}\}$ consists of one point, then $C_0(X) = C(X) \cong \mathbb{C}$.

If A is commutative, there exists a locally compact Hausdorff space X so that $A \cong C_0(X)$ as C*-algebras. It is also interesting to note that such a space X is compact if and only if A is unital. This correspondence between the algebraic properties of A and the topological properties of X is one of many similar consequences of the commutative version of the Gelfand-Naimark Theorem, which states that there is a, so called, duality between the category of locally compact Hausdorff spaces and the category of commutative C*-algebras. It is then natural to ask whether this correspondence generalizes to the case of noncommutative C*-algebras. This question leads to the subject of noncommutative geometry, which is an active research area.

As a final example, one readily verifies that the set of $n \times n$-matrices over a C*-algebra A is also a C*-algebra with matrix operations and the operator norm. This algebra will be denoted $M_n(A)$. Formally, this can be defined as $M_n(\mathbb{C}) \otimes A$.

A convenient criterion for invertability of elements in a C*-algebra is the following.

Proposition 4.1.6 *Let $a \in A$ satisfy $\|a\| < 1$, then $1 - a$ is invertible with inverse $\sum_{n=0}^{\infty} a^n$.*

Proof. Note first that $x_N = \sum_{n=0}^{N-1} a^n$ converges to some $x \in A$ because it is Cauchy. Indeed, for $M \leq N$,

$$\|x_N - x_M\| \leq \|a\|^M \sum_{n=0}^{N-M-1} \|a\|^n \leq \|a\|^M \frac{1}{1 - \|a\|} \to 0, \text{ as } M, N \to \infty.$$

Moreover, since $x_N(1-a) = 1 - a^N = (1-a)x_N$, we have

$$\|x(1-a) - 1\| = \|(1-a)x - 1\| = \lim_N \|a^N\| \leq \lim_N \|a\|^N = 0.$$

In other words, $(1-a)^{-1} = x = \sum_{n=0}^{\infty} a^n$. □

4.1.1 The Spectral Theorem and Functional Calculus

A theorem that is used frequently in operator theory is the spectral theorem for normal operators, which generalizes the finite dimensional version for Hermitian matrices. A bounded operator $T \in \mathcal{B}(H)$ is said to be normal if $T^*T = TT^*$.

Denote by $C^*(T)$ the C*-subalgebra of $\mathcal{B}(H)$ generated by polynomials in T and T^*. Note that since a C*-algebra is complete by definition, $C^*(T)$ includes all limits of Cauchy sequences of polynomials in $C^*(T)$. If T is normal, the multiplication on $C^*(T)$ is commutative.

The spectrum $\mathrm{spec}(T)$ of T is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \,\mathrm{Id}$ is not invertible. The spectrum generalizes the concept of eigenvalues of finite dimensional matrices in the sense that if λ is an eigenvalue of T, then $T - \lambda \,\mathrm{Id}$ is not injective and thus not invertible. Therefore the spectrum contains the eigenvalues of an operator. However, all elements of the spectrum need not be eigenvalues. It can be shown that the spectrum of a bounded operator is compact and nonempty (Theorem 3.6 of [5]).

Certain properties of operators can be related to properties of their spectrum. For example if $T \in \mathcal{B}(H)$

- is self-adjoint, i.e. $T = T^*$, then $\mathrm{spec}(T) \subseteq \mathbb{R}$
- is unitary, i.e. $T^{-1} = T^*$, then $\mathrm{spec}(T) \subseteq S^1$
- is positive, i.e. $T = S^*S$ for some $S \in \mathcal{B}(H)$, then $\mathrm{spec}(T) \subseteq \mathbb{R}_{\geq 0}$

Moreover, the converse of these statements are true under the assumption that T is normal.

Theorem 4.1.7 (Spectral Theorem for Normal Operators, Theorem 1.1.11 of [8]) *Let $T \in \mathcal{B}(H)$ be a normal operator. Then the spectral map*

$$C^*(T) \to C(\mathrm{spec}(T)), \text{ defined by } T \mapsto (z \mapsto z)$$

is an isomorphism.

An important consequence of the spectral theorem is that we may sometimes define continuous functions of operators, e.g. $|T|$ or $\ln(T)$. This is achieved by functional calculus.

Definition 4.1.8 (Functional Calculus) Let $T \in \mathcal{B}(H)$ be a normal operator. For a continuous function $f \in C(\mathrm{spec}(T))$, we denote by $f(T)$ the preimage of f under the spectral map.

Example 4.1.9: For any operator $T \in \mathcal{B}(H)$, we have mentioned that the spectrum of T^*T is contained in the nonnegative real numbers. The map $f(z) = \sqrt{z}$ is therefore well-defined and continuous on $\mathrm{spec}(T^*T)$. Hence $|T| := \sqrt{T^*T}$ is well-defined for any operator. Now, since $|T|^2$ maps to the function $f(z)^2 = z$, which has preimage T^*T under the spectral map, we have $|T|^2 = T^*T$ as expected.

A way to practically deal with functions of operators is to use the fact that the C*-subalgebra generated by $z \mapsto z$ equals $C(\mathrm{spec}(T))$ by the Stone-Weierstrass theorem (Theroem 8.1 of [5]). It follows that any function $f \in C(\mathrm{spec}(T))$ can be represented as $\lim_n p_n$, where p_n is a polynomial in z and $\overline{z}$. We may therefore think of $f(T)$ as $\lim_n p_n(T) \in C^*(T)$. This point of view turns out to be useful in the proof of the following propositions.

Proposition 4.1.10 *Let $(T_n) \subseteq \mathcal{B}(H)$ denote a sequence of bounded operators with norm-limit T. If f is continuous on a compact set K that contains the spectrum of*

T and the spectrum of each T_n, then

$$\|f(T_n) - f(T)\| \to 0.$$

Proof. Fix $\varepsilon > 0$ and pick a polynomial p so that $\sup_{z\in K} |f(z)-p(z)| < \frac{\varepsilon}{3}$ (Theroem 8.1 of [5]). Pick $N \geq 1$ such that, for $n \geq N$, we have $\|p(T_n) - p(T)\| < \frac{\varepsilon}{3}$. Now, for $n \geq N$

$$\begin{aligned}\|f(T_n) - f(T)\| &\leq \|f(T_n) - p(T_n)\| + \|p(T_n) - p(T)\| + \|p(T) - f(T)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

□

Proposition 4.1.11 *Let $T, S \in \mathcal{B}(H)$ and assume T is self-adjoint and $ST = TS$. Then $Sf(T) = f(T)S$ for all $f \in C(\text{spec}(T))$.*

Proof. By assumption S commutes with T. Induction proves that S commutes with T^n and it follows that S commutes with polynomials in T. Now, fix a function $f \in C(\text{spec}(T))$. By the Stone-Weierstrass theorem (Theorem 8.1 of [5]), there is a sequence of polynomials (p_n) such that $\|f - p_n\| < 1/n$ and therefore we have $\|f(T) - p_n(T)\| < 1/n$. Now,

$$\begin{aligned}\|Sf(T) - f(T)S\| &= \|Sf(T) - Sp_n(T) + p_n(T)S - f(T)S\| \\ &\leq \|S\|\|f(T) - p_n(T)\| + \|f(T) - p_n(T)\|\|S\| \\ &< \frac{2\|S\|}{n} \to 0.\end{aligned}$$

This implies $\|Sf(T) - f(T)S\| = 0$, i.e. $Sf(T) = f(T)S$. □

4.1.2 Unitaries and Projections

The K-groups will consist of equivalence classes of unitaries or projections. We make precise these notions below.

Definition 4.1.12 (Unitaries over a C*-algebra) Let A be a unital C*-algebra. A unitary **in** A is an element u of A such that

$$uu^* = u^*u = 1.$$

A unitary **over** A is a unitary u in $M_n(A)$, for some $n \geq 0$. We say that two unitaries u and v in A are homotopic if there is a continuous path of unitaries in A, joining u to v. The homotopy class u is denoted $[u]$.

Example 4.1.13: Any unitary $u \in M_n(\mathbb{C})$ can diagonalized, i.e. $u = vDv^*$ with D a diagonal matrix and v a unitary. Furthermore, since each eigenvalue of u is contained in the unit circle, rotating each eigenvalue to 1 defines a continuous path of unitaries. Thus

$$[u] = [vDv^*] = [vIv^*] = [vv^*] = [I].$$

Definition 4.1.14 (Projections over a C*-algebra) Let A be a C*-algebra. A projection **in** A is an element p of A such that

$$p^2 = p^* = p.$$

A projection **over** A is a projection p in $M_n(A)$, for some $n \geq 0$.

Definition 4.1.15 Let p and q be projections in A

- We say that p and q are homotopic if there is a continuous path of projections in A, joining p to q. The homotopy class p is denoted $[p]$.
- We say that p and q are unitarily equivalent if there is a unitary $u \in A$ such that $q = upu^*$.

Example 4.1.16: Let p and q be projections in $M_n(\mathbb{C})$ of rank $0 \leq k \leq n$. Take a basis $\{e_i\}_{i=1}^k$ for the image of p and extend it to a basis $\{e_i\}_{i=1}^n$ for $\mathbb{C}^n$. Construct a basis $\{f_i\}_{i=1}^n$ analogously for q. The matrix representations of p and q is their respective bases are then

$$\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

where I_k is the identity matrix in $M_k(\mathbb{C})$. Now, consider the unitary change of basis matrix $u \in M_n(\mathbb{C})$. Then $q = upu^*$. Hence all projections in $M_n(\mathbb{C})$ of the same rank are unitarily equivalent.

Clearly, p and upu^* have the same rank, since $x \mapsto ux$ is an isomorphism from $\mathrm{Im}(p)$ to $\mathrm{Im}(upu^*)$. Hence $\mathrm{rank}(p) = \mathrm{rank}(q)$ if and only if p and q is unitarily equivalent.

4.1.3 Pullback in the Category of C*-algebras

In order to study the K-homology of amalgamated free products, we will introduce the corresponding notion in the category of C*-algebras.

Given three C*-algebras A_i, $i \in \{0, 1, 2\}$, and surjective $*$-homomorphisms

$$\begin{array}{ccc} & & A_1 \\ & & \downarrow{\scriptstyle \alpha_1} \\ A_2 & \xrightarrow{\alpha_2} & A_0, \end{array}$$

we define the pullback of the above diagram by

$$P := \{(a_1, a_2) \in A_1 \oplus A_2 : \alpha_1(a_1) = \alpha_2(a_2)\}$$

This concept is related to amalgamated free products of groups via the following observation.

Given groups G_i, $i \in \{0, 1, 2\}$, as well as injective group homomorphisms

$$\begin{array}{ccc} G_0 & \xrightarrow{j_1} & G_1 \\ \downarrow{\scriptstyle j_2} & & \\ G_2 & & \end{array}$$

we define maps on the n-skeletons $Bj_i^n : (BG_0)^n \to (BG_i)^n$ by $Bj_i^n([g_1, \ldots, g_k]) = [j_i(g_1), \ldots, j_i(g_k)]$. Since j_i is injective, we have

$$\begin{aligned}
& Bj_i^n([g_1, \ldots, g_k]) = Bj_i^n([g'_1, \ldots, g'_k]) && (\text{in } BG_i) \\
\implies & [xj_i(g_1), \ldots, xj_i(g_k)] = [j_i(g'_1), \ldots, j_i(g'_k)] && (\text{in } EG_i) \\
\implies & xj_i(g_l) = j_i(g'_l),\ \forall l && (\text{in } G_i) \\
\implies & x = j_i(g'_l g_l^{-1}) \in \mathrm{im}(j_i) \\
\implies & [x'g_1, \ldots, x'g_k] = [g'_1, \ldots, g'_k] && (\text{in } BG_0),
\end{aligned}$$

where $j_i(x') = x$. So the induced maps are injective. Hence $(BG_0)^n$ is a CW-subcomplex of $(BG_i)^n$. Now, consider the maps $\alpha_i^n : C((BG_i)^n) \to C((BG_0)^n)$ obtained by taking a map $(BG_i)^n \to \mathbb{C}$ to its restriction to $(BG_0)^n$. Since CW-complexes are normal, these maps are surjective (by the Tietze extension theorem, Theorem 35.1 of [12]) and fit into the following diagram.

$$\begin{array}{ccc}
 & & C((BG_1)^n) \\
 & & \big\downarrow{\scriptstyle \alpha_1^n} \\
C((BG_2)^n) & \xrightarrow{\alpha_2^n} & C((BG_0)^n)
\end{array}$$

The pullback of this diagram consists of pairs of functions (f_1, f_2) contained in $C((BG_1)^n) \oplus C((BG_2)^n)$ such that f_1 and f_2 have the same restriction to $(BG_0)^n$ or, equivalently, functions on $(BG_1)^n \vee_{(BG_0)^n} (BG_2)^n$. Note the similarity to Lemma 2.5.2.

