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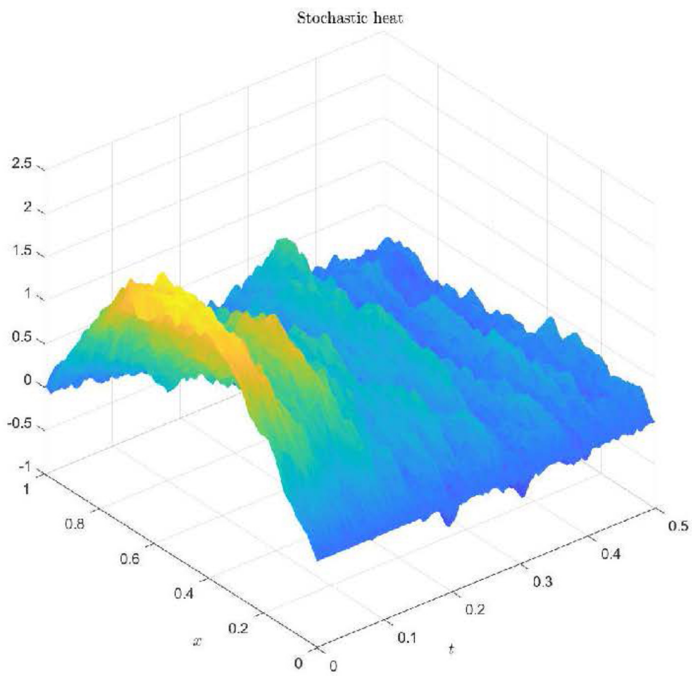

Statistical inference for the stochastic heat equation

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Statistical inference for the stochastic heat equation

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Abstract

This thesis is concerned with two p-variation type estimators for the parameters θ (diffusivity) and σ^2 (noise size) of the stochastic heat equation, proposed by Cialenco and Huang [3]. The theory of the stochastic heat equation is reviewed. The theory of p-variation type estimators is reviewed and p-variation type estimators relating to the stochastic heat equation are introduced. These estimators are investigated numerically. From the simulation results, it is conjectured that the estimators for θ as well as the estimators for σ^2 converge in mean and root mean square with convergence rate 1/2.

Keywords: Stochastic heat equation, p-variation type estimators, statistical inference, stochastic PDEs.

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Introduction

We study statistical estimators for the parameters of the stochastic heat equation on a finite interval. The stochastic heat equation is given by

$$\mathrm{d}u(t,x) = \theta \frac{\partial^2}{\partial x^2} u(t,x)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t),$$

where W is a cylindrical Wiener process and the positive parameters θ and σ are called the thermal diffusivity and the noise size, respectively. We will consider the stochastic heat equation on the interval $[0, L]$ with homogeneous Dirichlet boundary conditions and initial condition $u(0, x) = 0$. The exact mathematical meaning of this equation will be described in Chapter 2, but intuitively the solution u describes the temperature of some rod of finite length L, inside a room where the temperature fluctuates somewhat (this is described by the term $\sigma\,\mathrm{d}W(t)$). Good overviews of the theory of stochastic partial differential equations can be found in [11] and [8].

The current theory of estimators of the parameters θ and σ is overviewed in [2]. In short, most of the work is based on a so called spectral approach where one assumes that the solution-path $u(t, x_0)$ of the stochastic heat equation is known for some time-span $t \in [t_0, t_1]$ at a fixed spatial point x_0. In this thesis we will however be concerned with the "opposite" situation. The estimators considered will be calculated from the solution at a fixed time t_0.

We will investigate two families of estimators, $\{\theta_n\}_{n=2}^{\infty}$ for θ and $\{\sigma_n\}_{n=2}^{\infty}$ for σ, proposed by Cialenco and Huang [3]. The estimators are based on the calculation of the 2-variation of the solution of the stochastic heat equation fixed at some specific time t_0. In fact the estimators are determined from the value of the solution on a space grid $0 = x_1 < x_2 < \ldots < x_n = L$ at the time t_0. The estimators proposed by Cialenco and Huang are potentially interesting because they are very easy to calculate.

The main result of this thesis is showing numerically that the estimators θ_n and σ_n both seem to converge in mean and root mean square with convergence rate $1/2$.

In Chapter 2 we introduce the stochastic heat equation and the underlying theoretical basis. In Chapter 3 we describe the statistical estimators on which the thesis is built. Finally, in Chapter 4 we describe the computer simulations used for the exploration of the properties of the statistical estimators and we present the main results of this thesis.

2

The stochastic heat equation

In this chapter we review some facts about the deterministic heat equation, and develop the stochastic heat equation. The material in this chapter is heavily influenced by the book [11], and so the interested reader is referred there for more in-depth knowledge.

We denote by $L^2(a,b)$ the space of square-integrable functions from the interval (a,b) to $\mathbb{R}$, similarly $L^2(\mathbb{R})$ is the space of square-integrable functions from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, we denote by $H^r(a,b)$ the Sobolev space of square-integrable functions from the interval (a,b) to $\mathbb{R}$ with weak derivatives up to and including order r and analogously $H^r(\mathbb{R})$ is the Sobolev space of square-integrable functions from $\mathbb{R}$ to $\mathbb{R}$ with weak derivatives of order r. We note that $L^2(a,b)$, $L^2(\mathbb{R})$, $H^r(a,b)$ and $H^r(\mathbb{R})$ are Hilbert spaces. Furthermore, we will denote by $H_0^1(a,b)$ the set of functions in $H^1(a,b)$ satisfying homogeneous Dirichlet boundary conditions, i.e.

$$H_0^1(a,b) = \{f \in H^1(a,b) : f(a) = f(b) = 0\}.$$

2.1 Semigroups of linear operators

This section deals with semigroups, a useful theoretical tool which we will use to write down the solution of the stochastic heat equation in Section 2.3.

Let H be a Hilbert space equipped with a norm $\|\cdot\|$. We denote by $\mathcal{L}(H)$ the set of all linear operators on H, i.e. the set of all linear maps from H to itself. Recall that the operator norm on $\mathcal{L}(H)$ is given by

$$\|T\|_{\mathcal{L}(H)} := \sup_{h\in H, h\neq 0} \frac{\|T(h)\|}{\|h\|},$$

for operators $T \in \mathcal{L}(H)$.

Definition 1. Given a Hilbert space H, a function $S : \mathbb{R}_{\geq 0} \to \mathcal{L}(H)$ is called a *semigroup of linear operators* if the following two conditions hold:

- $S(0) = I$, where I is the identity operator on X.
- $S(t_1 + t_2) = S(t_1)S(t_2)$ for any $t_1, t_2 \geq 0$.

If furthermore $S(\cdot)h : \mathbb{R}_{\geq 0} \to H$ is continuous for all $h \in H$ then S is called a C_0 *semigroup of linear operators* or a *strongly continuous semigroup of linear operators.* Finally, if S is strongly continuous and

$$\|S(t) - I\|_{\mathcal{L}(H)} \to_{t\to 0^+} 0,$$

then S is a *uniformly continuous semigroup of linear operators.*

Next, we consider a sort of derivative of a semigroup, as the following definition makes rigorous.

Definition 2. Given a Hilbert space H and a semigroup of linear operators S into $\mathcal{L}(H)$ the *infinitesimal generator* of S is the operator $-A : \mathcal{D}(A) \to H$ defined by

$$-Ah = \lim_{t\to 0^+} \frac{S(t)h - h}{t}.$$

Here the domain $\mathcal{D}(A)$ of A consists of all $h \in H$ for which the above limit exists.

There is a bijective correspondence between uniformly continuous semigroups of linear operators and bounded linear operators:

Theorem 3 (From [5])**.** *Let H be a Hilbert space. If $-A \in \mathcal{L}(H)$ is bounded then it is possible to define a semigroup of linear operators $S : \mathbb{R}_{\geq 0} \to \mathcal{L}(H)$ by*

$$S(t)h = \mathrm{e}^{-tA}h := \left(\sum_{k=0}^{\infty} \frac{t^k(-A)^k}{k!}\right) h.$$

Conversely if $S : \mathbb{R}_{\geq 0} \to \mathcal{L}(H)$ is a uniformly continuous semigroup of linear operators, then the infinitesimal generator of S is a bounded linear operator, with the whole of H as domain.

Next we show an example of an infinitesimal generator of a semigroup of linear operators.

