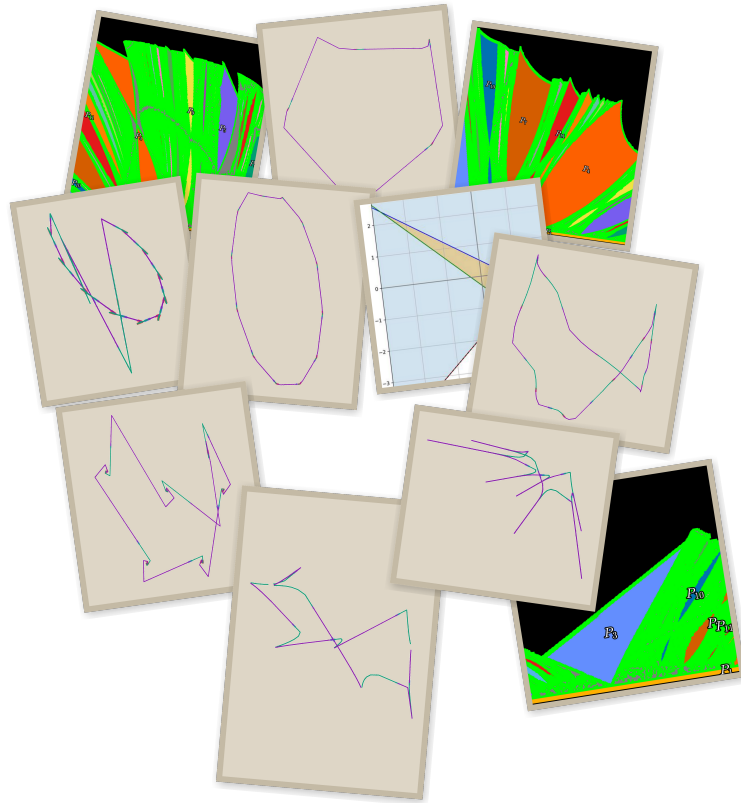




CHALMERS
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Bifurcations of Invariant Closed Curves in the Piecewise-Linear 3D Border Collision Normal Form

Master's thesis in Systems, Control and Mechatronics

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CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2026
www.chalmers.se

MASTER'S THESIS 2026

**An Overview over
Stability Related Bifurcations of ICCs in a
Piecewise-Linear Setting**

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AI Disclosure: This thesis uses AI tools for coding, language refinement, translation, reference validation, and grammar and spelling correction. The intellectual responsibility for the content of this thesis remains entirely with the author.

Cover: Cover: A collage of discoveries made throughout this thesis — invariant closed curves, bifurcation diagrams, and stability regions gathered like specimens on an expedition, in the spirit of a 19th-century naturalist returning from an African safari with a scattered stack of polaroids: lions, elephants, giraffes, each snapshot a different creature encountered along the way.

Typeset in L^AT_EX

Printed by Chalmers Reproservice

Gothenburg, Sweden 2026

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Abstract (english)

Invariant closed curves (ICCs) have received increasing attention in the theory of smooth dynamical systems in recent years, with Neimark–Sacker bifurcations being one of the most prominent mechanisms through which they are created. This thesis investigates whether the theory developed for such invariant curves in the smooth setting can be transferred to piecewise-linear systems, and, where it cannot, identifies the mechanisms responsible for the resulting differences.

The investigation is carried out for the piecewise-linear three-dimensional Border Collision Normal Form (BCNF), a piecewise-affine map that serves as a canonical local model for systems exhibiting border collision bifurcations. Within this framework, an attracting resonant ICC is understood as being composed of a saddle cycle, a stable node or focus cycle, and the connecting unstable one-dimensional set of the saddle. Building on this notion, the thesis focuses on the three elementary bifurcations associated with eigenvalues leaving the unit circle under parameter perturbation. The thesis examines for each possibility – leaving the unit circle via "+1", "-1" or as a complex conjugated pair anywhere through the unit circle – how an ICC responds when its constituent cycles cross such boundaries in close succession, comparing the resulting behavior to the corresponding smooth bifurcation scenario in order to identify both parallels and fundamental differences.

A key difference is that all (non-border-collision) stability-related bifurcations in the BCNF are inherently degenerate; consequently, the stability of newly created invariant sets is governed by a global, border-collision-induced mechanism rather than by a local, smooth-theory-style criterion.

As a byproduct of this investigation, a broad numerical exploration of a slice of the BCNF's parameter space is conducted to catalogue the wide variety of geometric shapes that ICCs can take on in the nonsmooth setting – ranging from needle- and star-shaped curves to tightly wound spirals – and to trace these distinctive shapes back to the underlying eigenvalue structure of the constituent cycles and their proximity to the switching manifold.

The results are obtained using the numerical tools PeriodScanner, for the detection and stability classification of periodic orbits, and Manifis, for the high-precision computation of their one-dimensional invariant sets of their saddle cycles.

Keywords: piecewise-linear systems, nonsmooth dynamical systems, invariant closed curves, Neimark–Sacker bifurcation, flip bifurcation, fold bifurcation, bifurcation theory, stability analysis, discrete-time dynamical systems

Abstract (deutsch)

Invariante geschlossene Kurven (ICCs) haben in den letzten Jahren in der Theorie glatter dynamischer Systeme zunehmend Aufmerksamkeit erhalten, wobei Neimark–Sacker-Bifurkationen einer der prominentesten Mechanismen sind, durch die sie entstehen. Diese Arbeit untersucht, ob die für solche invarianten Kurven im glatten Fall entwickelte Theorie auf stückweise-lineare Systeme übertragen werden kann, und identifiziert dort, wo dies nicht möglich ist, die Mechanismen, die für die entstehenden Unterschiede verantwortlich sind.

Die Untersuchung wird für die stückweise-lineare dreidimensionale Border Collision Normal Form (BCNF) durchgeführt, eine stückweise-affine Abbildung, die als kanonisches lokales Modell für Systeme mit Border-Collision-Bifurkationen dient. In diesem Rahmen wird eine attraktierende resonante ICC als aus einem Sattelzyklus, einem stabilen Knoten- oder Fokuszyklus sowie der verbindenden instabilen eindimensionalen Menge des Sattels bestehend verstanden. Aufbauend auf diesem Verständnis konzentriert sich die Arbeit auf die drei elementaren Bifurkationen, die mit Eigenwerten verbunden sind, die den Einheitskreis unter Parameterstörung verlassen. Für jede Möglichkeit – das Verlassen des Einheitskreises über "+1", "-1" oder als komplex konjugiertes Paar an einer beliebigen Stelle des Einheitskreises – untersucht die Arbeit, wie eine ICC reagiert, wenn ihre konstituierenden Zyklen solche Grenzen in rascher Folge überschreiten, und vergleicht das resultierende Verhalten mit dem entsprechenden glatten Bifurkationsszenario, um sowohl Parallelen als auch grundlegende Unterschiede zu identifizieren.

Ein zentraler Unterschied besteht darin, dass alle (nicht-Border-Collision-bezogenen) stabilitätsrelevanten Bifurkationen in der BCNF inhärent entartet sind; folglich wird die Stabilität neu entstehender invarianter Mengen durch einen globalen, durch Border-Collisions induzierten Mechanismus bestimmt und nicht durch ein lokales, an der glatten Theorie orientiertes Kriterium.

Als Nebenprodukt dieser Untersuchung wird eine umfangreiche numerische Exploration eines Ausschnitts des Parameterraums der BCNF durchgeführt, um die große Vielfalt geometrischer Formen zu katalogisieren, die ICCs im nichtglatten Fall annehmen können – von nadel- und sternförmigen Kurven bis hin zu eng gewundenen Spiralen – und um diese charakteristischen Formen auf die zugrunde liegende Eigenwertstruktur der konstituierenden Zyklen sowie deren Nähe zur Schaltmannigfaltigkeit zurückzuführen.

Die Ergebnisse wurden mit den numerischen Werkzeugen PeriodScanner, zur Detektion und Stabilitätsklassifikation periodischer Orbits, und Manifis, zur hochpräzisen Berechnung der eindimensionalen invarianten Mengen ihrer Sattelzyklen, erzielt.

Keywords: piecewise-linear systems, nonsmooth dynamical systems, invariant closed curves, Neimark–Sacker bifurcation, flip bifurcation, fold bifurcation, bifurcation theory, stability analysis, discrete-time dynamical systems

Abstract (svensk)

Invarianta slutna kurvor (ICCs) har på senare år fått ökad uppmärksamhet inom teorin för glatta dynamiska system, där Neimark–Sacker-bifurkationer utgör en av de mest framträdande mekanismerna genom vilka de uppstår. Denna avhandling undersöker huruvida den teori som utvecklats för sådana invarianta kurvor i det glatta fallet kan överföras till styckvis-linjära system, och identifierar, i de fall detta inte är möjligt, de mekanismer som ligger bakom de resulterande skillnaderna.

Undersökningen genomförs för den styckvis-linjära tredimensionella Border Collision Normal Form (BCNF), en styckvis-affin avbildning som fungerar som en kanonisk lokal modell för system som uppvisar border collision-bifurkationer. Inom detta ramverk förstås en attraherande resonant ICC som sammansatt av en sadelcykel, en stabil nod- eller fokuscykel, samt sadelns förbindande instabila endimensionella mängd. Med denna förståelse som utgångspunkt fokuserar avhandlingen på de tre elementära bifurkationer som är förknippade med egenvärden som lämnar enhetscirkeln vid parameterstörning. För varje möjlighet – att lämna enhetscirkeln via "+1", "-1" eller som ett komplext konjugerat par var som helst på enhetscirkeln – undersöker avhandlingen hur en ICC reagerar när dess ingående cykler korsar sådana gränser i snabb följd, och jämför det resulterande beteendet med motsvarande glatta bifurkationsscenario för att identifiera såväl paralleller som fundamentala skillnader. En central skillnad är att alla (icke border collision-relaterade) stabilitetsrelaterade bifurkationer i BCNF är inneboende degenererade; följaktligen styrs stabiliteten hos nyskapade invarianta mängder av en global, border collision-inducerad mekanism snarare än av ett lokalt kriterium av den typ som är vanligt inom den glatta teorin. Som en biprodukt av denna undersökning genomförs en omfattande numerisk utforskning av ett snitt av BCNF:s parameterutrymme för att katalogisera den stora variationen av geometriska former som ICCs kan anta i det icke-glatta fallet – allt från nål- och stjärnformade kurvor till hårt hopvirade spiraler – och för att spåra dessa särpräglade former tillbaka till de ingående cyklernas underliggande egenvärdesstruktur samt deras närhet till switchningsmångfalden.

Resultaten har erhållits med hjälp av de numeriska verktygen PeriodScanner, för detektion och stabilitetsklassificering av periodiska banor, samt Manifis, för högprecisionsberäkning av sadelcyklernas endimensionella invarianta mängder.

Keywords: piecewise-linear systems, nonsmooth dynamical systems, invariant closed curves, Neimark–Sacker bifurcation, flip bifurcation, fold bifurcation, bifurcation theory, stability analysis, discrete-time dynamical systems

Acknowledgements

I would like to thank my supervisor, Viktor Avrutin, for the guidance, valuable discussions, and continuous support throughout the development of this thesis, and for introducing me to the study of bifurcations in the border collision normal form maps.

I would also like to thank my second supervisor, Emmanuel Dean, for his valuable feedback and support during this thesis.

I am also grateful to Dennis Güney for developing and providing Manifis. His tool was indispensable for the numerical investigation of the bifurcation scenarios presented in this work.

Finally, I would like to thank my father and friends for their patience and support throughout the writing of this thesis.

Sven Görner, Gothenburg, August 2026

List of Acronyms

Below is the list of acronyms that have been used throughout this thesis listed in alphabetical order:

BCNF	Border Collision Normal Form
ICC	Invariant Closed Curve

List of Indices and Parameters

Below is the nomenclature of indices and parameters that have been used throughout this thesis.

Indices

L, R	Indices for left or right hand side branch of investigated map
n or $n + 1$	Index for discrete time step
b	Index of switching manifold (border)

Parameters

τ	Parameter of 3D BCNF
σ	Parameter of 3D BCNF
δ	Parameter of 3D BCNF
μ	Parameter of 3D BCNF

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1

Introduction

This chapter introduces the research field in which this thesis is situated and motivates the study of the three-dimensional Border Collision Normal Form.

1.1 What is Nonlinear Dynamics?

Nonlinear dynamics is the study of the behavior of dynamical systems whose response is not proportional to their inputs. Although linear systems constitute a special subclass of nonlinear systems, they exhibit only a limited range of dynamical behavior. In contrast, nonlinear systems in general display a much richer spectrum of phenomena, including stable equilibria, periodic oscillations, quasiperiodic motion, and chaos. A defining characteristic of many nonlinear systems is their sensitivity to initial conditions, where even arbitrarily small perturbations can lead to fundamentally different long-term behavior.

The origins of nonlinear dynamics can be traced back to the late nineteenth century. While investigating the three-body problem, Henri Poincaré discovered that even deterministic systems can exhibit highly complex and seemingly unpredictable behavior, laying the foundation for what would later become chaos theory. In the following decades, researchers such as Aleksandr Lyapunov developed fundamental mathematical tools for analyzing the stability of nonlinear dynamical systems, establishing many of the concepts that continue to shape the field today.

1.1.1 What is a Bifurcation?

A bifurcation is a qualitative change in the behavior of a parameterized dynamical system caused by a continuous infinitesimal variation of one or more system parameters. While the system evolves smoothly as the parameter changes, its long-term dynamics may suddenly change at certain critical parameter values, called *bifurcation points*. Such changes may include the creation or destruction of equilibria, the onset of periodic motion, or the transition to chaotic dynamics. Consequently, bifurcation theory provides a systematic framework for understanding how the qualitative behavior of dynamical systems depends on their parameters.

1.1.2 Which Objects Can Experience a Bifurcation?

Bifurcations can affect all robust invariant objects that determine the long-term behavior of a dynamical system. The most fundamental objects are *fixed points* (or equilibria), whose stability or existence may change as parameters vary. Likewise, *periodic orbits* can be created, destroyed, or undergo changes in stability through bifurcations. More complex invariant sets, such as *invariant closed curves (ICCs)*, and *chaotic attractors*, may also experience bifurcations that alter their geometry or stability. Understanding how these invariant objects change under parameter variation is one of the central objectives of bifurcation theory.

1.2 Why Investigate the 3D Border Collision Normal Form?

Since its beginnings, nonlinear dynamics has evolved into a highly interdisciplinary research area, with applications in meteorology, physics, ecology, medicine, economics, engineering, and many other disciplines. A common feature of many real-world systems is the presence of switching mechanisms or decision processes, where the governing dynamics change once a certain threshold is crossed. Such behavior is naturally modeled by piecewise-smooth or piecewise-linear systems possessing a border (also known as a switching manifold), across which the dynamics change depending on which side of the border a system state lies. This mechanism makes such models significantly more representative of practical applications than globally smooth systems. The introduction of such a border gives rise to an entirely new class of phenomena, triggered when fixed points collide with the border; these phenomena are known as border-collision bifurcations. To investigate border collisions systematically, normal forms were introduced via appropriate linearization methods. The linear three-dimensional Border Collision Normal Form (BCNF) accordingly provides an ideal framework for studying a wide range of interesting bifurcation behavior. As a piecewise-linear map, it captures the essential dynamics associated with threshold-induced switching while remaining sufficiently simple to permit a detailed mathematical analysis.

In recent years, considerable attention has been devoted to the study of invariant closed curve structures (ICCs) in smooth dynamical systems. A natural next step is to investigate whether the insights and techniques developed for smooth systems can be extended to the nonsmooth setting. This transition, however, is not straightforward: as mentioned above, piecewise-smooth systems possess mathematical features absent in smooth systems, such as discontinuities or a loss of differentiability. These characteristics give rise to new dynamical phenomena that require dedicated analytical and numerical methods.

Consequently, the BCNF serves as a fundamental model for investigating the existence and properties of invariant closed curve structures in the nonsmooth setting.

1.3 Research question

This thesis aims to provide answers to the following questions:

1. To what extent can the established theory of stability related bifurcations of ICCs in smooth 3D maps be extended to piecewise-smooth 3D maps?
2. What mathematical obstacles arise when applying smooth bifurcation theory to ICCs in nonsmooth systems, and how can these obstacles be characterized?
3. How can ICC structures in piecewise-linear 3D maps look like?
4. What can stability related bifurcation mechanisms of ICCs look like in piecewise-linear 3D maps?

A more detailed account of this goals, together with its mathematical foundations, is given in Chapter 3.

1.4 Thesis Scope

(1) Piecewise-Linearity: The investigation presented in this thesis is restricted to the piecewise-linear three-dimensional Border Collision Normal Form (BCNF). By their very nature, normal forms provide a local approximation of a broad class of dynamical systems. In particular, every piecewise-continuous map can be approximated by an appropriate normal form sufficiently close to a border-collision bifurcation. Consequently, the results obtained are not limited to the specific normal form considered here but are locally representative of a much wider class of systems. The piecewise-linear structure of the BCNF, together with its invertibility, significantly simplifies the analysis without sacrificing generality.

(2) Dimensionality: Furthermore, the three-dimensional BCNF employed in this thesis is a special case of the n -dimensional normal forms derived by Patra and Banerjee [6], which themselves extend the n -dimensional formulation introduced by di Bernardo et al. [2]. As a result, many of the insights and methods developed in this thesis are expected to generalize naturally to higher-dimensional systems.

(3) Stability: This thesis is further restricted to the investigation of stability-related bifurcations (also referred to as *basic* bifurcations throughout this thesis). This restriction is motivated by the objective of maintaining comparability with the existing theory of ICC bifurcations in smooth dynamical systems. One of the primary goals of this thesis is to transfer established knowledge of ICC bifurcations from the smooth to the piecewise-smooth setting. Since border-collision bifurcations have no counterpart in smooth dynamical systems, they cannot be included in such a comparison. Consequently, border-collision bifurcations are considered only insofar as they are directly related to the stability-related bifurcations under investigation.

(4) Descriptivity: A final restriction is, that this thesis is limited to the identification and description of bifurcation phenomena. While the occurrence and characteristics of the observed bifurcations are documented, explaining the underlying dynamical mechanisms lies beyond the scope of this work. In particular, no detailed

1. Introduction

investigation of the topology or geometry of the surrounding piecewise-smooth invariant surrounding 2D manifolds is undertaken to account for the observed bifurcation behavior.

2

State of the Art

The investigation of the linear 3D Border Collision Normal Form requires concepts from nonlinear dynamics, bifurcation theory, and the theory of piecewise-linear systems. This chapter reviews the state of the art and introduces the mathematical framework on which the subsequent analysis is based. The results and definitions presented here provide the necessary foundation for the theoretical and numerical investigations carried out in this thesis.

2.1 Theory

2.1.1 Border Collision Normal Form (BCNF)

This section describes the mathematical foundations needed to follow the investigations carried out in the remainder of this thesis. Starting from the general framework of piecewise-smooth dynamical systems and in particular the linear 3D border-collision normal form (BCNF) studied throughout this work. It introduces the fixed-point stability conditions and the resulting stability triangle, the classical bifurcation types (fold, flip, and Neimark–Sacker) together with some of their nonsmooth, border-collision analogues, and the theory of invariant closed curves (ICCs) – how they are composed, how they bifurcate in a smooth setting, and the vocabulary (shrinking points, resonance tongues, multiplier notation) used to describe them. Together, these concepts form the mathematical vocabulary and toolset on which the results presented in the following chapters are built.

2.1.1.1 Introducing the piecewise linear 3D BCNF

Let:

$$\vec{x}_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}, \vec{\mu} = \begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

and let the border (switching manifold) be given by:

$$x_b : x = 0.$$

The linear 3D Border Collision Normal Form (BCNF) is the continuous, piecewise affine map:

$$\vec{x}_{n+1} = F_\mu(\vec{x}_n) = \begin{cases} A_L \vec{x}_n + \vec{\mu}, & \text{if } x_n \leq 0, \\ A_R \vec{x}_n + \vec{\mu}, & \text{if } x_n \geq 0. \end{cases} \quad (2.1)$$

where the real-valued 3×3 matrices A_L and A_R are:

$$A_L = \begin{pmatrix} \tau_L & 1 & 0 \\ -\sigma_L & 0 & 1 \\ \delta_L & 0 & 0 \end{pmatrix}, \quad A_R = \begin{pmatrix} \tau_R & 1 & 0 \\ -\sigma_R & 0 & 1 \\ \delta_R & 0 & 0 \end{pmatrix}.$$

The 2D Border Collision Normal Form was originally introduced by Nusse and Yorke to study border-collision bifurcations in piecewise-smooth dynamical systems [1]. It provides a canonical representation of the dynamics near a switching boundary, where the system's behavior changes due to the collision of a robust invariant object with a discontinuity boundary. The formulation was later generalized to higher dimensions by di Bernardo [5], establishing the BCNF as a fundamental tool for the analysis of piecewise-smooth systems in n dimensions [2]. The specific three-dimensional case considered in this thesis was developed by Patra and Banerjee [6]. Throughout, this thesis relies heavily on the notation and conventions introduced by Simpson [4], who has extensively investigated bifurcations in piecewise-smooth systems.

The specific companion-matrix structure of the BCNF is not an arbitrary choice but results from transforming a general piecewise-linear system into a canonical form, i.e. the minimal representation of the essential local dynamics. This transformation reduces the number of independent parameters while preserving the qualitative behavior of the system, making the resulting representation particularly well suited to mathematical analysis. Continuity of the map across the switching manifold is ensured by applying the same translation vector $\vec{\mu}$ on both sides of the border, and by choosing the matrix structure such that the two linear regions differ only in their parameter sets $(\tau_L, \sigma_L, \delta_L)$ and $(\tau_R, \sigma_R, \delta_R)$.

Before turning to more exotic robust invariant objects, it is instructive to first consider the simplest case: the bifurcations of fixed points. The stability of a fixed point is determined by the location of the eigenvalues of the corresponding system matrix (A_L or A_R , depending on which side of the border the fixed point lies); for a discrete map, stability requires all eigenvalues to lie inside the unit circle of the complex plane (Schur–Cohn criterion). Further details are given in 2.1.2.

Since the stability of a fixed point is thus governed by its eigenvalues, it is natural to ask which parameters determine them. It has been shown by Nusse and Yorke [1] and later by di Bernardo [2] that the three parameters introduced above – the trace τ , the sum of the principal second-order minors σ , and the determinant δ – are precisely the minimal information required to determine these eigenvalues on either side of the border. Together they form the simple characteristic polynomial

$$p(\lambda) = \lambda^3 - \tau\lambda^2 + \sigma\lambda - \delta,$$

whose roots are exactly the eigenvalues of the considered fixed point. Different combinations of (τ, σ, δ) may therefore give rise to different configurations of stable and unstable fixed points, periodic solutions, or even more complex robust invariant structures.

Although the BCNF family are strongly simplified models, it has been proven, that they are capable of reproducing a wide range of complex phenomena observed in applications, including robust chaos [3] and intricate bifurcation structures. Building

on this, Avrutin, Görner, and Sushko [28] recently showed that chaotic attractors in discontinuous two-dimensional piecewise-smooth maps can undergo border collision bifurcations that significantly alter their shape and structure, in some cases producing additional bands that preserve the original attractor’s structure; they further demonstrate that such a bifurcation can act in combination with an expansion bifurcation, transforming the attractor’s topology – for instance, turning a one-dimensional attractor into a fractal or full-measure one.

A further important remark is that, since the considered 3D BCNF is piecewise linear, all observed bifurcations are necessarily either border collision or degenerate variants. *Degenerate* means that rather than occurring at a single bifurcation point, they occur throughout an entire *bifurcation interval*, filled with infinitely many periodic and aperiodic solutions existing exactly at the bifurcation parameter value. Consequently, all robust invariant objects created by such a bifurcation begin to exist only at the edges of this interval (at some small distance from it), rather than growing continuously from a single, exact bifurcation point. More information on that in 2.1.2, where all the stability related bifurcations are explained in more detail.

2.1.1.2 Prior investigations of the 3D BCNF

Building on the two-dimensional theory, several studies have extended the border-collision framework specifically to three dimensions. De, Dutta, Banerjee and Roy [7] were among the first to systematically study a three-dimensional, continuous, piecewise smooth map of this type, deriving conditions for nonsmooth period-doubling, saddle-node, and Neimark–Sacker bifurcations, and showing that so-called dangerous border-collision bifurcations can also occur in this setting. Subsequent work by Patra and Banerjee refined this picture: Patra and Banerjee [8] examined the bifurcation of quasiperiodic orbits in the 3D piecewise-linear map, Patra [9] identified parameter regions giving rise to multiple coexisting attractors following a single border collision, and Patra and Banerjee [6] extended the classical two-dimensional robust-chaos result of Banerjee, Yorke and Grebogi [3] to three dimensions, deriving explicit conditions for robust chaos in the 3D case. More recently, Muni and Banerjee [10] investigated bifurcations of mode-locked periodic orbits in three-dimensional maps, extending analogous results known from the two-dimensional BCNF. The concept of a dangerous border-collision bifurcation itself, originally restricted to the collapse of a fixed point’s basin of attraction, was generalized to periodic and chaotic orbits by Avrutin, Zhusubaliyev, Saha, Banerjee, Sushko and Gardini [15], a mechanism that De et al. [7] had already observed occurring within the 3D normal form itself. Despite these advances, the existing literature remains focused primarily on the description of individual bifurcation mechanisms. A unified framework for the basic bifurcations of ICCs in piecewise-smooth maps is still lacking. In particular, it remains unclear to what extent results from smooth bifurcation theory of ICCs extend to the nonsmooth setting, which genuinely nonsmooth obstacles prevent such extensions, and under what additional conditions those obstacles can be overcome. This thesis addresses this gap by developing a systematic overview of the principal bifurcation types of ICCs in piecewise-smooth maps.

2.1.2 The stability triangle of a fixed point

For a fixed point of the linear map $F_{\mu,R}(\vec{x}_n) = A_R \vec{x}_n + \vec{\mu}$ (or, analogously, $F_{\mu,L}(\vec{x}_n) = A_L \vec{x}_n + \vec{\mu}$) to be stable, all eigenvalues of the corresponding system matrix must lie within the unit circle. Since the eigenvalues are the roots of the characteristic polynomial $p(\lambda) = \lambda^3 - \tau\lambda^2 + \sigma\lambda - \delta$, this stability condition can be translated directly into inequalities on the parameters τ , σ , and δ via the Schur–Cohn (Jury) criterion. When one fixes $\delta_R = 0$, the characteristic polynomial of the right hand side factorizes as $p(\lambda) = \lambda(\lambda^2 - \tau_R\lambda + \sigma_R)$, so that one eigenvalue is trivially zero and the stability of the fixed point is governed entirely by the remaining quadratic factor. In the resulting (τ_R, σ_R) -plane, the region of stability takes the shape of a triangle – commonly referred to as the *stability triangle* – bounded by three straight lines, each of which corresponds to a different type of local bifurcation of the fixed point:

- $\sigma_R = 1$: a pair of complex-conjugate eigenvalues crosses the unit circle, giving rise to a (degenerate) Neimark–Sacker-type bifurcation;
- $\sigma_R = \tau_R - 1$: an eigenvalue crosses through $\lambda = +1$, corresponding to a (degenerate) transcritical/fold (saddle-node)/pitchfork-type (" +1 ") bifurcation;
- $\sigma_R = -\tau_R - 1$: an eigenvalue crosses through $\lambda = -1$, corresponding to a (degenerate) period-doubling (flip) bifurcation.

Note that due to $\delta_R = 0$ condition (1) is just $\sigma_R = 1$ which results in an exactly horizontal Neimark–Sacker line of the stability triangle in figure 2.1. These three eigenvalue-crossing scenarios – through +1, through –1, and through the unit circle as a complex-conjugate pair (the Neimark–Sacker boundary) – are what this thesis considers the *basic bifurcation types* of a robust invariant object, and the subsequent analysis will focus exclusively on bifurcations associated with one of these three crossings. These three stability related bifurcation conditions, and their role in classifying the local bifurcation structure of piecewise-linear normal forms, were established by Banerjee and Grebogi in their analysis of border-collision bifurcations in piecewise smooth maps [16], and were later carried over explicitly to the three-dimensional setting by De, Dutta, Banerjee and Roy [7], whose derivation of the nonsmooth period-doubling, saddle-node, and Neimark–Sacker bifurcation conditions for the 3D BCNF this section directly builds on.

Figure 2.1 illustrates the resulting stability triangle in the τ_R - σ_R Poincaré slice for $\mu = 1.0$ and $\delta_R = 0.00$. The light blue region indicates where the fixed point exists at all (i.e. lies on the correct side of the switching manifold for the given μ), while the shaded triangular region (orange) marks the parameter combinations for which the fixed point is also stable. The pink arrows additionally show representative parameter variation directions: displacing the parameters from inside the triangle across any one of its three edges forces the fixed point through the corresponding bifurcation – upward through the blue line induces a degenerate Neimark–Sacker bifurcation, while the two diagonal directions through the red and green lines induce a degenerate +1 bifurcation and a degenerate period-doubling (flip) bifurcation, respectively.

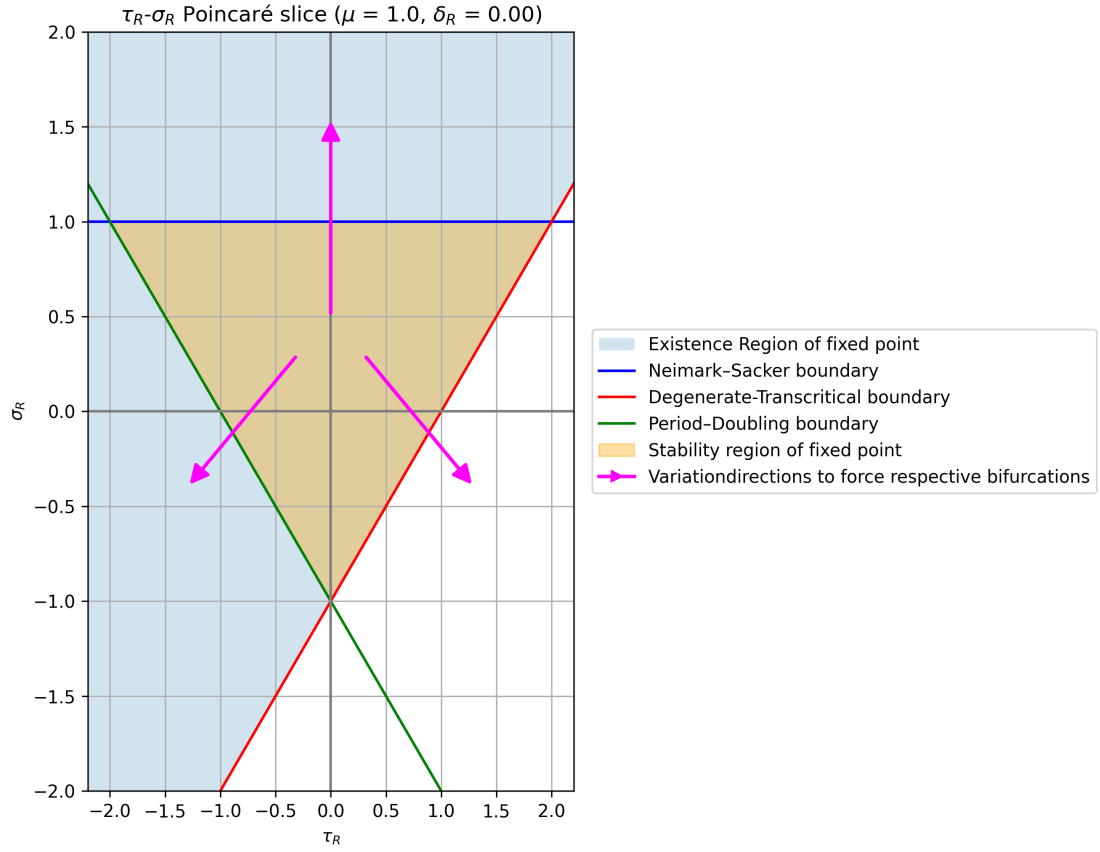


Figure 2.1: The stability triangle of the fixed point in the τ_R - σ_R Poincaré slice for $\mu = 1.0$, $\delta_R = 0.00$. Existence region (light blue), stability region (orange), and the three bifurcation boundaries – Neimark–Sacker (blue), degenerate-transcritical (red), and period-doubling (green) with representative parameter-variation directions illustrating how crossing each boundary forces the respective bifurcation.

2.1.3 Fold (and BC-Fold) bifurcation

A traditional smooth *fold* (or saddle-node) bifurcation is one of the most fundamental bifurcation types in the theory of dynamical systems: as a parameter is varied, two fixed points – one stable, one unstable – approach each other, collide, and annihilate one another, so that on one side of the bifurcation a stable/unstable pair of fixed points exists, while on the other side no nearby fixed point remains at all. In smooth systems this collision occurs at a generic point in phase space, with no special significance attached to its location; what matters is only that the Jacobian of the map develops a real eigenvalue equal to $+1$ at the colliding fixed point.