In Section 5.6, we will use the above to relate the K-homology of $B(G *_K H)$ to the K-homology of BG, BG, and BK.

4.2 The K_0 Group

We are now ready to define the K-groups. We begin by defining K_0 for a unital C*-algebra and then extend this an arbitrary C*-algebra. The results and proofs of this section are heavily inspired by Section 4.1 of [8].

Definition 4.2.1 (K_0 of a unital C*-algebra) For a unital C*-algebra A, let $K_0(A)$ denote the free abelian group generated by homotopy classes of projections $[p]$ over A with the following relations.

$$\begin{aligned}
& [0] = 0, && \text{for any zero matrix over } A \text{ and} \\
& [p] + [q] = [p \oplus q], && \text{for any projections over } A
\end{aligned}$$

Any element of $K_0(A)$ can be expressed as $[p] - [q]$ with $p, q \in M_n(A)$, for some $n \geq 0$. Indeed, a general element of the free abelian group generated by homotopy classes of projections over A can be written $\sum n_i[p_i]$, for finitely many $n_i \in \mathbb{Z} \setminus \{0\}$. By adding zero matrices of appropriate size, we may assume each p_i is contained in some $M_n(A)$. Thus, in $K_0(A)$,

$$\sum n_i[p_i] = \sum_{n_i > 0} |n_i|[p_i] - \sum_{n_i < 0} |n_i|[p_i] = \left[\bigoplus_{n_i > 0} \left(\bigoplus_{k=1}^{|n_i|} p_i \right) \right] - \left[\bigoplus_{n_i < 0} \left(\bigoplus_{k=1}^{|n_i|} p_i \right) \right]$$

One may also note that K_0 is a covariant functor. A unital $*$-homomorphism $\varphi : A \to B$ acts on each entry of a projection p over A. It is straight forward to show that $\varphi(p)$ is a projection over B. We denote the induced homomorphism by $\varphi_* : K_0(A) \to K_0(B)$ and note that $\varphi \mapsto \varphi_*$ respects composition and $(\mathrm{Id}_A)_* = \mathrm{Id}_{K_0(A)}$. Moreover, if $\varphi, \psi : A \to B$, then $(\varphi + \psi)_* = \varphi_* + \psi_*$. Using this property, it can be shown that the natural maps $A_j \to A_1 \oplus A_2$ induce isomorphisms $K_0(A_1) \oplus K_0(A_2) \cong K_0(A_1 \oplus A_2)$.

It is useful to note that homotopic projections are unitarily equivalent. This follows from the next proposition.

Proposition 4.2.2 (Proposition 4.1.7 of [8]) *Let p and q be projections in a unital C*-algebra A and assume $\|p - q\| < 1$. Then there is a unitary $u \in A$ such that $q = upu^*$.*

Proof. Consider $x = qp + (1-q)(1-p)$. Then

$$\begin{aligned} xp &= qp^2 + (1-q)(p-p^2) \\ &= qp + 0 \\ &= q^2p + (q-q^2)(1-p) \\ &= qx \end{aligned}$$

and

$$x - 1 = 2qp - p - q = 2qp - 2q^2 - p + q = (2q-1)(p-q).$$

It follows that $\|x - 1\| \leq \|2q - 1\| \cdot \|p - q\| < 1$, which implies that x is invertible. Define $u := x(x^*x)^{-1/2}$ by functional calculus. Note that by taking adjoints, $xp = qx \implies px^* = x^*q$ and consequently $x^*xp = x^*qx = px^*x$.

Now, Proposition 4.1.11 implies

$$up = x(x^*x)^{-1/2}p = xp(x^*x)^{-1/2} = qx(x^*x)^{-1/2} = qu.$$

□

Corollary 4.2.3 (corllary 4.1.8 of [8])**.** *Let $t \mapsto p_t$ be a path of projections in a unital C*-algebra A. Then there is a path of unitaries $t \mapsto u_t$ in A such that $p_t = u_t p_0 u_t^*,\ \forall t \in [0,1]$.*

Proof. Subdivide $[0,1]$ into open intervals U_i, such that for each i, $\|p_t - p_{t'}\| < 1$, $\forall t, t' \in U_i$. By compactness, finitely many (say n) such U_i cover $[0,1]$. In particular, there is an i such that $0 \in U_i$. We note that $t_1 := \sup U_i \notin U_i$. However, $t_1 \in U_j$ for some $j \neq i$. Clearly, $U_i \cap U_j \neq \emptyset$ and we pick $t_0 \in U_i \cap U_j$. By the assumption on the intervals and the preceding proposition, there is a path of unitaries $\{\widetilde{u}_t\}$ such that $\widetilde{u}_t p_0 \widetilde{u}_t^* = p_t$ for $0 \leq t \leq t_0$ as well as a path of unitaries $\{\hat{u}_t\}$ such that $\hat{u}_t p_{t_0} \hat{u}_t^* = p_t$ for $t_0 \leq t \leq t_1$. Thus we obtain a path of unitaries $\{u_t\}$ such that $u_t p_{t_0} u_t^* = p_t$ for $0 \leq t \leq t_1$.

We may repeat the procedure with 0 replaced by t_1. We note that this process will terminate since $[t_1, 1]$ is covered by at most $n-1$ intervals, which is strictly less than n. □

While it is not true that unitarily equivalent projections are necessarily homotopic, they still define the same element in K_0 (Corollary 4.2.5). This statement - as well as many others related to K-theory - is proven by embedding a projection $p \in M_n(A)$ in $M_{n+k}(A)$, for $k > 0$. Heuristically, the statements are true if we give the projections enough leeway.

Lemma 4.2.4 (Lemma 4.1.7 of [8]) *Let A be a unital C*-algebra. If $u \in A$ is a unitary, then $u \oplus u^*$ is unitary and connected to the identity by a path of unitaries.*

Proof. Consider the path of unitaries in $M_2(A)$

$$t \mapsto \begin{bmatrix} \cos(\pi t/2)u & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2)u^* \end{bmatrix}.$$

This path implements a homotopy between $u \oplus u^*$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Repeating the same argument with $u = 1$ proves the claim. □

Corollary 4.2.5. *Let A be a unital C*-algebra. If p and q are unitarily equivalent projections over A, with $q = upu^*$. Then $[p] = [q]$ in $K_0(A)$*

Proof. Since $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$ and its adjoint are homotopic to the identity, we have

$$[p] = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} upu^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} = [q].$$

□

Example 4.2.6: In Example 4.1.16, we noted that two projections p and q over $\mathbb{C}$ are unitarily equivalent - and therefore define the same element in K_0 by Corollary 4.2.5 - if they have the same rank. Conversely, suppose p and q are connected by a path of projections over $\mathbb{C}$. Then Corollary 4.2.3 implies that there exists a unitary u over $\mathbb{C}$ such that $q = upu^*$. It follows from Example 4.1.16 that $\operatorname{rank}(p) = \operatorname{rank}(q)$. Hence a class $[p]$ in $K_0(\mathbb{C})$ is uniquely determined by its rank. It follows that $K_0(\mathbb{C}) \cong \mathbb{Z}$.

4.2.1 K_0 for non-unital algebras

To define K_0 for non-unital C*-algebras, we will consider the unitalization of A. Unitalization is a canonical way of adjoining a unit to any C*-algebra. Define $\widetilde{A} = A \times \mathbb{C}$ with multiplication given by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. We may think of A as a subalgebra of $\widetilde{A}$ since $A \cong A \times \{0\}$. Now, consider the quotient $\widetilde{A}/A \cong \mathbb{C}$; the quotient map induces a map $q_* : K_0(\widetilde{A}) \to K_0(\mathbb{C}) \cong \mathbb{Z}$. For a non-unital C*-algebra A, we define $K_0(A) := \ker(q_*)$.

Example 4.2.7: Consider $A = C_0(X)$, where X is locally compact, non-compact, and Hausdorff. The unitalization $\widetilde{A}$ is isomorphic to $C(\widetilde{X})$, where $\widetilde{X}$ is the one-point compactification of X. The image under q_* of any class $[p]$ of projections in $K_0(C(\widetilde{X}))$ is its evaluation at $x_\infty \in \widetilde{X} \setminus X$. Hence $[p] - [q]$ belongs to $K_0(C_0(X))$ if p and q agree at infinity.

4.3 The K_1 Group

There are several equivalent definitions of the K_1-group of C*-algebra A. In this section we will mention two definitions; one in terms of unitaries over A, which resembles the above definition of $K_0(A)$, as well as one in terms of K_0.

Definition 4.3.1 (K_1 of a unital C*-algebra) For a unital C*-algebra A, let $K_1(A)$ denote the free abelian group generated by homotopy classes of unitaries $[u]$ over A with the following relations.

$$\begin{aligned} &[\mathrm{Id}] = 0, && \text{for any identity matrix over } A \text{ and} \\ &[u] + [v] = [u \oplus v], && \text{for any unitaries over } A \end{aligned}$$

We note that in the same way as K_0, K_1 is a functor. The following proposition shows that an element of $K_1(A)$ can be represented as $[u]$ for some unitary over A.

Proposition 4.3.2 (Page 106 of [8]) *Let u and v be unitaries in $M_n(A)$. Then*

$$[u] + [v] = [uv] = [vu]$$

Proof. Fix $v \in M_n(A)$. Since $0 = [\mathrm{Id}]$, we have $[v] = [v \oplus \mathrm{Id}]$. Furthermore, note that

$$t \mapsto \begin{bmatrix} v\cos^2(\pi t/2) + \sin^2(\pi t/2) & 2\sin(\pi t)(v-1) \\ 2\sin(\pi t)(v-1) & \cos^2(\pi t/2) + v\sin^2(\pi t/2) \end{bmatrix}$$

defines a path of unitaries connecting $v \oplus \mathrm{Id}$ with $\mathrm{Id} \oplus v$. It follows, for any $u \in M_n(A)$, that $[uv] = [uv \oplus \mathrm{Id}] = [(u \oplus \mathrm{Id})(v \oplus \mathrm{Id})] = [(u \oplus \mathrm{Id})(\mathrm{Id} \oplus v)] = [u \oplus v] = [u] + [v]$. The second equality follows by switching the labels of u and v. □

From this proposition, it follows that $-[u] = [u^{-1}]$. Hence any element in $K_1(A)$ is represented by a formal sum of homotopy classes of unitaries over A or, equivalently, the homotopy class of the product of all those elements. Similarly to the corresponding discussion of K_0, we may add identity matrices so that all unitaries are contained in the same $M_n(A)$.

Defining K_1 for non-unital algebras can be done similarly to K_0, i.e. $K_1(A) := \ker(K_1(\widetilde{A} \to K_1(\mathbb{C}))$. However, since $K_1(\mathbb{C}) = 0$ (Example 4.1.13), we simply obtain $K_1(A) = K_1(\widetilde{A})$.

Lastly, we will comment on the second definition of K_1, which was mentioned in the introduction of this section and is valid for unital C*-algebras. For a C*-algebra A, we define the suspension SA of A by the set of continuous functions $f : [0,1] \to A$ with $f(0) = f(1) = 0$ or equivalently $C_0(0,1) \otimes A$. A slight inconvenience is that SA is non-unital even when A is unital. Therefore we must interpret $K_0(SA)$ as the kernel of $K_0(\widetilde{SA}) \to \mathbb{Z}$.

Proposition 4.3.3 *Let A be a C*-algebra, then*

$$K_0(SA) \cong K_1(A)$$

A proof for the equivalence of the definitions can be found in [8] (Proposition 4.8.2).

4.4 Some Properties of K-Theory

We will now discuss some properties of K-theory. For example, K-theory turns out to satisfy the axioms of a generalized cohomology theory when considered as a functor taking a compact Hausdorff X to $K_*(C_0(X))$. In fact, K-theory was initially defined for locally compact spaces and we will now discuss this definition very briefly.

4.4.1 K-Theory of Topological Spaces

A complex vector bundle (E, X, p, F) is a fiber bundle where the fiber F is a finite dimensional complex vector space. Moreover, some further requirements - related to compatibility with the linear structure of the vector space - are imposed. For simplicity, we restrict to the case where X is a connected compact Hausdorff space. The isomorphism classes of complex vector bundles $\mathrm{Vect}(X)$ over X form a monoid, where the operation is given by taking direct product of the fibers over each point of X. $K^0(X)$ is defined as formal differences of elements in $\mathrm{Vect}(X)$. Higher K-theory is then defined as $K^n(X) := K^0(S^nX)$, where S^nX is the one-point compactification of $R^n \times X$. Using this definition, one can prove that $K^*(X)$ coincides with the operator K-theory of $C(X)$ discussed above (see Theorem 1.4,15 of [1]). The isomorphism $K_0(C(X)) \to K^0(X)$ is given by taking $[p]$ to the vector bundle defined by the range of $p(x)$ for each $x \in X$.

