Degree Project in Computer Science and Engineering

Second cycle, 30 credits

Free Groups in Homotopy Type Theory

Free Groups as Filtered Colimits

DAVID WEISSKOPF HOLMQVIST

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DAVID WEISSKOPF HOLMQVIST

CHALMERS
UNIVERSITY OF TECHNOLOGY

Master's Program, Computer Science
Date: January 30, 2026

Supervisor: Thierry Coquand
Examiner: Christian Sattler
Chalmers Institute of Technology

Abstract

While the construction of the free group on an arbitrary set is trivial in classical set-theory, in absence of decidable equality this is not so. This is the case in univalent mathematics, where not every set a priori has this property. Existing work uses the Church-Rosser property to get around this, while we propose a different appraoch. First, the free group is constructed for finite sets in the expected manner. This construction is then used as, secondly, the free group of an arbitrary set X is constructed as a filtered colimit of finite sets with maps into X. This necessitates showing that this comma category of finite sets is filtered, where coequalizers are given by induction on the cardinality of finite sets, as opposed to the conventional approach using quotients. Finally, we show that the resulting free group functor is indeed the left Kan extension. We formalize the resulting univalent construction in Agda.

Keywords

Homotopy Type Theory, Algebra, Group Theory, Constructive Mathematics, Proof Assistants

Acknowledgments

I want to thank my supervisor Thierry Coquand for our meetings and his patience, for Christian Sattler for interesting input (in particular the alternative approach as discussed in Future Work), and Nils-Daniel Andersson for his help regarding Agda. I also want to thank my dear friend Armaan Rashid for providing great support and himself, generously, as a great reservoire of knowledge.

Gothenburg, January 2026
David Weisskopf Holmqvist

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Chapter 1
Introduction

Groups are a fundamental concept in mathematics [1, 2], capturing the ideas of symmetry and transformation. Given a set X, the free group on X is the most general group generated by X, satisfying only the group axioms. Classically, the free group can be constructed using words in $X \cup X^{-1}$, with reduction rules that rely on the ability to decide when two elements are equal.

In homotopy type theory (HoTT) [3, 4, 5], however, the situation is more subtle. Types can have higher-dimensional structure, and, crucial in this context, equality in sets (0-types) is not necessarily decidable. This lack of decidable equality poses a challenge for constructing free groups, as many classical arguments and reduction procedures depend on comparing elements for equality.

For example, in a common construction of the free group in the classical setting, one reduces words by canceling adjacent inverse pairs xx^{-1}, which requires knowing whether two elements are equal ($y \stackrel{?}{=} x^{-1}$ in xy). In HoTT, this is not always possible, so alternative approaches are needed.

Previous work, such as Escardó's construction in the TypeTopology library [6], adapts classical ideas to the univalent setting by using quotients of lists and reduction relations, thus avoiding the need for decidable equality. It uses a technique first given in [7] published in 1988; Escardó notes that it is remarkable that no decidable equality is required with this approach.

In this thesis we present an alternative approach, using a categorical framework and reasoning. The free group on an arbitrary set X is constructed as a filtered colimit [8, 9] of finite sets mapping into X, sending (A, f) to the free group on A. This relies on the presence of the free group functor on finite sets, which in HoTT is relatively straightforward since finite sets have decidable equality.

Our construction is formalized in Agda [10] and is designed to be both constructive and compatible with the univalent foundations of HoTT. This approach provides a new perspective on free groups in HoTT, while opening the door to similar constructions for other algebraic structures. To generalize this more robustly one might want to use reasoning from functorial semantics, and employ a strategy as described in Chapter 5.

The thesis is organized as follows:

- In Chapter 2, we review the necessary background in HoTT and discuss related work on free groups.
- In Chapter 3, we present the construction of the free group, first for finite sets and then for arbitrary sets via filtered colimits.

- In Chapter 4, we discuss the implications and potential generalizations of our approach.
- In Chapter 5, we outline directions for further research.

We hope this work contributes to the understanding of algebraic structures in HoTT and demonstrates the utility of categorical methods in the study of free groups.

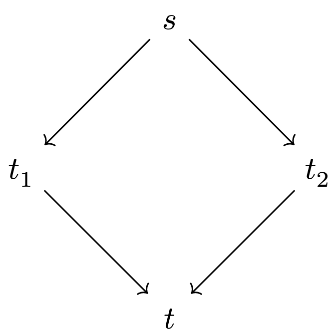
Chapter 2
Context & Related Work

The construction of free groups in HoTT is complicated by the lack of decidable equality for arbitrary sets. In classical mathematics, as already mentioned, the free group on a set X can be constructed using words in $X \cup X^{-1}$, but this approach does not directly generalize to the univalent setting.

Previous work, such as Escardó's construction in the TypeTopology library [6], provides a quotient-based approach that avoids the need for decidable equality, based on [7] and the Church-Rosser property [11]. As expected, s reduces to t ($s \rhd t$), where $s, t : F(A) :\equiv [2 \times A]$, given that

$$\sum_{\substack{u,v:F(A),\\ x:A}} (s = u +\!\!+ \, x :: x^{-1} :: v) \times (t = u +\!\!+ \, v).$$

They go on to prove that the Church-Rosser property holds for this relation; that is, given $s \rhd t_1$ and $s \rhd t_2$, t_1 is already equal to t_2, or there exists a common term t such that $t_1 \rhd t$ and $t_2 \rhd t$. This can be visualized as follows:



The actual type of the free group is then $[2 \times A]$ quotiented by the symmetric, reflexive and transitive closure of $\rhd$.

Our approach instead leverages the machinery of filtered colimits: we construct the free group on X as a colimit over the category of finite sets mapping into X. Note that in a classical setting, one can work with a colimit over the category of finite *subsets* of X, but the constructive/univalent approach with arbitrary sets (mapping into X) is equally valid and is what we are forced to do in the absence of decidable equality.

Filtered colimits are an important concept in category theory, especially because they interact well with finite limits. In this thesis, however, we do not rely on this property in our explicit construction, or in verifying the group laws and universal property for the free group. While the commutation of filtered colimits with finite limits provides a general categorical reasoning for such constructions, our development is self-contained and highlights the explicit nature of the construction.

This approach is formalized for the most part in Agda, and is amenable to further generalization, as discussed in the future work section.

We now briefly explain the challenge that homotopy type theory (HoTT) presents for the construction of free groups. In HoTT, types can have different *truncation levels*, which describe the complexity of their identity types. Of particular interest for this thesis are sets (also called 0-types): types where, for any two elements x, y, the type $x = y$ is a proposition (i.e., there is at most one way to witness their equality).

Unlike in classical set theory, equality in sets in HoTT is not necessarily decidable. That is, given $x, y : A$ for a set A, we cannot in general decide whether $x = y$ or $x \neq y$. This lack of decidable equality poses a challenge for constructing algebraic objects such as free groups, since many classical constructions rely on being able to compare elements for equality. In our case, the conventional, classical construction of the free group on some set depends on canceling adjacent inverse pairs. Given a word $w_1 x y w_2$, we need to be able to tell whether $y = x^{-1}$ in order to either reduce to $w_1 w_2$, or do nothing. As we are in the framework of HoTT, where decidable equality is not necessarily available for an arbitrary set, an alternative approach is needed.

Unlike in Escardó's construction, our approach instead uses categorical reasoning and filtered colimits, as detailed below.

A filtering of a category $\mathcal{C}$ consists of:
- An object in $\mathcal{C}$.
- For any objects X, Y in $\mathcal{C}$, morphisms $f_X : X \to Z$, $f_Y : Y \to Z$;
- For any parallel morphisms $f, g : X \rightrightarrows Y$, a morphism $h : Y \to Z$ such that $hf = hg$.

Note that in the univalent context one would usually truncate these conditions to propositions. We do not truncate here because it allows us to make use of some concrete data given by the satisfaction of these conditions later on. From here on, when we say a category is filtered, what we really mean is the notion given here.

Intuitively, a filtered category is one in which we can always find a common upper bound for objects and morphisms. Filtered colimits are particularly well-behaved: their key property is that they commute with finite limits in $\mathbf{Set}$ (and other categories), though our construction does not rely on this property explicitly.

Consider the comma category $\mathbf{FinSet} \downarrow X$, whose objects are pairs (A, f) with A a finite set and $f : A \to X$. This category is filtered: the finite coproducts are obvious, and the coequalizers are usually constructed using quotients. For the latter we use a different approach than usual, and construct them by induction on the cardinality of finite sets. We then consider the diagram $J : (\mathbf{FinSet} \downarrow X) \to \mathbf{Grp}$ sending (A, f) to the free group $F^{\Phi}(A)$ and morphisms to the induced group homomorphisms. Since finite sets have decidable equality, the construction of this diagram is straightforward. Our approach defines the free group on any set X as the colimit of this diagram:

$$F(X) = \operatorname*{colim}_{(A,f)\colon \mathbf{FinSet} \downarrow X} \big(J(A, f) = F^{\Phi}(A)\big).$$

We conclude by showing that this object indeed is the colimit, and that F is the left Kan extension of F^{Φ} along the inclusion $\iota : \mathbf{FinSet} \to \mathbf{Set}$.

As far as the author knows, this approach is novel in the univalent setting, especially given the formalization in Agda.

Chapter 3
Construction

We proceed by giving the construction of the free group on any set; in particular, we are splitting this into two steps:
1. We first construct the free group on any *finite* set.
2. We then construct the free group on any set X, as the filtered colimit of $J : (\mathbf{FinSet} \downarrow X) \to \mathbf{Grp}$.

3.1. Free Group on Finite Set A

We denote $F^\Phi : \mathbf{FinSet} \to \mathbf{Grp}$ as the functor we are endeavouring to construct, to distinguish it from the later one, $F : \mathbf{Set} \to \mathbf{Grp}$, for arbitrary sets.

We take letters to be in the coproduct $A + A$, where $\iota_l(x)$ is simply the embedding of x, and $\iota_r(x)$ is x^{-1}, where inversion is defined as expected. The underlying set of $F^\Phi(A)$ we define to be $[A + A]$; this works because truncation-level 0 is closed under $[-]$.

Definition 3.1. *(Reducedness of words)*
A word w in $F^\Phi(A)$ is said to be fully reduced if there are no reducible adjacent letters in it. More concretely, we define it as follows by induction:

$$\begin{aligned} \text{is-red}([]) &:= \mathbb{1} \\ \text{is-red}(x :: w) &:= \text{is-good}(x, w) \times \text{is-red}(w) \end{aligned}$$

$$\begin{aligned} \text{is-good}(x, []) &:= \mathbb{1} \\ \text{is-good}(x, y :: w) &:= y \neq x^{-1} \end{aligned}$$

All of the operations defined on $F^\Phi(A)$ from here on have to respect this property. Note that is-red is a family of propositions.

Definition 3.2. *(Unit of $F^\Phi(A)$)*
The unit $1 : F^\Phi(A)$ is defined to be the empty list $[]$.

Note that 1 is trivially a reduced word.

Definition 3.3. *(Consing letters onto words)*
Given a letter x and word w, we define the *consing* of x onto w, denoted $x \bullet w$, as follows:

$$x \bullet [] := x$$

$$x \bullet (y :: w) := \begin{cases} w & \text{if } y = x^{-1} \\ x :: y :: w & \text{otherwise} \end{cases}$$

Lemma 3.4. *(Consing preserves reducedness)*
If w is reduced then so is $x \bullet w$.

Proof. If $w \doteq []$, this holds trivially. Given $w \doteq y :: w$, if $y = x^{-1}$ this reduces to w which we know to be reduced; if $y \neq x^{-1}$ then this is precisely what we need in order for $x :: y :: w$ to be reduced. □

Lemma 3.5. *(Conditional equality of consing)*
If $x :: w$ is reduced, then $x \bullet w = x :: w$.

Proof. If $w \doteq []$, this holds trivially. On the other hand, if $w \doteq y :: w$, we then know $x :: y :: w$ to be reduced, giving us $y \neq x^{-1}$. □

Definition 3.6. *(Multiplication)*
Given two words w_1, w_2, we define their multiplication $w_1 w_2$:

$$[]w_2 := w_2$$

$$(x :: w_1)w_2 := x \bullet w_1 w_2$$

Lemma 3.7. *(Multiplication preserves reducedness)*
If w_1 and w_2 are reduced then so is $w_1 w_2$.

