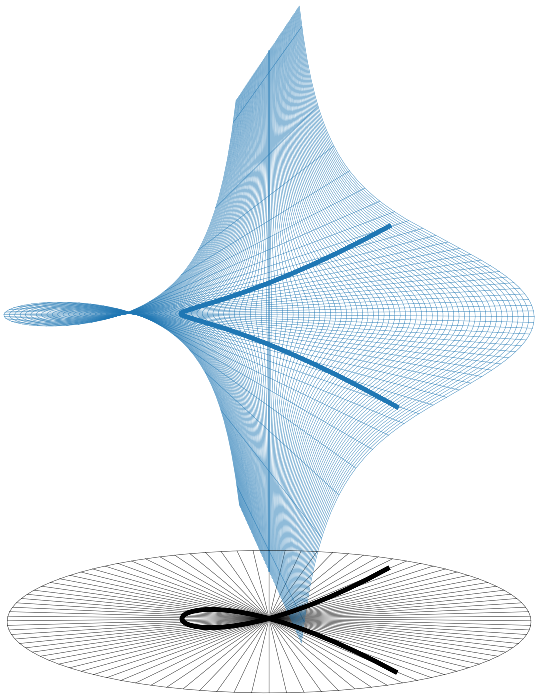

Regularisation of Feynman integrals on complexified configuration spaces

Master's thesis in Physics

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CHALMERS UNIVERSITY OF TECHNOLOGY
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Cover: A visualisation of the blowup at the origin of $\mathbb{R}^2$ and the strict transform of the variety defined by $y^2 - x^2(x-1) = 0$.

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Abstract

We present a regularisation procedure for divergent configuration space Feynman integrals coming from a complexified euclidean scalar quantum field theory on a complex manifold X. The inspiration for the thesis as well as the setting is provided by recent works of Ceyhan and Marcolli that proposes a construction of the configuration space and a complex generalisation of a Feynman amplitude dictated by a Feynman graph Γ, reminiscent of a set of Feynman rules in physics. Furthermore, Ceyhan and Marcolli describe a compactification of the configuration space of a given graph as an iterated sequence of blowups along certain diagonals in a product space where the amplitude associated to the graph in general has non-integrable singularities. We identify the possibility of the amplitude also having singularities at infinity and propose a construction, complementary to that of Ceyhan and Marcolli, with the desired result that the singular locus of the pullback of the amplitude constitutes a normal crossings divisor. This property allows for the application of techniques from the theory of currents in complex analysis. We consider a regularisation of the divergent integral, which has a Laurent series expansion in the regularisation parameter with current coefficients. We define the degree of divergence as the leading order of the expansion. We go through the regularisation procedure for three explicit Feynman graphs, with $X = \mathbb{CP}^D$. We give upper bounds for their respective degrees of divergence and for one of the graphs, in the special case $D = 2$, we show that the leading order term vanishes.

Keywords: regularisation, divergent integrals, configuration space, blowups, currents, meromorphic continuation.

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Contents

1

Introduction

In recent years, quantum field theory has seen a resurgence of interest, though, it has been from mathematicians rather than from physicists, due to developments towards understanding the apparent geometric and algebraic structure underlying *regularisation* and *renormalisation* of divergent integrals in perturbation theory.

In perturbative quantum field theory, probability amplitudes for interactions of quantum fields are recovered as perturbative series, where by computing sufficiently many terms in the expansion one may achieve the desired accuracy. There are many subtleties here, but in a nutshell, this amounts to computing increasingly complicated integrals of correlation functions between particle states over some number of copies of space-time known as *configuration space*, or, more commonly, the dual *momentum space*. The saving grace when doing perturbative quantum field theory is a graphical formalism largely credited to physicist Richard Feynman, which translates any given perturbative expansion into a formal sum of graphs, known as *Feynman diagrams*. The process of going back and forth between an integral and its associated graph is made simple by a set of *Feynman rules*, unique to any quantum field theory, but easy to derive. The graphs are much more intuitive, and far less notationally taxing to work with, and so one tends to begin any calculation in the graphical formalism and then translate the graphs one wishes to compute into integrals using the rules. Perhaps due to the interchangeable nature of the integrals and associated Feynman diagrams, the integrals have come to be referred to as *Feynman integrals*.

The reader who is somewhat familiar with quantum field theory might also be familiar with a certain problem of infinities appearing in perturbation theory, and the seemingly miraculous countermeasure to this known as renormalisation. Renormalisation is the process of consitently removing infinities from a quantum field theory, in such a way that all that is left are finite numbers that capture the true physical phenomena. Associated to renormalisation is the procedure known as regularisation, in which a divergent integral is modified with a *regulator*, an auxiliary parameter that for some values makes the integral converge. In terms of this parameter, an asymptotic behaviour can be characterised, which informs how to perform the renormalisation if possible.

In this thesis we look at a certain regularisation procedure for divergent Feynman integral coming from a *complexified* euclidean massless scalar quantum field theory. Scalar theories are the prototypical quantum field theories, where one has done away with many embellishments of a more general quantum field theory. Masslessness refers to a property of the fields, and is another simplifying restriction. Euclidean refers to the metric signature of the space in which the quantum fields ”live”. It is not the most common choice in physics since Einstein’s theory of special relativity postulates that space-time has lorentzian signature. Lastly, complexified refers to the fact that we consider a mathematical generalisation of what might be called a physical quantum field theory. Here spacetime will be a complex manifold, in particular, *complex projective D-space* $\mathbb{CP}^D$, and the Feynman rules will associate a complex differential form to any given Feynman diagram. For this reason and others, the subject of this thesis is detached from much in way of physical relevance or application. The argument for this departure from physics is the techniques that we are able to employ, that are better understood in the complex setting. Regardless of our efforts ultimately being useful in physics or not, it can be said that similar integrals to the ones we will end up looking at, have been studied in the context of string theory by, e.g., Witten [Wit18].

We conduct a case study, where we look at a three different Feynman diagrams and their associated integrals. The complexified setting which we consider was proposed by Ceyhan and Marcolli in [CM12a; CM12b], who investigate connections between the regularisation and renormalisation of such a construction and the theory of motives in algebraic geometry. For a given Feynman graph, Ceyhan and Marcolli construct the Feynman integral in terms of a Feynman amplitude to be integrated over a configuration space associated to the graph. The configuration space will generically resemble an n-fold cartesian product of the manifold representing spacetime, where n is the number of vertices of the graph. Moreover, a general Feynman amplitude will have non-integrable singularities along certain diagonals of this product, referred to as *ultraviolet divergences*, also dictated by the Feynman graph. Ceyhan and Marcolli present a way of compactifying the complement of these diagonals, such that the singular locus of the Feynman amplitude becomes a reasonable *divisor* (see § 2.1.5).

Given that the singular locus of the pullback of the Feynman amplitude to the compactified configuration space constitutes a *normal crossings divisor*, that is, a divisor which is locally either smooth or is a union of coordinate hyperplanes, the Feynman integral is well suited for a certain regularisation procedure recently studied by Lennartsson in [Len20]. However, by following the construction of Ceyhan and Marcolli for some explicit cases, we observe that the Feynman amplitudes may, in addition to the ultraviolet divergences, have non-integrable singularities at infinity, referred to as *infrared divergences*. Furthermore, in neighbourhoods of these infrared divergences, the compactification does not produce a divisor with the desired properties. Thus, the Ceyhan-Marcolli construction requires some complementing adjustments for us to be able to perform the aforementioned regularisation. Inspired by the initial compactification of configuration space, which is obtained through an iterated sequence of a particular type of mappings called *blowups* (see § 2.2), we propose making further blowups until the singular locus of the Feynman amplitude becomes a normal crossings divisor also in neighbourhoods of the infrared singularities. This is then done on a case-by-case basis, allowing us to complete the regularisation procedure for the three integrals in our case study. Moreover, by our case-by-case approach, we gather some evidence to support the existence of a method for dealing with the infrared divergences for a general Feynman integral.

1.1 Outline of thesis

The thesis is organised as follows. Chapter 2 contains the mathematical ideas underlying the rest of the thesis. We begin by giving a very brief introduction to complex analysis in several variables, followed by some aspects of complex geometry. We go on to introduce *complex-analytic varieties* and *divisors*, and end the chapter with a more detailed description of our two main tools, *blowups* and *currents*.

In Chapter 3 we outline a process of regularisation of certain divergent integrals associated to a complexified scalar quantum field theory. We begin with a presentation of the Ceyhan-Marcolli construction of the configuration space and compactification thereof, for a given (Feynman) graph Γ. We then go on to present the regularisation of the integral associated to Γ. We show how one can define a current-valued function from a divergent Feynman integral. This function is a function of a complex variable λ. A priori the current is only defined for $\mathfrak{Re}\,\lambda \gg 1$, however, we obtain a meromorphic current-valued function in all of $\mathbb{C}$ by way of meromorphic continuation. Furthermore, there is a Laurent series expansion at $\lambda = 0$ with current coefficients, from which we can define a *degree of divergence* κ.

Chapter 4 is a case study of three different Feynman graphs, Γ_1, Γ_2 and Γ_3, associated to a scalar quantum field theory on $\mathbb{CP}^D$. We carry out the Ceyhan-Marcolli construction, along with our own modifications, resulting in an amplitude with singularities on normal crossings divisor for each of the three graphs. We produce upper bounds for the respective degrees of divergence and present explicit regularisations for Γ_1 and Γ_2. Lastly, we show that the leading order coefficient in the Laurent series expansion for Γ_1 vanishes in the special case $D = 2$.

In Chapter 5 we make some concluding remarks. In particular, we discuss the possibility of a general procedure for modifying the Ceyhan-Marcolli construction, supported by the results from the case study.

2

Mathematical preliminaries

We assume the reader is familiar with elementary concepts from algebra, as well as some knowledge of basic differential geometry, topology and single variable complex analysis. Sometimes we will recall the definitions of basic concepts, other times we will not. If some term is invoked without being given some sort of definition or reference, it is regarded as common knowledge and will be easy to look up for the reader wanting to refresh his or her memory. For a more thorough review of the topics below see, e.g., [GH78].

2.1 Complex analysis in several variables

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a continuously differentiable function, and $z = (z_1, \ldots, z_n)$, with $z_j = x_j + \mathrm{i}\, y_j$, be coordinates on $\mathbb{C}^n$. The function f is *holomorphic* if the map

$$z_j \mapsto f(z_1, \ldots, z_j, \ldots, z_n)$$

is holomorphic in the single variable sense, i.e., if it satisfies the *Cauchy-Riemann* equations, compactly written as

$$\frac{\partial f}{\partial \bar{z}_j} = 0,$$

where

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}\Big).$$

Like in the single variable case, holomorphic functions are analytic and vice versa, and we will use the terms interchangeably. We say that a function is *anti-holomorphic* if the complex conjugate is holomorphic. Furthermore, we define a *meromorphic function* $f : \mathbb{C}^n \to \mathbb{C}$ as locally given by a quotient of two holomorphic functions without common factors.

From the coordinate functions z and their complex conjugates $\bar{z}$ we may define the holomorphic and anti-holomorphic 1-forms, respectively, as

$$\mathrm{d}z_j = \mathrm{d}x_j + \mathrm{i}\,\mathrm{d}y_j \quad \text{and} \quad \mathrm{d}\bar{z}_j = \mathrm{d}x_j - \mathrm{i}\,\mathrm{d}y_j \quad \text{for } 1 \leq j \leq n,$$

where d is the exterior derivative. Taking the exterior derivative on a function $f : \mathbb{C}^n \to \mathbb{C}$ we have

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j.$$

We define the *Dolbeault* and *conjugate Dolbeault operators* $\bar{\partial}$ and ∂, respectively, by

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j \quad \text{and} \quad \partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\mathrm{d}z_j,$$

such that $\mathrm{d} = \partial + \bar{\partial}$. Equivalently to the definition above, a function f is holomorphic if and only if it satisfies $\bar{\partial} f = 0$.

A mapping $f = (f_1, \ldots, f_n) : U \to V$ for $U \subseteq \mathbb{C}^m$ and $V \subseteq \mathbb{C}^n$ open, is holomorphic if each $f_j : \mathbb{C}^m \to \mathbb{C}$, for $1 \leq j \leq n$, is a holomorphic function.

2.1.1 Complex manifolds

One defines a complex (differentiable) manifold in analogy with the real case except that the transition maps are *(bi)holomorphic*, i.e., holomorphic and with a holomorphic inverse. Local coordinates on a complex manifold are referred to as *holomorphic coordinates*. Complex manifolds are naturally even-dimensional, with the *complex dimension* (abbreviated $\dim_{\mathbb{C}}$) defined as half the *real* dimension. We will henceforth refer to complex dimension simply as dimension unless otherwise specified. A mapping g between two complex manifolds is holomorphic if, when expressed in local coordinates, it is a holomorphic mapping between open sets in $\mathbb{C}^n$.

Example 2.1. A prime example of a complex manifold is the *complex projective n-space* $\mathbb{CP}^n$. It is defined as the set of equivalence classes $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where for $z, w \in \mathbb{C}^{n+1} \setminus \{0\}$, $z \sim w$ if $\exists \lambda \in \mathbb{C}^*$ such that $v = \lambda w$. Given a point $p \in \mathbb{CP}^n$, any representative in the equivalence class of tuples corresponding to p is called a *set of homogenous coordinates for* p, which we denote by $[X] = [X_0 : \ldots : X_n]$.

$\mathbb{CP}^n$ has an open covering by the n-dimensional affine spaces $U_j = \mathbb{CP}^n \setminus \{X_j = 0\}$ for $j = 0, \ldots, n$. For any point $p \in \mathbb{CP}^n$, at least one component $X_j \neq 0$, whence $p \in U_j$ for some j. Furthermore, we have the mappings $\varphi_j : U_j \to \mathbb{C}^n$, defined by

$$\varphi_j : [X_0 : \ldots : X_n] \mapsto (\frac{X_0}{X_j}, \ldots, \widehat{\frac{X_j}{X_j}}, \ldots, \frac{X_n}{X_j}) \coloneqq (x_1, \ldots, x_n) \in \mathbb{C}^n, \tag{2.1}$$

where the hat means that the expression should be omitted. The collection $\{(U_j, \varphi_j)\}$ constitutes an atlas for $\mathbb{CP}^n$; it is a simple exercise to show that the transition maps $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are biholomorphic, concluding that $\mathbb{CP}^n$ defines a complex manifold. Furthermore, it can be shown that $\mathbb{CP}^n$ is compact.

For complex manifolds there are different notions of the *tangent space*. Let M be a complex manifold, with $\dim_{\mathbb{C}} M = n$, and consider a point $p \in M$ and a set of local holomorphic coordinates $z = (z_1, \dots, z_n)$, with $z_j = x_j + \mathrm{i}\, y_j$, in a neighbourhood U of p. Naturally we can consider the *real tangent space* to M at p, i.e., the real $2n$-dimensional vector space given as

$$T_{\mathbb{R},p}(M) = \mathrm{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\}_{j=1}^{n} .$$

Furthermore we can consider what is called the *complexified tangent space* to M at p, $T_{\mathbb{C},p}M = T_{\mathbb{R},p}M \otimes_{\mathbb{R}} \mathbb{C}$ which is a $\mathbb{C}$-linear vector space of complex dimension $2n$.

Lastly, we may consider the *holomorphic* resp. *anti-holomorphic tangent space* to M at p, given by

$$T_p^{1,0} M = \mathrm{Span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_j} \right\}_{j=1}^{n} \quad \text{and} \quad T_p^{0,1} M = \mathrm{Span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_j} \right\}_{j=1}^{n} ,$$

where we have that $T_{\mathbb{C},p}M = T_p^{1,0}M \oplus T_p^{0,1}M$. The anti-holomorphic tangent space is obtained from the holomorphic tangent space via the operation of conjugation, sending $\partial/\partial z_j$ to $\partial/\partial \bar{z}_j$ and vice versa.

The dual space to $T_p^{1,0}M$ ($T_p^{0,1}M$) is called the *(anti-)holomorphic cotangent space* to M at p. It is spanned by the (anti-)holomorphic 1-forms $\{\mathrm{d}z_j\}$ ($\{\mathrm{d}\bar{z}_j\}$), satisfying

$$\mathrm{d}z_i\Big(\frac{\partial}{\partial z_j}\Big) = \delta_{ij}, \quad \mathrm{d}\bar{z}_i\Big(\frac{\partial}{\partial \bar{z}_j}\Big) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

2.1.2 Complex vector bundles

Recall that a *fiber bundle*, denoted E, $E \to M$ or $\pi : E \to M$, is a tuple of spaces (E, M, F), where E is called the *total space*, M the *base space* and F the *fiber*, together with a map $\pi : E \to M$ called the *bundle projection*, which satisfies the condition of *local triviality*. This means that for each point $p \in M$ there exists an open neighbourhood $U \subset M$ of p such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$, such that the following diagram commutes,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\mathrm{proj}_1"] \\ & U \end{tikzcd} \tag{2.2}$$

where $\mathrm{proj}_1 : U \times F \to U$ is the natural projection onto the first factor. Notice that φ is *fiber preserving*, i.e., $\varphi|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times F$. The set of all such pairs (U, φ) above is called a *local trivialisation* of the bundle.

A *(global) section* of a fiber bundle $\pi : E \to M$ is a continuous map

$$\sigma : M \to E$$

such that

$$\pi(\sigma(p)) = p \ \forall p \in M.$$

Let $U \subset M$ open such that (2.2) holds. The functions $U \to F$ can be identified with the sections of $\pi^{-1}(U) \to U$. In fact, if $s : U \to F$ is a function, then $p \mapsto \varphi^{-1}(p, s(p))$ is a section of $\pi^{-1}(U) \to U$ and if σ is a section of $\pi^{-1}(U) \to U$, then $p \mapsto \mathrm{proj}_2 \circ \varphi \circ \sigma(p)$ is a function $U \to F$. Thus, local sections of E can be identified with functions to F.

A *complex vector bundle* is a fiber bundle whose fibers are complex vector spaces such that if $\varphi : \pi^{-1}(U) \to U \times F$ is a local trivialisation, then for any $p \in U$, $\varphi|_{\pi^{-1}(p)} \to \{p\} \times F$ is a $\mathbb{C}$-linear isomorphism. Note that $F \simeq \mathbb{C}^k$, where k is called the *rank* of E. A *holomorphic vector bundle* is a complex vector bundle over a complex manifold M such that the total space E is a complex manifold and the bundle projection π is a holomorphic mapping. A local section $\sigma : U \to E$ is a holomorphic local section if σ is a holomorphic mapping.

There is a prototypical vector bundle associated to any smooth manifold M, namely the *tangent bundle* TM. It is obtained as the disjoint union of tangent spaces, where a point $(p, v) \in TM$ consists of a point $p \in M$ together with a vector $v \in T_pM$. The bundle projection is the natural projection $\pi : (p, v) \mapsto p$ and the chart maps φ_α, from the atlas of charts $\{(U_\alpha, \varphi_\alpha)\}$ on M, give rise to local trivialisations on TM. Linear algebra constructions, such as tensor products, exterior products and dualisation of vector spaces, can be made fiber-wise on vector bundles, yielding new vector bundles. Thus, tensor fields and higher order differential forms can be defined on exterior powers of the tangent and cotangent bundle to a smooth manifolds, and analogously on complex manifolds.

If M is a complex manifold, in addition to a complexified tangent bundle there is a canonical complex vector bundle, called the *(anti-)holomorphic tangent bundle* $T^{1,0}M$ ($T^{0,1}M$), given as the disjoint union of (anti-)holomorphic tangent spaces $T^{1,0}_pM$ ($T^{0,1}_pM$) to M. Similarly, we may construct the dual *(anti-)holomorphic cotangent bundle* $T^*_{1,0}M$ ($T^*_{0,1}M$). The complexified tangent bundle and cotangent bundle split into direct sums of the holomorphic and anti-holomorphic tangent and cotangent bundles, respectively. This splitting induces a bi-grading on the exterior powers of the tangent and cotangent bundles, replacing the notion of differential p-forms on smooth manifolds, with differential (p, q)-forms. On a complex manifold, the exterior derivative d, uniquely defined as a (linear) map between exterior powers of cotangent bundles of a smooth manifold,

$$\mathrm{d} : \bigwedge^{p} T^*M \to \bigwedge^{p+1} T^*M,$$

taking p-forms to $(p+1)$-forms, has a decomposition into the Dolbeault and conjugate Dolbeault operators $\bar{\partial}$ and ∂, respectively, mapping (p, q)-forms to $(p, q+1)$-forms and $(p+1, q)$-forms, respectively. The notion of holomorphic functions, extends to differential $(p, 0)$-forms, where such a form α is said to be holomorphic if and only if $\bar{\partial}\alpha = 0$.

2.1.3 Line bundles

Vector bundles of rank 1 are known as *line bundles.* For any holomorphic line bundle $\pi : L \to M$ there is an open cover $\{U_\alpha\}$ and local trivialisations

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}.$$

From the local trivialisations we define the *transition maps* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ by

$$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}.$$

Remark. By definition $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{C} \to U_\alpha \cap U_\beta \times \mathbb{C}$. However, for each $z \in U_\alpha \cap U_\beta$, $\varphi_\alpha \circ \varphi_\beta^{-1}(z, \cdot) : \mathbb{C} \to \mathbb{C}$ is a $\mathbb{C}$-linear isomorphism, i.e., it is multiplication by a non-zero complex number. This number is $g_{\alpha,\beta}(z)$.

The maps $g_{\alpha\beta}$ are holomorphic, and satisfy

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1, \quad g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1. \tag{2.3}$$

Given such a collection of holomorphic non-vanishing functions $\{g_{\alpha\beta}\}$ on $U_\alpha \cap U_\beta$ we can construct a line bundle L with transition functions $\{g_{\alpha\beta}\}$ by taking the union over all α of $U_\alpha \times \mathbb{C}$ and identifying $\{z\} \times \mathbb{C}$ in $U_\alpha \times \mathbb{C}$ and $U_\beta \times \mathbb{C}$ via multiplication by $g_{\alpha\beta}(z)$.

Example 2.2. There is a natural holomorphic line bundle over $\mathbb{CP}^n$ called the *hyperplane bundle* $\mathcal{O}(1)$. Consider the open cover $\{U_j\}$ of $\mathbb{CP}^n$, where $U_j = \mathbb{CP}^n \setminus \{X_j = 0\}$ for $0 \leq j \leq n$. On the overlaps $U_i \cap U_j$ we define transition maps $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$ as

$$g_{ij}([X_0 : \ldots : X_n]) = \frac{X_i}{X_j}.$$

The g_{ij}'s are holomorphic and non-zero on $U_i \cap U_j$. The line bundle obtained from this collection of functions is called the hyperplane bundle $\mathcal{O}(1)$. The global sections of $\mathcal{O}(1)$ are given by homogeneous polynomials on $\mathbb{CP}^n$ of degree 1.

2.1.4 Complex analytic varieties

Complex analytic varieties constitute a generalisation of complex manifolds, to what might be described as complex manifolds that may contain singularities. We proceed with the definition.

Definition 2.3 (Complex analytic variety [GH78, p. 12])**.** A subset V of an open set $U \subset \mathbb{C}^n$ is an *(complex) analytic variety* in U if, for any $p \in U$, there exists a neighbourhood W of p in U such that $V \cap W$ is the common zero-set of a finite collection of holomorphic functions on W.

A point $z \in V$ is a *regular point* if there is a neighbourhood of z where V is a complex manifold. The set of regular points, denoted V_{reg}, is dense in V. The complement $V \backslash V_{\text{reg}} = V_{\text{sing}}$ is the set of *singular points*. We say that an analytic variety V is *irreducible* if its regular locus V_{reg} is connected. An analytic variety has a unique decomposition into irreducible varieties. We will henceforth refer to complex analytic varieties as simply analytic varieties, and in some cases, where there is no risk of confusion, simply as varieties.

Definition 2.4 (Analytic subvariety)**.** An *analytic subvariety* V of a complex manifold M is a subset given everywhere locally as the zero set of a finite collection of holomorphic functions.

The *dimension* of an irreducible analytic variety is defined as the dimension of its regular (non-singular) locus, regarded as a complex manifold. The *codimension* of a variety V with respect to an ambient manifold M is $\dim M - \dim V$.

An analytic subvariety to a manifold is called a *locally complete intersection* if, locally, the minimal number of functions needed to define the variety, equals the codimension.

2.1.5 Divisors and associated line bundles

This section closely follows [GH78, Chapter 1, § 1]. We introduce the notion of a divisor on a complex manifold, and some of its properties; in particular its connection with line bundles.

Let M be a complex manifold. An analytic subvariety $V \subset M$ of codimension 1 is called an *analytic hypersurface*. Such a hypersurface is a locally complete intersection, so for each point $p \in V$, there exists a neighborhood of p in M such that V is given as the zero locus of a holomorphic function f, called a *local defining function*. f is unique up to multiplication by a function non-vanishing at p. Any holomorphic function g defined at p vanishing on V is divisible by f in a neighborhood of p. An analytic hypersurface V can be expressed uniquely as the union of irreducible analytic hypersurfaces. We have the following definition.

