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Analysis and performance of three-mode qutrit rotationally symmetric bosonic codes

Master's thesis in Microtechnology and nanoscience - MCCX04

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Gothenburg, Sweden 2026
www.chalmers.se

Master's thesis 2026

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Master's Thesis 2026
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Acknowledgements, dedications, and similar personal statements in this thesis, reflect the author's own views. The work of the thesis has been carried out without the involvement of large language models or similar AI-tools.

Cover illustration:
Pascals triangle for the instance of trinomial distributions. The different shades of gray represent single-mode basis states in a RSB qutrit code.

Typeset in LaTeX
Printed by Chalmers Reproservice
Gothenburg, Sweden 2026

Abstract

Encoding information in bosonic quantum states has an advantage over traditional two-level systems since they inhabit Hilbert spaces of infinite dimensions. This makes bosonic codes into very resourceful carriers of quantum information which can be exploited in error correcting codes. Thus, making bosonic codes a viable option in the realisation of fault tolerant quantum computers.

In this report we begin by adapting a general formulation of multi-mode d-dimensional Rotational symmetrical bosonic (RSB) codes from Ref. [1] and by setting the dimension to $d = 3$ results in a general qutrit code space in three modes. Before the code is further specified, the operations of beam splitters, describing linear passive optical system, is analysed from definitions in Ref. [1] and solutions for how excitations transforms under such evolutions are found analytically from an algebraic structure.

We then specify the coefficients of a code in two separate methods. First, the single-mode RSB basis states are projected onto a state characterized by a binomial distribution. The result is rejected from a preliminary analysis of the symmetries in the resulting single-mode states. Another method is then carried out by arbitrary truncating the sums of general RSB states and enforcing conditions of orthonormality and equal mean photon number $\langle \hat{n} \rangle$ of each single-mode basis state. The result is considered to be a good candidate as an error correctable code.

The code, denoted as "the arbitrary code" is further analysed with a theorem which defines error correctable codes known as the KL-conditions. From this analytical study, we find that the code is correctable for single photon loss errors but not for dephasing errors.

An analytical study of the "arbitrary code", in terms of the KL conditions, is followed by a numerical estimation of its performance against noise in the form of photon loss and dephasing, individually. The performance is estimated in terms of near-optimal entanglement fidelity.

Inspired by the analysis of symmetries in the single-mode basis states and the KL conditions for the full code, we find a third approach experimentally, leading to a new framework for the construction and classification of d-nomial RSB codes. The new frame work also makes it clear that the qutrit codes that has been studied, are in fact characterized by states from trinomial distributions and are therefore denoted as trinomial codes.

Acknowledgements

This project has been the best part of my education at university and I would like to express my gratitude to Giulia Ferrini and Debjyoti Biswas for the support, appreciation and also some constructive and fun debating on quantum information.

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1 Background

The task of quantum computers is to process information, under the principles of quantum mechanics, in order to solve problems of different kinds that classical computers would solve in an exponentially larger time. Example of such computations are the Quantum Fourier transform and Shor's algorithm for finding prime factors [2]. During such computations, the information must be able to withstand any unwanted influence of its environment in the form of noise. The evolution of the information, in this noisy environment, defines an average lifetime, when the information can be useable. This process can be compared to that of diffusion where any initial correlations in a system decays with time. In order to prolong the lifetime, requires the use of a Quantum error correction (QEC) algorithm which constantly monitors the information, evaluates whether any error has occurred and enact with appropriate operations to restore the information.

In quantum computations, each part of the information is encoded into quantum states which functions as logical codewords. If the lifetime is enhanced, as the effective noise is lowered from some QEC process, the performance of the logical code is said to increase. However, a code can only be corrected from a noise process to a certain degree, if any at all. Constructing high performance codes is a crucial part in the realization of so-called fault-tolerant quantum computers.

One family of codes of interest are bosonic codes, where the quantum information is encoded into Fock states of one or several harmonic resonators. Bosonic codes have shown robustness to noise in the form of photon gain/loss and dephasing errors, and are promising candidates for fault-tolerant quantum computers [3], [4], [5], [6]. One type of bosonic codes are the RSB codes [3], which are constructed around Fock states with photon numbers that are multiples of some symmetry order number N.

A recent theoretical study [1] on multi-mode RSB codes has shown some promising and interesting features. The study is based on group theoretical concepts that are used to define logical gates which can be implemented as simple physical operations, like those describing the action of beam splitters. With these concepts, the authors show how to construct a logical RSB code space, composed of bosonic states from one or several modes. The performance of such codes are then computed numerically to qualify their resilience against noise.

The motivation for the report is to act as a continuation of the study mentioned above [1]. Both in terms of exploring the performance of three-mode codes compared to single-mode codes, and also to contribute to a less studied area of quantum information.

Our aim is to first construct a three-mode qutrit RSB code. Secondly, we investigate, analytically, if it qualifies as a QEC code. If so, for which conditions is it viable and optimal? Thirdly, will be to compare its performance with corresponding single-mode codes while also characterizing any features of interest in the performance which might differ from the single-mode case. While there have been several studies on qutrit codes, both in single and multiple modes [7], [8], [9], there seems to be no studies on three-mode qutrit bosonic codes.

The study will be entirely theoretical and will not attempt to explain how the code can be realized, if possible, in any physical form. A full set of gates, required to perform quantum computations on the code, will not be constructed. Nor will any complete algorithms, using such set of gates, be constructed.

2 Theory

In this section we present some of the key concepts that are used in the report.

A set of basic definitions and notations are first brought up as repetition for the readers who are somewhat familiar with quantum information and open quantum system, but also as a foundation for the unfamiliar readers.

This is then followed by a few sections on main underlying concepts such as bosonic codes, quantum operations and fidelity.

2.1 Basic definitions & notations

All states in quantum mechanics are related to very restricted, so called, Hilbert spaces. The elements of these vector spaces can, for example, be in the form of n-dimensional complex vectors or continuous functions on some domain. The dimension of such vector spaces may be finite or infinite. In the bra-ket notation, a state $|\psi\rangle$ may be represented as a linear combination of some orthonormal computational basis $\{|e_i\rangle\}$ such that

$$\langle e_i|e_j\rangle = \delta_{ij} \tag{2.1}$$

$$\sum_i |e_i\rangle\langle e_i| = \mathbb{1} \tag{2.2}$$

$$|\psi\rangle = \sum_i |e_i\rangle\langle e_i|\psi\rangle = \sum_i c_i|e_i\rangle \tag{2.3}$$

$$\sum_i |c_i|^2 = 1, \tag{2.4}$$

where the function $\langle\cdot|\cdot\rangle$ is called an inner product, c_i are complex coefficients and the last relation is the resulting l^2-norm squared, $||\cdot||^2_{l^2}$, of $|\psi\rangle$.

When states $|\psi^A\rangle$ and $|\phi^B\rangle$ from different vector spaces A and B are composed into a joint product state of a joint vector space $A \otimes B$ the following notations are equivalent

$$|\Psi^{AB}\rangle = |\psi^A\rangle \otimes |\phi^B\rangle \tag{2.5}$$

$$|\Psi^{AB}\rangle = |\psi^A\rangle|\phi^B\rangle = |\phi^B\rangle|\psi^A\rangle \tag{2.6}$$

$$|\psi^A\rangle|\phi^B\rangle = |\psi^A\phi^B\rangle. \tag{2.7}$$

Note that the states $|\psi^A\rangle$ and $|\psi^B\rangle$ commutes. However, in some situations, the joint states are represented in the same computational basis, for example $\{|0\rangle, |1\rangle\}$. In those cases, the order matters. For example

$$|\psi^A\rangle|\phi^B\rangle = |0\rangle|1\rangle \tag{2.8}$$

$$|\psi^A\rangle|\phi^B\rangle \neq |1\rangle|0\rangle. \tag{2.9}$$

The different notations can be helpful to jump between in various situations. Similarly, for operators $\hat{a}_A$ and $\hat{a}_B^\dagger$ acting on the composed vector space of A and B, the following notations are equivalent

$$\hat{a}_A = \hat{a} \otimes \mathbb{1} \tag{2.10}$$

$$\hat{a}_B^\dagger = \mathbb{1} \otimes \hat{a}^\dagger \tag{2.11}$$

$$\hat{a}_A\hat{a}_B^\dagger = \hat{a}_B^\dagger\hat{a}_A. \tag{2.12}$$

However, $\hat{a}_A\hat{a}_A^\dagger \neq \hat{a}_A^\dagger\hat{a}_A$, since the commutator $\left[\hat{a}_A, \hat{a}_B^\dagger\right] = \delta_{AB}\mathbb{1}$.

The density operator $\hat{\rho}$ acting on a Hilbert space is a projection of the ensemble of states in a Hilbert space. It is a Hermitian, *positive operator* such that for any state $|\phi\rangle$,

$$\rho^\dagger = \rho \tag{2.13}$$
$$0 \leq \langle\phi|\,\rho\,|\phi\rangle \in \mathbb{R}. \tag{2.14}$$

If each state $|\psi_i\rangle$ corresponds to the probability p_i the density operator is defined as

$$\rho \equiv \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|\,, \tag{2.15}$$

and if there is only one outcome in the ensemble then the density operator is said to represent a pure state denoted as

$$\rho = |\psi\rangle\,\langle\psi|\,. \tag{2.16}$$

The density operator will sometimes be referred to as the "state" of a system.

The projection of the ensemble is used when taking estimates $\langle\hat{O}\rangle$ of some observable $\hat{O}$. This is carried out by the means of the *trace*, denoted as $\mathrm{Tr}\{\cdot\}$. The trace acts on linear operators and projects the sum of all diagonal entries of the operator in its argument. It is defined as

$$\mathrm{Tr}\left\{\hat{O}\right\} \equiv \sum_o \langle o|\,\hat{O}\,|o\rangle\,, \tag{2.17}$$

where $\{|o\rangle\}$ is a computational basis for the operator $\hat{O}$. The trace of any density operator is always unity and the trace of the density operator squared is 1 for pure states and less than one for mixed states such that

$$\mathrm{Tr}\,\{\hat{\rho}\} = 1 \tag{2.18}$$
$$\mathrm{Tr}\left\{\hat{\rho}^2\right\} \leq 1. \tag{2.19}$$

The estimate of an observable $\hat{O}$ is defined as

$$\langle\hat{O}\rangle = \mathrm{Tr}\left\{\hat{O}\hat{\rho}\right\} = \mathrm{Tr}\left\{\hat{\rho}\hat{O}\right\} \tag{2.20}$$

where the last equality shows the invariance of the trace under cyclical permutations of its argument.

The partial trace is an operation which reduces the dimension of an operator by applying the ordinary trace on one of its subsystems. For the density operator ρ^{AB} composed by two density operators ρ^A and ρ^B acting on vector spaces denoted as A and B respectively, the partial trace over subsystem B gives the reduced density operator of subsystem A such that

$$\rho^{AB} = \rho^A \otimes \rho^B \tag{2.21}$$
$$\mathrm{Tr}_B\left(\rho^{AB}\right) = \rho^A\mathrm{Tr}\left(\rho^B\right) = \rho^A. \tag{2.22}$$

An important property of the partial trace is that it is possible to find situations where it operates on a pure product state, leaving a reduced state which is mixed.

The process of *purification* is, in contrast to the reduced trace, the opposite operation, where two states from different vector spaces are joint into a pure state. For example if $\rho^A = \sum_i p_i |i^A\rangle \langle i^A|$ is a density matrix with orthonormal states of a vector space denoted A then by introducing a fictitious reference system R represented by the density operator $\rho^R = \sum_i p_i |i^R\rangle \langle i^R|$, with the same eigen values and states as ρ^A, the purification of ρ^A is then defined as the pure state

$$|\Psi^{AR}\rangle = \sum_i \sqrt{p_i} |i^A\rangle |i^R\rangle \tag{2.23}$$

$$\rho^{AR} = \sum_{ij} \sqrt{p_i p_j} |i^A\rangle |i^R\rangle \langle j^A| \langle j^R| . \tag{2.24}$$

This definition is related to Schmidt decomposition.

Theorem 2.1 (Schmidt decomposition) *If $|\Psi^{AB}\rangle$ is a pure state of a composed system of vector spaces A and B, then there exists orthonormal states $|i^A\rangle \in A$ and $|i^B\rangle \in B$ such that*

$$|\Psi^{AB}\rangle = \sum_i \lambda_i |i^A\rangle |i^B\rangle , \tag{2.25}$$

where $0 \leq \lambda_i \in \mathbb{R}$ are called Schmidt coefficients, satisfying $\sum_i \lambda_i^2 = 1$.

2.2 QEC codes

We proceed by introducing some basic concepts on QEC codes.

A qubit is a two dimensional logical code space representation of a Hilbert space. The most simple example is the logical code space $\{|0_L\rangle = |0\rangle, \ |1_L\rangle = |1\rangle\}$, where $|0\rangle = (1,0)^T, \ |1\rangle = (0,1)^T$ are two orthonormal computational basis states and T denotes the transpose. The code functions as a carrier of information in its simplest form, together with operators which acts as modulators of the information.

A common situation in quantum computers is the event of a spontaneous error experienced by a quantum state. For example, a qubit is initially in a state of $|0\rangle$ and at some point it either undergoes a bit-flip error, represented by the Pauli operator $\hat{X}$ with a probability p or it stays in its initial state with probability $1-p$.

$$\begin{array}{lcl} |\psi_0\rangle = |0\rangle & \overset{1-p}{\rightarrow} & |0\rangle \text{ no error} \\ & \underset{p}{\searrow} & \\ & & |1\rangle \text{ bit-flip error} \end{array}$$

The events of errors can be said to distort the encoded information in the system and the example portraits the stochastic nature of noise which the code is evolving under.

In order to control the errors, a QEC protocol is required for the code to be useable under quantum computations. A usual approach is in the form of redundancy where several quantum systems from different Hilbert spaces can be composed into more complex logical code spaces. For example, a two dimensional *repetition code* made from several identical physical qubits can be represented by the logical code space

$$|0_L\rangle = |0000...0\rangle \overset{\text{error}}{\rightarrow} |0100...0\rangle \tag{2.26}$$

$$|1_L\rangle = |1111...1\rangle \overset{\text{error}}{\rightarrow} |1011...1\rangle , \tag{2.27}$$

where the second physical qubit experiences a bit-flip error. The redundancy makes it easy to distinguish the error such that an appropriate correction can be made. However, the resources in terms of high number of physical qubits is expensive and complicated to realize. An alternative is to use bosonic codes instead, where information is encoded into bosonic quantum states. These states are unlimited in their number of configurations which makes them into effective alternatives in the realizations of fault tolerant quantum computers.

2.3 Bosonic codes

In the bosonic regime, the computational basis states are represented by superpositions of Fock states $|n\rangle$ which describes the number of photons in a system. The number of photons are represented by excitations in the form of *creation operators* $\hat{a}^\dagger$ acting on Fock states as

$$\hat{a}^\dagger |0\rangle = |1\rangle \tag{2.28}$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{2.29}$$

where $|0\rangle$ is the *vacuum state* and n is an arbitrary photon number. The hermitian conjugate of the creation operator is the *annihilation operator* $\hat{a}$ which is defined as

$$\hat{a}\,|0\rangle = 0 \tag{2.30}$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{2.31}$$

The commutation relations of the two operators are

$$\left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1} \tag{2.32}$$

$$[\hat{a}, \hat{a}] = 0 \tag{2.33}$$

$$\left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0. \tag{2.34}$$

An example of a bosonic systems could be a weakly coupled microwave cavity where the free Hamiltonian $\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a}$ is in the form of a quantized harmonic oscillator of infinite levels where $\hbar$ is the reduced Planck constant and ω is the eigen frequency of the oscillator. In general, such bosonic states has the form of Eq. (2.3).

A bosonic, logical code space could, for example, be represented by the 2-dimensional *cat code* [10]

$$|\pm_L\rangle = \frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}. \tag{2.35}$$

$|\alpha\rangle$ is the commonly known *coherent state*

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \tag{2.36}$$

where α is a complex amplitude and n is the number of photons represented by the Fock state.

The dimension d or d_L of the logical code space is 2 for qubits, but can be higher, in which case the term qubit is exchanged by *qutrit* for three dimensions and for d-dimensional codes in general, the term *qudit* is used. A bosonic qutrit repetition code of a composite system made from three bosonic state spaces can be described by the logical code space

$$|0_L\rangle = |000\rangle \tag{2.37}$$
$$|1_L\rangle = |111\rangle \tag{2.38}$$
$$|2_L\rangle = |222\rangle \,. \tag{2.39}$$

This multi-mode system is represented by individual annihilation and creation operators $(\hat{a}_i, \hat{a}_i^\dagger)$ for each subsystem denoted with i. Their commutation relations are described as

$$[\hat{a}_i, \hat{a}_j] = 0 \tag{2.40}$$
$$\left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0 \tag{2.41}$$
$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \mathbb{1}\delta_{ij}. \tag{2.42}$$

A beam splitter is a passive optical device, consisting of a semi-transparent mirror, which both reflects and transmits incoming electromagnetic waves. This is illustrated in the simple circuit in Fig. 2.1. This can be used to alter the population of photons between quantum subsystems through unitary operations. Such dynamics are based on interacting Hamiltonians, constructed from annihilation and creation operators on the form $\phi(\hat{a}\hat{b}^\dagger \pm \hat{b}\hat{a}^\dagger)$, where $(\hat{a}^\dagger, \hat{a})$ and $(\hat{b}^\dagger, \hat{b})$ are creation and annihilation operators for two separate subsystems (modes) and ϕ is some coupling constant. The evolution of the interacting Hamiltonian is represented by an unitary operator $\hat{U}_{BS}$ which describes the dynamics of excitations between the two modes.

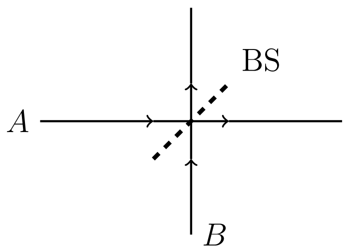

Figure 2.1: Diagram over a passive circuit, consisting of a beam splitter (BS) and two input states of subsystem A and B.

2.4 RSB-structure of single-mode state space

In this section we introduce a certain family of bosonic codes known as RSB codes. We present this by deriving its structure under their representation of general Pauli-operators. RSB states are similar to general bosonic states but where the only Fock states involved are multiples of some integer number.