Example 4. Consider the homogeneous heat equation on the real line

$$\frac{\partial}{\partial t}u(t,x) - \frac{\partial}{\partial x^2}u(t,x) = 0,$$

with initial condition

$$u(0,x) = u_0(x).$$

The solutions are expressed in terms of the heat kernel $K(t,x) = \exp(-x^2/4t)(4\pi t)^{-1/2}$ as the convolution

$$u(t,x) = \int_{-\infty}^{\infty} u_0(y)K(t,x-y)\,\mathrm{d}y.$$

It is a standard result from Fourier analysis that if we take $u_0 \in L^2(\mathbb{R})$, then for any fixed $t \geq 0$ we have $u(t,\cdot) \in L^2(\mathbb{R})$. We note further that the mapping that takes u_0 to $u(t,\cdot)$ is linear since the integral is linear. Hence, we can introduce $S : \mathbb{R}_{\geq 0} \to \mathcal{L}(L^2(\mathbb{R}))$, defined by

$$S(t)u_0 = u(t,\cdot),$$

where u is the solution to the heat equation corresponding to the initial condition u_0. S is a semigroup of linear operators. The first condition of Definition 1 is satisfied since

$S(0)$ maps $u_0 \in L^2(\mathbb{R})$ to $u(0,\cdot) = u_0$. To show that the second condition is satisfied requires a little more work. For any $t_1, t_2 \geq 0$, $x \in \mathbb{R}$ and $u_0 \in L^2(\mathbb{R})$ we have

$$\begin{aligned}(S(t_1)S(t_2)u_0)(x) &= S(t_1)\int_{-\infty}^{\infty} u_0(y)K(t_2, x-y)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(y)K(t_2, z-y)\,\mathrm{d}y\, K(t_1, x-z)\,\mathrm{d}z.\end{aligned}$$

We observe that

$$\begin{aligned}K(t_2, z-y)K(t_1, x-z) &= \exp(-(z-y)^2/4t_2)(4\pi t_2)^{-1/2}\exp(-(x-z)^2/4t_1)(4\pi t_1)^{-1/2} \\ &= \exp\left(-\frac{t_1(z-y)^2 + t_2(x-z)^2}{4t_1t_2}\right)\left(4\pi\sqrt{t_1t_2}\right)^{-1}\end{aligned}$$

and

$$\begin{aligned}t_1(z-y)^2 + t_2(x-z)^2 = (t_1+t_2)(z - (t_1y + t_2x)/(t_1+t_2))^2 \\ + \frac{t_1t_2}{t_1+t_2}(y-x)^2.\end{aligned}$$

Now let

$$f(z) = (t_1+t_2)^{1/2}(4\pi t_1t_2)^{-1/2}\exp\left(-\frac{(t_1+t_2)(z-(t_1y+t_2x)/(t_1+t_2))^2}{4t_1t_2}\right)$$

be the probability density function of a normal distribution with mean $(t_1y+t_2x)/(t_1+t_2)$ and variance $2t_1t_2/(t_1+t_2)$. We then have

$$\begin{aligned}\exp\left(-\frac{t_1(z-y)^2 + t_2(x-z)^2}{4t_1t_2}\right) &= \sqrt{\frac{4\pi t_1t_2}{t_1+t_2}}f(z)\exp\left(-\frac{(y-x)^2}{4(t_1+t_2)}\right) \\ &= 4\pi\sqrt{t_1t_2}f(z)K(t_1+t_2, y-x),\end{aligned}$$

and we know that the integral of f over $\mathbb{R}$ is equal to 1. Putting all this together, we find

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(y)K(t_2, z-y)\,\mathrm{d}y\, K(t_1, x-z)\,\mathrm{d}z &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(y)K(t_2, z-y)K(t_1, x-z)\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} u_0(y)K(t_1+t_2, x-y)\,\mathrm{d}y \\ &= (S(t_1+t_2)u_0)(x),\end{aligned}$$

where the change of integration order can be motivated by the Fubini–Tonelli theorem. Hence S is a semigroup of linear operators.

Next we calculate the infinitesimal generator of S. We have that

$$\lim_{t\to 0^+}\frac{S(t)u_0 - u_0}{t} = \lim_{t\to 0^+}\frac{u(t,\cdot) - u(0,\cdot)}{t} = u_t(0,\cdot) = u_{xx}(0,\cdot) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}u_0,$$

so the infinitesimal generator is given by $-A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$. We note that $-A$ is linear and that we can write the heat equation in the form

$$u_t(t,\cdot) = -Au(t,\cdot), u(0,\cdot) = u_0,$$

which is a linear ODE on the Hilbert space $H^2(\mathbb{R})$. For more on this line of thought, see Section 2.3.

Note that the operator $-A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in the preceding example does not have domain $L^2(\mathbb{R})$ since there are non-differentiable functions in $L^2(\mathbb{R})$. Therefore Theorem 3 shows that the related semigroup S can't be uniformly continuous. It is often the case that Theorem 3 has too restrictive assumptions - in particular that A needs to be everywhere defined and bounded.

There is however a correspondence between strongly continuous semigroups and a different class of operators A as described by the following two theorems. This will cover the operators considered in the remaining parts of this thesis.

Theorem 5 (From [5]). *The infinitesimal generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Theorem 6 (From [11]). *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and assume that the linear operator $A : \mathcal{D}(A) \subset H \to H$ has a complete orthonormal set of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ with corresponding eigenvalues $\lambda_j > 0$, such that $\lambda_{j+1} \geq \lambda_j$ for all $j \in \mathbb{N}$. Then we can define a semigroup of linear operators $S : \mathbb{R}_{\geq 0} \to \mathcal{L}(H)$ by*

$$S(t)x = \mathrm{e}^{-tA}x := \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} \langle x, \varphi_j \rangle \varphi_j.$$

Furthermore, the infinitesimal generator of S is $-A$.

2.2 Stochastic partial differential equations

In this section we introduce the concept of a stochastic partial differential equation (SPDE). We will be quite restrictive and mostly introduce concepts of use later in this thesis. For a more general and in depth treatise we refer to for example [11] or [8].

Definition 7. If H is a separable Hilbert space with orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$, then a *cylindrical Wiener process* is the stochastic process

$$\sum_{j\in\mathbb{N}} \varphi_j B_j(t),$$

where the B_j's are independent Brownian motions.

Note that it is not obvious that the sum in the definition should converge in some meaningful way. It is however possible to show this, using the theory of Hilbert-Schmidt-inclusions (see for instance [11]). We will skip this for the sake of brevity. Similarly we do not show the meaningfulness of the following definitions rigorously, but introduce the important concepts to be used later on.

Definition 8. If H is a separable Hilbert space with orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$, then

$$\mathrm{HS}(H) = \{T \in \mathcal{L}(H) : \sum_{j\in\mathbb{N}} ||T\varphi_j||_H^2 < \infty\}$$

is the set of *Hilbert-Schmidt operators* on H.

Definition 9. Let B be a Brownian motion and let γ be a real-valued step function on $[0,T]$ where $T > 0$, i.e. γ has the form

$$\gamma(t) = \sum_{j=1}^{J-1} \alpha_j 1_{[t_j,t_{j+1})}(t),$$

for some partition $\{t_j\}_{j=1}^J$ of $[0,T]$ and real constants α_j. The *Wiener integral* of γ with respect to B is defined as

$$\int_0^T \gamma(t)\,\mathrm{d}B(t) = \sum_{j=1}^{J-1} \alpha_j(B(t_{j+1}) - B(t_j)).$$

Definition 10. Let B be a Brownian motion on an underlying probability triple $(\Omega, \mathcal{F}, \mathbf{P})$ and let $g \in L^2(0,T)$ for some $T > 0$. The *Wiener integral* of g with respect to B is defined as the $L^2(\Omega)$-limit

$$\int_0^T g(t)\,\mathrm{d}B(t) = \lim_{k\to\infty} \int_0^T \gamma_k(t)\,\mathrm{d}B(t),$$

where $\{\gamma_k\}_{k=1}^\infty$ is a sequence of step functions converging in $L^2(0,L)$ to g.

For a proof of the above integral being well-defined we refer to Kuo [9]. In [9] we also find the following often used fact.

Theorem 11. *Let $g \in L^2(0,T)$. Then $\int_0^T g(t)\,\mathrm{d}B(t)$ is a normally distributed random variable with mean 0 and variance $\int_0^T g(t)^2\,\mathrm{d}t$.*

Given a Hilbert space H we can, analogously to Definitions 9 and 10, define the Wiener integral of a H-valued function on $[0,T]$ as the limit of Wiener integrals of so called simple H-valued functions which are very much like the γ in Definition 9 except that the α_j's are in H.