In nonsmooth, piecewise-linear systems such as the BCNF, an analogous but distinct bifurcation exists – the *border-collision fold* – in which two fixed points collide with each other exactly on the switching manifold x_b itself, rather than at a generic point in phase space. As the parameter is varied towards its critical value, the two fixed points approach x_b ; at the critical value they coincide exactly on the border, and beyond it both cease to exist – exactly mirroring the pair-creation/annihilation character of the classical fold. The qualitative outcome (a pair of fixed points on one side, none on the other) is therefore the same as in the smooth case; only the underlying mechanism differs, being tied to the simultaneous collision of the two fixed points with the switching manifold rather than to a purely local eigenvalue condition at an arbitrary point in phase space.

Since this mechanism is tied to the collision of fixed points with a border, it is worth emphasizing that a fixed point can undergo many different types of behavior upon colliding with a border; the BC-fold described here is only one of several such border-collision scenarios.

A further important remark concerns terminology: the fold bifurcation is not the only bifurcation associated with an eigenvalue crossing the $+1$ stability boundary mentioned above – more generally, one may speak of $+1$ bifurcations, of which the fold variety is a special case. As mentioned as a note in 2.1.1.1 the stability related bifurcations (non border collision bifurcations) are in *degenerate* form due to the piecewise-linear nature of the system. Accordingly, it has become convention to speak of " $+1$ bifurcations of the fold variety" and, or in the considered case, "*degenerate* $+1$ bifurcations of the fold variety." In some figures throughout this thesis, only one of the possible $+1$ bifurcations is used as a label for the corresponding bifurcation curve. However, the text explicitly identifies a specific type of $+1$ bifurcation only when it has been rigorously established that the observed bifurcation is indeed of that type.

Figure 2.2 shows schematic bifurcation diagrams of the 3 different possible fold bifurcation variants (smooth - border collision - degenerate " $+1$ " type fold).

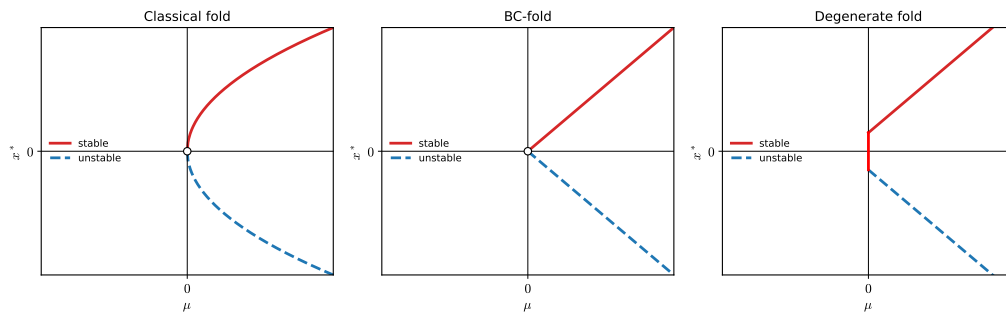


Figure 2.2: Qualitative bifurcation diagrams for a fold bifurcation. Dashed blue branches denote unstable fixed points, solid red branches denote stable fixed points. Left: smooth case, where a stable/unstable pair of fixed points is created past the bifurcation. Center: border collision case, where the stable and unstable pair of fixed points is created at the critical border in a piecewise-linear map. Right: degenerate case, where the stable and unstable fixpoints are created after a bifurcation intervall filled with infinite periodic and aperiodic cycles.

2.1.4 Flip (period-doubling) bifurcation

2.1.4.1 Standard flip bifurcation

A traditional flip bifurcation occurs when one of the eigenvalues of a fixed point crosses through $\lambda = -1$, corresponding to the period-doubling boundary $\sigma_R = -\tau_R - 1$ in the stability triangle. Two variants are possible. In the *supercritical* case, the fixed point loses its stability at the bifurcation, and a stable period-2 orbit is born continuously around it, so that the trajectory smoothly transitions onto this new stable periodic orbit as the parameter is varied. In the *subcritical* case, an unstable period-2 orbit already surrounds the (still stable) fixed point prior to the bifurcation; this unstable orbit shrinks and collapses onto the fixed point exactly at the critical parameter value, so that only an unstable fixed point remains afterwards. Every variant of the flip bifurcation – classical, border-collision, or degenerate – occurs in either a subcritical or a supercritical form, depending on the stability of the invariant object undergoing the bifurcation. For completeness, it should be quickly mentioned that although border-collision flips exist, they are of no further relevance to this investigation. Of considerably greater relevance, however, is the *degenerate* flip bifurcation, in which a single bifurcation point is replaced by an entire bifurcation interval containing infinitely many periodic and aperiodic solutions. Since this case is of particular importance for the present work, it is treated in detail in 2.1.4.2.

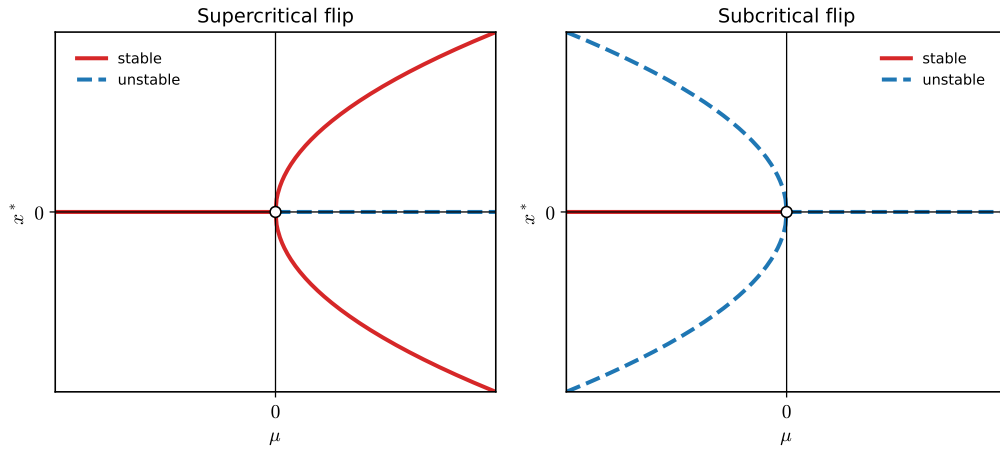


Figure 2.3: Qualitative bifurcation diagrams for a flip (period-doubling) bifurcation. Dashed blue branches denote unstable orbits, solid red branches denote stable orbits. Left: supercritical case, where a stable period-2 orbit is born continuously as the fixed point loses stability. Right: subcritical case, where an unstable period-2 orbit collapses onto the fixed point at the bifurcation and an unstable fixed point remains.

2.1.4.2 Degenerate flip bifurcation

Since the BCNF considered in this work is piecewise linear rather than smooth, a period-doubling (flip) bifurcation is also not coming in the two smooth case scenarios described above. Owing to the system's piecewise-linear structure, a stability related (non border collision flip) comes in degenerate form. Besides the supercritical and

subcritical degenerate cases shown in fig. 2.4, there exist two more cases – the degenerate flip bifurcations that lead to chaos (See fig. 2.5). These are relevant, because they add to the possibilities how the investigated ICC flip scenario can fail, even though everything seems to work out.

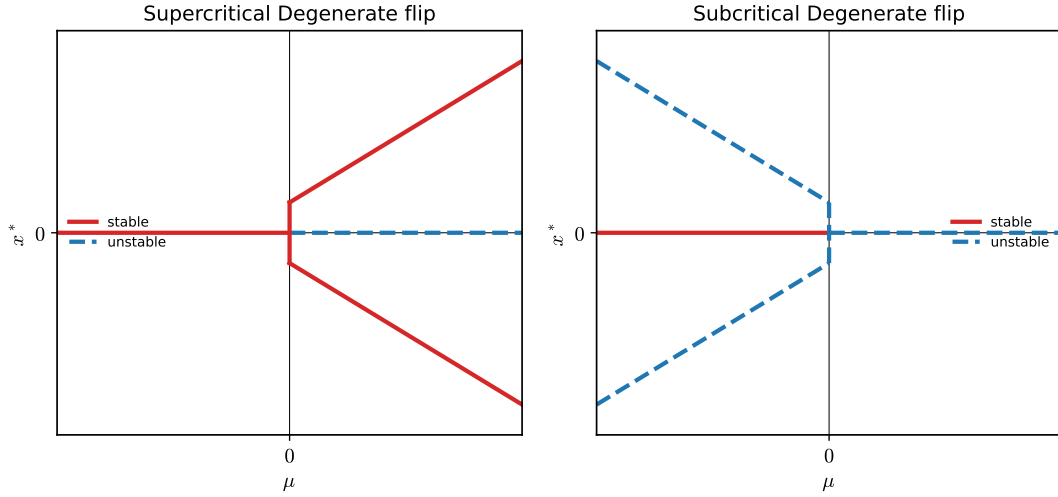


Figure 2.4: Qualitative bifurcation diagrams for the degenerate flip bifurcation, covering the two standard cases possible in the piecewise-linear (nonsmooth) setting. Left: supercritical case, stable period-2 orbit is born from a stable fixed point at a distance after bifurcation interval. Right: subcritical case, unstable period-2 orbit collapsing at a distance. After bifurcation interval only unstable fixed point remains.

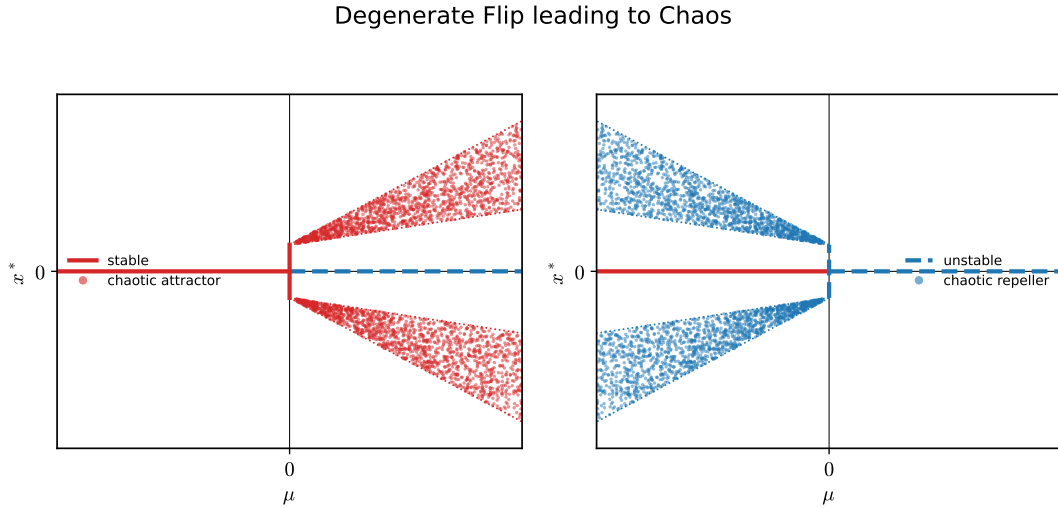


Figure 2.5: Qualitative bifurcation diagrams for the degenerate flip bifurcation, covering the two possible cases leading to chaos in the piecewise-linear (nonsmooth) setting. Left: attractor case, chaotic 2-band attractor is born from a stable fixed point at a distance after bifurcation interval. Right: repeller case, chaotic 2-band repeller collapsing at a distance. After bifurcation interval only unstable fixed point remains.

2.1.5 The Neimark–Sacker bifurcation in the piecewise-linear setting

A classical Neimark–Sacker bifurcation in a smooth map occurs when a pair of complex-conjugate eigenvalues of a fixed point crosses the unit circle continuously as a parameter is varied; generically (away from strong resonances) this gives rise to an invariant circle on which the dynamics is either quasiperiodic or mode-locked to a periodic orbit. Whether this invariant circle is attracting (supercritical case) or repelling (subcritical case) is not determined by the linear eigenvalue condition $|\lambda| = 1$ alone – this condition only signals that a bifurcation occurs, not of which type. Criticality is instead governed by the cubic term of the Poincaré–Birkhoff normal form (the first Lyapunov coefficient) [29], which is precisely the term that bends the locally unstable (or stable) spiral back onto itself at a finite radius, either inward toward an attracting circle or outward away from a repelling one.

In the BCNF, the situation differs in two related respects. First, as with the degenerate bifurcation phenomena discussed for the flip bifurcation in 2.1.4.2, the invariant curve does not grow from the fixed point at the smooth, square-root rate familiar from the classical Neimark–Sacker bifurcation; instead, owing to the piecewise-linear scaling structure of the BCNF, any invariant set born through this bifurcation grows *linearly* in the bifurcation parameter. Second, and more fundamentally, the very mechanism that decides criticality in the smooth case is absent here altogether. As long as a trajectory remains on one side of the switching manifold, the BCNF acts there *exactly* linearly, with no polynomial correction of any order as in the smooth case – and a purely linear map with $|\lambda| > 1$ admits no mechanism to curb the exponential growth of a spiral at any finite radius, since doing so would require precisely the cubic (or higher-order) term that a linear map does not, and by construction cannot, possess. Consequently, no local analogue of the smooth super-/subcritical distinction exists at the level of the individual linear pieces of the BCNF: whether an emerging invariant object is bent into an attracting or a repelling structure is decided entirely by a *global* mechanism – the collision of the growing spiral with the switching manifold and its reinjection into the other linear region – rather than by a local Taylor-series computation at the fixed point. As will be seen below, this global mechanism carries no obligation to reproduce the smooth scenario’s outcome, and this is precisely what distinguishes the piecewise-linear case from its smooth counterpart.

A further consequence of this same piecewise-linear structure concerns the bifurcation point itself. Recall that the degenerate flip bifurcation gives rise to an entire interval filled with periodic *and* aperiodic solutions exactly at the bifurcation point (see Fig. 2.4). Because the Neimark–Sacker bifurcation is inherently two-dimensional, its degenerate counterpart is no longer a one-dimensional “bifurcation interval” as in the degenerate flip case, but a *bifurcation plate*: according to Sushko and Gardini [21], depending on whether the rotation number is rational (periodic case) or irrational (aperiodic case), this is a filled, two-dimensional region containing infinitely many nested periodic polygons or aperiodic ellipses, respectively. Throughout this thesis, this bifurcation is referred to as the *degenerate Neimark–Sacker* bifurcation, although in parts of the literature – including [21] itself – it is also known as the

centerpoint bifurcation, or simply the *center bifurcation*.

Immediately beyond the bifurcation point, the global mechanism described above resolves in one of (at least) two qualitatively different ways, both documented by Sushko and Gardini. In the *supercritical* case [21], the outermost curve surrounding the bifurcation plate persists as a robust invariant object. In the periodic case, this object is not a smooth ellipse but a topological circle composed of one-dimensional manifolds that need not follow any simple geometric shape or direct path; for this reason, it is more precisely referred to as an *invariant closed curve* (ICC), the precise composition of which is made rigorous in 2.1.8.1. In the *subcritical* case, by contrast, Sushko and Gardini [22] show that the trajectory is instead thrown directly onto a chaotic attractor immediately upon destabilization of the fixed point – a one-piece chaotic attractor in one of their examples, and a two-piece chaotic attractor in another. This is the direct Neimark–Sacker analogue of the chaos-producing case already encountered for the degenerate flip bifurcation, confirming that a direct, genuinely nonsmooth route to chaos is not a peculiarity of the flip case, but a general feature of degenerate bifurcations in piecewise-linear systems. Practically, this means for the upcoming investigation of degenerate Neimark-Sacker bifurcations of ICCs, that even when every classical eigenvalue condition for a Neimark–Sacker bifurcation of the two constitutive cycles appear to be satisfied, the bifurcation actually encountered may still turn out to be of this chaos-producing type rather than the textbook, ICC-producing one.

It is against this background that the results of Simpson and Meiss [17] should be read. They were the first to systematically study the nonsmooth invariant closed curves arising from the supercritical case described above, showing that, despite the presence of the switching manifold, the resulting dynamics is often strikingly similar to the smooth Neimark–Sacker scenario: immediately beyond the bifurcation, an attracting ICC is guaranteed to be born, on which the dynamics is again either quasiperiodic or mode-locked. For the 3D BCNF considered in this thesis, the conditions under which such a nonsmooth Neimark–Sacker bifurcation occurs were derived explicitly by De, Dutta, Banerjee and Roy [7], building directly on this planar theory. Both results, however, describe only the supercritical branch of the dichotomy established above; since no local cubic-type mechanism constrains criticality in a piecewise-linear system, there is a priori no reason to expect their guarantee of attractivity to hold uniformly across the parameter space of the 3D BCNF, rather than being one of (at least) two possible outcomes. This mirrors the conclusions reached by Keller [26] for the piecewise-linear flip bifurcations of ICCs, where satisfying the classical eigenvalue conditions was likewise shown to be necessary, but far from sufficient, for the textbook scenario to actually occur.

2.1.6 Resonance (Arnold) tongues and their sausage-string geometry

As in the smooth case, the mode-locked solutions that occur near a (nonsmooth) Neimark–Sacker bifurcation may organize themselves into *resonance tongues* (also called Arnold tongues): regions of parameter space within which a stable periodic solution with a fixed rotation number exists. A qualitative sketch is given in Fig. 2.6. The intricate geometry of Arnold tongues in a one-dimensional, piecewise-linear circle map was first observed by Yang and Hao [20], who found that each tongue can take the form of a chain of distinct lens-shaped regions strung together – a structure commonly described as *sausage-string* or *lens-chain* shaped. Simpson and Meiss [17, 18, 19] developed a general theory of this phenomenon for N -dimensional, piecewise-smooth, continuous maps: within a tongue, the stable periodic solution coexists with an unstable periodic solution of the same period and rotation number, and the two boundaries of each lens correspond to *border-collision fold bifurcations* (as opposed to ordinary folds in the smooth case), at which the two orbits collide and annihilate as one point of each periodic orbit hits the switching manifold. This is, in fact, quite intuitive: since a pair of periodic orbits of the given period exists inside the lens and no such pair exists outside it, the boundary itself can only be fold-type behavior.

As a minor side note, Simpson and Meiss [18] prove that this is not the whole story at a non-terminating shrinking point: the intersection of two such border-collision fold bifurcations there forces $(I - M_S)$ to be singular as well – that is, a degenerate eigenvalue-+1 bifurcation of the cycle’s stability matrix necessarily passes through the very same point (their Cor. 8, building on Lem. 7).

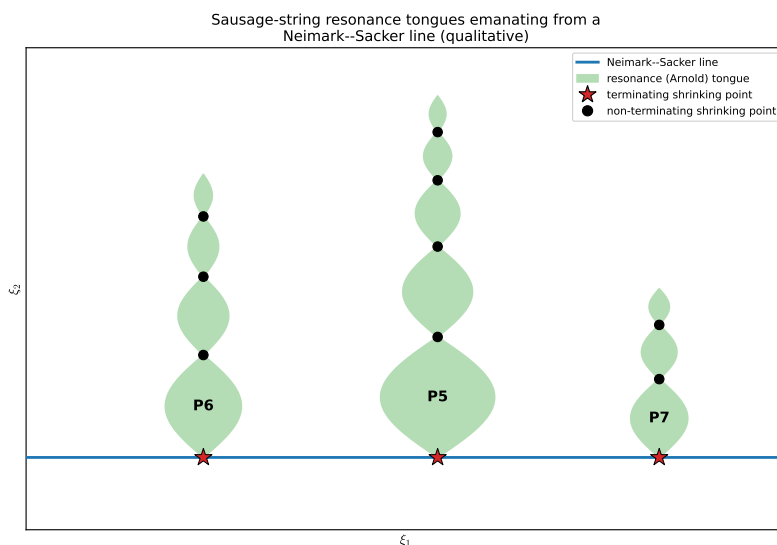


Figure 2.6: Qualitative sketch of a Neimark–Sacker line with a sausage-string chain of resonance (Arnold) tongues emanating from it. The chain originates at a *terminating shrinking point* (red star), where it meets the primary Neimark–Sacker bifurcation curve, and repeatedly pinches to zero width at a sequence of *non-terminating shrinking points* (black circles) between neighboring lenses.

2.1.7 Shrinking points (waist points)

Each lens in the chain eventually narrows and closes off at a pair of codimension-two points where its width shrinks to exactly zero. Following the original terminology of Yang and Hao [20], who called these points *waist points*, Simpson and Meiss [17, 18] instead adopted the name *shrinking points* for the same objects, and this is the term that has become standard in the subsequent literature; *shrinking point* and *waist point* therefore refer to one and the same phenomenon; Throughout this thesis though only the more modern term *shrinking point* will be used.

At a shrinking point, the periodic orbit possesses *two* points lying simultaneously on the switching manifold rather than the single colliding point found at an ordinary border-collision fold, making a shrinking point a genuinely codimension-two phenomenon that requires the variation of (at least) two parameters to unfold.

Simpson and Meiss [18] distinguish two qualitatively different types of shrinking point:

- **Non-terminating shrinking points**, which lie in the interior of a sausage-chain. Here, four distinct border-collision fold curves pass through the shrinking point pairwise tangentially, and a *second* sausage-shaped tongue emanates from the point – this is precisely the mechanism by which one sausage of the chain narrows to zero width while the neighboring sausage simultaneously opens up, producing the repeated widening-and-narrowing sausage-string pattern.
- **Terminating shrinking points**, which lie at the end of a sausage-chain. These coincide with a *center bifurcation* (a degenerate Neimark–Sacker bifurcation with rational rotation number) of the fixed point itself, as studied by Sushko and Gardini [21], rather than with the collision of two coexisting periodic orbits.

(See fig.2.6) At either type of shrinking point, Simpson and Meiss show that the map possesses an invariant polygon – planar at a terminating shrinking point, but generically *nonplanar* at a non-terminating one whenever the phase space dimension exceeds two, as is the case for the 3D BCNF studied in this thesis – on which the dynamics is conjugate to a rigid rotation. This invariant-polygon structure, together with the codimension-two unfolding that governs the local geometry of the resonance tongues around shrinking points, is central to the analysis of mode-locked solutions carried out later in this thesis.

2.1.8 Invariant Closed Curves and where to find them

2.1.8.1 What an ICC actually consists of

Before proceeding, it is worth making precise what an invariant closed curve (ICC) is built from, since the term is used loosely in much of the literature. As mentioned, for example, in [23], an ICC in a Poincaré section (or, equivalently, for a discrete-time map such as the BCNF) corresponds to a two-frequency torus in the phase space of the underlying continuous-time system. Two qualitatively different cases are distinguished depending on the nature of the dynamics on the curve:

- If the dynamics on the ICC is periodic, it is called a *mode-locked ICC* (or *resonant torus*). In the simplest case, such an ICC consists of exactly *two* periodic orbits of the same period and rotation number, together with a connecting one-dimensional manifold¹ whose points asymptotically converge onto one of them. More complicated resonant tori may contain several such pairs.
- If the dynamics on the ICC is quasiperiodic, it is called a *quasiperiodic ICC* (or *ergodic torus*, sometimes also a *drift ring*); in this case there are no periodic points on the curve at all, and the ICC is instead defined as the closure of any one of the quasiperiodic orbits it contains. According to the authors of [23], this case is best regarded as an edge case of the periodic variant above, rather than as a separate phenomenon in its own right.

This description of a mode-locked ICC as the union of cycles, and the connecting *one-dimensional* manifold of the saddle is not specific to two dimensions: Muni and Banerjee [10] give the same characterization directly for maps of three or more dimensions, describing the resulting torus as “a closed loop in discrete time, which contains stable and unstable cycles of the same periodicity, and the unstable manifolds of the saddle” – precisely the attractive situation relevant to the 3D BCNF studied in this thesis.

¹Strictly speaking, the term *manifold* is a slight abuse of language in the present non-smooth setting: the classical notion of an (un)stable manifold presupposes a differentiable map, and the piecewise-linear BCNF is only continuous, not differentiable, across the border, so the invariant set associated with a saddle cycle need not be a manifold in the mathematically precise sense. A more accurate term would be *invariant set of the saddle cycle*. For readability, and because the object plays exactly the role its smooth-theory namesake would, we nonetheless retain the familiar term *manifold* throughout this thesis.

In two dimensions however, a mode-locked ICC of the type described above is necessarily either fully attracting or fully repelling: an *attracting* ICC is built from a stable node/focus cycle and a saddle cycle, connected by the saddle’s unstable manifold, whose points converge onto the stable cycle; a *repelling* ICC is built analogously from an unstable node/focus cycle and a saddle cycle, connected instead by the saddle’s stable manifold. This dichotomy is a consequence of the two directions available at each point of a planar saddle – one attracting, one repelling – with the curve itself occupying the repelling (or attracting) direction. In three dimensions, however, a saddle cycle has an additional, third direction that is not involved in constructing the curve at all, and which may independently be attracting or repelling. This makes it necessary to introduce a *saddle ICC*, directly analogous to a saddle fixed point or cycle: a curve that is neither fully attracting nor fully repelling, but instead attracts along one direction while repelling along another.

Since only attracting curves are of interest for the majority of this thesis, unless stated otherwise, “node” should always be understood to mean “stable node,” and the discussion below is restricted to the attracting case. To describe an attracting ICC, one therefore needs to examine the unstable manifold of the saddle that forms the saddle–(stable) node/focus connection making up the curve. Since the intuitive notion of an unstable manifold – “the points that are pushed away” – is too imprecise to work with, the manifold is instead defined in backward time: the unstable manifold consists of those points that converge to the saddle under backward iteration. Near the saddle cycle itself, this manifold is tangent to the corresponding unstable eigenvector, but as one moves away from the cycle it generally curves and can extend arbitrarily far through phase space under the nonlinear (or, in the BCNF, piecewise-linear) action of the map. Whenever a saddle cycle in the BCNF has only a single unstable eigenvalue – the generic case relevant to the mode-locked ICCs discussed here – this unstable manifold is a genuinely *one-dimensional* curve (a single invariant line through phase space) connecting the saddle to the corresponding stable node/focus cycle.

Shortly beyond the (degenerate) Neimark–Sacker line, before the linear growth described in section 2.1.5 has carried the curve far from the fixed point, the ICC still appears almost perfectly circular and tame – the piecewise nature of the underlying map is barely visible, and the curve closely resembles the invariant circle of a classical, smooth Neimark–Sacker bifurcation (see Fig. 2.7). Only as the parameters are pushed further from the Neimark–Sacker origin does the curve’s true, deformed character – distinct cycles in three-dimensional phase space, connected by the geometrically complex manifolds just described – become visually apparent.

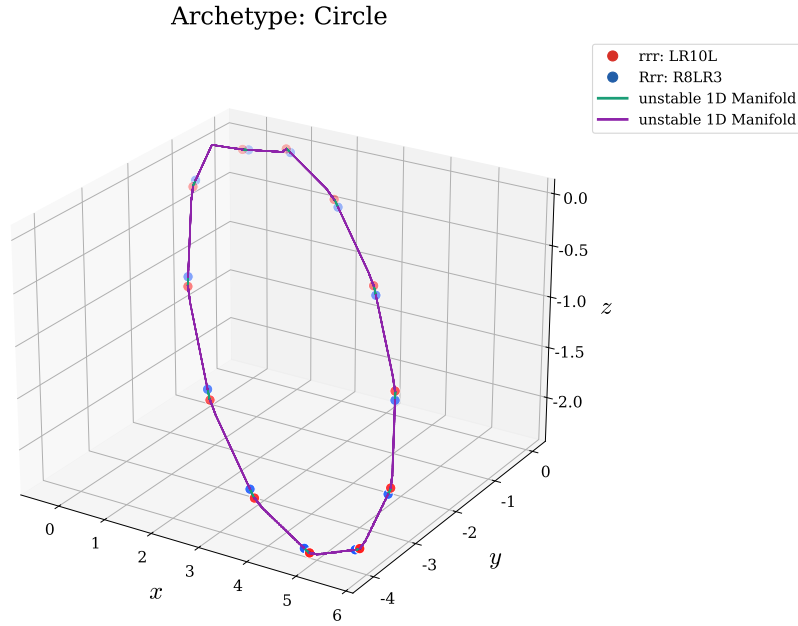


Figure 2.7: Period 12 ICC found relatively close beyond the Neimark–Sacker line consisting of a stable node cycle with three real valued eigenvalues (rrr) and an unstable Saddle cycle with three real valued eigenvalues (Rrr).

2.1.8.2 Nomenclature for cycles and their itineraries

To describe the cycles that make up an ICC concisely, it is useful to fix two complementary pieces of notation: one for the stability type of a cycle’s eigenvalues, and one for its itinerary relative to the switching manifold.

2.1.8.2.1 Eigenvalue notation. A period- n cycle of the 3D BCNF has three multipliers (the eigenvalues of its stability matrix A). We classify each multiplier along two independent axes and encode this in a three-letter string, one letter per multiplier:

- **Case (stability):** a *lowercase* letter denotes a multiplier with modulus less than one (a stable/contracting direction), while an *uppercase* letter denotes a multiplier with modulus greater than one (an unstable/expanding direction).
- **Letter (type):** the letter **r** (or **R**) denotes a real multiplier, while **c** (or **C**) denotes a multiplier belonging to a complex-conjugate pair. Since complex multipliers of a real matrix necessarily occur in conjugate pairs sharing the same modulus, they are always written as two adjacent, identically-cased letters (**cc** or **CC**) – a complex-conjugate pair cannot be split into a stable and an unstable multiplier.

For example, **Rcc** denotes a saddle cycle with one unstable real multiplier and one stable complex-conjugate pair – a saddle-focus of the type relevant to the mode-locked ICCs discussed above – while **rrr** denotes a fully stable node and **RRR** a fully unstable node.

2.1.8.2.2 Itinerary (mapping sequence) notation. Independently of the multiplier type, each point of a period- n cycle lies on one side of the switching manifold, and can therefore be labelled L or R according to whether A_L or A_R is applied there following [18]. Writing out the sequence of labels around one full period gives a word such as LLLRR; where a symbol repeats consecutively, we abbreviate it with an exponent, e.g. LLLRR = L^3R^2 . This itinerary notation characterizes the admissible cycles that make up an ICC alongside their multiplier type.

Note that this word is *shift-invariant*: since a cycle has no distinguished starting point, the itinerary is only defined up to cyclic permutation – L^3R^2 , RL^3R , and R^2L^3 , for instance, all describe the same period-5 orbit, merely read starting from a different point along it. Which cyclic representative is written down therefore depends purely on an arbitrary choice of where along the curve one begins counting, not on any property of the cycle itself.

2.1.9 Bifurcations of Invariant Closed Curves

2.1.9.1 Bifurcations of ICCs in smooth Systems

Once an invariant closed curve has been created—whether through a classical Neimark–Sacker bifurcation in a smooth map or through its nonsmooth analogue in the BCNF—the curve itself may undergo further bifurcations as the parameters are varied. This applies to both ergodic and resonant ICCs. For resonant ICCs, however, the corresponding bifurcation scenario consists of two steps, as each of the constituent cycles must undergo the respective bifurcation. There is no reason to expect these two bifurcations to occur simultaneously; rather, they are generally expected to occur in close succession. Consequently, additional bifurcations may occur between the two events and interrupt the intended bifurcation scenario.

For the complete bifurcation scenario of a resonant ICC to occur, both constituent cycles must therefore undergo the respective bifurcation in close succession and without any intervening bifurcation. The only exception is the fold bifurcation, for which a pair of cycles is inherently involved in the bifurcation itself.

In the smooth setting, three qualitatively different scenarios are classically distinguished. First, a mode-locked ICC can lose its periodic structure through a fold (saddle-node) bifurcation of the underlying saddle-node/focus cycle pair, at which point the dynamics on the curve becomes quasiperiodic (an ergodic torus) rather than periodic; conversely, crossing back into a resonance tongue restores mode-locking. Second, the curve can lose its smoothness altogether and break down via homoclinic tangencies between the stable and unstable manifolds of the cycles residing on it – a mechanism whose foundational treatment for smooth systems goes back to Afraimovich and Shilnikov [11], and which has since been documented directly for closed invariant curves in piecewise-smooth maps as well [12, 13, 14], typically marking a route to chaos. Third, the curve itself can undergo a *torus-doubling bifurcation*, in direct analogy to the period-doubling of a fixed point, in which the ICC loses stability and a new, longer invariant curve is born around it – a phenomenon whose first systematic description in smooth flows goes back to Arneodo, Couillet and Spiegel [24], who reported cascades of successive period-doublings of tori analogous to the Feigenbaum cascade of periodic orbits.

Of particular relevance to this thesis is the doubling scenario, which Avrutin and collaborators have shown to be considerably richer than a single mechanism. Biswas, Banerjee, Avrutin and Sushko [23] identify *two* topologically distinct types of ICC doubling that have been repeatedly reported in the literature:

- (a) the original ICC splits into *two disjoint* closed curves, with the dynamics alternating (“flipping”) between the two loops on successive iterations; and
- (b) the original ICC is replaced by a *single* closed curve of twice the length, which the dynamics traverses fully before returning to its starting point.

Both scenarios are consistent with the doubling of the cycles residing on the ICC – since, as noted in 2.1.8.1, a mode-locked ICC is built from a stable and a saddle cycle connected by the saddle’s unstable manifold, a flip bifurcation of these cycles is the natural mechanism driving the doubling of the curve itself. Crucially, Biswas, Banerjee, Avrutin and Sushko [23] demonstrate that the methods previously used to distinguish and predict which of the two doubling types will occur are not always reliable, and propose a refined criterion for doing so. A concrete instance of this doubling phenomenon in an essentially smooth (delay-differential, hybrid) system is documented by Zhusubaliyev, Avrutin and Medvedev [25], who observe the doubling of a closed invariant curve on a two-dimensional invariant manifold in a model of an impulsive Goodwin oscillator with delay, and clarify the role that the delay plays in creating the phase-space configuration necessary for the doubling to occur.

