Evolutionary dynamics of spatial games

Master's thesis in Complex Adaptive Systems

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Abstract

Game theory is a subject that can be applied to many fields of study, for instance biology and economics. In this report, we look at two known games in game theory, Prisoner's dilemma and Hawk dove. In contrast to previous research on these two games, every agent is located in two-dimensional space on the unit square. This spatial case on game theory changes the dynamics of the games. For example, we found that cooperating is a superior strategy in Prisoner's dilemma, for some combinations of parameters. For reference, in the normal Prisoner's dilemma, the strategy defect dominates the strategy to cooperate. It is also found that the initial state of the simulations has a big impact on the results obtained. The simulations made are computationally demanding. An attempt was therefore made to make approximations that could replace the spatial model. The result from the approximations shows that it is not good enough to replace the spatial model. However, steps has been made that can help future researchers to make an approximation that works well.

Acknowledgements

I wish to express my gratitude to Philip Gerlee, who has guided me through this project.

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1
Background

The concept of game theory was invented by John von Neumann and Oskat Morganstern, with the goal to design a model to study human behavior [Now06]. Game theory is a widely used concept, with applications in for instance economy and biology. In the most simple form of game theory, there are two players and two strategies. Each player will get rewarded depending on which strategies are played. This means that there are 4 different combinations of rewards, and these combinations are often represented in a payoff matrix. If no player can benefit from changing their strategy, the state is called Nash equilibrium. The concept of game theory can be expanded to evolutionary game theory. In evolutionary game theory, a group of players are interacting in a game. Each individual have a fixed strategy, and will receive a payoff, depending on the interactions with the other players. A higher payoff will lead the a higher chance of reproduction, similar to the process of natural selection.

Altruistic behavior can lead to advantages in the evolutionary process. This is easy to confirm by looking at how some animals behave, or most clearly how humans behave. It is therefore interesting to look at how altruistic behavior plays out for evolutionary dynamics. For a starter, it is well know that altruistic behavior is not appropriate for a simple version of a social dilemma. However, for more sophisticated models, altruism can be favorable. This has been looked at in several papers [Min98][Chr04]. In these papers, they analyse the conditions for altruism to prevail, when the agents are placed on a two dimensional grid. One of their results was that the benefit to others by altruistic behavior divided by the cost of altruism has to be a certain value, for altruism to be a successful strategy.

1.1 Evolutionary dynamics

In a game with only two strategies A and B, the payoff function can be written as a payoff matrix:

$$
\begin{array}{c|cc}
 & A & B \\
A & a & b \\
B & c & d
\end{array}
$$

Table 1.1: A and B are the two different strategies. A gets a in payoff if faced against A, and b in payoff if faced against B. B gets c in payoff if faced against A, and d in payoff if faced against B.

By introducing x_A and x_B, the frequency of respective strategy in the population, we can write the expected payoff for A-players and B-players as

$$f_A = ax_A + bx_B \tag{1.1}$$

$$f_B = cx_A + dx_B. \tag{1.2}$$

The equations hold true for the non-spatial case, with the assumption that the probability of interacting with an A agent is x_A, and x_B for B agents. Furthermore, we introduce the rate of change for x_A and x_B, $\dot{x}_A$ and $\dot{x}_B$, with the following equations [Pet78]:

$$\dot{x}_A = x_A(f_A(x_A, x_B) - \Phi) \tag{1.3}$$

$$\dot{x}_B = x_B(f_B(x_A, x_B) - \Phi) \tag{1.4}$$

where Φ is the average fitness for the agents, which can be written as $\Phi = x_A f_A(x_A, x_B) + x_B f_B(x_A, x_B)$. As we only have two strategies, $x_A + x_B = 1$. With this information, we can rewrite equation 1.3 and 1.4 to one equation, with $x_A = x$ and $x_B = 1 - x$

$$\dot{x} = x(1 - x)(f_A(x) - f_B(x)). \tag{1.5}$$

With the payoff matrix shown in table 1.1, we can rewrite equation 1.5 to

$$\dot{x} = x(1 - x)[(a - b - c + d)x + b - d] \tag{1.6}$$

where the solution for $\dot{x} = 0$ is shown in equation 1.7. This solution is interesting, because $\dot{x} = 0$ is a stable equilibrium.

$$x^* = \frac{d - b}{a - b - c + d} \tag{1.7}$$

There are in total 5 different types of dynamics of 1.6, which depends on the entries of the payoff matrix.

(i) If strategy A always is superior to strategy B, as it is when $a > c$ and $b > d$, we say that A dominates B. This means that if two players are facing each other, it is always better to play A, no matter what the other player are playing. Another way to see it, is that the average payoff for an A agent will always exceed that of a B agent. Only one of the inequalities $a > c$ and $b > d$ has to be strict for A to dominate B

(ii) B dominates A, if $a < c$ and $b < d$. This is a mirror image to case (i). As in case (i), only one of the inequalities has to be strict.¨

(iii) If $a > c$ and $b < d$, then A and B are bistable. If two players where to face each other, the highest fitness is obtained by playing the same strategy as the other agent. For selection dynamics, the initial condition is crucial. If the frequency of A-players at $t = 0$ is less than that of the unstable equilibrium shown in equation 1.7, $x(0) < x^*$, the system will converge to only B. If $x(0) > x^*$, the system will converge to only A.

(iv) if $a < c$ and $b > d$, A and B stably coexist. Here, the highest fitness is obtained by playing the opposite strategy as the opponent does. The equilibrium at x^* is in this case stable.

(v) If $a = c$ and $b = d$, A and B are neutral. If two players where to face each other, regardless of the choice of strategy, both players would end up with the same payoff.

1.2 Prisoner's dilemma

Two criminals carry out a crime and gets arrested for it. They are taken to an interrogation room by the police and get questioned one by one, without knowing how the other criminal will act. They both have two options, remain silent or confess. If both remain silent, they get 1 year in prison each. If one remains silent and one confess, the person who remained silent gets 10 years, while the one who confessed goes free. If both confess, they get 7 years each.

Prisoner's dilemma is an example in game theory where the rational choose is to not cooperate. However, if no one cooperates, the average payoff becomes the worst. As we saw in section 1.1, this is an example where the strategy to defect dominates the strategy to cooperate. The generalised payoff matrix for Prisoner's dilemma is shown in Table 1.2.

	cooperator	defector
cooperator	$b - c$	$-c$
defector	b	0

Table 1.2: Payoff matrix for Prisoner's dilemma

where b is benefit for others by remaining silent and c is the cost of remaining silent. The payoff matrix can be made non-negative, by adding c to each element. This results is the adjusted payoff matrix shown in Table 1.3

	cooperator	defector
cooperator	b	0
defector	$b + c$	c

Table 1.3: Adjusted payoff matrix for Prisoner's dilemma

For Prisoner's dilemma, the replicator equation has two fixed points. One stable equilibrium when every agent is a defector, and one unstable equilibrium when all agents are cooperators.

1.3 Hawk dove

In the Hawk dove game, two birds compete over a resource b. If a dove meets another dove, they split the resource and get $\frac{b}{2}$ each. If a dove meets a hawk, the hawk gets the whole resource and the dove gets nothing. Lastly, if a hawk meets a hawk, they have to fight for the resource with a total cost c, and they both end up with $\frac{b-c}{2}$. The payoff matrix for the Hawk dove game is shown in Table 1.4.

	Hawk	Dove
Hawk	$\frac{b-c}{2}$	b
Dove	$0,$	$\frac{b}{2}$

Table 1.4: The payoff matrix for the Hawk dove game

If $b > c$, Hawk dove shows similar dynamics as Prisoner's dilemma. It is therefore often assumed that $c > b$. If the opponent plays hawk, the highest fitness is obtained by playing dove. If the opponent plays dove, the highest fitness is obtained by playing hawk, under the assumption that $c > b$. This is equal to case (iv) in section 1.1.

The payoff matrix can be made non-negative, by adding $\frac{c}{2}$ to each element. By doing this, we get the adjusted payoff matrix shown in Table 1.5

	Hawk	Dove
Hawk	$\frac{b}{2}$	$b + \frac{c}{2}$
Dove	$\frac{c}{2},$	$\frac{b+c}{2}$

Table 1.5: Adjusted payoff matrix for the Hawk dove game

We can use expression for $\dot{x}$, shown in equation 1.7, to classify the fixed point. We have two unstable equilibriums: one when all agents are doves, and one when all agents are hawks. Furthermore, $\dot{x}$ will be equal to zero when $x = \frac{(\frac{b+c}{2})-(b+\frac{c}{2})}{(\frac{b}{2})-(b+\frac{c}{2})-(\frac{c}{2})+(\frac{b+c}{2})} = \frac{b}{c}$, where x is the frequency of hawks. Note that the classifications of the fixed points are based on the assumption that $c > b$.

1.4 Fixation probability

Evolution in finite population that is stochastic, can be described with the Morans process. In each time step, and individual is selected for reproduction, with a probability proportional to the individuals fitness. The offspring will replace an random individual in the population, which means that the population size is constant. The individuals will have the same average lifespan, but individuals who has higher fitness will have a higher chance to reproduce.

In the Moran process when the dynamics are stochastic, one has to characterise the dynamics in more ways than just listing the fixed points. One such measure is the fixation probability. Considering a population of two types of agents, A and B. The probability that all A or B takes over the population is called the fixation probability for A or B. Equation 11 in [Arn07] gives the following formula for the fixation probability.

$$\phi_k = \frac{\sum_{i=0}^{k-1} \Pi_{j=1}^{i} \frac{P_j^-}{P_j^+}}{\sum_{i=0}^{N-1} \Pi_{j=1}^{i} \frac{P_j^-}{P_j^+}} \tag{1.8}$$

Where ϕ_k is the fixation probability for agents of type A, k is the number of agents of type A at $t = 0$, N is the population size, P_j^+ is the probability that the number of agents of type A will increase from j to $j+1$, and P_j^- is the chance that the number of agents of type A will decrease from j to $j-1$. P_j^+ is given by

$$P_j^+ = \frac{(1 - w + w\pi_A)j}{(1 - w + w\pi_A)j + (1 - w + w\pi_B)(N - j)} \cdot \frac{N - j}{N} \tag{1.9}$$

where π_A is the fitness for an agent of type A, π_B is the fitness for an agent of type B and w is a weight factor, that has to do with weak and strong selection. P_j^- is given by

$$P_j^- = \frac{(1 - w + w\pi_B)(N - j)}{(1 - w + w\pi_A)j + (1 - w + w\pi_B)(N - j)} \cdot \frac{j}{N}. \tag{1.10}$$

Note that $(N - j)$ is the amount of agents of type B. The first part of $P_j^{+/-}$ is the chance that an agent of type A or B will be chosen for reproduction, and the second part is the chance that an agent of type B or A will be removed.

