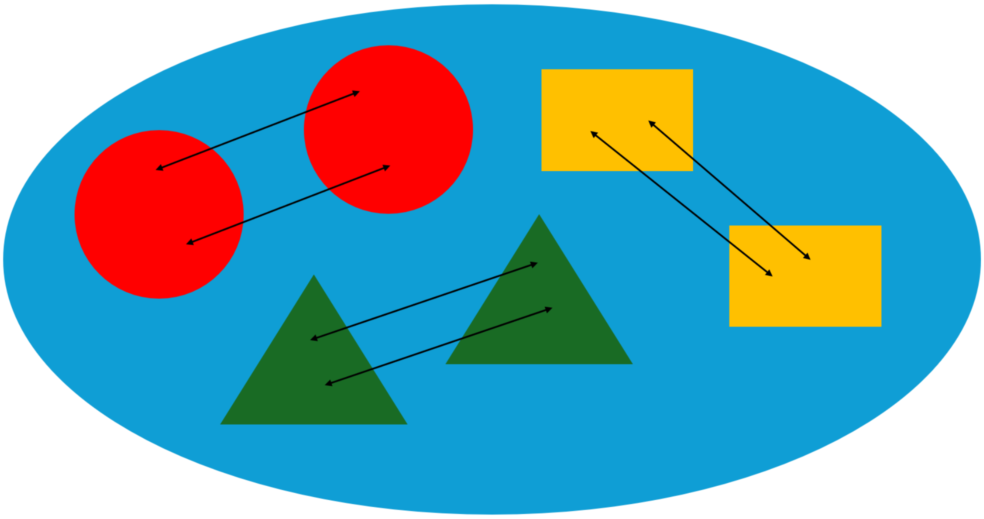

Partial actions: Restriction and Globalization

Master's thesis in Engineering Mathematics and Computational Science

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Abstract

Partial actions generalize group actions by allowing the domain of the maps induced by each group element to be subspaces rather than the entire space. Often, partial actions arise from restricting a global action to a subspace, which means they are globalizable. It is a good question to ask whether every partial action is globalizable. In the category of sets, every partial action admits a unique globalization. In the topological setting, however, some properties may be lost in the globalization process. That is, a topological partial action on a Hausdorff space may have an enveloping action that is not Hausdorff. Moreover, a special class of partial actions of a finite group, those with the decomposition property, are guaranteed to have a Hausdorff enveloping space.

Keywords: partial action. globalization, Hausdorff space, decomposition property.

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Danai Kiewwan, Gothenburg, June 2025

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Introduction

1.1 Background

The concept of a partial dynamical system, or partial action, appears in various areas of mathematics and can be traced back to the late 19th century in the study of differential equations. Specifically, consider a vector field $X : U \to \mathbb{R}^n$ defined on an open subset $U \subseteq \mathbb{R}^n$, the initial value problem

$$f'(t) = X(f(t)) \quad \text{and} \quad f(0) = x_0$$

has a unique solution f_{x_0} for each $x_0 \in U$, which is defined on some (maximal) neigborhood of $0 \in \mathbb{R}$. If we denote this solution by $\phi_t(x_0) = f_{x_0}(t)$, then ϕ acts as a diffeomorphism between open subsets of U, meaning it is only partially defined. Furthermore, the composition property $\phi_t \circ \phi_s = \phi_{t+s}$ holds whenever the composition is well-defined. These characteristics define a *partial dynamical system*.

A basic example of a partial action occurs when a global dynamical system is restricted to an open subset, a process known as restriction. Systems formed this way are called globalizable. This leads to the natural question of whether every partial dynamical system can be extended to a global one and whether the extended system retains the same local and global properties under certain conditions.

1.2 Aim

This master's thesis studies the notion of a partial action, which is a generalization of group action, with a focus on restriction and globalization. In Chapter 2, we study the concept of a partial action on sets, where we explore essential properties and develop the ideas of restriction and globalization. We also prove that a globalization exists and is unique. In Chapter 3, we move on to the topological setting, where we adapt the notion of a partial action to topological spaces. We study restriction and globalization in this context and investigate the conditions that guarantee that the globalization is Hausdorff. In Chapter 4, we study the decomposition property for partial actions of finite groups.

2

Partial actions

Group actions are powerful tools used to study various mathematical structures and have broad applications across many areas of mathematics [9]. Partial actions generalize group actions by allowing each map induced by a group element to be defined on a subspace rather than the entire space. This chapter focuses on partial actions on sets, a concept introduced and developed by Exel. It also presents foundational results concerning globalization [6].

2.1 Partial actions

The simplest mathematical structures are sets, so it is a good idea to develop the notion of partial actions on sets where there is no additional structure before moving on to more complex structures.
Throughout this chapter, we will consider a fixed group G with the identity element denoted as 1, along with a set X.

Definition 2.1.1 A *partial action* of G on X is a pair

$$\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$$

consisting of a collection $(D_g)_{g\in G}$ of subsets of X, and a collection $(\theta_g)_{g\in G}$ of maps,

$$\theta_g : D_{g^{-1}} \to D_{g^1},$$

such that

(i) $D_1 = X$, and θ_1 is the identity map, and
(ii) if $\theta_g(\theta_h(x))$ is defined then so is $\theta_{gh}(x)$ and $\theta_g(\theta_h(x)) = \theta_{gh}(x)$ for all $g, h \in G$.

If $D_g = X$ for all $g \in G$, then we call θ a *global action*.

In condition (ii), we can consider $\theta_g(\theta_h(x))$ as the composition $\theta_g \circ \theta_h$, which is not defined in the usual way because the image of θ_h may not be contained in the domain of θ_g. Specifically, x must belong to $D_{h^{-1}}$, and $\theta_h(x)$ must be in $D_{g^{-1}}$. In other words, the domain of $\theta_g \circ \theta_h$ is the set

$$\{x \in D_{h^{-1}} : \theta_h(x) \in D_{g^{-1}}\} = \theta_h^{-1}(D_{g^{-1}}) = \theta_h^{-1}(D_h \cap D_{g^{-1}}). \tag{2.1.2}$$

For x in said set, we define $(\theta_g \circ \theta_h)(x) = \theta_g(\theta_h(x))$. Moreover, we say that θ_{gh} is an *extension* of $\theta_g \circ \theta_h$ if condition (ii) is satisfied.

Basically, when a group acts on a set, each element in the group serves as a permutation of the set, that is, a bijection. For partial actions, we get just bijections on their respective domains.

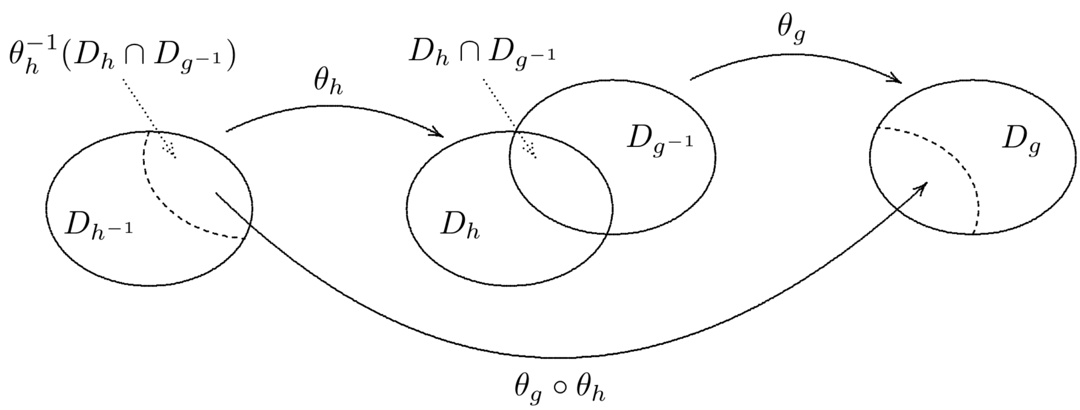


Figure 2.1: Diagram. Composing partially defined functions.

Proposition 2.1.3 Given a partial action θ of G on X, each θ_g is a bijection from $D_{g^{-1}}$ onto D_g and, moreover, $\theta_{g^{-1}} = \theta_g^{-1}$.

Proof. By (2.1.1.ii) and (2.1.1.i), we have that $\theta_{g^{-1}} \circ \theta_g(x) = \theta_1(x) = x,$ for all $x \in D_{g^{-1}}$ and $\theta_g \circ \theta_{g^{-1}}(x) = \theta_1(x) = x,$ for all $x \in D_g$. □

Proposition 2.1.4 Let $(D_g)_{g\in G}$ be a collection of subsets of X, and let $(\theta_g)_{g\in G}$ be a collection of maps

$$\theta_g : D_{g^{-1}} \to D_g.$$

Then, $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ is a partial action of G on X if and only if the following conditions are satisfied:

(i) $D_1 = X$, and θ_1 is the identity map.
(ii) $\theta_g(D_{g^{-1}} \cap D_h) \subseteq D_{gh}$ for all $g, h \in G$.
(iii) $\theta_g(\theta_h(x)) = \theta_{gh}(x)$, for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$ and all $g, h \in G$.

Proof. Suppose θ is a partial action. Item (i) is immediate. For condition (ii), we know from (2.1.2) that the domain of $\theta_g \circ \theta_h$ is $\theta_h^{-1}(D_h \cap D_{g^{-1}})$, and by (2.1.1.ii), the map is extended by θ_{gh}, whose domain is $D_{(gh)^{-1}}$, we then have that

$$\theta_h^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}}. \tag{*}$$

By substituting h with g^{-1} and g with h^{-1} in (*), we obtain $\theta_{g^{-1}}^{-1}(D_{g^{-1}} \cap D_h) \subseteq D_{gh}$. Since θ_g is a bijection (2.1.2), $\theta_{g^{-1}}^{-1}(D_{g^{-1}} \cap D_h) = \theta_g(D_{g^{-1}} \cap D_h) \subseteq D_{gh}$.

Next, let $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$, then $\theta_h(x)$ and $\theta_{gh}(x)$ are defined, hence

$$\theta_h(x) \in \theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}}) \overset{\text{(ii)}}{\subseteq} D_{h(gh)^{-1}} = D_{g^{-1}}.$$

Therefore, $\theta_h(x)$ lies in the domain of θ_g, that is, x lies in the domain of $\theta_g \circ \theta_h$. By (2.1.1.ii), we get that $\theta_g \circ \theta_h(x) = \theta_g(\theta_h(x)) = \theta_{gh}(x)$. This proves condition (iii).