Two further claims made by Biswas, Banerjee, Avrutin and Sushko [23] are of particular importance for the three-dimensional setting of this thesis. First, they show that in 3D the doubling bifurcation takes place within a two-dimensional attracting manifold in which the ICC itself lies. Second, they show that it is precisely the topology of this surrounding manifold that decides which of the two doubling types occurs: a *cylindrical* topology forces case (a), the creation of two disjoint loops, whereas a *Möbius-strip* topology forces case (b), the creation of a single, double-length loop. To determine this topology in practice, they evaluate the eigenvectors associated with the doubling eigenvalue at closely spaced points along the ICC and sum the angles between consecutive eigenvectors; taken modulo 2π , this sum converges either to 0 or to π . A sum converging to 0 (mod 2π) indicates a cylindrical (untwisted) manifold and therefore loop-doubling, whereas a sum converging to π (mod 2π) indicates a Möbius-strip (once-twisted) manifold and therefore length-doubling.

2.1.9.2 Flip Bifurcations of ICCs in piecewise linear Systems

While 2.1.9.1 outlined the classical smooth-system framework for ICC doubling – the conditions (A1)–(A4) on the radial, tangential, and transverse multipliers of the node and saddle cycle, together with the cylinder-vs-Möbius topology criterion that distinguishes loop doubling from length doubling – a natural question is whether this framework carries over directly to the piecewise-linear setting of the 3D BCNF studied in this thesis. This question was investigated in detail by Keller [26], who examined whether a genuine, admissible ICC flip (doubling) bifurcation – one that reproduces the node-doubles-first or saddle-doubles-first scenario exactly as predicted by the smooth theory – can actually be realized within the continuous 3D BCNF.

The central finding of that investigation is that satisfying the classical conditions

(A1)–(A4) is *necessary but far from sufficient* in the piecewise-linear setting, and that a correctly-behaving ICC flip bifurcation is, in practice, an exceedingly rare occurrence. Across every one of the three continuous 3D-BCNF scenarios systematically explored by Keller – in each of which the base-period cycles satisfied the smooth spectral assumptions (A1)–(A3) – the predicted doubling failed to materialize correctly, for one of several distinct reasons specific to piecewise-linear systems:

- **Incorrect stability of the doubled cycle.** The degenerate flip bifurcation curve associated with the doubled-period cycle can lie entirely *virtual* – i.e., outside the admissibility region of that cycle. The cycle then only becomes admissible *after* its relevant eigenvalue has already crossed -1 , so it inherits an extra unstable eigendirection immediately upon coming into existence, rather than emerging purely stable (or as a clean saddle) as the smooth theory predicts.
- **Border-collision-induced stability transitions.** A doubled-period cycle can undergo a flip *border-collision* bifurcation, at which its symbolic sequence changes discontinuously. Because the monodromy matrix depends directly on the symbolic sequence, this produces a discontinuous jump in the eigenvalue spectrum even though the orbit points themselves vary continuously – a mechanism with no analogue in smooth systems.
- **Admissibility-bounded doubled cycles.** Rather than persisting as a stable, freely-existing doubled-period cycle, the admissible parameter region of the doubled cycle can be squeezed between two degenerate flip bifurcation curves, so that immediately beyond this narrow window the system transitions directly into chaotic dynamics instead of completing the doubling scenario.
- **Cascading virtual flip bifurcations.** In some cases, a cycle already satisfies the spectral condition for a further degenerate flip immediately upon entering its admissibility region, with the associated bifurcation curve itself remaining virtual; Keller observes evidence that this can trigger the effectively simultaneous, degenerate creation of an entire cascade of higher-period cycles (period-32, -64, -128, ...) at a single parameter value, rather than a single, clean doubling step.

In other words, even when the local eigenvalue conditions (A1)–(A4) are satisfied exactly as the smooth theory requires, the global interplay of admissibility regions, symbolic sequence transitions, and border-collision geometry in the piecewise-linear BCNF obstructs the doubling in every scenario examined. A genuinely well-behaved ICC flip bifurcation – producing an admissible doubled-period node/saddle pair with the correct stability configuration, exactly as in the smooth node-doubles-first or saddle-doubles-first scenario – is therefore not the generic outcome one should expect in this setting, but rather a special, fine-tuned occurrence that the classical eigenvalue criteria alone cannot guarantee. Notably, Keller further shows that this obstruction is a consequence of continuity itself rather than a fundamental impossibility: introducing a small discontinuity at the switching manifold (via a discontinuous extension of the 3D-BCNF-based map) can relax the admissibility constraints responsible for the failure and restore genuine ICC doubling in parameter regions where the continuous system could not sustain it.

2.1.9.3 Border Collision Fold of ICCs

Border-collision fold bifurcations of the cycles residing on an ICC are not merely a peripheral phenomenon but the very mechanism that bounds the resonance tongues discussed in 2.1.6: as already noted there, the sausage-string boundaries of a tongue in the piecewise-linear setting are formed by border-collision folds rather than by classical folds. Recalling the composition of a mode-locked ICC from 2.1.8.1, every such curve is built from a saddle-node (or saddle-focus) pair, connected by the one-dimensional unstable manifold of the saddle. This has an immediate and rather unforgiving consequence: whenever this saddle-node/focus pair itself passes through a border-collision fold and annihilates, the entire ICC is destroyed along with it – without a saddle, there is no unstable manifold to speak of, and without a stable/saddle pair in the first place, there is no mode-locked ICC at all. In this sense, the border-collision fold acts on the level of the whole curve exactly symmetrical as it does on the level of a single pair of fixed points (See Sec. 2.1.3), or the level of a single pair of cycles. Zhusubaliyev, Avrutin, Sushko and Gardini [27] study the border collision mechanisms of resonant invariant closed curves, showing that a border-collision fold bifurcation is only one of the possible border collision scenarios a resonant invariant closed curve can undergo.

A particularly delicate instance of this mechanism occurs at the non-terminating shrinking points introduced in 2.1.6, where two border-collision fold boundaries meet at the same point in parameter space, marking the end of a periodicity sausage. As also discussed in the context of the period-7 tongue investigated in this thesis, Simpson and Meiss [18] prove that this coincidence is never purely incidental: at such a point, a degenerate eigenvalue-+1 condition is forced to pass through the very same parameter value as well (their Corollary 8, building on Lemma 7). Two border-collision folds meeting at a single point is therefore not an independent, additional coincidence on top of the shrinking point condition, but a direct algebraic consequence of it.

2.2 Technical Resources and Applications

2.2.1 What is AnT?

AnT (short for *AnT* 4.669, a name referencing the Feigenbaum constant 4.669... associated with period-doubling cascades) is a software tool for the simulation and analysis of dynamical systems. It was developed at the Institute of Parallel and Distributed Systems (IPVS), University of Stuttgart, primarily by **V. Avrutin, R. Lammert, M. Schanz and G. Wackenhut**, with development beginning in the late 1990s and the first publications describing the tool appearing in the early 2000s.

AnT was designed with a deliberately general architecture so as to support a wide range of dynamical system classes, including time-discrete maps (in particular piecewise-defined, non-smooth systems), ordinary and time-delay differential equations, as well as cellular and hybrid systems. The user specifies the system function together with its parameters and initial conditions; the software then numerically iterates

or integrates the system and applies one of several *investigation methods* to the resulting trajectories. These methods include the detection and stability classification of periodic orbits (via their eigenvalues), the computation of Lyapunov exponents, Poincaré sections, and the symbolic-image method for determining invariant sets, stable and unstable manifolds, and basins of attraction. A central feature is the ability to perform one-, two-, or higher-dimensional parameter *scans*, in which an investigation method is applied systematically over a grid of parameter values – the basic principle underlying bifurcation diagrams.

AnT is implemented in C++, runs on multiple platforms (Linux and Windows), provides a GTK-based graphical user interface, and uses Gnuplot for visualising results. It is distributed as open source under the GNU General Public License.

2.2.2 What is PeriodScanner?

PeriodScanner is a Python-based software tool for locating periodic orbits of a discrete (possibly piecewise-defined) map and automatically classifying their stability. It was developed by **Sven Görner** in 2024 under the supervision of V. Avrutin at the Institute for Systems Theory and Automatic Control (IST), University of Stuttgart.

The tool is built around a multi-start Newton-type root search combined with density-based clustering. The mathematical model under investigation is supplied by the user through an abstract `Model` interface, which requires only the system dimension, the (possibly piecewise-defined) map F , and its analytic Jacobian on each region – the piecewise-linear *border Collision Normal Form* model used throughout this thesis is one such implementation. To locate period- p orbits, the search domain is discretised into a regular grid of initial guesses; for every grid point, `scipy.optimize.fsolve` (a hybrid Powell/Newton method) is used to solve the fixed-point equation

$$F^p(x) - x = 0 \tag{2.2}$$

and only solutions whose residual falls below a prescribed tolerance are retained. Since many initial guesses typically converge to the same underlying orbit (or to cyclic shifts of it), the resulting set of candidate points is grouped using the DBSCAN clustering algorithm, yielding one representative point per distinct periodic orbit. Starting from this representative, the complete orbit is reconstructed by forward iteration of F .

For each orbit, the exact Jacobian of the p -th iterate is assembled as the chain-rule product of the (piecewise-constant) Jacobians along the orbit, and its eigenvalues and eigenvectors are computed. Each eigenvalue is classified as stable or unstable ($|\lambda| < 1$ or $|\lambda| \geq 1$) and as real or complex, which is used to automatically generate a descriptive label for the orbit (e.g. **Rrr**, **CCr**) indicating, letter by letter, the type and stability of each eigendirection – the naming convention used for the periodic orbits throughout this thesis. The resulting orbits, together with their coordinates, parameters, and eigenstructure, are exported to per-cycle data files. PeriodScanner additionally provides an optional graphical user interface as well as a headless batch mode for automated scans across multiple periods and parameter settings.

2.2.3 What is Manifis?

Manifis is a Python-based tool for computing the (typically one-dimensional) stable or unstable manifold of a periodic point of a – possibly piecewise-defined – map, by growing it outward from the periodic point along the corresponding eigendirection. It was developed by **Dennis Güney** in 2024 under the supervision of V. Avrutin at the Institute for Systems Theory and Automatic Control (IST), University of Stuttgart.

Given a periodic orbit, the exact Jacobian along the orbit is assembled in the same way as for the periodic-orbit search itself, but its eigenvalues and eigenvectors are computed using the `mpmath` arbitrary-precision library rather than standard floating-point arithmetic, since the exponential growth of the manifold over many iterations quickly exhausts double precision. The orbit is first classified by counting the number of eigenvalues with modulus greater than and less than one, to confirm that it is a genuine saddle (as opposed to a node or a focus, for which manifold growth is not meaningful). A short initial segment is then seeded along the relevant (stable or unstable) eigenvector at a small offset from the periodic point, in the direction and branch specified by the user.

The manifold is grown iteratively by mapping the current set of points forward under the map (or backward, via an explicitly defined inverse, for stable manifolds). Whenever two consecutive image points are farther apart than the resolution required in the corresponding region of phase space, a new point is inserted by mapping the midpoint of the corresponding *preimages* – rather than interpolating the images directly – and this refinement is applied recursively until the target resolution is met; conversely, points that become closer together than necessary are pruned again, so that the point density adapts to the locally required resolution instead of growing without bound. Because the underlying map is piecewise-defined with a switching manifold, consecutive points can straddle this boundary; such discontinuity jumps are detected explicitly, points generated from different branches of the map are tracked through dedicated bookkeeping objects, and the exact location of the switching boundary is resolved to a prescribed precision. The resulting branch is written to a data file in the same high-precision format used throughout this thesis, can optionally be plotted while it is being computed, and the overall computation can be parallelized across multiple parameter values using Python’s `multiprocessing` module.

3

Goals and Tasks

As already mentioned in Section 1.3, the four research questions this thesis aims to answer are:

1. To what extent can the established theory of stability-related bifurcations of ICCs in smooth 3D maps be extended to piecewise-smooth 3D maps?
2. What mathematical obstacles arise when applying smooth bifurcation theory to ICCs in nonsmooth systems, and how can these obstacles be characterized?
3. What can ICC structures in piecewise-linear 3D maps look like?
4. What can the stability-related bifurcation mechanisms of ICCs look like in piecewise-linear 3D maps?

(1) Smooth to piecewise-smooth To answer this question, the thesis first aims to build an understanding of how ICC bifurcation mechanisms work in smooth systems. The corresponding task is to carry out an extensive literature review of which bifurcations of ICCs have already been investigated; Section 2.1.9 compiles the resulting body of knowledge on bifurcations of ICCs in smooth systems. Building on this review, the next task is to transfer this smooth theory to a nonsmooth (piecewise-linear) setting: the thesis aims to reproduce the stability-related bifurcation behavior observed in smooth systems within a piecewise-linear environment.

(2) Obstacles The thesis also aims to identify the obstacles that arise when following this approach. On the theoretical side, for example, the method used by Biswas, Banerjee, Avrutin and Sushko [23] to compute the surrounding two-dimensional manifold that determines which variant of ICC doubling occurs is not readily applicable to nonsmooth systems, since there is currently no established definition of how to connect the individual pieces of this surrounding manifold across the kink points introduced by the switching manifold. On the practical side, the thesis also aims to identify how common or uncommon the various ICC bifurcation scenarios are within a piecewise-linear setting. This requires developing efficient investigation methods capable of covering a substantially wider parameter range, with the goal of surveying as much of the parameter space as is feasible.

(3) ICC structures To answer this question, the thesis aims to provide a non-exhaustive collection of ICC archetypes found at different locations in parameter space (close to the Neimark–Sacker line, in the middle of individual sausages, in different sausages along a sausage chain, close to sausage boundaries, etc.). This collection is intended to illustrate a diverse spectrum of geometries of such ICCs, ranging from tame, near-circular curves to highly intricate structures.

(4) Observed bifurcation behavior To address this final question, the thesis aims to provide a non-exhaustive overview of the bifurcation behavior encountered while investigating the three stability-related bifurcation mechanisms of ICCs. Considering also the archetype collection from question (3), the different shapes of such bifurcation behaviour of a given ICC can be understood more easily.

4

Methodology

As already mentioned in Section 1.3, the four research questions this thesis aims to answer are:

1. To what extent can the established theory of stability-related bifurcations of ICCs in smooth 3D maps be extended to piecewise-smooth 3D maps?
2. What mathematical obstacles arise when applying smooth bifurcation theory to ICCs in nonsmooth systems, and how can these obstacles be characterized?
3. What can ICC structures in piecewise-linear 3D maps look like?
4. What can the stability-related bifurcation mechanisms of ICCs look like in piecewise-linear 3D maps?

Figure 4.1 gives a graphical overview of the four research goals derived from these questions, together with their main task from chapter 3. Building on this overview, the remainder of this chapter addresses each of the four goals in turn: for every goal, its main task is broken down into a small set of concrete subtasks, and the investigation methods used to carry out these subtasks – and thereby ultimately reach the corresponding research goal – are outlined.

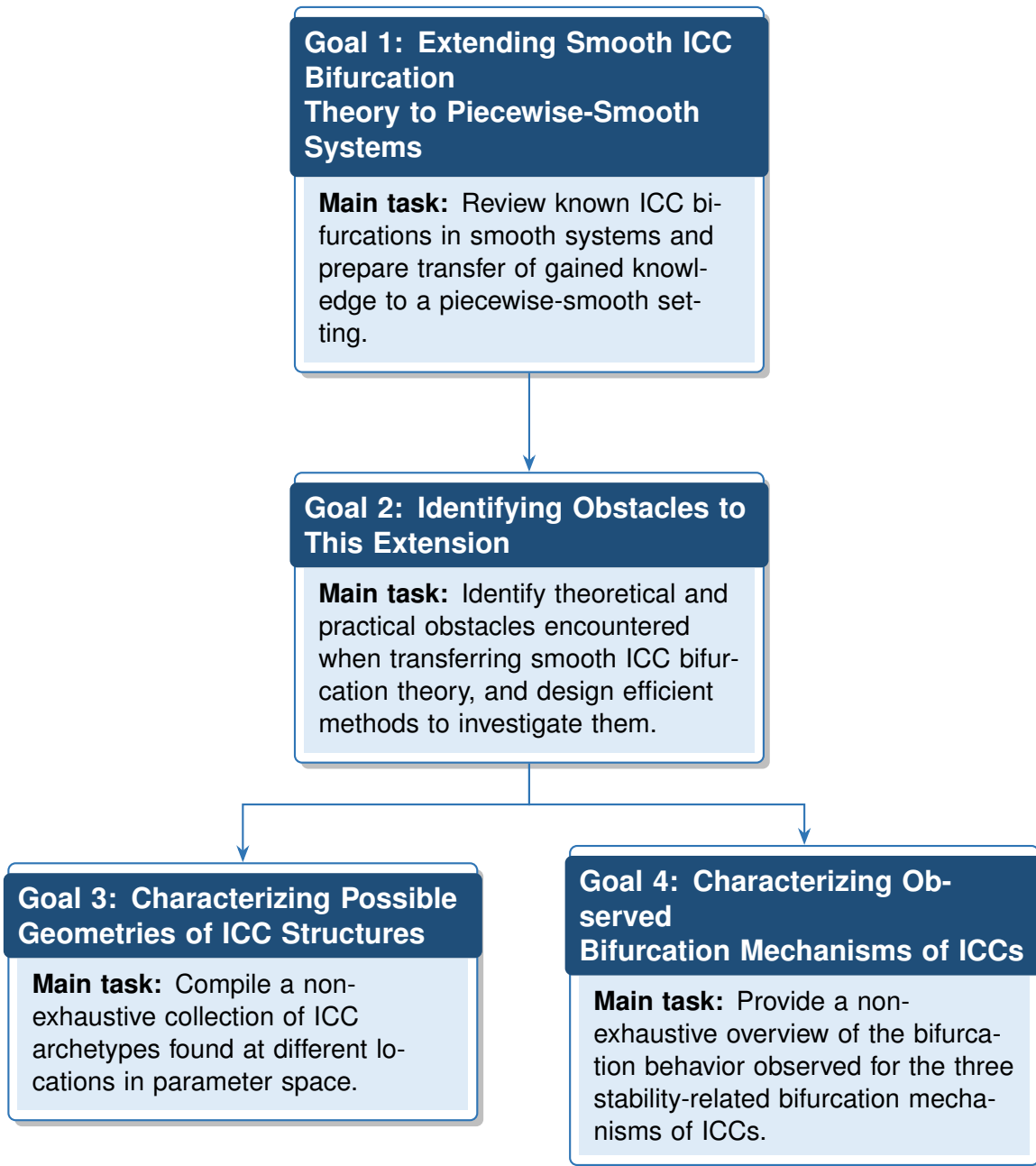


Figure 4.1: Overview of the four research goals of this thesis and their main tasks.

4.1 Goal 1: Extending Smooth ICC Bifurcation Theory to Piecewise-Smooth Systems

4.1.1 Literature Review of ICC Bifurcations in Smooth Systems

The corresponding subtask consists of surveying the published literature for prior investigations of ICC bifurcations in smooth dynamical systems. The results of this survey are compiled in Section 2.1.9, which serves as the theoretical foundation for everything that follows in this thesis.

4.1.2 Selecting a Piecewise-Smooth System for the Transfer

The motivation for this choice was already given in detail in Section 2.1.1.1; only a short recap is given here. The 3D BCNF was selected because piecewise-smooth systems are considerably more applicable to real-world problems than purely smooth ones, because the system needs to remain tractable enough to investigate thoroughly, and because canonical normal forms represent minimal configurations that are particularly easy to analyze. Choosing a normal form does not come at the cost of generality, since any sufficiently small neighborhood around a bifurcation of a more general piecewise-smooth system can be linearized to such a normal form. As this reasoning has already been laid out in full in Section 2.1.1.1, it is not repeated in further detail here.

4.1.3 Identifying Relevant Bifurcation Mechanisms to Transfer

This subtask was likewise already addressed in Section 1.4. Since the goal is to transfer bifurcation behavior from the smooth to the nonsmooth setting, only those bifurcations that also exist in the smooth case – that is, the stability-related bifurcations – are selected for investigation. Border-collision bifurcations themselves are not smooth phenomena and are therefore not a target of the transfer in their own right; however, whenever a border-collision bifurcation is directly related to one of the selected stability-related mechanisms (as is the case, for instance, for the border-collision fold boundaries of sausage structures discussed in section 2.1.3), it is naturally included as well.

Taken together, these latter two subtasks answer the question posed by Goal 1: whether, and how, stability-related bifurcation theory from the smooth setting carries over to the piecewise-linear 3D BCNF. Actually *reproducing* this transferred behavior in full, however, is not a separate subtask of Goal 1 – it is precisely the subject of Goals 3 and 4, which investigate the resulting ICC structures and their bifurcation mechanisms in detail. This is also why Goal 1 is placed hierarchically above Goals 3 and 4 in the block diagram of Figure 4.1: only once it is established *which* mechanisms are being transferred and *to which* system, can the investigation of Goals 3 and 4 meaningfully begin.

4.2 Goal 2: Identifying Obstacles to This Extension

Before any meaningful experimentation can begin, the experiments themselves must be designed around a number of obstacles that stand in the way of transferring smooth ICC bifurcation theory to the piecewise-linear setting. These obstacles fall into two categories: theoretical obstacles, which concern the applicability of existing analysis methods themselves, and practical obstacles, which concern how feasible it is to actually find the phenomena of interest within the available parameter space.

4.2.1 Theoretical Obstacles

One theoretical obstacle identified in the course of this thesis concerns the two-dimensional manifold used in the smooth-case theory to determine which type of ICC doubling occurs (see Section 2.1.9.2). In the nonsmooth setting, this surrounding manifold is itself not smooth, since it must be pieced together across the switching manifold; as a consequence, the rotation of the eigendirections along it – the very quantity used to distinguish a cylindrical from a Möbius-strip topology – is no longer well defined in general. **Solution:** rather than attempting to resolve this open problem, this thesis does not attempt to compute the 2D manifold for nonsmooth ICCs, and consequently does not attempt to differentiate between the two doubling scenarios. This restriction of scope is discussed in more detail in 1.4 (4).

A second theoretical obstacle concerns the choice of $\delta_{L,R}$. Throughout much of the existing literature on the stability triangle, and in the early chapters of this thesis, $\delta_{L,R} = 0$ is chosen for convenience: it removes two of the seven system parameters and yields a conveniently symmetric stability triangle in the τ - σ plane. However, this choice is not justified on any deeper grounds and artificially restricts the investigation to a lower-dimensional slice of the full parameter space. **Solution:** this thesis instead explicitly investigates how varying $\delta \neq 0$ affects the subsequent investigation, rather than fixing it to zero out of pure convenience.

4.2.2 Practical Obstacles

A key practical obstacle is that, at the outset of this thesis, it is simply unknown how common or uncommon the stability-related ICC bifurcations of interest are within the investigated parameter space – whether they occur throughout large regions of parameter space or are confined to comparatively small, hard-to-find corners of it. **Solution:** this thesis employs a high volume of randomly sampled parameter configurations to identify configurations that exhibit, or lie close to, the stability-related bifurcation behavior under investigation. This approach is referred to throughout this thesis as *shotgun exploration*.

4.2.3 Experimental Design

The first step in designing the experiments used throughout this thesis is to select a suitable parameter plane to investigate. The remaining parameters were chosen such that μ can always be taken to be positive, and, for the majority of this thesis, $\mu = 1$, as already explained in 2.1.2. Of the remaining parameter combinations, the τ - σ plane was selected for investigation; the reasoning behind this choice is given in 5.1.1 and illustrated in Fig. 5.4.

Building on this choice, the first concrete experiment of this thesis investigates how the “R”-side stability triangle changes under variation of δ_R . The goal of this experiment is to determine whether the two δ parameters can indeed be chosen freely – as assumed by the solution to the second theoretical obstacle above – and, if so, what consequences this freedom has for the design of the experiments that follow.

An immediate follow up experiment looks into what the respective two-dimensional bifurcation diagrams look like. As a direct consequence of the first part of the experiment, τ_R and σ_R are subjected to a coordinate transformation; the object under investigation is therefore the two-dimensional bifurcation diagram in the resulting $\tau_{R,\text{param}}\text{-}\sigma_{R,\text{param}}$ -slice of the seven-dimensional parameter sphere. Because it needs to be verified that changing $\delta \neq 0$ still yields a meaningful two-dimensional $\tau_{R,\text{param}}\text{-}\sigma_{R,\text{param}}$ bifurcation diagram. This bifurcation diagrams are calculated with *AnT* described in Section 2.2.1 and then compared to the corresponding $\delta \neq 0$ -stability triangle.

4.3 Goal 3: Characterizing the Possible Geometries of ICC Structures

Building on the results of the δ -variation experiments from Goal 2, this goal is addressed using the *shotgun exploration* approach already introduced there.

4.3.1 Experimental Design

Following the conclusions of the δ -variation experiment, δ_R is varied over the previously identified, meaningful interval $[-1, 1]$ in steps of 0.1, for a fixed set of L -side parameters $(\tau_L, \sigma_L, \delta_L)$ and a fixed $\mu = 1$.

Within this plane, the emerging periodicity regions (or Arnold tongues; see Sec. 2.1.6) are calculated with *AnT* and explored via *shotgun exploration*: a random pair of $\tau_{R,\text{param}}, \sigma_{R,\text{param}}$ coordinates is selected, and the cycles existing at that parameter point are computed using the *PeriodScanner* tool described in Section 2.2.2. The unstable manifolds of the corresponding saddle cycles, which together with the associated node or focus cycles form the resulting ICCs, are then computed using the *Manifis* tool described in Section 2.2.3. The ICCs obtained in this way are finally grouped into archetypes based on their geometric similarity.

4.3.2 Automation

Since this is an inherently repetitive experimental design – carried out with a fixed step size and relying on the law of large numbers to make otherwise improbable configurations more likely to be found the more often the experiment is repeated – an automated pipeline that runs the experiment repeatedly was chosen as the preferred approach. In addition to making the investigation feasible at all, given the sheer number of repetitions required, this automation also substantially reduced the potential for human-induced errors compared to running each instance of the experiment by hand.

4.4 Goal 4: Characterizing the Observed Bifurcation Mechanisms of ICCs

4.4.1 Experimental Design

In parallel to classifying the ICCs found through the *shotgun exploration*, each resulting ICC is also passed to a custom-built Jupyter notebook for semi-automatic bifurcation-boundary analysis. For every cycle making up an ICC, the analytical boundary curves for each of the three basic bifurcation types, as well as the real/-complex eigenvalue boundary, are computed. Using this notebook, each of the L -side parameters can then be varied interactively until the specific bifurcation-boundary configuration required for the bifurcation type under investigation is reached.

4.4.2 Case Studies of the Three Bifurcation Mechanisms

For the fold and flip bifurcations, this experimental approach was applied directly and yielded usable results. For the Neimark–Sacker bifurcation, however, this approach did not produce any results within the investigated parameter range. An additional literature review was therefore carried out, aimed at finding prior work investigating Neimark–Sacker bifurcations of periodic cycles in the 3D BCNF specifically. This is why the approach taken for the Neimark–Sacker bifurcation in this thesis deviates from the standard experimental design described above.

4.4.3 Relating Geometric Complexity Before and After Bifurcation

The results of the Goal 3 experiments help explain why the results of the Goal 4 experiments are not the clean, textbook-style examples one might expect. Since the geometry of ICCs in the investigated piecewise-linear setting can deviate immensely from the simple circle of the smooth case – and, as Section 5.2 will show, tends to become more intricate the further the chosen parameter pair lies from a shrinking point – the resulting ICCs observed after bifurcation can be considerably more elaborate than a smooth-case analogy would suggest. This is compounded by the fact that a single ICC bifurcation is not a single event, but consists of two consecutive cycle bifurcations of the same kind, one for each of the two cycles making up the

ICC, occurring one after the other rather than simultaneously and without any other interrupting bifurcation in between.

5

Theory Validation and Results

5.1 Investigation of the stability triangle

In Section 2.1.2, the concept of the stability triangle was introduced, and Fig. 2.1 showed this triangle for the special case $\delta_R = 0$. It was also motivated there why this is the customary way to construct the stability triangle: setting $\delta_R = 0$ simplifies the characteristic polynomial from $p(\lambda) = \lambda^3 - \tau\lambda^2 + \sigma\lambda - \delta$ to $p(\lambda) = \lambda(\lambda^2 - \tau_R\lambda + \sigma_R)$, reducing the problem to the stability of a single quadratic factor. However, since δ is, in principle, just as freely choosable a parameter as τ and σ , restricting it to zero purely for convenience is not justified on general grounds. Even granting that $\delta_{L,R} \neq 0$ is a reasonable case to consider, it has so far remained an open question how the stability triangle is affected once this restriction is lifted. The results presented in this section address exactly this question (Goal 2). The hypothesis is that delta can be chosen as freely as all the other parameters. To test this hypothesis the right stability triangle will be calculated with varying δ_R parameter values of stepsize 0.1 as described in 4.2.3.

5.1.1 Mathematical implications

For $\delta_R = 0$, the boundaries of the stability triangle presented in Section 2.1.2 were given by the three simple conditions

- $\sigma_R = 1$: a pair of complex-conjugate eigenvalues crosses the unit circle, giving rise to a (degenerate) Neimark–Sacker-type bifurcation;
- $\sigma_R = \tau_R - 1$: an eigenvalue crosses through $\lambda = +1$, corresponding to a (degenerate) transcritical/fold-type bifurcation;
- $\sigma_R = -\tau_R - 1$: an eigenvalue crosses through $\lambda = -1$, corresponding to a (degenerate) period-doubling (flip) bifurcation.

Once δ_R is allowed to be nonzero, these three simple conditions must be replaced by the full Schur–Cohn (Jury) stability conditions for the general cubic characteristic polynomial $p(\lambda) = \lambda^3 - \tau_R\lambda^2 + \sigma_R\lambda - \delta_R$. All three eigenvalues of the fixed point lie strictly inside the unit circle if and only if

$$|\delta_R| < 1, \tag{5.1}$$

$$1 - \tau_R + \sigma_R - \delta_R > 0, \tag{5.2}$$

$$1 + \tau_R + \sigma_R + \delta_R > 0, \tag{5.3}$$

$$|\sigma_R - \delta_R\tau_R| < 1 - \delta_R^2. \tag{5.4}$$

Three of these four conditions correspond, at equality, to a specific type of eigenvalue crossing, and therefore to a specific bifurcation boundary of the (now δ_R -dependent) stability region, while the fourth – condition Eq. (5.1) – plays a qualitatively different role addressed separately below.

- Equality in (5.2), i.e. $\sigma_R = \tau_R - 1 + \delta_R$, corresponds to a real eigenvalue crossing $\lambda = +1$ – the fold/transcritical-type boundary, now shifted by $+\delta_R$ relative to the $\delta_R = 0$ case.
- Equality in (5.3), i.e. $\sigma_R = -\tau_R - 1 - \delta_R$, corresponds to a real eigenvalue crossing $\lambda = -1$ – the period-doubling (flip) boundary, shifted by $-\delta_R$.
- Equality in (5.4) corresponds to a pair of complex-conjugate eigenvalues crossing the unit circle – the Neimark–Sacker-type boundary. Crucially, condition (5.4) is an absolute-value (two-sided) condition, and therefore yields *two* boundary lines rather than the single line $\sigma_R = 1$ found in the $\delta_R = 0$ case:

$$\sigma_R = \delta_R \tau_R + (1 - \delta_R^2) \quad \text{and} \quad \sigma_R = \delta_R \tau_R - (1 - \delta_R^2). \quad (5.5)$$

Two qualitative changes relative to the $\delta_R = 0$ case are therefore immediately apparent. First, the fold and flip boundaries are no longer symmetric about τ_R but are displaced in σ_R by $+\delta_R$ and $-\delta_R$ respectively. Second, and more significantly, the single Neimark–Sacker line present at $\delta_R = 0$ splits into two $\delta_R \tau_R$ -tilted lines once $\delta_R \neq 0$, so that the stability triangle threatens to deform into a stability *quadrilateral* (or, for suitable parameter combinations, an even more complex bounded region) as δ_R is varied away from zero.