To calculate the fixation probability, the fraction $\frac{P_j^-}{P_j^+}$ has to be calculated. This is done by dividing equation 1.10 by equation 1.9.

$$\frac{P_j^-}{P_j^+} = \frac{1 - w + w\pi_B}{1 - w + w\pi_A} \tag{1.11}$$

1.5 Spatial effects

Changing the environment to spatial from non-spatial creates interesting effects. In the book Evolutionary Dynamics - Exploring the Equations of Life [Now06], spatial effects in Prisoner's dilemma are explored. Note that their game setup is not the same as in this report. It does however give an indication of what can happen in the spatial case.
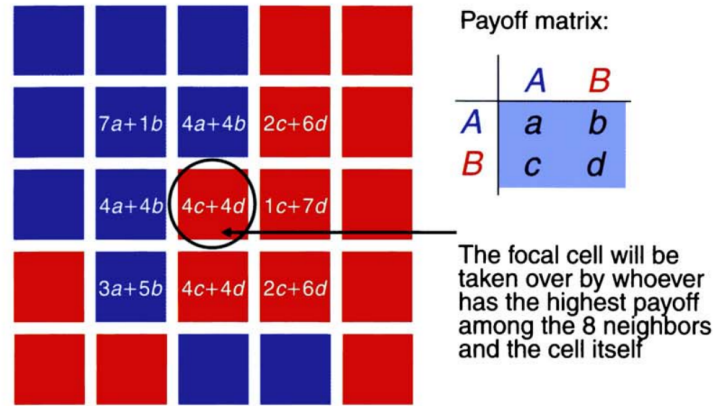
Figure 9.1 The rules of spatial games. Each cell plays the game with all of its neighbors. In this example, we use a square lattice and the Moore neighborhood, where each cell has 8 neighbors. The payoff for each player is evaluated. Subsequently each player compares its own payoff with that of its neighbors and adopts the strategy of whoever has the highest score. The fate of each cell depends on the state of all 25 cells in the 5 × 5 square that is centered around the cell.

Figure 1.1: Picture taken from the book "Evolutionary Dynamics - Exploring the Equations of Life", [Now06].

In figure 1.1, we can see a brief explanation of their spatial model. The model consists of a grid, where each individual interacts with its 8 neighbors. They go on to create some simulations of this. For the simulations, they use the following payoff matrix

$$
\begin{array}{c c c}
 & C & D \\
C & 1 & 0 \\
D & b & \epsilon
\end{array}
$$

where C is the strategy to cooperate, D is the strategy to defect and ϵ is a small value greater then zero. They do simulations for different values of b, on a 100 by 100 grid. For $b < 1.55$, a majority of the cells played C and for $b > 1.65$, a majority played D.

This result is interesting, because we saw in the non-spatial case that the strategy to defect dominates the strategy to cooperate. In the spatial case however, we saw that in some cases, the best strategy is to cooperate.

2

Method

2.1 Simulation setup

In our spatial model, every individual is placed on a two dimensional plane. This plane is a square, with the side lengths 1. It is not possible for an individual to be outside this square. Before the Morans process is started, every individual is given a position in the square. The positions are chosen uniformly at random for each individual.

We will now introduce all the model variables. There are two variables that determine the outcome of an interaction on a given distance. For Prisoner's dilemma, these are benefit(b) and cost(c), where $b > c$. The payoff for the different scenarios in Prisoner's dilemma are shown in Table 1.3. For Hawk dove, the two variables are the value of the resource(b) and the cost of fight(c), where $c > b$. The payoff for the different scenarios are shown in Table 1.5.

The variable that determines how far away an agent places its offspring is called d_{RD}, short for replicate distance. d_{RD} can vary between 0 and $\frac{1}{\sqrt{2}}$, because the furthest distance from an individual placed at $(0.5, 0.5)$ is $\frac{1}{\sqrt{2}}$. The angle that the offspring is placed at is uniformly random chosen over the angles that puts the offspring in the 1×1 square.

d_L is a variable that determines how much an individual interacts with other individual. An interactions effect on an individuals fitness is given by $a \cdot e^{\frac{d}{d_L}}$, where d is the euclidean distance between the two individuals in the interaction, and a is the outcome taken from the relevant payoff matrix.

In each iteration of the Morans process, every individual is moved from its position with a euclidean distance d_{RM}, short for random movement. As for d_{RD}, d_{RM} can vary between 0 and $\frac{1}{\sqrt{2}}$, and the angle is random chosen for each individual so that the individual is in the 1×1 square.

2.2 The Morans process

When a population has been initialised the Morans process will be executed in iterations. It will run until the max number of iterations is reached, or until all individuals are of the same type. the Morans process consists of 3 steps.

i) An individual is removed. The individual that will be removed is chosen randomly. This is shown in figure 2.1, where a population of 11 individuals has been initialised, and one of the individuals has been chosen to be removed.
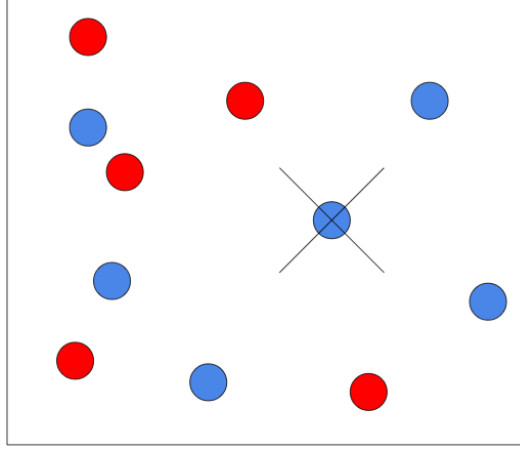
Figure 2.1: Step 1 in the the Morans process. The blue dots represents agents with strategy A, and the red dots represents agents with strategy B. The agent with the cross has been chosen to be removed.

ii) An individual is replicated. The probability that a individual is replicated is equal to that individuals fitness divided by the total fitness of the population. This probability can be written as

$$\frac{1 - w + w \cdot \Pi_k}{\sum_{i=1}^{N}(1 - w + w \cdot \Pi_i)} \tag{2.1}$$

where N is the population size and Π_k is the payoff for agent k. Π_k can be written as

$$\Pi_k = \sum_{i=1, i \neq k}^{N} e^{\frac{-d_{ki}}{d_L}} p_{ki} \tag{2.2}$$

where d_{ik} is the distance between agent i and agent k and p_{ki} is the payoff for agent k while interacting with agent i.

The replicated individual is placed at an euclidean distance d_{RD} from the individual that was chosen for replication, and the replicated individual will also be of the same type as the individual that was chosen for replication. The angle at which the replicated individual is placed from the individual chosen for replication is chosen uniformly at random, between all angles that will place the new individual in the one by one square. This is shown in figure 2.2.
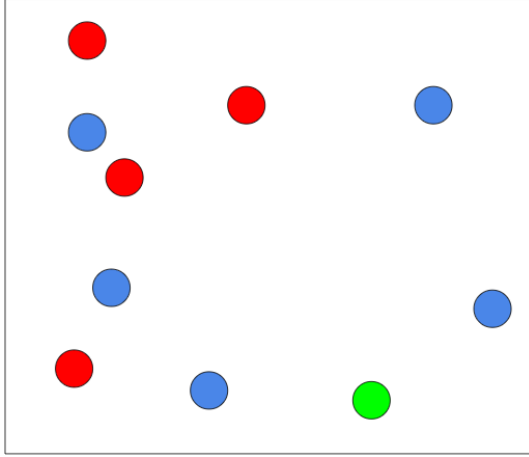

(a) Step 2a. The green agent has been chosen for replication.
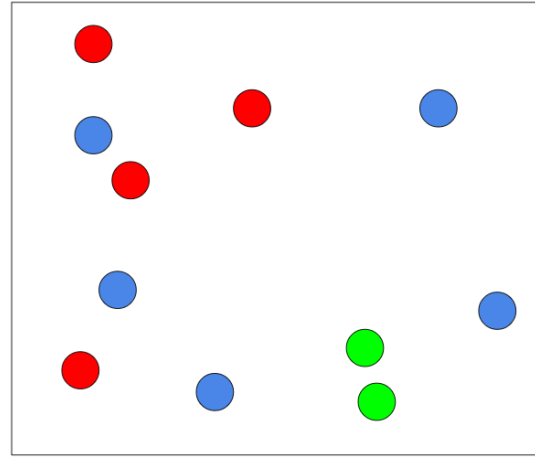
(b) Step 2b. The green agent replicates

Figure 2.2: Step 2. A previous blue agent, green in the figures, has been chosen for replication. This agent replicates itself and creates another blue agent at the euclidean distance d_{RD} from itself.

iii) This step only happens if the variable determining the random movement is not equal to zero, $d_{RM} \neq 0$. Then all individuals are moved at an euclidean distance d_{RM}, and as above, the angle is random. This is shown in figure 2.3.
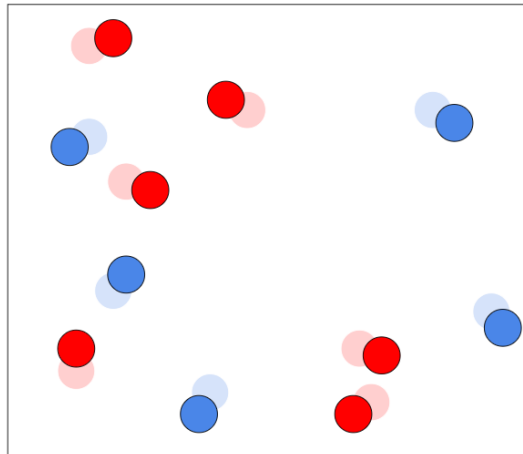

Figure 2.3: Step 3 in the Morans process. Every agents moves a euclidean distance d_{RM}. The faded circles represents the previous positions.

2.3   Fixation time

The fixation time is defined as the number of iterations in the Morans process it takes until all agents are of the same type. The fixation time is important in multiple ways. One important thing is that the time it takes to run the simulations increases approximately linearly with the fixation time. Because of this, there is a limit of 10^4 iterations on all simulations.