Definition 2.5 (Divisor)**.** A *divisor* $\mathfrak{D}$ on a complex manifold M is a locally finite formal integer linear combination

$$\mathfrak{D} = \sum_j a_j \cdot V_j, \quad a_j \in \mathbb{Z},$$

of irreducible analytic hypersurfaces of M. If all of the coefficients a_j are non-negative, the divisor is said to be *effective*.

Remark. We will henceforth consider M to be compact, for which local finiteness implies finiteness.

We define the *support* $|\mathfrak{D}|$ *of a divisor* $\mathfrak{D}$ as the union of irreducible hypersurfaces $\bigcup_j V_j$ of $\mathfrak{D}$. In this thesis we will often refer to analytic hypersurfaces as divisors and vice versa. Strictly speaking we are then identifying a hypersurface V with the effective divisor $\sum_j V_j$, where the V_j's are the irreducible components of V.

Example 2.6. Let $f : M \to \mathbb{C}$ be a holomorphic function. Let V be the zero-locus of f with V_j the irreducible components of V. Let a_j be the order of vanishing of f along $(V_j)_{\text{reg}}$. The divisor

$$\sum_j a_j \cdot V_j$$

is called the *divisor of* f and is denoted $\operatorname{div}(f)$.

There is a connection between effective divisors $\mathfrak{D}$ on complex manifolds M and holomorphic line bundles $\pi : L \to M$ with a global holomorphic section. To see this, assume that we have a holomorphic line bundle $L \to M$, and a global holomorphic section σ of L. Then we have an associated effective divisor $\mathfrak{D}$ obtained as follows. The support of $\mathfrak{D}$ will be the zero locus of the section σ.

The section can be associated with a collection of holomorphic functions f_α on a open cover $\{U_\alpha\}$ of M. In U_α we let $\mathfrak{D}|_{U_\alpha} = \operatorname{div}(f_\alpha)$. This defines a global divisor $\mathfrak{D}$, since in $U_\alpha \cap U_\beta$, $f_\alpha = f_\beta \cdot h$, where h is a non-zero holomorphic function.

Conversely, if we have an effective divisor $\mathfrak{D}$ in M, we can choose an open covering U_α and holomorphic functions f_α such that $\mathfrak{D}|_{U_\alpha} = \operatorname{div}(f_\alpha)$. On the overlaps $U_\alpha \cap U_\beta$ we obtain non-zero holomorphic functions f_α / f_β. As described in § 2.1.3, such a set of functions can be regarded as the transition maps of a line bundle. The line bundle obtained is referred to as the *associated line bundle to* $\mathfrak{D}$ and denoted by $[\mathfrak{D}]$. In addition, the functions $\{f_\alpha\}$ will constitute a global section of the line bundle.

We end this section with a definition of a certain type of divisor.

Definition 2.7. A *normal crossings divisor* is a divisor which locally looks like a union of coordinate hyperplanes.

2.2 Blowups

A *blowup* is a particular type of transformation of a variety X producing a new variety Y called the *blowup of* X. Conceptually, a blowup replaces a subvariety V of X with *the directions pointing into* V, formalised by the *projectivised normal bundle* to V in X.

Associated to the blowup is a projection $\pi : Y \to X$, sometimes called the *blowup map*. A blowup is completely detemined by its *center* C which is a subvariety of X. The center is the locus over which the projection $\pi : Y \to X$ is not an isomorphism. The inverse image of the center, $\pi^{-1}(C)$ is called the *exceptional locus* or *exceptional divisor* of the blowup and is denoted by $\mathrm{Exc}(\pi)$. We have the following definitions which are useful when discussing blowups.

Definition 2.8. A map $f : X' \to X$ is a *modification* if it is a proper holomorphic mapping, such that for some lower-dimensional subvariety $Z \subset X$, $f : X' \setminus f^{-1}(Z) \to X \setminus Z$ is biholomorphism.

Remark. Notice that f is a biholomorphism outside of a set of (Lebesgue) measure 0.

Definition 2.9. Consider a modification $f : Y \to X$ with exceptional set $Z \subset X$ and let V be a subvariety in X. We define the *total transform of* V as $\mathrm{Tot}(V) := f^{-1}(V)$. The *strict transform* of V is defined as

$$\mathrm{Strict}(V) := \overline{f^{-1}(V \setminus Z)} = \overline{f^{-1}(V) \setminus f^{-1}(Z)}.$$

Remark. A blowup is a typical example of a modification. Furthermore, any composition of blowups constitutes a modification. The exceptional set of a blowup π is its center C, thus

$$\mathrm{Strict}(V) = \overline{\pi^{-1}(V) \setminus \pi^{-1}(C)} = \overline{\mathrm{Tot}(V) \setminus \mathrm{Exc}(\pi)}.$$

The aim of the following two sections is to give a practical description of the blowup construction, which is one that we will make use of in the rest of this thesis. We will also attempt to put this hands-on description into perspective by looking at a less hands-on, but more visual description of the blowup.

2.2.1 Blowup of the origin in $\mathbb{C}^2$

One of the simplest examples of a blowup is the blowup of the origin in $\mathbb{C}^2$. We define the blowup of the origin in $\mathbb{C}^2$,

$$\mathrm{Bl}_0\mathbb{C}^2 = \{(z, w, [t_0 : t_1]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : zt_1 - wt_0 = 0\}. \tag{2.4}$$

With $\Pi : \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ the natural projection, we let $\pi = \Pi|_{\mathrm{Bl}_0\mathbb{C}^2}$, where we have that $\mathrm{Exc}(\pi) = \{0\} \times \mathbb{CP}^1$. We have following proposition.

Proposition 2.10. *The following two items give alternative descriptions of the blowup* $\mathrm{Bl}_0\mathbb{C}^2$.

(i) *The closure in $\mathbb{C}^2 \times \mathbb{CP}^1$ of the graph of the function $f : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ given by*

$$f(z, w) = [z : w].$$

(ii) The complex two-dimensional manifold obtained by the glueing of $\mathbb{C}^2_{(u,v)}$ and $\mathbb{C}^2_{(\zeta,\eta)}$ subject to the constraints

$$u = \zeta\eta, \quad v = \frac{1}{\eta}.$$

Proof. We begin by showing that (i) is a alternative definition of (2.4). Consider the graph

$$G = \{\underbrace{(z, w, f(z, w))}_{=(z,w,[z:w])} \in (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{CP}^1\}.$$

For a point $(z, w, [z : w]) \in G$, clearly $zw - wz = 0$, so $G \subseteq \mathrm{Bl}_0\mathbb{C}^2$. Since $\mathrm{Bl}_0\mathbb{C}^2$ is closed, it follows that $\overline{G} \subseteq \mathrm{Bl}_0\mathbb{C}^2$. We want to show that $\mathrm{Bl}_0\mathbb{C}^2 \subseteq \overline{G}$. Consider a point $(z, w, [t_0 : t_1]) \in \mathrm{Bl}_0\mathbb{C}^2$ and assume without loss of generality that $t_0 \neq 0$; then $w = z(t_1/t_0)$. If $z \neq 0$, $f(z, w) = [z : w] = [z : z(t_1/t_0)] = [t_0 : t_1]$, which implies that $(z, w, [t_0 : t_1]) \in G$. If $z = 0$ then also $w = 0$ so our point in $\mathrm{Bl}_0\mathbb{C}^2$ is $(0, 0, [t_0 : t_1])$. Consider a sequence $(z_j)_{j=1}^{\infty}$, where $z_j \neq 0\ \forall j$, and such that $z_j \to 0$ for $j \to \infty$. Setting $w_j = z_j(t_1/t_0)$ we have that $(z_j, w_j, [z_j, w_j]) = (z_j, w_j, [t_0 : t_1]) \in G\ \forall j$. Taking the limit $(z_j, w_j, [t_0 : t_1]) \to (0, 0, [t_0 : t_1])$ we see that a sequence in G converges to our point in $\mathrm{Bl}_0\mathbb{C}^2$, which implies that $\mathrm{Bl}_0\mathbb{C}^2 \subseteq \overline{G}$.

Now we want to show that (ii) is equivalent to (2.4). We emphasise that $\mathrm{Bl}_0\mathbb{C}^2$ is a manifold of complex dimension 2, since $\dim_{\mathbb{C}} \mathbb{C}^2 \times \mathbb{CP}^1 = 3$ and $\mathrm{Bl}_0\mathbb{C}^2$ is $\mathbb{C}^2 \times \mathbb{CP}^1$ subject to one (complex) constraint. We would like to find a covering of $\mathrm{Bl}_0\mathbb{C}^2$ by charts, that will be isomorphic to $\mathbb{C}^2$. Let $(z, w, [t_0 : t_1]) \in \mathrm{Bl}_0\mathbb{C}^2$, and assume $t_0 \neq 0$. Then we have

$$zt_1 - wt_0 = 0 \overset{t_0 \neq 0}{\Longrightarrow} z\frac{t_1}{t_0} - w = 0.$$

Let $u = z$ and $v = t_1/t_0$ and consider

$$\pi_1 : (u, uv, [1 : v]) \mapsto (u, v).$$

We see that (u, v) are local coordinates for $\mathrm{Bl}_0\mathbb{C}^2$ in the chart where $t_0 \neq 0$.

A similar argument can be made for $t_1 \neq 0$, where we have

$$zt_1 - wt_0 = 0 \overset{t_1 \neq 0}{\Longrightarrow} z - w\frac{t_1}{t_0} = 0.$$

Letting $\zeta = w$ and $\eta = t_0/t_1$ we have

$$\pi_2 : (\zeta\eta, \zeta, [\eta : 1]) \mapsto (\zeta, \eta),$$

where we see that (ζ, η) are local coordinates for $\mathrm{Bl}_0\mathbb{C}^2$ in the chart where $t_1 \neq 0$. The charts are clearly compatible, i.e., the transition maps $\pi_1 \circ \pi_2^{-1}$ and $\pi_2 \circ \pi_1^{-1}$ are homeomorphisms on the intersection of charts; furthermore, we have

$$\pi_1 \circ \pi_2^{-1} : (\zeta, \eta) \mapsto (\zeta\eta, \frac{1}{\eta}) = (u, v),$$

which tells us that (ii) $\iff$ (2.4). $\square$

With the description (ii) of $\mathrm{Bl}_0\mathbb{C}^2$, in the local coordinates (u, v), $\mathrm{Exc}(\pi) = \{u = 0\}$ and, in the local coordinates (ζ, η), $\mathrm{Exc}(\pi) = \{\zeta = 0\}$.

Remark. While (2.4) and (i) arguably give us a clearer image of the blowup than (ii), (ii) is more practical when doing explicit computations involving blowups, as we will see.

2.2.2 Blowup along a subvariety

Let us be a bit more general than in the example above. For a complex manifold X of dimension n and a smooth analytic subvariety Z of codimension κ, we can consider the blow-up of X along Z. In this less explicit case we can still adopt different viewpoints. Below we will consider both a *local picture*, well suited for computation but gives little intuition about the geometry of the blowup, and a *global picture* which is more visual but, for our purposes below, less practical.

Local construction

Consider a point $p \in Z$ and a neighbourhood B of p. We choose local coordinates in B, $z = (z_1, \ldots, z_n)$ such that p corresponds to $(0, \ldots, 0)$, and moreover such that, locally, $Z = \{z_1 = \ldots = z_\kappa = 0\}$.

Now we consider the product $B \times \mathbb{CP}^{\kappa-1}$ with homogeneous coordinates $[t] = [t_1 : \ldots : t_\kappa]$ on $\mathbb{CP}^{\kappa-1}$. The blowup of X along Z is given locally (in B) as

$$\mathrm{Bl}_Z B = \{(z, [t]) \in B \times \mathbb{CP}^{\kappa-1} : z_i t_j = z_j t_i, 1 \leq i, j \leq \kappa\}.$$

Remark: This is very similar to the case of the blowup of a point in $\mathbb{C}^2$. Note, however, that the constraint is only on the coordinates $(z_1, \ldots, z_\kappa)$ (and $[t]$); the coordinates $(z_{\kappa+1}, \ldots, z_n)$ can take on any values.

There is a natural projection $\Pi : B \times \mathbb{CP}^{\kappa-1} \to B$. We can consider the restriction $\pi := \Pi|_{\mathrm{Bl}_Z B}$, and consider the pullback of the tuple $\pi^*(z_1, \ldots, z_\kappa)$. We note that in general, the pullback to a blow-up, there is locally always at least one component that divides the others. To see this, let $(z_1, \ldots, z_n, [t_1 : \ldots : t_\kappa]) \in \mathrm{Bl}_Z B \subset B \times \mathbb{CP}^{\kappa-1}$, and assume without loss of generality that $t_1 \neq 0$. Then, $\forall j = 1, \ldots, \kappa$, $z_j = z_1(t_j/t_1)$ on $\mathrm{Bl}_Z B \setminus \{t_1 \neq 0\}$, i.e., z_1 divides z_j. It follows that the ideal generated by the components of $\pi^*(z_1, \ldots, z_\kappa)$ is locally generated by a single function. These local functions define the exceptional divisor $\mathrm{Exc}(\pi)$.

Now, we want to describe the line bundle associated to $\mathrm{Exc}(\pi)$. Let $\mathcal{O}(-1) \to \mathbb{CP}^{\kappa-1}$ be the dual bundle of the hyperplane bundle $\mathcal{O}(1)$ discussed in Example 2.2. $\mathcal{O}(-1)$ is called the *tautological line bundle*. We can extend it trivially to a line bundle over $B \times \mathbb{CP}^{\kappa-1}$. One can check that $\mathcal{O}(-1)|_{\mathrm{Bl}_Z B} = [\mathrm{Exc}(\pi)]$. The local defining functions for $\mathrm{Exc}(\pi)$

discussed above will constitute a global section of this bundle. We present the following commutative diagram, see, e.g., [And+21], which summarises the above.

$$\begin{tikzcd}
\mathcal{O}(-1)|_{\mathrm{Exc}(\pi)} \arrow[r, hook] \arrow[d] & \mathcal{O}(-1)|_{\mathrm{Bl}_Z B} \arrow[r, hook] \arrow[d] & \mathcal{O}(-1) \arrow[d] \\
\mathrm{Exc}(\pi) \arrow[r, hook] \arrow[d, "\pi|_{\mathrm{Exc}(\pi)}"] & \mathrm{Bl}_Z B \arrow[r, hook] \arrow[d, "\pi"] & B \times \mathbb{CP}^{\kappa-1} \arrow[dl, "\Pi"] \\
Z \arrow[r, hook] & B &
\end{tikzcd}$$

The blowup $\mathrm{Bl}_Z B$ may be described in terms of local coordinates $(\zeta_1, \ldots, \zeta_n)$, with the chart maps

$$\pi_j : (\zeta_1, \ldots, \zeta_n) \mapsto (\zeta_j \zeta_1, \ldots, \zeta_j \zeta_{j-1}, \zeta_j, \zeta_j \zeta_{j+1}, \ldots, \zeta_j \zeta_\kappa, \zeta_{\kappa+1}, \ldots, \zeta_n) = (z_1, \ldots, z_n) \in B,$$

defined in the neighbourhoods $\mathrm{Bl}_Z B \cap \{t_j \neq 0\}$, for $1 \leq j \leq \kappa$. This a direct generalisation of (ii) in the example of the blowup of $\mathbb{C}^2$ at the origin, where we showed that the following,

$$\pi_1 : (u, v) \mapsto (u, uv) \quad \pi_2 : (\xi, \eta) \mapsto (\xi\eta, \xi),$$

is a valid description. Note that we here omit the points in $\mathbb{CP}^{\kappa-1}$, $[1 : v]$ and $[\eta : 1]$, which are determined by (u, v) resp. (ξ, η).

The blow-up construction turns out to be independent on the choice of local coordinates, each object in the above diagram for different sets of coordinates will be related through biholomorphisms. Thus the charts corresponding to this local blow-up construction will ”glue” together to form a unique global object.

Global construction

We here present, without proof, a global description of $\mathrm{Bl}_Z X$. This construction is similar to (i) of Proposition 2.10. For the following global construction, we have some additional requirements on X, namely the following: we assume there exists a holomorphic vector bundle $E \to X$ of rank N, as well as a global holomorphic section σ of E such that σ defines the subvariety Z. Blowing up along a smooth subvariety of codimension κ means that we need at least κ local defining functions, i.e., we require $N \geq \kappa$. Locally σ is an N-tuple, $\sigma = (\sigma_1, \ldots, \sigma_N)$. Our requirement is that we are able to choose κ elements of σ that act as the coordinates $z_1, \ldots, z_\kappa$ do in the local description. We allow for different sets of κ elements in different points on X, which corresponds to changes of local coordinates in the local picture.

Remark: This additional condition is weaker than it might a priori seem. For instance, such a pair, (E, σ), always exists if X is projective. This follows from the Cartan–Serre–Grothendieck theorem, see, e.g., [Laz04, Theorem 1.2.6. (iii)].

Now, consider the *projective bundle associated to* E, denoted $\mathbb{P}(E) \to X$. It is a fiber bundle with fiber at $p \in X$ given by the projectivisation $\mathbb{P}(E_p)$ of the fiber E_p over p, i.e., $\mathbb{P}(E_p)$ is $E_p \setminus \{0\}$ modulo multiplication by a non-zero scalar.

For $p \in X \setminus Z$, $\sigma(p) \neq 0$ in E_p and we let $[\sigma] \in \mathbb{P}(E_p)$ be the corresponding equivalence class. The blowup is given by the closure of the graph $G = \{(p, [\sigma(p)]) \in \mathbb{P}(E) : p \in X \setminus Z\}$ inside $\mathbb{P}(E)$, i.e., $\overline{G} = \mathrm{Bl}_Z X$.

Like in the local picture, we have a summarising commutative diagram, again see, e.g., [And+21].

$$\begin{tikzcd}
\mathcal{O}_E(-1)|_{\mathrm{Exc}(\pi)} \arrow[r, hook] \arrow[d] & \mathcal{O}_E(-1)|_{\mathrm{Bl}_Z X} \arrow[r, hook] \arrow[d] & \mathcal{O}_E(-1) \arrow[d] \\
\mathrm{Exc}(\pi) \arrow[r, hook] \arrow[d, "\pi|_{\mathrm{Exc}(\pi)}"] & \mathrm{Bl}_Z X \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}(E) \arrow[dl, "\Pi"] \\
Z \arrow[r, hook] & X &
\end{tikzcd}$$

Here $\mathcal{O}_E(-1)$ is the tautological line bundle associated to the projective bundle $\mathbb{P}(E)$, that we will not discuss in detail.

2.2.3 Connection to resolution of singularities and Hironaka's theorem

This section is tangential and not directly important to the subject of this thesis. It is here with the aim of giving some background to blowups and how they can be very useful. A prime example of the applicability of blowups is Hironaka's theorem on the *resolution of singularities of an algebraic variety over a field of characteristic* 0 [Hir64], which is a famous theorem in algebraic geometry. In this section we will replace the word algebraic with analytic (for which the statement of the theorem also holds) and consider $\mathbb{C}$ to be our field of characteristic 0. What is meant by a *resolution of singularities* of an analytic variety? Consider an analytic variety X. A resolution of singularities of X is a complex manifold X' together with a proper surjective holomorphic mapping $\epsilon : X' \to X$ such that

$$\epsilon|_{X' \setminus \epsilon^{-1}(X_{\mathrm{sing}})} : X' \setminus \epsilon^{-1}(X_{\mathrm{sing}}) \to X_{\mathrm{reg}}$$

is a biholomorphism.

Hironaka's theorem asserts, in particular, that there always exists a resolution of singularities, for any complex analytic variety, obtained as an iterated sequence of blowups along smooth centers. Hironaka's proof is notoriously complicated, and while subsequent authors have made efforts towards simplifying the arguments of the proof, we will not present any version of the proof in this thesis. For the interested reader, the author can recommend the more pedagogical, but still demanding disclosure of the proof by H. Hauser [Hau03]. One can consider the case of resolution of singularities for an analytic variety

X, where X is embedded in some smooth ambient variety W. Hironaka's theorem then states that there exists a modification ϵ from a smooth variety W' onto W such that

- $X' \coloneqq \mathrm{Strict}(X)$ is smooth and transverse to $\mathrm{Exc}(\epsilon)$ in W',
- ϵ restricted to X' is a resolution of the singularities of X,
- W' is obtained by repeated blowups of smooth, closed subvarieties of W, each transverse to the exceptional divisor of previous blowups.

2.3 Currents

A *current* is a generalisation of a distribution in the following sense. A distribution T on a smooth manifold X is a linear functional on the set of compactly supported smooth functions on X. These functions are called *test functions*. One defines the *support* of a distribution T as the smallest closed subset $\mathrm{supp}(T)$ such that the distribution evaluated on all test functions with compact support in the complement of $\mathrm{supp}(T)$ vanishes. In an analogous manner, a current is a functional on smooth, compactly supported differential forms, so called *test forms*. The support of a current is defined in the same way as for distributions. A real current has a degree p and *acts on* (takes as its argument) test forms of complementary degree $n-p$.

Assume now that X is a complex n-dimensional manifold. The space of linear functionals on test forms of bi-degree $(n-p, n-q)$ is the space of currents of bi-degree (p, q). We use the following notation: let T be a current of bi-degree (p, q) and let ξ be a test form of complementary bi-degree. We denote the action of T on ξ as $\langle T, \xi \rangle$.

A current of maximal or top (bi-)degree acts on test forms of degree $(0, 0)$, which are simply test functions, i.e., top degree currents are distributions.

Two commonly occurring types of currents are the following. If we have a (p, q)-form T on X whose coefficients are in $\mathcal{L}^1_{\mathrm{loc}}(X)$ (locally integrable), then T defines a (p, q)-current

$$\langle T, \xi \rangle \coloneqq \int_X T \wedge \xi, \text{ for } \xi \in \mathscr{D}^{n-p,n-q}(X),$$

where $\mathscr{D}^{n-p,n-q}(X)$ denotes the space of test forms of bi-degree $(n-p, n-q)$. By abuse of notation we write T for both the form and the associated current. Now, let $V \subset X$ be an analytic subvariety of (complex) codimension κ. There is a *current of integration* $[V]$, due to the work of Lelong [Lel57], defined as

$$\langle [V], \xi \rangle \coloneqq \int_V \xi, \text{ for } \xi \in \mathscr{D}^{n-\kappa,n-\kappa}(X).$$

When we define differential operators acting on differential forms, such as the exterior derivative and Dolbeault operator, by the dual nature of currents to differential forms, these operators are implicitly defined on currents. For instance, the Dolbeault operator $\bar{\partial}$ acting on a (p,q)-current T defines a $(p,q+1)$-current whose action on a test form ξ is as follows

$$\langle \bar{\partial}T, \xi \rangle = (-1)^{p+q+1} \langle T, \bar{\partial}\xi \rangle, \text{ for } \xi \in \mathscr{D}^{n-p,n-(q+1)}(X). \tag{2.5}$$

Furthermore, we have the following definition.

Definition 2.11. Let X and Y be complex manifolds, $f : X \to Y$ a proper holomorphic mapping and T a (p,q)-current on X. The *push-forward* f_*T is the current on Y defined by

$$\langle f_*T, \xi \rangle \coloneqq \langle T, f^*\xi \rangle, \text{ for } \xi \in \mathscr{D}^{n-p,n-q}(Y),$$

where $n = \dim X$.

Remark. If $r = \dim Y - \dim X$, then f_*T has bi-degree $(r+p, r+q)$.

Defining a product of currents is ambiguous except for in certain specific cases. However, for T as above and β a smooth (p',q')-form on X we can define the *wedge product* $T \wedge \beta$ by

$$\langle T \wedge \beta, \xi \rangle \coloneqq \langle T, \beta \wedge \xi \rangle, \text{ for } \xi \in \mathscr{D}^{n-p-p',n-q-q'}(X). \tag{2.6}$$

Note, in particular, that if β is a smooth function, we have

$$\langle \beta T, \xi \rangle = \langle T, \beta\xi \rangle.$$

2.3.1 Residue and principal value currents

Many have contributed to the efforts of generalising the classical theory of residues to several complex variables. We will try to give a short introduction to residues in higher dimensions, with the aim of giving some context to the techniques that we will apply to divergent Feynman integrals.

In single variable complex analysis, recall that the *residue* of a meromorphic function is defined as the coefficient of the (-1)-order term of the function's Laurent series expansion at a given pole. The residue is proportional to the contour integral of the function enclosing said pole. This relationship is known as the *residue theorem.* The residue theorem is a useful tool for computing integrals and infinite series, available in single variable complex analysis. Naturally, a possible generalisation to multi-variable complex analysis would be considered of well worth. A naive approach to defining a residue for meromorphic functions of several complex variables, with the goal of obtaining a residue theorem in analogy with the single variable case, may encounter problems due to, e.g., the fact that curves do not bound domains, and that the polar set of a meromorphic function may be non-compact. A way to circumvent these issues is to associate to a meromorphic function a *residue current*, instead of a number.