In practice, however, only one of these two Neimark–Sacker branches is ever relevant, and the analysis throughout this thesis therefore deliberately retains only the positive branch,

$$\sigma_R = \delta_R \tau_R + (1 - \delta_R^2), \quad (5.6)$$

while discarding the negative branch

$$\sigma_R = \delta_R \tau_R - (1 - \delta_R^2). \quad (5.7)$$

This is not merely a convenient simplification carried over from the $\delta_R = 0$ case, but can be justified rigorously for every admissible value of δ_R . The fold and flip boundaries intersect at

$$(\tau_R, \sigma_R) = (-\delta_R, -1), \quad (5.8)$$

and direct substitution shows that the negative Neimark–Sacker branch passes through exactly this same point. Moreover, computing the difference between the fold boundary and the negative branch, and between the flip boundary and the negative branch, respectively, gives

$$(\tau_R - 1 + \delta_R) - (\delta_R \tau_R - 1 + \delta_R^2) = (1 - \delta_R)(\tau_R + \delta_R), \quad (5.9)$$

$$(-\tau_R - 1 - \delta_R) - (\delta_R \tau_R - 1 + \delta_R^2) = -(1 + \delta_R)(\tau_R + \delta_R). \quad (5.10)$$

As will be shown below, condition (5.1) requires $|\delta_R| < 1$ for the stability triangle to exist at all; both prefactors $(1 - \delta_R)$ and $(1 + \delta_R)$ are therefore strictly positive whenever the triangle is nonempty. Consequently, the first expression is non-negative

for $\tau_R \geq -\delta_R$, and the second is non-negative for $\tau_R \leq -\delta_R$ – that is, the fold boundary lies at or above the negative Neimark–Sacker branch everywhere to the right of the shared vertex, and the flip boundary lies at or above it everywhere to the left. Since all three lines involved are straight, this dominance, once established locally at the vertex, holds globally for all τ_R . The negative Neimark–Sacker branch therefore touches the stability region at exactly one point – the same vertex at which the fold and flip boundaries already meet – and lies strictly outside the region everywhere else. It never contributes an active edge to the stability polygon and can consequently be omitted without loss of generality, for any admissible δ_R , not merely in the special case $\delta_R = 0$.

Figure 5.1 illustrates this for the representative value $\delta_R = -0.4$: the left panel shows the stability triangle using only the positive Neimark–Sacker branch, as done throughout this thesis, while the right panel additionally overlays the negative branch (dashed), confirming that it remains tangent to the region at the fold–flip vertex and never cuts into its interior.

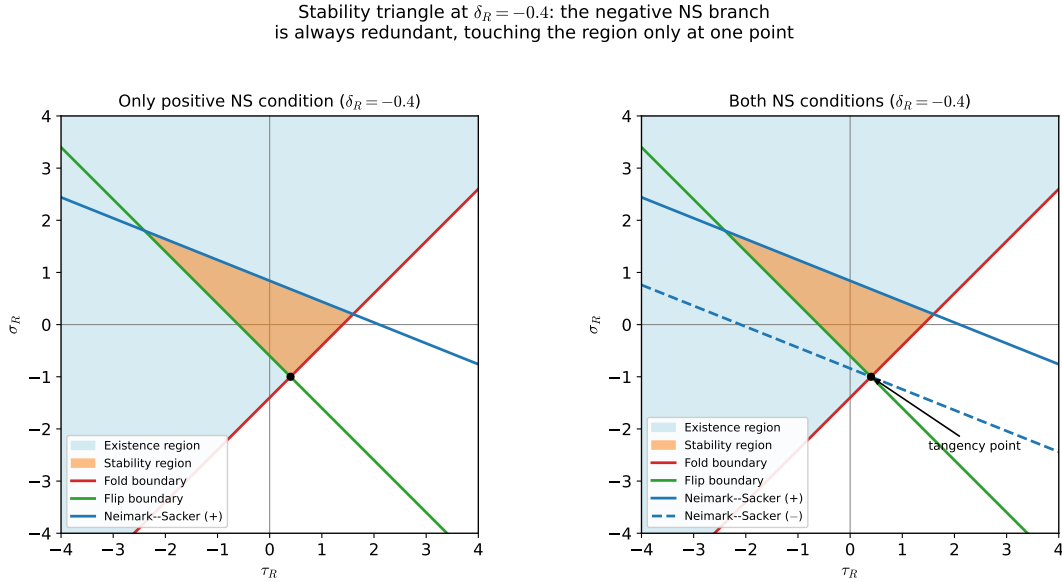


Figure 5.1: Stability region in the τ_R - σ_R plane for $\delta_R = -0.4$. Left: only the positive Neimark–Sacker condition is shown, matching the convention used throughout the thesis. Right: both branches of the Neimark–Sacker condition are shown; the negative branch (dashed) passes exactly through the same vertex where the fold and flip boundaries meet, and lies strictly outside the stability region everywhere else – confirming that it never contributes an active boundary and can safely be omitted.

Having dealt with conditions (5.2)–(5.4), we now turn to condition (5.1), which was deliberately set aside above and plays a fundamentally different role. Unlike the fold, flip, and Neimark–Sacker conditions, which each describe a specific eigenvalue crossing and hence trace out a line in the (τ_R, σ_R) -plane for fixed δ_R , condition (5.1) does not itself define any such boundary: it simply states that $|\delta_R| < 1$ is required for *any* stable fixed point to exist, for *any* choice of τ_R, σ_R whatsoever. Rather than cutting through the stability triangle, it constrains the admissible range of

δ_R for which the triangle exists at all – and it is precisely this bound, rather than a modelling convenience, that motivates why δ_R is kept within the open interval $(-1, 1)$ throughout almost all of this thesis.

The structural importance of this bound becomes clear once the behaviour of the positive Neimark–Sacker boundary is examined as $\delta_R \rightarrow \pm 1$. Comparing the three boundary expressions,

$$\sigma_R = \tau_R - 1 + \delta_R \quad (\text{fold}), \quad \sigma_R = -\tau_R - 1 - \delta_R \quad (\text{flip}), \quad \sigma_R = \delta_R \tau_R + 1 - \delta_R^2 \quad (\text{NS+}) \quad (5.11)$$

shows that at $\delta_R = +1$ the Neimark–Sacker boundary coincides *identically* with the fold boundary (both reduce to $\sigma_R = \tau_R$), while at $\delta_R = -1$ it coincides identically with the flip boundary (both reduce to $\sigma_R = -\tau_R$). In either case, two of the three boundaries that together bound the stability triangle collapse onto a single line, so that the enclosed area – and with it the entire stability triangle – is destroyed at exactly $|\delta_R| = 1$.

Figures 5.2 and 5.3 illustrate this collapse for $\delta_R = +0.9, +1.0$ and $\delta_R = -0.9, -1.0$, respectively. In both sequences, the stability triangle visibly shrinks and is completely destroyed exactly at $|\delta_R| = 1$, confirming that condition (5.1) marks a genuine structural limit of the stability region rather than merely a convenient modelling choice.

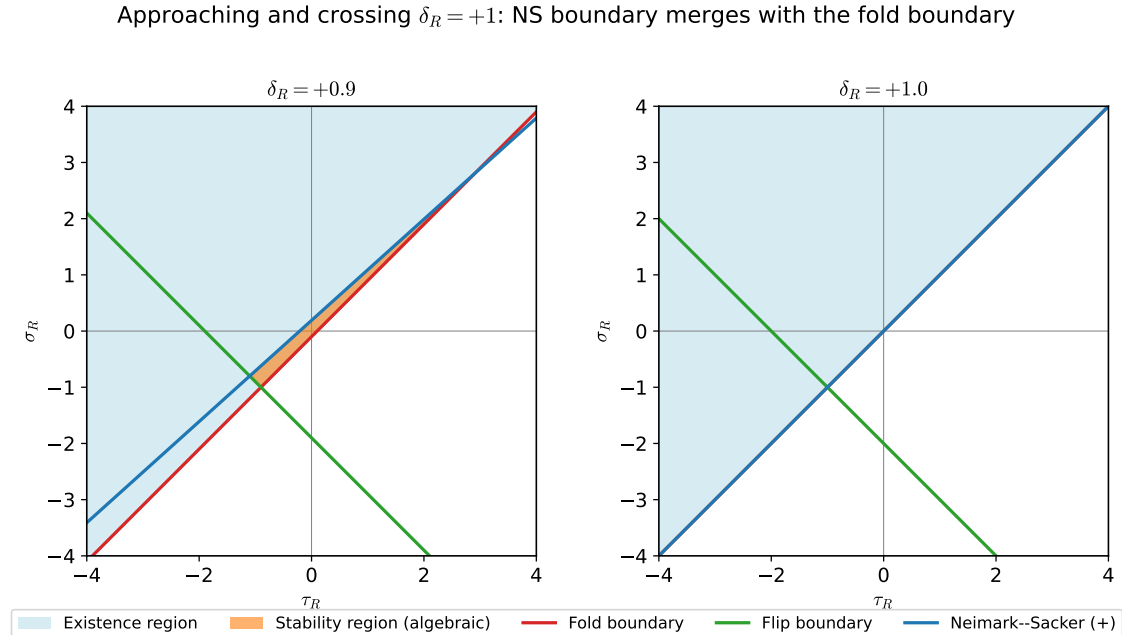


Figure 5.2: The stability triangle for $\delta_R = +0.9, +1.0$. The Neimark–Sacker boundary (blue) merges with the fold boundary (red) exactly at $\delta_R = +1$, destroying the stability triangle.

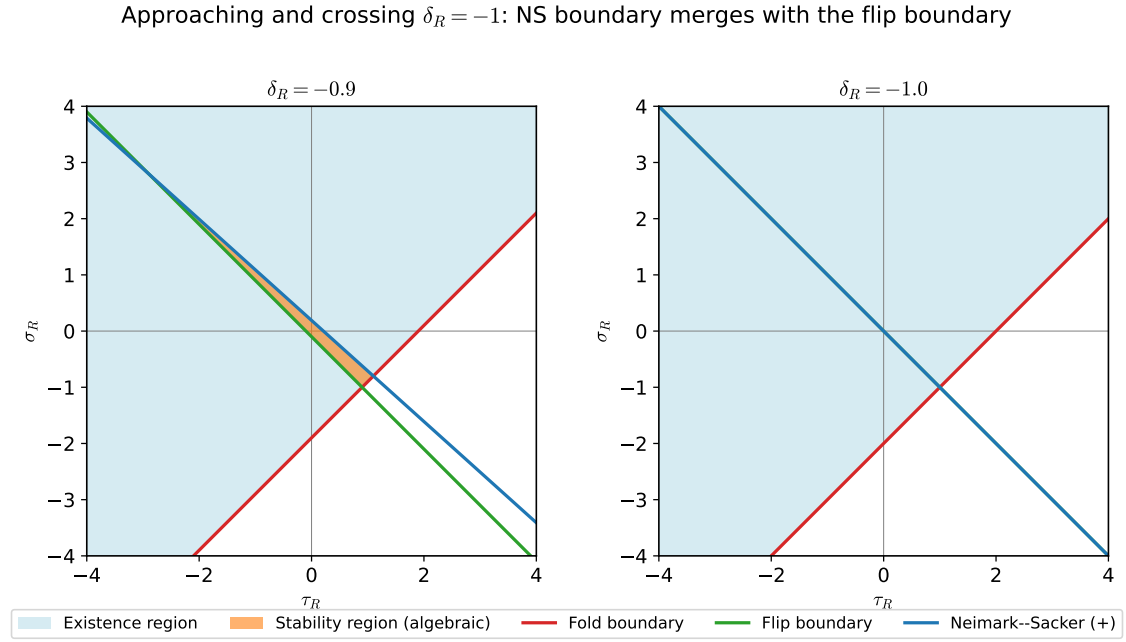


Figure 5.3: The stability triangle for $\delta_R = -0.9, -1.0$. The Neimark-Sacker boundary (blue) merges with the flip boundary (green) exactly at $\delta_R = -1$, destroying the stability triangle.

One further subtlety is worth noting before moving on: the shape of the existence region itself depends on the sign of μ . For $\mu > 0$, the existence region occupies the half-plane above the fold boundary (as shown in all figures above and used throughout this thesis), whereas for $\mu < 0$ it would instead occupy the half-plane below it (and vice versa for the L-parameters). This is precisely the reason why μ is fixed to $\mu = 1$ throughout the majority of this thesis: doing so fixes the existence region unambiguously to the one shown here, avoiding an additional sign-dependent case distinction.

Taken together, the four Jury conditions and their interplay are best appreciated by stepping back from the fixed- δ_R slices considered so far and viewing all three bifurcation conditions simultaneously, as surfaces in the full three-dimensional $(\tau_R, \sigma_R, \delta_R)$ parameter space. Figure 5.4 shows exactly this, following the visualization style of Banerjee. Because the saddle-node and period-doubling conditions are linear in $(\tau_R, \sigma_R, \delta_R)$, they form flat planes (a, b), whereas the Neimark-Sacker condition is quadratic in δ_R and therefore forms a genuinely curved, saddle-shaped surface (c).

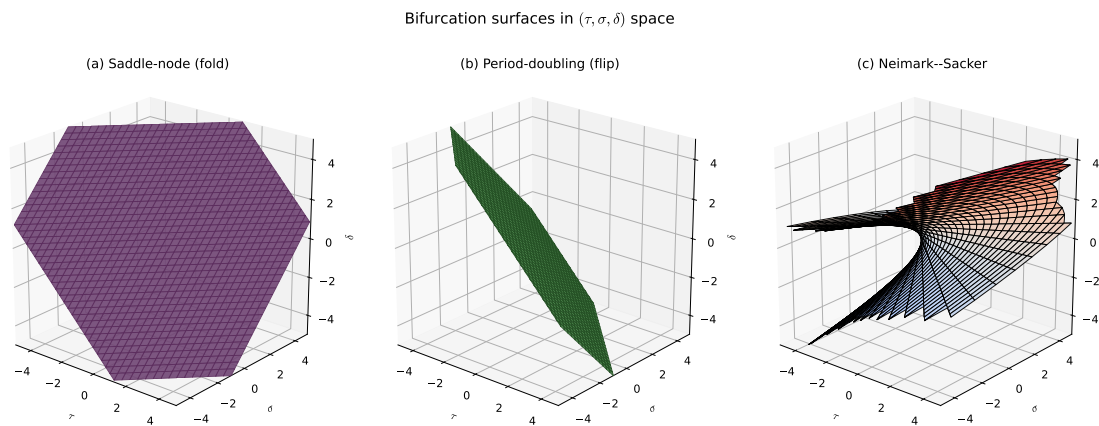


Figure 5.4: The saddle-node (fold), period-doubling (flip), and Neimark–Sacker bifurcation conditions, plotted as surfaces in the full three-dimensional $(\tau_R, \sigma_R, \delta_R)$ parameter space, following the visualization style of Banerjee. Because the saddle-node and period-doubling conditions are linear in $(\tau_R, \sigma_R, \delta_R)$, they form flat planes (a, b), whereas the Neimark–Sacker condition is quadratic in δ_R and therefore forms a genuinely curved, saddle-shaped surface (c).

This full three-dimensional picture also motivates why this thesis focuses primarily on the τ_R – σ_R plane rather than on the τ_R – δ_R or σ_R – δ_R planes: a fixed- δ_R slice cuts all three surfaces in straight lines (two exactly, one – the Neimark–Sacker surface – through a plane that intersects it in a line since the surface is only quadratic in δ_R and linear in τ_R, σ_R separately), reproducing the familiar and easily interpretable stability triangle discussed throughout this chapter. A fixed- τ_R or fixed- σ_R slice, by contrast, cuts through the curved Neimark–Sacker surface at a generic angle, and the resulting boundary in the remaining two-dimensional plane is a considerably less transparent, curved body rather than a simple polygon – offering none of the interpretive clarity of the stability triangle while still requiring the same underlying algebra.

5.1.2 Implications for the Investigations of ICCs

The figures presented in Section 5.1.1 show that in the $\delta_{L,R} \neq 0$ case, the stability triangle becomes skewed. Section 2.1.8 showed that a Neimark–Sacker bifurcation guarantees the existence of an ICC in close proximity beyond it, and that this guarantee holds even in the nonsmooth case. This makes the region immediately beyond the (now skewed) Neimark–Sacker line especially promising for locating ICCs for further investigation. However, since this line is skewed whenever $\delta_{L,R} \neq 0$, a coordinate transformation is required in order to scan a region that follows the line consistently:

$$\tau_R = \tau_{R,\text{param}} - \delta_R \sigma_{R,\text{param}}, \quad (5.12)$$

$$\sigma_R = (1 - \delta_R^2) + \delta_R \tau_{R,\text{param}} + \sigma_{R,\text{param}}. \quad (5.13)$$

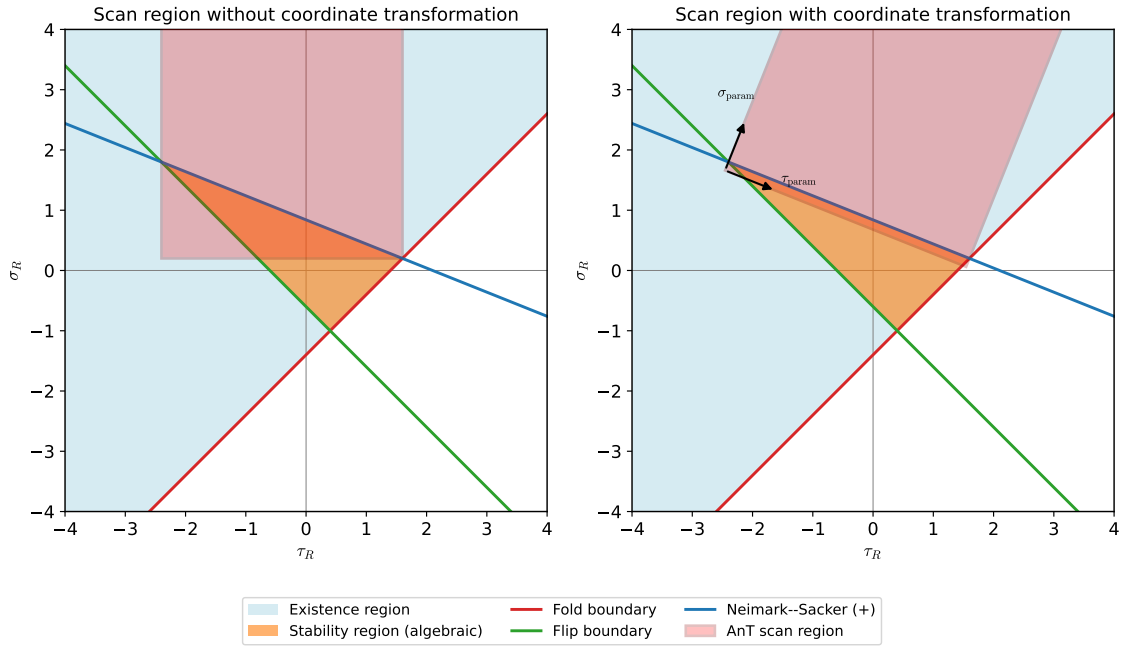


Figure 5.5: The AnT scan region sitting just above the Neimark–Sacker boundary (blue) of the stability triangle for the skewed $\delta_R = -0.4$ case. Left: without coordinate transformation. Right: with coordinate transformation.

The parameters used for the AnT scans presented below are summarized in Table 5.1; τ_R and σ_R are swept within the transformed scan region shown in Figure 5.5, while the remaining parameters are held fixed at the values given.

τ_L	-0.1
σ_L	0.2
δ_L	0.2
μ	1.0

Table 5.1: Fixed parameters used to produce Figure 5.6.

In Figure 5.6, each periodicity region in the bottom row is mapped to its corresponding stability triangle directly above it. The hypothesis to test here is that the $\tau_{R,\text{param}}\text{--}\sigma_{R,\text{param}}$ still yields meaningful results for $\delta_r \neq 0$. Notably, the stability triangles themselves are symmetric for equal distances from $\delta_R = 0$, whereas the periodicity regions look meaningful, but markedly different at those same distances. It is worth keeping in mind that each of these images is only a two-dimensional slice through a seven-dimensional parameter space, and can therefore be understood as a cross-section of a seven-dimensional "periodicity sphere". Interpreted this way, one might reasonably expect the surface of this parameter sphere to be smooth, and that small deviations in individual parameters would leave its overall geometry largely unchanged. Figure 5.6 shows, however, that the geometry changes noticeably even for parameter variations as small as those used here. This holds true for every parameter in general, though the sensitivity of the geometry to variation is, in general, specific to each individual parameter.

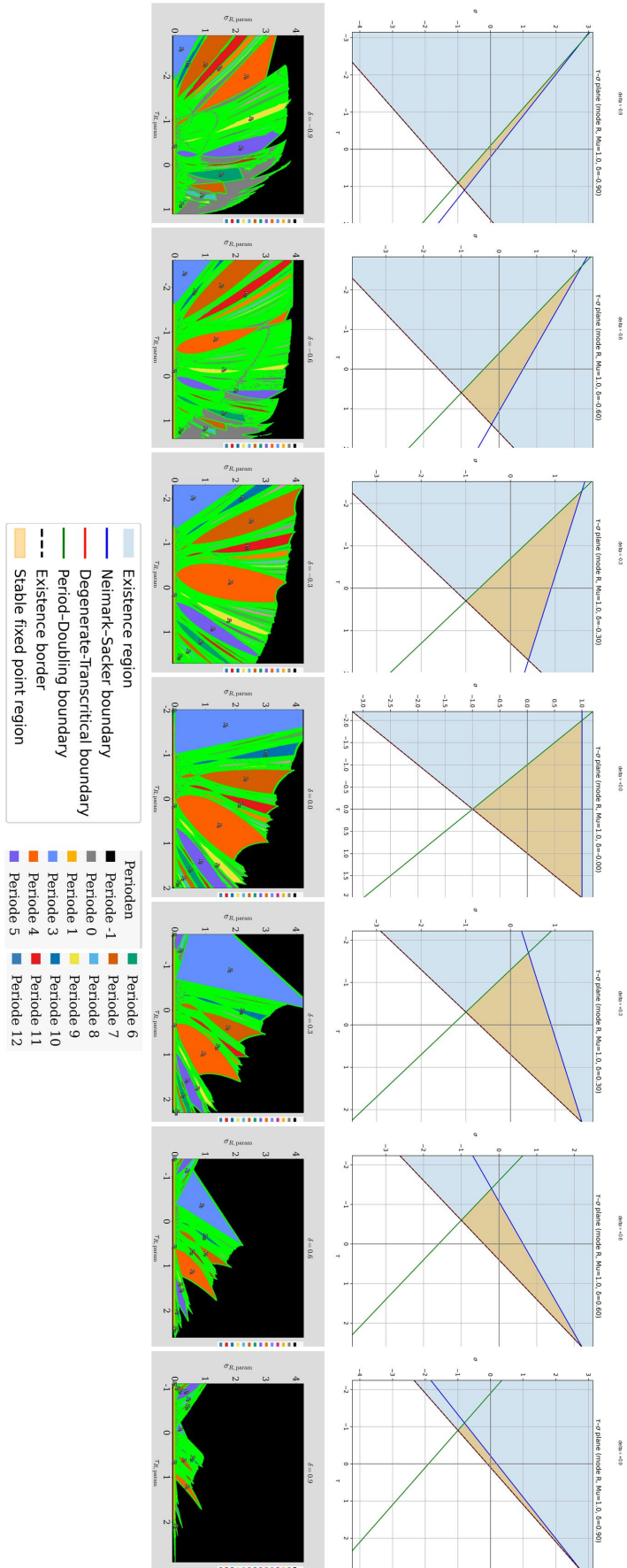


Figure 5.6: Matched evolution of the stability triangle (top row) and the corresponding periodicity-tongue structure (bottom row) as δ_R is swept symmetrically from -0.9 to $+0.9$ in steps of 0.3 .

Note that the experiment was originally conducted with a step size of 0.1 and the images presented in fig. 5.6 are compacted to a step size of 0.3. All the images for all the investigated parameter values are included in the appendix in section A.1.

It is precisely these results that motivate the choice of the $\tau_{R,\text{param}}-\sigma_{R,\text{param}}$ plane as the parameter space in which the bifurcations are investigated. Furthermore, the interval $(-1, 1)$ is chosen as the range over which δ is varied in the subsequent experiments. As will be shown in Section 5.4, it is entirely possible to consider values of δ outside this interval. However, doing so destroys the classical stability triangle interpretation. For this reason, values of δ outside $(-1, 1)$ are not considered elsewhere in this thesis, with the sole exception of the example presented in that section.

5.2 Results of Classification via Shotgun Exploration

There are only three distinct eigenvalue configurations in which ICCs can occur:

1. Both the node and the saddle have only real eigenvalues.
2. Either the node or the saddle cycle has a complex-conjugate pair of eigenvalues, while the other has only real eigenvalues.
3. Both the node and the saddle have a complex-conjugate pair of eigenvalues.

As described in Section 4.3.1, the objective of this specific investigation is to assess the geometric complexity of ICCs in the piecewise-linear setting. The underlying hypothesis is that piecewise-linear ICCs exhibit a substantially greater diversity of geometric structures than their smooth counterparts and that the complexity or non-complexity of the eigenvalues of the participating cycles have only minor influence on the overall shape of a piecewise-smooth ICC.

To test this hypothesis, the observations presented in Section 5.1.2 were used to guide the range selection, modification, and analysis of the right-hand-side parameters. As in the previous investigations, two-dimensional bifurcation diagrams were computed using *AnT* (introduced in Section 2.2.1), while Arnold tongues were explored through the random *shotgun exploration* approach described in Section 4.3.1.

The results of the *shotgun exploration* indicate, however, that the specific eigenvalue configuration has only a minor influence on the overall appearance, or “tameness”, of an ICC. In the following sections, a non-exhaustive collection of archetypes identified during the parameter exploration is presented, along with brief descriptions of the characteristic features that distinguish each case.

In several instances, visually similar structures were found across different eigenvalue configurations. This is expected, since the one-dimensional manifold connecting the two cycles must alternate as $\cdots \rightarrow r \rightarrow R \rightarrow r \rightarrow R \rightarrow \cdots$ along the ICC, regardless of whether the remaining (radial and transversal) eigenvalues happen to be real or complex. Such cases were deliberately included nonetheless, since they demonstrate that the qualitative shape of an archetype is indeed unaffected by this choice, as predicted.

τ_L	-0.1	$\tau_{R,(\text{param})}$	<i>varied</i>
σ_L	0.2	$\sigma_{R,(\text{param})}$	<i>varied</i>
δ_L	0.2	δ_R	<i>varied</i>
μ		1.0	

Table 5.2: Parameters used to produce Figures in section 5.2.

Note: The τ_R or $(\tau_{R,\text{param}})$ and σ_R or $(\sigma_{R,\text{param}})$ parameters are rounded to two digits. The same is true for the eigenvalues in the eigenvalue-tables below the figures.

5.2.1 Tame Geometries

This first group collects archetypes whose ICC remains close to an ideal simple geometry: a simple, non-self-intersecting closed curve with no/few visible anomalies, differing from the classical invariant circle. Note that even if something appears circular it is composed of straight connecting segments rather than a smooth arc, since this is a piecewise-linear map. It appears to be located in a single plane.

5.2.1.1 Circle

The node and saddle cycles are interleaved so uniformly around the ICC that, at first glance, the resulting polygon is almost indistinguishable from a smooth circle. The two connecting manifold branches form a single, gently curved loop without majorly visible kinks or self-intersections. Moreover, the ICC appears to lie almost entirely within one plane.

Figure 5.7 illustrates that the vast majority of the cycle points are mapped by the right-hand-side map branch, while only a small number are mapped by the left-hand-side map. Consequently, the example shown is located almost entirely within a single plane, with only the upper section exhibiting slight deviations from planarity. The ICC shown in Figure 5.7 also lies close to a border-collision fold (BC-fold) bifurcation, as the constituent cycles are in close proximity and could annihilate on the switching manifold under a small parameter perturbation. A further observation can be made by inspecting Table 5.3: the eigenvalue λ_3 is remarkably close to zero – close enough, in fact, to round to zero at the displayed precision.

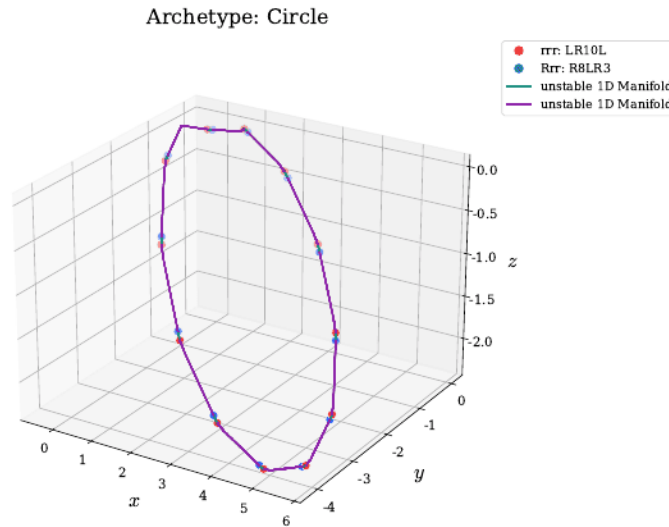


Figure 5.7: Period-12 ICC of the Circle archetype, found at $\delta_R = -0.4$, $\tau_R = 1.38$ ($\tau_{R,param} = 1.36$), $\sigma_R = 0.34$ ($\sigma_{R,param} = 0.04$).

Cycle	λ_1	λ_2	λ_3
rrr	0.39	-0.12	-0.00
Rrr	1.38	-0.06	0.00

Table 5.3: Eigenvalues of the cycles composing the *Circle* archetype ICC.

5.2.1.2 Polygon

Similar to the Circle archetype, 5.8 shows an ICC which mostly lies in one plane, with a second section of three circle pints being into an other plane. The two kink points can be interpreted as the intersection of those planes. An other similarity between those two archetypes is that the two respective cycle points are also quite close to each other and therefore in the vicinity of a fold bifurcation. The cycles themselves however are much more apart from the switching manifold.

And also table 5.4 tells a familiar story to the circle archetype: some of the eigenvalues are remarkably close to zero. (less than 10% away from the center of the unit circle)

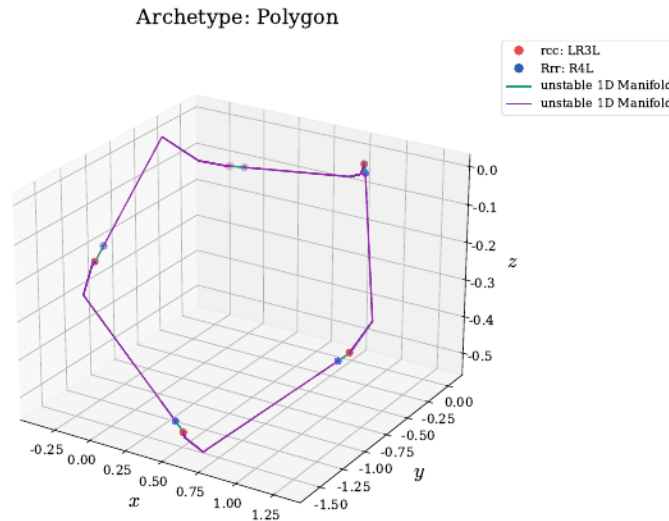


Figure 5.8: Period-5 ICC of the Polygon archetype, found at $\delta_R = -0.4$, $\tau_R = 0.36$ ($\tau_{R,param} = 0.25$), $\sigma_R = 1.02$ ($\sigma_{R,param} = 0.28$).

Cycle	λ_1	λ_2	λ_3
rcc	-0.06	$0.15 + 0.15i$	$0.15 - 0.15i$
Rrr	1.70	0.04	0.08

Table 5.4: Eigenvalues of the cycles composing the *Polygon* archetype ICC.

5.2.1.3 One-Kink

Also closely resembling the textbook circle is the one-kink archetype shown in Figure 5.9. As in the previous examples, the majority of cycle points are mapped by the right-hand-side branch of the BCNF, causing most of the ICC to lie within a single plane.

In this case, however, the parameter δ_R is chosen to be exactly 0.0, corresponding to the horizontal special case of the stability triangle. As discussed in Section 2.1.2, this choice simplifies the characteristic polynomial to

$$p(\lambda) = \lambda(\lambda^2 - \tau_R\lambda + \sigma_R),$$

implying that one eigenvalue is exactly zero rather than merely close to zero (see Table 5.5).

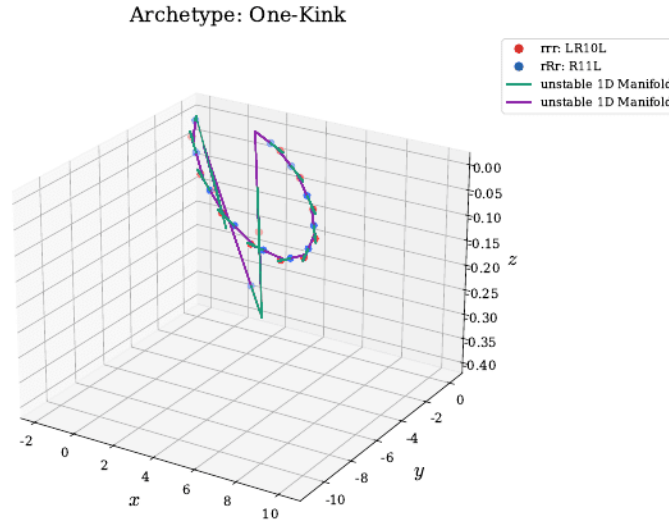


Figure 5.9: Period-12 ICC of the One-Kink archetype, found at $\delta_R = 0.0$, $\tau_R = 1.85$ ($\tau_{R,param} = 1.85$), $\sigma_R = 1.08$ ($\sigma_{R,param} = 0.08$).

Cycle	λ_1	λ_2	λ_3
rrr	-0.12	-0.50	0.00
rRr	-0.10	3.42	0.00

Table 5.5: Eigenvalues of the cycles composing the *One-Kink* archetype ICC.