Conversely, assume that conditions (i)–(iii) hold true. By (iii), we have that $\theta_g(\theta_{g^{-1}}(x)) = x$, for all $x \in D_g$ and $\theta_{g^{-1}}(\theta_g(x)) = x$, for all $x \in D_{g^{-1}}$, that is, $\theta_{g^{-1}} = \theta_g^{-1}$. Let x be in the domain of $\theta_g \circ \theta_h$, then

$$x \overset{(2.1.2)}{\in} \theta_h^{-1}(D_h \cap D_{g^{-1}}) = \theta_{h^{-1}}(D_h \cap D_{g^{-1}}) \overset{(ii)}{\subseteq} D_{h^{-1}g^{-1}} = D_{(gh)^{-1}}.$$

Hence, the domain of $\theta_g \circ \theta_h$ is contained in the domain of θ_{gh} which means that θ_{gh} extends $\theta_g \circ \theta_h$. □

Condition (ii) is intuitive because an element in $D_{g^{-1}} \cap D_h$ is one that has been mapped to by some element under h, and can be further mapped under g. The extension property then ensures that the image of these elements can be mapped to by the same element under gh, that is, the image lies in D_{gh}.

Example 2.1.5 Let the group be $\mathbb{Z}$ (the integers under addition) and the set X be

$$X = (3\mathbb{Z} + 1) \cup (3\mathbb{Z} + 2).$$

Define a pair $\theta = ((D_n)_{n\in\mathbb{Z}}, (\theta_n)_{n\in\mathbb{Z}})$, where the sets D_n are given by

$$D_n = \begin{cases} 3\mathbb{Z}+1, & \text{if } n \equiv 2 \pmod 3, \\ 3\mathbb{Z}+2, & \text{if } n \equiv 1 \pmod 3, \\ X, & \text{otherwise.} \end{cases}$$

The maps $\theta_n : D_{-n} \to D_n$ are defined by

$$\theta_n(m) = m + n.$$

Then, θ is a partial action of $\mathbb{Z}$ on X.

In fact, item (2.1.4.i) is obvious by the definitions. If $-m \not\equiv n \pmod 3$, then $D_{-m} \cap D_n = \emptyset$. On the other hand, if $-m \equiv n \pmod 3$, we get $D_{m+n} = X$. So (2.1.4.ii) is satisfied. Lastly, item (2.1.4.iii) is trivial.

Example 2.1.6 Let $X = \{0,1\}^G$ and consider the usual Bernoulli shift of G on X. Note that $X \cong \mathbb{P}(G)$, the power set of G. The Bernoulli action is then

$$\beta_g(E) = gE$$

for all $g \in G$ and all $E \in X$. Set $\Omega = \{E \in \mathbb{P}(G) : 1 \in E\}$, and let $D_g = \{E \in \mathbb{P}(G) : 1, g \in E\}$. Then $\beta_g(D_{g^{-1}}) = D_g$ for all $g \in G$. Therefore, β_g restricts to a partial action $\theta = ((D_g)_{g\in G}, (\beta_g|_{D_{g^{-1}}})_{g\in G}$ of G on Ω.

We can see that D_1 is clearly Ω, hence (2.1.1.i). Suppose $\beta_g(\beta_h(E))$ is defined, then $1, g^{-1} \in \beta_h(E)$. This means that there exists $e \in E$ such that $he = g^{-1} \iff e = h^{-1}g^{-1}$, that is, $h^{-1}g^{-1} \in E$ which means that $\beta_{gh}(E)$ is defined. This proves (2.1.1.ii).

Example 2.1.7 Consider the initial value problem $y' = 1 + y^2$ and $y(0) = x_0$. The solution to this problem is $y_{x_0}(t) = \tan(t + \tan^{-1}(x_0))$ defined on

$$\left(-\frac{\pi}{2} - \tan^{-1}(x_0),\ \frac{\pi}{2} - \tan^{-1}(x_0)\right) := S_{x_0}.$$

For $t \in \mathbb{R}$, set $X_{-t} = \{x \in \mathbb{R} : t \in S_x\}$, and define a function $\theta_t : X_{-t} \to X_t$ by $\theta_t(x) = y_x(t)$. We claim that $\theta = ((X_t)_{t\in\mathbb{R}}, (\theta_t)_{t\in\mathbb{R}})$ is a partial action of $\mathbb{R}$ on $\mathbb{R}$.

It is clear that $0 \in S_x$, for all $x \in \mathbb{R}$, hence $X_0 = \mathbb{R}$. Observe that $\theta_0(x) = y_x(0) = x$, hence (2.1.1.i). To show (2.1.1.ii), for all $t, s \in \mathbb{R}$, suppose $\theta_s \circ \theta_t(x)$ is defined, then $x \in X_{-t}$ and $\theta_t(x) \in X_{-s}$, which means that $t \in S_x$ and $s \in S_{\theta_t(x)}$ implying that $t + s \in S_x$. Moreover,

$$\theta_s \circ \theta_t(x) = \theta_s(\theta_t(x)) = \theta_s(\tan(t + \tan^{-1}(x))) = \tan(s + t + \tan^{-1}(x)) = \theta_{s+t}(x).$$

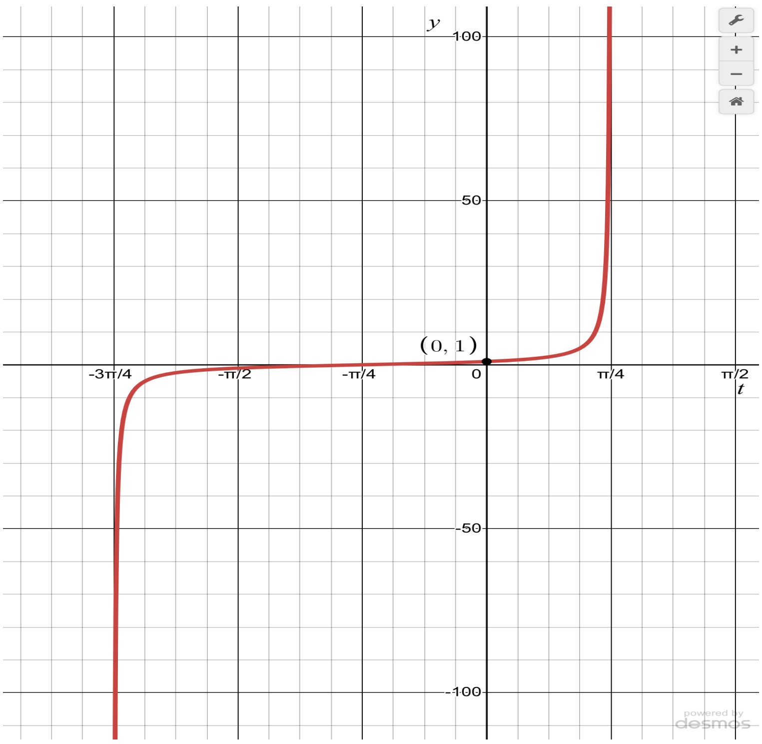


Figure 2.2: Graph of $y_1(t)$ where $x_0 = 1$ defined on the time interval $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$.

We can make condition (2.1.4.ii) stronger by replacing inclusion with equality.

Proposition 2.1.8 Let $\theta = (\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of G on X. Then,

$$\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}, \text{ for all } g, h \in G.$$

Proof. Since $\theta_g(D_{g^{-1}} \cap D_h) \subseteq \theta_g(D_{g^{-1}}) = D_g$, we have that

$$\theta_g(D_{g^{-1}} \cap D_h) \overset{(2.1.4.ii)}{\subseteq} D_g \cap D_{gh}.$$

We then apply $\theta_{g^{-1}}$ to both sides to get $D_{g^{-1}} \cap D_h \subseteq \theta_{g^{-1}}(D_g \cap D_{gh})$. After replacing g with g^{-1} and h with gh, we obtain $D_g \cap D_{gh} \subseteq \theta_g(D_{g^{-1}} \cap D_h)$. □

Definition 2.1.9 Suppose that, for each $i = 1, 2$, we are given a partial action $\theta^{(i)} = ((D_g^{(i)})_{g\in G}, (\theta_g^{(i)})_{g\in G})$ of G on a set $X^{(i)}$. A map

$$\phi : X^{(1)} \to X^{(2)}$$

is be said to be $G-$*equivariant* when the following conditions are satisfied:
(i) $\phi(D_g^{(1)}) \subseteq D_g^{(2)}$ for all $g \in G$, and
(ii) $\phi(\theta_g^{(1)}(x)) = \theta_g^{(2)}(\phi(x))$, for all $x \in D_{g^{-1}}^{(1)}$ and all $g \in G$.
If ϕ is bijective and its inverse ϕ^{-1} is also G-equivariant, we call ϕ an *equivalence of partial actions*. In this case, we say that $\theta^{(1)}$ and $\theta^{(2)}$ are *equivalent*.

Note that a G-equivariant bijection is not necessarily an equivalence, as condition (i) may not hold for its inverse. The necessary and sufficient condition for a G-equivariant bijection to be an equivalence is that equality holds in (i). However, when dealing with global actions, we only need to consider (ii), since (i) is always true.

Example 2.1.10 Let θ^X and θ^Y be partial actions of $\mathbb{Z}_2$ on X and Y, respectively, where $X = Y = \{1, 2, 3, 4, 5, 6\}$, $X_1 = \{1, 6\}, Y_1 = \{1, 2, 5, 6\}$, and $\theta_1^X : X_1 \to X_1$ and $\theta_1^Y : Y_1 \to Y_1$ are given by $\theta_1^{(X)}(t) = \theta_1^{(Y)}(t) = 7 - t$ for all t in their domains. We have that the identity map $\mathrm{id}_X : X \to Y$ is $G-$equivariant but its inverse $\mathrm{id}_Y : Y \to X$ is not since $2 \in Y_1$ but $i(2) = 2 \notin X_1$. Thus, $\theta^{(X)}$ and $\theta^{(Y)}$ are not equivalent.

Definition 2.1.11 Let $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ be a partial action of G on X. The *graph* of θ is defined to be the set

$$\mathrm{Graph}(\theta) = \{(y, g, x) \in X \times G \times X : x \in D_{g^{-1}}, \text{ and } \theta_g(x) = y\}.$$

The graph of a partial action shows which pairs of elements in a set are connected through the action of a group element. This makes it a useful tool for understanding how partial actions work.

