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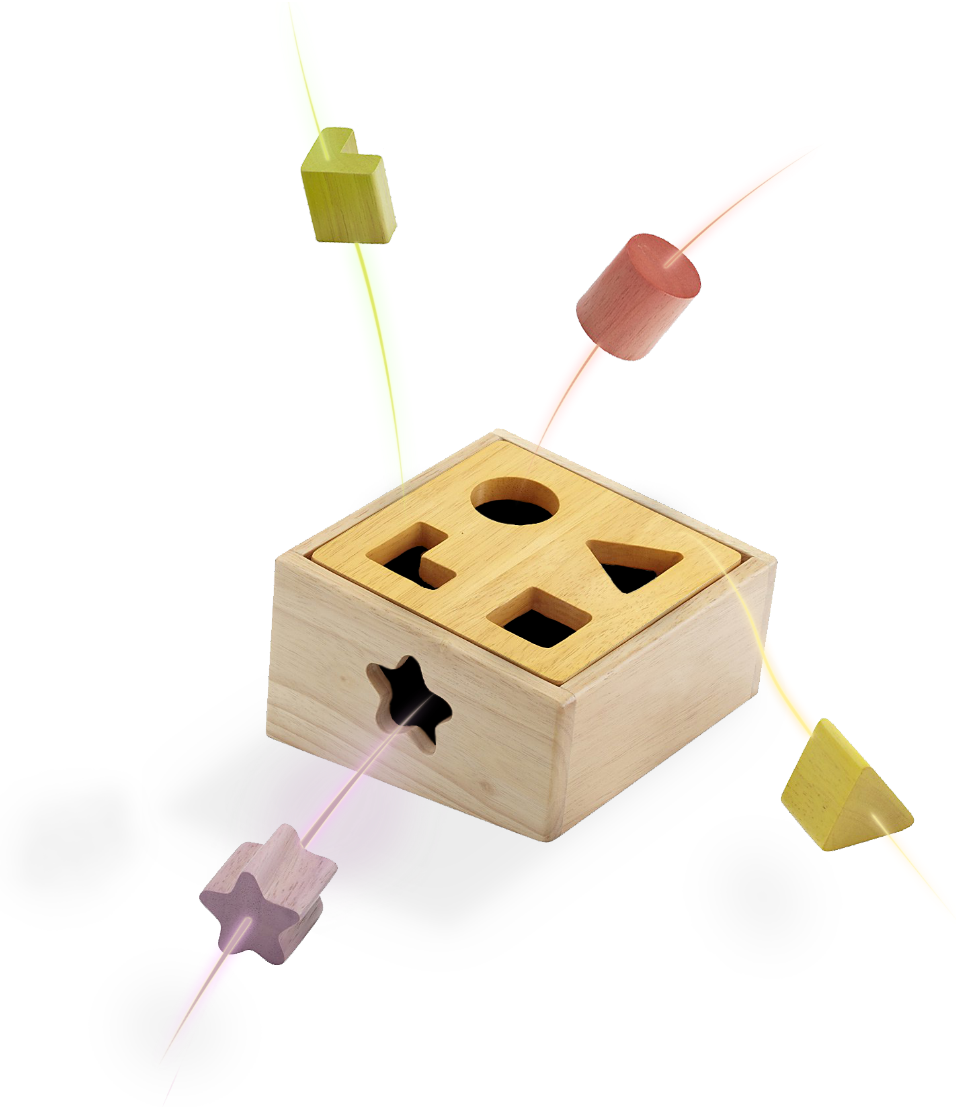

Extremal Black Holes and Nilpotent Orbits

Thesis for the degree Master of Science in Physics and Astronomy

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Abstract

Einstein's equations in general relativity are a set of highly non-linear differential equations. During the 1980's, Breitenlohner and Maison developed techniques to study stationary solutions to them by exploiting hidden symmetries revealed after dimensional reduction. These methods are applicable in general when seeking solutions allowing for one or more Killing vectors. When reducing a gravity theory down to three dimensions the field content can be dualized into a gravity theory coupled to a non-linear sigma model on a symmetric space $\mathcal{G}/\mathcal{H}$. This formulation is manifestly invariant under the Lie groups $\mathcal{G}$ and $\mathcal{H}$ of global, respectively local, transformations which can be used to generate new solutions from known seed solutions. More recent developments, motivated by supersymmetric string theory, has focused on solution classification through the nilpotent orbits of $\mathcal{G}$ as these correspond to certain black hole solutions (so called BPS solutions).

This has so far been done for symmetry groups of finite dimensions. This thesis provides a background to the current attempts to generalize this classification in terms of nilpotent orbits to the infinite dimensional affine Kac-Moody algebras, where it is physically expected but not yet understood. These algebras arise from the hidden infinite dimensional symmetries revealed when reducing down to two dimensions and are thus relevant for black hole solutions with two commuting Killing vectors.

The thesis covers the basics of dimensional reduction with the solution-generating techniques, nilpotent orbits and their classification, affine Kac-Moody Algebras and includes a *Mathematica*-package developed to study conjugation in the affine Lie algebras $\mathfrak{sl}_n^+$ and $\mathfrak{g}_2^+$. It aims at providing a pedagogical introduction and thus bridging the gap between master students and current research.

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Erik Widén, Gothenburg 10/2/14

Contents

1

Introduction

We begin with an introduction and motivation to this thesis aimed to be accessible also to a reader outside the field. We then provide an outline for the following chapters before the actual presentation of the material.

Ever since Einstein published his theory about general relativity in 1915 physicists have looked upon gravity in a completely new manner. In a history relying on Newtonian mechanics the developments were truly groundbreaking and still today, after an entire century of research and experimental evidence, the ideas are so dazzling that many find them hard to believe at first encounter.

The most revolutionary insight of Einstein is that gravity is not to be considered as a force, like those we consider in mechanics, but rather as a consequence of the geometry we live in. We speak about this geometry in terms of a spacetime, the composite object of space and time, and gravity is a manifestation of its curvature.

A first understanding of what a curved geometry is can be gained by imagining an ant on a ball. The ant wandering around on the surface experiences the world as a two dimensional space which, on first sight, does not differ so much from the flat surface of a table. Should the ant draw a circle and measure the ratio between the radius and the circumference, however, it would become evident to the ant that this space differs from the flat table. From this it is clear that the curvature is an intrinsic property of a space and that it is related to the notion of distance. In differential geometry, the language of relativity, distances are described in terms of a metric g, a symmetric matrix, and the curvature by the related Riemann tensor R. The geometry of spacetime and thereby gravity is thus described by the metric g.

The equations which govern the laws for g are the famous Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

and they relate the left hand side, which describes the geometry, with the right hand side, which describes matter. We call a metric g which satisfies these equations for a solution and the set of all such metrics is known as the solution space.

Although extremely beautiful in their simple presentation, the Einstein equations in four spacetime dimensions are a set of ten highly non-linear differential equations and are thus extremely hard to solve. Circumventing this problem by exploiting symmetries to generate new solutions from known ditto is the underlaying task of which this thesis is a part.

Shortly after the birth of general relativity curious solutions to the Einstein equations were found which possessed curvature singularities. These were given the well-known name black holes after the fact that they have what is called an event horizon surrounding the singularity. From beyond this horizon nothing can return, not even light, whence the black hole appears as black. Though argued in the beginning to be a quirk of the theory there is now empirical evidence of existing black holes in the universe, the closest one in the center of our galaxy.

Among the rather few analytical black hole solutions that have been found so far, the simplest ones are the *Schwarzschild* solution which describes a non-rotating, spherically symmetric black hole, the *Reissner-Nordström* solution describing a non-rotating, spherically symmetric black hole with electric and magnetic[1]charge, the *Kerr* solution describing a rotating, axially symmetric black hole without charge and the *Kerr-Newman* solution which is like a Kerr black hole with electromagnetic charge.

The Schwarzschild and Reissner-Nordström solutions are especially interesting for this thesis. The former because it will turn out to be a good starting point for finding new solutions and the latter because it is the simplest example of a black hole that may be *extremal*. Charged black holes actually have two event horizons and in the special case when these coincide the black hole is defined as extremal.

Symmetries

The bearing idea for this thesis and the long-going project of which it is a part is to exploit symmetries to avoid dealing with the Einstein equations directly. Symmetries is the guiding principle in most theoretical physics of today and it is in this context used in much wider sense than in common language. What in physics is meant by a symmetry is nicely put by Hermann Weyl: "*a thing is symmetrical if there is something we can do to it so that after we have done it, it looks the same as it did before*". A butterfly is symmetric under a mirror reflection, a square is symmetric under a rotation of 90° and a sphere is symmetric under any rotation, which is called a continuous symmetry. In this sense also equations can possess symmetries if there is some mathematical operation we can do on them after which they look the same as they did before. What we refer to when using the word symmetry in this thesis is mainly either *i*) geometrical symmetries of the spacetimes described by the solutions or *ii*) symmetries of the governing equations. The latter means that there is one

[1]Magnetic charges have so far never been found in nature but are still interesting from a theoretical point of view.

or more of these mathematical operations, called symmetry transformations, which leaves the equations unchanged. Such a transformation of the Einstein equations may change the metric g but only in such a way that the equation still holds, i.e. looks the same. We say that the equation is invariant under the transformation. The new metric, let us denote it g', must therefore also be a solution. Schematically,

$$(\text{equation true for } g) \quad \xrightarrow{\text{symmetry transformation}} \quad (\text{equation true for } g')$$

If we know a solution (which we call a seed solution) we can thus generate a new one if only we know some apt symmetry transformation. This is the beauty of the idea as we hence can find new solutions to the Einstein equations without actually having to solve them.

The Einstein equations possess symmetries already as they are, the whole relativity theory is invariant under general coordinate transformations, but this thesis focuses on a method to enhance these symmetries further. Through a concept called dimensional reduction one can reveal what is referred to as *hidden symmetries* which provide the transformations to find new solutions. We will come back to this shortly.

In a wide range of theories in physics there is an important symmetry called supersymmetry. This means that all governing equations are symmetric under a transformation between the two particle types that have been found in nature, bosons and fermions. Different theories allow for a different number of supersymmetry transformations which usually is labeled with the letter $\mathcal{N}$ (typically $\mathcal{N} = 1, 2, 4, 8$). Experimental evidence for this symmetry has however not yet been found but there are many reasons to believe that a fundamental theory of the four forces in nature, the electromagnetic force, the weak and strong nuclear forces and gravitation, should possess this supersymmetry.

The most promising candidate for such a unifying theory is supersymmetric string theory, although much research remains to answer this big question. In string theory, mathematics conspires to demand a ten dimensional spacetime to ensure a consistent theory, instead of the four dimensions described in the original general relativity. In order to explain these extra dimensions, which have so far never been observed, it is assumed that these are compact and very small. This means that they only become relevant at extremely high energies, way beyond the reach of mankind in the foreseeable future. By taking the low energy limit of string theory, meaning neglecting different aspects which become irrelevant at lower energies, one can find a wide range of supersymmetric gravity theories depending on how the limit is taken. These theories live in different numbers of dimensions and by studying them one can gain a lot of insights, both about the more complex string theory but also about relativity theory in general. It is thus of great interest to investigate gravity in many different numbers of spacetime dimensions although our everyday appreciation of reality truly is four dimensional.

There is in supergravity a special kind of black hole solutions which are called BPS solutions. They play an important role for understanding these theories as they preserve some of the supersymmetry. Furthermore, the condition which defines the BPS property is also a condition of extremality which bring an extra motivation to study specifically extremal black holes.

Dimensional reduction and the relevance of nilpotent orbits

The method of dimensional reduction considers solutions which have some sort of geometrical symmetry. This can be utilized in an appropriate coordinate system such that one or more coordinates become redundant. It is then possible to employ the mathematical trick of making this coordinate into a compact dimension and neglect it. In such a way the number of spacetime dimensions is reduced. This process involves a splitting of all the objects in the original theory and after all this has been consistently done one ends up with a theory in a spacetime with one dimension less. This theory may look a bit different due to the made splits and, with some additional mathematical reformulations, it can in a very obvious way be shown that it possesses extra symmetries which were not apparent in the original theory. In technical jargon, one preforms a Kaluza-Klein compactification down to three dimensions in which all vector content can be dualized into scalars which describes a non-linear sigma model on a coset space $\mathcal{G}/\mathcal{H}$ symmetric under the global action of $\mathcal{G}$. These hidden symmetries now provide transformations and can be used to generate new solutions.

A special case of these hidden symmetries occur when the dimensional reduction is carried out all the way down to two dimensions. The symmetries in this setting are vastly enhanced to involve what is called infinite dimensional symmetries. These are described in terms of affine Kac-Moody algebras.

It was Geroch who first observed these infinite dimensional symmetries during the 1970's and his work was then brought into a more group theoretical[2] description by Julia. This approach was carried on by the pioneering work of Breitenlohner and Maison during the 1980's. More recent developments have been lead by researchers as Bossard, Nicolai and Pioline among many others. Their study of how the symmetry transformations structure the solution space has revolved a lot around the concept of nilpotent[3] orbits.

The set of all solutions that can be reached by transformations from a given seed solution are called an orbit. To understand the structure of the solution space it therefore comes down to understand orbits. Specifically for extremal black holes, this amounts to study the subset of *nilpotent* orbits as the mathematical property of nilpotency in a certain way corresponds to the property of being extremal. These orbits have been extensively studied by mathematicians, especially due to their prominent role in what is called representation theory, and a lot on this subject has already be done for extremal black holes. In particular [1], [15] and [18] have provided a foundation for this thesis. However, these studies have limited their scope to finite dimensional symmetries and there is now ongoing research on the attempts to generalize the present results to the infinite dimensional Kac-Moody algebras in two dimensional theories. Not much on this has yet been published and it is in precisely this gap where this thesis attempts to provide a pedagogical introduction. This

[2] Group theory is the mathematical framework for handling the symmetry transformations.

[3] An operator X is nilpotent if there exists an integer n such that $X^n = 0$.

work has additionally led to the development of two *Mathematica*-packages for calculation of adjoint actions in some Kac-Moody algebras intended to be used as a tool for future work in this direction.

1.1 Outline and Prerequisites

The purpose of this thesis is to provide an overall introduction to the research on nilpotent orbits of infinite dimensional algebras in dimensionally reduced gravity theory. Accordingly it is roughly divided into two halves where the first four chapters are concerned with the concepts in the context of finite dimensional symmetries. The second half then takes the step into the setting of infinite dimensional symmetries and their algebras.

The outline of the thesis is as follows. The second chapter on sigma models on symmetric spaces gives a description of the mathematical setting of dimensionally reduced gravity theories. The intention is to give an understanding of the structures involved without any specific reference to gravity and without going into too much detail. It is then followed by chapter three on the dimensional reduction of gravity theories and the use of hidden symmetries. The chapter is repeatedly exemplified by calculations in pure gravity reduced from four to three dimensions and alternates between example and general comments. The chapter ends with the important charge matrix which leads to the motivation on the study of nilpotent orbits. This is the subject of chapter four which once again focuses on giving a mathematical background without so much specific references to the gravity context. The second half of the chapter is however more concerned with the applications in physics and also contains examples of the use in minimal supergravity in five dimensions.

Chapter five is the first to introduce the infinite dimensional symmetries which is done through the compactification of four dimensional pure gravity to two dimensions. With this motivation the basics of affine Kac-Moody algebras are covered in chapter six and ends with a short introduction to affine orbits and the current state of research. The seventh and final chapter contains documentation for the two developed *Mathematica*-packages. A few appendices follow the thesis to clarify some concepts and calculations and contain a presentation of some group elements which conjugate between the simple and highest root vectors in the algebras $\mathfrak{sl}_3, \mathfrak{sl}_4, \mathfrak{sl}_5$ and $\mathfrak{g}_2$ as potentially interesting information for future work.

Prerequisites

This thesis is intended to be as accessible as possible to a master student in theoretical physics. This means that a basic knowledge of group theory, differential geometry, gauge theory and general relativity from now on is assumed but that many calculations are done explicitly and most concept are introduced in way to also catch readers less acquainted with these subjects.

2

Sigma Models on Symmetric Spaces

This chapter will provide the necessary mathematical background and notation for the description of dimensionally reduced gravity theories reformulated to sigma models. We begin with the definition of symmetric spaces, which is the structure of the target manifolds in all the relevant sigma models, and describe their geometry. Most importantly we construct a metric which will be the central object in the Lagrangians of the compactified gravity theories. From this we move on to discuss the sigma models themselves and illustrate them through the examples of models on $SL(2,\mathbb{R})/SO(2)$ and $SL(2,\mathbb{R})/SO(1,1)$. These will reoccur in the examples through out the following chapters.

The presentation is focused on the mathematical aspects and the connection to physics will be done first in the next chapter.

2.1 Symmetric Spaces

A **symmetric space** $\mathcal{G}/\mathcal{H}$ for a Lie group $\mathcal{G}$ is a space where there exists an involution σ on $\mathcal{G}$, i.e. an automorphism on the group squaring to the identity, where $\mathcal{H}$ is a subgroup of the σ-invariant subset of $\mathcal{G}$. The involution has the identity as a fixed point and induces thereby an involution[1] σ on the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. The invariant subset $\mathfrak{h} = \{Z \in \mathfrak{g} \mid \sigma(Z) = Z\}$ forms a subalgebra and is the Lie algebra to the subgroup $\mathcal{H}$. Being an involution, σ has the eigenvalues ± 1 with eigenspaces $\mathfrak{h}, \mathfrak{m}$ respectively and $\mathfrak{g}$ splits into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, as a direct sum of vector spaces. Moreover, for σ to generate a symmetric space it has to fulfill three criteria regarding $\mathfrak{h}$ and its complement $\mathfrak{m}$:

$$\begin{aligned} [\mathfrak{h},\mathfrak{h}] &\subset \mathfrak{h} \\ [\mathfrak{h},\mathfrak{m}] &\subset \mathfrak{m} \\ [\mathfrak{m},\mathfrak{m}] &\subset \mathfrak{h} \ , \end{aligned} \tag{2.1}$$

[1]We denote the induced involution with the same symbol.

i.e. $\mathfrak{h}$ must be a subalgebra, $\mathcal{G}/\mathcal{H}$ must be reductive and the complement must bracket into the subalgebra. Furthermore, the two subspaces $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal in $\mathfrak{g}$ with respect to the Killing form, following from that the Killing form respects any automorphism. Note that since $\mathfrak{h}$ is not an ideal in $\mathfrak{g}$ the subgroup $\mathcal{H}$ is not normal in $\mathcal{G}$ and, accordingly, the symmetric space $\mathcal{G}/\mathcal{H}$ does not form a group. In the following $\mathcal{G}/\mathcal{H}$ will be referred to as both a symmetric space, coset space and sometimes quotient space.

A special case of symmetric spaces is when the subgroup $\mathcal{H}$ is the maximally compact subgroup $\mathcal{K}$ of $\mathcal{G}$. There is a unique (up to conjugation by $\mathcal{H}$) involution θ on $\mathfrak{g}$ which yields this construction called the **Cartan involution** [15]. It splits the algebra into

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\theta(\mathfrak{k}) = \mathfrak{k}$ is the corresponding Lie algebra to $\mathcal{K}$ and where, of course, $\mathfrak{k}$ and $\mathfrak{p}$ satisfy the corresponding relations to (2.1). This is called a **Cartan decomposition**. Expressed in the Chevalley basis, the action of θ is

$$\theta(H_i) = -H_i, \quad \theta(E_i) = -F_i, \quad \theta(F_i) = -E_i.$$

Example 1. One of the simplest examples of a symmetric space is $SL(2,\mathbb{R})/SO(1,1)$ for which the involution on the Lie algebra is $\sigma(E) = F, \sigma(H) = -H$. The invariant subspace $\mathfrak{h} = \text{span}\{E+F\}$ is indeed the Lie algebra for the subgroup $SO(1,1)$ and with $\mathfrak{m} = \mathbb{R}(E-F) \oplus \mathbb{R}H$ the relations (2.1) are readily checked.

Example 2. Changing the involutive automorphisms in the example above to the Cartan involution we have the action $\theta(E) = -F, \theta(H) = -H$ and get the symmetric space $SL(2,\mathbb{R})/SO(2)$ with $\mathfrak{k} = \text{span}\{E-F\}$, the generator of the maximal compact subgroup $SO(2)$, and $\mathfrak{p} = \mathbb{R}(E+F) \oplus \mathbb{R}H$.

The involution σ can be used to construct a **generalized transpose** on the group elements which will turn out very useful in the following. For a group element[2] $g = \exp[Z]$ in $\mathcal{G}$ with Z in $\mathfrak{g}$, the generalized transpose is defined as

$$g^{\mathcal{T}} = \exp[Z]^{\mathcal{T}} := \exp[-\sigma(Z)]. \tag{2.2}$$

For the special case of the Cartan involution and the matrix groups $SL(n)$, this construction yields the ordinary matrix transpose. A general involution σ generalizes in this way the defining property of $SO(n)$, the maximal compact subgroup of $SL(n)$, to the subgroup $\mathcal{H}$ defined by σ. Thus, for an element h in $\mathcal{H}$ we have

$$h^{\mathcal{T}} = h^{-1} \tag{2.3}$$

and it is this property which will be exploited below.

[2]For the definition(s) of and associated discussion about the exponential map between Lie algebras and Lie groups, see appendix A.

Additionally, the generalized transpose satisfies

$$\begin{aligned}(g^{-1})^{\mathcal{T}} &= (\exp[-Z])^{\mathcal{T}} = \exp[\sigma(Z)] = (g^{\mathcal{T}})^{-1} \\ (gh)^{\mathcal{T}} &= h^{\mathcal{T}} g^{\mathcal{T}},\end{aligned} \tag{2.4}$$

where the second property can be proven by use of the Baker-Campbell-Hausdorff formula. On the Lie algebra level the generalized transpose acts as

$$Z^{\mathcal{T}} = -\sigma(Z). \tag{2.5}$$

2.1.1 Geometrical structure

The symmetric space $\mathcal{G}/\mathcal{H}$ is a manifold which can be endowed with a metric. In fact, the common use of the Killing form as a metric on a Lie group $\mathcal{G}$ can be transferred to the symmetric space $\mathcal{G}/\mathcal{H}$ and the corresponding unique Levi-Civita connection has the property that its geodesics and the one-parameter subgroups coincide. Let us look more closely into how these concepts are defined.

A metric on $\mathcal{G}$

As the metric on the coset space is deduced from the metric on $\mathcal{G}$ we begin with reviewing that construction and introducing the notation.

A Lie group $\mathcal{G}$ defines a diffeomorphism on itself through the left-action[3] $L_{\cdot} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ as in

$$L_g f = gf \qquad g, f \in \mathcal{G} \tag{2.6}$$

which induces the pushforward $L_{g*} : T_f\mathcal{G} \to T_{gf}\mathcal{G}$ on the tangent bundle. We define a **left-invariant vector field** V to be a vector field on $\mathcal{G}$ which satisfies

$$V\Big|_{gf} = L_{g*}V\Big|_{f} \qquad \forall g, f \in \mathcal{G}\ . \tag{2.7}$$

There is a one-to-one correspondence between left-invariant vector fields on $\mathcal{G}$ and the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ and any left-invariant vector field V is defined through equation (2.7) by its value at the identity $V\Big|_{\mathrm{id}}$. As the Lie bracket satisfies [4]

$$\varphi_*\left([V, W]\right) = [\varphi_* V, \varphi_* W]$$

[3]and also analogously through the right-action.

[4]*Proof:* Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism and $f \in \mathcal{F}(\mathcal{N})$ be a function on $\mathcal{N}$. Then

$$\begin{aligned}(\varphi_*[V, W])f &= [V, W](f \circ \varphi) = V\big[W[f \circ \varphi]\big] - W\big[x[f \circ \varphi]\big] \\ &= V\big[(\varphi_* W)[f] \circ \varphi\big] - W\big[(\varphi_* V)[f] \circ \varphi\big] \\ &= \varphi_* V\big[(\varphi_* W)[f]\big] - \varphi_* W\big[(\varphi_* V)[f]\big] \\ &= [\varphi_* V, \varphi_* W] \qquad \square\end{aligned}$$

for any diffeomorphism φ we have in particular that

$$L_{g*}\left([V,W]\Big|_f\right) = \left[L_{a*}V\Big|_f, L_{g*}W\Big|_f\right] = [V,W]\Big|_{gf}. \tag{2.8}$$

By use of the induced left-action we can construct the **Maurer-Cartan form** ω as a Lie algebra valued 1-form on the Lie group, i.e.

$$\begin{aligned}\omega_g : T_g\mathcal{G} &\to \mathfrak{g}\\ V &\mapsto L_{g^{-1}*}V\end{aligned}$$

where the isomorphy $T_{\mathrm{id}}\mathcal{G} \cong \mathfrak{g}$ is silently taken as an identification. A common way to denote the Maurer-Cartan form is $\omega_g = g^{-1}\mathrm{d}g$ and should be interpreted as $\mathrm{d}g$ being the identity operator on the tangent space and g^{-1} the pushforward induced by left translation, that is

$$g^{-1}\mathrm{d}g\Big|_g = L_{g^{-1}*}\Big[\partial_i \otimes \mathrm{d}\phi^i\Big] : T_g\mathcal{G} \to T_{\mathrm{id}}\mathcal{G}$$

where $g \in \mathcal{G}$ and ϕ^i are coordinates on the group. The notation originates from the study of matrix groups in which $\mathrm{d}g$ denotes taking the exterior derivative of each matrix element in g.

We can use the Maurer-Cartan form to use any non-degenerate symmetric bilinear form $\mathrm{B}_\rho : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g}$ to define a metric γ on $\mathcal{G}$ through

$$\gamma(V,W) = \mathrm{B}_\rho\Big(\omega_g(V), \omega_g(W)\Big) \quad V,W \in T_g\mathcal{G}. \tag{2.9}$$

Defining the bilinear form B_ρ as the trace of the product of its two arguments calculated in the representation ρ, the cyclic property of the trace furthermore makes the metric invariant under the adjoint action of G. In particular, for ρ being the adjoint representation of $\mathfrak{g}$ we get the **Killing form**

$$\mathrm{B}_{\mathrm{ad}}(X,Y) = \mathrm{K}(X,Y) := \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \qquad X,Y \in \mathfrak{g}.$$

As proven in appendix A, the Killing form is invariant under any automorphisms on $\mathfrak{g}$ and thus in particular the induced left-action. This makes the $\mathcal{G}$-left-action an isometry of the metric γ.

Induced metric on $\mathcal{G}/\mathcal{H}$

With the metric (2.9) on $\mathcal{G}$ we now turn to the coset space $\mathcal{G}/\mathcal{H}$. As the Maurer-Cartan form takes its values in $\mathfrak{g}$ we can split it into two parts

$$\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}, \tag{2.10}$$

where $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{m}}$ are the projections onto $\mathfrak{h}$ and $\mathfrak{m}$ respectively[5]. This can be written by the use of the involution σ

$$\begin{aligned} \omega_{\mathfrak{h}} &:= \frac{1}{2}\left(\omega + \sigma(\omega)\right) \\ \omega_{\mathfrak{m}} &:= \frac{1}{2}\left(\omega - \sigma(\omega)\right). \end{aligned} \tag{2.11}$$

As $\mathfrak{m}$ contains the generators for $\mathcal{G}/\mathcal{H}$ it is natural to try to define the metric as something in the direction of the sketch

$$\gamma_{\text{"}\mathcal{G}/\mathcal{H}\text{"}}(V,W) = \mathrm{B}_\rho\left(\omega_{\mathfrak{m},g}(V), \omega_{\mathfrak{m},g}(W)\right), \quad V,W \in T_g\mathcal{G}. \tag{sketch}$$

This loosely denoted expression could be interpreted as a metric on $\mathcal{G}/\mathcal{H}$ but where its action is defined at any, but a single, representative g of $g\mathcal{H}$ which also is an element of $\mathcal{G}$. For this to make any sense there cannot be any dependence on the particular representative such that the metric must be invariant under the right-action of $\mathcal{H}$. That this is indeed the case will be motivated below.

A more refined way of defining the metric on the coset space is to view $\mathcal{G}$ as a principal fiber bundle with $\mathcal{H}$-fibers,

$$\mathcal{H} \hookrightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{G}/\mathcal{H}.$$

The base manifold is then $\mathcal{G}/\mathcal{H}$ and choosing particular representatives for elements in $\mathcal{G}/\mathcal{H}$ amounts to choosing a section $s : \mathcal{G}/\mathcal{H} \to \mathcal{G}$ (also called choosing a gauge). The pullback of the Maurer-Cartan form by this section, $s^*\omega$, also splits as in equation (2.10) and it can be shown [27] that the $\mathfrak{m}$-projected part transforms as

$$(s_2^*\omega_{\mathfrak{m}}) \to \mathrm{Ad}_{h^{-1}}(s_1^*\omega_{\mathfrak{m}}) \tag{2.12}$$

when moving between two sections $s_1 = s_2 h$ with transition map $h \in \mathcal{H}$. This transformation rule makes $s^*\omega_{\mathfrak{m}}$ appropriate to use in B_ρ as the latter is invariant under the adjoint action of $\mathcal{G}$ and thereby of $\mathcal{H}$. Hence, the more precise definition of the metric

$$\gamma_{\mathcal{G}/\mathcal{H}}(V,W) = \mathrm{B}_\rho\left(s^*\omega_{\mathfrak{m},g}(V), s^*\omega_{\mathfrak{m},g}(W)\right), \qquad V,W \in T_{g\mathcal{H}}\mathcal{G}/\mathcal{H} \tag{2.13}$$

is independent on the particular section and can be used as a metric on $\mathcal{G}/\mathcal{H}$.

A way of justifying the proposed sketch of the metric on $\mathcal{G}/\mathcal{H}$ is to restrict the attention to matrix groups[6] and study the right-action of an h in $\mathcal{H}$ by employing the notation $\omega_g = g^{-1}\mathrm{d}g$:

$$g^{-1}\mathrm{d}g \to (gh)^{-1}\mathrm{d}(gh) = h^{-1}\mathrm{d}h + h^{-1}g^{-1}\mathrm{d}gh.$$

From this we find

$$\omega_{\mathfrak{m},g} = \frac{1}{2}\Big(g^{-1}\mathrm{d}g - \sigma(g^{-1}\mathrm{d}g)\Big) \to \frac{1}{2}\Big(h^{-1}\mathrm{d}h + h^{-1}g^{-1}\mathrm{d}gh - \sigma(h^{-1}\mathrm{d}h + h^{-1}g^{-1}\mathrm{d}gh)\Big)$$

[5]The part $\omega_{\mathfrak{m}}$ is often denoted with a P in the literature and when we switch focus to the applications in gravity we will also adopt this convention.

[6]As these will be the groups of relevance through out this thesis this restriction causes no problem.

and since $h^{-1}\mathrm{d}h \in \mathfrak{h}$ and $\sigma(h^{-1}\omega h) = h^{-1}\sigma(\omega)h$, which straightforwardly can be proven by the use of the Baker-Hausdorff-Campbell formula and the σ-invariance of $\mathfrak{h}$, we get

$$\omega_{\mathfrak{m}} \to h^{-1}\omega_{\mathfrak{m}}h,$$

under the right-action of $\mathcal{H}$. This conjugating transformation corresponds to equation (2.12) which ensures that the metric in the equation marked (sketch) is independent of the particular coset representative.