The reason $*$ is a subscript for operator K-theory and a superscript for topological K-theory is that the former is a covariant functor while the latter is a contravariant functor. However, $X \mapsto K_*(C_0(X))$ is, of course, also contravariant since $X \mapsto C_0(X)$ is contravariant. In order to add to the confusion, K-homology - defined in the next chapter - will be denoted $K_*(X)$ for a locally compact Hausdorff space X and $K^*(A)$ for a C*-algebra A. The placement of $*$ depends on the functoriality as above. Thus it is important to take into account which category the input argument of an expression lies in.

4.4.2 Homotopy Invariance

We have previously defined homotopy of maps between topological spaces. This notion can be extended to C*-algebras.

Definition 4.4.1 (Homotopy of $*$-Homomorphisms) A homotopy of $*$-homomorphisms is a family $\varphi_t : A \to B$ of $*$-homomorphisms that is point-norm continuous. That is, for each $a \in A$, the map $t \mapsto \varphi_t(a)$ is continuous in the norm topology. Two $*$-homomorphisms $\varphi, \psi : A \to B$ are homotopic if there is a homotopy of $*$-homomorphisms connecting them.

If two maps $\varphi, \psi : A \to B$ are homotopic, we obtain - for any projection p over A - a path from $\varphi(p)$ to $\psi(p)$ so that $[\varphi(p)] = [\psi(p)]$ in $K_0(B)$. Using the fact that $K_1(A) \cong K_0(SA)$, the argument shows that homotopic maps induce the same map on K-theory.

When X and Y are a compact Hausdorff spaces. Two maps $f_0, f_1 : X \to Y$ are homotopic via $f_t : X \to Y$ (in the sense of topological spaces) if and only if the induced maps from $C(Y)$ to $C(X)$ are homotopic via f_t^* (in the sense of C*-algebras).

The same statement is also true for locally compact Hausdorff spaces X, Y, given that we take the morphisms in this category to be continuous proper [1] maps $U \to Y$, where U is an open subset of X.

Definition 4.4.2 (Contractible C*-algebra) We say that a C*-algebra A is contractible if the identity map $\mathrm{Id} : A \to A$ is homotopic to the zero map $a \mapsto 0$.

It follows from this definition that a contractible C*-algebra has trivial K-theory. However, one must note that even though a topological space X is contractible, $C_0(X)$ may be non-contractible. Indeed, $C_0(0,1) \cong S\mathbb{C}$ and consequently $K_1(C_0(0,1)) \cong K_1(S\mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. Here we use Bott periodicity (Theorem 4.4.11), which states that $K_1(SA) \cong K_0(A)$ for any C*-algebra.

Definition 4.4.3 We say that a $*$-homomorphism $\varphi : A \to B$ is a homotopy equivalence if there is a $\psi : B \to A$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are both homotopic to the identity.

By homotopy invariance and functoriality, a homotopy equivalence induces an isomorphism on K-theory.

4.4.3 Stability

In this section, we study how K-theory behaves with regard to colimits. The main result is that K-theory is stable, i.e. taking the tensor product with compact operators does not change K-theory.

Lemma 4.4.4 (Lemma 4.2.4 of [8]) *For a unital C*-algebra A and any $n \geq 1$, the stability map $A \to M_n(A)$*

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

induces an isomorphism on K-theory.

Proof. Considering a class of $K_0(M_n(A))$ as a class of $K_0(A)$ defines an inverse of the map induced by the stability map. Now, $K_1(A) = K_0(SA)$ and $SM_n(A) = C_0(0,1) \otimes M_n(\mathbb{C}) \otimes A = M_n(\mathbb{C}) \otimes C_0(0,1) \otimes A = M_n(SA)$ implies that $K_1(A) = K_0(SA) \to K_0(M_n(SA)) \cong K_1(M_n(A))$ is also an isomorphism. □

Example 4.4.5 ($K_*(M_n(\mathbb{C}))$): As we have seen in Example 4.2.6, $K_0(\mathbb{C}) \cong \mathbb{Z}$ and it follows from Example 4.1.13 that $K_1(\mathbb{C}) = 0$. The above proposition thus yields, for any $n \geq 1$,

$$K_0(M_n(\mathbb{C})) \cong \mathbb{Z} \text{ and } K_1(M_n(\mathbb{C})) = 0.$$

Proposition 4.4.6 (Proposition 4.1.15 of [8]) *Let $A_1 \subseteq A_2 \subseteq \ldots$ be a sequence of inclusions of C*-subalgebras of A such that the union $\cup A_i$ is dense in A. Then the inclusions $A_i \to A$ induce an isomorphism*

$$\varinjlim K_*(A_i) \cong K_*(A).$$

[1] A map $f : X \to Y$ is called proper if $K \subseteq Y$ being compact implies that $f^{-1}(K)$ is compact.

Proof. Surjectivity: Fix a projection p in $M_n(A)$. Since $\cup A_i$ is dense in A, there exists an $a \in M_n(A_j)$ for some j such that $\|a - p\| < \frac{1}{2}$. We may assume a is self-adjoint, otherwise replace a by $\frac{a+a^*}{2}$. We want to show that spectrum of a is contained in $(-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$. To this end, note that for any $\mu \in \mathbb{R}$, we have $\|p - \mu\| = \max(|\mu|, |\mu - 1|) \geq 1/2$, where $\mu := \mu \,\mathrm{Id}$. Hence, with

$$x := \frac{a - \mu}{2\|p - \mu\|},$$

we have

$$\|x\| \leq \frac{\|a - p\| + \|p - \mu\|}{2\|p - \mu\|} < \frac{1}{4\|p - \mu\|} + \frac{1}{2} \leq 1.$$

By Proposition 4.1.6, $x \pm 1$ is invertible. Equivalently, for $\mu \geq \frac{1}{2}$

$$a - \mu \pm 2\|p - \mu\| = a - \mu \pm 2\mu$$

is invertible and consequently $\mathrm{spec}(a) \subseteq (-\frac{1}{2}, \frac{3}{2})$. Furthermore, note that

$$\|a - \frac{1}{2}\| \leq \|a - p\| + \|p - \frac{1}{2}\| < \frac{1}{2} + \frac{1}{2} = 1.$$

Hence, $\frac{1}{2} \notin \mathrm{spec}(a)$ follows from Proposition 4.1.6.

Now, the indicator function $f = \chi_{(\frac{1}{2}, \infty)}$ is continuous on $\mathrm{spec}(A)$. Therefore, we may define a projection $f(a) \in M_n(A_j)$ by functional calculus. Since $\|z - f\| < \frac{1}{2}$, we have $\|f(a) - a\| < \frac{1}{2}$. Now,

$$\|p - f(a)\| \leq \|p - a\| + \|a - f(a)\| < 1.$$

By Proposition 4.2.2, p and $f(a)$ are unitarily equivalent and thus $[p] = [f(a)]$ in $K_0(A)$, which proves surjectivity.

Injectivity: Assume that two projections $p_0, p_1 \in M_n(A_j)$ are connected by a path of projections $t \mapsto p_t$ in $M_n(A)$. Then, there exists a path $t \mapsto a_t$ in $M_n(A_k)$ for some $k \geq j$ such that $a_0 = p_0$, $a_1 = p_1$, and $\|a_t - p_t\| < \frac{1}{2}$ for each t. Similarly to the above case, we may assume a_t is self-adjoint and define $f(a_t)$ for each t. Proposition 4.1.10 ensures continuity of the path $t \mapsto f(a_t)$. Since f is the identity on $\mathrm{spec}(p_0) = \mathrm{spec}(p_1) = \{0, 1\}$, we have $[p_0] = [p_1]$ in $K_0(A_k)$. □

Example 4.4.7 (Stability of K-Theory): Denote by $\mathbb{K}$ the compact operators on a separable Hilbert space. Every finite rank operator F is compact so that $M_n(\mathbb{C}) \subseteq \mathbb{K}$ for each $n \geq 1$. Conversely, every compact operator K is the limit of a sequence of finite rank operators, i.e. $\mathbb{K} \subseteq \overline{\cup M_n(\mathbb{C})}$. Hence $\mathbb{K} = \overline{\cup M_n(\mathbb{C})}$. It follows that

$$K_0(\mathbb{K}) \cong \varinjlim K_0(M_n(\mathbb{C})) \cong \varinjlim \mathbb{Z} \cong \mathbb{Z} \text{ and similarly, } K_1(\mathbb{K}) \cong 0.$$

The same argument is valid with $M_n(\mathbb{C})$ replaced by $A \otimes M_n(\mathbb{C})$. Then we find

$$K_*(A \otimes \mathbb{K}) \cong K_*(A).$$

This is known as the stability property of K-theory.

4.4.4 Bott Periodicity and the Six Term Exact Sequence

When calculating the K-theory of C*-algebra A, it is often useful to consider exact sequences relating the $K_0(A)$ with known K-groups. A natural way of finding such exact sequences is the six term exact sequence.

Theorem 4.4.8 *Given an exact sequence $0 \to J \to A \to A/J \to 0$ of C*-algebras, there are maps $\partial : K_*(A/J) \to K_{*+1}(J)$ making the following sequence is exact.*

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J) \end{array}$$

The above statement can and will be made more explicit. Namely, the maps will be defined. There are several different approaches of describing the maps and proving exactness of the sequence. We will describe one such way, indicate what needs to be proved, and provide references for the full proofs.

Proposition 4.4.9 *The functor $A \to K_*(A)$ is half-exact, i.e. if*

$$0 \to J \xrightarrow{i} A \xrightarrow{p} A/J \to 0$$

is an exact sequence of C-algebras, then*

$$K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{p_*} K_0(A/J) \quad \textit{and} \quad K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{p_*} K_1(A/J)$$

are exact.

In Section 4.3 of [8], the exactness of the left sequence is proved by considering relative K-theory groups, showing that $K_0(A, A/J) \to K_0(A) \to K_0(A/J)$ is exact as well as $K_0(A, A/J) \cong K_0(J)$. A more direct approach is taken in the proof of Theorem 6.3.2 of [15], employing results similar to Corollary 4.2.3 and Lemma 4.2.4.

The exactness of the right sequence follows by recalling that $K_1(A) \cong K_0(SA)$ and noting that $A \to SA = C_0(0,1) \otimes A$ is an exact functor.

In order to describe the map $K_1(A/J) \to K_0(J)$, we introduce the mapping cone of a surjective $*$-homomorphism.

Definition 4.4.10 (Mapping Cone) Let $\varphi : A \to B$ be a surjective $*$-homomorphism. The mapping cone of φ is the C*-subalgebra of

$$A \times \{f : [0,1] \to B \text{ continuous}\}$$

consisting of pairs (a, f) with $f(0) = 0$ and $f(1) = \varphi(a)$.

Let J be an ideal of the C*-algebra A. Denote by $C(A, A/J)$ the mapping cone of the quotient map $p : A \to A/J$. We note that there is a natural map $S(A/J) \to C(A, A/J)$ given by $f \mapsto (0, f)$ that fits into the following exact sequence.

$$\xi : 0 \to S(A/J) \to C(A, A/J) \to A \to 0$$

Applying K_0, we find that $K_1(A/J) \to K_0(C(A, A/J)) \to K_0(A)$ is exact. Furthermore, $a \to (a, 0)$ induces an isomorphism $K_0(J) \to K_0(C(A, A/J))$ (Proposition

4.5.3 of [8]). Since the composition of maps $K_0(J) \to K_0(C(A, A/J)) \to K_0(A)$ is induced by the inclusion $a \mapsto (a, 0) \mapsto a$, it follows that

$$K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{p_*} K_0(A/J)$$

is exact.

We must also show that the exact sequence can be extended to the left by $p_* : K_1(A) \to K_1(A/J)$. The argument is similar to the above. Let $C := C(A, A/J)$ and let $Z := C(C, A)$ be the mapping cone of the map $\psi : C \to A$, $(a, f) \mapsto a$. Note that $0 \to SA \to Z \to C \to 0$ is exact since the kernel of ψ is naturally isomorphic to SA. Thus

$$\eta : K_1(A) \to K_0(Z) \to K_0(J)$$

is exact.

The elements of Z are of the form (c, f), where $c = (a, g)$ with $a \in A$ and $g : [0, 1] \to A/J$. Hence there is a natural injection of $S(A/J)$ into Z, given by $g \mapsto ((0, g), 0)$. The quotient $Z/S(A/J)$ is isomorphic to the cone CA, i.e. the C*-algebra of continuous maps $f : [0, 1] \to A$ with $f(0) = 0$. Since CA is contractible via $H_t(f)(x) = f(tx)$, we find $K_1(A/J) = K_0(S(A/J)) \cong K_0(Z)$, which implies

$$\eta' : K_1(A) \to K_1(A/J) \to K_0(J)$$

Moreover, we note that the map $K_1(A/J) \to K_0(J)$ in η' is induced by $f \mapsto ((0, f), 0) \mapsto (0, f)$, i.e. the left map of ξ. Therefore,

$$\zeta : K_1(A) \to K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{p_*} K_0(A/J)$$

is exact.