Definition 2.1.12 Given a partial action $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ of G on X, we will say that a given subset $Y \subseteq X$ is *invariant* under θ, if

$$\theta_g(Y \cap D_{g^{-1}}) \subseteq Y, \text{ for all } g \in G.$$

Proposition 2.1.13 Given a partial action $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ of G on X, and an invariant subset $X' \subseteq X$, let

$$D'_g = X' \cap D_g, \text{ for all } g \in G,$$

and let θ'_g be the restriction of θ_g to $D'_{g^{-1}}$. Then

(i) $\theta' = ((D'_g)_{g\in G}, (\theta'_g)_{g\in G})$ is a partial action of G on X', and

(ii) the inclusion $X' \hookrightarrow X$ is a G-equivariant map.

Proof. For (i), from the assumption we get that each θ'_g maps $D'_{g^{-1}}$ into D'_g for all $g \in G$. $D'_1 = X' \cap X = X'$ and θ'_1 is evidently an identity map, hence (2.1.4.i). Next, for all g and h in G we have that,

$$\theta'_g(D'_{g^{-1}} \cap D'_h) \subseteq \theta_g(X' \cap D_{g^{-1}}) \cap \theta_g(D_{g^{-1}} \cap D_h) \overset{(2.1.4.ii)}{\subseteq} X' \cap D_{gh} = D'_{gh},$$

which gives (2.1.4.ii). Next, let $g, h \in G$ and $x \in D'_{h^{-1}} \cap D'_{(gh)^{-1}} = X' \cap D_{h^{-1}} \cap D_{(gh)^{-1}}$. Thus

$$\theta'_g(\theta'_h(x)) = \theta'_g(\theta_h(x)) = \theta_g(\theta_h(x)) \overset{(2.1.4.iii)}{=} \theta_{gh}(x) = \theta'_{gh}(x),$$

as $\theta'_g(\theta_h(x))$ is defined due to (2.1.1.ii) and the fact that X' is invariant, hence (2.1.4.iii).

For (ii), note that $\iota(D'_g) = D'_g = D_g \cap X' \subseteq D_g$, and thus

$$\iota(\theta'_g(x)) = \iota(\theta_g(x)) = \theta_g(x) = \theta_g(\iota(x)), \text{ for all } x \in D'_{g^{-1}}$$

which means that the inclusion map is G-invariant. □

2.2 Restriction and Globalization

One simple way to construct non-trivial examples of partial actions is by restricting a global action to subsets that are not necessarily invariant. Let us describe this process in more detail.

Consider a global action η of a group G on a set Y. Now, suppose that we have a subset X of Y and we want to define an action of G on X by restricting η. Naturally, for this restriction to yield a well-defined global action, the subset X must be invariant under η. However, even if X is not invariant under η, we can still define a partial action of G on X. For each $g \in G$, we set the domain of the partial action to be

$$D_g = \eta_g(X) \cap X. \tag{2.2.1}$$

Furthermore, After observing that $\eta_g(D_{g^{-1}}) = D_g$, we can define the maps

$$\theta_g : D_{g^{-1}} \to D_g$$

where θ_g is the restriction of η_g to $D_{g^{-1}}$, for each $g \in G$.

Remark 2.2.2 We can verify that $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$, as defined above, is indeed a partial action. First, it is easy to show (2.1.4.i) as $D_1 = \eta_1(X) \cap X = X$ and $\theta_1 = \eta_1$

is the identity map restricted to X. Next,

$$\begin{aligned}
\theta_g(D_{g^{-1}} \cap D_h) &= \eta_g(D_{g^{-1}} \cap D_h) \\
&= \eta_g\big(\eta_{g^{-1}}(X) \cap \eta_h(X) \cap X\big) && \text{(by 2.2.1)} \\
&= \eta_g(\eta_{g^{-1}}(X)) \cap \eta_g(\eta_h(X)) \cap \eta_g(X) \\
&= X \cap \eta_{gh}(X) \cap \eta_g(X) && \text{(by 2.1.4.iii)} \\
&= D_{gh} \cap \eta_g(X) && \text{(by 2.2.1)} \\
&\subseteq D_{gh}.
\end{aligned}$$

Therefore, (2.1.4.ii) is satisfied. Lastly, (2.1.4.iii) is evident.

Definition 2.2.3 Let η be a global action of G on Y, and let X be a subset of Y. The partial action θ of G on X defined above is called the *restriction* of the global action η to X.

Example 2.2.4 Consider $G = PSL(2, \mathbb{C})$ the group of complex projective transformations on the complex projective line $\mathbb{CP}^1 \cong S^2$, and let $X = \mathbb{C}$ be the complex plane. The partial action is given by the fractional linear transformations on the complex plane, the so-called Möbius transformations. That is, for

$$g = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in PSL(2, \mathbb{C}),$$

we have the domains

$$D_g = \begin{cases} \mathbb{C}\backslash\{a/c\} & \text{if } c \neq 0, \\ \mathbb{C} & \text{if } c = 0 \end{cases}$$

and

$$\theta_g : D_{g^{-1}} \to D_g, \text{ given by } \theta_g(z) = \frac{az+b}{cz+d}.$$

The Möbius transformation can be defined on the extended complex plane, so this is an example of restricting global action into subset $\mathbb{C} \subseteq \hat{\mathbb{C}}$. For given $g = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$ and the corresponding Möbius transformation η_g, we have that $\eta_g(\mathbb{C}) = \mathbb{C}\backslash\{a/c\}$ if $c = 0$ as the element that is mapped to $\frac{a}{c}$ is infinity.

Proposition 2.2.5. Let η be a global action of a group G on a set Y and let θ be its restriction to a subset $X \subseteq Y$. Then

$$\text{Graph}(\theta) = \text{Graph}(\eta) \cap (X \times G \times X).$$

Proof. The inclusion of the set on the left in the set on the right is obvious. Let $(y, g, x) \in \text{Graph}(\eta) \cap (X \times G \times X)$, then $x, y \in X,\ g \in G$ and $\eta_g(x) = y$. Since $x, y \in X,\ \eta_{g^{-1}}(y) = x$, we have that $x \in \eta_{g^{-1}}(X) \cap X = D_{g^{-1}}$, hence the inclusion of the set on the right in the set on the left. □

From a different perspective, one could begin with a partial action θ of G on X and investigate whether there exists a global action η on a larger set Y that contains X, such that θ is the restriction of η. If such an action η exists, the *orbit* of X within Y, defined as

$$\mathrm{Orb}(X) := \bigcup_{g\in G} \eta_g(X),$$

will naturally be invariant under η. Moreover, the restriction of η to $\mathrm{Orb}(X)$ forms another global action, which when further restricted to X, gives the original partial action θ.

Definition 2.2.6. Let η be a global action of G on a set Y, and let θ be the partial action obtained by restricting η to a subset $X \subseteq Y$. If the orbit of X coincides with Y, we will say that η is a *globalization* or an *enveloping action* for θ.

Theorem 2.2.7.(Abadie) Every partial action admits a globalization, which is unique in the following sense: if θ is a partial action of G on X, and we are given globalizations $\eta^{(i)}$ acting on sets $Y^{(i)}$, for $i = 1, 2$ then there exists an equivalence

$$\phi : Y^{(1)} \to Y^{(2)},$$

such that ϕ coincides with the identity on X.

Proof. Let $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ be a partial action of G on X. We want for each g, the corresponding map to be defined for all elements in the set X. This way all maps are defined on the entire X. We can start with $G \times X$ where (g, x) represents the element that x will be mapped to under g. However, we have to group some elements together by defining a relation on $G \times X$ by

$$(g, x) \sim (h, y) \iff x \in D_{g^{-1}h}, \text{ and } \theta_{h^{-1}g}(x) = y.$$

This relation is evidently reflexive and symmetric. For transitivity, assume that $(g, x) \sim (h, y) \sim (k, z)$, we have that $y \in D_{h^{-1}k}$, and $\theta_{k^{-1}h}(y) = z$. We also have that $x \in D_{g^{-1}h}$ and

$$x = \theta_{g^{-1}h}(y) \in \theta_{g^{-1}h}(D_{h^{-1}g} \cap D_{h^{-1}k}) \overset{(2.1.8)}{=} D_{g^{-1}h} \cap D_{g^{-1}k}.$$

Therefore,

$$\theta_{k^{-1}g}(x) \overset{(2.1.4.\text{iii})}{=} \theta_{k^{-1}h}(\theta_{h^{-1}g}(x)) = \theta_{k^{-1}h}(y) = z,$$

that is, $(g, x) \sim (k, z)$. Hence, it is an equivalence relation. Let $\widetilde{X} = G \times X/\sim$ be the quotient of $G \times X$ by this equivalence relation, and denote the equivalence class of (g, x) by $[g, x]$.

Next, we define the embedding map $\iota : X \to \widetilde{X}$, by $\iota(x) = [1, x]$ which is obviously injective. We can now define a global action on $\widetilde{X}$. Consider $\tau_g([h, x]) = [gh, x]$ for all $g, h \in G$, and all $x \in X$. We will show that $\tau = ((\widetilde{X})_{g\in G}, (\tau_g)_{g\in G})$ is a global action of G on $\tilde{X}$. Suppose $[h, x] = [f, y] \in \widetilde{X}$, then $x \in D_{h^{-1}f}$ and $\theta_{f^{-1}h}(x) = y$, which is equivalent to, $x \in D_{(gh)^{-1}(gf)}$ and $\theta_{(gf)^{-1}(gh)}(x) = y$, that is,

$$\tau_g([h, x]) = [gh, x] = [gf, y] = \tau_g([f, y]).$$

Therefore, τ is compatible with $\sim$. Moreover, for all $[h,x] \in \widetilde{X}$, and all $g, s \in G$, $\tau_1([h,x]) = [1h,x] = [h,x]$, and $\tau_g(\tau_s([h,x]) = \tau_g([sh,x]) = [gsh,x] = \tau_{gs}([h,x])$. Then, we need to show that the restriction of τ on $\iota(X)$ is our original partial action. First, we will show that

$$\tau_g(\iota(X)) \cap \iota(X) = \iota(D_g).$$

This is because an element of the form $[1,x] \in \iota(X)$ lies in $\tau_g(\iota(X))$ if and only if $[1,x] = [g,y]$ for some y in X, that is, $x \in D_g$, and $\theta_{g^{-1}}(x) = y$. Thus, $[1,x] \in \iota(D_g)$. Observe that

$$(1, \theta_g(x)) \sim (g,x),$$

then

$$\iota(\theta_g(x)) = [1, \theta_g(x)] = [g,x] = \tau_g([1,x]) = \tau_g(\iota(x)),$$

that is $\iota \circ \theta_g = \tau_g \circ \iota$. If we identify $x \in X$ with $[1,x] \in \iota(X)$, we get that $\theta_g = \tau_g$, on $D_{g^{-1}}$ for all $g \in G$. This verifies that θ corresponds to the restriction of τ to X. Since $\tau_g([1,x]) = [g,x]$, we also see that the orbit of $\iota(X)$ coincides with $\widetilde{X}$, so τ is indeed a globalization of θ.

