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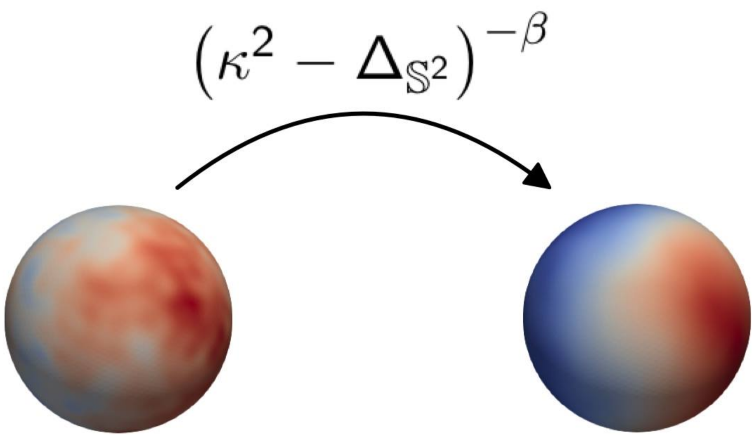

Generation of Gaussian random fields on the sphere

A surface finite element method for a type of fractional elliptic stochastic partial differential equation on the sphere

Erik Jansson

Department of Mathematical Sciences
Division of Applied Mathematics and Statistics
CHALMERS UNIVERSITY OF TECHNOLOGY
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Thesis for the Degree of Master of Science

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Department of Mathematical Sciences
Division of Applied Mathematics and Statistics
Chalmers University of Technology and University of Gothenburg
SE – 412 96 Gothenburg, Sweden
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Erik Jansson

Supervisor: Annika Lang, Mathematical Sciences
Examiner: Annika Lang, Mathematical Sciences

Master's Thesis 2019:NN
Mathematical sciences
Chalmers University of Technology
SE-412 96 Gothenburg
Telephone +46 31 772 1000

Cover: Illustration of solving the field-generating SPDE

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Abstract

In order to approximate solutions to a fractional elliptic SPDE used to generate random fields on the sphere with Matérn covariance, a sinc quadrature approach combined with a surface finite element method is used. The right hand side of the equation is replaced with trace class noise on the sphere. Error bounds in $L^2(\Omega; L^2(\mathbb{S}^2))$-norm are proved using energy estimates and an Aubin–Nitsche duality type argument. $\mathbb{P}$-almost surely asymptotic error estimates are shown. Some properties of the exact solution to both trace class right hand side and white noise right hand side are discussed and the relation between the mean square differentiability and the parameters of the equation are established. Some properties of white noise on the sphere are considered. The method is implemented in FEniCS and parts of the algorithm is verified which agree with the theoretical results.

Keywords: Isotropic Gaussian random fields, stochastic partial differential equations, surface finite elements, fractional equations, sinc quadrature.

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Erik Jansson, Gothenburg, December 9, 2019

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1

Introduction

Modelling with Gaussian random fields is an important area of interest in for instance spatial statistics, environmental sciences and cosmology. Because of this, methods for simulation of these objects have long been a subject of study. Whittle noted in 1963 [1] that there is a connection between random fields with a certain type of covariance function and solutions of stochastic partial differential equations, that is to say, partial differential equations with certain random components, of the form

$$(\kappa^2 - \Delta)^{\beta} u = \mathcal{W},$$

where $\kappa, \beta > 0$ and $\mathcal{W}$ denotes white noise. This approach was further developed by for instance [2], [3] and [4]. In [2], a way to define Gaussian random fields with similar covariance properties on the sphere was given, namely, one *defines* the field not through the use of a certain covariance function but as the stationary solution of the equation

$$(\kappa^2 - \Delta_{\mathbb{S}^2})^{\beta} u = \mathcal{W}. \tag{1.1}$$

In this thesis, we intend to develop a finite element method for this equation, which is essentially the premise in [3]. However, since the intention is to do this in the special setting on the sphere, there are additional difficulties. For instance, it is to us unknown how we in practice easily can simulate white noise on the sphere. In order to avoid tackling this problem, we replace $\mathcal{W}$ with another type of noise, which we denote by $\overline{\mathcal{W}}$. This noise will be of so called *trace class*, which is feasible to simulate using truncated spherical harmonics.

It is furthermore not possible to deal with the fractional part of the equation in the same step as performing the finite element discretisation, due to the additional need for a discretisation of the sphere. In order to arrive at a method, we employ the sinc quadrature approach of [5], which will result in the need of solving several subproblems. The solutions to these subproblems are then summed together to obtain an approximate solution to the equation. Each of these subproblems will be solved using the surface finite element (SFEM) method of [6]. Moreover, we will prove a $L^2(\Omega; L^2(\mathbb{S}^2))$-error estimate of this method and verify it using simulations. The error will be bounded by

$$\|u - u_h^{\ell}\|_{L^2(\Omega, L^2(\Omega))} \leq c_1(k)\sqrt{\mathrm{Tr}(Q)} + c_2(k)\left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\mathrm{Tr}(Q)}\right),$$

where $c_1(k)$ and $c_2(k)$ are parameters dependent on the sinc quadrature step size, $\mathfrak{K}$ is the truncation parameter of the Karhunen–Loève expansion of the sphere, h is the mesh size of the discretised sphere, α is a parameter determining the smoothness of the right hand side, Q is the covariance operator of the right hand side and $D(\mathfrak{K}, A_l)$ is a parameter which is determined by the angular power spectrum of the right hand side and the truncation parameter.

We are not aware of that surface finite elements have been combined with a sinc quadrature approach to solve fractional-type stochastic elliptic partial differential equations previously.

In order to facilitate the implementation work, a lot of overhead work was saved by implementing the methods in the finite-element solver FEniCS[1]. However, the computations involved are still relatively time-consuming, and therefore, it is fortunate that we were granted access to the computational resources of the Department of Mathematical Sciences at Chalmers University of Technology.

There are several possible directions for further work. One is to focus on a non Karhunen–Loève approach to generation of the noise on the sphere, since this then would allow for generalisation to more arbitrary domains. The SFEM theory is in [6] developed for general smooth surfaces, with or without boundary, and the method of [5] requires only that the appropriate Gelfand triples are selected. Also of interest would be to consider fields in which the parameters κ and β are spatially varying or even stochastic processes themselves, since non-stationary models could be interesting for modelling purposes. Another possible area of further research could be to apply the theory of evolving domains in [6] to the problem of generating random fields on the sphere, to see if it might be possible to consider Gaussian random fields on randomly growing spheres. This would require studying spectral approaches to generating isotropic Gaussian random fields on evolving domains.

An aspect completely omitted in this thesis is the statistical considerations. It would be interesting to develop methods for estimating the parameters κ, β and A_l from data and explore different meteorological, cosmological and environmental applications. It would also be of interest to obtain a fast implementation of the method. There exist packages in compiled languages, such as C or FORTRAN, designed for fast evaluations of spherical harmonic expansions. This could be combined with a study of how to mesh the sphere efficiently to obtain a more efficient implementation.

Chapter 2 is intended as a very brief review of the necessary theory for the latter parts of the thesis as well as a way to clarify the notation. Apart from some general background of functional analysis and probability, we focus mainly on the geometry of the sphere and elements of calculus on it. Furthermore, the theory of random fields as well as the theory of spherical harmonics and their use in the study of random fields on the sphere are discussed.

In Chapter 3, we present a modification of the noise in the right hand side of Equation (1.1) which allows for its exact solution. We treat this modification in

[1] https://fenicsproject.org/

detail and give the necessary assumptions for a solution to exist. We then solve the equation, discuss its properties and provide a simulation scheme for it. The chapter is concluded with a brief remark on the unmodified equation and its solution as well as some regularity properties of white noise.

In Chapter 4, we discuss the surface finite element method for the modified fractional-type stochastic elliptic partial differential equation. The chapter begins with a discussion on the discretisation of the sphere. We then give a simulation method of the modified noise on the discretised sphere, and prove the error of this for different regularity properties of the noise. Subsequently, the SFEM method is developed in detail, and error bounds are proved.

In Chapter 5, we discuss the generation of the discretised sphere and the implementation of the methods described in the earlier parts of the thesis.

In Chapter 6, we present some numerical simulations made to showcase the algorithms and verify error bounds. We solve both Poisson's equation and the fractional equation, with a deterministic right-hand side, on the sphere. We thereafter verify the convergence rates of the approximation scheme of the noise using a Monte-Carlo approach.

2

Theoretical background

This chapter summarises the theoretical preliminaries needed for the rest of the report. The first part of the chapter gives the necessary background in functional analysis, operator theory and probability. In the second part, the geometry of the sphere and calculus on it is briefly outlined, whereupon random fields and their generation by use of spherical harmonics is described.

2.1 Functional analysis

In this section we introduce the necessary theory and notation in order to be able to discuss concepts such as operators and function spaces. The reader already familiar with functional analysis is referred to section 2.2. The presentation is mainly based on [7] but also [8]. It is assumed that the reader is familiar with the basics of functional analysis and probability.

2.1.1 Banach and Hilbert space theory

The theory of Hilbert and Banach spaces will be used extensively throughout this thesis. The reader is assumed to have some basic familiarity with the subject matter, but certain definitions are given in this section so that the notation is clear.

Definition 2.1.1 (Separable Hilbert space)**.** A Hilbert space U is said to be *separable* if there is a countable set dense in U.

Definition 2.1.2 (Basis of Hilbert space)**.** Let $(U, (\cdot,\cdot)_U)$ be a real Hilbert space. A sequence $(e_k)_{k=1}^{\infty} \subset U$ is a *basis* if there exists for every $u \in U$ a sequence of real-valued numbers $(c_k)_{k=1}^{\infty}$ such that

$$u = \sum_{k=1}^{\infty} c_k e_k.$$

The basis is said to be *orthonormal* if $(e_i, e_j)_U = 1$ if $i = j$ and zero else.

Definition 2.1.3 (Dual of Hilbert space)**.** Let U be a real Hilbert space. The dual of U, denoted by U^*, is the space of all bounded linear functionals $\phi : U \to \mathbb{R}$. The *duality pairing* is given by $\langle u, x\rangle_{U\times U^*} = x(u)$, $x \in U^*$, $u \in U$.

One can use the Riesz representation theorem to show that for any $x \in U^*$, there is a unique $u_x \in U$ such that for every $u \in U$ it holds that $x(u) = \langle u, x\rangle_{U\times U^*} = (u_x, u)_U$. Note that this defines an isomorphism between U and U^* so we may conclude that U^* may be identified with U.

Definition 2.1.4 (Compact embeddings)**.** Let $(U, \|\cdot\|_U), (V, \|\cdot\|_V)$ be two Banach spaces such that $U \subseteq V$. The space U is *compactly embedded* in V, denoted by $U \hookrightarrow_c V$ if

1. There is a constant $c > 0$ such that $\|u\|_V \leq c\|u\|_U$ for every $u \in U$. This means that U is *continuously embedded* into V
2. The embedding operator $i : U \to V$ is a compact operator.

As it will be used in later parts of the thesis, we need to introduce the concepts of *Gelfand triples*. These triples are introduced to obtain a convenient setting in which we work with partial differential equations. Gelfand triples are also needed to obtain the appropriate function space setting for the discretization of fractional operators introduced later in the text.

Definition 2.1.5 (Gelfand triples)**.** Let U, V be separable Hilbert spaces such that

$$U \hookrightarrow V \cong V^* \hookrightarrow U^*,$$

where all the embeddings are dense, and it holds for every $f \in V$ and every $u \in U$ that

$$\langle f, u\rangle_{U^*\times U} = (f, u)_V.$$

This means that using the embedding, U is identified with a dense subspace of V. The space V in turn, is identified with its dual, which is identified with a dense subspace of U^* using the embedding. Note here that V acts on the elements of U using the V-inner product, and not the inner product of U.

2.1.2 Operators

In this subsection the necessary definitions and results from operator theory will be introduced. The treatment is based mainly on [7].

Definition 2.1.6 (Linear, bounded operator)**.** A *linear, bounded operator* $A : U \to V$ is a transformation between two normed spaces U and V, such that there exists $M > 0$ such that

$$\|Au\|_V \leq M\|u\|_U.$$

By $L(U, V)$ we denote the space of all linear, bounded operators between U and V. This space is equipped with a norm called the *operator norm*, given by

$$\|A\|_{L(U,V)} = \sup_{\phi\in U} \frac{\|A\phi\|_V}{\|\phi\|_U}.$$

Note that $L(U, V)$ is a Banach space equipped with the operator norm if V is a Banach space.

Definition 2.1.7 (Domain of operator)**.** The *domain* of an operator $A : U \to V$, denoted $\mathrm{Dom}(A)$, is the subspace of U on which A is defined, that is to say, finite.

Definition 2.1.8 (Densely defined operator)**.** An operator from U to V is said to *be densely defined* if $\mathrm{Dom}(A) \subset U$ is dense in U.

Definition 2.1.9 (Range of operator)**.** The *range* of an operator $A \in L(U, V)$ is defined as the image of all the elements in the domain of A.

Definition 2.1.10 (Inverse of operator)**.** The inverse of a bijective operator $A \in L(U, V)$, denoted by A^{-1}, is the operator which maps $v \in V$ to a unique element $u \in U$, such that $Au = v$. This is denoted by $A^{-1}v = u$.

Definition 2.1.11 (Compact operator)**.** A linear operator $A : U \to V$ is a *compact operator* if for every bounded sequence $(u_i)_{i=1}^{\infty} \subset U$, $(Au_i)_{i=1}^{\infty}$ contains a convergent subsequence.

Definition 2.1.12 (Closed operator)**.** More general than a bounded operator, a linear operator $T : U \to V$ is said to be *closed* if for every sequence $(u_n)_{n=1}^{\infty} \subset \mathrm{Dom}(T)$, which converges to $u \in U$ such that $Tu_j \to v \in V$, it holds that $u \in \mathrm{Dom}(T)$ and $Tu = v$.

Definition 2.1.13 (Adjoint of operator)**.** Let U and V be Banach spaces. Let $A : \mathrm{Dom}(A) \subset U \to V$ be a densely defined linear operator. Its *adjoint* is given by $A^* : \mathrm{Dom}(A^*) \subset V^* \to U^*$, where U^* and V^* denote the dual of U and V, respectively. Here

$$\mathrm{Dom}(A^*) = \{x_v \in V^* : \exists c > 0 \text{ such that } \forall u \in \mathrm{Dom}(A) \; : \; x_v(Au) \leq c\|u\|_U\},$$

and the action of the adjoint operator is defined by noting that by the Hahn–Banach theorem, the operator $u : \mathrm{Dom}(A) \to \mathbb{R}$ for which it holds that $f(u) = x_v(Au)$ may be extend to an operator defined on all of U, and the adjoint is the operator which maps x_v to this extension.

If instead $(X, (\cdot, \cdot)_X)$ and $(Y, (\cdot, \cdot)_Y)$ are Hilbert spaces, the adjoint of an operator $A \in L(X; Y)$, is the operator $A^* \in L(Y^*; X^*)$ given by

$$T^* y^*(x) = y^*(Tx).$$

Definition 2.1.14 (Symmetric and self-adjoint operator)**.** Let U be a Hilbert space, and let A be a densely defined operator $A : \mathrm{Dom}(A) \subset U \to U$. A is said to be *symmetric* if for any $u, v \in \mathrm{Dom}(A)$, $(Au, v)_U = (u, Av)_U$. This implies that $\mathrm{Dom}(A) \subset \mathrm{Dom}(A^*)$. A symmetric operator T is *self-adjoint* with adjoint $A^* = A$ if $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$.

Equipped with these definitions, we can state a theorem, which will give conditions for an operator on a Hilbert space to have an eigenbasis.

Theorem 2.1.1 (Spectral theorem)**.**
Let A be a compact self-adjoint densely defined operator on a Hilbert space U such

that $-A$ is positive definite and the inverse of $-A$ is a compact operator. Then, there is an orthonormal basis of U consisting of the eigenvectors of $-A$ with real eigenvalues, that is to say, there is a sequence $(e_k, \alpha_k)_{k=1}^{\infty} \subset U \times \mathbb{R}$ such that $(\alpha_k)_{k=1}^{\infty}$ is an increasing sequence of non-negative real numbers and $(e_k)_{k=1}^{\infty}$ is an orthonormal basis of U, and it holds that

$$-Ae_k = \alpha_k e_k.$$

Equipped with this theorem, we can rigorously define fractional powers of an operator.

Definition 2.1.15 (Fractional powers)**.** Let A be a compact, self-adjoint operator on a Hilbert space U such that $-A$ is positive definite and has a compact inverse. Let $\nu \in \mathbb{R}$. The fractional power of $-A$ is defined by

$$(-A)^{\nu/2}\phi = \sum_{k=1}^{\infty} \alpha_k^{\nu/2}(\phi, e_k)_U e_k,$$

for every $\phi \in \mathrm{Dom}\left((-A)^{\nu/2}\right)$ where

$$\mathrm{Dom}\left((-A)^{\nu/2}\right) = \left\{\phi \in U : \|\phi\|_{U^\nu}^2 = \sum_{k=1}^{\infty} \alpha_k^{\nu}(\phi, e_k)_U^2 < \infty\right\}. \tag{2.1}$$

Definition 2.1.16 (Nuclear operator)**.** The set of nuclear operators $L_N(U, V) \subset L(U, V)$ is given by all $A \in L(U, V)$ such that there are sequences $(v_k)_{k=1}^{\infty} \subset V$ and $(u_k^*)_{k=1}^{\infty} \subset U^*$ for which it holds that:

1. For every $\phi \in U$ that $A\phi = \sum_{k=1}^{\infty} u_k^*(\phi)v_k$.
2. $\|A\|_{L_N(U,V)} = \sum_{k=1}^{\infty} \|v_k\|_V \|u_k^*\|_{U^*} < \infty$.

Definition 2.1.17 (Trace)**.** Let U be a separable Hilbert space and denote by $(e_i)_{i=1}^{\infty}$ the basis of U. We define the *trace* of an operator $A \in L(U, U)$ being self-adjoint and positive definite by

$$\mathrm{Tr}(A) = \sum_{i=1}^{\infty} (Ae_i, e_i)_U \,.$$

If the trace is finite the operator is said to be of *trace class.*

In order to discuss the inverse of an operator with more rigour, as we later will consider the fractional inverse of operators, we introduce the *resolvent set.*

Definition 2.1.18 (Resolvent set)**.** Let U be a Banach space and let $T : U \to U$. We say that λ is a *regular value* of T if it holds that

$$T_\lambda = T - \lambda I_U,$$

where I_U is the identity operator on U, is an injective operator. Furthermore, it should hold that the inverse of T_λ, denoted by $R(\lambda, T)$, exists and is a linear, bounded operator with range dense in U. Then, the *resolvent set*, which can be shown to be open, is given by all the regular values of T.

In order to define for instance the covariance of Hilbert space-valued random variables, we must introduce tensor product spaces.

Definition 2.1.19 (Tensor product space)**.** Let U and V be Hilbert spaces that are both real or both complex. The algebraic tensor product $U\overline{\otimes}V$ is given by the vector space of elements

$$\sum_{i=1}^{n} \phi_i \otimes \psi_i, \phi_i \in U, \psi_i \in V, i = 1, ..., n,$$

with the equivalence relations

$$\begin{aligned}
(\phi_1 + \phi_2) \otimes \psi_1 &= \phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_1, \\
\phi_1 \otimes (\psi_1 + \psi_2) &= \phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_1, \\
(\alpha\phi_1) \otimes \psi_1 &= \phi_1 \otimes (\alpha\psi_1) = \alpha(\phi_1 \otimes \psi),
\end{aligned}$$

for every $\alpha \in \mathbb{R}$. The inner product on this space is given by

$$(\phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2)_{U\otimes V} = (\phi_1, \phi_2)_U (\psi_1, \psi_2)_V,$$

and the completion of $U\overline{\otimes}V$ with respect to the Hilbert space norm is called the *Hilbert tensor product space* and is denoted by $U \otimes V$.

2.1.3 Basics of Lebesgue–Bochner space theory

In the theory of elliptic partial differential equations, the study of Banach space valued functions arises naturally. These functions will also be of use in the study of stochastic partial differential equations (SPDEs).

A function space setting for studying such objects is therefore desirable. Let therefore $(B, \|\cdot\|_B)$ be a Banach space and let D be some set. The space of *Bochner integrable* functions $f : D \to B$, denoted by $L^p(D; B)$, $p \in [1, \infty)$ contains functions such that

$$\begin{aligned}
\|f\|_{L^p(D;B)} &:= \left(\int_D \|f(t)\|_B^p \, \mathrm{d}t\right)^{1/p} < +\infty \text{ for } 1 \leq p < \infty, \\
\|f\|_{L^\infty(D;B)} &:= \operatorname{ess\,sup}_{d\in D} \|f(d)\|_B < +\infty.
\end{aligned}$$

The set D could for instance be a sample space Ω in the case of SPDEs or some time-interval in the case of parabolic PDEs. This definition can be extended in the natural way to encompass Sobolev–Bochner spaces as well, if needed. An interesting property is that if B is reflexive, then $L^p(D, B)^* = L^{p'}(D, B^*)$ where p' is the Hölder conjugate of p. In this case, the duality pairing is given by

$$\langle F, f \rangle_{L^p(D,B^*)\times L^p(D,B)} = \int_D \langle F(t), f(t) \rangle_{B^*,B} \, \mathrm{d}t.$$

It is possible to rigorously define integrability as well as constructing the integral in an analogue way to the regular Lebesgue integral by building from simple functions. However, that presentation is of limited applicability in this text and is therefore omitted. The interested reader is referred to Section 6.A.1. in [8].

2.2 Geometry and analysis on the unit sphere

In this section we outline the necessary basics from geometry and analysis on the sphere. We follow the outline of the first sections in [6], adapted to the specific case of the sphere.

Definition 2.2.1 (The sphere)**.** The *sphere*, denoted by $\mathbb{S}^2$, is defined by

$$\mathbb{S}^2 = \left\{x \in \mathbb{R}^3 \ : (x, x)_{\mathbb{R}^3} = 1\right\},$$

where $(\cdot, \cdot)_{\mathbb{R}^3}$ is the Euclidean inner product. The distance function used on the sphere is the geodesic distance, or great-circle distance, given by

$$d(x, y) = \arccos((x, y)_{\mathbb{R}^3}), \ x, y \in \mathbb{S}^2.$$

With this metric, $\mathbb{S}^2$ is a compact metric space. The outward unit normal of the sphere is given by

$$\nu(x) = \frac{x}{\|x\|}. \tag{2.2}$$

The unit sphere may be parametrised by

$$(x(\phi, \theta), y(\phi, \theta), z(\phi, \theta)) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \ (\theta, \phi) \in [0, \pi] \times [0, 2\pi). \tag{2.3}$$

In order to discuss differentiability rigorously, we need to define certain concepts from differential geometry.

Definition 2.2.2 (Charts and atlases)**.** Let I be some index set. An *atlas* on $\mathbb{S}^2$ is a collection of pairs $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$, where $\bigcup_{\alpha \in I} U$ covers $\mathbb{S}^2$ and ϕ_α are smooth functions $\phi_\alpha : U_\alpha \to \mathbb{R}^2$. The *transition map* is given by

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta)} \colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta).$$

Definition 2.2.3 (Differentiable functions)**.** Let (U, ϕ) be a chart such that the open set U, which contains x, is a subset of $\mathbb{S}^2$ and ϕ is a function from U to $\mathbb{R}^2$. Then $f : \mathbb{S}^2 \to \mathbb{R}$ is differentiable in x if and only if it holds that

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R},$$

is differentiable at $\phi(x)$. If for every point $x \in \mathbb{S}^2$, f is continuously differentiable $k \in \mathbb{N}$ times, f is said to belong to $C^k(\mathbb{S}^2)$.

Definition 2.2.4 (Tangential gradient)**.** Let $f : \mathbb{S}^2 \to \mathbb{R}$ be differentiable at $x \in \mathbb{S}^2$. The *tangential gradient* of f in x is given by

$$\nabla_{\mathbb{S}^2} f = \nabla f(x) - (\nabla f(x), \nu(x))_{\mathbb{R}^3}\, \nu(x), \tag{2.4}$$

where ∇f is the regular gradient in $\mathbb{R}^3$ of f, extended to a neighbourhood of $\mathbb{S}^2$. For notational convenience, let

$$(P(x))_{ij} = \delta_{ij} - \nu_i \nu_j,$$

where $i, j = 1, 2, 3$, ν is given by (2.2) and $\delta_{ij} = 1$ if $i = j$ and zero else. It is possible to write

$$\nabla_{\mathbb{S}^2} f = P(x) \nabla f(x).$$

As in ordinary multivariate calculus, a function is said to be once differentiable, denoted by $f \in C^1(\mathbb{S}^2)$, if every component of the vector $\nabla_{\mathbb{S}^2} f := (\underline{D}_1 f, \underline{D}_2 f, \underline{D}_3 f)$ is continuous. Each component of the nabla vector $\nabla_{\mathbb{S}^2}$ is denoted by $\underline{D}_i$.

Since we consider second-order elliptic partial differential equations on the sphere, we include the definition of the Laplace–Beltrami operator on the sphere.

Definition 2.2.5 (The Laplace–Beltrami operator on the sphere)**.** Assume that $f \in C^2(\mathbb{S}^2)$. Then

$$\Delta_{\mathbb{S}^2} f = (\nabla_{\mathbb{S}^2}, \nabla_{\mathbb{S}^2} f)_{\mathbb{R}^3} = \mathrm{Div}(\nabla_{\mathbb{S}^2} f).$$

In the local coordinates defined by (2.3), the Laplace–Beltrami operator on the sphere is given by

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}. \tag{2.5}$$

Definition 2.2.6 (Hölder continuity)**.** A function $f : \mathbb{S}^2 \to \mathbb{R}$ is said to be uniformly *Hölder continuous* with exponent α if it satisfies

$$[f]_\alpha = \sup_{x, y \in \mathbb{S}^2, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

The space of all uniformly Hölder continuous functions with exponent α equipped with the norm $\|f\|_{C^{0,\alpha}(\mathbb{S}^2)} = \sup_{\mathbb{S}^2} |f| + [f]_\alpha$ is denoted by $C^{0,\alpha}(\mathbb{S}^2)$.

Definition 2.2.7 (Weingarten map, exchange of derivatives and mean curvature)**.** The *extended Weingarten map* $\mathcal{H} : \mathbb{S}^2 \to \mathbb{R}^{3 \times 3}$ is given by

$$\mathcal{H}_{ij} = \underline{D}_i \nu_j = \underline{D}_i \frac{x_j}{\|x\|} = \delta_{ij} - x_i x_j.$$

In the special case of the sphere, the Weingarten map is given by:

$$\mathcal{H} = \begin{bmatrix} 1-x_1^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & 1-x_2^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & 1-x_3^2 \end{bmatrix}.$$

The need for the Weingarten map arises for instance when exchanging tangential derivatives. If $f \in C^2(\mathbb{S}^2)$ then it holds that

$$\underline{D}_i\underline{D}_j f = \underline{D}_j\underline{D}_i f + (\mathcal{H}\nabla_{\mathbb{S}^2} f)_j\nu_i - (\mathcal{H}\nabla_{\mathbb{S}^2} f)_i\nu_j.$$

Definition 2.2.8 ($L^p(\mathbb{S}^2)$-spaces)**.** On the sphere we use the surface measure $\mathrm{d}A = \sin(\theta)\ \mathrm{d}\theta\ \mathrm{d}\phi$. For $p \in [1, \infty]$, $L^p(\mathbb{S}^2)$ denotes the space of functions $f : \mathbb{S}^2 \to \mathbb{R}$ which are measurable with respect to $\mathrm{d}A$, and for which

$$\|f\|_{L^p(\mathbb{S}^2)} = \left(\int_{\mathbb{S}^2} |f|^p\ \mathrm{d}A\right)^{1/p} < +\infty,$$
$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{\mathbb{S}^2} |f| < +\infty.$$

These spaces are Banach spaces, and if $p = 2$, which is the main case of interest, the space is a Hilbert space with inner product

$$(f, g)_{L^p(\mathbb{S}^2)} = \int_{\mathbb{S}^2} fg\ \mathrm{d}A,$$

for $f, g \in L^2(\mathbb{S}^2)$

We have the following integration by parts formula [6, Theorem 2.10]

$$\int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} f\ \mathrm{d}A = 2\int_{\mathbb{S}^2} f\nu\ \mathrm{d}A,$$

where ν is the unit normal defined in (2.2). Since we in the later parts of the thesis will work with partial differential equations, we need to have an appropriate function space setting. These spaces are known as *Sobolev spaces*, and there are several ways to introduce Sobolev spaces on the sphere, for instance, in [6], a definition in analouge with the case of open sets in a Euclidean space is given, using weak derivatives to define integer-order Sobolev spaces. For completeness and readability, we provide this definition as well.