To see that the leftmost map of ζ is induced by p, we will show that $SA \to Z$, $f \mapsto ((0, 0), f)$ is homotopic to the composition of $p : SA \to S(A/J)$ and $S(A/J) \to Z$, $g \mapsto ((0, g), 0)$. In other words we will show that the element $((0, 0), f)$ is homotopic to $((0, p \circ f), 0)$. To this end, consider the homotopy $h_t = ((f(1 - t), p \circ f_t'), f_t'')$, where

$$f_t'(x) = \begin{cases} 0, & \text{if } x \in [0, t) \\ f(x - t), & \text{if } x \in [t, 1] \end{cases}$$

and

$$f_t''(x) = \begin{cases} 0, & \text{if } x \in [0, 1 - t) \\ f(2 - x - t), & \text{if } x \in [1 - t, 1]. \end{cases}$$

Specifically note that, for each t, we have $f_t'(1) = f(1 - t)$ and $f_t''(1) = f(1 - t)$ so that the homotopy lies in Z. Moreover, $h_0 = ((0, p \circ f), 0)$ and $h_1 = ((0, 0), f')$, where $f'(x) = f(1 - x)$. A similar homotopy; namely,

$$t \mapsto ((2f(1 - t), 2p(f(1 - t))), f_t' + f_t'')$$

connects $((0, 0), f)$ to $((0, 0), f')$.

The above result together with Proposition 4.4.9 implies that

$$K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{p_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{p_*} K_0(A/J)$$

is exact. Setting $K_n(A) := K_0(S^n A)$, we may iterate this construction to obtain a long exact sequence

$$\ldots \xrightarrow{p_*} K_{n+1}(A/J) \xrightarrow{\partial} K_n(J) \xrightarrow{i_*} K_n(A) \xrightarrow{p_*} K_n(A/J) \xrightarrow{\partial} K_{n-1}(J) \xrightarrow{i_*} \ldots$$

ending at $K_0(A/J)$.

To prove Theorem 4.4.8, it remains to show that $K_2(A) := K_0(S^2 A) \cong K_1(SA)$ is ismorpic to $K_0(A)$. This is known as the Bott Periodicity Theorem.

Theorem 4.4.11 (Bott Periodicity Theorem) *The map*

$$K_0(A) \to K_1(SA), \quad [p] \mapsto [1 + p(z-1)]$$

is an isomorphism.

We are viewing $z \mapsto 1 + p(z-1)$ as a function f from the unit circle in $\mathbb{C}$ to A (or the unitalization of A if A is non-unital) with $f(1) \in \mathbb{C}$, i.e. an element of the unitalization of SA. In Theorem 4.9.1 of [8], this statement is proved by first developing the machinery of products on K-theory, which allows for an elegant proof. In Chapter 9 of [15], the proof is obtained by considering approximations of elements in the unitalization of SA. The latter has the advantage of being more direct.

Remark 4.4.12: By considering the unitalization exact sequence

$$0 \to A \to \widetilde{A} \to \mathbb{C} \to 0$$

of a non-unital C*-algebra A, Theorem 4.4.8 and the five lemma (See Exercise 1.3.3 of [16]) imply that Lemma 4.4.4 extends to non-unital C*-algebras, i.e. the stabilization maps induce isomorphisms for any C*-algebra.

4.5 Group C*-Algebra of Finite Groups

We will now calculate the K-theory of the group C*-algebra of a finite group. This calculation will play a crucial role in the determining the K-homology of classifying spaces of finite groups.

For a finite group G, the group C*-algebra $C^*(G)$ is simply the group algebra $\mathbb{C}[G]$ [2]. That is, formal linear combinations $\sum_{g \in G} c_g g$, $c_g \in \mathbb{C}$, or equivalently functions $G \to \mathbb{C}$. The multiplication in $\mathbb{C}[G]$ is induced by the multiplication on G. It is a standard fact of representation theory that $\mathbb{C}[G] \cong \oplus_\pi \mathrm{End}(V_\pi)$, where the direct sum is taken over all irreducible representation of G. Since $\mathrm{End}(V_\pi)$ can be identified

[2]For infinite groups, $\mathbb{C}[G]$ is not complete. One must then choose a norm and take the completion to obtain a C*-algebra. Different norms yield different completions.

with a matrix algebra, we have $K_0(\mathrm{End}(V_\pi)) \cong \mathbb{Z}$ and $K_1(\mathrm{End}(V_\pi)) \cong 0$ by Example 4.4.5. Now,

$$K_0(C^*(G)) \cong K_0(\oplus_\pi \mathrm{End}(V_\pi)) = \oplus_\pi \mathbb{Z} = R(G)$$
$$K_1(C^*(G)) = 0,$$

where $R(G)$ is the representation ring of G. Thats is, the ring with representations of G as elements and direct sum and tensor product as operations. In order to make the isomorphism from $K_0(C^*(G))$ to $R(G)$ explicit, we note that the map $\mathbb{C}[G] \to \oplus_\pi \mathrm{End}(V_\pi)$ is given by $x \to \oplus_\pi \pi(x)$. Since the $K_0(\mathbb{C}) \to \mathbb{Z}$ is given by $[p] \mapsto \mathrm{rank}(p)$, a class $[p] \in K_0(C^*(G))$ is mapped to $\oplus_\pi \mathrm{rank}(\pi(p))$.

4.5.1 Restriction Maps and the Augmentation Ideal

Fix a subgroup H of finite group G and consider the restriction map $\mathrm{res}^H_G : R(G) \to R(H)$, given by taking a representation of G to its restriction to H. When $H = 0$, we have $R(H) = \mathbb{Z}$ and $\mathrm{res}^H_G(V) = \dim V$ is called the augmentation map. The kernel I_G of the augmentation map is called the augmentation ideal.

If two representation of G have the same dimension, then they will certainly have the same dimension when restricted to H. Hence the image of I_G under res^H_G is contained in I_H. The notation $\mathrm{im}(\mathrm{res}^H_G)$ will refer to the image of the restriction $\mathrm{res}^H_G : I_G \to I_H$.

4.5.2 Equivariant K-Theory

In [10], an extension of K-theory to G-spaces, called equivariant K-theory, is considered. We denote the equivariant K-theory of X by $K^*_G(X)$. In this thesis, we will only need the equivariant K-theory of a point with G finite. In this setting, it can be shown that $K^*_G(X) \cong K_*(C^*(G))$ (Theorem 11.7.1 of [4]).

5

K-Homology

In this chapter we begin by briefly introducing some theory of K-homology, which allows us to state and prove the main results of the thesis. The main results - namely, the calculation of the K-homology of finite group and of certain infinite groups - are found in Section 5.5 and 5.6.

5.1 Fredholm Modules

The Kasparov version of K-homology, which we will focus on in this thesis, is defined in terms of Fredholm modules. Below we present some fundamentals of Fredholm modules.

Definition 5.1.1 (Ungraded Fredholm Module) Let A be a separable C*-algebra. An ungraded Fredholm module over A is given by a triple $[H, \rho, F]$, where H is a separable Hilbert space, $\rho : A \to \mathcal{B}(H)$ is a representation of A, and F is an operator on H such that the following operators are compact for each $a \in A$.

$$(F - F^*)\rho(a), \quad (F^2 - 1)\rho(a), \quad F\rho(a) - \rho(a)F. \tag{5.1}$$

When the representation is clear from context, a Fredholm module may be written $[H, F]$.

Example 5.1.2: We consider a Fredholm module over $C(S^1)$. Take $H = L^2(S^1)$ and let $\rho : C(S^1) \to \mathcal{B}(L^2(S^1))$ be defined by $g \mapsto M_g$, where $M_g f = gf$. Recall that $\{z^n\}_{n\in\mathbb{Z}}$ is a Hilbert space basis for $L^2(S^1)$. Let $H^2(S^1)$ denote the Hardy space, i.e. the subspace spanned by $\{z^n\}_{n\geq 0}$, and denote by P the projection onto $H^2(S^1)$. We claim that $L^2(S^1)$, $g \mapsto M_g$, and $F := 2P - 1$ define a Fredholm modules over $C(S^1)$.

In order to prove the claim, we need to verify that F satisfies the three compactness properties above. Since P is self-adjoint, it follows that F is also self-adjoint. Moreover, $F^2 = (2P-1)^2 = 4P^2 - 4P + 1 = 1$. It remains to show that $FM_g - M_g F$ is compact for any $g \in C(S^1)$. Note initially that the set of such g form a subalgebra of $C(S^1)$, which we will denote by $\mathcal{T}$. This can shown in a straight forward manner by appealing to the fact that the compact operators define a closed C*-subalgebra of $\mathcal{B}(H)$ (Theorem 3.4 and Corollary 3.6 of [5]). Indeed, algebraic manipulations show that $\mathcal{T}$ is closed under addition, multiplication, and taking adjoints. Additionally, if $\{g_n\}_{n\geq 0} \subseteq \mathcal{T}$ and $g_n \to g \in C(S^1)$, then $FM_{g_n} - M_{g_n}F \to FM_g - M_g F$.

Furthermore, $z \in \mathcal{T}$ since

$$\begin{aligned}(FM_z - M_zF)\sum_{n\in\mathbb{Z}} a_n z^n &= 2P\sum_{n\in\mathbb{Z}} a_n z^{n+1} - 2M_z\sum_{n\geq 0} a_n z^n \\ &= 2\sum_{n\geq -1} a_n z^{n+1} - 2\sum_{n\geq 0} a_n z^{n+1} \\ &= 2a_{-1}\end{aligned}$$

proves that $FM_z - M_zF$ is a rank one operator. Since z generates $C(S^1)$ as a C*-algebra by the Stone-Weierstrass theorem (Theroem 8.1 of [5]), we conclude that $\mathcal{T} = C(S^1)$.

A graded Fredholm module is given by the same data as an ungraded Fredholm module, with some additional information. Namely,

- H is graded. That is, $H = H_+ \oplus H_-$,
- F is odd, i.e. $F = \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}$, and
- $\rho = \rho_+ \oplus \rho_-$, where $\rho_\pm : A \to H_\pm$.

Remark 5.1.3: Using the relations in the definition of an ungraded Fredholm module, it is possible to derive relations for U and V similar to 5.1.

As a conclusion to our discussion of Fredholm modules, we shall introduce some equivalence relations of Fredholm modules.

Definition 5.1.4 (Unitary Equivalence of Fredholm Modules) Let $[H, \rho, F]$ be a Fredholm module and $U : H' \to H$ be a unitary isomorphism, which preserves the grading if a grading exists. Then $[H', U^*\rho U, U^*FU]$ is a Fredholm module and said is to be unitarily equivalent to $[H, \rho, F]$

Definition 5.1.5 (Operator Homotopy) Let $[H, \rho, F_t]$, $t \in [0, 1]$ be a family of Fredholm modules. If $t \mapsto F_t$ is continuous in the norm topology, we call this family an operator homotopy and say that F_0 and F_1 are operator homotopic.

5.2 The K-Homology Groups

We may now define the K-homology groups. We will not give definitions for a general C*-algebra. Instead, we focus on the case that is relevant to the main results, namely topological spaces with compact support (see definition 5.2.4).

Definition 5.2.1 (K^0 of a unital C*-algebra) For a unital C*-algebra A, let $K^0(A)$ denote the free abelian group generated by unitary equivalence classes of graded Fredholm modules $[H, F]$ over A with the following relations.

$$\begin{aligned}&[H, F] = [H, F'], && \text{if } F \text{ and } F' \text{ are operator homotopic and} \\ &[H, F] + [H', F'] = [H \oplus H', F \oplus F'], && \text{for any Fredholm modules over } A\end{aligned}$$

Remark 5.2.2: The definition of $K^1(A)$ is almost identical. The only difference is that $K^1(A)$ consists of equivalence classes of ungraded Fredholm modules. In fact, it is possible to define K-homology groups for all $p \geq -1$. However, K-homology

enjoys a periodicity similar to that of K-theory (Actually, the K-homology of a C*-algebra A can be defined as the K-theory of certain dual algebra of A, see Chapter 5 of [8]). Therefore we omit the definition of the other groups.

Note that the relations imply that the zero module $[0, 0]$ is the zero element of the K-homology groups. It is - however - possible to strengthen this statement. We shall say that a Fredholm module is degenerate if the elements

$$(F - F^*)\rho(a), \quad (F^2 - 1)\rho(a), \quad F\rho(a) - \rho(a)F.$$

are zero for each $a \in A$.