What we more or less have done here is to define a connection on the principal fiber bundle $\mathcal{G}$. As $\omega_{\mathfrak{h}}$ is parallel to the fibers $\mathcal{H}$ it takes its values in the *vertical tangent space*, to adopt the general nomenclature, and its kernel is the *horizontal tangent space* in which $\omega_{\mathfrak{m}}$ lives. Although we gloss over a lot of the details here, we can use $\omega_{\mathfrak{h}}$ as a connection 1-form and define an exterior covariant derivative acting on a Lie algebra-valued 1-form[7] ψ as

$$D\psi := \mathrm{d}\psi + \frac{1}{2}[\omega_{\mathfrak{h}} \wedge \psi]. \tag{2.14}$$

Here $[\omega_{\mathfrak{h}} \wedge \psi]$ denotes[8] the commutator of Lie algebra-valued forms defined through

$$[\xi \wedge \eta] := \xi \wedge \eta - (-1)^{pq}\eta \wedge \xi$$

for p-form ξ and q-form η. For the special case of of 1-forms in the exterior covariant derivative we can also write equation (2.14) as

$$D\psi(V,W) = \mathrm{d}\psi(V,W) + \frac{1}{2}\Big([\omega_{\mathfrak{h}}(V), \psi(W)] - [\omega_H(W), \psi(V)]\Big).$$

In this context we also note that the so called structure equation for the Maurer-Cartan form reads

$$\mathrm{d}\omega + \omega \wedge \omega = 0$$

which yields similar conditions for $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{m}}$, where in particular

$$\mathrm{d}\omega_{\mathfrak{m}} + [\omega_{\mathfrak{h}} \wedge \omega_{\mathfrak{m}}] = 0, \tag{2.15}$$

that is

$$D\omega_{\mathfrak{m}} = 0.$$

Analogous expressions hold also for the forms pulled back by a section $s : \mathcal{G}/\mathcal{H} \to \mathcal{G}$.

[7] The form must actually be *horizontal* for this expression to hold, which means that it has to vanish when acting on a vector within the vertical tangent space. This holds however in all occasions this derivative occurs in this thesis.

[8] This commutation operator is also often notated as $[\xi, \eta]$ but to minimize the potential confusion with the Lie bracket we stick to $[\xi \wedge \eta]$.

Geodesics on the coset space

As will be shown in section 3.3.2, black hole solutions depending on one variable correspond to geodesics in the coset space why we devote a few words to the topic.

As familiar, geodesics are curves along which their corresponding tangent vector is parallel transported and are hence defined in terms of a connection ∇. Through out this thesis we will use the Levi-Civita connection corresponding to the metric on $\mathcal{G}$ (or $\mathcal{G}/\mathcal{H}$), i.e. the unique connection defined by demanding metric compatibility and vanishing torsion

$$\nabla_X g(V,W) = g(\nabla_X V, W) + g(V, \nabla_X W)$$
$$\nabla_V W - \nabla_W V - [V,W] = 0,$$

for vector fields V, W and X on $\mathcal{G}$. A part from ensuring integral curves of one-parameter subgroups to be geodesics, as proved in appendix A, this choice also provides a link between geodesics in $\mathcal{G}$ and geodesics in $\mathcal{G}/\mathcal{H}$ as all geodesics with respect to the Levi-Civita connection in $\mathcal{G}/\mathcal{H}$ is given by the projection onto $\mathcal{G}/\mathcal{H}$ of a geodesic in $\mathcal{G}$ with a tangent vector in $\mathfrak{m}$ at some point [15, p. 41]. When studying geodesics in $\mathcal{G}/\mathcal{H}$ further on, we thus can use the standard exponential map in $\mathcal{G}$ on vectors in $\mathfrak{m}$.

2.2 Sigma Models on Symmetric Spaces

A sigma model is based on scalar maps ϕ^i from a D-dimensional manifold $\mathcal{X}$ equipped with a metric g, called the base manifold, to a target manifold $\mathcal{M}$ with metric γ. The $\phi^i(x)$:s are functions on $\mathcal{X}$ with its coordinates x^μ, and constitute in turn the coordinates on $\mathcal{M}$. We will always denote the base manifold indices with Greek letters, also in less than four dimensions, and the target manifold indices with Latin letters. The model is described by the action [22, p. 132]

$$S = \int_{\mathcal{X}} \mathrm{d}^D x \sqrt{g} g^{\mu\nu}(x) \partial_\mu \phi^i(x) \partial_\nu \phi^j(x) \gamma_{ij}\big(\phi(x)\big). \tag{2.16}$$

We see that the Lagrangian is in fact the pullback of the target metric to the base manifold. The name sigma model originates from when the type of models first was introduced in particle physics but carries no longer any meaning beyond the mere name.

In the relevant sigma models arising in dimensionally reduced gravity theories, the base manifold $\mathcal{X}$ will be a more or less dimensionally reduced version of spacetime and the target manifold $\mathcal{M}$ will be some coset space $\mathcal{G}/\mathcal{H}$ of the form discussed in section 2.1. The particular coset space will depend on both the theory and the reduction procedure as will be discussed further below.

In order to connect the action (2.16) to the coset space objects defined above we must work out the coordinate mappings ϕ^i. Depending on the topology of $\mathcal{G}/\mathcal{H}$ there might or might not exist global coordinates and might or might not exist a global decomposition of

$\mathcal{G}$ suitable for the description of $\mathcal{G}/\mathcal{H}$. In the special case of $\mathcal{H}$ being the maximal compact subgroup $\mathcal{K}$ there is a general result stating that

$$\mathcal{G} = \mathcal{K}\exp[\mathfrak{p}] \tag{2.17}$$

where $\mathcal{G}/\mathcal{K}$ is homoemorphic to $\mathfrak{p}$ through the exponential mapping. Additionally, each connected semisimple Lie group $\mathcal{G}$ with maximal compact subgroup $\mathcal{K}$ can be decomposed as

$$\mathcal{G} = \mathcal{K}\mathcal{A}\mathcal{N} \quad \text{or equivalently} \quad \mathcal{G} = \mathcal{N}\mathcal{A}\mathcal{K},$$

where $\mathcal{A}$ is an abelian subgroup generated by the non-compact elements of a Cartan subalgebra[9] , i.e. the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, and $\mathcal{N}$ is a nilpotent subgroup generated by the positive root vectors (alternatively the negative root vectors). This is called the **Iwasawa decomposition**. As the parameters of $\mathfrak{p}$ provide global coordinates for $\mathcal{G}/\mathcal{K}$ through the decomposition (2.17), so does a parametrization of $\mathcal{G}/\mathcal{K} = \mathcal{N}\mathcal{A}$ in the Iwasawa decomposition [15, p. 47][22, p. 41].

2.2.1 Parametrization

Let us illustrate these more general comments more concretely and parametrize the coset space. We exemplify each step with the two simple but physically relevant spaces $SL(2,\mathbb{R})/SO(2)$ and $SL(2,\mathbb{R})/SO(1,1)$ and through them develop some general tactics. In the former case we have $\mathcal{K} = SO(2)$ and we can employ the unique Iwasawa decomposition for each group element g in $\mathcal{G}$,

$$g = nak, \quad n \in \mathcal{N}, a \in \mathcal{A}, k \in \mathcal{K}.$$

Explicitly, in the defining matrix representation we have

$$\mathfrak{n} = \text{span}\{E\} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \mathfrak{a} = \text{span}\{H\} = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$
$$\mathfrak{k} = \text{span}\{E - F\} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

such that we can write an arbitrary element V of $\mathcal{G}$ who is also representative of $\mathcal{G}/\mathcal{K}$ as

$$V = nak \in [na] \in G/K = SL(2,\mathbb{R})/SO(2)$$

with

$$\begin{aligned} V &= \exp[\chi E]\exp[\phi H]k \\ &= \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\phi/2} & 0 \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix} k = \begin{pmatrix} \mathrm{e}^{\phi/2} & \mathrm{e}^{-\phi/2}\chi \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix} k\ . \end{aligned} \tag{2.18}$$

[9] A Cartan subalgebra is a maximal set of commuting generators.

The two dimensional space $SL(2,\mathbb{R})/SO(2)$ is thus parametrized by the two coordinates $\phi^i = (\phi, \chi)$, which in the sigma model are spacetime dependent functions $\phi^i(x)$. Bigger coset spaces naturally require more scalar fields as coordinates. As any choice of k changes the representative but preserves the particular coset, it can be chosen freely and the most convenient choice is $k = \mathrm{id}$. This fixes a section in the fiber bundle picture or, equivalently, the choice of gauge. The form (2.18) is particularly simple when trying to read off the fields ϕ^i. However, when transforming V with a group element g, this property is lost if we do not make sure that we stay within the gauge. To respect the gauge choice we let a compensating gauge transformation $k\Big(g, V(x)\Big)$ accompany g, depending on both g and V. Acting from the right k ensures that the form (2.18) is restored even after the transformation:

$$V \to gVk(g,x), \qquad g \in \mathcal{G}, k \in \mathcal{K}. \tag{2.19}$$

We from now on abbreviate the dependence of k on V to directly depend on x.

In practise, however, it is very difficult to find the correct compensating transformation k and in order to circumvent it we can make use of the generalized transpose (2.2) and construct

$$M = VV^{\mathcal{T}}. \tag{2.20}$$

Under the transformation (2.19) this group element M has the desirable transformation law

$$M \to gVkk^{\mathcal{T}}V^{\mathcal{T}}g^{\mathcal{T}} \overset{(2.3)}{=} gVV^{\mathcal{T}}g^{\mathcal{T}} = gMg^{\mathcal{T}}, \tag{2.21}$$

which eliminates any question about k. Furthermore, the Maurer-Cartan form ω_M at M turns out to be related in a very preferable way to the Maurer-Cartan form ω_V as

$$\begin{aligned}
\omega_M &= M^{-1}\mathrm{d}M = (VV^{\mathcal{T}})^{-1}\,\mathrm{d}(VV^{\mathcal{T}}) \\
&= (V^{\mathcal{T}})^{-1}V^{-1}\,\mathrm{d}VV^{\mathcal{T}} + (V^{\mathcal{T}})^{-1}\mathrm{d}(V^{\mathcal{T}}) \\
&= (V^{\mathcal{T}})^{-1}\left[V^{-1}\mathrm{d}V + \mathrm{d}V^{\mathcal{T}}(V^{\mathcal{T}})^{-1}\right]V^{\mathcal{T}} \\
&\overset{(2.4)}{=} (V^{\mathcal{T}})^{-1}\left[\omega_V + (V^{-1}\mathrm{d}V)^{\mathcal{T}})\right]V^{\mathcal{T}} \\
&\overset{(2.5)}{=} (V^{\mathcal{T}})^{-1}\left[\omega_V - \theta(\omega_V)\right]V^{\mathcal{T}} \\
&\overset{(2.11)}{=} (V^{\mathcal{T}})^{-1}\,2\omega_{\mathfrak{m},V}\,V^{\mathcal{T}}.
\end{aligned} \tag{2.22}$$

This is especially neat as the coset metric γ appears in the action (2.16) and we have

$$\mathrm{B}(\omega_M, \omega_M) \overset{(2.22)}{=} 4\,\mathrm{B}(\omega_{\mathfrak{m},V}, \omega_{\mathfrak{m},V}) = 4\gamma_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j \tag{2.23}$$

where we used the conjugation invariance and bilinearity of B. Acting on a base manifold vector push-forwarded by the coordinate maps ϕ^i this yields

$$\omega_M\left(\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial}{\partial\phi^i}\right) = M^{-1}\frac{\partial M}{\partial\phi^j}\mathrm{d}\phi^j\left(\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial}{\partial\phi^i}\right) = M^{-1}\frac{\partial M}{\partial\phi^i}\frac{\partial\phi^i}{\partial x^\mu} = M^{-1}\partial_\mu M$$

and evaluated in some representation ρ we can now rewrite the action (2.16) as

$$S = \int_{\mathcal{X}} \mathrm{d}^D x \sqrt{g} g^{\mu\nu} \frac{c_\rho}{4} \operatorname{Tr}\left[M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\right], \tag{2.24}$$

where c_ρ is a representation dependent constant appearing in the formulation of B as the trace. Thus we can work with M directly and do not have to bother finding any compensating transformation.

Explicitly for $SL(2,\mathbb{R})/SO(2)$ and V as in (2.18) we have

$$g^{\mu\nu}\frac{1}{4}\operatorname{Tr}\left[M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\right] = \frac{1}{2}\left((\partial\phi)^2 + \mathrm{e}^{-2\phi}(\partial\chi)^2\right), \tag{2.25}$$

where $(\partial\phi)^2 = \partial_\mu\phi\partial^\mu\phi$.

This sigma model adapted way of parametrizing the coset space carries over also for cases where $\mathcal{H}$ is not the maximal compact subgroup $\mathcal{K}$ with only minor complications. Illustrated by the coset space $SL(2,\mathbb{R})/SO(1,1)$, we can still write a coset representative as

$$\begin{aligned} V &= nah, \quad n\in\mathcal{N}, a\in\mathcal{A}, h\in\mathcal{H}=SO(1,1),\\ V &= \begin{pmatrix} \mathrm{e}^{\phi/2} & \mathrm{e}^{-\phi/2}\chi \\ 0 & \mathrm{e}^{-\phi/2}\end{pmatrix} h \end{aligned} \tag{2.26}$$

but this decomposition is no longer unique and global, as is the Iwasawa decomposition above. We recall the involution σ from example 1 with its invariant subspace $\operatorname{span}\{E+F\} = \mathfrak{h}$.

We still can make sure to stay within this parametrization with the transformation law (2.19) (but with a $h(g,x)$ instead of $k(g,x)$ of course) and once again this calls for a way of avoiding the explicit $h(x)$. We construct an M but in the case of non-compact subgroups $\mathcal{H}$ we may alter the definition slightly:

$$M = VV^{\mathcal{T}}\eta.$$

Here we have included η which is the metric preserved by the subgroup $\mathcal{H} = SO(1{,}1)$, i.e.

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defined from

$$h\eta h^T = \eta, \quad \forall h\in SO(1,1).$$

^{T} denotes ordinary matrix transpose and employing it further on we limit our scope to matrix groups. Including η in the definition of M does not alter the relation (2.22) in the metric as any constant matrix to the right and left of $VV^{\mathcal{T}}$ cancel in that construction.

However, its inclusion actually throws M out of $\mathcal{G}$ as η is not an element there of. This might appear worrying but as equation (2.22) holds and stripping down to $VV^{\mathcal{T}}$ is simple in calculations there is no actual problem. The point is to make M symmetric in the matrix sense:

$$M = V\eta V^T$$

since

$$V^{\mathcal{T}} = \eta V^T \eta^{-1},$$

which follows from the defining property of the general transpose.

Using the parametrization of equation (2.26) we find the metric on the coset model $SL(2,\mathbb{R})/SO(1,1)$ to be

$$g^{\mu\nu}\frac{1}{4}\,\mathrm{Tr}\left[M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\right] = \frac{1}{2}\left((\partial\phi)^2 - \mathrm{e}^{-2\phi}(\partial\chi)^2\right). \qquad (2.27)$$

We note that it is identical to the metric of $SL(2,\mathbb{R})/SO(2)$ in equation (2.25) apart from the sign between the two terms.

Summary

The general work flow learned from these two examples are

- parametrize a big enough part of the coset space by a coset representative V
- construct an $M = AVV^{\mathcal{T}}B$, where A and B are suitable constant matrices, satisfying the relation (2.22)
- write Lagrangian in terms of the Maurer-Cartan form of M in the coset metric.

3

Gravity Theories and Dimensional Reduction

In this section the concept of dimensional reduction will be given a rather detailed explanation and the relevant Lagrangians with corresponding equations of motion will be presented. The resulting Lagrangians will be cast into the forms of the non-linear sigma models presented in the previous chapter and the former hidden but now manifest symmetries will be discussed and exemplified. The dependence on the compactification details of precisely which symmetry is found will be discussed, in particular the compact/non-compactness dependence on timelike or spacelike compactification and the general procedure of finding a suitable representative in the coset space. We then discuss the transformations between solutions more generally and end the chapter with an account on the important charge matrix which will motivate the further study of nilpotent orbits.

3.1 General Concept

Gravity theories based on general relativity exhibit invariance under general coordinate transformations and possess no *a priori* preferred coordinate system. These symmetries are manifestly built into the theories but as special solutions to the Einstein equations were studied, first by Ehlers in [10], additional symmetries were revealed which are not explicitly seen in the original formulations of the theories. Because of this, these symmetries are said to be "hidden" and in order to find them and make them explicit a reformulation of the original theory is needed, based on some symmetry restrictions imposed on the solutions. More concretely, one assumes the existence of one or more Killing vectors, as is the case for e.g. stationary and axially symmetric solutions. These isometries allow for the method of Kaluza-Klein **compactification**. As the isometries ensure independence of the coordinates along the Killing vectors there is no loss of information in making these directions compact, i.e. imposing an equivalence relation for the compactified coordinates $x^{\tilde{\mu}} \sim x^{\tilde{\mu}} + l^{\tilde{\mu}}$ where

$l^{\tilde{\mu}}$ is the length of the dimension expressed in the particular coordinate system and a ˜ over the index denotes that the dimension is compact[1]. In theories which assume existing extra compact dimensions, such as string theory and quantum gravity, the non-apparent dependence on them are explained through their smallness which in a Fourier expansion excludes all but the zero-mode at accessible energies. In our case though, there is no need of such motivations as the independence on the compact dimensions is already assumed through the initial symmetry criteria. We thus have reduced the number of dimensions in our problem, whence the name **dimensional reduction**.

Starting from D dimensions and having compactified n dimensions, the original D dimensional objects split up into smaller ones; there is not room for a D dimensional object in a $(D-n)$ dimensional spacetime. As we still demand invariance under general coordinate transformations this put narrow frames around how to carry out these splits. The new $(D-n)$ dimensional objects should have tensorial behaviours and the resulting $(D-n)$ dimensional theory should respect the original symmetries. With a suitable splitting of the metric a D dimensional gravity theory reduces into a $(D-n)$ dimensional gravity theory coupled to additional fields resulting from the splits of the metric and the original matter fields. The precise field content naturally depends on the number of dimensions and the original theory.

When reducing down to the special case of *three* dimensions the very useful opportunity arises to **dualize** vectors into scalars. Although not true in general for arbitrary fields in three dimension, the field content arising from dimensionally reduced gravity theories can entirely be expressed by a set of scalars and the resulting action can be identified as a non-linear sigma model. Suddenly with this rewritten action, the "hidden" symmetries become apparent; the sigma model explicitly exhibit some model dependent invariance which previously could not be seen.

Once these symmetries are established the field is open to exploit them. Starting from a known seed solution formulated into this sigma model we can transform it with any element of the symmetry group. As the Lagrangian is completely invariant under this operation the solution property is left untouched and we can thus generate new solutions to the Einstein equations without actually having to solve them. This is a remarkable possibility and the true strength and gain with the method!

3.1.1 Infinite dimensional hidden symmetries

The existence of hidden symmetries goes even further. If one is reducing all the way down to two dimensions there are usually more than one way of getting there. Depending on the route and in what steps the dualizations are made, one can arrive at slightly different two dimensional versions of the theory, each with its own symmetry. The full symmetry of the theory is thus exposed first when the cross-action of these transformations are studied, i.e. letting the symmetry transformations of one version act on another, and it turns out that

[1] This equivalence relation simply makes the dimension to a circle but there are of course less trivial ways of making dimensions compact.

the resulting symmetries are infinite dimensional, realizing affine Kac-Moody algebras. We will however save this for a later chapter and stay in three dimensions for the time being.

Let us now illustrate the general description through the simplest example.

3.2 Dimensional Reduction of Pure 4D Gravity

Let us start with the Einstein-Hilbert action of pure gravity in four dimensions

$$S^{(4)}_{\text{E-H}} = \int \mathrm{d}^4x \, \sqrt{g^{(4)}} R^{(4)}, \tag{3.1}$$

where $g = |\det g_{\mu\nu}|$ and R is the Ricci scalar. The index $^{(4)}$ denotes the dimension in order to differentiate between the compactified versions of an object. We now assume the existence of a Killing vector and with this isometry we can compactify along its dimension without loosing any information in the theory. The resulting theory will depend on whether the Killing vector is space- or timelike but the calculations are almost identical. We thus wait to specify this until it actually makes a difference.

We begin with an ansatz for how the dimensional reduction splits up the four dimensional metric. The choice of how to do this while preserving the symmetries for the new objects is not obvious on first sight. It has, however, been done before and learning from that we make the following choice and will motivate it in a moment:

$$g^{(4)}_{\alpha\beta} \doteq \begin{pmatrix} g_{33}^{-1} g_{\mu\nu} + g_{33}\mathcal{A}_\mu\mathcal{A}_\nu & g_{33}\mathcal{A}_\mu \\ g_{33}\mathcal{A}_\nu & g_{33} \end{pmatrix}, \tag{3.2}$$

where

$$g_{33} = \mathrm{e}^{\phi}$$

is called the dilaton and $g_{\mu\nu}$ will turn out to be the three dimensional metric. $\mathcal{A}_\mu$ is the obtained vector field from the four dimensional metric's $g^{(4)}_{\mu 3}$ components. In this example we drop the indices $^{(3)}$ indicating the three dimensional quantities to avoid clutter. Hence $\mu = 0, 1, 2$ and all objects without a "dimensional index" are three dimensional. Note that since we have not yet specified whether the compactified direction is space- or timelike but chosen its coordinate to be x^3, we have not yet chosen the convention of which coordinate is timelike. For a spacelike Killing vector and thus a spacelike x^3-coordinate we choose the signature $(-, +, +, +)$ with the zero coordinate for time while for a timelike Killing vector, we switch the convention around and choose signature $(-, -, -, +)$ as we compactify along x^3. The important difference between these cases is the resulting signature of the three dimensional metric $g_{\mu\nu}$; spacelike compactification gives Lorentzian signature while timelike compactification yields Riemannian.

There are two reasons to why this ansatz is the preferable choice. Firstly, the listed objects resulting from it transform properly under general coordinate transformations:

- $g_{\mu\nu}$ is independent of the coordinate in the compactified dimension and transforms as a 2-tensor under coordinate transformations in the *uncompactified* dimensions. Under coordinate transformations in the *compactified* dimension which depends on the compactified dimension, $g_{\mu\nu}$ scales with a constant factor.

- $\mathcal{A}_\mu$ is also independent of the compactified coordinate. It transforms as a Maxwell field, i.e. as a vector under uncompactified coordinate transformations and with a $U(1)$ gauge under compactified coordinate transformations.

- g_{33} behaves as a scalar under uncompactified coordinate transformations and scales with a constant factor under compactified coordinate transformations.

Secondly, this choice of ansatz takes the four dimensional action (3.1) into the compactified three dimensional action

$$S^{(3)}_{\text{E-H}} = \int \mathrm{d}^3x \left(\sqrt{g}R - \frac{\sqrt{g}}{2}(\partial\phi)^2 - \frac{\sqrt{g}}{4}\mathrm{e}^{2\phi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \right), \tag{3.3}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$. I.e. the compactified action is a three dimensional Einstein-Hilbert action with an additional scalar field and a Maxwell field. That this is indeed true is not easily seen but requires a rather lengthy calculation. We do not present it here but it can be found in [26, p. 24] in full detail.

3.2.1 Dualization

Having found the expression for the three dimensional action we can now exploit the special feature of three dimensions that 2-forms dualize to 1-forms. This means that we can express the field strength $\mathcal{F}_{\mu\nu}$ in terms of a scalar. We do this through thinking of $\mathcal{F}_{\mu\nu}$ as an independent variable, i.e. forgetting that it is the exterior derivative of $\mathcal{A}_\mu$. Instead we encode this information into the Lagrangian by the use of a Lagrangian multiplier χ with the constraint that $\mathcal{F}_{\mu\nu}$ should satisfy the Bianchi identity

$$\partial_{[\rho}\mathcal{F}_{\mu\nu]} = 0. \tag{3.4}$$

As this is normally a consequence of $\mathcal{F}_{\mu\nu}$ being exact, such a constraint captures the same information about $\mathcal{F}_{\mu\nu}$ even if we now make $\mathcal{F}_{\mu\nu}$ an arbitrary 2-form. Adding the constraint with a suitable factor we can write the Lagrangian as

$$\mathcal{L}^{(3)}_{\text{E-H}} = \sqrt{g}R - \frac{\sqrt{g}}{2}(\partial\phi)^2 - \frac{\sqrt{g}}{4}\mathrm{e}^{2\phi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\sqrt{g}}{2}\chi\epsilon^{\mu\nu\rho}\partial_\mu\mathcal{F}_{\nu\rho}, \tag{3.5}$$

where ϵ denotes the Levi-Civita tensor[2]. This Lagrangian carries precisely the same information as the one used in the action (3.3); it is just reformulated. The equation of

[2] We will use ϵ for the Levi-Civita *tensor* and ε for the Levi-Civita *symbol*.

motion for the scalar multiplier χ gives obviously back the constraint (3.4). The equation of motion for $\mathcal{F}_{\mu\nu}$ is as usual found from the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\mathcal{F}_{\mu\nu}} - \partial_\rho \frac{\delta\mathcal{L}}{\delta(\partial_\rho\mathcal{F}_{\mu\nu})} = -2\frac{\sqrt{g}}{4}\mathrm{e}^{2\phi}\mathcal{F}^{\mu\nu} - \frac{1}{2}\partial_\rho(\sqrt{g}\chi\epsilon^{\rho\mu\nu}) = 0$$

and since $\sqrt{g}\epsilon^{\rho\mu\nu} = \varepsilon^{\rho\mu\nu}$ is spacetime independent we get

$$\mathcal{F}^{\mu\nu} = -\mathrm{e}^{-2\phi}\epsilon^{\rho\mu\nu}\partial_\rho\chi. \tag{3.6}$$

Note that this expresses the dual relationship $\mathcal{F} = -\mathrm{e}^{-2\phi}\star\mathrm{d}\chi$. Substituting[3] equation (3.6) back into the Lagrangian (3.5) the second last term becomes

$$-\frac{\sqrt{g}}{4}\mathrm{e}^{2\phi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{\sqrt{g}}{4}\mathrm{e}^{-2\phi}\epsilon^{\rho\mu\nu}\epsilon_{\lambda\mu\nu}\partial_\rho\chi\partial^\lambda\chi = (-1)^{s+1}\frac{\sqrt{g}}{2}\mathrm{e}^{-2\phi}(\partial\chi)^2$$

where we in the last step have used

$$\epsilon^{\rho\mu\nu}\epsilon_{\lambda\mu\nu} = (-1)^s\, 2\,\delta^\rho_\lambda$$

and s depends on the signature of $g_{\mu\nu}$:

$$s = \begin{cases} 0 & g_{\mu\nu} \text{ Riemannian (Killing vector timelike)} \\ 1 & g_{\mu\nu} \text{ Lorentzian (Killing vector spacelike)} \end{cases}.$$

The last term becomes

$$\begin{aligned}\frac{\sqrt{g}}{2}\chi\epsilon^{\mu\nu\rho}\partial_\mu\mathcal{F}_{\nu\rho} &= \frac{\sqrt{g}}{2}\chi\epsilon^{\mu\nu\rho}\partial_\mu(-\mathrm{e}^{-2\phi}\epsilon_{\lambda\nu\rho}\partial^\lambda\chi) \\ &= \partial_\mu\Big(\frac{\sqrt{g}}{2}\chi\epsilon^{\mu\nu\rho}\Big)\mathrm{e}^{-2\phi}\epsilon_{\lambda\nu\rho}\partial^\lambda\chi) + \text{total divergence} \\ &= (-1)^s\sqrt{g}\mathrm{e}^{-2\phi}(\partial\chi)^2\end{aligned}$$

and thus our final Lagrangian is

$$\mathcal{L}^{(3)}_{\text{E-H}} = \sqrt{g}\Big(R - \frac{1}{2}\Big((\partial\phi)^2 - (-1)^s\mathrm{e}^{-2\phi}(\partial\chi)^2\Big)\Big). \tag{3.7}$$

The scalar part is familiar and for $s = 1$ we recognize it as the sigma model for $SL(2,\mathbb{R})/SO(2)$ as written in equation (2.25) and its action (2.24) ($c_\rho = 1$) while $s = 0$ provides precisely the sign flip which turns the sigma model into $SL(2,\mathbb{R})/SO(1,1)$, as in equation (2.27). Thus we have found that the action in both cases can be written as

$$S = \int_{\mathcal{X}} \mathrm{d}^D x\sqrt{g}\Big(R - g^{\mu\nu}\frac{1}{4}\,\mathrm{Tr}\left[M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\right]\Big).$$

[3]This is perfectly allowed since the equation of motion (3.6) for $\mathcal{F}$ is algebraic, i.e. not a differential equation [23, p. 37].

3.3 Reduced Gravity Theory

The example of dimensional reduction from four to three dimensions in pure gravity exhibits many of the general features of the method, although the starting theory may have more fields than just the metric and D may be larger than four.

The general picture is a starting theory in D dimensions containing the metric, a number of vector fields and additional scalars. This theory is normally the bosonic part of a supersymmetric gravity theory, e.g. such which may be obtained from low energy limits of different string theory compactifications. With the existence of n commuting Killing vectors, for which their action on the matter fields also is zero, the compactification on a n-torus[4] down to $(D-n)$ dimensions can be made. This naturally now includes similar splits of the vector fields as we did with the metric in the preceding example. Reformulating the vector fields into form language, each p-form existing in a $\tilde{D}$ dimensional theory will give rise to one p-form and one $(p-1)$-form in the $(\tilde{D}-1)$ dimensional theory[5]. For each step in the reduction we thus get a rapidly growing set of objects in addition to the total n new vector fields and $\frac{n(n+1)}{2}$ new scalars which originate from the metric.

The set of scalar fields in the $(D-n)$ dimensional theory is now in general invariant under some global transformation under which the resulting vector content transforms linearly.

In this manner one may obtain a great number of four dimensional theories as one sooner or later arrives at $(D-n)=4$. Among these we only study those theories in which the scalars in this four dimensional version already constitute a sigma model on some symmetric space $\mathcal{G}_4/\mathcal{H}_4$ with the semisimple Lie group $\mathcal{G}_4$. We further more demand the vector fields to transform under a representation $\mathfrak{l}_4$ of the same group [1, 15].

When continuing down to the special case of $D-n=3$ all these vector fields can be dualized to scalars, just as done in the example above, and the resulting full set of scalars exhibit a global invariance under a Lie group $\mathcal{G}$ of which the "original" symmetry group $\mathcal{G}_4$ is a subgroup. The field content originating from different parts of the four dimensional Lagrangian may here interplay and enlarge the symmetry to a much bigger group than $\mathcal{G}_4$. All in all the Lie algebra $\mathfrak{g}$ consists of [1]

$$\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{g}_4 \oplus (\mathbf{2} \otimes \mathfrak{l}_4) \tag{3.8}$$

where $\mathfrak{sl}(2,\mathbb{R})$ corresponds to the Ehlers group we have seen explicitly in the example above.