Definition 2.2.9 (Weak derivatives on $\mathbb{S}^2$)**.** A function $f \in L^1(\mathbb{S}^2)$ is said to be weakly differentiable with weak derivative in direction i, where $i \in \{1, 2, 3\}$, denoted by $\partial^i f \in L^1(\mathbb{S}^2)$ if for every function $\phi \in C^1(\mathbb{S}^2)$ with compact support $K \subset \mathbb{S}^2$ it holds that

$$\int_{\mathbb{S}^2} f\underline{D}_i\phi\ \mathrm{d}A = -\int_{\mathbb{S}^2} \partial^i f\phi\ \mathrm{d}A + 2\int_{\mathbb{S}^2} f\phi\nu_i\ \mathrm{d}A.$$

Definition 2.2.10 (Sobolev spaces on $\mathbb{S}^2$)**.** For $k \in \mathbb{N}_0$ and $1 \le p \le \infty$ the Sobolev spaces on $\mathbb{S}^2$, $W^{k,p}(\mathbb{S}^2)$, consist of all integrable functions $f : \mathbb{S}^2 \to \mathbb{R}$ such that

$$\partial^i f \in L^p(\mathbb{S}^2) \text{ for every } 0 \le |i| \le k,$$

where $|i|$ denotes the usual multi-index notation, that is to say, in this case a k-tuple $i = (i_1, ..., i_k)$ where every component is a non-negative integer, for which it holds that $|i| = i_1 + \cdots + i_k$. It it also possible to use multi-indices to denote higher order derivatives, $\partial^i = \partial_{x_1}^{i_1}\partial_{x_2}^{i_2}, ..., \partial_{x_k}^{i_k}$, where $\partial_{x_l}^{i_l} = \frac{\partial^{i_l}}{\partial_{x_l}^{i_l}}$, that is to say, the lth derivative of order i_l.

A convenient shorthand for the useful special case of $k = 2$ is given by $H^k(\mathbb{S}^2) = W^{k,2}(\mathbb{S}^2)$. Equipped with the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{|i|\le k} \|\partial^i f\|_{L^p(\mathbb{S}^2)}^p\right)^{1/p},$$
$$\|f\|_{W^{k,\infty}} = \max_{|i|\le k} \|\partial^i f\|_{L^\infty(\mathbb{S}^2)}.$$

The spaces are Banach spaces, and $H^k(\mathbb{S}^2)$ is a Hilbert space with inner product

$$(f, g)_{H^k(\mathbb{S}^2)} = \sum_{|i|\le k} (\partial^i f, \partial^i g)_{L^p(\mathbb{S}^2)}.$$

Remark 2.2.1. Note that if $u \in H^1(\mathbb{S}^2)$, then $u \in L^2(\mathbb{S}^2)$ since

$$\|u\|_{L^2(\mathbb{S}^2)}^2 \le \|u\|_{L^2}(\mathbb{S}^2)^2 + \|\nabla_{\mathbb{S}^2} u\|_{L^2(\mathbb{S}^2)} < +\infty.$$

We need to define Sobolev spaces for non-integer and negative smoothness indices as well, for instance to obtain the necessary Hilbert triple setting needed for the quadrature expansion of the elliptic operators introduced later in the report. We therefore introduce the necessary definitions.

Since we need a more general setting than functions, we have to introduce the notion of a distribution. These might be rigorously constructed and have a rich theory, which is beyond the scope of this thesis since the definition of distributions on Riemannian manifolds is less straightforward than on $\mathbb{R}^n$. Intuitively, we may think of distributions as the elements of the dual space of $C^\infty(\mathbb{S}^2)$.

Definition 2.2.11 (Bessel potential)**.** Let $s \in \mathbb{R}$. In general, s may be complex with positive real part, but in this setting we restrict s to the real line. The *Bessel potential* on $L^p(\mathbb{S}^2)$, $p \in [1, \infty)$ is given by

$$\left(I_{L^p(\mathbb{S}^2)} - \Delta_{\mathbb{S}^2}\right)^{-s/2} = \mathcal{B}_{L^p(\mathbb{S}^2)}(s).$$

where $I_{L^p}(\mathbb{S}^2)$ is the identity operator on $L^p(\mathbb{S}^2)$. The Bessel potential is an example of a so-called *pseudodifferential operator*.

The Bessel potential can be represented using Fourier analysis or functional calculus, but the precise definition will be omitted in this thesis. We can now define what we mean by Sobolev spaces with smoothness indices $s \in \mathbb{R}$.

Definition 2.2.12 (Sobolev spaces with real smoothness index)**.** Let $s \in \mathbb{R}$ and let $p \in [1, \infty)$, then the *Sobolev space* of smoothness index s is defined by

$$W^{s,p}(\mathbb{S}^2) = \mathcal{B}_{L^p(\mathbb{S}^2)}(s) L^p(\mathbb{S}^2). \tag{2.6}$$

If s>0, the norm will be given by

$$\|f\|_{W^{s,p}(\mathbb{S}^2)} := \|\mathcal{B}_{L^p(\mathbb{S}^2)}(-s) f\|_{L^p(\mathbb{S}^2)}.$$

Note that we will omit the detailed theory behind this definition, but for $s < 0$, we think of $W^{s,p}(\mathbb{S}^2)$ as a set of distributions of the form

$$W^{s,p}(\mathbb{S}^2) \ni u = \mathcal{B}_{L^p(\mathbb{S}^2)}(-2k)v, \ v \in W^{2k+s,p}(\mathbb{S}^2),$$

where $k \in \mathbb{N}$ is such that $2k + s > 0$. In this case, the norm is given by

$$\|u\|_{W^{s,p}(\mathbb{S}^2)} = \|v\|_{W^{2k+s,p}(\mathbb{S}^2)}.$$

As above, we will mainly consider the special case of $p = 2$ and use the notation $W^{s,2}(\mathbb{S}^2) = H^s(\mathbb{S}^2)$. If $s = 0$, it holds that $W^{0,p}(\mathbb{S}^2) = L^p(\mathbb{S}^2)$. We can also view Bessel potentials as mappings between Sobolev spaces according to

$$B_{L^p(\mathbb{S}^2)}(-s) : H^t(\mathbb{S}^2) \to H^{t-s}(\mathbb{S}^2),$$

where $t, s \in \mathbb{R}$.

Note that for integer order smoothness, these spaces will coincide with the earlier definition of integer order smoothness Sobolev spaces, see for instance Section 7.59 in [9].

In order to be able to discuss existence and uniqueness of fractional elliptic partial differential equations, we need to introduce the appropriate function spaces.

Definition 2.2.13 (The spaces $\dot{H}^{2\beta}$)**.** Let H be a Hilbert space. The action of a fractional power of an operator $L : H \to H^*$, L^β, where $\beta \in (0, 1)$, will be well defined on the space $\dot{H}^{2\beta}$ which is given by

$$\dot{H}^{2\beta} = \mathrm{Dom}(L^\beta) = \left\{ \psi \in H : \|\psi\|_{2\beta}^2 := \|L^\beta \psi\|_H^2 = \sum_{j=0}^{\infty} \lambda_j^{2\beta} (\psi, e_j)^2 < \infty \right\}.$$

As in the well-known case of domains in $\mathbb{R}^n$, we have Sobolev embeddings on $\mathbb{S}^2$ as well. We state a version of the Sobolev embedding theorem, from Theorem 3.19 in [10]. It should be noted that this is not the most general version of the theorem.

Theorem 2.2.1 (Sobolev embeddings on $\mathbb{S}^2$)**.**
It holds that:

1. *For any $q \in [0, 3)$ and $p \geq 1$ such that $1/p > 1/q - 1/3$, the embedding $W^{1,p}(\mathbb{S}^2)$ in $L^p(\mathbb{S}^2)$ is compact.*

2. For any $q > 0$, the embedding of $W^{1,q}(\mathbb{S}^2)$ into $C^0(\mathbb{S}^2)$ is compact.

For completeness we state two inequalities used extensively in the later parts of the thesis.

Lemma 2.2.1 (Hölder's inequality)**.**
Let $p \in [1, \infty]$ and let $p' = p/(p-1)$ denote the Hölder conjugate *of p. Note that if $p = \infty$, then $p' = 1$. Let $f \in L^{p'}(\mathbb{S}^2)$ and let $g \in L^p(\mathbb{S}^2)$. It holds that*

$$\int_{\mathbb{S}^2} fg \, \mathrm{d}A \leq \|g\|_{L^p(\mathbb{S}^2)} \, \|f\|_{L^{p'}(\mathbb{S}^2)}. \tag{2.7}$$

Lemma 2.2.2 (Young's product inequality)**.**
Let $a, b \in \mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$. Let $p \in [1, \infty)$ and denote by p' the Hölder conjugate of p. It then holds that

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

If $p = 2$, then $p' = 2$ and the inequality is of the form that is later used extensively,

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2},$$

for any $\epsilon > 0$.

2.3 Elements from probability theory

2.3.1 Probability backgrounds

Throughout this section, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability triple, meaning that Ω is the sample space of all outcomes, $\mathcal{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure, meaning that $\mathbb{P}(\Omega) = 1$ in addition to the other axioms that define a measure. With the basic notation clarified, we can now introduce some necessary basics from probability theory.

Definition 2.3.1 (Random variables, density)**.** A real-valued *random variable* is a $\mathcal{B}(\mathbb{R})$-measurable mapping $X : \Omega \to \mathbb{R}$ where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ–algebra of $\mathbb{R}$. The *distribution* is given by the push-forward measure $X_*\mathbb{P}(E) = \mathbb{P}(X \in E), E \in \mathcal{B}(\mathbb{R})$. The *cumulative distribution function* of a random variable will be given by

$$F_X(x) = \mathbb{P}(X \in (-\infty, x])) = \mathbb{P}\left(X^{-1}\left((-\infty, x]\right)\right),$$

and the *probability density function* is given by the Radon–Nikodym derivative of $\mathbb{P}$ with respect to the Lebesgue measure on $\mathbb{R}$ and is denoted by f_X.

Definition 2.3.2 (Mean and covariance)**.** The *mean* of a random variable is defined by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\ \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}} x\ \mathrm{d}F_X(x),$$

which in the case of continuous random variables further simplifies to the well-known expression $\int_{\mathbb{R}} x f_X(x)\ \mathrm{d}x$. The *covariance* of two random variables X and Y is given by

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])\,(Y - \mathbb{E}[Y])].$$

Definition 2.3.3 (Gaussian random variable)**.** A random variable X is said to be *Gaussian* with mean m and variance σ^2 if the density f is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}.$$

By standard normal random variable one denotes the special case of $m = 0, \sigma = 1$. Furthermore, an n-dimensional random vector $(X_1, ..., X_n)$ is said to be multivariate Gaussian if for every selection of $(\alpha_i)_1^n \subset \mathbb{R}^n$ it holds that

$$\sum_{i=1}^{n} \alpha_i X_i,$$

is Gaussian. In the case of a multivariate Gaussian random vector, the *mean vector* is given by

$$m_j = \mathbb{E}[X_j],\ j = 1, ..., n,$$

and the *covariance matrix*, C is given by

$$C_{ij} = \mathrm{Cov}[X_i, X_j],\ 1 \leq i, j \leq n.$$

There are two lemmas which are of use in the later parts of the thesis, namely Chebyshev's inequality and the Borel–Cantelli lemma which will be used when proving $\mathbb{P} - a.s$ asymptotic error estimates in 4.4.

Lemma 2.3.1 (Chebyshev's inequality)**.**
Let $f \in L^p(\Omega; L^2(\mathbb{S}^2))$ It holds that

$$\mathbb{P}(\|f\|_{L^2(\mathbb{S}^2)} > \epsilon) \leq \epsilon^{-p} \leq \mathbb{E}[\|f\|^p_{L^2(\mathbb{S}^2)}].$$

Lemma 2.3.2 (Borell–Cantelli)**.**
Let $(E_i)_{i=1}^{\infty}$ be events such that

$$\sum_{i=1}^{\infty} \mathbb{P}(E_i) < \infty.$$

It then holds that

$$\mathbb{P}(\limsup_{i \to \infty} E_i) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} \bigcup_{j \geq i}^{\infty} E_j \right) = 0.$$

This should be interpreted as the probability of an event occurring infinitely many times is zero.

2.3.2 Random variables in Banach and Hilbert spaces

Before beginning with the definitions in the case of random variables taking values in Banach spaces, it is good to note that a lot of the theory carries over from real-valued random variables.

In analogue with the real-valued case by a random variable X, we denote a measurable mapping taking values in a Banach space, and by its distribution the push-forward measure. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, then so is the distribution on $(B, \mathcal{B}(B))$. As in the real-valued case, we can define the expectation by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) \, \mathrm{d}\mathbb{P}(\omega) = \int_{B} x \, \mathrm{d}F_X(x),$$

if the random variable is Bochner–integrable. The higher order moments require a bit more care to introduce properly.

Definition 2.3.4. Let $(B, \|\cdot\|_B)$ be a Hilbert space and let $X, Y \in L^2(\Omega; B)$. The covariance is an element of $B \otimes B$, which is the tensor product space of Definition 2.1.19. This element is given by

$$\mathrm{Cov}\,[X] := \mathbb{E}\left[(X - \mathbb{E}[X]) \otimes (X - \mathbb{E}[X])\right].$$

The variance is given by

$$\begin{aligned} \mathrm{Var}[X] &= \mathbb{E}\left[\|(X - \mathbb{E}[X]) \otimes (X - \mathbb{E}[X])\|_{B\otimes B}\right] \\ &= \mathbb{E}[\|X - \mathbb{E}[X]\|_B^2] = \|X - \mathbb{E}[X]\|^2_{L^2(\Omega;B)}. \end{aligned}$$

There is a unique operator $Q \in L_N^+(B)$ so that

$$(\mathrm{Cov}[X], \phi \otimes \psi)_{B\otimes B} = (Q\phi, \psi)_B,$$

for $\phi, \psi \in B$.

Note that this theoretical framework for instance gives meaning to expressions such as

$$\left(\mathbb{E}[\|X - Y\|^2_{L^2(\mathcal{D})}]\right)^{1/2} = \left(\int_{\Omega} \|X - Y\|^2_{L^2(\mathcal{D})} \mathrm{d}\mathbb{P}\right)^{1/2},$$

where $\mathcal{D}$ is some domain and Ω is a probability space. In the Bochner framework, this is the norm in the space $L^2(\Omega; L^2(\mathcal{D}))$.

2.3.3 Random fields

A stochastic process for which the index set is a more general object than a subset of $\mathbb{R}$ is known as a random field. In this thesis, we will consider random fields on the sphere.

Definition 2.3.5. A mapping $Z : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ that is $\mathcal{F} \otimes \mathcal{B}(\mathbb{S}^2)$-measurable is called a real-valued random field on the unit sphere, where $\mathcal{B}(\mathbb{S}^2)$ denotes the Borel σ–algebra of $\mathbb{S}^2$.

The mean function $m : \mathbb{S}^2 \to \mathbb{R}$ is given by

$$m(x) = \mathbb{E}[Z(x)] = \int_\Omega Z(\omega, x) \, \mathrm{d}\mathbb{P}(\omega).$$

The covariance function $C : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ is for $x, y \in \mathbb{S}^2$ given by

$$C(x, y) = \operatorname{Cov}(Z(x), Z(y)).$$

The field is said to be *centred* if the mean function is equal to zero for every $x \in \mathbb{S}^2$.

A random field Z is said to be *strongly isotropic* if for every $k \in \mathbb{N}, (x_1, ..., x_k) \in \mathbb{S}^2$ and $g \in SO(3)$ it holds that,

$$(Z(x_1), ..., Z(x_k)) =_{\text{law}} (Z(g(x_1)), ..., Z(g(x_k)))\,,$$

where $SO(3)$ denotes the rotation group of all rotations in $\mathbb{R}^3$ around the origin equipped with the operation of composition.

A random field Z is said to be *weakly isotropic* for $n \geq 2$ if $\mathbb{E}[|Z(x)|^n] < \infty$ for every $x \in \mathbb{S}^2$ and every $k = 1, ..., n$, $x_1, ..., x_k \in \mathbb{S}^2$ and $g \in SO(3)$ it holds that

$$\mathbb{E}[Z(x_1) \cdots Z(x_k)] = \mathbb{E}[Z(g(x_1)) \cdots Z(g(x_k))].$$

A random field Z is Gaussian if for every $k \in \mathbb{N}, (x_1, ..., x_k) \in \mathbb{S}^2$ it holds that $(Z(x_1), ..., Z(x_k))$ is *multivariate Gaussian.* A Gaussian random field is hereafter abbreviated by GRF.

In the special case of a GRF, it can be shown that the condition of the field being 2-weakly isotropic is equivalent with the field being strongly isotropic. If the random field is isotropic, the covariance function will be a function of the distance between two points, that is to say,

$$C(x, y) = C(d(x, y)),$$

where $d(\cdot, \cdot)$ denotes the geodesic distance. It is also necessary to introduce the concept of *modification.*

Definition 2.3.6 (Modification)**.** Let Z_1, Z_2 be two random fields on $\mathbb{S}^2$. If for every $x \in \mathbb{S}^2$ it holds that

$$\mathbb{P}(Z_1(x) = Z_2(x)) = 1,$$

then Z_1 is a *modification* of Z_2. If two fields are modifications of each other, then fields will have the same law. If it holds that the field $Z : \mathbb{S}^2 \to \mathbb{R}$ has a modification with surely continuous sample paths, then the modification is known as a *continuous modification.*

In order to give meaning to *smoothness* of random fields, we introduce the concepts of mean-square continuity and differentiability.

Definition 2.3.7 (Mean-square continuity)**.** Let Z be a random field on $\mathbb{S}^2$. Let $(y_i)_{i=1}^{\infty} \subset \mathbb{S}^2$ be a sequence of points on $\mathbb{S}^2$ for which there is a point $y \in \mathbb{S}^2$ such that

$$\lim_{i\to\infty} d(y, y_i) = 0.$$

The field is said to be *mean-square continuous* if

$$Z(y_i) \xrightarrow{L^2} Z(y),$$

as $i \to \infty$, that is to say, $Z(y_i)$ converges in mean square to $Z(y)$, which means that

$$\lim_{i\to\infty} \mathbb{E}[(Z(y_i) - Z(y))^2] = 0.$$

Just as we can define continuity in mean-square sense, we can define mean-square differentiability. In order to do this, we employ the definition of [11]. In order to define mean-square differentiability we must first define the concept of a great-circle on the sphere. For every $\theta \in [0, \pi)$, there will be a circle which is obtained by considering all $\phi \in [0, 2\pi)$. A great-circle $\mathbb{X}$ is an isometric isomorphism to $\mathbb{S}^1$, and $\mathbb{S}^1$ is isometrically isomorphic with $[0, 2\pi)$ if the distance on $[0, 2\pi)$ is defined by $d(a, b) = \max(|a - b|, 2\pi - |a - b|)$. According to [11], there is therefore a distance-preserving mapping $\phi : \mathbb{X} \to [0, 2\pi)$.

Definition 2.3.8 (Mean-square differentiability)**.** Let Z be a random field on $\mathbb{S}^2$. Let $\mathbb{X}$ be a great-circle on $\mathbb{S}^2$. Denote for any $x \in \mathbb{S}^2$ by $Z_{\mathbb{X}}(\phi(x))$ the restriction of Z to $\mathbb{X}$ at x. Z is mean-square differentiable at x if there exists a random field $Z^1_{\mathbb{X}}(\phi(x))$

$$\frac{Z_{\mathbb{X}}(\phi(x) + \epsilon) - Z(x)}{\epsilon} \xrightarrow{L^2} Z^1_{\mathbb{X}}(x),$$

as $\epsilon \to 0$. If the process is mean-square differentiable at every point $x \in \mathbb{S}^2$, it is said to be *mean-square differentiable*. Note that if the field is isotropic, it suffices to have mean square differentiability at one great-circle in order to have mean-square differentiability. It is possible to define higher-order derivatives in the same manner.

Mean-square properties can often be deduced from the covariance behaviour of the field. Note here that mean-square properties in general say nothing about sample path properties.

A very useful special case of a random field on the sphere is the so-called *white noise*. In order to properly introduce these objects we first give the definition of a generalised Gaussian field.

Definition 2.3.9 (Generalised Gaussian field)**.** A measurable function $Z : \mathbb{S}^2 \to \mathbb{R}$ is a *generalised Gaussian field* if for any set of test functions $\{\phi_i \in L^2(\mathbb{S}^2), i = 1, ..., n\}$ the vector $\{(Z, \phi_i)_{L^2(\mathbb{S}^2)}, i = 1, ..., n\} \in \mathbb{R}^n$ is jointly Gaussian.

Definition 2.3.10 (Gaussian white noise on $\mathbb{S}^2$)**.** *Gaussian white noise* on $\mathbb{S}^2$, denoted by $\mathcal{W}$, is a generalised Gaussian field which is zero-mean and bounded in $L^2(\mathbb{S}^2)$-norm. In addition it holds that for any set of test functions $\{\phi_i \in L^2(\mathbb{S}^2), i = 1, ..., n\}$, the vector $\{(\mathcal{W}, \phi_i), i = 1, ..., n\}$, apart from being jointly Gaussian with mean zero, has covariance given by

$$\operatorname{Cov}\Big((\mathcal{W}, \phi_i)_{L^2(\mathbb{S}^2)}, (\mathcal{W}, \phi_j)_{L^2(\mathbb{S}^2)}\Big) = (\phi_i, \phi_j)_{L^2(\mathbb{S}^2)}.$$

When the random field does not use the sphere as index set but rather a subset of the Euclidean space, one can specify a special class of isotropic centred Gaussian random fields by giving a special covariance function, which for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ is given by

$$C_\nu(\|\mathbf{x} - \mathbf{y}\|) = \operatorname{Cov}[Z(\mathbf{y}), Z(\mathbf{x})] = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \Big(\alpha\|\mathbf{x} - \mathbf{y}\|\Big)^\nu K_\nu\Big(\alpha\|\mathbf{x} - \mathbf{y}\|\Big),$$

where K_ν is the modified Bessel function of the second kind. The parameters σ^2 and α are positive, and ν is known as the smoothness parameter which influences the mean-square continuity properties of the field by determining the differentiability properties of the covariance function.

On the sphere, however, it is not possible to specify this type of fields in this way, instead one *defines* them as the solution of the fractional elliptic stochastic partial differential equation of the form [2]

$$\mathcal{L}^\beta u = \Big(\kappa^2 - \Delta_{\mathbb{S}^2}\Big)^\beta u = \mathcal{W}, \tag{2.8}$$

where in our case, $\beta \in (1/2, 1]$, $\kappa > 0$ and $\mathcal{W}$ denotes white noise on the sphere.

However, as we later shall see, white noise on the sphere has some pathological properties that make it less than ideal to work with from a computational point of view. For instance, it is not feasible in practice to use truncated Karhunen–Loève expansions of the field. Our solution to this problem is to not consider $\mathcal{W}$ but rather a field with properties more suitable for our needs.

2.4 Linear elliptic PDEs on the sphere

In this section we establish some necessary properties of elliptic PDEs on the sphere. In principle, the 3-sphere is not different from $\mathbb{R}^2$, and hence, many properties that hold for elliptic PDEs and Sobolev spaces on $\mathbb{R}^n$ map over by using charts and partition of unity arguments. Throughout this section, let $L = -\Delta_{\mathbb{S}^2} + X$, where X is a general first-order linear differential operator with smooth coefficients, which will be the case in our applications. This means that

$$X = \sum_{k=1}^{\infty} \partial^i b_i + c,$$

where one usually assumes some sufficient smoothness of the coefficients b_i and c, typically that $b_i, c \in L^\infty(\mathbb{S}^2)$ or $C^\infty(\mathbb{S}^2)$. We will omit any discussion on the properties of the coefficients, and refer to for instance [8] or [12] for an extensive treatment of the subject.

In order to see that the operator $\kappa^2 - \Delta_{\mathbb{S}^2}$ can be written on the form $-\Delta_{\mathbb{S}^2} + X$, one sets $X = \kappa^2 I_{H^1(\mathbb{S}^2)}$. Since $\kappa^2 I_{H^1(\mathbb{S}^2)}$ is a zero-order differential operator which trivially is a first order differential operator with smooth coefficients, we indeed have a differential operator on this form.

Note that in our case, L will be viewed as an operator that maps from $H^1(\mathbb{S}^2)$ to $H^{-1}(\mathbb{S}^2)$.

Existence and uniqueness follow directly from the use of the Lax–Milgram theorem. Recall that we require that the bilinear form $a : H^1(\mathbb{S}^2) \times H^1(\mathbb{S}^2) \to \mathbb{R}$ associated with the operator L is:

1. Bounded, meaning that there is a constant $C > 0$ such that $|a(u, v)| \leq C\|u\|_{H^1(\mathbb{S}^2)}\|v\|_{H^1(\mathbb{S}^2)}$ for every $u, v \in H^1(\mathbb{S}^2)$.
2. Coercive, that is to say that there is a constant $c > 0$ such that $a(u, u) \geq c\|u\|^2_{H^1(\mathbb{S}^2)}$ for every $u \in H^1(\mathbb{S}^2)$.

Then, for any $v \in H^1(\mathbb{S}^2)$, and every $f \in H^{-1}(\mathbb{S}^2)$, we have that there is a unique $u \in H^1(\mathbb{S}^2)$ such that

$$a(u, v) = f(v),$$

in other words, we have existence and uniqueness. By Chapter 5.1 in [12] it holds for $u \in H^1(\mathbb{S}^2)$ and $Lu = f \in H^{k-1}(\mathbb{S}^2)$, $k \geq 0$, that $u \in H^{k+1}(\mathbb{S}^2)$. This means, as an example, that if $f \in L^2(\mathbb{S}^2)$ and $u \in H^1(\mathbb{S}^2)$ is the solution of Poisson's equation, then we know that we will have two weak derivatives of u. This will come into use in for instance error estimates in the later treatment of the numerical methods. We have the following estimate for some $c > 0$

$$\|u\|^2_{H^{k+1}(\mathbb{S}^2)} \leq c\|f\|^2_{H^{k-1}(\mathbb{S}^2)} + c\|u\|^2_{H^k(\mathbb{S}^2)}. \tag{2.9}$$

Note that this in the case of $k = 1$ reduces to the estimate

$$\|u\|_{H^2(\mathbb{S}^2)} \leq c\|f\|_{L^2(\mathbb{S}^2)} + c\|u\|^2_{H^1(\mathbb{S}^2)}.$$

Note that neither of these regularity estimates is directly applicable to the later error analysis. In our case, the bilinear form that appears is coercive. We will use this observation to estimate $\|u\|_{H^1(\mathbb{S}^2)}$. Note that the coercivity of the bilinear form

$a : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ implies that

$$c\|u\|^2_{H^1} \leq a(u, u).$$

Recall now that $u \in H^1(\mathbb{S}^2)$ is the weak solution of the problem, meaning that, it holds that $a(u, u) = (f, u)_{L^2}$. By Cauchy-Schwartz and then Young's product inequality (Lemma 2.2.2), for an $\epsilon > 0$

$$\begin{aligned} c\|u\|_{H^1}^2 &\leq a(u,u) = (f,u)_{L^2(\mathbb{S}^2)} \leq \|f\|_{L^2(\mathbb{S}^2)}\|u\|_{L^2(\mathbb{S}^2)} \\ &\leq \frac{1}{2\epsilon}\|f\|_{L^2(\mathbb{S}^2)}^2 + \frac{\epsilon}{2}\|u\|_{L^2(\mathbb{S}^2)}^2 \leq \frac{1}{2\epsilon}\|f\|_{L^2(\mathbb{S}^2)}^2 + \frac{\epsilon}{2}\|u\|_{H^1(\mathbb{S}^2)}^2, \end{aligned}$$

where in the final step the fact that the $H^1(\mathbb{S}^2)$-norm bounds the $L^2(\mathbb{S}^2)$-norm is used. Selecting $\epsilon = c$, and rearranging the terms yields that

$$\frac{c}{2}\|u\|_{H^1}^2 \leq \frac{1}{2c}\|f\|_{L^2(\mathbb{S}^2)}^2,$$

so it holds that

$$\|u\|_{H^1}^2 \leq \frac{1}{c^2}\|f\|_{L^2(\mathbb{S}^2)}^2,$$

and hence we can obtain the final regularity estimate,

$$\|u\|_{H^2(\mathbb{S}^2)} \leq c\|f\|_{L^2(\mathbb{S}^2)}. \tag{2.10}$$

2.5 Spherical harmonics

In our later presentation, in which we will consider stochastic partial differential equations on the sphere which will be driven by random fields, it is desirable to obtain Karhunen–Loève expansions of these fields. For that we need a basis for $L^2(\mathbb{S}^2)$. We first introduce the spherical harmonics on $\mathbb{S}^2$. In order to do this, we first introduce the Legendre polynomials. The treatment closely follows that of [13], to which the reader is referred to for proofs of the statements.