Proof. If $w_1 \doteq []$, then $[]w_2 \doteq w_2$, which we assume to be reduced. Otherwise, we want reducedness of

$$(x :: w_1)w_2 \doteq x \bullet w_1 w_2,$$

given to us by inductive hypothesis and Lemma 3.4. □

Definition 3.8. *(Inversion)*
Given a word w, we define its inverse w^{-1} as follows:

$$[]^{-1} := []$$

$$(x :: w)^{-1} := w^{-1} x^{-1}$$

Lemma 3.9. *(Inversion preserves reducedness)*
If w is reduced then so is w^{-1}.

Proof. $[]^{-1} \doteq []$ is trivially reduced. Otherwise, we want reducedness of $w^{-1}x^{-1}$, given to us by inductive hypothesis and Lemma 3.7. □

We proceed to show that $F^{\Phi}(A)$ in fact is a group. Reducedness of words is always assumed from here on.

Lemma 3.10. *(Left & Right Unit)*
1 is a left and right unit of multiplication: $1w = w = w1$.

Proof. The left case $1w \doteq w$ holds by computation.

For the right case, when $w \doteq []$, we have

$$[]1 \doteq 1 \doteq [].$$

Else, note that $(x :: w)1 \doteq x \bullet w1$, and so by inductive hypothesis and Lemma 3.5 we are done. □

Lemma 3.11. *(Consing twice)*
If $y = x^{-1}$ then $x \bullet y \bullet w = w$.

Proof. When $w \doteq []$, then

$$\begin{aligned} x \bullet y \bullet [] &\doteq x \bullet y :: [] \\ &= []. \end{aligned}$$

Otherwise we want $x \bullet y \bullet (z :: w) = z :: w$. Consider when $z = y^{-1}$; we have $z = \left(x^{-1}\right)^{-1} = x$, and so

$$\begin{aligned} x \bullet y \bullet (z :: w) &= x \bullet w \\ &= z \bullet w \\ &= z :: w. \end{aligned}$$

When $z \neq y^{-1}$ it is the case that $x \bullet y :: z :: w = z :: w$. □

Lemma 3.12. *(Associativity of consing and multiplication)*
It is the case that $x \bullet (w_1 w_2) = (x \bullet w_1) w_2$.

Proof. When $w_1 \doteq []$, this holds by computation. If $w_1 \doteq y :: w_1$, we first consider the case when $y = x^{-1}$; by Lemma 3.11 we get

$$x \bullet (y \bullet w_1 w_2) = (x \bullet y :: w_1) w_2.$$

If $y \neq x^{-1}$, by computation it is the case that

$$(x :: y :: w_1) w_2 \doteq x \bullet (y \bullet w_1 w_2).$$

□

Lemma 3.13. *(Associativity of multiplication)*
It is the case that $w_1(w_2 w_3) = (w_1 w_2) w_3$.

Proof. If $w_1 \doteq []$ this is trivial. Else if $w_1 \doteq x :: w_1$, this is provided by Lemma 3.12 as well as the inductive hypothesis, as follows:

$$\begin{aligned}(x :: w_1)(w_2 w_3) &\doteq x \bullet (w_1(w_2 w_3)) \\ &= x \bullet ((w_1 w_2) w_3) \\ &= (x \bullet (w_1 w_2)) w_3 \\ &\doteq ((x :: w_1) w_2) w_3.\end{aligned}$$

□

Lemma 3.14. *(Left & Right Inversion)*
We have $w^{-1}w = 1 = ww^{-1}$.

Proof. Beginninig with $w^{-1}w = 1$, if $w \doteq []$, this holds trivially. When $w \doteq x :: w$, the proof is given by Lemma 3.13 and the inductive hypothesis, like so:

$$\begin{aligned}(x :: w)^{-1}(x :: w) &\doteq (w^{-1}x^{-1})(x :: w) \\ &= w^{-1}(x^{-1} \bullet (x :: w)) \\ &= w^{-1}w \\ &= 1.\end{aligned}$$

For the other case of $ww^{-1} = 1$, if $w \doteq []$, this is again trivial. When $w \doteq x :: w$, the proof is again given by Lemma 3.13 and the inductive hypothesis, like so:

$$\begin{aligned}(x :: w)(x :: w)^{-1} &\doteq (x :: w)(w^{-1}x^{-1}) \\ &\doteq x \bullet (w(w^{-1}x^{-1})) \\ &= x \bullet ((ww^{-1})x^{-1}) \\ &= x \bullet x^{-1} \\ &= 1.\end{aligned}$$

□

Theorem 3.15. *($F^{\Phi}(A)$ is a group)*
The set $F^{\Phi}(A) :\equiv \sum_{w:\,\mathrm{list}_{A+A}} \text{is-red}(w)$ where the unit is given in Definition 3.2, multiplication in Definition 3.6, and inversion in Definition 3.8, is a group.

Proof. This is witnessed by Lemma 3.13, Lemma 3.10 and Lemma 3.14, in combination with $\sum$-induction and the fact that is-red is a family of propositions. □

We proceed by showing that $F^{\Phi}(A)$ indeed is the free group on A.

Definition 3.16. *(Sending A to $F^{\Phi}(A)$)*
The map $\eta_A^{\Phi} : A \to F^{\Phi}(A)$ is straight-forwardly given by

$$x \mapsto [x].$$

This is trivially a reduced word. Note also that η_A^{Φ} is a monomorphism, due to injectivity of introduction rules/constructors.

Definition 3.17. *(Extending map into group)*
Given $f : A \to G$ from a finite set into a group G, we define the extension of f, denoted $\overline{f}^{\Phi} : F^{\Phi}(A) \to G$ as follows:

$$\begin{aligned} [] &\mapsto 1 \\ xw &\mapsto f(x)\overline{f}^{\Phi}(w) \\ x^{-1}w &\mapsto f(x)^{-1}\overline{f}^{\Phi}(w). \end{aligned}$$

Lemma 3.18. *(Pseudo-functoriality of $\overline{f}^{\Phi}$)*
It is the case that

$$\overline{f}^{\Phi}(x \bullet w) = \overline{f}^{\Phi}(x)\overline{f}^{\Phi}(w).$$

Proof. When $w \doteq []$, we have

$$\begin{aligned} \overline{f}^{\Phi}(x \bullet []) &\doteq \overline{f}^{\Phi}(x) \\ &= \overline{f}^{\Phi}(x)1 \\ &= \overline{f}^{\Phi}(x)\overline{f}^{\Phi}([]). \end{aligned}$$

Otherwise, when $w \doteq y :: w$, consider first when $y = x^{-1}$. We have

$$\begin{aligned} \overline{f}^{\Phi}(x \bullet (y :: w)) &\doteq \overline{f}^{\Phi}(x \bullet (x^{-1} :: w)) \\ &= \overline{f}^{\Phi}(w) \\ &= 1\overline{f}^{\Phi}(w) \\ &= f(x)f^{-1}\overline{f}^{\Phi}(w) \\ &\doteq \overline{f}^{\Phi}(x)\overline{f}^{\Phi}(y :: w). \end{aligned}$$

Lastly, if $y \neq x^{-1}$ we have

$$\begin{aligned} \overline{f}^{\Phi}(x \bullet (y :: w)) &= \overline{f}^{\Phi}(x :: y :: w) \\ &\doteq f(x)\overline{f}^{\Phi}(y :: w) \\ &\doteq \overline{f}^{\Phi}(x)\overline{f}^{\Phi}(y :: w). \end{aligned}$$

□

Lemma 3.19. *($\overline{f}^{\Phi}$ is a group homomorphism)*
It is the case that $\overline{f}^{\Phi}(w_1 w_2) = \overline{f}^{\Phi}(w_1)\overline{f}^{\Phi}(w_2)$.

Proof. When $w_1 \doteq []$, we have

$$\begin{aligned}\overline{f}^{\Phi}([]w_2) &\doteq \overline{f}^{\Phi}(w_2)\\ &= 1\overline{f}^{\Phi}(w_2)\\ &= \overline{f}^{\Phi}([])\overline{f}^{\Phi}(w_2).\end{aligned}$$

When $w_1 \doteq x :: w_1$, by Lemma 3.18, Lemma 3.47 and the inductive hypothesis, we have

$$\begin{aligned}\overline{f}^{\Phi}((x :: w_1)w_2) &\doteq \overline{f}^{\Phi}(x \bullet w_1 w_2)\\ &= \overline{f}^{\Phi}(x)\overline{f}^{\Phi}(w_1 w_2)\\ &= \overline{f}^{\Phi}(x)\overline{f}^{\Phi}(w_1)\overline{f}^{\Phi}(w_2)\\ &\doteq \overline{f}^{\Phi}(x :: w_1)\overline{f}^{\Phi}(w_2).\end{aligned}$$

□

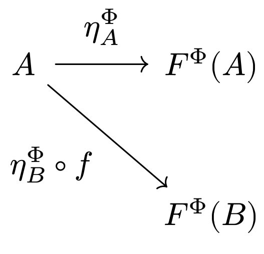
Theorem 3.20. *($F^{\Phi}(A)$ is the free group on A)*
$F^{\Phi}(A)$ is the universal group generated by A; any other map $f : A \to G$ into a group factors uniquely through η^{Φ}_A.

Proof. Given such a map $f : A \to G$, we obtain $\overline{f}^{\Phi} : F^{\Phi}(A) \to G$ as in Definition 3.17. $\overline{f}^{\Phi}$ is a homomorphism due to Lemma 3.19.

$\overline{f}^{\Phi}$ indeed lets f factor through η^{Φ}_A, because $\overline{f}^{\Phi}\big(\eta^{\Phi}_A(x)\big) \doteq \overline{f}^{\Phi}(x) = f(x)$. Given any other homomorphism $u' : F^{\Phi}(A)$ such that $u' \circ \eta^{\Phi}_A = f$ we can identify $\overline{f}^{\Phi}$ and u'. Firstly, $\overline{f}^{\Phi}$ and u' clearly identify at $[] \doteq 1$; otherwise we have $\overline{f}^{\Phi}(x :: w) \doteq f(x)\overline{f}^{\Phi}(w) = u'\big(\eta^{\Phi}_A(x)\big)u'(w) = u'(xw)$ by the inductive hypothesis and that u' also is a factor of f. □

Finally we want to give the map component of the functor $F^{\Phi} : \mathbf{FinSet} \to \mathbf{Grp}$.

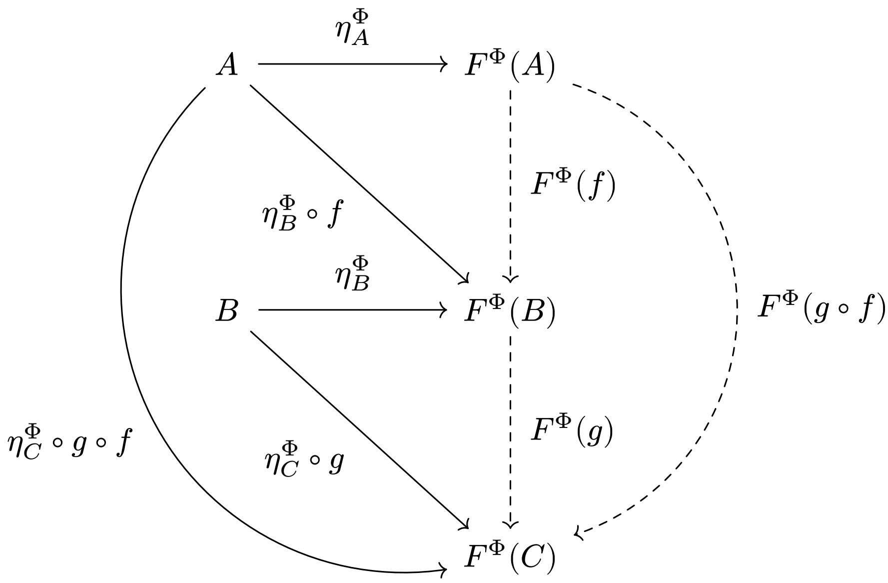
Definition 3.21. *(Action of F^{Φ} on maps)*
Given any map $f : \mathrm{Hom}_{\mathbf{FinSet}}(A, B)$ we construct a homomorphism $F^{\Phi}(f) : \mathrm{Hom}_{\mathbf{Grp}}(F^{\Phi}(A), F^{\Phi}(B))$. Consider the following diagram:



By Theorem 3.20 we thus get the homomorphism $F^{\Phi}(f) : F^{\Phi}(A) \to F^{\Phi}(B)$.