Uniqueness is left to be shown. Suppose we have another globalization η of θ, acting on the set $Y \supseteq X$, define a map $\phi : G \times X \to Y$ by $\phi(g,x) = \eta_g(x) \in Y$, and observe that

$$\phi(g,x) = \phi(h,y) \iff \eta_g(x) = \eta_h(y) \iff x = \eta_{g^{-1}h}(y).$$

We can conclude that $x \in \eta_{g^{-1}h}(X) \cap X = D_{g^{-1}h}$, and $y = \eta_{h^{-1}g}(x) = \theta_{h^{-1}g}(x)$, meaning that $(g,x) \sim (h,y)$. Therefore, the map ϕ, induced the quotient map $\tilde{\phi} : \tilde{X} \to Y$ which is given by

$$\tilde{\phi}([g,x]) = \eta_g(x)$$

This map is clearly injective. As Y is a globalization, it must coincide with the orbit of X under η. Let $y \in Y$, there exist $g \in G, x \in X$ such that $y = \eta_g(x)$, which implies that $\tilde{\phi}$ is surjective. We now show that the map $\tilde{\phi}$ is G-equivariant. We can skip (2.1.9.i) since we are dealing with global actions. For all $g, h \in G$, and all $x \in X$, we have that

$$\tilde{\phi}(\tau_g([h,x])) = \tilde{\phi}([gh,x]) = \eta_{gh}(x) \overset{(2.1.4.\text{iii})}{=} \eta_g(\eta_h(x)) = \eta_g(\tilde{\phi}([h,x])).$$

Moreover, $\tilde{\phi}([1,x]) = \eta_1(x) = x$ for all $[1,x] \in \iota(X)$, that is, the map coincides with the identity on $\iota(X)$ in $\widetilde{X}$ and Y. As a consequence, if we identify $[g,x] \in \widetilde{X}$ with $\eta_g(x)$, we get that τ and $\eta \in Y$ are indeed the same global actions. Hence, the partial action θ admits a unique globalization. □

We can also generalize the process of restricting a global action to a non-invariant subset to the case in which the action is, itself, partial. Suppose we have a partial action of G on a set Y,

$$\eta = ((Y_g)_{g \in G}, (\eta_g)_{g \in G})$$

Let $X \subseteq Y$ be a subset which is not necessarily invariant. An easy way to do this is to construct the globalization $\tilde{\eta}$ on $\widetilde{Y} \supseteq Y$. Since $X \subseteq Y \subseteq \widetilde{Y}$, we then restrict $\tilde{\eta}$ to X to get a partial action of G on X by (2.2.2).

3

Topological partial actions

Building on the notion of partial actions on sets, we now extend the discussion to topological spaces. In this chapter, we introduce the concept of topological partial actions and provide several illustrative examples. We then show that every topological partial action admits a globalization, uniquely characterized by a universal property [1]. An interesting phenomenon is that even when a partial action is defined on a Hausdorff space, its enveloping action may act on a non-Hausdorff space. A necessary and sufficient condition to ensure that the enveloping space is Hausdorff is that the graph of the action is closed.

3.1 Partial actions on topological spaces

In topological settings, continuity is important as it preserves many topological characteristics, among them compactness, connectedness, separability, and the limits of convergent sequences. Before we can define a partial action on a topological space, we need to ensure it is compatible with the group, that is, the group must be equipped with a topology. This leads to the definition of a topological group [10].

Definition 3.1.1. A *topological group* consists of a group G and a topology on it for which the multiplication map and the inversion map are continuous.

Remark 3.1.2. Suppose that G is a topological group. For every $g \in G$, the right translation map $\rho_g(h) = hg$, and the left translation map $\lambda_g(h) = gh$ are homeomorphisms of G onto itself, with inverses $\lambda_{g^{-1}}, \rho_{g^{-1}}$, respectively. In fact, $\iota_g : G \to G \times G$ given by $x \mapsto (x, g)$ is continuous and ρ_g is the composition of ι_g and the multiplication map which is continuous. Continuity of the left translation can be shown similarly.

Definition 3.1.3 A *topological partial action* of a topological group G on a topological space X is a pair

$$\theta = ((X_g)_{g\in G}, (\theta_g)_{g\in G})$$

consisting of a collection $(D_g)_{g\in G}$ of open subsets of X, and a collection $(\theta_g)_{g\in G}$ of homeomorphisms,

$$\theta_g : X_{g^{-1}} \to X_g,$$

such that it is a partial action on the underlying set X, the set $\Gamma_\theta = \{(g, x) \in G \times X : g \in G, x \in X_{g^{-1}}\}$ is open in $G \times X$, and the function $\theta : \Gamma_\theta \to X$ given

by $\theta(g,x) = \theta_g(x)$ is continuous. When this continuity condition is satisfied, we say that the partial action is continuous.
If $D_g = X$ for all $g \in G$, then we call θ a *topological global action*. In this case we have $\Gamma_\theta = G \times X$.

Example 3.1.4 Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $X = \{(x,y) \in \mathbb{C} \times \mathbb{C} : |x| = |y| = 1, \text{ and } \mathrm{Re}(x) > 0\}$. For $z \in S^1$, define $X_z = \{(x,y) \in X : \mathrm{Re}(\frac{x}{z}) > 0\}$ and a function $\alpha_z : X_{z^{-1}} \to X_z$ given by $\alpha_z((x,y)) = (zx, y)$. We claim that $\alpha = ((X_z)_{z \in S^1}, (\alpha_z)_{z \in S^1})$ is a topological partial action of S^1 on X.

Indeed, each X_z is open and each α_z is continuous. Suppose $t, s \in S^1$, then we get the followings;

- Item (2.1.4.i) is apparent.
- For (2.1.4.ii), let $(x,y) \in \alpha_t(X_{t^{-1}} \cap X_s)$, this is true only when $\alpha_t((z,y)) = (tz, y) = (x,y)$ for some $(z,y) \in X_{t^{-1}} \cap X_s$, that is, $\mathrm{Re}(zt), \mathrm{Re}\left(\frac{z}{s}\right) > 0$. This means that $(x,y) \in X_{ts}$ since $\mathrm{Re}(\frac{x}{ts}) = \mathrm{Re}(\frac{z}{s}) > 0$.
- For (2.1.4.iii), let $(x,y) \in X_{t^{-1}} \cap X_{t^{-1}s^{-1}}$, then $\alpha_s \circ \alpha_t((x,y)) = \alpha_s(tx, y) = (stx, y) = \alpha_{st}(x,y)$.
- Let $(t,x,y) \in \Gamma_\alpha$, then $x = e^{i\theta^*}$, $\theta^* \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We then have that $(t,x,y) \in$

  $$\left\{\left(e^{i\theta_1}, e^{i\theta_2}, z\right); -\frac{\pi}{2} - \theta^* + \epsilon < \theta_1 < \frac{\pi}{2} - \theta^* - \epsilon,\ \theta^* - \epsilon < \theta_2 < \theta^* + \epsilon,\ |z| = 1\right\}$$

  which is open and contained in Γ_α where ϵ is sufficiently small. Moreover, α is obviously continuous.

Example 3.1.5. Let X be a topological space and let $h : X \to X$ be a homeomorphism. If $\{(X_t, \alpha_t)\}_{t \in \mathbb{R}}$ is given by $X_t = X$ and $\alpha_t = h^t$ when $t \in \mathbb{Z}$, and $X_t = \emptyset$ and $\alpha_t = \emptyset$ when $t \notin \mathbb{Z}$. Then $\Gamma_\alpha = \mathbb{Z} \times X$ and the function $\alpha : \Gamma_\alpha \to X$ given by $\alpha(t,x) = h^t(x)$, is a topological partial action of $\mathbb{R}_d$ on X.

The following example shows that topological partial actions arise naturally in differential geometry.

Example 3.1.6. The flow of a diffentiable vector field is a partial action. More-precisely, consider a smooth vector field $\mathbf{v} : X \to TX$ on a manifold X, and for $x \in X$ let γ_x be the corresponding integral curve through x (i.e.: $\gamma_x(0) = x$), defined on its maximal interval (a_x, b_x). Let us define, for $t \in \mathbb{R} : X_{-t} = \{x \in X : t \in (a_x, b_x)\}$, $\alpha_t : X_{-t} \to X_t$ such that $\alpha_t(x) = \gamma_x(t)$, and $\alpha = ((X_t)_{t \in \mathbb{R}}, (\alpha_t)_{t \in \mathbb{R}})$. We claim that α is a partial action of $\mathbb{R}$ on X.

To show that this is indeed a partial action, let $t, s \in \mathbb{R}$, so we get the followings;

- X_t is open by the the existence and uniqueness theorem, and α_t is a diffeomorphism, then α_t and $\alpha_{t^{-1}}$ are continuous. Hence, α_t is a homeomorphism.
- Since $0 \in (a_x, b_x)$, for all $x \in X$, we have $X_0 = X$. Additionally, $\alpha_0(x) = \gamma_x(0) = x = \mathrm{id}_X(x)$, for all $x \in X$, hence (2.1.4.i).

- Let $x \in \alpha_t(X_{-t} \cap X_s)$, then $\alpha_t(y) = \gamma_y(t) = x,$ for some $y \in X_{-t} \cap X_s$, that is, $t, -s \in (a_y, b_y)$ and $\gamma_x(-t) = y, \gamma_x(-t-s) = \gamma_y(-s) = z \in X_{-s}$. Therefore, $-t-s \in (a_x, b_x),$ that is $x \in X_{t+s}$, hence (2.1.4.ii)
- Condition (2.1.4.iii) is true from the fact that X is a smooth vector field since trajectories (integral curves) do not intersect.
- Let $(t, x) \in \Gamma_a,$ then $t \in (a_x, b_x)$ and $x \in X_{-t}$. There exist t_{lower} and t_{upper} such that $t_{lower} < t_{upper}$ and $\gamma_z(s)$ is defined for all $z \in X_{-t}$ and all $s \in (t_{lower}, t_{upper})$. Hence, we have that $(t_{lower,}t_{upper}) \times X_{-t}$ is an open subset of $\mathbb{R} \times X$ containing (t, x), so Γ_α is open in $\mathbb{R} \times X$.

