On the Localization Equations of Topologically Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory in Five Dimensions

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Abstract

In recent articles, topologically twisted N=4 supersymmetric Yang-Mills theory on a four-manifold of the form $V = W \times \mathbb{R}^+$ or $V = W \times I$ were considered. W here is a Riemannian three-manifold, and a suitable set of boundary conditions apply to the endpoints of I (or $\mathbb{R}^+$). In the special case where $W = S^3$, spherically symmetric solutions where obtained to the localization equations. For large interval lengths, these consist of pairwise occurring (non gauge-equivalent) solutions, which then coincide for a certain critical interval length, only to disappear if it decreases below this critical value. Only for the instance were the interval length is of critical value was an exact analytical solution obtained.

The only feasible explanation for this is that there exist a tunneling between the solutions in one solution-pair as one goes to five dimensions. This will be shown to be the case in this thesis.

A five-dimensional version of the previously mentioned theory on $\mathbb{R} \times S^3 \times I$ is considered, and the localization equations of this theory obtained. An analytical expression of this five-dimensional supersymmetric field configuration has not been possible to obtain, similarly to the case in four dimensions, but the solution is instead obtained as a series expansion in terms of an infinitesimal parameter ε stating how much the solutions differ from the exactly solvable static case for critical interval length in four dimensions, where we have stationary solutions in five dimensions as well.

Keywords: Topological Field Theory, Maximal Supersymmetry, Yang-Mills Theory, Topological Twisting, High Energy Particle Physics.

Acknowledgements

Contents

To those who do not know mathematics
it is difficult to get across a real feeling as to the beauty,
the deepest beauty, of nature ...
If you want to learn about nature, to appreciate nature,
it is necessary to understand the language that she speaks in.
– Richard P. Feynman

Chapter 1

Introduction

Since the ancient Greeks, mankind has strived to understand the fundamental laws that govern our universe. It started with observations and experiments where the outcome was explained in the language of mathematics, and thus theories of nature were formed. As our understanding of the world increases, more complicated theories are required to explain the diversity of nature. These in turn pose great demands of our ability in mathematics, and the ability to formulate properties of nature in mathematical terms.

Physics and mathematics are thus thoroughly intertwined, and knowledge in one field can lead to advances in the other. If mathematics is the language in which nature speaks, physics is the ability to properly interpret what she says. This is why the combination of knowledge in physics as well as mathematics is so powerful, and our insight in how the world should behave can lead us to solve problems that from a purely mathematical point of view might be deemed incredibly hard, or even impossible. This is part of the beauty of the field of theoretical physics, where the absolute fundamentals of our universe can be studied.

The theories that have been most successful in accurately describing the world are so-called *field theories*. Common for these theories are that we can represent observable quantities by fields on spacetime, the evolution of which can be described by the Lagrangian formalism.

Such field theories exists for all four fundamental forces of nature, the electro-magnetic force, the strong nuclear force, weak interaction and gravity. The standard model of particle physics, which has had great success in satisfactorily describing all the forces of nature with the exception of gravity in one field theory and then we have Einsteins theory of general relativity describing gravity.

An important concept here is the so-called *gauge theories*, where the Lagrangian is invariant under some continuous group of local transformations, which is called a *gauge group*. A gauge group is necessarily a Lie group. For example, Maxwell's theory of electromagnetism is a gauge theory with the gauge group $U(1)$. (This theory is invariant under multiplication with a phase.) Cases where the gauge group is non-abelian are called *Yang-Mills theories*. For example, the standard model of particle physics is such a theory, since it has the gauge group $SU(3) \times SU(2) \times U(1)$. Thus the field of Yang-Mills theories is

an interesting research topic since our world seems to have an innate non-abelian structure. (See [1].) Yang-Mills theory is also included in the list of "Millennium Prize Problems" of the Clay Mathematics Institute [2].