Theorem 3.22. *(F^{Φ} is a functor)*
The constructed map on finite sets (Theorem 3.15) and maps between them (Definition 3.21) is a functor from $\mathbf{FinSet} \to \mathbf{Grp}$.

Proof. Consider the following diagram:



It is a simple computation to show that $F^\Phi(g) \circ F^\Phi(f) \circ \eta_A^\Phi = \eta_C^\Phi \circ g \circ f$ in order to use uniqueness of $F^\Phi(g \circ f)$. □

Lemma 3.23. *(F^Φ is faithful)*
The free group functor $F^\Phi : \mathbf{FinSet} \to \mathbf{Grp}$ is faithful.

Proof. Assume that $F^\Phi(f) = F^\Phi(g)$ for some $f, g : A \to B$ between finite sets. Given some $x : A$ we want to show that $f(x) = g(x)$. By Theorem 3.22 we get that

$$[f(x)] \doteq \overline{\eta_B^\Phi \circ f}^\Phi([x]) \doteq F^\Phi(f, [x])$$
$$\doteq F^\Phi(f, [y]) \doteq \overline{\eta_B^\Phi \circ f}^\Phi([y]) \doteq [f(y)]$$

and so we're done. □

3.2. Free Group on Set X

Given an arbitrary set X without the assumption of decidable equality, we cannot proceed like in the previous Section 3.1. We instead construct the free group as a filtered colimit.

3.2.1. Comma Category $\mathbf{FinSet} \downarrow X$

Recall the comma category $\mathbf{FinSet} \downarrow X$ (also category of elements of the presheaf $\mathrm{Hom}_{\mathbf{Set}}(\iota(-), X)$). As objects we have pairs (A, f) where A is a finite set, and $f : A \to X$; a morphism from $(A, f) \to (B, g)$ is a map $h : A \to B$ such that $f = g \circ h$. Because maps into sets are sets, equalities between morphisms reduce to equalities between the maps.

In particular, this category is filtered. To show this, our approach is going to be the following:

1. For each of the coproduct and coequalizer cases we are going to give only the objects and morphisms in $\mathbf{FinSet}$;
2. We show that these objects and morphisms are the coproduct/coequalizer in $\mathbf{Set}$;
3. Finally we go back to $\mathbf{FinSet} \downarrow X$ to show filteredness.

> **Lemma 3.24.** *($\mathbf{FinSet} \downarrow X$ has an object)*
> There is an object (A, f) in $\mathbf{FinSet} \downarrow X$.

Proof. We simply take $A := \emptyset$, and let $f :=\, ¡ : \emptyset \to X$. □

> **Lemma 3.25.** *($\mathbf{FinSet} \downarrow X$ has upper bounds of objects)*
> For any $(A, f), (B, g)$ there exists an object (C, h) and morphisms $\iota_A : (A, f) \to (C, h), \iota_B : (B, g) \to (C, h)$.

Proof. We pick $A + B$ in $\mathbf{FinSet}$, along with the appropriate inclusions ι_A, ι_B. This is clearly the coproduct in $\mathbf{Set}$ as well, and so we use its universal property to acquire a map $f + g : A + B \to X$ so that $(f + g) \circ \iota_A = f$ and $(f + g) \circ \iota_B = g$. □

> **Lemma 3.26.** *(Decidable subtypes of finite types are finite)*
> Given a finite type A, and a decidable subtype $P(x)$ (valued in Prop) for all $x : A$, $\sum_{x:A} P(x)$ is a finite type as well.

Proof. Because A is finite, we get that $A \simeq [n]$.

Consider when $n \doteq 0$, and thus $A \simeq \emptyset$; $\sum_{x:\emptyset} P(x)$ is also empty, and thus equivalent to $[0]$.

When $n \doteq n + 1$, note first that $P \circ \iota_{[n]}$ is a decidable subtype of $[n]$ valued in Prop; thus we get that $\sum_{x:[n]} P\left(\iota_{[n]}(x)\right) \simeq [m]$. Note that

$$\sum_{x:[n+1]} P(x) \simeq \sum_{x:[n]} P\left(\iota_{[n]}(x)\right) + P(\iota_{\mathbb{1}}(\star))$$

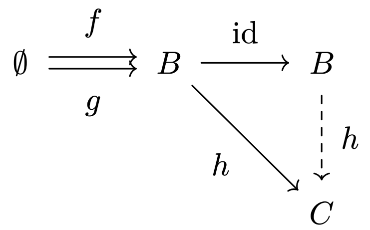
and so we may consider whether $P(\iota_{\mathbb{1}}(\star))$.

If $p : P(\iota_{\mathbb{1}}(\star))$, then we let $\sum_{x:[n+1]} P(x) \simeq [m + 1]$, and otherwise just $\sum_{x:[n+1]} P(x) \simeq [m]$. □

> **Lemma 3.27.** *(Finite coequalizer in $\mathbf{Set}$)*
> For any two maps $f, g : A \to B$ between finite sets, there is a finite coequalizer $h : B \to C$ (meaning C is finite) such that this is the coequalizer in $\mathbf{Set}$.

Proof. We are going to use the fact that $A \simeq [n]$ for some $n : \mathbb{N}$, to give a coequalizer $h : B \to C$ in $\mathbf{Set}$ (where C is finite).

When $n \doteq 0$, because any two maps out of $\emptyset$ are identifiable, there is nothing to coequalize in f, g. The coequalizer is $\mathrm{id} : B \to B$, and clearly any other coequalizer will uniquely factor through this.



If $n \doteq 1$, because $(\mathbb{1} \to B) \simeq B$, f and g are each picking out one $b : B$; let them be denoted as b_f, b_g. Now, if $b_f = b_g$, then $f = g$, so there is nothing to coequalize; the coequalizer is $\mathrm{id} : B \to B$ again. On the other hand, if $p : b_f \neq b_g$, they need to be coequalized in some finite set; let this be $\sum_{b:B} b \neq b_g$. This is finite because decidable subtypes of finite sets are finite as well (Lemma 3.26).

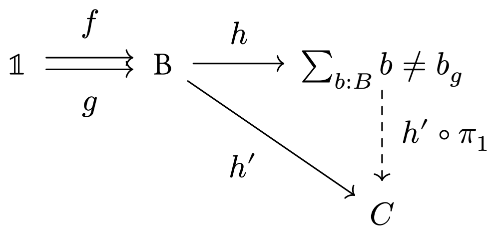
The coequalizer $h : B \to \sum_{b:B} b \neq b_g$ itself we define as follows:

$$b_g \mapsto (b_f, p)$$
$$b, q : b \neq b_g \mapsto (b, q)$$

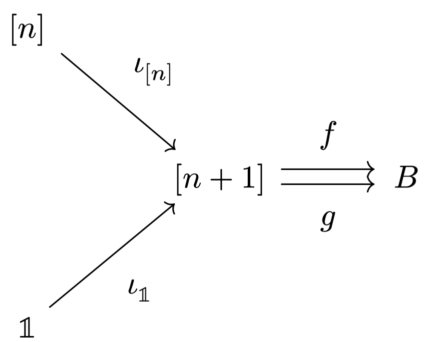
It certainly sends b_f, b_g to the same first component, and we disregard the second because it is a proposition.

Given any other coequalizer $h' : B \to C$, we set the factor to $u := h' \circ \pi_1$. For any $b = b_g$ we have $h'(b_g) = h'(b_f) = h'(\pi_1(h(b_g)))$; conversely, for any $b \neq b_g$, we have $h'(b) = h'(\pi_1(h(b)))$.

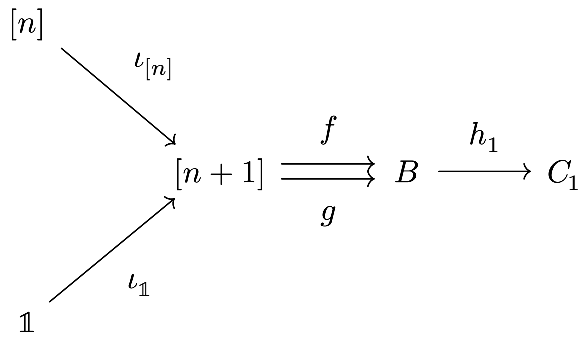
To identify u with any other factor u' such that $u' \circ h = h' = u \circ h$, it suffices to show that h is surjective and thus an epimorphism (right-cancellative). The fiber over any (b, p) is given by b, and precisely because $p : b \neq b_g$ it is the case that $h(b) = (b, p)$.



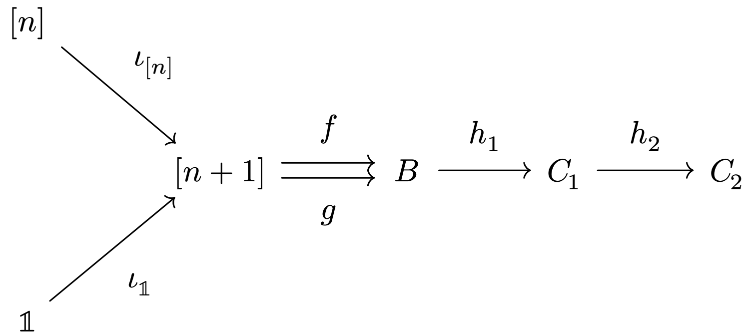
Finally, if $n \doteq n + 1$, consider the following diagram:



By inductive hypothesis we get a coequalizer $h_1 : B \to C_1$ for $f \circ \iota_{[n]}$ and $g \circ \iota_{[n]}$:

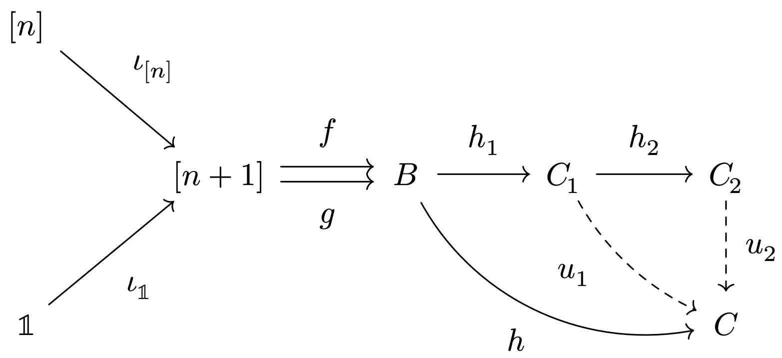


Due to having already shown the $n \doteq 1$ case, we get a coequalizer $h_2 : C_1 \to C_2$ for $h_1 \circ f \circ \iota_{\mathbb{1}}$ and $h_1 \circ g \circ \iota_{\mathbb{1}}$:



We pick $h_2 \circ h_1$ as our coequalizer for f and g, because we've shown that both $h_2 \circ h_1 \circ f$ and $h_2 \circ h_1 \circ g$ agree on $\iota_{\mathbb{1}}$ and $\iota_{[n]}$.

Given any other morphism $h : B \to C$ such that $h \circ f = h \circ g$, by the universal property for h_1 we get a unique factor $u_1 : C_1 \to C$ so that $u_1 \circ h_1 = h$. Then, by the universal property for h_2 we get a unique factor $u_2 : C_2 \to C$ such that $u_2 \circ h_2 = u_1$.