If z is a complex coordinate in $\mathbb{C}$ and ξ is a test form on $\mathbb{C}$, then the limit

$$\lim_{\epsilon\to 0} \int\limits_{|z|^2>\epsilon} \frac{\xi}{z^k}$$

exists and defines a current. This current is known as the *principal value current* associated to the meromorphic function $1/z^k$. One generalisation of this is a current defined from $1/g$, where g is a holomorphic function on $\mathbb{C}^n$, called the *principal value current of* $1/g$, due to Herrera and Lieberman [HL71], presented in the following theorem.

Theorem 2.12 (Theorem 7.1 in [HL71])**.** *Let g be a holomorphic function in $\mathbb{C}^n$, not identically zero, and ξ a top degree test form. Then the following limit exists and defines a current*

$$\lim_{\epsilon\to 0} \int\limits_{|g|>\epsilon} \frac{\xi}{g} =: \langle [1/g], \xi \rangle. \tag{2.7}$$

Herrera and Lieberman go on to define the *residue current of* $1/g$ as $\bar{\partial}[1/g]$, where $\bar{\partial}[1/g]$ is defined by duality as described above in (2.5). Since $[1/g]$ is a $(0,0)$-current, $\bar{\partial}[1/g]$ is a $(0,1)$-current acting on $(n, n-1)$-forms. We have that

$$\langle \bar{\partial}[1/g], \xi \rangle = \lim_{\epsilon\to 0} \int\limits_{|g|=\epsilon} \frac{\xi}{g}. \tag{2.8}$$

Let us show this. We have that

$$\begin{aligned}
\langle \bar{\partial}[1/g], \xi \rangle &= \langle [1/g], \bar{\partial}\xi \rangle \\
&= \lim_{\epsilon\to 0} \int\limits_{|g|>\epsilon} \frac{1}{g}\bar{\partial}\xi \\
&= \lim_{\epsilon\to 0} \int\limits_{|g|>\epsilon} \bar{\partial}(\frac{\xi}{g}) - \lim_{\epsilon\to 0} \int\limits_{|g|>\epsilon} \bar{\partial}\frac{1}{g} \wedge \xi.
\end{aligned}$$

The function $1/g$ is holomorphic on the chain of integration for each $\epsilon > 0$, whence $\bar{\partial}(1/g)$ in the integrand of the second term vanishes. Furthermore, since ξ/g is a $(n, n-1)$-form, we have

$$\partial(\frac{\xi}{g}) = 0 \implies \mathrm{d}(\frac{\xi}{g}) = (\partial + \bar{\partial})(\frac{\xi}{g}) = \bar{\partial}(\frac{\xi}{g}).$$

Thus,

$$
\begin{aligned}
\langle \bar\partial[1/g], \xi\rangle &= \lim_{\epsilon\to 0} \int\limits_{|g|>\epsilon} \bar\partial(\frac{\xi}{g}) \\
&= \lim_{\epsilon\to 0} \int\limits_{|g|>\epsilon} \mathrm{d}(\frac{\xi}{g}) \\
&= \lim_{\epsilon\to 0} \int\limits_{|g|=\epsilon} \frac{\xi}{g},
\end{aligned}
$$

where in the last step we used Stokes' theorem.

Remark. For a holomorphic function g on $\mathbb{C}$ with a zero at the origin, and with $\xi = f\,\mathrm{d}z$ with $f \equiv 1$ in a neighbourhood of 0, from (2.8) one gets $\langle \bar\partial[1/g], \xi\rangle = 2\pi\mathrm{i}\,\mathrm{Res}_{\{z=0\}} 1/g$, where $\mathrm{Res}_{\{z=0\}} 1/g$ is the classical residue of $1/g$ at $z = 0$.

Since $\bar\partial(1/g) = 0$ outside of $\{g = 0\}$, we see that $\mathrm{supp}(\bar\partial[1/g]) \subseteq \{g = 0\}$, i.e., the support of the residue current is contained within the analytic hypersurface defined by the zero-locus of g.

There is an alternative way to define $[1/g]$ and $\bar\partial[1/g]$ based on analytic continuation. This approach is the one we will use when studying Feynman integrals below. For $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,\lambda \gg 1$, the integral

$$
\int\limits_X \frac{|g|^{2\lambda}\xi}{g}, \tag{2.9}
$$

is convergent and defines an analytic function of λ. One can show that this function has an analytic continuation to $\mathfrak{Re}\,\lambda > -\epsilon$, for some $\epsilon > 0$. The value at $\lambda = 0$ defines the action of a current on ξ. It turns out that this current is $[1/g]$.

In a similar vein as above it follows that $\langle \bar\partial[1/g], \xi\rangle$ is the value at $\lambda = 0$ of

$$
\int\limits_X \frac{\bar\partial|g|^{2\lambda}\xi}{g}.
$$

We will in this thesis consider currents defined from divergent integrals similar to the principal value currents in (2.9).

3

Divergent integrals

In this chapter we present the Ceyhan-Marcolli construction, based on the work by Li. Furthermore, we show how this construction together with current techniques give rise to a regularisation procedure for divergent Feynman integrals.

3.1 Overview of the construction of Ceyhan & Marcolli

Li [Li09] descibes a procedure to construct what is referred to as a *wonderful compactification* of the complement of a certain collection of subvarieties by way of iterated blowups. Ceyhan and Marcolli [CM12a; CM12b] propose an application of Li's results to *configuration spaces* (see below) associated to Feynman graphs coming from (euclidean) scalar quantum field theories. Following [CM12a, § 2], we present this construction of Ceyhan and Marcolli.

Let Γ be a (Feynman) graph, and X some complex manifold which can be regarded as the space-time of some (euclidean) massless scalar quantum field theory. We denote by $\mathbf{E}_\Gamma$ and $\mathbf{V}_\Gamma$ the set of edges and vertices, respectively, in Γ. Furthermore we define the boundary map

$$\partial_\Gamma : \mathbf{E}_\Gamma \to \mathbf{V}_\Gamma^2/\mathfrak{S}_2, \tag{3.1}$$

that assigns to an edge its endpoints; $\mathfrak{S}_2$ denotes the symmetric group on a set of two elements. With the definition in (3.1) the endpoints are defined up to ordering in the symmetric product $\mathbf{V}_\Gamma^2/\mathfrak{S}_2$, whence we implicitly consider Γ to be unoriented.

Let $X^{\mathbf{V}_\Gamma}$ be the cartesian $|\mathbf{V}_\Gamma|^{\text{th}}$ power of X. Denote the points in $X^{\mathbf{V}_\Gamma}$ by $(x_v)_{v\in\mathbf{V}_\Gamma}$. The diagonal Δ_e, associated to the edge $e \in \mathbf{E}_\Gamma$, is given by

$$\Delta_e = \{(x_v)_{v\in\mathbf{V}_\Gamma} : x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}.$$

We define *the configuration space* $\mathrm{Conf}_\Gamma(X)$ *of* Γ *in* X as the complement of the diagonals associated to the edges of Γ in $X^{\mathbf{V}_\Gamma}$, i.e.,

$$\mathrm{Conf}_\Gamma(X) = X^{\mathbf{V}_\Gamma} \setminus \bigcup_{e\in\mathbf{E}_\Gamma} \Delta_e.$$

(a) Looping edge (b) Multiple edge

Figure 3.1: Two examples of loop diagrams.

We assume that our graph Γ has no *multiple* or *looping edges*, examples of which are shown in Figure 3.1. Multiple edges are not *seen* in the construction of configuration space or its compactification, they are only made visible when we consider the associated Feynman integral. Thus, if our graph has multiple edges or not does not matter until we consider Feynman integrals, and for simplicity we assume it does not. Looping edges, however, we do not allow for the simple reason that the construction cannot handle them; a looping edge e_{loop} has a single endpoint vertex, whence, by definition, the corresponding diagonal $\Delta_{e_{\text{loop}}} \cong X^{\mathbf{V}_\Gamma}$, and consequently $\mathrm{Conf}_\Gamma(X) = \varnothing$.

Remark. There is no real reason to disregard looping edges, as they tend to appear when computing radiative corrections in perturbative quantum field theory. However, the treatment of such graphs, in this setting, would call for a different method.

A *subgraph* $\gamma \subseteq \Gamma$ is a subset of vertices and edges in Γ, such that for every edge $e \in \mathbf{E}_\gamma$ the endpoints $\partial_\Gamma(e) \in \mathbf{V}_\gamma$. Two vertices $v, v' \in \mathbf{V}_\Gamma$ are said to be *connected* if Γ contains a path (sequence of edges) between v and v'. A graph is connected if every pair of vertices is connected.

Definition 3.1 (Induced subgraph)**.** An *induced subgraph* γ of Γ, is a subgraph such that two vertices $v, v' \in \mathbf{V}_\gamma$ are connected by an edge $e \in \mathbf{E}_\gamma$ if and only if they are connected by an edge $e \in \mathbf{E}_\Gamma$. We denote the set of all connected, induced subgraphs of Γ by $\mathbf{SG}(\Gamma)$. We denote the set of subgraphs of Γ that are unions of disjoint, connected, induced subgraphs by $\widehat{\mathbf{SG}}(\Gamma)$.

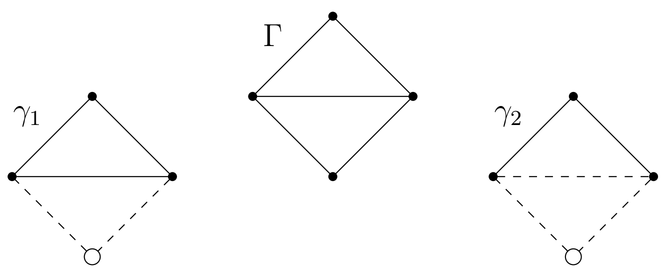

Figure 3.2: Example and counterexample of an induced subgraph. Solid lines and filled circles indicate inclusion in the graph, dashed lines and unfilled circles indicate inclusion in $\Gamma \setminus \gamma_i$. $\gamma_1 \subset \Gamma$ is an induced subgraph; $\gamma_2 \subset \Gamma$ is *not* an induced subgraph.

An example and a counterexample of an induced subgraph is presented in Figure 3.2. For an induced subgraph $\gamma \subset \Gamma$, we define the *corresponding diagonal* as

$$\Delta_\gamma = \{(x_{v_1}, \ldots, x_{v_n}) \in X^{\mathbf{V}_\Gamma} : x_{v_i} = x_{v_j} \ \forall v_i, v_j \in \mathbf{V}_\gamma\}, \tag{3.2}$$

and the corresponding *polydiagonal* as

$$\hat{\Delta}_\gamma = \{(x_{v_1}, \ldots, x_{v_n}) \in X^{\mathbf{V}_\Gamma} : x_v = x_{v'} \text{ for } \{v, v'\} = \partial_\gamma(e), e \in \mathbf{E}_\gamma\}. \tag{3.3}$$

Note that, when a subgraph γ is connected, $\Delta_\gamma = \hat{\Delta}_\gamma$. We have the following definition.

Definition 3.2. A graph Γ is *biconnected* (*2-vertex-connected*) if it cannot be made disconnected by the removal of a single vertex, along with any incident edges for said vertex.

From the configuration spaces of Feynman graphs, Ceyhan and Marcolli construct *wonderful compactifications* in the framework developed by Li [Li09], from families of polydiagonals $\hat{\Delta}_\gamma$ as defined in (3.3). In the setting of [Li09], a *simple arrangement* $\mathcal{S}$ of a complex manifold X is a finite collection of non-singular subvarieties, $\{S_i\}$, with the following properties

- $\mathcal{S}$ is closed under (non-empty) intersections.
- Any two $S_i, S_j \in \mathcal{S}$ *intersects cleanly*, that is along a non-singular subvariety, with the tangent bundle of the intersection equal to the intersection of the restrictions of the respective tangent bundles.

Definition 3.3. A *building set* $\mathcal{G}$ for a simple arrangement $\mathcal{S}$ is a subset $\mathcal{G}$ of $\mathcal{S}$ such that $\forall S \in \mathcal{S} \setminus \mathcal{G}$ the minimal elements $G \in \mathcal{G}$ with $G \supseteq S$ intersects transversely with intersection S.

Let $\mathcal{G}$ be a finite collection of non-singular subvarieties. If the set of all possible intersections of elements in $\mathcal{G}$ forms a simple arrangement $\mathcal{S}$, and $\mathcal{G}$ constitutes a building set of $\mathcal{S}$, we say that $\mathcal{S}$ is the *simple arrangement induced by* $\mathcal{G}$.

The reason for introducing simple arrangements and building sets, is that the configuration space associated to a Feynman diagram Γ admits a wonderful compactification, the construction of which is determined by the building set of a simple arrangement of polydiagonals corresponding to subgraphs $\gamma \subseteq \Gamma$. The following three theorems, that we state without proof, connect Feynman diagrams in configuration space with the aforementioned simple arrangements of polydiagonals.

Theorem 3.4 (Lemma 5 in [CM12a]). *For a given graph Γ, the collection*

$$\mathcal{S}_\Gamma = \{\hat{\Delta}_\gamma : \gamma \in \widehat{\mathbf{SG}}(\Gamma)\}, \tag{3.4}$$

is a simple arrangement of (polydiagonal) subvarieties in $X^{\mathbf{V}_\Gamma}$*.*

Theorem 3.5 (Proposition 1 in [CM12a])**.** *For a given graph* Γ*, the set*

$$\mathcal{G}_\Gamma = \{\Delta_\gamma : \gamma \subseteq \mathbf{SG}(\Gamma) \textit{ induced, biconnected } \} \tag{3.5}$$

is a building set for the arrangement $\mathcal{S}_\Gamma$ *in* (3.4)*.*

Theorem 3.6 (Lemma 7 in [CM12a])**.** *We have that*

$$\mathrm{Conf}_\Gamma(X) = X^{\mathbf{V}_\Gamma} \setminus \bigcup_{\Delta_\gamma \in \mathcal{G}_\Gamma} \Delta_\gamma,$$

where $\mathcal{G}_\Gamma$ *is as in Theorem 3.5.*

From these results and [Li09, Theorem 1.2] Ceyhan and Marcolli [CM12a] assert that there is a *wonderful compactification* $\overline{\mathrm{Conf}_\Gamma}(X)$ of the configuration space $\mathrm{Conf}_\Gamma(X)$ given as the closure of the image of $\mathrm{Conf}_\Gamma(X)$ under the inclusion

$$\iota : \mathrm{Conf}_\Gamma(X) \hookrightarrow \prod_{\Delta_\gamma \in \mathcal{G}_\Gamma} \mathrm{Bl}_{\Delta_\gamma} X^{\mathbf{V}_\Gamma}. \tag{3.6}$$

Moreover, one of the main results (Theorem 1.3) in Li's paper gives an alternative, constructive description of $\overline{\mathrm{Conf}_\Gamma}(X)$ as the result of an iterated sequence of blowups as follows:

- Enumerate the set $\mathcal{G}_\Gamma = \{\Delta_{\gamma_1}, \dots, \Delta_{\gamma_N}\}$ such that whenever there is a containment $\gamma_i \supset \gamma_j$ ($\implies \Delta_{\gamma_i} \subset \Delta_{\gamma_j}$), the order of the corresponding indices is $i < j$.
- Define $Y^{(0)} = X^{\mathbf{V}_\Gamma}$.
- For $k = 1, \dots, N$, let $Y^{(k)}$ be the blow-up of $Y^{(k-1)}$ along the iterated strict transform of Δ_{Γ_k}.
- Theorem 1.3 and Proposition 2.13 of [Li09] then shows that $Y^{(N)}$ is isomorphic to the wonderful compactification $\overline{\mathrm{Conf}_\Gamma}(X)$.

Now with algorithms for constructing the configuration space and its wonderful compactification, the only thing left is a set of *Feynman rules*, i.e., a consistent way of translating the Feynman graphs to integral expressions. Such a set of rules, associated to a massless scalar quantum field theory on X, $\dim_{\mathbb{C}} X = D$, are provided by Ceyhan and Marcolli; the *(complexified) Feynman amplituded associated to the graph* Γ is given by

$$\omega_\Gamma = \prod_{e \in \mathbf{E}_\Gamma} \frac{1}{\|x_{s(e)} - x_{t(e)}\|^{2(D-1)}} \bigwedge_{v \in \mathbf{V}_\Gamma} \mathrm{d}x_v \wedge \mathrm{d}\bar{x}_v. \tag{3.7}$$

where $\|\cdot\|$ is the *euclidean norm on* X and $\{s(e), t(e)\} = \partial_\Gamma(e)$. We will consider $X = \mathbb{CP}^D$ and regard $\|\cdot\|$ as the norm on a chosen $\mathbb{C}^D \subset \mathbb{CP}^D$; $x_{s(e)}$ and $x_{t(e)}$, for each edge e, are the standard coordinates on that $\mathbb{C}^D$. The chain of integration associated to ω_Γ in (3.7) is $(\mathbb{C}^D)^{\mathbf{V}_\Gamma}$. We may allow for Γ to have multiple edges. As stated above, this does not affect the construction of configuration space and its wonderful compactification. However, if we have multiple edges, say N edges between vertices v_1 and v_2, then the factor $\|x_{v_1} - x_{v_2}\|^{2(D-1)}$ in ω_Γ is replaced with $\|x_{v_1} - x_{v_2}\|^{2N(D-1)}$.

Remark. We call the Feynman amplitude complexified since it is intended to be integrated over the complex manifold $X^{\mathbf{V}_\Gamma}$. We adopt the terminology of Ceyhan and Marcolli in [CM12b], who describe two different constructions related to configuration space Feynman integrals. One is corresponds to a *physical* Feynman integral, where the amplitude is defined over the real locus of $X^{\mathbf{V}_\Gamma}$. The other, which they refer to as *complexified* and is the one presented above, is a mathematical generalisation that shows both similarities and differences with respect to the physical case. The reason we look at the complexified case, and not the physical case, is, straightforwardly, the fact that the mathematical techniques that can be applied to the two cases differs significantly, and where we commit to techniques suited for complexified case. An interest in the application of these techniques is the determining factor, more so than the physical relevance of the model we analyse.

Remark. One should note that the complexified construction in [CM12b] also involves a particular doubling of the ambient space, $X^{\mathbf{V}_\Gamma}$ is replace by $X^{\mathbf{V}_\Gamma} \times X^{\mathbf{V}_\Gamma}$, and a projection map to one of the factors. This is done in such a way that the wonderful compactification described above is contained within a larger space; moreover, the chain of integration for ω_Γ is $X^{\mathbf{V}_\Gamma} \times \{p\} \subset X^{\mathbf{V}_\Gamma} \times X^{\mathbf{V}_\Gamma}$, where p is a point. We have omitted this detail in our construction since it has no impact on the work in this thesis.

3.1.1 Infrared divergences

A feature that does not seem to be properly addressed in [CM12a; CM12b], at least as far as the author is aware, is the possibility of singularities *not* associated to the diagonals in configuration space. As we will come to find in Chapter 4 when looking at some explicit cases, the amplitudes corresponding to certain Feynman graphs over projective space are singular at *hyperplanes at infinity* in $\mathrm{Conf}_\Gamma(\mathbb{CP}^D)$. These singularities, which we refer to as *infrared singularities*, need also be handled for us to be able to regularise the integral using the techniques in [Len20] that we briefly recall in § 3.2 below. Here we do not have an algorithmic way to proceed, in contrast to how we construct the wonderful compactification of configuration space. Instead we take the final steps on more of a case by case basis. We will see that there appears to be structure behind what we do, and we shall hint towards a more algorithmic approach, although we will not be giving any proofs of such. Our goal is to find a modification of $\overline{\mathrm{Conf}}_\Gamma(\mathbb{CP}^D)$ such that the variety defined by the singular set of the Feynman amplitude is a normal crossings divisor. This is somewhat parallel to what was discussed in § 2.2.3.

3.2 Regularisation of divergent integrals

In this section, we describe a treatment of certain divergent integrals that, in particular, can be applied to the Feynman integrals introduced in § 3.1, explicit cases of which are studied in Chapter 4. We call this treatment a *regularisation*, since it entails constructing a modification of a given integral, where we introduce an auxiliary parameter that can be used to describe the nature of the divergence, mirroring many of the regularisation techniques used in physics.

Central to this thesis are what Lennartsson [Len20] calls *quasi-meromorphic forms*, defined below.

Definition 3.7 (Quasi-meromorphic form)**.** A *quasi-meromorphic form* on a complex manifold M is a singular differential form α that can be written locally as

$$\alpha = \frac{\tilde{\alpha}}{f\bar{g}},$$

where $\tilde{\alpha}$ is a smooth differential form and where f and g are holomorphic functions not identically zero. A quasi-meromorphic form α is said to be in $\mathcal{E}(*\bar{*}\mathfrak{D})$ if there is a hypersurface $\mathfrak{D} \subset M$ such that $\{fg = 0\} \subseteq \mathfrak{D}$. We denote by $\mathcal{QM}(M)$ the set of quasi-meromorphic forms on M.

We define the *polar set* $P(\alpha)$ of $\alpha \in \mathcal{QM}(M)$ to be the set of points where α is not smooth. Furthermore, we define $P^{1,0}(\alpha) \subseteq P(\alpha)$ as the *complement* of the set of points $p \in P(\alpha)$ for which there exists a holomorphic function $g \not\equiv 0$ in a neighbourhood of p, such that $\bar{g}\alpha$ is smooth there. Similarly, we define $P^{0,1}(\alpha) \subseteq P(\alpha)$ to be the complement of the set of points $p \in P(\alpha)$ for which there exists a holomorphic function $f \not\equiv 0$ in a neighbourhood of p, such that $f\alpha$ is smooth. The sets $P^{1,0}(\alpha)$ and $P^{0,1}(\alpha)$ are said to be the set of *holomorphic singularities* and *anti-holomorphic singularities*, respectively, of α. By the definition of α we have that $P(\alpha) = P^{1,0}(\alpha) \cup P^{0,1}(\alpha)$. Note that $P^{1,0}(\alpha) \cap P^{0,1}(\alpha)$ need not be empty, it is simply the set where α has both holomorphic and anti-holomorphic poles.

Henceforth, we consider the case where α is a quasi-meromorphic as in Definition 3.7, but where everywhere locally on M, $f = g$, i.e.,

$$\alpha = \frac{\tilde{\alpha}}{|f|^2};$$

thus $P^{1,0}(\alpha) = P^{0,1}(\alpha) = P(\alpha)$. If $\alpha \in \mathcal{E}(*\bar{*}\mathfrak{D})$ then $P(\alpha)$ is a hypersurface contained in $\mathfrak{D}$. In this setting Lennartsson [Len20] proposes a way of quantifying how "bad" the singularity of α is. This is done as follows. We define $H(\alpha)_0 = M$, $H(\alpha)_1$ to be the codimension 1 components of $P(\alpha)$, and, for $k = 2, \dots, D$, where $D = \dim_{\mathbb{C}} M$, $H(\alpha)_k$ is defined to be $\left(H(\alpha)_{k-1}\right)_{\text{sing}}$ along with the all the components of $H(\alpha)_{k-1}$ with codimension $\geq k$. Then we have a sequence of inclusions

$$H(\alpha)_D \subset H(\alpha)_{D-1} \subset \cdots \subset H(\alpha)_1 \subset H(\alpha)_0.$$

Remark. This construction is an example of what is known as a *stratification* of M.

We define the integer $\kappa(\alpha)$ to be the largest number k such that $H(\omega)_k$ is non-empty. If the hypersurface $\mathfrak{D}$ has normal crossings, there is a more straightforward alternative description of $\kappa(\alpha)$. In a neighbourhood of any point $p \in M$ there are local coordinates $(z_1, \dots, z_D)$ such that the hypersurface $\mathfrak{D} = \{z_1 \cdots z_k = 0\}$. There is then a multi-index $J = (J_1, \dots, J_D)$ such that $|z^J|^2\alpha$; here $z^J = z_1^{J_1} \cdots z_D^{J_D}$. With the minimal choice of non-zero entries of J such that $|z^J|^2\alpha$ is smooth at p, we define

$$\kappa_p(\alpha) \coloneqq \#\{j : J_j \neq 0\}, \tag{3.8}$$

and then we have

$$\kappa(\alpha) = \max_{p \in M} \kappa_p(\alpha). \tag{3.9}$$

In this description κ is given by the maximal number of intersecting hyperplanes in a local description of $\mathfrak{D}$.

Let $\sigma : M \to L$ be a holomorphic section of some line bundle L such that $\mathfrak{D} = \operatorname{div}(\sigma)$. For $\alpha \in \mathcal{E}(*\bar{*}\mathfrak{D})$ and $\lambda \in \mathbb{C}$ such that $\mathfrak{Re}\,\lambda \gg 1$, $\|\sigma\|^{2\lambda}$ is locally integrable; here $\|\cdot\|$ is some choice of hermitian metric on L. We refer to $\|\sigma\|^{2\lambda}\alpha$ as a *regularisation of* α. For a test function ξ, we let

$$F_\xi(\lambda) = \int_M \|\sigma\|^{2\lambda} \alpha \wedge \xi. \tag{3.10}$$

Theorem 3.8 (Theorem 2.3 in [Len20])**.** *For $\alpha \in \mathcal{E}(*\bar{*}\mathfrak{D})$, where $\mathfrak{D} \subset M$ is an analytic hypersurface with normal crossings, the function* (3.10) *has a meromorphic continuation to $\mathbb{C}$. Furthermore the possible poles of F_ξ are contained in $\mathbb{Q}$ and the order of the pole at the origin is $\leq \kappa(\alpha)$.*

Remark. As stated by Theorem 3.8, note that κ is an upper bound and not the final word on the degree of the singularity of the Feynman integral. This will become clear when we look at explicit cases of Feynman integrals in Chapter 4.