The global symmetries of nature are of great interest in physics as well. In addition to symmetries of space, such as rotational- and translational invariance, we can have another form of symmetry, namely *Supersymmetry*. Supersymmetry relates to each fermion a bosonic superpartner. This is done by essentially requiring the Lagrangian to be invariant (up to a total derivative) under the supersymmetry transformation with a parameter ε that relates the variation of the bosonic fields to the fermionic fields and vice versa. By adding this "extra symmetry" to the theory, one can for example solve the naturalness problem in the standard model. (The naturalness problem can loosely speaking be described as follows; in order for the standard model to be accurate, very specific requirements of the Higgs mass appears, but there is no good explanation for why it should be so small compared to e.g. the Planck scale or GUT scale. This is more natural with supersymmetry.)

1.1 Maximally Supersymmetric Yang-Mills theories

Here, we have considered $\mathcal{N} = 4$ super Yang-Mills theory, i.e. a Yang-Mills theory with four times the minimal amount of supersymmetry, meaning that there here are four supersymmetry generators, which is realized by adding an index to the supersymmetry generator and allowing this to take the values $1, 2, 3, 4$. This is the maximal amount of supersymmetry possible without being forced to consider gravity as well, since every supersymmetry generator can be seen as lowering helicity by $\frac{1}{2}$, so with 4 generators, if one starts at helicity $+1$, one gets precisely down to -1. The next possible number of supersymmetry generators in the theory is 8, which then would force us to take particles of helicity ± 2, that is, gravitons, into the spectrum. Hence $\mathcal{N} = 4$ super Yang-Mills theory is sometimes referred to as maximally supersymmetric Yang-Mills theory. This theory has some interesting mathematical properties, as for example being exactly scale invariant, so that the β-function is identically 0, meaning that to all orders of pertubation theory, the fermionic and bosonic contributions cancel out the quadratic divergencies that appear, and the theory is ultraviolet finite. This is the first known example of a four-dimensional field theory with this property [3].

1.2 Topological Field Theories

A topological field theory is a field theory where the correlation functions do not depend on the metric of space-time, i.e. they are topological invariants. This means that they will be unaffected by continuous deformations of space-time and is thus highly convenient.

For any topological field theory with an action I with some supersymmetry Q, computations can, under favorable conditions, be localized on configurations that obey $\{Q,\zeta\} = 0$ for all

fermion fields ζ. This is done by adding a suitable term to the action, which is Q-exact, on the form

$$I' = I - \frac{1}{\epsilon} \left\{ Q, \int_V \text{Tr} \left(\sum_\zeta \zeta \left\{ Q, \zeta \right\} \right) \right\}. \tag{1.1}$$

This integral diverges as $1/\epsilon$ unless the localization equations are satisfied, that is $\{Q,\zeta\} = 0 \; \forall \; \zeta$. Recall that we in the path integral formalism have an expression for the path integral on the form $\int \mathcal{D}... \; e^{-I'}$. This means that the path integral will be reduced to a calculation on configurations that obey the localization equations since it will tend to zero if $\{Q,\zeta\} \neq 0 \; \forall \; \zeta$.

This integral diverges as $1/\epsilon$ unless the localization equations are satisfied, that is $\{Q,\zeta\} = 0 \; \forall \; \zeta$. Thus the infinite-dimensional path integral has been reduced to an intergral over fields satisfying the supersymmetry equations.

1.3 Motivation for the Thesis

This thesis builds on the paper "Fivebranes and Knots" by Edward Witten, published in March of 2011, [4], as well as the paper "Boundary conditions for GL-twisted N=4 SYM" by Måns Henningson [5].

By performing a topological twisting of $\mathcal{N} = 4$ super Yang-Mills theory (obtained from ten dimensions by dimensional reduction) on $V = \mathbb{R}^+ \times W$ for suitable boundary conditions on W, a topological field theory is obtained [4].