The structure of a bosonic code space, with single-mode d-dimensional states of N-order rotational symmetry, can be derived by first defining a corresponding Z Pauli-operator on the code space and from this, construct a computational basis which spans the single-mode code space.

The Z-operator on a vector space of dimension d, can be written in its computational basis

$$|k\rangle = \sum_{n}^{\infty} c_n^k |n\rangle \,, \quad k \in \{0, 1, ..., d-1\}, \tag{2.43}$$

as [11]

$$\hat{Z}_d = \sum_{k=0}^{d-1} \omega_d^k |k\rangle \langle k| \tag{2.44}$$

where $\omega_d = \exp\{i2\pi/d\}$. Thus, for any $j \in (0, ..., d-1)$, Eq. (2.44) satisfies

$$\hat{Z}_d |k\rangle = e^{i\frac{2\pi k}{d}} |k\rangle . \tag{2.45}$$

Note that $\exp\{i2\pi k/d\}$ is a global phase for the k:th state. For a dimension $d = 2$ (qubit) the global phase is either ± 1.

A more generalized definition of a Z-operator for N-order rotational codes, also found in Ref. [11] is

$$\hat{Z}_d = \hat{R}_{dN} = e^{i\frac{2\pi}{dN}\hat{n}}, \tag{2.46}$$

where $\hat{n} = \hat{a}^\dagger \hat{a}$ is the number operator. If the computational basis states are initially considered to be on the general form of bosonic states $|k\rangle = \sum_n^\infty c_n^k |n\rangle$, then for the specific instance $k = 0$, the definitions from Eqs. (2.45) and (2.46) for the Z-operator are satisfied if

$$\hat{Z}_d |k=0\rangle = \sum_n^\infty c_n^0 e^{i\frac{2\pi}{dN}n} |n\rangle = \sum_n^\infty c_n^0 (+1) |n\rangle . \tag{2.47}$$

Note that the expression in the middle generates different phases for each Fock state, while the expression on the right only carries the global phase $+1$. The solution for the coefficients c_n^0 must be such that $c_n^0 = 0$ for all $n \neq dNl$ where l is a non-negative integer. This makes it possible to write the state as $|0_N\rangle = \sum_m^\infty c_{mdN} |mdN\rangle$, where the state is now denoted as a N-order RSB basis state, with a subscript N. By doing the same procedure for all basis states ($k = 0, .., d-1$), the general form of the RSB computational basis states become

$$|k_N\rangle = \sum_{m_k=0}^\infty c_{(dm_k+k)N} |(dm_k + k)N\rangle , \quad k = (0, ..., d-1). \tag{2.48}$$

The RSB representation is invariant under N-fold rotations such that

$$e^{i\frac{2\pi}{N}\hat{n}} |k_N\rangle = +1 |k_N\rangle . \tag{2.49}$$

2.5 Open quantum systems & quantum operations

A physical quantum system is always engaged with its environment in some way and therefore differs from closed systems. Important quantities such as energy, excitations and probability, of an open system are therefore not necessarily conserved. Only by including whatever environment that a system is coupled with, can the complete physical picture be analysed.

If a principal system of interest, denoted S, with a density operator ρ^S and its environment, denoted by R, with a corresponding density operator ρ^R, the full picture, denoted by RS, can then be written as the product state

$$\rho^{RS} = \rho^R \otimes \rho^S. \tag{2.50}$$

This composition guarantees that a physical process between the system and its environment can be described by unitary operations $\hat{U}$ acting on the composite system RS. Under such circumstances, the full system RS is described as a closed system.

A unitary operation acting on both the system and the environment can be described as [2]

$$\rho^{R'S'} = \hat{U}(\rho^R \otimes \rho^S)\hat{U}^\dagger. \tag{2.51}$$

The reduced density operator of the system $\rho^{S'}$ after the operation can be obtained by applying the partial trace over the environment

$$\rho^{S'} = \mathrm{Tr}_R\left\{\hat{U}(\rho^R \otimes \rho^S)\hat{U}^\dagger\right\} = \mathrm{Tr}_R\left\{\rho^{R'S'}\right\}. \tag{2.52}$$

By defining an orthonormal basis $\{|k^R\rangle\}$ for the environment and assuming that the initial state of the environment is the pure state $|0^R\rangle$, the partial trace over the environment yields the reduced density operator after the process as

$$\begin{aligned}
\mathrm{Tr}_R\{\rho^{R'S'}\} &= \sum_k \langle k^R|\,\hat{U}(|0^R\rangle\,\langle 0^R| \otimes \rho^S)\hat{U}^\dagger\,|k^R\rangle \\
&= \sum_k \langle k^R|\,\hat{U}\,|0^R\rangle\,\langle 0^R|\sum_i p_i\,|\phi_i^S\rangle\,\langle\phi_i^S|\,\hat{U}^\dagger\,|k^R\rangle \\
&= \sum_{k,i} p_i\,\langle k^R|\,\hat{U}\,|0^R\rangle\,|\phi_i^S\rangle\,\langle\phi_i^S|\,\langle 0^R|\,\hat{U}^\dagger\,|k^R\rangle \\
&= \sum_k \langle k^R|\,\hat{U}\,|0^R\rangle\,\rho^S\,\langle 0^R|\,\hat{U}^\dagger\,|k^R\rangle \\
&= \sum_k \hat{E}_k\rho^S\hat{E}_k^\dagger = \rho^{S'},
\end{aligned}$$

where $\sum_k \hat{E}_k\rho^S\hat{E}_k^\dagger$ is called a operator-sum representation of a quantum operation $\mathcal{E}^S(\rho^S)$ and $\{\hat{E}_k\}$ are *operation elements* of $\mathcal{E}^S$. If the density operator after the process $\mathcal{E}^S(\rho^S) = \rho^{S'}$ also satisfies the conditions of a density matrix, the operation $\mathcal{E}^S$ is said to be *trace preservative.* Additionally, if the operation $[\mathbb{1}^R \otimes \mathcal{E}^S](\rho^{RS})$ is also positive, then $\mathcal{E}^S$ is said to be *completely positive.* A completely positive map is sometimes referred to as a *Kraus representation.*

$\mathrm{Tr}\{\mathcal{E}^S(\rho^S)\} = 1$, implies that the operation elements satisfy a completeness relation

$$\sum_k \hat{E}_k^\dagger\hat{E}_k = \mathbb{1}. \tag{2.53}$$

These type of operations which stochastically describes the dynamics of a system from an initial state are called *channels.* The name "channel" origins from classical information theory and refers to the process enacting on an information source which is being transmitted over some noisy channel of communication [12].

2.6 Fidelity

In order to quantify how close a state ρ is related to another state σ, a common measure to use is the *Fidelity* [2], [13], defined as

$$F(\rho,\sigma) = \mathrm{Tr}\left\{\sqrt{\sigma^{1/2}\rho\sigma^{1/2}}\right\}. \tag{2.54}$$

If $\rho = \sigma$ then the fidelity is 1. Eq. (2.54) has the property of being symmetric in its argument

$$F(\rho, \sigma) = F(\sigma, \rho), \tag{2.55}$$

and its range is defined as

$$0 \leq F(\rho, \sigma) \leq 1 \tag{2.56}$$

Fidelity can also be used to measure how close a state ρ, after some process represented by a channel $\mathcal{E}(\rho)$, is to its initial state. In this situation, the fidelity $F(\rho, \mathcal{E}(\rho))$ is a value of preservation of quantum information in the state ρ from the process. As such, it becomes a useful tool in analysing quantum codes under the influence of noise.

Since the world of quantum mechanics and quantum information often contains the presence of entanglement, there is a need for a measure of preservation of information which also includes the preservations of such entanglements under some process. If the initial state of a system S is purified, with a reference system R, into the entangled state $|\Psi^{RS}\rangle$ followed by the trace preservative and positive channel $\mathcal{E}^{RS}(\cdot) = [\mathbb{1}^R \otimes \mathcal{E}^S](\cdot)$, which only acts non-trivially on the system, the entanglement fidelity [2], [13] is defined as

$$F_{ent}\left(|\Psi^{RS}\rangle\,,\ \mathcal{E}^{RS}\right) = \mathrm{Tr}\left\{|\Psi^{RS}\rangle\langle\Psi^{RS}|\,\rho^{RS'}\right\} \tag{2.57}$$

$$= \langle\Psi^{RS}|\left[\mathcal{I}^R \otimes \mathcal{E}^S\right]\left(|\Psi^{RS}\rangle\langle\Psi^{RS}|\right)|\Psi^{RS}\rangle\,. \tag{2.58}$$

The fidelity does not depend on how the state $|\Psi^{RS}\rangle$ is purified and the reference system can be a fictitious system.

Now, consider a channel $\mathcal{E}^S$ consisting of a noise process $\mathcal{N}^S$ acting on the system S in relation to some environment, causing some kind of distortions in the information of the system. Followed by another process $\mathcal{R}^S$, representing some error correcting operation on the distorted system S. If the process denoted as $\left[\mathbb{1}^R \otimes \mathcal{R}^S \circ \mathcal{N}^S\right](|\Psi^{RS}\rangle\langle\Psi^{RS}|) = \rho^{RS'}$, where $\mathcal{E}^S$ has been replaced by $\mathcal{R}^S \circ \mathcal{N}^S$, results in a poor error correction, then the corresponding entanglement fidelity, using Eq. (2.58), will be of low value. If the process instead results in an ideal case such that $\rho^{RS'} = \rho^{RS}$ then the entanglement fidelity equals 1.

Under this framework it is possible to try different concepts of codes, noise channels and recovery operations and compare the resulting ability to preserve information, including entanglements, with a measure. In this report, the framework which has just been demonstrated, will be used to qualify the resilience, of certain codes, towards noise in terms of photon loss errors and dephasing errors.

With a noise channel and a code at hand, the only part that needs to be defined is the recovery $\mathcal{R}^S \sim \{\hat{R}_l\}$. One way to do so is through optimization of the entanglement fidelity [14] in the recovery such that

$$F^{\mathrm{Opt}} = \max_{\mathcal{R}^S} F_{ent}(|\Psi^{RS}\rangle\,, \mathcal{R}^S \circ \mathcal{N}^S). \tag{2.59}$$

This represents a measure of the best possible performance of a certain code against some type of noise. However, since this kind of analysis is surely made numerically, the computational cost will become enormous when the involved Hilbert spaces are large in dimensions. Therefore, in this report, an approximation of the optimal entanglement fidelity will be used which does not rely on optimization.

This approximation, called the *near-optimal entanglement fidelity* [14], consists of a certain near-optimal recovery operation called the *Transpose channel* [15]. Its corresponding operational elements are defined as

$$\hat{R}_l = \hat{P}_L \hat{N}_l \mathcal{N}\left(\hat{P}_L\right)^{-1/2}, \tag{2.60}$$

where

$$\hat{P}_L = \sum_{j=0}^{d-1} |j_L\rangle \langle j_L| \tag{2.61}$$

is the projector of the logical code space and $\{\hat{N}_l\}$ are the operational elements for a specific noise channel $\mathcal{N}$ such that

$$\mathcal{N}(\hat{P}_L) = \sum_k \hat{N}_k \hat{P}_L \hat{N}_k^\dagger. \tag{2.62}$$

The inverse of $\mathcal{N}(\cdot)$ is taken as the Moore-Penrose pseudo inverse. One corresponding expression for the near-optimal fidelity under these conditions, which will be applied for numerical analysis in this report, is

$$\tilde{F}^{\text{Opt}} = \frac{1}{d_L^2} \sum_{i,j} \left| \text{Tr}\left\{\hat{R}_i \hat{N}_j\right\} \right|^2, \tag{2.63}$$

where d_L is the dimension of the logical code space.

Ref. [14] also contains a lemma, representing an analytical relationship between the optimal entanglement fidelity and the near-optimal entanglement fidelity for Transpose recovery channels as the upper and lower bound

$$\frac{1}{2}\left(1 - \tilde{F}^{Opt}\right) \leq \left(1 - F^{Opt}\right) \leq \left(1 - \tilde{F}^{Opt}\right). \tag{2.64}$$

The complement to the fidelity, $1 - F$, is denoted as the *infidelity*.

2.7 Knill-Laflamme conditions

The numerical estimate of information preservation, using the near-optimal entanglement fidelity gives an extensive picture of the performance of a code in terms of some coupling parameter between the code space and its environment. However, it might say nothing about the preservation under specific, restricted, types of errors. For example, will a specific bosonic code of interest stand resilient under a noise channel containing only single photon loss errors? Establishing such properties can be useful in preliminary classifications of a code.

A complementary, analytical, approach to the near-optimal entanglement fidelity can be used, with a well known theorem [16], [2] in order to estimate the capability of a code of interest. This theorem, which is referred to the Knill-Laflamme (KL)conditions throughout the report, can be stated as

Theorem 2.2 (QEC conditions) *Let $C = \{|j_L\rangle\,,\ j \in (0, 1, ..., d-1)\}$ be a code space of dimension d, containing logical codewords and suppose $\mathcal{Q} \sim \{\hat{E}_k\}$ is a quantum operation of error operations acting on the code space C, then a sufficient and necessary condition for the existence of a recovery operation $\mathcal{R}$, which corrects for $\mathcal{Q}$ on C, is*

$$\langle j_L| \hat{E}_k^\dagger \hat{E}_l |j'_L\rangle = \alpha_{kl}\delta_{jj'}, \tag{2.65}$$

where α is a hermitian matrix called the QEC matrix.

3 Methodology

3.1 Logical codewords in general and beam splitters

The logical codewords for a general d-dimensional RSB code with d modes can be described with the following expression, found in Ref. [1].

$$|j_L\rangle = \hat{U}_{BS} \sum_{\{m_i\}=0}^{\infty} f_{m_0,\ldots,m_{d-1}} \bigotimes_{l=j}^{j\oplus d-1} |(dm_l + l)N\rangle \tag{3.1}$$

where N is a symmetry order number, $f_{m_0,\ldots,m_{d-1}}$ are coefficients that can freely be chosen such that they satisfy normalization of the codewords, $k \oplus d-1 = k+d-1 \mod d$ and

$$\hat{U}_{BS} = \exp\left\{ i \sum_{\substack{j,k=1 \\ j<k}}^{d} \left(\theta^-_{jk} \hat{G}^-_{jk} + \theta^+_{jk} \hat{G}^+_{jk} \right) \right\} \tag{3.2}$$

is a passive linear operation generated by beam splitters which runs over all pairs of modes (j,k) where $\theta^{\pm}_{jk}$ are real valued parameters, $\hat{G}^-_{jk} = i(\hat{a}^\dagger_j \hat{a}_k - \hat{a}^\dagger_k \hat{a}_j)$ and $\hat{G}^+_{jk} = (\hat{a}^\dagger_j \hat{a}_k + \hat{a}^\dagger_k \hat{a}_j)$.

Since we are considering qutrit codes in this report we set the dimension to $d = 3$ in Eq. (3.1), which results in the general logical codewords

$$\begin{aligned}
|0_L\rangle &= \hat{U}_{BS} \sum_{m_0,m_1,m_2}^{\infty} f_{m_0,m_1,m_2} \bigotimes_{j=0}^{2} |(3m_j + j)N\rangle \\
&= \hat{U}_{BS} \sum_{m_0,m_1,m_2}^{\infty} f_{m_0,m_1,m_2} |(3m_0)N\rangle |(3m_1+1)N\rangle |(3m_2+2)N\rangle \\
|1_L\rangle &= \hat{U}_{BS} \sum_{m_0,m_1,m_2}^{\infty} f_{m_0,m_1,m_2} \bigotimes_{j=1}^{0} |(3m_j + j)N\rangle \\
&= \hat{U}_{BS} \sum_{m_0,m_1,m_2}^{\infty} f_{m_0,m_1,m_2} |(3m_1+1)N\rangle |(3m_2+2)N\rangle |(3m_0)N\rangle \\
|2_L\rangle &= \hat{U}_{BS} \sum_{m_0,m_1,m_2}^{\infty} f_{m_0,m_1,m_2} \bigotimes_{j=2}^{1} |(3m_j + j)N\rangle \\
&= \hat{U}_{BS} \sum_{m_0,m_1,m_2}^{\infty} f_{m_0,m_1,m_2} |(3m_2+2)N\rangle |3m_0N\rangle |(3m_1+1)N\rangle,
\end{aligned}$$

where j in the tensor products runs periodically over (0, 1, 2). All the codewords are just permutations of the same three single-mode states. Since the coefficients f_{m_0,m_1,m_2} are chosen freely [1], as long as the codewords are orthonormal, we write them as

$$f_{m_0,m_1,m_2} = c_{(3m_0)N} c_{(3m_1+1)N} c_{(3m_2+2)N},$$

From this we identify the sums over coefficients and Fock states in the logical codewords to have the same RSB structure as in Eq. (2.48). The resulting logical code words with subscript L can therefore be written in terms of the single-mode basis states $|k_N\rangle$ with subscript N and the beam splitter operator as

$$|0_L\rangle = \hat{U}_{BS}|012_N\rangle \tag{3.3}$$
$$|1_L\rangle = \hat{U}_{BS}|120_N\rangle \tag{3.4}$$
$$|2_L\rangle = \hat{U}_{BS}|201_N\rangle\,. \tag{3.5}$$

From here on, our main tasks is to specify the coefficients of the single-mode basis states, define the beam splitter operator and how it relates excitations and de-excitations between the beam splitter picture and the normal picture.

In the normal picture where the beam splitter operator can be considered to be the identity $\hat{U}_{BS} = \mathbb{1}$, the annihilation operators for the three modes are just $\{\hat{a}_1,\ \hat{a}_2,\ \hat{a}_3\}$, while in the beam splitter picture, the annihilation operators are transformed as $\hat{a}_i \mapsto \hat{b}_i = \hat{U}_{BS}\hat{a}_i\hat{U}^\dagger_{BS}$, where i denotes the subspace or equivalently, the corresponding mode. The following procedure can be found, in detail, in Appendix A.