2.4 Approximation

In this section we attempt to construct an approximation of the spatial Moran process. We proceed by approximating the interaction function by a piecewise constant function. Instead of previous $e^{\frac{-d}{d_L}}$, it can be written as a constant I_N for $d \leq n$ and as another constant I_f for $d > n$. n is chosen such as $e^{\frac{n}{d_L}} = 0.5 \rightarrow n = -ln(0.5) \cdot d_L$
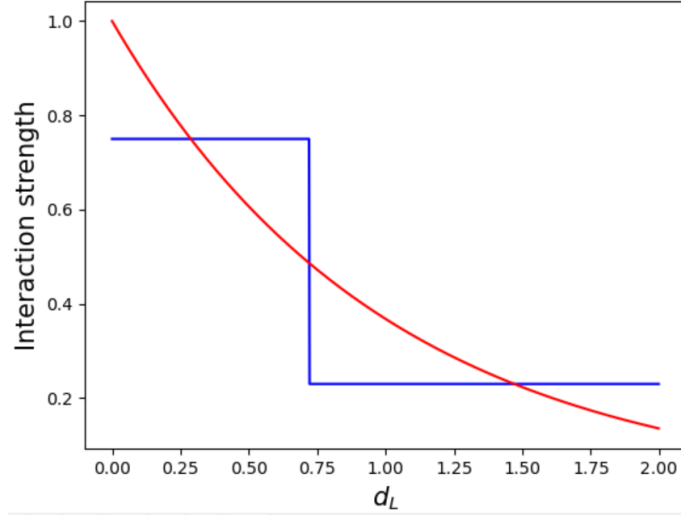
Figure 2.4: The red curve is interaction strength on the form $e^{\frac{-d}{d_L}}$. The blue line is the approximation of the interaction strength.

I_n is chosen as the average over the interval $[0, n]$. The average can be can be calculated to

$$I_n = \frac{\int_0^n e^{-\frac{d}{d_L}} dd}{n} = \frac{-0.5}{ln(0.5)}. \tag{2.3}$$

Calculating I_f introduces a new problem, which is deciding an upper limit on the distance when taking the average. The maximum value of the upper limit that would be reasonable is $\sqrt{2}$, as that is the largest possible distance between any two agents in the one by one square. However, as the cases where the distance is close to $\sqrt{2}$ are rare, a better result might be acquired by taking an upper limit less than $\sqrt{2}$. For simplicity reasons, the upper limit has been chosen to 1 in this report. I_f then becomes

$$I_f = \frac{\int_n^1 e^{-\frac{d}{d_L}} dd}{1 - n} = \frac{d_L}{1 - n}(e^{-\frac{n}{d_L}} - e^{-\frac{1}{d_L}}). \tag{2.4}$$

Given two types of agents, A and B, the average fitness $\bar{\pi}$ for an agent of type A and B can be written as in equation 2.5 and equation 2.6. π in the formulas stand for payoffs from the payoff matrices. $\bar{N}_{ij}$ denotes the average number of j agents

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close to an i agent, where close means a euclidean distance less than or equal to n. $\bar{F}_{ij}$ denotes the average number of j agents far from an i agent, where far means a euclidean distance greater than n. The average payoffs can now be written:

$$\bar{\pi}_A = I_n(\pi_{AA}\bar{N}_{AA} + \pi_{AB}\bar{N}_{AB}) + I_f(\pi_{AA}\bar{F}_{AA} + \pi_{AB}\bar{F}_{AB}) \tag{2.5}$$

$$\bar{\pi}_B = I_n(\pi_{BB}\bar{N}_{BB} + \pi_{BA}\bar{N}_{BA}) + I_f(\pi_{BB}\bar{F}_{BB} + \pi_{BA}\bar{F}_{BA}). \tag{2.6}$$

In equation 2.5 and equation 2.6, there are eight unknown variables. By introducing two new variables, A and B, which are the number of agents of respective type, we can come up with the following equations. $N_{AA}+F_{AA} = A - 1$, because the average number of close agents added to the average number of agents far away, has to be equal to the number of agents at any distance, minus one, as the observed agent has to be excluded. For B, the same argument can be used, $N_{BB}+F_{BB} = B - 1$. Also, we can obtain $N_{AB}+F_{AB} = B$, $N_{BA}+F_{BA} = A$, with similar reasoning, but the minus one is gone, because we have two different types of agents. Lastly, we have that $\frac{N_{AB}}{B} = \frac{N_{BA}}{A}$.

In conclusion, we will end up with three unknown variables, which for example can be N_{AA}, N_{BB} and N_{AB}. These variables will be approximated for different configurations in the Result section.

3
Result

In this section we present results on how the model parameters affect the dynamics of the model. An example of a simulation of the spatial Moran process is shown in 3.1, where we can see how to agents have clustered together in the plane.
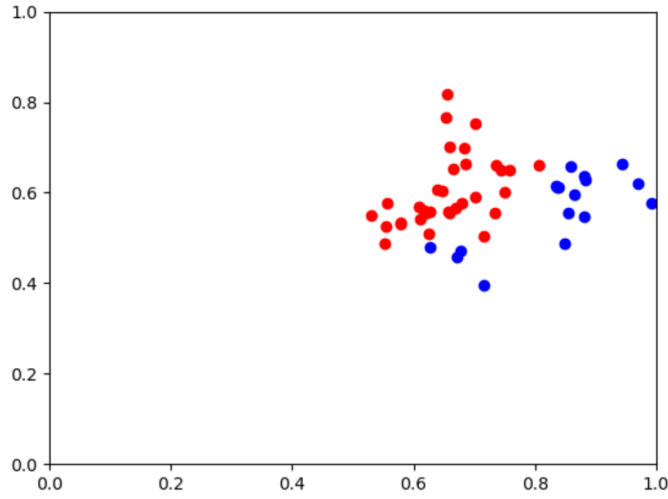
Figure 3.1: the Morans process after 7000 iterations, for Prisoner's dilemma. The red dots represents defectors, and the blue dots represents cooperators.

3.1 The fixation probability depends on d_L, d_{RD} and d_{RM}

In this section, results are shown on how the fixation probability and long term frequency depend on the variables d_L, d_{RD} and d_{RM}, for Hawk dove and Prisoner's dilemma. This is done by simulating the model for 10^4 iterations, 1000 times for each data point and then taking the average. The limit of 10^4 iterations had to be set, as it would take to long time to generate the result otherwise. If every run finishes under 10^4 iterations, the measurement is labeled as fixation probability. Otherwise, it is labeled as long term frequency.

The values for the variables that were constant were chosen so that the fixation probability and long term frequency would change significantly. The values for Prisoner's dilemma are show in table 3.1, and the values for Hawk dove are shown in table 3.2.

notation	value	meaning
d_L	1	exponential constant
d_{RD}	0.05	replicate distance
d_{RM}	0	random movement
b	30	benefit
c	1	cost
w	1	weight factor
N_c	25	start amount of cooperators
N_d	25	start amount of defectors

Table 3.1: The base case of variables for Prisoner's dilemma

notation	value	meaning
d_L	1	exponential constant
d_{RD}	0.05	replicate distance
d_{RM}	0	random movement
b	1	benefit
c	2	cost
w	1	weight factor
N_d	25	start amount of doves
N_h	25	start amount of hawks

Table 3.2: The base case of variables for Hawk dove

An exponential curve has also been fitted to the data points for the different cases, on the form $a_1 e^{a_2 x} + a_3$.
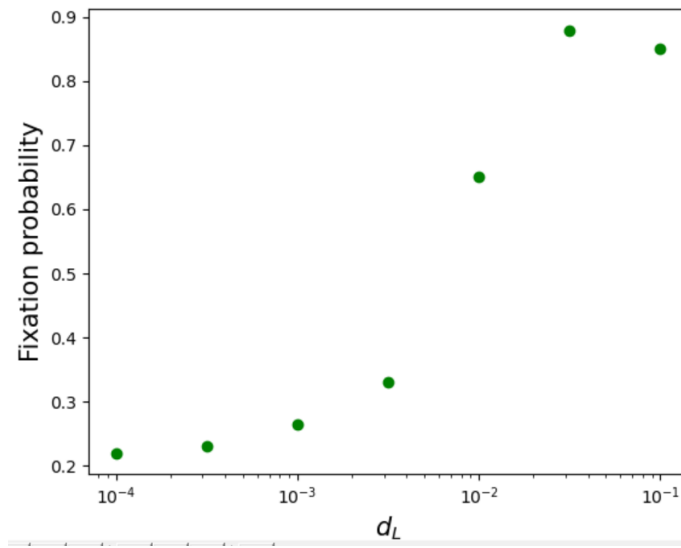
3.1.1 Prisoner's dilemma

For Prisoner's dilemma, the chance that the Morans process will converge before 10^4 iterations can be approximated to one (see appendix A.1). The y-axis in the results shown in this section is therefore labeled as a fixation probability. The constant a_3 is set to the fixation probability given by equation 1.8. The reason for this is that when any of the parameters d_L, d_{RD} or d_{RM} is infinity, the positions of the agents does not matter, so it is similar to the non-spatial case.

Below in this section, it is shown how the fixation probability depends on the parameters d_L, d_{RD} and d_{RM}. For d_L, an additional figure with logarithmic scale has been created, to see how small values of d_L affects the fixation probability. Nothing special happens when d_{RD} and d_{RM} are low, so the additional plot has not been created for those parameters.
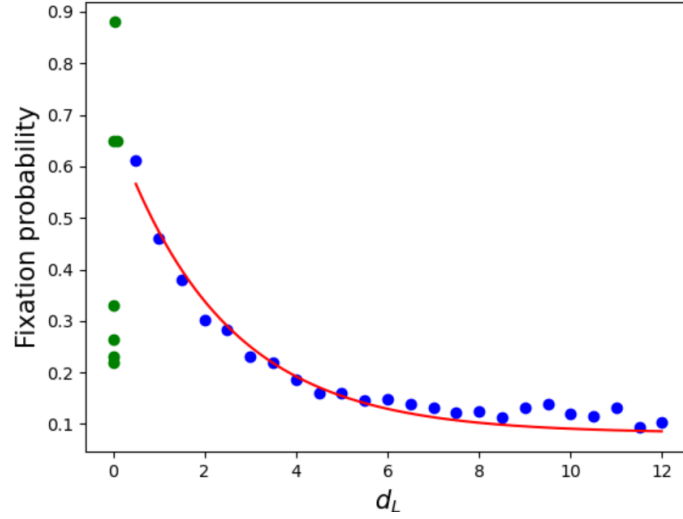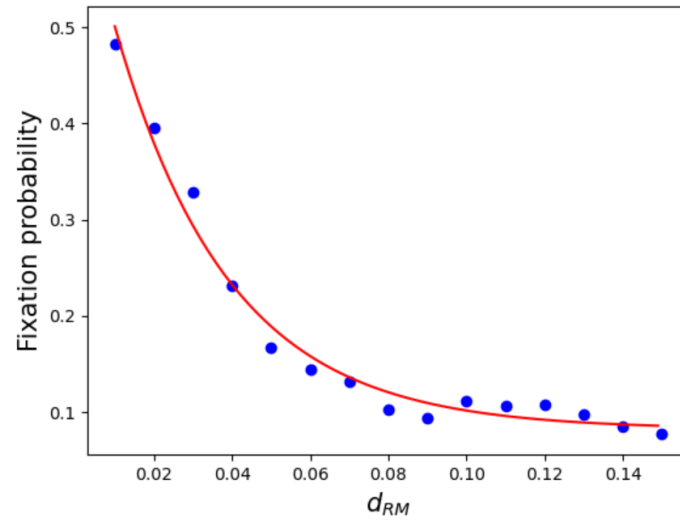

Figure 3.2: The blue dots is Prisoner's dilemma with the parameters: $b = 30$, $c = 1$, $d_{RM}=0$, $runs = 1000$, $d_{RD}=0.05$, iterations=10^4, $w = 1$, $N_c = 25$, $N_d = 25$. The red line is a exponential fitted exponential curve on the form $a_1 e^{a_2 x} + a_3$. $a_1 = 0.597$, $a_2 = -0.423$, $a_3 = 0.0823$.