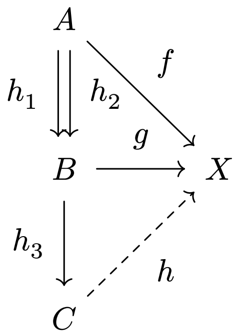


Assume any other factor $u' : C_2 \to C$ such that $u' \circ h_2 \circ h_1 = h$. Uniqueness of u_1 gives us that $u' \circ h_2 = u_1$, which in turn gives us $u' = u_2$ by uniqueness of u_2. □

Lemma 3.28. *(**FinSet** $\downarrow X$ has upper bounds of morphisms)*
For any two parallel morphisms $h_1, h_2 : (A, f) \rightrightarrows (B, g)$ there exists a morphism $h_3 : (B, g) \to (C, h)$ such that $h_3 \circ h_1 = h_3 \circ h_2$.

Proof. Due to Lemma 3.27 , we get a coequalizer (in **Set**) $h_3 : B \to C$ so that $h_3 \circ h_1 = h_3 \circ h_2$. Because $g \circ h_1 = f = g \circ h_2$, by the universal property we get a unique factor

$h : C \to X$ so that $h \circ h_3 = g$, precisely what we need to make this a coequalizer in $\mathbf{FinSet} \downarrow X$.



□

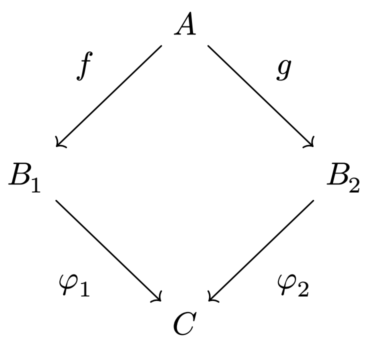
Theorem 3.29. *(Filteredness of* $\mathbf{FinSet} \downarrow X$*)*
The category $\mathbf{FinSet} \downarrow X$ is filtered.

Proof. This follows from Lemma 3.24, Lemma 3.25 and Lemma 3.28. □

3.2.2. Some Properties Of Filtered Categories

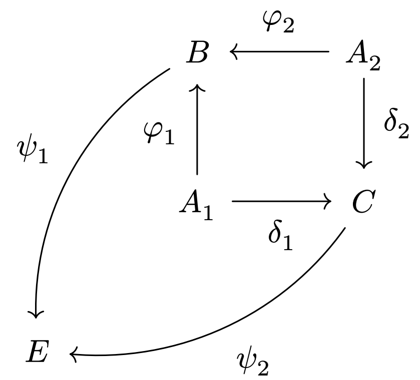
We proceed by showing two key properties of filtered categories in general.

Lemma 3.30. *(Filtered diamond completion)*
Given any maps $f : A \to B_1, g : A \to B_2$ in a filtered category, there is an object C along with maps $\varphi_1 : B_1 \to C, \varphi_2 : B_2 \to C$ so that the following diagram commutes:



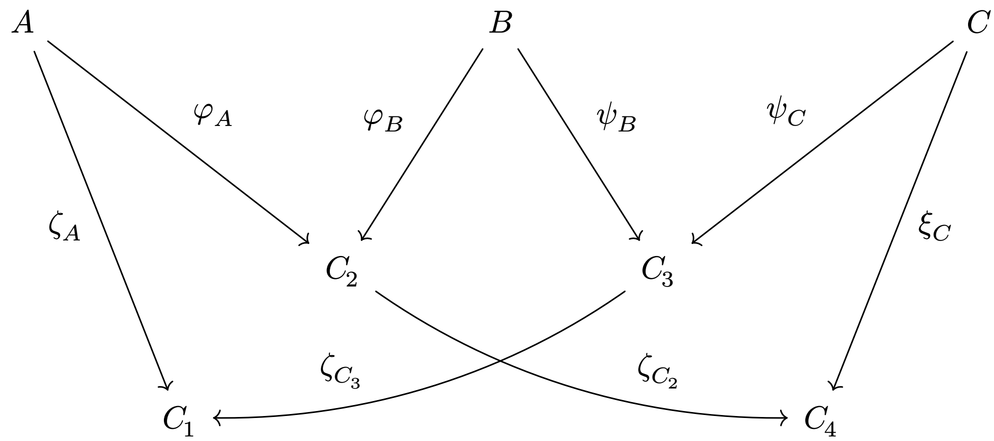
Proof. We begin by acquiring an upper bound E_1 and maps $\varphi_1 : B_1 \to E_1, \varphi_2 : B_2 \to E_1$. But then we have parallel maps $\varphi_1 \circ f, \varphi_2 \circ g : A \rightrightarrows E_1$, and so get a coequalizer $\gamma : E_1 \to E_2$. Thus we pick E_2 as the object and maps $\gamma \circ \varphi_1, \gamma \circ \varphi_2$. □

Lemma 3.31. *(Filtered scissors completion)*
For any maps $\varphi_1 : A_1 \to B, \delta_1 : A_1 \to C, \varphi_2 : A_2 \to B$ and $\delta_2 : A_2 \to C$ in a filtered category, there is an object E along with maps $\psi_1 : B_1 \to E, \psi_2 : B_2 \to E$ so that the following diagram commutes:



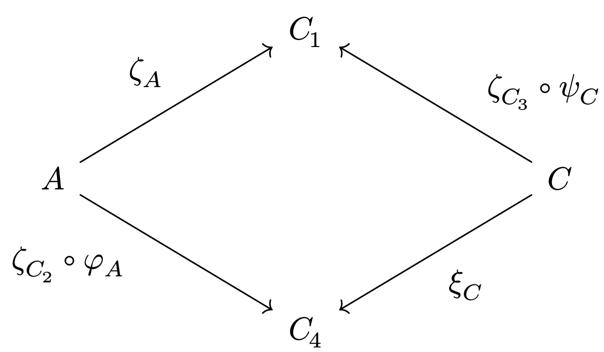
Proof. By Lemma 3.30 we can complete the diamond for A_1 and get maps $\phi_B : B \to E_1, \phi_C : C \to E_1$ so that $\phi_B \circ \varphi_1 = \phi_C \circ \delta_1$. This then gives us two parallel maps $\phi_B \circ \varphi_2, \phi_C \circ \delta_2 : A_2 \rightrightarrows E_1$, so we acquire a coequalizer $\gamma : E_1 \to E_2$. □

Lemma 3.32. *(Filtered diamond-scissors completion)*
Given any filtered diagram

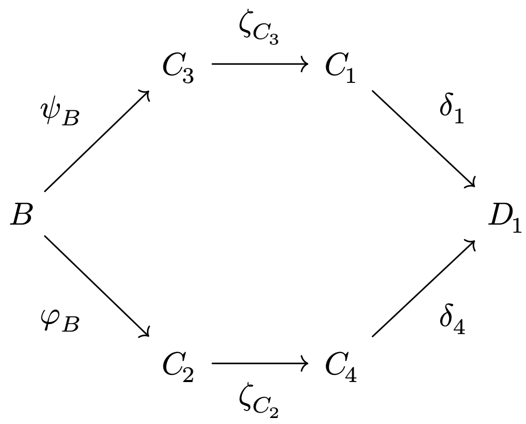


we can find a completing object D and maps from C_1, C_2, C_3 and C_4 into D so that the resulting diagram commutes.

Proof. One first uses Lemma 3.31 on the following subdiagram:



We acquire maps $\delta_1 : C_1 \to D_1$ and $\delta_4 : C_4 \to D_1$. The only non-commuting subdiagram left is the inner one starting at B:



But because we are in a filtered category we can obtain a coequalizer for these maps and we're done.

□

3.2.3. Construction of $F(X)$

Consider the diagram $J : (\mathbf{FinSet} \downarrow X) \to \mathbf{Grp}$, defined as follows:

$$\begin{array}{ccc} (A, f) & \longmapsto & F^{\Phi}(A) \\ h \big\downarrow & \longmapsto & \big\downarrow F^{\Phi}(h) \\ (B, g) & \longmapsto & F^{\Phi}(B) \end{array}$$

Our aim is to construct the free group on X as precisely the filtered colimit of this diagram:

$$\operatorname*{colim}_{(A,f) \colon \mathbf{FinSet} \downarrow X} (J(A, f) = F^{\Phi}(A)).$$

Consider the following type:

$$F^{\mathrm{P}}(X) :\equiv \sum_{A \colon \mathbf{FinSet}} (A \to X) \times F^{\Phi}(A).$$

This is essentially $\mathbf{FinSet} \downarrow X$ where the objects are decorated with a term in the free group constructed in Section 3.1. Next consider the relation on this type given by

$$\begin{gathered} (A, f, u) \sim (B, g, v) \\ :\equiv \\ \left\| \sum_{\substack{C \colon \mathbf{FinSet}, \\ h : C \to X}} \sum_{\substack{\iota_A : (A,f) \to (C,h), \\ \iota_B : (B,g) \to (C,h)}} F^{\Phi}(\iota_A, u) = F^{\Phi}(\iota_B, v) \right\| \end{gathered}$$

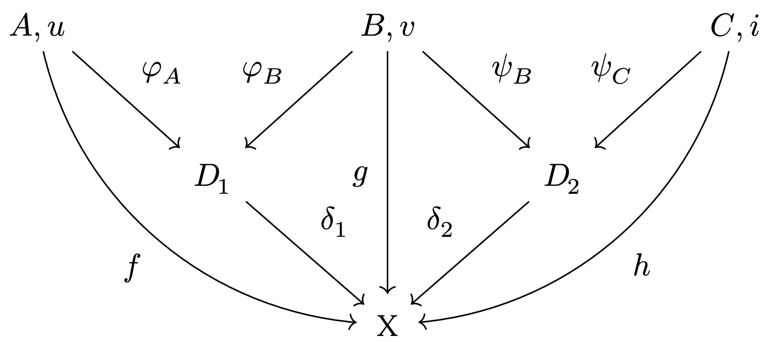
where $\|-\|$ is the type former for propositional truncation. We can think of this as *filtering* the terms u and v into a common finite set C so that they coincide there.

Lemma 3.33. *($\sim$ is an equivalence relation)*
$\sim$ is reflexive, symmetric and transitive.

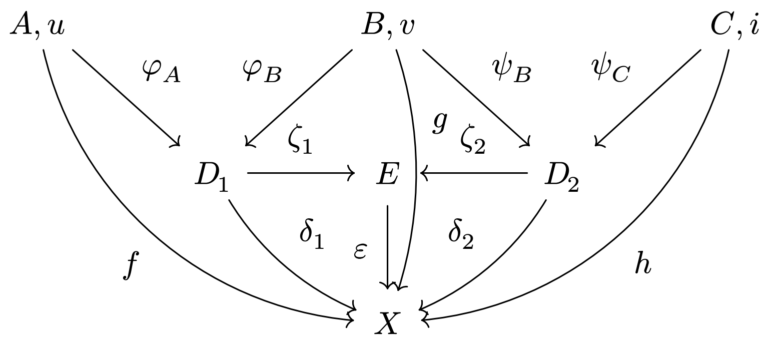
Proof. It is clearly the case that $(A, f, u) \sim (A, f, u)$ by picking (A, f) along with the identity morphism(s).

If $(A, f, u) \sim (B, g, v)$, all the data for $(B, g, v) \sim (A, f, u)$ is already given.

Assume that $(A, f, u) \sim (B, g, v)$ and $(B, g, v) \sim (C, h, i)$ and consider the following diagram:



In particular we have that $F^{\Phi}(\varphi_A, u) = F^{\Phi}(\varphi_B, v)$ and $F^{\Phi}(\psi_B, v) = F^{\Phi}(\psi_C, i)$. By Lemma 3.30 we can extend it as follows:



We pick E along with maps $\zeta_1 \circ \varphi_A$ and $\zeta_2 \circ \psi_C$; due to functoriality in combination with $\zeta_1 \circ \varphi_B = \zeta_2 \circ \psi_B$ we get

$$
\begin{aligned}
F^{\Phi}(\zeta_1 \circ \varphi_A, u) &= F^{\Phi}(\zeta_1, F^{\Phi}(\varphi_A, u)) \\
&= F^{\Phi}(\zeta_1, F^{\Phi}(\varphi_B, v)) \\
&= F^{\Phi}(\zeta_1 \circ \varphi_B, v) \\
&= F^{\Phi}(\zeta_2 \circ \psi_B, v) \\
&= F^{\Phi}(\zeta_2, F^{\Phi}(\psi_B, v)) \\
&= F^{\Phi}(\zeta_2, F^{\Phi}(\psi_C, i)) \\
&= F^{\Phi}(\zeta_2 \circ \psi_C, i).
\end{aligned}
$$

□

We proceed by constructing "raw" group operations on $F^{\mathrm{P}}(X)$ to then show that these are group operations up to $\sim$.