5.2.1.3.1 A degenerate parameter as the origin of the single kink. The pronounced kink that interrupts an otherwise smooth, near-planar loop in the *One Kink* archetype is a direct, exact consequence of a degenerate model parameter rather than a generic geometric feature. For this parameter set, $d_R = 0$, and since

the z -component of the map is given by

$$z_{n+1} = \begin{cases} d_L x_n, & x_n < 0, \\ d_R x_n, & x_n \geq 0, \end{cases} \quad (5.14)$$

setting $d_R = 0$ means that *every* point mapped from the right half of phase space ($x_n \geq 0$) is sent to $z_{n+1} = 0$, irrespective of its incoming z -value. Equivalently, the Jacobian on the right side is singular, which is also reflected in the condition number of the period-map Jacobian being numerically infinite for both cycles involved.

This is confirmed directly in the periodic-point data: of the twelve points making up the **rrr** cycle, nine lie on the right side ($x > 0$) and all nine have $z = 0$ to within floating-point precision. Only the two points that briefly cross into the left half ($x < 0$, where $d_L = 0.2 \neq 0$) acquire a nonzero z -coordinate – e.g. the point at $x = -0.7279$ maps to $z = d_L \cdot (-0.7279) = -0.1456$, matching the data exactly – before z is immediately reset to 0 as soon as the orbit re-enters the right half. The same signature appears in the manifold data: 83–84 % of all points on both branches of the unstable 1D manifold have $z \approx 0$, with the remaining points exhibiting nonzero z concentrated almost exclusively in a narrow band around the switching manifold ($x \in [-2, 1.3]$).

The large, unremarkable circular arc therefore corresponds to the long stretch of the orbit confined to the right half, where the singular map collapses all trajectories onto the plane $z = 0$; the single kink is the brief window, during the short passage through the regular left half, in which the third dimension is momentarily reintroduced before being erased again on the very next step.

5.2.2 Well-behaved Geometries

This second group collects archetypes whose ICC is no longer a simple skewed planar or bi-planar polygonal loop in phase space, but instead develops a distinctive, repeating local motif at each saddle-node connection. A variety of spikes, curls, or waves can be observed, while the ICC remains fully organized and free of confusing substructure.

5.2.2.1 Sticks

In some parts around the stable cycle points the connecting manifold branches run back and forth in thin, parallel straight segments between each saddle-node pair, giving the ICC partly the appearance of a loose arrangement of disconnected sticks rather than a continuous closed curve as can be seen in figure 5.10. A close up around such a section is given in 5.11 to see the straight lines running in parallel better.

As with the previous archetypes, Table 5.6 shows λ_3 sitting remarkably close to zero for both involved cycles.

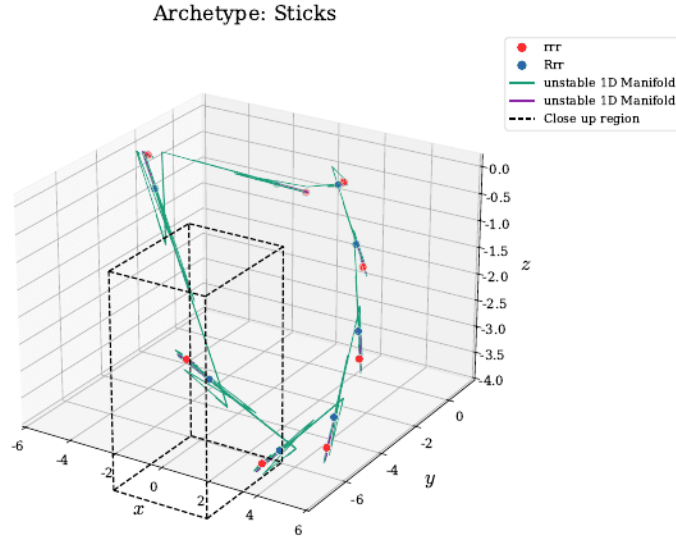


Figure 5.10: Period-8 ICC of the Sticks archetype, found at $\delta_R = -0.7$, $\tau_R = 1.63$ ($\tau_{R,\text{param}} = 1.00$), $\sigma_R = 0.70$ ($\sigma_{R,\text{param}} = 0.89$).

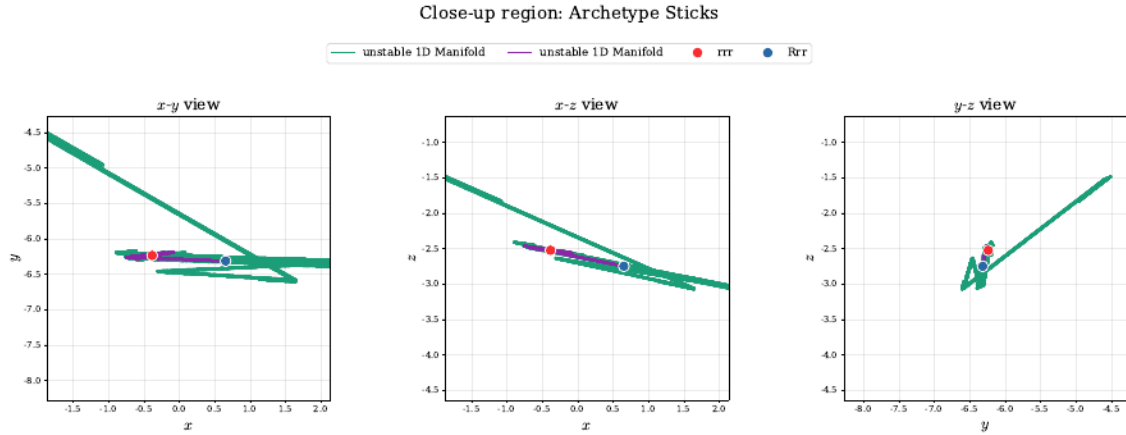


Figure 5.11: Close-up of the phase-space region highlighted by the dashed cube in the Sticks archetype (fig. 5.10), shown as three orthogonal projections (x - y , x - z , y - z). The cube contains exactly one point of the stable period-8 cycle **rrr** (red) and one point of the unstable period-8 cycle **Rrr** (blue); the teal and purple point clouds are the sections of the two unstable 1D manifold branches that pass through this region, repeatedly running back and forth in near-parallel segments before continuing on to the neighbouring cycle points.

Cycle	λ_1	λ_2	λ_3
Rrr	2.16	-0.29	-0.01
rrr	-0.64	-0.55	-0.00

Table 5.6: Eigenvalues of the cycles composing the *Sticks* archetype ICC.

5.2.2.1.1 Verification of the Sticks ICC. Since the close up pictures in fig. 5.11 are quite hard to read and therefore no easy visual verification can be done, there was a separate numerical verification. Both cycles were confirmed to reproduce their tabulated points to machine precision. rrr has eigenvalues $[-0.637, -0.551, -0.0038]$ – all real with $|\lambda| < 1$, making it a genuine stable *node* – while Rrr is a saddle with a strong real unstable eigenvalue ($\lambda = 2.155$). Tracing both branches of the saddle’s unstable manifold shows both being captured by the basin of this node, though to differing degrees of resolution: one branch of the unstable 1D manifold of the saddle has fully locked onto the periodic orbit, its trailing points cycling exactly and repeatedly through the index sequence 4, 5, 6, 7, 0, 1, 2, 3, . . . with residuals consistently below 0.005; the other branch, computed over roughly two orders of magnitude more iterations, shows a clearly decreasing mean distance to the same node cycle over its trailing blocks (from ≈ 0.59 down to ≈ 0.45) and remains confined to the same spatial scale as the node throughout, but had not yet reached the same tight resolution by the point the computation was terminated. Since a branch once captured by a stable periodic orbit necessarily traces out its full cycle under continued iteration, both branches jointly contribute to closing the loop around the entire 8-cycle, confirming that this parameter location produces a genuine ICC.

5.2.2.2 Curly

A particularly striking example is the large ICC shown in Figure 5.12, which occupies a phase-space cube with a diagonal of approximately 85.23. In this case, the one-dimensional manifold spirals toward the focus within the plane spanned by the complex-conjugate eigenvalues, giving rise to a pronounced curling of the connecting branches. Owing to the large extent of the ICC, the manifold follows long, straight segments while bending toward the focus, making the spiral structure particularly prominent.

Once again, one eigenvalue of each constituent cycle is remarkably close to zero. As shown in Table 5.7, these are $\lambda_{2,3}$ for the node and saddle cycle, respectively.

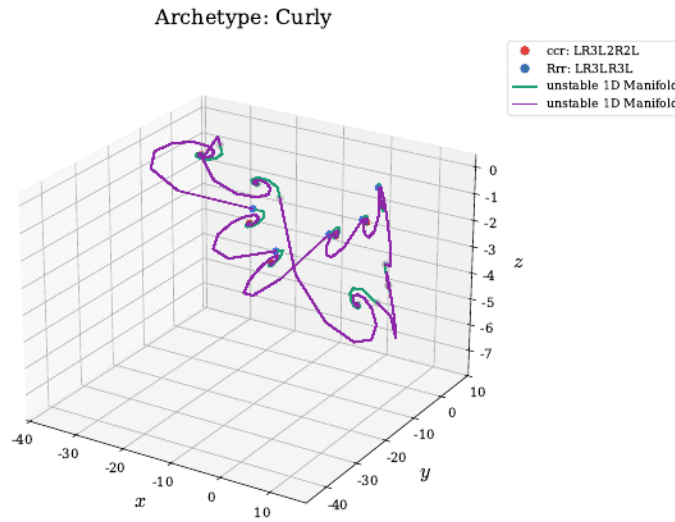


Figure 5.12: Period-9 ICC of the Curly archetype, found at $\delta_R = -0.1$, $\tau_R = 1.69$ ($\tau_{R,\text{param}} = 1.47$), $\sigma_R = 3.07$ ($\sigma_{R,\text{param}} = 2.23$).

Cycle	λ_1	λ_2	λ_3
ccr	$0.60 + 0.45i$	$0.60 - 0.45i$	-0.00
Rrr	7.73	0.00	0.44

Table 5.7: Eigenvalues of the cycles composing the *Curly* archetype ICC.

5.2.2.3 Spiky

The ICC shown in Figure 5.13 is characterized by manifold branches that connect consecutive cycle points through sharp, narrow zigzags rather than smooth or gently curved appearing segments. As a result, the ICC exhibits a jagged, spiky appearance while still forming a single closed loop. This example is particularly noteworthy because, as in the circle and polygon archetypes, only three cycle points in total are mapped by the left-hand-side branch of the BCNF. In contrast to those archetypes,

however, the resulting kink is propagated throughout the entire ICC, producing a genuinely three-dimensional structure rather than one that lies predominantly within two single planes.

Table 5.8 once again reveals the familiar pattern of one eigenvalue per constituent cycle being remarkably close to zero. Unlike the previously discussed archetypes, however, approximately half of the eigenvalues are negative. Since the ICC does not exhibit pronounced spiral structures, these negative eigenvalues are associated only with the radial and transversal directions rather than with the ICC constructing tangent dynamics.

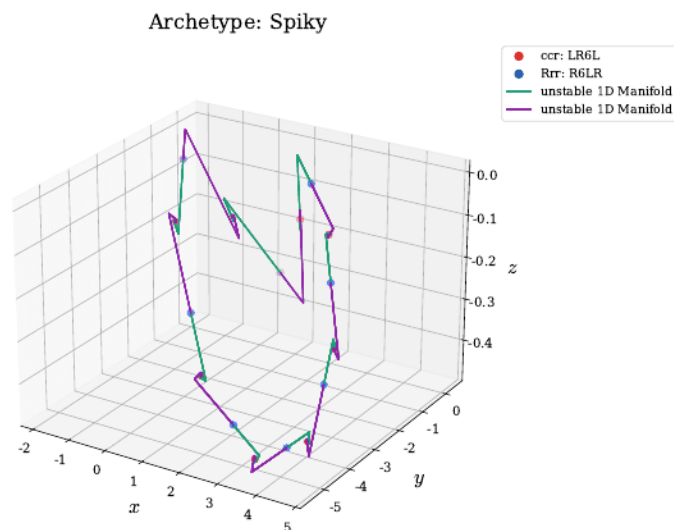


Figure 5.13: Period-8 ICC of the Spiky archetype, found at $\delta_R = -0.1$, $\tau_R = 1.62$ ($\tau_{R,param} = 1.59$), $\sigma_R = 1.09$ ($\sigma_{R,param} = 0.26$).

Cycle	λ_1	λ_2	λ_3
ccr	$-0.15 + 0.20i$	$-0.15 - 0.20i$	0.00
Rrr	3.70	0.00	-0.09

Table 5.8: Eigenvalues of the cycles composing the *Spiky* archetype ICC.

5.2.2.4 Curly-Spiky-Mix

Similar to the Spiky archetype, the ICC shown in Figure 5.14 exhibits sharp kink points along its manifold connections. As in both the Curly and the Spiky archetype, the connection is formed between a saddle and a focus. In this case, however, the unstable manifold spirals toward the focus within the plane spanned by the complex-conjugate eigenvalues like in the Curly archetype. As a consequence, the manifold crosses the switching manifold several times during a single connection, producing multiple kinks per connection instead of just one like in the Spiky archetype.

A comparison of its eigenvalue structure with the Spiky archetype one further reveals a striking similarity. As shown in Table 5.9, the same two eigenvalues are again remarkably close to zero, and the same three eigenvalues are negative. Unlike the Spiky archetype, however, the spiraling behavior of the unstable manifold directly involves the complex-conjugate eigenvalue pair. Consequently, the negative eigenvalues are no longer confined to the radial and transversal directions but instead become an integral part of the constituent dynamics that shape the ICC geometry.

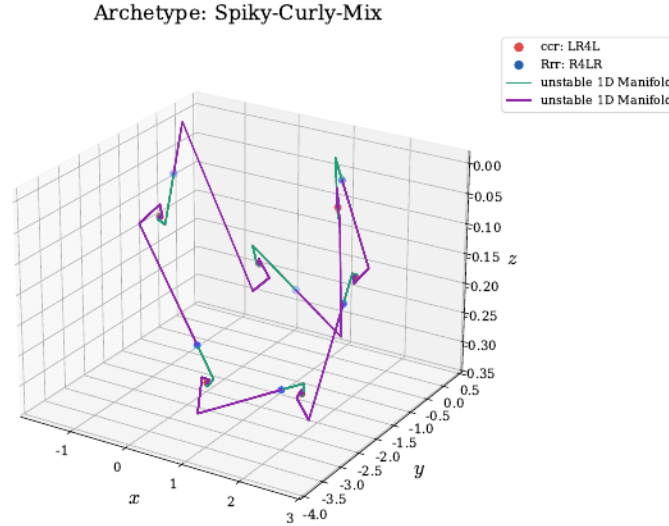


Figure 5.14: Period-6 ICC of the Curly-Spiky-Mix archetype, found at $\delta_R = -0.1$, $\tau_R = 1.30$ ($\tau_{R,\text{param}} = 1.25$), $\sigma_R = 1.30$ ($\sigma_{R,\text{param}} = 0.43$).

Cycle	λ_1	λ_2	λ_3
ccr	$-0.06 + 0.35i$	$-0.06 - 0.35i$	0.00
Rrr	3.58	0.00	-0.01

Table 5.9: Eigenvalues of the cycles composing the *Spike-Curl-Mix* archetype ICC.

5.2.2.5 Rubberband

Figure 5.15 presents another archetype closely related to the Spiky configuration. The key difference is the substantially larger kink angle, which gives rise to a smooth, wave-like appearance rather than the sharp, pointed spikes observed previously. In contrast to the tame geometries, the ICC no longer resembles a planar object but instead occupies a gently curved, well-behaved half-pipe-shaped region of phase space.

The eigenvalue structure follows the familiar pattern observed in the preceding examples. As shown in Table 5.10, two eigenvalues are again remarkably close to zero, with one of them approaching zero from the negative side.

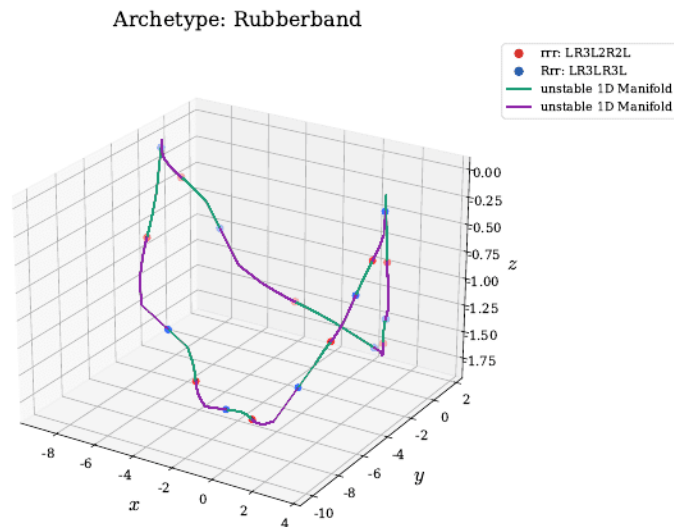


Figure 5.15: Period-9 ICC of the Rubberband archetype, found at $\delta_R = -0.4$, $\tau_R = 1.30$ ($\tau_{R,param} = 0.46$), $\sigma_R = 2.74$ ($\sigma_{R,param} = 2.09$).

Cycle	λ_1	λ_2	λ_3
rrr	0.46	0.29	-0.00
Rrr	8.48	0.00	0.18

Table 5.10: Eigenvalues of the cycles composing the *Rubberband* archetype ICC.

5.2.2.5.1 Border proximity as the origin of sharp corners versus smooth looking curvature. The qualitative difference in appearance between for example the *Curly Spike Mix* and the *Rubberband* archetypes can be traced directly back to a single geometric quantity: the distance of the closest periodic point of either cycle to the switching manifold $x = 0$. As long as a trajectory remains on one side of the border, the BCNF acts there exactly affinely, and an affine map sends straight lines to straight lines. Consequently, every segment of an ICC or manifold branch between two consecutive border crossings is not merely curve-like but an exact, straight line segment in $\mathbb{R}^3$; what visually appears as a curved trajectory is in reality a polygonal path assembled from such segments, with a kink occurring at every point where the border is crossed.

The length of these individual straight segments is governed by how close the relevant part of the invariant object passes to the border: the closer a trajectory approaches $x = 0$, the sooner it crosses back into the other linear region, and the shorter the resulting segment. For *Curly Spike Mix*, the closest periodic point to the border lies at $|x| \approx 0.207$, only $\approx 5.4\%$ of the total extent of the two involved cycles ($x \in [-1.32, 2.47]$) – points in this region therefore travel a comparatively long distance before the next border crossing, producing few, long, sharply angled segments and hence the visually pronounced spikes and edges. For *Rubberband*, by contrast, the

closest periodic point lies at $|x| \approx 0.107$, but relative to the much larger extent of this configuration ($x \in [-8.92, 3.53]$) this amounts to only $\approx 0.86\%$ of the total range – nearly an order of magnitude closer in relative terms. Trajectories near this point cross the border almost immediately, yielding many short segments with only small changes in direction between them. The resulting polygonal path therefore approximates a smooth curve much more closely, in the same way that a polygon with many short sides approximates a circle far better than one with few long sides. Beyond segment length, a second geometric factor shapes the visual sharpness of a kink: the angle at which consecutive straight segments meet at a border crossing. Since the manifold is straight on each side of $x = 0$ and the map is continuous but non-differentiable across the border, the turning angle at a given kink is fixed by the local Jacobians J_L and J_R acting on the incoming direction, independent of segment length. The number of distinct, recurring turning angles per period is likewise a combinatorial property: it equals twice the number of separate L -blocks in the itinerary of the organizing cycle, since each such block contributes one entry crossing and one exit crossing with its own local geometry. For *Curly Spike Mix* for example, the manifold converges onto P6-0-ccr, whose itinerary L^2R^4 contains a single L -block and therefore yields exactly two recurring turning angles, 135.63° and 157.92° , i.e. deviations of 44.4° and 22.1° from a straight line (180°). For *Rubberband*, the manifold instead remains organized around P9-1-Rrr, whose itinerary $L^1R^3L^1R^3$ splits into two separate L -blocks and consequently produces four recurring turning angles, 141.78° , 151.96° , 154.99° , and 171.54° , i.e. deviations of only 38.2° , 28.0° , 25.0° , and 8.5° . Even the sharpest kink occurring in *Rubberband* (141.78°) is softer than the sharpest kink in *Curly Spike Mix* (135.63°), and its softest kink is nearly indistinguishable from a straight line. The angle of attack against the border is therefore consistently softer throughout *Rubberband*, reinforcing – independently of the segment-length effect discussed above – the visual impression of a smooth curve rather than a faceted, spike-like one.

5.2.2.6 Wavy

Visually identical to the Rubberband archetype as fig. 5.16 shows, but listed separately here because it is formed by an rcc/Rrr connection instead of an rrr/Rrr connection, proving its existence even if the radial and transversal eigendirections become complex.

Table 5.11 is strikingly similar to the eigenvalue table of archetype *rubberband*, with again two eigenvalues being very close to zero and one of them approaching zero from the negative side.

Besides that, everything said about the appearance of *rubberband* archetype in 5.2.2.5.1 also applies to the *wavy* archetype as well.

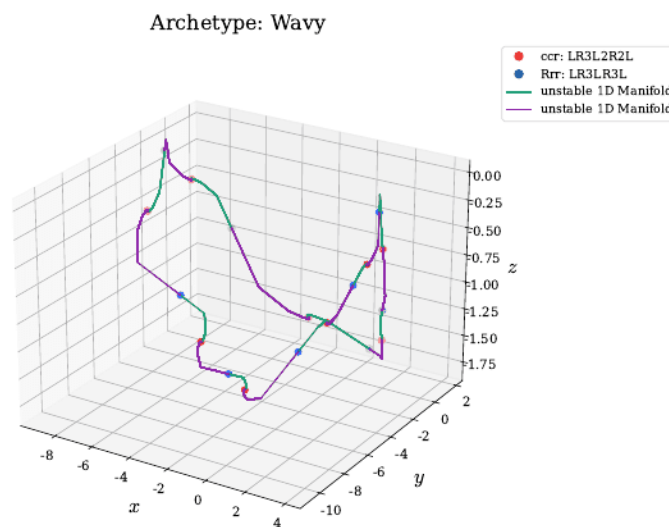


Figure 5.16: Period-9 ICC of the Wavy archetype, found at $\delta_R = -0.3$, $\tau_R = 1.32$ ($\tau_{R,param} = 0.73$), $\sigma_R = 2.64$ ($\sigma_{R,param} = 1.95$).

Cycle	λ_1	λ_2	λ_3
ccr	$0.32 + 0.24i$	$0.32 - 0.24i$	-0.00
Rrr	6.97	0.18	0.00

Table 5.11: Eigenvalues of the cycles composing the *Wavy* archetype ICC.

5.2.3 Non-Standard Geometries

This final group collects archetypes whose ICC geometries depart qualitatively from the well-behaved cases above: rather than a single, easily identifiable repeating motif, one or both connecting manifold branches develop a genuinely dense, hard-to-distinguish substructure at some or all of the foci on the ICC.

5.2.3.1 Black Hole

This is one of the archetypes where the definition of an ICC gets a little overstretched.

The organizing focus P6-1-ccr is distinguished from the foci encountered in the previous archetypes by how close its complex-conjugate eigenvalue pair lies to the unit circle (See Tab. 5.12). From the eigenvalues $\lambda_{1,2} = 0.55534719 \pm 0.82478633i$ of the period-6 Floquet matrix,

$$|\lambda_{1,2}| = 0.994325, \quad 1 - |\lambda_{1,2}| = 0.005675,$$

so the pair sits only 0.57% inside the unit circle – the cycle is stable, but only barely so, placing the system close to a Neimark–Sacker threshold. The associated rotation per period is $\arg(\lambda_{1,2}) \approx 56.0^\circ$.

Since the local contraction rate toward the focus is exactly $|\lambda_{1,2}|$ per period, the distance to P6-1-ccr shrinks by only 0.57% per period. The number of periods required to reduce this distance by a given factor k follows from $n = \ln(k)/\ln(1/|\lambda_{1,2}|)$: halving the distance already requires ≈ 122 periods (≈ 732 iterations), a tenfold reduction ≈ 405 periods (≈ 2428 iterations), and a hundredfold reduction ≈ 809 periods (≈ 4855 iterations). This is orders of magnitude slower than, for instance, the focus organizing *Curly Spike Mix* ($|\lambda_{1,2}| \approx 0.352$), where a single period already reduces the distance to roughly a third.

This weak attraction is directly observable in Figure 5.17, where the unstable manifold of the saddle approaches the stable focus in a manner reminiscent of matter spiraling toward a black hole and forming an accretion disk. This characteristic geometry motivates the name of the archetype, *Black Hole*. At each saddle–focus connection, one manifold branch surrounds the respective focus in a similar fashion, giving the ICC the appearance of a series of small black holes strung along its path rather than a sequence of points connected by a single clean arc. It is worth noting, however, that the apparent failure of the manifold to reach the focus is purely a consequence of the finite number of iterations shown. With sufficiently many iterations and computational resources, the manifold would eventually be seen to collapse into the focus, thereby satisfying the condition that the ICC connects its focus and saddle points.

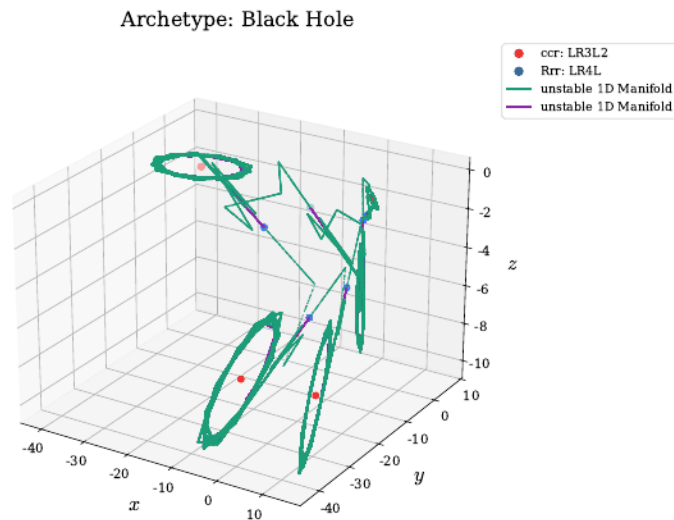


Figure 5.17: Period-6 ICC of the Black Hole archetype, found at $\delta_R = -0.7$, $\tau_R = 2.34$ ($\tau_{R,\text{param}} = 0.73$), $\sigma_R = 2.31$ ($\sigma_{R,\text{param}} = 2.31$).

Cycle	λ_1	λ_2	λ_3
Rrr	1.81	-0.27	-0.02
ccr	$0.56 + 0.82i$	$0.56 - 0.82i$	-0.00

Table 5.12: Eigenvalues of the cycles composing the *Black Hole* archetype ICC.

5.2.3.2 Double Manifold Swirls

Closely related to the *black hole* archetype is this archetype shown in 5.18. Both connecting unstable manifold branches simultaneously develop a dense, tightly wound swirl at every stable focus. However, in contrast to the marginally stable focus organizing the Black Hole archetype, the period-8 focus underlying *Double Manifold Swirl* possesses a complex-conjugate eigenvalue pair that lies substantially further from the unit circle (See Tab. 5.13). From the eigenvalues $\lambda_{1,2} = -0.690973 \pm 0.591975i$ of the period-8 Floquet matrix,

$$|\lambda_{1,2}| = 0.909878, \quad 1 - |\lambda_{1,2}| = 0.090122,$$

so the pair sits 9.01% inside the unit circle – roughly sixteen times further from marginal stability than the 0.57% found for the Black Hole period-6 focus. The associated rotation per period is also markedly larger, $\arg(\lambda_{1,2}) \approx 139.4^\circ$ compared to 56.0° , so the manifold advances nearly three-quarters of a full revolution per period rather than only one-sixth.

Since the local contraction rate toward the focus is exactly $|\lambda_{1,2}|$ per period, this directly translates into a proportionally faster inward spiral: halving the distance

to the focus points requires only ≈ 7.3 periods (≈ 59 iterations), a tenfold reduction ≈ 24.4 periods (≈ 195 iterations), and a hundredfold reduction ≈ 48.8 periods (≈ 390 iterations) – about sixteen times faster, per reduction factor, than the corresponding figures for Black Hole. The markedly larger distance from the unit circle thus accounts both for the visibly tighter, faster-converging spiral and for the stronger rotational component that gives the *Double Manifold Swirl* archetype its name.

Figure 5.19 and Table 5.14 show the same archetype again produced by an entirely complex saddle-focus connection, proving that even introducing a second pair of complex conjugate eigenvalues still can produce the same archetype.

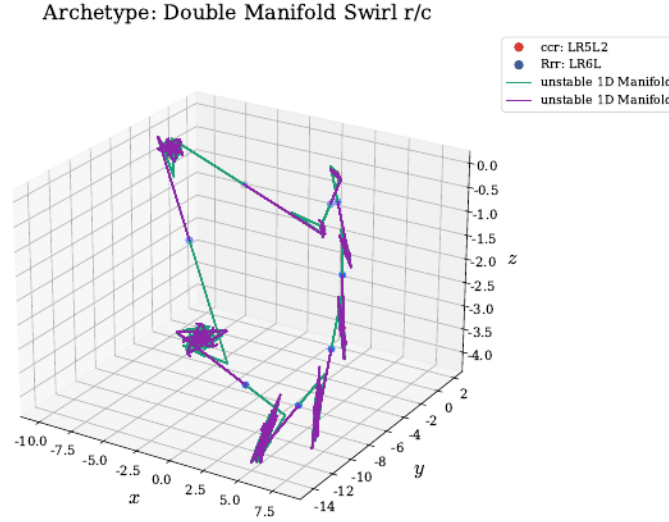


Figure 5.18: Period-8 ICC of the Double Manifold Swirls archetype (real saddle variant), found at $\delta_R = -0.5$, $\tau_R = 2.01$ ($\tau_{R,param} = 1.41$), $\sigma_R = 1.26$ ($\sigma_{R,param} = 1.21$).

Cycle	λ_1	λ_2	λ_3
Rrr	2.04	-0.35	-0.00
ccr	$-0.69 + 0.59i$	$-0.69 - 0.59i$	-0.00

Table 5.13: Eigenvalues of the cycles composing the *Double Manifold Swirl rc* archetype ICC.

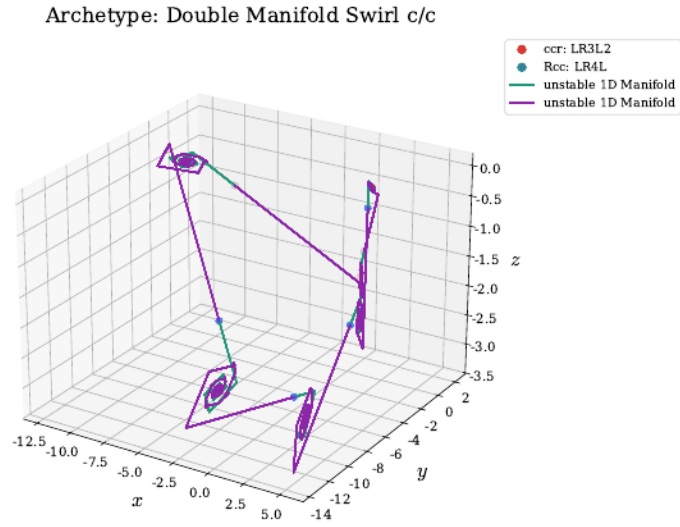


Figure 5.19: Period-6 ICC of the Double Manifold Swirls archetype (complex saddle variant), found at $\delta_R = -0.6$, $\tau_R = 2.00$ ($\tau_{R,\text{param}} = 0.85$), $\sigma_R = 2.05$ ($\sigma_{R,\text{param}} = 1.92$).

Cycle	λ_1	λ_2	λ_3
Rcc	1.72	$-0.05 + 0.01i$	$-0.05 - 0.01i$
ccr	$-0.01 + 0.82i$	$-0.01 - 0.82i$	-0.00

Table 5.14: Eigenvalues of the cycles composing the *Double Manifold Swirl cc* archetype ICC.

5.2.3.3 Needle

A saddle-focus connection in which both branches of the manifold enter the focus in close proximity and at roughly the same angle of incidence as one can see in figure 5.20. Both branches remain close together for some time, giving the ICC a needle- or thorn-shaped appearance.

Again the eigenvalue table 5.15 shows both λ_3 eigenvalues remarkable close to zero, with one of them approaching from the negative side.