Figure 3.3: The green dots from figure 3.3 in log scale
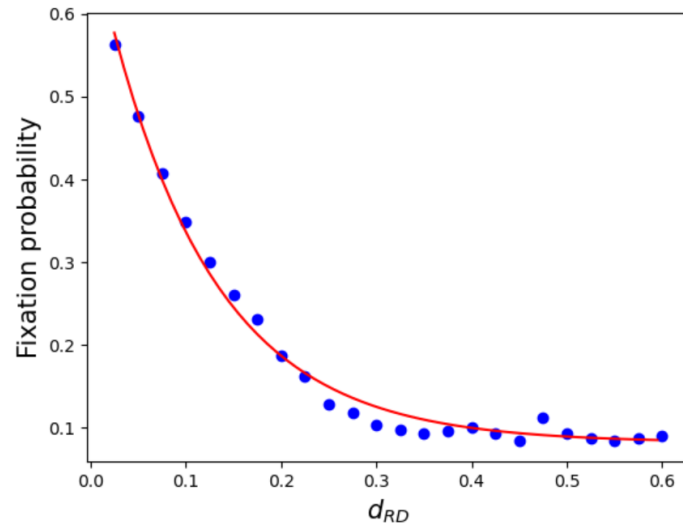
**Figure 3.4:** The blue dots is Prisoner's dilemma with the parameters: $b = 30$, $c = 1$, $d_{RM}=0$, $runs = 1000$, $d_L=1$, iterations=10^4, $w = 1$, $N_c = 25$, $N_d = 25$. The red line is a exponential fitted exponential curve on the form $a_1 e^{a_2 x} + a_3$. $a_1 = 0.617$, $a_2 = -8.861$, $a_3 = 0.0823$.

Figure 3.5: The blue dots is Prisoner's dilemma with the parameters: $b = 30$, $c = 1$, $d_{RD} = 0.05$, $runs = 1000$, $d_L=1$, iterations=10^4, $w = 1$, $N_c = 25$, $N_d = 25$. The red line is a exponential fitted exponential curve on the form $a_1 e^{a_2 x} + a_3$. $a_1 = 0.589$, $a_2 = -34.14$, $a_3 = 0.0823$.

3.1.2 Prisoner's dilemma with weak selection

In this section, the result from section 3.1.1 has been replicated, but with $w = 0.1$ and $w = 0.01$. Note that the fixation probability is quite similar to when $w = 1$.
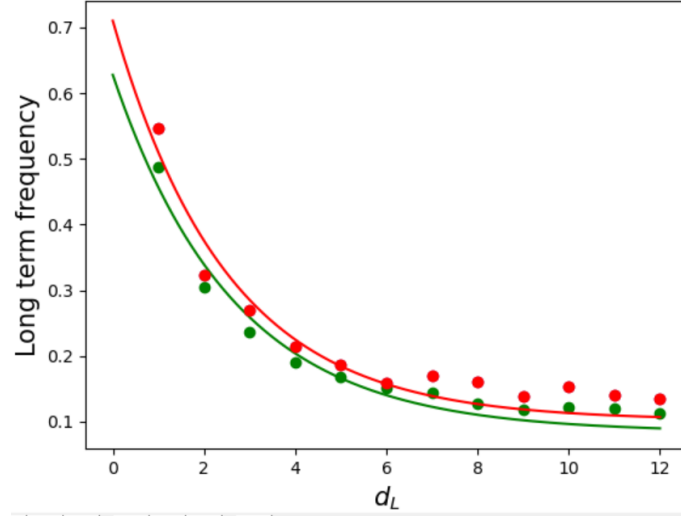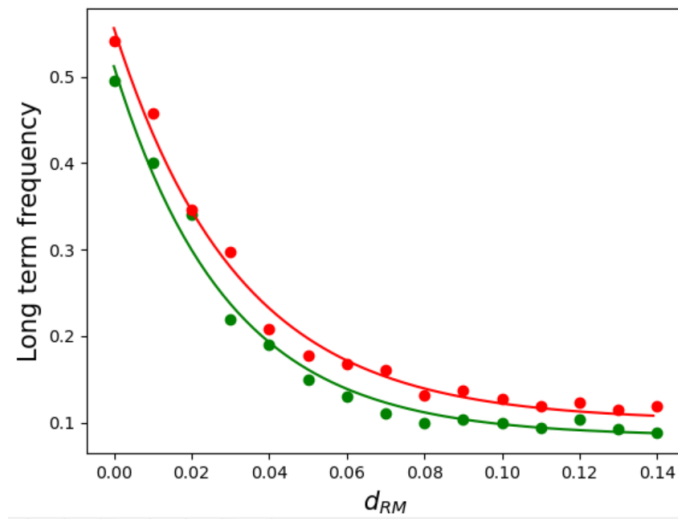

Figure 3.6: Same setup as for in figure 3.3, but $w = 0.1$ for the green line, and $w = 0.01$ for the red line. The green fitted curve has the parameters $a_1 = 0.6082$, $a_2 = -0.3995$, $a_3 = 0.0840$ and the red line has the parameters $a_1 = 0.5438$, $a_2 = -0.3780$, $a_3 = 0.1019$.
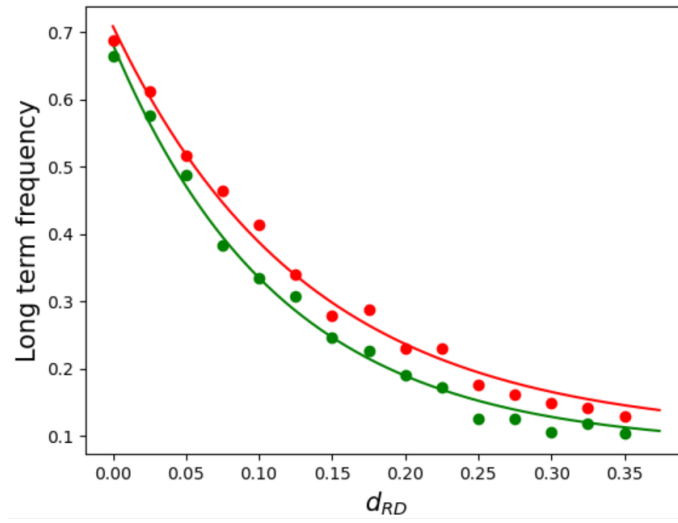
Figure 3.7: Same setup as for in figure 3.8, but $w = 0.1$ for the green line, and $w = 0.01$ for the red line. The green fitted curve has the parameters $a_1 = 0.5982$, $a_2 = -8.6882$, $a_3 = 0.0840$ and the red line has the parameters $a_1 = 0.6072$, $a_2 = -7.5324$, $a_3 = 0.1019$.

Figure 3.8: Same setup as for in figure 3.8, but $w = 0.1$ for the green line, and $w = 0.01$ for the red line. The green fitted curve has the parameters $a_1 = 0.4281$, $a_2 = -34.2487$, $a_3 = 0.0840$ and the red line has the parameters $a_1 = 0.4542$, $a_2 = -31.2254$, $a_3 = 0.1019$.
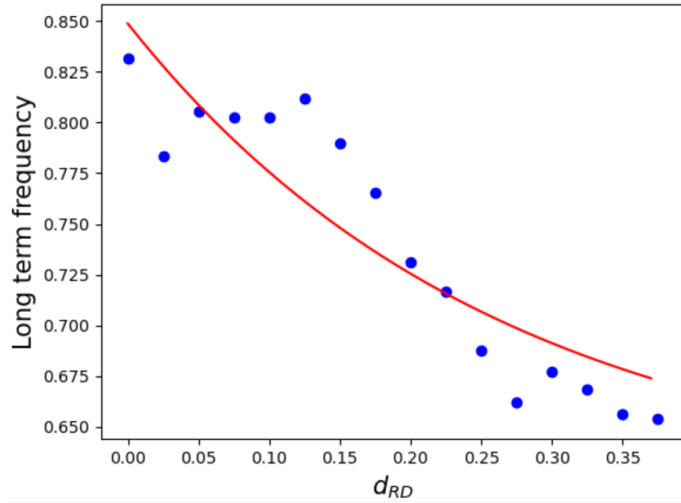
3.1.3 Hawk dove