The dimensionally reduced Lagrangian does now contain gravity coupled to a non-linear sigma model on the coset space $\mathcal{G}/\mathcal{H}$ where $\mathcal{H}$ is a real-form, compact or non-compact, of the maximally compact subgroup $\mathcal{K}$ of $\mathcal{G}$. The Lagrangian can be written precisely on the form

$$\mathcal{L} = \sqrt{g}\left(R - \frac{1}{4}\operatorname{Tr}\left[M^{-1}\partial_\mu M\ M^{-1}\partial^\mu M\right]\right) \tag{3.9}$$

[4]This is the simplest way of compactifying n dimensions and while there are many other possible compact manifolds, this is what we exclusively deal with in this thesis.

[5]We will not show the calculations here but a good example is provided in [23, p. 8].

where now M is an element of $\mathcal{G}/\mathcal{H}$ constructed just as in section 2.2. Some examples of these coset spaces are given in table 3.1 together with their corresponding theories.

Table 3.1: Some examples given by Breitenlohner and Maison in [21] of the resulting coset spaces in compactification to four and three dimensions for some gravitation theories.

Coset space		Theory
$D \to 4$	$D \to 3$	
$GL(n)/SO(n)$	$\dfrac{SL(n+2)}{SO(2,n)}$	gravity in $D = n+4$ dimensions
$U(1)/U(1)$	$\dfrac{SU(2,1)}{S(U(1,2)\times U(1))}$	Einstein-Maxwell $N = 2$ supersymmetry
$\dfrac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$	$\dfrac{SO(8,2)}{SO(6,2)\times SO(2)}$	$N = 4$ supergravity
$\dfrac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$	$\dfrac{SO(8{,}8)}{SO(6{,}2)\times SO(2{,}6)}$	$N = 4$ supergravity + supersymmetric Maxwell, $D = 10$ supergravity
$E_{7(+7)}/SU(8)$	$E_{8(+8)}/SO^{*}(16)$	$N = 8$ supergravity, $D = 11$ supergravity

At this point we introduce some new notation to align somewhat with leading literature. Recalling the split of the Maurer-Cartan form in equations (2.11), we rename the split of the coset representative form $V^{-1}\mathrm{d}V$ as

$$Q := \frac{1}{2}\Big(V^{-1}\mathrm{d}V + \sigma(V^{-1}\mathrm{d}V)\Big) \qquad Q_\mu = \frac{1}{2}\Big(V^{-1}\partial_\mu V + \sigma(V^{-1}\partial_\mu V)\Big)$$
$$P := \frac{1}{2}\Big(V^{-1}\mathrm{d}V - \sigma(V^{-1}\mathrm{d}V)\Big) \qquad P_\mu = \frac{1}{2}\Big(V^{-1}\partial_\mu V - \sigma(V^{-1}\partial_\mu V)\Big),$$

such that

$$V^{-1}\mathrm{d}V = Q + P.$$

Q thus corresponds to $\omega_{\mathfrak{h}}$ and P corresponds to $\omega_{\mathfrak{m}}$ in our former notation. Remembering also how we rewrote the metric in equation (2.23), the Lagrangian in (3.9) can equally well be written as

$$\mathcal{L} = \sqrt{g}\,(R - g^{\mu\nu}\,\mathrm{Tr}\,[P_\mu\, P_\nu])\,.$$

For future reference we also write the Lagrangian as a differential form expressed in terms of P. With $d = (D-n)$ dimensions in the reduced spacetime we get[6]

$$\mathcal{L} = R\star 1 - \mathrm{Tr}\,(P \wedge \star P)\,, \tag{3.10}$$

[6]Recall that $\star 1 = \epsilon = \sqrt{g}d^d x$

which follows from the small calculation

$$\begin{aligned}
P \wedge \star P &= \frac{1}{(d-1)!}\, P_\rho \epsilon^\lambda{}_{\mu_1 \cdots \mu_{d-1}} P_\lambda \,\mathrm{d}x^\rho \wedge \cdots \wedge \mathrm{d}x^{\mu_{d-1}} \wedge \mathrm{d}x^{\mu_1} \\
&\overset{(*)}{=} \frac{1}{(d-1)!}\, P^\lambda P_\lambda \,\frac{1}{d}\, \epsilon_{\rho\mu_1 \cdots \mu_{d-1}} \,\mathrm{d}x^\rho \wedge \cdots \wedge \mathrm{d}x^{\mu_{d-1}} \wedge \mathrm{d}x^{\mu_1} \\
&= P^\lambda P_\lambda \,\sqrt{g} d^d x,
\end{aligned}$$

where we in $(*)$ have used that the anti-symmetry of ϵ and the wedge products forces $\lambda = \rho$ in each term, and then compensated the overcounting in the following expression with the factor $\frac{1}{d}$.

3.3.1 Equations of motion

The equations of motion derived from the gravity and sigma model Lagrangian (3.9) are

$$R_{\mu\nu} = \frac{1}{4}\,\mathrm{Tr}\left[M^{-1}\partial_\mu M\; M^{-1}\partial_\nu M\right] \tag{3.11}$$

$$\nabla^\mu \left(M^{-1}\partial_\mu M\right) = 0 \tag{3.12}$$

where the first line is obtained through varying the metric and is thus the Einstein equations in the reduced theory and the second line comes from the variation of M. Equation (3.12) actually expresses a conserved current which we will come back to in the section 3.5 about the charge matrix.

In the new notation and starting from the form Lagrangian these equations read

$$R_{\mu\nu} = \mathrm{Tr}\left(P_\mu P_\nu\right) \tag{3.13}$$

$$\mathrm{d}P + [Q, P] = 0. \tag{3.14}$$

The latter can also be expressed as

$$\mathrm{d}\star\left((V^{\mathcal{T}})^{-1} P V^{\mathcal{T}}\right) = 0 \tag{3.15}$$

which also follows from equation (3.12), since for any 1-form ψ we have $\mathrm{d}\star\psi = \nabla_\lambda \psi^\lambda \epsilon$ and equation (2.22) gives $M^{-1}\mathrm{d}M = 2(V^{\mathcal{T}})^{-1} P V^{\mathcal{T}}$.

3.3.2 Pure 4D gravity and the Schwarzschild representative

After these general statements we now illustrate the work of finding a coset representative for a certain solution. We here follow [21, p. 27] closely, preforming the explicit calculations to find the representative for Schwarzschild solution in four dimensional pure gravity reduced to $D = 3$. We thus continue the path of the example in section 3.2. We found the reduced theory to express three dimensional gravity coupled to an $SL(2, \mathbb{R})$-sigma model, either $SL(2, \mathbb{R})/SO(2)$ or $SL(2, \mathbb{R})/SO(1, 1)$, as described by the Lagrangian (3.7). We choose now to have compactified the timelike dimension giving a Riemannian three dimensional metric ($s = 0$) and the sigma model target space $SL(2,\mathbb{R})/SO(1,1)$. With this observation we make use of the coset parametrization presented in section 2.2.1 and write the action on the form as equation (2.24):

$$S = \int \mathrm{d}^3x\sqrt{g}\bigg(R - g^{\mu\nu}\frac{1}{4}\,\mathrm{Tr}\left[M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M\right]\bigg). \tag{3.16}$$

As we seek a coset representative of the Schwarzschild solution we do a spherical symmetric ansatz[7]

$$-\,ds^2_{(3)} = dr^2 + f^2(r)d\Omega^2 \tag{3.17}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the standard metric on S^2 and f is a scalar function depending on the radial coordinate r. The field equations (3.11) and (3.12) derived from the Lagrangian in (3.16) reads with the ansatz (3.17)

$$\frac{1}{f^2}\left(\frac{d}{dr}\Big(f\frac{df}{dr}\Big) - 1\right) = 0 \tag{3.18}$$

$$\frac{1}{4}\,\mathrm{Tr}\left[(M^{-1}\frac{dM}{dr})^2\right] = R_{rr} = -\frac{2}{f}\frac{d^2f}{dr^2} \tag{3.19}$$

$$\frac{d}{dr}\left(f^2M^{-1}\frac{dM}{dr}\right) = 0\ . \tag{3.20}$$

The first two are components of the Einstein equations $R_{\mu\nu} = \frac{c}{4}\,\mathrm{Tr}[M^{-1}\partial_\mu M\, M^{-1}\partial_\nu M]$ and the third is the r-component of equation (3.12). The Ricci component R_{rr} is directly computed from

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & f^2 & 0 \\ 0 & 0 & f^2\sin^2\theta \end{pmatrix} \quad ; \qquad R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}$$

$$R_{rr} = \partial_\mu\Gamma^\mu_{rr} - \partial_r\Gamma^\mu_{\mu r} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{rr} - \Gamma^\mu_{r\lambda}\Gamma^\lambda_{\mu r}$$

where

$$\begin{aligned}
\Gamma^\mu_{rr} &= \frac{g^{\mu\lambda}}{2}\left(2\partial_r g_{\lambda r} - \partial_\lambda g_{rr}\right) = 0 \\
\Gamma^\mu_{\mu r} &= \frac{g^{\mu\lambda}}{2}\partial_r g_{\mu\lambda} = \frac{1}{2}\left\{f^{-2}(f^2)' + f^{-2}\sin^{-2}\theta\,(f^2\sin^2\theta)'\right\} \\
&= f^{-2}(f^2)' \\
\Gamma^\lambda_{\mu r} &= \frac{g^{\lambda\delta}}{2}\partial_r g_{\delta\mu} = \frac{1}{2}f^{-2}(f^2)'(\delta^\theta_\mu\delta^\lambda_\theta + \delta^\varphi_\mu\delta^\lambda_\varphi)
\end{aligned}$$

[7]Remember the signature $(-,-,-,+)$

where a prime denotes $\frac{d}{dr}$ and θ, φ as indices denotes the coordinates. We find that

$$\Gamma^{\mu}_{r\lambda}\Gamma^{\lambda}_{\mu r} = \frac{1}{4}(\delta^{\theta}_{\mu}\delta^{\lambda}_{\theta} + \delta^{\varphi}_{\mu}\delta^{\lambda}_{\varphi})(\delta^{\theta}_{\lambda}\delta^{\mu}_{\theta} + \delta^{\varphi}_{\lambda}\delta^{\mu}_{\varphi})\left[f^{-2}(f^2)'\right]^2 = \frac{1}{2}\left[f^{-2}(f^2)'\right]^2$$

such that

$$\begin{aligned} R_{rr} &= -\left[f^{-2}(f^2)'\right]' - \frac{1}{2}\left[f^{-2}(f^2)'\right]^2 = -\left[-2f^{-2}(f')^2 + 2f^{-1}f''\right] - \frac{1}{2}f^{-4}\Big((f^2)'\Big)^2 \\ &= 2f^{-2}(f')^2 - 2f^{-1}f'' - df^{-2}(f')^2 = -2f^{-1}f''. \end{aligned}$$

We can find an expression for f by solving equation (3.18):

$$\begin{aligned} ff' = r + \tilde{c} &\implies fdf = (r+\tilde{c})dr \\ &\implies f^2 = (r-r_0)^2 + c \end{aligned} \tag{3.21}$$

with the two integration constants r_0 and c. This plugged back into the expression for the Ricci component yields

$$R_{rr} = -2\frac{c}{f^4}\ . \tag{3.22}$$

To recast equation (3.20) we define the parameter

$$\tau(r) = -\int_r^\infty \frac{1}{f^2(s)}\,\mathrm{d}s \tag{3.23}$$

and substitute into (3.20) with $\frac{d}{dr} = \frac{d\tau}{dr}\frac{d}{d\tau} = \frac{1}{f^2(r)}\frac{d}{d\tau}$. We get

$$\frac{d}{d\tau}\left(\hat{M}^{-1}\frac{d}{d\tau}\hat{M}\right) = 0 \tag{3.24}$$

where $\hat{M}\Big(\tau(r)\Big) = M(r)$, such that $\frac{d\hat{M}(\tau(r))}{d\tau} = f^2(r)\frac{dM(r)}{dr}$. This is a geodesic equation for $\hat{M}$ in G/K[8] as concluded in the following.

τ is a parameter describing a curve in G/K and $\hat{M}^{-1}\frac{d}{d\tau}\hat{M}$ is the Lie algebra element corresponding to the tangent vector T to the curve, related through the pushforward of the left-action

$$\hat{M}^{-1}\frac{d\hat{M}}{d\tau} = L_{\hat{M}^{-1}*}\left(T\Big|_{\hat{M}(\tau)}\right) := \hat{T}(\tau) \in T_e(G/K) \quad \forall\tau\ .$$

For the tangent vector directly, $\frac{dT}{d\tau}$ is not defined as there is nothing said how to compare the different tangent spaces. Here though, we have

$$\frac{d}{d\tau}\left(\hat{M}^{-1}\frac{d\hat{M}}{d\tau}\right) = \frac{d}{d\tau}\hat{T}(\tau) = 0$$

[8] Or rather $\hat{M}\eta^{-1} = \hat{V}\hat{V}^{\mathcal{T}}$ but we leave out this trivial conversion in the notation

where $\hat{T}$ always stays in $T_e(G/K)$ such that the comparison in the derivative can be made. The fact that $\hat{T}(\tau) = \hat{T}$ is constant for all τ implies conversely that $T(\tau)$ is defined as the left-invariant vector field constructed from $\hat{T}$, i.e.

$$T(\tau) = T\Big|_{\hat{M}(\tau)} = L_{\hat{M}(\tau)*}\hat{T} \ .$$

There is a one-to-one relation between left-invariant vector fields and one-parameter subgroups in a Lie group and, as mentioned in section A, the one-parameter subgroups form geodesics with respect to the Levi-Civita connection of the Killing form metric. Hence equation (3.24) states directly that its solution $\hat{M}(\tau)$ is a geodesic.

Another approach to see that this is a geodesic equation is to vary the Lagrangian in equation (3.16) with respect to ϕ^i. This calculation can be found in appendix B.

This last step in spelling the equation of motion (3.20) as a geodesic equation relies on the fact that our metric only has dependence on one coordinate. In such a case this dependence can always be recast by a suitable choice of parameter. The picture changes if we would be interested of only axially symmetric solutions whereby we would have a dependence on two parameters.

The explicit geodesic equations

To solve the geodesic equations we need the Christoffel symbols of the coset space. We find the metric components from equation

$$d\phi^i d\phi^j \gamma_{ij} = \frac{1}{4}\,\mathrm{Tr}\left[(M^{-1}\frac{dM}{dr})^2\right] \ .$$

We find

$$M = VV^{\mathcal{T}}\eta = \begin{pmatrix} \mathrm{e}^{\phi/2} & \mathrm{e}^{-\phi/2}\chi \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\phi/2} & 0 \\ -\mathrm{e}^{-\phi/2}\chi & \mathrm{e}^{-\phi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\phi} - \mathrm{e}^{-\phi}\chi^2 & -\mathrm{e}^{-\phi}\chi \\ -\mathrm{e}^{-\phi}\chi & -\mathrm{e}^{-\phi} \end{pmatrix}$$

$$dM = \begin{pmatrix} d\phi\mathrm{e}^{\phi} + (d\phi\chi^2 - 2\chi d\chi)\mathrm{e}^{-\phi} & (d\phi\chi - d\chi)\mathrm{e}^{-\phi} \\ (d\phi\chi - d\chi)\mathrm{e}^{-\phi} & d\phi\mathrm{e}^{-\phi} \end{pmatrix}$$

$$dM^{-1} = \begin{pmatrix} -d\phi\mathrm{e}^{-\phi} & -(d\chi - d\phi\chi)\mathrm{e}^{-\phi} \\ -(d\chi - d\phi\chi)\mathrm{e}^{-\phi} & -d\phi\mathrm{e}^{\phi} + (2\chi d\chi - d\phi\chi^2)\mathrm{e}^{-\phi} \end{pmatrix}$$

such that

$$\begin{aligned} -\frac{1}{4}\,\mathrm{Tr}\left[(M^{-1}\frac{dM}{dr})^2\right] &= \frac{1}{4}\,\mathrm{Tr}\left[dM^{-1}dM\right] \\ &= \frac{1}{4}\,\mathrm{Tr}\begin{pmatrix} -d\phi^2 + \mathrm{e}^{-2\phi}d\chi^2 & 0 \\ 0 & -d\phi^2 + \mathrm{e}^{-2\phi}\chi^2 \end{pmatrix} \\ &= -\frac{1}{2}\left[d\phi^2 - \mathrm{e}^{-2\phi}d\chi^2\right] \ . \end{aligned}$$

Now, direct calculations from the definition of the Christoffel symbols[9]

$$\tilde{\Gamma}^k_{ij} = \frac{\gamma^{kl}}{2}(\partial_i \gamma_{lj} + \partial_j \gamma_{il} - \partial_l \gamma_{ij})$$

and the coset metric

$$\gamma_{ij} \doteq \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{e}^{-2\phi} \end{pmatrix} \qquad \gamma^{ij} \doteq 2\begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{e}^{2\phi} \end{pmatrix}$$

gives

$$\begin{aligned}
\tilde{\Gamma}^1_{11} &= \tilde{\Gamma}^1_{12} = \tilde{\Gamma}^2_{11} = \tilde{\Gamma}^2_{22} = 0 \\
\tilde{\Gamma}^1_{22} &= \frac{\gamma^{11}}{2}(0 + 0 - \partial_1 \gamma_{22}) = -\mathrm{e}^{-2\phi} \\
\tilde{\Gamma}^2_{12} &= \frac{\gamma^{22}}{2}(\partial_1 \gamma_{22} + 0 + 0) = \mathrm{e}^{2\phi}\frac{-2}{2}\mathrm{e}^{-2\phi} = -1 \,.
\end{aligned}$$

Denoting $\frac{d}{d\tau}$ with dots, the geodesic equations become

$$\begin{cases} \ddot{\hat{\phi}} - \mathrm{e}^{-2\hat{\phi}}(\dot{\hat{\chi}})^2 = 0 \\ \ddot{\hat{\chi}} - 2\dot{\hat{\phi}}\dot{\hat{\chi}} = 0 \end{cases} \tag{3.25}$$

which for the Schwarzschild solution, where $\hat{\chi} = 0$, we find the solution $\hat{\phi}(\tau) = A\tau + B$, for constants A and B. Asymptotically flat boundary conditions as r tends to infinity translates into $\hat{\phi} \to 0$ as $\tau \to 0$, whence $B = 0$.

Writing out the geodesic equation (3.24) in components we find

$$\begin{aligned}
&\frac{d}{d\tau}\left(\hat{M}^{-1}\frac{d}{d\tau}\hat{M}\right) \\
&= \begin{pmatrix} \ddot{\hat{\phi}} - \left(\dot{\hat{\chi}}^2 + \hat{\chi}\ddot{\hat{\chi}} - 2\hat{\chi}\dot{\hat{\chi}}\dot{\hat{\phi}}\right)\mathrm{e}^{-2\hat{\phi}} & \left(2\dot{\hat{\chi}}\dot{\hat{\phi}} - \ddot{\hat{\chi}}\right)\mathrm{e}^{-2\hat{\phi}} \\ \ddot{\hat{\chi}} + \left(2\hat{\chi}\dot{\hat{\chi}}^2 + \hat{\chi}^2\ddot{\hat{\chi}} - 2\hat{\chi}^2\dot{\hat{\chi}}\dot{\hat{\phi}}\right)\mathrm{e}^{-2\hat{\phi}} - 2(\dot{\hat{\chi}}\dot{\hat{\phi}} + \hat{\chi}\ddot{\hat{\phi}}) & \left(\dot{\hat{\chi}}^2 + \hat{\chi}\ddot{\hat{\chi}} - 2\hat{\chi}\dot{\hat{\chi}}\dot{\hat{\phi}}\right)\mathrm{e}^{-2\hat{\phi}} - \ddot{\hat{\phi}} \end{pmatrix} \\
&= 0
\end{aligned}$$

which upon substitution of the upper right corner component equation into the others boils down to precisely the explicit geodesic equations (3.25). That only two equations are independent are to be expected as M is symmetric and based on an element of the Lie group and thus has fixed determinant.

[9]We denote the Christoffel symbols belonging to the coset space connection with a tilde.

Geodesic implications

Knowing that $\hat{\phi}^i(\tau)$ forms a geodesic we recall that the norm of the tangent vector along any geodesic is constant. This can be used in the equation of motion (3.19) as

$$\frac{d\hat{\phi}^i}{d\tau}\frac{d\hat{\phi}^j}{d\tau}\gamma_{ij} = f^4(r)\frac{d\phi^i}{dr}\frac{d\phi^j}{dr}\gamma_{ij} = f^4(r)\frac{1}{4}\operatorname{Tr}\left[(M^{-1}\partial_r M)^2\right] = f^4(r)R_{rr} \stackrel{!}{=} 2v^2 \qquad (3.26)$$

for some constant v^2. Recalling equation (3.22) see that $c = -v^2$ and thus that $f^2(r) = (r-r_0)^2 - v^2$. In general, v^2 can have any sign due to the possibility of an indefinite coset metric. Here, however, we have $v^2 > 0$.

The parameter τ can thus be calculated as

$$\tau(r) = -\int_r^\infty \frac{1}{f^2(s)}\,\mathrm{d}s = \frac{1}{2v}\ln\left|\frac{r-r_0-v}{r-r_0+v}\right| \; .$$

Finding the coset representative

The coset representative V can be found either by solving the geodesic equation for ϕ and χ, as done above for $\chi = 0$, or through an ansatz. Since the Schwarzschild solution is so simple the latter is quick and easy. We know that $\hat{M}(\tau)\eta^{-1}$ forms a geodesic in G/K and can thus be written as an exponential of some tangent vector $T|_{\hat{M}_0}$ starting at some $\hat{M}_0$ at $\tau = 0$. Since $\tau = 0$ corresponds to the asymptotically flat Minkowski space where $\phi = 0$, it is immediately seen that $\hat{M}_0$ is the identity. Thus

$$\hat{M}(\tau)\eta^{-1} = \exp\left[\tau T\Big|_{\mathrm{id}}\right] = \begin{pmatrix} \mathrm{e}^{\phi} + \mathrm{e}^{-\phi}\chi^2 & \mathrm{e}^{-\phi}\chi \\ \mathrm{e}^{-\phi}\chi & \mathrm{e}^{-\phi} \end{pmatrix} = VV^{\mathcal{T}}$$

from which the tangent vector for Schwarzschild must be

$$\hat{M}_{\mathrm{S}}(\tau) = \exp\left[\tau T\Big|_{\mathrm{id}}\right]\eta = \exp\left[\phi H\right]\eta \; , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

Now we can directly see that $\phi \propto \tau$ and with proportionality constant A

$$V_{\mathrm{S}} = \exp\left[\frac{\phi}{2}H\right] = \exp\left[\frac{A\tau}{2}H\right] \; . \qquad (3.27)$$

A can be determined from equation (3.19) as

$$\operatorname{Tr}\left[(\hat{M}^{-1}\frac{d}{d\tau}\hat{M})^2\right] = -\operatorname{Tr}\left[\frac{d\hat{M}^{-1}}{d\tau}\frac{d\hat{M}}{d\tau}\right] = A^2\operatorname{Tr}\left[H^2\right] = 4\cdot 2v^2$$

$$\Longrightarrow$$

$$A = \overset{+}{\scriptstyle(-)}2v \; .$$

Finally, we conclude

$$V_{\mathrm{S}} = \exp\left[\frac{1}{2}\ln\left(\frac{r-r_0-v}{r-r_0+v}\right)H\right] \tag{3.28}$$

$$= \begin{pmatrix} \sqrt{\frac{r-r_0-v}{r-r_0+v}} & 0 \\ 0 & \sqrt{\frac{r-r_0+v}{r-r_0-v}} \end{pmatrix} \tag{3.29}$$

for $r > r_0 + v$. As the last step we can identify $v = m$ by just writing out the four dimensional metric, in which we recall $g_{33} = \mathrm{e}^{\phi}$ and $B_\mu = 0$,

$$\begin{aligned} ds^2_{(4)} &= -g_{33}^{-1} ds^2_{(3)} + g_{33} dt^2 \\ &= \left(\frac{r-r_0-v}{r-r_0+v}\right)^{-1}(-dr^2 - f^2(r)d\Omega^2) + \left(\frac{r-r_0-v}{r-r_0+v}\right)dt^2 = \left\{ \begin{array}{c} \tilde{r} = r - r_0 - v \\ d\tilde{r} = dr \end{array} \right\} \\ &= -\left(1-\frac{2v}{\tilde{r}}\right)^{-1}\left(d\tilde{r}^2 + \left[(\tilde{r}^2 - v^2) - v^2\right]d\Omega^2\right) + \left(1-\frac{2v}{\tilde{r}}\right)dt^2 \\ &= \left(1-\frac{2v}{\tilde{r}}\right)dt^2 - \left(1-\frac{2v}{\tilde{r}}\right)^{-1}d\tilde{r}^2 - \tilde{r}^2 d\Omega^2\ . \\ &\stackrel{!}{=} \left(1-\frac{2m}{\tilde{r}}\right)dt^2 - \left(1-\frac{2m}{\tilde{r}}\right)^{-1}d\tilde{r}^2 - \tilde{r}^2 d\Omega^2\ . \end{aligned}$$

The last step can equally be done with the boundary conditions for V which in this case is

$$\mathrm{e}^{\phi(r)} \xrightarrow[r\to\infty]{} 1 - \frac{2m}{r} + \cdots$$

This is the route one has to take when seeking new solutions.

We have now found the coset representative of the Schwarzschild solution in our chosen gauge and we have seen that it indeed gives the right four dimensional metric. This can now be used as a seed solution.

3.3.3 *Example:* 4D Einstein-Maxwell theory and the Reissner-Nordström solution

Before moving on with the solution generating transformations we can apply what we have learned also to the spherically symmetric Reissner-Nordström solution in Einstein-Maxwell theory, i.e. gravity coupled to electromagnetism. This special solution turns out to involve very similar calculations as done above but will not preform them entirely. Instead we only present the minimal amount to illustrate some later observations.

The starting Lagrangian is

$$\mathcal{L}^{(4)} = \sqrt{g^{(4)}}\left(\frac{1}{4}R^{(4)} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right),$$

where, of course, F is the electromagnetic field strength. We reuse the metric ansatz (3.2) from the Schwarzschild example but incorporate the spherical symmetry from the beginning, hence no cross-terms,

$$g^{(4)}_{\alpha\beta} \doteq \begin{pmatrix} e^{-\phi} g_{\mu\nu} & \\ 0 & e^{\phi} \end{pmatrix}.$$

As also the Reissner-Nordström solution is static we following the paved road all along to the three dimensional metric and take

$$-ds^2_{(3)} = dr^2 + f^2(r) d\Omega^2$$

from above as well. Thus the Einstein-Hilbert part of the reduced Lagrangian looks exactly as before with the scalar χ set to zero.

The Maxwell potential A in $F = \mathrm{d}A$ does also split but as we are looking for a static solution, only the component $A^\alpha(r)$ where $\alpha = 4$ is non-vanishing[10]. We denote this component simply as $A := A^4(r)$ as we will make no further reference to the potential. For F this translates to only one non-zero component $F_{r4} = \partial_r A$. All in all we get the three dimensional Lagrangian

$$\mathcal{L}^{(3)} = \frac{\sqrt{g}}{4}\Big(R^{(3)} - \frac{1}{2}\Big(\partial_r\phi\partial^r\phi - 4e^{-2\phi}\partial_r A \partial^r A\Big)\Big) \tag{3.30}$$

where the factor $e^{-2\phi}$ in front of the A-term comes from $\sqrt{g^{(4)}} = e^{-2\phi}\sqrt{g}$. We have once again found the sigma model $SL(2)/SO(1,1)$.

We have just covered in detail how to find the equations of motion in this model and restated in this context we are to solve

$$\begin{cases} \ddot{\hat{\phi}} - 2e^{-2\hat{\phi}}(\dot{\hat{A}})^2 = 0 \\ \ddot{\hat{A}} - \dot{\hat{\phi}}\dot{\hat{A}} = 0 \end{cases}$$

where the dots once again denote derivatives with respect to τ, the parameter introduced in equation (3.23), and we made the identification $2A = \chi$ compared to prior equations. These equations are subjected to the boundary conditions

$$e^{\phi(r)} \xrightarrow[r\to\infty]{} 1 - \frac{2m}{r} + \cdots$$
$$A(r) \xrightarrow[r\to\infty]{} -\frac{q}{r} + \cdots$$

where m denotes the mass and q the electric charge of the black hole. These equations are solvable although it requires quite some algebra not so relevant for our purposes[11]. What

[10] Remember that we have chosen the fourth component to represent time in the four dimensional theory and let the indices $\alpha, \beta = 1, \cdots, 4$ and $\mu, \nu = 1, 2, 3$.

[11] A full derivation can be found in chapter 4 in [26].

matters most to us here is the equation (3.26) for the constant v^2 which was introduced as a consequence of the geodesic and its constant tangent vector norm. With the coset metric in the Lagrangian (3.30) we have

$$\gamma_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} = \frac{1}{2}\dot{\phi}^2 - 2\mathrm{e}^{-2\phi}\dot{A}^2 = 2v^2. \tag{3.31}$$

In order to simplify the calculations we write

$$\Delta = \mathrm{e}^{\phi}$$

and use $\frac{d}{d\tau} = f^2(r)\frac{d}{dr}$ to rewrite the boundary conditions in τ. Recalling that $f^2(r) = (r - r_0)^2 + c$ we find

$$\begin{aligned} \dot{\Delta}\Big|_{\tau\to\infty} &= f^2(r)\left(\frac{2m}{r^2} + \cdots\right) \\ \dot{A}\Big|_{\tau\to\infty} &= f^2(r)\left(\frac{q}{r^2} + \cdots\right) \end{aligned}$$

such that

$$\begin{aligned} \Delta(\tau) &\xrightarrow[\tau\to 0]{} 1 \\ \dot{\Delta}(\tau) &\xrightarrow[\tau\to 0]{} 2m \\ \dot{A}(\tau) &\xrightarrow[\tau\to 0]{} q. \end{aligned}$$

Equation (3.31) thus goes to

$$\frac{1}{\Delta^2}\dot{\Delta}^2 - \frac{4}{\Delta^2}\dot{A}^2 = 4v^2 \qquad \xrightarrow[\tau\to 0]{} \qquad m^2 - q^2 = v^2. \tag{3.32}$$

We see here that v^2 is directly linked to the mass and electrical charge; a fact we will come back to in section 3.5 about the charge matrix.

3.3.4 *Example:* Five dimensional minimal supergravity

To motivate the interest of $\mathfrak{g}_2$ in later chapters and the *Mathematica*-packages we include a glimpse also on the dimensional reduction of five dimensional minimal supergravity. This additionally gives a flavor of the procedures in more complicated theories and serves as an example with vector content without the simplifying assumptions we made for the Reissner-Nordström solution.