Lennartsson [Len20] goes on to show that the meromorphic continuation of F_ξ has a Laurent series expansion in λ around $\lambda = 0$

$$F_\xi(\lambda) = \frac{1}{\lambda^{\kappa(\alpha)}} \langle \mu_{\kappa(\alpha)}, \xi \rangle + \cdots + \frac{1}{\lambda} \langle \mu_1, \xi \rangle + \langle \mu_0, \xi \rangle + \mathcal{O}(|\lambda|), \tag{3.11}$$

where μ_j for $j \leq \kappa(\alpha)$ are top (bi-)degree currents. The current μ_j has support in the set where j irreducible components of $\mathfrak{D}$ intersect. It follows that $\mu_0|_{M\setminus|\mathfrak{D}|} = \alpha$ as currents in $M \setminus |\mathfrak{D}|$. Thus, μ_0 is a current extension of α across $|\mathfrak{D}|$.

There is an ambiguity when defining $F_\xi(\lambda)$, in the choice of section σ and metric $\|\cdot\|$. A priori, thus, the μ_j's depend on these choices, and indeed they do. However, as showed by Lennartsson [Len20], $\mu_{\kappa(\alpha)}$ in (3.11) is independent on the choice of metric and section. To emphasise this we give the following definition.

Definition 3.9 (Canonical current)**.** For $\alpha \in \mathcal{E}(*\bar{*}\mathfrak{D})$, where $\mathfrak{D}$ is a normal crossings divisor, we call $\mu_{\kappa(\alpha)}$ the *canonical current* associated to α, and denote it by $\{\alpha\}$.

Now, assume that $\alpha \wedge \xi$ has compact support in a polydisc $\Delta \subset M$, where we have local coordinates $z = (z_1, \dots, z_D)$, such that

$$\alpha \wedge \xi = \frac{\psi}{|z^J|^2} \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where ψ is smooth and compactly supported, J is a non-negative multi-index and $\mathfrak{D} \cap \Delta = \{z_1 \cdots z_{\kappa(\alpha)} = 0\}$. Following [Len20, § 2.3], one finds that the canonical current $\{\alpha\}$ is given by

$$\langle \{\alpha\}, \xi \rangle = \frac{1}{(J - 1_J)!^2} \int\limits_{\{z_1 = \cdots = z_{\kappa(\alpha)} = 0\}} \frac{\partial^{2(J-1_J)} \psi}{\partial z^{J-1_J} \partial \bar{z}^{J-1_J}} \mathrm{d}z' \wedge \mathrm{d}\bar{z}', \tag{3.12}$$

where $\mathrm{d}z' \wedge \mathrm{d}\bar{z}' = \mathrm{d}z_{\kappa(\alpha)+1} \wedge \mathrm{d}\bar{z}_{\kappa(\alpha)+1} \wedge \cdots \wedge \mathrm{d}z_D \wedge \mathrm{d}\bar{z}_D$, 1_J is a mutli-index such that

$$(1_J)_j = \begin{cases} 1 \text{ for } J_j \neq 0, \\ 0 \text{ for } J_j = 0, \end{cases}$$

and

$$(J - 1_J)! = (J_1 - 1_1)! \cdots (J_D - 1_D)!.$$

Remark. We see from (3.12), that the support of the canonical current $\{\alpha\}$ is indeed contained in the codimension $\kappa(\alpha)$ component of $\mathfrak{D} \cap \Delta$.

If M is compact, then $\xi = 1$ is a test function on M, and it is natural to let

$$\int_M \alpha := \langle \mu_0, 1 \rangle. \tag{3.13}$$

However, the definition of (3.13) depends on choices. In local coordinates on M, μ_j is given explicitly, see, e.g., (3.12). Using a partition of unity $\{\rho_\iota\}$, one can in principal calculate μ_j. In fact, if ρ_ι has support in a coordinate chart, then $\rho_\iota \cdot \mu_j$ can be explicitly computed, and

$$\langle \mu_j, \xi \rangle = \langle \sum_\iota \rho_\iota \cdot \mu_j, \xi \rangle = \sum_\iota \langle \rho_\iota \cdot \mu_j, \xi \rangle. \tag{3.14}$$

3.2.1 Regularisation of divergent Feynman integrals

Our object of interest are divergent integrals, formally given as

$$\int_X \omega. \tag{3.15}$$

We obtain these divergent integrals from Feynman diagrams, the details of which are given in § 3.1. The integrand ω will generically be a smooth form of top degree on X but with singularities along a subvariety H of codimension ≥ 1, possibly not of pure codimension. Thus, a priori ω neither defines a differential form nor a current on X. However, since it is smooth on $X \setminus H$, ω defines a current on $X \setminus H$. We want to find an extension of ω as a current from $X \setminus H$ to X. To make use of the techniques in the previous section, we make the following two assumptions.

1. There is a (holomorphic) vector bundle $E \to X$ and a global holomorphic section $\sigma : X \to E$ defining the singular locus of ω.

2. For $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,\lambda$ sufficiently large $\|\sigma\|^{2\lambda}\omega$ is smooth, where $\|\cdot\|$ is some metric on E.

Under these assumptions, for any test function ξ, the function

$$F_\xi(\lambda) = \int_X \|\sigma\|^{2\lambda}\omega \wedge \xi,$$

is analytic if $\mathfrak{Re}\,\lambda \gg 1$.

Let $\pi : Y \to X$ be a modification. Since π is proper, if ξ is a test function on X, then $\pi^*\xi$ is a test function on Y. With π we can consider the pullback of $\|\sigma\|^{2\lambda}\omega$ to Y, which is smooth for $\mathfrak{Re}\lambda \gg 1$, and thereby defines a current on Y. For $\mathfrak{Re}\,\lambda \gg 1$, $\|\sigma\|^{2\lambda}\omega$ is locally integrable, and since π is a biholomorphism outside a set of measure 0, we have

$$\int_X \|\sigma\|^{2\lambda}\omega \wedge \xi = \int_Y \|\pi^*\sigma\|^{2\lambda}\pi^*\omega \wedge \pi^*\xi. \tag{3.16}$$

Moreover, assuming that π is such that the preimage of the locus of divergence of ω in X is a normal crossings divisor $\mathfrak{D} \subset Y$, then $\pi^*\omega$ will be singular on $\mathfrak{D}$. Thus, Theorem 3.8 is applicable to the right-hand side of (3.16) which implies that there is a meromorphic continuation of the right-hand-side of (3.16) to $\mathbb{C}$. However, by uniqueness of meromorphic continuations, this means that this is a meromorphic continuation of the left-hand side of (3.16), i.e., of $F_\xi(\lambda)$, as well.

Consider the Laurent series expansion at $\lambda = 0$ of the right-hand side of (3.16),

$$\int_Y \|\pi^*\sigma\|^{2\lambda}\pi^*\omega \wedge \pi^*\xi = \frac{1}{\lambda^{\kappa(\pi^*\omega)}}\langle \mu_{\kappa(\pi^*\omega)}, \pi^*\xi\rangle + \cdots + \frac{1}{\lambda}\langle \mu_1, \pi^*\xi\rangle + \langle \mu_0, \pi^*\xi\rangle + \mathcal{O}(|\lambda|),$$

where μ_j, for $j = 0, \ldots, \kappa(\pi^*\omega)$ are currents on Y. We can regard the coefficients as push-forwards of the currents μ_j to X acting on test functions ξ on X via Definition 2.11. We then have an expression for the expansion of the meromorphic continuation of $F_\xi(\lambda)$ at zero,

$$\int_X \|\sigma\|^{2\lambda}\omega \wedge \xi = \frac{1}{\lambda^{\kappa(\omega)}}\langle \pi_*\mu_{\kappa(\omega)}, \xi\rangle + \cdots + \frac{1}{\lambda}\langle \pi_*\mu_1, \xi\rangle + \langle \pi_*\mu_0, \xi\rangle + \mathcal{O}(|\lambda|) \tag{3.17}$$

where $\pi_*\mu_j$, for $j = 0, \ldots, \kappa(\pi^*\omega)$ are currents on X, and where we define $\kappa(\omega) = \kappa(\pi^*\omega)$. Notice that μ_j, for $j \geq 1$, have support in H. As in the previous section, thus, $\pi_*\mu_0$ is a current extension of ω to X. It is natural to define (3.15) as $\langle \pi_*\mu_0, 1\rangle$; $\pi_*\mu_0$ is a finite part of ω, however, it depends on the choice of metric and the section.

4

A Case Study

In this chapter we present case studies for the three different Feynman diagrams and their associated integrals. Starting off with the simplest possible graph, we will be thorough in our analysis. Moving on to the other two cases, much of the treatment will be similar and therefore we will present fewer intermediate steps.

4.1 Case One: Propagator

We begin by considering the, unquestionably simple, Feynman diagram Γ_1 in Figure 4.1 consisting of only two vertices and a single connecting edge, contributing to the perturbative amplitude of a process in a scalar quantum field theory on the space $X = \mathbb{CP}^D$. We call Γ_1 a *propagator* after the physics interpretation of such a graph.

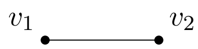

Figure 4.1: Feynman graph Γ_1.

Following the construction in [CM12a; CM12b], summarised in § 3.1, we see that $X^{\mathbf{V}_{\Gamma_1}} = \mathbb{CP}^D \times \mathbb{CP}^D$, and the configuration space is simply

$$\mathrm{Conf}_{\Gamma_1}(X) \cong \mathbb{CP}^D \times \mathbb{CP}^D \setminus \Delta = \{([X],[Y]) \in \mathbb{CP}^D_{[X]} \times \mathbb{CP}^D_{[Y]} : [X] \neq [Y]\}. \tag{4.1}$$

Moving to the wonderful compactification of the configuration space, we note that since Γ_1 has no induced proper subgraphs, i.e., $\mathbf{SG}(\Gamma_1) = \widehat{\mathbf{SG}}(\Gamma_1) = \{\Gamma_1\}$; this will make virtually all of the graph theoretical considerations void. The simple arrangement of (poly)diagonal subvarieties in $X^{\mathbf{V}_{\Gamma_1}}$ becomes

$$\mathcal{S}_{\Gamma_1} = \{\hat{\Delta}_\gamma : \gamma \in \widehat{\mathbf{SG}}(\Gamma_1)\} = \{\Delta\},$$

since the only induced subgraph is Γ_1 itself which is connected and only contains one edge, whence $\hat{\Delta}_{\Gamma_1} = \Delta_{\Gamma_1} = \Delta$. The building set for the simple arrangement $\mathcal{S}_{\Gamma_1}$ is

$$\mathcal{G}_{\Gamma_1} = \{\Delta_\gamma : \gamma \subseteq \mathbf{SG}(\Gamma_1) \text{ induced, biconnected}\},$$

see Theorem 3.5, and trivially becomes $\mathcal{G}_{\Gamma_1} = \{\Delta\}$. The wonderful compactification $\overline{\mathrm{Conf}}_{\Gamma_1}(X)$ is given by the closure of $\mathrm{Conf}_{\Gamma_1}(X)$ in the blowup $\mathrm{Bl}_\Delta X^{\mathbf{V}_{\Gamma_1}}$ which is just $\mathrm{Bl}_\Delta X^{\mathbf{V}_{\Gamma_1}}$ itself.

Take $(X, Y) = ([X_0 : \ldots : X_D], [Y_0 : \ldots : Y_D])$ to be homogeneous coordinates on $\mathbb{CP}^D \times \mathbb{CP}^D$. Now consider the chart defined by $\{X_0 \neq 0\} \times \{Y_0 \neq 0\} \cong \mathbb{C}^D \times \mathbb{C}^D \cong \mathbb{C}^{2D}$ with local coordinates

$$(1, x_1, \ldots, x_D, 1, y_1, \ldots, y_D) \coloneqq \Big(1, \frac{X_1}{X_0}, \ldots, \frac{X_D}{X_0}, 1, \frac{Y_1}{Y_0}, \ldots, \frac{Y_D}{Y_0}\Big).$$

In these local coordinates, $(x, y) = (x_1, \ldots, x_D, y_1, \ldots, y_D)$, the Feynman amplitude associated to Γ_1, according to (3.7), is given by

$$\omega_{\Gamma_1} = \frac{1}{\|x - y\|^{2(D-1)}} \mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}. \tag{4.2}$$

Let

$$\pi_1 : \mathrm{Bl}_\Delta X^{\mathbf{V}_{\Gamma_1}} \to X^{\mathbf{V}_{\Gamma_1}} \tag{4.3}$$

be the blowup along the diagonal Δ.

Proposition 4.1. *The pullback $\pi_1^* \omega_{\Gamma_1}$ is smooth in $\pi_1^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})$. In particular, ω_{Γ_1} is locally integrable in $\{X_0 \neq 0, Y_0 \neq 0\} \subset \mathbb{CP}^D \times \mathbb{CP}^D$.*

Proof. To make the local description of the blowup simpler we make the following change of coordinates, $(x, y) \mapsto (z, w)$ where $z_i = x_i - y_i$ and $w_i = x_i + y_i$ for $1 \leq i \leq D$. A straight forward calculation tells us that the Jacobian determinant of the change of coordinates $\det J_{\mathbb{C}} = 2^{-D}$, whence, in the coordinates (z, w), the Feynman amplitude becomes

$$\omega_{\Gamma_1} = \frac{1}{\|z\|^{2(D-1)}} \frac{1}{2^{2D}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}. \tag{4.4}$$

We will now blow up along $z = 0$. In view of § 2.2.2 above, in one chart in the blowup we have

$$\pi_1 : (\zeta_1, \ldots, \zeta_D, \eta_1, \ldots, \eta_D) \mapsto (\zeta_1, \zeta_1\zeta_2, \ldots, \zeta_1\zeta_D, \eta_1, \ldots, \eta_D) = (z_1, \ldots, z_D, w_1, \ldots, w_D).$$

The pullback of ω_{Γ_1} to this chart of the blowup is given by

$$\begin{aligned} \pi_1^* \omega_{\Gamma_1} &= \frac{1}{|\zeta_1|^{2(D-1)} \left(1 + |\zeta_2|^2 + \cdots + |\zeta_D|^2\right)^{D-1}} \frac{1}{2^{2D}} |\zeta|^{2(D-1)} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \\ &= \frac{1}{\left(1 + |\zeta_2|^2 + \cdots + |\zeta_D|^2\right)^{D-1}} \frac{1}{2^{2D}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}. \end{aligned} \tag{4.5}$$

We see that on the exceptional divisor $\mathrm{Exc}(\pi_1)$, defined in this chart by $\{\zeta_1 = 0\}$, the pullback of ω_{Γ_1} is smooth. Thus, we see that the singularity of ω_{Γ_1} on the diagonal was in fact integrable, since it disappeared completely in the blowup. The other charts in the blowup along $z = 0$, are handeled in the same way. □

4.1.1 Infrared Divergences of ω_{Γ_1}

Any hyperplane of a projective space may be singled out as the *hyperplane at infinity* H_∞. However, our choice of local coordinates (x, y) above forces the choice of hyperplane at infinity in $\mathbb{CP}^D$ to be $H_\infty = \{X_0 = 0\}$, and $H_\infty = \{Y_0 = 0\}$ in the second copy of $\mathbb{CP}^D$. We generalize the notion of hyperplane at infinity to the product space $\mathbb{CP}^D \times \mathbb{CP}^D$, and refer to $H_\infty \times \mathbb{CP}^D$ and $\mathbb{CP}^D \times H_\infty$ as hyperplanes at infinity in $\mathbb{CP}^D \times \mathbb{CP}^D$.

To study the behaviour of ω_{Γ_1} in a neighbourhood of the hyperplanes at infinity, $H_\infty \times \mathbb{CP}^D$ and $\mathbb{CP}^D \times H_\infty$, there are three types of points $(p, q) \in \mathbb{CP}^D \times \mathbb{CP}^D$ to consider.

(A) : $p \in H_\infty$, $q \notin H_\infty$.

(B) : $p = q \in H_\infty$.

(C) : $p, q \in H_\infty$, $p \neq q$.

By symmetry we can have (A) represent also the case $q \in H_\infty$, $p \notin H_\infty$. We have the following proposition.

Proposition 4.2. *In a neighbourhood of a point of type (A) there are local coordinates such that*

$$\omega_{\Gamma_1} = \frac{1}{|x_1|^4} \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}}{\left(|1 - x_1 y_1|^2 + \sum_{j=2}^{D} |x_j - x_1 y_j|^2\right)^{D-1}}, \tag{4.6}$$

where the point lies on $x_1 = 0$.

In a neighbourhood of a point of type (B) there are local coordinates such that

$$\omega_{\Gamma_1} = \frac{1}{|x_1 y_1|^4} \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}}{\left(|y_1 - x_1|^2 + \sum_{j=2}^{D} |x_j y_1 - y_j x_1|^2\right)^{D-1}}, \tag{4.7}$$

where the point satisfies $x_j = y_j$ for $1 \leq j \leq D$ where $x_1 = y_1 = 0$.

In a neighbourhood of a point of type (C) there are local coordinates such that

$$\omega_{\Gamma_1} = \frac{1}{|x_1 y_1|^4} \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}}{\left(|y_1 - x_1 y_2|^2 + |x_2 y_1 - x_1|^2 + \sum_{j=3}^{D} |x_j y_1 - y_j x_1|^2\right)^{D-1}}, \tag{4.8}$$

where the point lies on $x_1 = y_1 = 0$.

Proof. To study the behaviour of ω_{Γ_1} in a neighbourhood of a point of type (A), we can consider the local coordinates

$$(\tilde{x}_1, 1, \tilde{x}_2, \dots, \tilde{x}_D, 1, y_1, \dots, y_D) \coloneqq \Big(\frac{X_0}{X_1}, 1, \frac{X_2}{X_1}, \dots, \frac{X_D}{X_1}, 1, \frac{Y_1}{Y_0}, \dots, \frac{Y_D}{Y_0}\Big). \tag{4.9}$$

We note that, in the coordinates $(\tilde{x}, y)$, $H_\infty \times \mathbb{CP}^D = \{\tilde{x}_1 = 0\}$. The transition map from the (x, y) to the $(\tilde{x}, y)$ chart is given by

$$x_1 = \frac{1}{\tilde{x}_1}, \quad x_i = \frac{\tilde{x}_i}{\tilde{x}_1} \quad \text{for } 2 \leq i \leq D. \tag{4.10}$$

The holomorphic D-form $\mathrm{d}x$ transforms as

$$\begin{aligned} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_D &= -\frac{1}{\tilde{x}_1^2}\mathrm{d}\tilde{x}_1 \wedge \Big(\frac{1}{\tilde{x}_1}\mathrm{d}\tilde{x}_2 - \frac{\tilde{x}_2}{\tilde{x}_1^2}\mathrm{d}\tilde{x}_1\Big) \wedge \dots \wedge \Big(\frac{1}{\tilde{x}_1}\mathrm{d}\tilde{x}_D - \frac{\tilde{x}_D}{\tilde{x}_1^2}\mathrm{d}\tilde{x}_1\Big) \\ &= -\frac{1}{\tilde{x}_1^{D+1}}\mathrm{d}\tilde{x}_1 \wedge \dots \wedge \mathrm{d}\tilde{x}_D. \end{aligned}$$

The volume form $\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}$ becomes

$$\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y} = \frac{1}{|\tilde{x}_1|^{2(D+1)}}\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}.$$

In the new coordinates, the Feynman amplitude is given by

$$\begin{aligned} \omega_{\Gamma_1} &= \frac{1}{\big(|\frac{1}{\tilde{x}_1} - y_1|^2 + |\frac{\tilde{x}_2}{\tilde{x}_1} - y_2|^2 + \dots + |\frac{\tilde{x}_D}{\tilde{x}_1} - y_D|^2\big)^{D-1}} \frac{1}{|\tilde{x}_1|^{2(D+1)}}\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y} \\ &= \frac{1}{\big(|1 - \tilde{x}_1 y_1|^2 + |\tilde{x}_2 - \tilde{x}_1 y_2|^2 + \dots + |\tilde{x}_D - \tilde{x}_1 y_D|^2\big)^{D-1}} \frac{1}{|\tilde{x}_1|^4}\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}. \end{aligned}$$

Now, we want to consider ω_{Γ_1} in a neighbourhood of a point of type (B). Starting out in the coordinates (x, y) as before, we make the change from (x, y) to $(\tilde{x}, \tilde{y})$ by (4.10) in both x and y. In these coordinates we have that $H_\infty \times H_\infty = \{\tilde{x}_1 = \tilde{y}_1 = 0\}$. The Feynman amplitude then becomes

$$\begin{aligned} \omega_{\Gamma_1} &= \frac{1}{\big(|\frac{1}{\tilde{x}_1} - \frac{1}{\tilde{y}_1}|^2 + |\frac{\tilde{x}_2}{\tilde{x}_1} - \frac{\tilde{y}_2}{\tilde{y}_1}|^2 + \dots + |\frac{\tilde{x}_D}{\tilde{x}_1} - \frac{\tilde{y}_D}{\tilde{y}_1}|^2\big)^{D-1}} \frac{1}{|\tilde{x}_1\tilde{y}_1|^{2(D+1)}}\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\bar{\tilde{y}} \\ &= \frac{1}{\big(|\tilde{y}_1 - \tilde{x}_1|^2 + |\tilde{x}_2\tilde{y}_1 - \tilde{y}_2\tilde{x}_1|^2 + \dots + |\tilde{x}_D\tilde{y}_1 - \tilde{y}_D\tilde{x}_1|^2\big)^{D-1}} \frac{1}{|\tilde{x}_1\tilde{y}_1|^4}\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\bar{\tilde{y}}. \end{aligned}$$

Lastly, we want to consider ω_{Γ_1} in a neighbourhood of a point of type (C). As a representative, we may consider the following coordinates,

$$(\tilde{x}_1, 1, \tilde{x}_2, \dots, \tilde{x}_D, \tilde{y}_1, \tilde{y}_2, 1, \tilde{y}_3, \dots, \tilde{y}_D) = \Big(\frac{X_0}{X_1}, 1, \frac{X_2}{X_1}, \dots, \frac{X_D}{X_1}, \frac{Y_0}{Y_2}, \frac{Y_1}{Y_2}, 1, \frac{Y_3}{Y_2}, \dots, \frac{Y_D}{Y_2}\Big).$$

The transition map $x \mapsto \tilde{x}$ is the same as in (4.10), furthermore

$$y_1 = \frac{\tilde{y}_2}{\tilde{y}_1}, \quad y_2 = \frac{1}{\tilde{y}_1}, \quad y_j = \frac{\tilde{y}_j}{\tilde{y}_1} \quad \text{for } 3 \leq j \leq D, \tag{4.11}$$

whence,

$$\mathrm{d}y \wedge \mathrm{d}\bar{y} = \frac{1}{|\tilde{y}_1|^{2(D+1)}} \mathrm{d}\tilde{y} \wedge \mathrm{d}\bar{\tilde{y}}.$$

In these coordinates the Feynman amplitude becomes

$$\begin{aligned}\omega_{\Gamma_1} &= \frac{1}{|\tilde{x}_1\tilde{y}_1|^{2(D+1)}} \frac{\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\bar{\tilde{y}}}{\left(|\frac{1}{\tilde{x}_1} - \frac{\tilde{y}_2}{\tilde{y}_1}|^2 + |\frac{\tilde{x}_2}{\tilde{x}_1} - \frac{1}{\tilde{y}_1}|^2 + |\frac{\tilde{x}_3}{\tilde{x}_1} - \frac{\tilde{y}_3}{\tilde{y}_1}|^2 + \cdots + |\frac{\tilde{x}_D}{\tilde{x}_1} - \frac{\tilde{y}_D}{\tilde{y}_1}|^2\right)^{D-1}} \\ &= \frac{1}{|\tilde{x}_1\tilde{y}_1|^4} \frac{\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\bar{\tilde{y}}}{\left(|\tilde{y}_1 - \tilde{x}_1\tilde{y}_2|^2 + |\tilde{x}_2\tilde{y}_1 - \tilde{x}_1|^2 + |\tilde{x}_3\tilde{y}_1 - \tilde{y}_3\tilde{x}_1|^2 + \cdots + |\tilde{x}_D\tilde{y}_1 - \tilde{y}_D\tilde{x}_1|^2\right)^{D-1}}\end{aligned}$$

□

By Proposition 4.2 we see that we have non-integrable singularities of ω_{Γ_1} along the hyperplanes at infinity $H_\infty \times \mathbb{CP}^D$ and $\mathbb{CP}^D \times H_\infty$. Indeed, $1/|x_1|^4$ is not locally integrable.