Definition 12. Let H be a separable Hilbert space and let W be a H-valued cylindrical Wiener process. Let G be a $\mathrm{HS}(H)$-valued function on $\mathbb{R}_{\geq 0}$. The *Wiener integral* of G is defined by

$$\int_0^t G(s)\,\mathrm{d}W(s) = \sum_{j=1}^\infty \int_0^t G(s)\varphi_j\,\mathrm{d}B_j(s),$$

where the latter integrals are ordinary Wiener integrals. By definition, the integral is H-valued. Note that if for every $s \in (0,t)$ G can be decomposed as

$$G(s)h = \sum_{k=1}^\infty \lambda_k(s)\,\langle h, \varphi_k\rangle\,\varphi_k,$$

then

$$\int_0^t G(s)\,\mathrm{d}W(s) = \sum_{j=1}^\infty \varphi_k \int_0^t \lambda_j(s)\,\mathrm{d}B_j(s).$$

Now we are ready to define a type of stochastic partial differential equation.

Definition 13. An *Itô stochastic semi-linear evolution equation driven by additive noise* is of the form

$$\mathrm{d}u = (-Au + f)\,\mathrm{d}t + G\,\mathrm{d}W(t), \tag{2.1}$$

where $u(0) = u_0 \in H$, $f : \mathbb{R}_{\geq 0} \to H$, $G : \mathbb{R}_{\geq 0} \to \mathrm{HS}(H)$, W is a H-valued cylindrical Wiener process and A is a linear operator satisfying the assumptions in Theorem 6.

There are different ways to consider solutions to such equations. The most straightforward is probably the strong solution.

Definition 14. A *strong solution* on $[0, T]$ to (2.1) is a predictable H-valued stochastic process u such that for every $t \in [0, T]$

$$u(t) = u_0 + \int_0^t (-Au(s) + f(s))\,\mathrm{d}s + \int_0^t G(s)\,\mathrm{d}W(s)$$

holds $\mathbf{P}$-a.s..

The strong solution is as the name hints quite strong, in fact it is often too strong to be useful. In the next section we introduce a different type of solution, called the *mild solution*.

2.3 Mild solutions of semi-linear evolution equations

This section will cover one way of considering solutions to the stochastic heat equation.

We start once again by taking a step back and considering deterministic equations.

Definition 15. If H is a Hilbert space, a *semi-linear evolution equation* on H is of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f, \quad u(0) = u_0 \in H, \tag{2.2}$$

where A is a linear operator satisfying the assumptions in Theorem 6 and f is a function from $\mathbb{R}_{\geq 0}$ to H.

We note that there is a semigroup S generated by $-A$ guaranteed by Theorem 6.

Definition 16. With the same assumptions as in Definition 15, a *mild solution* of (2.2) on $[0, T]$ is a continuous function $u : [0, T] \to H$ which satisfies for every $t \in [0, T]$ that

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s. \tag{2.3}$$

Formula (2.3) is referred to as the *variation of constants* formula.

Similarly to the above, we define the mild solution of the stochastic equation (2.1).

Definition 17. Let H be a separable Hilbert space. A *mild solution* of (2.1) on $[0,T]$ is a predictable H-valued stochastic process such that for every $t \in [0,T]$

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s + \int_0^t S(t-s)G(s)\,\mathrm{d}W(s), \tag{2.4}$$

where S is the semigroup generated by $-A$.

2.4 The heat equation on a finite interval

Before we take a closer look at the stochastic heat equation, we will recall some basic facts about the deterministic heat equation.

In what follows, we will gloss over some of the more theoretical results. The interested reader is referred to [11].

We consider the heat equation on a finite interval $[0,L]$ with homogeneous Dirichlet boundary conditions and trivial initial condition,

$$\begin{aligned} u_t - \theta u_{xx} &= f, \quad t > 0, \quad x \in [0,L], \\ u(t,0) = u(t,L) &= 0 \quad t > 0, \\ u(0,x) &= 0 \quad x \in [0,L], \end{aligned} \tag{2.5}$$

where θ is a positive parameter called the thermal diffusivity and f is a real-valued function on $\mathbb{R}_{\geq 0} \times [0,L]$ called the forcing term.

In order to solve the heat equation (2.5), one can make use of the fact that the differential operator $-A = \theta \frac{\partial^2}{\partial x^2}$ is decomposable. An orthonormal basis of $\mathcal{D}(A) = H^2(0,L) \cap H_0^1(0,L)$ consisting eigenfunctions of $-A$ is $\{\varphi_j(x)\}_{j=1}^{\infty} = \{\sqrt{2/L}\sin(\pi j x/L)\}_{j=1}^{\infty}$. The corresponding eigenvalues are $\lambda_j = j^2\pi^2\theta/L^2$. We let $H = L^2(0,L)$ equipped with the usual inner product denoted by $\langle \cdot, \cdot \rangle$. Following Theorem 6, the semigroup of linear operators having infinitesimal generator $-A$ is defined by

$$S(t)v = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} \langle v, \varphi_j \rangle \, \varphi_j(x),$$

and the mild solution to (2.5) is given by

$$u(t,x) = \int_0^t S(t-s)f(s)\,\mathrm{d}s. \tag{2.6}$$

Another way to obtain this solution is to look at the spectral decomposition of u,

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \sum_{j=1}^{\infty}\langle \frac{\partial}{\partial t}u(t),\varphi_j\rangle\,\varphi_j \\ &= \sum_{j=1}^{\infty}\langle \frac{\partial^2}{\partial x^2}u(t)+f(t),\varphi_j\rangle\,\varphi_j(x).\end{aligned} \tag{2.7}$$

We note further that since the φ_j's are eigenfunctions of $\frac{\partial^2}{\partial x^2}$ and we have homogeneous Dirichlet boundary conditions that by twice integrating by parts

$$\begin{aligned}\langle \frac{\partial^2 u(t)}{\partial x^2},\varphi_j\rangle &= \langle u(t),\frac{\partial^2\varphi_j}{\partial x^2}\rangle \\ &= \lambda_j\,\langle u(t),\varphi_j\rangle\,.\end{aligned}$$

Now, since the φ_j's are orthogonal in $L^2(0,L)$, we must have that Equality (2.7) holds in every j separately. Thus, after introducing the notation $g_j = \langle g,\varphi_j\rangle$ we obtain for any $j\in\mathbb{N}$

$$\left(\frac{\partial u(t)}{\partial t}\right)_j = \lambda_j u(t)_j + f_j.$$

And after changing the order of differentiation and integration in the first term we get

$$\frac{\partial u(t)_j}{\partial t} = \lambda_j u(t)_j + f_j,$$

which is a first order ordinary differential equation on $L^2[0,L]$ and is solvable by the variation of constants method. Using the initial condition

$$u(0)_j = 0$$

we obtain

$$u(t)_j = \mathrm{e}^{\lambda_j t}\int_0^t f_j(s)\mathrm{e}^{-\lambda_j s}\,\mathrm{d}s.$$

Hence, the solution can be written as

$$u(t,x) = \sum_{j=1}^{\infty}\varphi_j \mathrm{e}^{\lambda_j t}\int_0^t \langle f,\varphi_j\rangle\,(s)\mathrm{e}^{-\lambda_j s}\,\mathrm{d}s. \tag{2.8}$$

which corresponds to the previously obtained solution (2.6).

Next, we turn to the stochastic version of the heat equation.

2.5 The stochastic heat equation on $[0, L]$

We note that one way to rewrite the heat equation (2.5) is by using the higher level of abstraction as seen in previous sections. This yields the equation

$$u_t + Au = f, \tag{2.9}$$

where $-A = \theta \frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

The stochastic analogue to this $L^2(0, L)$-valued equation is in the next definition.
Definition 18. The *stochastic heat equation* is the Itô stochastic semi-linear equation given by

$$\mathrm{d}u + Au\,\mathrm{d}t = \sigma\,\mathrm{d}W(t), \tag{2.10}$$

where the domain of $-A = \theta \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is given by $\mathcal{D}(A) = H^2(0, L) \cap H_0^1(0, L)$, θ is a positive parameter called the thermal diffusivity, σ is a positive parameter called the noise size and W is a $L^2(0, L)$-valued cylindrical Wiener process. We impose the initial condition $u(0) = u_0 \in L^2(0, L)$.

By Definition 17, a mild solution of (2.10) on $[0, T]$ satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\sigma W(t)$$

where $S(t) = \mathrm{e}^{-At}$ as defined in Theorem 6 and the integral is defined by Definition 12.

We note that the dependence on u_0 is exactly like in the deterministic case, therefore we will later be only concerned with the case $u_0 = 0$ since it simplifies matters and still captures all the randomness of the stochastic heat equation.