For Hawk dove, the long term frequency is measured by the average amount of doves after 10^4 iterations of the Morans process. The reason that long term frequency is measured instead of fixation probability, is that the fixation time is much higher, because of the dynamics of Hawk dove. The constant a_3 is obtained by setting $d_L = \infty$. By averaging over 2000 runs, the long term frequency was given to $a_3 = 0.62078$. Using equation 1.8 to calculate the fixation probability, 0.814 is obtained for the non-spatial case. The reason these two values differ, is that 0.62078 is calculated with the limit of 10^4 iterations. After 10^4 iterations, only 40.8% has converged to either only doves or only haws. Looking at only the 40.8% that converged, the fixation probability is 0.819, which is close to the value obtained from equation 1.8.
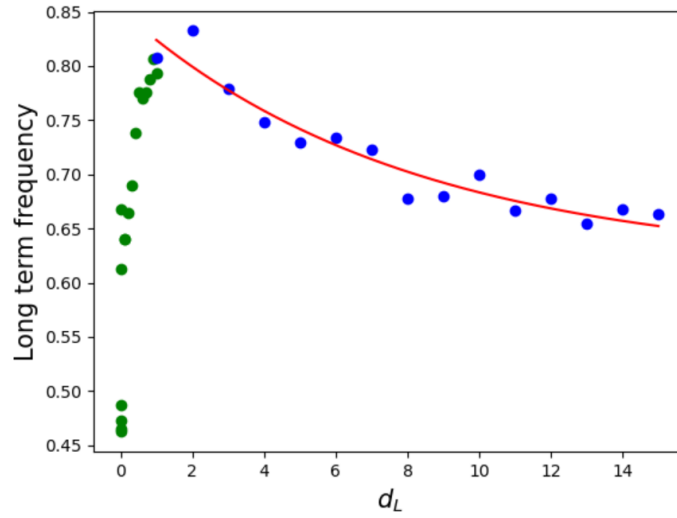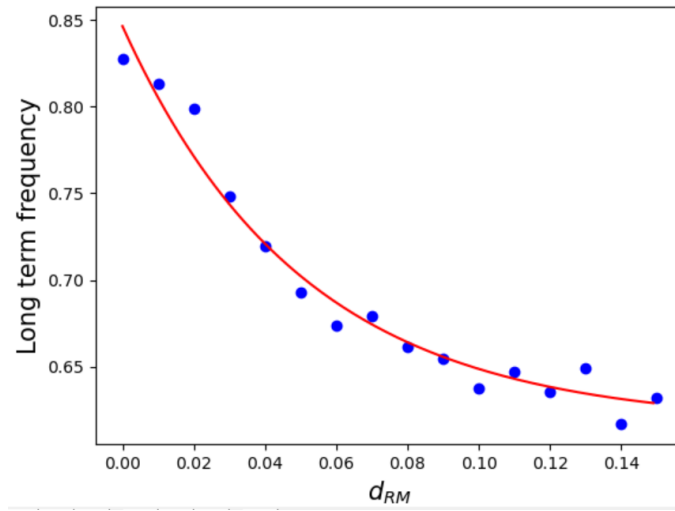

Figure 3.9: The blue dots are Hawk dove for $d_L \geq 1$ and the green dots are Hawk dove for $d_L \leq 1$. The parameters are: $b = 1$, $c = 2$, $d_{RM}=0$, $runs = 1000$, $d_{RD}=0.05$, iterations$=10^4$, $w = 1$, $N_d = 25$, $N_h = 25$. The red line is an exponential fitted exponential curve to the blue dots on the form $a_1 e^{a_2 x} + a_3$. $a_1 = 0.233$, $a_2 = -0.131$, $a_3 = 0.62078$.
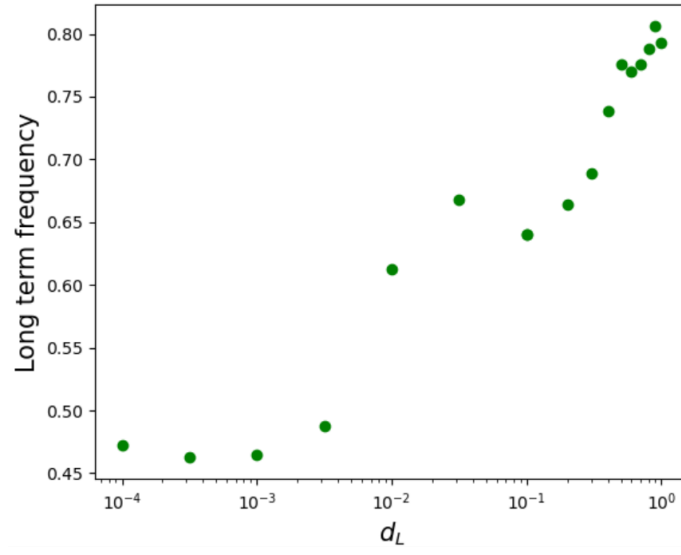
Figure 3.10: The green dots from figure 3.10 in log scale

Figure 3.11: The blue dots are Hawk dove with the parameters: $b = 1$, $c = 2$, d_{RM}=0, runs=1000, d_L=1, iterations=10^4, $w = 1$, $N_d = 25$, $N_h = 25$. The red line is an exponential fitted exponential curve to the blue dots on the form $a_1 e^{a_2 x} + a_3$. $a_1 = 0.228$, $a_2 = -3.898$, $a_3 = 0.62078$.

Figure 3.12: The blue dots are Hawk dove with the parameters: $b = 1$, $c = 2$, d_{RM}=0, runs=1000, d_L=1, iterations=10^4, $w = 1$, $N_d = 25$, $N_h = 25$. The red line is an exponential fitted exponential curve to the blue dots on the form $a_1 e^{a_2 x} + a_3$. $a_1 = 0.227$, $a_2 = -20.666$, $a_3 = 0.62078$.

3.2 Fixation time

The fixation time is measured for both Prisoner's dilemma and Hawk dove, as function of d_L, d_{RD} and d_{RM}. Also, the probability that the the iterations will be below 10^4 is measured. It turned out that the fixation time for Prisoner's dilemma is around 850 iterations in average, almost regardless of the parameters d_L, d_{RD} and d_{RM}. The only exception is when d_L is low, where the fixation time goes down to 200 iterations for the measured cases. The result for fixation time for Prisoner's dilemma is therefore put in appendix A.1.
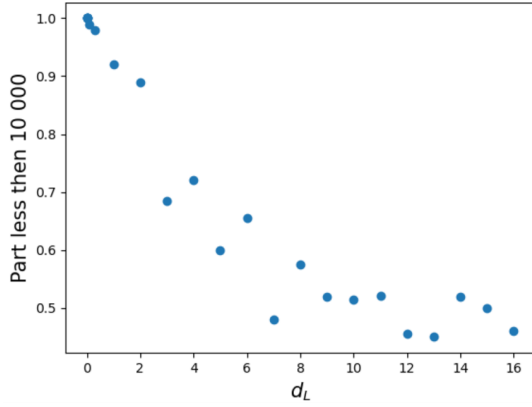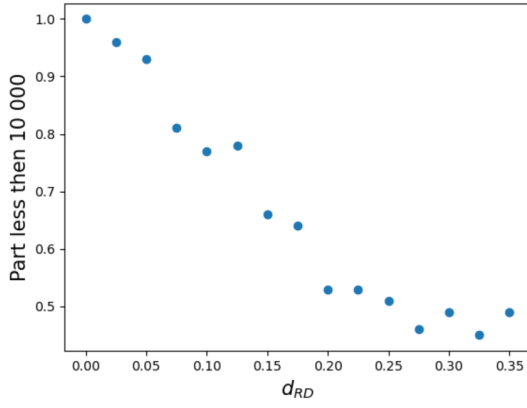
(a) The fraction that converge before 10^4 iterations, as a function of d_L
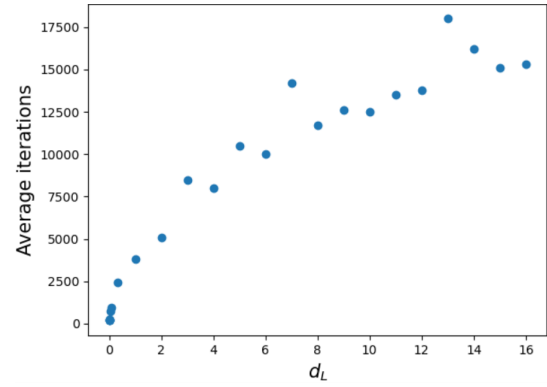
(b) The average amount of iterations as a function of d_L

Figure 3.13: The results in the two figures comes from the same measurement of Hawk dove. The parameters are: $b = 1$, $c = 2$, $d_{RM}{=}0$, $runs = 100$, $d_{RD}{=}0.05$, $w = 1$, $N_d = 25$, $N_h = 25$.

(a) The fraction that converge before 10^4 iterations, as a function of d_{RD}

(b) The average amount of iterations as a function of d_{RD}

Figure 3.14: The results in the two figures comes from the same measurement of Hawk dove. The parameters are: $b = 1$, $c = 2$, d_{RM}=0, $runs = 100$, d_L=1, $w = 1$, $N_d = 25$, $N_h = 25$.

(a) The fraction that converge before 10^4 iterations, as a funtion of d_{RM}
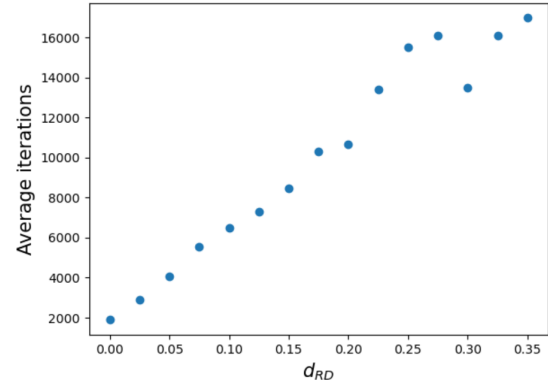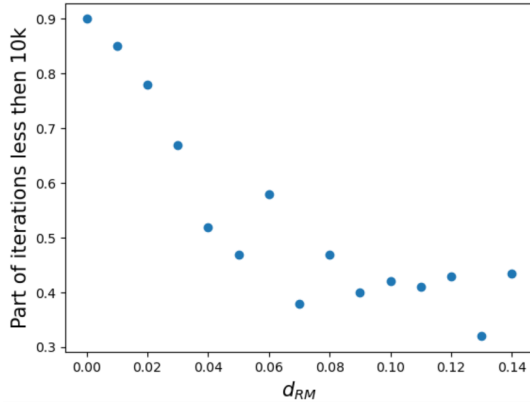
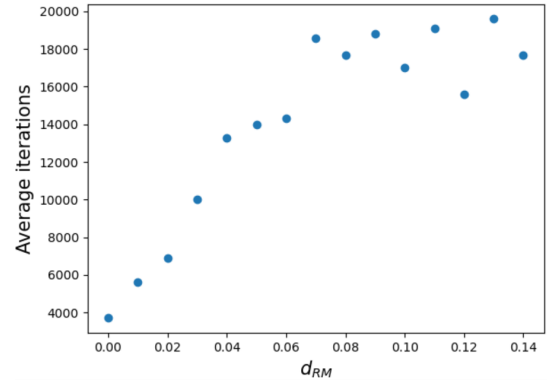**(b)** The average amount of iterations as a function of d_{RM}

Figure 3.15: The results in the two figures comes from the same measurement of Hawk dove. The parameters are: $b = 1$, $c = 2$, d_{RD}=0.05, $runs = 100$, d_L=1, $w = 1$, $N_d = 25$, $N_h = 25$.