Proposition 5.2.3 (Proposition 8.2.8 of [**?**]) *The class of a degenerate Fredholm module $[H, \rho, F]$ over A is zero in $K^n(A)$ for all n.*

Proof. Consider $[H_\infty, \rho_\infty, F_\infty] = [\oplus_{n=1}^\infty H, \oplus_{n=1}^\infty \rho, \oplus_{n=1}^\infty F]$. Clearly,

$$(F_\infty - F_\infty^*)\rho_\infty(a), \quad (F_\infty^2 - 1)\rho_\infty(a), \quad F_\infty\rho_\infty(a) - \rho_\infty(a)F_\infty.$$

are all still zero for each $a \in A$. Hence $[H_\infty, \rho_\infty, F_\infty]$ is a Fredholm module over A. Now,

$$[H_\infty, \rho_\infty, F_\infty] + [H, \rho, F] = [H_\infty \oplus H, \rho_\infty \oplus \rho, F_\infty \oplus F] = [H_\infty, \rho_\infty, F_\infty].$$

Thus $[H, \rho, F] = 0$. □

The above style of proof - where one derives "$1 = 0$" from "$\infty = \infty + 1$" - is called an Eilenberg swindle and is useful for proving certain statements about K-theory and K-homology.

Similar to K-theory, we denote the n-th K-homology of $C(X)$ by $K_n(X)$. We are now interested in extending the definition of K-homology to a certain class of non-compact spaces. This definition will differ slightly from the case of K-theory.

Definition 5.2.4 (K-Homology with Compact Support) Let $Y = \cup Y_n$ be a non-compact increasing union of compact spaces Y_n. Then the K-homology of Y is given by $K_*(Y) := \varinjlim K_*(Y_n)$.

The reason this definition is chosen is that we are interested in the K-homology of CW-complexes of finite type, such spaces give rise a natural increasing union of compact spaces; namely, the n-skeletons X^n.

5.2.1 Some properties of K-homology

Some of the properties that we discussed for K-theory are essentially true by definition for K-homology. For example, Homotopy invariance has a short proof (Theorem 9.3.3 of [8]) and Definition 5.2.4 can be seen as an analog to Proposition 4.4.6. However, note that Definition 5.2.4 uses spaces as input while Proposition 4.4.6 uses C*-algebras. If this was not the case, the functoriality would be reversed.

However, K-homology introduces a technical difficulty, which is not present in the study of K-theory. Specifically, for an exact sequence of C*-algebras

$$0 \to J \xrightarrow{i} A \xrightarrow{p} A/J \to 0,$$

where we assume A is unital, there need not exist an exact sequence on K-homology

$$\begin{array}{ccccc} K^0(A/J) & \xrightarrow{p^*} & K^0(A) & \xrightarrow{i^*} & K^0(J) \\ \uparrow & & & & \downarrow \\ K^1(J) & \xleftarrow[i^*]{} & K^1(A) & \xleftarrow[p^*]{} & K^1(A/J). \end{array} \tag{5.2}$$

However, if we add the additional assumption that $p : A \to A/J$ is split by a completely positive map [1], then there are boundary maps $\partial : K^*(J) \to K^{*+1}(A/J)$ that make sequence 5.2 exact.

Sequence 5.2 is derived in Chapter 5 of [8] by considering an alternate model for K-homology and then showing that the models yield groups isomorphic to the ones defined above. In particular $K^*(A)$ is defined as $K_{*+1}(\mathcal{D}(A))$, the K-theory of a certain dual algebra of A. Various properties of K-homology are more easily proved using a specific model. We shall not say much more about this, as it will not be required to formulate the main results of the thesis.

5.3 The Fredholm Index

The Fredholm index has a simple definition in terms of analytical quantities but turns out to have interesting topological properties. In fact, the Atiyah-Singer index theorem [3] allows for calculation of the Fredholm index of elliptic operators using topological quantities.

Definition 5.3.1 (Fredholm index) Let $F \in \mathcal{B}(H)$. F is called a Fredholm operator if it has finite-dimensional kernel and cokernel. If F is a Fredholm operator, then the Fredholm index of F is defined by

$$\mathrm{Ind}(F) = \dim\ker(F) - \dim\mathrm{coker}(F) \in \mathbb{Z}.$$

Denote the set of Fredholm operators in $\mathcal{B}(H)$ by $\mathcal{F}(H)$.

To exemplify the nature of the topological properties of the Fredholm index, we include the following proposition.

Proposition 5.3.2 (Proposition 2.1.6 of [8]) *Let F be a Fredholm operator.*

- *$\mathcal{F}(H)$ is open in the norm-topology on $\mathcal{B}(H)$ and the Fredholm index is continuous on this set. Hence the index of any $F' \in \mathcal{F}(H)$, joined by a path in $\mathcal{F}(H)$ to F, satisfies* $\mathrm{Ind}(F') = \mathrm{Ind}(F)$.
- *The Fredholm index is stable under compact perturbations, i.e. if K is any compact operator, then* $\mathrm{Ind}(F + K) = \mathrm{Ind}(F)$

Remark 5.3.3: When $F = \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}$ is an odd operator, we denote by $\mathrm{Ind}(F)$ the index of U.

[1] A completely positive map is a bounded linear map $\sigma : A/J \to A$ such that $\sigma(1) = 1$ and $\sum_{i,j} b_i^* \sigma(a_i^* a_j) b_j$ is positive for any $n \geq 1$, $a_1, \ldots, a_n \in A/J$, and $b_1, \ldots, b_n \in A$.

5.4 Index Pairing and The Universal Coefficient Theorem

As described above, the Fredholm index provides a map from $\mathcal{F}(H)$ to $\mathbb{Z}$. This map can utilized to define a pairing of K-theory and K-homology. That is, a map $K^0(A) \times K_0(A) \to \mathbb{Z}$ or, equivalently, a map $K^0(A) \to \mathrm{Hom}_{\mathbb{Z}}(K_0(A), \mathbb{Z})$. We assume that A is unital.

Recall that if $[H, \rho, F] \in K^0(A)$, then

$$H = H_+ \oplus H_-,\ \rho = \rho_+ \oplus \rho_-,\ \text{and } F = \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}.$$

Hence, given $[H, \rho, F] \in K^0(A)$ and $[p] \in K_0(A)$, it is possible to show that $P_-(1 \oplus U)P_+ : \mathbb{C}^n \oplus H_+ \to \mathbb{C}^n \oplus H_-$, where $P_\pm = (1 \oplus \rho_\pm(p))$ and $p \in M_n(A)$, is unitary modulo compact operators [2] by using the relations in Definition 5.1.1. It follows that $P_-(1 \oplus U)P_+$ is a Fredholm operator. Thus the index map $([H, \rho, F], [p]) \mapsto \mathrm{Ind}(P_-(1 \oplus U)P_+)$ is a well-defined map and it can be checked that it is a homomorphism.

It is possible to define a similar map $K^1(A) \to \mathrm{Hom}_{\mathbb{Z}}(K_1(A), \mathbb{Z})$. However, since $\mathrm{Hom}_{\mathbb{Z}}(K_1(A), \mathbb{Z}) = 0$ in the case which is the focus of this thesis, we will omit the description of this map.

Our main interest in the above maps is the following theorem

Theorem 5.4.1 (The Universal Coefficient Theorem, Theorem 7.6.1 of [8]) *Let A be a separable commutative C^*-algebra. Then the sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to K^{*+1}(A) \to \mathrm{Hom}_{\mathbb{Z}}(K_{*+1}(A), \mathbb{Z}) \to 0,$$

where the right map is the index map, is exact.

A proof of this theorem can be found in Section 7.6 of [8].

5.5 Classifying Spaces of Finite Groups

We have arrived at the first main result of the thesis. Namely, the calculation of the K-homology of classifying spaces of finite groups. The approach is based on the results of [10]. Specifically, the following special case of the Cocompletion Theorem (Theorem 0.2 of [10]) plays an important role.

Theorem 5.5.1 *Let G be a finite group. Let X be a finite G-CW-complex. Moreover, let I denote the augmentation ideal.*

Then there is a short exact sequence

$$0 \to \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(K_G^{*+1}(X)/I^n \cdot K_G^{*+1}(X), \mathbb{Z}) \to K_*(EG \times_G X) \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(K_G^*(X)/I^n \cdot K_G^*(X), \mathbb{Z}) \to 0.$$

[2] u is unitary modulo compact operators if $uu^* - 1$ and $u^*u - 1$ are compact

The theorem above can be proved by appealing to the completion theorem, which provides a pro-isomorphism $\{K_G^*(X)/I^n\} \to \{K_*((EG \times_G X)^n)\}$. In [2], the completion theorem is proven directly for the circle group and then extended to an n-torus by induction. The case of the unitary group is reduced to its maximal torus and finally, by embedding any compact Lie group into the unitary group, the theorem is proven for any such group. Furthermore, the Universal Coefficient Theorem (Theorem 5.4.1) states that

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_G^{*+1}((EG \times_G X)^n), \mathbb{Z}) \to K_*((EG \times_G X)^n) \\ \to \mathrm{Hom}_{\mathbb{Z}}(K_G^*((EG \times_G X)^n), \mathbb{Z}) \to 0.$$

is exact. Therefore, applying $\varinjlim$ and using Lemma 3.5.2 proves the theorem.

Now, $X = \{\mathrm{pt}\}$ is a finite G-CW-complex. Then $EG \times_G X = BG$ so that

$$K_G^0(X) \cong R(G) \quad \text{and} \quad K_G^1(X) = 0$$

in accordance with section 4.5.2. Thus the exact sequence of Theorem 5.5.1 reduces to

$$K_*(BG) = \begin{cases} \varinjlim \mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}), & \text{if } * = 0 \\ \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}), & \text{if } * = 1. \end{cases} \tag{5.3}$$

It remains to evaluate the above expressions. Firstly, we will consider the case $G = \mathbb{Z}/2$, where I^n assumes an explicit form.

Example 5.5.2 ($K_*(B\mathbb{Z}/2)$): $R(\mathbb{Z}/2) \cong \mathbb{Z}[x]/\langle x^2 - 1\rangle$, $I = \langle 1 - x\rangle$. Since $R(\mathbb{Z}/2)$ is generated by 1 and $1 - x$ as a $\mathbb{Z}$-module, $R(\mathbb{Z}/2)/I \cong \mathbb{Z}$ as $\mathbb{Z}$-modules. To find I^n, we may note that I^2 is generated by $(1-x)^2 = 1 - 2x + x^2 = 2(1-x)$. By induction, $I^n = \langle 2^{n-1}(1-x)\rangle$. We conclude that the quotient $R(G)/I^n$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2^{n-1}$ as modules.

To evaluate $\mathrm{Hom}_{\mathbb{Z}}$ and $\mathrm{Ext}^1_{\mathbb{Z}}$ we use the fact that they both commute with direct sums together with Proposition 3.2.4 and Proposition 3.4.4 to obtain, for all $n \geq 1$, that

$$\mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \cong \mathbb{Z}, \text{ and } \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \cong \mathbb{Z}/2^{n-1}.$$

Finally, taking the colimits, we conclude that

$$K_0(B\mathbb{Z}/2) \cong \mathbb{Z} \quad \text{and} \quad K_1(B\mathbb{Z}/2) \cong \mathbb{Z}/2^\infty$$

For an arbitrary finite group, the structure of I is typically more complicated. However, we may apply the following lemma, which is proven in [11].

Theorem 5.5.3 (Structure of $\{I_G/I_G^{n+1}\}$, Theorem 3.5 of [11]) *Let G be a finite group. Denote by $\mathcal{P}(G)$ the set of primes dividing $|G|$. For each $p \in \mathcal{P}(G)$, let G_p denote a p-Sylow subgroup.*

(a) For every $p \in \mathcal{P}(G)$ let $\mathrm{im}(\mathrm{res}_G^{G_p})$ be the image of $\mathrm{res}_G^{G_p} : I_G \to I_{G_p}$. There is a pro-isomorphism of pro-groups

$$\{I_G/I_G^{n+1}\} \xrightarrow{\cong} \prod_{p \in \mathcal{P}(G)} \{\mathrm{im}(\mathrm{res}_G^{G_p})/p^n \cdot \mathrm{im}(\mathrm{res}_G^{G_p})\}.$$

(b) There is an isomorphim of pro-groups

$$\{\mathbb{Z}\} \oplus \{I_G/I_G^n\} \xrightarrow{\cong} \{R(G)/I_{G^n}\}.$$

To exemplify the usefulness of this lemma, we will calculate the K-homology of the symmetric group on a set of three elements $K_*(BS_3)$.