The localization equations of the four dimensional theory were obtained in [4], and maximally symmetric solutions to these were obtained in [5] in the $V = W \times I$ case with $W = S^3$. These solutions show up as non-gauge equivalent pairs that cancel below a certain critical interval length. Thus it was speculated that these were connected by a solution in five dimensions. This has been shown to be the case in this thesis. The localization equations have been obtained for a general W in [4], and are herein analyzed for the maximally symmetric case.

1.4 Structure of the Thesis

Below, the outlay of the thesis is presented to facilitate for the reader. It is worth noting that the calculations made in chapter 2 are highly schematic and non-detailed in order to not loose focus of our primary result here, that is, solving the localization equations in five dimensions. For more detailed calculations and explanations, see chapters 1-3 in [4].

The first chapter in this work will be a quick review of the content in [4] that is needed for this thesis to remain self-contained to obtain the localization equations in four dimensions. The starting point is here the Jones polynomial of a knot, that can be obtained from the Chern-Simons action for a gauge theory with gauge group G on the three manifold W.

This can then be related to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions for suitable boundary conditions on W [6]. These boundary conditions turn out to be the ones of the D3-NS5-system of type IIB string theory. A topological twisting of the $\mathcal{N} = 4$ Yang-Mills theory will then be performed to obtain a topological field theory, whose localization equations then will be simplified by S-dualizing the theory.

In the following chapter, a review of the results from [5] are made, and these are further analyzed. Chapter 4 further reviews [4] where the five-dimensional localization equations are obtained by a T-duality of the previously obtained theory. These five-dimensional localization equations are then solved in the maximally symmetric case.

Chapter 2

Obtaining a Topological Field Theory

A knot is an embedding of a circle in a three-dimensional space. Usually one here talks about $\mathbb{R}^3$, but below we will consider the case of S^3. To every knot, K, in $\mathbb{R}^3$ or S^3 one can associate a Jones polynomial, denoted $\mathcal{J}(q;K)$ in [4]. The Jones polynomial is a Laurent polynomial in one variable q with integer coefficients. There are many ways of constructing the Jones polynomial for a knot, but if one wishes to make the three-dimensional symmetry manifest, it requires a construction starting from a three-dimensional gauge theory with a Chern-Simons action [7].

The Chern-Simons action for a gauge theory with gauge group G and gauge field A on the three manifold W is given by

$$I = \frac{k}{4\pi} \int_W \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{2.1}$$

where k is an integer for topological reasons.

Now assume K is an oriented loop embedded in W. Given a representation R of G, we can then define the Wilson loop operator by:

$$\mathcal{W}(K,R) = -\text{Tr}_R P \exp(-\oint_K A). \tag{2.2}$$

It turns out that the Jones polynomial, as well as the generalizations of it can be computed as an expectation value of Wilson operators. Exactly how this is done differs depending on the exact appearance of G and W.

In [4], one wishes to use nonperturbative string theory/field theory dualities to three-dimensional Chern-Simons gauge theory. However, there is no good way of realizing this directly. Thus one must first use the recent insight that the path integral of the Chern-Simons theory on a three-manifold W can be expressed as a path integral of $\mathcal{N} = 4$ super Yang-Mills theory on a half-space $V = W \times \mathbb{R}_+$, where $\mathbb{R}_+$ is parameterized by $y : y \geq 0$ [6]. Here, any knots in W will be represented by Wilson operators in the boundary of V. This path integral of $\mathcal{N} = 4$ can however be dualized by standard dualities and this will lead to a higher-dimensional description.

Firstly, in order to relate the $\mathcal{N} = 4$ path integral to a Chern-Simons path integral on W, one needs to use the right boundary conditions on W. These boundary conditions turns out to be those of the D3-NS5 system of type IIB superstring theory in the presence of a theta angle. In type IIB superstring theory, we have two two-form gauge fields, one originating from the R-R sector of the closed string and one from the NS-NS-sector. Thus strings can carry two types of charges, and similarly we will here have two types of branes. The D-branes that couple to the charges originating from the R-R-sector (here in addition to the two-form there is one zero-form and one four-form, which gives us stable D1-, D3-, and D5-branes), and the NS5-brane, which couple to the gauge field originating from the NS-NS-sector [8].