Finally we state the important existence and uniqueness property of the mild solution.
Theorem 19 (See [11])**.** *The stochastic heat equation* (2.10) *has a unique mild solution on* $[0, T]$ *for any* $T > 0$*. The uniqueness here is on the Banach space of* $L^2(0, L)$*-valued predictable processes on* $[0, T]$*, with the norm*

$$\|v\| = \sup_{t \in [0,T]} \left(\mathbf{E} \left[\|v(t)\|_{L^2(0,L)}^2 \right] \right)^{1/2}.$$

3

p-variation type estimators for the stochastic heat equation

The setting for this chapter is the stochastic heat equation on the interval $[0, L]$ as developed in Chapter 2

$$\mathrm{d}u(t,x) = \theta u_{xx}(t,x)\mathrm{d}t + \sigma \mathrm{d}W(t,x), \tag{3.1}$$

with

$$u(0,x) = 0$$

and homogeneous Dirichlet boundary conditions. We will consider sampling the solution of this equation at some fixed time t_0. Our goal will be to, from this sample, estimate one of the parameters θ and σ, taking the other to be known. We will investigate two of the estimators proposed by Cialenco and Huang in 2017 [3]. These estimators are based on the computation of the so-called *p-variation* of a stochastic process.

Three key facts will be the following, and they will be made precise and more general in the coming sections.

- First, one can describe the solution of the stochastic heat equation (3.1) at a fixed time t_0 as the sum $u(t_0, x) = \frac{\sigma}{\sqrt{2\theta}} B(x) + G(x)$, where $B(x)$ is a Brownian motion on $[0, L]$ and $G(x)$ is a smooth Gaussian process on $[0, L]$.
- Furthermore, the p-variation (to be defined in Definition 28 of a stochastic process is not changed if one perturbs it with some smooth process. This means that the p-variation of $u(t_0, x)$ is the same as the p-variation of $\frac{\sigma}{\sqrt{2\theta}} B(x)$.
- Finally, the 2-variation of a Brownian motion is easy to compute, in fact the following holds: if a is some real parameter, then $aB(x)$ has 2-variation $a^2 L$.

We will see that there is a very straightforward way of estimating the 2-variation of the solution $u(t_0, x)$ from a sample. The practical thing about this is that the above three key facts combined show that by estimating the 2-variation of $u(t_0, x)$ we estimate the quantity $\frac{\sigma^2}{2\theta} L$ and hence if we assume one of the parameters θ or σ to be known then we can estimate the other.

We begin this chapter by stating an already described result, which is in fact a theorem by Cialenco and Huang [3].

Definition 20. A modification of a stochastic process Y on A is another stochastic process $\tilde{Y}$ on A such that $\mathbf{P}(Y(x) = \tilde{Y}(x)) = 1$ for each $x \in A$.

Theorem 21 (From [3])**.** *Let $u(t,x)$ be the solution to the stochastic heat equation* (3.1)*. For every $t_0 > 0$ there is a Brownian motion B and a Gaussian process G on* $[0, L]$*, such that G has a $C^\infty(0, L)$ modification and*

$$u(t_0, x) = \frac{\sigma}{\sqrt{2\theta}} B(x) + G(x),$$

for all x in $[0, L]$*.*

3.1 Sampling of random processes on $[0, L]$

In this section we lay the foundation for later sections by defining some key terms.

Definition 22. A *partition* π of $[0, L]$ is a finite set of ordered points in $[0, L]$ which contains 0 and L. I.e. a partition π of $[0, L]$ is of the form

$$\pi = \{0 = x_0 < x_1 < ... < x_{\text{final}} = L\}.$$

As previously mentioned, we will be interested in the solution of the stochastic heat equation fixed at some time t_0. That is the stochastic process $u(t_0, x)$ where x ranges over $[0, L]$. We will sample $u(t_0, x)$ on some partition of $[0, L]$. In order to clear up some nomenclature, we provide the following definitions. Throughout this section we will assume that Y is a real-valued stochastic process on $[0, L]$. We will denote the underlying probability triple by $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 23. A *discretization* of Y on a partition

$$\pi = \{0 = x_0 < x_1 < ... < x_{\text{final}} = L\}$$

is the random vector

$$(Y(0), Y(x_1), ..., Y(L)).$$

Definition 24. A *sample path* of Y is one realization of Y. Note that for every $x \in [0, L]$, $Y(x)$ is a random variable. A sample path of Y is a function $Y(\cdot, \omega) : [0, L] \to \mathbb{R}$, where $\omega \in \Omega$ is fixed.

Definition 25. A *discrete sample* or simply *sample* of Y is a vector consisting of the evaluations of a sample path of Y on a partition of $[0, L]$. In other words, a sample of Y has the form

$$(Y(0, \omega), Y(x_1, \omega), ..., Y(L, \omega)),$$

where $\omega \in \Omega$ is fixed.

If one were to sample the temperature of an idealized iron rod, obeying the stochastic heat equation, at a specific time t_0 and a finite amount of spots, then the measurements would form a sample of $u(t_0, \cdot)$ in the sense of the preceding definition. We note that there is no randomness ”left” in such a sample – it is simply a vector of real numbers.

Definition 26. Given a discretization $(Y(0), Y(x_1), ..., Y(L)) : \Omega \to \mathbb{R}^m$, a *point estimator* or just *estimator* is the composition of any function W on $\mathbb{R}^m$ with the discretization, i.e. the estimator is the random variable $W(Y(0), Y(x_1), ..., Y(L))$.

Clearly the above definition is far too broad to cover much of interest. Our goal is to estimate a parameter of a probability distribution corresponding to the solution of a stochastic heat equation. The following definition relates estimators to parameters.

Definition 27. A *consistent sequence of estimators* of a parameter η is a sequence $(W_n)_{n\in\mathbb{N}}$ of estimators of η, which converges almost surely to η.

Here it is of worth to note that some authors (e.g. [1]) define a consistent sequence of estimators to be a sequence which converges in probability to the parameter. We opt for a stronger definition, because the sequences of estimators ((3.2) and (3.3)) we will see later, actually satisfy it (Theorem 31).

3.2 p-variation of a stochastic process

Definition 28. Let Y be a real-valued stochastic process on $[0, L]$. The *p-variation* of Y, denoted $V^p[Y]$, is the a.s. limit

$$V^p[Y] := \lim_{k\to\infty} \sum_{j=1}^{k} \left| Y\left(\frac{j}{k}L\right) - Y\left(\frac{j-1}{k}L\right) \right|^p .$$

We add a couple of remarks with respect to this definition.

- The 2-variation of a stochastic process is often called its *quadratic variation* and similarly the 3-variation is sometimes called its *cubic variation*, the 4-variation its *quartic variation*, etc. We opt here for the more flexible notation p-variation, but see the next remark for a potential pitfall.
- The above definition of p-variation is **not** the same one that is used in analysis. There one typically (for the origin of this definition, see Wiener [12]) defines the p-variation of a function f on $[0, L]$ to be

$$\sup_{\pi} \left(\sum_{j=1}^{|\pi|-1} |f(x_{\pi,j}) - f(x_{\pi,j-1})|^p \right)^{1/p} ,$$

  where the supremum ranges over all partitions π of $[0, L]$. We note that this type of 2-variation is a.s. infinite for a Brownian motion path - this was shown by Lévy [10].
- Recall Definition 22 of a partition of $[0, L]$. We define the *mesh* (often *mesh size*) of a partition π to be the length of the largest interval between two adjacent points in π and denote this quantity by $m(\pi)$. Hence a uniform partition π is defined by satisfying $m(\pi) = L/(|\pi| - 1)$ where $|\pi|$ is the number of points in π.

Note that in the definition of p-variation, we consider the stochastic process Y on uniform partitions of $[0, L]$. We denote these partitions by π_k and note that they are of size $k+1$, so that $m(\pi_k) = L/k$. One might wonder if one can relax the definition of p-variation to non-uniform partitions of $[0, L]$ and to sequences of partitions π_k with various convergence speeds of the mesh $m(\pi_k) \to 0$. We provide an important example to show that this is not possible, at least not with the greatest generality. In this thesis we will be interested in the 2-variation of a Brownian motion B on $[0, L]$, a subject that was first studied by Lévy [10]. Let π_k be a sequence of partitions of $[0, L]$. It was shown by Dudley [4] that if $m(\pi_k) = \mathrm{o}(1/\log(k))$, then

$$\lim_{k\to\infty} \sum_{j=1}^{|\pi_k|-1} (B(x_{k,j}) - B(x_{k,j-1}))^2 = L \quad \text{a.s.},$$

where of course $x_{k,j}$ is in the partition π_k for all j and k. On the other hand, it was shown by Fernandez de la Vega [6] that there are sequences of partitions satisfying $m(\pi_k) = \mathrm{O}(1/\log(k))$ such that

$$\sum_{j=1}^{|\pi_k|-1} (B(x_{k,j}) - B(x_{k,j-1}))^2$$

does not converge a.s. to L.
Hence it was imperative in Definition 28 that $m(\pi_k) = \mathrm{o}(1/\log(k))$.