Definition 2.5.1 (Legendre polynomials, Legendre functions)**.** By Rodrigues formula the *Legendre polynomials* are given by

$$P_l(\mu) = 2^{-l}\frac{1}{l!}\frac{\partial^l}{\partial\mu^l}(\mu^2-1)^l,$$

where $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mu \in [-1,1]$. Given the Legendre polynomials, it is possible to obtain the *associated Legendre functions* for $m = 0, ..., l$ by

$$P_{lm} = (-1)^m(1-\mu^2)^{m/2}\frac{\partial^m}{\partial\mu^m}P_l(\mu).$$

Definition 2.5.2 (Surface spherical harmonics)**.** The surface spherical harmonics $Y_{l,m} : [0,\pi] \times [0,2\pi) \to \mathbb{C}$ are given by

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{l,m}(cos(\theta))e^{im\phi},$$

where $l \in \mathbb{N}_0, m = 0, ..., l$ and $(\theta,\phi) \in [0,\pi] \times [0,2\pi)$. and $Y_{l,m} := (-1)^m\,\overline{Y_{l,-m}}$ for $l \in \mathbb{N}$, $m = -l, ..., -1$.

Theorem 2.5.1 (Properties of the spherical harmonics)**.**
In this theorem we collect some properties of the spherical harmonics.

1. *The spherical harmonics form a basis of $L^2(\mathbb{S}^2)$, so every real-valued function $f \in L^2(\mathbb{S}^2)$ admits a spherical harmonic expansion*

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} Y_{l,m},$$

where $f_{l,m} = (-1)^m \overline{f_{l,-m}}$.

2. *The spherical harmonics are eigenfunctions of the Laplace–Beltrami operator, that is to say $\Delta_{\mathbb{S}^2} Y_{l,m} = -(l(l+1))Y_{l,m}$.*

3. *It holds that*

$$Y_{l,m} = (-1)^m \overline{Y_{l,-m}(x)},$$

for every $l \in \mathbb{N}_0, m = -l, ..., l$.

4. *The spherical harmonics are orthonormal in $L^2(\mathbb{S}^2)$, that is to say,*

$$\int_{\mathbb{S}^2} Y_{l,m} Y_{l',m'} \mathrm{d}\ A(x) = \delta_{l,l'} \delta_{m,m'}.$$

5. *For every $x, y \in \mathbb{S}^2$ it holds that*

$$\sum_{m=-l}^{l} Y_{l,m}(x) \overline{Y_{l,m}}(y) = \frac{2l+1}{4\pi} P_l((x,y)_{\mathbb{R}^3}).$$

6. *It turns out that, denoting by $\mathcal{H}_l(\mathbb{S}^2)$ the span of the spherical harmonics for a fixed l, that*

$$L^2(\mathbb{S}^2) = \bigoplus_{l \in \mathbb{N}_0} \mathcal{H}_l(\mathbb{S}^2).$$

Theorem 2.5.2 (Properties of 2-weakly isotropic random fields)**.**
Let Z be a 2-weakly isotropic RF. Then

1. $$\mathbb{P}\left(\int_{\mathbb{S}^2} Z(x)^2 \ \mathrm{d}A < \infty \right) = 1.$$

2. *The field Z admits a Karhunen–Loève expansion,*

$$Z(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{l,m} Y_{l,m}(x),$$

where $a_{l,m} := \int_{\mathbb{S}^2} Z(y) \overline{Y_{l,m}}(y) \mathrm{d}\ A(y)$.

3. *The Karhunen–Loève expansion converges both in* $L^2(\Omega \times \mathbb{S}^2, \mathbb{R})$,

$$\lim_{L\to\infty} \mathbb{E}\left[\int_{\mathbb{S}^2} \left(Z(y) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(y)\right)^2 dA(y)\right] = 0,$$

as well as in $L^2(\Omega, \mathbb{R})$ *for any fixed* $y \in \mathbb{S}^2$, *that is to say that,*

$$\lim_{L\to\infty} \mathbb{E}\left[\left(Z(y) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m}(y)\right)^2\right] = 0.$$

This means that every 2-weakly isotropic random field is an element of $L^2(\Omega; L^2(\mathbb{S}^2))$. In order to obtain a more efficient simulation method, one can use properties of the coefficients appearing in the Karhunen–Loève expansion. Let therefore

$$\mathbb{A} = \{a_{l,m}, l \in \mathbb{N}_0, m = -l, ..., l\}.$$

The following theorem summarises some useful properties of the elements of $\mathbb{A}$.

Theorem 2.5.3 (Properties of $\mathbb{A}$)**.**
Let Z *be a strongly isotropic random field on* $\mathbb{S}^2$ *with Karhunen–Loève cofficients* $\mathbb{A}$. *The coefficients of the sequence* $\mathbb{A}$ *are all centred random variables apart from* $a_{0,0}$. *Furthermore, there exists a sequence* $(A_l)_{l\in\mathbb{N}_0}$ *of nonnegative real numbers, known as the* angular power spectrum *such that for all pairs* $l_1, l_2 \in \mathbb{N}$ *and* $m_i = -l_i, ..., l_i$, $i = 1, 2$, *it holds that*

$$\mathbb{E}[a_{l_1,m_1}\overline{a_{l_2,m_2}}] = A_{l_1}\delta_{l_1,l_2}\delta_{m_1,m_2}, \tag{2.11}$$

where $\delta_{x,y} = 1$ *if* $x = y$ *and zero otherwise. For the first element* $a_{0,0}$, *it holds instead that*

$$\mathbb{E}[a_{0,0}\overline{a_{l,m}}] = (A_0 + \mathbb{E}[a_{0,0}]^2)\delta_{0,l_2}\delta_{0,m_2}$$

The random variables $a_{l,m}$ *and* $a_{l,-m}$ *satisfy for* $l \in \mathbb{N}$ *and* $m = 1, ..., l$ *that*

$$a_{l,m} = (-1)^m\overline{a_{l,-m}}.$$

In the case of strongly isotropic Gaussian random fields, we have more properties. Denote by

$$\mathbb{A}_+ := (a_{l,m}, l \in \mathbb{N}_0, m = 0, ..., l).$$

Theorem 2.5.4 (Properties of isotropic Gaussian random fields)**.**
Let Z *be a 2-weakly isotropic Gaussian random field and let the setting be as in Theorem 2.5.2. It then holds that:*

1. *The elements of* $\mathbb{A}_+$ *are independent, centred, complex-valued Gaussian random variables for which it for* $m > 0$ *holds that the real part of* $a_{l,m}$, $\mathfrak{R}[a_{l,m}]$ *and the imaginary part of* $a_{l,m}$, $\mathfrak{Im}[a_{l,m}]$ *are uncorrelated and equal in law, with variance*
$$\mathrm{Var}(\mathfrak{R}[a_{l,m}]) = \mathrm{Var}(\mathfrak{Im}[a_{l,m}]) = A_l/2.$$

2. *If* $m = 0$, *the elements of* $\mathbb{A}_+$ *are real-valued with variance* A_l *if* $l > 0$, *while* $a_{0,0}$ *is distributed as* $\mathcal{N}(2\sqrt{\pi}\ \mathbb{E}[Z], A_0)$.

3. *The elements of* $\mathbb{A} \setminus \mathbb{A}_+$, *which are the elements of* $\mathbb{A}$ *with* $m < 0$, *are obtained from those of* $\mathbb{A}_+$ *by the relation*
$$a_{l,m} = (-1)^m \overline{a_{l,-m}}.$$

We can use the above theory to rewrite the series expansion of an isotropic Gaussian random field.

Theorem 2.5.5.
Let Z *be a centred isotropic Gaussian random field. For* $l \in \mathbb{N}_0, m = -l, ..., l$, *and* $\theta \in [0, \pi]$, *set*

$$L_{l,m}(\theta) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos(\theta)). \tag{2.12}$$

It then holds in law that

$$Z(x) = \sum_{l=0}^{\infty} \sqrt{A_l} z_{l,0}^1 L_{l,0}(\theta) + \sqrt{2A_l} \sum_{m=1}^{l} L_{l,m}(\theta) \left(z_{l,m}^1 \cos(m\phi) + z_{l,m}^2 \sin(m\phi) \right),$$

where $\{(z_{l,m}^1, z_{l,m}^2), l \in \mathbb{N}_0, m = 1, ..., l\}$ *is a sequence of independent standard normal random variables such that* $z_{l,0}^2 = 0$ *for every* $l \in \mathbb{N}_0$.

3

Solving the equation

3.1 Trace class field as right hand side

We have so far assumed our noise to be Gaussian white noise. As previously hinted upon, there are issues with white noise, one being that it is difficult to approximate white noise since its covariance operator is given by the identity operator I, and it hence holds that

$$\mathrm{Tr}(I) = \infty,$$

which implies that the L^2-norm is infinite. This means that simulation by means of truncated spherical harmonics is not feasible in practice, since there will be no observed convergence of the truncation method.

Furthermore, by Theorem 2 in [14], it is not possible to define a field with Matérn covariance function by specifying the covariance function on the sphere, but rather, as done by [2], one must define the Matérn field on the sphere as the solution to the SPDE in Equation (2.8).

For this reason, by replacing $\mathcal{W}$ with a different type of noise $\overline{\mathcal{W}}$, we will obtain a different type of solution to the equation. We make some assumptions on $\overline{\mathcal{W}}$, namely that:

1. The covariance operator Q has the property that $\mathrm{Tr}(Q) < \infty$, and hence it holds that $\overline{\mathcal{W}} \in L^2(\Omega, L^2(\mathbb{S}^2))$.

2. The power spectrum decays algebraically with order $\alpha > 2$, meaning that there exists a constant $C > 0$ and a $l_0 \in \mathbb{N}$ such that for every $l > l_0$,

$$A_l \leq C l^{-\alpha}.$$

By using this, we simplify the problem at hand. The equation of interest is now given by

$$\mathcal{L}^\beta u = \overline{\mathcal{W}} = Q^{1/2}\mathcal{W}.$$

This implies that the distribution of the solution to Equation (2.8) with the right-hand side replaced by $\overline{\mathcal{W}}$ is given by

$$u \sim \mathcal{N}(0, \mathcal{L}^{-\beta} Q \mathcal{L}^{-\beta}).$$

We can, using the theory outlined in section 2.5, give a Karhunen–Loève expansion of the noise.

$$\overline{\mathcal{W}}(y) = \sum_{l=0}^{\infty} \left(\sqrt{A_l} z_{l,0}^1 L_{l,0}(\theta) + \sqrt{2A_l} \sum_{m=1}^{l} \left[L_{l,m}(\theta) \left((z_{l,m}^1 \cos(m\phi) + z_{l,m}^2 \sin(m\phi) \right) \right] \right),$$

where $y \in \mathbb{S}^2$ and $\{(z_{l,m}^1, z_{l,m}^2), l \in \mathbb{N}_0, m = 0, ..., l\}$ is a sequence of independent standard normal random variables such that $z_{l,0}^2 = 0$ for all l and $L_{l,m}$ are given by Equation (2.12). We can approximate $\overline{\mathcal{W}}$ on $\mathbb{S}^2$ by truncating the series with a suitable $\mathfrak{K} \in \mathbb{N}$ so that

$$\overline{\mathcal{W}}(y)_{\mathfrak{K}} = \sum_{l=0}^{\mathfrak{K}} \left(\sqrt{A_l} z_{l,0}^1 L_{l,0}(\theta) + \sqrt{2A_l} \sum_{m=1}^{l} \left[L_{l,m}(\theta) \left(z_{l,m}^1 \cos(m\phi) + z_{l,m}^2 \sin(m\phi) \right) \right] \right). \tag{3.1}$$

3.2 Properties of the solution of $\mathcal{L}^\beta u = \overline{\mathcal{W}}$

In this section we collect some properties of the solution field of $\mathcal{L}^\beta u = (\kappa^2 - \Delta_{\mathbb{S}^2})^\beta u = \overline{\mathcal{W}}$. It is already noted that the distribution of this field will be $\mathcal{N}(0, \mathcal{L}^{-\beta} Q \mathcal{L}^{-\beta})$, where Q is the covariance operator of $\overline{\mathcal{W}}$.

We begin by considering the covariance function C of u, or the covariance between two points $x, y \in \mathbb{S}^2$. Note that we may, according to Section 2.5 expand $\overline{\mathcal{W}}$ using the Karhunen–Loève theorem according to

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}.$$

It holds that $Y_{l,m}$ are eigenfunctions to the spherical Laplacian, with eigenvalues $-l(l+1)$. Hence, $Y_{l,m}$ are also eigenfunctions to $\mathcal{L}$. The eigenvalues in this case can be found by applying $\mathcal{L}$ to $Y_{l,m}$, according to

$$\mathcal{L} Y_{l,m} = (\kappa^2 + \Delta_{\mathbb{S}^2}) Y_{l,m} = (\kappa^2 + (l+1)l) Y_{l,m},$$

so we see that the eigenvalue of $Y_{l,m}$ is $\kappa^2 + (l+1)l$. It also holds that the multiplicity of these eigenvalues is equal to $2l + 1$. This implies that the eigenfunctions of $\mathcal{L}^{-1}$ also will be $Y_{l,m}$, and the eigenvalues will be $(\kappa^2 + (l+1)l)^{-1}$. According to Definition 2.1.15, it holds, since $a_{l,m} = (\overline{\mathcal{W}}, Y_{l,m})_{L^2(\mathbb{S}^2;\mathbb{C})}$, that

$$u = \mathcal{L}^{-\beta} \mathcal{W} = \sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-\beta} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}. \tag{3.2}$$

The covariance kernel is therefore, utilising that $\mathbb{E}[a_{l_1,m_1} \overline{a_{l_2,m_2}}] = A_{l_1} \delta_{l_1,l_2} \delta_{m_1,m_2}$, given by,

$$\begin{aligned}
C(x,y) &= \mathbb{E}\left[u(x)\overline{u(y)}\right] \\
&= \mathbb{E}\left[\sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-\beta} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(u), \overline{\sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-\beta} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(y)} \right] \\
&= \sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-2\beta} A_l \sum_{m=-l}^{l} Y_{l,m}(x) \overline{Y_{l,m}(y)}.
\end{aligned}$$

Now, recall that $\sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(y)} = \frac{2l+1}{4\pi} P_l(\cos(d(x,y)))$. Hence, we see that the covariance function only depends on the geodesic distance of x and y, and therefore, the field u is a centred isotropic Gaussian random field. We can now apply Theorem 2.5.4 to equation (3.2) so that we can rewrite it according to

$$\begin{aligned} u &= \sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-\beta} \left(a_{l,0} Y_{l,0} + \sum_{m=1}^{l} a_{l,m} Y_{l,m} + a_{l,-m} Y_{l,-m} \right) \\ &= \sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-\beta} \left(a_{l,0} Y_{l,0} + \sum_{m=1}^{l} a_{l,m} Y_{l,m} + (-1)^m \overline{a_{l,m}} \overline{Y_{l,m}} \right) \\ &= \sum_{l=0}^{\infty} (\kappa^2 + l(l+1))^{-\beta} \left(a_{l,0} L_{l,0} + \sum_{m=1}^{l} 2\Re(a_{l,m} Y_{l,m}) \right), \end{aligned}$$

and hence, using the proof of Lemma 5.1 in [13], which relates $a_{l,m}$ to A_l we have in law that

$$\begin{aligned} u(\theta, \phi) = \sum_{l=0}^{\infty} \left(\kappa^2 + l(l+1)\right)^{-\beta} \Bigg(& \sqrt{A_l} z^1_{l,0} L_{l,0}(\theta) \\ & + \sqrt{2A_l} \sum_{m=1}^{l} L_{l,m}(\theta)(z^1_{l,m} \cos(m\phi) + z^2_{l,m} \sin(m\phi)) \Bigg), \end{aligned} \tag{3.3}$$

where $\{(z^1_{l,m}, z^2_{l,m}), l \in \mathbb{N}_0, m = 0, ..., l\}$ is a sequence of independent standard normal random variables such that $z^2_{l,0} = 0$ for every $l \in \mathbb{N}_0$.

We wish to elaborate on some sample properties of the solution field u. As shown in Section 4 of [13], one can relate the algebraic decay of the angular power spectrum with the $\mathbb{P}$-almost sure Hölder continuity of sample paths of an isotropic GRF. For completeness, we state this relation, which comes from Theorem 4.5 in [13].

Theorem 3.2.1.
Let Z be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_l, l \in \mathbb{N}_0)$. If for some $\xi \in (0, 2]$, the angular power spectrum satisfies

$$\sum_{l=0}^{\infty} A_l l^{1+\xi} < \infty,$$

then there exists a continuous modification of Z that is Hölder continuous with exponent $\mathfrak{g}$ for every $\mathfrak{g} < \xi/2$.

Let us try to apply this to our solution field u. We already know that the angular power spectrum of $\overline{\mathcal{W}}$ satisfies the assumption of summability, since we have assumed the algebraic decay condition that $A_l \leq Cl^{-\alpha}$ for some $l > l_0$. Hence we see that

$$\sum_{l=0}^{\infty} A_l l^{1+\xi} = \sum_{l=0}^{l_0} A_l l^{1+\xi} + \sum_{l=l_0+1}^{\infty} A_l l^{1+\xi}.$$

The first of the two terms will be finite for well-behaved power spectra. The second can be bounded by

$$\sum_{l=l_0+1}^{\infty} A_l l^{1+\xi} \leq C \sum_{l=l_0+1}^{\infty} l^{-\alpha+1+\xi},$$

which is finite if and only if $\xi < \alpha - 2$. We can now view the power spectrum of the solution field u as being given by

$$A_{u,l} = \frac{A_l}{(\kappa^2 + (l+1)l)^{2\beta}}.$$

We see that this is the case since inserting it into Equation (3.3) yields

$$u(\theta, \phi) = \sum_{l=0}^{\infty} \sqrt{A_{u,l}} z_{l,0}^1 L_{l,0}(\theta) + \sqrt{2A_{u,l}} \sum_{m=1}^{l} L_{l,m}(\theta)(z_{l,m}^1 \cos(m\phi) + z_{l,m}^2 \sin(m\phi)).$$

Now, $A_{u,l}$ also decays algebraically, since

$$A_{u,l} \leq C \frac{l^{-\alpha}}{(\kappa^2 + (l+1)l)^{2\beta}} \leq C \frac{l^{-\alpha}}{l^{4\beta}} = C l^{-(\alpha+4\beta)},$$

for every $l > l_0$. Hence, we see that the summability condition in Theorem 3.2.1 is satisfied for every $\xi < \alpha + 4\beta - 2$. As an example, selecting $A_l = (l+1)^{-3}$ and $\beta = 0.8$, would yield that the solution is $\mathfrak{g}$-Hölder continuous for every $\mathfrak{g} < 1$.

Note here that u can be viewed as the field obtained by a reweighting of the angular power spectrum of $\overline{\mathcal{W}}$. One can directly observe the regularising effect of the inverse operator $\mathcal{L}^{-\beta}$, since the exponent of the algebraic decay assumption becomes $-(\alpha + 4\beta)$. It is not difficult to show that if the exponent is larger than 6, the field is in $H^2(\mathbb{S}^2)$. In our case, we have that $\alpha + 4\beta > 6$ if $\beta = 1$ and $\alpha > 2$.

Note that we can determine some mean-square properties of the solution u. Let us for the moment being consider a general random field Z with an angular power spectrum which decays algebraically with order $2 + \epsilon$ with $\epsilon > 0$. By [15, Theorem A], it holds that Z is k times mean-square differentiable if $\epsilon > 2k$. In the case of u, we have seen that the angular power spectrum decays algebraically with order $\alpha + 4\beta$, where $\alpha > 2$. Hence, we conclude that u is k times integrable where k is the largest integer such that $\alpha + 4\beta - 2 > 2k$. If we were to relate this to the mean-square differentiability of the random fields on $\mathbb{R}^n$ with Matérn covariance of [3], which were differentiable $2k$ times, with $4\beta - n = 2k$, we see that we obtain a quite similar relationship.

3.3 Simulation of the solution of $\mathcal{L}^\beta u = \overline{\mathcal{W}}$ by truncation

Note that Equation (3.3) directly yields a simulation method for approximating u, simply by truncating the series expansion at a certain $l = \mathfrak{K}$, that is to say,

$$u_{\mathfrak{K}}(\theta, \phi) = \sum_{l=0}^{\mathfrak{K}} \Bigg(\sqrt{A_{l,u}} z_{l,0}^1 L_{l,0}(\theta) + \sqrt{2A_{l,u}} \sum_{m=1}^{l} L_{l,m}(\theta)(z_{l,m}^1 \cos(m\phi) + z_{l,m}^2 \sin(m\phi)) \Bigg). \tag{3.4}$$

The error of this method can be directly determined by the use of the results of [13], since u can be viewed as the field obtained by a reweighting of the angular power spectrum A_l of the right hand side.

Theorem 3.3.1 (Error of truncating spherical harmonic expansions)**.**
Let $u = \mathcal{L}^{-\beta}\overline{\mathcal{W}}$. Assume that $\overline{\mathcal{W}}$ satisfies the algebraic decay condition with $\alpha > 2$ for all $l > l_0$, that is to say, there is a parameter $C > 0$ such that for every $l > l_0$,

$$A_l \leq C l^{-\alpha}.$$

Denote by $u_{\mathfrak{K}}$ the truncated series in Equation (3.4), where $\mathfrak{K} > l_0$. It then holds that

$$\|u - u_{\mathfrak{K}}\|_{L^2(\Omega; L^2(\mathbb{S}^2))} \leq C \sqrt{\left(\frac{2}{\alpha + 4\beta - 2} + \frac{1}{\alpha + 4\beta - 1} \right) \mathfrak{K}^{-(\alpha + 4\beta - 2)/2}}.$$

Proof. Note that we can view u as a spherical harmonic expansion, but with angular power spectrum given not by A_l but by $\frac{A_l}{(\kappa^2 + (l+1)l)^{2\beta}}$. This implies that u will satisfy the algebraic decay condition. In order to determine with which parameter, we see that

$$\frac{A_l}{(\kappa^2 + (l+1)l)^{2\beta}} \leq C l^{-\alpha} \frac{1}{(\kappa^2 + (l+1)l)^{2\beta}} \leq C l^{-(\alpha + 4\beta)},$$

for $l > l_0$. Hence the algebraic decay parameter is $\alpha + 4\beta$, and therefore, using Proposition 5.2 of [13] and the definition of the Bochner norm and Equation (3.3) and (3.4), it holds that

$$\|u - u_{\mathfrak{K}}\|_{L^2(\Omega; L^2(\mathbb{S}^2))}^2 \leq C \left(\frac{2}{\alpha + 4\beta - 2} + \frac{1}{\alpha + 4\beta - 1} \right) \mathfrak{K}^{-(\alpha + 4\beta - 2)},$$

which proves the theorem. □

3.4 Solving the equation with white noise right hand side

In order to determine what the angular power spectrum of $\mathcal{W}$ is, we note that

$a_{l,m} = (\mathcal{W}, Y_{lm})_{L^2(\mathbb{S}^2)}$. It holds from the definition of white noise on the sphere that selecting any number of spherical harmonics, the resulting vector of $a_{l_1,m_1}, ..., a_{l_n,m_n}$ should have covariance

$$\mathrm{Cov}(a_{l_1,m_1}, a_{l_2,m_2}) = (Y_{l_j,m_i}, Y_{l_i,m_i})_{L^2(\mathbb{S}^2)} = \delta_{l_j,l_i}\delta_{m_j,m_i},$$

and hence, we see that for every $a_{l,m}$ and $a_{l',m'}$, it holds that

$$A_{\mathcal{W},l} = \mathbb{E}[a_{l,m}^2] = 1. \tag{3.5}$$

This means that we can view white noise as the field having power spectrum $A_{\mathcal{W},l} = 1$ for every $l \in \mathbb{N}_0$. We will now consider the solution of the equation

$$(\kappa^2 - \Delta_{\mathbb{S}^2})^\beta u = \mathcal{W}.$$

In the earlier part of this section, we observed the regularising effect of the inverse operator, namely that the angular power spectrum of the solution was given by

$$A_{l,u} = \frac{A_l}{(\kappa^2 + l(l+1))^{2\beta}},$$

where A_l is the power spectrum of $\overline{\mathcal{W}}$. The same thing holds in this case, so that the angular power spectrum of $u = \mathcal{L}^{-\beta}\mathcal{W}$ will be given by

$$A_{l,u} = \frac{1}{(\kappa^2 + l(l+1))^{2\beta}},$$

Note that we directly see that it must hold that $\beta \in (1/2, 1]$ for u to be in $L^2(\Omega; L^2(\mathbb{S}^2))$. The solution u itself will be given by

$$u(\theta,\phi) = \sum_{l=0}^{\infty} \sqrt{A_{u,l}} z_{l,0}^1 L_{l,0}(\theta) + \sqrt{2A_{u,l}} \sum_{m=1}^{l} L_{l,m}(\theta)(z_{l,m}^1 \cos(m\phi) + z_{l,m}^2 \sin(m\phi)),$$

where $(\theta,\phi) \in [0,\pi] \times [0, 2\pi)$ and $\{(z_{l,m}^1, z_{l,m}^2), l \in \mathbb{N}_0, m = 0, \cdots, l\}$ with $z_{l,0}^2 = 0$ for every $l \in \mathbb{N}_0$.

The field u is a solution of the equation which according to [2] defines Gaussian random fields with Matérn covariance on the sphere. The covariance is given by

$$\mathrm{Cov}(u(x), u(y)) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \frac{1}{((l+1)l + \kappa^2)^{2\beta}} P_l\left((x,y)_{\mathbb{R}^3}\right).$$

Note that we can directly apply the truncation-based simulation to u to obtain an approximation of Gaussian random fields on the sphere with Matérn covariance, meaning that it is possible to generate fields of this type without applying any finite elements methods.