4.1.2 Pullback of ω_{Γ_1} to $\overline{\mathrm{Conf}}_{\Gamma_1}(\mathbb{CP}^D)$

By Proposition 4.1, $\pi_1^*\omega_{\Gamma_1}$ is smooth in $\pi_1^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})$. It remains to consider $\pi_1^*\omega_{\Gamma_1}$ in $\pi_1^{-1}(U)$, where U is a neighbourhood of a point of type (A), (B) or (C). Since π_1 is a biholomorphism outside $\mathrm{Exc}(\pi_1) = \pi_1^{-1}(\Delta)$, if U is a small neighbourhood of a point of type (A) or (C), then $\pi_1^*\omega_{\Gamma_1}$ is given by (4.6) and (4.8), respectively, in $\pi_1^{-1}(U) = U$. Notice that $(H_\infty \times \mathbb{CP}^D) \cap (\mathbb{CP}^D \times H_\infty) = H_\infty \times H_\infty$.

Proposition 4.3. *Let U be a neighbourhood of a point of type (B), such that* (4.7) *holds. Each point in $\pi_1^{-1}(U)$ has a coordinate neighbourhood such that $\pi_1^*\omega_{\Gamma_1}$ is either of the form*

$$\pi_1^*\omega_{\Gamma_1} = \frac{1}{|\zeta_1 + \eta_1|^4|\zeta_1 - \eta_1|^4} \frac{\frac{1}{2^{2D-8}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left(1 + \frac{1}{4}\sum_{j=2}^{D} |\zeta_j\eta_1 - \eta_j|^2\right)^{D-1}}, \tag{4.12}$$

or

$$\pi_1^*\omega_{\Gamma_1} = \frac{1}{|\zeta_2\zeta_1 + \eta_1|^4|\zeta_2\zeta_1 - \eta_1|^4} \frac{\frac{1}{2^{2D-8}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left(|\zeta_1|^2 + \frac{1}{4}|\eta_1 - \eta_2\zeta_1|^2 + \frac{1}{4}\sum_{j=3}^{D} |\zeta_j\eta_1 - \eta_j\zeta_1|^2\right)^{D-1}}. \tag{4.13}$$

In the former case, $\{\zeta_1 = 0\} = \mathrm{Exc}(\pi_1)$ and $\pi_1^{-1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. In the latter case, $\{\zeta_2 = 0\} = \mathrm{Exc}(\pi_1)$, and $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$.

Proof. Consider ω_{Γ_1} in U. Starting out with the expression (4.7), we proceed by making the following change of variables, made in the proof of Proposition 4.1, setting $z_j = x_j - y_j$, $w_j = x_j + y_j$ for $1 \leq j \leq D$. In these coordinates $\Delta = \{z_1 = \cdots = z_D = 0\}$, $H_\infty \times \mathbb{CP}^D =$

$\{z_1 + w_1 = 0\}$ and $\mathbb{CP}^D \times H_\infty = \{z_1 - w_1 = 0\}$; thus, a point of type (B) satisfies $z_1 = \cdots = z_D = w_1 = 0$. We obtain the following expression for the Feynman amplitude

$$\omega_{\Gamma_1} = \frac{\frac{1}{2^{2D-8}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}}{|z_1 + w_1|^4 |z_1 - w_1|^4 \left(|z_1|^2 + \frac{1}{4} \sum_{j=2}^{D} |z_j w_1 - w_j z_1|^2\right)^{D-1}},$$

where we used that $x_j = (z_j + w_j)/2$, $y_j = (-z_j + w_j)/2$ and

$$x_i y_j - y_i x_j = \frac{1}{2}(z_i w_j - w_i z_j).$$

Again we want to consider the pullback of ω_{Γ_1} to the blowup. Recall from § 2.2.2 that the blowup along $z = 0$ is defined from D charts given by

$$\pi_1 : (\zeta, \eta) \mapsto (\zeta_{\{j\}}, \eta) = (z, w)$$

for $1 \leq j \leq D$, where we define

$$\zeta_{\{j\}} \coloneqq (\zeta_j \zeta_1, \ldots, \zeta_j \zeta_{j-1}, \zeta_j, \zeta_j \zeta_{j+1}, \ldots, \zeta_j \zeta_D). \tag{4.14}$$

Given our explicit choice of hyperplanes at infinity in $\mathbb{CP}^D \times \mathbb{CP}^D$ the chart

$$\pi_1 : (\zeta, \eta) \mapsto (\zeta_{\{1\}}, \eta) = (z, w) \tag{4.15}$$

distinguishes itself from the rest. In this chart we have

$$\begin{aligned} \pi_1^* \omega_{\Gamma_1} &= \frac{\frac{1}{2^{2D-8}} |\zeta_1|^{2(D-1)} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{|\zeta_1 + \eta_1|^4 |\zeta_1 - \eta_1|^4 \left(|\zeta_1|^2 + \frac{1}{4} \sum_{j=2}^{D} |\zeta_1 \zeta_j \eta_1 - \eta_j \zeta_1|^2\right)^{D-1}} \\ &= \frac{\frac{1}{2^{2D-8}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{|\zeta_1 + \eta_1|^4 |\zeta_1 - \eta_1|^4 \left(1 + \frac{1}{4} \sum_{j=2}^{D} |\zeta_j \eta_1 - \eta_j|^2\right)^{D-1}}. \end{aligned}$$

Note that $\{\zeta_1 = 0\} = \mathrm{Exc}(\pi_1)$ and $\pi_1^{-1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$.

Now, we consider the pullback of ω_{Γ_1} in the chart

$$\pi_1 : (\zeta, \eta) \mapsto (\zeta_{\{2\}}, \eta),$$

which, by symmetry, is representative of the charts $2 \leq j \leq D$. We have

$$\begin{aligned}
\pi_1^*\omega_{\Gamma_1} &= \frac{\frac{1}{2^{2D-8}}|\zeta_2|^{2(D-1)}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left(|\zeta_2\zeta_1|^2 + \frac{1}{4}|\zeta_2\eta_1 - \eta_2\zeta_2\zeta_1|^2 + \frac{1}{4}\sum_{j=3}^{D}|\zeta_2\zeta_j\eta_1 - \eta_j\zeta_2\zeta_1|^2\right)^{D-1}} \times \\
&\quad \times \frac{1}{|\zeta_2\zeta_1 + \eta_1|^4|\zeta_2\zeta_1 - \eta_1|^4} \\
&= \frac{1}{|\zeta_2\zeta_1 + \eta_1|^4|\zeta_2\zeta_1 - \eta_1|^4} \frac{\frac{1}{2^{2D-8}}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left(|\zeta_1|^2 + \frac{1}{4}|\eta_1 - \eta_2\zeta_1|^2 + \frac{1}{4}\sum_{j=3}^{D}|\zeta_j\eta_1 - \eta_j\zeta_1|^2\right)^{D-1}}.
\end{aligned}$$

Note that $\{\zeta_2 = 0\} = \mathrm{Exc}(\pi_1)$, and $\pi_1^{-1}(H_\infty \times H_\infty) = \{\zeta_1\zeta_2 + \eta_1 = \zeta_1\zeta_2 - \eta_1 = 0\} = \{\eta_1 = \zeta_1 = 0\} \cup \{\eta_1 = \zeta_2 = 0\}$. Hence, $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. ☐

4.1.3 Blowing up $\overline{\mathrm{Conf}}_{\Gamma_1}(\mathbb{CP}^D)$ along $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)$

We note that, $\pi_1^*\omega_{\Gamma_1} \notin \mathcal{E}(*\bar{*}\mathfrak{D})$ for any normal crossings divisor $\mathfrak{D}$. In fact, consider (4.13), where $\mathrm{Exc}(\pi_1) = \{\zeta_2 = 0\}$ and $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$; moreover, the strict transforms of the hyperplanes at infinity are $\{\zeta_2\zeta_1 \pm \eta_1 = 0\}$, respectively. While both hyperplanes are smooth, which we deduce from the fact that they constitute graphs, their common zero locus $\{\eta_1 = \zeta_1\zeta_2 = 0\}$ contains, in itself, a normal crossings singularity, $\{\zeta_1\zeta_2 = 0\}$. Therefore, the strict transforms of the hyperplanes at infinity cannot be a normal crossings divisor. Furthermore, one has an additional singularity at $\zeta_1 = \eta_1 = 0$ from the factor

$$\left(|\zeta_1|^2 + \frac{1}{4}|\eta_1 - \eta_2\zeta_1|^2 + \frac{1}{4}\sum_{j=3}^{D}|\zeta_j\eta_1 - \eta_j\zeta_1|^2\right)^{D-1}.$$

To handle this, we will do an additional blowup. Let

$$\pi_2 : \mathrm{Bl}_{\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)}\overline{\mathrm{Conf}}_{\Gamma_1}(\mathbb{CP}^D) \longrightarrow \overline{\mathrm{Conf}}_{\Gamma_1}(\mathbb{CP}^D), \tag{4.16}$$

be the blowup of $\overline{\mathrm{Conf}}_{\Gamma_1}(\mathbb{CP}^D)$ along $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)$, and let $\pi = \pi_1 \circ \pi_2$.

Theorem 4.4. *The divisor*

$$\mathfrak{D} = \mathrm{Exc}(\pi_2) + \mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) + \mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty),$$

has normal crossings, and $\pi^*\omega_{\Gamma_1} \in \mathcal{E}(*\bar{*}\mathfrak{D})$, with $\kappa(\omega_{\Gamma_1}) = 3$.

Proof. Notice that $\pi_2^{-1}(\pi_1^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})) = \pi_1^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})$. Thus, by Proposition 4.1, $\pi^*\omega_{\Gamma_1}$ is smooth in $\pi^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})$. Moreover, $|\mathfrak{D}| \cap \pi^{-1}(\{X_0 \neq 0, Y_0 \neq 0\}) = \varnothing$.

Let U be a neighbourhood of a point of type (A), such that (4.6) holds in U, $\Delta \cap U = \varnothing$, and $(\mathbb{CP}^D \times H_\infty) \cap U = \varnothing$. Then $\pi^{-1}(U) = U$. Hence, $|\mathfrak{D}| \cap \pi^{-1}(U) = U \cap (H_\infty \times \mathbb{CP}^D)$, which is clearly has normal crossings since it is smooth. Moreover, $\pi^*\omega_{\Gamma_1}$ is given by (4.6), whence $\pi^*\omega_{\Gamma_1} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ in $\pi^{-1}(U)$.

Let U be a neighbourhood of a point of type (B), such that (4.7) holds in U. In $\pi_1^{-1}(U)$, $\pi_1^*\omega_{\Gamma_1}$ is given by Proposition 4.3. Let U_1 be a neighbourhood of a point in $\pi_1^{-1}(U)$, such that $\pi_1^*\omega_{\Gamma_1}$ is given by (4.12). Then, $U_1 \cap \mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \varnothing$, thus, $\pi = \pi_1$ here and $\pi^*\omega_{\Gamma_1}$ is given by (4.12). Moreover, $\mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) = \{\zeta_1 + \eta_1 = 0\}$ and $\mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty) = \{\zeta_1 - \eta_1 = 0\}$. Thus $\mathfrak{D}$ has normal crossings in U_1, and $\pi^*\omega_{\Gamma_1} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ there.

Now, let U_2 be a neighbourhood of a point in $\pi_1^{-1}(U)$ such that $\pi_1^*\omega_{\Gamma_1}$ is given by (4.13). Recall that here $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. The blowup along $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)$ in U_2 is given by the following

$$\pi_2 : (u, v) \mapsto (u_1, \dots, u_D, u_1 v_1, v_2, \dots, v_D) = (\zeta, \eta), \tag{4.17}$$

$$\pi_2 : (u, v) \mapsto (v_1 u_1, u_2, \dots, u_D, v_1, \dots, v_D) = (\zeta, \eta). \tag{4.18}$$

Consider $\pi^*\omega_{\Gamma_1}$, i.e., the pullback of $\pi_1^*\omega_{\Gamma_1}$, to the chart defined in (4.17),

$$\begin{aligned}
\pi^*\omega_{\Gamma_1} &= \frac{\frac{1}{2^{2D-8}}|u_1|^2 \mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{\left(|u_1|^2 + \frac{1}{4}|u_1 v_1 - v_2 u_1|^2 + \frac{1}{4}\sum_{j=3}^{D} |u_j u_1 v_1 - v_j u_1|^2\right)^{D-1}} \times \\
&\quad \times \frac{1}{|u_2 u_1 + u_1 v_1|^4 |u_2 u_1 - u_1 v_1|^4} \\
&= \frac{1}{|u_2 + v_1|^4 |u_2 - v_1|^4 |u_1|^{2(D+2)}} \frac{\frac{1}{2^{2D-8}} \mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{\left(1 + \frac{1}{4}|v_1 - v_2|^2 + \frac{1}{4}\sum_{j=3}^{D} |u_j v_1 - v_j|^2\right)^{D-1}}.
\end{aligned} \tag{4.19}$$

Here $\mathrm{Exc}(\pi_2) = \{u_1 = 0\}$, $\mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) = \{u_2 + v_1 = 0\}$ and $\mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty) = \{u_2 - v_1 = 0\}$. It follows that $\mathfrak{D}$ has normal crossings in this chart and we see that $\pi^*\omega_{\Gamma_1} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ here.

The corresponding computation for the chart (4.18) yields

$$\begin{aligned}\pi^*\omega_{\Gamma_1} &= \frac{\frac{1}{2^{2D-8}}|v_1|^2 \mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{\left(|v_1 u_1|^2 + \frac{1}{4}|v_1 - v_2 v_1 u_1|^2 + \frac{1}{4}\sum_{j=3}^{D} |u_j v_1 - v_j v_1 u_1|^2\right)^{D-1}} \times \\ &\quad \times \frac{1}{|u_2 v_1 u_1 + v_1|^4 |u_2 v_1 u_1 - v_1|^4} \\ &= \frac{1}{|u_2 u_1 + 1|^4 |u_2 u_1 - 1|^4 |v_1|^{2(D+2)}} \frac{\frac{1}{2^{2D-8}} \mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{\left(|u_1|^2 + \frac{1}{4}|1 - v_2 u_1|^2 + \frac{1}{4}\sum_{j=3}^{D} |u_j - v_j u_1|^2\right)^{D-1}}.\end{aligned} \tag{4.20}$$

Here $\mathrm{Exc}(\pi_2) = \{v_1 = 0\}$, $\mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) = \{u_2 u_1 + 1 = 0\}$ and $\mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty) = \{u_2 u_1 - 1 = 0\}$. The two surfaces $\{u_1 u_2 \pm 1 = 0\}$ do not intersect in this chart, and their respective intersections with $\{v_1 = 0\}$ have normal crossings. Thus $\mathfrak{D}$ has normal crossings in this chart. To see that $\pi^*\omega_{\Gamma_1}$ is in $\mathcal{E}(*\bar{*}\mathfrak{D})$ in this chart, we notice that the factor

$$\left(|u_1|^2 + \frac{1}{4}|1 - v_2 u_1|^2 + \frac{1}{4}\sum_{j=3}^{D} |u_j - v_j u_1|^2\right)^{D-1}$$

is nowhere vanishing.

Let U be a neighbourhood of a point of type (C), such that (4.8) holds in U and $\Delta \cap U = \varnothing$. Then $\pi_1^{-1}(U) = U$, so $\pi_1^*\omega_{\Gamma_1}$ is given by (4.8) and $\pi^{-1}(U) = \pi_2^{-1}(U)$. The blowup π_2 is again given by (4.17) and (4.18) with (ζ, η) on the right-hand side replaced by (x, y). Beginning with (4.17) we have

$$\begin{aligned}\pi_2^*\omega_{\Gamma_1} &= \frac{1}{|u_1^2 v_1|^4} \frac{|u_1|^2 \mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{\left(|u_1 v_1 - u_1 v_2|^2 + |u_2 u_1 v_1 - u_1|^2 + \sum_{j=3}^{D} |u_j u_1 v_1 - v_j u_1|^2\right)^{D-1}} \\ &= \frac{1}{|u_1|^{2(D+2)}|v_1|^4} \frac{\mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{\left(|v_1 - v_2|^2 + |u_2 v_1 - 1|^2 + \sum_{j=3}^{D} |u_j v_1 - v_j|^2\right)^{D-1}}.\end{aligned} \tag{4.21}$$

Here $\mathrm{Exc}(\pi_2) = \{u_1 = 0\}$, $\mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D)$ is not visible in this chart and $\mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty) = \{v_1 = 0\}$, hence $\mathfrak{D}$ has normal crossings here. Moreover, the expression

$$\left(|v_1 - v_2|^2 + |u_2 v_1 - 1|^2 + \sum_{j=3}^{D} |u_j v_1 - v_j|^2\right)^{D-1}$$

is non-vanishing in a neighbourhood of $|\mathfrak{D}|$ so that $\pi^*\omega_{\Gamma_1} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ here.

Similarly for (4.18) we have

$$\begin{aligned}\pi_2^*\omega_{\Gamma_1} &= \frac{1}{|u_1v_1^2|^4}\frac{|v_1|^2\mathrm{d}u\wedge\mathrm{d}\bar{u}\wedge\mathrm{d}v\wedge\mathrm{d}\bar{v}}{\left(|v_1-v_1u_1v_2|^2+|u_2v_1-v_1u_1|^2+\sum_{j=3}^{D}|u_jv_1-v_jv_1u_1|^2\right)^{D-1}}\\ &= \frac{1}{|u_1|^4|v_1|^{2(D+2)}}\frac{\mathrm{d}u\wedge\mathrm{d}\bar{u}\wedge\mathrm{d}v\wedge\mathrm{d}\bar{v}}{\left(|1-u_1v_2|^2+|u_2-u_1|^2+\sum_{j=3}^{D}|u_j-v_ju_1|^2\right)^{D-1}}.\end{aligned} \tag{4.22}$$

In the same way as above we see that $\mathfrak{D}$ has normal crossings and $\pi^*\omega_{\Gamma_1}\in\mathcal{E}(*\bar{*}\mathfrak{D})$.

Since $\mathfrak{D}$ has three irreducible components, $\kappa(\pi^*\omega_{\Gamma_1})\leq 3$. However, in view of (4.19), we see that $\kappa(\pi^*\omega_{\Gamma_1})=3$. We recall from § 3.2.1 that we define $\kappa(\omega_{\Gamma_1})=\kappa(\pi^*\omega_{\Gamma_1})$. □

It is instructive to verify this result with the definition of κ by way of locally considering stratifications of the blowup, as described in § 3.2, since this is the more general approach. For (4.12) we have, with $P(\pi^*\omega_{\Gamma_1}) = P^{1,0}(\pi^*\omega_{\Gamma_1}) = P^{0,1}(\pi^*\omega_{\Gamma_1}) = P^{1,0}(\pi^*\omega_{\Gamma_1})\cap P^{0,1}(\pi^*\omega_{\Gamma_1})$,

$$P(\pi^*\omega_{\Gamma_1})=\{\zeta_1+\eta_1=0\}\cup\{\zeta_1-\eta_1=0\}.$$

The codimension 1 components of $P(\pi^*\omega_{\Gamma_1})$ are just $P(\pi^*\omega_{\Gamma_1})$ itself, whence $H(\pi^*\omega_{\Gamma_1})_1 = P(\pi^*\omega_{\Gamma_1})$. Furthermore, $H(\pi^*\omega_{\Gamma_1})_2=(H(\pi^*\omega_{\Gamma_1})_1)_{\text{sing}}=\{\zeta_1=\eta_1=0\}$. Since $H(\pi^*\omega_{\Gamma_1})_2$ is smooth, the sequence terminates.

For (4.19) we have

$$P(\pi^*\omega_{\Gamma_1})=\{u_2+v_1=0\}\cup\{u_2-v_1=0\}\cup\{u_1=0\}.$$

Again $H(\pi^*\omega_{\Gamma_1})_1=P(\pi^*\omega_{\Gamma_1})$, and $(H(\pi^*\omega_{\Gamma_1})_1)_{\text{sing}}$ is given by the pairwise intersections of the hyperplanes in $P(\pi^*\omega_{\Gamma_1})$, i.e.,

$$H(\pi^*\omega_{\Gamma_1})_2=\{u_2+v_1=u_1=0\}\cup\{u_2-v_1=u_1=0\}\cup\{u_2=v_1=0\}.$$

In this case $H(\omega_{\Gamma_1})_2$ is not smooth, however, $H(\omega_{\Gamma_1})_3=(H(\omega_{\Gamma_1})_2)_{\text{sing}}=\{u_1=u_2=v_1=0\}$ is, so $\kappa(\omega_{\Gamma_1})=3$.

For (4.20), we have

$$H(\omega_{\Gamma_1})_1=P(\pi^*\omega)=\{u_1u_2+1=0\}\cup\{u_1u_2-1=0\}\cup\{v_1=0\},$$

furthermore, since $\{u_1u_2+1=0\}\cap\{u_1u_2-1=0\}=\varnothing$ for u_1,u_2 sufficiently small,

$$H(\pi^*\omega_{\Gamma_1})_2=\{u_1u_2+1=v_1=0\}\cup\{u_1u_2-1=v_1=0\},$$

which is a union of smooth disjoint varieties, whence $\kappa(\pi^*\omega_{\Gamma_1})=2$. Lastly, for both (4.21) and (4.22), $H(\pi^*\omega_{\Gamma_1})_1=P(\pi^*\omega_{\Gamma_1})=\{u_1=0\}\cup\{v_1=0\}$, and $H(\pi^*\omega_{\Gamma_1})_2=\{u_1=v_1=0\}$ which is smooth, whence $\kappa(\pi^*\omega_{\Gamma_1})=2$. The maximum value of $\kappa(\pi^*\omega_{\Gamma_1})$ in any chart was 3, which is consistent with the result obtained above.

4.1.4 Regularisation of the integral associated to Γ_1

We are now ready to reap the rewards of the above calculations. The modified integral of the pullback amplitude over the blowup has properties that allow us to employ regularisation methods from the theory of residue currents. We can be concrete in the regularisation of the integral associated to Γ_1 with the following choice of line bundle over $\mathbb{CP}^D \times \mathbb{CP}^D$, metric and section. Recall the hyperplane bundle $\mathcal{O}(1) \to \mathbb{CP}^D$ from Example 2.2; from it we can construct the following (holomorphic) line bundle on the product manifold $\mathbb{CP}^D \times \mathbb{CP}^D$, denoted

$$\mathcal{O}(1) \boxtimes \mathcal{O}(1) \to \mathbb{CP}^D \times \mathbb{CP}^D. \tag{4.23}$$

There are natural projections $\mathrm{proj}_1, \mathrm{proj}_2 : \mathbb{CP}^D \times \mathbb{CP}^D \to \mathbb{CP}^D$ onto the first and second factor, respectively, of the product. With $\mathcal{O}(1)$ a line bundle on each copy of $\mathbb{CP}^D$, we define $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ to be the tensor product $\mathrm{proj}_1^*\mathcal{O}(1) \otimes \mathrm{proj}_2^*\mathcal{O}(1)$, which constitutes a (holomorphic) line bundle on $\mathbb{CP}^D \times \mathbb{CP}^D$.

There is a section $\sigma : \mathbb{CP}^D \times \mathbb{CP}^D \to \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ that vanishes precisely on the subvariety $(H_\infty \times \mathbb{CP}^D) \cup (\mathbb{CP}^D \times H_\infty)$, given in homogenous coordinates by

$$\sigma = X_0 Y_0.$$

Recall that ω_{Γ_1} was integrable on the diagonal, hence, we do not need to dampen the amplitude there. Furthermore, we can take the metric on $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ to be the product metric

$$\|\sigma\|^2 = \|X_0\|^2_{\mathcal{O}(1)} \cdot \|Y_0\|^2_{\mathcal{O}(1)} \tag{4.24}$$

where, for any 1-homogeneous polynomial $p(X) : \mathbb{CP}^D \to \mathcal{O}(1)$, let

$$\|p(X)\|^2_{\mathcal{O}(1)} = \frac{|p(X)|^2}{|X|^2},$$

which defines as a metric on $\mathcal{O}(1)$. In the affine coordinates (x, y) on $\{X_0 \neq 0, Y_0 \neq 0\} \subset \mathbb{CP}^D \times \mathbb{CP}^D$ we have

$$\|\sigma\|^2 = \frac{|X_0|^2}{|X|^2} \cdot \frac{|Y_0|^2}{|Y|^2} = \frac{1}{1+|x|^2} \cdot \frac{1}{1+|y|^2}.$$

With the above choices, we obtain an explicit regularised integral, which, since $\{X_0 \neq 0, Y_0 \neq 0\}$ is dense in $\mathbb{CP}^D \times \mathbb{CP}^D$, can be written as

$$F_\xi(\lambda) = \int_{\mathbb{C}^D \times \mathbb{C}^D} \Big|\frac{1}{1+|x|^2}\Big|^\lambda \Big|\frac{1}{1+|y|^2}\Big|^\lambda \omega_{\Gamma_1} \wedge \xi. \tag{4.25}$$

Corollary 4.5. *The map $\lambda \mapsto F_\xi(\lambda)$, a priori defined for $\mathfrak{Re}\,\lambda \gg 1$, has a meromorphic continuation to all of $\mathbb{C}$, with a pole at $\lambda = 0$ of order ≤ 3.*

We note, in (4.19), that $\pi^*\omega_{\Gamma_1} \cdot (|u_2 + v_1|^4 |u_2 - v_1|^4 |u_1|^{2(D+2)})$ is smooth and independent on u_1. In view of (3.12), it seems that the contribution to $\langle\{\pi^*\omega_{\Gamma_1}, 1\rangle$ vanishes. However, we need to take into account the coordinate dependencies of a partition of unity, as in (3.14). In § 4.4 we show that $\langle\{\pi^*\omega_{\Gamma_1}\}, 1\rangle = 0$ in dimension $D = 2$.