Example 5.5.4: The symmetric group $G = S_3$ has three irreducible representations. These are the trivial representation π_0, the sign representation sgn, and a 2-dimensional representation π_2. To find the representation ring of S_3, we must check the multiplication of the representations. By direct calculation, we have $\pi_0 \otimes \pi = \pi$, for any representation π, and $\mathrm{sgn} \otimes \mathrm{sgn} = \pi_0$. Using the character table of S_3 (Table 5.1), we must also have $\mathrm{sgn} \otimes \pi_2 = \pi_2$ and $\pi_2 \otimes \pi_2 = \pi_2 \oplus \pi_0 \oplus \mathrm{sgn}$. More compactly, this can be written

$$R(G) \cong \mathbb{Z}[X,Y]/\langle X^2 - 1, XY - Y, Y^2 - X - Y - 1\rangle$$

The augmentation ideal I_G is generated by $1 - X$ and $2 - Y$. We may take $G_2 = \{\mathrm{Id}, (12)\}$ and $G_3 = \{\mathrm{Id}, (123), (132)\}$ as Sylow subgroups and we shall now find the images of the restriction maps to I_{G_2} and I_{G_3}.

Consider first $\mathrm{res}_G^{G_2}$. Similarly to Example 5.5.2, we see $R(G_2) \cong \mathbb{Z}[x]/\langle x^2 - 1\rangle$, $I_{G_2} = \langle 1 - x\rangle$. The restriction of the trivial representation $\mathrm{res}_G^{G_2}(\pi_0)$ is clearly trivial. Now, $\mathrm{sgn}(12) = -1$ so that $\mathrm{res}_G^{G_2}(\mathrm{sgn}) = x$ is the nontrivial representation of $G_2 \cong \mathbb{Z}/2$. Since $\pi_0 - \mathrm{sgn} = 1 - X \in I_G$, it follows that $\mathrm{im}(\mathrm{res}_G^{G_2}) = \langle 1 - x\rangle = I_{G_2}$.

For G_3, the sign representation is mapped to the trivial representation. However, we find from the character table of S_3 that the trace of $\pi_2(\mathrm{Id})$ is 2 and $\mathrm{Tr}(\pi_2(123)) = \mathrm{Tr}(\pi_2(132)) = -1$. Thus π_2 must be a direct sum of the two nontrivial representations r_1 and r_2 of G_3. That is

$$\mathrm{im}(\mathrm{res}_G^{G_3}) = \langle \mathrm{res}_G^{G_3}(1 - X), \mathrm{res}_G^{G_3}(2 - Y)\rangle = \langle 2 - r_1 - r_2\rangle$$

Hence, as $\mathbb{Z}$-modules, $\mathrm{im}(\mathrm{res}_G^{G_2}) \cong \mathrm{im}(\mathrm{res}_G^{G_3}) \cong \mathbb{Z}$. Now, from Theorem 5.5.3, it follows that

$$\{R(G)/I_G^n\} \cong \{\mathbb{Z}\} \times \{\mathbb{Z}/2^{n-1}\} \times \{\mathbb{Z}/3^{n-1}\}.$$

It now remains to use equation 5.3, i.e. evaluate $\mathrm{Hom}_{\mathbb{Z}}$, $\mathrm{Ext}^1_{\mathbb{Z}}$, and $\varinjlim$ to find

$$K_0(BS_3) \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z}/2^{n-1} \times \mathbb{Z}/3^{n-1}) = \mathbb{Z} \text{ and}$$
$$K_1(BS_3) \cong \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z}/2^{n-1} \times \mathbb{Z}/3^{n-1}) = \mathbb{Z}/2^\infty \times \mathbb{Z}/3^\infty.$$

Table 5.1: Character table of S_3

	Id	(123)	(12)
π_0	1	1	1
sgn	1	1	-1
π_2	2	-1	0

Returning to the general theory, we note that $\mathrm{im}(\mathrm{res}_G^{G_p})/p^n \cdot \mathrm{im}(\mathrm{res}_G^{G_p}) \cong \mathrm{im}(\mathrm{res}_G^{G_p}) \otimes \mathbb{Z}/p^n$ as $\mathbb{Z}$-modules and that $\mathrm{im}(\mathrm{res}_G^{G_p}) \subseteq R(G_p)$ is finitely generated as a free $\mathbb{Z}$-module. The following lemma determines the rank of $\mathrm{im}(\mathrm{res}_G^{G_p})$. The outline of this argument is found in the proof of Theorem 0.3 of [11] but the following proof fills in some details.

Lemma 5.5.5 *Let G be a finite group. The rank of* $\mathrm{im}(\mathrm{res}_G^{G_p})$ *is* $r(p) = |\operatorname{conj}_p(G)|$, *the number of conjugacy classes consisting of elements whose order are p^d for some integer $d \geq 1$.*

Proof. Let $\mathrm{class}_{\mathbb{C}}(H)$ denote the complex vector space of class functions on a group H. A class function $H \to \mathbb{C}$ is a function, which is constant on each of the conjugacy classes of H. Consider the following commutative diagram.

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{Z}} R(G) & \xrightarrow{\mathrm{res}_G^{G_p}} & \mathbb{C} \otimes_{\mathbb{Z}} R(G_p) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{class}_{\mathbb{C}}(G) & \xrightarrow{\mathrm{res}_G^{G_p}} & \mathrm{class}_{\mathbb{C}}(G_p). \end{array}$$

The vertical arrow are given by taking the character of a representation and the bottom horizontal map takes a class function $G \to \mathbb{C}$ to its restriction to G_p. $\mathrm{class}_{\mathbb{C}}(G)$ is spanned by indicator functions $\chi_{[g]}$, where $[g]$ is the conjugacy class of $g \in G$. We claim that $\mathrm{res}_G^{G_p}(\chi_{[g]}) \neq 0$ if and only if $|g| = p^d$ for some $d \geq 0$.

To prove the "if"-statement, we assume that $|g| \neq p^d$ for all $d \geq 0$. Then $g \notin G_p$ and consequently $[g] \cap G_p = \emptyset$ since all elements of $[g]$ have the same order. It follows that $\chi_{[g]}(h) = 0$ for all $h \in G_p$.

Conversely, if $|g| = p^d$ for some $d \geq 0$ then the group generated by g has order p^d. By Zorn's lemma, g is contained in a p-Sylow subgroup. Since all p-Sylow subgroups are conjugate, there exists $h \in G$ such that $hgh^{-1} \in G_p$. Thus $\mathrm{res}_G^{G_p}(\chi_{[g]})(hgh^{-1}) = 1$ so that $\mathrm{res}_G^{G_p}(\chi_{[g]}) \neq 0$.

It follows that $\mathrm{im}(\mathrm{res}_G^{G_p} : \mathrm{class}_{\mathbb{C}}(G) \to \mathrm{class}_{\mathbb{C}}(G_p))$ has dimension $r(p) + 1$ (the added 1 corresponds to $\chi_{[1]}$). since $R(G)$ and $R(G_p)$ are isomorphic to $\mathbb{Z}[1] \oplus I_G$ and $\mathbb{Z}[1] \oplus I_{G_p}$, respectively, $\mathrm{im}(\mathrm{res}_G^{G_p})$ has rank $r(p)$. □

It is then straight forward to prove the following theorem.

Theorem 5.5.6 (K-Homology of Classifying Spaces of Finite Groups) *Let G be a finite group. Denote by $\mathcal{P}(G)$ the set of primes dividing $|G|$ and by $r(p) = |\operatorname{conj}_p(G)|$ the number of conjugacy classes consisting of elements whose order are p^d for some positive integer d. Then there are isomorphisms*

$$K_0(BG) \cong \mathbb{Z} \quad \textit{and} \quad K_1(BG) \cong \prod_{p \in \mathcal{P}(G)} (\mathbb{Z}/p^\infty)^{r(p)}$$

Proof. We begin by calculating $K_0(BG)$ using equation (5.3). Note that by Theorem 5.5.3, $R(G)/I_G^n$ is isomorphic to $\mathbb{Z} \oplus I_G/I_G^n$ for each $n \geq 1$. Furthermore, $\{I_G/I_G^n\}$ is pro-isomorphic to a pro-system of torsion groups. Thus

$$\mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \cong \mathbb{Z}, \quad \forall n \geq 1$$

and

$$K_0(BG) \cong \varprojlim \mathbb{Z} \cong \mathbb{Z}.$$

Similarly, for $K_1(BG)$, we find

$$\mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \cong \prod_{p \in \mathcal{P}(G)} (\mathbb{Z}/p^n)^{r(p)}, \quad \forall n \geq 1.$$

It follows that

$$K_1(BG) \cong \varinjlim \prod_{p \in \mathcal{P}(G)} (\mathbb{Z}/p^n)^{r(p)} \cong \prod_{p \in \mathcal{P}(G)} (\mathbb{Z}/p^\infty)^{r(p)}.$$

□

5.6 Classifying Spaces of Amalgams

Theorem 5.5.6 can be extended to certain infinite groups using the theory of amalgamated free products, which we have discussed in section 2.5. Recall that if we are given groups G, H, and K with inclusions $i : K \to G$ and $j : K \to H$, we may form the amalgam $\Gamma = G *_K H$ by identifying the image of i and j with each other in $G * H$.

It has been shown that if all groups are finite, $B\Gamma$ is homotopy equivalent to $BG \vee_{BK} BH$ (Lemma 2.5.2). We will use this result to prove that if there is a splitting of the inclusion $K \to G$, then the K-homology of $B\Gamma$ may be expressed in terms of the K-homology of its finite constituents.

Let A_1, A_2, and A_0 be C*-algebras and assume that there are surjections $\alpha_1 : A_1 \to A_0$ and $\alpha_2 : A_2 \to A_0$. Denote by P the pullback of the following diagram

$$\begin{array}{ccc} P & \xrightarrow{p_1} & A_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \alpha_1} \\ A_2 & \xrightarrow{\alpha_2} & A_0 \end{array}$$

Theorem 5.6.1 (Mayer-Vietoris Sequence) *With the above notation there is an exact sequence*

$$\begin{array}{ccccc} K^0(A_0) & \xrightarrow{\alpha_1^* \oplus (-\alpha_2^*)} & K^0(A_1) \oplus K^0(A_2) & \xrightarrow{p_1^* + p_2^*} & K^0(P) \\ \uparrow & & & & \downarrow \\ K^1(P) & \xleftarrow[p_1^* + p_2^*]{} & K^1(A_1) \oplus K^1(A_2) & \xleftarrow[\alpha_1^* \oplus (-\alpha_2^*)]{} & K^1(A_0) \end{array}$$

The following proof is based on the ideas found in Section 21.2 of [4].

Proof. Consider $Z = \{(h_1, h_2) \in CA_1 \oplus CA_2 : \alpha_1(h_1(1)) = \alpha_2(h_2(1))\}$ and the surjective map $\psi : Z \to SA_0$ given by taking (h_1, h_2) to

$$\psi(h_1, h_2) = \begin{cases} \alpha_1(h_1(2t)), & \text{if } t \in [0, \frac{1}{2}] \\ \alpha_2(h_2(2 - 2t)), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

The kernel of ψ is the direct sum of cones $C \ker \alpha_1 \oplus C \ker \alpha_2$, which is contractible. Hence, ψ induces an isomorphism on K-homology.