3.4.1 A note on regularity of white noise on the sphere

It does not make sense to view $\mathcal{W}$ as an element of $L^2(\Omega; L^2(\mathbb{S}^2))$, since the algebraic decay condition is not satisfied for the angular power spectrum A_l. However, recall Definition 2.2.12 of the $H^{-\omega}(\mathbb{S}^2)$-norm, where $\omega > 0$. For $f \in H^{-\omega}(\mathbb{S}^2)$, it holds that

$$\|f\|_{H^{-\omega}(\mathbb{S}^2)} = \|u\|_{H^{2k-\omega}(\mathbb{S}^2)},$$

where $k \in \mathbb{N}$ is such that $2k - \omega > 0$, and u is the unique element of $H^{2k-\omega}(\mathbb{S}^2)$ such that

$$f = \mathcal{B}_{L^2(\mathbb{S}^2)}(-2k)u.$$

Recall that $f = \mathcal{B}_{L^2(\mathbb{S}^2)}(-2k)u = (1 - \Delta_{\mathbb{S}^2})^k u$. This means that $u = (1 - \Delta_{\mathbb{S}^2})^{-k} f = \mathcal{B}_{L^2(\mathbb{S}^2)}(2k)f$. By the definition of the Sobolev norm,

$$\|u\|_{H^{2k-\omega}(\mathbb{S}^2)} = \|\mathcal{B}_{L^2(\mathbb{S}^2)}(\omega - 2k)u\|_{L^2(\mathbb{S}^2)}.$$

In other words,

$$\|f\|_{H^{-\omega}(\mathbb{S}^2)} = \|\mathcal{B}_{L^2(\mathbb{S}^2)}(\omega - 2k)\mathcal{B}_{L^2(\mathbb{S}^2)}(2k)f\|_{L^2(\mathbb{S}^2)},$$

which gives that

$$\|f\|_{H^{-\omega}(\mathbb{S}^2)} = \|\mathcal{B}_{L^2(\mathbb{S}^2)}(\omega - 2k)\mathcal{B}_{L^2(\mathbb{S}^2)}(2k)f\|_{L^2(\mathbb{S}^2)} = \|\mathcal{B}_{L^2(\mathbb{S}^2)}(\omega)f\|_{L^2(\mathbb{S}^2)}.$$

We now determine the largest possible value of ω. Let $u \in L^2(\Omega; L^2(\mathbb{S}^2))$ be given by

$$u = \mathcal{B}_{L^2(\mathbb{S}^2)}(\omega)f,$$

and consequently,

$$u = (1 - \Delta_{\mathbb{S}^2})^{-\omega/2} \sum_{l=0}^{\infty} \sum_{m=-l^l} a_{l,m} Y_{l,m} = \sum_{l=0}^{\infty} \sum_{m=-l^l} \frac{a_{l,m}}{(1 + l(l+1))^{\omega/2}} Y_{l,m}.$$

Let us now verify for which ω it holds that $u \in L^2(\Omega; L^2(\mathbb{S}^2))$.

$$\|u\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))} = \mathbb{E}\left[\left\| \sum_{l=0^\infty} \sum_{m=-l^l} \frac{a_{l,m}}{(1 + l(l+1))^{\omega/2}} Y_{l,m} \right\|^2\right].$$

By the triangle inequality and using the orthogonality of the spherical harmonics to see that the cross-terms cancel out, we obtain

$$\begin{aligned}
\|u\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))} &\leq \sum_{l=0}^{\infty} \left(\frac{1}{(1 + l(l+1))^{\omega}} \|Y_{l,0}\|^2 + 2 \sum_{m=1}^{l} \frac{1}{(1 + (l+1)l)^{\omega}} \|Y_{l,m}\|^2 \right) \\
&= \sum_{l=0}^{\infty} \frac{2l+1}{(1 + l(l+1))^{\omega}}.
\end{aligned}$$

This sum converges for every $\omega > 1$, and hence we conclude that $u \in L^2(\Omega; L^2(\mathbb{S}^2))$ for every $\omega > 1$. Therefore, $\mathcal{W} \in L^2(\Omega; H^{-1-\epsilon}(\mathbb{S}^2))$ for every $\epsilon > 0$.

Previously, we viewed the operator $\mathcal{L}$ as a mapping from $H^1(\mathbb{S}^2)$ to $H^{-1}(\mathbb{S}^2)$. The above work shows that if we were to consider the SFEM method of this thesis with white noise instead, we would not only run into the computational issue of truncating the Karhunen–Loève expansion of white noise, but also the issue of not having the Gelfand triple setting of $H^1(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2) \hookrightarrow H^{-1}(\mathbb{S}^2)$. This means that many of the regularity estimates and methods used in the error analysis will not work.

It should be noted that Equation (2.8) is not the only type of fractional elliptic SPDE used to generate random fields. In certain cases, see for instance [3] and [4] the Laplacian might be replaced by a more general second-order elliptic differential operator, or the right-hand side of the equation might be more complicated, for instance containing terms for which a Karhunen–Loève based simulation is nonfeasible. Therefore, it is desirable to develop a method which allows for the numerical solution of more general problems. We will combine the surface finite elements of [6] with the fractional elliptic equation quadrature approach of [5] in order to develop a FEM method for problems of the type of (2.8).

4

Fractional elliptic surface finite elements

In this part we present the surface finite element method of [6] adapted to the sphere and show that the approach to fractional elliptic SPDEs from [3] and [2] can be adapted to the setting on the sphere, using only a modification of the right-hand side in the SPDE given by Equation (2.8).

4.1 Triangulation of the sphere

In order to obtain a surface finite element method on the sphere (SFEM), it is necessary to first develop a way to discretize the sphere. The idea is to approximate $\mathbb{S}^2$ with a piecewise polygonal surface. The sphere will be approximated by non-degenerate triangles. The vertices of these will be on $\mathbb{S}^2$. For two triangles T and $\tilde{T}$, it holds that either $\tilde{T} \cap T = \varnothing$ or their intersection is their common edge or point. $\mathcal{T}_h$ denotes the set of triangles making up the discretised sphere,

$$\mathbb{S}^2_h := \bigcup_{T_j \in \mathcal{T}_h} T_j.$$

Here h is the size of the largest triangle, which is defined as the *in-ball radius*, that is to say, the diameter of the smallest circle such that T can be inscribed in it.

In order to continue, it is necessary to introduce some theoretical framework.

Definition 4.1.1 (Oriented distance function)**.** Let $G = B_1(0) \subset \mathbb{R}^3$ denote the unit ball. Note that $\mathbb{S}^2 = \partial G$. The oriented distance function $\mathfrak{d} : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\mathfrak{d}(x) = \|x\| - 1. \tag{4.1}$$

This gives rise to the following lemma, [6, Lemma 2.8].

Lemma 4.1.1.
Let

$$U_\delta := \left\{ x \in \mathbb{R}^3 : |\mathfrak{d}(x)| < \delta \right\}.$$

Note that U_δ is a spherical shell with inner radius $1-\delta$ and outer radius $1+\delta$. It holds that

1. $\mathfrak{d} \in C^\infty(U_\delta)$*.*

2. For every point $x \in U_\delta$ *there exists a unique point* $a(x) \in \mathbb{S}^2$ *such that*

$$x = a(x) + \mathfrak{d}(x)\nu(a(x)).$$

This yields a global coordinate system for us to work in when developing a surface finite element method.

4.1.1 Triangulation, extension scheme

It holds for the discretized sphere $\mathbb{S}_h^2$ that

$$\mathbb{S}_h^2 \subset U_1.$$

According to [6], for every $x \in U_1$ it is possible to write

$$x = a(x) + \mathfrak{d}(x)\nu(a(x)),$$

hence it is possible to *lift* a function η defined on $\mathbb{S}_h^2$ to a function η^ℓ defined on $\mathbb{S}^2$ by

$$\eta^\ell(a) = \eta(x(a)), \ a \in \mathbb{S}^2,$$

where $x(a)$ denotes the solution of the equation

$$x = a + \mathfrak{d}(x)\nu(a). \tag{4.2}$$

Note that the lift is linear, since if

$$\eta = \sum_{i=1}^n \xi_i \eta_i,$$

where $\eta_i : \mathbb{S}_h^2 \to \mathbb{R}$ and $\xi_i \in \mathbb{R}$, then

$$\eta^\ell(a) = \eta(x(a)) = \sum_{i=1}^n \xi_i \eta_i(x(a)).$$

It then holds that $\eta_i^\ell(a) = \eta_i(x(a))$, so

$$\eta^\ell(a) = \sum_{i=1}^n \xi_i \eta_i^\ell(a).$$

Likewise, for a function $\varsigma : \mathbb{S}^2 \to \mathbb{R}$ we define the projected function $\varsigma^{-\ell} : \mathbb{S}_h^2 \to \mathbb{R}$ by

$$\varsigma^{-\ell}(x) = \varsigma(a(x)), \ x \in \mathbb{S}_h^2,$$

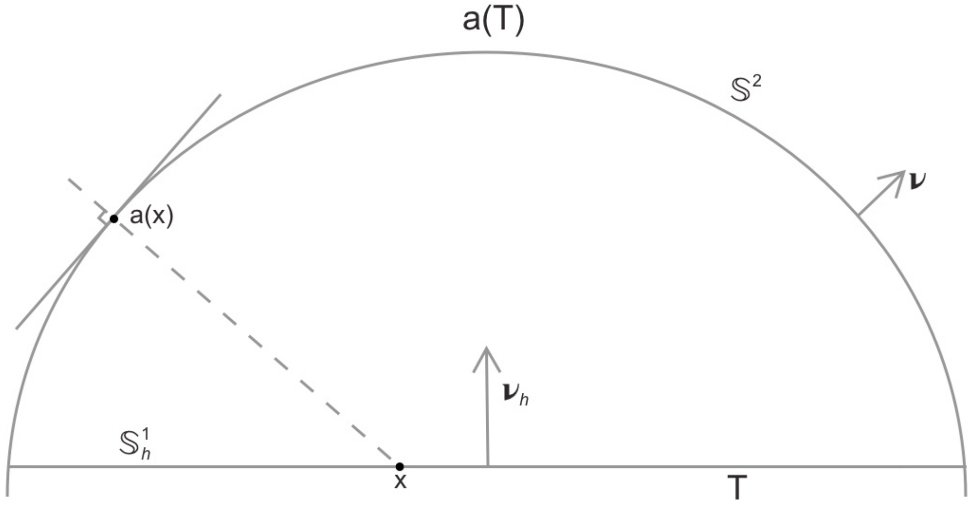


Figure 4.1: Illustration of lift on $\mathbb{S}^1$.

where $a(x)$ is given by solving $x = a + \mathfrak{d}(x)\nu(a)$. Note that it is possible to view the point on $\mathbb{S}^2$ obtained by the lift as a mapping $a : \mathbb{S}^2_h \to \mathbb{S}^2$, which will be given by

$$a(x) = \frac{x}{\|x\|_{\mathbb{R}^3}}.$$

In the case of the inverse lift, we can view the point obtained on $\mathbb{S}^2$ as a mapping $x : \mathbb{S}^2 \to \mathbb{S}^2_h$.

For every $T \in \mathcal{T}_h$, we define a lifted triangle $T^\ell \subset \mathbb{S}^2$ by $T^\ell = a(T)$. The procedure is illustrated on the circle in figure 4.1.

The *tangential gradient* of a function $\eta : \mathbb{S}^2_h \to \mathbb{R}$ is defined in a pointwise sense by

$$\nabla_{\mathbb{S}^2_h}\eta(x) = P_h(x)(I - \mathfrak{d}(x)\mathfrak{H}(x))\nabla_{\mathbb{S}^2}\eta^l(a(x)), \tag{4.3}$$

where $(P_h)_{ij} = \delta_{ij} - \nu_{h,i}\nu_{h,j}$ and $\mathfrak{H}$ is the Hessian of the oriented distance function $\mathfrak{d}$, which is given by

$$\mathfrak{H} = \frac{1}{\|x\|}\begin{bmatrix} 1 - \frac{x_1^2}{\sqrt{\|x\|}} & -\frac{x_1x_2}{\sqrt{\|x\|}} & -\frac{x_1x_3}{\sqrt{\|x\|}} \\ -\frac{x_1x_2}{\sqrt{\|x\|}} & 1 - \frac{x_2^2}{\sqrt{\|x\|}} & -\frac{x_2x_3}{\sqrt{\|x\|}} \\ -\frac{x_1x_3}{\sqrt{\|x\|}} & -\frac{x_2x_3}{\sqrt{\|x\|}} & 1 - \frac{x_3^2}{\sqrt{\|x\|}} \end{bmatrix}.$$

In order to be able to discretize problems defined on $\mathbb{S}^2_h$, we will have to define the following finite element spaces

Definition 4.1.2 (Finite element spaces)**.** Let the *finite element spaces* S_h be defined by

$$S_h := \left\{\phi_h \in C^0(\mathbb{S}^2) : \phi_h|_T \in \mathcal{P}^1(T), T \in \mathcal{T}_h\right\} \subset H^1(\mathbb{S}^2_h).$$

where $\mathcal{P}^1(T)$ denotes the set of all polynomials of degree at most one. The lifted finite element space is defined by

$$S_h^\ell = \left\{ \varphi_h = \phi_h^\ell : \phi_h \in S_h \right\} \subset H^1(\mathbb{S}^2).$$

4.1.1.1 Interpolation methods

In our error analysis, we will need to resort to some interpolation methods. We define the interpolation in the same way as done in [6, Lemma 4.3]. Note first that by the Sobolev embedding theorem, a function $u \in H^2(\mathbb{S}^2)$ will also be in $C^0(\mathbb{S}^2)$.

Denote the N nodes of $\mathbb{S}_h^2$ by $(x_1, ..., x_N)$. For every $T \in \mathcal{T}_h$, it holds that the nodes are on $\mathbb{S}^2$. We now construct $\tilde{I}_h u \in S_h \subset H^1(\mathbb{S}_h^2)$ by first setting

$$\tilde{I}_h u(x_i) = u(x_i),$$

and then performing linear interpolation using the basis functions of S_h. This means that we can view $\tilde{I}_h$ as a mapping $\tilde{I}_h H^2(\mathbb{S}^2) \to S_h$.

Now, we define $I_h H^2(\mathbb{S}^2) \to S_h^l \subset H^1(\mathbb{S}^2)$ by lifting the interpolated function $\tilde{I}_h u \in S_h$ onto $\mathbb{S}^2$ using Equation (4.2).

Lemma 4.1.2 (Interpolation estimate)**.**
Let $\eta \in H^2(\mathbb{S}^2)$. Then for the interpolation $I_h \eta$, it holds that

$$\|\eta - I_h\eta\|_{L^2(\mathbb{S}^2)} + h\|\nabla_{\mathbb{S}^2}(\eta - I_h\eta)\|_{L^2(\mathbb{S}^2)} \leq ch^2 \left(h\|\nabla_{\mathbb{S}^2}\eta\|_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2}\eta\|_{L^2(\mathbb{S}^2)} \right).$$

This lemma is not directly applicable in the later error analysis in which quantities such as $\|\eta - I_h\eta\|_U$ where U is either $H^1(\mathbb{S}^2)$ or $L^2(\mathbb{S}^2)$ will arise. Therefore, we will use Lemma 4.1.2 to obtain estimates for these errors.

Lemma 4.1.3 ($L^2(\mathbb{S}^2)$ interpolation estimate)**.**
Let $\eta \in H^2(\mathbb{S}^2)$. Then for the interpolation as $I_h \eta$, it holds that

$$\|\eta - I_h\eta\|_{L^2(\mathbb{S}^2)} \leq \sqrt{2}ch^2\|\eta\|_{H^2(\mathbb{S}^2)}.$$

Proof. Note first that by Lemma 4.1.2,

$$\begin{aligned}
\|\eta - I_h\eta\|^2_{L^2(\mathbb{S}^2)} &\leq \left(\|\eta - I_h\eta\|_{L^2(\mathbb{S}^2)} + h\|\nabla_{\mathbb{S}^2}(\eta - I_h\eta)\|_{L^2(\mathbb{S}^2)} \right)^2 \\
&\leq ch^4 \left(h\|\nabla_{\mathbb{S}^2}\eta\|_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2}\eta\|_{L^2(\mathbb{S}^2)} \right)^2,
\end{aligned}$$

which by Young's product inequality (Lemma 2.2.2) and noting that $h < 1$, can be rewritten as

$$\|\eta - I_h\eta\|^2_{L^2(\mathbb{S}^2)} \leq 2ch^4 \left(\|\nabla_{\mathbb{S}^2}\eta\|^2_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2}\eta\|^2_{L^2(\mathbb{S}^2)} \right).$$

Finally, we can add the remaining part of the $H^2(\mathbb{S}^2)$-norm to the right-hand side in the above inequality to obtain that

$$\|\eta - I_h\eta\|^2_{L^2(\mathbb{S}^2)} \leq 2ch^4\|\eta\|^2_{H^2(\mathbb{S}^2)},$$

and the inequality follows. □

Lemma 4.1.4 ($H^1(\mathbb{S}^2)$ interpolation estimate)**.**
Let $\eta \in H^2(\mathbb{S}^2)$. Then for the interpolation $I_h\eta$, it holds that

$$\|\eta - I_h\eta\|_{H^1(\mathbb{S}^2)} \leq ch\|\eta\|_{H^2(\mathbb{S}^2)}.$$

Proof. Note first that by Lemma 4.1.2, it holds that

$$\|\nabla_{\mathbb{S}^2}(\eta - I_h\eta)\|_{L^2(\mathbb{S}^2)} \leq ch(h\|\nabla_{\mathbb{S}^2}(\eta - I_h\eta)\|_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2}(\eta - I_h\eta)\|_{L^2(\mathbb{S}^2)}).$$

Using the fact that $h < 1$, and the same Young's product inequality trick as above together with adding the remaining part of the $H^2(\mathbb{S}^2)$-norm, we see that

$$\|\nabla_{\mathbb{S}^2}(\eta - I_h\eta)\|^2_{L^2(\mathbb{S}^2)} \leq ch^2\|\eta\|^2_{H^2(\mathbb{S}^2)}.$$

Note now that by the definition of the $H^1(\mathbb{S}^2)$-norm and Lemma 4.1.3, it holds that

$$\begin{aligned}&\|\eta - I_h\eta\|^2_{H^1(\mathbb{S}^2)}\\ &= \|\eta - I_h\eta\|^2_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2}(\eta - I_h\eta)\|^2_{L^2(\mathbb{S}^2)} \leq c(h^2 + h^4)\|\eta\|^2_{H^2(\mathbb{S}^2)} \leq ch^2\|\eta\|_{H^2(\mathbb{S}^2)},\end{aligned}$$

so the claim follows. □

4.1.1.2 Technical lemmas

We now introduce a few technical lemmas necessary for the theoretical development and later convergence analysis. We refer to [6] for proofs of the lemmas in the case of general Riemannian manifolds, apart from the following lemma, which is from Theorem 3.21 in [10].

Lemma 4.1.5 (Poincaré's inequality)**.**
Let $p \in [1, \infty)$. There exists $c(p) > 0$ such that for every $f \in W^{1,p}(\mathbb{S}^2)$ it holds that

$$\left(\int_{\mathbb{S}^2} |f - c_f|^p \ \mathrm{d}A\right)^{1/p} \leq c\|\nabla_{\mathbb{S}^2} f\|_{L^p(\mathbb{S}^2)},$$

where $c_f = \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} f \mathrm{d}A$. Note in particular, that if $\int_{\mathbb{S}^2} f \ \mathrm{d}A = 0$ it holds that

$$\|f\|_{L^p(\mathbb{S}^2)} \leq c\|\nabla_{\mathbb{S}^2} f\|_{L^p(\mathbb{S}^2)}.$$

Proof. We begin the proof with a quick note from functional analysis. Recall that for $L^p(\mathbb{R}^n)$, elements of $(L^p)^*(R^n)$ can be identified with an element in $g \in L^{p'}(\mathbb{R}^n)$ using an integral representation, $\int_{\mathbb{R}^n} gf\mathrm{d}x$, $f \in L^p(\mathbb{R}^n)$, which by Hölder's inequality is well-defined and continuous. Here p' denotes the Hölder conjugate of p.

Suppose the above is false on $\mathbb{S}^2$. Then there must exist a sequence of natural numbers k, and corresponding sequence $u_k \in W^{1,p}(\mathbb{S}^2)$ such that

$$\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)} > k\|\nabla_{\mathbb{S}^2} u_k\|_{L^p(\mathbb{S}^2)}.$$

Let

$$v_k = \frac{u_k - c_{u_k}}{\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}}.$$

It then holds that $\|v_k\|_{L^p(\mathbb{S}^2)} = 1$ since

$$\|v_k\|_{L^p(\mathbb{S}^2)} = \frac{1}{\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}} \left(\int_{\mathbb{S}^2} |u_k - c_{u_k}|^p \, \mathrm{d}A\right)^{1/p} = \frac{\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}}{\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}} = 1.$$

Furthermore, it holds that

$$\begin{aligned}
c_{v_k} &= \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} v_k \mathrm{d}A = \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} \frac{u_k - c_{u_k}}{\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}} \,\mathrm{d}A \\
&= \frac{1}{|\mathbb{S}^2|\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}} \left(\int_{\mathbb{S}^2} u_k \mathrm{d}A - \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2}\left[\int_{\mathbb{S}^2} u_k \mathrm{d}A\right]\mathrm{d}A\right) \\
&= \frac{1}{|\mathbb{S}^2|\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}} \left(\int_{\mathbb{S}^2} u_k \mathrm{d}A - \int_{\mathbb{S}^2} u_k \mathrm{d}A\right) = 0.
\end{aligned}$$

Note that $W^{1,p}(\mathbb{S}^2)$ embeds into $L^p(\mathbb{S}^2)$, so there exists a subsequence $v_{k_j} \in W^{1,p}$ and $v \in L^p(\mathbb{S}^2)$ such that $v_{k_j} \to v$ in $L^p(\mathbb{S}^2)$-norm, and it holds that

$$\left(\int_{\mathbb{S}^2} v^{p'} \mathrm{d}A\right)^{1/p'} = \lim_{j\to\infty}\left(\int_{\mathbb{S}^2} v_{k_j}^{p'} \mathrm{d}A\right)^{1/p'} = 1. \tag{4.4}$$

and since $c_{v_{k_j}} = 0$, $c_v = 0$. Let us now consider the gradient of v_k. It holds that

$$\|\nabla_{\mathbb{S}^2} v_k\|_{L^p(\mathbb{S}^2)} = \frac{1}{\|u_k - c_{u_k}\|_{L^p(\mathbb{S}^2)}}\|\nabla u_k\|_{L^p(\mathbb{S}^2)} \le \frac{1}{k},$$

by our contradiction assumption. We can furthermore identify $L^p(\mathbb{S}^2)$ with the dual of $L^{p'}(\mathbb{S}^2)$ by a Hölder's inequality argument, and see that it holds that if a sequence $(f_k) \subset L^p(\mathbb{S}^2)$ is bounded, then by Theorem 1.19 of [8], there will be a subsequence f_{k_l} that converges weakly to some $f \in L^p(\mathbb{S}^2)$, and hence, since our sequence $\nabla_{\mathbb{S}^2} v_k$ is bounded, we can extract a subsequence ∇v_{k_l} that converges weakly to some $\eta \in L^p(\mathbb{S}^2)$, that is to say, it holds that

$$\int_{\mathbb{S}^2} \eta\phi \mathrm{d}A = \lim_{k_l\to\infty}\int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} v_{k_l}\phi \mathrm{d}A = \lim_{k_l\to\infty} -\int_{\mathbb{S}^2} v_{k_l}\nabla_{\mathbb{S}^2}\phi \mathrm{d}A = -\int_{\mathbb{S}^2} v\nabla_{\mathbb{S}^2}\phi \mathrm{d}A,$$

for every $\phi \in C^1(\mathbb{S}^2)$, so it holds that $v \in W^{1,p}(\mathbb{S}^2)$, and furthermore, since we bounded the gradient by $1/k$, it holds that $\nabla_{\mathbb{S}^2} v = 0$. Hence v is constant since the sphere is a connected manifold. Since $c_v = 0$, the only possibility is that $v = 0$, and hence $\|v\|_{L^p(\mathbb{S}^2)} = 0$, but that cannot be true since we showed that $\|v\|_{L^p(\mathbb{S}^2)} = 1$, and hence the contradiction assumption must be false and the inequality follows. □

Lemma 4.1.6.
Let $f, g \in C^2(\mathbb{S}^2)$. It then holds that

$$\int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} f \nabla_{\mathbb{S}^2} g \ \mathrm{d}A = -\int_{\mathbb{S}^2} f \Delta_{\mathbb{S}^2} g \ \mathrm{d}A.$$

Note that this implies that we for any $f, g \in H^1(\mathbb{S}^2)$ can define a bilinear form for the Laplace–Beltrami operator given by $a(u, v) = \int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} f \nabla_{\mathbb{S}^2} g \ \mathrm{d}A$, where $\nabla_{\mathbb{S}^2}$ is to be understood in the sense of weak derivatives.

4.1.2 A brief note on surface finite elements

Denote by $(x_1, ..., x_N)$ the nodes of $\mathbb{S}_h^2$. The finite element space S_h is spanned by the nodal basis which is given by

$$\chi_j \in S_h, \ \chi_j(x_k) = \delta_{j,k},$$

for $k = 1, ..., N$ and $j = 1..., N$. Every function $u_h \in S_h$ can be written as

$$u_h(x) = \sum_{j=1}^{N} \alpha_j \chi_j(x), \ x \in \mathbb{S}_h^2.$$

where $\alpha_j, j = 1, ..., N$ are real numbers and N is the number of nodes. This is what in practice is used to obtain solutions. If we for instance were to consider Poisson's equation, for some appropriate $f_h \in L^2(\mathbb{S}_h^2)$, it holds that the discrete weak solution $u_h \in S_h$ satisfies

$$\int_{\mathbb{S}_h^2} \left(\nabla_{\mathbb{S}_h^2} u_h, \nabla_{\mathbb{S}_h^2} v_h\right)_{\mathbb{R}^3} \ \mathrm{d}A_h = \int_{\mathbb{S}_h^2} f_h v_h \ \mathrm{d}A_h, \tag{4.5}$$

for every $v_h \in S_h$. Inserting the expression $u_j = \sum_{j=1}^{J} \alpha_j \chi_j$ and setting v_h to one of the nodal basis functions χ_k, we obtain due to the orthogonality of the basis functions that the linear system of equations is given by

$$\mathcal{S}\alpha = F,$$

with

$$\mathcal{S}_{jk} = \int_{\mathbb{S}_h^2} \left(\nabla_{\mathbb{S}_h^2} \chi_j, \nabla_{\mathbb{S}_h^2} \chi_k\right)_{\mathbb{R}^3} \mathrm{d}A_h,$$

where $j, k = 1, .., J$, $\alpha = (\alpha_1, ...\alpha_j)$ and $F_k = \int_{\mathbb{S}_h^2} f_h \chi_k \mathrm{d}A_h, k = 1, ..., J$. In our case, we will obtain a similar system of equations that must be solved, stemming from that we will have a more general elliptic operator, yielding both a stiffness matrix $\mathcal{S}$ as well as a mass matrix but this is in principle not significantly more complicated. It is this step of the numerical solution that FEniCS may assist us with, so that we only need to supply the bilinear and linear form. The rest is done automatically by FEniCS.

4.2 Approximation of the noise on $\mathbb{S}^2_h$

We must also approximate the right hand side of the equation. In this section we achieve this goal by using interpolation, and we estimate the introduced error.

In order to arrive at an approximation of $\overline{\mathcal{W}}$ that is defined on $\mathbb{S}^2_h$, we begin with the truncated series $\overline{\mathcal{W}}_{\mathfrak{K}}$, which was introduced in Section 3.1. We evaluate $\overline{\mathcal{W}}_{\mathfrak{K}}$ in every vertex of $\mathbb{S}^2_h$, that is to say, every $y \in \mathbb{S}^2_h \cap \mathbb{S}^2$. We then interpolate on the finite element basis of $\mathbb{S}^2_h$, so that we obtain for $y \in \mathbb{S}^2_h$

$$\begin{aligned}&\widetilde{\mathcal{W}}(y)_{\mathfrak{K},h} = \\ &\tilde{I}_h\Big(\sum_{l=0}^{\mathfrak{K}}\Big(\sqrt{A_l} z^1_{l,0} L_{l,0}(\theta) + \sqrt{2A_l}\sum_{m=1}^{l}\Big[L_{l,m}(\theta)\Big(z^1_{l,m}\cos(m\phi) + z^2_{l,m}\sin(m\phi)\Big)\Big]\Big)\Big),\end{aligned}$$

where $\tilde{I}_h$ denotes the interpolation as in Section 4.1.1.1. In order to compare $\widetilde{\mathcal{W}}_{\mathfrak{K},h}$ and $\overline{\mathcal{W}}$, we must lift $\widetilde{\mathcal{W}}_{\mathfrak{K},h}$ so that we obtain

$$\begin{aligned}&\widetilde{\mathcal{W}}(y)^{\ell}_{\mathfrak{K},h} = \\ &I_h\left(\sum_{l=0}^{\mathfrak{K}}\left(\sqrt{A_l} z^1_{l,0} L_{l,0}(\theta) + \sqrt{2A_l}\sum_{m=1}^{l}\Big[L_{l,m}(\theta)\left((z^1_{l,m}\cos(m\phi) + z^2_{l,m}\sin(m\phi)\right)\Big]\right)\right).\end{aligned}$$

The error consists of two parts, a truncation error and an interpolation error. Hence, we split the error and treat them one by one.

$$\left\|\overline{\mathcal{W}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \le \left\|\overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathfrak{K}}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} + \left\|\overline{\mathcal{W}}_{\mathfrak{K}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))}.$$

The first term in the above sum is bounded using Proposition 4.2.2 in [13], which states that

$$\left\|\overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathfrak{K}}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \le \hat{C}\mathfrak{K}^{\frac{-(\alpha-2)}{2}},$$

for $\mathfrak{K} > l_0$.