We start from the Lagrangian for $D = 5$ minimal supergravity

$$\mathcal{L}^{(5)} = R^{(5)}\star 1 - \frac{1}{2}\star F^{(5)} \wedge F^{(5)} - \frac{1}{3\sqrt{3}}F^{(5)} \wedge F^{(5)} \wedge A^{(5)} \tag{3.33}$$

where $F^{(5)} = \mathrm{d}A^{(5)}$. We assume a spacelike Killing field and compactify and reduce along its direction through the five dimensional ansätze [18]

$$\begin{aligned} ds^2_{(5)} &= \mathrm{e}^{\frac{1}{\sqrt{3}\phi_1}} ds^2_{(4)} + \mathrm{e}^{-\frac{2}{\sqrt{3}}\phi_1}(dz + \mathcal{A})^2 \\ A^{(5)} &= A^{(4)} + \chi_1 dz. \end{aligned}$$

This yields the four dimensional Lagrangian [6]

$$\begin{aligned} \mathcal{L}^{(4)} = R^{(4)} \star 1 &- \frac{1}{2}\star d\phi_1 \wedge d\phi_1 - \frac{1}{2}\mathrm{e}^{\frac{2}{\sqrt{3}}\phi_1}\star d\chi_1 \wedge d\chi_1 - \frac{1}{2}\mathrm{e}^{-\sqrt{3}\phi_1}\star\mathcal{F}^{(4)} \wedge \mathcal{F}^{(4)} \\ &- \frac{1}{2}\mathrm{e}^{-\frac{1}{\sqrt{3}}\phi_1}\star F_1^{(4)} \wedge F_1^{(4)} + \frac{1}{\sqrt{3}}\chi_1 dA^{(4)} \wedge dA^{(4)}, \end{aligned}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$. Continuing the process under the assumption of the existence of a timelike Killing vector commuting with the one already exploited, we reach a three dimensional theory by the ansatz

$$\begin{aligned} ds^2_{(4)} &= \mathrm{e}^{\phi_2} ds^2_{(3)} - \mathrm{e}^{-\phi_2}(dt + \omega_3)^2, \\ \mathcal{A}^{(4)} &= \mathcal{A}^{(3)} + \xi dt, \\ A^{(4)} &= A^{(3)} + \chi_2 dt. \end{aligned}$$

We can also go directly from five to three dimension through the ansätze [11]

$$ds^2_{(5)} = \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1 + \phi_2} ds^2_{(3)} + \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1 - \phi_2}(\mathrm{d}\psi + \mathcal{A}^2)^2 - \mathrm{e}^{\frac{-2}{\sqrt{3}}\phi_1}(\mathrm{d}t + \chi_1 \mathrm{d}\psi + \mathcal{A}^1)^2 \tag{3.34}$$

$$A^{(5)} = A + \chi_3 \mathrm{d}z_4 + \chi_2 \mathrm{d}z_5. \tag{3.35}$$

Continuing with this latter expression we can write the three dimensional field strengths as

$$\begin{aligned} \mathcal{F} &= \mathrm{d}\chi_1 & F^1 &= \mathrm{d}\chi_2 \\ \mathcal{F}^1 &= \mathrm{d}\mathcal{A}^1 + \mathcal{A}^2 \wedge \mathrm{d}\chi_1 & F^2 &= \mathrm{d}\chi_3 - \chi_1 \mathrm{d}\chi_2 \\ \mathcal{F}^2 &= \mathrm{d}\mathcal{A}^2 & F &= \mathrm{d}A - \mathrm{d}\chi_2 \wedge (\mathcal{A}^1 - \chi_1\mathcal{A}^2) - \mathrm{d}\chi_3 \wedge \mathcal{A}^2 \end{aligned}$$

which can be dualized to the 1-forms

$$\begin{aligned} G_4 &:= \mathrm{e}^{\sqrt{3}\phi_1 - \phi_2}\star F =: \mathrm{d}\chi_4 + \frac{1}{\sqrt{3}}(\chi_2 - \chi_3 \mathrm{d}\chi_2) \\ G_5 &:= -\mathrm{e}^{-\sqrt{3}\phi_1 - \phi_2}\star F^1 =: \mathrm{d}\chi_5 - \chi_2 \mathrm{d}\chi_4 + \frac{\chi_2}{3\sqrt{3}}(\chi_3 - \chi_2 \mathrm{d}\chi_3) \end{aligned}$$

$$G_6 := \mathrm{e}^{-2\phi_2}\star F^2 =: \mathrm{d}\chi_6 - \chi_1 \mathrm{d}\chi_5 + (\chi_1\chi_2 - \chi_3)\mathrm{d}\chi_4 + \frac{1}{3\sqrt{3}}(-\chi_1\chi_2 + \chi_3)(\chi_3 \mathrm{d}\chi_2 - \chi_2 \mathrm{d}\chi_3).$$

In terms of this scalar content we find the three dimensional Lagrangian to be

$$\begin{aligned}\mathcal{L} = R\star 1 &- \frac{1}{2}(\star\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_1 + \star\mathrm{d}\phi_2 \wedge \mathrm{d}\phi_2) + \frac{1}{2}\mathrm{e}^{-\sqrt{3}\phi_1+\phi_2}\star\mathrm{d}\chi_1 \wedge \mathrm{d}\phi_1 \\ &+ \frac{1}{2}\mathrm{e}^{\frac{2}{\sqrt{3}}\phi_1}\star\mathrm{d}\chi_2 \wedge \mathrm{d}\chi_2 - \frac{1}{2}\mathrm{e}^{-\frac{1}{\sqrt{3}}\phi_1+\phi_2}\star(\mathrm{d}\chi_3 - \chi_1\mathrm{d}\chi_2) \wedge (\mathrm{d}\chi_3 - \chi_2\mathrm{d}\chi_2) \\ &+ \frac{1}{2}\mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1+\phi_2}\star G_4 \wedge G_4 - \frac{1}{2}\mathrm{e}^{\sqrt{3}\phi_1+\phi_2}\star G_5 \wedge G_5 \\ &+ \frac{1}{2}\mathrm{e}^{2\phi_2}\star G_6 \wedge G_6.\end{aligned} \tag{3.36}$$

Once again we find gravity coupled to a sigma model, this time parametrized by the eight scalars $\phi_1, \phi_2, \chi_1, \cdots, \phi_6$. The coset space describing this model is the mentioned $\mathcal{G}_{2(2)}/SO(2{,}2)$ and one can actually write the exponents in this expressions as the six positive roots α_i of $\mathfrak{g}_2$ dotted with the dilatons (ϕ_1, ϕ_2) [11].

We will return to this model when discussing the physical relevance of the orbit structure in chapter 4.

3.4 Solution-generating Transformations and Orbits

We have now reach the point where we can actually discuss the use of the method of dimensional reduction. We begin with the types of solutions we are concerned with.

The solutions considered are in general asymptotically flat which in this context refers to the criteria of Misner. These require that the there is a function r on the four dimensional spacetime which tends to infinity at spatial infinity and that $\partial_\mu r \partial^\mu r \to 1$ in the same limit. Moreover, each component of the Riemann tensor in any vierbein frame must tend to zero as $\mathcal{O}(r^{-3})$ as $r \to \infty$ and the components of any Maxwell field strength must likewise go as $\mathcal{O}(r^{-2})$. The coset representative in $\mathcal{G}_4$, consisting of the scalars in the four dimensional theory, should go as $\mathbb{1} + \mathcal{O}(r^{-1})$ which all together forces V in the compactified theory to also go as

$$V = \mathbb{1} + \mathcal{O}(r^{-1}) \qquad r \to \infty. \tag{3.37}$$

Additionally, in order to have well defined charges when compactifying along the time coordinate, the Killing vector κ must leave the function r invariant and satisfy the two conditions $\epsilon^{\mu\nu\rho\sigma}\kappa_\nu\partial_\rho\kappa_\sigma \sim \mathcal{O}(r^{-2}$ and $-\kappa_\mu\kappa^\mu = 1 + \mathcal{O}(r^{-1})$.

The focus on asymptotically flat solutions restricts the set of relevant transformations to only involve such elements that preserves the boundary conditions. From the coset condition (3.37) we see specifically that these constitute the group $\mathcal{H}$ as these act as the identity element on the coset space.

Transformation of the fields

We will not explicitly preform any transformation between different solutions in this thesis. It is however nice to see how the transformations in principle are carried out. We thus look at an example transformation in the sigma model $SL(2,\mathbb{R})/SO(1,1)$

$$V = \begin{pmatrix} \Delta^{1/2} & \chi\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} \longrightarrow gVh$$

where again $\Delta = \mathrm{e}^{\phi}$ and g and h are elements of $\mathcal{G}$ and $\mathcal{H}$ respectively. To preserve the boundary conditions we choose g to actually lie in $\mathcal{H}$ and study now the infinitesimal action on the fields. With $E+F$ as the generator of $\mathfrak{h}$ and the parameter c to parameterize the compensating gauge transformation we get

$$\delta V = (E+F)\phi + cV(E+F) = \begin{pmatrix} 0 & \Delta^{-1/2} \\ \Delta^{1/2} & \chi\Delta^{-1/4} \end{pmatrix} + c\begin{pmatrix} \chi\Delta^{-1/2} & \Delta^{1/4} \\ \Delta^{-1/2} & 0 \end{pmatrix}.$$

We see immediately that $c = -\Delta$ and have

$$\delta V = \begin{pmatrix} -\chi\Delta^{1/2} & \Delta^{-1/2} - \Delta^{3/2} \\ 0 & \chi\Delta^{-1/2} \end{pmatrix}. \tag{3.38}$$

We now do the same small calculation but apply the infinitesimal transformations directly on the fields in V:

$$\delta V = \begin{pmatrix} \frac{1}{2}\Delta^{-1/2}\delta\Delta & \Delta^{-1/2}\delta\chi - \frac{1}{2}\chi\Delta^{-3/2}\delta\Delta \\ 0 & -\frac{1}{2}\Delta^{-3/2}\delta\Delta \end{pmatrix}.$$

Comparing this to equation (3.38) we can get the transformation laws of the fields

$$\begin{aligned} \delta\Delta &= -2\chi\Delta \\ \delta\chi &= 1 - \Delta^2 - \chi^2. \end{aligned}$$

It is now clearer than ever that this is a non-linear sigma model.

In this way it is possible to find the transformation laws of the fields also in more complicated theories and thus the transformation of the spacetime metric, although the calculations might be a lot more involved. In order to recover the metric in the full theory the compactification procedure must be reversed, a process sometimes called “oxidizing”.

The question of orbit structure and the BMG theorem

The next question to address is which solutions are good as seeds and which parts of the solution space can be reached from each one of these, i.e. what is the orbit structure of the solution space? Our first observation is that empty Minkowski space cannot be used as a seed solution as its coset representative $V_{\text{Min}} = \mathbb{1}$ is a fixed point. This is however neither trivial nor true in the infinite dimensional symmetry situation below. Here, however, we must use other seed solutions.

A very important answer to this question is provided by a theorem due to Breitenlohner, Maison and Gibbons (BMG) [3, 15].

Theorem 3.4.1 (BMG Theorem)**.** *Any static single black hole solution in four dimensions with non-degenerate horizon can be reached through some transformation in $\mathcal{H}$ applied on the Schwarzschild solution.*

It follows from this theorem that all static single black hole solutions are spherically symmetric which in turn makes all these solutions dependent only on one coordinate. Such solutions are also called **cohomogeneity-one** solutions and we will pay specific attention to these later. In the same paper BMG also present a corresponding theorem for stationary, axisymmetric single black holes where instead the Kerr solution of a rotating black hole suffices as seed solution.

The next step in answering the question about how the solution space splits up leads us directly in to the subject of orbits. However, we postpone the mathematical background just a moment to devote a section to what is called the charge matrix, which also will provide a motivation to why we are particularly interested in *nilpotent* orbits.

3.5 The Charge Matrix and Motivation to Nilpotency

We mentioned in section 3.3.1 about the equations of motion that there is a conserved charge corresponding to the global symmetry of the sigma model. It was expressed in the equivalent equations

$$\nabla^\mu(M^{-1}\partial_\mu M) = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}\star\Big((V^{\mathcal{T}})^{-1}PV^{\mathcal{T}}\Big) = 0.$$

We now turn our attention to this charge for the rest of this section and will see that it is a crucial object. To a large extent it defines the solution, above all for cohomogeneity-one solutions, it contains most of the scalar charges as observed from infinity and its norm defines the extremality of the solution. This last property will lead to a nilpotency criteria which in the motivates the study of nilpotent orbits in this context. We mainly follow [1], [15] and [21] in this section.

We begin by defining the conserved current as the Lie algebra-valued 1-form

$$J := \frac{1}{2}M^{-1}\partial_\mu M \mathrm{d}x^\mu = (V^{\mathcal{T}})^{-1}PV^{\mathcal{T}}$$

and the corresponding charge, **the charge matrix** $\mathscr{C}$, as

$$\mathscr{C} := \frac{1}{4\pi}\int_{\partial\Omega}\star J \quad \in \mathfrak{g}. \tag{3.39}$$

In the compactified theory there might be no time but the integral in (3.39) is nevertheless independent of the particular hypersurface $\partial\Omega$, (as long as the volume $\Omega \subset \mathcal{X}$ contains all singularities and topological non-trivialities). We can thus talk about a conserved charge in some sense, independent of any spacetime coordinates [1, p. 9].

As we have imposed asymptotic flatness which implied that $V \to \mathbb{1}$ in spatial infinity, $\mathscr{C}$ can easily be computed from the values of P. As a general assumption we have that

$$P = \mathscr{C}\frac{1}{r^2} + \mathcal{O}(r^{-3}). \tag{3.40}$$

It can equally well be obtained through an expansion of $M = M_0 + \frac{1}{r}M_1 + \mathcal{O}(r^{-2})$ where

$$\mathscr{C} = \frac{1}{2}M_0^{-1}M_1.$$

From the relation to P it is thus clear that $\mathscr{C}$ must be an element of $\mathfrak{m}$.

Recalling the general comments on dimensionally reduced gravity theories in section 3.3 we stated the composition of the Lie algebra $\mathfrak{g}$ to the full symmetry group $\mathcal{G}$ in equation 3.8. The $\mathfrak{m}$-part decomposes as [1]

$$\mathfrak{m} \cong \Big(\mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2)\Big) \oplus \mathfrak{l}_4 \oplus (\mathfrak{g}_4 \ominus \mathfrak{h}_4)$$

and this allows for a split of the charge matrix into the conserved charges of the four dimensional theory. The Komar mass and the Komar NUT charge (see appendix C) correspond to the $\mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2)$-part while the electromagnetic charges fall into the $\mathfrak{l}_4$ part. The scalar charges lie in $\mathfrak{l}_4 \ominus \mathfrak{h}_4$. It is notable, however, that the potential angular momentum is missing here and that it in fact lies in the next order in the expansion of P in equation 3.40. The charge matrix is thus completely unaffected by any change of the angular momentum.

The charge matrix for cohomogeneity-one solutions

Let us for a moment focus on cohomogeneity-one solutions. In these cases

$$\hat{M} = \exp[\tau 2\mathscr{C}]\,,$$

as we saw in the Schwarzschild-example above, and with the tangent vector $T(\tau) = \frac{\partial}{\partial\tau}$ to $\hat{M}(\tau)$ we have

$$\begin{aligned}\hat{M}^{-1}\mathrm{d}\hat{M}\frac{\partial}{\partial\tau} &= L_{\hat{M}^{-1}*}\Big[\partial_i \otimes \phi^i\left(\frac{\partial}{\partial\tau}\right)\Big] = L_{\hat{M}^{-1}*}\Big[\partial_i \otimes \phi^i(T^i\partial_i)\Big]\\ &= L_{\hat{M}^{-1}*}\Big[T^i\partial_i\Big] = T\Big|_{\mathrm{id}} = 2\mathscr{C}\end{aligned}$$

since T is left-invariant. The factor of 2 is there in order to have $\frac{1}{4\pi}\int \star J = \mathscr{C}$. We thus see that the charge matrix is in fact the tangent vector which defines the geodesic which the solution constitutes. In the example we exploited the constant norm of such a tangent vectors and set

$$\frac{1}{4}\,\mathrm{Tr}\left[(\hat{M}^{-1}\frac{d}{d\tau}\hat{M})^2\right] = 2v^2 \tag{3.41}$$

which for $\hat{M} = \exp[\tau 2\mathscr{C}]$ implies

$$\mathrm{Tr}\left[\mathscr{C}^2\right] = 2v^2\,.$$

This is really explicitly the squared norm of $\mathscr{C}$ with respect to the Killing form metric, or in some sense the speed of the geodesic.

3.5.1 Transformations and the relevance of nilpotent orbits

It is an important observation that the left hand side of equation (3.41) is invariant under $\mathcal{G}$-transformations. This makes the value of v^2 into a constant of the entire orbit of solutions. Since the metric on $\mathfrak{m}$ is indefinite the constant v^2 can take any sign. It turns out, however, that solutions with negative values of v^2 describe spacetimes with naked singularities and are not considered. Recalling the result $m^2 - q^2 = v^2$ of equation (3.32) in the Reissner-Nordström example, section 3.3.3, we see that the vanishing of v^2 exactly corresponds to the extremality condition for the Reissner-Nordström metric. This is in fact a general result and v^2 is therefore called the **extremality parameter**, or sometimes the BPS[12] parameter. Non-rotating extremal black holes are thus described by charge matrices with vanishing norm and, equivalently, null geodesics in the case of cohomogeneity-one solutions [1, 15].

This has important implications when paired together with the theorem of BMG and central role of the Schwarzschild metric as a seed solution. To see this we first need the explicit form of the Schwarzschild charge matrix.

Explicit calculation of the Schwarzschild charge matrix

The current for the Schwarzschild solution is calculated by use of the coset representative in equation (3.28). We have from equation (3.27)

$$V^{-1}\mathrm{d}V = \mathrm{e}^{-v\tau H}\ vH\mathrm{d}\tau\ \mathrm{e}^{v\tau H} = mH\,\mathrm{d}\tau = mH\,\frac{1}{f^2}\mathrm{d}r.$$

Since $\sigma(H) = -H$ we have for this particular case

$$J = VPV^{-1} = V\frac{1}{2}\left(V^{-1}\mathrm{d}V - \sigma(V^{-1}\mathrm{d}V)\right)V^{-1} = V(V^{-1}\mathrm{d}V)V^{-1} = mH\,\frac{1}{f^2}\mathrm{d}r$$

and

$$\mathscr{C} = \frac{1}{4\pi}\int_{\partial\Omega}\star J = \frac{1}{4\pi}\int_{\partial\Omega}\frac{mH}{f^2}\sqrt{g}\,\delta^1_\mu\varepsilon^\mu{}_{\rho\sigma}\mathrm{d}x^\rho\otimes\mathrm{d}x^\sigma \tag{3.42}$$

$$= \frac{1}{4\pi}\int_{\partial\Omega} mH\ \sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi = \frac{1}{4\pi}\int_{\partial\Omega} mH\ \sin\theta d\theta d\varphi \tag{3.43}$$

$$= mH \tag{3.44}$$

where the last step assumes $\partial\Omega$ to be a sphere at infinity. This also matches the equation

$$\mathrm{Tr}[\mathscr{C}^2] = 2v^2 \qquad \text{with} \qquad v = m.$$

[12] The name BPS originates from a bound in supersymmetric field theories found by Bogomol'nyi, Prasad and Sommerfeld where the mass of a state saturates an inequality with the central charges of the supersymmetric algebra. In the context of supergravity this amounts to a saturation of a similar inequality between the mass and the conserved charges of the theory. These solutions are of special interest as they allow for the existence of a covariantly constant Killing spinor through which they preserve some of the supersymmetry in the theory.

The characteristic equation for $\mathscr{C}$

With the explicit charge matrix of the Schwarzschild solution, Bossard, Nicolai and Stelle observe in [1] that the fundamental representation of the Lie algebra $\mathfrak{g}$ admits a 3-grading[13]

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with respect to the Lie algebra element H, the Cartan element of the $\mathfrak{sl}_2$-part of the $\mathfrak{g}$-decomposition (3.8). In this, the element H takes the form $\mathrm{diag}(1, 0, -1)$ whence it in this representation satisfies

$$H^3 = H. \tag{3.45}$$

If we normalize the extremality parameter according to

$$v^2 = \frac{\mathrm{Tr}[\mathscr{C}^2]}{\mathrm{Tr}[H^2]},$$

which for the Schwarzschild solution corresponds to $v^2 = m^2$ as we just noted that $\mathscr{C} = mH$, equation (3.45) can be written as

$$\mathscr{C}^3 - v^2\mathscr{C} = 0. \tag{3.46}$$

This is what is referred to as the **characteristic equation** for the charge matrix.

We learned above that extremal black holes share the property of $v^2 = 0$ and, as such, they must all have a charge matrix obeying the **nilpotency criteria**

$$\mathscr{C}^3 = 0.$$

We hence can state the important fact that *all non-rotating extremal black holes, such as BPS solutions, have a nilpotent charge matrix.* From this we are led directly into the study of nilpotent orbits as the adjoint action by $\mathcal{G}$ on $\mathscr{C}$ gives the set

$$\mathcal{O}_{\mathscr{C}} = \{g\mathscr{C}g^{-1} \mid g \in \mathcal{G}\}.$$

Thus, we now delve into the mathematical descriptions of these objects.

[13] or a 5-grading in case of supergravity theories based on real forms of E_8 for which similar results as the presented follow.

4

Nilpotent Orbits

In this section a brief mathematical background on nilpotent orbits is provided. After the necessary definitions, the classification concepts of nilpotent orbits in complex and real semisimple Lie algebras are presented, followed by a discussion about the relevant orbits for the physical models. The material here is mostly a condensate of the relevant parts of [5] but, instead of only being strictly streamlined for the context of dimensionally reduced gravity, it also is intended to provide a bit more general insights to the structures of nilpotent orbits.

4.1 Definition

Let $\mathcal{G}$ be a Lie group with corresponding Lie algebra $\mathfrak{g}$.

In general, an operator O is called nilpotent if there exists a natural number n such that $O^n = 0$. In the case of Lie algebra elements the definition of nilpotency is based on the adjoint action on the algebra itself. This is also the case for the notion of semisimplicity of operators.

Definition 4.1.1. *An element X in a Lie algebra $\mathfrak{g}$ is called* ***nilpotent*** *if it, regarded as an endomorphism on $\mathfrak{g}$ through the adjoint action, is nilpotent. I.e. X is nilpotent in $\mathfrak{g}$ if there exists a natural number n such that*

$$\mathrm{ad}_X^n = 0.$$

Similarly, an element H in $\mathfrak{g}$ is called ***semisimple*** *if it, regarded as an endomorphism on $\mathfrak{g}$ through the adjoint action, is semisimple*[1]*.*

[1] Recall that an operator H is semisimple if each subspace invariant under H has an H-invariant complement.

A group element g of $\mathcal{G}$ acts on $\mathfrak{g}$ through the adjoint representation, denoted Ad_g, and an adjoint **orbit** $\mathcal{O}_Z$ of an element Z in $\mathfrak{g}$ is defined as the set $\{\mathrm{Ad}_g Z | g \in \mathcal{G}\}$. For a nilpotent element we have the definition:

Definition 4.1.2. *A **nilpotent orbit** $\mathcal{O}_X$ in $\mathfrak{g}$ is the adjoint orbit of some nilpotent element $X \in \mathfrak{g}$, i.e. $\mathcal{O}_X = \{\mathrm{Ad}_g X | g \in \mathcal{G}\} = \mathcal{G}_{\mathrm{Ad}}(X)$ [2]. A corresponding construction $\mathcal{O}_H$ for a semisimple element H is called a **semisimple orbit**.*

It follows from the conjugating adjoint action that any element of $\mathcal{O}_X$ is nilpotent, which is why the name is adequate. Furthermore, any adjoint orbit $\mathcal{O}_Z$ is a homogeneous complex space isomorphic to $\mathcal{G}_{\mathrm{Ad}}/\mathcal{G}^Z_{\mathrm{Ad}}$, where $\mathcal{G}^Z_{\mathrm{Ad}}$ is the stabilizer of Z in $\mathcal{G}_{\mathrm{Ad}}$. As a manifold it has the dimension $\dim \mathcal{O}_Z = \dim \mathfrak{g} - \dim \mathfrak{g}^Z$, where $\mathfrak{g}^Z$ denotes the centralizer of Z in $\mathfrak{g}$. Although an adjoint orbit is a subset of $\mathfrak{g}$ it is not a sub*space* as it, in general, is not closed under vector addition. However, if a nilpotent orbit contains X it also contains all scalar multiples of X.

4.2 Classification in Complex Lie Algebras

It has been shown that the nilpotent orbits in a semisimple Lie algebra $\mathfrak{g}$ are finitely many and there are developed methods to classify them in both the complex semisimple Lie algebras and in their split real forms. We will now briefly review these methods in the mentioned order.

4.2.1 Complex semisimple case

The real work horse in the classification of nilpotent orbits is the following theorem of Jacobson and Morozov.

Theorem 4.2.1 (Jacobson-Morozov). *Any non-zero nilpotent element X of a complex semisimple Lie algebra $\mathfrak{g}$ is part of a $\mathfrak{sl}(2, \mathbb{C})$ subalgebra $\{H, X, Y\}$ of $\mathfrak{g}$, where H is the semisimple generator and X, Y are the positive and negative root generators, respectively.*

Such a subalgebra is called a **standard triple** and its elements are also referred to as the neutral, nilpositive and nilnegative element. Any two standard triples with the same nilpositive element are conjugate, by a theorem of Kostant, and there is a one-to-one map between conjugacy classes of nilpotent elements and conjugacy classes of standard triples. Furthermore, there is a natural one-to-one mapping from nilpotent orbits to a certain subset of the set of semisimple orbits established through these triples, $(\mathcal{O}_X \to \{H, X, Y\} \to \mathcal{O}_H)$. These semisimple orbits are called **distinguished**. In this relation, we can always choose H to be an element of a Cartan subalgebra in such a way that the action of the simple roots only yields the values $\alpha(H) \in \{0, 1, 2\}$. This provides a labeling of each node in

[2] The adjoint group $\mathcal{G}_{\mathrm{Ad}}$ is simply the center-free version of $\mathcal{G}$ as the center always acts trivially in the adjoint action.

the Dynkin diagram with the corresponding value $\alpha(H)$ and such a diagram is called a **weighted Dynkin diagram**. It is proven that each nilpotent orbit corresponds to a unique weighted Dynkin diagram and there is thus at most $3^{\operatorname{rank}\mathfrak{g}}$ nilpotent orbits in a Lie algebra $\mathfrak{g}$. However, not all labels of the nodes with the numbers $0, 1, 2$ give a weighted Dynkin diagram so the number of nilpotent orbits in $\mathfrak{g}$ is in general less.

4.2.2 Bala-Carter and general simple Lie algebras

Even though the weighted Dynkin diagrams provide a neat way of classifying the nilpotent orbits there is still the problem of finding which labels actually constitute such a diagram. For the classical algebras there are rather simple algorithms based on partitions of n, the dimension of the defining representatino, with only minor complications in some cases. These rely on the existence of Jordan forms and are not applicable to all semisimple Lie algebras. There is, however, a more general method devised by Bala and Carter. To present it we need some notation for the decomposable structure of a Lie algebra.

Let Φ denote the set of roots corresponding to a semisimple Lie algebra $\mathfrak{g}$ with some choice of a Cartan subalgebra $\mathfrak{C}$ and let the subsets $\Phi^+ \subset \Phi$ and $\Delta \subset \Phi^+$ denote the set of positive roots and the set of simple roots, respectively. With a fixed $\mathfrak{C}$ there is always a root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{C} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha, \tag{4.1}$$

where the subspaces $\mathfrak{g}_\alpha$ are the eigenspaces of the elements in H, i.e.

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g} \,|\, [H, Z] = \alpha(H)Z,\, H \in \mathfrak{C}\}.$$

The root space decomposition will now be used to define three types of subalgebras which are the basic pieces in the Bala-Carter method.

A **Borel subalgebra** $\mathfrak{b}$ of a semisimple Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra and it has the following property [5, p. 32]

Lemma 4.2.2. *A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$can always be decomposed as $\mathfrak{b} = \mathfrak{C} \oplus \mathfrak{n}$ where $\mathfrak{C}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilradical[3] of $\mathfrak{b}$, consisting of exactly the nilpotent elements of $\mathfrak{b}$. There is, furthermore, always a possible choice of Φ^+ such that $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.*

We call a subalgebra $\mathfrak{p}$ containing a Borel subalgebra for a **parabolic subalgebra**. We can clarify the structure of the different parabolic subalgebras in $\mathfrak{g}$ by choosing a subset $\Theta \subseteq \Delta$ and construct a parabolic subalgebra $\mathfrak{p}_\Theta$ spanned by the generators corresponding to the roots $\alpha \in \Phi^+$ and $-\alpha \in \Theta$, together with $\mathfrak{C}$ and all their commutators. This will always give a parabolic subalgebra. We denote the full root system generated by Θ as $\langle\Theta\rangle$.

Now, any parabolic subalgebra is $\mathcal{G}$-conjugate to some other on the form $\mathfrak{p}_\Theta$ and two such algebras $\mathfrak{p}_\Theta$ and $\mathfrak{p}_{\Theta'}$ are conjugate if and only if $\Theta = \Theta'$. This gives $2^{\operatorname{rank}[\mathfrak{g},\mathfrak{g}]}$ conjugacy

[3] The nilradical is the maximal nilpotent ideal.

classes of parabolic subalgebras from the possible choices of Θ. Moreover, for any $\mathfrak{p}_\Theta$ there is a **Levi decomposition** which splits $\mathfrak{p}_\Theta$ into

$$\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta,$$

where

$$\mathfrak{l}_\Theta = \bigoplus_{\alpha \in \langle\Theta\rangle} \mathfrak{g}_\alpha$$
$$\mathfrak{n}_\Theta = \bigoplus_{\alpha \in (\Phi^+ \setminus \Theta)} \mathfrak{g}_\alpha.$$

$\mathfrak{l}_\Theta$ is called a **Levi subalgebra** of $\mathfrak{g}$ while $\mathfrak{n}_\Theta$ is the nilradical of $\mathfrak{p}_\Theta$. Two Levi subalgebras $\mathfrak{l}_\Theta$ and $\mathfrak{l}_{\Theta'}$ are $\mathcal{G}$-conjugate if and only if the corresponding root systems $\langle\Theta\rangle$ and $\langle\Theta'\rangle$ are Weyl-conjugate [5, p. 51].

The Bala-Carter procedure goes one step deeper in the decompositions and look at parabolic subalgebras of the semisimple part of Levi subalgebras, i.e. $\mathfrak{p}_\mathfrak{l} \subseteq [\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$, and their corresponding Levi splitting into $\mathfrak{p}_\mathfrak{l} = \mathfrak{l}_\mathfrak{l} \oplus \mathfrak{n}_\mathfrak{l}$. Some of these parabolic (sub)subalgebras $\mathfrak{p}_\mathfrak{l}$ has a particular property and are called **distinguished** in $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$ if and only if

$$\dim \mathfrak{l}_\mathfrak{l} = \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]).$$

This is the same as saying that $\mathfrak{p}_\mathfrak{l}$ is distinguished if and only if $\dim \mathfrak{l}_\mathfrak{l}$ equals the number of *indecomposable* root generators in $\mathfrak{n}_\mathfrak{l}$. It is these pairs of $(\mathfrak{l}_\Theta, \mathfrak{p}_\mathfrak{l})$ which are the central objects for the main result of Bala and Carter.