Now, denote the restriction of ψ to $SA_1 \oplus SA_2$ by $k : SA_1 \oplus SA_2 \to SA_0$. We will show that the induced map $k_* : K^n(SA_0) \to K^n(SA_1) \oplus K^n(SA_2)$ is equal to $(\alpha_{1*}, -\alpha_{2*})$, or equivalently $k_* + (0, \alpha_{2*}) = (\alpha_{1*}, 0)$. To this end, recall that the stabilization maps

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

induce an isomorphism on K-homology (Proposition **??**). Furthermore, conjugation with the unitary

$$R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

defines a path from

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \quad \text{to} \quad a \mapsto \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$$

It follows that the map

$$(h_1, h_2) \mapsto \begin{bmatrix} k(h_1, h_2) & 0 \\ 0 & \alpha_2(h_2) \end{bmatrix} =: M$$

is homotopic to $k_* + (0, \alpha_{2*})$. Now, consider the reparametrization $M'(t)$ of $M(t)$ given by

$$\begin{bmatrix} \alpha_1(h_1(3t)) & 0 \\ 0 & \alpha_2(h_2(3t)) \end{bmatrix} \text{ on } [0, \frac{1}{3}]$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ on } [\frac{1}{3}, \frac{2}{3}]$$
$$\begin{bmatrix} \alpha_2(h_2(3-3t)) & 0 \\ 0 & 0 \end{bmatrix} \text{ on } [\frac{2}{3}, 1],$$

which is homotopic via conjugation by

$$R'(t) = \begin{cases} R(0), & \text{if } t \in [0, \frac{1}{3}] \\ R(3t-1), & \text{if } t \in [\frac{1}{3}, \frac{2}{3}] \\ R(1), & \text{if } t \in [\frac{2}{3}, 1]. \end{cases}$$

to $M''(t)$ with

$$M''(t) = M'(t), \ t \in [0, \frac{2}{3}]$$
$$M''(t) = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_2(h_2(3-3t)) \end{bmatrix} \text{ on } [\frac{2}{3}, 1].$$

It remains to show that M'' induces same map as α_1 on K-homology. We will find a homotopy joining the bottom right part of M'' to zero. Indeed, note that this part is of the form

$$g \mapsto f, \text{ where } f(s) = \begin{cases} g(2s), & \text{if } s \in [0, \frac{1}{2}] \\ g(2-2s), & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$

for some $g \in CA_0$. A homotopy to 0 is given by

$$g \mapsto f_t, \text{ where } f_t(s) = \begin{cases} g(2s-t), & \text{if } s \in [\frac{1}{2}t, \frac{1}{2}] \\ g(2-2s+t), & \text{if } s \in [\frac{1}{2}, 1-\frac{1}{2}t] \\ 0, & \text{otherwise.} \end{cases}$$

Now, consider the surjection $\mathrm{ev}_1 : Z \to P$, $(h_1, h_2) \mapsto (h_1(1), h_2(1))$ with kernel $SA_1 \oplus SA_2$. Taking K-homology yields

$$\begin{array}{ccccccccc} \ldots \longrightarrow & K^n(P) & \longrightarrow & K^n(Z) & \longrightarrow & K^n(SA_1 \oplus SA_2) & \xrightarrow{\partial} & K^{n+1}(P) & \longrightarrow \ldots \\ & & & \cong\uparrow{\scriptstyle \psi_*} & & \cong\uparrow & & & \\ & & & K^n(SA_0) & \xrightarrow{k_*} & K^n(SA_1) \oplus K^n(SA_2) & & & \end{array}$$

Using $K^{n+1}(P) \cong K^n(SP)$ and noting that ∂ is induced by (p_1, p_2) under these identification finishes the proof. □

As was explained in Section 4.1.3, two injective group homomorphisms $\varphi : K \to G$ and $\psi : K \to H$ give rise to the following pullback diagram.

$$\begin{array}{ccc} C(BG^n \vee_{BK^n} BH^n) & \xrightarrow{p_1^n} & C(BG^n) \\ \downarrow{\scriptstyle p_2^n} & & \downarrow{\scriptstyle \alpha_1^n} \\ C(BH^n) & \xrightarrow{\alpha_2^n} & C(BK^n) \end{array}$$

If we assume that α_1^n splits for each n - for example, if φ splits - then α_1^{n*} also splits. Then, considering the induced six term exact sequence on K-homology

$$\begin{array}{ccccc} K_0(BK^n) & \xrightarrow{\alpha_1^{n*} \oplus (-\alpha_2^{n*})} & K_0(BG^n) \oplus K_0(BH^n) & \longrightarrow & K_0(BG^n \vee_{BK^n} BH^n) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(BG^n \vee_{BK^n} BH^n) & \longleftarrow & K_1(BG^n) \oplus K_1(BH^n) & \xleftarrow[\alpha_1^{n*} \oplus (-\alpha_2^{n*})]{} & K_1(BK^n), \end{array}$$

we have $\operatorname{im} \partial = \ker(\alpha_1^* \oplus (-\alpha_2^*)) \subseteq \ker \alpha_1^* = 0$. Thus

$$0 \to K_i(BK^n) \to K_i(BG^n) \oplus K_i(BH^n) \to K_i(BG^n \vee_{BK^n} BH^n) \to 0$$

is exact for $i \in \{0, 1\}$. We would like to take the colimit of this sequence. The colimit of the first and second term is $K_i(BK)$ and $K_i(BG) \oplus K_i(BH)$, respectively. For the third term to have colimit $K_i(B(G *_K H))$, we will show that $BG^n \vee_{BK^n} BH^n$ is homeomorphic to $(BG \vee_{BK} BH)^n$ for each n. Then, since $BG \vee_{BK} BH$ is homotopy equivalent to $B(G *_K H)$, we have derived an exact sequence which allows the K-homology of amalgams to be calculated in terms of its factors.

Formally, $BG^n \vee_{BK^n} BH^n$ consists of k-cells with $k \leq n$ in BG or BH under the identification of cells in $BK^n \subseteq BG^n$ with the corresponding cells in $BK^n \subseteq BH^n$. A cell in $(BG \vee_{BK} BH)^n$ is given by a k-cell in $BG \vee_{BK} BH$ with $k \leq n$, i.e. a k-cell in $BG \coprod BH$ with cells in $BK \subseteq BG$ and $BK \subseteq BH$ identified with one another. However, as subsets of BG^n and BH^n, we have $BK^n = BK$. Therefore, we obtain

a bijective correspondence of cells between $BG^n \vee_{BK^n} BH^n$ and $(BG \vee_{BK} BH)^n$. This correspondence defines a continuous map, which is a homeomorphism since both spaces are compact and Hausdorff.

Summarizing the above, we have

Lemma 5.6.2 *Let $\varphi : K \to G$ and $\psi : K \to H$ be injective group homomorphisms and assume that φ is split by $\theta : G \to K$. Then*

$$0 \to K_i(BK) \xrightarrow{\varphi_* \oplus (-\psi_*)} K_i(BG) \oplus K_i(BH) \to K_i(B(G *_K H)) \to 0$$

is exact for $i \in \{0, 1\}$

Now, since φ_* is split by θ_*, we may continue this line of reasoning and obtain a more explicit expression for $K_i(B(G *_K H))$.

Theorem 5.6.3 *Let $\varphi : K \to G$ and $\psi : K \to H$ be injective group homomorphisms and assume that φ is split by $\theta : G \to K$. Then*

$$K_i(B(G *_K H)) \cong (K_i(BG)/K_i(BK)) \oplus K_i(BH)$$

Proof. Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i(BK) & \xrightarrow{\varphi_* \oplus (-\psi_*)} & K_i(BG) \oplus K_i(BH) & \xrightarrow{\beta} & K_i(B(G *_K H)) & \longrightarrow & 0 \\ & & & & \downarrow \alpha & & \downarrow = & & \\ 0 & \longrightarrow & & & (K_i(BG)/K_i(BK)) \oplus K_i(BH) & \xrightarrow{\beta'} & K_i(B(G *_K H)) & \longrightarrow & 0 \end{array}$$

α takes (x, y) to $([x], y + \psi_* \circ \theta_*(x))$, where $[x]$ is the equivalence class of x in the quotient $K_i(BG)/K_i(BK)$. Note that α is surjective and that $\alpha(x, y) = 0$ exactly when $(x, y) \in \operatorname{im}(\varphi_* \oplus (-\psi_*)) = \ker \beta$. Hence we may define a map $\beta' : (K_i(BG)/K_i(BK)) \oplus K_i(BH) \to K_i(B(G *_K H))$ by first lifting an element via α and then applying β. By the above, β' is injective. Surjectivity of β' follows from the surjectivity of β. □

Since free products split by the trivial morphism $G \to 0$, we immediately obtain the following.

Corollary 5.6.4. *For any groups G and H*

$$K_*(B(G * H)) \cong K_*(BG) \oplus_{K_*(\{\mathrm{pt}\})} K_*(BH)$$

Other than free products we may also apply Theorem 5.6.3 to $SL(2, \mathbb{Z})$.

Example 5.6.5: In Example 2.5.4, we showed that $SL(2, \mathbb{Z}) \cong \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$. Using Theorem 5.5.6, we obtain

$$\begin{aligned} &K_0(B\mathbb{Z}/2) \cong \mathbb{Z},\ K_1(B\mathbb{Z}/2) \cong \mathbb{Z}/2^\infty \\ &K_0(B\mathbb{Z}/4) \cong \mathbb{Z},\ K_1(B\mathbb{Z}/4) \cong (\mathbb{Z}/2^\infty)^3 \\ &K_0(B\mathbb{Z}/6) \cong \mathbb{Z},\ K_1(B\mathbb{Z}/6) \cong (\mathbb{Z}/3^\infty)^2 \times (\mathbb{Z}/2^\infty). \end{aligned}$$

Moreover, since the inclusion $i : \mathbb{Z}/2 \to \mathbb{Z}/6$ splits, it follows from Theorem 5.6.3 that

$$K_0(BSL(2, \mathbb{Z})) \cong \mathbb{Z} \text{ and } K_1(BSL(2, \mathbb{Z})) \cong (\mathbb{Z}/2^\infty)^3 \times (\mathbb{Z}/3^\infty)^2$$

6

Analytic Surgery

In this chapter we discuss a possible future application of the results of Chapter 5. In particular, we explain to what extent homological methods can be applied to the problem of calculating the analytic structure group and what further knowledge is required.

6.1 The Assembly Map

One of the main components of analytical surgery is the assembly map μ. The assembly map $\mu : K_*(BG) \to K_*(C^*(G))$ is map from the K-homology of BG to the K-theory of $C^*(G)$. In general, it is not easy to define. However, when G is a finite group, it can be shown that $\mu : K_0(BG) \to K_0(C^*(G))$ takes $[H, F]$ to $\mathrm{Ind}(F)[1]$. Since $K_1(C^*(G)) = 0$, the map $\mu : K_1(BG) \to K_1(C^*(G)) = 0$ is trivial.

6.1.1 The Assembly Map for Amalgams

In this section, it is verified that the assembly map for amalgams is compatible with the assembly maps of its factors. This ensures that the K-homology of classifying spaces of amalgams with finite factors can be treated by considering the factors when studying the assembly map.

As above, let $\varphi : K \to G$ and $\psi : K \to H$ be injective group homomorphisms. Moreover, let $i : G \to \Gamma := G *_K H$ and $j : H \to \Gamma$ denote the natural inclusions. In [13], the following six term exact sequence is derived.

$$\begin{array}{ccccc} K_0(C^*(K)) & \xrightarrow{\varphi_* \oplus (-\psi_*)} & K_0(C^*(G)) \oplus K_0(C^*(H)) & \xrightarrow{i_* + j_*} & K_0(C^*(\Gamma)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\Gamma)) & \xleftarrow[i_* + j_*]{} & K_1(C^*(G)) \oplus K_1(C^*(H)) & \xleftarrow[\varphi_* \oplus (-\psi_*)]{} & K_1(C^*(K)) \end{array}$$

If we also assume that φ is split, then we obtain two short exact sequences

$$0 \to K_n(C^*(K)) \xrightarrow{\varphi_* \oplus (-\psi_*)} K_n(C^*(G)) \oplus K_n(C^*(H)) \xrightarrow{i_* + j_*} K_n(C^*(\Gamma)) \to 0,$$

where $n \in \{0, 1\}$.

Now, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n(BK) & \longrightarrow & K_n(BG) \oplus K_n(BH) & \longrightarrow & K_n(B\Gamma) & \longrightarrow & 0 \\ & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \\ 0 & \longrightarrow & K_n(C^*(K)) & \longrightarrow & K_n(C^*(G)) \oplus K_n(C^*(H)) & \longrightarrow & K_n(C^*(\Gamma)) & \longrightarrow & 0 \end{array}$$

The commutativity of this diagram follows from the naturality of the assembly map [6]. That is, if $\alpha : G_1 \to G_2$ is a group homomorphism, then

$$\begin{array}{ccc} K_n(BG_1) & \xrightarrow{\alpha_*} & K_n(BG_2) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ K_n(C^*(G_1)) & \xrightarrow{\alpha_*} & K_n(C^*(G_2)) \end{array}$$

commutes. In this sense, the assembly map $\mu : K_n(B\Gamma) \to K_n(C^*(\Gamma))$ can be understood in terms the assembly map for G and H.

6.2 The Analytic Surgery Exact Sequence

The assembly maps can be placed into an exact sequence [9].

$$\begin{array}{ccccc} K_0(BG) & \xrightarrow{\mu} & K_0(C^*(G)) & \xrightarrow{r} & \mathcal{S}_1(G) \\ \uparrow & & & & \downarrow \\ \mathcal{S}_0(G) & \xleftarrow{r} & K_1(C^*(G)) & \xleftarrow{\mu} & K_1(BG) \end{array} \tag{6.1}$$

This sequence is called the analytic surgery exact sequence. The analytic structure groups $\mathcal{S}_*(G)$ measure the extent to which the assembly maps μ deviate from being isomorphisms. Specifically, if $\mathcal{S}_*(G)$ are both zero, then μ are isomorphisms. However, for finite groups μ is never an isomorphism and we may view $\mathcal{S}_*(G)$ as an explanation of why this is the case. For example, if $\mu : K_0(BG) \to K_0(C^*(G))$ is not injective, the difference of two elements with the same image can be used to define an element in $\mathcal{S}_0(G)$ that explains why these distinct elements map to same element under μ.