In order to find an expression for the transformation, we express the conjugation $\hat{U}_{BS}\hat{a}_i\hat{U}^\dagger_{BS}$ as an Baker-Campbell-Haussdorff (BCH) expansion. The operator is first written as $\hat{U}_{BS} = \exp(i\hat{A})$ where

$$i\hat{A} = i\sum_{\substack{j,k=1\\ j<k}}^{3}\Big(i\theta^-_{jk}(\hat{a}^\dagger_j\hat{a}_k - \hat{a}^\dagger_k\hat{a}_j) + \theta^+_{jk}(\hat{a}^\dagger_j\hat{a}_k + \hat{a}^\dagger_k\hat{a}_j)\Big) \tag{3.6}$$

is an anti-hermitian operator such that $(i\hat{A})^\dagger = -i\hat{A}$. Then, the expansion becomes

$$\hat{U}_{BS}\hat{a}_i\hat{U}^\dagger_{BS} = e^{\hat{A}}\hat{a}_i e^{-\hat{A}} = \hat{a}_i + \Big[i\hat{A}, a_i\Big] + \frac{1}{2!}\Big[i\hat{A}, \Big[i\hat{A}, a_i\Big]\Big] + ... \tag{3.7}$$

When we investigate a couple of the terms explicitly, we do not end up with a definite result, where for example all terms become zero after a certain order. Neither does any periodical pattern occur. A solution to this is therefore sought by investigating the commutation relation of the terms in $i\hat{A}$. By re-writing $i\hat{A}$ in Eq. (3.6) as

$$i\hat{A} \equiv i\sum_{\substack{j,k=1;\\ j<k}}^{3}\Big(z_{jk}\hat{a}^\dagger_j\hat{a}_k + \bar{z}_{jk}\hat{a}^\dagger_k\hat{a}_j\Big)\,, \tag{3.8}$$

where $z_{jk} = \theta^+_{jk} + i\theta^-_{jk}$, the three resulting terms are denoted as

$$\hat{A}_{12} = i\Big(z_{12}\hat{a}^\dagger_1\hat{a}_2 + \bar{z}_{12}\hat{a}^\dagger_2\hat{a}_1\Big) \tag{3.9}$$
$$\hat{A}_{13} = i\Big(z_{13}\hat{a}^\dagger_1\hat{a}_3 + \bar{z}_{13}\hat{a}^\dagger_3\hat{a}_1\Big) \tag{3.10}$$
$$\hat{A}_{23} = i\Big(z_{23}\hat{a}^\dagger_2\hat{a}_3 + \bar{z}_{23}\hat{a}^\dagger_2\hat{a}_3\Big)\,, \tag{3.11}$$

which has the following commutation relations

$$\Big[\hat{A}_{12}, \hat{A}_{13}\Big] = \big(-\bar{z}_{12}z_{13}\hat{a}^\dagger_2\hat{a}_3 + z_{12}\bar{z}_{13}\hat{a}^\dagger_3\hat{a}_2\big) \tag{3.12}$$
$$\Big[\hat{A}_{13}, \hat{A}_{23}\Big] = \big(-z_{13}\bar{z}_{23}\hat{a}^\dagger_1\hat{a}_2 + \bar{z}_{13}z_{23}\hat{a}^\dagger_2\hat{a}_1\big) \tag{3.13}$$
$$\Big[\hat{A}_{23}, \hat{A}_{12}\Big] = \big(+z_{12}z_{23}\hat{a}^\dagger_1\hat{a}_3 - \bar{z}_{12}\bar{z}_{23}\hat{a}^\dagger_3\hat{a}_1\big). \tag{3.14}$$

We now make the following choice

$$-\bar{z}_{12}z_{13} \equiv iz_{23} \tag{3.15}$$

$$z_{12}\bar{z}_{13} \equiv i\bar{z}_{23} \tag{3.16}$$

$$-z_{13}\bar{z}_{23} \equiv iz_{12} \tag{3.17}$$

$$\bar{z}_{13}z_{23} \equiv i\bar{z}_{12} \tag{3.18}$$

$$z_{12}z_{23} \equiv iz_{13} \tag{3.19}$$

$$-\bar{z}_{12}\bar{z}_{23} \equiv i\bar{z}_{13}. \tag{3.20}$$

The result of this is that the commutators in Eqs. (3.12)-(3.14), show relations of an algebraic structure such that

$$\left[\hat{A}_{12}, \hat{A}_{13}\right] = \hat{A}_{23} \tag{3.21}$$

$$\left[\hat{A}_{13}, \hat{A}_{23}\right] = \hat{A}_{12} \tag{3.22}$$

$$\left[\hat{A}_{23}, \hat{A}_{12}\right] = \hat{A}_{13}. \tag{3.23}$$

The Eqs. (3.15)-(3.20) also gives the relations

$$|z_{12}|^2 = |z_{13}|^2 = |z_{23}|^2 = 1, \tag{3.24}$$

by multiplying and adding the equations together in different ways. We then use these solutions of the coefficients z_{jk} in the BCH expansion of Eq. (3.7), which then reduces to the much simpler form

$$\hat{b}_i = \hat{a}_i + i\frac{\sin(\sqrt{3})}{\sqrt{3}}\left[\hat{A}, \hat{a}_i\right] + \frac{(\cos(\sqrt{3}) - 1)}{3}\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right], \tag{3.25}$$

where the commutators can be expressed as

$$\begin{aligned}\left[\hat{A}, \hat{a}_i\right] = -((&\delta_{i1}\left(z_{12}\hat{a}_2 + z_{13}\hat{a}_3\right) \\ &+\delta_{i2}\left(z_{23}\hat{a}_3 + \bar{z}_{12}\hat{a}_1\right) \\ &+\delta_{i3}\left(\bar{z}_{13}\hat{a}_1 + \bar{z}_{23}\hat{a}_2\right))\end{aligned} \tag{3.26}$$

$$\begin{aligned}\left[\hat{A}, [A, \hat{a}_i]\right] = \quad &\delta_{i1}\left(2\hat{a}_1 - iz_{12}\hat{a}_2 + iz_{13}\hat{a}_3\right) \\ &+\delta_{i2}\left(i\bar{z}_{12}\hat{a}_1 + 2\hat{a}_2 - iz_{23}\hat{a}_3\right) \\ &+\delta_{i3}\left(-i\bar{z}_{13}\hat{a}_1 + i\bar{z}_{23}\hat{a}_2 + 2\hat{a}_3\right),\end{aligned} \tag{3.27}$$

where δ_{ik} is the Kronecker delta. The results of the transformed annihilation operators are then written in vector-matrix form as

$$\hat{\boldsymbol{b}} = (\mathbb{1} + i\beta\boldsymbol{O} + \mu\boldsymbol{E})\,\hat{\boldsymbol{a}} \tag{3.28}$$

$$\boldsymbol{O} = -\begin{pmatrix} 0 & z_{12} & z_{13} \\ \bar{z}_{12} & 0 & z_{23} \\ \bar{z}_{13} & \bar{z}_{23} & 0 \end{pmatrix} \tag{3.29}$$

$$\boldsymbol{E} = \begin{pmatrix} 2 & -iz_{12} & iz_{13} \\ i\bar{z}_{12} & 2 & -iz_{23} \\ -i\bar{z}_{13} & i\bar{z}_{23} & 2 \end{pmatrix}, \tag{3.30}$$

where $\boldsymbol{O}$ and $\boldsymbol{E}$ are matrices for odd and even terms respectively in the expansion, $\hat{\boldsymbol{b}} = (\hat{b}_1,\ \hat{b}_2,\ \hat{b}_3)^T$, $\hat{\boldsymbol{a}} = (\hat{a}_1,\ \hat{a}_2,\ \hat{a}_3)^T$, $\mu = (\cos(\sqrt{3}) - 1)/3$, $\beta = \sin(\sqrt{3})/\sqrt{3}$. The result is further reduced as

$$\hat{\boldsymbol{b}} = \begin{pmatrix} (1+2\mu) & -iz_{12}\sigma & iz_{13}\Delta \\ i\bar{z}_{12}\Delta & (1+2\mu) & -iz_{23}\sigma \\ -i\bar{z}_{13}\sigma & i\bar{z}_{23}\Delta & (1+2\mu) \end{pmatrix} \hat{\boldsymbol{a}}, \tag{3.31}$$

where $\sigma = \mu + \beta$ and $\Delta = \mu - \beta$. From this transformation, we can write each operator $\hat{b}_i$ as a linear combination of annihilation operators

$$\hat{b}_i = \sum_{l=1}^{3} \xi_{il} \hat{a}_l \tag{3.32}$$

where ξ_{il} is an element in the transformation matrix of Eq. (3.31). Deciding on the coefficients z_{jk} will be done numerically in a later stage.

This proves that a solution exists for the transformation of the annihilation operators into the beam splitter picture. We now proceed with specifying the coefficients of the logical codewords.

3.2 Method 1: Projection on binomial distribution

One method we use to specify the single-mode basis states, described in Refs. [3], [11], is by constructing projectors of each single-mode RSB state as

$$\Pi_{dN}^{kN} = \sum_{s=0}^{\infty} |(sd+k)N\rangle \langle (sd+k)N| \tag{3.33}$$

and apply this to a *primitive state* $|\Theta\rangle$. The primitive state can for example be a coherent state $|\Theta\rangle = |\alpha\rangle$ or a binomial state $|\Theta\rangle = |\Theta_{\text{bin}}^{N,K}\rangle$. In these cases the resulting codes are called *cat-codes* and *binomial-codes* respectively. On this form, the states in general are written as

$$|k_N\rangle = \frac{dN}{\sqrt{\mathcal{N}_k}} \Pi_{dN}^{kN} |\Theta\rangle \tag{3.34}$$

where $\mathcal{N}_k$ is a normalization constant.

An attempt to construct a binomial qutrit code, in the framework described above, calls for a binomial primitive state, which is found in Ref. [17] on the form

$$|\Theta_{\text{bin}}^{N,K}\rangle = \frac{1}{\sqrt{2^{K-1}}} \sum_{m=0}^{K} \sqrt{\binom{K}{m}} |mN\rangle . \tag{3.35}$$

From this, we find that the single-mode qutrit states in Eq. (3.34) become

$$\begin{aligned}
|k_N\rangle &= \frac{3N}{\sqrt{\mathcal{N}_k}}\frac{1}{\sqrt{2^{K-1}}}\sum_{s=0}^{\infty}|(s3+k)N\rangle\langle(s3+k)N|\sum_{m=0}^{K}\sqrt{\binom{K}{m}}|mN\rangle \\
&= \frac{3N}{\sqrt{\mathcal{N}_k}}\frac{1}{\sqrt{2^{K-1}}}\sum_{s=0}^{\infty}\sum_{m=0}^{K}\sqrt{\binom{K}{m}}|(s3+k)N\rangle\,\delta_{s3+k,m} \\
&= \frac{3N}{\sqrt{\mathcal{N}_k}}\frac{1}{\sqrt{2^{K-1}}}\sum_{s=0}^{\lfloor (K-k)/3\rfloor}\sqrt{\binom{K}{s3+k}}|(s3+k)N\rangle \\
&= \frac{1}{\sqrt{\tilde{\mathcal{N}}_k}}\sum_{s=0}^{\lfloor (K-k)/3\rfloor}\sqrt{\binom{K}{s3+k}}|(s3+k)N\rangle\,,
\end{aligned}$$

where all pre-factors are reduced into a single normalization constant $\tilde{\mathcal{N}}_k$ and $\lfloor\cdot\rfloor$ is the floor function. The codewords are verified for the so called 0N qubit code [3] when $K=1$ and for the common general binomial qubit code [5] when $K=2$.

For a qutrit code with $K=2$, the upper limit in the sum is 0 for all k which is a consequence of $d>K$. The resulting codewords become

$$|0_N\rangle = \frac{1}{\sqrt{\tilde{\mathcal{N}}_0}}|0\rangle \tag{3.36}$$

$$|1_N\rangle = \frac{1}{\sqrt{\tilde{\mathcal{N}}_1}}|N\rangle \tag{3.37}$$

$$|2_N\rangle = \frac{1}{\sqrt{\tilde{\mathcal{N}}_2}}|2N\rangle\,, \tag{3.38}$$

with normalization constants $\tilde{\mathcal{N}}_k = 1,\ \forall\ k$.

Before we encode the single-mode states into the full logical code and analyse it under the KL conditions, we find it appropriate to analyse the single-mode instances first. It will become clear further on, that the relation $\langle k_N|\,\hat{a}^\dagger\hat{a}\,|k_N\rangle$, which gives the mean photon number of the k:th state and at the same time being one of the KL conditions, both for errors of photon loss and dephasing, is an important property which will follow into the fully encoded code. Since the mean photon number, in this case, is not the same for the single-mode states $\{|0\rangle, |N\rangle, |2N\rangle\}$, for any N, the KL conditions for any set of errors will not be satisfied. The set of single-mode states in Eqs. (3.36)-(3.38) is therefore considered to result in a low performing code.

By repeating the procedure above with $K=3$ and $K=4$, the same problem occurs. Working with binomial primitive state does not seem to be very straight forward, at least when the dimension is 3.

3.3 Method 2: Arbitrary code by truncation of sums

We now take another approach for finding a set of single-mode states, by using the general form of the single-mode qutrit RSB states, $|k_N\rangle = \sum_{m=0}^{\infty} c_{(3m+k)N}\,|(3m+k)N\rangle$ and try to find the smallest sets of codes by truncating the terms in the sums as much as possible. If the sums only contain one term, the same problem, with different mean photon numbers, happens as in the attempt to find a code from a binomial primitive state. With two terms however, the states become

$$\begin{aligned}|0_N\rangle &= c_0\,|0\rangle + c_{3N}\,|3N\rangle\\ |1_N\rangle &= c_N\,|N\rangle + c_{4N}\,|4N\rangle\\ |2_N\rangle &= c_{2N}\,|2N\rangle + c_{5N}\,|5N\rangle\,.\end{aligned}$$

By enforcing equal mean photon number $\bar{n}$, and orthonormality, the following 6 equations emerge

$$\begin{aligned}|c_{3N}|^2 3N &= \bar{n}\\ |c_N|^2 N + |c_{4N}|^2 4N &= \bar{n}\\ |c_{2N}|^2 2N + |c_{5N}|^2 5N &= \bar{n}\\ |c_0|^2 + |c_{3N}|^2 &= 1\\ |c_N|^2 + |c_{4N}|^2 &= 1\\ |c_{2N}|^2 + |c_{5N}|^2 &= 1,\end{aligned}$$

which has the general solutions

$$\begin{aligned}|c_0|^2 &= 1 - \frac{\bar{n}}{3N}\\ |c_N|^2 &= \frac{4}{3} - \frac{\bar{n}}{3N}\\ |c_{2N}|^2 &= \frac{5}{3} - \frac{\bar{n}}{3N}\\ |c_{3N}|^2 &= \frac{\bar{n}}{3N}\\ |c_{4N}|^2 &= \frac{\bar{n}}{3N} - \frac{1}{3}\\ |c_{5N}|^2 &= \frac{\bar{n}}{3N} - \frac{2}{3}.\end{aligned}$$

This also gives the upper and lower bound $2/3 \leq \bar{n}/3N \leq 1$ since the modulus of the amplitudes are non-negative. From this boundary one can choose the modulus for the amplitudes and the complex phase of the amplitudes can be chosen freely.

For example, by choosing $\bar{n}/3N = 2/3$ the states will have the lowest possible photon number. And by assuming that the truncation of the sums, which was done initially, also affects the mean photon number of the final encoded codewords, then this choice would represent an infinitely large family of the smallest possible 3-dimensional RSB codes which satisfy orthonormality and equal mean photon numbers for all single-mode states. By setting the phase of all coefficients to zero results in the following single-mode states

$$|0_N\rangle = \frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|3N\rangle \tag{3.39}$$

$$|1_N\rangle = \sqrt{\frac{2}{3}}\,|N\rangle + \frac{1}{\sqrt{3}}\,|4N\rangle \tag{3.40}$$

$$|2_N\rangle = |2N\rangle\,. \tag{3.41}$$

Since we consider these single-mode states to possess some favourable properties by design, they are supposed to meet the requirements of a well performing code against errors. We therefore accept the states for further analysis in the three-mode code space and will next be analysed analytically under Theorem 2.2 (KL-conditions) for correctability, up to first order in photon loss and dephasing. If it is found to be correctable up to first order, it will probably classify as a high performing code. Then, in the next section, we numerically analyse its performance under noise in terms of the near-optimal entanglement fidelity.

The full analytical analysis can be found in Appendix B. We start out by analysing the full, three-mode code in the normal picture where $\hat{U}_{BS} = \mathbb{1}$ with an error set $\{\mathbb{1},\ \hat{a}_i,\ \hat{a}_i^\dagger \hat{a}_i\}$ of photon loss errors $\hat{a}_i^n$ and dephasing errors $(\hat{a}_i^\dagger \hat{a}_i)^n$ up to first order $n \leq 1$. The main task is to test all combinations $\langle j_L | \hat{E}_k^\dagger \hat{E}_l | j_L' \rangle$, where $\hat{E}_k$ and $\hat{E}_l$ are error operators from the error set. Even if the error set is rather small, the amount of unique combinations are 441. In order to handle such an amount of calculations, we reduce the combinations into general forms. Firstly, by noting that the logical three-mode code words are permutations of $|012_N\rangle$, they can be written as

$$|j_L\rangle = \hat{U}_{BS} \bigotimes_{i=0}^{2} |(j \oplus i)_N\rangle\,, \tag{3.42}$$

where $\oplus$ is addition modulo 3. And secondly, instead of calculating every unique combination $\langle j_L | \hat{E}_k^\dagger \hat{E}_l | j_L' \rangle$, one whole class of combinations are calculated at a time. The process also exploits the fact that the error set contains only single-errors. This has the effect that $\langle j_L | \hat{E}_k^\dagger \hat{E}_l | j_L' \rangle$ will only act on two of the three subsystems at most. And since the single-mode states are orthonormal, the results will always contain a Kronecker delta which reduces the possible outcomes considerably.

We immediately find that for the particular combination $(\hat{E}_k, \hat{E}_l) = (\hat{a}_p^\dagger \hat{a}_p, \hat{a}_p^\dagger \hat{a}_p)$, where p denotes the subsystem, the KL conditions are not satisfied for all $j,\ j'$. This means that the code is not able to correct for dephasing errors.

It is noted that this would not have been the case if the single-mode code space had satisfied equal second moments $\langle \hat{n}^2 \rangle_k$ for all the single-mode states $|k_N\rangle$.

However, going through all combinations $(\hat{E}_k, \hat{E}_l)$ from the reduced error set of only photon loss up to first order, all combinations are satisfying the KL conditions for $N = 1$ or N being an even number.