For the second of these estimates, we can first note that

$$\left\|\overline{\mathcal{W}}_{\mathfrak{K}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} = \left(\int_\Omega \left\|\overline{\mathcal{W}}_{\mathfrak{K}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|^2_{L^2(\mathbb{S}^2))}\, \mathrm{d}\mathbb{P}\right)^{1/2}.$$

We can apply the interpolation estimates from Lemma 4.1.2 to

$$\left(\int_\Omega \left\|\overline{\mathcal{W}}_{\mathfrak{K}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|^2_{L^2(\mathbb{S}^2))}\, \mathrm{d}\mathbb{P}\right)^{1/2},$$

in order to obtain that

$$\left\|\overline{\mathcal{W}}_{\mathfrak{K}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \le \tilde{c}h^2\left(\|\overline{\mathcal{W}}_{\mathfrak{K}}\|_{L^2(\Omega;H^2(\mathbb{S}^2))}\right).$$

Note here that the truncated Karhunen–Loève expansion indeed will be in $H^2(\mathbb{S}^2)$, since all the functions appearing in the finite sum in turn are in $H^2(\mathbb{S}^2)$ so for any $\omega \in \Omega$, the sum itself will be in $H^2(\mathbb{S}^2)$.

We shall estimate the $L^2(\Omega; H^2(\mathbb{S}^2))$-norm of $\overline{\mathcal{W}}_{\mathfrak{K}}$.

In order to do this, note that in the proof of Proposition 4.2.2 of [13], the theorem used to obtain Equation (3.1), one begins with the expression

$$\overline{\mathcal{W}}_{\mathfrak{K}} = \sum_{l=0}^{\mathfrak{K}} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}.$$

It hence follows that

$$\|\overline{\mathcal{W}}_{\mathfrak{K}}\|^2_{L^2(\Omega;H^2(\mathbb{S}^2))} = \left\| \sum_{l=0}^{\mathfrak{K}} \sum_{m=-l}^{l} a_{l,m} Y_{l,m} \right\|^2_{L^2(\Omega;H^2(\mathbb{S}^2))} = \mathbb{E}\left[\left\| \sum_{l=0}^{\mathfrak{K}} \sum_{m=-l}^{l} a_{l,m} Y_{l,m} \right\|^2_{H^2(\mathbb{S}^2)} \right].$$

We rewrite this using the triangle inequality and the relation $a_{l,-m} = (-1)^m \overline{a_{l,m}}$ to obtain that

$$\begin{aligned} &\|\overline{\mathcal{W}}_{\mathfrak{K}}\|^2_{L^2(\Omega;H^2(\mathbb{S}^2))} \\ &\leq \mathbb{E}\left[\Big(\sum_{l=0}^{\mathfrak{K}} a_{l,0} \|Y_{l,0}\|_{H^2(\Omega)} + \sum_{m=1}^{l} \overline{a_{l,m}} \|\overline{Y_{l,m}}\|_{H^2(\mathbb{S}^2)} + a_{l,m} \|Y_{l,m}\|_{H^2(\mathbb{S}^2)} \Big)^2 \right]. \end{aligned}$$

When expanding the square, we note that by Equation (2.11), that the cross-terms will vanish, leaving us with

$$\|\overline{\mathcal{W}}_{\mathfrak{K}}\|^2_{L^2(\Omega;H^2(\mathbb{S}^2))} \leq \sum_{l=0}^{\mathfrak{K}} \left(A_l \|Y_{l,0}\|^2_{H^2(\mathbb{S}^2)} + 2A_l \sum_{m=1}^{l} \|Y_{l,m}\|^2_{H^2(\mathbb{S}^2)} \right). \tag{4.6}$$

Note now that for a given function $f \in H^2(\mathbb{S}^2)$ it holds that

$$\|f\|^2_{H^2(\mathbb{S}^2)} = \|f\|^2_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2} f\|^2_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2} f\|^2_{L^2(\mathbb{S}^2)}.$$

It is possible to rewrite the middle term according to

$$\|\nabla_{\mathbb{S}^2} f\|^2_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} f, \nabla_{\mathbb{S}^2} f)_{\mathbb{R}^3} \quad \mathrm{d}A = -\int_{\mathbb{S}^2} f \Delta_{\mathbb{S}^2} f \quad \mathrm{d}A.$$

Applying first Hölder's and then Young's product inequality, we obtain that

$$\|\nabla_{\mathbb{S}^2} f\|^2_{L^2(\mathbb{S}^2)} \leq \frac{1}{2} \left(\|f\|^2_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2} f\|^2_{L^2(\mathbb{S}^2)} \right),$$

and therefore, we see that

$$\|f\|^2_{H^2(\mathbb{S}^2)} \leq \frac{3}{2} \left(\|f\|^2_{L^2(\mathbb{S}^2)} + \|\Delta_{\mathbb{S}^2} f\|^2_{L^2(\mathbb{S}^2)} \right).$$

The $L^2(\mathbb{S}^2)$-norm of $Y_{l,m}$ is 1, and it holds that

$$\Delta_{\mathbb{S}^2} Y_{l,m} = -(l+1) l Y_{l,m},$$

which implies that

$$\|\Delta_{\mathbb{S}^2} Y_{l,m}\|^2_{L^2(\mathbb{S}^2)} = l^2 (l+1)^2.$$

Hence we estimate the squared $H^2(\mathbb{S}^2)$-norm of $Y_{l,m}$ by

$$\|Y_{l,m}\|^2_{H^2(\mathbb{S}^2)} \leq \frac{3}{2}\left(1 + l^2(l+1)^2\right).$$

Returning to Equation (4.6), inserting the above estimate, we get

$$\begin{aligned}\|\overline{\mathcal{W}}_{\mathfrak{K}}\|^2_{L^2(\Omega;H^2(\mathbb{S}^2))} &\leq \frac{3}{2}\sum_{l=0}^{\mathfrak{K}}\left(A_l(1 + l^2(l+1)^2) + 2A_l\sum_{m=1}^{l}(1 + l^2(l+1)^2)\right) \\ &= \frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) = D(\mathfrak{K}, A_l).\end{aligned}$$

and hence, we see that the constant will depend on both $\mathfrak{K}$ and the angular power spectrum. In applications, it is possible to estimate the angular power spectrum and hence determine the size of the constant. Hence, we have proved the following lemma.

Lemma 4.2.1 (Error of the noise approximation)**.**
The error of the truncated, interpolated and lifted noise will be given by

$$\left\|\overline{\mathcal{W}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq c\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2.$$

Using this lemma, we see that we can control the error of this approximation in two different ways, since the angular power spectrum is to be viewed as fixed. The first is to control the number of terms in the truncated series, the second is to refine the sphere. A possible drawback with an approach based on Karhunen–Loève expansions is its lack of generalizability. It is not the case for any manifold that we have an explicit eigenbasis available, so in more complicated cases in which there may be no nice setting such as the one provided by the spherical harmonics, this approach might not work at all.

We shall now estimate $D(\mathfrak{K}, A_l)$. Note that it would be possible to make the additional assumption of the angular power spectrum decaying fast enough so that we obtain convergence also in $H^2(\mathbb{S}^2)$-norm for the non-truncated field, but this is less than ideal since it is a restricting assumption, so for the moment, we assume that $\alpha \leq 5$.

By assumption, $\mathfrak{K} > l_0$, which implies that

$$\begin{aligned}D(\mathfrak{K}, A_l) &= \frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) \\ &= \frac{3}{2}\sum_{l=0}^{l_0} A_l\left(1 + 2l + (l^2 + 2l^3)(l+1)^2\right) + \frac{3}{2}\sum_{l=l_0+1}^{\mathfrak{K}} A_l\left(1 + 2l + (l^2 + 2l^3)(l+1)^2\right).\end{aligned}$$

The sum $\frac{3}{2}\sum_{l=0}^{l_0} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) = D_{\overline{\mathcal{W}}}$ is a constant depending on the field, in particular the angular power spectrum of the field, but it is not dependent on the truncation parameter $\mathfrak{K}$. For the remaining term, it holds that

$$\frac{3}{2}\sum_{l=l_0+1}^{\mathfrak{K}} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) \leq C\sum_{l=l_0+1}^{\mathfrak{K}} l^{\alpha-5},$$

by the assumption of algebraic decay of the angular power spectrum for every $l > l_0$. Now, since $\alpha \leq 5$, we can estimate all the terms in the above sum by

$$l^{\alpha-5} \leq \mathfrak{K}^{\alpha-5},$$

and hence,

$$C \sum_{l=l_0+1}^{\mathfrak{K}} l^{\alpha-5} \leq C \sum_{l=1}^{\mathfrak{K}} l^{\alpha-5} \leq C\mathfrak{K}^{6-\alpha}.$$

This implies, for $\alpha \leq 5$ that

$$D(\mathfrak{K}, A_l)h^2 \leq D_{\overline{\mathcal{W}}} h^2 + C\mathfrak{K}^{6-\alpha} h^2.$$

If instead $\alpha > 5$, we do the same thing up so that we obtain

$$D(\mathfrak{K}, A_l) \leq D_{\overline{\mathcal{W}}} + C \sum_{l=l_0+1}^{\mathfrak{K}} \frac{1}{l^{\alpha-5}}.$$

Note now that

$$\sum_{l=l_0+1}^{\mathfrak{K}} \frac{1}{l^{\alpha-5}} \leq \sum_{l=1}^{\mathfrak{K}} \frac{1}{l^{\alpha-5}}.$$

Note that the values of this sum for different $\mathfrak{K}$ and values of $\alpha - 5$ is known as the *Generalized harmonic numbers* and are denoted by $H_{\mathfrak{K},\alpha-5}$. Let us try to determine how fast these numbers grow. For $\alpha = 5$, we see that $H_{\mathfrak{K},0}$ grow linearly. For $\alpha = 6$ it holds that it holds that $H_{\mathfrak{K},1} \sim \log(\mathfrak{K})$ since

$$H_{\mathfrak{K},1} \leq 1 + \int_1^{\mathfrak{K}} \frac{1}{x} \,\mathrm{d}x = 1 + \log(\mathfrak{K}).$$

For the intermediate values of $\alpha \in (5, 6)$, we have that the constants will grow faster than the logarithm, but still slower than when $\alpha = 5$. We hence approximate the growth with linear growth.

If $\alpha > 6$, we instead have that the sum

$$\sum_{k=1}^{\infty} \frac{1}{l^{\alpha-5}} = \zeta(\alpha - 5),$$

where ζ denotes Riemann's zeta function which means that $H_{\mathfrak{K},\alpha-6}$ approaches a constant. In Table 4.1 we summarize the different estimates of $D(\mathfrak{K}, A_l)$.

	$\alpha \in (2, 5)$	$\alpha \in [5, 6)$	$\alpha = 6$	$\alpha > 6$
$D(\mathfrak{K}, A_l)h^2 \leq$	$D_{\overline{\mathcal{W}}}h^2 + C\mathfrak{K}^{6-\alpha}h^2$	$D_{\overline{\mathcal{W}}}h^2 + C\mathfrak{K}h^2$	$D_{\overline{\mathcal{W}}}h^2 + C\log(\mathfrak{K})h^2$	Ch^2

Table 4.1: Different estimates for different regularity of $\overline{\mathcal{W}}$.

Consider now $\alpha \in (2, 5]$. If we wish to balance

$$C\mathfrak{K}^{6-\alpha}h^2,$$

with $\mathfrak{K}^{-\frac{\alpha-2}{2}}$, we can solve the equation

$$\mathfrak{K}^{6-\alpha}h^2 = \mathfrak{K}^{-\frac{\alpha-2}{2}},$$

to see that the selection $h = \mathfrak{K}^{\frac{\alpha-10}{4}}$ yields the desired behaviour. For the other values of α, the parameters can also be balanced against each other in an analogue way.

4.3 FEM for the problem $\mathcal{L}^{\beta}(u) = \overline{\mathcal{W}}$

In order to solve equation (2.8), we need to develop a finite element discretization. In essence, we follow the outline of [3], but adapted to the setting of the sphere. Recall that our problem is given by

$$\mathcal{L}^{\beta}(u) = \overline{\mathcal{W}},$$

and note that we view the operator $\mathcal{L}$ as a mapping from $H^1(\mathbb{S}^2)$ to $H^{-1}(\mathbb{S}^2)$.

In order to arrive at a quadrature method to approximate the fractional inverse, as done in [5], we must ascertain that we have an appropriate Gelfand triple setting, that is to say, meaning that the operator must map from $H^1(\mathbb{S}^2)$ to $H^{-1}(\mathbb{S}^2)$, and that the solutions and test functions are in $H^1(\mathbb{S}^2)$. Furthermore, we must have an embedding $H^1(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2) \hookrightarrow H^{-1}(\mathbb{S}^2)$ such that the embeddings are dense and continuous. Following Proposition 3.22 in [10], we have that this indeed is the case, and hence, the appropriate Gelfand triple setting is

$$H^1(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2) \hookrightarrow H^{-1}(\mathbb{S}^2).$$

Note now that it is possible to write

$$u = \mathcal{L}^{-\beta}\overline{\mathcal{W}}.$$

According to [5], we can write the inverse fractional operator as a so-called Dunford–Taylor integral if $H^1(\mathbb{S}^2)$ is densely and compactly contained in $L^2(\mathbb{S}^2)$, if there is a constant c_0 such that

$$\|u\|_{L^2(\mathbb{S}^2)} \leq c_0 \|u\|_{H^1(\mathbb{S}^2)},$$

and if the operator $\mathcal{L}$ is unbounded, positive definite and symmetric.

We just verified the first condition, that $H^1(\mathbb{S}^2)$ is densely and compactly contained in $L^2(\mathbb{S}^2)$, and, since the $L^2(\mathbb{S}^2)$-norm can be bounded by the $H^1(\mathbb{S}^2)$-norm, the second condition is also met. The assumptions on the operator can be verified one by one:

I. **Unboundedness**: The operator $\mathcal{L}$ is an unbounded operator since the Laplace–Beltrami operator on $L^2(\mathbb{S}^2)$ is an unbounded operator, and hence, $\kappa^2 - \Delta_{\mathbb{S}^2} = \mathcal{L}$ is also unbounded.

II. **Symmetric**: The operator is symmetric, since by the definition of $\mathcal{L}$, it holds that

$$(\mathcal{L}u, v)_{L^2(\mathbb{S}^2)} = \kappa^2 (u, v)_{L^2(\mathbb{S}^2)} + (\nabla_{\mathbb{S}^2} u, \nabla_{\mathbb{S}^2} v)_{L^2(\mathbb{S}^2)} = (u, \mathcal{L}v)_{L^2(\mathbb{S}^2)} .$$

III. **Positive definite**: By inserting $u = v$ in the above equation, it is seen that

$$(\mathcal{L}u, u)_{L^2(\mathbb{S}^2)} = \kappa^2 \|u\|^2_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2} u\|^2_{L^2(\mathbb{S}^2)} \geq 0.$$

We have verified the assumptions on the Gelfand triple $H^1(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2) \hookrightarrow H^{-1}(\mathbb{S}^2)$ and the operator $\mathcal{L}$, so by Theorem 2.1 in [5], it holds that

$$\mathcal{L}^{-\beta} = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{\infty} e^{2\beta y} \left(I_{H^1(\mathbb{S}^2)} + e^{2y}\mathcal{L} \right)^{-1} \mathrm{d}y. \tag{4.7}$$

One can then partition the range of y into an equidistant grid, and following [5] approximate the integral in (4.7) using a sinc quadrature approach resulting in,

$$\mathcal{L}^{-\beta} \approx \mathfrak{Q}_k^{\beta} = \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=-K^-}^{K^+} e^{2\beta y_l} \left(I_{H^1(\mathbb{S}^2)} + e^{2y_l}\mathcal{L} \right)^{-1}, \tag{4.8}$$

where k is the step size of the quadrature, $y_l = kl$, and

$$K^+ = \left\lceil \frac{\pi^2}{4(1-\beta)k^2} \right\rceil, \quad K^- = \left\lceil \frac{\pi^2}{4\beta k^2} \right\rceil. \tag{4.9}$$

Here $\lceil \cdot \rceil$ denotes rounding up to the closest integer. This means that we approximate the solution $u = \mathcal{L}^{-\beta}\overline{\mathcal{W}}$ with

$$u_{\mathfrak{Q},k} = \mathfrak{Q}_k^{\beta}\overline{\mathcal{W}} = \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=-K^-}^{K^+} e^{2\beta y_l} \left(I_{H^1(\mathbb{S}^2)} + e^{2y_l}\mathcal{L} \right)^{-1} \overline{\mathcal{W}},$$

where $\left(I_{H^1(\mathbb{S}^2)} + e^{2y_l}\mathcal{L} \right)^{-1} \overline{\mathcal{W}}$ is obtained by solving the subproblems given by

$$u_l + e^{2y_l}\mathcal{L}u_l = \left(1 + e^{2y_l}\kappa^2\right) u_l - e^{2y_l}\Delta_{\mathbb{S}^2} u_l = \mathfrak{L}_l u_l = \overline{\mathcal{W}}.$$

In order to derive the appropriate forms for these subproblems, let $u, v \in C^2(\mathbb{S}^2)$. Then, it follows that

$$\int_{\mathbb{S}^2} v\mathfrak{L}_l u \;\mathrm{d}A = \left(1 + e^{2y_l}\kappa^2\right) \int_{\mathbb{S}^2} uv \;\mathrm{d}A + e^{2y_l} \int_{\mathbb{S}^2} \nabla_{\mathbb{S}^2} u \nabla_{\mathbb{S}^2} v \;\mathrm{d}A.$$

We see that the bilinear form of the lth subproblem, $\mathfrak{a}_{\mathbb{S}^2,l} : H^1(\mathbb{S}^2) \times H^1(\mathbb{S}^2) \to \mathbb{R}$, is given by

$$\mathfrak{a}_{\mathbb{S}^2,l}(u, v) = \left(1 + e^{2y_l}\kappa^2\right) (u, v)_{L^2(\mathbb{S}^2)} + e^{2y_l} (\nabla_{\mathbb{S}^2} u, \nabla_{\mathbb{S}^2} v)_{L^2(\mathbb{S}^2)} , \tag{4.10}$$

for $u, v \in H^1(\mathbb{S}^2)$. This means that the weak formulation of the subproblem is given by:

Find $u \in H^1(\mathbb{S}^2)$ such that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u,v) = (\overline{\mathcal{W}}, v)_{L^2(\mathbb{S}^2)},$$

for every $v \in H^1(\mathbb{S}^2)$.

This formulation can be used to discretize the subproblems. We begin by defining

$$\mathfrak{L}_{\mathbb{S}_h^2,l} : H^1(\mathbb{S}_h^2) \to H^{-1}(\mathbb{S}_h^2),$$

using its bilinear form, which is given by

$$\mathfrak{a}_{\mathbb{S}_h^2,l}(u,v) = \left(1 + e^{2y_l}\kappa^2\right)(u,v)_{L^2(\mathbb{S}_h^2)} + e^{2y_l}\left(\nabla_{\mathbb{S}_h^2} u, \nabla_{\mathbb{S}_h^2} v\right)_{L^2(\mathbb{S}_h^2)},$$

for $u, v \in H^1(\mathbb{S}_h^2)$. Let $\alpha_l = (1 + e^{2y_l}\kappa^2)$ and $\beta_l = e^{2y_l}$. The weak formulation of the subproblems on the discretized sphere, which was introduced in Section 2.2, is hence given by:

Find $u \in H^1(\mathbb{S}_h^2)$ such that

$$\mathfrak{a}_{\mathbb{S}_h^2,l}(u,v) = (\widetilde{\mathcal{W}}_{\mathfrak{K},h}, v)_{L^2(\mathbb{S}_h^2)}, \tag{4.11}$$

for every $v \in H^1(\mathbb{S}_h^2)$. Here $\widetilde{\mathcal{W}}_{\mathfrak{K},h}$ denotes the noise on the discretized sphere.

The subproblem can be discretzed in the ordinary finite element manner, since it is a nice example of a linear elliptic SPDE. We hence let $u \in S_h$ and note that we obtain

$$\mathfrak{a}_{\mathbb{S}_h^2,l}(u,v) = (\widetilde{\mathcal{W}}_{\mathfrak{K},h}, v)_{L^2(\mathbb{S}_h^2)}, \tag{4.12}$$

for $v \in S_h$. These subproblems can be solved in parallel to improve speed.

4.4 Error analysis

In this section we will prove an estimate of the strong error of our FEM solution. Note that in all the following work, the bookkeeping of constants will be held to a minimum and we will use c to denote any constant appearing in our estimates. The quantity of interest is the difference $u - u_h^\ell$, where u_h^ℓ denotes the FEM solution on $\mathbb{S}_h^2$ lifted to $\mathbb{S}^2$ by Equation (4.2). Note that it holds that

$$u - u_h^\ell = u - u_{\mathfrak{Q},k} + u_{\mathfrak{Q},k} - u_h^\ell$$

where $u_{\mathfrak{Q},k}$ is the function introduced by approximating the inverse fractional operator $\mathcal{L}^{-\beta}$ with a quadrature with k nodes, according to Equation (4.8). Note that

$$u_{\mathfrak{Q},k} = \mathfrak{Q}_k^{-\beta}\overline{\mathcal{W}} = \frac{2k\sin(\pi\beta)}{\pi}\sum_{l=-K^-}^{K^+} e^{2\beta y_l}\left(I_{H^1(\mathbb{S}^2)} + e^{2y_l}\mathcal{L}\right)^{-1}\overline{\mathcal{W}}$$

where K^+ and K^- are as in (4.9). We can rewrite this in a more convenient manner as

$$u_{\mathfrak{Q},k} = \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=-K^-}^{K^+} e^{2\beta y_l} u_l,$$

where u_l is the solution of the problem given by:

Find $u_l \in H^1(\mathbb{S}^2)$ such that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u_l, v) = \int_{\mathbb{S}^2} \overline{\mathcal{W}} v \, \mathrm{d}A,$$

and that u_h is given by

$$u_h = \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=-K^-}^{K^+} e^{2\beta y_l} u_{l,h},$$

where $u_{l,h}$ is the finite element solution of Equation (4.12).

This means, that we can rewrite the difference of $u_{\mathfrak{Q},k} - u_h^\ell$ by

$$u_{\mathfrak{Q},k} - u_h^\ell = \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=-K^-}^{K^+} e^{2\beta y_l} \left(u_l - u_{l,h}^\ell\right), \tag{4.13}$$

where $u_{l,h}^\ell$ denotes the lifted finite element approximation of the solution to the subproblems.

Hence, there are two errors we need to consider at this stage. The first is the quadrature error, and the second is the sum of the $K^+ + K^- + 1$ finite element errors, stemming from the subproblems. We will first consider the quadrature error.

4.4.1 Quadrature error

The first step in the proof is to estimate the error between scalars raised to a fractional power and the quadrature evaluated not with an operator but with the same scalar values. This will yield a necessary estimate so that we later can estimate the actual error using Fourier series expansions. Let the setting be as in Section 2.5. Following [5], by the assumptions on the operator and the Hilbert spaces, there will be an orthonormal eigenbasis in $H^{-1}(\mathbb{S}^2)$ of $\mathcal{L}^{-1}$ given by (μ_i, Ψ_i) where μ_0 denotes the largest eigenvalue of $\mathcal{L}^{-1}$. The eigenvalues of $\mathcal{L}$ are given by

$$\lambda_i := \frac{1}{\mu_i}.$$

By results from operator theory and as noted in [5], the eigenvalues μ_i are non-increasing with limit zero. It also holds that the functions

$$\widetilde{\Psi}_i := \mu_i^{1/2} \Psi_i$$

are an orthonormal basis of $L^2(\mathbb{S}^2)$. The following lemma is due to Lemma 3.4, Remark 3.1 and Theorem 3.5 in [5].

Lemma 4.4.1 (Scalar error estimate)**.**
Let, assuming that $\lambda > \lambda_0$*,*

$$Q^\beta(\lambda) = \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=K^-}^{K^+} e^{2\beta y_l} \left(1 + e^{2y_l}\lambda\right)^{-1},$$

where $y_l = kl$*, and*

$$K^+ = \left\lceil \frac{\pi^2}{4(1-\beta)k^2} \right\rceil, \quad K^- = \left\lceil \frac{\pi^2}{4\beta k^2} \right\rceil.$$

It holds that

$$\begin{aligned}
|e(\lambda)| &= |\lambda^{-\beta} - Q^\beta(\lambda)| \\
&\le \frac{2\sin(\pi\beta)}{\pi} \left(\frac{1}{2\beta} + \frac{1}{(2-2\beta)\lambda_0} \right) \left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)} \right),
\end{aligned}$$

which, as noted in Remark 3.1 of [5], asymptotically, as $k \to 0$*, behaves like*

$$\frac{2\sin(\pi\beta)}{\pi} \left(\frac{1}{\beta} + \frac{1}{(1-\beta)\lambda_0} \right).$$

Note now that for a given $f \in L^2(\mathbb{S}^2)$, we can expand it using the eigenpairs of $\mathcal{L}^{-1}$ according to

$$f = \sum_{i=1}^{\infty} (f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)} \widetilde{\Psi}_i,$$

and hence it holds that

$$\mathcal{L}^{-\beta} f = \mathcal{L}^{-\beta} \sum_{i=1}^{\infty} (f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)} \widetilde{\Psi}_i = \sum_{i=1}^{\infty} \mu_i^\beta (f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)} \widetilde{\Psi}_i.$$

We also note that

$$\begin{aligned}
\mathfrak{Q}_k^\beta f &= \mathfrak{Q}_k^\beta \sum_{i=1}^{\infty} (f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)} \widetilde{\Psi}_i \\
&= \sum_{i=1}^{\infty} (f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)} \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=K^-}^{K^+} e^{2\beta y_l} \left(1 + e^{2y_l}\mathcal{L}\right)^{-1} \widetilde{\Psi}_i.
\end{aligned}$$

We now need to determine what is obtained by applying $(1 + e^{2y_l}\mathcal{L})^{-1}$ to $\widetilde{\Psi}_i$. Note that $(1 + e^{2y_l}\mathcal{L})\,\widetilde{\Psi}_i = \widetilde{\Psi}_i + e^{2y_l}\lambda_i\widetilde{\Psi}_i$, and hence, using that $\lambda_i = 1/\mu_i$

$$\left(1 + e^{2y_l}\mathcal{L}\right)^{-1} \widetilde{\Psi}_i = \left(1 + e^{2y_l}\frac{1}{\mu_i}\right)^{-1} \widetilde{\Psi}_i.$$

Hence, we see that

$$\mathfrak{Q}_k^\beta f = \sum_{i=1}^{\infty} (f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)} \left(1 + e^{2y_l}\frac{1}{\mu_i}\right)^{-1} \widetilde{\Psi}_i,$$

so it holds that

$$\mathfrak{L}^{-\beta}f - \mathfrak{Q}_k^{\beta}f = \sum_{i=1}^{\infty} e(\mu_i^{-1})(f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)}\widetilde{\Psi}_i,$$

where $e(\mu_i^{-1}) = \mu_i^{\beta} - Q^{\beta}(\mu_i^{-1})$. Now, in order to bound

$$\|\mathfrak{L}^{-\beta}f - \mathfrak{Q}_k^{\beta}f\|_{L^2(\mathbb{S}^2)},$$

we see that

$$\begin{aligned}&\|\sum_{i=1}^{\infty} e(\mu_i^{-1})(f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)}\widetilde{\Psi}_i\|^2_{L^2(\mathbb{S}^2)}\\ &\quad= \left(\sum_{i=1}^{\infty} e(\mu_i^{-1})(f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)}\widetilde{\Psi}_i, \sum_{i=1}^{\infty} e(\mu_i^{-1})(f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)}\widetilde{\Psi}_i\right)_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Note that $\widetilde{\Psi}_i$ and $\widetilde{\Psi}_j$ are orthonormal, so the cross terms will be zero and we obtain that

$$\|\sum_{i=1}^{\infty} e(\mu_i^{-1})(f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)}\widetilde{\Psi}_i\|^2_{L^2(\mathbb{S}^2)} = \sum_{i=1}^{\infty} e(\mu_i^{-1})^2(f, \widetilde{\Psi}_i)^2_{L^2(\mathbb{S}^2)}.$$

We then apply Lemma 4.4.1 to estimate $e(\mu_i^{-1})^2$ and use the fact that $\|f\|^2_{L^2(\mathbb{S}^2)} = \sum_{i=1}^{\infty}(f, \widetilde{\Psi}_i)^2_{L^2(\mathbb{S}^2)}$ to arrive at the final estimate

$$\|\sum_{i=1}^{\infty} e(\mu_i^{-1})(f, \widetilde{\Psi}_i)_{L^2(\mathbb{S}^2)}\widetilde{\Psi}_i\|^2_{L^2(\mathbb{S}^2)} = \sum_{i=1}^{\infty} e(\mu_i^{-1})^2(f, \widetilde{\Psi}_i)^2_{L^2(\mathbb{S}^2)} \tag{4.14}$$

$$\leq \left[\frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{\mu_0}{(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right)\right]^2 \|f\|^2_{L^2(\mathbb{S}^2)}. \tag{4.15}$$

We can use these observations to prove the following lemma.