3.2 Restriction and Globalization

In the previous chapter with sets, we explored how nontrivial examples of partial actions could be constructed by restricting a global action to subsets that are not necessarily invariant. While this approach is straightforward in the context of sets, it can similarly be applied in the category of topological spaces, with the added consideration of ensuring that each bijection is a homeomorphism and each domain is an open set.

Let η be a topological action of a group G on Y and X be an open subset of Y. Consider θ, the restriction of η to X where $X_g = \eta_g(X) \cap X$ and $\theta_g = \eta_g|_{X_{g^{-1}}}$ for all $g \in G$. Next, we prove that the restriction θ defines a topological partial action.

Proposition 3.2.1. Let η be a topological global action of G on Y, and let X be an open subset of Y. The restriction of η to X is a topological partial action.

Proof. Let θ be the restriction of η to X. We get by (2.2.2.) that it is a partial action. Since η_g is a homeomorphism, it is an open map, that is, $\eta_g(X)$ is open. Thus, $X_g = \eta_g(X) \cap X$ is open, and $\theta_g = \eta_g|_{X^{g^{-1}}}$ is continuous for all $g \in G$. As $\eta_g(X_{g^{-1}}) = X_g,$ we also have $\theta_g^{-1} = \theta_{g^{-1}}$ which implies that θ_g is a homeomorphism for all $g \in G$.

Next, we claim that

$$\Gamma_\theta = \bigcup_{g \in G} \eta^{-1}(X_g) \cap (G \times X)$$

which is an open set in $G \times X$. We prove the claim by letting $(v, x) \in \Gamma_\theta$, then we have that $x \in X_{v^{-1}},$ that is $\eta_v(x) = \eta(v, x) \in \eta^{-1}(X_v)$.

Conversely, let $(v, x) \in \bigcup_{g \in G} \eta^{-1}(X_g) \cap (G \times X)$, then there exists $w \in G$ such that $(v, x) \in \eta^{-1}(X_w) \cap (G \times X)$. Hence, $\eta(v, x) \in X_w \subseteq X$. Let $\eta(v, x) = y,$ that is $\eta(v^{-1}, y) = x = \eta_{v^{-1}}(y) \in \eta_{v^{-1}}(X)$. Thus, $x \in X \cap \eta_{v^{-1}}(X) = X_{v^{-1}}$, that is, $(v, x) \in \Gamma_\theta$. Consequently, for any open set $U \subseteq X$, and $U = V \cap X$ for some open set $V \subseteq Y$. we have $\theta^{-1}(U) = \eta^{-1}(V \cap X) \cap \Gamma_\theta$ which is an open set in Γ_θ. Hence, the map $\theta : \Gamma_\theta \to X$ is continuous. □

One interesting fact is that the integral curves of a vector field on a compact manifold X are defined on the entire $\mathbb{R}$. This is a particular case of the following result.

Propositioin 3.2.2. Let $\theta = ((X)_{g\in G}, (\theta_g)_{g\in G})$ be a topological partial action of G on a compact space X. Then, there exists an open subgroup H of G such that θ restricted to H is global action. In particular, if G is connected, θ is a global action.

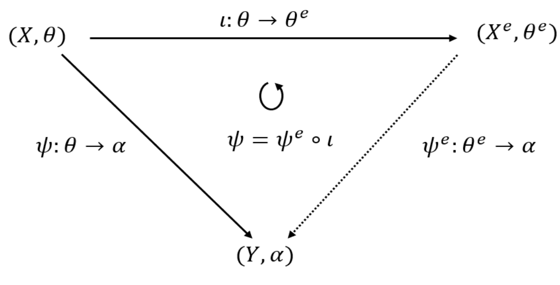
Proof. Let $A_x = \{g \in G : x \in X_{g^{-1}}\}$, and $A = \bigcap_{x\in X} A_x$. It is clear that $1 \in A$ and if $s, t \in A$, then $st \in A$, that is, A is a submonoid of G. Since Γ_θ is open in $G \times X$, there exist for every $x \in X$ open neighborhoods $U_x \subseteq X$ of x and $V_x \subseteq G$ of 1 such that $V_x \times U_x \subseteq \Gamma_\alpha$, and $V_x = V_x^{-1}$. The latter is because if $V_x \neq V_x^{-1}$, we can pick $W_x = V_x \cap V_x^{-1}$ which is open in X. By compactness of X, there exist $x_1, ...x_n \in X$ such that $X = \bigcup_{j=1}^{n} U_{x_j}$. Let $V = \bigcap_{j=1}^{n} V_{x_j}$. By the choice of our Vx we get that V is a symmetric subset of A. Next, $H = \bigcup_{n=1}^{\infty} V^n$ is an open subgroup of G contained in A. In particular, let us assume that G is connected. Observe that gH which is open by (3.1.2.) and $gH \cap H = \emptyset$ for all $g \notin H$, that is, $\bigcup_{g\notin H} gH = G\backslash H$ is also open. Hence, H is an open and closed subgroup of G. This is only true when $H = G$. □

We know that every partial action on a set admits a globalization, but in the topological setting, we need to ensure the continuity of the global action that contains the original partial action. Moreover, we characterize the topology of the globalization space by a universal property.

Definition 3.2.3. Let α and β be topological actions of G on X and Y, respectively. We call a continuous map $\phi : X \to Y$ a *morphism* $\alpha \to \beta$ if ϕ is $G-$equivariant.

Theorem 3.2.4 (Abadie). Let $\theta = ((X_g)_{g\in G}, (\theta_g)_{g\in G})$ be a topological partial action of G on X. Then, there exists a topological global action $\theta^e = ((X_g^e)_{g\in G}, (\theta_g^e)_{g\in G})$ of G on X^e, and $\iota : \theta \to \theta^e$ is a morphism, such that for any morphism $\psi : \theta \to \alpha$, where α is a topological global action of G on Y, there exists a morphism $\psi^e : \theta^e \to \alpha$ such that $\psi^e(\iota(x)) = \psi(x),$ for all $x \in X$. Moreover;

(i) $\iota(X)$ is open in X^e.
(ii) $\iota : X \to \iota(X)$ is a homeomorphism.
(iii) X^e is the θ^e-orbit of $\iota(X)$.



Proof. Define the global action $\gamma : G \times (G \times X) \to G \times X$ such that

$$\gamma(s, t, x) = \gamma_s(t, x) = (st, x), \text{ for all } s, t \in G, \text{ and all } x \in X.$$

We endow $G \times X$ with the product topology. Let U and V be open subsets of G and X, respectively. Let $A = \{(s, t) : st \in U\}$ open in $G \times G$. Thus, $\gamma^{-1}(U \times V) =$

$A \times V$ which is open in $G \times (G \times X)$. Hence, γ is continuous. Moreover, define an equivalence relation $\sim$ on $G \times X$ by

$$(s,x) \sim (t,y) \iff x \in X_{s^{-1}t} \text{ and } \theta_{t^{-1}s}(x) = y.$$

Let $X^e = (G \times X)/\sim$ endowed with the quotient topology, and $q : G \times X \to X^e$ be the quotient map. As we have seen in the category of sets, γ is compatible with the $\sim$ relation. We can then define a global action $\theta^e : G \times X^e \to X^e$ given by

$$\theta^e_s(q(t,x)) = q(st,x) = q(\gamma_s(t,x)).$$

Next, observe that for all $t \in G$, and all $U \subseteq X^e$ we have that $q^{-1}(\theta^e_t(U)) = \gamma_t(q^{-1}(U))$. This implies that each θ^e_t is a homeomorphism. We can show next that this action is continuous. Define $\iota : X \to X^e$ such that $\iota(x) = q(1,x)$. Since the inclusion $X \hookrightarrow G \times X$ given by $x \mapsto (1,x)$ is continuous, we have that ι also is. Let us first show that ι is an open map. Suppose $U \subseteq X$ be an open subset. Consider $q^{-1}(\iota(U)) = \{(t,x) : (t,x) \sim (1,y) \text{ for some } y \in U\} = \{(t,x) : \theta_t(x) \in U\} = \theta^{-1}(U)$, which is open in Γ_θ due continuity of θ and hence open in $G \times X$ because Γ_θ is open. Next, we show that q is an open map. If $U \subseteq G$ and $V \subseteq X$ are open subsets, we have $q(U \times V) = \bigcup_{t \in U} q(\gamma_t(\{1\} \times V)) = \bigcup_{t \in U} \theta^e_t(\iota(V))$ which is open because ι is open and every θ^e_t is a homeomorphism. Now, let $W \subseteq X^e$ be an open subset. Since $\theta^e \circ (\mathrm{id} \times q) = q \circ \gamma : G \times (G \times X) \to X^e$ is continuous, we have that $(\mathrm{id} \times q)^{-1}((\theta^e)^{-1}(W)) = \gamma^{-1}(q^{-1}(W))$ is an open subset of $G \times (G \times X)$. Since $\mathrm{id} \times q$ is open and surjective, $(\theta^e)^{-1}(W) = (\mathrm{id} \times q)((\mathrm{id} \times q)^{-1}((\theta^e)^{-1}(W))) = (\mathrm{id} \times q)(\gamma^{-1}((q^{-1}(W))))$ which is open.