Theorem 4.2.3 (Bala-Carter)**.** *There is a natural one-to-one correspondence between nilpotent orbits of* $\mathfrak{g}$ *and* $\mathcal{G}$*-conjugacy classes of pairs* $(\mathfrak{l}_\Theta, \mathfrak{p}_\mathfrak{l})$ *where* $\mathfrak{l}_\Theta$ *is a Levi subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{p}_\mathfrak{l}$ *is a distinguished parabolic subalgebra of* $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$.

The full decomposition procedure in the Bala-Carter method can be summarized as follows.

$$\begin{array}{l}
\mathfrak{g} \supseteq \mathfrak{p}_\Theta \\
\qquad\downarrow \\
\qquad \mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta \\
\qquad\qquad\quad\downarrow \\
\qquad\qquad [\mathfrak{l}_\Theta, \mathfrak{l}_\Theta] \supseteq \mathfrak{p}_\mathfrak{l} \\
\qquad\qquad\qquad\qquad\quad\downarrow \\
\qquad\qquad\qquad\qquad \mathfrak{p}_\mathfrak{l} = \mathfrak{l}_\mathfrak{l} \oplus \mathfrak{n}_\mathfrak{l} \\
\qquad\qquad\qquad\qquad\qquad\quad\downarrow \\
\qquad\qquad\qquad\qquad\qquad \mathfrak{p}_\mathfrak{l} \text{ distinguished iff} \\
\qquad\qquad\qquad\qquad\qquad \dim \mathfrak{l}_\mathfrak{l} = \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}])
\end{array}$$

To exhibit its workings we now apply it to two simple examples.

Example 3 (Bala-Carter method on $\mathfrak{sl}_3$). $\mathfrak{sl}_3$ has four conjugacy classes of parabolic subalgebras for a fixed Borel algebra $\mathfrak{b}$, corresponding to the choices $\Theta = \{\}, \{\alpha_1\}, \{\alpha_2\}, \{\alpha_1, \alpha_2\} = \Delta$ of the simple roots α_1 and α_2. Any parabolic subalgebra $\mathfrak{p}$ is conjugate to $\mathfrak{p}_\Theta$ for one and only one of these choices and there are thus $4 = 2^{\mathrm{rank}[\mathfrak{sl}_3,\mathfrak{sl}_3]}$ conjugacy classes of parabolic subalgebras. For the choices of Θ there are however only three Weyl conjugacy classes of $\langle\Theta\rangle$, since $\langle\{\alpha_i\}\rangle, i = 1, 2$, are W-conjugate, and correspondingly there are three conjugacy classes of Levi subalgebras $\mathfrak{l}_\Theta$. The possible choices are illustrated in figure 4.1. We now need to find all the distinguished parabolic (sub)subalgebras of these Levi subalgebras and we do it systematically for each choice of Θ.

1. $\Theta = \{\}$**:** The minimal Levi subalgebra $\mathfrak{l}_\Theta = \mathfrak{C}$ has only one parabolic subalgebra $\mathfrak{p}_\mathfrak{l}$ which is always distinguished. This is immediate from the trivial calculation

$$[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta] = 0 \implies \mathfrak{p}_\mathfrak{l} = 0 \implies \dim \mathfrak{l}_\mathfrak{l} = \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) = 0. \qquad \boxed{\text{dist.}}$$

2. $\Theta = \{\alpha_i\}$**:** As mentioned the two choices of $i = 1, 2$ yields the same $\mathfrak{l}_\Theta$ with $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta] \cong \mathfrak{sl}_2$ and the possible choices of $\mathfrak{p}_\mathfrak{l} = \mathrm{span}\{E_{\alpha_i}, H_i\}, \mathrm{span}\{E_{\alpha_i}, H_i, F_{\alpha_i}\}$. The corresponding Levi (sub)subalgebras are

- $\mathfrak{l}_\mathfrak{l} = \mathrm{span}\{H_i\}$ with complement $\mathfrak{n}_\mathfrak{l} = \mathrm{span}\{E_{\alpha_i}\}$ which satisfies

$$\dim \mathfrak{l}_\mathfrak{l} = 1 = \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) \qquad \boxed{\text{dist.}}$$

- $\mathfrak{l}_\mathfrak{l} \cong \mathfrak{sl}_2$ and $\mathfrak{n}_\mathfrak{l} = 0$, thus not distinguished.

3. $\Theta = \Delta$**:** The maximal Levi subalgebra $\mathfrak{l}_\Theta = \mathfrak{g}$ gives $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta] = \mathfrak{sl}_3$ with the possible parabolic (sub)subalgebras $\mathfrak{p}_\mathfrak{l} = \mathfrak{p}_\Theta$ for all choices of Θ. We denote the subsets of simple roots which labels the subsubalgebras $\mathfrak{p}_\mathfrak{l}$ as $\Theta_\mathfrak{l}$. The Levi decompositions are

- $\Theta_\mathfrak{l} = \{\}$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{C}$; $\mathfrak{n}_\mathfrak{l} = \mathrm{span}\{E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+\alpha_2}\}$ implying

$$\dim \mathfrak{l}_\mathfrak{l} = 2 \text{ and } \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) = 3 - 1 = 2. \qquad \boxed{\text{dist.}}$$

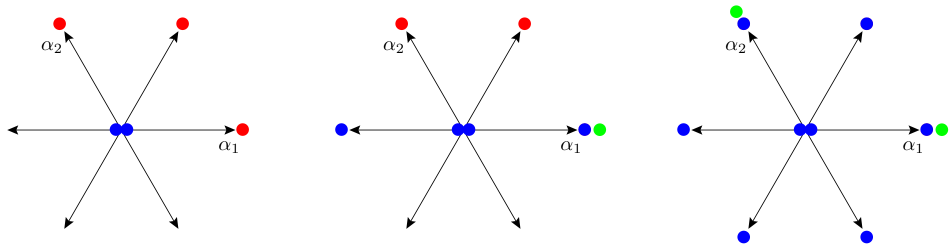
- $\Theta_\mathfrak{l} = \{\alpha_i\}$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{C} \oplus \mathrm{span}\{E_{\alpha_i}, F_{\alpha_i}\}$; $\mathfrak{n}_\mathfrak{l} = \mathrm{span}\{E_{\alpha_j}, E_{\alpha_i+\alpha_j}\}$ with i,$j \in \{1,2\}$ and $i \neq j$. This yields

$$\dim \mathfrak{l}_\mathfrak{l} = 4 \neq \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) = 2 - 0 = 2,$$

  which is not distinguished.

- $\Theta_\mathfrak{l} = \Delta$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{sl}_3$; $\mathfrak{n}_\mathfrak{l} = 0$, i.e. not distinguished.

In total we find three pairs $(\mathfrak{l}_\Theta, \mathfrak{p}_\mathfrak{l})$ with distinguished $\mathfrak{p}_\mathfrak{l}$:s and thus three nilpotent orbits.



(a) $\Theta = \{\}$ **(b)** $\Theta = \{\alpha_1\}$ **(c)** $\Theta = \{\alpha_1, \alpha_2\}$

Figure 4.1: Root diagrams of $\mathfrak{sl}_3$ with the four decompositions made in example 3. The choice $\Theta = \{\alpha_2\}$ is just a reflection of $\Theta = \{\alpha_1\}$. The subset Θ of simple roots is marked with green dots, the Levi subalgebra $\mathfrak{p}_\Theta$ with blue dots and the nilradical $\mathfrak{n}_\Theta$ with red dots.

Example 4 (Bala-Carter method on $\mathfrak{g}_2$). Carrying out the same procedure for $\mathfrak{g}_2$ we also find four conjugacy classes $\mathfrak{p}_\Theta$ corresponding to $\Theta = \{\}, \{\alpha_1\}, \{\alpha_2\}, \{\alpha_1, \alpha_2\} = \Delta$ where α_1 (α_2) is the short (long) simple root, as can be seen in figure 4.2. In this case we also have four conjugacy classes of Levi subalgebras as no $\langle\Theta\rangle$ is Weyl conjugate to another and they are represented with a subfigure each in figure 4.2. We list the Levi decompositions of their parabolic (sub)subalgebras, as done for $\mathfrak{sl}_3$.

1. $\Theta = \{\}$**:** The minimal Levi subalgebra has one distinguished parabolic subalgebra, as seen above. $\boxed{\text{dist.}}$

2. $\Theta = \{\alpha_i\}$**:** Although the two values of $i = 1, 2$ now give distinct conjugacy classes of $\mathfrak{l}_\Theta = \mathfrak{C} \oplus \mathrm{span}\{E_{\alpha_i}, F_{\alpha_i}\}$, the calculations are the same and completely analogous to the corresponding case 2 in the example 3 for $\mathfrak{sl}_3$ above. Hence we find two distinguished parabolic (sub)subalgebras, one for each i. $\boxed{2\times \text{dist.}}$

3. $\Theta = \Delta$**:** As for the $\mathfrak{sl}_3$-example we find the maximal Levi subalgebra $\mathfrak{l}_\Theta = \mathfrak{g}$ with $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta] = \mathfrak{g}_2$ and parabolic (sub)subalgebras $\mathfrak{p}_\mathfrak{l} = \mathfrak{p}_{\Theta_\mathfrak{l}}$ for all $\Theta_\mathfrak{l}$:s, with the corresponding Levi decompositions

- $\Theta_\mathfrak{l} = \{\}$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{C}$ and $\mathfrak{n}_\mathfrak{l} = \mathrm{span}\{E_\alpha \mid \alpha \in \Phi^+\}$ implying

$$\dim \mathfrak{l}_\mathfrak{l} = 2 \quad \text{and} \quad \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) = 6 - 4 = 2. \qquad \boxed{\text{dist.}}$$

- $\Theta_\mathfrak{l} = \{\alpha_1\}$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{C} \oplus \mathrm{span}\{E_{\alpha_1}, F_{\alpha_1}\}$ and $\mathfrak{n}_\mathfrak{l} = \mathrm{span}\{E_{\alpha_2}, E_{\alpha_2+i\alpha_1}, E_{2\alpha_2+3\alpha_1} \mid i = 1, 2, 3\}$. This yields

$$\dim \mathfrak{l}_\mathfrak{l} = 4 \quad \text{and} \quad \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) = 5 - 1 = 4, \qquad \boxed{\text{dist.}}$$

- $\Theta_\mathfrak{l} = \{\alpha_2\}$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{C} \oplus \mathrm{span}\{E_{\alpha_2}, F_{\alpha_2}\}$ and $\mathfrak{n}_\mathfrak{l} = \mathrm{span}\{E_{\alpha_1}, E_{\alpha_2+i\alpha_1}, E_{2\alpha_2+3\alpha_1} \mid i = 1, 2, 3\}$ giving

$$\dim \mathfrak{l}_\mathfrak{l} = 4 \quad \text{and} \quad \dim(\mathfrak{n}_\mathfrak{l}/[\mathfrak{n}_\mathfrak{l}, \mathfrak{n}_\mathfrak{l}]) = 5 - 3 = 2$$

  and no distinguished parabolic (sub)subalgebra.

- $\Theta_\mathfrak{l} = \Delta$: $\mathfrak{l}_\mathfrak{l} = \mathfrak{g}_2$ and $\mathfrak{n}_\mathfrak{l} = 0$, i.e. not distinguished.

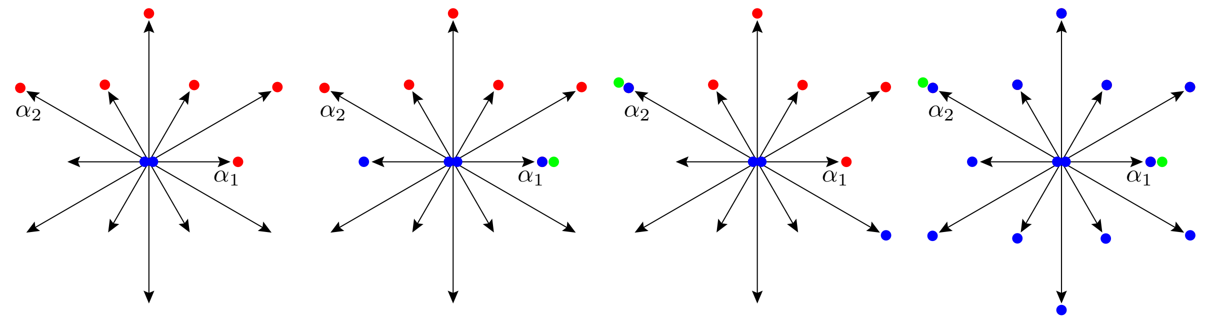
In short:

$\Theta_\mathfrak{l}$	$\mathfrak{l}_\mathfrak{l}$	$\mathfrak{n}_\mathfrak{l}$	$\dim \mathfrak{l}_\mathfrak{l}$	$\dim(\frac{\mathfrak{n}_\mathfrak{l}}{[\mathfrak{n}_\mathfrak{l},\mathfrak{n}_\mathfrak{l}]})$	dist.
$\{\}$	$\mathfrak{C}$	$\mathrm{s}\{E_\alpha \mid \alpha \in \Phi^+\}$	2	2	•
$\{\alpha_1\}$	$\mathfrak{C} \oplus \mathrm{s}\{E_{\alpha_1}, F_{\alpha_1}\}$	$\mathrm{s}\{E_{\alpha_2}, E_{\alpha_2+i\alpha_1}, E_{2\alpha_2+3\alpha_1} \mid i = 1,2,3\}$	4	5-1=4	•
$\{\alpha_2\}$:	$\mathfrak{C} \oplus \mathrm{s}\{E_{\alpha_2}, F_{\alpha_2}\}$	$\mathrm{s}\{E_{\alpha_1}, E_{\alpha_2+i\alpha_1}, E_{2\alpha_2+3\alpha_1} \mid i = 1,2,3\}$	4	5-3 =2	
Δ	$\mathfrak{g}_2$	0	14	0	

*where $\mathrm{s}\{\cdot\}$ denotes $\mathrm{span}\{\cdot\}$

Thus, in total we find five nilpotent orbits in $\mathfrak{g}_2$ and their corresponding Dynkin labels are

$$\big(\alpha_1(H), \alpha_2(H)\big) = (0{,}0),\ (1{,}0),\ (0{,}1),\ (2{,}0) \text{ and } (2{,}2). \tag{4.2}$$



(a) $\Theta = \{\}$ **(b)** $\Theta = \{\alpha_1\}$ **(c)** $\Theta = \{\alpha_2\}$ **(d)** $\Theta = \{\alpha_1, \alpha_2\}$

Figure 4.2: Root diagrams of $\mathfrak{g}_2$ with the four decompositions made in example 4. The subset Θ of simple roots is marked with green dots, the Levi subalgebra $\mathfrak{p}_\Theta$ with blue dots and the nilradical $\mathfrak{n}_\Theta$ with red dots.

4.3 Classification in Real Algebras

So far the classification methods have been concerned with complex Lie algebras $\mathfrak{g}_\mathbb{C}$. Our interest lies primarily in the split real forms $\mathfrak{g}_\mathbb{R}$ of these and to classify the nilpotent orbits in them it requires a bit more theory. It turns out that the Jacobson-Morozov theorem 4.2.1 carries over to the real case. Moreover, given a Cartan involution θ this standard triple is conjugate to one for which the conditions (2.1)

$$[\mathfrak{k}_\mathbb{R}, \mathfrak{k}_\mathbb{R}] \subset \mathfrak{k}_\mathbb{R}\,, \qquad [\mathfrak{k}_\mathbb{R}, \mathfrak{p}_\mathbb{R}] \subset \mathfrak{p}_\mathbb{R}\,, \qquad [\mathfrak{p}_\mathbb{R}, \mathfrak{p}_\mathbb{R}] \subset \mathfrak{k}_\mathbb{R}\,,$$

holds. Here $\mathfrak{k}_\mathbb{R}$ denotes the Lie algebra of the maximally compact subgroup $\mathcal{K}_\mathbb{R}$ of $\mathcal{G}_\mathbb{R}$ on which the Cartan involution acts as the identity and $\mathfrak{p}_\mathbb{R} = \mathfrak{g}_\mathbb{R} \ominus \mathfrak{k}_\mathbb{R}$. Such a triple is called a **Cayley triple**. The key point in the classification is to use the Cartan decomposition $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$ and the corresponding complexifications $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$ as there is a bijection between the nilpotent orbits in $\mathfrak{g}_\mathbb{R}$ and the nilpotent orbits of $\mathcal{K}^\mathbb{C}$ in $\mathfrak{p}_\mathbb{C}$. However, in order to take advantage of that one needs the neutral element to be part of $\mathfrak{k}_\mathbb{C}$ and all vectors in the triple to be θ-eigenvectors. This is accomplished through the **Cayley transform** of a Cayley triple $\{H, X, Y\}$:

$$\{H', X', Y'\} = \{\mathrm{i}(X - Y), \tfrac{1}{2}(X + Y + \mathrm{i}H), \tfrac{1}{2}(X + Y - \mathrm{i}H)\}$$

This triple lives in $\mathfrak{g}_\mathbb{C}$ with $H' \in \mathfrak{k}_\mathbb{C}$ and the other two in $\mathfrak{p}_\mathbb{C}$. This last property makes a standard triple to what is called *normal* in [5].

As stated is every standard triple in $\mathfrak{g}_\mathbb{R}$ conjugate to a Cayley triple and it can now be shown that the Cayley transformations of two Cayley triples in $\mathfrak{g}_\mathbb{R}$ with the same nilpotent element $X_\mathbb{R}$ are $\mathcal{K}_\mathbb{C}$ conjugate. Moreover, any two standard triples in $\mathfrak{g}_\mathbb{C}$ with the same nilpositive element $X_\mathbb{C} \in \mathfrak{p}_\mathbb{C}$ are related by a $\mathcal{K}_\mathbb{C}$-conjugation and if they are conjugated to be the Cayley transformations of two Cayley triples, these latter two are related by a $\mathcal{G}_\mathbb{R}$ conjugation. Schematically,

$$\begin{array}{ccc} \{H_\mathbb{R}, X_\mathbb{R}, Y_\mathbb{R}\} & \xrightarrow{\text{Cayley transf.}} & \{H_\mathbb{C}, X_\mathbb{C}, Y_\mathbb{C}\} \\ \mathcal{G}_\mathbb{R} \updownarrow & & \mathcal{K}_\mathbb{C} \updownarrow \\ \{H'_\mathbb{R}, X_\mathbb{R}, Y'_\mathbb{R}\} & \xrightarrow{\text{Cayley transf.}} & \{H'_\mathbb{C}, X'_\mathbb{C}, Y'_\mathbb{C}\} \end{array}$$

and

$$\begin{array}{ccc} \{H_\mathbb{R}, X_\mathbb{R}, Y_\mathbb{R}\} & \xleftarrow{\text{inv. Cayley transf.}} & \{H_\mathbb{C}, X_\mathbb{C}, Y_\mathbb{C}\} \\ \mathcal{G}_\mathbb{R} \updownarrow & & \mathcal{K}_\mathbb{C} \updownarrow \\ \{H'_\mathbb{R}, X'_\mathbb{R}, Y'_\mathbb{R}\} & \xleftarrow{\text{inv. Cayley transf.}} & \{H'_\mathbb{C}, X_\mathbb{C}, Y'_\mathbb{C}\}. \end{array}$$

This provide the foundation to the proof of the following theorem.

Theorem 4.3.1 (Sekiguchi)**.** *There is a natural one-to-one correspondence between nilpotent $\mathcal{G}_\mathbb{R}$-orbits in $\mathfrak{g}_\mathbb{R}$ and nilpotent $\mathcal{K}_\mathbb{C}$-orbits in $\mathfrak{p}_\mathbb{C}$. This correspondence sends the zero orbit to the zero orbit and the orbit through the nilpositive element of a Cayley triple to the one through the nilpositive element of its Cayley transform.*

To study nilpotent orbits we can thus go back to the methods developed for complex algebras and study of the structure of $\mathcal{K}_\mathbb{C}$ orbits in $\mathfrak{p}_\mathbb{C}$.

An additional remark to this theorem is that it has been proved that this bijection preserves the partial ordering of orbits, which will be discussed further in section 4.5, in all classical algebras $\mathfrak{g}_\mathbb{R}$. It is however not known whether this holds also in the exceptional cases.

Example 5 ($G_{2(2)}$-orbits)*.* We have seen the orbit structure of G_2-orbits in $\mathfrak{g}_2$ by employing the Bala-Carter method and it is instructive to see how this splits up when considering the orbits of the real form $G_{2(2)}$. As we have seen that $G_{2(2)}/SO(2,2)$ is the relevant coset space for five dimensional minimal supergravity this is particularly interesting. This study has been carried out in detail in [18].

Let us denote the maximal compact subgroup of $G_{2(2)}$ as $\mathcal{K}$ and its Lie algebra with $\mathfrak{k}$. According to theorem 4.3.1 above we can thus study the orbits of $\mathcal{K}_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{g}_2 = \mathfrak{p}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$. To track these we introduce an additional labeling which [18] call the β-label, in addition to the weighted Dynkin labels $\big(\alpha_1(H), \alpha_2(H)\big)$ of the orbits in G_2 we presented in equation (4.2). These β-labels corresponds to the weighted Dynkin labels of $\mathfrak{p}_{\mathbb{C}}$ obtained through a choice of a Cartan subalgebra and simple roots for $\mathfrak{p}_{\mathbb{C}}$. These can be found in the math literature (e.g. [7]) and with them the orbit structure of $G_{2(2)}$ is displayed in table 4.1 as presented by [18]. We note that the G_2-orbit with labels (2,0) splits up into two distinct orbits in $G_{2(2)}$.

Table 4.1: The five nonzero $G_{2(2)}$-orbits. Note the split of the G_2-orbit corresponding to α-labels (2,0) [18].

$G_{2(2)}$-orbit	α-labels	β-labels	$\dim(G_{2(2)} \cdot x)$
$\mathcal{O}_1$	$(1,0)$	$(1,1)$	6
$\mathcal{O}_2$	$(0,1)$	$(1,3)$	8
$\mathcal{O}_3$	$(2,0)$	$(2,2)$	10
$\mathcal{O}_4$	$(2,0)$	$(0,4)$	10
$\mathcal{O}_5$	$(2,2)$	$(4,8)$	12

4.3.1 $\mathcal{H}$-orbits in $\mathfrak{m}$

Although nilpotent orbits in complex and real Lie algebras can and have been classified, this is not exactly what arose when we studied the transformation of the charge matrix. As we are primarily interested in those transformations that preserves the boundary condition of an asymptotically flat spacetime we are limited to the transformations of the subgroup $\mathcal{H}$. What thus really matters in the context of extremal black holes are the orbits of $\mathcal{H}$ in $\mathfrak{m}$. This is not a trivial restriction as there might be elements in $\mathcal{G}$ which connect two elements in $\mathfrak{m}$ while they are missing in $\mathcal{H}$. $\mathcal{G}$-orbits may thus split into two or more $\mathcal{H}$-orbits.

The problem is unfortunately also non-trivial and it is in fact not yet solved in full generality. There are however different techniques developed to tackle this, as e.g. in [2, 28, 12], and [15] supplies a summary. In short one considers a decomposition of $\mathfrak{g}$ into representations of $\mathfrak{sl}_2$ corresponding to a normal standard triple with semisimple element H. These are further decomposed according to the H-grading of these. Now, any standard triple for a given nilpositive element can be chosen such that H lies in $\mathfrak{h}\cap\mathfrak{p}$, the intersection of the Lie algebra to $\mathcal{H}$ and the Cartan involution invariant subspace $\mathfrak{p}$. By noting that two semisimple elements $H, H' \in \mathfrak{h}\cap\mathfrak{p}$ in two standard triples are $\mathcal{H}$-conjugate if and only if the simple roots $\alpha_{\mathcal{H}}$ of $\mathcal{H}$ satisfies $\alpha_{\mathcal{H}}(H) = \alpha_{\mathcal{H}}(H)$, the problem is reduced to study the nilpotent orbits in $\mathfrak{m}_2$, the part of $\mathfrak{m}$ with eigenvalue 2 in the mentioned H-grading.

In spite of all these efforts a full classification is missing and one often has to resort to case-by-case studies.

As an example of these splits we can continue example 5 and look at the orbits of the subgroup $\mathcal{H}$ which in this case is $SO(2{,}2)$.

Example 6 ($SO(2{,}2)$-orbits in $G_{2(2)}$). When we restrict the adjoint action on $\mathfrak{g}_{2(2)}$ to the denominator group $\mathcal{H} = SO(2{,}2)$ in $G_{2(2)}/SO(2{,}2)$ some of the $\mathcal{G}_{2(2)}$-orbits split. Adapting the notation to table 4.1, it is in [18] found that $\mathcal{O}_3$ splits into two orbits which we denote $\mathcal{O}_{3\mathcal{H}}$ and $\mathcal{O}'_{3\mathcal{H}}$ respectively and the same thing happens to $\mathcal{O}_4$ which splits into $\mathcal{O}_{4\mathcal{H}}$ and $\mathcal{O}'_{3\mathcal{H}}$. Choosing standard triples $\{H, X, Y\}$ with X as representative for each one of these orbits it is possible to define a third label, called the γ‘-label in [18], by applying the same simple roots as in the β-labels to the semisimple elements H. With these labels the table of the $G_{2(2)}$-orbits can be extended as in table 4.2.

Table 4.2: The splittings of the G_2-orbits when going to the real form $G_{2(2)}$ and then restricting the adjoint action to $\mathcal{H} = SO(2{,}2)$ [18].

G_2-orbit	α-label	$G_{2(2)}$-orbit	β-label	$\mathcal{H} = SO(2{,}2)$-orbit	γ-label
$\mathcal{O}_{1G_2}$	(1,0)	$\mathcal{O}_1$	(1,1)	$\mathcal{O}_1$	
$\mathcal{O}_{2G_2}$	(0,1)	$\mathcal{O}_2$	(1,3)	$\mathcal{O}_2$	
$\mathcal{O}_{3G_2}$	(2,0)	$\mathcal{O}_3$	(2,2)	$\mathcal{O}_{3\mathcal{H}}$	(0,4)
				$\mathcal{O}'_{3\mathcal{H}}$	(2,2)
		$\mathcal{O}_4$	(0,4)	$\mathcal{O}_{4\mathcal{H}}$	(0,4)
				$\mathcal{O}'_{4\mathcal{H}}$	(2,2)
$\mathcal{O}_{4G_2}$	(2,2)	$\mathcal{O}_5$	(4,8)	$\mathcal{O}_5$	

4.4 Nilpotent Orbits as Black Hole Solutions

Nilpotent orbits have been classified in complex and real Lie algebras but the setting that matters the most for the solution space of extremal black holes is not fully solved. The orbits of subgroup $\mathcal{H}$, which preserves the asymptotic flatness of the solutions, lack a full classification, but we have just seen that they nevertheless alter the orbit structure. This means that there is as of today no complete description of the extremal solution orbits. Furthermore, not all $\mathcal{H}$-orbits correspond to physically relevant solutions. This is also the case in our example of five dimensional minimal supergravity and the $\mathcal{H}$-orbits in $G_{2(2)}$ presented above. In [18], each orbit in $\mathcal{H}$-column of table 4.2 is systematically investigated. It is shown that only $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}'_{3\mathcal{H}}$ and $\mathcal{O}_{4\mathcal{H}}$ correspond to physical solutions. $\mathcal{O}_{3\mathcal{H}}$ and $\mathcal{O}'_{4\mathcal{H}}$ are ruled out by studying a well-known quartic polynomial of the charges in $\mathcal{N} = 8$ supergravity which is invariant under the group $E_{7(7)}$. This polynomial can be written also in the dimensionally reduced theory as long it is a consistent truncation of the mentioned theory. The study of the asymptotic values of the polynomial divulges $\mathcal{O}_{3\mathcal{H}}$ and $\mathcal{O}'_{4\mathcal{H}}$ to have naked curvature singularities. The exclusion of $\mathcal{O}_5$ is simpler as it turns out that the nilpotency degree of its representatives is seven which is too high to satisfy the characteristic equation for the charge matrix (3.46). There are believes ([2]) that it is a general fact that the β- and γ-labels coincide for physical orbits and the findings of [18] support this idea. Another notable result in [18] is that the biggest of the physical orbits, $\mathcal{O}_{4\mathcal{H}}$, contains non-supersymmetric solutions. This is in accordance with the subject of the next section.

From this we learn that the orbit structure in the Lie algebras cannot be applied as is onto the solution space of extremal black holes. Care must be taken when each orbit is analyzed in the specific gravity theory. Additionally we may remark that the $\mathcal{H}$-orbits may split further when opposing conditions, as the requirement of no NUT-charge [15].

4.5 Partial Ordering and the Minimal Orbit

An important property with a physical realisation is that the set of nilpotent orbits possesses a partial ordering $\mathcal{O} \leq \mathcal{O}'$. It is based on the Zariski closure[4] operation $\overline{\mathcal{O}}$ such that

$$\mathcal{O} \leq \mathcal{O}' \iff \overline{\mathcal{O}} \subseteq \overline{\mathcal{O}'}.$$

A smaller orbit is therefore always contained in the closure of a bigger one and is thus smaller in dimension. Conversely, the closure of an orbit contains all of the smaller ones. There always exists a **principal orbit** which is the largest orbit in the partial ordering, hence containing all other orbits in its closure. Moreover, there are two more canonical orbits labeled as the **subregular orbit** and the **minimal orbit**. They are the second largest and second smallest orbits, respectively.

The physical relevance of the partial ordering is what follows. Every real orbit $\mathcal{O}$ maps to a family of black holes as do all other orbits that are contained in its closure. These orbits describe special cases of the black hole solutions coming from $\mathcal{O}$. In particular, [1] states that the partial ordering in fact corresponds to an ordering of the black holes according to there BPS-degree. The amount of symmetry of the black hole family increases the smaller its corresponding orbit is and this gives the minimal orbit a special role physically. It corresponds to the black hole family with the highest BPS-degree and is thus particularly interesting [15].

An example of this is e.g. found in $D = 4, \mathcal{N} = 8$ supergravity deduced from eleven dimensional supergravity compactified on a 7 torus. The symmetry group of this theory

[4]For a minimal comment on Zariski topology, see appendix D.

is E_7 and the number of supercharges is 32. The compactified theory contains 28 Maxwell fields A^I_μ which gives a 56 dimensional lattice $\Gamma \cong \mathbb{Z}^{56}$ of electric and magnetic charges. The lattice Γ is preserved by the symplectic group $Sp(56,\mathbb{Z})$ and the actual symmetry group is

$$E_{7(7)}(\mathbb{Z}) = \{g \in E_{7(7)}(\mathbb{R}) \mid g\Gamma = \Gamma\} = E_{7(7)}(\mathbb{R}) \cap Sp(56,\mathbb{Z}).$$

Solutions to the field equations include black hole solutions with charge $\gamma = (p^I, q_I) \in \Gamma$ but not all of these solutions preserves the supersymmetry. It turns out that the $E_{7(7)}$-orbit of half BPS solutions $\gamma_{\frac{1}{2}}$, preserving $\frac{1}{2}32$ of the supercharges, corresponds precisely to the minimal orbit.