A simple lemma following Definition 28 is the following:

Lemma 29. *If Y is a stochastic process on $[0, L]$ with $V^p[Y] = C$, for some p, then $V^p[aY] = |a|^p C$ for any real constant a.*

The following theorem is essential for our development of estimators for the parameters θ and σ of the stochastic heat equation.

Theorem 30 (From [3])**.** *If Y is a stochastic process on $[0, L]$ such that there is some $p > 1$ with $0 < V^p[Y] < \infty$, and if S is a stochastic process on $[0, L]$ which has C^1 sample paths, then*

$$V^p[Y + S] = V^p[Y].$$

Now we return to our main setting – letting $u(t_0, \cdot)$ be the solution to the stochastic heat equation (3.1) at some fixed $t_0 > 0$. We see that by combining Theorem 30, Theorem 21 and Lemma 29 with the previously mentioned result by Dudley [4] that the 2-variation of a Brownian motion on $[0, L]$ equals L, we find

$$V^2[u(t_0, \cdot)] = V^2[\frac{\sigma}{\sqrt{2\theta}} B] = \frac{\sigma^2 L}{2\theta}.$$

The parameters θ and σ^2 of the stochastic heat equation can hence be estimated by the following estimators [3]:

$$\theta_n = \frac{L\sigma^2}{2\sum_{j=1}^{n}(u(t_0,x_j)-u(t_0,x_{j-1}))^2}, \tag{3.2}$$

$$\sigma_n^2 = \frac{2\theta\sum_{j=1}^{n}(u(t_0,x_j)-u(t_0,x_{j-1}))^2}{L}, \tag{3.3}$$

where $x_j = \frac{j}{n}L$.

The following theorem is evident from the definition of p-variation.
Theorem 31 (From [3])**.** *The estimators* (3.2) *and* (3.3) *are consistent.*

It is also of interest to note that both estimators are asymptotically normal.
Theorem 32 (From [3])**.** *The following convergences in distribution hold:*

$$\lim_{n\to\infty}\sqrt{n}(\theta_n-\theta) = \mathcal{N}(0,2\theta^2),$$

$$\lim_{n\to\infty}\sqrt{n}(\sigma_n^2-\sigma^2) = \mathcal{N}(0,2\sigma^4).$$

4

Computer simulations

We implement an approximate solution of the stochastic heat equation in MATLAB. Since we are only interested in the solution at a fixed time t_0, we will only be simulating fixed time solutions.

4.1 Spectral approximation of the fixed time solution

We consider a mild solution (Definition 17) u to the stochastic heat equation (3.1) on the interval $[0, L]$ with homogeneous Dirichlet boundary conditions and initial condition $u(0, x) = 0$ for every $x \in [0, L]$.

As we saw in Chapter 2, $u(t_0, x)$ for a fixed time t_0 can be viewed as a function in $H^2(0, L) \cap H_0^1(0, L)$ and decomposed as

$$u(t_0, x) = \sum_{j=1}^{\infty} \langle u(t_0), \varphi_j \rangle \, \varphi_j,$$

where the φ_j's form the orthonormal basis of $H^2(0, L) \cap H_0^1(0, L)$ presented in Chapter 2, i.e. for every j

$$\varphi_j(x) = \sqrt{2/L} \sin(\pi j x / L).$$

We denote the corresponding eigenvalues of the operator $-\theta \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ by $\lambda_j = j^2 \pi^2 \theta / L^2$.

Furthermore by virtue of being a mild solution to the stochastic heat equation, u satisfies

$$u(t_0) = S(t_0) u(0) + \int_0^{t_0} S(t_0 - t) \sigma \, \mathrm{d}W(t),$$

with $S(t)$ defined by Theorem 6 as

$$S(t)v = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} \langle v, \varphi_j \rangle \varphi_j.$$

And since $u(0) = 0$ we are left with

$$u(t_0) = \sigma \int_0^{t_0} S(t_0 - t) \, \mathrm{d}W(t),$$

which by definition of the integral equals

$$\sigma \sum_{k=1}^{\infty} \int_0^{t_0} S(t_0 - t)\varphi_k \, \mathrm{d}B_k(t) = \sigma \sum_{k=1}^{\infty} \varphi_k \int_0^{t_0} \mathrm{e}^{-\lambda_k (t_0 - t)} \, \mathrm{d}B_k(t),$$

where we applied $S(t_0 - t)$ and made use of the fact that the φ_j's are orthonormal. Now, by Theorem 11 we can evaluate the integrals above as follows

$$\int_0^{t_0} \mathrm{e}^{-\lambda_k (t_0 - t)} \, \mathrm{d}B_k(t) = \mathrm{e}^{-\lambda_k t_0} \int_0^{t_0} \mathrm{e}^{\lambda_k t} \, \mathrm{d}B_k(t) = \mathrm{e}^{-\lambda_k t_0} Y_k,$$

where for every k, Y_k is normally distributed with mean 0 and variance $(\exp(2\lambda_k t_0) - 1)/(2\lambda_k)$. Hence we can summarize by writing

$$u(t_0) = \sum_{k=1}^{\infty} Z_k \varphi_k,$$

where each $Z_k = \sigma \exp(-\lambda_k t_0) Y_k$ is normally distributed with mean 0 and variance $\sigma^2(1 - \exp(-2\lambda_k t_0))/(2\lambda_k)$. In order to simulate the solution of the stochastic heat equation we can thus truncate the above sum at some finite number of terms K:

$$u_{\text{approx.}}(t_0) = \sum_{k=1}^{K} Z_k \varphi_k. \tag{4.1}$$

To get a sense of the error introduced by this truncation we calculate

$$\begin{aligned} \mathbf{E}\left(\|u(t_0) - u_{\text{approx.}}(t_0)\|_{L^2}^2 \right) &= \mathbf{E}\left(\left\| \sum_{k=K+1}^{\infty} Z_k \varphi_k \right\|_{L^2}^2 \right) \\ &= \sum_{k,l=K+1}^{\infty} \mathbf{E}(Z_k Z_l) \langle \varphi_k, \varphi_l \rangle_{L^2} \\ &= \sigma^2 \sum_{k=K+1}^{\infty} \frac{1 - \exp(-2\lambda_k t_0)}{2\lambda_k} \end{aligned}$$

and a crude upper bound for this is found by

$$\begin{aligned} \sigma^2 \sum_{k=K+1}^{\infty} \frac{1}{2\lambda_k} (1 - \exp(-2\lambda_k t_0)) &\leq \sigma^2 \sum_{k=K+1}^{\infty} \frac{1}{(2\lambda_k)} \\ &\leq \sigma^2 \int_K^{\infty} \frac{1}{(2\lambda_k)} \, \mathrm{d}k \\ &= \frac{\sigma^2 L^2}{\pi^2 \theta K}. \end{aligned}$$

The precise algorithms we use are described next. First for approximating the value of u

at a specific spatial point $x_0 \in [0, L]$.

Algorithm 1: Spectral approximation of the solution to the stochastic heat equation at a fixed time t_0 and a fixed point x_0 in space

Data: The fixed time $t_0 > 0$, the interval length $L > 0$, the fixed point in space $x_0 \in [0, L]$, the parameters $\theta > 0$ and $\sigma > 0$, the number of terms $K \in \mathbb{N}$ in the spectral approximation

Result: An approximation of the solution $u(t_0, x_0)$

```
initialize u = 0;
generate size K vector phi with entries phi[k] = (2/L)^{1/2} · sin(π · k · x_0/L);
generate size K vector lambda with entries lambda[k] = k^2 · π^2 · θ/L^2;
generate size K vector z of standard normally distributed numbers;
for k=1:K do
    u = u + σ · z[k] · phi[k] · (1 − exp(−2 · lambda[k] · t_0))^{1/2}/(2 · lambda[k])^{1/2};
end
return u;
```

Next, we show the algorithm used for estimating the parameters θ and σ in the way described in Chapter 3. We use the Monte Carlo method in order to approximate the expected value of the estimators and also to get a sense of their distributions.