4.2 Case Two: One-loop diagram

Now we want to look at the graph Γ_2 presented in Figure 4.2, which can be seen as simply a modification to the amplitude associated to Γ_1, since the configuration space and wonderful compactification are the same for the two graphs.

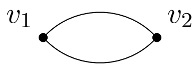

Figure 4.2: Feynman graph Γ_2.

The only thing distinguishing the setup for Γ_2 from Γ_1 is the associated amplitude, which now is given by

$$\omega_{\Gamma_2} = \frac{1}{\|x-y\|^{4(D-1)}} \mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}. \tag{4.26}$$

As above, let $X = \mathbb{CP}^D$ and let

$$\pi_1 : \mathrm{Bl}_\Delta X^{\mathbf{V}_{\Gamma_2}} \to X^{\mathbf{V}_{\Gamma_2}},$$

be the blowup of $X^{\mathbf{V}_{\Gamma_2}} = \mathbb{CP}^D \times \mathbb{CP}^D$ along the diagonal Δ.

Proposition 4.6. *In $\pi_1^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})$, there are local coordinates (ζ, η) such that*

$$\pi_1^*\omega_{\Gamma_2} = \frac{1}{|\zeta_1|^{2(D-1)}(1 + \sum_{j=2}^{D} |\zeta_j|^2)^{2(D-1)}} \frac{1}{2^{2D}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}. \tag{4.27}$$

Proof. In coordinates where the diagonal is given by $z_1 = \cdots = z_D = 0$

$$\omega_{\Gamma_2} = \frac{1}{\|z\|^{4(D-1)}} \frac{1}{2^{2D}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}.$$

The pullback to the wonderful compactification, in the chart where the exceptional divisor is given by $\{\zeta_1 = 0\}$, is

$$\pi_1^*\omega_{\Gamma_2} = \frac{1}{|\zeta_1|^{2(D-1)}(1 + \sum_{j=2}^{D} |\zeta_j|^2)^{2(D-1)}} \frac{1}{2^{2D}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}.$$

□

In contrast to Γ_1, the pullback amplitude $\pi_1^*\omega_{\Gamma_2}$ is singular along the $\mathrm{Exc}(\pi_1)$. Let us consider the amplitude in a neighborhood of the hyperplanes at infinity. We proceed by again considering points of type (A), (B) and (C) as in § 4.1.1. We present the following proposition without proof, since the proof is nearly identical to the proof of Proposition 4.2.

Proposition 4.7. *In a neighbourhood of a point of type (A) there are local coordinates such that*

$$\omega_{\Gamma_2} = |x_1|^{2(D-3)} \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}}{\left(|1 - x_1 y_1|^2 + \sum_{j=2}^{D} |x_j - x_1 y_j|^2\right)^{2(D-1)}} \tag{4.28}$$

where the point lies on $x_1 = 0$.

In a neighbourhood of a point type (B) there are local coordinates such that

$$\omega_{\Gamma_2} = |x_1 y_1|^{2(D-3)} \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}}{\left(|y_1 - x_1|^2 + \sum_{j=2}^{D} |x_j y_1 - y_j x_1|^2\right)^{2(D-1)}}, \tag{4.29}$$

where the point satisfies $x_j = y_j$ for $1 \leq j \leq D$ where $x_1 = y_1 = 0$.

In a neighbourhood of a point of type (C) there are local coordinates such that

$$\omega_{\Gamma_2} = |x_1 y_1|^{2(D-3)} \frac{\mathrm{d}x \wedge \mathrm{d}\bar{x} \wedge \mathrm{d}y \wedge \mathrm{d}\bar{y}}{\left(|y_1 - x_1 y_2|^2 + |x_2 y_1 - x_1|^2 + \sum_{j=3}^{D} |x_j y_1 - y_j x_1|^2\right)^{2(D-1)}} \tag{4.30}$$

where the point lies on $x_1 = y_1 = 0$.

We see that the expressions (4.28) to (4.30) have different characteristics for $D = 2$ (disregarding the case $D = 1$) and $D \geq 3$. We will henceforth assume that $D \geq 3$ such that the factors $|x_1(y_1)|^{2(D-3)}$ remain in the numerators of ω_{Γ_2}; the case $D = 2$ a priori needs its own separate analysis.

4.2.1 Pullback of ω_{Γ_2} to $\overline{\mathrm{Conf}}_{\Gamma_2}(\mathbb{CP}^D)$

By Proposition 4.6, in $\{X_0 \neq 0, Y_0 \neq 0\}$, there are local coordinates such that $\pi_1^*\omega_{\Gamma_2}$ is given by (4.27). For a complete description of the pullback of ω_{Γ_2} to the wonderful compactification, it remains for us to consider $\pi_1^*\omega_{\Gamma_2}$ in $\pi_1^{-1}(U)$, where U is a neighbourhood of a point of type (A), (B) or (C). Since π_1 is a biholomorphism outside $\mathrm{Exc}(\pi_1)$, if U is a small neighbourhood of a point of type (A) or (C), then $\pi_1^*\omega_{\Gamma_2}$ is given by (4.28) and (4.30), respectively, in $\pi_1^{-1}(U) = U$.

Proposition 4.8. *Let U be a neighbourhood of a point of type (B), such that* (4.29) *holds. Each point in $\pi_1^{-1}(U)$ has a coordinate neighbourhood such that $\pi_1^*\omega_{\Gamma_1}$ is either of the form*

$$\pi_1^*\omega_{\Gamma_2} = \frac{|(\zeta_1+\eta_1)(\eta_1-\zeta_1)|^{2(D-3)}}{|\zeta_1|^{2(D-1)}} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\eta\wedge\mathrm{d}\bar{\eta}}{\left(1+\frac{1}{4}\sum_{j=2}^{D}|\zeta_j\eta_1-\eta_j|^2\right)^{2D-1}}, \tag{4.31}$$

or

$$\pi_1^*\omega_{\Gamma_2} = \frac{|(\zeta_2\zeta_1+\eta_1)(\zeta_2\zeta_1-\eta_1)|^{2(D-3)}}{|\zeta_2|^{2(D-1)}} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\eta\wedge\mathrm{d}\bar{\eta}}{\left(|\zeta_1|^2+\frac{1}{4}|\eta_1-\eta_2\zeta_1|^2+\frac{1}{4}\sum_{j=3}^{D}|\zeta_j\eta_1-\eta_j\zeta_1|^2\right)^{2(D-1)}}. \tag{4.32}$$

In the former case, $\mathrm{Exc}(\pi_1) = \{\zeta_1 = 0\}$ *and* $\pi_1^{-1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$*. In the latter case,* $\mathrm{Exc}(\pi_1) = \{\zeta_2 = 0\}$ *and* $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$.

Proof. Consider ω_{Γ_2} in U. Starting out with the expression (4.29), we go to a set of local coordinates (z, w), as in the proof of Proposition 4.3, such that $\Delta = \{z = 0\}$, $H_\infty \times \mathbb{CP}^D = \{z_1 + w_1 = 0\}$ and $\mathbb{CP}^D \times H_\infty = \{z_1 - w_1 = 0\}$. We obtain the following expression

$$\omega_{\Gamma_2} = |(z_1+w_1)(z_1-w_1)|^{2(D-3)} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}z\wedge\mathrm{d}\bar{z}\wedge\mathrm{d}w\wedge\mathrm{d}\bar{w}}{\left(|z_1|^2+\frac{1}{4}\sum_{j=2}^{D}|z_jw_1-w_jz_1|^2\right)^{2(D-1)}}. \tag{4.33}$$

We want to consider the pullback $\pi_1^*\omega_{\Gamma_2}$ of ω_{Γ_2} in U to the blowup along $z = 0$, where, again, we have the two distinguished charts given by (4.15) and (4.17), respectively. For the former we find

$$\pi_1^*\omega_{\Gamma_2} = \frac{|(\zeta_1+\eta_1)(\eta_1-\zeta_1)|^{2(D-3)}}{|\zeta_1|^{2(D-1)}} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\eta\wedge\mathrm{d}\bar{\eta}}{\left(1+\frac{1}{4}\sum_{j=2}^{D}|\zeta_j\eta_1-\eta_j|^2\right)^{2D-1}},$$

noting that $\mathrm{Exc}(\pi_1) = \{\zeta_1 = 0\}$ and $\pi_1^{-1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. For the latter chart we find

$$\pi_1^*\omega_{\Gamma_2} = \frac{|(\zeta_2\zeta_1+\eta_1)(\zeta_2\zeta_1-\eta_1)|^{2(D-3)}}{|\zeta_2|^{2(D-1)}} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\eta\wedge\mathrm{d}\bar{\eta}}{\left(|\zeta_1|^2+\frac{1}{4}|\eta_1-\eta_2\zeta_1|^2+\frac{1}{4}\sum_{j=3}^{D}|\zeta_j\eta_1-\eta_j\zeta_1|^2\right)^{2(D-1)}},$$

noting that $\mathrm{Exc}(\pi_1) = \{\zeta_2 = 0\}$ and $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. □

4.2.2 Blowing up $\overline{\mathrm{Conf}}_{\Gamma_2}(\mathbb{CP}^D)$ along $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)$

We note, as above, that $\pi_1^*\omega_{\Gamma_2} \notin \mathcal{E}(*\bar{*}\mathfrak{D})$ for any normal crossings divisor $\mathfrak{D}$, which becomes clear in view of (4.32) where we have the factor in the denominator

$$\left(|\zeta_1|^2 + \frac{1}{4}|\eta_1 - \eta_2\zeta_1|^2 + \frac{1}{4}\sum_{j=3}^{D}|\zeta_j\eta_1 - \eta_j\zeta_1|^2\right)^{2(D-1)}$$

which vanishes on $\zeta_1 = \eta_1 = 0$.

Again, we let

$$\pi_2 : \mathrm{Bl}_{\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)}\overline{\mathrm{Conf}}_{\Gamma_2}(\mathbb{CP}^D) \longrightarrow \overline{\mathrm{Conf}}_{\Gamma_2}(\mathbb{CP}^D),$$

be the blowup of $\overline{\mathrm{Conf}}_{\Gamma_2}(\mathbb{CP}^D)$ along $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)$, and let $\pi = \pi_1 \circ \pi_2$.

Theorem 4.9. *Assume that $D \geq 3$. The divisor*

$$\mathfrak{D} = \mathrm{Exc}(\pi) = \mathrm{Exc}(\pi_2) + \mathrm{Strict}_{\pi_2}(\mathrm{Exc}(\pi_1))$$

has normal crossings and the pullback $\pi^*\omega_{\Gamma_2} \in \mathcal{E}(*\bar{*}\mathfrak{D})$, with $\kappa(\omega_{\Gamma_1}) = 2$.

Proof. We note that $\pi^{-1}(\{X_0 \neq 0, Y_0 \neq 0\}) = \pi_1^{-1}(\{X_0 \neq 0, Y_0 \neq 0)$, whence, $\pi^*\omega_{\Gamma_2}$ is given by (4.27) in $\pi^{-1}(\{X_0 \neq 0, Y_0 \neq 0)$. Here, $\mathfrak{D} = \mathrm{Exc}(\pi) = \mathrm{Exc}(\pi_1) = \{\zeta_1 = 0\}$, since $\mathrm{Strict}_{\pi_1}(H_\infty \times \mathbb{CP}^D) = \mathrm{Strict}_{\pi_1}(\mathbb{CP}^D \times H_\infty) = \mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \varnothing$, whence, $\mathrm{Exc}(\pi_2) = \varnothing$. Clearly, $\mathfrak{D}$ has normal crossings, and $\pi^*\omega_{\Gamma_2} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ in $\pi^{-1}(\{X_0 \neq 0, Y_0 \neq 0\})$.

Let U be a neighbourhood of a point of type (A), such that $U \cap \Delta = \varnothing$ and $U \cap (\mathbb{CP}^D \times H_\infty) \neq 0$. Then $\pi^{-1}(U) = U$ and $|\mathfrak{D}| \cap U = \varnothing$. Furthermore, $\pi^*\omega_{\Gamma_2}$ is given by (4.28) which is smooth in U.

Now, let U a neighbourhood of a point of type (B), such that (4.29) holds in U. Going to (diagonal) coordinates where ω_{Γ_2} is given by (4.33), recall that $\Delta = \{z = 0\}$, $H_\infty \times \mathbb{CP}^D = \{z_1 + w_1 = 0\}$ and $\mathbb{CP}^D \times H_\infty = \{z_1 - w_1 = 0\}$. From Proposition 4.8 we know that every point in $\pi_1^{-1}(U)$ will have a neighbourhood such that $\pi_1^*\omega_{\Gamma_2}$ is given by either (4.31) or (4.32). Let U_1 be a neighbourhood such that (4.31) holds for $\pi_1^*\omega_{\Gamma_2}$. Recall that $\mathrm{Exc}(\pi_1) = \{\zeta_1 = 0\}$ and $\pi_1^{-1}(H_\infty \times H_\infty)$, hence, $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \varnothing$ in U_1. Thus, $\pi = \pi_1$ and, moreover, $\pi_1^*\omega_{\Gamma_2} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ where $\mathfrak{D}$ has normal crossings in U_1.

Let U_2 be a neighbourhood such that (4.32) holds. Recall that $\mathrm{Exc}(\pi_1) = \{\zeta_2 = 0\}$ and $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. We want to consider $\pi^*\omega_{\Gamma_2}$. The blowup of U_2 along $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty)$ is given, as in the proof of Theorem 4.4, by the two charts (4.17) and

(4.18), respectively. For the former case we have

$$\pi^*\omega_{\Gamma_2} = \frac{|(u_2+v_1)(u_2-v_1)|^{2(D-3)}}{|u_2|^{2(D-1)}|u_1|^6} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}u\wedge\mathrm{d}\bar{u}\wedge\mathrm{d}v\wedge\mathrm{d}\bar{v}}{\big(1+\frac{1}{4}|v_1-v_2|^2+\frac{1}{4}\sum_{j=3}^{D}|u_jv_1-v_j|^2\big)^{2(D-1)}}. \tag{4.34}$$

Here $\mathrm{Exc}(\pi_2) = \{u_1 = 0\}$ and $\mathrm{Strict}_{\pi_2}(\mathrm{Exc}(\pi_1)) = \{u_2 = 0\}$. It follows $\mathfrak{D}$ has normal crossings and, moreover, we see that $\pi^*\omega_{\Gamma_2} \in \mathcal{E}(*\bar{*}\mathfrak{D})$.

For the latter case we have

$$\pi^*\omega_{\Gamma_2} = \frac{|(u_1u_2+1)(u_1u_2-1)|^{2(D-3)}}{|u_2|^{2(D-1)}|v_1|^6} \frac{\frac{1}{2^{6(D-2)}}\mathrm{d}u\wedge\mathrm{d}\bar{u}\wedge\mathrm{d}v\wedge\mathrm{d}\bar{v}}{\big(|u_1|^2+\frac{1}{4}|1-v_2u_1|^2+\frac{1}{4}\sum_{j=3}^{D}|u_j-v_ju_1|^2\big)^{2(D-1)}}. \tag{4.35}$$

Here $\mathrm{Exc}(\pi_2) = \{v_1 = 0\}$ and $\mathrm{Strict}_{\pi_2}(\mathrm{Exc}(\pi_1)) = \{u_2 = 0\}$. Again, it follows that $\mathfrak{D}$ has normal crossings and we see that $\pi^*\omega_{\Gamma_2} \in \mathcal{E}(*\bar{*}\mathfrak{D})$.

Lastly, let U be a neighbourhood of a point of type (C), such that (4.30) holds and $U\cap\Delta = \varnothing$. Notice that $\pi_1^{-1}(U) = U$, whence, $\pi_1^*\omega_{\Gamma_2}$ is still given by (4.30) and $\pi^{-1}(U) = \pi_2^{-1}(U)$. The blowup is again given by the two charts (4.17) and (4.18), with (ζ,η) on the right-hand side replaced by (x,y). In the chart (4.17) we have

$$\pi^*\omega_{\Gamma_2} = \frac{|u_1|^{2(D-3)}}{|v_1|^6} \frac{\mathrm{d}u\wedge\mathrm{d}\bar{u}\wedge\mathrm{d}v\wedge\mathrm{d}\bar{v}}{\big(|1-u_1v_2|^2+|u_2-u_1|^2+\sum_{j=3}^{D}|u_j-v_ju_1|^2\big)^{2(D-1)}}. \tag{4.36}$$

Here $\mathrm{Exc}(\pi_2) = \{u_1 = 0\}$ and $\mathrm{Strict}_{\pi_2}(\mathrm{Exc}(\pi_1)) = \pi^{-1}(\Delta) = \varnothing$, thus, $\mathfrak{D}$ trivially has normal crossings and, moreover, we see that $\pi^*\omega_{\Gamma_2} \in \mathcal{E}(*\bar{*}\mathfrak{D})$. By symmetry, the above holds also in (4.18), here with $\mathrm{Exc}(\pi_2) = \{v_1 = 0\}$.

Since $\mathfrak{D}$ has two irreducible components, $\kappa(\pi^*\omega_{\Gamma_2}) \le 2$. However, in view of (4.34) and (4.35), we see that $\kappa(\pi^*\omega_{\Gamma_2}) = 2$. Again, we recall from § 3.2.1 the definition $\kappa(\omega_{\Gamma_2}) = \kappa(\pi^*\omega_{\Gamma_2})$. □

4.2.3 Regularisation of the integral associated to Γ_2

We would again like to present some explicit choices for the regularisation of the integral of ω_{Γ_2}. Since ω_{Γ_2} is singular on the diagonal, which has codimension D, as well as the intersection of hyperplanes at infinity, $H_\infty \times H_\infty$, we need to do things differently compared to § 4.1.3. To this end, we let

$$\|\sigma\|^2 = \frac{|X - \frac{X\cdot\bar{Y}}{|Y|^2}Y|^2}{|X|^2}\cdot\Big(\frac{|X_0|^2}{|X|^2}+\frac{|Y_0|^2}{|Y|^2}\Big).$$

The factor

$$\frac{|X - \frac{X \cdot \bar{Y}}{|Y|^2} Y|^2}{|X|^2}$$

is in fact the norm squared of a section s of a vector bundle $E \to \mathbb{CP}^D \times \mathbb{CP}^D$, such that $\Delta = \{s = 0\}$, see [Ber91].

Corollary 4.10. *The map*

$$\lambda \longmapsto \int\limits_{\mathbb{CP}^D \times \mathbb{CP}^D} \|\sigma\|^{2\lambda} \omega_{\Gamma_2} \wedge \xi,$$

where ξ is a test function on $\mathbb{CP}^D \times \mathbb{CP}^D$, a priori defined for $\mathfrak{Re}\,\lambda \gg 1$, has a meromorphic continuation to all of $\mathbb{C}$, with a pole at $\lambda = 0$ of order ≤ 2.

4.3 Case Three: Three-point interaction

Due to the explosive increase in types of neighbourhoods of points in the configuration space of Γ_3 which increase further with blowups, we will be less exhaustive, as compared to our study of Γ_1 and Γ_2, in our treatment of Γ_3.

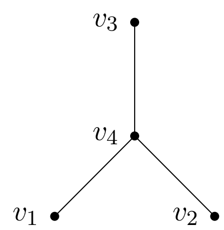

Figure 4.3: Feynman graph Γ_3.

Assuming that our underlying quantum field theory allows for three point interactions at tree level, i.e., the interaction Hamiltonian contains a term $\propto \phi^3$, the diagram in Figure 4.3 can arise. We label the outer vertices v_1, v_2 and v_3, in some order, and the middle vertex v_4, and the edges between v_j and v_4 as e_j. The configuration space of Γ_3 over $\mathbb{CP}^D$ is given by

$$\mathrm{Conf}_{\Gamma_3}(\mathbb{CP}^D) = (\mathbb{CP}^D)^{\times 4} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3), \tag{4.37}$$

where Δ_j is the diagonal corresponding to the edge e_j.

The simple arrangement of polydiagonal subvarieties associated with Γ_3 in $(\mathbb{CP}^D)^{\times 4}$ is given by

$$\mathcal{S}_{\Gamma_3} = \{\Delta_1, \Delta_2, \Delta_3, \Delta_1 \cap \Delta_2, \Delta_1 \cap \Delta_3, \Delta_2 \cap \Delta_3, \Delta_1 \cap \Delta_2 \cap \Delta_3\}.$$

The building set for $\mathcal{S}_{\Gamma_3}$, given in Theorem 3.5, is then the set of simple diagonals, i.e.,

$$\mathcal{G}_{\Gamma_3} = \{\Delta_1, \Delta_2, \Delta_3\}, \tag{4.38}$$

since these are the only elements of $\mathcal{S}_{\Gamma_3}$ corresponding to bi-connected subgraphs. There is no preferred ordering of the diagonals in $\mathcal{G}_{\Gamma_3}$ since there appears no inclusions of induced bi-connected subgraphs of Γ_3. Thus the iterated blowup description of the wonderful compactification of $\mathrm{Conf}_{\Gamma_3}(\mathbb{CP}^D)$ can be done in any order. For convenience, we choose the ordering $\mathcal{G}_{\Gamma_3} = \{\Delta_1, \Delta_2, \Delta_3\}$. We take $Y^{(0)} = (\mathbb{CP}^D)^{\times 4}$, and let $\pi_1^{(k)} : Y^{(k)} \to Y^{(k-1)}$ be the blowup of $Y^{(k-1)}$ along the iterated strict transform of Δ_k, for $k = 1, 2, 3$. Then $Y^{(3)} = \overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D)$. We let $\pi_1 = \pi_1^{(1)} \circ \pi_1^{(2)} \circ \pi_1^{(3)}$.

Let $X^{(j)}$ be a set of homogeneous coordinates on the j^{th} copy of $\mathbb{CP}^D$ in the product $(\mathbb{CP}^D)^{\times 4}$, for $j = 1, 2, 3, 4$. We associate $X^{(j)}$ with the vertex v_j in Γ_3. For each $j = 1, 2, 3, 4$ we define

$$x^{(j)} = (x_1^{(j)}, \dots, x_D^{(j)}) = \left(\frac{X_1^{(j)}}{X_0^{(j)}}, \dots, \frac{X_D^{(j)}}{X_0^{(j)}} \right).$$

Then $x = (x^{(1)}, \dots, x^{(4)})$ is a set of coordinates on $\{X_0^{(j)} \neq 0 \text{ for } j = 1, \dots, 4\} \subset (\mathbb{CP}^D)^{\times 4}$. In these coordinates, the diagonal Δ_j is given by $x^{(j)} = x^{(4)}$. From (3.7) we find that the Feynman amplitude associated to Γ_3 is then given by

$$\omega_{\Gamma_3} = \frac{\mathrm{d}x^{(1)} \wedge \mathrm{d}\bar{x}^{(1)} \wedge \cdots \wedge \mathrm{d}x^{(4)} \wedge \mathrm{d}\bar{x}^{(4)}}{\left(\|x^{(1)} - x^{(4)}\| \|x^{(2)} - x^{(4)}\| \|x^{(3)} - x^{(4)}\|\right)^{2(D-1)}}. \tag{4.39}$$

We make the following coordinate transformation

$$\begin{pmatrix} z^{(1)} \\ z^{(2)} \\ z^{(3)} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ x^{(4)} \end{pmatrix}, \tag{4.40}$$

with inverse

$$\begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ x^{(4)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^{(1)} \\ z^{(2)} \\ z^{(3)} \\ w \end{pmatrix} \tag{4.41}$$

and Jacobian determinant 1. Note that the 1's in (4.40) and (4.41) represent $D \times D$ identity matrices. In the coordinates $(z^{(1)}, z^{(2)}, z^{(3)}, w)$ the diagonals are given by $\Delta_j = \{z_1^{(j)} = \cdots = z_D^{(j)} = 0\}$, and the Feynman amplitude is given by

$$\omega_{\Gamma_3} = \frac{\mathrm{d}z^{(1)} \wedge \mathrm{d}\bar{z}^{(1)} \wedge \mathrm{d}z^{(2)} \wedge \mathrm{d}\bar{z}^{(2)} \wedge \mathrm{d}z^{(3)} \wedge \mathrm{d}\bar{z}^{(3)} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}}{\left(\|z^{(1)}\| \|z^{(2)}\| \|z^{(3)}\|\right)^{2(D-1)}}. \tag{4.42}$$

We conduct the iterated blowup $\pi_1 = \pi_1^{(1)} \circ \pi_1^{(2)} \circ \pi_1^{(3)}$ in $\{X_0^{(j)} \neq 0 \text{ for } j = 1, 2, 3, 4\}$ as follows. The blowup along the diagonal Δ_k is given by the charts

$$\pi_1^{(k)} : (z^{(1)}, \dots, z^{(k-1)}, \zeta^{(k)}, z^{(k+1)}, \dots, z^{(3)}, \eta) \mapsto (z^{(1)}, \dots, z^{(k-1)}, \zeta_{\{j_k\}}^{(k)}, z^{(k+1)}, \dots, z^{(3)}, \eta)$$

for $1 \leq k \leq 3$ and $1 \leq j_k \leq D$. The subscript $\{j_k\}$ in $\zeta^{(k)}_{\{j_k\}}$ again refers to the definition (4.14); each value of $1 \leq j_k \leq D$ corresponds to a distinct chart. For ease of notation, we keep the names of the coordinate functions corresponding to diagonals that are mapped identically with $\pi_1^{(k)}$. The strict transform of Δ_k in the blowup along Δ_ℓ, for $k \neq \ell$, is generically isomorphic to Δ_k, thus we can treat each blowup separately and on equal footing as above.