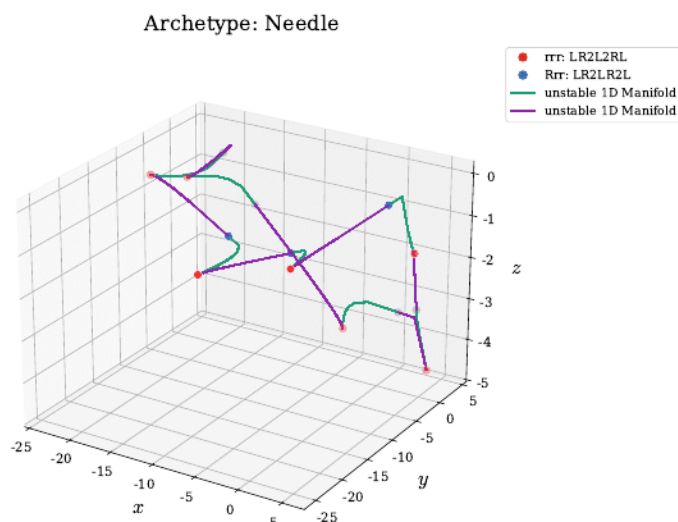


Figure 5.20: Period-7 ICC of the Needle archetype, found at $\delta_R = -0.1$, $\tau_R = -0.62$ ($\tau_{R,param} = -0.93$), $\sigma_R = 4.19$ ($\sigma_{R,param} = 3.11$).

Cycle	λ_1	λ_2	λ_3
rrr	0.71	0.49	-0.00
Rrr	6.97	0.69	0.00

Table 5.15: Eigenvalues of the cycles composing the *Needle* archetype ICC.

5.2.3.4 Star

Figure 5.21 shows a spectacular extreme case of the Needle archetype, in which the two manifold branches run side by side for most of their length entering the node points from the same side, instead of running through it. The two cycles are so far apart that the manifold must hop back and forth between the inner and outer cycle on every turn, giving the ICC its star-shaped appearance, with manifold branches resembling individual rays. In fact are the cycles so far apart that the manifold calculation terminated before the node cycle was reached in most cases. However, similar to the *black hole* archetype this is just an issue of calculation power and patience.

And again the eigenvalue table 5.16 shows both λ_3 eigenvalues remarkable close to zero.

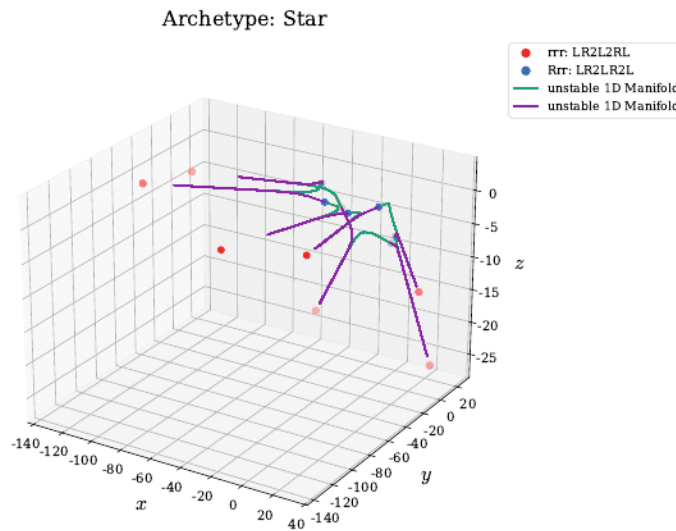


Figure 5.21: Period-7 ICC of the Star archetype, found at $\delta_R = 0.1$, $\tau_R = -0.27$ ($\tau_{R,param} = 0.03$), $\sigma_R = 3.96$ ($\sigma_{R,param} = 2.97$).

Cycle	λ_1	λ_2	λ_3
rrr	0.95	0.41	0.00
Rrr	5.15	0.83	0.00

Table 5.16: Eigenvalues of the cycles composing the *Star* archetype ICC.

5.2.3.4.1 Dominant eigendirection and phase-space scale as the origin of the Needle and Star patterns. Although one might intuitively think that for the *Needle* and *Star* archetypes a pair of complex-conjugated eigenvalues needs to be involved, the situation is different. Both *Needle* and *Star* are built from an **Rrr/rrr** saddle-node connection: all three eigenvalues of the stable cycle are real, so no complex-conjugate pair – and hence no rotational component – is present. Since the classical fact that every trajectory converging onto a stable node becomes asymptotically tangent to its *dominant* (slowest-decaying) eigendirection [29] applies equally to purely real systems, both directions of the saddle’s unstable manifold, terminating at the same node, are increasingly funneled onto this single common real eigendirection as they approach it – producing the straight, non-rotating rays characteristic of both archetypes, in contrast to the spiraling approach of for example Black Hole.

Table 5.17 lists the relevant eigenvalues for both cases. Two effects, both governed by this eigenvalue structure, jointly determine whether the resulting pattern appears as a Needle or a Star. First, the *ratio* between the sub-dominant and the dominant

eigenvalue modulus controls how quickly both manifold directions align onto the common eigendirection: at 0.688 for the Needle example, this ratio is considerably closer to 1 than the 0.438 found for the Star example, so alignment in the Needle case only completes very close to the node, leaving a clearly visible angle between both manifold directions over most of their approach; for Star, the smaller ratio causes the two branches to coincide much earlier, producing a long, near-parallel stretch. Second, the dominant eigenvalue’s own *distance from the unit circle* governs how far the trajectory travels along this direction before finally vanishing into the node: at $|\lambda| \approx 0.947$ (5.3 % from the unit circle), Star’s dominant eigenvalue decays far more slowly than Needle’s ($|\lambda| \approx 0.714$, 28.6 % from the unit circle), so the manifold covers a correspondingly much greater distance along the common direction before being absorbed – producing the long, pronounced arms of the Star pattern as opposed to the short, barely extended spike of the Needle pattern.

A third, independent factor reinforces both of these effects: the sheer phase-space scale of the two archetypes. Measured by the bounding-box diagonal of the two periodic cycles, Needle spans ≈ 42.0 and Star spans ≈ 233.8 , compared to only ≈ 17.2 for Rubberband as an example – roughly $2.4\times$ and $13.6\times$ larger, respectively. This substantial difference in absolute geometric extent compounds the eigenvalue-driven effects described above: quite apart from how quickly both unstable manifold directions align or how slowly the dominant direction decays, the sheer size of the underlying attractor means that any given ray, once formed, is stretched across a correspondingly larger region of phase space – most dramatically so for Star, whose combined eigenvalue structure and phase-space scale together produce its characteristic long, thin, star-like arms.

	Needle	Star	(Rubberband, for scale)
dominant $ \lambda $	0.7137	0.9470	–
distance to unit circle (dominant)	0.2863	0.0530	–
sub-dominant $ \lambda $	0.4911	0.4146	–
ratio sub-dominant/dominant	0.688	0.438	–
bounding-box diagonal (cycles)	≈ 42.0	≈ 233.8	≈ 17.2

Table 5.17: Eigenvalue structure and phase-space scale of the Needle and Star archetypes, compared to Rubberband.

5.2.4 When ICC construction fails

The previous subsections of this section have shown a lot of different archetypes of ICCs found via the described method. However sometimes there is also the case of two cycles existing in a saddle-node (or saddle-focus) connection, with the saddle having exactly one unstable manifold and still no ICC is constructed.

5.2.4.1 Coexistence

At the given parameter location, *two* remarkable saddle-focus pairs were identified: a period-5 pair (Rrr/ccr) and a period-11 pair (also Rrr/ccr). Both saddles possess a substantial real unstable eigenvalue -2.55 and 2.21 , respectively (See Table 5.19) –, and both are paired with a locally stable focus via an unstable manifold. Superficially, this configuration seems to work out perfectly, but investigating figure 5.22 shows that none of the two looks like the ICCs we had before. Instead the unstable manifolds of both saddles appeared, upon visual inspection, to wander irregularly through the phase space between them rather than settling onto either designated focus, suggesting the two heteroclinic-like connections might tangle into a chaotic attractor instead of forming clean ICCs.

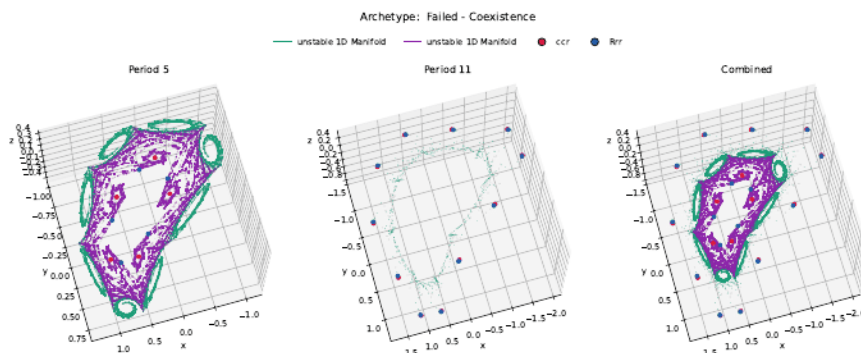


Figure 5.22: Unstable 1D manifold of Period-5/11 saddles not constructing an ICC, found at $\delta_R = 0.3$, $\tau_R = -1.52$ ($\tau_{R,param} = -1.5$), $\sigma_R = 0.54$ ($\sigma_{R,param} = 0.08$).

5.2.4.1.1 Ruling out chaos via the Lyapunov spectrum. To test the chaos hypothesis rigorously, the full Lyapunov spectrum was computed via Benettin’s method, exploiting the fact that the tangent map along any BCNF trajectory is given exactly by J_L or J_R with no linearization error. Starting from an orthonormal frame $Q_0 = I_3$, trajectory and frame are propagated jointly, $x_{n+1} = J_s(x_n)x_n + \mu e_1$, $V_{n+1} = J_s(x_n)Q_n$, with periodic QR-renormalization $V_{n+1} = Q_{n+1}R_{n+1}$ used to accumulate the exponents $\chi_i = \frac{1}{N-N_0} \sum \log |(R_{n+1})_{ii}|$. Applying this over $N = 3 \times 10^5$ iterations ($N_0 = 5000$ transient) from three independent seeds – the unstable eigendirection at P5 Rrr, at P11 Rrr, and an unrelated generic point – yields, in every case, the same spectrum:

Seed	χ_1	χ_2	χ_3
unstable eigendirection at P5 Rrr	-0.00798	-0.00797	-1.08014
unstable eigendirection at P11 Rrr	-0.00798	-0.00797	-1.08014
generic point (3.7, -2.1, 0.9)	-0.00798	-0.00797	-1.08014

Table 5.18: Lyapunov spectrum of the *Failed* – *Coexistence* configuration, computed from three independent initial conditions.

Every exponent is negative, and in particular $\chi_1 < 0$ rules out chaos outright, since a positive leading exponent is a necessary condition for sensitive dependence on initial conditions. The agreement across three unrelated seeds further shows that all trajectories are drawn into a single, common, non-chaotic attracting set.

5.2.4.1.2 Resolution: both manifolds are captured by a third, unrelated P8 ICC. It is not uncommon for cycles of different periods to coexist; it is even a necessary requirement for mechanisms such as the flip bifurcation, where a cycle and another of double its period exist simultaneously by construction. Here, however, none of the three periods is a multiple of another – 5, 11, and 8 share no such relationship, and 5 and 11 are in fact prime – so this coexistence cannot be attributed to any such bifurcation mechanism; the three saddle-focus pairs are, as far as can be determined, structurally unrelated candidates competing for the same trajectories.

The measured rate $\chi_{1,2} \approx -0.00798$ matches neither the theoretical contraction rate of P5 ccr ($\ln |\lambda_{1,2}|/5 = -0.00618$) nor of P11 ccr ($\ln |\lambda_{1,2}|/11 = -0.02107$). Systematically testing the post-transient trajectory for periodicity reveals an exact period-8 orbit, subsequently confirmed with PeriodScanner as the period-8 pair (Rrr/ccr) (Table 5.19), whose focus eigenvalues $|\lambda_{1,2}| = 0.9382$ give $\ln(0.9382)/8 = -0.00798$ – an exact match. Neither the period-5 nor the period-11 pair therefore forms its own ICC; instead, *both* unstable manifolds are captured by the basin of the third, entirely independent saddle-focus pair of period 8, which alone organizes into a genuine ICC.

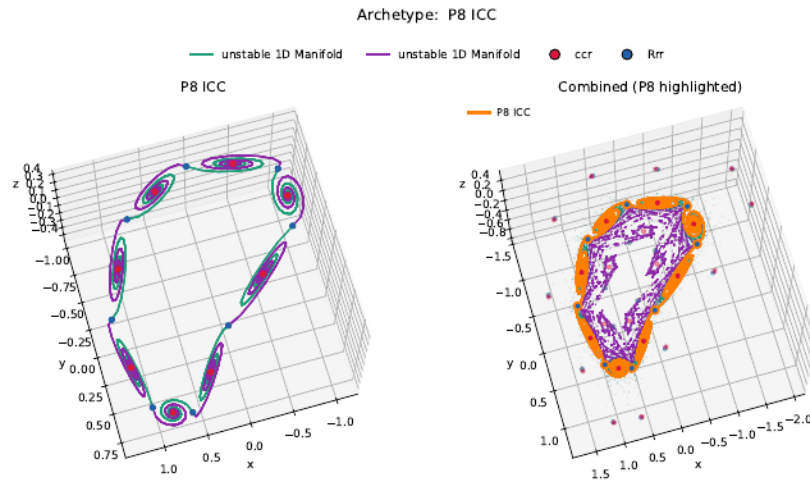


Figure 5.23: The period-8 ICC that both the period-5 and period-11 unstable manifolds converge to instead: shown alone (left) and highlighted within the combined phase portrait (right), where it stands out clearly against the non-converging period-5/11 structures.

Cycle	λ_1	λ_2	λ_3
Rrr (P5)	2.55	0.42	0.00
ccr (P5)	$0.30 + 0.92i$	$0.30 - 0.92i$	0.00
Rrr (P8)	4.20	0.24	0.00
ccr (P8)	$0.84 + 0.41i$	$0.84 - 0.41i$	0.00
Rrr (P11)	2.21	0.37	0.00
ccr (P11)	$0.52 + 0.60i$	$0.52 - 0.60i$	0.00

Table 5.19: Eigenvalues of the cycles composing the *Failed – Coexistence* archetype (P5, P8, and P11 saddle-focus connections).

5.2.4.1.3 Why the weaker attractor wins. That P8 captures both manifolds is at first counter-intuitive, since its focus is not the strongest local attractor among the three candidates:

Focus	$ \lambda_{1,2} $	local contraction	closest approach to border $ x _{\min}$
P5 ccr	0.970	medium	0.1069
P8 ccr	0.938	weakest	0.3262
P11 ccr	0.793	strongest	0.0113

Table 5.20: Local contraction rate versus distance from the switching border for the three competing foci.

P11 ccr contracts locally faster than P8 ccr, yet it is P8 ccr that captures the global manifolds. The resolution lies not in local contraction strength but in the

size of the true basin of attraction. Linear stability, and with it the eigenvalue-based contraction rate, is only valid as long as a trajectory follows exactly the same itinerary as the periodic orbit – i.e. as long as no perturbation flips the sign of x at any point along the cycle. Since P11 ccr approaches the border to within $|x|_{\min} = 0.0113$, an arbitrarily small transversal perturbation at that point is sufficient to flip the local itinerary and eject the trajectory from the linear neighborhood entirely, regardless of how strongly it would otherwise contract; its basin is, in practice, vanishingly thin along this direction. P8 ccr, by contrast, stays at least 0.3262 away from the border at every point of its cycle – almost thirty times further than P11 ccr – giving it a comparatively wide, robust basin. A macroscopic, one-dimensional unstable manifold sweeping through phase space has a realistic chance of entering this wide basin, but is essentially unable to thread the needle-thin basin of P11 ccr (or, to a lesser extent, of P5 ccr, whose own margin of 0.1069 is likewise small). P8 therefore does not "pull harder" than its competitors; it simply is the only one of the three candidates whose basin is wide enough to actually be reached.

5.2.4.2 ICC-like Chaotic Attractor

Fig. 5.24 shows the two branches of the one-dimensional unstable manifold of a period-6 saddle cycle. The first branch behaves exactly as expected from – it converges monotonically onto the stable focus cycle, forming a simple heteroclinic connection between the saddle and the sink that was already seen a few times in previous archetypes. The second branch, however, exhibits markedly different behaviour. Rather than settling onto an other focus point, it repeatedly passes close to the images of neighbouring saddle cycle points, giving the visual impression of a heteroclinic network in which the manifold splits and re-converges onto each of these saddle points in turn. This can be seen in even greater detail in the close up pictures of figure 5.25

A closer numerical investigation shows that this impression is misleading: the branch does not converge onto any of the neighbouring saddle cycle points. Instead, it remains permanently bounded and recurrent, densely tracing out a thin, ring-shaped structure that at first glance resembles an ICC but, on closer inspection, is a genuinely chaotic attractor. This was confirmed by computing the Lyapunov spectrum of the underlying border-collision normal form directly, using Benettin’s QR-based algorithm applied to the piecewise-constant Jacobian J_L/J_R along a long, freely iterated orbit seeded near the saddle’s unstable eigendirection (after discarding an initial transient of 3×10^5 iterations). The resulting spectrum, averaged over 10^6 – 2×10^6 iterations and cross-validated against the mean logarithmic Jacobian determinant, is summarised in Table 5.21.

Quantity	Value
λ_1	+0.0321
λ_2	−0.3499
λ_3	−0.7068
$\sum_i \lambda_i$	−1.0246
Kaplan–Yorke dimension D_{KY}	≈ 1.092

Table 5.21: Lyapunov exponents and Kaplan–Yorke dimension estimate of the chaotic attractor traced out by the second branch of the unstable 1D manifold of the saddle cycle.

The presence of a strictly positive leading exponent λ_1 , robust across independently perturbed initial conditions and across integration lengths, rules out both a quasi-periodic ICC ($\lambda_1 = 0$) and convergence onto a periodic orbit ($\lambda_1 < 0$) as explanations for the observed structure. Since the map is piecewise linear, the stretching along the unstable direction must be compensated by a folding mechanism to keep the orbit bounded; this folding is provided directly by the border-switching between J_L and J_R . Combined with the strong transverse contraction encoded in λ_2 and λ_3 (See Table 5.22), this stretch-and-fold action produces a self-similar, Cantor-like transverse structure, so that the attractor is fractal rather than a smooth curve, despite its Kaplan–Yorke dimension estimation lying only marginally above one.

This mechanism, whereby a chaotic attractor acquires a fractal, band-like structure as a direct consequence of its extension across the switching manifold, was recently

investigated systematically by Avrutin, Görner and Sushko [28], who demonstrated the underlying border-collision bifurcation mechanism for chaotic attractors in the (only 2D) BCNF. The present three-dimensional case might be a higher-dimensional manifestation of the same phenomenon. Note that since border-collision bifurcations of chaotic attractors were not within the scope of this thesis, this phenomenon was not investigated in further detail.

Archetype: Failed ICC-like Fractal Chaotic Attractor

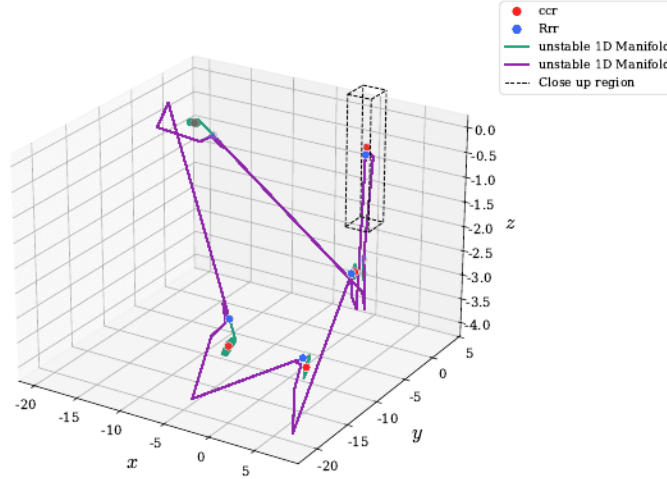


Figure 5.24: Period-6 failed ICC-like invariant object, found at $\delta_R = -0.5$, $\tau_R = 2.21$ ($\tau_{R,param} = 1.15$), $\sigma_R = 2.29$ ($\sigma_{R,param} = 2.12$). One branch of the one-dimensional unstable manifold of the saddle cycle **Rrr** (blue) converges onto the stable focus cycle **ccr** (red) as expected from smooth bifurcation theory (teal). The other branch (purple) does not terminate on any of the neighbouring saddle-cycle images; instead it densely traces out a thin fractal chaotic attractor (leading Lyapunov exponent $\lambda_1 \approx +0.032$, Kaplan–Yorke dimension $D_{KY} \approx 1.09$), visually mimicking an ICC without actually being one.

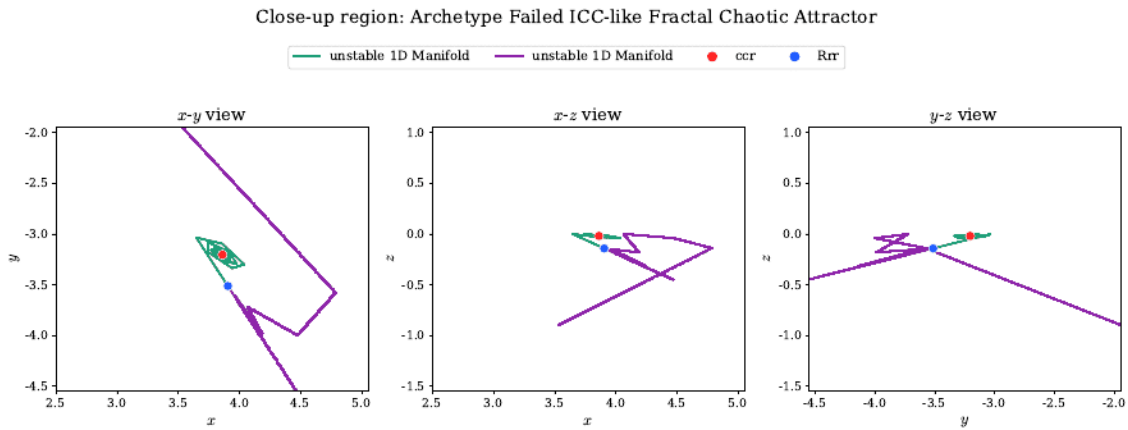


Figure 5.25: Close-up of the phase-space region highlighted by the dashed cube in Fig. 5.24, shown as three orthogonal projections (x - y , x - z , y - z). The cube contains exactly one point of the unstable saddle cycle **Rrr** (blue) and one point of the stable focus cycle **ccr** (red). The teal point cloud is the branch of the unstable 1D manifold that converges cleanly onto the node cycle, while the purple point cloud reveals the finely folded, near-parallel band structure of the second branch — the fingerprint of its underlying fractal, chaotic nature — rather than a simple tangential passage toward another saddle-cycle point as would be expected for a genuine ICC (see 2.1.8.1).

Cycle	λ_1	λ_2	λ_3
Rrr	1.53	0.02	0.08
ccr	$0.21 + 0.89i$	$0.21 - 0.89i$	-0.00

Table 5.22: Eigenvalues of the cycles composing the *Failed – Swirls* archetype ICC.

5.3 Results of Crossing the BC-Fold Boundary Through a Non-Terminating Shrinking Point

This section presents a concrete numerical confirmation, within the piecewise-linear 3D BCNF, of the non-terminating shrinking point mechanism described theoretically in 2.1.7: as one sausage of a resonance-tongue chain narrows to zero width, a *second* sausage emanates from the same point. The mode-locked ICC transition between two sausages (here between sausage 2 and 3 see fig. 5.26) is not a continuous process but can be more likely interpreted as destruction and immediate rebirth. The example investigated here is a period-7 ICC, tracked along a trajectory through a non-terminating shrinking point using the notations introduced in 2.1.8.2 (lowercase = stable, uppercase = unstable, $\mathbf{r}$ = real, $\mathbf{c}$ = complex-conjugate pair). The underlying hypothesis being that the expected behavior Simpson and Meiss describe in [18] can be reproduced.

τ_L	-0.1	$\tau_{R,(param)}$	<i>varied</i>
σ_L	0.2	$\sigma_{R,(param)}$	<i>varied</i>
δ_L	0.2	δ_R	-0.1
μ		1.0	

Table 5.23: Parameters used to produce Figures 5.26 to 5.30.

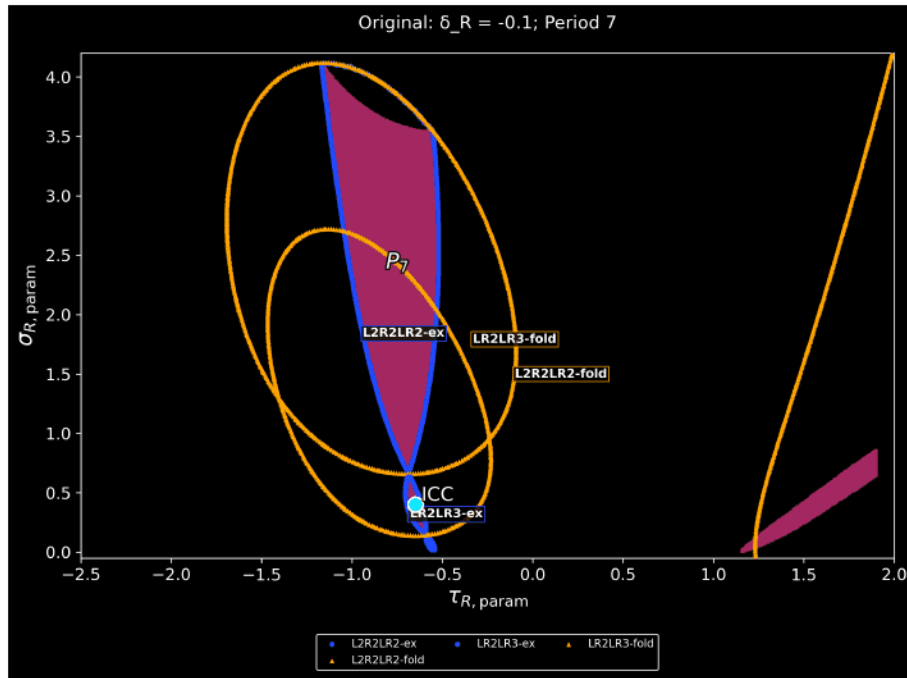


Figure 5.26: Periodicity region with marked outlines in which ICCs exist and where the "+1"-condition is satisfied for each of the cycles. Including Period-7 ICC marker in second sausage at starting point ($\tau_{R,param} = -0.65/\sigma_{R,param} = 0.4$)

As a side note, it is known that whenever two border-collision fold (BC-fold) boundaries intersect at a shrinking point, a $+1$ -curve must also pass through that point [18]. However, this result does not imply that shrinking points are the only locations at which a $+1$ -curve can intersect a sausage.

Figure 5.26 illustrates this phenomenon. The final sausage in the sausage chain appears as though a bite has been taken from its upper portion. This indentation is caused by a $+1$ -curve intersecting the sausage before the corresponding shrinking point is reached, thereby terminating the sausage chain prematurely.

(1) Far before the bifurcation. Far from the shrinking point, the period-7 ICC consists of a stable node cycle of type **rrr** and a saddle cycle of type **Rrr**, separated by a comfortable distance in phase space. Both cycles have purely real eigenvalues, and the resulting ICC is unremarkable – a well-behaved, clearly separated saddle-node connection.

insert reference to archetypes

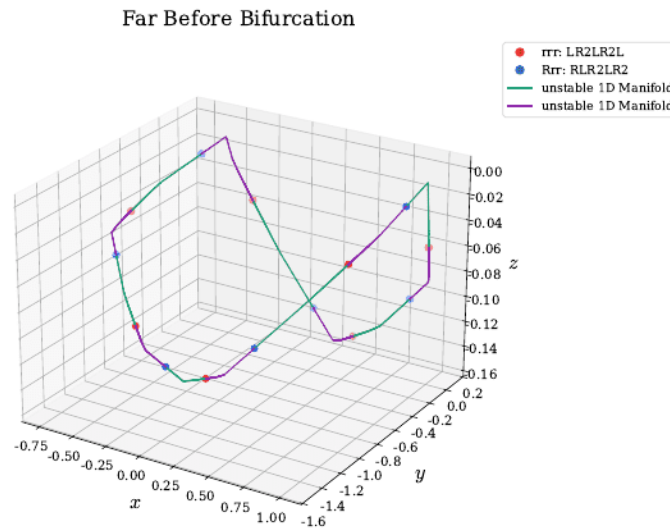


Figure 5.27: Period-7 ICC far before the border-collision fold bifurcation ($\tau_{R,param} = -0.65/\sigma_{R,param} = 0.4$): node (**rrr**) and saddle (**Rrr**) cycles at a clear distance from one another.

(2) Shortly before the bifurcation. As the relevant parameters are moved towards the shrinking point, the node and saddle cycles approach each other, their separation in phase space shrinking steadily while both retain their eigenvalue types (**rrr** and **Rrr**, respectively). No qualitative change in stability has occurred yet. The two cycles are simply being driven towards each other on a collision course.

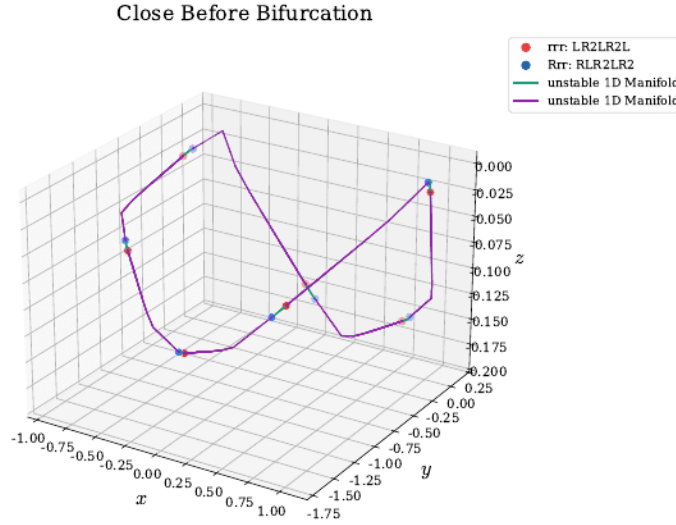


Figure 5.28: Period-7 ICC shortly before the bifurcation ($\tau_{R,param} = -0.7/\sigma_{R,param} = 0.5$): the node (rrr) and saddle (Rrr) cycles have moved close together, but retain their respective eigenvalue classes.

(3) At the bifurcation point. Before turning to the shrinking point itself, it is worth restating what ordinary (codimension-1) BC-fold behavior looks like: whenever a single parameter path crosses *either* of the two fold boundaries bounding a resonance tongue, one point of the corresponding stable cycle and one point of the corresponding saddle cycle are driven onto the switching manifold together, where the two cycles collide and annihilate each other entirely. This is the generic mechanism responsible for destroying a mode-locked ICC at the edge of its tongue, and it is exactly what is observed here as well.

The non-terminating shrinking point investigated in this section is simply the particular point in parameter space at which *both* BC-fold boundaries bounding the tongue happen to intersect. Or to be more precise: pairwise intersect, since they are continued through the shrinking point to form the edges of the new sausage. Importantly, this intersection does not change the underlying annihilation mechanism at all – the cycles still collide and annihilate at the switching manifold exactly as in the codimension-1 case, but this is the reason that raises the bifurcation codimension from one to two in this case.

Fig. 5.26 shows the "fold"-boundaries of both cycles, i.e., the loci in parameter space where the respective cycle satisfies the “+1” eigenvalue condition: inside each boundary the corresponding cycle has not experienced a fold bifurcation, and crossing it triggers the “+1” behavior described in 2.1.3 for that cycle. Closer inspection of Fig. 5.26 shows that each of these boundaries passes *exactly* through one such non-terminating shrinking point. This has an important consequence: although the ICC as a whole is unambiguously destroyed at this point, the two colliding cycles are not treated symmetrically by the bifurcation – one of the two cycles is annihilated outright, while the other persists beyond the shrinking point. For the persisting cy-

cle, passing through the shrinking point has two simultaneous effects: its itinerary (its “word”, the LR-sequence) changes by exactly one letter – precisely the word change triggered by the crossing of the two BC-fold boundaries – and its eigenvalue sequence changes from stable to saddle as a result of the “+1” crossing. This, in turn, implies that the newly created partner cycle on the other side of the sausage change must be stable.

The full set of conditions characterizing this non-terminating shrinking point in the period-7 tongue is therefore

$$\det(I - M_S) = 0, \quad x^{(k)} = 0, \quad x^{(l)} = 0, \quad \lambda_{+1}(M_S) = 1, \quad (5.15)$$

where $\lambda_{+1}(\cdot)$ denotes the eigenvalue of the corresponding monodromy matrix that crosses the unit circle at +1. Note that the last condition is not independent of the first in the strict algebraic sense – since $\det(I - M_S) = 0$ already implies $\lambda_{+1}(M_S) = 1$ – but it serves as the concrete diagnostic identifying *which* second point of the cycle (l , five steps ahead of k) is responsible for the codimension-two coincidence. It is precisely this third, additional condition that forces the persisting cycle’s stability type to change: satisfying the +1 crossing turns the purely real **rrr** node into an **Rrr** saddle – the persisting cycle observed in stages (4) and (5) above. The companion cycle, which does not satisfy this additional condition, is instead the one annihilated at the shrinking point.

(4) Shortly after the bifurcation. Immediately beyond the bifurcation point, a *new* ICC appears. But, consistent with the non-terminating shrinking point mechanism, the new stable cycle is *not* of the same eigenvalue type as the one that was destroyed. Instead, it now takes the form **ccr**: a stable complex-conjugate pair together with one stable real eigenvalue, replacing the purely real **Rrr** saddle seen before the bifurcation. The saddle seen in the new sausage is the former (**rrr**) that experienced the “+1” boundary crossing. This new node and saddle are still close together immediately after the collision, forming a new, distinct ICC.