Algorithm 2: Approximation of the p-variation type estimators to the parameters of the stochastic heat equation

Data: The fixed time $t_0 > 0$, the interval length $L > 0$, the parameters $\theta > 0$ and $\sigma > 0$, the number $n \in \mathbb{N}$ of spatial points in $[0, L]$ being used, the number of terms $K \in \mathbb{N}$ in the spectral approximation, the number $M \in \mathbb{N}$ of Monte Carlo simulations

Result: M approximations of the estimators θ_n and σ_n^2 as defined in (3.2) and (3.3)

```
initialize vectors estimatedThetas and estimatedSigma2s of size M;
generate size K × n matrix phi with entries phi[k, i] = (2/L)^{1/2} · sin(π · k · i/n);
generate size K vector lambda with entries lambda[k] = k^2 · π^2 · θ/L^2;
generate size K × n matrix factors with entries
 factors[k, i] = σ · phi[k, i] · (1 − exp(−2 · lambda[k] · t_0))^{1/2}/(2 · lambda[k])^{1/2};
for j=1:M do
    generate size K (row) vector z of standard normally distributed numbers;
    calculate size n vector of solutions u = z · factors;
    estimatedThetas[j] = L · σ^2/2/(∑(u[1 : n − 1] − u[2 : n]).^2);
    estimatedSigma2s[j] = 2 · θ · (∑(u[1 : n − 1] − u[2 : n]).^2)/L;
end
return estimatedThetas and estimatedSigma2s;
```

Note that in Algorithm 2 we calculate u by a vector matrix multiplication which corresponds to doing the explicit summation of Algorithm 1 n times at once.

4.2 Simulation results

In this section we present some results obtained from running Algorithm 2 with various input data.

In Figures 4.1, 4.2, 4.3 and 4.4 we observe the distributions of the approximated estimators for two different values of the amount of spatial sample points n. As n increases we should see a tendency toward a normal distribution as indicated by Theorem 32, and this seems clear from the figures. In Figures 4.1 and 4.2 where the amount n of spatial points is low the distributions look non-normal, while in Figures 4.3 and 4.4 where n is higher, the distributions look distinctly normal. The normal distributions were fitted by the MATLAB function `histfit`. Looking further into the simulations behind Figures 4.3 and 4.4 we note that Theorem 32 predicts an approximate variance of $2\theta^2/n$ for θ_n and $2\sigma^4/n$ for σ_n^2. This would mean with our input $\theta = \sigma = 1$ and $n = 512$ that both estimators should have a variance around $3.9 \cdot 10^{-3}$ and indeed when we calculate the fitted variance using the MATLAB function `normfit` we find a value of $4.0 \cdot 10^{-3}$ for the variance of the approximated θ_n's and $3.9 \cdot 10^{-3}$ for the variance of the approximated σ_n^2's. Satisfied with this and now somewhat confident that our algorithm and program work as intended we proceed to tackle the question of whether we can say something about the speed at which θ_n converges to θ and σ_n^2 to σ^2 respectively.

We are interested in the L^1 and L^2 convergences of the estimators, also known as the convergence in mean and in root mean square. Thus we use Algorithm 2 to estimate $\mathbf{E}(|\theta_n - \theta|)$, $\mathbf{E}((\theta_n - \theta)^2)^{1/2}$, $\mathbf{E}(|\sigma_n - \sigma|)$ and $\mathbf{E}((\sigma_n - \sigma)^2)^{1/2}$. In Figures 4.5 and 4.6 we see the convergence in mean of the estimators while in Figures 4.7 and 4.8 we see the convergence in root mean square.

From our simulations we conjecture convergence rates of $1/2$ for both θ_n and σ_n in both L^1 and L^2.

In order to strengthen our hypothesis we might simulate convergence plots with a broader range of parameter values. We present two such plots in Figures 4.9 and 4.10.

4.2.1 Conclusion

We have confirmed numerically some of the theoretical results of Cialenco and Huang [3]. We have numerically estimated the convergence rates of θ_n and σ_n to be $1/2$ both in L^1 and in L^2. Further research is needed to confirm this theoretically.

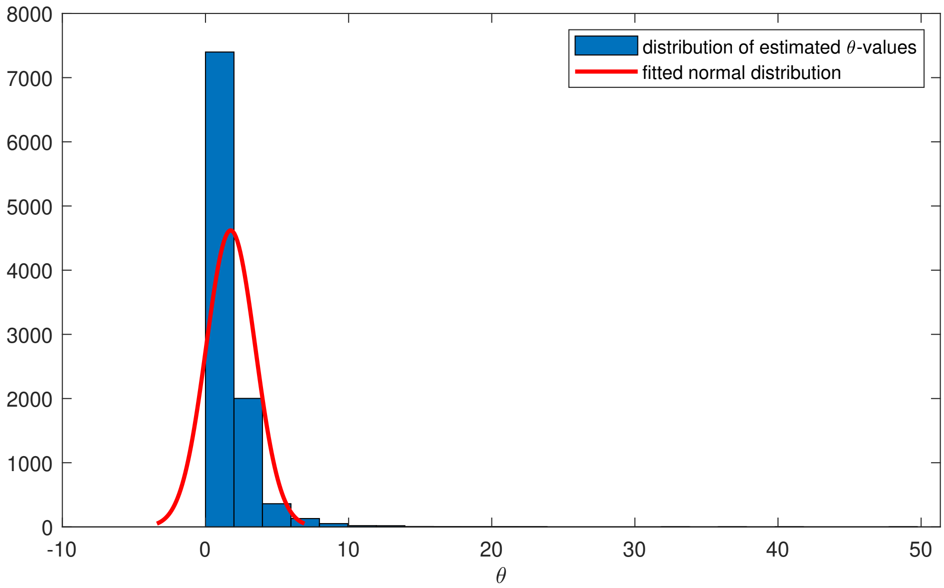

Figure 4.1: Histogram showing the distribution of 10^4 Monte Carlo samples of approximations of the estimator θ_n, based on 8 spatial points, along with a fitted normal distribution.

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	8	10^5	10^4

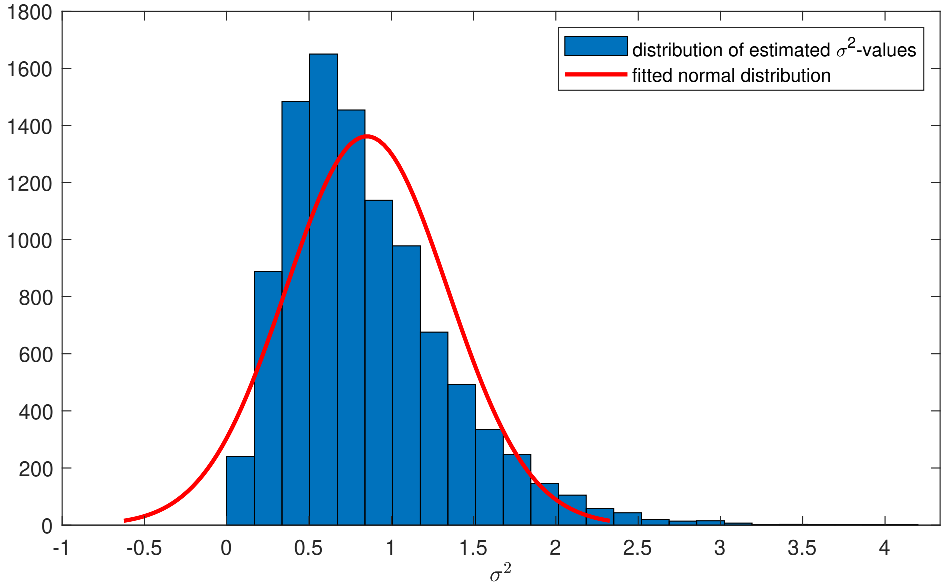

Figure 4.2: Histogram showing the distribution of 10^4 Monte Carlo samples of approximations of the estimator σ_n^2, based on 8 spatial points, along with a fitted normal distribution.

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	8	10^5	10^4

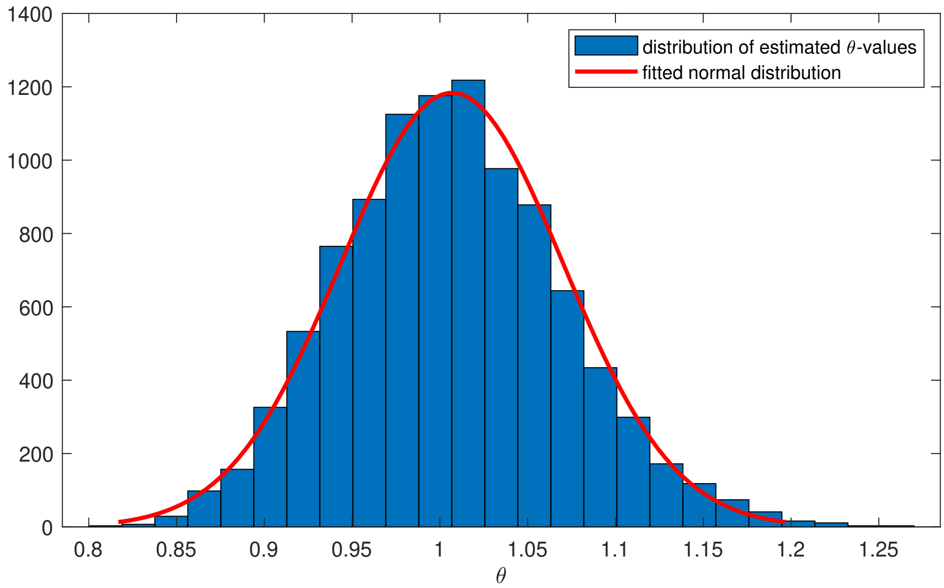

Figure 4.3: Histogram showing the distribution of 10^4 Monte Carlo samples of approximations of the estimator θ_n, based on 512 spatial points, along with a fitted normal distribution.