3.3   Approximation

In section 2.4, we concluded that with N_{AA}, N_{BB} and N_{AB} can be numerically approximated. With the values of these three variables, it is possible to simulate the dynamics much faster than in the spatial Morans proces. It turns out that the average over many runs of the fractions $\frac{N_{AA}}{A-1}$, $\frac{N_{BB}}{B-1}$ and $\frac{N_{AB}}{B}$, can be approximated quite well by a piecewise linear function. An example of this is shown in figure 3.16.
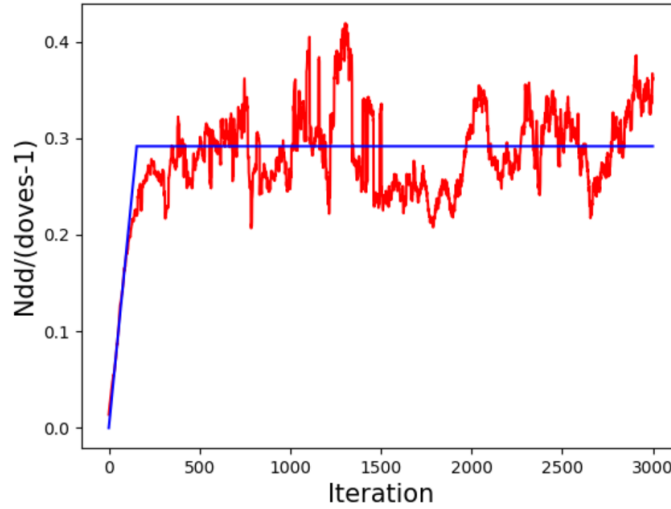
Figure 3.16: The red curve is the average $\frac{N_{DD}}{D-1}$ over 50 runs. The blue lines are two different approximated lines to the measured average.

A line can be written on the form $A_1 x + B_2$. For the first line in figure 3.16, the one with the high slope, $B_2 = 0$. For the other line in figure 3.16, $A_1 = 0$.

The two lines has to be approximated for each set of parameters. The values of the lines has been approximated for some sets of parameters values. The result of this is shown in table 3.3 and table 3.4. In each case in the tables, all the variables are the same as in section 3.1, except for the change written in the tables. The variable n that determines the interaction strength is not written in that section however. For all the cases, n has been set to $n = -ln(\frac{1}{2})d_L$.

In the tables, N^1 corresponds to A_1 for the first line and N^2 corresponds to B_2 for the second line. AFP is the approximated fixation probability for simulations made with the approximate method. MFP is the measured fixation probability calculated with the spatial Morans process. An important thing to note from the tables below, is that the approximated fixation probability does not match the measured fixation probability that well.

setup	N_{dd}^1	N_{dd}^2	N_{hh}^1	N_{hh}^2	N_{dh}^1	N_{dh}^2	AFP	MFP
$d_L = 0.1$	0.0013	0.2807	0.0010	0.3217	0.0004	0.1777	0.998	0.640
$d_L = 0.3$	0.0025	0.8037	0.0024	0.8250	0.0010	0.6113	0.967	0.738

Table 3.3: Measured values for Hawk dove.

setup	N_{cc}^1	N_{cc}^2	N_{dd}^1	N_{dd}^2	N_{cd}^1	N_{cd}^2	AFP	MFP
$d_L = 0.1$	0.0024	0.2894	0.0014	0.3369	0.0005	0.2217	0.998	0.88
$d_L = 0.01$	0.00007	0.0051	0.00003	0.0059	0.00001	0.0026	0.900	0.68
$d_L = 0.3$	0.0021	0.8092	0.0017	0.6864	0.0013	0.5437	0.835	0.815

Table 3.4: Measured values for Prisoner's dilemma.

3.4 Special case of small d_L and small d_{RD}

When d_L and d_{RD} are small enough, the fixation probability can be determined largely by looking at the fitness at $t = 0$. The reason is, that at $t = 1$, the agent that was chosen for replication, and the replicated agent, will have very high fitness. Then at $t = 2$, the chance that one of those two individuals are chosen for replication is almost 100%. The same argument can be made for every upcoming iteration, which means that the individual type that are replicated at $t = 0$ will be the only type after enough iterations. This phenomenon is shown in figure 3.17.

Figure 3.17 is made for Hawk dove, but the phenomenon holds true for Prisoner's dilemma as well. The x axis in figure 3.17 is initial fitness fraction for doves, which is the part of the total fitness that comes from the type that the fixation probability is measured for, which in figure 3.17 is doves.
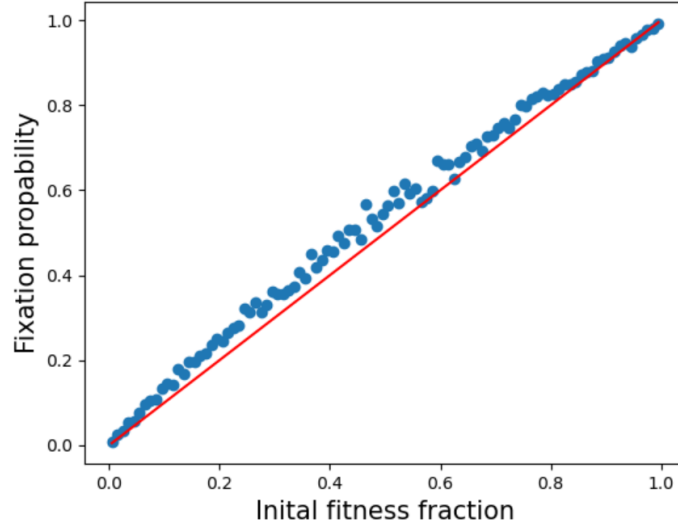
Figure 3.17: On the x-axis is the part of the total fitness that comes from doves at $t = 0$. The y-axis is the fixation probability for doves. The red line, initial fitness fraction equal to fixation probability, is just for reference. The parameters are $b = 1$, $c = 2$, total runs$=10^6$, $d_L = 0.00434$, $d_{RD} = 0.0001$, $d_{RM} = 0$, $N_d = 25$, $N_h = 25$.

Denoting the initial fitness fraction for strategy A, F_0^A, we can can write F_0^A as

$$F_0^A = \frac{\sum_{i=1}^{N_A} \Pi_i^A}{\sum_{i=1}^{N_A} \Pi_i^A + \sum_{i=1}^{N_B} \Pi_i^B} \tag{3.1}$$

where N_A is the number of agents of type A at $t = 0$, N_B is the number of agents of type B at $t = 0$, Π_i^A is the payoff for for agent i of type A at $t = 0$ and Π_i^B is the payoff for for agent i of type B at $t = 0$.

A way to measure how strong the initial state is related to the fixation probability, is to measure the mean square error(MSE) between the blue dots and the corresponding initial fraction fitness on the red line, shown in figure 3.17. MSE is calculated for different combinations of d_L and d_{RD} in the following two subsections.

Another interesting thing occurs when d_L is small enough. The initial fitness of the two agents who are closest to each other will be approximately equal to the total fitness of all agents. A way to measure this effect is to measure the chance that the total fitness of one of the agent types is more than 90% of the total fitness, or less than 10%. This has been done, and is shown for both Prisoner's dilemma and Hawk dove, in figure 3.18.
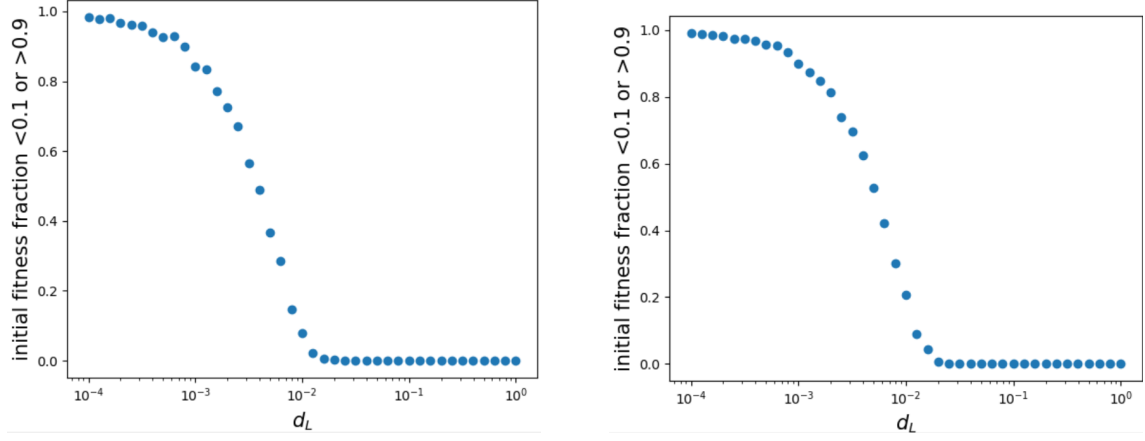

(a) The chance that the initial fitness for all doves will be less than 10% or more than 90% of the total fitness as a function of d_L.

(b) The chance that the initial fitness for all cooperates will be less than 10% or more than 90% of the total fitness as a function of d_L.

Figure 3.18: Figure (a) represents Hawk dove and figure (b) represents Prisoner's dilemma. For Hawk dove, the parameters are $b = 1$, $c = 2$, $runs = 1000$, $w = 1$. For Prisoner's dilemma, the parameters are $b = 30$, $c = 1$, $runs = 1000$, $w = 1$. In both cases, there are 25 agents of each type.

3.4.1 Hawk dove
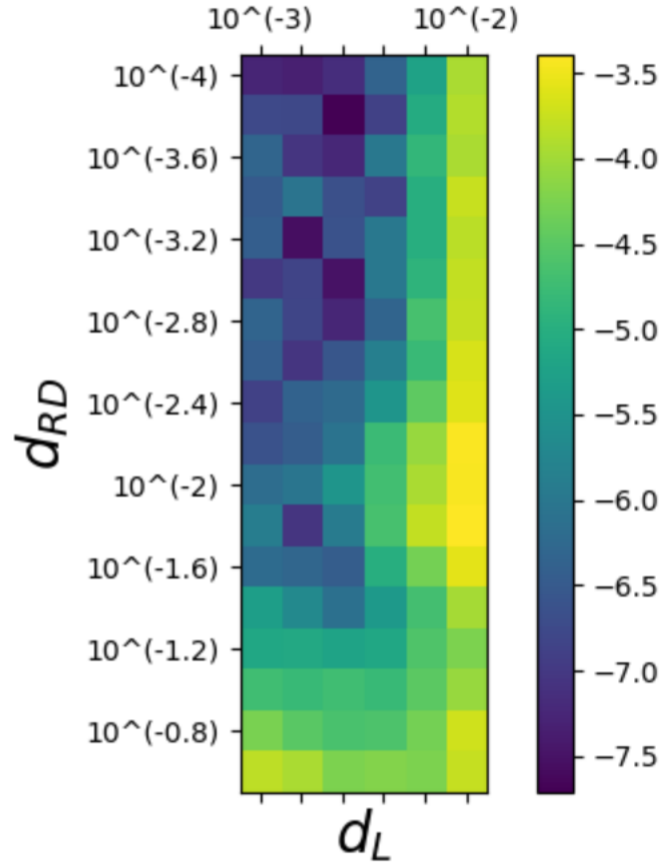
Figure 3.19: Each square represents the log(MSE) for different combinations of d_L and d_{RD}. The axes are logarithmic, where d_L range from 10^{-3} to 10^{-2}, and d_{RD} range from 10^{-4} to $10^{-0.6}$. The parameters are: $b = 1$, $c = 2$, d_{RM}=0, $runs = 10^4$, iterations=10^4, $N_d = 25$, $N_h = 25$.