Showing that ι is a morphism, we can see that $\iota(X_t) \subseteq X = X^e_t$, for all $t \in G$. Suppose $x \in X_{t^{-1}}$, then

$$\iota(\theta_t(x)) = q(1,\theta_t(x)) = q(t,x) = q(\gamma_t(1,x)) = \theta^e_t(q(1,x)) = \theta^e_t(\iota(x)).$$

Next, let $\alpha : G \times Y \to Y$ be a topological global action of G on Y, and let $\psi : X \to Y$ be a continuous function. Consider a map $\psi' : G \times X \to Y$ given by $\psi'(t,x) = \alpha_t(\psi(x))$. Therefore, we have that $\psi'(\gamma_t(h,x)) = \psi'(th,x) = \alpha_{th}(\psi(x)) \overset{(2.1.4.iii)}{=} \alpha_t(\alpha_h(x)) = \alpha_t(\psi'(h,x))$. Hence, ψ' is a morphism $\gamma \to \alpha$ since $\psi' = \alpha \circ (\mathrm{id} \times \psi)$ which is continuous and ψ' and α are global. Moreover, suppose $\psi : \theta \to \alpha$ is also a morphism. If $(r,x) \sim (s,y)$ in $G \times X$, then we have that

$$\alpha_{s^{-1}}(\psi'(r,x)) = \alpha_{s^{-1}}(\alpha_r(\psi(x))) \overset{(2.1.4.iii)}{=} \alpha_{s^{-1}r}(\psi(x)) = \psi(\theta_{s^{-1}r}(x)) = \psi(y).$$

Thus $\psi'(r,x) = \alpha_s(\psi(y)) = \psi'(s,y)$, that is ψ' is compatible with $\sim$. We can define a map $\psi^e : X^e \to Y$ by $\psi^e(q(t,x)) = \alpha_t(\psi(x))$ for all $t \in G$, and all $x \in X$. Let $U \subseteq Y$ be an open subset, hence we have that $(\psi^e)^{-1}(U) = q((\mathrm{id} \times \psi)^{-1}(\alpha^{-1}(U)))$ which is open because q is an open map and α is continuous. Therefore, ψ^e is continuous. Observe that

$$\psi^e(\theta^e_t(q(h,x))) = \psi^e(q(th,x)) = \alpha_{th}(\psi(x)) \overset{(2.1.4.iii)}{=} \alpha_t(\alpha_h(\psi(x)) = \alpha_t(\psi^e(q(h,x))),$$

so ψ^e is a morphism $\alpha^e \to \beta$ and $\psi^e\iota(x) = \psi^e(q(1,x)) = \psi(x)$.

Lastly, items (i) and (ii) comes immediately from the fact that ι is an open map. Item (iii) is clear as $q(t,x) = \theta^e_t(\iota(x))$. □

Note that θ^e in Theorem 3.2.4 is a globalization or an enveloping action for θ which is unique by the universal property.

Definition 3.2.5. Let θ be a topological partial action of G on X and θ^e be its enveloping action on X^e provided by Theorem 3.3.4. We call X^e the enveloping space of X and ψ^e the enveloping morphism of ψ.

Example 3.2.6. Consider the partial action α of $\mathbb{Z}_2 = \{1, -1\}$ with the discrete topology on the unit interval $X = [0, 1]$, given by $\alpha_1 = \mathrm{id}_X$ and $\alpha_{-1} = \mathrm{id}_V$, where $V = (a, 1]$ for some $a > 0$. Let $\alpha^e : G \times X^e \to X^e$ be the enveloping action of α. We have that X^e is the topological quotient space of $\{1, -1\} \times [0, 1]$ by identifying for each $t \in V$ the point $(1, t)$ with $(-1, t)$ denoted by t. The action α^e is given by

$$\alpha^e_1(x) = \mathrm{id}_{X^e} \text{ and } \alpha^e_{-1}(x) = \begin{cases} (-i, t) & ; x = (i, t) \text{ for some } t \in X \backslash V \\ x & ; x \in V \end{cases}.$$

Consider the points $(1, a)$ and $(-1, a)$ which are distinct. Their neighborhoods must contain (a, l) for some $l \leq 1$. Hence, we can not find two disjoint neighborhoods containing each of the two points, that is, X^e is not Hausdorff.

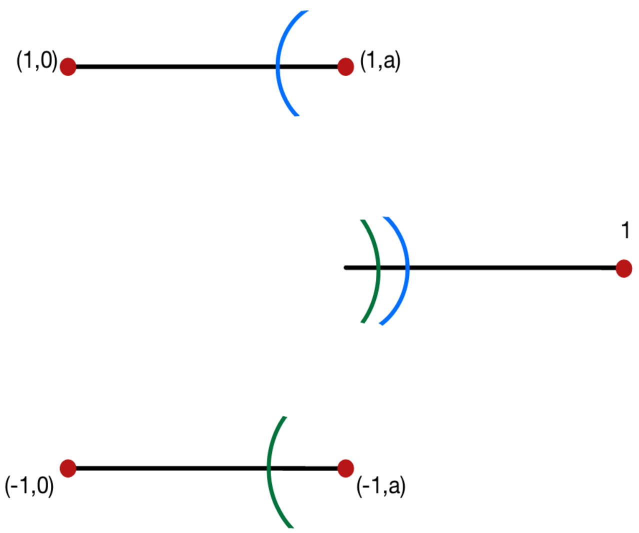


Figure 3.1: The enveloping space is not Hausdorff: the blue indicates a neighborhood of $(1, a)$, while the green indicates a neighborhood of $(-1, a)$.

As we have seen in the previous example, even though local properties of the original space are preserved, their global properties may differ; in this case, the enveloping space is not Hausdorff, although the original is.

Proposition 3.2.7. Let θ be a topological partial action of G on a Hausdorff space X. Then, its enveloping space is a Hausdorff space if and only if $\mathrm{Graph}(\theta)$ is closed in $X \times G \times X$.

Proof. Let X^e be the enveloping space of X and assume that it is a Hausdorff space. Let $\{(\theta(t_d, x_d), t_d, x_d)\}_d \subseteq \text{Graph}(\theta)$ be a net that converges to some element $(y, t, x) \in X \times G \times X$. Hence, $\theta(t_d, x_d) \to y \in X$. Since θ^e, is continuous, we have that $\theta^e(t_d, x_d) \to \theta^e(t, x)$. Therefore, $\theta^e(t, x)$ must be equal to $y \in X$, that is, $\theta(t, x)$ is defined and equal to y because of the uniqueness of limits in Hausdorff spaces. Hence, $(y, t, x) = (\theta^e(t, x), t, x) = (\theta(t, x), t, x) \in \text{Graph}(\theta)$.

On the other hand, let Graph(α) be a closed subset in $X \times G \times X$, and let $x^e, y^e \in X^e$. We have by (3.2.4.iii) that $x^e = \theta^e(t, x) = \theta^e_t(x)$ for some $t \in G$, and some $x \in X$ and $y^e = \theta^e(s, y) = \theta^e_s(y)$ for some $s \in G$, and some $y \in X$. Suppose that we cannot find a neighborhood of x^e and a neighborhood of y^e that are disjoint. Let $U, V \subseteq X$ be a neighborhood of x and a neighborhood of y, respectively. Since θ^e_t and θ^e_s are homeomorphisms, we have that $\theta^e_t(U) \cap \theta^e_s(V) \neq \emptyset$ as $\theta^e_t(U)$ is a neighborhood of x^e and $\theta^e_s(V)$ is a neighborhood of y^e. Let $a_{U,V} \in \theta^e_t(U) \cap \theta^e_s(V)$, that is, $a_{U,V} = \theta^e_t(x_{U,V}) = \theta^e_s(y_{U,V})$ for some $x_{U,V} \in U$, and some $y_{U,V} \in V$. Take the net $\{(y_{U,V}, s^{-1}t, x_{U,V})\} \subseteq \text{Graph}(\theta)$. This net converges to $(y, s^{-1}t, x)$. Since Graph(θ) is closed, we have that $(y, s^{-1}t, x) \in \text{Graph}(\theta)$. Therefore, $\theta_{s^{-1}t}(x) = y$, that is, $\theta^e_{s^{-1}t}(x) = y$,. Hence, $x^e = \theta^e_t(x) = \theta^e_s(y) = y^e$. We can conclude that X^e is Hausdorff. □

Remark 3.2.8. If G is a discrete group, then Graph(θ) is closed in $X \times G \times X$ if and only if $\text{Graph}(\theta_t) = \{(y, x) : x \in X_{t^{-1}} \text{ and } \theta_t(x) = y\}$ is closed in $X \times X$ for all $t \in G$.

4

Decomposable partial actions

Decomposable partial actions are a special type of action that are easier to understand because they can be broken down into simpler components. These components typically include a global action of a finite group, a translation, and a trivial action [2].

Consider a partial action of $\mathbb{Z}_2$ on a space X. This is equivalent to picking an open subset $U \subseteq X$ and a homeomorphism σ of order two on it. The restriction of this action on U is global, while the rest of the space $Y = X \backslash U$ is acted on trivially. That is to say, there is an equivariant topological extension $U \hookrightarrow X \twoheadrightarrow Y$.

When dealing with larger groups, we need to repeat this process many times. For example, an action of $\mathbb{Z}_3$ on a space X, we can choose two open subsets $U_1, U_2 \subseteq X$, and a homeomorphism $\sigma_1 : U_2 \to U_1$ such that $\sigma_1^3 = \mathrm{id}$ wherever the composition is well-defined. The restriction of this action to U is global, and we get the extension

$$U \hookrightarrow X \twoheadrightarrow Y,$$

where $Y = X \backslash U$. But the action on Y is not trivial. So, we can break it down further by letting $V_1 = U_1 \backslash U_2$ and $V_2 = U_2 \backslash U_1$. The homeomorphisms induced by σ_q and σ_2 exchange V_1 and V_2, while the compliment $Z = Y \backslash (V_1 \sqcup V_2)$ carries the trivial action. Therefore, we get an equivariant extension

$$V_1 \sqcup V_2 \hookrightarrow Y \twoheadrightarrow Z,$$

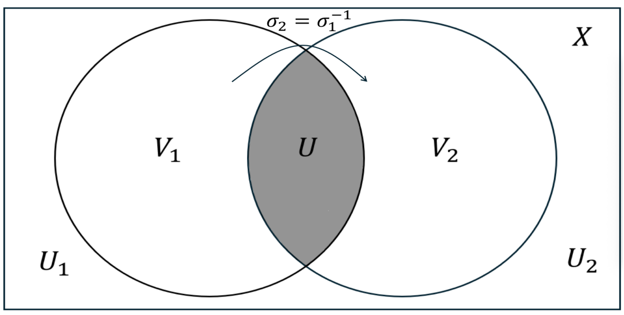


Figure 4.1: Decomposition of the action $\mathbb{Z}_3$ on X

where the action on $V_1 \sqcup V_2$ consists of a translation and a homeomophism of order three, and the action on Z is trivial. Therefore, we have simplified the problem:

instead of studying the full action of $\mathbb{Z}_3$ on X, we only need to understand how $\mathbb{Z}_3$ acts on U and on $V_1 \sqcup V_2$, along with the two extension sequences we have described.

In this chapter, we will investigate the common feature that the action on U and the action on $V_1 \sqcup V_2$ has which we call the *decomposition property.*

4.1 Decomposition property

We define the decomposition property of a topological partial action and study some of its basic features.
Throughout this chapter, we will consider a fixed finite group G, $n \in \mathbb{N}$, and a topological space X. We will also omit the adjective topological before "partial action" since we will only work on topological structures.