For $k = 1$ we have

$$\pi_1^{(1)*}\omega_{\Gamma_3} = \frac{\mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\bar{\zeta}^{(1)} \wedge z^{(2)} \wedge \mathrm{d}\bar{z}^{(2)} \wedge \mathrm{d}z^{(3)} \wedge \mathrm{d}\bar{z}^{(3)} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left(1 + |\zeta_1^{(1)}|^2 + \cdots + |\widehat{\zeta_{j_1}^{(1)}}|^2 + \cdots + |\zeta_D^{(1)}|^2\right)^{D-1} \left(\|z^{(2)}\|\|z^{(3)}\|\right)^{2(D-1)}},$$

which in particular shows that ω_{Γ_3} is locally integrable along the diagonal $\{z^{(1)} = 0\}$, just like in the case of Γ_1. By symmetry, the same is true for all diagonals. We have now shown the following proposition.

Proposition 4.11. *In $\pi_1^{-1}(\{X_0^{(j)} \neq 0$ for $j = 1, \ldots, 4\})$, $\pi_1^*\omega_{\Gamma_3}$ is smooth and locally given by*

$$\pi_1^*\omega_{\Gamma_3} = \frac{\mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\bar{\zeta}^{(1)} \wedge \mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\bar{\zeta}^{(2)} \wedge \mathrm{d}\zeta^{(3)} \wedge \mathrm{d}\bar{\zeta}^{(3)} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\prod_{k=1}^{3} \left(1 + |\zeta_1^{(k)}|^2 + \cdots + |\widehat{\zeta_{j_k}^{(k)}}|^2 + \cdots + |\zeta_D^{(k)}|^2\right)^{D-1}}. \tag{4.43}$$

In particular ω_{Γ_3} is locally integrable in $\{X_0^{(j)} \neq 0$ for $j = 1, \ldots, 4\}$.

4.3.1 Infrared divergences of ω_{Γ_3}

We have the following hyperplanes at infinity in $(\mathbb{CP}^D)^{\times 4}$,

$$H_j = \{X \in (\mathbb{CP}^D)^{\times 4} : X_0^{(j)} = 0\}, \tag{4.44}$$

for $j = 1, 2, 3, 4$. We consider ω_{Γ_3} in a neighbourhood U of a point $(p, p, p, p) \in (\mathbb{CP}^D)^{\times 4}$, corresponding to a point of type (B) in § 4.1 and § 4.2, where $p \in H_\infty$. The other kinds of points where ω_{Γ_3} has infrared divergences can be handled in a similar way as we did for Γ_1 above. Here we omit the details and focus on points of the form (p, p, p, p), since these are where ω_{Γ_3} behaves the worst.

We may assume that $p = [0 : 1 : 0 : \ldots : 0]$. We can express ω_{Γ_3} in the local coordinates $\tilde{x}$ in a neighbourhood of (p, p, p, p) as follows, by letting $\tilde{x} = (\tilde{x}^{(1)}, \ldots, \tilde{x}^{(4)})$ where

$$\tilde{x}^{(j)} = (\tilde{x}_1^{(j)}, 1, \tilde{x}_2^{(j)}, \ldots, \tilde{x}_D^{(j)}) = \left(\frac{X_0^{(j)}}{X_1^{(j)}}, 1, \frac{X_2^{(j)}}{X_1^{(j)}}, \ldots, \frac{X_D^{(j)}}{X_1^{(j)}}\right)$$

for $j = 1, \ldots, 4$. In a similar way as in the proof of Proposition 4.2 we find

$$\omega_{\Gamma_3} = \frac{\mathrm{d}\tilde{x}^{(1)} \wedge \mathrm{d}\bar{\tilde{x}}^{(1)} \wedge \cdots \wedge \mathrm{d}\tilde{x}^{(4)} \wedge \mathrm{d}\bar{\tilde{x}}^{(4)}}{|\tilde{x}_1^{(1)}\tilde{x}_1^{(2)}\tilde{x}_1^{(3)}|^4 |\tilde{x}_1^{(4)}|^{12} \prod_{k=1}^{3} \left(|\tilde{x}_1^{(4)} - \tilde{x}_1^{(k)}|^2 + \sum_{j=2}^{D} |\tilde{x}_j^{(k)}\tilde{x}_1^{(4)} - \tilde{x}_j^{(4)}\tilde{x}_1^{(k)}|^2\right)^{D-1}}. \tag{4.45}$$

Note, that in these coordinates $H_j = \{\tilde{x}_1^{(j)} = 0\}$ for $j = 1, \ldots, 4$. Applying the coordinate transform defined in (4.40) and (4.41) we obtain

$$\omega_{\Gamma_3} = \frac{\mathrm{d}z^{(1)} \wedge \mathrm{d}\bar{z}^{(1)} \wedge \mathrm{d}z^{(2)} \wedge \mathrm{d}\bar{z}^{(2)} \wedge \mathrm{d}z^{(3)} \wedge \mathrm{d}\bar{z}^{(3)} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}}{\left|(z_1^{(1)} + w_1)(z_1^{(2)} + w_1)(z_1^{(3)} + w_1)\right|^4 |w_1|^{12} \prod_{k=1}^{3} \left(\left|z_1^{(k)}\right|^2 + \sum_{j=2}^{D} \left|z_j^{(k)} w_1 - w_j z_1^{(k)}\right|^2\right)^{D-1}}. \tag{4.46}$$

In these coordinates $H_j = \{z_1^{(j)} + w_1 = 0\}$, for $j = 1, 2, 3$, $H_4 = \{w_1 = 0\}$, and $\Delta_k = \{z_1^{(k)} = \cdots = z_D^{(k)} = 0\}$, for $k = 1, 2, 3$.

Now we want to consider $\pi_1^* \omega_{\Gamma_3}$ in $\pi_1^{-1}(U)$. The wonderful compactification is locally given in terms of the charts

$$\pi_1 : (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \eta) \longmapsto (\zeta^{(1)}_{\{j_1\}}, \zeta^{(2)}_{\{j_2\}}, \zeta^{(3)}_{\{j_3\}}, \eta) = (z^{(1)}, z^{(2)}, z^{(3)}, w), \tag{4.47}$$

with $1 \leq j_k \leq D$ for $k = 1, 2, 3$, where we can represent any chart by its corresponding tuple (j_1, j_2, j_3). We see that there are D^3 charts in total, however, out of these there are only 2^3 distinct ones (with respect to $\pi_1^* \omega_{\Gamma_3}$), $j_k = 1$ and $j_k \neq 2$ (which we represent by $j_k = 2$) for each $k = 1, 2, 3$. Furthermore, by symmetry we only need to look at half of these charts since, e.g., $\pi_1^* \omega_{\Gamma_3}$ in the chart $J = (2, 1, 1)$ looks the same as in the charts $J = (1, 2, 1)$ and $J = (1, 1, 2)$ if we interchange $z^{(1)}$ and $z^{(2)}$, respectively $z^{(1)}$ and $z^{(3)}$. Thus our four representative charts can be taken to be

$$\{(1,1,1), (1,1,2), (1,2,2), (2,2,2)\}.$$

The pullback of ω_{Γ_3} in these four charts are as follows,

$(1, 1, 1)$:

$$\pi_1^* \omega_{\Gamma_3} = \frac{\mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\bar{\zeta}^{(1)} \wedge \mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\bar{\zeta}^{(2)} \wedge \mathrm{d}\zeta^{(3)} \wedge \mathrm{d}\bar{\zeta}^{(3)} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left|(\zeta_1^{(1)} + \eta_1)(\zeta_1^{(2)} + \eta_1)(\zeta_1^{(3)} + \eta_1)\right|^4 |\eta_1|^{12} \prod_{k=1}^{3} \left(1 + \sum_{j=2}^{D} \left|\zeta_j^{(k)} \eta_1 - \eta_j\right|^2\right)^{D-1}}, \tag{4.48}$$

$(1, 1, 2)$:

$$\begin{aligned}\pi_1^* \omega_{\Gamma_3} = {} & \frac{\mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\bar{\zeta}^{(1)} \wedge \mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\bar{\zeta}^{(2)} \wedge \mathrm{d}\zeta^{(3)} \wedge \mathrm{d}\bar{\zeta}^{(3)} \wedge \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}}{\left(\left|\zeta_1^{(3)}\right|^2 + \left|\eta_1 - \eta_2 \zeta_1^{(3)}\right|^2 + \sum_{j=3}^{D} \left|\zeta_j^{(3)} \eta_1 - \eta_j \zeta_1^{(3)}\right|^2\right)^{D-1}} \times \\ & \times \frac{1}{\left|(\zeta_1^{(1)} + \eta_1)(\zeta_1^{(2)} + \eta_1)(\zeta_2^{(3)} \zeta_1^{(3)} + \eta_1)\right|^4 |\eta_1|^{12} \prod_{k=1}^{2} \left(1 + \sum_{j=2}^{D} \left|\zeta_j^{(k)} \eta_1 - \eta_j\right|^2\right)^{D-1}},\end{aligned} \tag{4.49}$$

(1, 2, 2):

$$\pi_1^*\omega_{\Gamma_3} = \frac{\mathrm{d}\zeta^{(1)}\wedge\mathrm{d}\bar{\zeta}^{(1)}\wedge\mathrm{d}\zeta^{(2)}\wedge\mathrm{d}\bar{\zeta}^{(2)}\wedge\mathrm{d}\zeta^{(3)}\wedge\mathrm{d}\bar{\zeta}^{(3)}\wedge\mathrm{d}\eta\wedge\mathrm{d}\bar{\eta}}{\prod_{k=2}^{3}\big(|\zeta_1^{(k)}|^2+|\eta_1-\eta_2\zeta_1^{(k)}|^2+\sum_{j=3}^{D}|\zeta_j^{(k)}\eta_1-\eta_j\zeta_1^{(k)}|^2\big)^{D-1}}\times$$
$$\times\frac{1}{|(\zeta_1^{(1)}+\eta_1)(\zeta_2^{(2)}\zeta_1^{(2)}+\eta_1)(\zeta_2^{(3)}\zeta_1^{(3)}+\eta_1)|^4|\eta_1|^{12}(1+\sum_{j=2}^{D}|\zeta_j^{(1)}\eta_1-\eta_j|^2)^{D-1}}, \tag{4.50}$$

(2, 2, 2):

$$\pi_1^*\omega_{\Gamma_3} = \frac{\mathrm{d}\zeta^{(1)}\wedge\mathrm{d}\bar{\zeta}^{(1)}\wedge\mathrm{d}\zeta^{(2)}\wedge\mathrm{d}\bar{\zeta}^{(2)}\wedge\mathrm{d}\zeta^{(3)}\wedge\mathrm{d}\bar{\zeta}^{(3)}\wedge\mathrm{d}\eta\wedge\mathrm{d}\bar{\eta}}{\prod_{k=1}^{3}\big(|\zeta_1^{(k)}|^2+|\eta_1-\eta_2\zeta_1^{(k)}|^2+\sum_{j=3}^{D}|\zeta_j^{(k)}\eta_1-\eta_j\zeta_1^{(k)}|^2\big)^{D-1}}\times$$
$$\times\frac{1}{|(\zeta_2^{(1)}\zeta_1^{(1)}+\eta_1)(\zeta_2^{(2)}\zeta_1^{(2)}+\eta_1)(\zeta_2^{(3)}\zeta_1^{(3)}+\eta_1)|^4|\eta_1|^{12}}. \tag{4.51}$$

We note that only in the chart (1, 1, 1) we have $\pi_1^*\omega_{\Gamma_3} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ where $\mathfrak{D}$ is a normal crossings divisor.

4.3.2 Blowing up $\overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D)$

We here describe a further blowup of $\overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D)$ in which our amplitude will be in $\mathcal{E}(*\bar{*}\mathfrak{D})$ for a normal crossings divisor $\mathfrak{D}$. Recall that $\pi_1 : \overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D) \to \mathrm{Conf}_{\Gamma_3}(\mathbb{CP}^D)$ is the wonderful compactification of the configuration space associated with Γ_3. We have the following theorem.

Theorem 4.12. **a)** *With $H_{ij} = H_i \cap H_j$, $H_{ijk} = H_i \cap H_j \cap H_k$ and $H_{1234} = H_1 \cap H_2 \cap H_3 \cap H_4$, the collection*

$$\mathcal{G}_\infty = \{H_{14}, H_{24}, H_{34}, H_{124}, H_{134}, H_{234}, H_{1234}\} \tag{4.52}$$

is a building set of non-singular subvarieties of $(\mathbb{CP}^D)^{\times 4}$; the induced simple arrangement is $\mathcal{G}_\infty$ itself.

b) *The collection*

$$\mathcal{G}'_\infty = \{\mathrm{Strict}_{\pi_1}(S) : S \in \mathcal{G}_\infty\}$$

constitutes a building set, which is its own induced simple arrangement.

Proof. a) Clearly $\mathcal{G}_\infty$ is closed under intersections. Now the question is whether this set is a simple arrangement. By the aforementioned fact it is closed under non-empty intersections. Furthermore, for any two elements $S_1, S_2 \in \mathcal{G}_\infty$, such that $S_1 \cap S_2 \neq \varnothing$, and a point $X \in S_1 \cap S_2$ there are local coordinates in a neighbourhood of X such that S_1, S_2

are respectively given by intersections of coordinate hyperplanes. Let x be a set of local coordinates in a neighbourhood of X, and I and J tuples of indices such that (locally)

$$S_1 = \bigcap_{i\in I}\{x_i = 0\} \quad \text{and} \quad S_2 = \bigcap_{j\in J}\{x_j = 0\}.$$

The tangent spaces of S_1 and S_2 are then given by

$$T_X S_1 = \mathrm{Span}\left\{\frac{\partial}{\partial x_i} : i \in \{1,\dots,4D\}\backslash I\right\} \quad \text{and} \quad T_X S_2 = \mathrm{Span}\left\{\frac{\partial}{\partial x_j} : j \in \{1,\dots,4D\}\backslash J\right\},$$

respectively. The tangent space at X of the intersection $S_1 \cap S_2$ is given by

$$T_X(S_1\cap S_2) = \mathrm{Span}\left\{\frac{\partial}{\partial x_k} : k \in \{1,\dots,4D\} \setminus (I \cup J)\right\}.$$

It follows that

$$T_X(S_1\cap S_2) = T_X S_1|_{S_1\cap S_2} \cap T_X S_2|_{S_1\cap S_2}.$$

Since X was chosen arbitrarily the above holds everywhere on the intersection $S_1 \cap S_2$, whence S_1 and S_2 intersect cleanly. Since this is true for any such elements $S_1, S_2 \in \mathcal{G}_\infty$, we conclude that $\mathcal{G}_\infty$ is a simple arrangement. Furthermore, by the definition in § 3.1, $\mathcal{G}_\infty$ trivially is a building set.

b) Let, as above, $(p,p,p,p) \in H_{1234}$ and let $x^{(j)}$, for $j = 1,2,3,4$, be local coordinates in a neighbourhood of p such that $\{x_1^{(j)} = 0\} = \{X_0^{(j)} = 0\}$. We will show that $\mathcal{G}'_\infty \cap \pi_1^{-1}(U)$ is a building set in $\pi_1^{-1}(U)$ where U is a neighbourhood of (p,p,p,p). Neighbourhoods of other types of points on $\bigcup H_{j4}$ are handled in a similar way.

We change coordinates according to (4.40) and (4.41). Recall the description of π_1 in (4.47). The strict transform H_{k4} is

$$\mathrm{Strict}_{\pi_1}(H_{k4}) = \overline{\pi_1^{-1}(H_{k4}) \setminus \mathrm{Exc}(\pi_1)}.$$

In our (z,w) coordinates, we have

$$H_{k4} = \{z_1^{(k)} + w_1 = 0\} \cap \{w_1 = 0\} = \{z_1^{(k)} = w_1 = 0\}.$$

A direct computation gives

$$\mathrm{Strict}_{\pi_1}(H_{k4}) = \begin{cases} \overline{\{\zeta_{j_k}^{(k)}\zeta_1^{(k)} = \eta_1 = 0\} \setminus \{\zeta_{j_k}^{(k)} = 0\}} = \{\zeta_1^{(k)} = \eta_1 = 0\} \text{ for } j_k \neq 1 \\ \overline{\{\zeta_1^{(k)} = \eta_1 = 0\} \setminus \{\zeta_1^{(k)} = 0\}} = \varnothing \text{ for } j_k = 1. \end{cases}$$

Moreover, $\mathrm{Strict}_{\pi_1}(H_{k\ell 4}) = \{\zeta_1^{(k)} = \zeta_1^{(\ell)} = \eta_1 = 0\}$ if $j_k, j_\ell \neq 1$ and $\mathrm{Strict}_{\pi_1}(H_{k\ell 4}) = \varnothing$ otherwise, and similarly $\mathrm{Strict}_{\pi_1}(H_{1234}) = \{\zeta_1^{(1)} = \zeta_1^{(2)} = \zeta_3^{(3)} = \eta_1 = 0\}$ if $j_1, j_2, j_3 \neq 1$ and $\mathrm{Strict}_{\pi_1}(H_{1234}) = \varnothing$ otherwise.

We would like to show that $\mathcal{G}'_\infty \cap U = \{\mathrm{Strict}_{\pi_1}(S) : S \in \mathcal{G}_\infty\} \cap U$ is a simple arrangement. To check closedness and cleanness under non-empty intersections we only need to consider the intersections of strict transforms in the charts with the relevant j_k's not equal to 1.

Consider a chart (j_1, j_2, j_3) where $j_1, j_2 \neq 1$. We then have $\mathrm{Strict}_{\pi_1}(H_{14}) = \{\zeta_1^{(1)} = \eta_1 = 0\}$, $\mathrm{Strict}_{\pi_1}(H_{24}) = \{\zeta_1^{(2)} = \eta_1 = 0\}$ and $\mathrm{Strict}_{\pi_1}(H_{124}) = \{\zeta_1^{(1)} = \zeta_1^{(2)} = \eta_1 = 0\}$. Hence

$$\mathrm{Strict}_{\pi_1}(H_{14}) \cap \mathrm{Strict}_{\pi_1}(H_{24}) = \mathrm{Strict}_{\pi_1}(H_{124}).$$

With $H'_{14} = \mathrm{Strict}_{\pi_1}(H_{14})$ and $H'_{24} = \mathrm{Strict}_{\pi_1}(H_{24})$, by the same argument as in part a) above, we have that

$$T_X(H'_{14} \cap H'_{24}) = T_X H'_{14}|_{H'_{14} \cap H'_{24}} \cap T_X H'_{24}|_{H'_{14} \cap H'_{24}}.$$

Thus, H'_{14} and H'_{24} intersects cleanly, with intersection $H'_{124} = \mathrm{Strict}_{\pi_1}(H_{124})$. An analogous argument can be made for any two elements $S_1, S_2 \in \mathcal{G}'_\infty$, whence we conclude that $\mathcal{G}'_\infty$ is a simple arrangement. Again by the definition in § 3.1, $\mathcal{G}'_\infty$ is trivially a building set, which completes the proof. □

Theorem 4.13. *With $\mathcal{WC}_1 = \overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D)$, let*

$$\pi_2 : \mathcal{WC}_2 \to \mathcal{WC}_1$$

be the wonderful compactification of $\mathcal{WC}_1$ with respect to the simple arrangement $\mathcal{G}'_\infty$. There is a normal crossings divisor $\mathfrak{D}$ in $\mathcal{WC}_2$ such that

$$\pi_2^* \pi_1^* \omega_{\Gamma_3} \in \mathcal{E}(*\bar{*}\mathfrak{D}).$$

Proof. Let $\pi = \pi_1 \circ \pi_2$. We will give a sketch of the proof, looking only at $\pi^{-1}(U)$, where, as above, U is a neighbourhood of a point $(p, p, p, p) \in (\mathbb{CP}^D)^{\times 4}$ where $p \in H_\infty$. We adopt the notation and coordinates that we used above when we discussed ω_{Γ_3} in such a neighbourhood. We enumerate the sets $H'_\bullet \in \mathcal{G}'_\infty$, where $H'_\bullet = \mathrm{Strict}_{\pi_1}(H_\bullet)$, as follows

$$H'_1 = H'_{1234},\ H'_2 = H'_{124},\ H'_3 = H'_{134},\ H'_4 = H'_{234},\ H'_5 = H'_{14},\ H'_6 = H'_{24},\ H'_7 = H'_{34}, \tag{4.53}$$

such that

$$\mathcal{G}'_\infty = \{H'_1, H'_2, H'_3, H'_4, H'_5, H'_6, H'_7\}.$$

Hence, for any containment $H'_i \subset H'_j$ we have that $i < j$. Note that there is an ambiguity in the internal ordering of H'_{124}, H'_{134} and H'_{234}, as well as of H'_{14}, H'_{24} and H'_{34}, and that our particular choice has no impact on the resulting wonderful compactification. In fact, in [Li09, Theorem 1.3], one requires that the ordering of elements satisfies that the set $\{G_j \in \mathcal{G}'_\infty : 1 \leq j \leq N\}$ is a building set for every $N \leq 7$. It is easily verified that this is the case, independent of the sub-orderings of H'_2, H'_3 and H'_4, and of H'_5, H'_6 and H'_7. Following the iterated blowup procedure described in § 3.1 we obtain the wonderful compactification $\mathcal{WC}_2$ of $\overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D)$ as the result of a sequence of 7 blowups $Y^{(7)} \to \cdots \to Y^{(0)}$, where $Y^{(0)} = \overline{\mathrm{Conf}}_{\Gamma_3}(\mathbb{CP}^D)$, $Y^{(7)} = \mathcal{WC}_2$ and $Y^{(j)} \to Y^{(j-1)}$ is the blowup along the iterated strict transform of H'_j.

We have three levels of inclusions of elements of $\mathcal{G}'_\infty$,

$$H'_1 \subset \begin{cases} H'_2 \subset \begin{cases} H'_5 \\ H'_6 \end{cases} \\ H'_3 \subset \begin{cases} H'_5 \\ H'_7 \end{cases} \\ H'_4 \subset \begin{cases} H'_6 \\ H'_7 \end{cases} \end{cases} .$$

This structure of $\mathcal{G}'_\infty$ will result in the sequence of 7 iterated blowups locally looking like at most 3 blowups. To see this, recall the description of π_1 in charts, see (4.47). We consider the sequence of blowups, beginning in the neighbourhood of a point in $H'_1 \subset Y^{(0)}$ in the chart of $\pi_1 : Y^{(0)} \to (\mathbb{CP}^D)^{\times 4}$ labelled by (j_1, j_2, j_3), with $j_1, j_2, j_3 \neq 1$. In this chart $H'_1 = \{z_1^{(1)} = z_1^{(2)} = z_1^{(3)} = w_1 = 0\}$. The first blowup in the sequence,

$$\pi_2^{(1)} : Y^{(1)} \to Y^{(0)},$$

along H'_1, is then locally given by four charts,

$$\pi_2^{(1)} : (u^{(1)}, \dots, u^{(4)}) \mapsto \begin{cases} (u^{(1)}, u_1^{(1)} \odot_1 u^{(2)}, u_1^{(1)} \odot_1 u^{(3)}, u_1^{(1)} \odot_1 u^{(4)}) \\ \\ (u_1^{(2)} \odot_1 u^{(1)}, u^{(2)}, u_1^{(2)} \odot_1 u^{(3)}, u_1^{(2)} \odot_1 u^{(4)}) \\ \\ (u_1^{(3)} \odot_1 u^{(1)}, u_1^{(3)} \odot_1 u^{(2)}, u^{(3)}, u_1^{(3)} \odot_1 u^{(4)}) \\ \\ (u_1^{(4)} \odot_1 u^{(1)}, u_1^{(4)} \odot_1 u^{(2)}, u_1^{(4)} \odot_1 u^{(3)}, u^{(4)}) \end{cases} = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \eta), \tag{4.54}$$

since H'_1 is a codimension 4 variety. We define the symbol $\odot_j$ above as the product of the scalar to the left with the j^{th} component of the vector to the right. The reason for introducing such a symbol is purely for ease of notation, e.g., we write $u_k \odot_j u$ instead of $(u_1, \dots, u_{j-1}, u_k u_j, u_{j+1}, \dots, u_D)$. We will index these four charts in (4.54) by $k = 1, 2, 3, 4$.