Definition 3.34. *(Unit of $F^{\mathrm{P}}(X)$)*
We define $1 : F^{\mathrm{P}}(X)$ to be $(\emptyset, ¡, 1)$, where $¡ : \emptyset \to X$.

Definition 3.35. *(Inversion of $F^{\mathrm{P}}(X)$)*
Given (A, f, u) we define its inverse by

$$(A, f, u)^{-1} := (A, f, u^{-1}).$$

Definition 3.36. *(Multiplication of $F^{\mathrm{P}}(X)$)*
Given (A, f, u) and (B, g, v), by Theorem 3.29 we get an upper bound $(A + B, f + g)$. We thus define the multiplication:

$$(A, f, u)(B, g, v) := (A + B, f + g, F^{\Phi}(\iota_A, u)F^{\Phi}(\iota_B, v)).$$

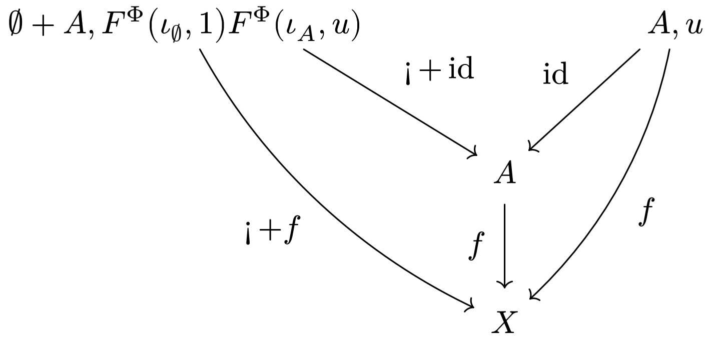
Lemma 3.37. *(Left & Right Unit ($\sim$))*
For any (A, f, u) it is the case that

$$\begin{aligned} 1(A, f, u) &\sim (A, f, u) \\ &\sim (A, f, u)1 \end{aligned}$$

Proof. For the left case, we want an upper bound for the objects

$$(\emptyset + A, ¡ + \mathrm{id}, F^{\Phi}(\iota_{\emptyset}, 1)F^{\Phi}(\iota_A, u)), (A, f, u).$$

We simply pick (A, f); from $\emptyset + A \to A$ we use $¡ + \mathrm{id}$ and id from $A \to A$; by functoriality and computation we get $F^{\Phi}(\iota_{\emptyset}, 1)F^{\Phi}(¡ + \mathrm{id} \circ \iota_A, u) = F^{\Phi}(\mathrm{id}, u)$. The maps into X are given by f and $¡ + f$.



For the right case we want an upper bound for the objects

$$(A + \emptyset, \mathrm{id} + ¡, F^{\Phi}(\iota_A, u)F^{\Phi}(\iota_{\emptyset}, 1)), (A, f, u)$$

which we arrive at by similar reasoning. □

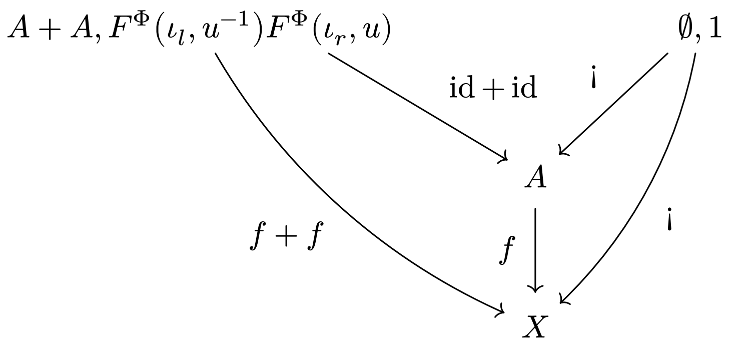
Lemma 3.38. *(Left & Right Inverse (∼))*
For any (A, f, u) it is the case that

$$(A, f, u)^{-1}(A, f, u) \sim 1 \sim (A, f, u)(A, f, u)^{-1}$$

Proof. An upper bound for

$$(A + A, F^{\Phi}(\iota_l, u^{-1})F^{\Phi}(\iota_r, u)), (\emptyset, 1)$$

is obtained by picking (A, f):



Functoriality and group laws again give us

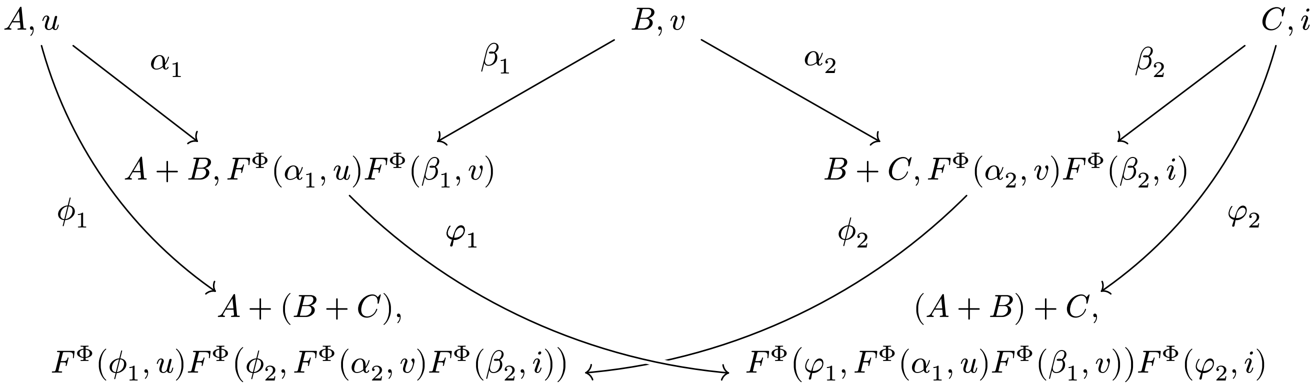
$$\begin{aligned} F^{\Phi}(\mathrm{id} + \mathrm{id}, F^{\Phi}(\iota_l, u^{-1})F^{\Phi}(\iota_r, u)) &= F^{\Phi}(\mathrm{id} + \mathrm{id} \circ \iota_l, u^{-1})F^{\Phi}(\mathrm{id} + \mathrm{id} \circ \iota_r, u) \\ &\doteq F^{\Phi}(\mathrm{id}, u^{-1})F^{\Phi}(\mathrm{id}, u) \\ &\doteq u^{-1}u \\ &= 1 \\ &= F^{\Phi}(¡, 1). \end{aligned}$$

The right case is worked out in a similar manner. □

Lemma 3.39. *(Associativity of Multiplication (∼))*
For any $(A, f, u), (B, g, v), (C, h, i)$ it is the case that

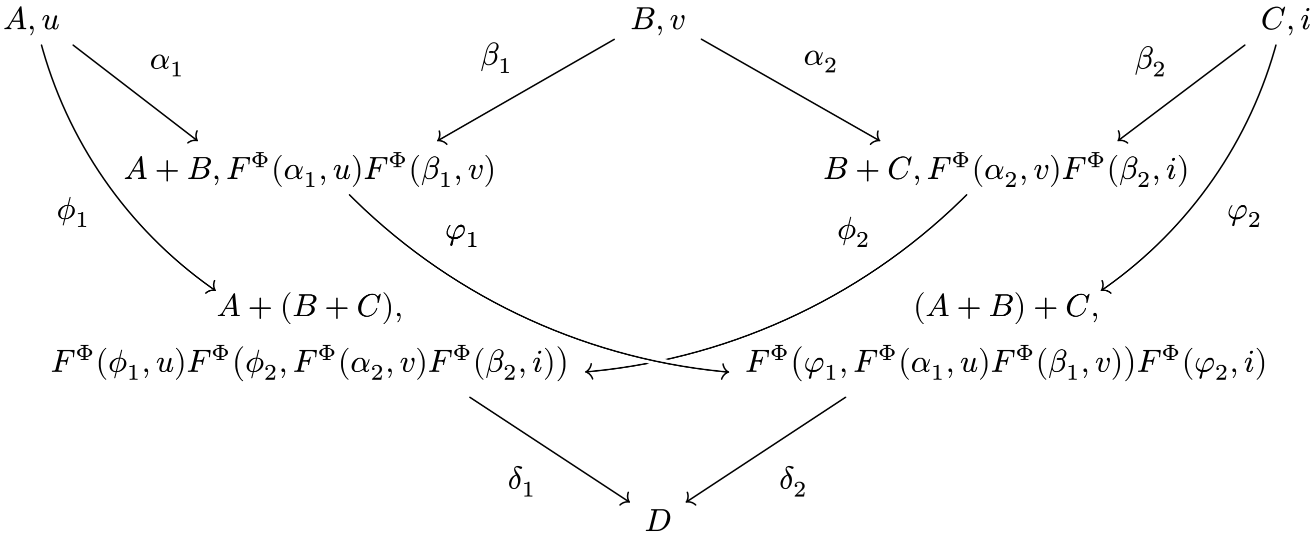
$$(A, f, u)\ \ ((B, g, v)(C, h, i)) \sim ((A, f, u)(B, g, v))\ \ (C, h, i).$$

Proof. Consider the following diagram:



Note that we omit the maps into X in the diagram to save space, and as they are easily worked out.

By Lemma 3.32 we get maps $\delta_1 : A + (B + C) \to D, \delta_2 : (A + B) + C$ so that the diagram commutes.



It remains to show that

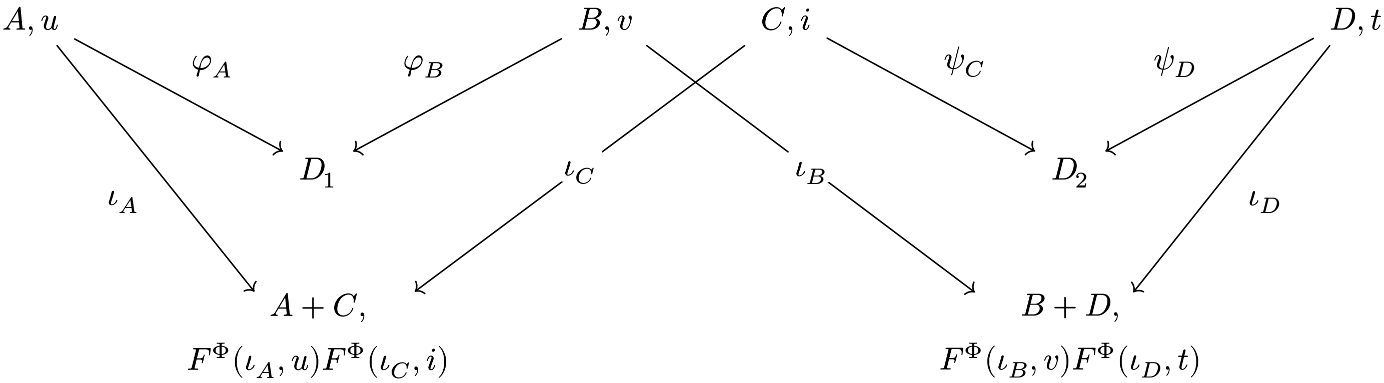
$$F^\Phi(\delta_1 \circ \phi_1, u)F^\Phi(\delta_1 \circ \phi_2 \circ f_2, v)F^\Phi(\delta_1 \circ \phi_2 \circ g_2, i)$$
$$=$$
$$F^\Phi(\delta_2 \circ \varphi_1 \circ f_1, u)F^\Phi(\delta_2 \circ \varphi_1 \circ g_1, v)F^\Phi(\delta_2 \circ \varphi_2, i)$$

which follows from the commutativity of the diagram. □

Lemma 3.40. *($\sim$ is a congruence on $F^{\mathrm{P}}(X)$)*
The equivalence relation $\sim$ is a congruence relation on $F^{\mathrm{P}}(X)$; for any $g_1 \sim g_2$ and $h_1 \sim h_2$ it is the case that

$$g_1 h_1 \sim g_2 h_2.$$

Proof. Consider the following diagram:



Again note the absence of the maps into X.