Lemma 4.4.2.
Let the setting be as in Lemma 4.4.1. Let $\overline{\mathcal{W}}$ be as in Section 3.1. It holds that

$$\|u - u_{\mathfrak{Q},k}\|_{L^2(\Omega;L^2(\mathbb{S}^2))} = \|\mathcal{L}^{-\beta}\,\overline{\mathcal{W}} - \mathfrak{Q}_k^{\beta}\,\overline{\mathcal{W}}\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \tag{4.16}$$

$$\leq \frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{\mu_0}{(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right)\|\overline{\mathcal{W}}\|_{L^2(\Omega;L^2(\mathbb{S}^2))}. \tag{4.17}$$

Proof. The proof is an application of the definition of the $L^2(\Omega; L^2(\mathbb{S}^2))$-norm and the estimates from Equation (4.15).

$$\begin{aligned}\|u - u_{\mathfrak{Q},k}\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} &= \int_\Omega \|u - u_{\mathfrak{Q},k}\|^2_{L^2(\mathbb{S}^2)}\,\mathrm{d}\mathbb{P}\\ &\leq \left[\frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{\mu_0}{(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right)\right]^2 \int_\Omega \|\overline{\mathcal{W}}\|^2_{L^2(\mathbb{S}^2)}\,\mathrm{d}\mathbb{P}\\ &\leq \left[\frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{\mu_0}{(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right)\right]^2 \|\overline{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))}.\end{aligned}$$

□

Noting that

$$\|u - u_h^\ell\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq \|u - u_{\mathfrak{Q},k}\|_{L^2(\Omega;L^2(\mathbb{S}^2))} + \|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\mathbb{S}^2))},$$

we see that the finite element error, $\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\mathbb{S}^2))}$ remains to be estimated.

4.4.2 Estimates for the finite element error

In this section we intend to prove the strong error convergence of the fractional SFEM method. We begin by collecting a few necessary preliminary estimates. We introduce the basic forms a and m,

$$m(u, v) = \int_{\mathbb{S}^2} uv \ \mathrm{d}A,$$

and

$$a(u, v) = \int_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} u, \nabla_{\mathbb{S}^2} v)_{\mathbb{R}^3} \,\mathrm{d}A,$$

for $u, v \in H^1(\mathbb{S}^2)$, and denote by $a_h(u_h, v_h)$ and $m_h(u_h, v_h)$ their discretised analogues on $H^1(\mathbb{S}_h^2)$. The following lemma (Lemma 4.7 from [6]), gives error estimates of the two basic forms a, m which make up the building blocks for the more complicated forms $\mathfrak{a}_{\mathbb{S}^2,l}$.

Lemma 4.4.3 (Geometric errors of basic forms)**.**
Recall the definition of the finite element space $S_h \subset H^1(\mathbb{S}_h^2)$ and its lifted analogue $S_h^\ell \subset H^1(\mathbb{S}^2)$. Let a be defined as above. For $u_h, v_h \in S_h$ and their lifted versions $u_h^\ell, v_h^\ell \in S_h^\ell$ it holds that there is a constant $c > 0$ such that

$$\left| a(u_h^\ell, v_h^\ell) - a_h(u_h, v_h) \right| \leq ch^2 \|\nabla_{\mathbb{S}^2} u_h^\ell\|_{L^2(\mathbb{S}^2)} \|\nabla_{\mathbb{S}^2} v_h^\ell\|_{L^2(\mathbb{S}^2)}. \tag{4.18}$$

$$\left| m(u_h^\ell, v_h^\ell) - m_h(u_h, v_h) \right| \leq ch^2 \|u_h^\ell\|_{L^2(\mathbb{S}^2)} \|v_h^\ell\|_{L^2(\mathbb{S}^2)}. \tag{4.19}$$

The proof of this lemma in the more general setting of smooth manifolds can be found in [6]. The proof does not in essence change in our case, but the key takeaway from the proof is that the constant c does not depend on h.

Equipped with these two lemmas, we can estimate $\mathfrak{a}_{\mathbb{S}^2,l}$ in a similar way.

Lemma 4.4.4 (Geometric error of the bilinear form $\mathfrak{a}_{\mathbb{S}^2,l}$)**.**
Let the setting be as in Lemma 4.4.3. It then holds that there is a constant $c > 0$ such that

$$\begin{aligned} &\left| \mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, v_h^\ell) - \mathfrak{a}_{\mathbb{S}_h^2,l}(u_h, v_h) \right| \\ &\leq c\gamma_l h^2 \left(\|u_h^\ell\|_{L^2(\mathbb{S}^2)} \|v_h^\ell\|_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2} u_h^\ell\|_{L^2(\mathbb{S}^2)} \|\nabla_{\mathbb{S}^2} v_h^\ell\|_{L^2(\mathbb{S}^2)} \right), \end{aligned}$$

where $\gamma_l = \max(\alpha_l, \beta_l)$.

The proof is an application of the above estimates and the triangle inequality. Note that it holds for a general function $g \in H^1(\mathbb{S}^2)$ that

$$\begin{aligned}
&\|\nabla_{\mathbb{S}^2} g\|_{L^2(\mathbb{S}^2)} \le \|g\|_{H^1(\mathbb{S}^2)},\\
&\|g\|_{L^2(\mathbb{S}^2)} \le \|g\|_{H^1(\mathbb{S}^2)},
\end{aligned}$$

so we can rewrite the estimate in Lemma 4.4.4 as

$$\left|\mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, v_h^\ell) - \mathfrak{a}_{\mathbb{S}_h^2,l}(u_h, v_h)\right| \le c\gamma_l h^2 \|u_h^\ell\|_{H^1(\mathbb{S}^2)} \|v_h^\ell\|_{H^1(\mathbb{S}^2)}.$$

Note that we can use Young's inequality for products to obtain that

$$\begin{aligned}
&c\gamma_l h^2 \left(\|u_h^\ell\|_{L^2(\mathbb{S}^2)}\|v_h^\ell\|_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2} u_h^\ell\|_{L^2(\mathbb{S}^2)}\|\nabla_{\mathbb{S}^2} v_h^\ell\|_{L^2(\mathbb{S}^2)}\right)\\
&\le \frac{c\gamma_l h^2}{2}\left(\|u_h^\ell\|^2_{L^2(\mathbb{S}^2)} + \|v_h^\ell\|^2_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2} u_h^\ell\|^2_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2} v_h^\ell\|^2_{L^2(\mathbb{S}^2)}\right)\\
&\le c\gamma_l h^2 \left(\|u_h^\ell\|^2_{H^1(\mathbb{S}^2)} + \|v_h^\ell\|^2_{H^1(\mathbb{S}^2)}\right).
\end{aligned}$$

Before we begin the error analysis in earnest, let us collect a few properties of the bilinear form $\mathfrak{a}_{\mathbb{S}_h^2,l} : H^1(\mathbb{S}_h^2) \times H^1(\mathbb{S}_h^2) \to \mathbb{R}$ and the problem at hand.

1. $\mathfrak{a}_{\mathbb{S}_h^2,l} : S_h \times S_h \to \mathbb{R}$ is coercive for h small enough since

   $$\begin{aligned}
   \mathfrak{a}_{\mathbb{S}_h^2,l}(u_h, u_h) &= \mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, u_h^\ell) - \left(\mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, u_h^\ell) - \mathfrak{a}_{\mathbb{S}_h^2,l}(u_h, u_h)\right)\\
   &\ge (\alpha_l + \beta_l - c\gamma_l h^2)\|u_h\|_{H^1(\mathbb{S}^2)} \ge \frac{1}{2}\|u_h\|_{H^1(\mathbb{S}^2)}.
   \end{aligned}$$

   This estimate of course also holds for the non-discretised subproblem. Note that by "small enough", we mean that ch will be around 1, since α_l and β_l will be of the same magnitude for reasonable values of k, κ and β.

2. It holds that the forms $\mathfrak{a}_{\mathbb{S}_h^2,l}$ and m_h are continuous forms by an application of Hölder's inequality. This holds for the forms in the non-discretised subproblem as well.

3. It holds from the existence results in Section 2.4 that there exists a unique solution $u \in H^1(\mathbb{S}^2)$ to the subproblems with deterministic right-hand side, that is to say, for every $f \in L^2(\mathbb{S}^2)$ there is a unique $u \in H^1(\mathbb{S}^2)$ such that,

   $$\mathfrak{a}_{\mathbb{S}^2,l}(u, v) = m(f, v) \text{ for every } v \in H^1(\mathbb{S}^2).$$

We can now begin our error analysis. We will first show estimates for the subproblems, where the right-hand side is any $f \in L^2(\mathbb{S}^2)$, later applying Bochner space theory to these estimates to obtain the SPDE estimates in $L^2(\Omega; L^2(\mathbb{S}^2))$-norm.

In order to estimate the $L^2(\mathbb{S}^2)$-norm of the error, we will use a duality argument, which often is called the *Aubin–Nitsche* trick. For this method to work as intended, we need to first obtain some $H^1(\mathbb{S}^2)$- or energy-norm estimates of the error.

Lemma 4.4.5 ($H^1(\mathbb{S}^2)$-error)**.**
Consider the problem of finding $u \in H^1(\mathbb{S}^2)$ such that for every $v \in H^1(\mathbb{S}^2)$ it holds that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u,v) = m(f,v), \tag{4.20}$$

for some $f \in L^2(\mathbb{S}^2)$. Denote by $u_h \in S_h$ the corresponding finite element solution, that is to say, the solution to the problem of finding $u_h \in S_h$ such that for every $v_h \in S_h$

$$\mathfrak{a}_{\mathbb{S}^2_h,l}(u_h,v_h) = m_h(F_h,v_h),$$

where F_h is the interpolation of f. It then holds that

$$\|u_h^\ell - u\|_{H^1(\mathbb{S}^2)} \le c\left(\gamma_l h\|f\|_{L^2(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)}\right).$$

Proof. By the regularity estimate (2.10), it holds that $u \in H^2(\mathbb{S}^2)$ and that there is c>0 such that $\|u\|_{H^2(\mathbb{S}^2)} \le c\|f\|_{L^2(\mathbb{S}^2)}$. Using the coercivity estimate, for h small enough, it holds that

$$\frac{1}{2}\|u_h^\ell - v_h^\ell\|^2_{H^1(\mathbb{S}^2)} \le \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h - v_h, u_h - v_h).$$

By adding and subtracting, we obtain that

$$\begin{aligned}\mathfrak{a}_{\mathbb{S}^2_h,l}(u_h - v_h, u_h - v_h) &= \mathfrak{a}_{\mathbb{S}^2,l}(u - v_h^\ell, u_h^\ell - v_h^\ell) + \mathfrak{a}_{\mathbb{S}^2,l}(v_h^\ell, u_h^\ell - v_h^\ell) - \mathfrak{a}_{\mathbb{S}^2_h,l}(v_h, u_h - v_h)\\ &\quad - \left(\mathfrak{a}_{\mathbb{S}^2,l}(u, u_h^\ell - v_h^\ell) - \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, u_h - v_h)\right).\end{aligned}$$

In the last term, we note that $u_h^\ell - v_h^\ell \in H^1(\mathbb{S}^2)$, as well as that $u_h - v_h \in S_h$, so we can use the weak solution to see that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u, u_h^\ell - v_h^\ell) - \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, u_h - v_h) = m(f, u_h^\ell - v_h^\ell) - m_h(F_h, u_h - v_h).$$

Adding and subtracting yields,

$$m(f - F_h^\ell, u_h^\ell - v_h^\ell) + m(F_h^\ell, u_h^\ell - v_h^\ell) - m_h(F_h, u_h - v_h).$$

We insert this expression to obtain that

$$\begin{aligned}&\frac{1}{2}\|u_h^\ell - v_h^\ell\|^2_{H^1(\mathbb{S}^2)}\\ &\quad\le \mathfrak{a}_{\mathbb{S}^2,l}(u - v_h^\ell, u_h^\ell - v_h^\ell) + \mathfrak{a}_{\mathbb{S}^2,l}(v_h^\ell, u_h^\ell - v_h^\ell) - \mathfrak{a}_{\mathbb{S}^2_h,l}(v_h, u_h - v_h)\\ &\quad - \left(m(f - F_h^\ell, u_h^\ell - v_h^\ell) + m(F_h^\ell, u_h^\ell - v_h^\ell) - m_h(F_h, u_h - v_h)\right).\end{aligned}$$

The first term is estimated by an application of Hölder's inequality,

$$\begin{aligned}\mathfrak{a}_{\mathbb{S}^2,l}(u - v_h^\ell, u_h^\ell - v_h^\ell) &= \alpha_l m(u - v_h^\ell, u_h^\ell - v_h^\ell) + \beta_l a(u - v_h^\ell, u_h^\ell - v_h^\ell)\\ &\le \gamma_l\left(\|u - v_h^\ell\|_{L^2(\mathbb{S}^2)}\|u_h^\ell - v_h^\ell\|_{L^2(\mathbb{S}^2)} + \|\nabla_{\mathbb{S}^2}(u - v_h^\ell)\|_{L^2(\mathbb{S}^2)}\|\nabla_{\mathbb{S}^2}(u_h^\ell - v_h^\ell)\|_{L^2(\mathbb{S}^2)}\right).\end{aligned}$$

For every element in $H^1(\mathbb{S}^2)$, we can estimate both $\|\cdot\|_{L^2(\mathbb{S}^2)}$ and $\|\nabla_{\mathbb{S}^2}(\cdot)\|_{L^2(\mathbb{S}^2)}$ with $\|\cdot\|_{H^1(\mathbb{S}^2)}$, so the above estimate becomes

$$\mathfrak{a}_{\mathbb{S}^2,l}(u - v_h^\ell, u_h^\ell - v_h^\ell) \le c\gamma_l \|u - v_h^\ell\|_{H^1(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)},$$

where $\gamma_l = \max(\alpha_l, \beta_l)$. In the same manner as above, applying Hölder's inequality to the fourth term allows for its estimation

$$\begin{aligned} & m(f - F_h^\ell, u_h^\ell - v_h^\ell) = \int_{\mathbb{S}^2} \left(f - F_h^\ell\right)\left(u_h^\ell - v_h^\ell\right) \, \mathrm{d}A \\ & \le \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{L^2(\mathbb{S}^2)} \le \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)}. \end{aligned}$$

Combining these two estimates with the estimates from Lemmas 4.4.3 and 4.4.4, we obtain that

$$\begin{aligned} & \frac{1}{2}\|u_h^\ell - v_h^\ell\|^2_{H^1(\mathbb{S}^2)} \\ & \le c\gamma_l \, \|u - v_h^\ell\|_{H^1(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)} + c\gamma_l \, h^2 \|v_h^\ell\|_{H^1(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)} \\ & + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)} + ch^2 \|F_h^\ell\|_{L^2(\mathbb{S}^2)} \|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)}. \end{aligned}$$

We divide by $\|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)}$, and arrive at the expression

$$\begin{aligned} & \frac{1}{2}\|u_h^\ell - v_h^\ell\|_{H^1(\mathbb{S}^2)} \le \\ & c\gamma_l \, \|u - v_h^\ell\|_{H^1(\mathbb{S}^2)} + c\gamma_l h^2 \|v_h^\ell\|_{H^1(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + ch^2 \|F_h^\ell\|_{L^2(\mathbb{S}^2)}. \end{aligned}$$

Since v_h is a general function in S_h, we let it be $\tilde{I}_h u$, so that $I_h u = v_h^\ell$, where I_h is the interpolant defined in Section 4.1.1.1. Note that by adding and subtracting u we can estimate $\|I_h u\|_{H^1(\mathbb{S}^2)}$ using the triangle inequality,

$$\|I_h u\|_{H^1(\mathbb{S}^2)} = \|u - I_h u - u\|_{H^1(\mathbb{S}^2)} \le \|u - I_h u\|_{H^1(\mathbb{S}^2)} + \|u\|_{H^1(\mathbb{S}^2)}.$$

In the same manner, we can bound $\|F_h^\ell\|_{L^2(\mathbb{S}^2)}$ by adding and subtracting f and applying the triangle inequality, so that we see that

$$\|F_h^\ell\|_{L^2(\mathbb{S}^2)} = \|F_h^\ell - f + f\|_{L^2(\mathbb{S}^2)} \le \|f\|_{L^2(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)}.$$

Inserting these estimates, we obtain that

$$\begin{aligned} & \frac{1}{2}\|u_h^\ell - I_h u\|_{H^1(\mathbb{S}^2)} \\ & \le c\gamma_l \|u - I_h u\|_{H^1(\mathbb{S}^2)} + c\gamma_l h^2 \|I_h u\|_{H^1(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + ch^2 \|F_h^\ell\|_{L^2(\mathbb{S}^2)} \\ & \le \gamma_l (1 + ch^2) \|u - I_h u\|_{H^1(\mathbb{S}^2)} + \gamma_l ch^2 \|u\|_{H^1(\mathbb{S}^2)} + (1 + ch^2) \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + ch^2 \|f\|_{L^2(\mathbb{S}^2)}. \end{aligned}$$

Lemma 4.1.4 is then used to bound $\|u - I_h u\|_{H^1(\mathbb{S}^2)}$, yielding

$$\begin{aligned} & \frac{1}{2}\|u_h^\ell - I_h u\|_{H^1(\mathbb{S}^2)} \\ & \le c\gamma_l h \|u\|_{H^2(\mathbb{S}^2)} + ch^2 \gamma_l \|u\|_{H^1(\mathbb{S}^2)} + (1 + ch^2) \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + ch^2 \|f\|_{L^2(\mathbb{S}^2)}. \end{aligned}$$

For any $u \in H^2(\mathbb{S}^2)$ it holds that $\|u\|_{H^1(\mathbb{S}^2)} \leq \|u\|_{H^2(\mathbb{S}^2)}$, which we apply to the expression above,

$$\frac{1}{2}\|u_h^\ell - I_h u\|_{H^1(\mathbb{S}^2)} \leq c\gamma_l h\|u\|_{H^2(\mathbb{S}^2)} + ch^2\|f\|_{L^2(\mathbb{S}^2)} + (1+ch^2)\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)}.$$

Recall that we are really interested in the error $\|u - u_h^\ell\|_{H^1(\mathbb{S}^2)}$. Hence, we can add and subtract $I_h u$ and use the triangle inequality so that the expression simplifies to

$$\|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} \leq \|u - I_h u\|_{H^1(\mathbb{S}^2)} + \|u_h^\ell - I_h u\|_{H^1(\mathbb{S}^2)}.$$

We apply the above estimate of $\frac{1}{2}\|u_h^\ell - I_h u\|_{H^1(\mathbb{S}^2)}$ and use the interpolation estimate from Lemma 4.1.4 to estimate $\|u - I_h u\|_{H^1(\mathbb{S}^2)}$ in order to see that

$$\begin{aligned}&\|u - u_h^\ell\|_{H^1(\mathbb{S}^2)}\\ &\leq c\gamma_l h\|u\|_{H^2(\mathbb{S}^2)} + ch^2\|f\|_{L^2(\mathbb{S}^2)} + (1+ch^2)\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + ch\|u\|_{H^2(\mathbb{S}^2)},\end{aligned}$$

which we can bound by

$$\|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} \leq c\left(\gamma_l h\|u\|_{H^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)}\right).$$

We can now use the regularity estimates from Equation (2.10), which tell us that $\|u\|_{H^2(\mathbb{S}^2)} \leq c\|f\|_{L^2(\mathbb{S}^2)}$,

$$\|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} \leq c\left(\gamma_l h\|f\|_{L^2(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)}\right). \qquad \square$$

Lemma 4.4.6 ($L^2(\mathbb{S}^2)$-error)**.**
Let the setting be the same as in Lemma 4.4.5. For the $L^2(\mathbb{S}^2)$-error between the lifted approximate FEM solution and the exact solution of Equation (4.20)*, it holds that*

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}^2 \leq c\gamma_l^2\left(\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)}\right).$$

Proof. In this proof, we will utilise the so-called Aubin–Nitsche trick to obtain certain estimates, which we then can deal with by using the Sobolev norm estimates from the previous lemma. The trick entails estimation of the solution of a related PDE with a special right-hand side. This PDE is given, for $v \in H^2(\mathbb{S}^2)$, by

$$-\mathfrak{L}_l v = u - u_h^\ell - \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2} u - u_h^\ell \, \mathrm{d}A. \tag{4.21}$$

By the regularity estimate from Equation (2.10), it holds that

$$\|v\|_{H^2(\mathbb{S}^2)} \leq c\left\|u - u_h^\ell - \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{S}^2}(u - u_h^\ell)\,\mathrm{d}A\right\|_{L^2(\mathbb{S}^2)} \leq c'\left\|u - u_h^\ell\right\|_{L^2(\mathbb{S}^2)}, \tag{4.22}$$

since

$$\left\| \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} u - u_h^\ell \, \mathrm{d}A \right\|_{L^2(\mathbb{S}^2)} \leq \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} \|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} \, \mathrm{d}A$$
$$\leq \|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}.$$

Hence it can be included in the constant c'. Note now that we can multiply Equation (4.21) with $u - u_h^\ell$ and integrate over $\mathbb{S}^2$ to obtain that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) = \|u - u_h^\ell\|^2_{L^2(\mathbb{S}^2)} - \frac{1}{|\mathbb{S}^2|} \left(\int_{\mathbb{S}^2} u - u_h^\ell \, \mathrm{d}A \right)^2.$$

Let us consider the term $\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v)$.

In the remainder of the proof we will denote the lift of the $\mathbb{S}^2_h$-interpolation of v by $I_h v$, as in Section 4.1.1.1. From the bilinearity of $\mathfrak{a}_{\mathbb{S}^2,l}$ it follows, by adding and subtracting, that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) = \mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v - I_h v) + \mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, I_h v)$$
$$= \mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v - I_h v) + \mathfrak{a}_{\mathbb{S}^2,l}(u, I_h v) - \mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, I_h v) - \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, \tilde{I}_h v) + \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, \tilde{I}_h v).$$

Note now, that by the weak formulation of the subproblems, it holds that $\mathfrak{a}_{\mathbb{S}^2,l}(u, \eta) = m(f, \eta)$ for all $\eta \in H^1(\mathbb{S}^2)$ and similarly, by the SFEM formulation, we have that $\mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, \eta_h) = m_h(F_h, \eta_h)$ for every $\eta_h \in S_h$. Note, that in the above expression, $I_h v \in H^1(\mathbb{S}^2)$, and $\tilde{I}_h v \in S_h$, so we substitute $\mathfrak{a}_{\mathbb{S}^2,l}(u, I_h v)$ with $m(f, I_h v)$ and $-\mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, \tilde{I}_h v)$ with $-m_h(F_h, \tilde{I}_h b)$ to obtain that

$$\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) =$$
$$\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v - I_h v) + m(f, I_h v) - m_h(F_h, \tilde{I}_h v) - \left(\mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, I_h v) - \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, \tilde{I}_h v) \right).$$

By adding and subtracting $m(F_h^\ell, I_h v)$, we can use the estimates from Lemma 4.4.3 obtaining that

$$\begin{aligned}
&\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) = \\
&\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v - I_h v) + m(f - F_h^\ell, I_h v) \\
&\qquad + \left(m(F_h^\ell, I_h v) - m_h(F_h, \tilde{I}_h v) \right) - \left(\mathfrak{a}_{\mathbb{S}^2,l}(u_h^\ell, I_h v) - \mathfrak{a}_{\mathbb{S}^2_h,l}(u_h, \tilde{I}_h v) \right) \\
&\leq \mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v - I_h v) + m(f - F_h^\ell, I_h v) + ch^2 \|F_h^\ell\|_{L^2(\mathbb{S}^2)} \|I_h v\|_{L^2(\mathbb{S}^2)} \\
&\qquad + c\gamma_l h^2 \|u_h^\ell\|_{H^1(\mathbb{S}^2)} \|I_h v\|_{H^1(\mathbb{S}^2)}.
\end{aligned}$$

Applying Hölder's inequality to the first two terms, and using that the $L^2(\mathbb{S}^2)$-norm is less than the $H^1(\mathbb{S}^2)$-norm in the second and third term, we arrive at the expression

$$\begin{aligned}
\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) &\leq c\gamma_l \|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} \|v - I_h v\|_{H^1(\mathbb{S}^2)} \\
&\quad + c \left(\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2 \|F_h^\ell\|_{L^2(\mathbb{S}^2)} + \gamma_l h^2 \|u_h^\ell\|_{H^1(\mathbb{S}^2)} \right) \|I_h v\|_{H^1(\mathbb{S}^2)}.
\end{aligned}$$

Note now that, by adding and subtracting and the triangle inequality,

$$\|I_h v\|_{H^1(\mathbb{S}^2)} \leq \|v - I_h v\|_{H^1(\mathbb{S}^2)} + \|v\|_{H^1(\mathbb{S}^2)}. \tag{4.23}$$

We apply the interpolation estimate from Lemma 4.1.2 to $\|v - I_h v\|_{H^1(\mathbb{S}^2)}$ to obtain that

$$\|v - I_h v\|_{H^1(\mathbb{S}^2)} \leq ch\|v\|_{H^2(\mathbb{S}^2)}.$$

Furthermore $\|v\|_{H^1(\mathbb{S}^2)} \leq \|v\|_{H^2(\mathbb{S}^2)}$ since $v \in H^2(\mathbb{S}^2)$ and the $H^2(\mathbb{S}^2)$-norm consists of adding something positive to the $H^1(\mathbb{S}^2)$-norm. Hence we can rewrite the estimate in Equation (4.23) as

$$\|I_h v\|_{H^1(\mathbb{S}^2)} \leq c\|v\|_{H^2(\mathbb{S}^2)}.$$

Applying the interpolation estimate and using the above reasoning, we see that

$$\begin{aligned}
&\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) \\
&\leq ch\gamma_l \|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} \|v\|_{H^2(\mathbb{S}^2)} \\
&\qquad + c\left(\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|F_h^\ell\|_{L^2(\mathbb{S}^2)} + \gamma_l h^2 \|u_h^\ell\|_{H^1(\mathbb{S}^2)}\right)\|v\|_{H^2(\mathbb{S}^2)}.
\end{aligned}$$

In order to simplify the expression, we bound $\|F_h^\ell\|_{L^2(\mathbb{S}^2)}$ by

$$\|F_h^\ell\|_{L^2(\mathbb{S}^2)} \leq \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}.$$

We can estimate $\|u_h^\ell\|_{H^1(\mathbb{S}^2)}$ by first nothing that

$$\|u_h^\ell\|_{H^1(\mathbb{S}^2)} \leq \|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} + \|u\|_{H^1(\mathbb{S}^2)}$$

and then using the fact that $\|u\|_{H^1(\mathbb{S}^2)} \leq c\|f\|_{L^2(\mathbb{S}^2)}$, so the above estimate can be rewritten as

$$\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) \leq c\gamma_l \|v\|_{H^2(\mathbb{S}^2)}\left(h\|u - u_h^\ell\|_{H^1(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)}\right).$$

We use Lemma 4.4.5 to estimate the $H^1(\mathbb{S}^2)$-norm of $u - u_h^\ell$, so it holds that

$$\begin{aligned}
\mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) &\leq c\gamma_l \|v\|_{H^2(\mathbb{S}^2)}\Big(h\left(h\gamma_l\|f\|_{L^2(\mathbb{S}^2)} + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)}\right) \\
&\qquad + \|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)}\Big) \leq \\
&c\gamma_l^2 \|v\|_{H^2(\mathbb{S}^2)}\left(\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)}\right).
\end{aligned}$$

We apply the regularity estimate from Equation (4.22) to $\|v\|_{H^2(\mathbb{S}^2)}$, in order to finally arrive at the estimate

$$\begin{aligned}
&\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}^2 - \frac{1}{|\mathbb{S}^2|}\left(\int_{\mathbb{S}^2} u - u_h^\ell \,\mathrm{d}A\right)^2 \\
&= \mathfrak{a}_{\mathbb{S}^2,l}(u - u_h^\ell, v) \\
&\leq c\gamma_l^2\left(\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)}\right)\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

By applying the Cauchy–Schwartz inequality and using that the sphere is a surface with finite area , we see that

$$\frac{1}{|\mathbb{S}^2|}\left(\int_{\mathbb{S}^2}(u - u_h^\ell)\,\mathrm{d}A\right)^2 \leq c\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}^2.$$

We rewrite the above estimate and divide by $\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}$ to obtain

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}^2 \leq c\gamma_l^2\left(\|f - F_h^\ell\|_{L^2(\mathbb{S}^2)} + h^2\|f\|_{L^2(\mathbb{S}^2)}\right),$$

which proves the lemma. □

Lemma 4.4.7 (Subproblem error).
Let u_l be the solution to the lth subproblem, and let $u_{h,l}$ denote the corresponding finite element approximation. The error of the subproblems given by equation (4.11) *can be estimated by*

$$\mathbb{E}[\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}]^{1/2} \leq c\gamma_l^2 \left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{Tr(Q)}\right),$$

where $\frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l \left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) = D(\mathfrak{K}, A_l)$ *and where Q is the covariance operator of the random field $\overline{\mathcal{W}}$, $\mathfrak{K}$ is the truncation parameter of the noise approximation, α is the decay parameter and h is the mesh size.*

Proof. By the definition of the $L^2(\Omega; L^2(\mathbb{S}^2))$-norm, it holds that

$$\mathbb{E}[\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}] = \int_{\Omega} \|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)} \, \mathrm{d}\mathbb{P}.$$

Using Lemma 4.4.6 to estimate $\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}$ we have that

$$\mathbb{E}[\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}] \leq c^2\gamma_l^4 \int_{\Omega} \left(\|\overline{\mathcal{W}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\|_{L^2(\mathbb{S}^2)} + h^2\|\overline{\mathcal{W}}\|_{L^2(\mathbb{S}^2)}\right)^2 \, \mathrm{d}\mathbb{P}.$$

which by using that $(a+b)^2 \leq 2a^2 + 2b^2$ for $a, b > 0$, can be further bounded by

$$\mathbb{E}[\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}] \leq 2c\gamma_l^4 \int_{\Omega} \|\overline{\mathcal{W}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\|^2_{L^2(\mathbb{S}^2)} + h^4\|\overline{\mathcal{W}}\|^2_{L^2(\mathbb{S}^2)} \, \mathrm{d}\mathbb{P}.$$

The above expression can be rewritten using the definition of the $L^2(\Omega; L^2(\mathbb{S}^2))$-norm,

$$\mathbb{E}[\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}] \leq c\gamma_l^4 \left(\|\overline{\mathcal{W}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} + h^4\|\overline{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))}\right).$$

Note now that

$$\|\overline{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} = \mathrm{Tr}(Q),$$

and furthermore, by what we showed in Section 3.1,

$$\|\overline{\mathcal{W}} - \widetilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq c\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2,$$

so we obtain that

$$\mathbb{E}[\|u^{\ell}_{h,l} - u_l\|^2_{L^2(\mathbb{S}^2)}]^{1/2} \leq c\gamma_l^2 \left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\mathrm{Tr}(Q)}\right)$$

where $\frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l \left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) = D(\mathfrak{K}, A_l)$. □

We now use the subproblem errors to obtain a bound for the strong error of the entire method.