In each chart we have different coordinate functions representing the exceptional divisor of the blowup, i.e., $\mathrm{Exc}(\pi_2^{(1)}) = \{u_1^{(k)} = 0\}$ for $k = 1, \dots, 4$. Consider, for example, the chart where $\mathrm{Exc}(\pi_2^{(1)}) = \{u_1^{(1)} = 0\}$. In this chart we find that the strict transforms of the H'_j's become

$$\begin{aligned} &\mathrm{Strict}_{\pi_2^{(1)}}(H'_1) = \varnothing, \quad \mathrm{Strict}_{\pi_2^{(1)}}(H'_2) = \varnothing, \quad \mathrm{Strict}_{\pi_2^{(1)}}(H'_3) = \varnothing, \\ &\mathrm{Strict}_{\pi_2^{(1)}}(H'_4) = \{u_1^{(2)} = u_1^{(3)} = u_1^{(4)} = 0\}, \quad \mathrm{Strict}_{\pi_2^{(1)}}(H'_5) = \varnothing, \\ &\mathrm{Strict}_{\pi_2^{(1)}}(H'_6) = \{u_1^{(2)} = u_1^{(4)} = 0\}, \quad \mathrm{Strict}_{\pi_2^{(1)}}(H'_7) = \{u_1^{(3)} = u_1^{(4)} = 0\}. \end{aligned}$$

Since $\mathrm{Strict}_{\pi_2^{(1)}}(H'_2)$ and $\mathrm{Strict}_{\pi_2^{(1)}}(H'_3)$ both vanish, the sequence of blowups effectively continues with the blowup

$$\pi_2^{(2)} : Y^{(2)} \to Y^{(1)}$$

along $\mathrm{Strict}_{\pi_2^{(1)}}(H_4')$. It is given by three charts,

$$\pi_2^{(2)} : (v^{(1)}, \ldots, v^{(4)}) \mapsto \begin{cases} (v_1^{(2)} \odot_1 v^{(1)}, v^{(2)}, v_1^{(2)} \odot_1 v^{(3)}, v_1^{(2)} \odot_1 v^{(4)}) \\ \\ (v_1^{(3)} \odot_1 v^{(1)}, v_1^{(3)} \odot_1 v^{(2)}, v^{(3)}, v_1^{(3)} \odot_1 v^{(4)}) \\ \\ (v_1^{(4)} \odot_1 v^{(1)}, v_1^{(4)} \odot_1 v^{(2)}, v_1^{(4)} \odot_1 v^{(3)}, v^{(4)}) \end{cases} = (u^{(1)}, \ldots, u^{(4)}), \tag{4.55}$$

since $\mathrm{Strict}_{\pi_2^{(1)}}(H_4')$ is a codimension 3 variety. We will index these three charts by $\ell = 2, 3, 4$.

Consider, for example, the chart where $\mathrm{Exc}(\pi_2^{(2)}) = \{v_1^{(2)} = 0\}$. Here we have that

$$\begin{aligned} &\mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_1') = \varnothing, \quad \mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_2') = \varnothing, \quad \mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_3') = \varnothing, \\ &\mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_4') = \varnothing, \quad \mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_5') = \varnothing, \quad \mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_6') = \varnothing, \\ &\mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_7') = \{v_1^{(3)} = v_1^{(4)} = 0\}. \end{aligned}$$

Thus, since $\mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_5')$ and $\mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_6')$ both vanish, we have a similar situation as we had for $\pi_2^{(2)}$. The sequence effectively continues with the blowup

$$\pi_2^{(3)} : Y^{(3)} \to Y^{(2)}$$

along $\mathrm{Strict}_{\pi_2^{(1)} \circ \pi_2^{(2)}}(H_7')$, defined by

$$\pi_2^{(3)} : (w^{(1)}, \ldots, w^{(4)}) \mapsto \begin{cases} (w_1^{(3)} \odot_1 w^{(1)}, w_1^{(3)} \odot_1 w^{(2)}, w^{(3)}, w_1^{(3)} \odot_1 w^{(4)}) \\ \\ (w_1^{(4)} \odot_1 w^{(1)}, w_1^{(4)} \odot_1 w^{(2)}, w_1^{(4)} \odot_1 w^{(3)}, w^{(4)}) \end{cases} = (v^{(1)}, \ldots, v^{(4)}). \tag{4.56}$$

We will index these two charts by $m = 3, 4$. With (4.56), the iterated sequence of blowups defining the wonderful compactification $\mathcal{WC}_2$ has terminated.

Now we will look at $\pi^*\omega_{\Gamma_3}$ in some representative charts, beginning with the chart $(j_1, j_2, j_3) = (2, 2, 2)$, $(k, \ell, m) = (1, 2, 3)$, where we recall that $\pi_1^*\omega_{\Gamma_3}$ in the chart $(2, 2, 2)$ is given by (4.51). We compute $\pi^*\omega_{\Gamma_3}$ by successively pulling back $\pi_1^*\omega_{\Gamma_1}$ under $\pi_2^{(1)}$, $\pi_2^{(2)}$, and $\pi_2^{(3)}$.

A straightforward albeit tedious computation yields the following,

$$
\begin{aligned}
\pi^*\omega_{\Gamma_3} = {} & \frac{\mathrm{d}w^{(1)} \wedge \mathrm{d}\bar{w}^{(1)} \wedge \mathrm{d}w^{(2)} \wedge \mathrm{d}\bar{w}^{(2)} \wedge \mathrm{d}w^{(3)} \wedge \mathrm{d}\bar{w}^{(3)} \wedge \mathrm{d}w^{(4)} \wedge \mathrm{d}\bar{w}^{(4)}}{\left(1 + |w_1^{(2)}w_1^{(3)}w_1^{(4)} - w_2^{(4)}|^2 + \sum_{j=3}^{D} |w_j^{(1)}w_1^{(2)}w_1^{(3)}w_1^{(4)} - w_j^{(4)}|^2\right)^{D-1}} \times \\
& \times \frac{1}{\left(1 + |w_1^{(3)}w_1^{(4)} - w_2^{(4)}|^2 + \sum_{j=3}^{D} |w_j^{(2)}w_1^{(3)}w_1^{(4)} - w_j^{(4)}|^2\right)^{D-1}} \times \\
& \times \frac{1}{\left(1 + |w_1^{(4)} - w_2^{(4)}|^2 + \sum_{j=3}^{D} |w_j^{(3)}w_1^{(4)} - w_j^{(4)}|^2\right)^{D-1}} \times \\
& \times \frac{1}{|w_1^{(1)}|^{6(D+2)}|w_1^{(2)}|^{4(D+3)}|w_1^{(3)}|^{2(D+6)}|w_1^{(4)}|^{12}|w_2^{(1)}w_2^{(2)}w_2^{(3)}|^4}.
\end{aligned}
\tag{4.57}
$$

We see that $\pi^*\omega_{\Gamma_3} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ for a suitable normal crossings divisor $\mathfrak{D}$ locally given by

$$
\mathfrak{D} = \{w_1^{(1)}w_1^{(2)}w_1^{(3)}w_1^{(4)}w_2^{(1)}w_2^{(2)}w_2^{(3)} = 0\}.
$$

The choices of charts above in any of the blowups $\pi_2^{(1)}, \pi_2^{(2)}$ will determine which of the succeeding blowups in the sequence become insubstantial and which do not. What we are left with is something that locally looks like up to three blowups. The reason it is *up to* and not exactly 3 blowups, is in part due to the fact that in the local description of every blowup, we always have a chart with exceptional divisor corresponding to the coordinate associated to the central vertex of Γ_3. In these charts all subsequent strict transforms of elements in $\mathcal{G}'_\infty$ vanish, and thus, the sequence of blowups terminates. In any of these charts, one can check that the pullback of ω_{Γ_3} has singularities on a normal crossings divisor. As a representative example of this, consider the pullback to the chart $(j_1, j_2, j_3) = (2, 2, 2)$ and $k = 4$; any choice of ℓ and m following $k = 4$ is superfluous since the sequence has terminated. Again, a straightforward but tedious computation yields

$$
\begin{aligned}
\pi^*\omega_{\Gamma_3} = {} & \frac{\mathrm{d}u^{(1)} \wedge \mathrm{d}\bar{u}^{(1)} \wedge \mathrm{d}u^{(2)} \wedge \mathrm{d}\bar{u}^{(2)} \wedge \mathrm{d}u^{(3)} \wedge \mathrm{d}\bar{u}^{(3)} \wedge \mathrm{d}u^{(4)} \wedge \mathrm{d}\bar{u}^{(4)}}{\prod_{k=1}^{3}\left(|u_1^{(k)}|^2 + |1 - u_2^{(4)}u_1^{(k)}|^2 + \sum_{j=3}^{D} |u_j^{(k)} - u_j^{(4)}u_1^{(k)}|^2\right)^{D-1}} \times \\
& \times \frac{1}{|(u_2^{(1)}u_1^{(1)} + 1)(u_2^{(2)}u_1^{(2)} + 1)(u_2^{(3)}u_1^{(3)} + 1)|^4 |u_1^{(4)}|^{6(D+2)}}.
\end{aligned}
$$

Since $u_1^{(k)} = 0$ and $1 - u_2^{(4)}u_1^{(k)} = 0$ never hold simultaneously, the expression

$$
\prod_{k=1}^{3}\left(|u_1^{(k)}|^2 + |1 - u_2^{(4)}u_1^{(k)}|^2 + \sum_{j=3}^{D} |u_j^{(k)} - u_j^{(4)}u_1^{(k)}|^2\right)^{D-1}
$$

is nowhere vanishing. Furthermore

$$\chi_1 = u_2^{(1)} u_1^{(1)} + 1,$$
$$\chi_2 = u_2^{(2)} u_1^{(2)} + 1,$$
$$\chi_3 = u_2^{(3)} u_1^{(3)} + 1,$$

together with $\chi_4 = u_1^{(1)}$, $\chi_5 = u_1^{(2)}$ and $\chi_6 = u_1^{(3)}$ defines new local coordinates. Thus we see that $\pi^*\omega_{\Gamma_3} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ where $\mathfrak{D}$ is locally given by $\mathfrak{D} = \{\chi_1\chi_2\chi_3 u_1^{(4)} = 0\}$. We settle with this sketch of the full proof, having outlined the different local pictures of $\mathcal{WC}_2$, and having shown that the theorem holds in neighbourhoods of the worst singularities of ω_{Γ_3}. □

4.4 The canonical current for Γ_1 in $D = 2$

Recall the Feynman amplitude ω_{Γ_1} given by (4.2) and the blowups π_1 and π_2 given by (4.3) and (4.16), respectively. By Theorem 4.4, $\pi^*\omega_{\Gamma_1} \in \mathcal{E}(*\bar{*}\mathfrak{D})$ where $\pi = \pi_1 \circ \pi_2$ and $\mathfrak{D} = \mathrm{Exc}(\pi_2) + \mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) + \mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty)$ is a normal crossings divisor. In view of § 3.2 we have a canonical current $\{\pi^*\omega_{\Gamma_1}\}$ associated to $\pi^*\omega_{\Gamma_1}$. We notice that

$$\mathrm{supp}(\{\pi^*\omega_{\Gamma_1}\}) \subseteq \pi^{-1}(\Delta \cap (H_\infty \times H_\infty)) = \mathrm{Exc}(\pi_2) \cap \mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) \cap \mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty).$$

In the special case $D = 2$, we observe that $\Delta \cap (H_\infty \times H_\infty) \simeq \mathbb{CP}^1$. We have the following proposition.

Proposition 4.14. *In the case $D = 2$, we have $\langle \{\pi^*\omega_{\Gamma_1}\}, 1\rangle = 0$ and the meromorphic function*

$$\lambda \longmapsto \int\limits_{\mathbb{C}^2 \times \mathbb{C}^2} \left|\frac{1}{1+|x|^2}\right|^\lambda \left|\frac{1}{1+|y|^2}\right|^\lambda \omega_{\Gamma_1}$$

has at most a double pole at $\lambda = 0$.

Proof. We first check that the second statement follows from the first. In view of (3.17) and Theorem 4.4, we have

$$\int\limits_{\mathbb{C}^2 \times \mathbb{C}^2} \left|\frac{1}{1+|x|^2}\right|^\lambda \left|\frac{1}{1+|y|^2}\right|^\lambda \omega_{\Gamma_1} = \frac{1}{\lambda^3}\langle \pi_*\mu_3, 1\rangle + \frac{1}{\lambda^2}\langle \pi_*\mu_2, 1\rangle + \frac{1}{\lambda}\langle \pi_*\mu_1, 1\rangle + \langle \pi_*\mu_0, 1\rangle + \mathcal{O}(|\lambda|),$$

where μ_j are the currents associated with $\pi^*\omega_{\Gamma_1}$, as described in § 3.2; in particular, $\mu_3 = \{\pi^*\omega_{\Gamma_1}\}$. Hence, if $\langle \{\pi^*\omega_{\Gamma_1}\}, 1\rangle = 0$, the order of the pole at $\lambda = 0$ is at most 2.

Consider the function

$$X = [X_0 : X_1 : X_2] \longmapsto \frac{|X_1|^2}{|X_1|^2 + |X_2|^2}$$

defined on $\mathbb{CP}^2 \setminus \{[1:0:0]\}$. We can regard it as a function on $\big(\mathbb{CP}^2 \setminus \{[1:0:0]\}\big) \times \mathbb{CP}^2$. Given our choice of hyperplanes at infinity in $\mathbb{CP}^2 \times \mathbb{CP}^2$, as $\{X_0 = 0\} \times \mathbb{CP}^2$ and $\mathbb{CP}^2 \times \{Y_0 = 0\}$, respectively, the function is clearly well-defined in a neighborhood of $\Delta \cap (H_\infty \times H_\infty) \supseteq \mathrm{supp}(\pi_*\mu_3)$. Furthermore, the function is smooth in a neighborhood of $\Delta \cap (H_\infty \times H_\infty)$.

It is well known that there are smooth (non-analytic) functions on the unit interval which are equal to 0 in a neighbourhood of 0 and 1 in a neighbourhood of 1. These are known as a *smooth transition from* 0 *to* 1. Let $\chi : [0,1] \to [0,1]$ be such a function. We then have a smooth partition of unity of a neighbourhood U of $\Delta \cap (H_\infty \times H_\infty)$ given by

$$\rho_1 = \chi\left(\frac{|X_1|^2}{|X_1|^2 + |X_2|^2}\right), \quad \rho_2 = 1 - \chi\left(\frac{|X_1|^2}{|X_1|^2 + |X_2|^2}\right).$$

Notice that $\rho_1 = 0$ in a neighbourhood of $([0:0:1],[0:0:1])$ and that $\rho_2 = 0$ in a neighbourhood of $([0:1:0],[0:1:0])$. Let

$$U_1 = U \setminus \{([0:0:1],[0:0:1])\} \quad \text{and} \quad U_2 = U \setminus \{([0:1:0],[0:1:0])\}.$$

Then $U \cap \operatorname{supp}(\rho_1) \subset U_1$ and $U \cap \operatorname{supp}(\rho_2) \subset U_2$. We want to show that

$$\langle \rho_1 \cdot \pi_* \mu_3, 1\rangle = 0 \quad \text{and} \quad \langle \rho_2 \cdot \pi_* \mu_3, 1\rangle = 0,$$

from which the proposition immediately follows. Furthermore, we note that

$$\begin{aligned} \langle \rho_1 \cdot \pi_* \mu_3, 1\rangle &= \langle \pi^* \rho_1 \cdot \mu_3, \underbrace{\pi^* 1}_{=1}\rangle \\ &= \langle \mu_3, \pi^* \rho_1\rangle, \end{aligned}$$

and similarly

$$\langle \rho_1 \cdot \pi_* \mu_3, 1\rangle = \langle \mu_3, \pi^* \rho_2\rangle.$$

Consider local coordinates in U_1 centred around $\{([0:1:0],[0:1:0])\}$ defined by

$$(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2) = \left(\frac{X_0}{X_1}, \frac{X_2}{X_1}, \frac{Y_0}{Y_1}, \frac{Y_2}{Y_1}\right)$$

Since $\operatorname{supp}(\pi_* \mu_3) \subseteq \Delta \cap (H_\infty \times H_\infty)$ and $U \cap \operatorname{supp}(\rho_1) \subseteq U_1$ we have that $\rho_1 \, \pi_* \mu_3$ has compact support in U_1 and can be computed in the local coordinates $(\tilde{x}, \tilde{y})$. In these coordinates, ρ_1 becomes

$$\rho_1 = \chi\left(\frac{1}{1 + |\tilde{x}_2|^2}\right).$$

Recall that these coordinates correspond to the coordinates defined in (4.9). We thus have the same transition map

$$x_1 = \frac{1}{\tilde{x}_1}, \; x_2 = \frac{\tilde{x}_2}{\tilde{x}_1}, \quad y_1 = \frac{1}{\tilde{y}_1}, \; y_2 = \frac{\tilde{y}_2}{\tilde{y}_1}.$$

The amplitude is given by the familiar expression, now with $D = 2$ explicitly,

$$\omega_{\Gamma_1} = \frac{\mathrm{d}\tilde{x} \wedge \mathrm{d}\bar{\tilde{x}} \wedge \mathrm{d}\tilde{y} \wedge \mathrm{d}\bar{\tilde{y}}}{|\tilde{x}_1 \tilde{y}_1|^4 (|\tilde{y}_1 - \tilde{x}_1|^2 + |\tilde{y}_1 \tilde{x}_2 - \tilde{x}_1 \tilde{y}_2|^2)}$$

Going to the coordinate system where the diagonal is given by the vanishing of $\{z_1 = z_2 = 0\}$, we have

$$\omega_{\Gamma_1} = \frac{1}{4} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w}}{|(z_1 + w_1)(z_1 - w_1)|^4 (|z_1|^2 + \frac{1}{4}|w_1 z_2 - z_1 w_2|^2)}$$

and

$$\rho_1 = \chi\left(\frac{1}{1 + \frac{1}{4}|z_2 + w_2|^2}\right).$$

The pullback $\pi_1^* \omega_{\Gamma_1}$ is described in Proposition 4.3. In the former case of Proposition 4.3 $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \varnothing$ and in the latter case we have $\pi_1^* \omega_{\Gamma_1}$ is given by (4.13) with

$D = 2$. Recall that here $\mathrm{Strict}_{\pi_1}(H_\infty \times H_\infty) = \{\zeta_1 = \eta_1 = 0\}$. Let us now consider $\pi_2^*\pi_1^*\omega_{\Gamma_1}$. As in the proof of Theorem 4.4, π_2 is given by (4.17) and (4.18) with $D = 2$. In the latter case we have $\mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) \cap \mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty) = \varnothing$, hence, we get no contribution to μ_3 in this chart. In the former case, the pullback of $\pi_1^*\omega_{\Gamma_1}$ is given by

$$\pi^*\omega_{\Gamma_1} = \frac{1}{4}\frac{\mathrm{d}u \wedge \mathrm{d}\bar{u} \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}}{|(u_2+v_1)(u_2-v_1)|^4|u_1|^8(1+\frac{1}{4}|v_1-v_2|^2)}.$$

Recall that here $\mathrm{Exc}(\pi_2) = \{u_1 = 0\}$, $\mathrm{Strict}_\pi(H_\infty \times \mathbb{CP}^D) = \{u_2 + v_1 = 0\}$ and $\mathrm{Strict}_\pi(\mathbb{CP}^D \times H_\infty) = \{u_2 - v_1 = 0\}$. Furthermore, in this chart

$$\pi^*\rho_1 = \chi\left(\frac{1}{1+\frac{1}{4}|u_2+v_2|^2}\right).$$

In view of (3.12), the integrand in the expression for $\langle \mu_3, \pi^*\rho_1\rangle$ is given by

$$\frac{\partial^2}{\partial(u_2+v_1)\partial(\bar{u}_2+\bar{v}_1)}\frac{\partial^2}{\partial(u_2-v_1)\partial(\bar{u}_2-\bar{v}_1)}\frac{\partial^6}{\partial u_1^3\partial\bar{u}_1^3}\chi\left(\frac{1}{1+\frac{1}{4}|u_2+v_2|^2}\right)\frac{1}{4}\frac{\mathrm{d}v_2\wedge\mathrm{d}\bar{v}_2}{1+\frac{1}{4}|v_1-v_2|^2},$$

which clearly vanishes since, for instance, the derivative with respect to u_1 acts upon an expression which is independent of u_1, whence $\langle \mu_3, \pi^*\rho_1\rangle = 0$.

Now, consider instead local coordinates in U_2 centred around $\{([0:0:1],[0:0:1])\}$ defined by

$$(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2) = \left(\frac{X_0}{X_2}, \frac{X_1}{X_2}, \frac{Y_0}{Y_2}, \frac{Y_1}{Y_2}\right)$$

By a similar argument as above, $\rho_2\,\pi_*\mu_3$ has compact support in U_2 and can thus be computed in its entirety in the local coordinates above. In these coordinates, ρ_2 becomes

$$\rho_2 = 1 - \chi\left(\frac{|\tilde{x}_2|^2}{1+|\tilde{x}_2|^2}\right)$$

The amplitude ω_{Γ_1} looks the same as above in these coordinates, and we find

$$\pi^*\rho_2 = 1 - \chi\left(\frac{\frac{1}{4}|u_2+v_2|^2}{1+\frac{1}{4}|u_2+v_2|^2}\right).$$

By a similar argument as above $\langle \rho_2 \cdot \pi_*\mu_3, 1\rangle = 0$ and we are done. $\square$

5

Concluding Remarks

We end this thesis with a free and somewhat speculative discussion on some of the results and techniques used. The freedom to choose a section and metric on the vector bundle over the chain of integration somewhat dissuades us from studying the non-leading order terms and in particular the finite part of the expansion, since we do not have a preferred set of choices. A possible parallel can be drawn to the process of renormalisation in physics. When renormalising a quantum field theory, one is also presented with an arbitrary choice, when interpreting the physical coupling constants in the renormalised Lagrangian. Usually working with momentum space coordinates, obtained by taking the Fourier transform, one specifies a *subtraction point momentum* which directly affects the value of the renormalised Feynman integral. Although an arbitrary choice can be made, the choice can be compared with the true value of the integral found in experiment, a crucial piece of data we omit when we relax the constraints put upon us by nature. These choices, in these two situations, might not be correlated at all, so we will steer clear of reading too much into possibly artificial connections.

There are a few interesting questions left unanswered, that the results of our case study has hinted towards the answers to. First and foremost is the possibility of there being an algorithmic approach for obtaining a modification $\pi : Y \to X^{\mathbf{V}_\Gamma}$ with a normal crossings divisor $\mathfrak{D} \subset Y$ such that $\pi^*\omega_\Gamma \in \mathcal{E}(*\bar{*}\mathfrak{D})$ for any given Feynman graph Γ. The author conjectures that, at least with $X = \mathbb{CP}^D$, for a given graph there is a distinguished simple arrangement of strict transformations of intersections of hyperplanes at infinity in the wonderful compactification of Ceyhan and Marcolli, such that the application of Li's theorem gives the desired results. We have seen that this is the case for each of the graphs Γ_1, Γ_2 and Γ_3. The main hindrance to us looking at even more examples of Feynman graphs is our local description of blowups that becomes overwhelmingly tedious the more complicated the graphs become. A proof of the successfulness of the conjectured algorithm would need to verify that the pullback of a general Feynman amplitude has singularities on a normal crossings divisor in a way that does not amount to checking the singular locus in each and every chart of the modification.

Say that the conjecture were to be true, one could ask what implications, if any, a second wonderful compactication of configuration space would have on the considerations in [CM12a]. However, since we have not looked at these aspects of the complexified quantum field theories in this thesis, any and all statements on the matter would be presumptuous.

Another interesting point, is that we have seen hints towards the vanishing of the canonical current for each of the graphs considered in the case study, and verified that it does indeed vanish in the special case of Γ_1, $D = 2$. Checking if the canonical current vanishes in some or all of the cases we have considered could be of interest, begging the question whether there is a way of more effectively determining the degree of divergence, other than of κ which is merely an upper bound.

In the case of Γ_3, by surveying the collection of local expressions for the pullback of ω_{Γ_3} to $\mathcal{WC}_2$ considered in § 4.3.2, we see from (4.57) that $\kappa(\omega_{\Gamma_3}) = 7$, and that contributions to the canonical current are found in the charts $(j_1, j_2, j_3), (k, \ell, m)$, where $j_1, j_2, j_3 \neq 1$ and $(k, \ell, m) \in P(\{1, 2, 3\})$. However, we again find that in each of these charts the "smooth part of the amplitude" is independent on $x_{j_1}^{(1)}, x_{j_2}^{(2)}$ and $x_{j_3}^{(3)}$. By the smooth part of the amplitude we mean the amplitude multiplied by a product of factors corresponding to the irreducible components of $\mathfrak{D}$. This independence of certain coordinates again points towards the possibility that $\langle \{\omega_{\Gamma_3}\}, 1 \rangle = 0$ in view of (3.12).

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