Because of its prominent role we now devote some special attention to the minimal orbit.

4.5.1 The minimal orbit

As stated above, there is in any simple Lie algebra $\mathfrak{g}$ a non-zero minimal nilpotent orbit $\mathcal{O}_{\min}$ which is contained in the closure of all other non-zero nilpotent orbits. It is the orbit of the nilpotent element corresponding to the highest root θ and is also denoted $\mathcal{O}_{E_\theta}$. The proof can e.g. be found in [5, pp. 61-62] and shows that any non-zero nilpotent orbit $\mathcal{O}_X$ is arbitrary close to E_θ by conjugating X in steps until its component along E_θ is arbitrary large compared to $X - E_\theta$ and then rescaling it to E_θ. By the partial ordering, the property of laying in the closure of all other nilpotent orbits is enough to conclude that $\mathcal{O}_{E^\theta}$ is the minimal nilpotent orbit.

This distinguished property is, as we will see in chapter 6, not generalizable to affine Lie algebras where there is no highest root. We have not yet seen these algebras nor how they arise in the compactification down to two dimensions but if we are to study a corresponding concept in these settings we ought to understand the minimal orbits in other terms here. In fact, very little is known about a minimal orbit in the affine Kac-Moody algebras but as a similar concept is expected from a physical perspective this is an important question in current research and calls for a better understanding of the minimal orbit also in finite dimensional algebras.

One approach in these efforts is to focus on the simple roots since, although the notion of a highest root is missing in affine algebras, these are still present. If one thus can understand the minimal orbit in terms of the simple roots that would be a good starting point to generalize the ideas to the infinite dimensional case. For this future project some work has been prepared in appendix F where conjugating elements between the highest root vector and the simple root vectors are found and presented for $\mathfrak{sl}_3, \mathfrak{sl}_4, \mathfrak{sl}_5$ and $\mathfrak{g}_2$.

5

Infinite Dimensional Symmetries Revealed

We will now take step into the world of infinite dimensional symmetry. Our first encounter is the same as the historical discovery of their relevance to dimensionally reduced gravity. By displaying the results of Geroch's work we will see how the algebra of $\mathfrak{sl}_2^+$, the infinite dimensional affine extension of $\mathfrak{sl}_2$, arise in four dimensional pure gravity reduced down to two dimensions. We then conclude with some remarks on the generality of these findings.

If there is two commuting Killing vectors in the four dimensional theory it is possible to dimensionally reduce down to two dimensions. This is e.g. the case of axially symmetric solutions. The reduction from four dimensions can be done in two ways, either in steps via the three dimensional theory or directly to two dimensions. Both ways result in a Lagrangian of the same form. However, they do not contain the same fields but are related to each other by a duality transformation, known under the name **Kramer-Neugebauer mappings**[1]. The first example of this was discovered by Geroch when he studied pure gravity in four dimensions and we will now look closer at this.

5.1 Dimensional Reduction of Pure 4D Gravity to Two Dimensions

We continue earlier examples by making a similar ansatz of the three dimensional metric as we did for the four dimensional metric in (3.2) in section 3.2. However, there is a crucial difference between the ansätze as the expected Kaluza-Klein vectors arising in such a split would carry $D-2$ degrees of freedom which now when $D = 2$ implies their vanishing.

[1]These mappings are in fact a wider concept out of which this is a special case.

Thus we are left with the simpler remains [16]

$$g^{(3)}_{\alpha\beta} \doteq \begin{pmatrix} f^2 g_{\mu\nu} & 0 \\ 0 & \rho^2 \end{pmatrix}.$$

Here f denotes the so called conformal factor which cannot be avoided in two dimensions. It is not related to the former function $f(r)^2$ in earlier sections.

The Lagrangian resulting from this split looks like [16, 8]

$$\mathcal{L}^{(2)}_{\mathrm{E}} = \rho\sqrt{g}\Big(R^{(2)} - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{-2\phi}\partial_\mu\chi\partial^\mu\chi\right) + 2f^{-1}\partial_\mu f\rho^{-1}\partial^\mu\rho\Big). \tag{5.1}$$

5.1.1 Dualization and the Kramer-Neugebauer mappings

The Lagrangian (5.1) allows for a dualization much like the one we preformed in three dimensions. First, however, we use the fact that any two dimensional metric is conformally flat which enables us to absorb what is needed in the conformal factor f such that $g_{\mu\nu} = \delta_{\mu\nu}$. Thereby we also get $\sqrt{g} = 1$. The relevant part of the Lagrangian is now

$$\mathcal{L}_{\mathrm{E,\ rel.\ part}} = \rho\frac{\delta^{\mu\nu}}{2}\left(\partial_\mu\phi\partial_\nu\phi + \mathrm{e}^{-2\phi}\partial_\mu\chi\partial_\nu\chi\right) \tag{5.2}$$

and we call this the Ehlers version of the Lagrangian. Starting the dualization we let $C_\mu = \partial_\mu\chi$ and incorporate this information in the Lagrangian multiplier $\tilde{\chi}\partial_\mu(\epsilon^{\mu\nu}C_\nu)$ such that

$$\tilde{\mathcal{L}} = \rho\frac{\delta^{\mu\nu}}{2}\left(\partial_\mu\phi\partial_\nu\phi + \mathrm{e}^{-2\phi}C_\mu C_\nu\right) + \tilde{\chi}\partial_\mu(\epsilon^{\mu\nu}C_\nu).$$

Varying with respect to $\tilde{\chi}$ gives the algebraical equation

$$C_\mu = \frac{\mathrm{e}^{2\phi}}{2}\delta_{\mu\nu}\epsilon^{\rho\nu}\partial_\nu\tilde{\chi}$$

which can be substituted back into the Lagrangian to yield

$$\tilde{\mathcal{L}} = \rho\frac{\delta^{\mu\nu}}{2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{\mathrm{e}^{2\phi}}{\rho^2}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}\right).$$

This expression is almost in the form of the Ehlers Lagrangian in (5.2) and we can in fact recreate it fully with the redefined fields

$$\mathrm{e}^{-2\tilde{\phi}} = \frac{\mathrm{e}^{2\phi}}{\rho^2} \qquad \tilde{\rho} = \rho \qquad \tilde{f} = f\rho^{1/4}\mathrm{e}^{\phi/4}.$$

These are the Kramer-Neugebauer mappings in this theory and with them we get the full Lagrangian

$$\tilde{\mathcal{L}}_{\mathrm{MM}} = \tilde{\rho}\Big(\frac{\delta^{\mu\nu}}{2}\left(\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \mathrm{e}^{-2\tilde{\phi}}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}\right) + 2\tilde{f}^{-1}\partial_\mu\tilde{f}\tilde{\rho}^{-1}\partial^\mu\tilde{\rho}\Big)$$

which is called the Matzner-Misner Lagrangian. If we would have dimensionally reduced directly down to two dimensions, this is the Lagrangian we would have found.

Both of the Ehlers and the Matzner-Misner versions of the Lagrangian exhibit an $SL(2,\mathbb{R})$-invariance. These are, however, two different SL-groups and the duality induces an action of the Ehlers $SL(2,\mathbb{R})$ on the Matzner-Misner fields and vice versa. Let us work out these transformations more in detail.

5.1.2 Transformations of the fields

We denote the Ehlers $SL(2,\mathbb{R})$ with Chevalley-Serre generators as is and the Matzner-Misner group as $\widetilde{SL}(2,\mathbb{R})$ with the generators $\tilde{E}, \tilde{H}, \tilde{F}$ accordingly. We now study the infinitesimal transformations by these groups on the fields $\Delta = \mathrm{e}^{\phi}, \chi, \tilde{\Delta} = \mathrm{e}^{\tilde{\phi}}$ and $\tilde{\chi}$ as induced by the action on the corresponding coset representatives V.[2] This includes the compensating gauge transformation on V whenever it is needed.

A small calculation gives that the Ehlers field representation transforms under the Ehlers $SL(2,\mathbb{R})$ as

$$\begin{aligned} \delta_E\Delta &= 0 & \qquad \delta_E &= -1 \\ \delta_H\Delta &= -2\Delta & \delta_H &= -2\chi \\ \delta_F\Delta &= 2\chi\Delta & \delta_F &= \chi^2 - \Delta^2 \end{aligned}$$

and the Matzner-Minser fields naturally satisfy exactly the same transformations under their $\tilde{SL}(2,\mathbb{R})$. So far it is just a repetition of the statement that these fields form $SL(2)$-representations and if one checks the commutation relations of these transformations one finds that E, H and F indeed form a Chevalley-Serre basis in this representation.

It becomes more interesting when looking at the commutators in between the $\mathfrak{sl}_2$- and $\tilde{\mathfrak{sl}}_2$-generators. Consider

$$\begin{aligned} \delta_{[\tilde{H},E]}\Delta &= \delta_{\tilde{H}}\delta_E\Delta - \delta_E\delta_{\tilde{H}}\Delta = -\delta_E\delta_{\tilde{H}}\left(\frac{\rho}{\tilde{\Delta}}\right) = -2\delta_E\Delta \\ \delta_{[H,\tilde{E}]}\Delta &= \delta_H\delta_{\tilde{E}}\Delta - \delta_{\tilde{E}}\delta_H\Delta = -\delta_{\tilde{E}}\delta_H\left(\frac{\rho}{\Delta}\right) = -2\delta_{\tilde{E}}\tilde{\Delta}. \end{aligned}$$

Repeating the same calculations for F and $\tilde{F}$ yields $[\tilde{H}, F] = 2F$ and $[H, \tilde{H}] = 2\tilde{F}$. We can thus see that these commutation relations gives the (generalized) Cartan matrix

$$A_{\mathfrak{sl}_2^+} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{5.3}$$

This is precisely the Cartan matrix for the affine extension $\mathfrak{sl}_2^+$ of $\mathfrak{sl}_2$. This is an infinite dimensional affine Kac-Moody algebra which will be the subject of the next chapter.

To really see the infinite dimension of this algebra we would need to look at the Matzner-Misner transformations on χ. Analogously to what we did above one can rewrite the

[2] Recall V in equation (2.18)

Ehlers field through the Kramer-Neugebauer mappings and calculate each transformation. However, acting repeatedly with the Matzner-Misner generators creates non-repeating expressions all dependent on Δ and χ. This never ends and by denoting the expressions as new field one gets an endless chain. In [21], Breitenlohner and Maison collect these in a generating function, a coset representative expanded in a power series of a parameter, which links this formulation of the sigma model to what is known as loop algebras. These will also be introduced in the next chapter.

The intertwining of the two $\mathfrak{sl}_2$-algebras we have just seen above is an example of a general property of the dimensional reduction down to two dimensions. All the theories relevant for the method have duality transformations which extend their symmetry groups to infinite dimensional versions.

6

Affine Kac-Moody Algebras

This chapter contains a basic introduction to affine Kac-Moody algebras and their construction from loop algebras. The central and double extensions are discussed followed by a short presentation of the root space. We then move on to some important formulas central to the orbit structure of these algebras. We conclude that section with some remarks on the implications on black hole solutions and the search for a corresponding concept to the minimal orbit in finite dimensional algebras.

Kac-Moody algebras are a generalization of the finite-dimensional semisimple Lie algebras and even though they usually are infinite-dimensional, they share a lot of properties. The generalization is done through the Cartan matrix which for the simple Lie algebras, we recall, is a $(r \times r)$-matrix fulfilling

$$A^{ii} = 2 \tag{6.1a}$$

$$A^{ij} = 0 \iff A^{ji} = 0 \tag{6.1b}$$

$$A^{ij} \in \mathbb{Z}_{-} \quad \text{for } i \neq j \tag{6.1c}$$

and

$$\det A > 0\,, \tag{6.1d}$$

where the last condition ensures A to be of rank r. It is also the requirement (6.1d) which is relaxed for Kac-Moody algebras such that the **generalized Cartan matrix** satisfies (6.1a)-(6.1c) and is allowed to be singular with rank $A \leq r$, and thus to have one or more zero eigenvalues. We will further on refer to both generalized and ordinary Cartan matrices as simply Cartan matrices.

Based on the definiteness of the Cartan matrix, it is possible to divide the Kac-Moody algebras into three main classes where the class of positive definite A:s contains all the finite simple Lie algebras, the class of positive semidefinite A:s are called the **affine Kac-Moody algebras** and the class of indefinite Cartan matrices goes under the natural name indefinite Kac-Moody algebras.

The affine Kac-Moody algebras thus have positive semidefinite Cartan matrices and, accordingly, they have precisely one zero eigenvalue. Hence, $\operatorname{rank} A = r - 1$ and from now on we choose the more convenient labeling where A is a rank r matrix of dimension $(r+1) \times (r+1)$. The requirement on A can equivalently be described as

$$\det A_{\{i\}} > 0 \quad \text{for } i = 0, \dots, r$$

where $\det A_{\{i\}}$ are the principal minors of A, i.e. the determinants of the matrices obtained by deleting the ith row and column from A.

6.1 Construction from Loop Algebras

Affine Kac-Moody algebras are often realized through extended loop algebras and it is also this construction which arise when dealing with the physics we just encountered the previous chapter.

A **loop algebra** $L\mathfrak{g} = \mathrm{Map}(S^1; \mathfrak{g})$ is the set of smooth maps from the unit circle to a finite-dimensional Lie algebra $\mathfrak{g}$ with a pointwisely defined bracket. The smooth maps can be expressed as (infinite) Laurent polynomials in the coordinate $z = \mathrm{e}^{\mathrm{i}\theta}$, or equivalently in trigonometric polynomials of θ, such that a basis for $L\mathfrak{g}$ can be written

$$\Big\{ T^a_m := T^a \otimes z^m \;\Big|\; \{T^a \mid a = 1, \cdots, \dim \mathfrak{g}\} \text{ forms a basis for } \mathfrak{g} \text{ and } z = \mathrm{e}^{\mathrm{i}\theta} \in S^1 \Big\}.$$

The Lie bracket is defined on the basis as

$$[T^a_m, T^b_n]_{L\mathfrak{g}} := [T^a, T^b]_{\mathfrak{g}} \otimes z^m z^n$$

and thereby on general loop algebra elements as

$$[X \otimes P(z), Y \otimes Q(z)]_{L\mathfrak{g}} = [X, Y]_{\mathfrak{g}} \otimes P(z)Q(z),$$

where $X, Y \in \mathfrak{g}$ and P and Q are (Laurent) polynomials of z.

6.1.1 Central extension and double extension

The loop algebra construction allows for a non-trivial central extension $\widehat{L\mathfrak{g}}$, i.e. the addition of an element C to $L\mathfrak{g}$ which commutes with the entire algebra without just being a simple sum of algebras, even in the cases where $\mathfrak{g}$ might be simple.

By definition, the **central extension** C has a vanishing commutator with all elements in $L\mathfrak{g}$ but it is added non-trivially by appearing at the right hand side in

$$[T^a_m, T^b_n]_{\widehat{L\mathfrak{g}}} = [T^a_m, T^b_n]_{L\mathfrak{g}} + m\, \delta_{m+n,0}\, \mathrm{B}(T^a, T^b)\, C, \tag{6.2}$$

where B is a symmetric invariant bilinear form on $\mathfrak{g}$. The central extensions of a loop algebra correspond precisely to such forms on $\mathfrak{g}$. Another way of expressing this is to define $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$

$$\omega(X,Y) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{B}\left(X(\theta), \frac{d}{d\theta}Y(\theta)\right) \mathrm{d}\theta.$$

Viewing $\widehat{L\mathfrak{g}}$ as the vector space sum $L\mathfrak{g}\oplus\mathbb{R}C$ the commutator of two elements $(X,\kappa),(Y,\lambda) \in \widehat{L\mathfrak{g}}$ is

$$\Big[(X,\kappa),(Y,\lambda)\Big] = \Big([X,Y],\,\omega(X,Y)\Big). \tag{6.3}$$

The equivalence between the commutators (6.2) and (6.3) is straight-forwardly checked. It is also common to express the commutation relation in terms of the residue of $P(z)Q'(z)$ at $z=0$, where the prime denotes derivation with respect to z, which of course is just the integral in the definition of ω.

For the bracket (6.3) to define a Lie algebra, ω must satisfy the condition

$$\omega\Big([X,Y],Z\Big) + \omega\Big([Y,Z],X\Big) + \omega\Big([Z,X],Y\Big) = 0$$

which in fact makes ω into a 2-cocycle[1] on $\widehat{L\mathcal{G}}$, the corresponding loop group to $\widehat{L\mathfrak{g}}$.

The central extension is, however, not enough to make a loop algebra into an affine Kac-Moody algebra. That demands the existence of a non-degenerate bilinear form cf. [22, p. 53] and [14, p. 103] which is impossible if we do not lift a degeneracy of the roots in the current setting, which soon will be explained more below in section 6.1.2 on the root system. It is done by introducing by hand an additional generator D together with the commutators

$$\begin{aligned} [C,D] &= 0 \\ [D,T^a_n] &= -[T^a_n,D] = nT^a_n. \end{aligned} \tag{6.4}$$

D is often called the **derivation** and D acts as $\frac{-id}{d\theta}$, or equivalently $z\frac{d}{dz}$, on the polynomial part of T^a_n. The resulting algebra, $\widetilde{L\mathfrak{g}}$, is called the double extension of $L\mathfrak{g}$ and since D does not occur on the right hand side of any bracket, the derived algebra of $\widetilde{L\mathfrak{g}}$ is

$$[\widetilde{L\mathfrak{g}},\widetilde{L\mathfrak{g}}] = \widehat{L\mathfrak{g}}\ .$$

The full doubly extended loop algebra can thus be written as the vector space sum

$$\widetilde{L\mathfrak{g}} = \mathbb{R}D \oplus \mathfrak{g}\otimes\mathcal{P}(z) \oplus \mathbb{R}C,$$

where $\mathcal{P}(z)$ is the set of all Laurent polynomials on S^1. It realizes the properties of an affine Kac-Moody algebra, which will be made a bit more explicit in the following section.

It should be mentioned that there are more ways of doing these extensions. What we have obtained here is the **untwisted** algebra. It is possible to define the maps from S^1 to $\mathfrak{g}$ involving automorphisms on $\mathfrak{g}$ upon a winding around the circle which give what is called **twisted** algebras. We will however not deal with these in the scope of this thesis.

[1] A p-cocycle is a closed p-form and in order for the central extension to be non-trivial it must not be a coboundary, that is it cannot be exact.

6.1.2 The root system

To construct the root system for $\widetilde{L\mathfrak{g}}$ we first find the analogue of the Chevalley-Serre basis and, particularly, the Cartan subalgebra in this basis. The latter consists of

$$\mathfrak{C} = \{H_0^i, C, D\}$$

where the H_0^i denotes the Cartan elements of $\mathfrak{g}$ paired with the constant polynomial. By equations (6.2) and (6.4), their brackets are checked as

$$[C, H_0^i] = [D, H_0^i] = [C, D] = 0$$

and from the commutation relations with the rest of the generators

$$\begin{aligned} &[H_0^i, E_n^{\bar\alpha}] = \bar\alpha^i E_n^{\bar\alpha}, \qquad && [C, E_n^{\bar\alpha}] = 0, \qquad && [D, E_n^{\alpha}] = nE_n^{\alpha} \\ &[H_0^i, H_n^j] = 0, && [C, H_n^j] = 0, && [D, H_n^j] = nH_n^j \end{aligned}$$

we find the roots

$$\begin{aligned} \alpha_i &= (\bar\alpha_i, 0, n) \qquad && \text{corresponding to } E_n^{\bar\alpha_i} \\ \alpha_H &= (0, 0, n) && \text{corresponding to } H_n^i\,, \end{aligned}$$

where $i = 1, 2, \ldots, r$ and $\bar\alpha_i$ are the roots of $\mathfrak{g}$, and thus a basis for $\mathfrak{C}^*$. We see here that the α_i:s are non-degenerate while α_H is r-degenerate and that the number of roots is infinite. We denote the set of roots as Φ and divide it into positive roots

$$\Phi^+ := \{\alpha_i = (\bar\alpha_i, 0, n) \mid i = 1, \ldots, r; n > 0\} \cup \{\alpha = (\bar\alpha, 0, 0) \mid \bar\alpha \in \bar\Phi^+\},$$

where $\bar\Phi^+$ is the set of positive roots of $\mathfrak{g}$, and the set of negative roots $\Phi^- = \Phi\backslash\Phi^+$. By writing

$$\delta = (0, 0, 1)$$

and denoting the highest root of $\mathfrak{g}$ as $\bar\theta$, we pick the subset of simple roots $\Delta \subset \Phi^+$ to be

$$\Delta := \{\alpha_i = (\bar\alpha_i, 0, 0) \mid i = 1, \ldots, r\} \cup \{\alpha_0 = \delta - \bar\theta = (-\bar\theta, 0, 1)\}$$

as all positive roots can be obtained as a sum of these with only positive coefficients.

Non-degenerate form

The finite case definition of the Killing form cannot be used for affine Kac-Moody algebras as the trace now runs over the infinite adjoint representation. Instead, one can impose the invariance requirement

$$\mathrm{K}([X, Y], Z) = \mathrm{K}(X, [Y, Z]) \tag{6.5}$$

on a symmetric bilinear form K on $\widetilde{L\mathfrak{g}}$ which is enough to actually find a non-degenerate form. Choosing the various basis elements of $\widetilde{L\mathfrak{g}}$ as X, Y and Z in equation (6.5), one finds

$$\begin{aligned} \mathrm{K}(T^a_m, T^b_n) &= \delta_{m+n,0}\bar{\mathrm{K}}(T^a, T^b) \\ \mathrm{K}(T^a_m, C) &= \mathrm{K}(T^a_m, D) = 0 \\ \mathrm{K}(C, D) &= 1 \end{aligned} \tag{6.6}$$

where $\bar{\mathrm{K}}$ is the Killing form of $\mathfrak{g}$.

Analogously to the finite-dimensional case, the restriction of K to $\mathfrak{C}$ defines a metric on the root space but in contrast to the Euclidean metrics of simple algebras, it is now of Lorentzian signature. By (6.6) we see that the scalar product in the root space of $\alpha = (\bar{\alpha}, k, n)$ and $\alpha = (\bar{\alpha}', k', n')$ is

$$(\alpha, \alpha') = (\bar{\alpha}, \bar{\alpha}') + kn' + k'n$$

which, due to the crossing of C and D, no longer is definite. However, as $k = k' = 0$ for all roots in Φ, the metric on the roots is actually semidefinite with only one direction with vanishing norm, namely

$$(\delta, \delta) = 0.$$

Accordingly δ is called the **null root**. This splits Φ in two non-intersecting subsets, the so called **real roots**

$$\Phi_{\mathrm{Re}} := \{\alpha \in \Phi \mid (\alpha, \alpha) > 0\} = \{\alpha = (\bar{\alpha}, 0, n) \mid n \in \mathbb{Z}\}$$

and the **imaginary roots**

$$\Phi_{\mathrm{Im}} := \{\alpha \in \Phi \mid (\alpha, \alpha) = 0\} = \{n\delta \mid n \in \mathbb{Z}\}.$$

These sets are also referred to as the **spacelike roots** and **lightlike roots**, respectively, and these more self-explanatory nomenclature will be used further on (still keeping the notation $\Phi_{\mathrm{Re}}, \Phi_{\mathrm{Im}}$ however to stick to praxis in the literature).

The explicit Cartan matrix

The elements of the Cartan matrix of $\widetilde{L\mathfrak{g}}$ can now be expressed in terms of the scalar product on the root space above:

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

which in particular gives

$$\begin{aligned} A_{ij} &= \bar{A}_{ij} \qquad \text{for } i,j \neq 0 \\ A_{0j} &= -2\frac{(\bar{\theta}, \bar{\alpha}_j)}{(\bar{\theta}, \bar{\theta})}. \end{aligned}$$

The Cartan matrix for our most relevant algebras are

$$A(\mathfrak{sl}_2^+) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

$$A(\mathfrak{sl}_3^+) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

$$A(\mathfrak{g}_2^+) = \begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

6.2 Nilpotent Orbits in Affine Algebras

We now generalize some of the concepts of orbits in chapter 4 to the case of affine Kac-Moody algebras.

6.2.1 The loop group and affine orbits

In order to speak about orbits we must first define the groups corresponding to the loop and affine algebras.

Starting with the loop algebras and their corresponding groups we say that a **loop group** $L\mathcal{G} = \mathrm{Map}(S^1; \mathcal{G})$ is the set of smooth maps from the unit circle to the Lie group $\mathcal{G}$ with a pointwise composition law. The Lie algebra is as usual the tangent space at the identity with the Lie bracket defined from the identification of the tangent vectors with left-invariant vector fields. This is precisely the loop algebras we have described in section 6.1. The exponential map is defined through the one-parameter subgroups $\gamma_Z : \mathbb{R} \to L\mathcal{G}$ where $\gamma_Z'(0) = Z \in L\mathfrak{g}$.[2]

The adjoint action of $L\mathcal{G}$ on $L\mathfrak{g}$ is immediately given by gZg^{-1} for g in $L\mathcal{G}$ and Z in $L\mathfrak{g}$.

The central extension

We denote the group corresponding to the centrally extended loop algebra $\widehat{L\mathfrak{g}}$ as $\widehat{L\mathcal{G}}$. When then considering the adjoint action of $\widehat{L\mathcal{G}}$ on $\widehat{L\mathfrak{g}}$ it is naturally enough to only work with the adjoint action of $L\mathcal{G}$ on $\widehat{L\mathfrak{g}}$, as the central extension commutes with everything by definition. On the element $\widehat{Z} = Z + \alpha C$ in $\widehat{L\mathfrak{g}}$, in which $Z \in L\mathfrak{g}$ and α is a scalar, this action is given by [25, p. 44]

$$\mathrm{Ad}_g(\widehat{Z}) := gZg^{-1} + \Big(\alpha - \langle g^{-1}g', Z\rangle\Big)\, C \tag{6.7}$$

[2]Although this definition is perfectly fine to generalize to loop groups, it should be noted that potential properties of $\mathcal{G}$, such that surjectivity of the exponential map, does not need to hold for $L\mathcal{G}$. See e.g. [25, p. 27-28].

where the prime denotes derivation with respect to θ and

$$\langle X, Y\rangle := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{B}\Big(X(\theta), Y(\theta)\Big)\,\mathrm{d}\theta, \qquad X, Y \in L\mathfrak{g},$$

with the symmetric invariant bilinear form B from equation (6.2).

The double extension

The adjoint action of an element g in the loop group $L\mathcal{G}$ on $\widetilde{Z} = Z + \alpha C + \beta D$ in the doubly extended algebra $\widetilde{L\mathfrak{g}}$ is given by [19]

$$\mathrm{Ad}_g(\widetilde{Z}) := gZg^{-1} - \beta Y + \Big(\alpha + \langle Z, Y\rangle - \frac{\beta}{2}\langle Y, Y\rangle\Big)C + \beta D \tag{6.8}$$

where $Y = g'g^{-1}$ is an element of $L\mathfrak{g}$.

6.2.2 Affine orbits and black hole solutions

There is yet a lot of research to be done when it comes to both the actual solution generating and the resulting solution orbits of the affine algebras. However, an important fact that has been known over thirty years (e.g. [13]) is that the full Geroch group we found in chapter 5 acts transitively on the set of axially symmetric solutions to the Einstein vacuum equations. This implies in turn that the Minkowski solution in fact can be used as a seed solution within this context, in contrast to what we found in the finite dimensional case [9]. The power of the solution generating technique is thus enhanced and similar properties are expected to hold also for more complicated settings. This provides extra motivation to try to generalize the methods of nilpotent orbits.

A more recent result is presented in [17] where the two-charge single-rotation JMaRT fuzzball in STU gravity[3] was proved to be smoothly connected to the Myers-Perry instanton by a transformation in the Geroch group.

There are more results but the picture calls for a more general understanding of the affine nilpotent orbits.

6.2.3 The search for a minimal orbit

As pointed out in section 4.5.1, the minimal orbit plays a significant role by corresponding to solutions with the highest BPS-degree, as e.g. the 1/2 BPS solutions in the mentioned $D = 4, \mathcal{N} = 8$ supergravity theory. It is well understood in finite dimensional Lie algebras but little is known about a corresponding structure in the affine Kac-Moody algebras. As there from a physical point of view are families of black holes with maximal BPS-degree

[3] The STU model is a consistent truncation of maximally supersymmetric supergravity with four types of electromagnetic fields.

also when the description is reduced to two dimensions there is however reason to expect the existence of such a structure. An understanding of what this could be and what implications it has is an important step to generalize the use of nilpotent orbits to gravity theories reduced to two dimensions.

As proposed in section 4.5.1 an entrance to this research is to study the relation between the minimal orbit and the simple roots in finite dimensional algebras from which hopefully some observations can be generalized to the affine cases. To provide a tool for such a work, this thesis provides two *Mathematica*-packages designed to easily calculate the adjoint actions in the affine algebras $\mathfrak{sl}_n^+$ and $\mathfrak{g}_2^+$. The following chapter includes the documentation of these.

7

Mathematica-packages

To facilitate the proposed future work of examining the conjugation between different simple root vectors in semisimple and affine Kac-Moody algebras, two Mathematica-packages have been developed to calculate the adjoint actions in the algebras $\mathfrak{sl}_n^+$ and $\mathfrak{g}_2^+$ and their semisimple counterparts. The names of the packages are **`affineSlConjugation`** *and* **`affineG2Conjugation`** *and their corresponding documentations now follow.*

In case of interest in the packages, please contact the author.

7.1 Documentation of `affineSlConjugation`

The *Mathematica*-package **`affineSlConjugation`** is a package with functions and symbol definitions to calculate the adjoint action and conjugation in $\mathfrak{sl}_n^+$ algebras. It features

- automatic definitions of the Chevalley-Serre basis for the horizontal algebra $\mathfrak{sl}_n$ as matrices
- a tensor product representation of the corresponding loop algebra of the form $X \otimes s^m$ with X as a Chevalley-Serre basis matrix in $\mathfrak{sl}_n$ and z as the complex loop parameter
- a basis for the affine extension with the symbols for the central extension and the "derivation" element
- functions to calculate commutators in the horizontal, loop and affine algebras
- matrix exponentiation from the loop algebra to the loop group
- functions to calculate the adjoint action of a group element on the corresponding algebra element for the horizontal and loop algebras

- functions to calculate the adjoint action of the exponentiation of an algebra element on another algebra element, in both the horizontal, loop and affine algebras

- some utility functions to manipulate *Mathematica*'s representation and visualization of the various objects.