From this general procedure of testing the KL conditions, we realize, that the combinations $\langle j_L | \hat{a}_p^\dagger \hat{a}_p | j_L' \rangle$, for all p, are satisfying as a consequence from the construction of the single-mode basis states with equal mean photon number $\langle n \rangle_p$. We also note that the satisfying combinations $\langle j_L | \hat{a}_p^\dagger | j_L' \rangle$ and $\langle j_L | \hat{a}_p | j_L' \rangle$ are zero for all $p,\ j$ and j'. These relations are, at first, not recognized as being connected to the coefficients of the single-mode basis states. However, it seems that they are related to the RSB structure of the single-mode state spaces. In that sense, they represent the orthogonality or the null space of each single-mode basis state. At this point, from using our general approach, it is apparent that each class of inner products that satisfies the KL conditions, is in fact corresponding to some symmetry over the code words $|j_L\rangle$ and that they are related to symmetries in the structures of the single-mode basis states.

Since the code was shown to be correctable for photon loss up to first order for the specified values of N in the normal picture, it might also be true in the beam splitter picture where $\hat{U}_{BS} \neq \mathbb{1}$. We therefore do the same procedure in the beam splitter picture where annihilation and creation operators are re-written as

$$\hat{U}_{BS}\hat{a}_i\hat{U}_{BS}^{\dagger} \equiv \hat{b}_i \tag{3.43}$$

$$\hat{U}_{BS}^{\dagger}\hat{a}_i\hat{U}_{BS} \equiv \hat{b}_i^{-}. \tag{3.44}$$

We derive these expressions in Appendix A where the transformed operators are re-written in terms of the normal annihilation and creation operators as

$$\hat{b}_i^{-} = \sum_{l=1}^{3} \xi_{il}^{-}\hat{a}_l,$$

where ξ_{il}^{-} is an element in the transformation matrix

$$\hat{\boldsymbol{b}}^{-} = \begin{pmatrix} 1+2\mu & -iz_{12}\Delta & iz_{13}\sigma \\ i\bar{z}_{12}\sigma & 1+2\mu & -iz_{23}\Delta \\ -i\bar{z}_{13}\Delta & i\bar{z}_{23}\sigma & 1+2\mu \end{pmatrix} \hat{\boldsymbol{a}}. \tag{3.45}$$

One of the resulting inner products $\langle j_L| \hat{E}_k^{\dagger}\hat{E}_l |j_L'\rangle$ is demonstrated here as an example.

$$\begin{aligned} \langle j_L| \hat{a}_p |j_L'\rangle &= \bigotimes_{i=0}^{2} \langle (j\oplus i)_N| \hat{U}_{BS}^{\dagger}\hat{a}_p\hat{U}_{BS} \bigotimes_{i'=0}^{2} |(j'\oplus i')_N\rangle \\ &= \bigotimes_{i=0}^{2} \langle (j\oplus i)_N| \hat{b}_p^{-} \bigotimes_{i'=0}^{2} |(j'\oplus i')_N\rangle \\ &= \bigotimes_{i=0}^{2} \langle (j\oplus i)_N| \sum_{l=1}^{3} \xi_{pl}^{-}\hat{a}_l \bigotimes_{i'=0}^{2} |(j'\oplus i')_N\rangle \\ &= \sum_{l=1}^{3} \xi_{pl}^{-} \langle (j\oplus l-1)_N| \hat{a} |(j'\oplus l-1)_N\rangle = 0. \end{aligned}$$

This is true for all p, j, j', $N = 1$ and N being an even number. Note that this clearly shows signs of symmetries. Even though the computations are becoming more complicated in this analysis, the same expressions, known from the previous analysis, in the normal picture, reoccurs in all combinations. This reduces the calculations into identifications of known symmetries. These symmetries, manifested in the coefficients of the single-mode state space, sparks a curiosity for a third type of method in constructing a code (see Section 3.5).

Further on, we once more find that the code does not satisfy the KL conditions for dephasing errors in the beam splitter picture. And again, this falls on the fact that there is a missing symmetry in the single-mode basis states in the form of equal second moments $\langle n^2\rangle$ for all states. We also conclude that the code is correctable for errors in terms of single photon loss.

While being inspired by all the symmetries to create another type of code, the code that was just analysed in terms of KL will next be subject for a numerical analysis.

3.4 Performance under coupling with a noisy environment

In this section, we analyse the performance of the arbitrary three-mode qutrit code numerically in terms of the near-optimal entanglement fidelity [14]. Computing the near-optimal entanglement fidelity for multi-mode Hilbert spaces is challenging since it requires to handle large sets of data which in normal situations would not fit in the Random access memory (RAM) or even on disc. In order to take on this challenge, we adapt a couple of special features in the scripts, which are made in the Python environment. First of all, most operators and vectors has been identified as being sparse objects, which made it extremely beneficial to define and work with these objects using the `scipy.sparse` library [18]. This converts objects as large as 1 Gb down to about 5-500 kb, making them fit in memory and on disc, while the methods for the mentioned library, at the same time, makes some operations very fast. Secondly, by storing the operators as sparse objects in the form of binary files, they take up minimal space on disc which makes them very fast to store and read. Thirdly, each part of the script is divided into separate environments where a minimal set of operators are occupying the memory at the same time. This prevents the memory from being flooded in the worst case and makes it possible to work on the much faster, but smaller, stack memory at best. The last point is that functions, for constructing annihilation operators, creation operators, number operators and the n:th power of these, has been made on explicit forms in minimal number of iterations to prevent the use of some methods in vane, such as matrix power, since they can take a lot of time to execute and may produce computational errors which are difficult to have control over.

The main parameters for the script are the beam splitter coefficients $\{z_{12},\ z_{13},\ z_{23}\}$, the truncation of the single-mode Fock spaces represented with the variable n_{Fock} and the truncation of the included orders of errors in the noise operators represented with the variable n_{Order}. For example, using truncation parameters $n_{\text{Fock}} = 2$ and $n_{\text{Order}} = 1$ means that any single-mode Fock state is represented by a vector of length 2 (including the vacuum state) and the set of errors will include four operators in total where one represents no error together with three operators, representing single photon loss or dephasing (one per subspace).

The set of error operators $\{\hat{N}_l\}$ corresponding to a noise channel $\mathcal{N}$ are either of loss or dephasing type. These errors are represented by the single-mode Kraus operators [1]

$$\hat{\mathcal{L}}_p^{(i)} = \sqrt{\frac{(1-e^{-\kappa_i t})^p}{p!}} e^{-\frac{1}{2}\kappa_i t \hat{a}_i^\dagger \hat{a}_i} \hat{a}_i^p \tag{3.46}$$

$$\hat{\mathcal{D}}_r^{(i)} = \sqrt{\frac{(\gamma_i t)^r}{r!}} e^{-\frac{1}{2}\gamma_i t (\hat{a}_i^\dagger \hat{a}_i)^2} (\hat{a}_i^\dagger \hat{a}_i)^r, \tag{3.47}$$

where $\hat{\mathcal{L}}_p^{(i)}$ and $\hat{\mathcal{D}}_r^{(i)}$ denotes loss errors and dephasing errors, acting on subsystem i, $p,\ r \leq n_{\text{Order}}$ are the order of error and κt and γt are the noise strengths of each channel type. Thus, the set of error operators for loss contains every combination on the form $\hat{N}_l \in \left\{\hat{\mathcal{L}}_{p_1}^{(1)}\hat{\mathcal{L}}_{p_2}^{(2)}\hat{\mathcal{L}}_{p_3}^{(3)}\right\}$, where p_i is the order of error for subspace i. For dephasing the corresponding set of operators are $\hat{N}_l \in \left\{\hat{\mathcal{D}}_{r_1}^{(1)}\hat{\mathcal{D}}_{r_2}^{(2)}\hat{\mathcal{D}}_{r_3}^{(3)}\right\}$.

The logical codewords $|j_L\rangle$ are written in terms of creation operators and the vacuum state $|000\rangle$ for the composed Hilbert space. An example is made for $|0_L\rangle$ below.

$$\begin{aligned}
|0_L\rangle &= \hat{U}_{BS}\left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|3N\rangle\right)\left(\sqrt{\frac{2}{3}}|N\rangle + \frac{1}{\sqrt{3}}|4N\rangle\right)|2N\rangle \\
&= \hat{U}_{BS}\left(\frac{1}{\sqrt{3}}\mathbb{1} + \sqrt{\frac{2}{3(3N)!}}(\hat{a}_1^\dagger)^{3N}\right)\left(\sqrt{\frac{2}{3(2N)!}}(\hat{a}_2^\dagger)^N + \frac{(\hat{a}_2^\dagger)^{4N}}{\sqrt{3(4N)!}}\right)\frac{(\hat{a}_3^\dagger)^{2N}}{\sqrt{(2N)!}}|000\rangle \\
&= \left(\frac{1}{\sqrt{3}}\mathbb{1} + \sqrt{\frac{2}{3(3N)!}}(\hat{b}_1^\dagger)^{3N}\right)\left(\sqrt{\frac{2}{3(2N)!}}(\hat{b}_2^\dagger)^N + \frac{(\hat{b}_2^\dagger)^{4N}}{\sqrt{3(4N)!}}\right)\frac{(\hat{b}_3^\dagger)^{2N}}{\sqrt{(2N)!}}\hat{U}_{BS}|000\rangle \\
&= \left(\frac{1}{\sqrt{3}}\mathbb{1} + \sqrt{\frac{2}{3(3N)!}}(\hat{b}_1^\dagger)^{3N}\right)\left(\sqrt{\frac{2}{3(2N)!}}(\hat{b}_2^\dagger)^N + \frac{(\hat{b}_2^\dagger)^{4N}}{\sqrt{3(4N)!}}\right)\frac{(\hat{b}_3^\dagger)^{2N}}{\sqrt{(2N)!}}|000\rangle ,
\end{aligned}$$

where creation operators are transformed into the beam splitter picture as $\hat{U}_{BS}\hat{a}_i^\dagger\hat{U}_{BS}^\dagger = \hat{b}_i^\dagger$ by inducing several identities $\hat{U}_{BS}^\dagger\hat{U}_{BS} = \mathbb{1}$ and in the last equality the operator $\hat{U}_{BS}$, acting on the vacuum, is identified as trivial since from the BCH lemma it is found that

$$\begin{aligned}
\hat{U}_{BS}|000\rangle\langle 000|\hat{U}_{BS}^\dagger &= |000\rangle\langle 000| + \left[i\sum_{\substack{j,k=1;\\ j<k}}^{3}\left(z_{jk}\hat{a}_j^\dagger\hat{a}_k + \bar{z}_{jk}\hat{a}_k^\dagger\hat{a}_j\right), |000\rangle\langle 000|\right] + \dots \\
&= |000\rangle\langle 000| ,
\end{aligned}$$

where every operator $\hat{a}_j^\dagger\hat{a}_k$ acting on the vacuum is zero.

From the example with $|0_L\rangle$ above, a possible value for the truncation parameter n_{Fock} can be found by looking at the products of all $\hat{b}^\dagger$-operators. If all prefactors are ignored for now, the example becomes

$$\begin{aligned}
|0_L\rangle &= \left(\mathbb{1} + (\hat{b}_1^\dagger)^{3N}\right)\left((\hat{b}_2^\dagger)^N + (\hat{b}_2^\dagger)^{4N}\right)(\hat{b}_3^\dagger)^{2N}|000\rangle \\
&= \left((\hat{b}_2^\dagger)^N(\hat{b}_3^\dagger)^{2N} + (\hat{b}_2^\dagger)^{4N}(\hat{b}_3^\dagger)^{2N} + (\hat{b}_1^\dagger)^{3N}(\hat{b}_2^\dagger)^N(\hat{b}_3^\dagger)^{2N} + (\hat{b}_1^\dagger)^{3N}(\hat{b}_2^\dagger)^{4N}(\hat{b}_3^\dagger)^{2N}\right)|000\rangle .
\end{aligned}$$

Since each $\hat{b}^\dagger$-operator is a linear combination $\hat{b}_i^\dagger = \sum_{l=1}^{3}\bar{\xi}_{il}\hat{a}_l^\dagger$, it is evident that the specific term above with the most excitations must be the last one. Without explicitly calculating this term, it is also evident that the term with highest excitation is on the form $(\hat{a}_i^\dagger)^{3N+4N+2N}$ for all i. It is thus likely, but not without doubt, that the Fock states of the single-mode Hilbert spaces can be truncated to $n_{\text{Fock}} = 9N + 1$. The extra 1 is for including the vacuum state. This estimate is considering that the codewords will only be acted on by annihilations and number operations (no excitations) in the computations for the near-optimal fidelity. It will shortly be clear if this estimate is reasonable or not. In the rest of the report, the symmetry order number N will be restricted to 2. The reason for not including more values is lack of time. It's also the smallest, even, value which should result in high performance, according to the analytical analysis in Section 3.3.

Before we decide on the remaining truncation parameter n_{Order}, the beam splitter coefficients z_{jk} are estimated from a grid search of the near-optimal entanglement fidelity. Since we deduced in Section 3.1 that the modulus of the coefficients were $|z_{jk}| = 1$, the parameters of the coefficients are decided in their corresponding phase angles $\{\phi_{12},\ \phi_{13},\ \phi_{23}\}$. However, from the relation $-\bar{z}_{12}z_{13} = iz_{23}$, one of the phases ϕ_{23} becomes a function of the other two, hence only two parameters needs to be varied. The grid search is seen in Fig. 3.1 where infidelities has been computed for dephasing errors and plotted from 144 points in the angles $(\phi_{12},\ \phi_{13}) \in [0, \pi) \times [0, \pi)$. The noise strengths $\gamma_1 t = \gamma_2 t = \gamma_3 t$ are set equal and to an arbitrary value $\gamma t = 0.01$. The positions of the maxima and minima in the result, are not changed for different n_{Fock} and n_{Order}, although the difference between the maxima and minima changes slightly with varying values. For the rest of the computations the phases are set to $\phi_{12} = \phi_{13} = \pi/3$ for minimized infidelity. When we do the same grid search for the loss channel, no clear patterns are perceived and is therefore considered being independent to the coefficients of the beam splitter.

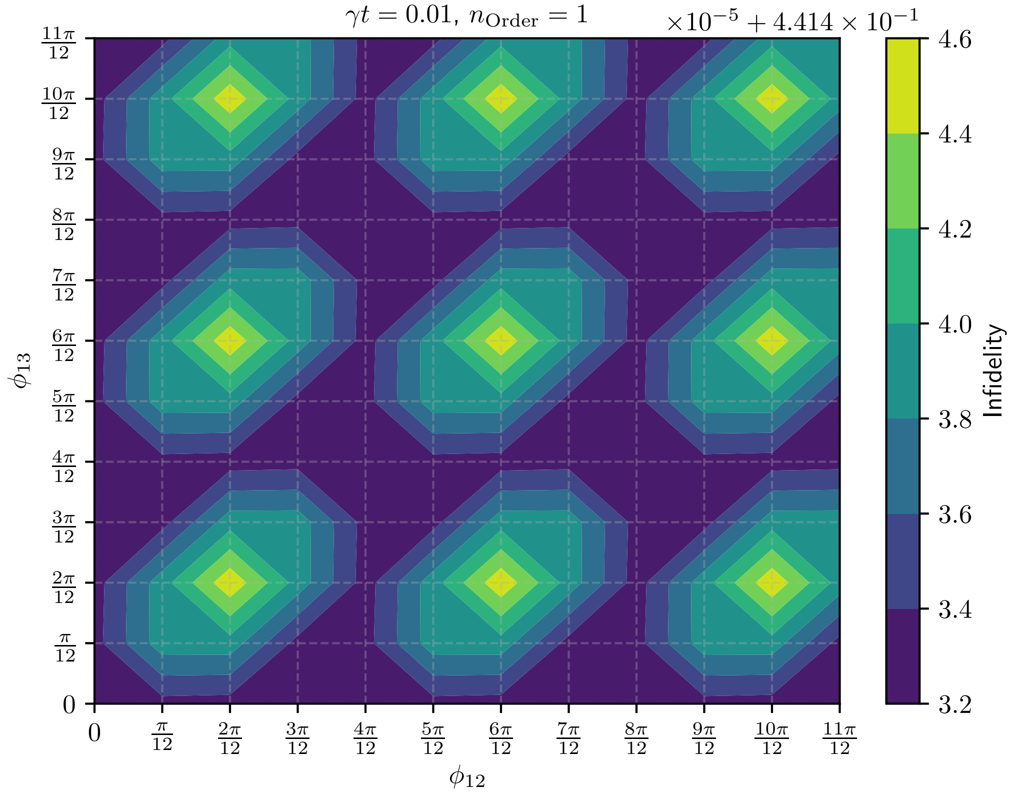

Figure 3.1: Contour map of the infidelity under dephasing channel from a grid search over angles (ϕ_{12}, ϕ_{13}).

In order to get a first impression of the infidelity in size and how it relates to n_{Order}, a couple of plots are made with various truncation values, demonstrated in Fig. 3.2. The curves seems to follow a log-log behaviour for lower noise strengths but deviates somewhat for higher noise strengths, either upwards or downwards depending on n_{Order}.

In order to see if the infidelity converge when n_{Order} becomes even larger, we compute the infidelity with respect to n_{Order} and fixed noise strengths $\kappa t = \gamma t = 0.1$. The results are presented in Fig. 3.3. It seems, from the results, that the infidelities do converge for higher n_{Order}. Estimates for where the infidelities converge in n_{Order}, are taken at $n_{\text{Order}} = 7$ for loss and $n_{\text{Order}} = 20$ for dephasing. At these points, the rate of convergence $|(1 - \tilde{F}^{\text{Opt}}_{n_{\text{Order}}}) - (1 - \tilde{F}^{\text{Opt}}_{n_{\text{Order}}+1})|/(1 - \tilde{F}^{\text{Opt}}_{n_{\text{Order}}})$ is less than 0.012% for loss and less than 0.6% for dephasing. In the large picture, where the infidelities span from 10^{-7} to 1, the

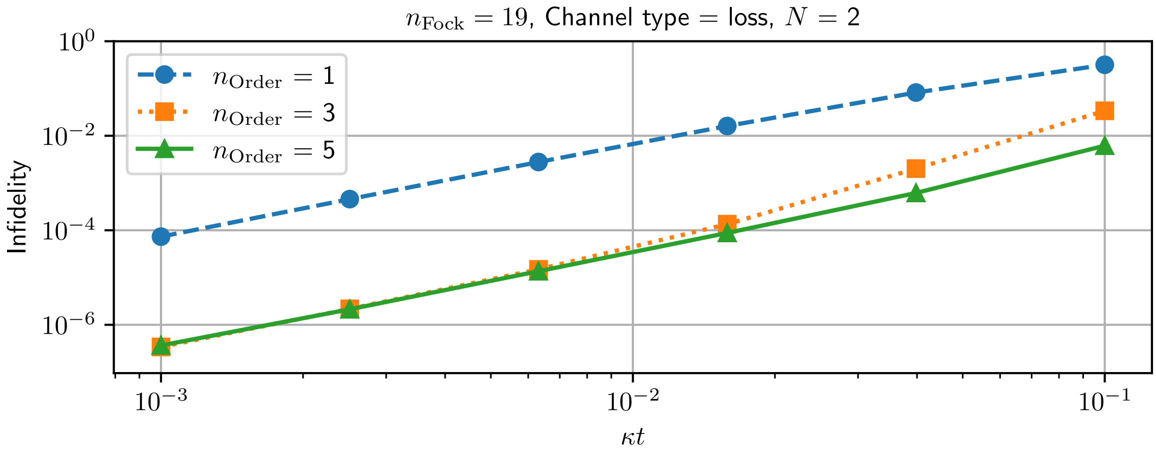

Figure 3.2: First plots of infidelities for different values of n_{Order}.