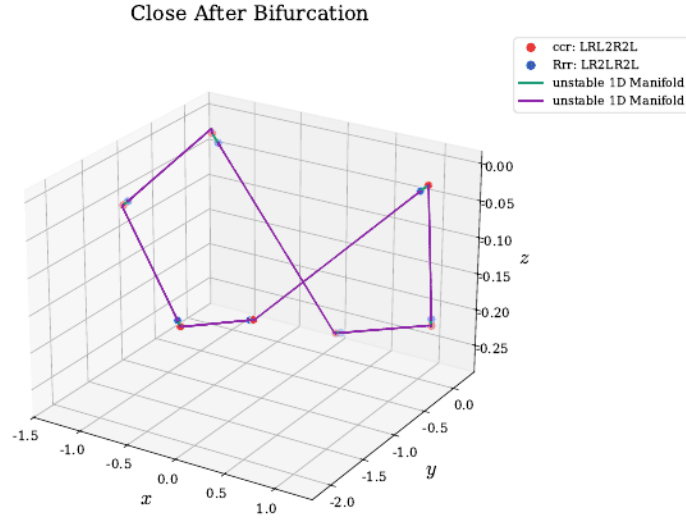


Figure 5.29: Newly created period-7 ICC shortly after the bifurcation ($\tau_{R,param} = -0.7/\sigma_{R,param} = 0.7$): the saddle-focus connection appeared with (**ccr**, a stable complex-conjugate pair plus one stable real valued eigenvalue) alongside the freshly converted saddle (**Rrr**), still close together immediately after the collision.

(5) Far after the bifurcation. Moving further away from the bifurcation point, the new **ccr**/**Rrr** saddle-focus pair separates again to a comfortable distance, forming a stable, still quite well-behaved ICC that belongs to the new resonance tongue – the second sausage above the non-terminating shrinking point.

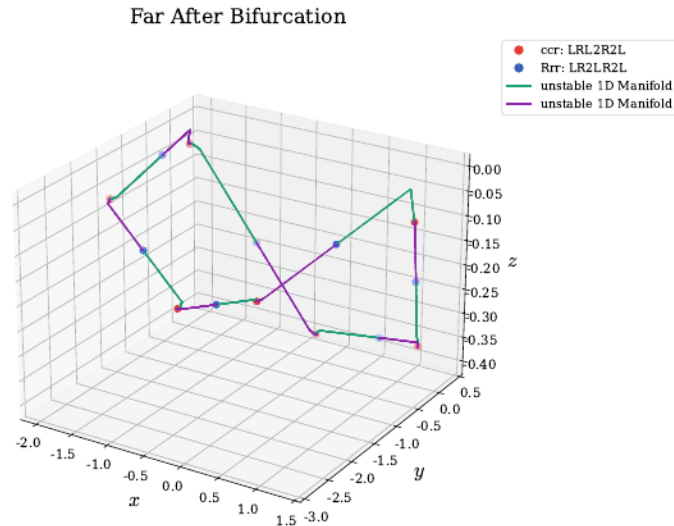


Figure 5.30: Period-7 ICC well after the bifurcation ($\tau_{R,param} = -0.7/\sigma_{R,param} = 1.0$): the new focus (**ccr**) and saddle (**Rrr**) cycles have separated to a clear distance, forming a stable ICC belonging to the newly opened resonance tongue.

This sequence – a real-multiplier node and saddle annihilating each other at the BC-fold boundary, immediately followed by the creation of a new saddle-focus pair whose focus has switched from a purely real multiplier to a complex-conjugate pair – is exactly the behavior expected from the non-terminating shrinking point mechanism established for resonance tongues by Simpson and Meiss [18]. Their results were, however, derived and illustrated primarily for planar (2D) piecewise-smooth maps. The sequence documented here therefore serves as a direct confirmation that the same qualitative mechanism also marks the transition between two neighboring sausages of a sausage-string chain in the three-dimensional, piecewise-*linear* setting of the BCNF studied in this thesis.

5.4 Results of Crossing the Neimark-Sacker Boundaries

For an ICC to cross a Neimark–Sacker boundary, it must first exist in precisely the form required for such a bifurcation: a saddle-focus connection in which both constituent cycles possess a pair of complex eigenvalues. This means there is exactly one admissible configuration in which such a scenario can occur – the original ICC composed of the cycles $\mathbf{rcc}$ and $\mathbf{Rcc}$, using the notation introduced in 2.1.8.2 (lower-case denotes stable, uppercase unstable; $\mathbf{r}$ a real eigenvalue, $\mathbf{c}$ a complex-conjugate pair). The real eigenvalues of both cycles must persist unchanged throughout the entire bifurcation: in the smooth Neimark–Sacker case they continue to underpin the original ICC, while the transition $\mathbf{cc} \rightarrow \mathbf{CC}$ causes several new ICCs to appear around the original one in the transversal-radial plane.

The hypothesis investigated here, derived directly from Goal 1 of this thesis, is that the same holds true for the degenerate Neimark–Sacker case in the piecewise-linear setting considered in this work. Since no example within the parameter range explored for this thesis was found in which an $\mathbf{rcc}/\mathbf{Rcc}$ -type ICC exists and undergoes two Neimark–Sacker bifurcations in close succession, an additional literature review was conducted with the aim of locating prior investigations of Neimark–Sacker bifurcations in the 3D BCNF. Muni and Banerjee [10] include a short section addressing precisely such a case and provide a set of parameters that sparked the investigation carried out here. However, since their interest lay only in establishing that a Neimark–Sacker bifurcation occurs at all, only the first of the two required bifurcations was investigated in [10]. The investigation carried out in this thesis goes one step further, additionally examining the subsequent Neimark–Sacker bifurcation and thereby completing the full ICC Neimark–Sacker scenario.

This section documents the numerical observations obtained for the discovered period-7 Neimark–Sacker scenario of an ICC in the 3D BCNF, structured into three stages: the ICC before any Neimark–Sacker bifurcation, the ICC after a single Neimark–Sacker bifurcation, and the ICC after both constituent cycles have undergone their respective bifurcation. The theoretical background for the terms and mechanisms used below – criticality, the absence of a local cubic-type restoring mechanism in the BCNF, and the composition of an ICC from periodic cycles and their manifolds – is given in 2.1.5 and 2.1.8.1 and is not repeated here.

(1) Before the Neimark–Sacker bifurcations. Figure 5.31 shows the period-7 ICC prior to either Neimark–Sacker bifurcation. It is composed of a fully stable cycle ccr (red) and a saddle cycle ccR (blue) whose single real eigenvalue is already unstable while its complex pair remains stable; the unstable 1D manifold of the saddle (teal/purple) connects the two cycles into a closed curve. As established in 2.1.8.1, a stable node/focus cycle together with a saddle cycle (with either a real or a complex unstable direction) and the saddle’s unstable 1D manifold constitutes precisely the ingredients of an attracting ICC; the configuration observed here is therefore attracting, matching this general composition rule.

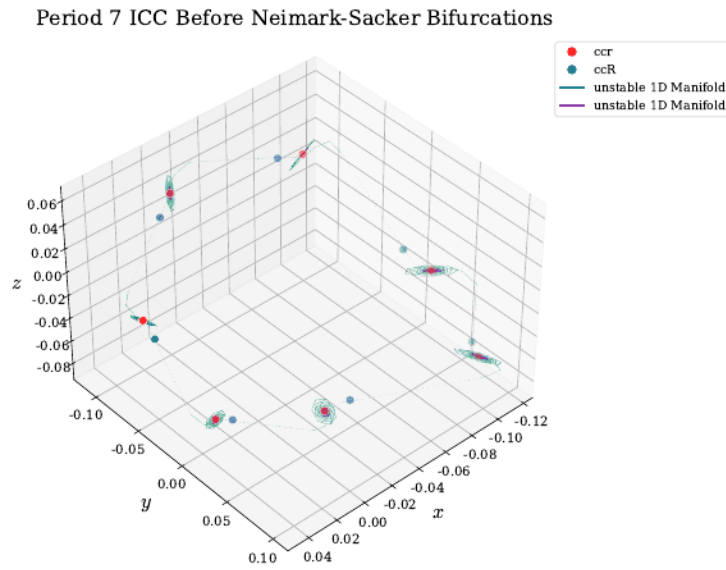


Figure 5.31: Period-7 ICC before the Neimark-Sacker bifurcations. The stable cycle ccr (red) coexists with the saddle cycle ccR (blue), whose single real eigenvalue is already unstable while its complex pair remains stable. The teal/purple branches show the saddle’s unstable 1D manifold in the tangential direction of the ICC.

(2) **After a single Neimark–Sacker bifurcation.** At the given parameter location a period-7 saddle-focus pair was identified: R_{cc} , possessing a real unstable eigenvalue and a stable complex-conjugate pair, and CCr (the former stable focus), possessing a real stable eigenvalue but a complex-conjugate pair whose modulus already exceeds unity after the first degenerate Neimark-Sacker bifurcation:

$$\lambda(R_{cc}) = 1.76989, \quad 0.52336 \pm 0.61911i \quad (|\lambda_{cc}| = 0.8107),$$

$$\lambda(CC_r) = 0.72900 \pm 0.80893i \quad (|\lambda_{CC}| = 1.0889), \quad 0.44755.$$

(See Table 5.24) Unlike every focus encountered in the previously discussed archetypes, CCr is therefore not a sink: its rotational eigendirection has already crossed the unit circle, placing the cycle just beyond a degenerate Neimark–Sacker-type threshold. Looking at the invariant set presented in 5.32 raised the suspicion that the unstable manifold of R_{cc} might, upon reaching the neighborhood of this already-destabilized focus, be driven directly into a chaotic attractor rather than organizing into a bounded invariant set.

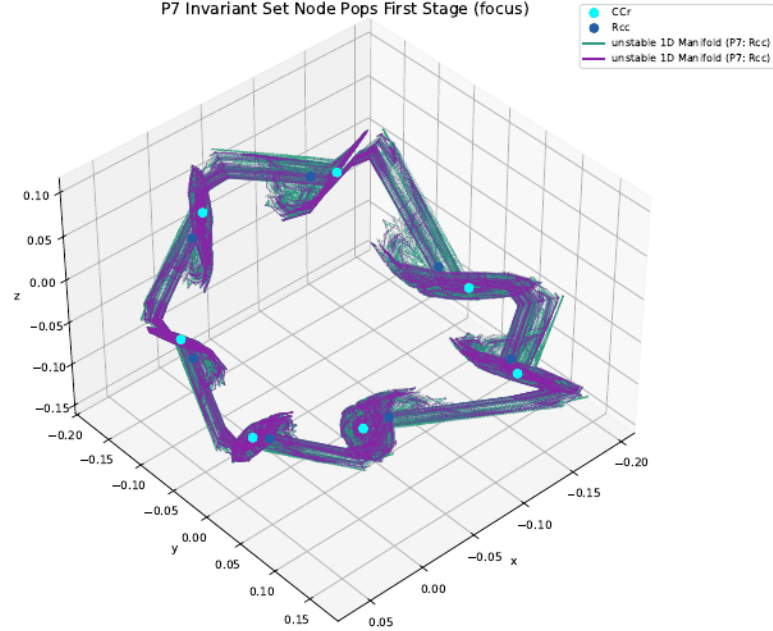


Figure 5.32: 1D unstable manifold of Period-7 saddle at the node-pops-first stage. The saddle cycle ccR (blue) is unchanged from Fig. 5.31; the former focus has undergone a Neimark-Sacker bifurcation, its complex eigenvalue pair crossing the unit circle and turning it from the stable cycle ccr into the unstable cycle CCr (cyan). The teal/purple branches show the slightly distorted tangential unstable 1D manifold of the saddle ccR .

Cycle	λ_1	λ_2	λ_3
Rcc (P7)	1.7699	$0.5234 + 0.6191i$	$0.5234 - 0.6191i$
CCr (P7)	$0.7290 + 0.8089i$	$0.7290 - 0.8089i$	0.4475

Table 5.24: Eigenvalues of the two period-7 cycles organizing the *P7 Invariant Set*. Both cycles reproduce their tabulated periodic points to machine precision under the corresponding BCNF matrices.

5.4.0.0.1 Ruling out chaos. The Lyapunov spectrum was computed via the same Benettin/QR procedure used previously, exploiting the exact piecewise-affine tangent map $J_{L/R}$. Because the leading exponent turned out to be extremely close to zero, convergence was tracked explicitly with increasing iteration count:

Iterations	χ_1	χ_2	χ_3
1 million	-0.001286	-0.005309	-0.026740
5 million	-0.001341	-0.005314	-0.026662
10 million	-0.001348	-0.005314	-0.026652
15 million	-0.001351	-0.005314	-0.026649

Table 5.25: Convergence of the Lyapunov spectrum with increasing sample length, seeded at the unstable eigendirection of Rcc.

The exponents stabilize cleanly and are confirmed by two further independent seeds run for 10 million iterations each:

Seed	χ_1	χ_2	χ_3
unstable eigendirection at Rcc	-0.00135	-0.00531	-0.02665
generic point (0.03, -0.05, 0.02)	-0.00136	-0.00531	-0.02664
perturbation at CCr (unstable focus)	-0.00134	-0.00531	-0.02666

Table 5.26: Lyapunov spectrum of the P7 configuration from three independent initial conditions, 10^7 iterations each.

Every exponent remains negative across all seeds and sample lengths, so $\chi_1 < 0$ rules out chaos despite the destabilized focus. Notably, χ_1 is almost an order of magnitude closer to zero than in the *Failed – Coexistence* configuration ($\chi_1 \approx -0.008$), placing the dynamics markedly nearer the critical threshold $\chi_1 = 0$ without crossing it.

5.4.0.0.2 Resolution: neither a periodic orbit nor a thin curve, but a quasi-periodic band. Two further tests were carried out to identify the actual attracting object. First, the post-transient trajectory was tested for periodicity by comparing x_n against x_{n+p} for candidate periods $p = 1, \dots, 59$; no near-exact match was found (the best candidate, $p = 46$, gave a mean separation of 0.0151, orders of magnitude above the machine-precision match of 0.000000 found for the analogous

test in the *Failed – Coexistence* case). The manifold therefore does not lock onto any periodic orbit.

Second, the transverse spread of the trajectory at a fixed location along the invariant set was compared between an early and a very late sampling window, to check whether the object is still slowly contracting onto a thin curve or has already reached its final extent:

Sampling window	σ_x	σ_y	σ_z
iterations 5.0×10^5 – 5.2×10^5	0.003609	0.003906	0.002394
iterations 2.98×10^6 – 3.0×10^6	0.003616	0.003914	0.002397

Table 5.27: Transverse spread of the trajectory near a fixed lobe of the invariant set, compared between an early and a very late sampling window (2.5×10^6 iterations apart).

The spread is, within numerical noise, identical between the two windows despite 2.5 million further iterations – the object has already reached its asymptotic transverse width by $\sim 5 \times 10^5$ iterations and does not contract further. Combined with the absence of any periodic match, this indicates that the unstable manifold does not converge to a periodic point at all, but instead settles onto a genuinely quasi-periodic, finite-width band: a structure that fills a two-dimensional-looking region transversally to its main extent rather than collapsing onto a one-dimensional curve or a single point.

This resolves the original question cleanly: the manifold is neither chaotic (Table 5.26) nor organized around a stable periodic orbit (Table 5.27 together with the periodicity test). It instead forms a robust, non-chaotic, but structurally irregular invariant object – sustained not by local contraction toward a sink, since CCr is itself unstable, but by the global piecewise-linear reinjection of the border-collision map. This is precisely why the object is termed an *invariant set* throughout this section rather than an ICC: unlike the smooth, one-dimensional closed curves seen in the archetypes collection, it possesses genuine transversal width and does not correspond to any single organizing periodic orbit.

Note, that this is more or less a reproduction of Muni and Banerjees pictures on that topic in [10], since the investigation here relies on their discovered parameter region.

(3) After both Neimark–Sacker bifurcations. Figure 5.33 shows the ICC after the saddle cycle has itself undergone a Neimark–Sacker bifurcation. Its complex eigenvalue pair has crossed the unit circle in addition to its already-unstable real eigenvalue, turning ccR into the fully unstable focus cycle CCR (blue). At the same time, the cycle CCr (cyan) is unchanged from Figure 5.32, but its role within the ICC has changed fundamentally: it now possesses a *stable* 1D manifold (teal/purple), converging onto the cycle rather than emanating from it, in place of the (nonexistent) unstable manifold examined in the previous stage.

The overall composition of the ICC has thus reversed compared to the initial stage: before either bifurcation, the ICC consisted of one stable node/focus cycle, one saddle cycle, and the saddle’s *unstable* 1D manifold; after both bifurcations, it instead consists of one *unstable* node/focus cycle (CCR), one saddle cycle (CCr), and a *stable* 1D manifold. By the same composition rule invoked above (2.1.8.1), this reversed configuration is again attracting: the flip from an unstable to a stable connecting manifold exactly compensates for the flip of the node/focus cycle from stable to unstable, so that stability of the ICC as a whole is restored, even though every one of its constituent pieces has switched stability type in the process.

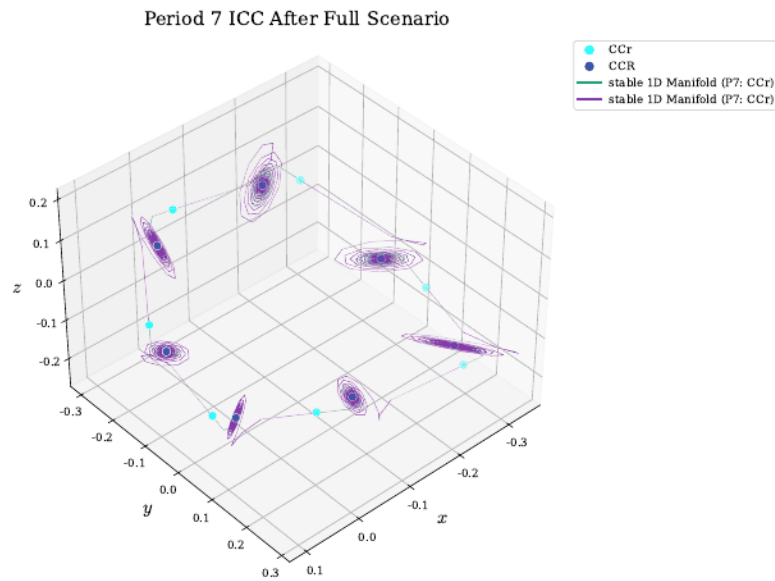


Figure 5.33: Period-7 ICC after the full Neimark-Sacker scenario. The former saddle ccR has itself undergone a Neimark-Sacker bifurcation and is now the doubly unstable cycle CCR (blue), while the cycle CCr (cyan) from Fig. 5.32 is unchanged. The teal/purple branches now show the *stable* 1D manifold of CCr , converging onto the cycle rather than emanating from it.

5.5 Results of Crossing the Flip Boundaries

This section presents a concrete numerical confirmation, within the piecewise-linear 3D BCNF, of the doubling mechanism described theoretically in 2.1.9.1: both cycles of an original ICC ($\mathbf{rrr}/\mathbf{Rrr}$) successfully undergo a period-doubling bifurcation, at which exactly one eigenvalue changes stability, giving birth to a new pair of cycles with the correct eigenvalue configuration to form a new, doubled-period ICC ($\mathbf{rrr}/\mathbf{Rrr}$), using the notation introduced in 2.1.8.2 (lowercase = stable, uppercase = unstable, $\mathbf{r}$ = real, $\mathbf{c}$ = complex-conjugate pair).

The underlying hypothesis is that one of the two ICC-doubling scenarios reported by the authors of [23]—namely, the cylindrical or the Möbius scenario—should also occur in the present setting in a degenerate form. This expectation persists despite the findings of Keller [26], which suggest that such bifurcations are exceedingly rare. Both of the examples presented in the following subsections were deliberately chosen and curated so that they exhibit the intermediate “node doubles first” step: after the first period-doubling bifurcation, the system passes through the intermediate configuration $\mathbf{Rrr}/\mathbf{Rrr}$, which can still support an ICC, since the eigenvalue that became unstable is not associated with the tangential eigendirection. Because it was the previously stable node that lost stability, the newly created double-period cycle must again be stable – though no guarantee exists as to whether its eigenvalues will be real or complex. 5.5.1 provides an example of the complementary case, in which an original saddle-focus ICC has the focus, rather than the node, create a doubled-period node cycle. Note also that for the carefully constructed P7 scenario 5.5.2, slightly different parameters may – and generically will – lead to more than one unstable eigenvalue after the flip, or to complex eigenvalues being involved as well.

According to Biswas, Banerjee, Avrutin and Sushko [23], an ICC can double in one of two orders: “saddle doubles first” or “node doubles first.” The numerical experiments however were only conducted for the “node doubles first.” doubling order. Unfortunately no parameter region was identified in which a successful “saddle doubles first” scenario could have been investigated.

The experimental design follows the procedure described in Section 4.3.1. Within the $\tau_{R,\text{param}} - \sigma_{R,\text{param}}$ parameter slice (i.e., the two-dimensional bifurcation diagram), the search focused on regions in which the cycles of an ICC undergo two consecutive flip bifurcations—one for each constituent cycle—without any intervening bifurcations. Multiple such regions were identified using the semi-automatic Jupyter notebook with interactive sliders. The corresponding parameter set was then investigated according to the three stages introduced above: before the first doubling, after the node cycle has doubled (while the saddle cycle remains undoubled), and after the complete doubling scenario has occurred. For all the stages the ICC was inspected in phase space as well to see if the ICC behaves as expected.

5.5.1 P5 failed scenario

This subsection examines a period-5 attempt at “node doubles first” doubling that fails to produce a persisting, admissible ICC on the other side of the two flip bi-

furcations, since the resulting saddle cycle after the second flip is created with two unstable eigenvalues instead of one. The scenario is still interesting, since the stable focus produces a stable node cycle after the bifurcation. The two bifurcation stages (before, first flip) will be shown in figures 5.34, 5.35 and their respective eigenvalue tables 5.29 and 5.30. This confirms again what Keller already claimed in [26], that the eigenvalue configuration of the doubled cycle is entirely unrelated to the cycle it originated from.

τ_L	0.1	$\tau_{R,param}$	<i>varied</i>
σ_L	-0.5	$\sigma_{R,param}$	<i>varied</i>
δ_L	0.4	δ_R	-0.5
μ	1.0		

Table 5.28: Parameters used to produce Figures 5.34 and 5.35.

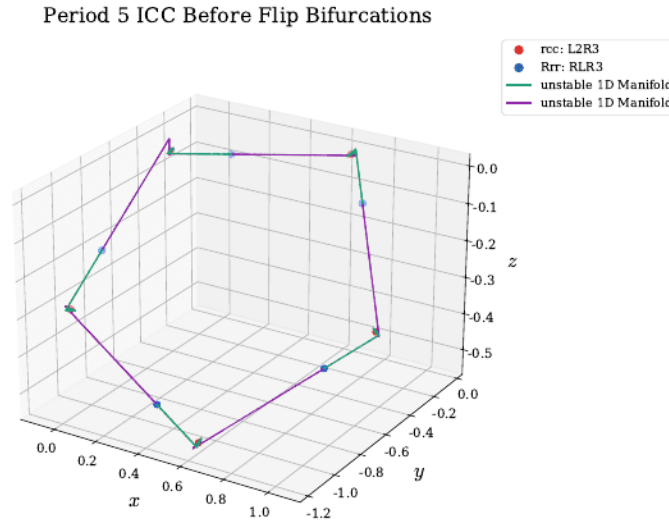


Figure 5.34: Period-5 ICC before the flip bifurcation ($\tau_{R,param} = 0.1/\sigma_{R,param} = 0.1$), consisting of the saddle cycle **Rrr** and the focus cycle **rcc**, connected by the unstable 1D manifold of the saddle.

Cycle	λ_1	λ_2	λ_3
Rrr (P5)	1.2314	-0.4557	-0.0446
rcc (P5)	-0.7960	$-0.0551 + 0.1486i$	$-0.0551 - 0.1486i$

Table 5.29: Eigenvalues of the **Rrr** saddle and **rcc** focus cycles.

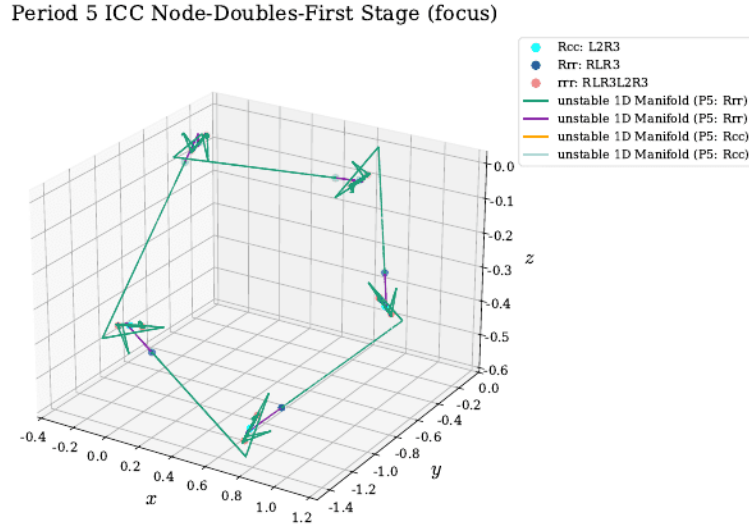


Figure 5.35: Period-5 ICC at the node-doubles-first stage ($\tau_{R,param} = 0.2/\sigma_{R,param} = 0.3$). The saddle cycle **Rrr** (blue) is unchanged from Fig. 5.34; the former focus has flipped from stable **rcc** to unstable **Rcc** (cyan), giving birth to a new stable period-10 cycle **rrr** (light red). The teal/purple manifold branches are the original saddle's unstable manifold in the tangential direction of the original ICC; the orange/light-blue branches are the new saddle's unstable manifold in the radial/transversal direction, connecting the ICC with the new period 10 node cycle.

Cycle	λ_1	λ_2	λ_3
Rrr (P5)	2.9646	0.4584	0.0005
Rcc (P5)	1.0997	$0.0016 + 0.0190i$	$0.0016 - 0.0190i$
rrr (P10)	-0.6642	0.3366	0.0022

Table 5.30: Eigenvalues of the cycles **Rrr** (P5), **Rcc** (P5), and **rrr** (P10).

5.5.2 P7 successful scenario

This subsection presents the curated period-7 case referenced above, in which the **rrr**/**Rrr** pair successfully completes the ICC length doubling scenario via the “node doubles first” path, producing a genuine, admissible **rrr**/**Rrr** ICC of twice the original period and twice the length. Figures 5.36 to 5.39 as well as the respective eigenvalue tables 5.32 to 5.34 show how the entire scenario plays out in the BCNF.

τ_L	0.7	$\tau_{R,param}$	<i>varied</i>
σ_L	0.4	$\sigma_{R,param}$	<i>varied</i>
δ_L	0.5	δ_R	-0.7
μ	1.0		

Table 5.31: Parameters used to produce Figures 5.36 to 5.38.

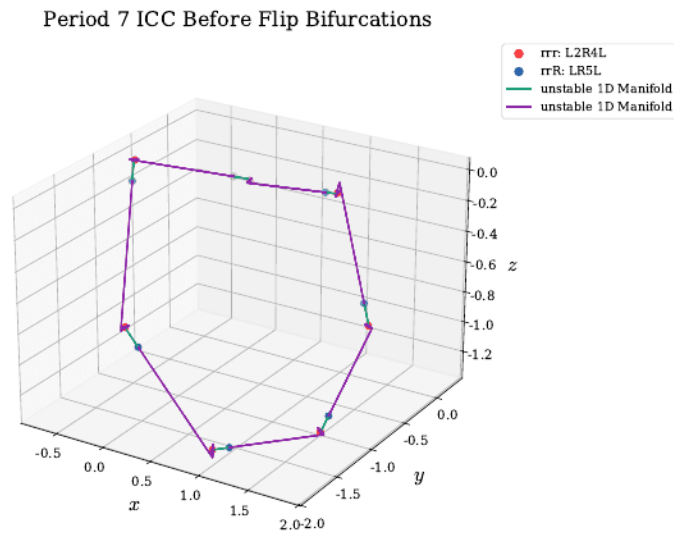


Figure 5.36: Period-7 ICC before the flip bifurcations ($\tau_{R,param} = 0.6/\sigma_{R,param} = 0.3$). The stable period-7 cycle **rrr** (red) coexists with the saddle cycle **rrR** (blue), which already possesses a single unstable direction. The teal/purple branches show the saddle’s unstable manifold in the tangential direction of the ICC.

Cycle	λ_1	λ_2	λ_3
rrR (P7)	-0.6576	0.0515	1.2409
rrr (P7)	-0.7978	-0.2653	0.1418

Table 5.32: Eigenvalues of the cycles **rrR** and **rrr** before the first flip bifurcation.

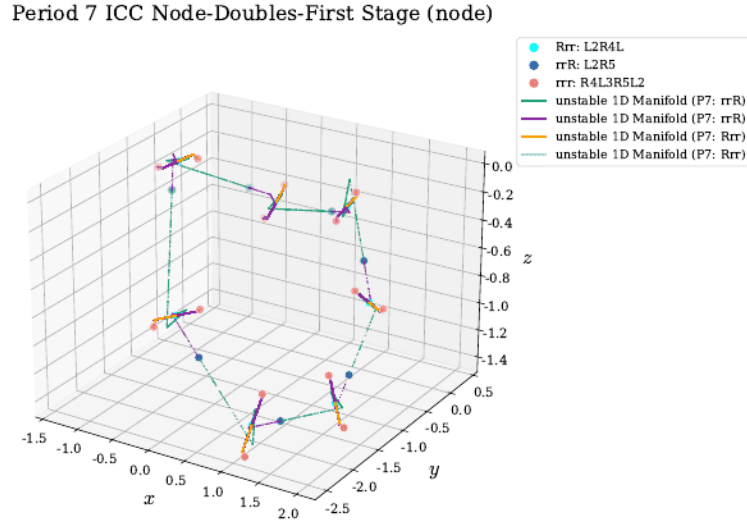


Figure 5.37: Period-7 ICC at the node-doubles-first stage ($\tau_{R,param} = 0.63/\sigma_{R,param} = 0.55$). The saddle cycle **rrR** (blue) is unchanged from Fig. 5.36; the former node has flipped from stable **rrr** to unstable **Rrr** (cyan), giving birth to a new stable period-14 cycle **rrr** (light red). The teal/purple manifold branches are the original saddle's unstable manifold in the tangential direction of the original ICC; the orange/light-blue branches are the new saddle's unstable manifold in the radial/transversal direction, connecting the ICC with the new period 14 node cycle.

Cycle	λ_1	λ_2	λ_3
Rrr (P7)	3.8103	0.7470	0.0006
Rrr (P7)	1.0993	0.0879	0.0093
rrr (P14)	-0.5799	0.6667	0.0033

Table 5.33: Eigenvalues of the period-7 cycles after the first flip, and the resulting period-14 cycle.

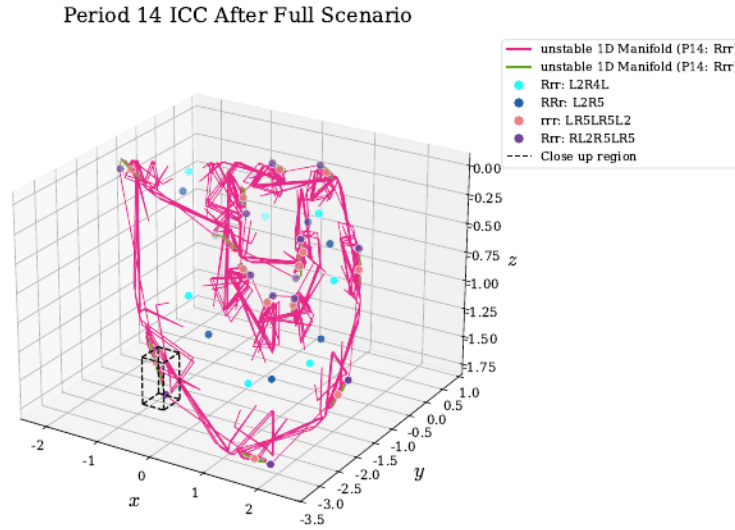


Figure 5.38: Period-14 ICC after the full double flip scenario ($\tau_{R,param} = 0.64/\sigma_{R,param} = 0.688$). The saddle cycle **rrR** (blue) from Fig. 5.37 has gained a second unstable direction and is now a twice as unstable cycle **RRr**; the weakly unstable cycle **Rrr** (cyan) and the stable period-14 node **rrr** (light red) are unchanged. A second, newly born saddle period-14 cycle **Rrr** (dark purple) has appeared with exactly one unstable eigenvalue; its unstable manifold (pink/olive branches) now forms the new, more intricate period-14 ICC. (The orange/light-blue connection manifolds have been left out for visibility reasons)

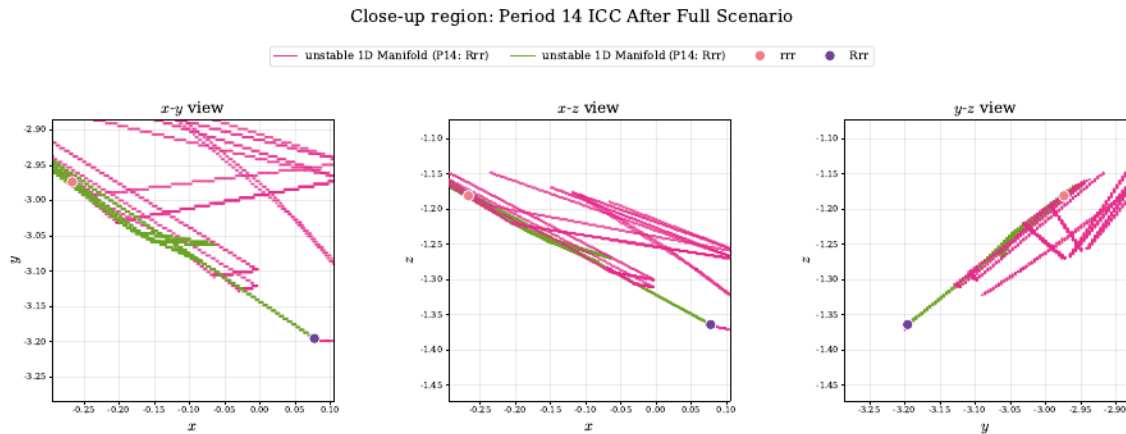


Figure 5.39: Close-up of the phase-space region highlighted by the dashed cube, shown as three orthogonal projections (x - y , x - z , y - z). The cube was chosen so that it contains exactly one point of the stable period-14 cycle **rrr** (coral) and one point of the unstable period-14 cycle **Rrr** (purple); the pink and olive point clouds are the sections of the two unstable 1D manifold branches of **Rrr** that pass through this region.