Theorem 4.4.1 (Strong error).
Let u denote the weak solution of the entire problem, given by equation (4.6)*, and*

denote by u_h^ℓ the lifted finite element solution of the problem. The strong error is bounded by

$$
\begin{aligned}
&\|u - u_h^\ell\|_{L^2(\Omega,L^2(\Omega))} \\
&\le \|u - u_{\mathfrak{Q},k}\|_{L^2(\Omega,L^2(\Omega))} + \|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} \\
&\le c_1(k)\sqrt{Tr(Q)} + c_2(k)\left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{Tr(Q)}\right),
\end{aligned}
$$

where

$$
c_1(k) = \frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{1}{\kappa^2(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right),
$$

and

$$
c_2(k) = c\frac{2k\sin(\pi\beta)}{\pi}e^{2\beta k K^+}(K^+ + K^- + 1)\left(1_{\kappa\ge 1}(1+\kappa^2 e^{2kK^+})^2 + 1_{\kappa<1}e^{4kK^+}\right),
$$

and

$$
D(\mathfrak{K}, A_l) = \frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right).
$$

Recall that $Tr(Q) = \sum_{l=0}^{\infty}(2l+1)A_l$.

Proof. Note that, with $u_{\mathfrak{Q},k}$ as above, it holds that

$$
\|u - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} = \|u - u_{\mathfrak{Q},k}\|_{L^2(\Omega;L^2(\Omega))} + \|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))}.
$$

As earlier proved in Lemma 4.4.2, the error $\|u - u_{\mathfrak{Q},k}\|_{L^2(\Omega;L^2(\Omega))}$ decays exponentially in k, the number of quadrature nodes, and is given by

$$
\|u_{\mathfrak{Q},k} - u\|_{L^2(\Omega;L^2(\Omega))} \le c(k)\|\mathcal{W}\|_{L^2(\Omega;L^2(\mathbb{S}^2))} = c(k)\sqrt{\mathrm{Tr}(Q)},
$$

where

$$
c(k) = \frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{\mu_0}{(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right).
$$

Using Equation (4.13) together with the triangle inequality, we obtain that

$$
\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} \le \frac{2k\sin(\pi\beta)}{\pi}\sum_{l=-K^-}^{K^+} e^{2\beta y_l}\left\|\left(u_l - u_{l,h}^\ell\right)\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))}.
$$

We can now apply Lemma 4.4.7 to obtain that

$$
\begin{aligned}
&\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} \\
&\quad\le \frac{2k\sin(\pi\beta)}{\pi}\sum_{l=-K^-}^{K^+} e^{2\beta y_l} c\gamma_l^2\left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\mathrm{Tr}(Q)}\right).
\end{aligned}
$$

Using that

$$e^{2\beta kl} \leq \max_{l=-K^-,\dots,K^+} e^{2\beta kl} = e^{2\beta kK^+},$$

we can estimate the above,

$$\begin{aligned}
&\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} \\
&\leq \frac{2k\sin(\pi\beta)}{\pi} e^{2\beta kK^+} \sum_{l=-K^-}^{K^+} c\gamma_l^2 \left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\operatorname{Tr}(Q)}\right).
\end{aligned}$$

Note now that

$$\begin{aligned}
\max_{l=-K^-,\dots,K}(\gamma_l^2) &= \max_{l=-K^-,\dots,K}(\max(\alpha_l, \beta_l)^2) \\
&= \max_{l=-K^-,\dots,K}(\max(1+\kappa^2 e^{2kl}, e^{2kl})) \\
&= 1_{\kappa\geq 1}(1+\kappa^2 e^{2kK^+})^2 + 1_{\kappa<1} e^{4kK^+}.
\end{aligned}$$

Here $1_{\kappa\geq 1} = 1$ if $\kappa \geq 1$, and 0 otherwise and $1_{\kappa<1} = 1$ if $\kappa < 1$ and 0 otherwise. We hence obtain

$$\begin{aligned}
&\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} \\
&\leq \mathfrak{c}(k, \beta, K^+) \sum_{l=-K^-}^{K^+} \left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\operatorname{Tr}(Q)}\right),
\end{aligned}$$

where $\mathfrak{c}(k, \beta, K^+) = c\frac{2k\sin(\pi\beta)}{\pi} e^{2\beta kK^+}\left(1_{\kappa\geq 1}(1+\kappa^2 e^{2kK^+})^2 + 1_{\kappa<1} e^{4kK^+}\right)$.

Recall that $D(\mathfrak{K}, A_l)$ does not depend on the summation index l, and therefore it can be factorised out of the sum. We use this to arrive at the final estimate,

$$\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\Omega;L^2(\Omega))} \tag{4.24}$$

$$\leq \mathfrak{c}(k, \beta, K^+)(K^+ + K^- + 1)\left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\operatorname{Tr}(Q)}\right). \tag{4.25}$$

We can now give the error estimate, combining Equation (4.24) and Lemma 4.4.2 to obtain that

$$\|u - u_h^\ell\|_{L^2(\Omega,L^2(\Omega))} \leq c_1(k)\sqrt{\operatorname{Tr}(Q)} + c_2(k)\left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\operatorname{Tr}(Q)}\right),$$

where

$$c_1(k) = \frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{\mu_0}{(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right),$$

and

$$c_2(k) = c\frac{2k\sin(\pi\beta)}{\pi} e^{2\beta kK^+}(K^+ + K^- + 1)\left(1_{\kappa\geq 1}(1+\kappa^2 e^{2kK^+})^2 + 1_{\kappa<1} e^{4kK^+}\right).$$

Recall now that $\mu_0 := 1/\lambda_0$ where λ_0 is the smallest eigenvalue of the operator $\mathcal{L} = \kappa^2 - \Delta_{\mathbb{S}^2}$. Note that the smallest eigenvalue of $-\Delta_{\mathbb{S}^2}$ is 0, so the smallest eigenvalue of $\mathcal{L}$ is κ^2, and hence $\mu_0 = \kappa^{-2}$, which proves our estimate. □

We also wish to give an asymptotic $\mathbb{P}$-a.s. error estimate. The idea is to first estimate the error in the $L^p(\Omega; L^2(\mathbb{S}^2))$-norm, and then use Chebyshev's inequality (Lemma 2.3.1) together with the Borel–Cantelli lemma (Lemma 2.3.2). We begin with the $L^p(\Omega; L^2(\mathbb{S}^2))$ estimates.

Theorem 4.4.2 ($L^p(\Omega; L^2(\mathbb{S}^2))$-error)**.**
Let the setting be as in Theorem 4.4.1. It holds for any $p \in [1, \infty)$ *that*

$$\begin{aligned}&\|u - u_h^\ell\|_{L^p(\Omega;L^2(\mathbb{S}^2))}\\ &\leq C_p \left(c_1(k)\sqrt{Tr(Q)} + c_2(k)\left(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{Tr(Q)} \right) \right),\end{aligned}$$

where

$$c_1(k) = \frac{2\sin(\pi\beta)}{\pi} \left(\frac{1}{2\beta} + \frac{1}{\kappa^2(2-2\beta)} \right) \left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)} \right),$$

and

$$c_2(k) = c\frac{2k\sin(\pi\beta)}{\pi} e^{2\beta k K^+}(K^+ + K^- + 1)\left(1_{\kappa\geq 1}(1 + \kappa^2 e^{2kK^+})^2 + 1_{\kappa<1}e^{4kK^+} \right),$$

and

$$D(\mathfrak{K}, A_l) = \frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l \left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2 \right),$$

and C_p *is a constant depending on* p *and* α*.*

Proof. If $p \leq 2$, it holds by Hölder's inequality that for any $f \in L^p(\mathbb{S}^2)$,

$$\|f\|_{L^p(\mathbb{S}^2)} \leq c_{\mathbb{S}^2}\|f\|_{L^2(\mathbb{S}^2)}.$$

We can use this immediately to see that the theorem follows for $p \leq 2$. We can also use this observation to see that if we show the theorem for every even $p \in [1, \infty)$, and then use the same Hölder's inequality observation as above for the intermediate values.

In order to show the theorem for the intermediate values, we apply [16, Proposition 2.17] which states that for a Gaussian variable X, it holds that

$$\|X\|^{2m}_{L^{2m}(\Omega;L^2(\mathbb{S}^2))} \leq C_m\|X\|^{2m}_{L^2(\Omega;L^2(\mathbb{S}^2))}.$$

Hence we must ascertain that the quantity $u - u_h^\ell$ is Gaussian. It holds that u is Gaussian, so if we can verify that u_h^ℓ also is Gaussian, we can continue. Note that u_h by definition will be a linear combination of solutions to the problem

$$\mathfrak{L}_{\mathbb{S}^2_h,l}u_{h,l} = \widetilde{\mathcal{W}}_{\mathfrak{K},h}.$$

This implies that we need to verify that $u_{h,l}$ is Gaussian. Note that

$$u_{h,l} = \mathfrak{L}^{-1}_{\mathbb{S}^2_h,l}\widetilde{\mathcal{W}}_{\mathfrak{K},h},$$

which in turn means that we only need to verify that $\widetilde{\mathcal{W}}_{\mathfrak{K},h}$ is Gaussian in order to ascertain that u_h is.

It holds that

$$\widetilde{\mathcal{W}}_{k,h} = \tilde{I}_h \overline{\mathcal{W}}_{\mathfrak{K}},$$

that is to say, it is a linear interpolation. The interpolation nodal points are the vertices of $\mathbb{S}^2_h$, which all lie on $\mathbb{S}^2$. Hence, by the Karhunen–Loève expansion of $\overline{\mathcal{W}}$, it holds that $\overline{\mathcal{W}}$ evaluated in the vertices are normal random variables, and since the interpolation evaluated in any point can be seen as a linear combination of the truncated field evaluated in the nodal points, it holds that $\widetilde{\mathcal{W}}_{k,h}$ is normal and hence u_h is normal.

Now, lifting a function is simply a rescaling of the linear combination, which preserves normality and therefore u_h^ℓ will be normal. We conclude that $u - u_h^\ell$ is Gaussian, and hence it holds that

$$\|u - u_h^\ell\|^{2m}_{L^{2m}(\Omega;L^2(\mathbb{S}^2))} \leq C_m \|u - u_h^\ell\|^{2m}_{L^2(\Omega;L^2(\mathbb{S}^2))}.$$

We apply Theorem 4.4.1 to this to see that

$$\begin{aligned} &\|u - u_h^\ell\|_{L^{2m}(\Omega;L^2(\mathbb{S}^2))} \\ &\leq C_m^{1/2m}\Big(c_1(k)\sqrt{\operatorname{Tr}(Q)} + c_2(k)\Big(\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2 + h^2\sqrt{\operatorname{Tr}(Q)}\Big)\Big), \end{aligned}$$

where

$$c_1(k) = \frac{2\sin(\pi\beta)}{\pi}\left(\frac{1}{2\beta} + \frac{1}{\kappa^2(2-2\beta)}\right)\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right),$$

and

$$c_2(k) = c\frac{2k\sin(\pi\beta)}{\pi} e^{2\beta k K^+}(K^+ + K^- + 1)\left(1_{\kappa\geq 1}(1+\kappa^2 e^{2kK^+})^2 + 1_{\kappa<1}e^{4kK^+}\right),$$

and

$$D(\mathfrak{K}, A_l) = \frac{3}{2}\sum_{l=0}^{\mathfrak{K}} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right).$$

Recall that for a random field, it holds that $\operatorname{Tr}(Q) = \sum_{l=0}^\infty (2l+1)A_l$. This completes the proof, since as previously noted we can for the intermediate values $p \in (2m-2, 2m)$ estimate the $L^p(\mathbb{S}^2)$-norm using Hölder's inequality. □

Hence, the rate of convergence is independent of p, which we use in the next theorem.

Remark 4.4.1. In what follows, we will assume that k is chosen to be so small that we can disregard the error stemming from the quadrature. Note that in practice, we will be able to select k as well in order to balance the error introduced by $c_2(k)$ and $c_1(k)$, but in what follows, recall that the error introduced by the quadrature through the constant $c_1(k)$ decays exponentially, which implies that it is of limited interest to also balance that error against the truncation and FEM error. Recall that we can estimate $D(A_l, \mathfrak{K})$ according to Table 4.1, and thereafter balance the FEM and Quadrature error to obtain that the entire error behaves as $\mathfrak{K}^{-(\alpha-2)/2}$. As an example, with $\alpha = 3$, and selecting $h^{-7/4}$ will allow us to see that the $L^p(\Omega; L^2(\mathbb{S}^2))$-error can be bounded by

$$\|u - u_h^\ell\|_{L^p(\Omega;L^2(\mathbb{S}^2))} \leq C_{p,k,\alpha}\mathfrak{K}^{-(\alpha-2)/2}.$$

Theorem 4.4.3.
Let the setting be as in Theorem 4.4.1. Assume that the parameters has been selected in accordance with Remark 4.4.1. It then holds that the series of successive fractional SFEM approximations u_h converges to the exact solution u $\mathbb{P}$-almost surely and for all $\beta < (\alpha-2)/2$, it holds that

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq \mathfrak{K}^{-\beta} \qquad \mathbb{P} - a.s..$$

Proof. Consider the probability of the event that the $L^2(\Omega)$-error is larger than $\mathfrak{K}^{-\beta}$, that is to say,

$$\mathbb{P}(\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} > \mathfrak{K}^{-\beta}).$$

By Chebyshev's inequality and Theorem 4.4.2 combined with Remark 4.4.1, it holds that

$$\begin{aligned}\mathbb{P}(\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} > \mathfrak{K}^{-\beta}) &\leq \mathfrak{K}^{\beta p}\mathbb{E}[\|u - u_h^\ell\|^p_{L^2(\mathbb{S}^2)}] \\ &= \|u - u_h^\ell\|^p_{L^p(\Omega;L^2(\mathbb{S}^2))} \leq C^p_{p,k,\alpha}\mathfrak{K}^{(\beta-(\alpha-2)/2)p}.\end{aligned}$$

We now wish to apply the Borel–Cantelli lemma. In order to do this, we must verify that the sum given by

$$\sum_{\mathfrak{K}=1}^{\infty} \mathbb{P}(\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} > \mathfrak{K}^{-\beta}),$$

converges. We first estimate the sum to obtain

$$\sum_{\mathfrak{K}=1}^{\infty} \mathbb{P}(\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} > \mathfrak{K}^{-\beta}) \leq C^p_{p,k,\alpha}\sum_{\mathfrak{K}=1}^{\infty}\mathfrak{K}^{(\beta-(\alpha-2)/2)p}.$$

The above sum converges for every $p > (\beta-(\alpha-2)/2)^{-1}$ and hence the Borel–Cantelli Lemma implies that

$$\limsup_{\mathfrak{K}\to\infty} \mathbb{P}(\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} > \mathfrak{K}^{-\beta}) = 0,$$

and the claim follows. □

5

Mesh generation and algorithms

5.1 Implementation

The actual triangulation of the sphere is obtained using the so-called icosahedron method, meaning that all but twelve vertices are the vertex in six triangles. The remaining vertices are each a vertex in five triangles. A discretization of this type is often referred to as an *icosphere*. In the case of this thesis, the icospheres are refined by so-called 2-frequency subdivision. It means that each edge is divided into two new edges. The new vertices are projected onto $\mathbb{S}^2$, and so, a new icosphere is obtained. The refinement procedure is illustrated in figure 5.1.

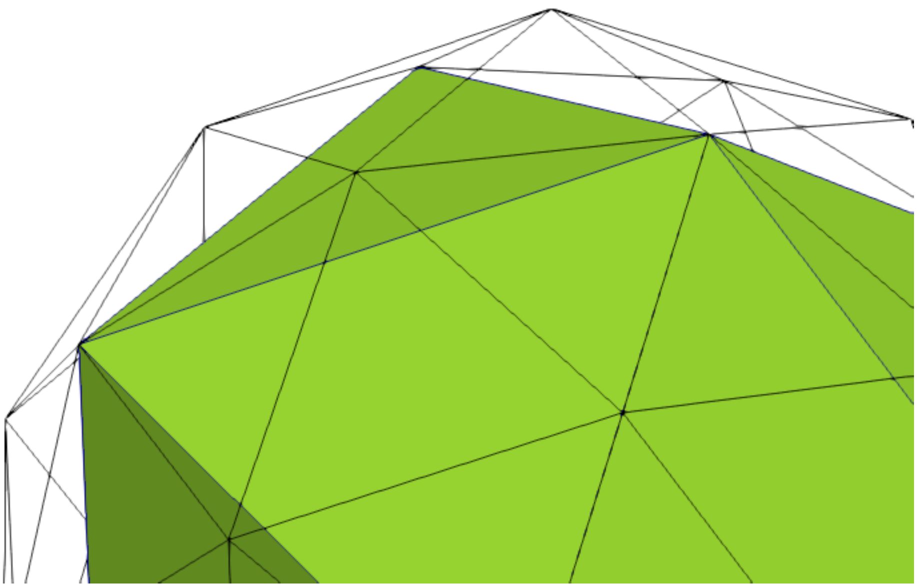
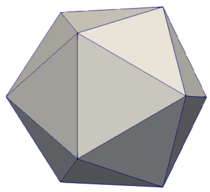
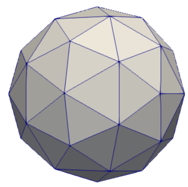


Figure 5.1: Illustration of the Icosphere 2-frequency refinement procedure. The green object is the simplest possible icosphere, and the black edges are its first refinement.

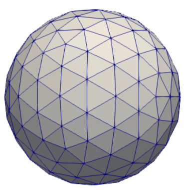
This also means that we refine the icospheres uniformly and that the sphere is meshed with similarly-sized triangles. The perks of doing this are that it is computationally fast compared to more advanced meshing methods, and due to the simple geometry of the sphere, there is no need for any more complicated meshing methods. See figure 5.2 for an example of meshes.



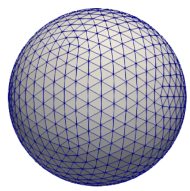
(a) Basic icosphere



(b) 2 refinements



(c) 4 refinements



(d) 9 refinements

Figure 5.2: Example meshes. Note that already with 4 refinements, the sphere is decently approximated

We can summarise the algorithm for numerically solving the field generating equation,

$$\left(\kappa^2 - \Delta_{\mathbb{S}^2}\right)^{\beta} = \overline{\mathcal{W}},$$

by considering each of the three steps, the field-generating step, the quadrature approximation of the fractional operator, and the interior SFEM step used in iteration of the quadrature. We give the algorithms one by one and discuss various challenges that occurred within the implementation work.

5.1.1 Field-generation algorithm

The field is generated following the procedure in section 3.1. From a numerical perspective, a naive implementation as done in this work is rather ineffective. This is due to several reasons, but one reason is that the functions $P_{l,m}$ will be difficult to approximate due to the large factorials that appear. Replacing the factorials with quadrature approximations of the Gamma function alleviates some of these problems, which allows us to calculate $P_{l,m}$ up until $l \approx 100$. In order to ascertain that the same random numbers were used when for instance performing a numerical simulation to verify the errors, the random coefficients are all generated first so that they may be reused when including more terms in the approximation.

Result: FEniCS approximation f of the right-hand side, $\overline{\mathcal{W}}_{\mathfrak{K},h}$
Input: Truncation parameter $\mathfrak{K}$, FEM space V, mesh $\mathbb{S}^2_h$, decay parameter α, random numbers **rands**
Extract the M vertices $((x_i, y_i, z_i))_{i=1}^M$ of $\mathbb{S}^2_h$;
Calculate $\phi = \arctan(y/x)$ and $\theta = \cos(z)$;
Set $\overline{\mathcal{W}}_{\mathfrak{K},h}$ in each point ;
for *every* ϕ_i, θ_i **do**
  $\overline{\mathcal{W}}_{\mathfrak{K},h}(\phi_i, \theta_i) = \sum_{l=0}^{\mathfrak{K}} \sqrt{A_l} z^1_{l,0} P_{l,0}(\cos(\theta)) + \sqrt{2A_l} \sum_{m=1}^{l} P_{l,m}(\cos(\theta))(z^1_{l,m}\cos(m\phi) + z^2_{l,m}\sin(m\phi))$;
end
Generate a FEniCS function, $f =$ **Function(V)**;
Set function vertex values to the calculated field values, f.**vector().[:]** $= \overline{\mathcal{W}}_{\mathfrak{K},h}$;

Algorithm 1: Field-generation algorithm

5.1.2 Algorithm for SFEM solution of subproblems

In this section we describe the implementation of the SFEM algorithm. In many ways, this is the simplest part of the problem due to the fact that it as far as FEniCS is concerned, is a very standard problem.

Result: The approximate numerical solution $u_{h,l}$ of equation (4.20)
Input: Mesh $\mathbb{S}^2_h$, iteration number l, quadrature step size k, right hand side f, FEniCS function space V
Initialise $\mathfrak{a}_{l,\mathbb{S}^2_h}$ according to equation (4.10);
Initialise $m(f) = \int_{\mathbb{S}^2} fv$ for any v in the FEniCS function space V ;
Initialise $u =$ **Function**(V) ;
Run $u_{h,l} =$ **Solve**$(\mathfrak{a}_{l,\mathbb{S}^2_h} = m(f), u)$;

Algorithm 2: Algorithm for solving the subproblems in FEniCS

5.1.3 Complete algorithm for numerical approximation of the solution

The algorithm for solving the entire fractional problem is relatively uncomplicated when the field generation and the subproblem algorithms have been implemented. It is essentially nothing more than solving $K^- + K^+ + 1$ subproblems and summing up the solutions.

Result: The approximate numerical solution u_h of Equation (2.8)
Input: Mesh size h, truncation parameter $\mathfrak{K}$, quadrature step size k, right hand side f
Initialise constants κ and β; Initialise K^-, K^+;
Import icosphere of desired mesh size;
Initialise FEniCS function space V by **V=FunctionSpace($\mathbb{S}^2_h$, FiniteElement('P1', $\mathbb{S}^2_h$.ufl_cell(),1))**;
Generate FEniCS function for placeholder of solution by $u_h =$ **Function**(V);
Create the approximate noise on the mesh using Algorithm 1;
Initialise the constant linear form $m(v) = \int_{\mathbb{S}^2} fv$ **for** $i = K^-, ..., K^+$ **do**
    Set $y = ik$;
    Solve subproblem according to Algorithm 2 to obtain $u_{h,l}$;
    Increment, $u_h = u_h + u_{h,l}$;
end
Normalise to obtain the solution $u_h = \frac{2k\sin(\pi\beta)}{\pi} u_h$;

Algorithm 3: Solving the field-generating equation

5.1.4 Integrating over $\mathbb{S}^2$ in FEniCS

In FEniCS there is no way of integrating directly over $\mathbb{S}^2$. However, given a discretization of the sphere, $\mathbb{S}^2_h$, we can view it as a parametrisation of the sphere. It is possible to perform a change of variables, namely the one generated by the "lift", that is to say, $x \mapsto x/\|x\|_{\mathbb{R}^3}$. We can then integrate over the discretised sphere, but scaling the integrand with the appropriate surface deformation Jacobian.

This Jacobian is rather complex to obtain, especially since it will be singular at certain points and in general difficult to work with since the actual surface deformation is rather involved. Using trial and error, we determined that a good approximation of the Jacobian is

$$J = r_h^{-2} |(N, n)_{\mathbb{R}^3}|,$$

where n is the deformed normal, N is the undeformed normal of the facets of $\mathbb{S}^2_h$ and $r_h(x)$ is the "radius" at $x \in \mathbb{S}^2_h$. In order to verify that this approximate Jacobian works decently enough, we perform an experiment of integrating over a sequence of discretised spheres. In table 5.1, the effect of including the Jacobian is illustrated. We see that it is necessary to take the variable change into consideration in the simple case of an area calculation. Not including the Jacobian when for instance calculating the error, will lead to this integration error dominating. This error can for instance impact convergence results and is therefore something that must be dealt with.