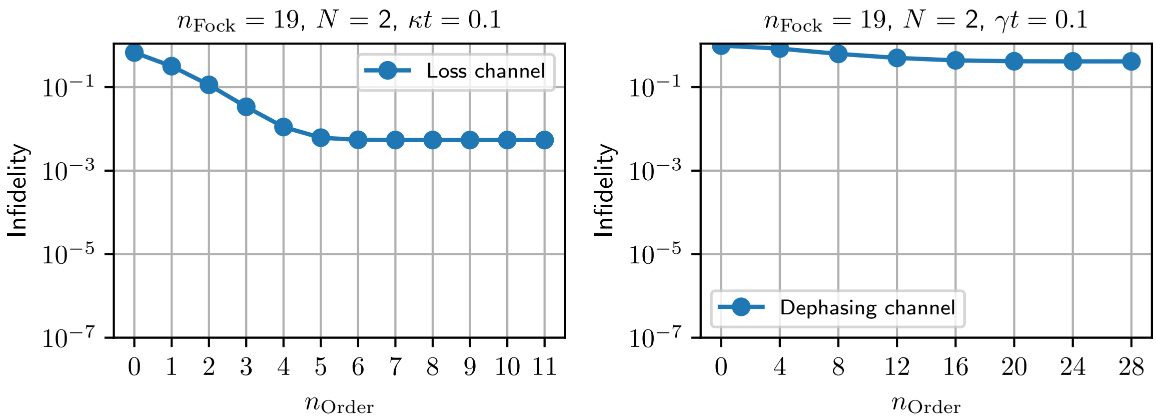

Figure 3.3: Infidelities with respect to n_{Order}.

difference from higher values of n_{Order} would not be perceived. In order to establish that

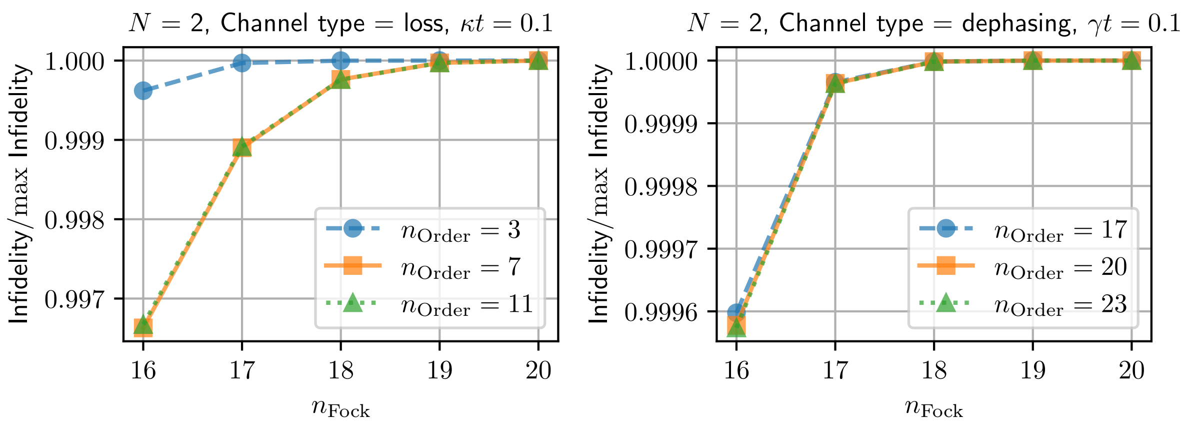

Figure 3.4: Infidelities with respect to n_{Fock} for different values of n_{Order}.

the choice of $n_{\text{Fock}} = 19$ is still valid, we also compute the infidelity with fixed values of n_{Order} while varying values of n_{Fock} around 19. The results are presented in Fig. 3.4 where the diagrams are normalized by dividing the infidelities with the highest value of infidelity, which results in cumulative distributions of convergence.

By looking at the vertical axes, it is evident that the infidelity does not change much for any of the chosen n_{Fock} but also that there is a pronounced convergence around $n_{\text{Fock}} = 19$. The estimate seems to be valid for the final computations.

3.5 Method 3: New framework for establishing codes

In this last section of the methodology, we find a new approach experimentally, which has not explicitly been found in any literature. It is an attempt to generalize the procedure of finding a RSB code, similar to binomial codes and to make very fast assessments of the performance of the code with respect to the KL conditions and the QEC matrix.

The procedure starts by identifying the modulus square of the coefficients $|c_{(ds+k)N}|^2$ of the single-mode states $|k_N\rangle$ in the corresponding Pascals triangle for a d-dimensional code space. Pascals triangle describes the entries in the d-nomial coefficient $\binom{K}{s}_d$. For $d = 2$, Pascals triangle gives the commonly known binomial coefficient, denoted as $\binom{K}{s}_2 \equiv \binom{K}{s}$, which is presented in the table below. Each row is indexed with K and each entry in a

Table 3.1: Pascals triangle for binomial distribution.

$$
\begin{array}{c}
1 \\
1\ 1 \\
1\ 2\ 1 \\
1\ 3\ \ 3\ 1 \\
1\ 4\ \ 6\ \ 4\ 1 \\
\vdots
\end{array}
$$

row is indexed with s. In order to demonstrate how to construct the most known binomial code for a two-dimensional code where the single-mode states are $|0_N\rangle = (|0\rangle + |2N\rangle)/\sqrt{2}$ and $|1_N\rangle = |N\rangle$, K is chosen to be 2 since the corresponding row has equally many entries as there are Fock-states in the states $\{|0_N\rangle\,,\ |1_N\rangle\}$ in total. Every second entry in the row is picked out for each state. Thus, for $|0_N\rangle$ the entries (1,1) are gathered and for $|1_N\rangle$ the remaining entry (2) is gathered. Each set of entries are then normalized such that $1 + 1 = \mathcal{N}_0$ and $2 = \mathcal{N}_1$. And finally, the corresponding coefficients modulus squared for each single-mode basis state is identified as

$$
\begin{aligned}
|c_0|^2 + |c_{2N}|^2 &= \frac{1}{\mathcal{N}_0} + \frac{1}{\mathcal{N}_0} = 1 \\
|c_N|^2 &= \frac{2}{\mathcal{N}_1} = 1,
\end{aligned}
$$

respectively. All coefficients can have any complex phase, but in this case they are left as real and non-negative which results in the binomial code that we wanted to show.

By using this procedure, it is possible to asses the potential of the resulting code by finding out how many symmetries in the form of equal moments the single-mode states have. Since each coefficient modulus square represents a probability $|c_{(ds+k)N}|^2 = p_{(ds+k)N}$ for a corresponding Fock state, the m:th moment for each of the states above is simply, from the classical form $\langle n^m \rangle = \sum_n n^m p_m$,

$$\langle n^m \rangle_0 = |c_0|^2 0^m + |c_{2N}|^2 (2N)^m = \frac{2^m N^m}{2}$$
$$\langle n^m \rangle_1 = |c_N|^2 N^m = N^m.$$

The expressions are only equal for $m = 1$. Therefore, we state that the binomial code has only one such symmetry tied to the KL conditions and the QEC matrix, just like with the arbitrary qutrit code in Section 3.3. It is therefore considered to perform well against photon loss errors. This claim is actually supported from Ref. [1] where the same single-mode binomial code is encoded into a two-mode qubit code and tested against noise.

For the three dimensional case, we find it very difficult to construct a code with the binomial distribution, which has any such symmetry, when using the same procedure. Instead, we extend Pascals triangle into the trinomial instance below.

Table 3.2: Trinomial instance of Pascals triangle.

				1				
			1	1	1			
		1	2	3	2	1		
	[1]	(3)	{6}	[7]	(6)	{3}	[1]	
1	4	10	16	19	16	10	4	1
				⋮				

As an example, the third row is targeted ($K = 3$) and the entries are grouped into sets of numbers for each single-mode state. Each group is denoted with [], () and {}. Normalization of each state gives

$$[] : 1 + 7 + 1 = \mathcal{N}_0$$
$$() : 3 + 6 = \mathcal{N}_1$$
$$\{\} : 6 + 3 = \mathcal{N}_2.$$

If all coefficients are set to be real and non-negative the corresponding states become

$$|0_N\rangle = \frac{1}{\sqrt{9}} |0\rangle + \sqrt{\frac{7}{9}} |3N\rangle + \frac{1}{\sqrt{9}} |6N\rangle \tag{3.48}$$

$$|1_N\rangle = \frac{1}{\sqrt{3}} |N\rangle + \sqrt{\frac{2}{3}} |4N\rangle \tag{3.49}$$

$$|2_N\rangle = \sqrt{\frac{2}{3}} |2N\rangle + \frac{1}{\sqrt{3}} |5N\rangle . \tag{3.50}$$

The m:th moment for each state is

$$\langle n^m \rangle_0 = \frac{7}{9}(3N)^m + \frac{1}{9}(6N)^m \tag{3.51}$$

$$\langle n^m \rangle_1 = \frac{1}{3}N^m + \frac{2}{3}(4N)^m \tag{3.52}$$

$$\langle n^m \rangle_2 = \frac{2}{3}(2N)^m + \frac{1}{3}(5N)^m. \tag{3.53}$$

We find that the equations are the same for $m \in \{1, 2\}$. Thus, carrying two symmetries. These symmetries will contribute to the QEC matrix and will probably result in a code which will be performing even better than the arbitrary qutrit code.

By doing the same procedure for the 1:st and 2:nd row $K \in (1,\ 2)$, the codes that emerges are in fact the trivial code

$$|0_N\rangle = |0\rangle \tag{3.54}$$
$$|1_N\rangle = |N\rangle \tag{3.55}$$
$$|2_N\rangle = |2N\rangle\,, \tag{3.56}$$

which caries no symmetries in terms of moments and is actually the same code that was constructed by projecting the RSB code space on the binomial distribution in Section 3.2. And for the second row ($K = 2$) the result is

$$|0_N\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|3N\rangle \tag{3.57}$$
$$|1_N\rangle = \sqrt{\frac{2}{3}}|N\rangle + \frac{1}{\sqrt{3}}|4N\rangle \tag{3.58}$$
$$|2_N\rangle = |2N\rangle\,, \tag{3.59}$$

which only carries one symmetry and happens to be the same as the arbitrary code in Section 3.3.

All these codes, which are three dimensional can therefore be considered to be trinomial codes.

4 Results

The main results from this report concerns the structure and performance against noise in the form of photon loss and dephasing of a three-mode qutrit RSB code which is denoted as the three-mode trinomial code of size $K = 2$, where K is corresponding to the 2:nd row in the trinomial instance of Pascals triangle. The framework on finding and analysing d-nomial codes is also a result of the report but will not be presented in this section. It can instead be found and studied in Section 3.5. The main code, of the study, is presented in a certain beam splitter picture, consisting of easy implementable operations in the form of passive, linear optics. The structure of these operations are also presented here.

The general form of the three-mode N-order RSB qutrit code space is

$$|0_L\rangle = \hat{U}_{BS}\,|012_N\rangle \tag{4.1}$$
$$|1_L\rangle = \hat{U}_{BS}\,|120_N\rangle \tag{4.2}$$
$$|2_L\rangle = \hat{U}_{BS}\,|201_N\rangle\,. \tag{4.3}$$

where $\hat{U}_{BS}$ is a unitary beam splitter operator. The kets are permutations of the following single-mode RSB states

$$|0_N\rangle = \frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|3N\rangle \tag{4.4}$$
$$|1_N\rangle = \sqrt{\frac{2}{3}}\,|N\rangle + \frac{1}{\sqrt{3}}\,|4N\rangle \tag{4.5}$$
$$|2_N\rangle = |2N\rangle\,. \tag{4.6}$$

These single-mode states were found from two different strategies. On the either hand from, arbitrary, truncating general RSB basis states $|k_N\rangle = \sum_{s=0}^{\infty} c_{(3s+k)N}\,|(3s+k)N\rangle$ shown in Section 3.3 and also from a developed framework where the entries of trinomial coefficients $\binom{K}{n}_3$ are projected and normalized in the three dimensional single-mode RSB space, demonstrated in Section 3.5.

The structure of the beam splitter transformations has the following transformation matrix

$$\hat{\boldsymbol{b}} = \begin{pmatrix} 1+2\mu & -iz_{12}\sigma & iz_{13}\Delta \\ i\bar{z}_{12}\Delta & 1+2\mu & -iz_{23}\sigma \\ -i\bar{z}_{13}\sigma & i\bar{z}_{23}\Delta & 1+2\mu \end{pmatrix} \hat{\boldsymbol{a}}, \tag{4.7}$$

where $\hat{\boldsymbol{b}} = (\hat{U}_{BS}\hat{a}_1\hat{U}_{BS}^{\dagger},\ \hat{U}_{BS}\hat{a}_2\hat{U}_{BS}^{\dagger},\ \hat{U}_{BS}\hat{a}_3\hat{U}_{BS}^{\dagger})^T$ are the annihilation operators in the beam splitter picture, $\hat{\boldsymbol{a}} = (\hat{a}_1, \hat{a}_2, \hat{a}_3)^T$, $\mu = (\cos(\sqrt{3})-1)/3$, $\beta = \sin(\sqrt{3})/\sqrt{3}$, $\sigma = \mu+\beta$ and $\Delta = \mu - \beta$. All the beam splitter coefficients z_{jk} have modulus of 1 in order to satisfy the algebraic structure that was constructed of the individual operators of the beam splitter operator in Section 3.1. The complex phases of the coefficients must also satisfy

$$-\bar{z}_{12}z_{13} = iz_{23}. \tag{4.8}$$

In the final numerical results, the coefficients $z_{jk} = \exp(i\phi_{jk})$ have been chosen, from the grid search in Section 3.4, to be $\phi_{12} = \phi_{13} = \pi/3$ and ϕ_{23} follows from Eq. (4.8).

The code was analytically found, in Section 3.3, to be correctable for the set of single photon errors $\{\mathbb{1},\ \hat{a}_i\}$, according to the KL conditions formulated in Theorem 2.2 in Section 2.7. On the other hand, the same was not found to be true for dephasing errors.

The performance against noise in terms of near-optimal entanglement fidelity $\tilde{F}^{\text{Opt}}$ is presented in Fig. 4.1. The left diagram shows the infidelity $1-\tilde{F}^{\text{Opt}}$ with respect to

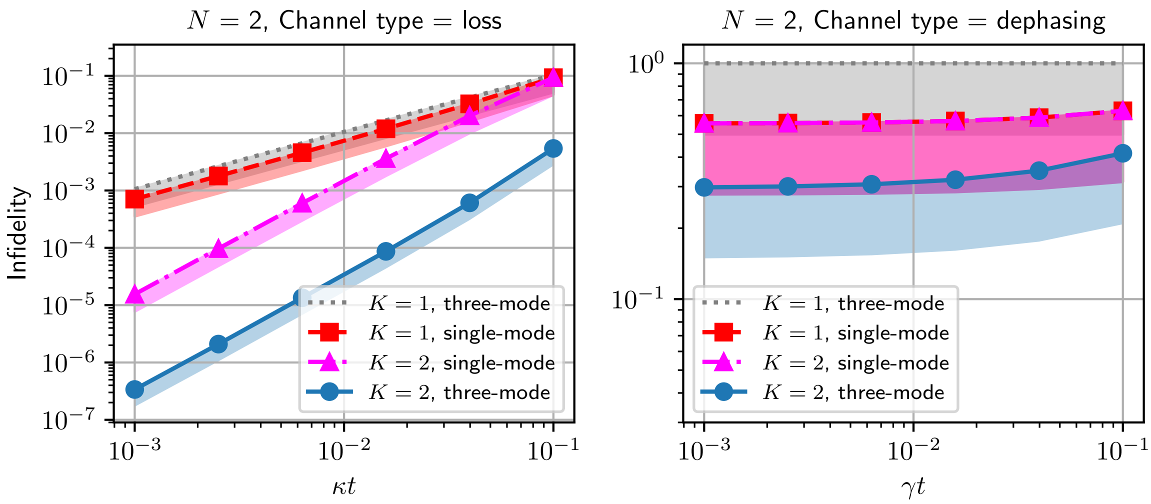

Figure 4.1: Near-optimal infidelity with respect to noise strengths κt for the loss channel to the left and γt for dephasing to the right. Noise strengths for each subspace are set equal such that $\kappa_1 t = \kappa_2 t = \kappa_3 t = \kappa t$ and $\gamma_1 t = \gamma_2 t = \gamma_3 t = \gamma t$. The blue solid curves represent the three-mode trinomial code which corresponds to the 2:nd row ($K = 2$) in Pascals triangle for trinomial coefficients. The gray dotted curves corresponds to the trivial instance of the three-mode trinomial code $|j_L\rangle = \{|012_N\rangle,\ |120_N\rangle,\ |201_N\rangle\}$ where $K = 1$. The magenta coloured, dash-dotted, curves corresponds to the single mode instances in Eqs. (4.4)-(4.6), where $K = 2$. The last curves in red, dashed lines corresponds to trivial instances of the single-mode code $|j_L\rangle = |jN\rangle$. Each curve is complemented with the shaded upper and lower bound of the optimal fidelity. The order number of the codes are set equally to $N = 2$.

noise strength from photon loss errors while the right diagram shows the infidelity from dephasing errors. The three-mode RSB qutrit code of interest is described in blue solid lines and is compared with three other examples.The curves are defined in terms of the number of modes involved, together with the value of K. Only the three-mode qutrit code with $K = 2$ includes the beam splitter operator. $K = 1$ corresponds to trivial instances where the single-mode basis states are $|k_N\rangle = |kN\rangle$, corresponding to the 1:st row in Pascals triangle for trinomial coefficients. The other codes with $K = 2$, incorporates the single mode basis states of Eqs. (4.4)-(4.6). In all examples, the order number has been chosen to be $N = 2$.