In figure 3.19, log(MSE) was highest when $d_L = 10^{-2}$ and $d_{RD} = 10^{-1.8}$. It is worth noting, that even in this case, the initial condition has a big impact on the fixation probability. In other words, around log(MSE)=-3.5 as we have for these parameters, are still quite small. As a reference, if the fixation probability would be 0.5, independent on the initial condition, the log(MSE) would be -2.5. For fixation probabilities other than 0.5, log(MSE)>-2.5. The case when $d_L = 10^{-2}$ and $d_{RD} = 10^{-1.8}$ is shown in figure 3.20
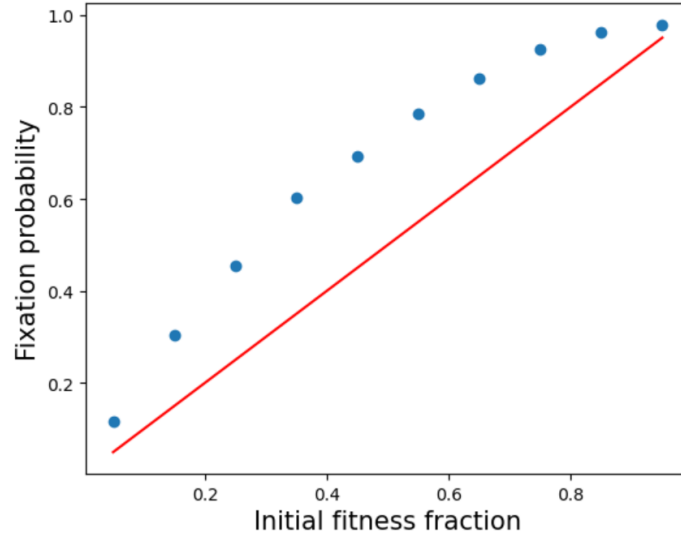
Figure 3.20: The measurement from figure 3.19 where $d_L = 10^{-2}$ and $d_{RD} = 10^{-1.8}$. The red line, initial fraction fitness equal to fixation probability, is just for reference.

3.4.2 Prisoner's dilemma
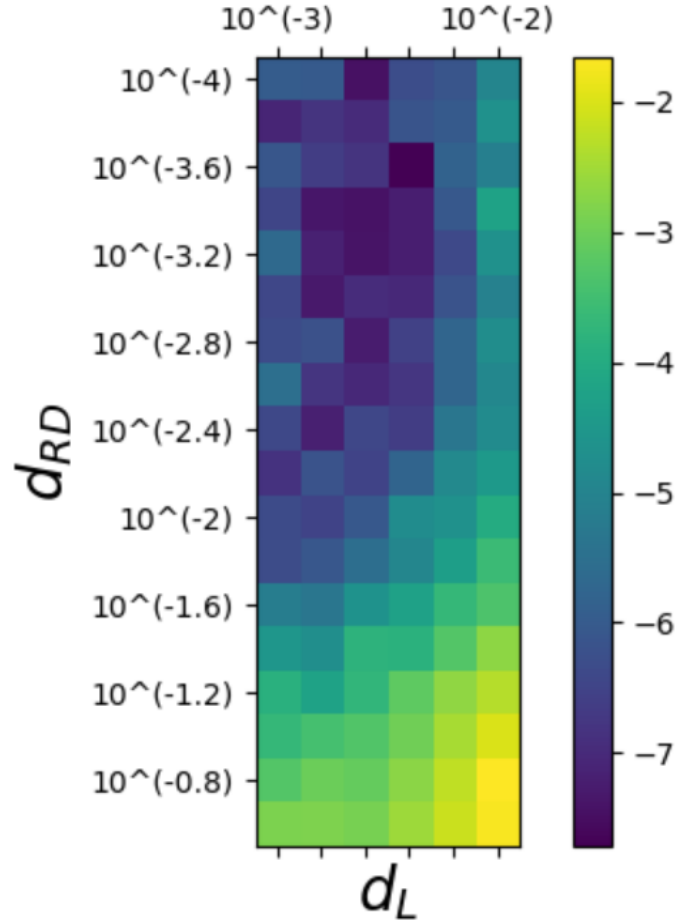

Figure 3.21: Each square represents the log(MSE) for different combinations of d_L and d_{RD}. The axes are logarithmic, where d_L range from 10^{-3} to 10^{-2}, and d_{RD} range from 10^{-4} to $10^{-0.6}$. The parameters are: $b = 30$, $c = 1$, $d_{RM}=0$, $runs = 10^4$, iterations$=10^4$, $N_c = 25$, $N_d = 25$.

In figure 3.21, log(MSE) was highest when $d_L = 10^{-2}$ and $d_{RD} = 10^{-0.8}$. As for Prisoner's dilemma, also in this case, there is a strong connection between the fixation probability and the initial fitness fraction. The case when $d_L = 10^{-2}$ and $d_{RD} = 10^{-0.8}$ is shown in figure 3.22.
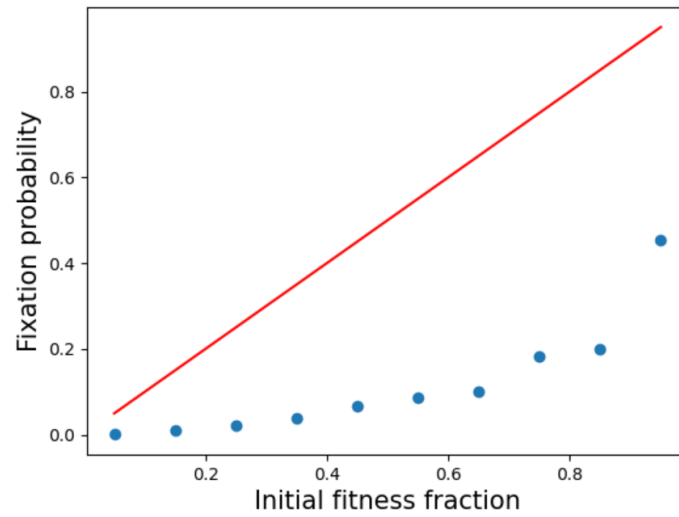
Figure 3.22: The measurement from figure 3.21 where $d_L = 10^{-2}$ and $d_{RD} = 10^{-0.8}$. The red line, initial fraction fitness equal to fixation probability, is just for reference.

4

Discussion

In the result, we presented how the fixation probability depends on the parameters d_L, d_{RD} and d_{RM}, and also how the fixation time depends on these three parameters. We proceeded with showing an approximation, with could replace the computational demanding simulation. Lastly, we showed what happens when d_L and d_{RD} are small.

4.1 The fixation probability depends on d_L, d_{RD} and d_{RM}

As d_L goes to zero, the fixation probability seems to approach 0.22 for Prisoner's dilemma, and 0.47 for Hawk dove. Our explanation to this, is that when d_L is small enough, the only thing that matters for the fixation probability, is the initial state. In this initial state, only the two closest agents have to be considered, as the payoff from that interaction will be much bigger then all other payoffs, because of the low d_L. If none of the two agents who are close to each other gets removed in the Morans process, these agents will most likely be responsible for every upcoming replication.

There are four different combinations of the closest 2 agents. For Prisoner's dilemma, these are $C-C$, $C-D$, $D-C$, $D-D$. Roughly, if $C-C$ is the closest, cooperators will fixate, and this state has approximately 25% chance of occurring. In any of the three other state, defectors will fixate. With this argument, 0.22 in fixation probability for low d_L in prisoners dilemma seams reasonable.

We could make the same argument for Hawk dove. According to the argument, when two doves are close to each other, doves will fixate. If a dove is close to a hawk, the hawk will have 4 times higher fitness then the dove. With this information, we could estimate the fixation probability for low d_L to $0.25 + 0.2 \cdot 0.5 = 0.35$, assuming that doves have 20% chance of fixating if the two closest agents are a hawk and a dove. This is quite a bit lower then the measured 0.47 fixation probability. The fixation probability for the different cases could quite easy be verified with some simulations.

While d_L increases, the positions of the agents matter less and less, with the setup equals the non-spatial case for $d_L = \infty$. This is shown in figure 3.10 and figure 3.3, where d_L is plotted against the fixation probability for Prisoner's dilemma and against long term frequency for Hawk dove.

As shown in the same figures, the fixation probability as a function d_L has a local maximum. For Prisoner's dilemma, this local maximum is measured at $d_L = 10^{-1.5}$,

and for Hawk dove it is at $d_L = 2$. To describe these local maximums, we can imagine two effects competing with each other, both lowering the fixation probability. One when d_L is low, where only the two closest agents matter, and one when d_L is high, making the dynamics as in the non-spatial case.

In the result in section 3.1, curves on the form $a_1 e^{a_2 x} + a_3$ has been fitted to the measured points. This exponential curve fits in general quite good to the measured points. However, only one of d_L, d_{RD} or d_{RM} is varied at the time. An interesting extension would be to fit a function that takes in more then one variable at the same time. This wasn't done in the report, because it requires a lot of computational power.

4.2 Special case of small d_L and small d_{RD}

With the goal to see how the initial condition impacts the fixation probability, log(MSE) was measured, shown in figure 3.19 and figure 3.21. From the figures, we can see that in general, log(MSE) increases as d_L and d_{RD} increases.

While d_L and d_{RD} changes, the fixation time will most likely change as well, which will impact the log(MSE). If the fixation time is low, the positions in the initial state will be more alike the positions in the final state, compared to if the fixation time is high. log(MSE) is therefore expected to increase as the fixation time increases.