Definition 4.1.1 We define the *space of n-tuples* of G to be

$$\mathcal{T}_n(G) = \{\tau \subseteq G : 1 \in \tau \text{ and } |\tau| = n\}.$$

For $g \in G$, we set $\mathcal{T}_n(G)_g = \{\tau \in \mathcal{T}_n(G) : g \in \tau\}$. There exists a canonical partial action $\mathbf{Lt}$ of G on $\mathcal{T}_n(G)$ with $\mathbf{Lt}_g : \mathcal{T}_n(G)_{g^{-1}} \to \mathcal{T}_n(G)_g$ given by $\mathbf{Lt}_g(\tau) = g\tau$ for all $\tau \in \mathcal{T}_n(G)_{g^{-1}}$.

We equip $\mathcal{T}_n(G)$ with the discrete topology.

Notation 4.1.2. Let $\theta = ((X_g)_{g\in G}, (\theta)_{g\in G})$ be a partial action of G on X. For $\tau \in \mathcal{T}_n(G)$, we write $X_\tau = \bigcap_{g\in\tau} X_g$ which is an open subset of X. For $g \in G$ and $\tau \in \mathcal{T}_n(G)_{g^{-1}}$, we have that $\theta_g(X_\tau) \overset{(2.1.8)}{=} X_{g\tau}$. Moreover, for $\tau \in \mathcal{T}_n(G)$, we write $G \cdot \tau \subseteq \mathcal{T}_n(G)$ for the orbit of τ with respect to the partial action $\mathbf{Lt}$ from (4.1.1), and we set $X_{G\cdot\tau} = \bigcup_{g\in\tau^{-1}} X_{g\tau}$.

Definition 4.1.3. Let $\theta = ((X_g)_{g\in G}, (\theta_g)_{g\in G})$ be a partial action on X. We say that θ has the *n-decomposition* property if

(i) $X = \bigcup_{\tau\in\mathcal{T}_n(G)} X_\tau$, and

(ii) $X_\tau \cap X_g = \emptyset$ for all $\tau \in \mathcal{T}_n(G)$ and all $g \in G$ such that $g \notin \tau$.

We say that θ has the *decomposition property* if it has the n-decomposition property for some $n \in \mathbb{N}$.

Remark 4.1.4. Item (4.1.3.ii) implies that $X_\tau \cap X_\sigma = \emptyset$ if $\tau, \sigma \in \mathcal{T}_n(G)$ are distinct. In particular, we think of X as a disjoint union of X_τ for all τ in the space of n-tuples. In fact, if $n > 1$, then θ has the n-decomposition property if and only if $X = \bigsqcup_{\tau\in\mathcal{T}_n(G)} X_\tau$ and $X_g = \bigsqcup_{\tau\in\mathcal{T}_n(G)_g} X_\tau$.

Example 4.1.5. Let θ be a partial action of G on X. We have the followings:

(i) θ has the 1-decomposition property if and only if $X_g = \emptyset$ for all $g \in G\backslash\{1\}$. This is the *trivial partial action* of G on X.

(ii) θ has the $|G|$-decomposition property if and only if θ is global.

Proposition 4.1.6. The partial action **Lt** of G on $\mathcal{T}_n(G)$ described in (4.1.1) has the n-decomposition property.

Proof. Let $\tau \in \mathcal{T}_n(G)$. It is clear that $\mathcal{T}_n(G)_\tau = \{\tau\}$. Since $\mathcal{T}_n(G) = \bigcup_{\tau \in \mathcal{T}_n(G)} \{\tau\}$, then it satisfies (4.1.3.i). Observe that

$$\mathcal{T}_n(G)_\tau \cap \mathcal{T}_n(G)_g = \{\tau\} \cap \{\sigma \in \mathcal{T}_n(G) : g \in \tau\} = \emptyset$$

for all $g \in G$ such that $g \notin \tau$, hence (4.1.3.ii). □

Lemma 4.1.7. Let $\tau \in \mathcal{T}_n(G)$ and set $H_\tau = \{h \in G : h\tau = \tau\}$. Then, the followings are true:

(i) H_τ is a finite subgroup of G, and $|H_\tau|$ divides n.

(ii) With $m_\tau = \frac{n}{|H_\tau|} - 1$, there exist $x_1^\tau, ..., x_{m_\tau}^\tau \in G$ distinct such that

$$\tau = H_\tau \sqcup H_\tau x_1^\tau \sqcup \cdots \sqcup H_\tau x_{m_\tau}^\tau. \tag{4.1.8}$$

.

(iii) If $y_1^\tau, ..., y_{m_\tau}^\tau \in G$ satisfy $\tau = H_\tau \sqcup H_\tau y_1^\tau \sqcup ... \sqcup H_\tau y_{m_\tau}^\tau$, then there exist a permutation $\sigma \in S_{m_\tau}$ and $h_1, ..., h_{m_\tau} \in H_\tau$ such that $y_j = h_j x_{\sigma(j)}$ for all j.

Proof. Let $g, h \in H_\tau$. Then, for each $f \in H_\tau$, we have have $gh^{-1}\tau = g\tau = \tau$, so H_τ is a subgroup of G. We also have that τ is H_τ-invariant because $h\tau = \tau$ for all $h \in H_\tau$ and $H_\tau \subseteq \tau$ since $1 \in \tau$. Hence, H_τ is finite. Since H_τ acts globally on the finite set τ, we have that τ is a disjoint union of H_τ orbits, that is $\tau = H_\tau x_0^\tau \sqcup H_\tau x_1^\tau \sqcup ... \sqcup H_\tau x_{m_\tau}^\tau$, where $m_\tau + 1$ is the cardinality of the orbit space, and $x_0^\tau, x_1^\tau, ..., x_{m_\tau}^\tau \in \tau$ are representatives. As H_τ must be equal to some $H_\tau x_i^\tau$ for some i, we asume that $x_0^\tau = 1$. Since H_τ is a left translation which is free, all of the orbits must have the same cardinality as H_τ. Therefore, $n = |\tau| = (m_\tau + 1)|H_\tau|$. We have proved items (i) and (ii).

Lastly, let $y_1^\tau, ..., y_{m_\tau}^\tau \in G$ be as described in (iii). Then, these elements define a decomposition of τ as a disjoint union of H_τ orbits. Therefore, after reordering them, they must match $x_1, ..., x_m$ modulo H_τ. □

Notation 4.1.9. For $\tau \in \mathcal{T}_n(G)$, we set $H_\tau = \{h \in G : h\tau = \tau\}$. By (4.1.7), we set $m_\tau = \frac{n}{|H_\tau|} - 1$ and fix elements $x_0^\tau = 1, x_1^\tau, ..., x_{m_\tau}^\tau \in G$ satisfying (4.1.8). We will simply write H, m and x_j for $j = 1, ..., m$ when we know τ from the context.

Let $\mathcal{O}_n(G)$ be the orbit space for the partial system described in (4.1.1). We denote by $\kappa : \mathcal{T}_n(G) \to \mathcal{O}_n(G)$ the canonical quotient map, and fix a global section $s : \mathcal{O}_n(G) \to \mathcal{T}_n(G)$ for it. For $z \in \mathcal{O}_n(G)$, we write τ_z for $s(z)$, H_z for H_{τ_z}, and m_z for m_{τ_z}.

Proposition 4.1.10. Let $\tau \in \mathcal{T}_n(G)$. Set $X = \{1, x_1, ...x_m\}$ to be the set of representatives of H_τ-classes. For $g \in G$, set $X_g = \{x \in X : g \in x^{-1}\tau\}$. Then,

(i) There is a unique $\sigma_g(x) \in X_g$ such that $g \in \sigma_g(x)^{-1}Hx$.

(ii) $((X_g)_{g \in G}, (\sigma_g)_{g \in G})$ is partial action of G on X.

Proof. Observe that $x_j^{-1}\tau \overset{(4.1.7.ii)}{=} \bigsqcup_{k=0}^m x_j^{-1}Hx_k$ for all j. Let $x_j \in X_{g^{-1}}$, then $g^{-1} \in x_j^{-1}\tau = \bigsqcup_{k=0}^m x_j^{-1}Hx_k$, that is, $g^{-1} = x_j^{-1}hx_i$ for some i and some $h \in H$. Hence,

$$g = x_i^{-1}h^{-1}x_j \in x_i^{-1}Hx_j \subseteq x_i^{-1}\tau \text{ and } x_i \in X_g.$$

Thus, σ_g maps x_j to x_i in this fashion proving (i).

As for item (ii), it is clear that $X_1 = X$. Let $x \in X_{g_2^{-1}}$ such that $\sigma_{g_2}(x) \in X_{g_1^{-1}}$. We have that $g_2^{-1} \in x^{-1}H\sigma_2(x)$ and $g_1^{-1} \in \sigma_{g_2}(x)^{-1}H\sigma_{g_1}(\sigma_{g_2}(x))$, that is $g_2^{-1} = x^{-1}h_1\sigma_{g_2}(x)$ and $g_1^{-1} = \sigma_{g_2}(x)^{-1}h_2\sigma_1(\sigma_2(x))$ for some $h_1, h_2 \in H$. Therefore,

$$(g_1g_2)^{-1} = x^{-1}h_1h_2\sigma_1(\sigma_2(x)) \in x^{-1}H\sigma_{g_1}(\sigma_{g_2}(x)) \subseteq x^{-1}\tau,$$

that is, $x \in X_{(g_1g_2)^{-1}}$, and $\sigma_{g_1g_2}(x) = \sigma_{g_1}(\sigma_{g_2}(x))$ by the uniqueness of the left member of the equality. □

Proposition 4.1.11. Let $\theta = ((X_g)_{g\in G}, (\theta_g)_{g\in G})$ be a partial action of G on X with the n-decomposition property, and let $\tau \in \mathcal{T}_n(G)$. Adopt the conventions from (4.1.9). Then, the followings are true:

(i) The restriction of $\theta|_H$ to X_τ is a global action.

(ii) The open set $X_{G\cdot\tau}$ is G-invariant, and for all $g \in G$, we have that

$$(X_{G\cdot\tau})_g = \begin{cases} \emptyset & \text{if } g \notin \tau^{-1}\cdot\tau \\ \bigcup_{0\le j\le m: g\in x_j^{-1}} X_{x_j^{-1}\tau} & \text{if } g \in \tau^{-1}\cdot\tau. \end{cases}$$

(iii) For $\sigma \in \mathcal{T}_n(G)$, we have $X_{G\cdot\sigma} \cap X_{G\cdot\tau} = \emptyset$ if $\sigma \notin G\cdot\tau$ and $X = \bigsqcup_{z\in\mathcal{O}_n(G)} X_{G\cdot\tau_z}$.