For photon loss, it can be seen that the three-mode RSB qutrit code of interest is outperforming the other examples by far. The slope of its curve is more or less the same for the single mode instance which is sharing the same single-mode basis states ($K = 2$). Similarly, the slope of the single-mode and three-mode instances of the trivial codes ($K = 1$) are about the same while both have a higher infidelity than the rest.

For dephasing channel, the comparison is less dramatical. The three-mode trivial instance, pictured in gray, shows almost no preservation of entanglement. Followed by the two identical results of the single-mode instances and the marginally better result of the three-mode trinomial code of interest. Note that theses results inhabits a much smaller range than for the results in the left diagram.

5 Analysis & discussion

Even if the analytical analysis, with respect to the KL conditions, and the numerical analysis in terms of the near-optimal entanglement fidelity brings different kinds of measures for resilience against noise, they do show signs of correlation in this report. The property of being correctable for photon loss errors up to first order, found in the analytical study, seems to have a striking contribution in the performance in terms of the near-optimal entanglement fidelity. While in the case for dephasing errors, the missing conditions in the analytical analysis do seem to correlate with rather poor performance.

It is noted that the KL conditions partially, rely on combinations on the form $\langle j_L| (\hat{a}_i^\dagger \hat{a}_i)^m |j_L\rangle$, which were identified as the m:th moments $\langle n^m\rangle_i$ in the single-mode states. They are therefore also related to statistical estimates such as variance and skewness of the distributions of photon numbers in the single-mode states. Thus, it might be reasonable to claim that the ideal case is when the distributions of photons represents the same curve for all states $|k_N\rangle$. The KL conditions were also noted to rely on symmetries on the form $\langle j_L| \hat{a}_i^\dagger |j_L'\rangle$ and $\langle j_L| \hat{a}_i |j_L'\rangle$ which were not directly related to the coefficients of the single-mode state spaces at first sight. But, seems to be a consequence from the RSB structure of the state spaces and their null spaces.

The numerical comparison, between the three examples and the arbitrary code, shows that the value of K varies the performance in terms of the slope of the infidelity with respect to noise strength. However, if this relationship is consistent for larger K cannot be determined from this limited set of examples. If it was the case, the slope could be related to the number of symmetries found in the structures of the single-mode basis state.

The three-mode code versus the single-mode code for instances where $K = 2$ is about two order of magnitude apart. Besides being an impressive feature for the three-mode code, the cause could have to do with the difference in modes but also that the three-mode instance incorporates the beam splitter operation. This pronounced difference is not observed for the two-mode qubit analysis in Ref. [1, Fig. 2].

For the instances where $K = 1$, the opposite relation is found. That is, the single-mode code seems to outperform the three-mode code, even if not by much. Thus, it seems like the evolution under noise has dis-similar structures when the codes contain one or no symmetries in terms of equal moments $\langle n^m\rangle_i$ in the single-mode basis states.

Even though the choice of coefficients in the beam splitter operator were set for highest entanglement fidelity, the improvements seemed to be small. This can be interpreted as, the choice of coefficients are not critical in an experimental realization, as long as the constraints $-\bar{z}_{12}z_{13} = iz_{23}$ and $|z_{jk}| = 1$ are satisfied. However, it has not been investigated if the variation in entanglement fidelity is similar for all values in noise strengths.

A few sources of errors can be noted for the numerical analysis. First of all, the near-optimal entanglement fidelity is an approximation of the optimal entanglement fidelity which has been displayed with the upper and lower bounds in Fig. 4.1. This however, still allows for a fair comparison between the different codes, presented in the diagrams. Secondly, the amount of error operators, used in the computations, were limited when deciding on the truncation parameter n_{Order}. Values of n_{Order} were taken as estimates without using elaborate procedures. Even though the estimates were motivated, they were only taken at one value in noise strength. And thirdly, some operations has been done with methods from the scipy library such as `sparse.linalg.expm()`, `linalg.pinv()` and `linalg.sqrtm()`. These are all approximations to some degree which have been taken for granted. It is however considered, that any deviations from analytical results in these methods are neglectable or part of the approximation in the near-optimal entanglement fidelity.

6 Conclusions

In this report, we have shown that a three-mode RSB qutrit code, under simple implementable operations, describing a passive linear optical system, can be formulated, accordingly with underlying concepts from Ref. [1].

From the construction and preliminary classification of such codes, one specific code under the name "the arbitrary code" was chosen for further analysis of its performance against noise. Firstly by an analytical study, where the formulations of Theorem 2.2, known as the KL conditions, were used to establish that the code was expected to correct for single photon losses, but not for dephasing errors. This was found to be the case for N being an even number but also for $N = 1$. And secondly, by numerically computing the near-optimal entanglement fidelity under photon loss errors and dephasing errors individually. The analytical and numerical analyses were found to be in agreement with each other since the performance under photon loss was high in relation to three other examples, while the performance under dephasing was not much better than the examples and relatively poor in general.

During the work of the report, it has been found that some of the inner products, which describes the KL conditions, defines symmetries with direct relations to the statistical distributions of the photon numbers in the single-mode basis states. In other words, symmetries found in the single-mode basis states, in the form of equal moments $\langle n^m \rangle_k$, for all single-mode states $|k_N\rangle$, follows into the fully encoded three-mode code space and the KL conditions in the form of $\langle j_L| (\hat{a}_i^\dagger \hat{a}_i)^m |j_L\rangle$. Constructing a code which contains a maximal amount of such symmetries is believed to be a necessity in order to achieve very high performance in terms of QEC. The amount of such symmetries have been found to be related to the value of K which describes the size or truncation of the trinomial distribution, used to construct single-mode basis states. It has been demonstrated, in the numerical results of the near-optimal entanglement fidelity, that the value of K sets the slope of the fidelity with respect to the noise strength in photon loss. In fact, the more structuring of a code, based on *any* kind of symmetries, might lead to improved performance. For example, it is believed that when the single-mode state space is formed into the RSB structure, it results in symmetries which satisfies the KL conditions of the form $\langle j_L| \hat{a}_i^\dagger |j'_L\rangle$ and $\langle j_L| \hat{a}_i |j'_L\rangle$. This seems to be related to the null spaces of the single-mode state spaces.

It was also noted in the analytical study, that the KL conditions were independent of the complex phases in the coefficients $c_{(3s+k)N}$. It is therefore interesting to further study if this independence is broken in some specific situation, for example when the error sets are containing errors of higher order in loss and/or dephasing, and if there are symmetries to be found in the complex phases, which improves the performance in such situations.

Inspired by the symmetries found from the single-mode states and the fact that the binomial code was not working for three dimensional instances, a final approach was developed by visiting the trinomial distribution. Instead of explicitly using the method of projectors on primitive states, we showed in Section 3.5 how Pascals triangle can be used instead. This method promotes the idea that binomial codes are special cases of d-nomial codes of dimension d. The method can be used to construct and analyse arbitrary d-nomial codes in a very simple way. The result is that any d-nomial code can be preliminary classified as being a promising code or not, with minimal efforts.

From this new procedure, it was also found that the code constructed with the binomial approach and "the arbitrary code" were both in fact trinomial codes, corresponding to the first ($K = 1$) and second ($K = 2$) row, respectively in Pascals triangle. We also showed from the same procedure that by applying the third row in the triangle, corresponding to $K = 3$, the resulting code contained the symmetry $\langle j_L| (\hat{a}_i^\dagger \hat{a}_i)^2 |j_L\rangle = \langle n^2\rangle_i$ which was missing in "the arbitrary code". It is therefore believed that using these single-mode sates would result in a three-mode code which would satisfy the KL conditions for dephasing up to first order as well.

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A Transformation of annihilation operators in the beam splitter picture

Before we define the transformations, $\hat{U}_{BS}$ is initially verified to be unitary. Since

$$\begin{aligned}\hat{U}_{BS} =& \exp\left\{ i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left(\theta^{-}_{jk}\hat{G}^{-}_{jk} + \theta^{+}_{jk}\hat{G}^{+}_{jk}\right)\right\}\\ =& \exp\left\{ i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left(i\theta^{-}_{jk}(\hat{a}^{\dagger}_{j}\hat{a}_{k} - \hat{a}^{\dagger}_{k}\hat{a}_{j}) + \theta^{+}_{jk}(\hat{a}^{\dagger}_{j}\hat{a}_{k} + \hat{a}^{\dagger}_{k}\hat{a}_{j})\right)\right\}\end{aligned}$$

then the hermitian conjugate is

$$\begin{aligned}\hat{U}^{\dagger}_{BS} =& \exp\left\{ -i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left((-i)\theta^{-}_{jk}(\hat{a}^{\dagger}_{k}\hat{a}_{j} - \hat{a}^{\dagger}_{j}\hat{a}_{k}) + \theta^{+}_{jk}(\hat{a}^{\dagger}_{k}\hat{a}_{j} + \hat{a}^{\dagger}_{j}\hat{a}_{k})\right)\right\}\\ =& \exp\left\{ -i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left(i\theta^{-}_{jk}(\hat{a}^{\dagger}_{j}\hat{a}_{k} - \hat{a}^{\dagger}_{k}\hat{a}_{j}) + \theta^{+}_{jk}(\hat{a}^{\dagger}_{k}\hat{a}_{j} + \hat{a}^{\dagger}_{j}\hat{a}_{k})\right)\right\},\end{aligned}$$

which also shows that the argument of $\hat{U}_{BS}$ is anti-hermitian.

Thus, by letting $i\hat{A} = i\sum_{j,k=1;\ j<k}^{d}\left(i\theta^{-}_{jk}(\hat{a}^{\dagger}_{j}\hat{a}_{k} - \hat{a}^{\dagger}_{k}\hat{a}_{j}) + \theta^{+}_{jk}(\hat{a}^{\dagger}_{j}\hat{a}_{k} + \hat{a}^{\dagger}_{k}\hat{a}_{j})\right)$, $\hat{U}_{BS}$ can be written as

$$\hat{U}_{BS} = e^{i\hat{A}} \tag{A.1}$$

$$\hat{U}^{\dagger}_{BS} = e^{-i\hat{A}}. \tag{A.2}$$

From this it can be shown, from the BCH theorem, that $\hat{U}_{BS}$ is unitary since for some $\hat{B}$

$$\begin{aligned}\log\left(e^{i\hat{A}}e^{-i\hat{A}}\right) =& \log\left(e^{\hat{B}}\right) = \hat{B}\\ &= i\hat{A} - i\hat{A} + \frac{1}{2}\left[i\hat{A}, -i\hat{A}\right] + \frac{1}{12}\left[i\hat{A},\left[i\hat{A}, -i\hat{A}\right]\right] + \frac{1}{12}\left[-i\hat{A},\left[-i\hat{A}, i\hat{A}\right]\right]...\\ &= 0,\end{aligned}$$

thus

$$e^{i\hat{A}}e^{-i\hat{A}} = e^{0} = \mathbb{1}.$$

It is therefore possible to express the transformation of the annihilation operators as the expansion

$$\begin{aligned}\hat{b}^{\pm}_{i} &= e^{\pm i\hat{A}}\hat{a}_{i}e^{\mp i\hat{A}}\\ &= \hat{a}_{i} + (\pm i)\left[\hat{A}, \hat{a}_{i}\right] + \frac{(\pm i)^{2}}{2!}\left[\hat{A},\left[\hat{A}, \hat{a}_{i}\right]\right] + ...\end{aligned}$$

accordingly with the BCH lemma. $\hat{b}_i^-$ can be considered being some inverse transformation of $\hat{a}_i$.

$i\hat{A}$ can also be written as

$$\begin{aligned}
&i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left(i\theta_{jk}^{-}(\hat{a}_j^{\dagger}\hat{a}_k - \hat{a}_k^{\dagger}\hat{a}_j) + \theta_{jk}^{+}(\hat{a}_j^{\dagger}\hat{a}_k + \hat{a}_k^{\dagger}\hat{a}_j) \right) \\
&= i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left((\theta_{jk}^{+} + i\theta_{jk}^{-})\hat{a}_j^{\dagger}\hat{a}_k + (\theta_{jk}^{+} - i\theta_{jk}^{-})\hat{a}_k^{\dagger}\hat{a}_j \right) \\
&\equiv i \sum_{\substack{j,k=1;\\ j<k}}^{d} \left(z_{jk}\hat{a}_j^{\dagger}\hat{a}_k + \bar{z}_{jk}\hat{a}_k^{\dagger}\hat{a}_j \right).
\end{aligned}$$

The commutation rules for $\hat{a}_j^{\dagger}\hat{a}_k, \hat{a}_k^{\dagger}\hat{a}_j$ and $\hat{a}_i$ are

$$\left[\hat{a}_j^{\dagger}\hat{a}_k, \hat{a}_i\right] = \left(\hat{a}_j^{\dagger}\left[\hat{a}_k, \hat{a}_i\right] + \left[\hat{a}_j^{\dagger}, \hat{a}_i\right]\hat{a}_k\right) = -\delta_{ji}\hat{a}_k \tag{A.3}$$

$$\left[\hat{a}_k^{\dagger}\hat{a}_j, \hat{a}_i\right] = \left(\hat{a}_k^{\dagger}\left[\hat{a}_j, \hat{a}_i\right] + \left[\hat{a}_k^{\dagger}, \hat{a}_i\right]\hat{a}_j\right) = -\delta_{ki}\hat{a}_j. \tag{A.4}$$

Initially, a solution for the BCH expansion will now be sought for by calculating the commutators term by term. The hope is, for example, to find a recursive pattern which has a closed expression or to find at which order, if any, where the commutators definitely are zero. Another approach could be to set all terms, above some order, to zero by choice, which would generate a set of useful restrictions for the complex amplitudes z_{jk}.

By using the commutation relations in Eqs. (A.3) and (A.4) above, the commutator in the first-order term, for a given i, in the BCH expansion above becomes

$$\begin{aligned}
\left[\hat{A}, \hat{a}_i\right] = \sum_{\substack{j,k=1;\\ j<k}}^{d} &\left[z_{jk}\hat{a}_j^{\dagger}\hat{a}_k + \bar{z}_{jk}\hat{a}_k^{\dagger}\hat{a}_j, a_i\right] = \\
&- (\delta_{i1}\left(z_{12}\hat{a}_2 + z_{13}\hat{a}_3\right) \\
&+ \ \delta_{i2}\left(z_{23}\hat{a}_3 + \bar{z}_{12}\hat{a}_1\right) \\
&+ \ \delta_{i3}\left(\bar{z}_{13}\hat{a}_1 + \bar{z}_{23}\hat{a}_2\right)).
\end{aligned}$$

If the decision is made to set this term to zero would result in nulling all complex coefficients z_{jk}. This would result in the absence of beam splitters.

The commutator in the second term of the expansion is

$$\begin{aligned}
\left[\hat{A},\left[\hat{A},\hat{a}_i\right]\right] = \sum_{\substack{j,k=1;\\ j<k}}^{d} & \left[z_{jk}\hat{a}_j^\dagger\hat{a}_k + \bar{z}_{jk}\hat{a}_k^\dagger\hat{a}_j, -(\delta_{i1}\left(z_{12}\hat{a}_2 + z_{13}\hat{a}_{13}\right))\right] \\
& + \left[z_{jk}\hat{a}_j^\dagger\hat{a}_k + \bar{z}_{jk}\hat{a}_k^\dagger\hat{a}_j, -(\delta_{i2}\left(z_{23}\hat{a}_3 + \bar{z}_{12}\hat{a}_1\right))\right] \\
& + \left[z_{jk}\hat{a}_j^\dagger\hat{a}_k + \bar{z}_{jk}\hat{a}_k^\dagger\hat{a}_j, -(\delta_{i3}\left(\bar{z}_{13}\hat{a}_1 + \bar{z}_{13}\hat{a}_2\right))\right] \\
= \quad & \dots \\
& = \delta_{i1}\left((|z_{12}|^2 + |z_{13}|^2)\hat{a}_1 + z_{13}\bar{z}_{23}\hat{a}_2 + (z_{23}z_{12})\hat{a}_3\right) \\
& + \delta_{i2}\left(z_{23}\bar{z}_{13}\hat{a}_1 + (|z_{12}|^2 + |z_{23}|^2)\hat{a}_2 + (\bar{z}_{12}z_{13})\hat{a}_3\right) \\
& + \delta_{i3}\left(\bar{z}_{12}\bar{z}_{23}\hat{a}_1 + (z_{12}\bar{z}_{13})\hat{a}_2 + (|z_{13}|^2 + |z_{23}|^2)\hat{a}_3\right).
\end{aligned}$$

By setting this term to zero, all complex amplitudes must be nulled again.