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	512	10^5	10^4

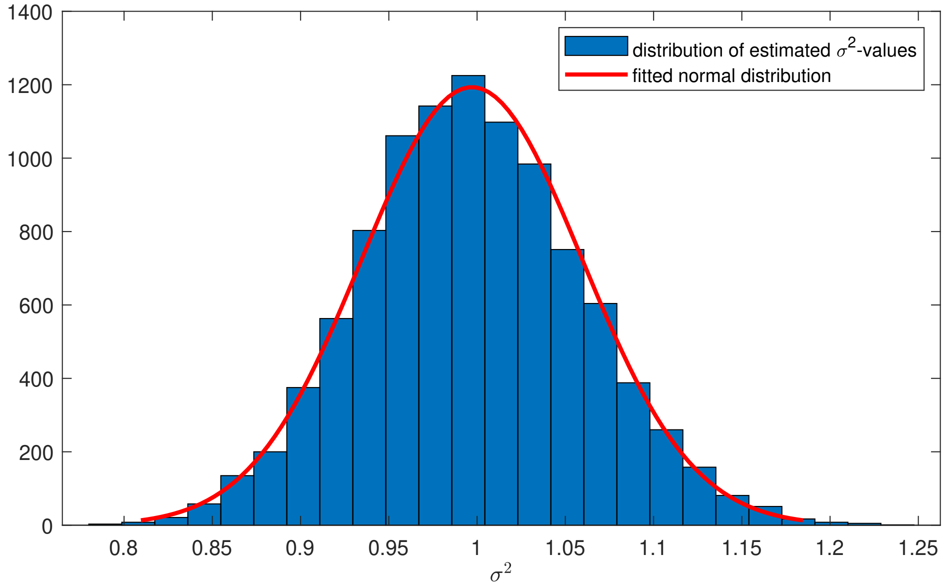

Figure 4.4: Histogram showing the distribution of 10^4 Monte Carlo samples of approximations of the estimator σ_n^2, based on 512 spatial points, along with a fitted normal distribution.