Proof. Since $H \subseteq \tau$, then for all $h \in H$ we have that $X_\tau \subseteq X_h$, that is, $X_\tau \cap X_h = X_\tau$. Moreover, $\theta_h(X_\tau) = X_{h\tau} = X_\tau$. We have that $\theta|_H$ induces a global action on A_τ which proves (i).

As for (ii), fix $g \in G$. We have by (4.1.3.ii) that $X_{g^{-1}} \cap X_\tau = \emptyset$ if $g^{-1} \notin \tau$, and $X_{g^{-1}} \cap X_\tau = X_\tau$ otherwise. In the latter case $\theta_g(X_\tau) = X_{g\tau}$ which means that $X_{G\cdot\tau}$ is invariant and there is a well-defined restricted partial action of G on it. Observe that if $g \in Hx_i$ for some representative x_i as in (4.1.9), we have that $g = hx_i$ for some $h \in H$, and

$$g^{-1} = x_ih^{-1} \implies g^{-1}\tau = x_i^{-1}h^{-1}\tau = x_i^{-1}\tau \tag{4.1.12}$$

For $g \in G$, we have that

$$(X_{G\cdot\tau})_g = X_{G\cdot\tau} \cap X_g \overset{(4.1.12)}{=} \bigcup_{j=0}^m X_{x_j^{-1}\tau} \cap X_g \overset{(4.1.3.ii)}{=} \bigcup_{0\le j\le m:\ g\in x_j^{-1}\tau} X_{x_j^{-1}\tau}.$$

In particular, if $g \notin$, then $(X_{G\cdot\tau})_g = \emptyset$.

We will prove the contrapositive of item (iii). Suppose that $X_{G\cdot\sigma} \cap X_{G\cdot\tau} \neq \emptyset$. Fix $g \in \tau^{-1}$ and $h \in \sigma^{-1}$ with $X_{g\tau} \cap X_{h\sigma} \neq \emptyset$. By (4.1.4), we have that $g\tau = h\sigma$, that is $h \in g\tau$ and $\sigma = (h^{-1}g)\tau \in G\cdot\tau$. Hence, we have that $X = \bigsqcup_{z\in\mathcal{O}_n(G)} X_{G\cdot\tau_z}$. □

Remark 4.1.12. Using (4.1.11.iv), many facts about decomposable partial actions can be understood by looking at G-invariant disjoint components $X_{G\cdot\tau_z}$. In practice, it is often enough to focus on just one tuple $\tau \in \mathcal{T}_n(G)$ and study the partial action it induces on $X_{G\cdot\tau}$.

4.2 Partial actions of finite groups

In this final section, we show that any partial action of a finite group can be expressed as an iterated extension of decomposable partial actions. Furthermore, we prove that a partial action of a finite group on a Hausdorff space with the decomposition property has a Hausdorff enveloping space.

Theorem 4.2.1.(Abadie, Gardella, Geffen) Let $\theta_g)_{g\in G})$ be a partial action of G on X. Then, there are canonical equivariant extensions

$$(D^{(k)}, \delta^{(k)}) \hookrightarrow (X^{(k)}, \theta^{(k)}) \twoheadrightarrow (X^{(k-1)}, \theta^{(k-1)}),$$

for $2 \leq k \leq |G|$, with $(X^{(|G|)}, \theta^{(|G|)}) = (X, \theta)$ and satisfying the following properties:
(i) $\delta^{(k)}$ has the $k-$decomposition property.
(ii) $X_\sigma^{(k)} = \emptyset$ for all $\sigma \in \mathcal{T}_{k+1}(G)$.
(iii) $\theta^{(1)}$ has the 1-decomposition property.
Thus, θ can be written canonically as an iterated extension of decomposable partial actions.

Proof. Set $n = |G|$ and take $(X^{(n)}, \theta^{(n)}) = (X, \theta)$. Define $D^{(n)} = \bigcap_{g\in G} X_g^{(n)}$, which is a G-invariant open subset of $X^{(n)}$. Let $\delta^{(n)} = \theta^{(n)}|_{D^{(n)}}$ which is global, and hence has the n-decomposition property. Let $X^{(n-1)} = X^{(n)} \backslash D^{(n)}$ Then, we have that $X_\sigma^{(n-1)} = \emptyset$ for all $\sigma \in \mathcal{T}_n(G) = \{G\}$. Suppose next that $(X^{(k)}, \theta^{(k)})$ has been constructed and satisfies item (ii). We will construct $(D^{(k)}, \delta^{(k)})$ and $(X^{(k-1)}, \theta^{(k-1)})$. We set $D^{(k)} = \bigcup_{\tau\in\mathcal{T}_k(G)} X_\tau^{(k)}$, which is a G-invariant open subset of $X^{(k)}$. Let $\delta^{(k)}$ be the induced action on it, we will show that $\delta^{(k)}$ has the k-decomposition property.

For $g \in G$, we have $D_g^{(k)} = \bigcup_{\tau\in\mathcal{T}_k(G)_g} X_\tau^{(k)}$, so $D_\tau^{(k)} = X_\tau^{(k)}$ for all $\tau \in \mathcal{T}_k(G)$. This means that $D^{(k)}$ satisfies (4.1.3.i). Next, let $\tau \in \mathcal{T}_k(G)$ and $g \notin \tau$. Set $\sigma = \tau \cup \{g\}$, which is a tuple in $\mathcal{T}_{k+1}(G)$. By condition (ii), we get that

$$\emptyset = X_\sigma^{(k)} = X_\tau^{(k)} \cap X_g^{(k)} = D_\tau^{(k)} \cap X_g^{(k)}.$$

Since $D_g^{(k)} \subseteq X_g^{(k)}$, we have that $D_\tau^{(k)} \cap D_g^{(k)} = \emptyset$, hence (4.1.3.ii). Therefore, $\delta^{(k)}$ has the k-decomposition property. Let $X^{(k-1)}$ denote $X^{(k)} \backslash D^{(k)}$, then $X_\tau^{(k-1)} = \emptyset$ for all $\tau \in \mathcal{T}_k(G)$. We have thus established conditions (i) and (ii) in the statement. Condition (iii) follows from taking $k = 1$ in condition (ii), since in this case we have $X_\tau^{(1)} = \emptyset$ for all $\tau \in \mathcal{T}_2(G)$ which is equivalent to $\theta^{(1)}$ having the 1-decomposition property by (4.1.5.i). This finishes the proof. □

Proposition 4.2.2. Let θ be a partial action of G on a Hausdorff space X with a decomposition property. Then, its enveloping space is a Hausdorff space.

Proof. Suppose θ has the n-decomposition property, then $X = \bigcup_{i=1}^m X_{\tau_i}$ and $X_{\tau_i} \cap X_g = \emptyset$ for all $g \notin \tau_i$. Note that $X \backslash X_{\tau_j} = \bigcup_{j\neq i} X_{\tau_j}$ which is open, that is, X_τ is closed and open. By (3.2.8), it is enough to show that for each $t \in G$ Graph(θ_t) is closed in $X \times X$ since the group is discrete. Let Graph$(\theta_t) \supseteq \{(\theta_t(x_d), x_d)\}_d$ be a net that converges to $(y, x) \in X \times X$. We have that $\{(x_d)\}_d \subseteq X_{t^{-1}}, \{(\theta_t(x_d)_d)\} \subseteq X_t$

and $x_d \to x \in X, \theta_t(x_d) \to y \in X$. By (4.1.4), we have that $X_t = \bigsqcup_{t \in \tau_i} X_{\tau_i}$ which is a finite union of closed sets, that, is X_t is closed. Therefore, $x \in X_{t^{-1}}$ and $y \in X_t$. Since X is Hausdorff and θ_t is continuous, we have that $\theta_t(x_d) \to \theta_t(x)$ and $\theta_t(x) = y$ which means that $(y, x) \in \text{Graph}(\theta_t)$, that is, it is closed in $X \times X$. We can deduce that the enveloping space of X is Hausdorff by (3.2.7). □

Example 4.2.3. Consider a partial action α of $\mathbb{Z}_3 = \{0, 1, 2\}$ on the space $X = [0, 1] \cup [2, 3]$ equipped with the subspace topology. The partial action is defined by

$$\alpha_0 = \text{id}_X, \ \alpha_1 : [0, 1] \to [2, 3], \ \alpha_1(x) = x + 2, \ \text{and } \alpha_2 : [2, 3] \to [0, 1], \ \alpha_2(x) = x - 2.$$

The action has 2-decomposition property and we get that the enveloping space X^e is homeomorphic to the interval $[0, 1] \cup [2, 3] \cup [4, 5]$, which is a Hausdorff space. Moreover, the enveloping action α^e on X^e is given by

$$\alpha_0^e = \text{id}_{X^e}, \quad \alpha_1^e(x) = x + 2 - 6 \left\lfloor \frac{x+2}{6} \right\rfloor, \quad \text{and} \quad \alpha_2^e(x) = x - 2 - 6 \left\lfloor \frac{x-2}{6} \right\rfloor.$$

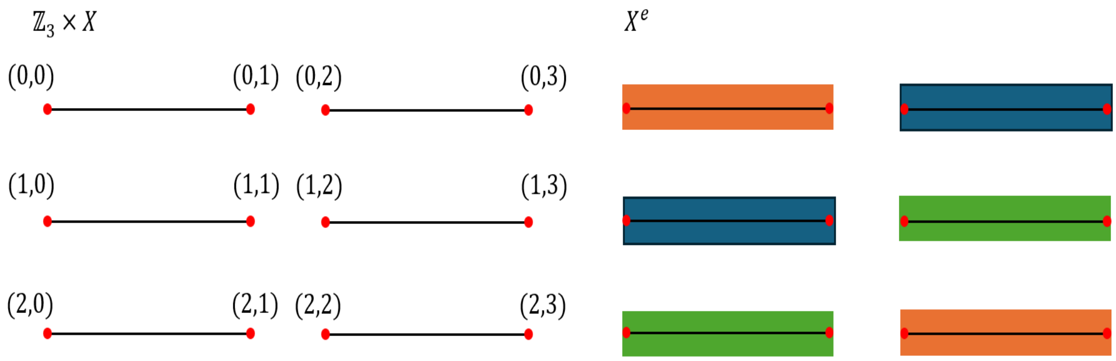


Figure 4.2: In the enveloping space X^e, intervals shaded in the same color represent regions that are identified with one another.

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