In order to find a solution to the expansion, the beam splitter operator $\hat{U}_{BS}$ is further investigated in a group theoretical approach. The three operators in the argument of the beam splitter operator have the following relations.

$$\begin{aligned}
\hat{A}_{12} &= i\left(z_{12}\hat{a}_1^\dagger\hat{a}_2 + \bar{z}_{12}\hat{a}_2^\dagger\hat{a}_1\right) \\
\hat{A}_{13} &= i\left(z_{13}\hat{a}_1^\dagger\hat{a}_3 + \bar{z}_{13}\hat{a}_3^\dagger\hat{a}_1\right) \\
\hat{A}_{23} &= i\left(z_{23}\hat{a}_2^\dagger\hat{a}_3 + \bar{z}_{23}\hat{a}_2^\dagger\hat{a}_3\right) \\
\left[\hat{A}_{12},\hat{A}_{13}\right] &= \left(-\bar{z}_{12}z_{13}\hat{a}_2^\dagger\hat{a}_3 + z_{12}\bar{z}_{13}\hat{a}_3^\dagger\hat{a}_2\right) \\
\left[\hat{A}_{13},\hat{A}_{23}\right] &= \left(-z_{13}\bar{z}_{23}\hat{a}_1^\dagger\hat{a}_2 + \bar{z}_{13}z_{23}\hat{a}_2^\dagger\hat{a}_1\right) \\
\left[\hat{A}_{23},\hat{A}_{12}\right] &= \left(+z_{12}z_{23}\hat{a}_1^\dagger\hat{a}_3 - \bar{z}_{12}\bar{z}_{23}\hat{a}_3^\dagger\hat{a}_1\right).
\end{aligned}$$

By choosing

$$-\bar{z}_{12}z_{13} = iz_{23} \tag{A.5}$$

$$z_{12}\bar{z}_{13} = i\bar{z}_{23} \tag{A.6}$$

$$-z_{13}\bar{z}_{23} = iz_{12} \tag{A.7}$$

$$\bar{z}_{13}z_{23} = i\bar{z}_{12} \tag{A.8}$$

$$z_{12}z_{23} = iz_{13} \tag{A.9}$$

$$-\bar{z}_{12}\bar{z}_{23} = i\bar{z}_{13}, \tag{A.10}$$

the commutators suddenly show relations of an algebraic structure such that

$$\left[\hat{A}_{12},\hat{A}_{13}\right] = \hat{A}_{23} \tag{A.11}$$

$$\left[\hat{A}_{13},\hat{A}_{23}\right] = \hat{A}_{12} \tag{A.12}$$

$$\left[\hat{A}_{23},\hat{A}_{12}\right] = \hat{A}_{13}. \tag{A.13}$$

From this, it is also easy to see that the set of commutators satisfy the Jacobi identity

$$\left[\hat{A}_{12},\left[\hat{A}_{13},\hat{A}_{23}\right]\right]+\left[\hat{A}_{23},\left[\hat{A}_{12},\hat{A}_{13}\right]\right]+\left[\hat{A}_{13},\left[\hat{A}_{23},\hat{A}_{12}\right]\right]=0. \tag{A.14}$$

From the chosen relations of the coefficients z_{jk} it is found that

$$\begin{aligned}
|z_{12}|^2=|z_{13}|^2=|z_{23}|^2&=1 && \text{(A.15)}\\
\theta_{13}^{-}\theta_{23}^{+}-\theta_{13}^{+}\theta_{23}^{-}&=\theta_{12}^{+} && \text{(A.16)}\\
\theta_{13}^{+}\theta_{23}^{+}+\theta_{13}^{-}\theta_{23}^{-}&=\theta_{12}^{-} && \text{(A.17)}\\
\theta_{12}^{+}\theta_{23}^{-}+\theta_{12}^{-}\theta_{23}^{+}&=\theta_{13}^{+} && \text{(A.18)}\\
\theta_{12}^{-}\theta_{23}^{-}-\theta_{12}^{+}\theta_{23}^{+}&=\theta_{13}^{-} && \text{(A.19)}\\
\theta_{12}^{-}\theta_{13}^{+}-\theta_{12}^{+}\theta_{13}^{-}&=\theta_{23}^{+} && \text{(A.20)}\\
\theta_{12}^{+}\theta_{13}^{+}+\theta_{12}^{-}\theta_{13}^{-}&=\theta_{23}^{-}, && \text{(A.21)}
\end{aligned}$$

where it is repeated, for clarity, that $z_{jk}=\theta_{jk}^{+}+i\theta_{jk}^{-}$.

Using the constraints of the coefficients in Eqs. (A.5)-(A.10) and (A.15), the second-order commutator in the BCH expansion is rewritten as

$$\begin{aligned}
\left[\hat{A},[A,\hat{a}_i]\right]&=\delta_{i1}\left(2\hat{a}_1-iz_{12}\hat{a}_2+iz_{13}\hat{a}_3\right)\\
&+\delta_{i2}\left(i\bar{z}_{12}\hat{a}_1+2\hat{a}_2-iz_{23}\hat{a}_3\right)\\
&+\delta_{i3}\left(-i\bar{z}_{13}\hat{a}_1+i\bar{z}_{23}\hat{a}_2+2\hat{a}_3\right).
\end{aligned}$$

And the next commutators from, order three through eight, becomes

$$\begin{aligned}
-3(&\delta_{i1}\left(z_{12}\hat{a}_2 + z_{13}\hat{a}_3\right)\\
+\ &\delta_{i2}\left(z_{23}\hat{a}_3 + \bar{z}_{12}\hat{a}_1\right)\\
+\ &\delta_{i3}\left(\bar{z}_{13}\hat{a}_1 + \bar{z}_{23}\hat{a}_2\right)),
\end{aligned}$$

$$\begin{aligned}
3(&\delta_{i1}\left(2\hat{a}_1 - iz_{12}\hat{a}_2 + iz_{13}\hat{a}_3\right)\\
+&\delta_{i2}\left(i\bar{z}_{12}\hat{a}_1 + 2\hat{a}_2 - iz_{23}\hat{a}_3\right)\\
+&\delta_{i3}\left(-i\bar{z}_{13}\hat{a}_1 + i\bar{z}_{23}\hat{a}_2 + 2\hat{a}_3\right)),
\end{aligned}$$

$$\begin{aligned}
-9(&\delta_{i1}\left(z_{12}\hat{a}_2 + z_{13}\hat{a}_3\right)\\
+\ &\delta_{i2}\left(z_{23}\hat{a}_3 + \bar{z}_{12}\hat{a}_1\right)\\
+\ &\delta_{i3}\left(\bar{z}_{13}\hat{a}_1 + \bar{z}_{23}\hat{a}_2\right)),
\end{aligned}$$

$$\begin{aligned}
9(&\delta_{i1}\left(2\hat{a}_1 - iz_{12}\hat{a}_2 + iz_{13}\hat{a}_3\right)\\
+&\delta_{i2}\left(i\bar{z}_{12}\hat{a}_1 + 2\hat{a}_2 - iz_{23}\hat{a}_3\right)\\
+&\delta_{i3}\left(-i\bar{z}_{13}\hat{a}_1 + i\bar{z}_{23}\hat{a}_2 + 2\hat{a}_3\right)),
\end{aligned}$$

$$\begin{aligned}
-27(&\delta_{i1}\left(z_{12}\hat{a}_2 + z_{13}\hat{a}_3\right)\\
+\ &\delta_{i2}\left(z_{23}\hat{a}_3 + \bar{z}_{12}\hat{a}_1\right)\\
+\ &\delta_{i3}\left(\bar{z}_{13}\hat{a}_1 + \bar{z}_{23}\hat{a}_2\right)),
\end{aligned}$$

$$\begin{aligned}
27(&\delta_{i1}\left(2\hat{a}_1 - iz_{12}\hat{a}_2 + iz_{13}\hat{a}_3\right)\\
+&\delta_{i2}\left(i\bar{z}_{12}\hat{a}_1 + 2\hat{a}_2 - iz_{23}\hat{a}_3\right)\\
+&\delta_{i3}\left(-i\bar{z}_{13}\hat{a}_1 + i\bar{z}_{23}\hat{a}_2 + 2\hat{a}_3\right)),
\end{aligned}$$

where all odd and even terms respectively only differ up to a constant.

The BCH expansion can now be written as

$$\begin{aligned}
\hat{b}^{\pm} &= \hat{a}_i + (\pm i)\left[\hat{A}, \hat{a}_i\right] + \frac{(\pm i)^2}{2!}\left[\hat{A}, [A, \hat{a}_i]\right] + \frac{(\pm i)^3}{3!}3\left[\hat{A}, \hat{a}_i\right]\\
&+ \frac{(\pm i)^4}{4!}3\left[\hat{A}, [A, \hat{a}_i]\right] + \frac{(\pm i)^5}{5!}3^2\left[\hat{A}, \hat{a}_i\right] + \ldots\\
&= \hat{a}_i + \left[\hat{A}, \hat{a}_i\right]\sum_{n=0}^{\infty}\frac{(\pm i)^{2n+1}}{(2n+1)!}3^n + \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\sum_{n=0}^{\infty}\frac{(\pm i)^{2n}}{(2n)!}\sqrt{3}^{2n} - \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\\
&= \hat{a}_i + \frac{\left[\hat{A}, \hat{a}_i\right]}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(\pm i)^{2n+1}\sqrt{3}^{2n+1}}{(2n+1)!} + \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\sum_{n=0}^{\infty}\frac{(\pm i)^{2n}\sqrt{3}^{2n}}{(2n)!} - \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\\
&= \hat{a}_i \pm \frac{i\left[\hat{A}, \hat{a}_i\right]}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}^{2n+1}}{(2n+1)!} + \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}^{2n}}{(2n)!} - \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\\
&= \hat{a}_i \pm \frac{i\left[\hat{A}, \hat{a}_i\right]}{\sqrt{3}}\sin(\sqrt{3}) + \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}\cos(\sqrt{3}) - \frac{\left[\hat{A}, \left[\hat{A}, \hat{a}_i\right]\right]}{3}.
\end{aligned}$$

If all instances of i are written on matrix form the expansion becomes

$$\hat{\boldsymbol{b}}^{\pm} = \left(\mathbb{1} \pm i\beta\boldsymbol{O} + \mu\boldsymbol{E}\right)\hat{\boldsymbol{a}} \tag{A.22}$$

$$\boldsymbol{O} = -\begin{pmatrix} 0 & z_{12} & z_{13} \\ \bar{z}_{12} & 0 & z_{23} \\ \bar{z}_{13} & \bar{z}_{23} & 0 \end{pmatrix} \tag{A.23}$$

$$\boldsymbol{E} = \begin{pmatrix} 2 & -iz_{12} & iz_{13} \\ i\bar{z}_{12} & 2 & -iz_{23} \\ -i\bar{z}_{13} & i\bar{z}_{23} & 2 \end{pmatrix}, \tag{A.24}$$

where $\boldsymbol{O}$ and $\boldsymbol{E}$ are the matrices for the odd and even terms respectively, $\mu = (\cos(\sqrt{3}) - 1)/3$ and $\beta = \sin(\sqrt{3})/\sqrt{3}$. The sum of the matrices in each case becomes

$$\hat{\boldsymbol{b}}^{+} = \begin{pmatrix} 1+2\mu & -iz_{12}\sigma & iz_{13}\Delta \\ i\bar{z}_{12}\Delta & 1+2\mu & -iz_{23}\sigma \\ -i\bar{z}_{13}\sigma & i\bar{z}_{23}\Delta & 1+2\mu \end{pmatrix}\hat{\boldsymbol{a}} \tag{A.25}$$

$$\hat{\boldsymbol{b}}^{-} = \begin{pmatrix} 1+2\mu & -iz_{12}\Delta & iz_{13}\sigma \\ i\bar{z}_{12}\sigma & 1+2\mu & -iz_{23}\Delta \\ -i\bar{z}_{13}\Delta & i\bar{z}_{23}\sigma & 1+2\mu \end{pmatrix}\hat{\boldsymbol{a}}, \tag{A.26}$$

where the abbreviations $\sigma = \mu + \beta$ and $\Delta = \mu - \beta$. It is noted that $1 + 2\mu \approx 0.23$, $\sigma \approx 0.18$ and $\Delta \approx -0.96$.

From this derivation, it is proven that there exists a solution that expresses the annihilation and creation operators in the beam splitter picture of three dimensions and three modes. However, it is not clear if this represents a unique solution, and if it is not unique, will other solutions contribute to better codes in terms of QEC. The constraints $|z_{jk}|^2 = 1$ for all (j, k) is a very limiting factor despite the fact that the set of satisfying values z_{jk} is infinite.

The notation of $(\hat{\boldsymbol{b}}^{+}, \hat{b}_i^{+})$ are reduced to $(\hat{\boldsymbol{b}}, \hat{b}_i)$ in the main parts of the report.

B Exact correctability of arbitrary three-mode qutrit code

Here follows a lengthy analysis of the arbitrary three-mode qutrit code in Section 3.3 under the theorem about exact correctable codes (Theorem 2.2). It starts by analysing the code in the normal picture and ends with the inclusion of the beam splitter operator.

B.1 Preliminary Knill-Laflamme conditions in the normal picture

In this section the three-mode RSB qutrit code

$$|0_L\rangle = \hat{U}_{BS}\,|012_N\rangle \tag{B.1}$$

$$|1_L\rangle = \hat{U}_{BS}\,|120_N\rangle \tag{B.2}$$

$$|2_L\rangle = \hat{U}_{BS}\,|201_N\rangle\,, \tag{B.3}$$

based on the arbitrary single-mode codewords

$$|0_N\rangle = \frac{1}{\sqrt{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|3N\rangle \tag{B.4}$$

$$|1_N\rangle = \sqrt{\frac{2}{3}}\,|N\rangle + \frac{1}{\sqrt{3}}\,|4N\rangle \tag{B.5}$$

$$|2_N\rangle = \,|2N\rangle\,, \tag{B.6}$$

will be checked against the KL conditions for single photon loss errors and single phase errors. This will initially, only concern the code in the normal picture such that the beam splitter operator $\hat{U}_{BS} = \mathbb{1}$ and $\hat{b}_i = \hat{b}_i^- = \hat{a}_i$. Definitions of $\hat{b}_i$ and $\hat{b}_i^-$ are found in Appendix A.

The error set in this case, corresponding to loss errors and dephasing errors up to first order, is $\hat{E}_k \in \{\mathbb{1},\ \hat{a}_i,\ \hat{a}_i^\dagger\hat{a}_i\}$ where $i \in \{1,\ 2,\ 3\}$ denotes the corresponding subspace. The KL conditions are defined as $\langle j_L|\,\hat{E}_k^\dagger\hat{E}_l\,|j_L'\rangle = \alpha_{kl}\delta_{jj'}$, where the logical codewords $|j_L\rangle$ are initially considered being in the normal picture and the constant α_{kl} is an element of a hermitian matrix, also known as the QEC matrix.

There are many combinations $(\hat{E}_k, \hat{E}_l)$ to try out together with all j and j', in order to verify if the code satisfy all the KL conditions, even with the rather small set of errors in this case. The combinations will therefore be divided into the following four classes

- $\hat{E}_k$ and $\hat{E}_l$ are the same type of error and on the same subspace, thus $k = l$
- $\hat{E}_k$ and $\hat{E}_l$ are different type of errors and on the same subspace
- $\hat{E}_k$ and $\hat{E}_l$ are the same type of error and on different subspaces
- $\hat{E}_k$ and $\hat{E}_l$ are different type of errors and on different subspaces.

It is also noted that since only one or two of the subspaces will be acted on, there will always be at least one inner product which renders $\langle j_L|\,\hat{E}_k^\dagger\hat{E}_l\,|j_L'\rangle = 0$ whenever $j \neq j'$. This reduces the number of combinations considerably such that all instances $j \neq j'$ can be ignored. And instead only look at instances when $j = j'$, where the inner products contain non-trivial operations. This is a consequence of having a multi-mode structure in the code words.

In general, it is possible to write the inner product as

$$\langle j_L| \hat{E}_k^\dagger \hat{E}_l |j_L'\rangle = \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| \hat{E}_k^\dagger \hat{E}_l \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle ,$$

where $\oplus$ denotes addition modulo 3 and $i+1$ denotes the subspaces. As a general example, if $\hat{E}_k$ and $\hat{E}_l$ are acting on subspaces k and l respectively, and $j = j'$, the expression above becomes

$$\sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus k-1)N} c_{(3s'+j\oplus k-1)N} \langle (3s + j \oplus k - 1)N| \hat{E}_k^\dagger |(3s' + j \oplus k - 1)N\rangle$$
$$\times \sum_{z,z'=0}^{1} \bar{c}_{(3z+j\oplus l-1)N} c_{(3z'+j\oplus l-1)N} \langle (3z + j \oplus l - 1)N| \hat{E}_l |(3z' + j \oplus l - 1)N\rangle .$$

The sums only goes to 1 since the code words are truncated to this point by construction.

Starting with the first class, it is trivial to see that the KL condition is satisfied for $(\hat{E}_k, \hat{E}_l) = (\mathbb{1}, \mathbb{1})$.

Next, for $(\hat{E}_k, \hat{E}_l) = (\hat{a}_p^\dagger \hat{a}_p, \hat{a}_p^\dagger \hat{a}_p)$, where p denotes the subsystem, the inner product can be written as

$$\begin{aligned}
\langle j_L| \hat{a}_p^\dagger \hat{a}_p \hat{a}_p^\dagger \hat{a}_p |j_L\rangle = & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} \\
& \times ((3s' + j \oplus p - 1)N)^2 \langle (3s + j \oplus p - 1)N|(3s' + j \oplus p - 1)N\rangle \\
= & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} \\
& \times ((3s' + j \oplus p - 1)N)^2 \delta_{(3s+j\oplus p-1)N,(3s'+j\oplus p-1)N} \\
= & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} ((3s' + j \oplus p - 1)N)^2 \delta_{s,s'} \\
= & \sum_{s=0}^{1} |c_{(3s+j\oplus p-1)N}|^2 ((3s + j \oplus p - 1)N)^2 \\
= & |c_{(j\oplus p-1)N}|^2 ((j \oplus p - 1)N)^2 + |c_{(3+j\oplus p-1)N}|^2 ((3 + j \oplus p - 1)N)^2.
\end{aligned}$$

For a given p, the expression must be the same for all j in order to satisfy the KL condition. However, $j \oplus p - 1$ is present in all factors and can have values $\{0, 1, 2\}$, so if all those expressions are the same for all $j \oplus p - 1$, they are the same for all j for any given p. Calculating for different values $j \oplus p - 1$ gives three cases

$$\begin{cases}
0 + |c_{3N}|^2 (3N)^2 = \frac{2}{3} 9N^2 & = 6N^2, \quad j \oplus p - 1 = 0 \\
|c_N|^2 N^2 + |c_{4N}|^2 16N^2 = \frac{2}{3} N^2 + \frac{1}{3} 16N^2 & = 6N^2, \quad j \oplus p - 1 = 1 \\
|c_{2N}|^2 4N^2 + |c_{5N}|^2 25N^2 = 4N^2 + 0 & = 4N^2, \quad j \oplus p - 1 = 2.
\end{cases}$$

The KL condition is therefore not satisfied for dephasing errors.