We take the upper bound for the multiplications to be $D_1 + D_2$ and first define the inclusion $\delta_1 : A + C \to D_1 + D_2$ as

$$\begin{aligned} \iota_A(a) &\mapsto \iota_{D_1}(\varphi_A(a)) \\ \iota_C(c) &\mapsto \iota_{D_2}(\psi_C(c)) \end{aligned}$$

and $\delta_2 : B + D \to D_1 + D_2$ as

$$\begin{aligned} \iota_B(b) &\mapsto \iota_{D_1}(\varphi_B(b)) \\ \iota_D(d) &\mapsto \iota_{D_2}(\psi_D(d)). \end{aligned}$$

By functoriality and computation in combination with $F^\Phi(\varphi_A, u) = F^\Phi(\varphi_B, v)$ and $F^\Phi(\psi_C, i) = F^\Phi(\psi_D, t)$ we get the equality of terms we want in $D_1 + D_2$:

$$\begin{aligned} F^\Phi(\delta_1, F^\Phi(\iota_A, u)F^\Phi(\iota_C, i)) &= F^\Phi(\delta_1 \circ \iota_A, u)F^\Phi(\delta_1 \circ \iota_C, i) \\ &\doteq F^\Phi\left(\iota_{D_1} \circ \varphi_A, u\right)F^\Phi\left(\iota_{D_2} \circ \psi_C, i\right) \\ &= F^\Phi\left(\iota_{D_1}, F^\Phi(\varphi_A, u)\right)F^\Phi\left(\iota_{D_2}, F^\Phi(\psi_C, i)\right) \\ &= F^\Phi\left(\iota_{D_1}, F^\Phi(\varphi_B, v)\right)F^\Phi\left(\iota_{D_2}, F^\Phi(\psi_D, t)\right) \\ &= F^\Phi\left(\iota_{D_1} \circ \varphi_B, v\right)F^\Phi\left(\iota_{D_2} \circ \psi_D, t\right) \\ &\doteq F^\Phi(\delta_2 \circ \iota_B, v)F^\Phi(\delta_2 \circ \iota_D, t) \\ &= F^\Phi(\delta_2, F^\Phi(\iota_B, v)F^\Phi(\iota_D, t)). \end{aligned}$$

□

As the underlying set of $F(X)$ we now use $F^{\mathrm{P}}(X)/{\sim}$.

Definition 3.41. *(Unit)*
The unit $1 : F(X)$ is defined to be $[(\emptyset,$¡$, 1)]$.

Definition 3.42. *(Inversion)*
Given any (A, f, u) we define its inversion as follows:

$$[(A,f,u)^{-1}]$$

Due to Lemma 3.40 this agrees on equivalent objects.

Definition 3.43. *(Multiplication)*
We define multiplication in two parts.

First, assume a fixed (A,f,u), to construct the map $[(A,f,u)](-) : F(X) \to F(X)$ as follows:

$$(B,g,v) \mapsto [(A,f,u)(B,g,v)].$$

Next, we want to define $(-)(-) : F(X) \to F(X) \to F(X)$; so we give

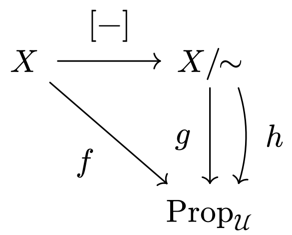
$$(A,f,u) \mapsto [(A,f,u)](-)$$

where the given map is the one constructed in the previous part.

Both the outer and inner maps send equivalent objects to the same term(s) due to Lemma 3.40.

Lemma 3.44. *(Surjectivity of* $[-]$*)*
For any $a : X/{\sim}$ there merely exists a $x : X$ such that $a = [x]$.

Proof. Consider the following triangle:



where

$$f(x) := \mathbb{1}$$
$$g(a) := \mathbb{1}$$
$$h(a) := \exists_{x:X} a = [x]$$

Both of g, h make the triangle commute, and by the universal property we have $\mathbb{1} = \exists_{x:X} a = [x]$. ☐

In particular, if we have $P : X/{\sim} \to \mathrm{Prop}_{\mathcal{U}}$ and a map $f : \prod_{x:X} P([x])$ then we can construct $f' : \prod_{x:X/\sim} P(x)$ with Lemma 3.44 (by transport). We will use Lemma 3.44 and this consequence silently from now on (in particular when showing that the group laws hold for this type).

Lemma 3.45. *(Left & Right Unit)*
For any $g : F(X)$ it is the case that

$$1g = g = g1.$$

Proof. Follows from Lemma 3.37. □

Lemma 3.46. *(Left & Right Inverse)*
For any $g : F(X)$ it is the case that

$$g^{-1}g = 1 = gg^{-1}.$$

Proof. Follows from Lemma 3.38. □

Lemma 3.47. *(Associativity of Multiplication)*
For any $g_1, g_2, g_3 : F(X)$ it is the case that

$$g_1(g_2g_3) = (g_1g_2)g_3.$$

Proof. Follows from Lemma 3.39. □

It remains to show that $F(X)$ in fact is the free group on X.

Definition 3.48. *(Sending X to $F(X)$)*
The map $\eta_X : X \to F(X)$ is given by

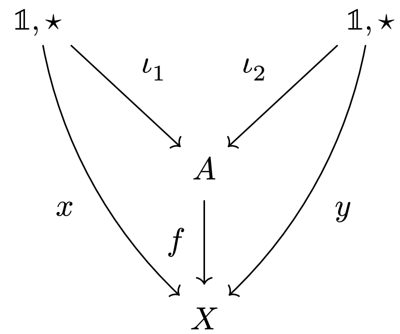
$$x \mapsto [(\mathbb{1}, x, \star)].$$

Theorem 3.49. *(η_X is injective)*
$\eta_X : X \to F(X)$ is injective.

Proof. Assume given $x, y : X$ that

$$\begin{aligned} \eta_X(x) = \eta_X(y) &\simeq [(\mathbb{1}, x, \star)] = [(\mathbb{1}, y, \star)] \\ &\simeq (\mathbb{1}, x, \star) \sim (\mathbb{1}, y, \star). \end{aligned}$$

Consider the given diagram:

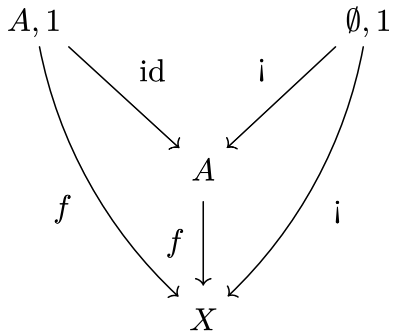


We have that $F^\Phi(\iota_1, \star) = F^\Phi(\iota_2, \star)$. Because $F^\Phi(\mathbb{1})$ is the trivial group, this gives us that $F^\Phi(\iota_1) = F^\Phi(\iota_2)$, and because F^Φ is faithful (Lemma 3.23) we get that $\iota_1 = \iota_2$. Because $f \circ \iota_1 = x$ and $f \circ \iota_2 = y$ this gives us $x = y$. □

Lemma 3.50. *(Underlying* 1*)*
For any A, f we have

$$(A, f, 1) \sim (\emptyset, ¡, 1).$$

Proof. We simply pick A as the upper bound, and the self-evident maps:



□

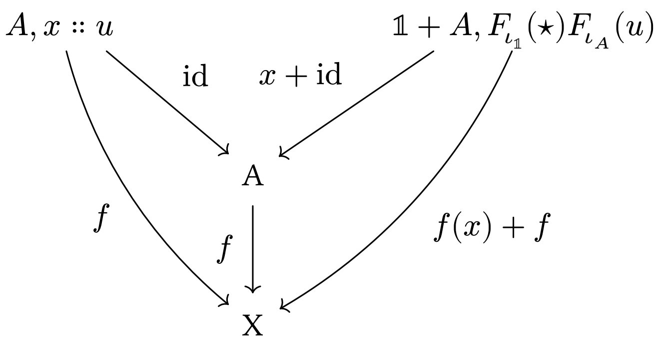
Lemma 3.51. *(Underlying multiplication)*
For any $A, f, x :: u$ it is the case that

$$(A, f, x :: u) \sim (\mathbb{1}, f(x), \star)(A, f, u)$$

Proof. We pick A again; the maps and their commutativity warrant some further explaining. First note that

$$(\mathbb{1}, f(x), \star)(A, f, u)$$
$$\doteq$$
$$\big(\mathbb{1} + A, f(x) + f, F^{\Phi}(\iota_1, \star)F^{\Phi}(\iota_A, u)\big).$$

The map from $\mathbb{1} + A \to A$ is given by $x + \text{id}$, and from $\mathbb{1} + A \to X$ by $f(x) + f$.



Showing that

$$F^{\Phi}(\text{id}, x :: u) = F^{\Phi}\big(x + \text{id}, F^{\Phi}(\iota_{\mathbb{1}}, \star)F^{\Phi}(\iota_A, u)\big)$$

reduces to showing

$$x :: u = xu$$

which is the case since $x :: u$ is reduced (Lemma 3.5). □

Lemma 3.52. *(Underlying* 1 *and multiplication)*
For any homomorphism $h : F(X) \to G$ it is the case that

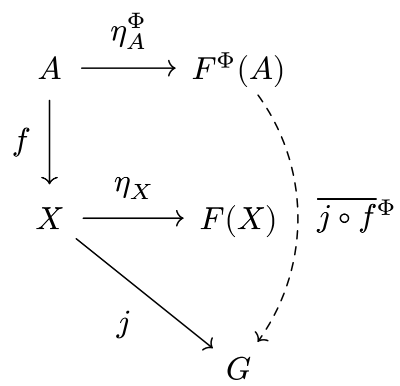
$$h([(A, f, 1)]) = 1$$

and

$$h([(A, f, x :: u)]) = h(\eta_X(f(x)))h([(A, f, u)]).$$

Proof. Due to Lemma 3.50 we get $[(A, f, 1)] = 1$, and so h has to send this to 1. Similarly, due to Lemma 3.51 we get that $[(A, f, x :: u)] = [(\mathbb{1}, f(x), \star)(A, f, u)]$, so the second equality holds by functoriality. □

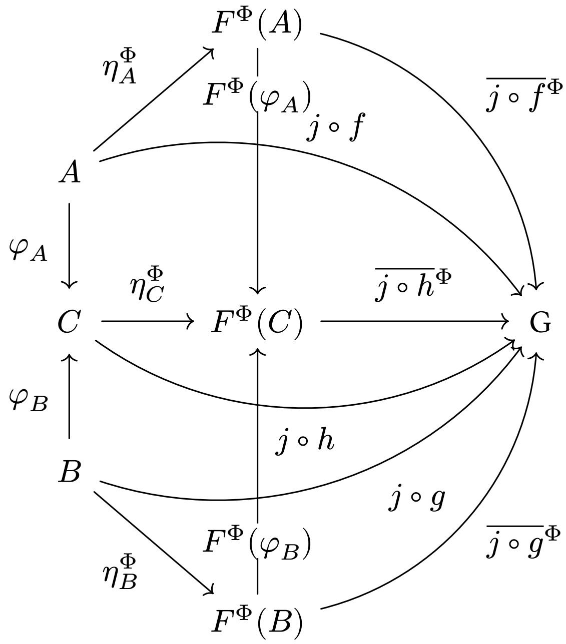
Definition 3.53. *(Extending map into group)*
Given any $j : X \to G$ into a group G we define $\overline{j} : F(X) \to G$. Consider the following diagram:



We give

$$[(A, f, u)] \mapsto \overline{j \circ f^\Phi}(u).$$

Given any $(A, f, u) \sim (B, g, v)$ we want that $\overline{j \circ f^\Phi}(u) = \overline{j \circ g^\Phi}(v)$. Let (C, h) be the upper bound of A and B, and consider the following commutative diagram:



Because $F^\Phi(\varphi_A, u) = F^\Phi(\varphi_B, v)$ it suffices to show that

$$\begin{aligned}\overline{j \circ f}^\Phi &= \overline{j \circ h}^\Phi \circ F^\Phi(\varphi_A) \\ &= \overline{j \circ h}^\Phi \circ F^\Phi(\varphi_B) \\ &= \overline{j \circ g}^\Phi.\end{aligned}$$

All of these maps are homomorphisms, so it suffices to show the equalities under postcomposition with η_A^Φ, η_B^Φ respectively. We show the reasoning for the first case ($\overline{j \circ f}^\Phi = \overline{j \circ h}^\Phi \circ F^\Phi(\varphi_A)$), since the second ($\overline{j \circ g}^\Phi = \overline{j \circ h}^\Phi \circ F^\Phi(\varphi_B)$) is analogous; consider

$$\begin{aligned}\overline{j \circ f}^\Phi \circ \eta_A^\Phi &= j \circ h \circ \varphi_A \\ &= \overline{j \circ h}^\Phi \circ \eta_C^\Phi \circ \varphi_A \\ &= \overline{j \circ h}^\Phi \circ F^\Phi(\varphi_A) \circ \eta_A^\Phi,\end{aligned}$$

making use of existing commutativity in the diagram.