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	512	10^5	10^4

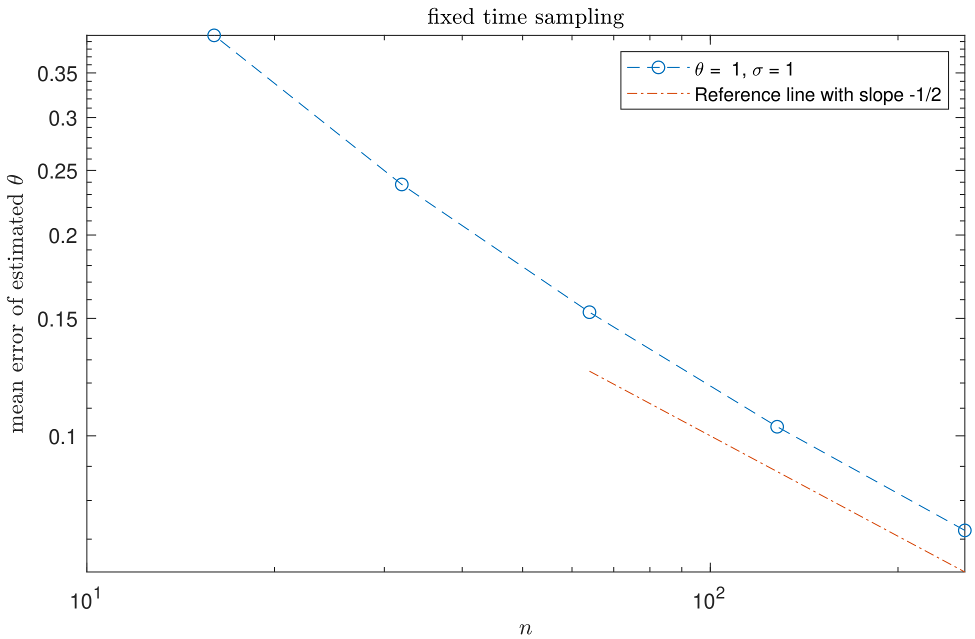

Figure 4.5: Convergence in mean of the estimators θ_n

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	$2^4 - 2^8$	10^5	10^4

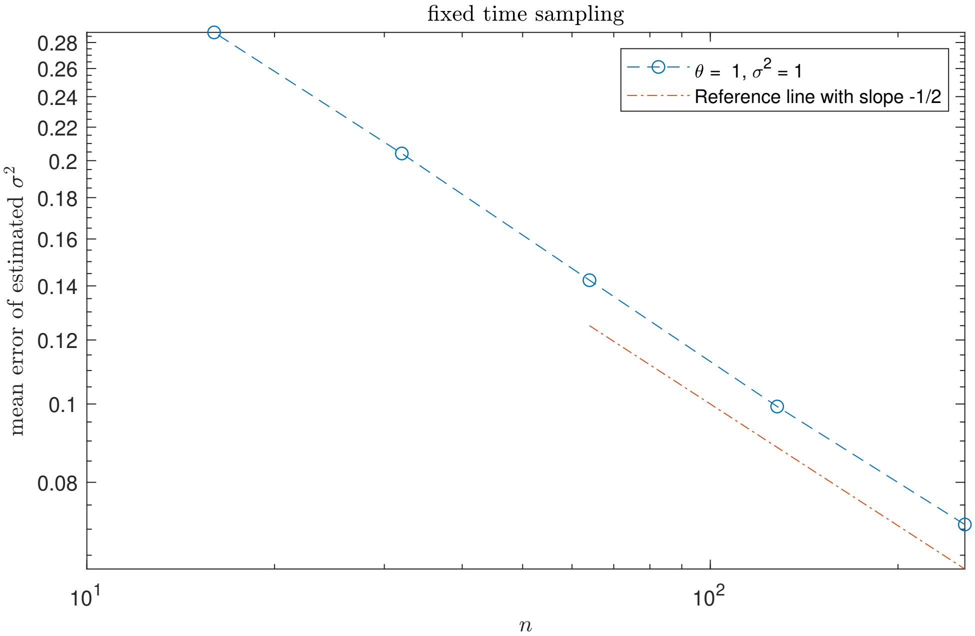

Figure 4.6: Convergence in mean of the estimators σ_n^2

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	$2^4 - 2^8$	10^5	10^4

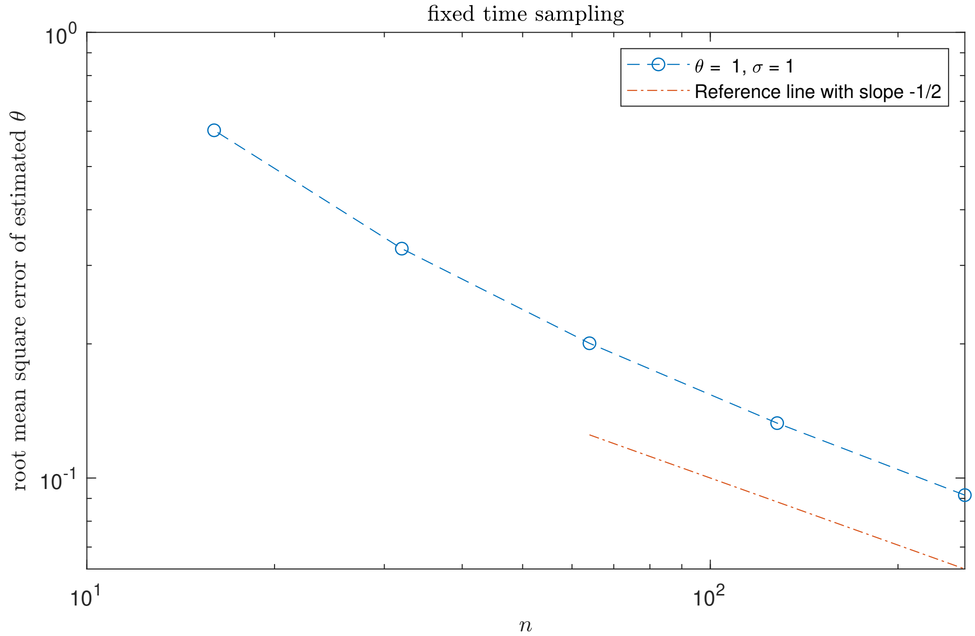

Figure 4.7: Convergence in root mean square of the estimators θ_n

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	$2^4 - 2^8$	10^5	10^4

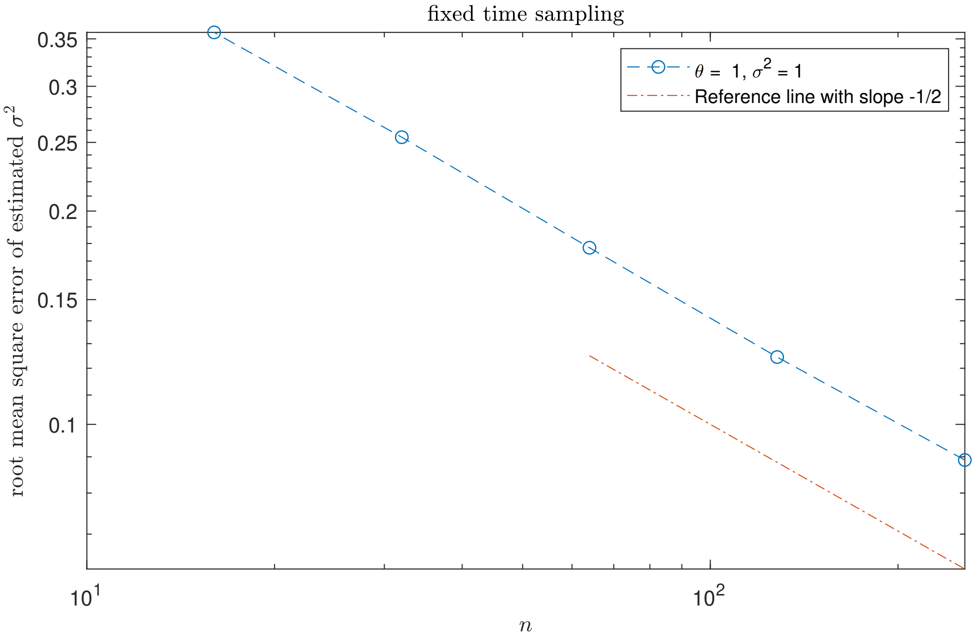

Figure 4.8: Convergence in root mean square of the estimators σ_n^2

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	1	1	$2^4 - 2^8$	10^5	10^4

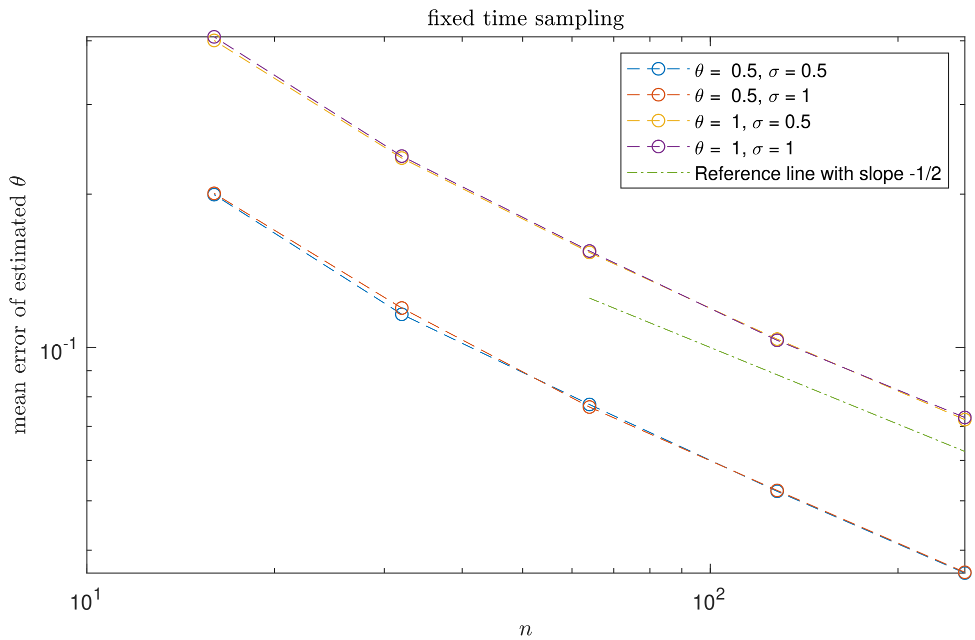

Figure 4.9: Convergence in mean of the estimators θ_n

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	0.5, 1	0.5, 1	$2^4 - 2^8$	10^5	10^4

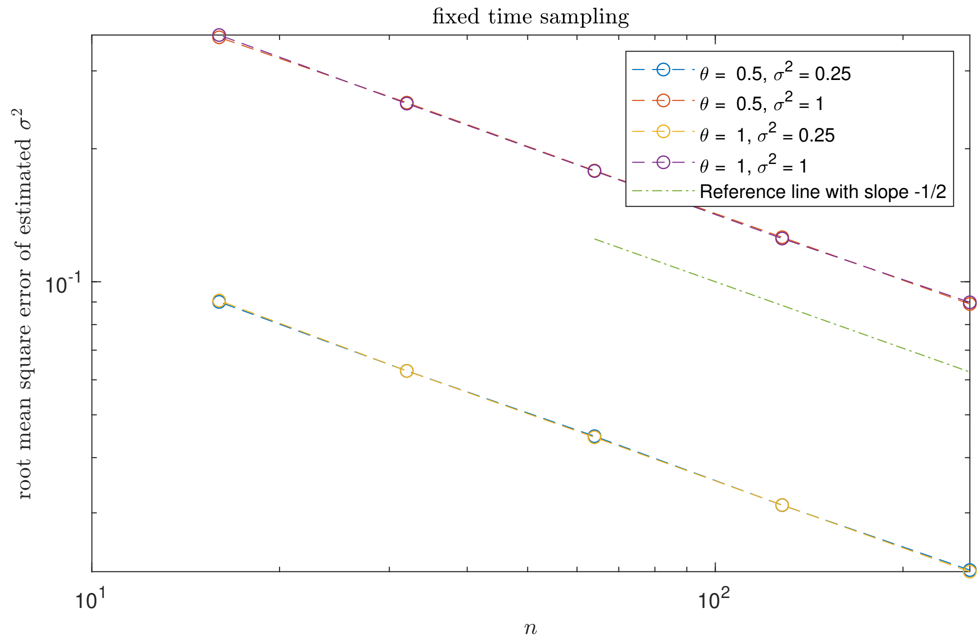

Figure 4.10: Convergence in root mean square of the estimators σ_n^2

Input to Algorithm 2						
t_0	L	θ	σ	n	K	M
1	1	0.5, 1	0.5, 1	$2^4 - 2^8$	10^5	10^4

Bibliography

[1] Casella, G. and Berger, R.L. (2002). *Statistical Inference (Second Edition).* Pacific Grove, Calif.: Duxbury.

[2] Cialenco, I. (2018). Statistical inference for SPDEs: an overview. *Stat. Inference Stoch. Processes.*, 21(2), 309-329. doi:10.1007/s11203-018-9177-9.

[3] Cialenco, I. and Huang, Y. (2017). *A note on parameter estimation for discretely sampled SPDEs.* arXiv:1710.01649.

[4] Dudley, R.M. (1973). Sample functions of the Gaussian process. *Ann. Probability*, 1(1), 66-103. doi:10.1214/aop/1176997026.

[5] Engel, K.-J. and Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations.* New York: Springer-Verlag. doi:10.1007/b97696

[6] Fernandez de la Vega, W. (1974). On almost sure convergence of quadratic Brownian variation. *Ann. Probability*, 2(3), 551-552. doi:10.1214/aop/1176996675.

[7] Frisk, D. (1990). *A Chalmers University of Technology Master's thesis template for LaTeX.* Unpublished.

[8] Khoshnevisan, D. (2014). Analysis of stochastic partial differential equations. *CBMS regional conference series in mathematics*, 119. Providence, R.I.: American mathematical society.

[9] Kuo, H.-H. (2006). *Introduction to Stochastic Integration.* New York: Springer. doi:10.1007/0-387-31057-6.

[10] Lévy, M.P. (1940). Le mouvement brownien plan. *American Journal of Mathematics*, 62(1), 487-550. doi:10.2307/2371467.

[11] Lord, G.L.; Powell, C.E. and Shardlow, T. (2014). *An Introduction to Computiational Stochastic PDEs.* New York: Cambridge University Press

[12] Wiener, N. (1924). The quadratic variation of a function and its Fourier coefficients. *Journal of Mathematics and Physics*, 3(2), 72-94. doi:10.1002/sapm19243272.