Continuing instead with the reduced set of errors $\{\mathbb{1}, \hat{a}_i\},\ i \in \{1,2,3\}$, the last combinations of the first class is

$$\begin{aligned}
\langle j_L|\, \hat{a}_p^\dagger \hat{a}_p \,|j_L\rangle = & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} \\
& \times ((3s'+j\oplus p-1)N) \langle (3s+j\oplus p-1)N|(3s'+j\oplus p-1)N\rangle \\
= & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} ((3s'+j\oplus p-1)N)\delta_{s,s'} \\
= & \sum_{s=0}^{1} |c_{(3s+j\oplus p-1)N}|^2 ((3s+j\oplus p-1)N) \\
= & |c_{(j\oplus p-1)N}|^2 ((j\oplus p-1)N)^2 + |c_{(3+j\oplus p-1)N}|^2((3+j\oplus p-1)N) \\
= & \begin{cases} 0 + |c_{3N}|^2 3N = \frac{2}{3}3N & = 2N, \quad j\oplus p-1 = 0 \\ |c_N|^2 N + |c_{4N}|^2 4N = \frac{2}{3}N + \frac{1}{3}4N & = 2N, \quad j\oplus p-1 = 1 \\ |c_{2N}|^2 2N + |c_{5N}|^2 5N = 4N^2 + 0 & = 2N, \quad j\oplus p-1 = 2. \end{cases}
\end{aligned}$$

This clearly satisfies the KL conditions and is a consequence that the arbitrary qutrit RSB code was constructed with equal mean photon number among the single-mode codewords $|k_N\rangle$. The first class is therefore verified.

For the second class, the only combinations are $(\hat{E}_k, \hat{E}_l) \in \{(\mathbb{1}, \hat{a}_p), (\hat{a}_p, \mathbb{1})\}$. The first becomes

$$\begin{aligned}
\langle j_L|\, \mathbb{1}\hat{a}_p \,|j_L\rangle = & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} \\
& \times \sqrt{((3s'+j\oplus p-1)N)}\, \langle (3s+j\oplus p-1)N|(3s'+j\oplus p-1)N-1\rangle \\
= & \sum_{s,s'=0}^{1} \bar{c}_{(3s+j\oplus p-1)N} c_{(3s'+j\oplus p-1)N} \\
& \times \sqrt{((3s'+j\oplus p-1)N)}\delta_{(3s+j\oplus p-1)N,(3s'+j\oplus p-1)N-1}.
\end{aligned}$$

Looking at the Kronecker delta, it is evident that if N is even then $(3s+j\oplus p-1)N$ is even while $(3s+j\oplus p-1)N-1$ is odd. In those cases the results are zero for all j and p. If $N=1$, it is sufficient to calculate the Kronecker deltas to find that

(s,s')	$\delta_{(3s+j\oplus p-1),(3s'-1+j\oplus p-1)}$
$(0,0)$	0
$(0,1)$	0
$(1,0)$	0
$(1,1)$	0

The result is also zero. Therefore, these combinations of $\langle j_L|\, \mathbb{1}\hat{a}_p \,|j_L\rangle$ with $N=1$ or N being an even number, satisfies the KL conditions. Higher odd numbers of N will not be considered.

The last combination in the class $\langle j_L|\, \hat{a}_p^\dagger \mathbb{1} \,|j_L\rangle$, renders a similar expression but with a $+1$ instead of -1 in the second argument of the Kronecker delta $\delta_{(3s+j\oplus p-1)N,(3s'+j\oplus p-1)N+1}$. Again, if N is even the delta is zero for all j and p. Checking for $N=1$ gives the table

(s, s')	$\delta_{(3s+j\oplus p-1),(3s'+1+j\oplus p-1)}$
$(0, 0)$	0
$(0, 1)$	0
$(1, 0)$	0
$(1, 1)$	0

This last combination is therefore also zero for $N = 1$ or an even number for all j and p. In fact, there are no combinations of (s, s') that renders the Kronecker non-zero since $3s = 3s' \pm 1$ has no solution if s and s' are integers.

The results from $\langle j_L| \hat{a}_p^\dagger |j_L\rangle$, $\langle j_L| \hat{a}_p |j_L\rangle$ and $\langle j_L| \hat{a}_p^\dagger \hat{a}_p |j_L\rangle$ can now be used in the last classes of combinations.

For instances with the same type of errors on different subspaces, there exists two sets of combinations. If p and q denotes different subspaces, the first combinations are

$$\begin{aligned}\langle j_L| a_p^\dagger a_q |j_L\rangle = &\left(\langle (j \oplus p - 1)_N| a^\dagger |(j \oplus p - 1)_N\rangle\right) \\ &\times \left(\langle (j \oplus q - 1)_N| a |(j \oplus q - 1)_N\rangle\right),\end{aligned}$$

where both inner products have already been calculated above and are zero for all $j,\ p \neq q$, even N and $N = 1$. And the other combinations, are

$$\begin{aligned}\langle j_L| a_p^\dagger a a^\dagger a_q |j_L\rangle = &\left(\langle (j \oplus p - 1)_N| a^\dagger a |(j \oplus p - 1)_N\rangle\right) \\ &\times \left(\langle (j \oplus q - 1)_N| a^\dagger a |(j \oplus q - 1)_N\rangle\right),\end{aligned}$$

where each inner product has already been calculated before, resulting in $(2N)^2$ for all $j,\ p \neq q$, even N and $N = 1$. Note that the last combinations are for dephasing errors which were previously excluded. They are calculated anyway for the sake of interest.

The last class of combinations, with different type of errors on different subspaces p and q, results in the inner products $\langle j_L| \hat{a}_p^\dagger \hat{a}_p \hat{a}_q |j_L\rangle$ and $\langle j_L| \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_q |j_L\rangle$. The first inner product results in

$$\langle j_L| \hat{a}_p^\dagger \hat{a}_p \hat{a}_q |j_L\rangle = \left(\langle (j \oplus p - 1)_N| a^\dagger a |(j \oplus p - 1)_N\rangle\right)\left(\langle (j \oplus q - 1)_N| a |(j \oplus q - 1)_N\rangle\right),$$

where the second inner product is known to be zero for all $j,\ q$, even N or $N = 1$. The same is found for the last combinations. These last combinations were actually considering an error set of mixed error types.

It is therefore concluded that, after having went through all classes consisting in total of 441 combinations, the code is correctable, up to one photon loss according to the KL condition.

From the generalizations, used for verifying the KL conditions, it seems that any three-mode qutrit N-order RSB code, where N is even or $N = 1$, will satisfy the KL conditions up to one single photon loss if the three following type of conditions are satisfied

$$\langle j_L| \hat{a}_p^\dagger |j_L\rangle = \langle j'_L| \hat{a}_p^\dagger |j'_L\rangle \tag{B.7}$$

$$\langle j_L| \hat{a}_p |j_L\rangle = \langle j'_L| \hat{a}_p |j'_L\rangle \tag{B.8}$$

$$\langle j_L| \hat{a}_p^\dagger \hat{a}_p |j_L\rangle = \langle j'_L| \hat{a}_p^\dagger \hat{a}_p |j'_L\rangle, \tag{B.9}$$

for all $j,\ j'$ and p, since these were the conditions which were used in all the instances above. The two first conditions (B.7), (B.8) were shown, for the given values of N, to be independent of the coefficients $c_{(3s+j)N}$ while the third relied on the fact that the mean photon number was equal for all single-mode states $|k_N\rangle$.

B.2 Knill-Laflamme conditions in the beam splitter picture

In the beam splitter picture the logical code words are acted on by the unitary operator $\hat{U}_{BS}$ such that $|j_L\rangle = \hat{U}_{BS} \bigotimes_{i=0}^{2} |(j \oplus i)_N\rangle$ where $\oplus$ is addition modulo 3. By again considering an error set of photon loss and dephasing up to first order, the first combinations $(\hat{E}_k, \hat{E}_l) = (\hat{a}_k, \hat{a}_l)$ gives the inner products

$$\begin{aligned}
\langle j_L| \hat{a}_k^\dagger \hat{a}_l |j_L'\rangle &= \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| \hat{U}_{BS}^\dagger \hat{a}_k^\dagger \hat{a}_l \hat{U}_{BS} \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle \\
&= \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| \hat{U}_{BS}^\dagger \hat{a}_k^\dagger \hat{U}_{BS} \hat{U}_{BS}^\dagger \hat{a}_l \hat{U}_{BS} \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle \\
&= \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| (\hat{b}_k^-)^\dagger \hat{b}_l^- \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle ,
\end{aligned}$$

where $\hat{b}_i^-$ refers to the transformation

$$\hat{\boldsymbol{b}}^- = \begin{pmatrix} 1+2\gamma & -iz_{12}\Delta & iz_{13}\sigma \\ i\bar{z}_{12}\sigma & 1+2\gamma & -iz_{23}\Delta \\ -i\bar{z}_{13}\Delta & i\bar{z}_{23}\sigma & 1+2\gamma \end{pmatrix} \hat{\boldsymbol{a}}, \tag{B.10}$$

which was defined in Appendix A. By using the expression above, $\hat{b}_k^-$ can be written as the linear combinations

$$\hat{b}_k^- = \sum_{n=1}^{3} \xi_{kn}^- \hat{a}_n, \tag{B.11}$$

where ξ_{kn}^- is the element from the transformation matrix at row k and column n.

Inserting this in the inner product above results in

$$\begin{aligned}
\langle j_L| \hat{a}_k^\dagger \hat{a}_l |j_L'\rangle &= \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| \sum_{n,m=1}^{3} \bar{\xi}_{kn}^- \xi_{lm}^- \hat{a}_n^\dagger \hat{a}_m \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle \\
&= \sum_{n=1}^{3} \bar{\xi}_{kn}^- \xi_{ln}^- \langle (j \oplus n-1)_N| \hat{a}^\dagger \hat{a} |(j' \oplus n-1)_N\rangle \\
&+ \sum_{n \neq m}^{3} \bar{\xi}_{kn}^- \xi_{lm}^- \langle (j \oplus n-1)_N| \hat{a}^\dagger |(j' \oplus n-1)_N\rangle \\
&\qquad \times \langle (j \oplus m-1)_N| \hat{a} |(j' \oplus m-1)_N\rangle ,
\end{aligned}$$

where the second sum contains inner products which are all known to be zero for all j, n, m, even N and $N=1$. For the first sum, it is also known that all inner products where $j \neq j'$ are zero, thus leaving the following expression

$$\langle j_L| \hat{a}_k^\dagger \hat{a}_l |j_L'\rangle = \sum_{n=1}^{3} \bar{\xi}_{kn}^- \xi_{ln}^- \langle (j \oplus n-1)_N| \hat{a}^\dagger \hat{a} |(j \oplus n-1)_N\rangle \delta_{jj'}.$$

Further, it is known that each inner product is equal to the mean photon number $\bar{n}$ of single-mode state $|(j \oplus n-1)_N\rangle$ which is the same for all $j \oplus n-1$ by construction, which results in

$$\langle j_L| \hat{a}_k^\dagger \hat{a}_l |j'_L\rangle = \bar{n}\delta_{jj'} \sum_{n=1}^{3} \bar{\xi}_{kn}^- \xi_{ln}^- = \begin{cases} \bar{n}\delta_{jj'} \sum_{n=1}^{3} |\xi_{kn}^-|^2, & k = l \\ \bar{n}\delta_{jj'} \sum_{n=1}^{3} \bar{\xi}_{kn}^- \xi_{ln}^-, & k \neq l \end{cases} .$$

The first case where $k = l$ gives the same result for all $j = j'$, k, even N and $N = 1$. Thus satisfying the diagonal conditions of the QEC matrix in the KL conditions. In the second case it is required to establish if by interchanging $k \leftrightarrow l$, results in the conjugate (since the QEC matrix is hermitian). Indeed, interchanging $k \leftrightarrow l$ leads to the conjugate $\bar{\xi}_{kn}^- \xi_{ln}^- \leftrightarrow \bar{\xi}_{ln}^- \xi_{kn}^- = (\bar{\xi}_{kn}^- \xi_{ln}^-)^*$, where $*$ refers to the conjugate. Therefore, $\langle j_L| \hat{a}_k^\dagger \hat{a}_l |j'_L\rangle$ satisfy the KL conditions for all k, l even N and $N = 1$.

The remaining combinations, in order to verify the KL conditions for the three-mode qutrit RSB code in the beam splitter picture, are $\langle j_L| \hat{a}_k^\dagger \mathbb{1} |j'_L\rangle$ and $\langle j_L| \mathbb{1} \hat{a}_k |j'_L\rangle$. Similarly to before, the first combinations become

$$\begin{aligned} \langle j_L| \hat{a}_k^\dagger |j'_L\rangle &= \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| \sum_{n=1}^{3} \bar{\xi}_{kn}^- \hat{a}_n^\dagger \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle \\ &= \sum_{n=1}^{3} \bar{\xi}_{kn}^- \langle (j \oplus n-1)_N| \hat{a}^\dagger |(j' \oplus n-1)_N\rangle , \end{aligned}$$

which is known, from before, to be zero for all j, j', n, even N and $N = 1$. And the same is found for $\langle j_L| \mathbb{1} \hat{a}_k |j'_L\rangle$.

The only excluded combinations are the trivial $\langle j_L| \mathbb{1}\mathbb{1} |j'_L\rangle$ which indeed satisfies KL. This concludes that in the beam splitter picture, the KL conditions, for only single photon loss errors, are satisfied.

It is found that the KL conditions are not satisfied for phase errors by looking at the following example. The combinations $(\hat{E}_k, \hat{E}_l) = (\hat{a}_k^\dagger \hat{a}_k, \hat{a}_l^\dagger \hat{a}_l)$, results in the inner product

$$\langle j_L| \hat{a}_k^\dagger \hat{a}_k \hat{a}_l^\dagger \hat{a}_l |j'_L\rangle = \bigotimes_{i=0}^{2} \langle (j \oplus i)_N| \sum_{n,m=1}^{3} \bar{\xi}_{kn}^- \xi_{km}^- \hat{a}_n^\dagger \hat{a}_m \sum_{n',m'=1}^{3} \bar{\xi}_{ln'}^- \xi_{lm'}^- \hat{a}_{n'}^\dagger \hat{a}_{m'} \bigotimes_{i'=0}^{2} |(j' \oplus i')_N\rangle .$$

The only terms that survive are those which has pairs of $\hat{a}_n^\dagger \hat{a}_m \hat{a}_{n'}^\dagger \hat{a}_{m'} = \hat{a}_n^\dagger \hat{a}_n \hat{a}_m^\dagger \hat{a}_m$ since the acting of a single annihilation or creation on an individual subspace is known to be zero. This leads to three different types of terms where the four indices (n, m, n', m') are either the same (n, n, n, n), paired as (n, n, m, m) or as (n, m, m, n). This results in three sums as

$$\begin{aligned}
\langle j_L|\,\hat{a}_k^\dagger\hat{a}_k\hat{a}_l^\dagger\hat{a}_l\,|j'_L\rangle =& \sum_{n=1}^{3}|\xi_{kn}^-|^2|\xi_{ln}^-|^2\bigotimes_{i=0}^{2}\langle(j\oplus i)_N|\,\hat{a}_n^\dagger\hat{a}_n\hat{a}_n^\dagger\hat{a}_n\bigotimes_{i'=0}^{2}|(j'\oplus i')_N\rangle \\
&+\sum_{\substack{n,m=1;\\ n\neq m}}^{3}|\xi_{kn}^-|^2|\xi_{lm}^-|^2\bigotimes_{i=0}^{2}\langle(j\oplus i)_N|\,\hat{a}_n^\dagger\hat{a}_n\hat{a}_m^\dagger\hat{a}_m\bigotimes_{i'=0}^{2}|(j'\oplus i')_N\rangle \\
&+\sum_{\substack{n,m=1;\\ n\neq m}}^{3}\bar{\xi}_{kn}^-\xi_{km}^-\bar{\xi}_{lm}^-\xi_{ln}^-\bigotimes_{i=0}^{2}\langle(j\oplus i)_N|\,\hat{a}_n^\dagger\hat{a}_m\hat{a}_m^\dagger\hat{a}_n\bigotimes_{i'=0}^{2}|(j'\oplus i')_N\rangle \\
=& \sum_{n=1}^{3}|\xi_{kn}^-|^2|\xi_{ln}^-|^2\,\langle(j\oplus n-1)_N|\,\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\,|(j'\oplus n-1)_N\rangle \\
&+\sum_{\substack{n,m=1;\\ n\neq m}}^{3}|\xi_{kn}^-|^2|\xi_{lm}^-|^2\,\langle(j\oplus n-1)_N|\,\hat{a}^\dagger\hat{a}\,|(j'\oplus n-1)_N\rangle \\
&\qquad\qquad\times\langle(j\oplus m-1)_N|\,\hat{a}^\dagger\hat{a}\,|(j'\oplus m-1)_N\rangle \\
&+\sum_{\substack{n,m=1;\\ n\neq m}}^{3}\bar{\xi}_{kn}^-\xi_{km}^-\bar{\xi}_{lm}^-\xi_{ln}^-\,\langle(j\oplus n-1)_N|\,\hat{a}^\dagger\hat{a}\,|(j'\oplus n-1)_N\rangle \\
&\qquad\qquad\times\langle(j\oplus m-1)_N|\,\hat{a}^\dagger\hat{a}+\mathbb{1}\,|(j'\oplus m-1)_N\rangle \\
=& \sum_{n=1}^{3}|\xi_{kn}^-|^2|\xi_{ln}^-|^2\delta_{jj'}\sum_{s=0}^{1}|c_{(3s+j\oplus n-1)N}|^2(3s+j\oplus n-1)^2N^2 \\
&+\sum_{\substack{n,m=1;\\ n\neq m}}^{3}|\xi_{kn}^-|^2|\xi_{lm}^-|^2\bar{n}^2\delta_{jj'} \\
&+\sum_{\substack{n,m=1;\\ n\neq m}}^{3}\bar{\xi}_{kn}^-\xi_{km}^-\bar{\xi}_{lm}^-\xi_{ln}^-\bar{n}(\bar{n}+1)\delta_{jj'},
\end{aligned}$$

where the inner products in the last two terms were identified to be the mean photon number $\bar{n}$ and $\bar{n}+1$ of each single-mode state $|k_N\rangle$, which is known to be the same for all k. All sums are equal to zero for $j\neq j'$ since there is always a trivial inner product from one subspace which is zero if $j\neq j'$, while in the cases where $j=j'$, it is evident that the last two sums are independent of j while the first sum is not independent of j. It is known from before that $\sum_{s=0}^{1}|c_{(3s+j\oplus n-1)N}|(3s+j\oplus n-1)^2N^2$, the second moment of state $|(j\oplus n-1)_N\rangle$, is not the same for different $j\oplus n-1$. Therefore, for a given k and l, $\langle j_L|\,\hat{a}_k^\dagger\hat{a}_k\hat{a}_l^\dagger\hat{a}_l\,|j'_L\rangle$ will not be the same for different $j=j'$, and will not satisfy the KL conditions for the same reason. The considered three-mode qutrit code is therefore not exactly correctable for phase errors in the beam splitter picture either.

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