Lemma 3.54. *($\bar{j}$ is a group homomorphism)*
It is the case that $\bar{j}([(A, f, u)][(B, g, v)]) = \bar{j}([(A, f, u)])\bar{j}([(B, g, v)])$.

Proof. To show that $\bar{j}$ is a group homomorphism, we are interested in the following equality:

$$
\begin{aligned}
\overline{j}([(A,f,u)][(B,g,v)]) &\doteq \overline{j}([A+B, f+g, F^{\Phi}(\iota_A,u)F^{\Phi}(\iota_B,v)]) \\
&\doteq \overline{j\circ f+g}^{\Phi}(F^{\Phi}(\iota_A,u)F^{\Phi}(\iota_B,v)) \\
&= \left(\overline{j\circ f+g}^{\Phi}\circ F^{\Phi}(\iota_A)\right)(u)\left(\overline{j\circ f+g}^{\Phi}\circ F^{\Phi}(\iota_B)\right)(v) \\
&\overset{?}{=} \overline{j\circ f}^{\Phi}(u)\overline{j\circ g}^{\Phi}(v) \\
&\doteq \overline{j}([(A,f,u)])\overline{j}([(B,g,v)]).
\end{aligned}
$$

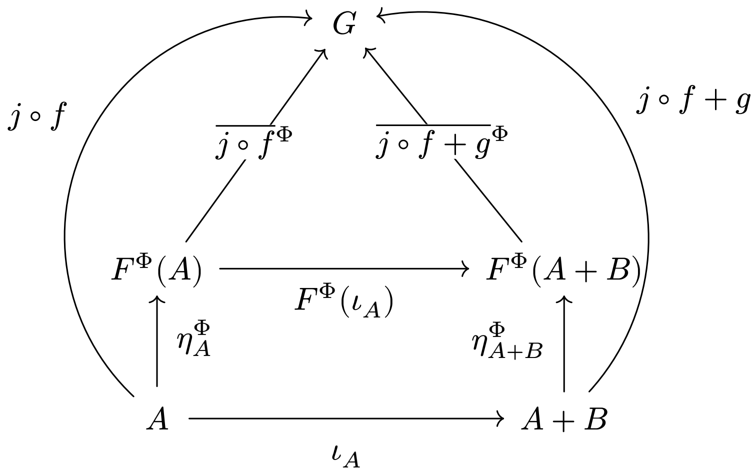
Thus, because the maps in question are homomorphisms, it suffices to show that

$$\overline{j\circ f+g}^{\Phi}\circ F^{\Phi}(\iota_A)\circ\eta_A^{\Phi} = \overline{j\circ f}^{\Phi}\circ\eta_A^{\Phi}$$

and

$$\overline{j\circ f+g}^{\Phi}\circ F^{\Phi}(\iota_B)\circ\eta_B^{\Phi} = \overline{j\circ g}^{\Phi}\circ\eta_B^{\Phi}$$

To make use of the universal property of the free group on finite sets, consider the following diagram:



By Theorem 3.20 we get the first desired equality:

$$
\begin{aligned}
\overline{j\circ f+g}^{\Phi}\circ F^{\Phi}(\iota_A)\circ\eta_A^{\Phi} &= \overline{j\circ f+g}^{\Phi}\circ\eta_{A+B}^{\Phi}\circ\iota_A \\
&= j\circ f+g\circ\iota_A \\
&\doteq j\circ f \\
&= \overline{j\circ f}^{\Phi}\circ\eta_A^{\Phi}.
\end{aligned}
$$

The other equality is given similarly. □

> **Theorem 3.55.** *($F(X)$ is the free group on X)*
> $F(X)$ is the universal group generated by X; any map $j : X \to G$ into a group factors uniquely through η_X.

Proof. Given any $j : X \to G$ into a group G, let $\overline{j} : F(X) \to G$ be as defined in Definition 3.53, being a group homomorphism (Lemma 3.54).

We finally show that $\bar{j}$ is unique. For any other $\bar{j}' : F(X) \to G$ such that $\bar{j}' \circ \eta_X = j$ we show that $\bar{j} = \bar{j}'$ by homotopy. Given $[(A, f, u)]$ we induct on u. If $u \doteq []$ then $\bar{j}$ and $\bar{j}'$ have to agree due to Lemma 3.50. If $u \doteq x :: u$ the left side computes to $j(f(x))\bar{j}([(A, f, u)])$. Due to Lemma 3.52 and the inductive hypothesis we get

$$\begin{aligned}\bar{j}'([A, f, x :: u]) &= \bar{j}'(\eta_X(f(x)))\bar{j}'([(A, f, u)]) \\ &= j(f(x))\bar{j}([(A, f, u)]).\end{aligned}$$

□

Definition 3.56. *(Action of F on maps)*
Given a map $h : E \to D$ between arbitrary sets, there is a group homomorphism $F(h) : F(E) \to F(D)$, constructed by the universal property for quotients. We define $F(h)$ by

$$(A, f, u) \mapsto [(A, h \circ f, u)]$$

Two objects $(A, f, u) \sim (B, g, v)$ are sent to equivalent objects $(A, h \circ f, u) \sim (B, h \circ g, v)$ by action of h on paths.

We have that

$$\begin{gathered}F(h, [(A, f, u)][(B, g, v)]) \\ = \\ F(h, [(A, f, u)])F(h, [(B, g, v)])\end{gathered}$$

by computation and the homotopy $h \circ f + g = h \circ f + h \circ g$:

$$\begin{aligned}F(h, [(A, f, u)][(B, g, v)]) &\doteq F(h, [(A + B, f + g, F^{\Phi}(\iota_A, u)F^{\Phi}(\iota_B, v))]) \\ &\doteq [(A + B, h \circ f + g, F^{\Phi}(\iota_A, u)F^{\Phi}(\iota_B, v))] \\ &= [(A + B, h \circ f + h \circ g, F^{\Phi}(\iota_A, u)F^{\Phi}(\iota_B, v))] \\ &\doteq [(A, h \circ f, u)][(B, h \circ g, v)] \\ &\doteq F(h, [(A, f, u)])F(h, [(B, g, v)]).\end{aligned}$$

Lemma 3.57. *(F is a functor)*
$F : \mathbf{Set} \to \mathbf{Grp}$ is a functor.

Proof. F clearly preserves both identities and composition due to it only acting on the inner maps f in $[(A, f, u)]$:

$$\begin{aligned}F(f \circ g, [(A, f, u)]) &\doteq [(A, f \circ g \circ f, u)] \\ &\doteq F(f, [(A, g \circ f, u)]) \\ &\doteq F(f \circ g, [(A, f, u)]).\end{aligned}$$

□

> **Theorem 3.58.** *($F(X)$ is the colimit)*
> $F(X)$ is the colimit of J.

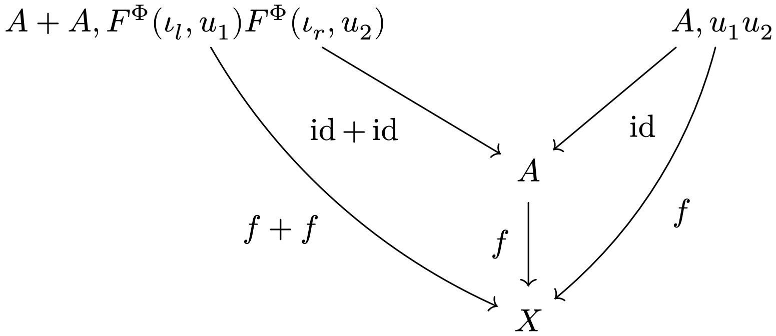
Proof. Given any (A, f) we define the leg $\varphi_{(A,f)} : F^\Phi(A) \to F(X)$ as

$$u \mapsto [(A, f, u)].$$

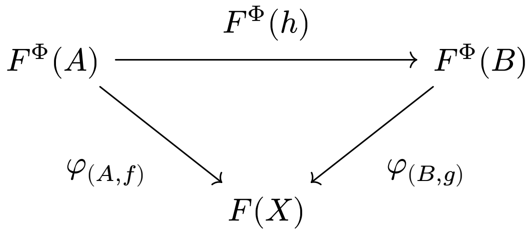
For this to be a homomorphism we require that

$$\begin{aligned}\varphi_{(A,f)}(u_1 u_2) &\doteq [(A, f, u_1 u_2)] \\ &\overset{?}{\doteq} [(A + A, f + f, F^\Phi(\iota_l, u_1) F^\Phi(\iota_r, u_2))] \\ &\doteq [(A, f, u_1)][(A, f, u_2)] \\ &\doteq \varphi_{(A,f)}(u_1)\varphi_{(A,f)}(u_2)\end{aligned}$$

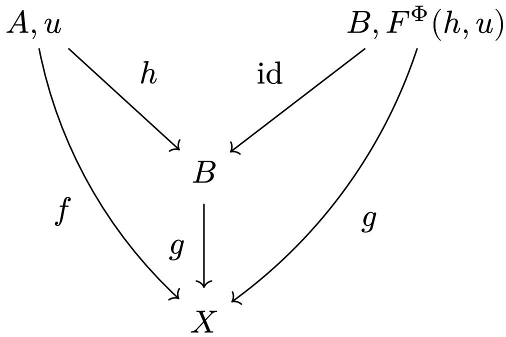
which we give by the following diagram:



Given any $h : (A, f) \to (B, g)$ we want the commutativity of the following diagram:



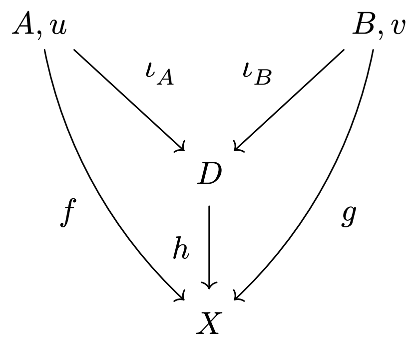
We want that $[(A, f, u)] = [(B, g, F^\Phi(h, u))]$. It is the case that $(A, f, u) \sim (B, g, F^\Phi(h, u))$, due to:



Given any other cocone (C, ϕ) we define $j : F(X) \to C$ by

$$(A, f, u) \mapsto \phi_{(A,f)}(u).$$

Given any $(A, f, u) \sim (B, g, v)$ as in:



where $F^{\Phi}(\iota_A, u) = F^{\Phi}(\iota_B, v)$ we want that

$$\begin{aligned} j([(A, f, u)]) &\doteq \phi_{(A,f)}(u) \\ &\stackrel{?}{\doteq} \phi_{(B,g)}(v) \\ &\doteq j([(B, g, v)]). \end{aligned}$$

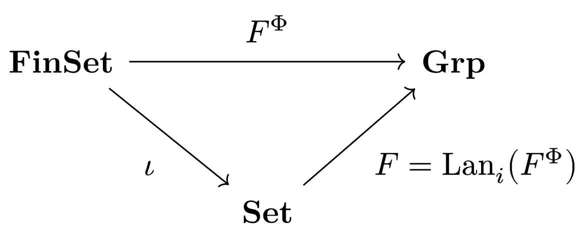
Note that we have $\phi_{(A,f)} = \phi_{(D,h)} \circ F^{\Phi}(\iota_A)$ and $\phi_{(B,g)} = \phi_{(D,h)} \circ F^{\Phi}(\iota_B)$, and so it suffices to have $F^{\Phi}(\iota_A, u) = F^{\Phi}(\iota_B, v)$ which we do.