Cycle	λ_1	λ_2	λ_3
RRr (P7)	6.0459	1.0008	0.0003
Rrr (P7)	1.4685	0.1041	0.0059
rrr (P14)	0.0034	-0.5275	-0.9939
Rrr (P14)	7.8130	0.0006	-0.5363

Table 5.34: Eigenvalues of the period 7 as well as the period 14 cycles after entire ICC flip scenario.

5.5.2.0.1 Verification of the P14 ICC. Both cycles were confirmed to reproduce their tabulated points to machine precision. The P14 rrr has eigenvalues $[0.00337, -0.52749, -0.99394]$ – all real with $|\lambda| < 1$, making it a genuine stable *node* – while the P14 Rrr is a saddle with a strong real unstable eigenvalue ($\lambda = 7.813$). Tracing both branches of the saddle’s unstable manifold confirms the mechanism required for a closed ICC: one branch the saddles unstable 1D manifold converges onto node point 10 (final distance 0.174), while the other branch converges onto node point 11 (final distance 0.074) – two distinct, cyclically *adjacent* points of the period-14 node ($\|x_{10} - x_{11}\| = 1.393$, small relative to the overall extent of the cycle), each approach consistent over the last 200 recorded points of its respective branch. Since the two branches dock onto neighboring points of the same periodic node, the manifold closes up around the entire 14-cycle into a single connected loop, irrespective of the convoluted path taken to get there (See fig. 5.39) – confirming that this parameter location indeed produces a genuine period-14 ICC from an ICC flip bifurcation.

6

Conclusion

In conclusion, the results of the conducted experiments show that the geometry of ICCs in the piecewise-linear setting is far more intricate than in the smooth case. It can also be concluded that finding successful ICC bifurcation scenarios is quite rare in comparison to the sheer number of different parameter configurations that were investigated. In total, across the initial shotgun exploration (approximately 150 randomly sampled configurations) and the subsequent, more targeted follow-up semi automatic investigations of the most promising candidates identified there (roughly 30 configurations each), on the order of 300 parameter configurations were examined. Of these, only a single fully successful ICC doubling scenario (of the Möbius-strip type) could be identified, alongside a handful of configurations exhibiting the degenerate eigenvalue-+1 condition. And not a single successful ICC Neimark–Sacker scenario was found via this method. Even when a boundary analysis yields a promising candidate at which a successful scenario might be expected to occur, there is still no guarantee that the invariant object born from the bifurcation of a cycle will have the correct eigenvalue configuration to actually persist.

This scarcity of successful scenarios is consistent with, and helps to further corroborate, the obstruction mechanisms identified by Keller [26]: incorrect stability of the doubled cycle, border-collision-induced stability transitions, admissibility-bounded doubled cycles, and cascading virtual flip bifurcations. Where Keller’s work established these obstructions analytically for a small number of carefully chosen scenarios, the broad, largely unguided shotgun exploration conducted in this thesis shows that their combined effect is severe enough to make a fully successful doubling scenario a genuinely rare outcome even across several hundred independently sampled parameter configurations – and, at least within the parameter budget of this thesis, to prevent a successful Neimark–Sacker scenario from being observed at all.

This thesis should therefore be understood primarily as a proof of concept. It built the necessary infrastructure and demonstrated that the investigation method employed does yield results – and, given sufficient time and effort, results that can come close to textbook-style examples. At the same time, it also showed that stability-related bifurcations of ICCs in piecewise-linear systems are comparatively rare events: it is far more common for the ICC to simply be destroyed, or for disruptive intermediate bifurcations to occur before both cycles involved can complete the full scenario successfully.

Beyond the bifurcation scenarios themselves, this thesis also contributes a first taxonomy of ICC geometries in the piecewise-linear 3D BCNF. The collection of archetypes compiled in Section 5.2 – grouped into tame geometries, well-behaved geometries, and non-standard geometries, and comprising around fifteen named

archetypes in total – did not exist in this form prior to this work, and should be regarded as a standalone contribution in its own right, independent of the specific bifurcation results discussed above.

Furthermore, this thesis showed that the δ parameters can, and should, be varied – but that beyond the interval $[-1, 1]$, the original interpretation of the parameter plane in terms of the stability triangle is lost.

Two open questions naturally follow from these findings and are left for future work. First, the theoretical obstacle identified in Goal 2 remains unresolved: no general definition currently exists for how to connect the pieces of the two-dimensional surrounding doubling-determining manifold across the kink points introduced by the switching manifold in a nonsmooth system, so the cylinder-versus-Möbius-strip criterion of Biswas, Banerjee, Avrutin and Sushko [23] could not be applied directly to explain the observed doubling behavior. Second, this thesis was restricted throughout to the *continuous* 3D BCNF; Keller [26] showed that introducing a small discontinuity at the switching manifold can relax exactly the admissibility constraints responsible for several of the obstructions observed here, and systematically investigating ICC bifurcations in such a discontinuous extension of the 3D BCNF is a natural next step.

7

Discussion

This chapter revisits the four research questions posed in Chapter 3 in light of the results obtained throughout this thesis, and discusses to what extent – and with what caveats – each of them can be considered answered.

7.1 To what extent can the established theory of stability-related bifurcations of ICCs in smooth 3D maps be extended to piecewise-smooth 3D maps?

The results of this thesis show that the extension is possible, but only partially, and not without qualitative changes to the underlying mechanisms. The general mechanism by which mode-locked ICCs are created, destroyed, and doubled does transfer: an ICC is still built from a saddle-node pair connected by the saddle's unstable manifold (2.1.8.1), resonance tongues might still organize themselves into sausage-string chains bounded by (now) border-collision folds, and non-terminating shrinking points still mark the transition between neighboring sausages, exactly as predicted by the planar theory of Simpson and Meiss [17, 18]. The single fully successful ICC doubling scenario documented in this thesis (Section 5.5.2) confirms that the smooth node-doubles-first/saddle-doubles-first framework of Biswas, Banerjee, Avrutin and Sushko [23] is, in principle, realizable in the piecewise-linear 3D BCNF as well. At the same time, the practical rarity of such successful scenarios (discussed further under Question 2 below) shows that this transfer is far more fragile than the smooth theory alone would suggest: satisfying the local eigenvalue conditions is necessary, but nowhere near sufficient, for the global picture to actually carry over.

A further open point concerns the doubling-type criterion itself. In the smooth theory, whether an ICC doubling produces two disjoint loops or a single loop of double length is decided by whether the two-dimensional manifold surrounding the ICC has a cylindrical or a Möbius-strip topology, diagnosed via the eigenvector rotation along the curve converging to 0 or $\pi \pmod{2\pi}$. Whether this criterion transfers to the piecewise-linear setting at all remains unresolved by this thesis (see Question 2).

7.2 What mathematical obstacles arise when applying smooth bifurcation theory to ICCs in nonsmooth systems, and how can these obstacles be characterized?

Several distinct obstacles were identified over the course of this thesis, at both the theoretical and the practical level.

On the theoretical side, the most significant obstacle is that the two-dimensional manifold used in the smooth theory to decide between the cylindrical and Möbius-strip doubling types is not, in general, a smooth object in the nonsmooth setting: it must be pieced together across the switching manifold, and no established definition currently exists for how to connect its individual pieces at the resulting kink points (Goal 2). As a direct consequence, this thesis was unable to explain, in these geometric terms, why the one successful doubling scenario found took the Möbius rather than the cylindrical form.

A second, more structural obstacle was clarified with reference to the work of Keller [26], who showed that satisfying the classical eigenvalue conditions (A1)–(A4) for an ICC doubling is necessary but far from sufficient in the piecewise-linear setting. Across every scenario Keller investigated in the continuous 3D BCNF, the predicted doubling was obstructed by one of four mechanisms: the doubled cycle acquiring the wrong stability because its degenerate flip curve is virtual; discontinuous, border-collision-induced jumps in the eigenvalue spectrum as the cycle’s symbolic sequence changes; the doubled cycle’s admissibility region being squeezed to nothing between two degenerate flip curves before it can stabilize; and cascades of simultaneous virtual flip bifurcations at higher periods. The shotgun exploration conducted in this thesis (Chapters 5.2, 5.5) provides independent, broad numerical evidence that is consistent with – and helps to further corroborate – Keller’s findings: out of roughly 300 parameter configurations investigated across the initial exploration and the subsequent, more targeted follow-up studies, only a single configuration produced a fully successful ICC doubling, and not a single configuration produced a successful Neimark–Sacker scenario. Where Keller’s result establishes analytically *why* these obstructions occur, the results of this thesis suggest just how severe their combined practical effect is once the search is no longer restricted to a small number of hand-picked scenarios.

On the practical side, a more basic obstacle is simply that, at the outset of this thesis, it was entirely unknown how common or rare successful stability-related ICC bifurcations are within the accessible parameter space. This motivated the shotgun-exploration methodology developed in this thesis (Goal 2) as a practical countermeasure, and the results above quantify, for the first time within this framework, just how rare such successful scenarios in fact are.

7.3 What can ICC structures in piecewise-linear 3D maps look like?

Section 5.2 answers this question directly by compiling a first, non-exhaustive taxonomy of ICC geometries observed in the piecewise-linear 3D BCNF, comprising around fifteen named archetypes grouped into three broad categories: *tame geometries* (Circle, Polygon, One-Kink), which remain close to what the smooth theory would predict; *well-behaved geometries* (Sticks, Curly, Spiky, Curly-Spiky-Mix, Rubberband, Wavy), which develop a consistent, repeating local motif at each saddle-node connection while remaining fully organized; and *non-standard geometries* (Black Hole, Double Manifold Swirls, Needle, Star), in which one or both connecting manifold branches develop genuinely dense, tangled substructure. This taxonomy shows that ICC geometry in the nonsmooth setting is very diverse. In the piecewise-linear BCNF, an ICC can range from a shape nearly indistinguishable from a smooth circle to highly intricate, self-winding structures, with the specific eigenvalue configuration of the underlying saddle-node pair (real, complex, or mixed) shown to have only a minor influence on this outcome. This taxonomy is offered as a standalone contribution of this thesis, independent of the specific bifurcation results discussed under the other research questions.

7.4 What can the stability-related bifurcation mechanisms of ICCs look like in piecewise-linear 3D maps?

Concrete examples of all three targeted stability-related bifurcation mechanisms of ICCs – fold (or in general $+1$), flip, and Neimark–Sacker – were investigated over the course of this thesis, with sharply varying degrees of success. The double border-collision fold plus “ $+1$ ” codimension 2 bifurcation at a non-terminating shrinking point was documented in full detail in Section 5.3, including numerical confirmation of the annihilation-and-rebirth mechanism (one cycle annihilated, the other persisting with a one-letter-changed itinerary and flipped stability) predicted by the theory of Simpson and Meiss [18]. The flip (ICC doubling) was likewise documented end-to-end for one successful period-7 scenario (Section 5.5), following the curated node-doubles-first path from the original ICC, through the intermediate **Rrr/Rrr** configuration, to the resulting doubled-period ICC – while simultaneously confirming, via the shotgun exploration results discussed under Question 2, that scenarios of this kind are the exception rather than the rule. The nonsmooth Neimark–Sacker scenario, by contrast, could not be observed in a fully successful form at all within the scope of parameter ranges chosen for this thesis. However with the parameters of paper [10] a successful double Neimark-Sacker scenario outside the chosen thesis parameter range was found.

Taken together, these findings show that the three stability-related bifurcation mechanisms of ICCs are far from equally accessible in the piecewise-linear 3D BCNF. The ICC fold appears to be the most robust of the three appearing at every non-

terminating shrinking point, the ICC flip occupies a fragile middle ground where success is possible but rare, and the ICC Neimark–Sacker scenario appears, at least based on the parameter budget explored in this thesis, to be the most fragile of all three – though whether this reflects a genuine structural obstruction, analogous to those characterized for the flip case by Keller [26], or simply an insufficiently large search so far, remains an open question for future work.

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A

Appendix

A.1 Delta Variation with stepsize 0.1

This section presents the complete set of results from the δ -variation experiment introduced in Sec. 5.1.2. While the main text discusses a representative subset with a step size of 0.3, the figures below cover the full parameter range $\delta \in [-0.9, 0.9]$ with a finer step size of 0.1, providing a more detailed view of how the stability triangle and the corresponding 2D bifurcation diagram evolve as δ is varied.

A. Appendix

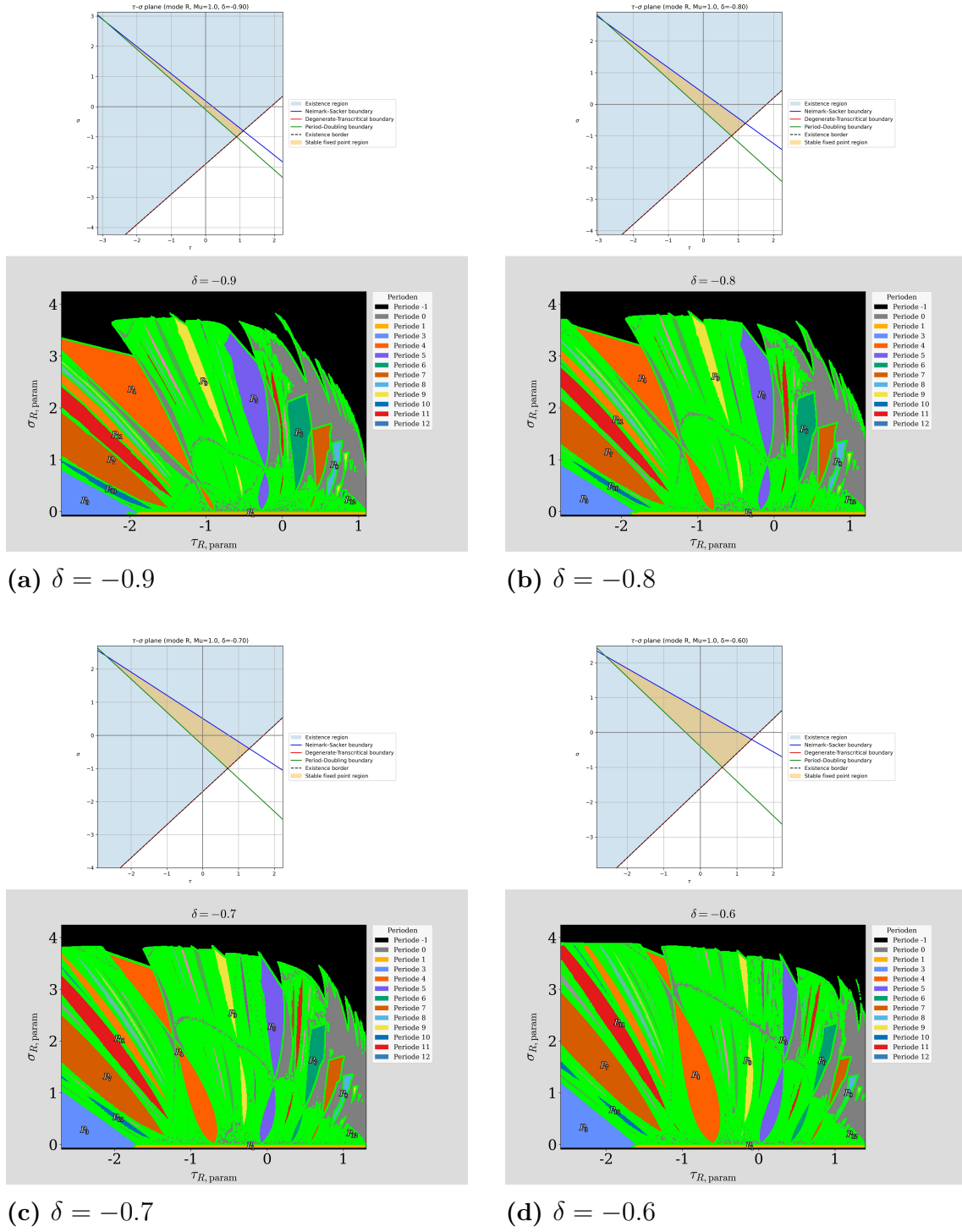


Figure A.1: Combined stability triangle and 2D bifurcation diagrams for $\delta \in [-0.9, -0.6]$.

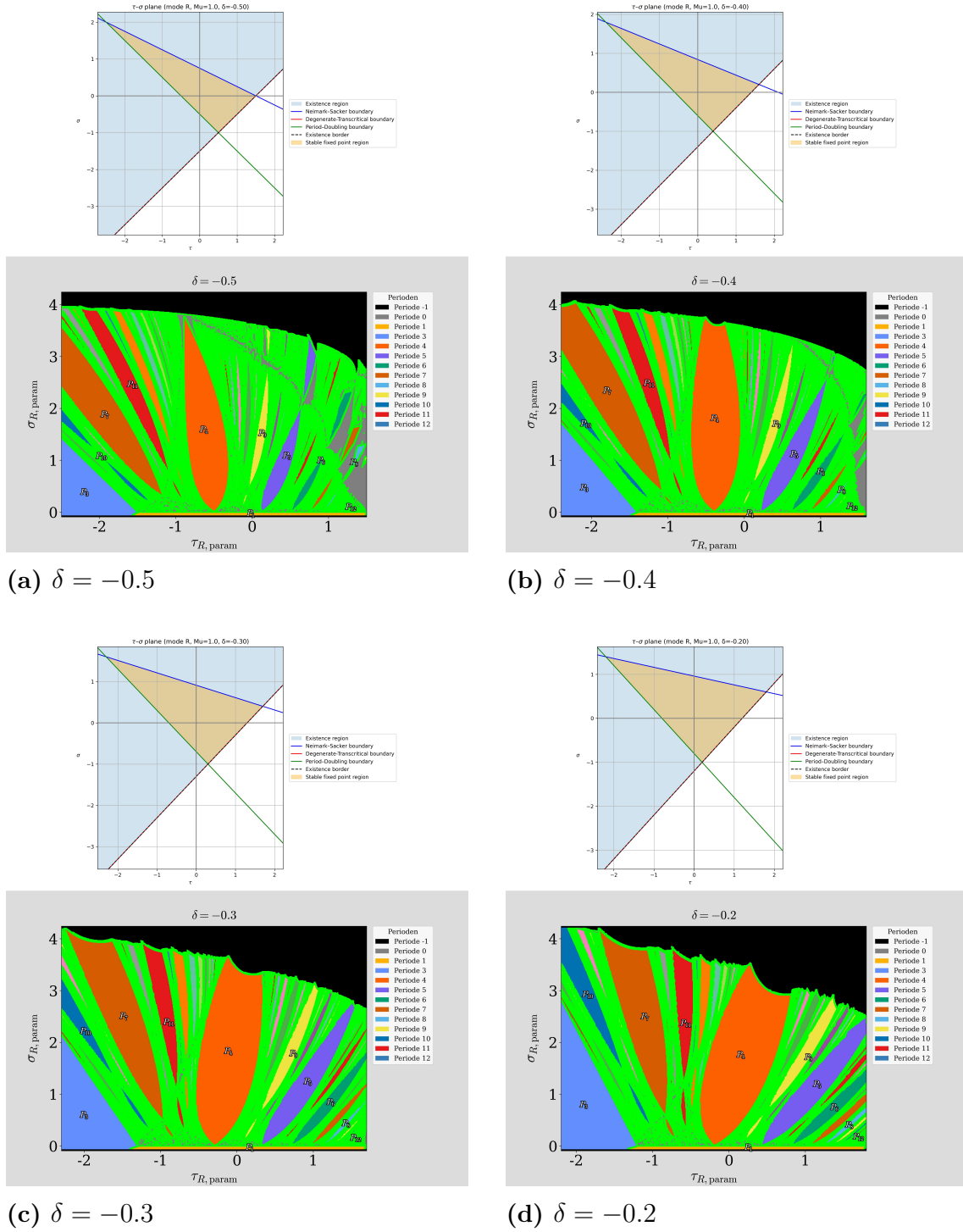


Figure A.2: Combined stability triangle and 2D bifurcation diagrams for $\delta \in [-0.5, -0.2]$.

A. Appendix

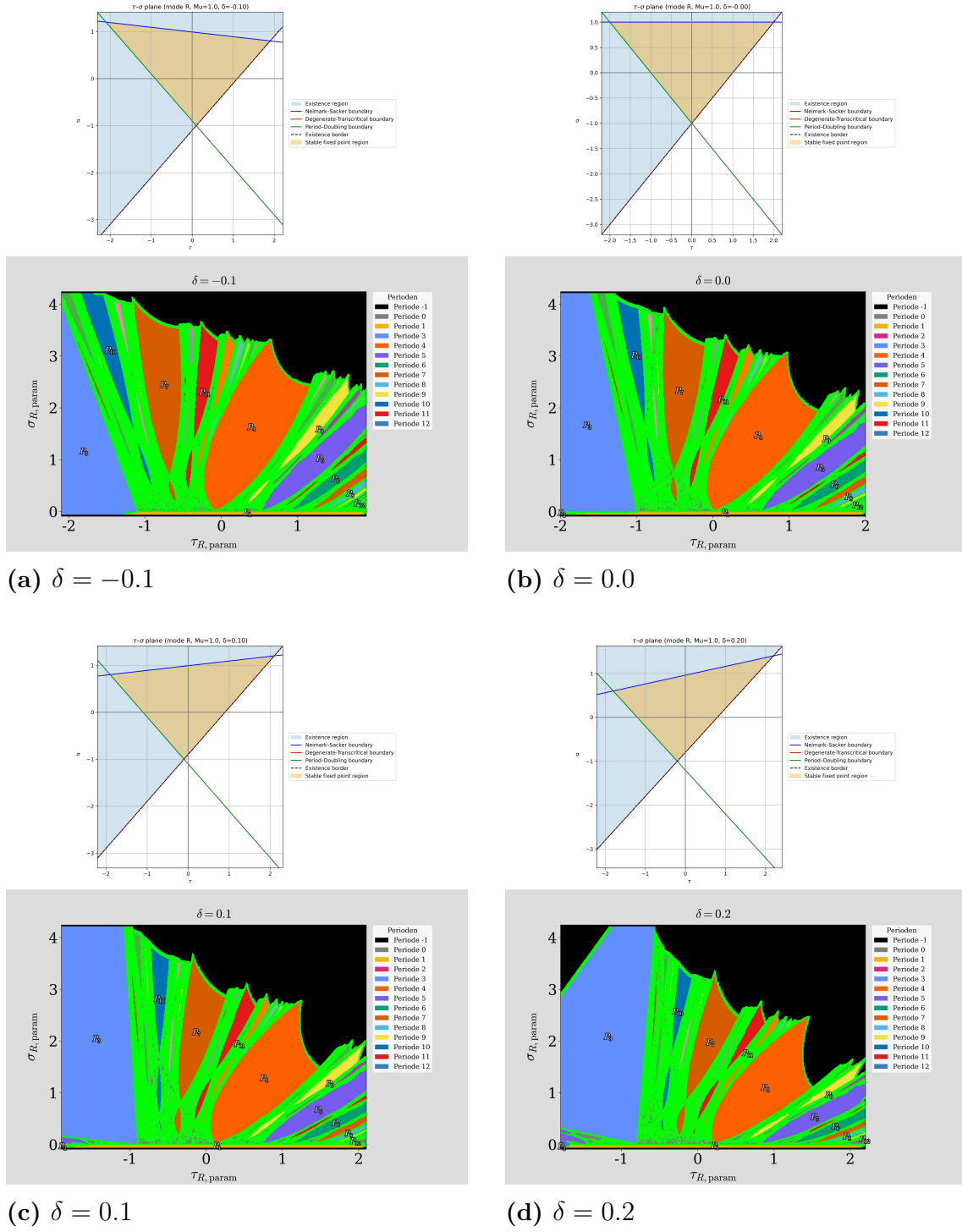


Figure A.3: Combined stability triangle and 2D bifurcation diagrams for $\delta \in [-0.1, 0.2]$.

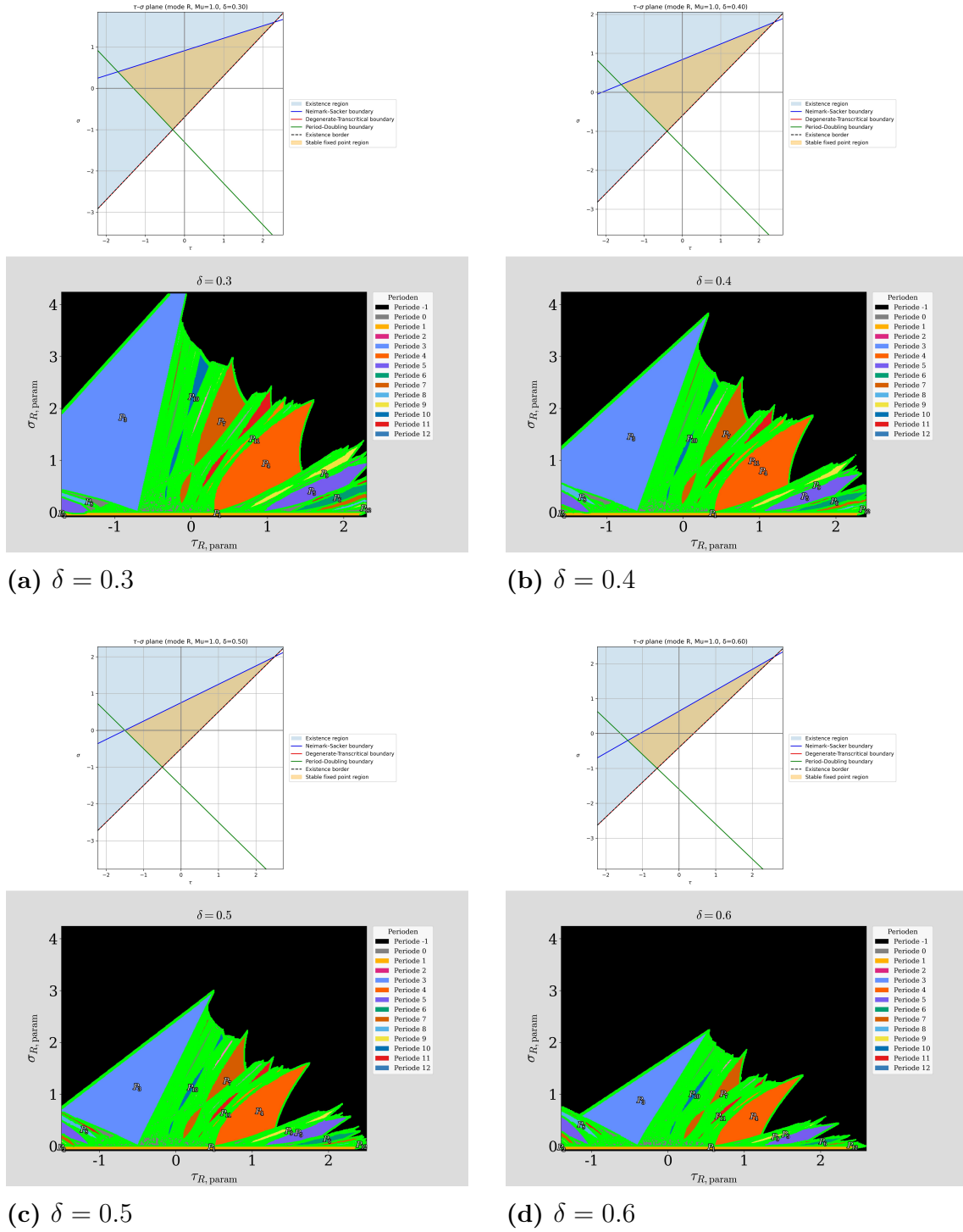


Figure A.4: Combined stability triangle and 2D bifurcation diagrams for $\delta \in [0.3, 0.6]$.

A. Appendix

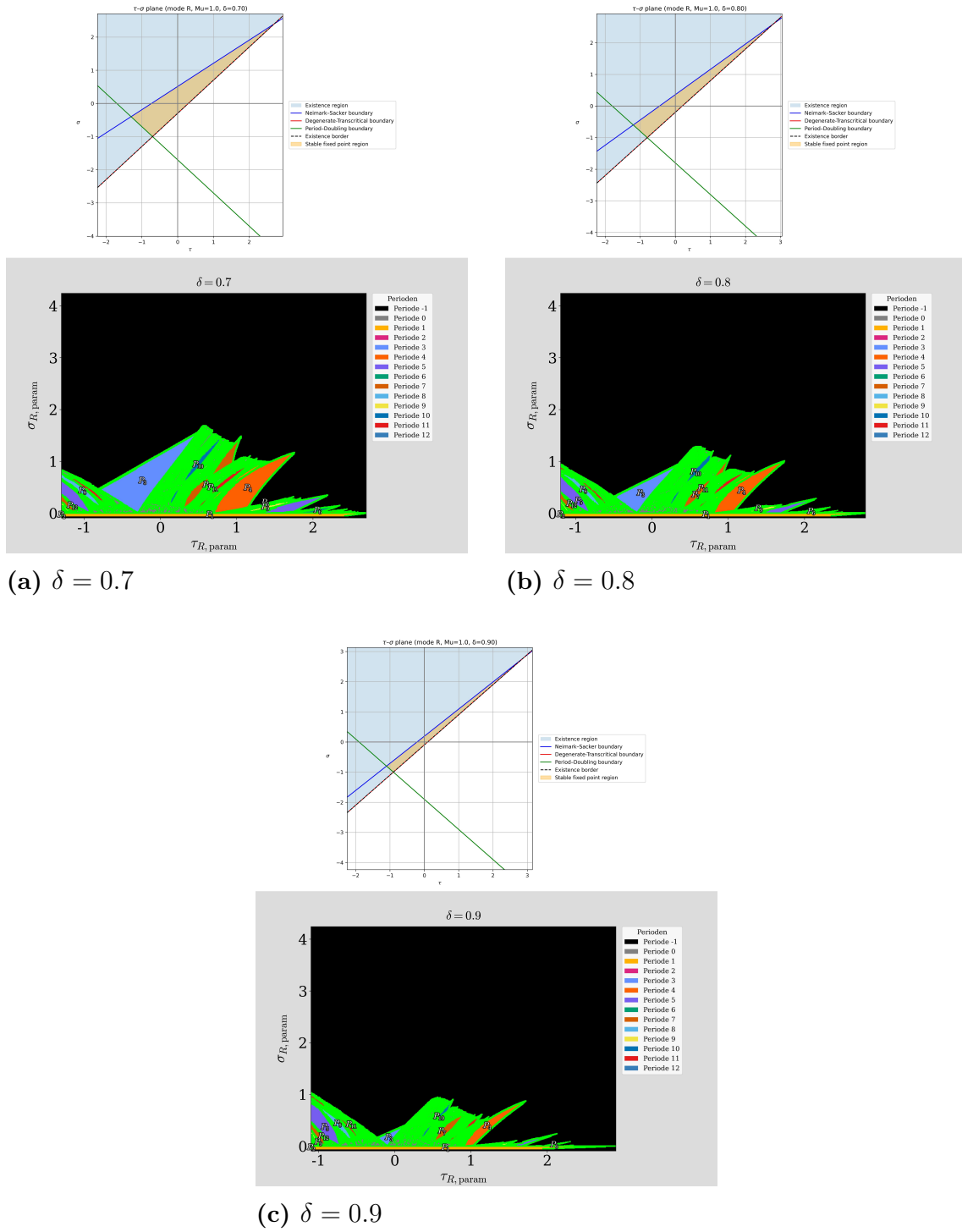


Figure A.5: Combined stability triangle and 2D bifurcation diagrams for $\delta \in [0.7, 0.9]$.

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