7.1.1 Basic usage

A session with the **`affineSlConjugation`**-package begins with calling the initialization function **`initializeSlAlgebra[n]`** which defines all the package's objects for $\mathfrak{sl}_{\mathtt{n}}$, its loop algebra and the affine extension. The horizontal algebra basis is defined and accessed mainly[1] by the lists **`Es⟦i⟧`**, **`Hs⟦i⟧`** and **`Fs⟦i⟧`**, the loop algebra basis as **`lE[i,m]`** ($=$ **`Es`**$[\![i]\!]\otimes$**`z`**$^{\mathtt{m}}$) etc. and the affine basis consists of the loop algebra basis elements together with the symbols **`Cen`** and **`Der`** which represent the central extension and the “derivation” element, respectively.

The general[2] rules for function names which exist in different versions are that functions which apply to the horizontal algebra are marked with an **`h`** or **`H`**, if at all, while the loop algebra objects are marked with **`l`** or **`L`**, and an **`a`** or **`A`** mark the names for the functions on the affine algebra. **`ct`** is often used as acronym for **`CircleTimes`** and for some of the utility functions a **`v`** is prepended to denote that the function is used for the visual representation. A few functions also have their functionality defined as rules and in these cases an **`F`** and an **`R`** are appended, respectively.

Commutators are calculated by the functions **`com`**, **`lCom`** and **`aCom`** and the adjoint actions by **`adg`**, **`lAdg`** and **`aAdg`** which take algebra generators as arguments and by **`adG`** and **`lAdG`** which take group element as arguments. This is documented more in detail in the function list below. To exponentiate a loop algebra element there is the function **`lMatrixExp`** and the Killing form exists for all three algebras as **`kill`**, **`lKill`** and **`aKill`**. The invariant bilinear form used to define the affine extension has the name $\boldsymbol{\omega}$ (or **`ωRes`**). A full list of the defined functions and objects follows below.

Example 7 (Basic usage).

```
In[1]:= initializeSlAlgebra[3]
In[2]:= $currentAlgebra
Out[2]= The currently initialized algebra is sl(3).
```

Horizontal algebra

```
In[3]:= hBasis
```

[1] It is also gathered in the list `basis`.

[2] There are however a few exceptions to these rules among the most notable are `com`, `adg` and `adG`.

Out[3]=

$$\left\{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\right\}$$

```
In[4]:= com[Es⟦1⟧, Fs⟦1⟧]
Out[4]=
```

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

```
In[5]:= adg[Es⟦1⟧, Fs⟦1⟧]
Out[5]=
```

$$\begin{pmatrix}1&-1&0\\1&-1&0\\0&0&0\end{pmatrix}$$

```
In[6]:= vDecompInHBasis[%]
```

Out[6]= $\{\mathtt{e}[1]\to -1, \mathtt{h}[1]\to 1, \mathtt{f}[1]\to 1\}$

Loop algebra

```
In[7]:= X =lE[1,3]
Out[7]=
```

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\otimes \mathtt{z}^3$$

```
In[8]:= Y =lF[1,-3] + lH[2,2]
Out[8]=
```

$$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\otimes\frac{1}{\mathtt{z}^3}+\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}\otimes \mathtt{z}^2$$

```
In[9]:= lCom[X,Y]
Out[9]=
```

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\otimes \mathtt{z}^5+\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\otimes 1$$

```
In[10]:= lAdg[X,Y]
Out[10]=
```

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathtt{z}^5 + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \frac{1}{\mathtt{z}^3} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathtt{z}^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \mathtt{z}^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes 1$$

Affine algebra

```
In[11]:= X = lE[1,3]
Out[11]=
```

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathtt{z}^3$$

```
In[12]:= Y = 3 Cen + Der + lF[1,-3] + lH[2,2]
Out[12]=
```

$$3\mathtt{Cen} + \mathtt{Der} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \frac{1}{\mathtt{z}^3} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \mathtt{z}^2$$

```
In[13]:= aCom[X,Y]
Out[13]=
```

$$-30\mathtt{Cen} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathtt{z}^5 - 3\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathtt{z}^3 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes 1$$

```
In[14]:= aAdg[X,Y]
Out[14]=
```

$$-27\mathtt{Cen} + \mathtt{Der} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes z^5 + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \frac{1}{z^3} - 4\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes z^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes z^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes 1$$

Some particularly useful utility functions

A few functions are provided to facilitate the work flow and are here referred to as utility functions. Among these the following might be of particular interest. A summarizing example follows the list.

- To decompose a general algebra element in the used bases, there are the functions **decompIn*Basis**, where ***** is either **H**, **L** or **A**. These give the decompositions as rules for the parameters in the variable list **params**. They are followed by sister functions with a **v** prepended in the function name (stands for 'visual') which drops rules which map to zero. In addition, there are **decompIn*BasisMatForm** (***** = **L**, **A**) which decompose elements into the sum of matrix expressions.
- To split a matrix containing functions of **z** into the tensor product between constant matrices and these functions, use **collapsedToCt**.
- If there is a wish to treat user defined symbols as scalars in the linear combinations of algebra elements, define these as being numeric with **makeSymbolCountAsNumeric**.
- To smash a linear combination into as few terms as possible, use **smashLinF**.
- The **affineSlConjugation**-package prints two-dimensional lists as matrices by default. Disable this through **alwaysPrintInMatrixForm[False]**.

Example 8 (Usage of some utility functions).

```
In[1]:= initializeSlAlgebra[3]
In[2]:= makeSymbolCountAsNumeric[a,b]
In[3]:= loopElem = a lE[1,3]+b lF[2,3]
Out[3]=
```

$$\mathrm{a}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{z}^3 + \mathrm{b}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \mathrm{z}^3$$

```
In[4]:= smashedLoopElem = smashLinF[loopElem]
Out[4]=
```

$$\begin{pmatrix} 0 & \mathrm{a} & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{b} & 0 \end{pmatrix} \otimes \mathrm{z}^3$$

```
In[5]:= ctToTimesF[smashedLoopElem]
        collapsedToCt[%]
```

```
Out[5]=
```

$$\begin{pmatrix} 0 & \mathtt{a}\,\mathtt{z}^3 & 0 \\ 0 & 0 & 0 \\ 0 & \mathtt{b}\,\mathtt{z}^3 & 0 \end{pmatrix}$$

```
Out[6]=
```

$$\begin{pmatrix} 0 & \mathtt{a} & 0 \\ 0 & 0 & 0 \\ 0 & \mathtt{b} & 0 \end{pmatrix} \otimes \mathtt{z}^3$$

```
In[7]:= vDecompInLBasis[smashedLoopElem]
```

`Out[7]=` $\{\mathtt{e}[1] \to \mathtt{a}, \mathtt{f}[2] \to \mathtt{b}\} \otimes \mathtt{z}^3$

```
In[8]:= decompInLBasisMatForm[smashedLoopElem]
Out[8]=
```

$$\mathtt{a}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathtt{z}^3 + \mathtt{b}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \mathtt{z}^3$$

Additional $\mathfrak{g}_2$ algebra

To initialize a parallel $\mathfrak{g}_2$ algebra duplicating all algebra elements, run the function **`initializeG2Objects`**. The object names are identical to the $\mathfrak{sl}_n$ related names but with the text **`g2`** prepended, e.g. **`lE`** gets the $\mathfrak{g}_2$ duplicate **`g2lE`**. The matrix representation used for $\mathfrak{g}_2$ is listed in appendix E.

It is also possible to use the **`initializeSlAlgebra`** function to define the $\mathfrak{g}_2$ algebra as the standard algebra. It is done by passing the string argument **`"g2Override"`** to **`initializeSlAlgebra`**.

7.1.2 List of user-available objects

Here follows a list of all user-available objects in **`affineSlConjugation`** ordered in subsections of intended use together with a short description.

Initialization

- **`$currentAlgebra`** contains the info of the current initialized algebra as a string.
- **`initializeSlAlgebra[n]`** initializes a $\mathfrak{sl}_n$-algebra, the corresponding loop algebra and its affine extension. Needs to be run to start using the package properly. It calls both **`initializeHBasis`** and **`initializeLBasis`**. An additional feature exists which makes $\mathfrak{g}_2$ the standard horizontal algebra. This is done by evoking **`initializeSlAlgebra`** with the string argument **`"g2Override"`**.

Horizontal algebra

- **Es** is a list with the E element matrices of the Chevalley-Serre basis. E.g. if the initialized algebra is $\mathfrak{sl}_2$,

$$\texttt{Es}[\![1]\!] = \begin{pmatrix} 01 \\ 00 \end{pmatrix}.$$

- **Fs** is a list with the F element matrices of the Chevalley-Serre basis.

- **genericHAlgElem** is a generic $\mathfrak{sl}_n$ element as a linear combination of the matrices in **hBasis** with the coefficients of **params**. I.e. $\sum_i^{n^2-1}$ **hBasis**[[**i**]] **params**[[**i**]], which for $\mathfrak{sl}_2$ is

$$\begin{pmatrix} \texttt{h}[1]\texttt{e}[1] \\ \texttt{f}[1] - \texttt{h}[1] \end{pmatrix}.$$

- **hBasis** as a list with all Chevalley-Serre basis elements ordered like {**Es**, **Hs**, **Fs**}.

- **Hs** is a list with the H element matrices of the Chevalley-Serre basis.

- **initializeHBasis[n]** defines the following objects of the defining representation of $\mathfrak{sl}_n$ as a horizontal algebra:
  - **Es**, **Hs**, **Fs** as lists with the Chevalley-Serre basis element matrices
  - **hBasis** as a list with all Chevalley-Serre basis elements ordered like {**Es**, **Hs**, **Fs**}
  - **params** as a list of parameters to be used together with the generators of the algebra
  - **genericHAlgElem** as a generic $\mathfrak{sl}_n$ element as a linear combination of the matrices in **hBasis** with the coefficients of **params**.

- **params** is a list of parameters {**e[1]**,..., **e[(n^2-n)/2)]**, **h[1]**,..., **h[n-1]**, **f[1]**,..., **f[(n^2-n)/2)]**} to be used together with the generators of the algebra.

- **randHElem** generates a random basis element of the current $\mathfrak{sl}_n$-algebra with basis coefficients between 0 and 1.

- **randIHElem** generates a random basis element of the current $\mathfrak{sl}_n$-algebra with basis coefficients as integers between and including -10 and 10.

Functions for the horizontal algebra

- **adg[X,Y]** requires two arguments, both $\mathfrak{sl}_n$ elements, and returns the adjoint action of the $SL(n)$ element equal to the exponentiation of arg1 on the $\mathfrak{sl}_n$ element arg2. I.e. **adg[X,Y]** $= \exp(\texttt{X})\texttt{Y}\exp(-\texttt{Y})$.

- **adG[g,X]** requires a $SL(n)$ group element as first argument and a $\mathfrak{sl}_n$ algebra element as second. Returns the adjoint action of the group element on the algebra element.
- **com[A,B]** is the commutator of the $\mathfrak{sl}_n$-algebra and takes two arguments as in **com[A,B]** $=$ **AB** $-$ **BA**.
- **kill[X,Y]** is the $\mathfrak{sl}_n$ Killing form and takes two arguments and returns $2n\,\mathrm{Tr}(\mathbf{XY})$. Normalization of **2n** can be altered by **setKillingFormNormalization**.
- **setKillingFormNormalization** has two forms: **setKillingFormNormalization[val]** sets the Killing form normalization for the horizontal algebra to **kill[X,Y] = val**$\cdot$ **Tr(X Y)**. **setKillingFormNormalization[X,Y,val]** sets the Killing form normalization such that **kill[X,Y]= val**.

Loop algebra and affine algebra

- **Cen** represents the central extension of the loop algebra and is only defined through its UpValues in the different functions which have the affine algebra as domain.
- **Der** represents the "derivation" element which extends the loop algebra and is only defined through its UpValues in the different functions which have the affine algebra as domain.
- **initializeLBasis** uses objects defined by **initializeHBasis[n]** to define a loop algebra basis of the form **X** $\otimes$ **z**$^{\mathtt{m}}$ with **X** in the $\mathfrak{sl}_n$-algebra and **z** $(= \mathrm{e}^{\mathrm{i}\theta})$ as the loop parameter. The objects are defined as functions **lE[i,m]**, **lH[i, m]**, **lF[i,m]** returning the i:th element of the **Es** (or **Hs** and **Fs**) tensor product with **z**$^{\mathtt{m}}$. The $\otimes$ (**\[CircleTimes]**) operator is not set to **KroneckerProduct** by default but gets its own defined rules by **initializeLBasis**.
- **lE[i,m]** takes two arguments and returns the element of the loop algebra basis $E_i \otimes z^m$.
- **lH[i,m]** takes two arguments and returns the element of the loop algebra basis $H_i \otimes z^m$.
- **lF[i,m]** takes two arguments and returns the element of the loop algebra basis $F_i \otimes z^m$.
- **randAElem** generates a random affine algebra element with $\mathfrak{sl}_n$ coefficients between 0 and 1 and an integer exponent of **z** in between -10 and 10. The coefficient for extensions are in between 0 and 1.
- **randIAElem** generates a random affine algebra element with integers as $\mathfrak{sl}_n$ coefficients between -10 and 10 and an integer exponent of **z** in between -10 and 10. The coefficient for extensions are also in between -10 and 10. The coefficient for extensions are in between 0 and 1.

- **randILElem** generates a random loop algebra element with integers as $\mathfrak{sl}_n$ coefficients between -10 and 10 and an integer exponent of **z** in between -10 and 10.
- **randLElem** generates a random loop algebra element with $\mathfrak{sl}_n$ coefficients between 0 and 1 and an integer exponent of **z** in between -10 and 10.
- **z** is the loop parameter also represented as **z** $= \exp(\mathrm{i}\theta)$.

Functions for the loop and affine algebras

- **ω[X,Y]** (\\[Omega][X,Y]) is the invariant bilinear form on the loop algebra. It takes two loop algebra elements and returns $\frac{1}{2\pi}\int\langle X,Y'\rangle\,\mathrm{d}\theta$.
- **ωRes[X,Y]** (\\[Omega]Res[X,Y]) is like the invariant bilinear form **ω** but calculated by use of the residue formula.
- **aAdg[X,Y]** takes two arguments, both in the affine algebra. **X** is projected on the loop algebra and **aAdg** returns the adjoint action of the loop group element $\exp\Big(\mathrm{proj}_{\mathrm{loop}}(\mathbf{X})\Big)$ on **Y**. **aAdg** is written according to the formula (6.8).
- **aCom[X,Y]** is the commutator of the affine algebra and takes two arguments, both in the affine algebra, and returns their commutator.
- **aElemToLElemF[X]** takes an affine algebra element and projects onto the loop algebra by setting **Cen** and **Der** to zero.
- **aKill[X,Y]** is the Killing form on the affine algebra. It takes two arguments, both in the affine algebra.
- **lAdg[X,Y]** takes two arguments, both in the loop algebra, and returns the adjoint action of the loop group element exp(**X**) on **Y**. **lAdg** is written according to the formula (6.7).
- **lAdG[g,Y]** takes a loop group element **g** as first argument and a loop algebra element **Y** as second argument and returns the adjoint action of **g** on **Y**.
- **lCom[X,Y]** is the commutator of the loop algebra and takes two arguments, both in the loop algebra, and returns their commutator.
- **lKill[X,Y]** is the Killing form on the loop algebra. It takes two arguments, both in the loop algebra.
- **lMatrixExp[X]** takes a loop algebra element and returns its matrix exponential. **NB:** **lMatrixExp** smashes **CircleTimes** to **Times** and returns the resulting group element as a plain matrix with **z**-dependent components.

7.1.3 affineG2Conjugation

The *Mathematica*-package **affineG2Conjugation** works exactly as the package **affineSlConjugation** and is thus covered by the same documentation except for these listed differences:

- there is no need of any initialization functions to be run and all such functions (beginning with **initialize...**) in the **affineSlConjugation**-package have no correspondence here
- there is naturally no $\mathfrak{g}_2$ "expansion pack"
- there is no string constant with the name **$currentAlgebra**.

A

On the Exponential Maps in Lie Groups

There are two different definitions of the exponential map on a Lie group $\mathcal{G}$ which often but not necessarily coincide. They are constructed from either the curves corresponding to the one-parameter subgroups or from the geodesics with respect to some metric on $\mathcal{G}$.

Definition I: based on one-parameter subgroups As proved in [20, p. 213-214] the integral curves of the left-invariant vector fields are precisely the one-parameter subgroups $\phi(\tau)$ in $\mathcal{G}$. This establishes the possibility to define an exponential map $\exp : T_g\mathcal{G} \to \mathcal{G}$ locally for some point g in $\mathcal{G}$. We take $g = \mathrm{id}$ as any other starting point can be reached by group multiplication:

$$\exp[X] := \phi_X(1) \qquad \text{where} \qquad \left.\frac{d\phi_X(\tau)}{d\tau}\right|_{\tau=0} = X \in T_{\mathrm{id}}\mathcal{G} \cong \mathfrak{g} \,. \tag{A.1}$$

This naturally generalizes to

$$\exp[\tau X] := \phi_X(\tau) \qquad \text{where} \qquad \left.\frac{d\phi_X(\tau)}{d\tau}\right|_{\tau=0} = X \in \mathfrak{g} \,. \tag{A.2}$$

and thus the exponential map traces out the one-parameter subgroup corresponding to the integral curve of X.

Definition II: based on geodesics For a smooth manifold $\mathcal{M}$ with an affine connection ∇ the notion of a geodesic $s(\tau)$ is defined as a curve along which its tangent vector is parallel transported, i.e. $\nabla_X X|_{s(\tau)} = 0$ for $X = \frac{d}{d\tau}s(\tau)$. The exponential map $\mathrm{Exp}_p : T_p\mathcal{M} \to \mathcal{M}$ at a point p in such a manifold can then be defined as

$$\mathrm{Exp}_p(X) := s(1) \qquad \text{where} \qquad \left.\frac{ds(\tau)}{d\tau}\right|_{\tau=0} = X \in T_p\mathcal{M} \tag{A.3}$$

which implies

$$\mathrm{Exp}_p(\tau X) = s(\tau) \qquad \text{where} \qquad \frac{ds(\tau)}{d\tau}\Big|_{\tau=0} = X \in T_p\mathcal{M} \ . \tag{A.4}$$

Hence, in a Lie group with an affine connection a tangent vector X at g defines two curves to which it is the tangent at g, the one-parameter subgroup $\phi_X(\tau)$ and the geodesic $s(\tau)$ and the two different exponential mappings exp and Exp are defined accordingly.

A.1 Equality between the Exponential Maps

Here we state the requirements for the two exponential maps to coincide and finish by proving that the Levi-Civita connection corresponding to the Killing form metric satisfies these. First we need a few definitions.

Definition A.1.1. *A connection ∇ on a Lie group $\mathcal{G}$ is a* ***left-invariant connection*** *if it for any left-invariant vector fields X and Y satisfies*

$$L_{g*}(\nabla_X Y) = \nabla_{L_{g*}X} L_{g*}Y \ , \tag{A.5}$$

for any $g \in \mathcal{G}$.

Definition A.1.2. *A left-invariant connection on a Lie group for which the one-parameter subgroup curves $\phi_X(\tau)$ and the geodesics $s(\tau)$ ($X = \frac{d}{d\tau}s(\tau)|_{\tau=0}$) coincide is called a* ***Cartan connection****.*

Thus, for any Cartan connection on a Lie group the two exponential mappings map to the same curves and we have

$$\exp = \mathrm{Exp} \ . \tag{A.6}$$

Any left-invariant connection is defined by its values at the identity and defines a $\mathbb{R}$-bilinear multiplication in the Lie algebra $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ through

$$\alpha(X, Y) := \nabla_X Y\Big|_{\mathrm{id}} \ . \tag{A.7}$$

This is in fact a one-to-one correspondence as any such multiplication also defines a unique connection [24, p. 71]. The requirement for a connection to be a Cartan connection translates to the requirement on the multiplication to be anti-symmetric. Any one-parameter subgroup is defined through the integral curve of some left-invariant vector field X. For this curve to simultaneously be a geodesic we must have

$$\nabla_X X\Big|_{\mathrm{id}} = \alpha(X, X) = 0 \ .$$

From this we can see that there exist a unique Cartan connection which is torsion-free, found by setting

$$\nabla_X Y\Big|_{\mathrm{id}} = \alpha(X,Y) = \frac{1}{2}[X,Y]\ . \tag{A.8}$$

There are of course other choices of the Cartan connection but as we here ultimately aim for a Levi-Civita connection of some metric we restrict our interest to this one. The question then is which metrics can give rise to such a connection and, in particular, whether the standard metric of the Killing form does so.

Definition A.1.3. *A metric g on a Lie group $\mathcal{G}$ is said to be a **left-invariant metric** if it satisfies*

$$g_f(X,Y) = g_{gf}(L_{g*}X, L_{g*}Y)$$

for any vectors $X, Y \in T_f\mathcal{G}$.

For a left-invariant metric to be covariantly conserved under a torsion-free Cartan connection it must also satisfy an additional invariance property. For any left-invariant vector fields X, Y and Z metric compatibility of the Cartan connection means

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = \frac{1}{2}g([X,Y],Z) + \frac{1}{2}g(Y,[X,Z])$$

and the left-invariance of the metric implies

$$g_g(Y\Big|_g, Z\Big|_g) = g_{\mathrm{id}}(Y\Big|_{\mathrm{id}}, Z\Big|_{\mathrm{id}}) = \text{const.}$$

such that

$$g([X,Y],Z) = g(X,[Y,Z])\ . \tag{A.9}$$

Restated, this equation is a requirement on all metrics of which the Levi-Civita connection is supposed to be a Cartan connection.

Lastly, we also state and prove the fact that the Levi-Civita connection deduced from a left-invariant metric is also left-invariant.

Proof: Let ∇ be the Levi-Civita connection of a left-invariant metric g on a Lie group $\mathcal{G}$. To clarify the logic we think of the left-translation $L_g : \mathcal{G} \to L_g\mathcal{G}$ as an isometry between the spaces $\mathcal{G}$ and $\mathcal{H} := L_g\mathcal{G}$ and temporarily simplify the notation by writing $\hat{X} := L_{a*}X$ for any $X \in \mathfrak{g}$. We want to prove the equation (A.5) which in this notation is

$$\widehat{\nabla_X Y} \stackrel{?}{=} \hat{\nabla}_{\hat{X}}\hat{Y} \tag{A.10}$$

where $\hat{\nabla}$ is the Levi-Civita connection of the metric $\hat{g}$ $(= L_a^* g = g)$ on $\mathcal{H}$. Define a connection $\tilde{\nabla}$ on $\mathcal{H}$ which does satisfies this, i.e. set

$$\tilde{\nabla}_{\hat{X}}\hat{Y} = \widehat{\nabla_X Y}\ . \tag{A.11}$$

Then

$$\tilde{\nabla}_{\hat{X}}\hat{g}(\hat{Y},\hat{Z}) = \hat{X}\hat{g}(\hat{Y},\hat{Z}) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{A.12}$$

$$= \hat{g}(\widehat{\nabla_X Y},\hat{Z}) + \hat{g}(\hat{Y},\widehat{\nabla_X Z}) = \hat{g}(\tilde{\nabla}_{\hat{X}}\hat{Y},\hat{Z}) + \hat{g}(\hat{Y},\tilde{\nabla}_{\hat{X}}\hat{Z}) \tag{A.13}$$

such that $\tilde{\nabla}$ is metric compatible. Moreover, $\tilde{\nabla}$ is also torsion-free since

$$\begin{gathered}\nabla_X Y - \nabla_Y X = [X,Y]\\ \Longrightarrow\\ (\nabla_X Y - \nabla_Y X)\widehat{\ } = \widehat{[X,Y]}\\ \Longrightarrow\\ \widehat{\nabla_X Y} - \widehat{\nabla_Y X} = [\hat{X},\hat{Y}]\\ \Longrightarrow\\ \tilde{\nabla}_{\hat{X}}\hat{Y} - \tilde{\nabla}_{\hat{Y}}\hat{X} = [\hat{X},\hat{Y}]\ ,\end{gathered}$$

where (long expression)$\widehat{\ }$ denotes a hat over the entire parenthesis. Hence $\tilde{\nabla}$ fulfills all the properties of the Levi-Civita connection of $\hat{g}$ and by uniqueness we must have $\tilde{\nabla} = \hat{\nabla}$, such that equation (A.10) is proven. The Levi-Civita connection of a left-invariant metric is left-invariant. □

A.1.1 The Killing form metric

For a semisimple Lie group $\mathcal{G}$, the Killing form is non-degenerate and defines a metric on $\mathcal{G}$ as in section 2.1.1. From the definition

$$\mathrm{K}(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \qquad X, Y \in \mathfrak{g} \tag{A.14}$$

it can be seen that the Killing form is invariant under all automorphisms φ on the Lie algebra as

$$[\varphi X, Y] = \varphi([X, \varphi^{-1}Y]) \implies \mathrm{ad}_{\varphi X} = \varphi \circ \mathrm{ad}_X \circ \varphi^{-1}$$

and the fact that the trace is cyclic. In particular, L_{g*} is an automorphism whence the Killing form as a metric is left-invariant.

The Levi-Civita connection deduced from the Killing form is thus also left-invariant and since it is torsion-free it satisfies $\nabla_X Y = \frac{1}{2}[X,Y]$. By uniqueness of the torsion-free Cartan connection we have for a semisimple Lie group that

$$\exp = \mathrm{Exp}$$

with respect to the Levi-Civita connection.

As a last check we note that the well-known invariance property of the Killing form

$$\mathrm{K}([X,Y],Z) = \mathrm{K}(X,[Y,Z])$$

is precisely equation (A.9).

B

Geodesic from ϕ-variation in the Derivation of the Schwarzschild Coset Representative

The relevant part of Lagrangian

$$\mathcal{L}_\sigma = \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \gamma_{ij}$$

gives the variation with respect to ϕ^k

$$\begin{aligned}
\frac{\delta \mathcal{L}_\sigma}{\delta \phi^k} \delta\phi^k &= 2\sqrt{g} g^{\mu\nu} \partial_\mu (\delta^i_k \delta\phi^k) \partial_\nu \phi^j \gamma_{ij} + \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \frac{\delta \gamma_{ij}}{\delta \phi^k} \delta\phi^k \\
&= \Big\{ -2\partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi^j \gamma_{ij} \right) + \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \frac{\partial \gamma_{ij}}{\partial \phi^k} \Big\} \delta\phi^k + \text{total derivative} \; .
\end{aligned}$$

We drop the total derivative and rewrite the last term with help of the identity[1]

$$\partial_k \gamma_{ij} = \gamma_{is} \tilde{\Gamma}^s_{jk} + \gamma_{js} \tilde{\Gamma}^s_{ik} \tag{B.1}$$

derived from

$$\begin{aligned}
\tilde{\Gamma}^k_{ij} &= \frac{\gamma^{kl}}{2} (\partial_i \gamma_{lj} + \partial_j \gamma_{il} - \partial_l \gamma_{ij}) \\
\Longrightarrow \quad 2\partial_i \gamma_{lj} &= 2\tilde{\Gamma}^k_{ij} \gamma_{kl} + 2\tilde{\Gamma}^k_{il} \gamma_{kj} \; .
\end{aligned} \tag{B.2}$$

Hence, the variation calculation continues as

$$\frac{\delta \mathcal{L}_\sigma}{\delta \phi^k} \delta\phi^k = \Big\{ -2\partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi^j \gamma_{ij} \right) + \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \underbrace{(\gamma_{is} \tilde{\Gamma}^s_{jk} + \gamma_{js} \tilde{\Gamma}^s_{ik})}_{2\gamma_{is}\tilde{\Gamma}^s_{jk}} \Big\} \delta\phi^k \tag{B.3}$$

[1] We denote the Christoffel symbols belonging to the coset space connection with a tilde.

For arbitrary variation we get

$$\partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi^j \gamma_{ij}\right) - \sqrt{g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \gamma_{is} \tilde{\Gamma}^s_{jk} = 0$$

The first term can be split in two by Leibniz

$$\partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi^j \gamma_{ij}\right) = \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi^j\right) \gamma_{ij} + \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi^j\right) \partial_\mu \gamma_{ij}$$

where the second term can be rewritten once again by use of equation (B.1)

$$\begin{aligned} \partial_\mu \gamma_{jk} &= \partial_\mu \phi^l \partial_l \gamma_{jk} = \partial_\mu \phi^l (\gamma_{sj} \tilde{\Gamma}^s_{lk} + \gamma_{sk} \tilde{\Gamma}^s_{jl}) \\ \overset{\text{(B.1)}}{\Longrightarrow} \sqrt{g} \partial^\mu \phi^j \partial_\mu \phi^l (\gamma_{sj} \tilde{\Gamma}^s_{lk} + \gamma_{sk} \tilde{\Gamma}^s_{jl}) &\overset{l \to i}{=} \sqrt{g} \partial_\mu \phi^i \partial^\mu \phi^j \gamma_{si} \tilde{\Gamma}^s_{jk} + \sqrt{g} \partial_\mu \phi^i \partial^\mu \phi^j \gamma_{sk} \tilde{\Gamma}^s_{ij} \ . \end{aligned}$$

We plug this back into equation (B.3) and find the equation of motion

$$\begin{gathered} -\partial_\mu (\sqrt{g} \partial^\mu \phi^j) \gamma_{jk} - \sqrt{g} \partial_\mu \phi^i \partial^\mu \phi^j \gamma_{sk} \tilde{\Gamma}^s_{ij} = 0 \\ \Longleftrightarrow \\ \partial_\mu (\sqrt{g} \partial^\mu \phi^l) + \sqrt{g} \partial_\mu \phi^i \partial^\mu \phi^j \tilde{\Gamma}^l_{ij} = 0 \ . \end{gathered} \tag{B.4}$$

In our ansatz we have dependencies only on r such that all spacetime indices μ goes to downstairs r (r-component of metric is 1). Switching once again to the parameter τ we get

$$\begin{gathered} \partial_\mu (\sqrt{g} \partial^\mu \phi^l) + \sqrt{g} \partial_\mu \phi^i \partial^\mu \phi^j \tilde{\Gamma}^l_{ij} = 0 \\ \longrightarrow \\ \partial_r (f^2 \partial_r \phi^l) + f^2 \partial_r \phi^i \partial_r \phi^j \tilde{\Gamma}^l_{ij} = 0 \\ \longrightarrow \\ \frac{d^2}{d\tau^2} \hat{\phi}^l + \frac{d\hat{\phi}^i}{d\tau} \frac{d\hat{\phi}^j}{d\tau} \tilde{\Gamma}^l_{ij} = 0 \end{gathered}$$

which is the well-known coordinate form of a geodesic in the coset with parameter τ.

C

Short on the Komar mass and Komar NUT charge

This appendix provides a short definition of the Komar mass and Komar NUT charge based mainly on [4].