Below, we describe how this sequence relates to the material of the previous chapter. The sequence

$$0 \to I^n \to R(G) \to R(G)/I^n \to 0$$

is used to calculate $K_*(BG)$. By applying $\mathrm{Hom}_{\mathbb{Z}}$ and $\varinjlim$, we obtain the following exact sequence.

$$\begin{aligned} \xi : 0 \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) &\xrightarrow{f} \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z}) \\ &\xrightarrow{g} \varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n, \mathbb{Z}) \xrightarrow{h} \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \to 0 \end{aligned}$$

Noting that there are isomorphisms

$$\begin{aligned} \alpha : K_0(BG) &\longrightarrow \varinjlim \mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}), \\ \beta : K_0(C^*(G)) &\longrightarrow R(G) \cong \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z}), \\ \gamma : K_1(BG) &\longrightarrow \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}), \end{aligned}$$

one is lead to the conjecture that ξ is isomorphic to the analytic surgery exact sequence. Towards a proof of this, we shall first prove the following proposition.

Proposition 6.2.1 *The following diagram commutes.*

$$\begin{array}{ccc} K_0(BG) & \xrightarrow{\quad\mu\quad} & K_0(C^*(G)) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \varinjlim \mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) & \xrightarrow{\quad f\quad} & \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z}) \end{array} \qquad (A)$$

Proof. The assembly map μ takes $[H, F] \in K_0(BG)$ to the K_0-class of the projection $\mathrm{Ind}(F)[1]$, where $1 \in C^*(G)$ is the identity. Under $C^*(G) \to \oplus_\pi \mathrm{End}(V_\pi)$, 1 is mapped to $\sum_\pi \mathrm{Id}_\pi$, where Id_π is the identity on $\mathrm{End}(V_\pi)$. Hence β maps $[1]$ to the homomorphism taking $\sum_\pi n_\pi \pi$ to $\sum_\pi(\mathrm{rank}(\mathrm{Id}_\pi) \cdot n_\pi)$. Furthermore, $\mathrm{rank}(\mathrm{Id}_\pi) = \dim(V_\pi)$, whence it follows that

$$\beta \circ \mu[H, F] = \left(\sum_\pi n_\pi \pi \mapsto \mathrm{Ind}(F) \cdot \sum_\pi \dim(V_\pi) n_\pi \right)$$

The map α is induced by the map : $K_0(BG^n) \to \mathrm{Hom}_{\mathbb{Z}}(K^0(BG^n), \mathbb{Z})$ taking $[H, F]$ to $\mathrm{Ind}_F = ([p] \mapsto \mathrm{Ind}(pFp))$ via the universal coefficient theorem. Since $\varinjlim \mathrm{Hom}_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \cong \mathbb{Z}$ is generated by the class $\varphi = ([\pi_0] \to 1)$ of the trivial representation, Ind_F is determined by $\mathrm{Ind}_F([\pi_0])$. Under the map of the completion theorem π_0 is mapped to $(EG^n \times \mathbb{C})/G \cong BG^n \times \mathbb{C}$, i.e. the trivial bundle. The trivial bundle corresponds to the identity projection when considering K-theory as projections. Therefore, $\mathrm{Ind}_F([\pi_0]) = \mathrm{Ind}(F)$. Moreover, for every representation $\pi \in R(G)$, the difference $\pi - \dim(V_\pi)\pi_0$ is in I. Therefore, $f(\varphi)$ is the composition of the projection map $p : R(G) \to R(G)/I^n \cong \mathbb{Z}[\pi_0] \oplus I/I^n$ with φ, which is given by

$$\sum_\pi n_\pi \pi = \sum_\pi n_\pi \dim(V_{\pi_i})\pi_0 + \sum_\pi n_\pi(\pi - \dim(V_\pi)\pi_0) \mapsto \sum_\pi n_\pi \dim(V_{\pi_i}) + 0$$

In conclusion,

$$f \circ \alpha[H, F] = \mathrm{Ind}(F) f(\varphi) = \mathrm{Ind}(F)\varphi \circ p = \beta \circ \mu.$$

□

Now, proving that (6.1) is isomorphic to ξ amounts to showing that there is a map $\psi : \mathcal{S}_0(G) \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n, \mathbb{Z})$ such that the following diagram commutes.

$$\begin{array}{ccccc} K_0(C^*(G)) & \xrightarrow{\quad r\quad} & \mathcal{S}_1(G) & \xrightarrow{\qquad d\qquad} & K_1(BG) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\gamma} \\ \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z}) & \xrightarrow{\quad g\quad} & \varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n, \mathbb{Z}) & \xrightarrow{\quad h\quad} & \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \end{array}$$

Since β and γ are isomorphisms, it will then follow from the five lemma (See Exercise 1.3.3 of [16]) that the map $\psi : \mathcal{S}_1(G) \to \varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n, \mathbb{Z})$ is an isomorphism. Moreover, we know that $K_1(C^*(G)) = 0$. Thus we may conclude - since diagram (A) is commutative - that the assembly map $\mu : K_0(BG) \to K_0(C^*(G))$ is injective, whence it follows that $\mathcal{S}_0(G) = 0$.

6.3 The $\mathbb{R}/\mathbb{Z}$-Index

Given two representations σ_1 and σ_2 of a discrete group G, there is a commutative diagram, which is related to the analytic surgery exact sequence [9]. Namely,

$$\begin{array}{ccccccc} \ldots & \longrightarrow & K_0(C^*(G)) & \longrightarrow & \mathcal{S}_1(G) & \longrightarrow & K_1(BG) \longrightarrow \ldots \\ & & \downarrow{\scriptstyle \mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2}} & & \downarrow{\scriptstyle \rho_{\sigma_1,\sigma_2}} & & \downarrow{\scriptstyle \mathrm{Ind}_{\sigma_1,\sigma_2}} \\ & & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array} \tag{6.2}$$

The map $\mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2}$ takes the class of a projection $p = \sum_{g\in G} a_g g \in C^*(G)$ to $\sum_{g\in G} a_g[\mathrm{Tr}(\sigma_1(g)) - \mathrm{Tr}(\sigma_2(g))]$ and the horizontal maps on the bottom row are the natural ones. The middle and right maps are much more difficult to describe and for the purpose of this thesis, it will suffice to know that there exists maps that make such a diagram commute. Recall that we are interested in showing that the following diagram commutes.

$$\begin{array}{ccccc} K_0(C^*(G)) & \longrightarrow & \mathcal{S}_1(G) & \longrightarrow & K_1(BG) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Hom}_{\mathbb{Z}}(R(G),\mathbb{Z}) & \longrightarrow & \varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n,\mathbb{Z}) & \longrightarrow & \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n,\mathbb{Z}) \end{array} \tag{6.3}$$

In order to find a map ψ such that diagram 6.3 commutes, we could use the maps ρ_{σ_1,σ_2} to define a map to $\varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n,\mathbb{Z})$. However, several properties must be checked in order for this map to be well-defined. In what follows, we will discuss the extent to which homological methods can be applied to verify that this map is well-defined.

Firstly, the codomain of ρ_{σ_1,σ_2} is $\mathbb{R}$, which - clearly - is not a subset of

$$\varinjlim \mathrm{Hom}_{\mathbb{Z}}(I^n,\mathbb{Z}) \cong (\oplus\mathbb{Z}) \oplus (\oplus_{p\in\mathcal{P}(G)}\mathbb{Z}[1/p]^{r(p)}).$$

However, since $K_1(BG)$ is a torsion group, the image of $\mathrm{Ind}_{\sigma_1,\sigma_2}$ is a torsion subgroup of $\mathbb{R}/\mathbb{Z}$ and hence a subgroup of $\mathbb{Q}/\mathbb{Z}$. By the commutativity of diagram (6.2), the image of ρ_{σ_1,σ_2} is contained in $\mathbb{Q}$. Since $K_1(BG)$ can be decomposed in terms of the p-Sylow subgroups of G, it is reasonable to conjecture that $\mathrm{Ind}_{\sigma_1,\sigma_2}$ respects this structure and that for suitably chosen pairs of representations (σ_1,σ_2), the image of $\mathrm{Ind}_{\sigma_1,\sigma_2}$ is contained in $\mathbb{Z}[1/p]$ for some $p \in \mathcal{P}(G)$. One may also note that if G is p-group, then $K_1(BG)$ is also a p-group and it follows that the image of ρ_{σ_1,σ_2} is contained in $\mathbb{Z}[1/p]$ for any choice of (σ_1,σ_2).

Moreover, $\mathrm{Hom}_{\mathbb{Z}}(R(G),\mathbb{Z}) \cong R(G)$. We may pick a module basis for $R(G)$ consisting of the trivial representation π_0 and elements $\sigma_{1,i} - \sigma_{2,i}$ in I. For example, the elements $\pi_i - \dim(\pi_i)\pi_0$ constitute such a basis. This observation allows us to prove the following proposition.

Proposition 6.3.1 *The following diagram commutes.*

$$\begin{array}{ccc} K_0(C^*(G)) & \xrightarrow{\beta} & R(G) \cong \oplus_i\mathbb{Z} \\ & \searrow^{\mathrm{Tr}_{\sigma_{1,i}} - \mathrm{Tr}_{\sigma_{2,i}}} & \downarrow{\scriptstyle \mathrm{Proj}} \\ & & \mathbb{Z} \end{array} \tag{6.4}$$

Proof. For any representation σ and $1 \leq k \leq \dim \sigma$, consider the rank 1 projection

$$p_k^\sigma = c_\sigma \sum_{g \in G} \sigma(g^{-1})_{kk} g \in C^*(G), \text{ where } c_\sigma = \frac{\dim \sigma}{|G|}.$$

While the explicit expression is somewhat complicated, this is simply the inverse Fourier transform of a matrix with a single one on the diagonal. Note that $[p_k^\sigma] = [p_l^\sigma]$ in $K_0(C^*(G))$ for any k, l. The projections $p_1^{\pi_i}$, with π_i irreducible, form a basis for $K_0(C^*(G))$. Therefore any class can be written $[p] = k_1[p_1^{\pi_1}] + \cdots + k_n[p_1^{\pi_n}]$ and it suffices to show that the diagram commutes for $[p_1^\pi]$ with π irreducible.

Evaluating $\varphi := \mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2}$ at p_k^σ, we find

$$\varphi(p_k^\sigma) = c_\sigma \sum_{g \in G} \sigma(g^{-1})_{kk} [\chi_1(g) - \chi_2(g)],$$

where χ_i is the character of σ_i. Summing over all $1 \leq k \leq \dim \sigma$, we find

$$\dim \sigma \cdot \varphi(p_k^\sigma) = c_\sigma \sum_{g \in G} \mathrm{Tr}\, \sigma(g^{-1}) [\chi_1(g) - \chi_2(g)].$$

In particular, for an irreducible representation π the orthogonality of characters implies that

$$\dim \pi \cdot \varphi(p_k^\pi) = c_\pi \sum_{g \in G} \overline{\mathrm{Tr}(\pi(g))} [\chi_1(g) - \chi_2(g)] = \dim \pi \cdot (m_{\pi,1} - m_{\pi,2}),$$

where $m_{\pi,i}$ is the multiplicity of π in σ_1.

In this basis, $\beta : K_0(C^*(G)) \to \oplus_i \mathbb{Z}$ is given by

$$[p] \mapsto \oplus_i [\mathrm{rank}(\sigma_{1,i}(p)) - \mathrm{rank}(\sigma_{2,i}(p))].$$

Now, $\sigma_{j,i} = \oplus m_{\pi_k,j} \pi_k$ and $\pi_k(p_1^\pi) = \delta_{\pi,\pi_k} p_1^\pi$. Since the rank is the same as the trace for a projection,

$$\mathrm{rank}(\sigma_{j,i}(p_1^\pi)) = \mathrm{Tr}(\sigma_{j,i}(p_1^\pi)) = \sum_k m_{\pi_k,j} \mathrm{Tr}(\pi_k(p_1^\pi)) = m_{\pi,j}.$$

□

If we could prove the analogous statement for the following diagram with suitable choices of $(\sigma_{1,i}, \sigma_{2,i})$

$$\begin{array}{ccc} K_1(BG) & \xrightarrow{\gamma} & \varinjlim \mathrm{Ext}^1_{\mathbb{Z}}(R(G)/I^n, \mathbb{Z}) \cong \prod_{p \in \mathcal{P}(G)} (\mathbb{Z}/p^\infty)^{r(p)} \\ & \searrow^{\mathrm{Ind}_{\sigma_{1,i},\sigma_{2,i}}} & \downarrow \mathrm{Proj} \\ & & \mathbb{R}/\mathbb{Z}, \end{array} \tag{6.5}$$

then the map ψ, defined as the product of the maps $\rho_{\sigma_{1,i},\sigma_{2,i}}$, would also commute. However, this statement cannot be proven by purely homological arguments or in another straight forward way.

In conclusion, further study of the $\mathrm{Ind}_{\sigma_1,\sigma_2}$-map is required to prove that ψ is well-defined.

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