For any other $j' : F(X) \to C$ such that $j' \circ \varphi_{(A,f)} = \phi_{(A,f)} = j \circ \varphi_{(A,f)}$ we can identify with j as follows:

$$\begin{aligned} j([(A, f, u)]) &\doteq j\big(\varphi_{(A,f)}(u)\big) \\ &= j'\big(\varphi_{(A,f)}(u)\big) \\ &\doteq j'([(A, f, u)]). \end{aligned}$$

□

Theorem 3.59.
$F : \mathbf{Set} \to \mathbf{Grp}$ is the left Kan extension of $F^{\Phi} : \mathbf{FinSet} \to \mathbf{Grp}$ along $\iota : \mathbf{FinSet} \to \mathbf{Set}$:



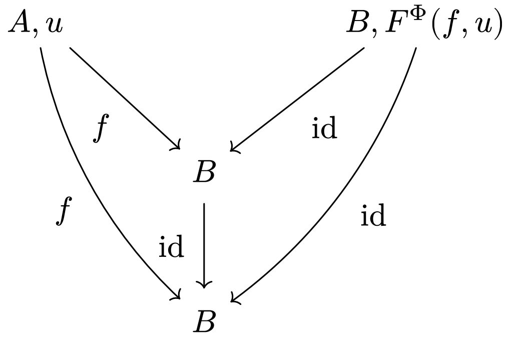
Proof. We begin by constructing the natural transformation $\alpha : F^{\Phi} \Rightarrow F\iota$, by simply reusing the legs from Theorem 3.58:

$$\alpha_A := \varphi_{(A,\,\mathrm{id})}.$$

This is a homomorphism because $\varphi_{(A,\,\mathrm{id})}$ is (Theorem 3.58). For this to be natural we need to show that $F(f) \circ \alpha_A = \alpha_B \circ F^{\Phi}(f)$, which amounts to showing that

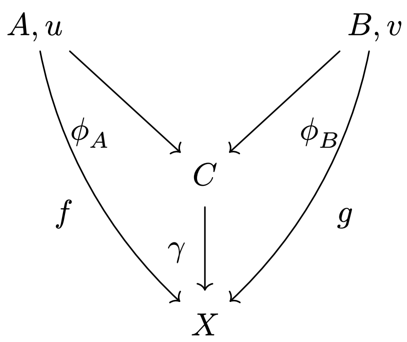
$$\begin{aligned} F(f, \alpha_A(u)) &\doteq F(f, [(A, \mathrm{id}, u)]) \\ &\doteq [(A, f, u)] \\ &\overset{?}{\doteq} [(B, \mathrm{id}, F^{\Phi}(f, u))] \\ &\doteq \alpha_B(F^{\Phi}(f, u)) \end{aligned}$$

which we give by the following diagram:



Assume another $G : \mathbf{Set} \to \mathbf{Grp}$ along with natural transformation $\eta : F^{\Phi} \Rightarrow G\iota$. To construct $\sigma : F \Rightarrow G$, consider $\sigma_X : F(X) \to G(X)$ where we are given $[(A, f, u)]$; applying $G(f) \circ \eta_A : F^{\Phi}(A) \to G(X)$ to u gives us what we want.

For any $(A, f, u) \sim (B, g, v)$ we need to show that $G(f, \eta_A(u)) = G(g, \eta_B(v))$. Notice first the upper bound:



We have the following chain of equalities:

$$\begin{aligned} G(f, \eta_A(u)) &= G(\gamma \circ \phi_A, \eta_A(u)) \\ &= G(\gamma, G(\phi_A, \eta_A(u))) \\ &= G(\gamma, \eta_C(F^{\Phi}(\phi_A, u))) \\ &= G(\gamma, \eta_C(F^{\Phi}(\phi_B, v))) \\ &= G(\gamma, G(\phi_B, \eta_B(v))) \\ &= G(\gamma \circ \phi_B, \eta_B(v)) \\ &= G(g, \eta_B(v)), \end{aligned}$$

using $\eta : F^{\Phi} \Rightarrow G\iota$ and the upper bound of A, B. This is a homomorphism because

$$\begin{aligned}
\sigma_X([(A,f,u)][(B,g,v)]) &\doteq \sigma_X([(A+B, f+g, F^{\Phi}(\iota_A,u)F^{\Phi}(\iota_B,v))]) \\
&\doteq G(f+g, \eta_{A+B}(F^{\Phi}(\iota_A,u)F^{\Phi}(\iota_B,v))) \\
&= G(f+g, \eta_{A+B}(F^{\Phi}(\iota_A,u))\eta_{A+B}(F^{\Phi}(\iota_B,v))) \\
&= G(f+g, G(\iota_A, \eta_A(u))G(\iota_B, \eta_B(v))) \\
&= G(f+g \circ \iota_A, \eta_A(u))G(f+g \circ \iota_B, \eta_B(v)) \\
&\doteq G(f, \eta_A(u))G(g, \eta_B(v)) \\
&\doteq \sigma_X([(A,f,u)])\sigma_X([(B,g,v)])
\end{aligned}$$

due to naturality of η and η_{A+B} being a homomorphism.

Naturality we give by $G(h) \circ \sigma_X = \sigma_Y \circ F(h)$ where $h : \mathrm{Hom}_{\mathbf{Set}}(X,Y)$, as follows:

$$\begin{aligned}
G(h, \sigma_X([(A,f,u)])) &\doteq G(h, G(f, \eta_A(u))) \\
&= G(h \circ f, \eta_A(h)) \\
&\doteq \sigma_Y([(A, h \circ f, u)]) \\
&\doteq \sigma_Y(F(h, [(A,f,u)])).
\end{aligned}$$

To proceed we want to show that $\sigma \circ \alpha = \eta$; to do so we give $\sigma_A \circ \alpha_A = \eta_A$:

$$\begin{aligned}
\sigma_A(\alpha_A(u)) &\doteq \sigma_A([(A, \mathrm{id}, u)]) \\
&\doteq G(\mathrm{id}, \eta_A(u)) \\
&= \eta_A(u).
\end{aligned}$$

We finish by showing that given any other $\sigma' : F \Rightarrow G$ such that $\sigma' \circ \alpha = \eta$, we have $\sigma = \sigma'$:

$$\begin{aligned}
\sigma_X([(A,f,u)]) &\doteq G(f, \eta_A(u)) \\
&= G(f, \sigma'_A(\alpha_A(u))) \\
&= \sigma'_X(F(f, \alpha_A(u))) \\
&\doteq \sigma'_X(F(f, [(A, \mathrm{id}, u)])) \\
&\doteq \sigma'_X([(A,f,u)]).
\end{aligned}$$

The reader may note that we have not given any “equality of equalities” here, justified by the fact that the maps in question are sets. □

Chapter 4
Discussion

In this thesis, we have constructed the free group on an arbitrary set in HoTT, using a categorical approach based on filtered colimits. This construction avoids the need for decidable equality, which is an unavoidable hurdle in the univalent setting, and instead leverages the structure of finite sets and their colimits.

One of the main contributions of this work is the explicit and constructive nature of the construction. Rather than relying on abstract categorical reasoning, we have provided direct proofs of the group laws and the universal property for the free group, both in the case of finite sets and in the general case. While this approach is very concrete, in the future we would like to generalize this to a more abstract approach, as outlined in Chapter 5.

The use of filtered colimits highlights the utility of categorical reasoning in constructive mathematics. By constructing the free group as the colimit of free groups on finite sets, we obtain a relatively modular framework. This method can potentially be generalized to other algebraic structures, such as rings, modules, or monoids, having the functor for the finite sets as a hypothesis.

Our formalization in Agda demonstrates the feasibility of this approach in practice. The process of formalization also revealed subtle points in the construction, such as the careful handling of morphisms in the filtered diagram as in Section 3.2.2, and the explicit construction of the colimit as in Theorem 3.58. These insights may be valuable for future work on formalizing other algebraic or categorical constructions in HoTT.

Finally, this work opens several avenues for further research. The categorical perspective adopted here suggests connections to universal algebra and functorial semantics, which may yield new insights into the structure of algebraic theories in HoTT (see Chapter 5).

In summary, this thesis demonstrates that categorical and constructive methods provide a robust foundation for algebraic constructions in HoTT, and that filtered colimits are a useful tool for overcoming the limitations imposed by the need for decidable equality. The results and techniques developed here hopefully advance the understanding of free groups in HoTT and may potentially contribute to integrating categorical methods with type theory.

Chapter 5
Future Work

In the future, one could explore an alternative to our approach, more abstract in nature, proceeding as follows:

First, filtered colimits commute with finite limits in $\mathbf{Set}$ (and in many other categories). This means that taking a finite limit (such as a pullback or equalizer) of a diagram of filtered colimits, yields the same result as first taking the finite limit in each diagram and then forming the filtered colimit.

Next, the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ creates filtered colimits. That is, given a filtered diagram of groups and group homomorphisms, the colimit in $\mathbf{Set}$ (of the underlying sets) can be equipped with a unique group structure making it the colimit in $\mathbf{Grp}$. This property allows us to lift constructions from sets to groups when working with filtered colimits.

Finally, note that X is a filtered colimit of finite sets. The universal property of this colimit gives the universal property of the group given by U creating filtered colimits.

Putting these facts together, we can argue as follows:
- Because filtered colimits commute with finite limits,
- If we can provide the free group for finite sets mapping into X,
- And because $U : \mathbf{Grp} \to \mathbf{Set}$ creates filtered colimits, we get the a unique group structure on X;
- Due to X being a filtered colimit of finite sets, the aforementioned group is indeed the free group on X.

This approach is more abstract, making the proof potentially more modular. One could investigate whether similar arguments apply to other algebraic structures (like rings, modules, etc.). This could lead to a broader generalization of the results and new insights into the structure of algebraic theories. In particular, one might extend this to locally presentable (or even accessible) categories and try to generalize this to arbitrary algebras.

Bibliography

[1] A. Cayley, "VII. On the Theory of Groups, as Depending on the Symbolic Equation θn =1," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 7, no. 42, pp. 40–47, Jan. 1854, doi: 10.1080/14786445408647421.

[2] W. Dyck, "Gruppentheoretische Studien," *Mathematische Annalen*, vol. 20, no. 1, pp. 1–44, Mar. 1882, doi: 10.1007/BF01443322.

[3] V. Voevodsky, "Univalent Foundations." [Online]. Available: https://github.com/vladimirias/Foundations

[4] T. Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[5] E. Riehl, "MATH 721: HOMOTOPY TYPE THEORY," [Online]. Available: https://github.com/emilyriehl/721

[6] M. H. Escardó and contributors, "TypeTopology." [Online]. Available: https://github.com/martinescardo/TypeTopology

[7] R. Mines, F. Richman, and W. Ruitenburg, *A Course in Constructive Algebra*. in Universitext. New York, NY: Springer, 1988. doi: 10.1007/978-1-4419-8640-5.

[8] S. Eilenberg and S. MacLane, "General Theory of Natural Equivalences," *Transactions of the American Mathematical Society*, vol. 58, pp. 231–294, 1945, doi: 10.1090/S0002-9947-1945-0013131-6.

[9] A. Grothendieck, "Sur Quelques Points d'algèbre Homologique, I," *Tohoku Mathematical Journal*, vol. 9, no. 2, pp. 119–221, Jan. 1957, doi: 10.2748/tmj/1178244839.

[10] U. Norell, "Towards a Practical Programming Language Based on Dependent Type Theory," SE-412 96 Göteborg, Sweden, 2007.

[11] A. Church and J. B. Rosser, "Some Properties of Conversion," *Transactions of the American Mathematical Society*, vol. 39, no. 3, pp. 472–482, 1936, doi: 10.1090/S0002-9947-1936-1501858-0.