Icosphere refinements	1	2	3	4	5	6	7	8	9
Surface area of sphere by integrating over mesh, without Jacobian compensation	9.57	11.67	12.15	12.33	12.41	12.46	12.49	12.51	12.52
Surface area of sphere by integrating over mesh, with Jacobian compensation	12.59	12.57	12.57	12.57	12.57	12.57	12.57	12.57	12.57

Table 5.1: Effect of including the Jacobian

5.2 Monte-Carlo methods and error estimation

When we calculate the strong error we calculate the Bochner-norm of the difference of the exact solution and our lifted approximated solution, which may, as previously noted, be seen as the expected value of the squared $L^2(\mathbb{S}^2)$-norm. It is in other words given by

$$\|u - u_h^l\|^p_{L^p(\Omega;L^2(\mathbb{S}^2))} = \mathbb{E}[\|u - u_h^l\|^p_{L^2(\mathbb{S}^2)}],$$

which famously can be estimated by

$$\hat{\mathbb{E}}_N[\|u - u_h^l\|^p_{L^2(\mathbb{S}^2)}] = \frac{1}{N}\sum_{i=1}^{N} \|u^{(i)} - u_h^{\ell\ (i)}\|^p_{L^2(\mathbb{S}^2)},$$

where $u^{(i)} - u_h^{\ell\ (i)}$, $i = 1, ..., N$ denotes the realisations of $u - u_h^\ell$. We shall first give the error introduced by the approximation of $\mathbb{E}$ by $\hat{\mathbb{E}}_N$. $\hat{\mathbb{E}}_N$ is an unbiased estimator of $\mathbb{E}$, meaning that

$$\mathbb{E}\left[\hat{\mathbb{E}}_N[\|u - u_h^l\|^p_{L^2(\mathbb{S}^2)}]\right] = \mathbb{E}[\|u - u_h^l\|^p_{L^2(\mathbb{S}^2)}]$$

Following Lemma 2.8.1 in [7], for any $N \in \mathbb{N}$ and $Y \in L^2(\Omega; L^2(\mathbb{S}^2))$, it holds that

$$\mathrm{Var}(\hat{\mathbb{E}}_N(Y))^{1/2} \leq \frac{1}{\sqrt{N}}\|Y\|_{L^2(\Omega;L^2(\mathbb{S}^2))},$$

and hence we see that the Monte-Carlo error is additive and of the form $\frac{1}{\sqrt{N}}$.

6

Numerical experiments

6.1 Verification of SFEM algorithm in FEniCS

A challenge with FEniCS is that it is a relatively high-level tool, which means that algorithms and implementations must be numerically verified to work before it is possible to move on to more complicated computational tasks. For that matter, we intend to verify the correctness of the implementation of SFEM by solving the problem

$$\Delta_{\mathbb{S}^2} u = 6Y_{2,2},$$

where $Y_{2,2}$ is the spherical harmonic function given by

$$Y_{2,2}(x, y, z) = \frac{\sqrt{15}}{4\sqrt{\pi}}(x^2 - y^2),$$

for $(x, y, z) \in \mathbb{S}^2$. According to the results in section 2.5, the solution to this equation is given by $Y_{2,2}$, since it is an eigenfunction of the spherical Laplacian. Furthermore, according to Theorem 4.9 in [6], the error will be given by

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq c_{Y_{2,2}}(h^2 + \|Y_{2,2} - I_h(Y_{2,2})\|_{L^2(\mathbb{S}^2)}),$$

where h is the maximum size of a triangle element in the triangulation of the sphere. If we apply the interpolation error estimate of the right-hand side approximation described in Section 4.1.1.1, the error should be given by

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq c_{Y_{2,2}} h^2.$$

The mesh used in this experiment is the icosphere mesh, moving from the most basic mesh to nine 2-frequency refinements. As we see in Figure 6.1, the algorithm indeed behaves as intended.

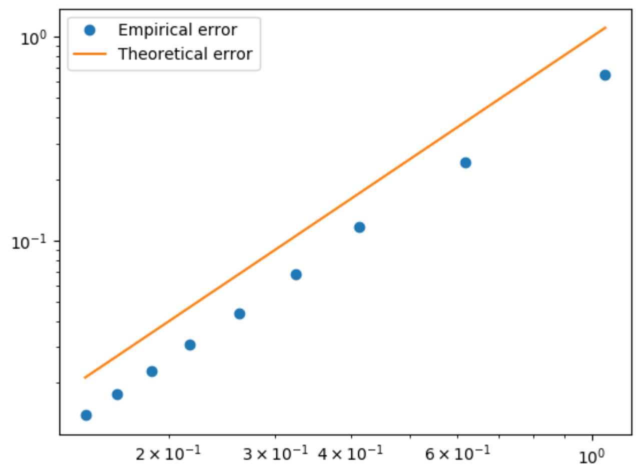


Figure 6.1: Error of solving Poisson's equation on $\mathbb{S}^2$.

6.2 Verification of the SFEM algorithm for fractional elliptic problems

In order to ascertain that the algorithm implemented in FEniCS behaves as intended, we try it out with a deterministic right-hand side before attempting to run the algorithm with the noise right-hand side. We select $\beta = 0.8$, $\kappa = 1$ and set the right-hand side to $Y_{2,2}$ as in the verification of the SFEM algorithm. We are hence considering the problem of

$$(1 - \Delta_{\mathbb{S}^2})^{0.8}\, u = Y_{2,2}.$$

Denote by u the exact solution the the above equation, let $u_{\mathfrak{Q},k}$ denote the result of applying the sinc quadrature to $Y_{2,2}$, and let u_h^ℓ denote the approximate surface finite element solution. We wish to determine the error. Note first that

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq \|u - u_{\mathfrak{Q},k}\|_{L^2(\mathbb{S}^2)} + \|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\mathbb{S}^2)}.$$

According to the results of Section 4.4, we see that

$$\|u - u_{\mathfrak{Q},k}\|_{L^2(\mathbb{S}^2)} \leq c_{Y(2,2)} c(k),$$

where $c(k) = C\left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)}\right)$ and C is a constant depending on β and κ. In order to estimate $\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\mathbb{S}^2)}$, we first note that according to Equation (4.13), it holds that

$$\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq \frac{2k\sin(\pi\beta)}{\pi} \sum_{l=-K^-}^{K^+} e^{2\beta y_l} \|u_l - u_{l,h}^\ell\|_{L^2(\mathbb{S}^2)},$$

where K^+ and K^- are as in Equation (4.9) and $y_j = jk$. According to Lemma 4.4.6, we have that

$$\|u_l - u_{l,h}^\ell\|_{L^2(\mathbb{S}^2)} \leq c_{Y_{2,2}} \gamma_l^2 h^2.$$

Hence, it holds that

$$\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq c_{Y_{2,2}} \frac{2k \sin(\pi\beta)}{\pi} (K^+ + K^- + 1) \max_{l=-K^- \dots K^+} (\gamma_l^2) h^2.$$

In our case, $\max_{l=-K^- \dots K^+}(\gamma_l^2) = (1 + \kappa^2 e^{2kK^+})^2$, so we obtain that

$$\|u_{\mathfrak{Q},k} - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq c_{Y_{2,2}} \frac{2k \sin(\pi\beta)}{\pi} (K^+ + K^- + 1)(1 + \kappa^2 e^{2kK^+})^2 h^2,$$

and hence, the total error will be given by

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)}$$
$$\leq c_{Y_{2,2}} \frac{2k \sin(\pi\beta)}{\pi} (K^+ + K^- + 1)(1 + \kappa^2 e^{2kK^+})^2 h^2 + \left(\frac{e^{-\pi^2/(4k)}}{\sinh(\pi^2/(4k))} + e^{-\pi^2/(2k)} \right).$$

Note that the error from k behaves exponentially, so we set $k = 1/10$. With our choice of parameters, the error should hence behave as

$$\|u - u_h^\ell\|_{L^2(\mathbb{S}^2)} \leq ch^2.$$

In practice, since we made a clever choice of right-hand side in the equation, we can find an explicit solution. Since $Y_{2,2}$ is an eigenfunction to $-\Delta_{\mathbb{S}^2}$ with eigenvalue 6, it is also an eigenfunction of $\kappa^2 - \Delta_{\mathbb{S}^2}$ with eigenvalue $\kappa^2 + 6$. Therefore, $(\kappa^2 - \Delta_{\mathbb{S}^2})^{-\beta} Y_{2,2} = \frac{1}{(\kappa^2+6)^\beta} Y_{2,2}$, which with our parameter choices will be equal to $\frac{1}{7^{0.8}} Y_{2,2}$, and hence the error can be calculated in the same manner as in the numerical verification of the SFEM algorithm, by using the built-in integration method of FEniCS. As we see in Figure 6.2, the numerical results agree with the theoretical behaviour.

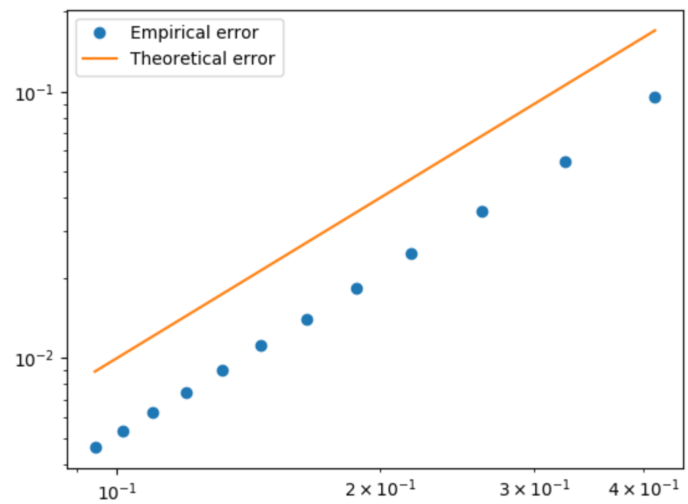


Figure 6.2: Error of fractional elliptic PDE.

6.3 Verification of noise generation algorithm

Let $\overline{\mathcal{W}}$ and $\widetilde{\mathcal{W}}_{\mathfrak{K}}$ be as in Section 3.1 and Section 4.2. We have shown in Section 4.2 that

$$\left\|\overline{\mathcal{W}} - \tilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq c\mathfrak{K}^{\frac{-(\alpha-2)}{2}} + D(\mathfrak{K}, A_l)h^2,$$

where the different values of $D(\mathfrak{K}, A_l)$ for different power spectra are given in Table 4.1. In order to numerically verify the error, we select an angular power spectrum of $A_l = (l+1)^{-3}$. This angular power spectrum satisfies an algebraic decay condition with $l_0 = 0$ and $\alpha = 3$, so we obtain that

$$D(\mathfrak{K}, A_l) \leq D_{\overline{\mathcal{W}}}h^2 + C\mathfrak{K}^3h^2.$$

Here $D_{\overline{\mathcal{W}}}h^2 = \frac{3h^2}{2}\sum_{l=0}^{l_0} A_l\left(1 + 2l + l^2(l+1)^2 + 2l^3(l+1)^2\right) = Ch^2$. Hence, the error behaves as

$$\left\|\overline{\mathcal{W}} - \tilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq c\left(\mathfrak{K}^{-1/2} + \mathfrak{K}^3h^2\right).$$

We now balance

$$\mathfrak{K}^3h^2,$$

with $\mathfrak{K}^{-\frac{1}{2}}$. In order to do this, we solve the equation

$$\mathfrak{K}^3h^2 = \mathfrak{K}^{-\frac{1}{2}},$$

to see that the selection $h = \mathfrak{K}^{-7/4}$ yields the desired behaviour. Hence, the error should behave as

$$\left\|\overline{\mathcal{W}} - \tilde{\mathcal{W}}^{\ell}_{\mathfrak{K},h}\right\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq c(\mathfrak{K}^{-1/2}).$$

Due to numerical considerations stemming from the availability of computational resources, we were forced to do this the other way around, by selecting $\mathfrak{K}$ given an h. As a result, we were forced to select a smaller $\mathfrak{K}$ than what would be ideal. We perform a Monte–Carlo estimate with $M = 500$ iterations the results of which can be seen in Figure 6.3. We see that the Monte–Carlo estimate of the error agrees with the theoretical results. The computations were fairly slow, and it should for completeness be noted that this highlights the need for a more efficient meshing of the sphere than the icosphere approach, since it would be more ideal to be able to select an h and then generate a meshing.

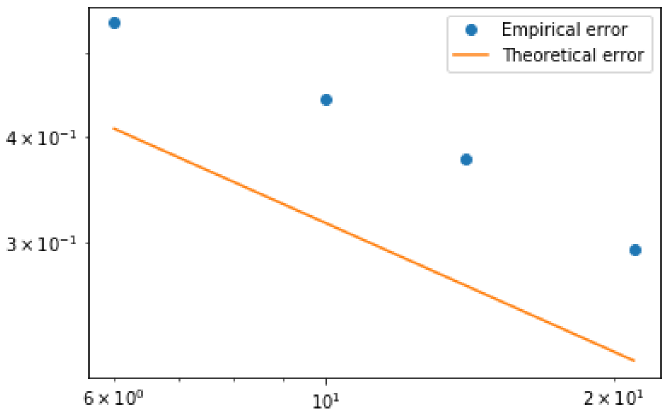


Figure 6.3: Monte-Carlo verification of the noise approximation algorithm.

All in all, since all parts of the algorithm seem to work as expected, we are confident that the entire algorithm should behave as expected having the necessary computational facilities available.

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A

Appendices

A.1 Source code

In this appendix we include the source code used to generate the images and verify parts of the algorithm. We first include functions that were used many times in different programs and therefore were collected in one file. We then include the verification programs written to test the algorithms.

A Tools and utilities

```
'''
In this section I have collected a lot of the little functions always used, so that they may be directly imported and does not have to be rewritten each time.
'''

from dolfin import *
import numpy as np
from scipy.special import gamma
set_log_active(False)
import meshio
import meshzoo

def meshgetter(n): #generates icosaspheres
    points, cells = meshzoo.icosa_sphere(n)
    name = f"icosa-sphere{n}.xdmf"
    meshio.write_points_cells(name, points, {"triangle": cells})


#this class is the normal of the sphere
class normal(UserExpression):
    def eval(self,value,x):
        value[0]=x[0]
        value[1]=x[1]
        value[2]=x[2]
    def value_shape(self):
        return(3,)

#this reads the meshes created by the mesh generator
def mesh_reader(low,high):
    surfaces=[] #placeholder
```

```
    global_normal=normal(degree=2) #create normal
    for i in range(low,high):
        temp=Mesh() #create temp mesh
        filename="./meshes/icosa-sphere"+str(i)+".xdmf" #build file name
        with XDMFFile(filename) as infile: #read mesh from file
            infile.read(temp)
        temp.init_cell_orientations(global_normal) #set orientation of mesh so that FEniCS can tell up from down
        surfaces.append(temp) #append
    return(surfaces) #return vector




#tests meshes to illustrate the need of including the Jacobian
def mesh_tester(surface):
    x_h = SpatialCoordinate(surface) #coordinates of s_h^2
    r_h = sqrt(inner(x_h,x_h))   #radius of s_h^2
    n = x_h/r_h # Deformed normal
    N = CellNormal(surface) # Un-deformed normal
    J =abs(inner(N,n))* ((1.0/r_h)**2) #approximate Jacobian
    print("Surface area of mesh = "+str(assemble(dot(N,N)*dx, form_compiler_parameters={"quadrature_degree": 5})))
    print("Exact area of sphere = "+str(4.0*np.pi)) #the exact area of the unit sphere
    print("Integrated area of sphere = "+str(assemble(J*dx, form_compiler_parameters={"quadrature_degree":5})))
    print("======")




#following are error calculation methods
def error_calcer_Y22(approx,surface):
    x_h=SpatialCoordinate(surface)
    r_h=sqrt(inner(x_h,x_h))
    n=x_h/r_h
    N=CellNormal(surface)
    J=(1.0/r_h)**2*abs(inner(N,n))
    x=x_h/sqrt(dot(x_h,x_h))
    u_exact=sqrt(15)*(x[0]**2-x[1]**2)/sqrt(16*pi)
    e_h=u_exact-approx
    return(sqrt(assemble(e_h**2*J*dx,form_compiler_parameters = {"quadrature_degree": 5})))


def error_calcer(a,b,surface):
    #when on same surface
    x_h=SpatialCoordinate(surface)
    r_h=sqrt(inner(x_h,x_h))
    n=x_h/r_h
    N=CellNormal(surface)
    J=(1.0/r_h)**2*abs(inner(N,n))
    e_h=a-b
```

```
    return(sqrt(assemble(e_h**2*J*dx,form_compiler_parameters = {"quadrature_degree": 5})))
def error_calcer_2c(sFS_coarse, surf_coarse,sFS_fine, surf_fine):
    x_h=SpatialCoordinate(surf_coarse)
    r_h=sqrt(inner(x_h,x_h))
    n=x_h/r_h
    N=CellNormal(surf_coarse)
    J=(1.0/r_h)**2*abs(inner(N,n))
    x=x_h/sqrt(dot(x_h,x_h))
    sFS_fine[0].set_allow_extrapolation(True)
    f_fine=project(sFS_fine[0],sFS_coarse[1])
    f_cors=project(sFS_coarse[0],sFS_coarse[1])
    e_h=f_fine-f_cors
    return(sqrt(assemble(e_h**2*J*dx(domain=sFS_coarse[1]),form_compiler_parameters = {"quadrature_degree": 5})))
   # return(norm(e_h*J))


#right hand side for equations when testing with deterministic RHS
def y2_rhs():
    class right_hand_side(UserExpression):
        def eval(self,value,x):
            value[0]=6*sqrt(15)*(x[0]**2-x[1]**2)/sqrt(16*pi)
        def value_shape(self):
            return()
    return(right_hand_side(degree=2))



def frac_solver(surface,f,beta,kappa,k,V): #solves fractional equation by sinc quadrature
    Kmin=int(np.ceil(pi**2/(4*beta*k**2))) #lower limit for j
    Kplus=int(np.ceil(pi**2/(4*(1-beta)*k**2))) #upper limit for j
    const=2*k*sin(pi*beta)/pi #constant to multiply it all by
    approx=Function(V) #preallocate output function
    u=TrialFunction(V) #set test and trial function
    v=TestFunction(V)
    w=Function(V) #function to hold solution of subproblems
    L=f*v*dx #linear form
    for j in range(-Kmin,Kplus+1):
        y=j*k
        a=u*v*dx+exp(2*y)*kappa**2*u*v*dx+exp(2*y)*dot(grad(u),grad(v))*dx #set bilinear form
        solve(a==L,w) #solve
        approx.vector()[:]=w.vector()[:]*exp(2*beta*y)+approx.vector()[:]
    approx.vector()[:]=approx.vector()[:]*const
    return(approx)

def rand_sequence(noterms): #generates a random sequence
   return(np.random.normal(0,1,[noterms,noterms,2]))

def naive_lpmv(m, v, x): #spec legendre polynomials
         poly = np.polynomial.legendre.Legendre([0]*v + [1])
         return poly.deriv(m)(x) * (1-x*x)**(m/2) * (-1)**m

```

```
def angular_spec(l,alpha): #test angular power spectrum.
    return((l+1)**(-alpha))


def plm(l,m,theta): #returns  functions L of expansion
    return(np.sqrt((2*l+1)*gamma(1+l-m)/4/np.pi/gamma(1+l+m))*naive_lpmv(m,l,np.cos(theta)))

def GRF(noterms,alpha,rands,x,y,z): #function to evaluate GRF in a point
    phi=np.arctan2(y,x) #convert to spherical coordinates
    theta=np.arccos(z)
    out=0.0
    for l in range(noterms):
        a=np.sqrt(angular_spec(l,alpha))*rands[l,0,0]*plm(l,0,theta) #first term
        b=np.sqrt(2*angular_spec(l,alpha))*sum([plm(l,m,theta)*(rands[l,m,0]*np.cos(m*phi)+rands[l,m,1]*np.sin(m*phi)) for m in range(1,l+1)]) #sum in m
        out=np.add(out,np.add(a,b)) #add together
    return(out) #return out

def angular_spec_solution(l,alpha,kappa,beta): #angular spec for solution
    return(angular_spec(l,alpha)/((kappa**2+l*(l+1))**(2*beta)))

def GRF_solution(noterms,alpha,rands,x,y,z):
    phi=np.arctan2(y,x)
    theta=np.arccos(z)
    out=0.0
    beta=0.6
    kappa=1
    for l in range(noterms):
        a=np.sqrt(angular_spec_solution(l,alpha,kappa,beta))*rands[l,0,0]*plm(l,0,theta)
        b=np.sqrt(2*angular_spec_solution(l,alpha,kappa,beta))*sum([plm(l,m,theta)*(rands[l,m,0]*np.cos(m*phi)+rands[l,m,1]*np.sin(m*phi)) for m in range(1,l+1)])
        out=np.add(out,np.add(a,b))
    return(out)



def grf_func(funcspace,mesh,noterms,alpha,rands,ret_sol): #returns GRF as fenics function
    dof_coordinates=funcspace.tabulate_dof_coordinates() #gets coordinates of mesh
    dof_coordinates=dof_coordinates.reshape((funcspace.dim(),mesh.geometry().dim()))
    dof_x=dof_coordinates[:,0]
    dof_y=dof_coordinates[:,1]
    dof_z=dof_coordinates[:,2]
    if(ret_sol==False): #returns RHS
        field_vec=GRF(noterms,alpha,rands,dof_x,dof_y,dof_z) #evaluate in points
        out=Function(funcspace) #preallocate function
```

```
        out.vector()[:]=field_vec #set solution nodal points to value of GRF in nodal points.
        return(out)
    elif(ret_sol==True): #retursn solution
        field_vec_sol=GRF_solution(noterms,alpha,rands,dof_x,dof_y,dof_z)
        sol=Function(funcspace)
        sol.vector()[:]=field_vec_sol
        return(sol)
```

B Poisson's equation on the sphere

```
from dolfin import * #import appropriate packages and functions
import numpy as np
import matplotlib
matplotlib.use("agg")
import matplotlib.pyplot as plt

from tools import mesh_reader, error_calcer_Y22



surfaces=mesh_reader(1,10) #import surfaces
n=len(surfaces)
errors=np.zeros([n,1]) #preallocate error and surface fectors
sizes=np.zeros([n,1])

class right_hand_side(UserExpression): #FEniCS class for RHS of equation
    def eval(self,value,x):
        value[0]=6*sqrt(15)*(x[0]**2-x[1]**2)/sqrt(16*pi) #value
    def value_shape(self):
        return() #shape (since return scalar value)
f=right_hand_side(degree=2) #preallocate

#file=File("./testim/poisson_ver.pvd") #uncomment if you want to save image in paraview

def verificator(surface):
    P1 = FiniteElement("P", surface.ufl_cell(), 1) #FEM
    C=FiniteElement("R",surface.ufl_cell(),0)  #additional FEM to deal with constant term
    V = FunctionSpace(surface,P1*C)
    # Define variational problem
    (u,c) = TrialFunction(V) #preallocate test and trial function
    (v,d) = TestFunction(V)
    a = inner(grad(u), grad(v))*dx+(c*v+u*d)*dx #preallocate forms
    L = f*v*dx
    # Compute solution
    w = Function(V)
    solve(a == L, w)
    (x,y)=w.split()
    #file<<x #uncomment if want to save file.
    size=surface.hmax()
    return([error_calcer_Y22(x,surface),size]) #return error and size
```

```
for i in range(n):  #run method for all meshes
  v=verificator(surfaces[i])
  errors[i]=v[0]
  sizes[i]=v[1]

#create nice figure

np.savetxt('errors_pover1.txt',errors)
np.savetxt('sizes_pover2.txt',sizes)
plt.loglog(sizes,errors,'o',sizes,sizes**2)
plt.legend(['Empirical error','Theoretical error'])
plt.savefig("yes.png")
```

C Verification of the sinc quadrature combined with SFEM

```
#import necessary libraries and functions
from dolfin import *
from numpy import pi,ceil, sin,zeros,savetxt
from tools import mesh_reader,y2_rhs
import matplotlib.pyplot as plt


#disable FEniCS output
set_log_active(False)
set_log_level(0)

#set parameters to obtain right equation
beta=0.8
kappa=1
k=0.1 #step size quadrature

Kmin=int(ceil(pi**2/(4*beta*k**2))) #lower sum limit
Kplus=int(ceil(pi**2/(4*(1-beta)*k**2))) #upper sum limit
const=2*k*sin(pi*beta)/pi #constant to scale the entire thing by

surfaces=mesh_reader(3,15) #import surfaces
n=len(surfaces)
errors=zeros([n,1]) #preallocate error and size vector
sizes=zeros([n,1])

def y2_rhs_fracver(): #FEniCS class for right-hand side
    class right_hand_side(UserExpression):
        def eval(self,value,x):
            value[0]=(kappa**2+6)**(beta)*sqrt(15)*(x[0]**2-x
    [1]**2)/sqrt(16*pi)
        def value_shape(self):
            return()
    return(right_hand_side(degree=2))

f=y2_rhs_fracver() #initalize RHS

def fracver(surface):
    P1=FiniteElement("P",surface.ufl_cell(),1)
    V=FunctionSpace(surface,P1) #initalize FEM spaces
    approx=Function(V) #output
    u=TrialFunction(V) #set test and trial functions
```

```
    v=TestFunction(V)
    w=Function(V) #holder for solution for each of the subproblems
    L=f*v*dx #set linear form
    for j in range(-Kmin,Kplus+1): #iterate..
        y=j*k
        a=exp(2*y)*kappa**2*u*v*dx+u*v*dx+exp(2*y)*dot(grad(u),grad
    (v))*dx #bilinear form
        solve(a==L,w) #solve
        approx.vector()[:]=w.vector()[:]*exp(2*beta*y)+approx.
    vector()[:] #add up
    approx.vector()[:]=approx.vector()[:]*const #scale entire sum
    return(approx)

approx_vector=[fracver(surface) for surface in surfaces] #apply to
    each surface

def e_c(approx,surface): #error calculator function
    x_h=SpatialCoordinate(surface) #coordinates of s_h2
    r_h=sqrt(inner(x_h,x_h)) #approximate radius
    n=x_h/r_h #normal of s^2
    N=CellNormal(surface) #normal of s_h^2
    J=(1.0/r_h)**2*abs(inner(N,n)) #approximate Jacobean
    x=x_h/sqrt(dot(x_h,x_h)) #lift
    u_exact=sqrt(15)*(x[0]**2-x[1]**2)/sqrt(16*pi) #exact solution
    e_h=u_exact-approx #integrate over difference to obtain error.
    return(sqrt(assemble(e_h**2*J*dx,form_compiler_parameters = {"
    quadrature_degree": 5})))


#calculate error and make nice plots
for i in range(n):
    errors[i]=e_c(approx_vector[i],surfaces[i])
    sizes[i]=surfaces[i].hmax()

ax=plt.loglog(sizes,errors,'o',sizes,sizes**2)
plt.legend(['Empirical error','Theoretical error'])
filename='conv'+'.png'
plt.savefig(filename)
#file=File("./testim/frac_test.pvd")
#file<<approx_vector[1]
```

D Monte Carlo method for checking convergence of noise approximation

```
#import necessary packages
import numpy as np
from scipy.special import lpmv,gamma
from dolfin import *
import numpy as np
from tools import mesh_reader, error_calcer, angular_spec, grf_func



#read surfaces
sf=[mesh_reader(10,11)[0],mesh_reader(24,25)[0],mesh_reader(50,51)
```

```
[0],mesh_reader(100,101)[0],mesh_reader(200,201)[0]]

#calculate numbers of terms
nt=[int(np.ceil(s.hmin()**(-4/7))) for s in sf]

#preallocate errors
m=len(sf)
errors=np.zeros(m)

#number of MC iterations
M=500
seed=1 #first seed for rep.
alpha=3 #alpha smoothies parameter
for j in range(M):
    print(j)
    rands=rand_sequence(128,seed+j)  #calculate new randoms
    for i in range(m):
        V=FunctionSpace(sf[i],FiniteElement("P",sf[i].ufl_cell(),1)) #set functionspace
        field=grf_func(V,sf[i],nt[i],alpha,rands) #calculate approx field
        rf_field=grf_func(V,sf[i],64,alpha,rands) #reference field
        errors[i]+=(np.max(np.abs(field.vector()[:]-rf_field.vector()[:]))) #calc error
errors/=M

#save txt and nice plots
np.savetxt('mc_error_ver.txt',errors)
plt.loglog(nt,errors,'o',nt,[n**(-1/2) for n in nt])
plt.legend(['Empirical error','Theoretical error'])
plt.savefig('MC_error.png')
```