The Komar mass is a definition of the total mass in a stationary, asymptotically flat four dimensional spacetime as a surface integral at spatial infinity. It can only be defined for stationary cases and coincides with the mass definition obtained from a Hamiltonian formulation based on time translational invariance. Hence it is the corresponding Noether charge. Denote the timelike Killing vector as κ and the Ricci tensor as $R^{\mu\nu}$, then define the vector

$$J^\mu := \kappa_\nu R^{\mu\nu}$$

which is a conserved current since

$$\nabla_\mu J^\mu = (\underbrace{\nabla_\mu \kappa_\nu}_{\text{anti.sym}}) R^{\mu\nu} + \kappa_\nu \nabla_\mu R^{\mu\nu} \overset{(*)}{=} \frac{1}{2}\kappa_\nu \nabla^\nu R = 0$$

where we used $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$ in $(*)$ and the fact that the derivative of the Ricci scalar along any Killing vector vanish. So J is conserved and writing

$$\kappa_\nu R^{\mu\nu} = \nabla_\mu \nabla^\nu \kappa^\mu,$$

which is true for any Killing vector, we see that the charge defined by the integral

$$Q = \int_\Sigma n_\mu J^\mu \sqrt{\gamma}\mathrm{d}^{n-1}x$$

is constant for any spacelike hypersurface Σ with normal vector n^μ and induced metric γ_{ij}. It admits the employment of Stoke's theorem on Σ by

$$\begin{aligned}\int\limits_{\Sigma} n_\mu J^\mu \sqrt{\gamma}\mathrm{d}^{n-1}x &= \int\limits_{\Sigma} n_\mu \nabla_\nu \underbrace{\nabla^\mu \kappa^\nu}_{\text{anti.sym}} \sqrt{\gamma}\mathrm{d}^{n-1}x \\ &\overset{\text{Stoke's}}{=} \int\limits_{\partial\Sigma} n_\mu \sigma_\nu \nabla^\mu \kappa^\nu \sqrt{\gamma^{(\partial\Sigma)}}\ \mathrm{d}^{(n-2)}x \ .\end{aligned}$$

With proper normalisation we find the Komar mass

$$m = \frac{1}{4\pi} \int\limits_{\partial\Sigma} n_\mu \sigma_\nu \nabla^\mu \kappa^\nu \sqrt{\gamma^{(\partial\Sigma)}}\ \mathrm{d}^{(n-2)}x \ .$$

This is naturally also the total energy of the spacetime.

Actually, the stationary requirement can be relaxed to an asymptotic stationarity as $r \to \infty$.

When considering a compactified four dimensional spacetime as a fibration over the three dimensional spatial space as base space, the Komar mass can also be expressed as [1]

$$m = \frac{1}{4\pi} \int\limits_{\partial\Sigma} s^* \star K$$

where s is a section in the fiber bundle and $K = \partial_\mu \kappa_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is the Komar 2-form. In this context the Komar NUT charge is defined as

$$n = \frac{1}{8\pi} \int\limits_{\partial\Sigma} s^* K.$$

D

Short Comment on the Closure Operation

Zariski topology is defined on affine spaces by the use of polynomial roots. If k is a(n algebraically closed) field and $k[x_1, \ldots, x_n]$ denotes the set of all polynomials over k with n variables, then the following defines the Zariski closed subsets of k^n:

$$V(S) = \{x \in k^n : f(x) = 0 \quad \forall f \in S, \text{ where } S \subset k[x_1, \ldots, x_n]\}.$$

The closure of an arbitrary set X can be defined in two different ways. First, it is the smallest closed set containing X. I interpret this as

$$\bar{X} = \{x \in k^n : f(x) = 0 \quad \forall f \in S\}$$

where $S \subset k[x_1, \ldots, x_n]$ is that set of polynomials which yields the smallest set $\bar{X}$ and fulfills $f(x) = 0\ \forall x \in X, f \in S$. Second definition of Zariski closure is

$$\bar{X} = \{x \in k^n : f(x) = 0 \quad \forall f \in k[x_1, \ldots, x_n] \text{ vanishing on } X\}$$

which involves all polynomials which vanish on X while the above can include any subset of these.

For the orbits, these closures define a partial ordering and the closure of an orbit contains the orbit together with all other orbits with smaller dimensions.

E

Representation of $\mathfrak{g}_2$

Here follows the representation used for all explicit calculations with the Chevalley-Serre basis in the $\mathfrak{g}_2$ algebra. It is also the one used for the packages **`affineSlConjugation`** and **`affineG2Conjugation`**.

$$
E_i = \overset{i=1}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}}, \overset{i=2}{\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0\\ 0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}}, \overset{i=\ldots}{\begin{pmatrix} 0&0&-1&0&0&0&0\\ 0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-2&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}}
$$

$$
\begin{pmatrix} 0&0&0&-4&0&0&0\\ 0&0&0&0&4&0&0\\ 0&0&0&0&0&4&0\\ 0&0&0&0&0&0&-4\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&-12&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&12\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&12&0\\ 0&0&0&0&0&0&12\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}
$$

$$
H_i = \overset{i=1}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}}, \overset{i=2}{\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&2&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-2&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&-1 \end{pmatrix}}
$$

$$F_i = \overset{i=1}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}}, \overset{i=2}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix}}, \overset{i=\dots}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0 \end{pmatrix}}$$

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -2&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&-2&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -3&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&3&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 3&0&0&0&0&0&0\\ 0&3&0&0&0&0&0 \end{pmatrix}$$

F

Some Elements Between Simple Root Vectors in the Minimal Orbit

As mentioned in the sections about the minimal orbit, a good starting point to generalize this concept to affine Kac-Moody algebras may be to describe the minimal orbit in terms of the simple roots. For the potential relevance in such a work, we present here some group elements which conjugate the highest root vector into the simple root vectors, together with their generators. We do this by solving the conjugations explicitly for the algebras $\mathfrak{sl}_3, \mathfrak{sl}_4, \mathfrak{sl}_5$ and exemplify with solutions also for $\mathfrak{g}_2$.

F.1 Explicit Conjugation from E_θ to Simple Root Vectors in $\mathfrak{sl}$ Algebras

More or less general solutions for $\mathfrak{sl}_3 : E_\theta \to E_1$

First, look at the conjugation of $E_\theta = E_3$ into E_1 in the defining representation. I.e. we seek group elements g_1 in $SL(3)$ which satisfies $g_1 E_\theta g_1^{-1} = E_1$. There are infinitely many of those as all matrices of the form

$$g_1 = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & 0 & g_{11} \\ 0 & -\frac{1}{g_{11}^2} & g_{33} \end{pmatrix} \in SL(3)$$

do the job. All[1] of the choices of g_1 seem to have single generators and the general logarithm has the form of

$$\log g_1 = \begin{pmatrix} \log g_{11} & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

[1]For $SL(3,\mathbb{R})$, the exponential map is not surjective, while it is for $SL(3,\mathbb{C})$.

where $*$ marks rather involved expressions of the non-zero components of g_1. We can thus see that neither F_1 nor F_3 are ever part of the generators and that $\log g_{11}$ always is the coefficient in front of H_1.

A few explicit examples of g_1:s with generators are listed here to reveal or exclude certain structures.

A "minimal" choice of g_1 in $\mathfrak{sl}_3$

The simplest choice for g_1, let us denote it $g_{1,\mathrm{min}}$, is to set all elements but g_{11} to zero. The general generator for this group element looks like

$$\log g_{1,\mathrm{min}} = \begin{pmatrix} \log(g_{11}) & 0 & 0 \\ 0 & \frac{1}{2}\left(\log\left(\frac{\mathrm{i}}{\sqrt{g_{11}}}\right) + \log\left(-\frac{\mathrm{i}}{\sqrt{g_{11}}}\right)\right) & \frac{1}{4}g_{11}^{3/2}\left(2\mathrm{i}\log\left(-\frac{\mathrm{i}}{\sqrt{g_{11}}}\right) + \mathrm{i}\log(g_{11}) + \pi\right) \\ 0 & -\frac{\mathrm{i}\left(-\mathrm{i}\pi + 2\log\left(-\frac{\mathrm{i}}{\sqrt{g_{11}}}\right) + \log(g_{11})\right)}{4g_{11}^{3/2}} & \frac{1}{2}\left(\log\left(\frac{\mathrm{i}}{\sqrt{g_{11}}}\right) + \log\left(-\frac{\mathrm{i}}{\sqrt{g_{11}}}\right)\right) \end{pmatrix}$$

which for the first three integer values of g_{11} evaluates to

$$\begin{aligned} g_{11} = 1: &\qquad \log g_{1,\mathrm{min}} = \frac{\pi}{2}E_2 - \frac{\pi}{2}F_2 \\ g_{11} = 2: &\qquad \log g_{1,\mathrm{min}} = \sqrt{2}\pi E_2 + \log(2)H_1 + \frac{\log(2)}{2}H_2 - \frac{\pi}{4\sqrt{2}}F_2 \\ g_{11} = 3: &\qquad \log g_{1,\mathrm{min}} = \frac{3\sqrt{3}\pi}{2}E_2 + \log(3)H_1 + \frac{\log(3)}{2}H_2 - \frac{\pi}{6\sqrt{3}}F_2. \end{aligned} \tag{F.1}$$

Thus, for the minimal choice of g_1, the generators do not contain any components in the directions of E_1, E_3, F_1 or F_3.

A "non-minimal" numerical example

For the sake of explicitness, we provide also a numerical example where all possible components of the generator are included. Namely, set $(g_{11}, g_{12}, g_{13}, g_{33}) = (2, 2, 1, 1)$ in g_1 above. Then the generator is given by

$$\begin{aligned} \log g_1 = \frac{1}{40}(19\pi + 42\log(2))E_1 + \pi E_2 + \frac{1}{10}(36\log(2) - 13\pi)E_3 \\ + \log(2)H_1 + \frac{1}{4}(\log(4) - \pi)H_2 - \frac{\pi}{8}F_2. \end{aligned}$$

More or less general solutions for $\mathfrak{sl}_3 : E_\theta \to E_2$

We redo what we have done above also for the conjugation to the other simple root vector E_2. The general group element g_2 in $g_2 E_\theta g_2^{-1} = E_2$ now looks like

$$g_2 = \begin{pmatrix} 0 & -\frac{1}{g_{21}^2} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & g_{21} \end{pmatrix}$$

with a generator of the form

$$\log g_2 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \log g_{21} \end{pmatrix},$$

where the components in the directions of F_2 and F_3 are missing. In this case it is the coefficient of H_2 that is easily given, namely $\log g_{21}$.

A "minimal" choice of g_2 in $\mathfrak{sl}3$

Setting all elements in g_2 but g_{21} to zero we get the generator

$$\log g_{2,\min} =$$
$$\begin{pmatrix} \frac{1}{2}\left(\log\left(\frac{\mathrm{i}}{\sqrt{g_{21}}}\right) + \log\left(-\frac{\mathrm{i}}{\sqrt{g_{21}}}\right)\right) & -\frac{\mathrm{i}\left(-\mathrm{i}\pi + 2\log\left(-\frac{\mathrm{i}}{\sqrt{g_{21}}}\right) + \log(g_{21})\right)}{4g_{21}^{3/2}} & 0 \\ \frac{1}{4}g_{21}^{3/2}\left(2\mathrm{i}\log\left(-\frac{\mathrm{i}}{\sqrt{g_{21}}}\right) + \mathrm{i}\log(g_{21}) + \pi\right) & \frac{1}{2}\left(\log\left(\frac{\mathrm{i}}{\sqrt{g_{21}}}\right) + \log\left(-\frac{\mathrm{i}}{\sqrt{g_{21}}}\right)\right) & 0 \\ 0 & 0 & \log(g_{21}) \end{pmatrix}.$$

This is very similar to the generator of the minimal choice $g_{1,\min}$ above and we can go between the matrices by interchanging the positions of the two block matrices, transpose and do the swap $g_{11} \leftrightarrow g_{21}$.

Some numerical examples yield

$$\begin{aligned} g_{21} = 1 : &\qquad \log g_{2,\min} = -\frac{\pi}{2}E_1 + \frac{\pi}{2}F_1 \\ g_{21} = 2 : &\qquad \log g_{2,\min} = -\frac{\pi}{4\sqrt{2}}E_1 - \frac{\log(2)}{2}H_1 - \log(2)H_2 + \sqrt{2}\pi F_1 \\ g_{21} = 3 : &\qquad \log g_{2,\min} = -\frac{\pi}{6\sqrt{3}}E_1 - \frac{\log(3)}{2}H_1 - \log(3)H_2 + \frac{3\sqrt{3}\pi}{2}F_1, \end{aligned}$$

which is identical to examples (F.1) of $g_{1,\min}$ under the swaps $X_1 \leftrightarrow -X_2$, where X denotes E, H and F.

More or less general solution for $\mathfrak{sl}_4 : E_\theta \to E_1$

We move on with a similar analysis for the algebra $\mathfrak{sl}_4$ and study the conjugations from the highest root vector into the simple ones. The general group element for the conjugation $g_1 E_\theta g_1^{-1} = E_1$ reads

$$g_1 = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & 0 & 0 & g_{11} \\ 0 & g_{32} & g_{33} & g_{34} \\ 0 & g_{42} & g_{43} & g_{44} \end{pmatrix} \in SL(4)$$

where the determinant condition from $SL(4)$ translates to

$$g_{11}^2 (g_{32} g_{43} - g_{33} g_{42}) \stackrel{!}{=} 1. \tag{F.2}$$

The general generator has the form

$$\log g_1 = \begin{pmatrix} \log g_{11} & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

which excludes generator components in the directions of F_1, F_4 and F_6.

Two “minimal” choices of g_1 in $\mathfrak{sl}_4$

When setting as much as possible to zero in g_1, we have two choices due to the two terms in the determinant condition (F.2). Choosing both g_{32} and g_{43} to be zero gives the general generator

$$\log g_{1,\mathrm{min1}} = \begin{pmatrix} \log g_{11} & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & \log g_{33} & * \\ 0 & * & * & * \end{pmatrix}$$

and choosing g_{33} and g_{42} to be zero instead, we find

$$\log g_{1,\mathrm{min2}} = \begin{pmatrix} \log g_{11} & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Two numerical examples of these "minimal" choices are

$$g_{1,\mathrm{min1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \log g_{1,\mathrm{min1}} = \frac{\pi}{2}\left(E_5 - F_5\right)$$

$$g_{1,\mathrm{min2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \log g_{1,\mathrm{min2}} = \frac{2\pi}{3\sqrt{3}}\left(-E_2 - E_3 + E_5 + F_2 + F_3 - F_5\right).$$

More or less general solution for $\mathfrak{sl}_4 : E_\theta \to E_2$

When conjugating to the next simple root vector, we find the general group element in $g_2 E_\theta g_2^{-1} = E_2$ to be

$$g_2 = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ 0 & 0 & 0 & g_{21} \\ 0 & g_{42} & g_{43} & g_{44} \end{pmatrix} \in SL(4)$$

where the determinant condition reads

$$g_{21}^2 (g_{12} g_{43} - g_{13} g_{42}) \overset{!}{=} 1.$$

The general generator is not immediately found by *Mathematica* but is given in terms of roots of very long expressions. We thus omit the form here.

Two "minimal" choices of g_2 in $\mathfrak{sl}_4$

As above, we have two choices when setting as much as possible to zero in g_2. Choosing both g_{12} and g_{43} to be zero gives the general generator

$$\log g_{2,\mathrm{min1}} = \begin{pmatrix} 0 & -\frac{\pi}{2\sqrt{2}g_{21}} & \frac{\pi g_{13}}{2\sqrt{2}} & -\frac{1}{4}\pi g_{13} g_{21} \\ \frac{\pi g_{21}}{2\sqrt{2}} & 0 & -\frac{1}{4}\pi g_{13} g_{21} & \frac{\pi g_{13} g_{21}^2}{2\sqrt{2}} \\ -\frac{\pi}{2\sqrt{2}g_{13}} & \frac{\pi}{4 g_{13} g_{21}} & 0 & \frac{\pi g_{21}}{2\sqrt{2}} \\ \frac{\pi}{4 g_{13} g_{21}} & -\frac{\pi}{2\sqrt{2} g_{13} g_{21}^2} & -\frac{\pi}{2\sqrt{2} g_{21}} & 0 \end{pmatrix}$$

with no components along H_i, $i = 1, 2, 3$. Choosing g_{13} and g_{42} to be zero instead, we find

$$\log g_{2,\mathrm{min2}} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

without components along E_i and F_i, $i = 2, 4, 5, 6$.

Two numerical examples of these “minimal” choices are

$$g_{2,\mathrm{min1}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \log g_{2,\mathrm{min1}} = \begin{pmatrix} 0 & -\frac{\pi}{2\sqrt{2}} & \frac{\pi}{2\sqrt{2}} & -\frac{\pi}{4} \\ \frac{\pi}{2\sqrt{2}} & 0 & -\frac{\pi}{4} & \frac{\pi}{2\sqrt{2}} \\ -\frac{\pi}{2\sqrt{2}} & \frac{\pi}{4} & 0 & \frac{\pi}{2\sqrt{2}} \\ \frac{\pi}{4} & -\frac{\pi}{2\sqrt{2}} & -\frac{\pi}{2\sqrt{2}} & 0 \end{pmatrix}$$

$$= \frac{\pi}{2\sqrt{2}}\Big(- E_1 - \frac{1}{\sqrt{2}}E_2 + E_3 + E_4 + E_5 - \frac{1}{\sqrt{2}}E_6 + F_1 + \frac{1}{\sqrt{2}}F_2 - F_3 - F_4 - F_5 + \frac{1}{\sqrt{2}}F_6\Big)$$

$$g_{2,\mathrm{min2}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \log g_{2,\mathrm{min2}} = \begin{pmatrix} 0 & -\frac{\pi}{2} & 0 & 0 \\ \frac{\pi}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\pi}{2} \\ 0 & 0 & -\frac{\pi}{2} & 0 \end{pmatrix} = \frac{\pi}{2}\left(-E_1 + E_3 + F_1 - F_3\right).$$

More or less general solution for $\mathfrak{sl}_4 : E_\theta \to E_3$

When conjugating $g_3 E_\theta g_3^{-1} = E_3$ we find the group element

$$g_3 = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ 0 & 0 & 0 & g_{31} \end{pmatrix} \in SL(4)$$

with the determinant condition

$$(g_3 g_{12} g_{23} - g_{13} g_{22}) g_{31}^2 \overset{!}{=} 1.$$

The general generator has the form

$$\log g_3 = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & \log g_{31} \end{pmatrix}$$

which excludes generator components in the directions of F_3, F_5 and F_6.

Two “minimal” choices of g_3 in $\mathfrak{sl}_4$

The two “minimal” choices are obtained either through setting $g_{1,2}$ and $g_{2,3}$ to zero where upon

$$\log g_{3,\mathrm{min1}} = \begin{pmatrix} * & 0 & * & 0 \\ 0 & \log g_{32} & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & \log g_{31} \end{pmatrix}$$

or through the vanishing of g_{13} and $g_{2,2}$, giving

$$\log g_{3,\mathrm{min2}} = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & \log g_{31} \end{pmatrix}.$$

Simple numerical examples of these are

$$g_{3,\mathrm{min1}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \log g_{3,\mathrm{min1}} = \begin{pmatrix} 0 & 0 & -\frac{\pi}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\pi}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{\pi}{2}\Big(- E_4 + F_4\Big)$$

$$g_{3,\mathrm{min2}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \log g_{3,\mathrm{min2}} = \begin{pmatrix} 0 & \frac{2\pi}{3\sqrt{3}} & -\frac{2\pi}{3\sqrt{3}} & 0 \\ -\frac{2\pi}{3\sqrt{3}} & 0 & \frac{2\pi}{3\sqrt{3}} & 0 \\ \frac{2\pi}{3\sqrt{3}} & -\frac{2\pi}{3\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{2\pi}{3\sqrt{3}}\left(E_1 + E_2 - E_4 - F_1 - F_2 + F_4\right).$$

More or less general solution for $\mathfrak{sl}_5 : E_\theta \to E_1$

We end the $\mathfrak{sl}$-series with $\mathfrak{sl}_5$ although the increasing possibilities make us only provide one example per conjugation. Following the order we first look at $g_1 E_\theta g_1^{-1} = E_1$ where

$$g_1 = \begin{pmatrix} g_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{11} \\ 0 & g_{32} & g_{33} & g_{34} & 0 \\ 0 & g_{42} & g_{43} & g_{44} & 0 \\ 0 & g_{52} & g_{53} & g_{54} & 0 \end{pmatrix} \in SL(5)$$

and

$$g_{11}^2\Big(g_{34}(g_{43}g_{52} - g_{42}g_{53}) + g_{33}(g_{42}g_{54} - g_{44}g_{52}) + g_{32}(g_{44}g_{53} - g_{43}g_{54})\Big) \overset{!}{=} 1,$$

with a generator of the form

$$\log g_1 = \begin{pmatrix} \log g_{11} & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{pmatrix}.$$

A "minimal" choice of g_1 in $\mathfrak{sl}_5$

As there are six choices of a "minimal" g_1, we only provide the choice of non-zero g_{11}, g_{34}, g_{43} and g_{52} here. It yields a generator of the form

$$\log g_{1,\min} = \begin{pmatrix} \log g_{11} & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & 0 \\ 0 & * & 0 & 0 & * \end{pmatrix}$$

and a numerical example of ones looks like

$$g_{1,\min} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \log g_{1,min} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{2} \\ 0 & 0 & 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & -\frac{\pi}{2} & 0 & 0 \\ 0 & -\frac{\pi}{2} & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{\pi}{2}\Big(E_3 + E_9 - F_3 - F_9\Big).$$

Also a sign flip for E_3 and F_3 generates the conjugating element.

More or less general solution for $\mathfrak{sl}_5 : E_\theta \to E_2$

Continuing with $g_2 E_\theta g_2^{-1} = E_2$ we find that

$$g_2 = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} & g_{15} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} \\ 0 & 0 & 0 & 0 & g_{21} \\ 0 & g_{42} & g_{43} & g_{44} & g_{45} \\ 0 & g_{52} & g_{53} & g_{54} & g_{55} \end{pmatrix} \in SL(5)$$

and

$$g_{21}^2\Big(g_{14}(g_{43}g_{52} - g_{42}g_{53}) + g_{13}(g_{42}g_{54} - g_{44}g_{52}) + g_{12}(g_{44}g_{53} - g_{43}g_{54})\Big) \stackrel{!}{=} 1.$$

A "minimal" choice of g_2 in $\mathfrak{sl}_5$

The presented "minimal" choice is non-zero g_{21}, g_{12}, g_{44} and g_{53} which yields a generator of the form

$$\log g_{2,\min} = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & \log g_{44} & 0 \\ 0 & 0 & * & 0 & * \end{pmatrix}.$$

A numerical example is

$$g_{2,\min} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \log g_{2,min} = \begin{pmatrix} 0 & -\frac{\pi}{2} & 0 & 0 & 0 \\ \frac{\pi}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\pi}{2} & 0 & 0 \end{pmatrix}$$
$$= \frac{\pi}{2}\Big(- E_1 + E_7 + F_1 - F_7\Big).$$

A sign flip of the entire generator yields the same conjugating element.

More or less general solution for $\mathfrak{sl}_5 : E_\theta \to E_3$

Moving on, we turn to $g_3 E_\theta g_3^{-1} = E_3$ where

$$g_3 = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} & 0 \\ 0 & g_{22} & g_{23} & g_{24} & 0 \\ g_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{31} \\ 0 & g_{52} & g_{53} & g_{54} & 0 \end{pmatrix} \in SL(5)$$

and

$$g_{31}^2\Big(g_{14}(g_{23}g_{52} - g_{22}g_{53}) + g_{13}(g_{22}g_{54} - g_{24}g_{52}) + g_{12}(g_{24}g_{53} - g_{23}g_{54})\Big) \stackrel{!}{=} 1.$$

A "minimal" choice of g_2 in $\mathfrak{sl}_5$

The single "minimal" choice presented is that with non-zero g_{31}, g_{22}, g_{13} and g_{54}. The corresponding generator has the form

$$\log g_{3,\min} = \begin{pmatrix} * & 0 & * & 0 & 0 \\ 0 & \log g_{22} & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

and as a numerical example we show

$$g_{3,\min} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \log g_{3,min} = \begin{pmatrix} 0 & 0 & \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{\pi}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\pi}{2} \\ 0 & 0 & 0 & \frac{\pi}{2} & 0 \end{pmatrix}$$
$$= \frac{\pi}{2}\Big(- E_4 + E_5 + F_4 - F_5\Big).$$

Swapping the sign for the entire generator still gives the same conjugating element.

More or less general solution for $\mathfrak{sl}_5 : E_\theta \to E_4$

The final simple root vector in $\mathfrak{sl}_5$ is E_4 and for $g_4 E_\theta g_4^{-1} = E_4$ the group element is given by

$$g_4 = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} & g_{15} \\ 0 & g_{22} & g_{23} & g_{24} & g_{25} \\ 0 & g_{32} & g_{33} & g_{34} & g_{35} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} \\ 0 & 0 & 0 & 0 & g_{41} \end{pmatrix} \in SL(5)$$

and

$$\Big(g_{14}(g_{23}g_{32} - g_{22}g_{33}) + g_{13}(g_{22}g_{34} - g_{24}g_{32}) + g_{12}(g_{24}g_{33} - g_{23}g_{34})\Big) g_{41}^2 \overset{!}{=} 1,$$

with a generator of the form

$$\log g_4 = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & 0 & \log g_{41} \end{pmatrix}.$$

A "minimal" choice of g_2 in $\mathfrak{sl}_5$

We choose the "minimal" group element by keeping g_{41}, g_{33}, g_{23} and g_{14} non-zero. It yields a generator of the form

$$\log g_{4,\min} = \begin{pmatrix} * & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & \log g_{41} \end{pmatrix}$$

where a numerical example of ones can look like

$$g_{4,\min} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \log g_{4,min} = \begin{pmatrix} 0 & 0 & 0 & -\frac{\pi}{2} & 0 \\ 0 & 0 & \frac{\pi}{2} & 0 & 0 \\ 0 & -\frac{\pi}{2} & 0 & 0 & 0 \\ \frac{\pi}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{\pi}{2}\Big(E_2 - E_8 - F_2 + F_8\Big).$$

A sign flip of both E_2 and F_2 gives the same conjugating element.

F.1.1 Summaries of numerical examples

Summary of numerical examples in $\mathfrak{sl}_3$

These are the conjugating elements g_i in $g_i E_\theta g_i^{-1} = E_i$ and $gE_2g^{-1} = E_1$:

$$\overset{i=1}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}, \quad \overset{i=2}{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which have the generators

$$\begin{aligned} \log(g_{\theta\to 1}) &= \frac{\pi}{2}\Big(E_2 - F_2\Big) \\ \log(g_{\theta\to 2}) &= \frac{\pi}{2}\Big(-E_1 + F_1\Big) \\ \log(g_{2\to 1}) &= \frac{2\pi}{3\sqrt{3}}\left(E_1 + E_2 - E_3 - F_1 - F_2 + F_3\right). \end{aligned}$$

Summary of numerical examples in $\mathfrak{sl}_4$

These are the conjugating elements g_i in $g_i E_\theta g_i^{-1} = E_i$:

$$\overset{i=1}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}, \quad \overset{i=\ldots}{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}, \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and their corresponding generators are

$$\begin{aligned} \log(g_{\theta\to 1}) &= \frac{\pi}{2}\left(E_5 - F_5\right) \\ \log(g_{\theta\to 2}) &= \frac{\pi}{2}\left(-E_1 + E_3 + F_1 - F_3\right) \\ \log(g_{\theta\to 3}) &= \frac{\pi}{2}\Big(-E_4 + F_4\Big). \end{aligned}$$

Summary of numerical examples in $\mathfrak{sl}_5$

These are the conjugating elements g_i in $g_i E_\theta g_i^{-1} = E_i$:

$$\overset{i=1}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}}, \quad \overset{i=\ldots}{\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the generators are

$$\begin{aligned}
\log(g_{\theta\to 1}) &= \frac{\pi}{2}\Big(E_3 + E_9 - F_3 - F_9\Big)\\
\log(g_{\theta\to 2}) &= \frac{\pi}{2}\Big(-E_1 + E_7 + F_1 - F_7\Big)\\
\log(g_{\theta\to 3}) &= \frac{\pi}{2}\Big(-E_4 + E_5 + F_4 - F_5\Big)\\
\log(g_{\theta\to 4}) &= \frac{\pi}{2}\Big(E_2 - E_8 - F_2 + F_8\Big).
\end{aligned}$$

F.2 Explicit Conjugation from E_θ to Simple Root Vectors in $\mathfrak{g}_2$

As concluded in section 4.2.2 about the nilpotent orbits of $\mathfrak{g}_2$, all the long root vectors are in the same orbit and thus in the minimal orbit as the highest root vector $E_\theta = E_6$ of $\mathfrak{g}_2$ is long. With the labels chosen in the representation used (see appendix E), the other two correspond to E_1 and E_5. This representation yields too big expressions for using the brute force method from above but as these root vectors, together with F_1, F_5, H_1 and a suitable choice in H, can be rescaled to form a $\mathfrak{sl}_3$ algebra we let us be inspired by the $\mathfrak{sl}_3$ examples.

An example of the conjugation $E_\theta \to E_1$

Rescaling the generators in $\mathfrak{g}_2$ and copying the result from the $\mathfrak{sl}_3$ calculation, we find that with

$$\tilde{g}_1 = \exp\left(\frac{\pi}{2}\Big(\frac{1}{12}E_5 - \frac{1}{3}F_5\Big)\right)$$

we get

$$\tilde{g}_1 E_\theta \tilde{g}_1^{-1} = 12E_1.$$

We can normalize this by an additional adjoint action based on one of the generators H_1 or H_2, thus making the conjugating element $g_1 = \exp(\alpha_j H_j)\tilde{g}_1$, where α_j is a coefficient to be determined and j is either 1 or 2. Both choices of j work but the generator for $j = 2$ is slightly cleaner and looks like

$$
\begin{aligned}
\log g_1 &= \log\Big(\exp\Big(\log(12^{1/3})H_2\Big)\ \tilde{g}_1\Big) \\
&= \begin{pmatrix}
-\frac{\log(12)}{6} & 0 & 0 & 0 & -\sqrt{3}\pi & 0 & 0 \\
0 & -\frac{\log(12)}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\log(12)}{6} & 0 & 0 & 0 & \sqrt{3}\pi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\pi}{4\sqrt{3}} & 0 & 0 & 0 & -\frac{\log(12)}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\log(12)}{3} & 0 \\
0 & 0 & -\frac{\pi}{4\sqrt{3}} & 0 & 0 & 0 & \frac{\log(12)}{6}
\end{pmatrix} \\
&= \frac{\pi}{4\sqrt{3}}E_5 - \frac{\log(12)}{2}H_1 - \frac{\log(12)}{6}H_2 - \frac{\pi}{12\sqrt{3}}F_5.
\end{aligned}
$$

An example of the conjugation $E_\theta \to E_5$

We proceed with the same method but in the case of $g_5 E_\theta g_5^{-1} = E_5$ there is no need of additional normalization. We have that

$$
g_5 = \exp\left(\frac{\pi}{2}\Big(-E_1 + F_1\Big)\right)
$$

does the trick, which explicitly reads

$$
\begin{aligned}
\log g_5 &= \\
&\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\pi}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{\pi}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\pi}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{\pi}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&= \frac{\pi}{2}\Big(-E_1 + F_1\Big).
\end{aligned}
$$

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