The log(MSE) is also affected by the number of runs that was averaged over, and the number of bins. Ideally, both these parameters would be set to as high as possible. What limits is the time it takes to create the result, and that time increases approximately linearly with the number of runs. For the number of bins, it has to be set so there is enough results in each bin the reflect the average.

In figure 3.17, which shows how the initial fitness affects the fixation probability, we can see something interesting. If we were to compute the derivative for the blue dots in the figure, this derivative would vary more for those dots who corresponds the initial fitness fraction around 0.5, than those dots who corresponds to initial fitness fraction around 0 or 1. This is true, because initial fitness fraction is less likely to be around 0.5, than it is to be around 0 or 1. This is also shown in figure 3.18, where we could see that that when L decreases, the chance that almost all fitness comes from one type of agent increases.

In many cases when d_L is low, for example in the range the range 10^{-4} to 10^{-3}, the initial fitness will in many cases come from only one of the agent types. If we look at the first bin while calculating log(MSE), which is between $0 - 0.1$, the log(MSE) calculation would compare with the average, 0.05. However, the average of the measured initial fitness fraction in this interval is much lower. Looking at the raw data, the fixation probability for the cases who end up in this bin is close to 0, for low d_L. The conclusion of this, is that log(MSE) increases when bins decreases.

If d_L would become lower than 10^{-4}, the log(MSE) would increase if all other parameters stay constant. The reason is, that the chance that the initial fitness fraction will be around 0.5 decreases so much, so it does not represent the average anymore. If d_L goes to zero, the chance that the initial fitness fraction would be around 0.5 would also go to zero.

4.3 Approximation

Looking at the results of the approximations made in section 3.3, it is apparent that the approximations does not match the measured values to good. There can be multiple reasons for this. Two possible reasons is discussed below

First of all, setting $e^{\frac{-d}{d_L}}$ to a function that only has two values may make the approximation not work. It could be tried to change the approximated function to take

three or more values. However, increasing the values the function can take would require more fractions like $\frac{N_{AA}}{A-1}$ to be estimated.

Looking at figure 3.16, where the fraction $\frac{N_{dd}}{doves-1}$ measured, we can see the average of the fraction varies between around 0.2 to 0.4 for the iterations that correspond to the second approximated line. For a single measurement, the fraction varies significantly more. Taking the average of many runs, may be a reason why to approximation doesn't work.

4.4   Simulation setup

While initialising positions of the agents, every agent are given a position in a 2d square with the side length 1. The shape of this area is chosen to be a square for convenient reasons. The shape could for example be chosen to be a circle instead, which could have an effect on the result. The side lengths of the square should not matter. Multiplying the side lengths with a constant, q, should be equal to dividing the parameters d_L, d_{RD} and d_{RM} with q.

An interesting change to the area would be to let the edges connect with each other. In this scenario, an agent on the position $(0.5, 0)$ would be at distance zero to an agent in the position $(0.5, 1)$. Furthermore, if an agent at position $(0.9, 0)$ where to move at a distance $(0.2, 0)$, the new position would be $(0.1, 0)$. Doing this, possible effects from having edges could be removed.

The effects of edges should increase as d_L, d_{RD} and d_{RM} increases. For example, d_{RD} and d_{RM} can not be greater then $\frac{1}{\sqrt{2}}$. If it were to be just below this value, something special might occur, for example that agents are grouped together in some areas.

In every simulation, the population size is 50. This was chosen arbitrary, but with the knowledge that increasing the population size would increase the time it takes to run the simulations significantly. The dynamics might change if the population size changes. Increasing the population size would for example decrease the average distance between agents, which could result in effects not discovered in this report. A larger population size would also make the dynamics less stochastic.

Furthermore, there are 25 agents of each type in the beginning of each set of iterations. This could be changed so that the starting amount of agents are not equal. The biggest effect of this would probably be that the fixation probability increases(decreases), if the number of agents the fixation probability is measured on increases(decreases)

While doing the simulations, a small error occurred which is good to know about. Looking at the payoff matrices in section 1.3, we can see that when making the payoff matrix for dove hawk non-negative, $\frac{c}{2}$ was added to each element. However, it would be possible to add $\frac{c-b}{2}$ to make the matrix non-negative, which is smaller then $\frac{c}{2}$. Adding $\frac{c-b}{2}$ instead of $\frac{c}{2}$, would be equal to adding $\frac{c}{2}$, and setting $w = \frac{2}{3}$, when $c = 2$ and $b = 1$. The value of w can be determined by picking two of the values from the payoff matrix, and the condition that the fraction between them

has to be equal. For example, we can pick $\frac{b}{2}$ and $\frac{c}{2}$ from table 1.5. The fraction is $\frac{\frac{b}{2}}{\frac{c}{2}} = \frac{b}{c} = \frac{1}{2}$. In the case where we only add $\frac{c}{2}$ to make it none negative, we get $\frac{1-w+w*0}{1-w+w*\frac{c-b}{2}} = \frac{1-w}{1-\frac{w}{2}}$. For this fraction to be equal to $\frac{1}{2}$, w has to be $\frac{2}{3}$. $w = \frac{2}{3}$ will be obtained regardless of the choice of elements from the payoff matrix.

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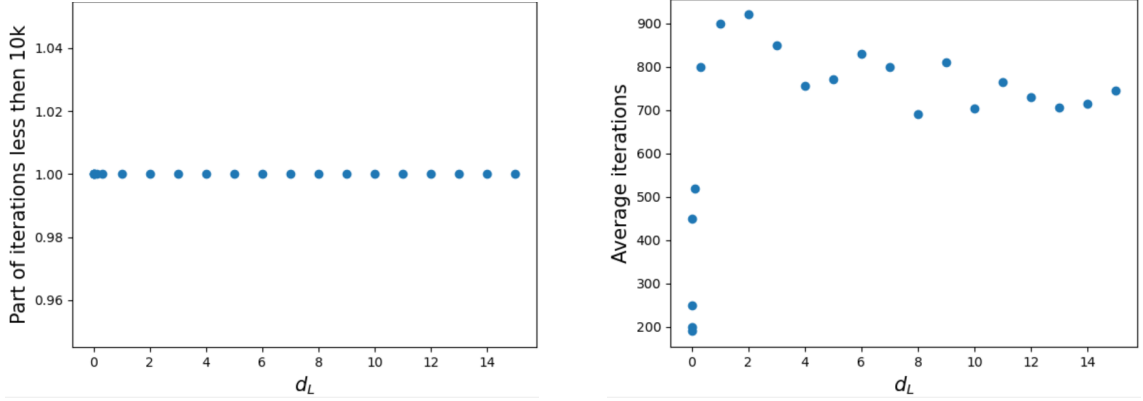
A
Appendix

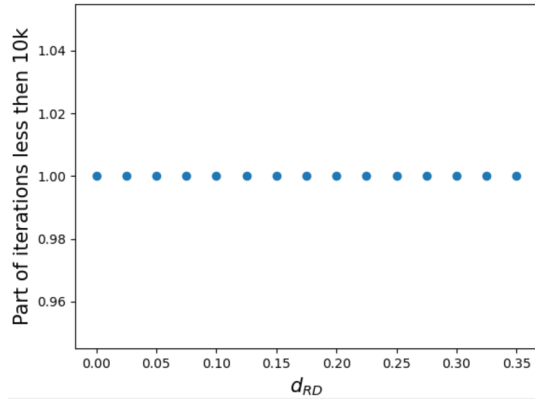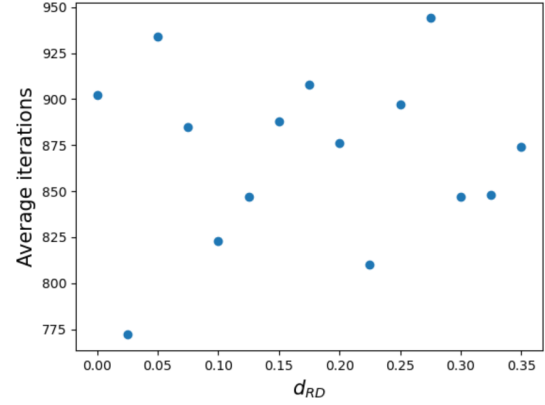## **A.1 Prisoner's dilemma fixation time**

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(a) The fraction that converge before 10^4 iterations as a function of d_L.

(b) The average amount of iterations as a function of d_L.

Figure A.1: The results in the two figures comes from the same measurement. The parameters are: $b = 30$, $c = 1$, d_{RD}=0.05, runs= 100, d_{RM}=0, $w = 1$, $N_c = 25$, $N_d = 25$.
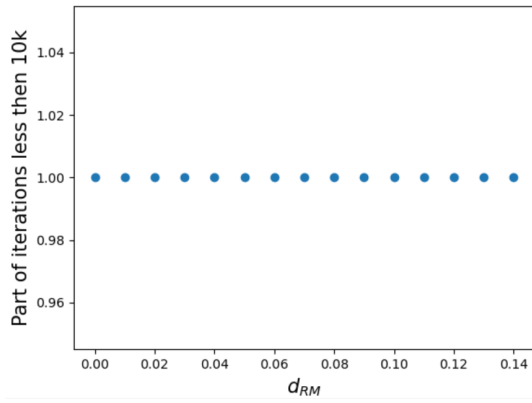
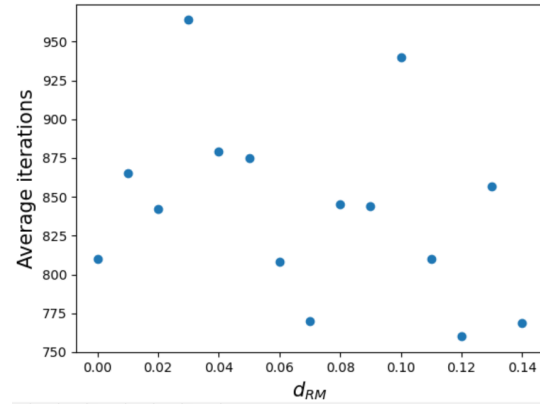**(a)** The fraction that converge before 10^4 iterations as a function of d_{RD}

(b) The average amount of iterations as a function of d_{RD}

Figure A.2: The results in the two figures comes from the same measurement. The parameters are: $b = 30$, $c = 1$, $d_{RM}=0$, runs= 100, $d_L=1$, $w = 1$, $N_c = 25$, $N_d = 25$.

(a) The fraction that converge before 10^4 iterations

(b) The average amount of iterations as a function of d_{RM}

Figure A.3: The results in the two figures comes from the same measurement. The parameters are: $b = 30$, $c = 1$, d_{RD}=0.05, runs=100, d_L=1, $w = 1$, $N_c = 25$, $N_d = 25$.