2.1   The D3-NS5-system

Consider $\mathbb{R}^{1,9}$, with coordinates $x^0, x^1, ...x^9$ and metric $(- + ...+)$. Now consider N D3-branes all supported at $x^4 = x^5 = ... = x^9 = 0$, ending on an NS5-brane, which is supported at $x^3 = x^7 = x^8 = x^9 = 0$. Thus in the four-dimensional space spanned by $x^0...x^4$, one sees that the D3-branes span the half-space $x^3 > 0$. The theory of the D3-branes is a $U(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry. Here, the NS5 brane provides a boundary condition that preserves half the supersymmetry, a so-called half-BPS boundary condition.

In type IIB superstring theory, we have the complex coupling parameter $\tau_{str} = \theta/2\pi + i/g_s$, that in the gauge theory becomes $\tau_{YM} = \theta/2\pi + 4\pi i/g_{YM}^2$. When the theta angle disappears, we simply get Neumann boundary conditions for the gauge fields in the gauge theory. However, since the D3-NS5-system is half-BPS for all values of τ_{str}, this means that from a gauge-theory point of view, the Neumann boundary-conditions must have a half-BPS generalization when $\theta \neq 0$.

The R-symmetry group for $\mathcal{N} = 4$ Yang-Mills theory is $SO(6)$ (or actually $Spin(6)$). To properly explain what the R symmetry group is, one needs to consider in detail what happens when we reduce the theory from 10 dimensions down to 4. Let A_M be the 10-dimensional notation of the gauge fields. Let us now relabel these fields, so that $A_\mu = A_M : M, \mu = 0,1,2,3$, $\phi_i = A_M : M = 4,5,6,7$ and $\sigma = A_8 + iA_9$. Then the R-symmetry group is the group that acts on the ϕ-fields as well as A_8, A_9.

From a ten-dimensional point of view, the boundary conditions arising from the presence of the NS5-brane is invariant under a subgroup of $SO(1,9)$ denoted by $\mathcal{U} = SO(1,2) \times SO(3)_{\vec{X}} \times SO(3)_{\vec{Y}}$, where the two copies of $SO(3)$ results from the splitting of the R-symmetry group caused by the presence of the NS5-brane. The supersymmetries of $\mathcal{N} = 4$ Yang-Mills theory transforms under $SO(1,9)$ as a spinor **16** of definite chirality, that is:

$$\Gamma_{0,1,...9}\varepsilon = \varepsilon, \tag{2.3}$$

where $\Gamma_I, I = 0,...9$ are the gamma matrices of $SO(1,9)$. Each factor in $\mathcal{U}$ has a two-dimensional representation denoted by **2**. The **16** of $SO(1,9)$ transform under $\mathcal{U}$ as two copies of the tensor-product $(\mathbf{2},\mathbf{2},\mathbf{2}) = V_8$. This is a real representation of $\mathcal{U}$ of dimension 8. Hence the supersymmetries transform as $V_8 \otimes V_2$, where V_2 is a two-dimensional real

vector space. We now need to find a base for V_2. This is found by considering elements in the Clifford algebra of $SO(1,9)$ that commutes with $\mathcal{U}$. They are generated by:

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

Let now $\varepsilon_0 \in V_2$ as $\binom{s}{t}$, and $\bar{\varepsilon}_0$ be defined as the row-vector $(t, -s)$. In any half-BPS boundary condition that is invariant under $\mathcal{U}$, we must have that the unbroken symmetries are on the form $V_8 \otimes \varepsilon_0$ for some $\varepsilon_0 \in V_2$. Scaling of ε_0 here is not of interest so we can simply choose

$$\varepsilon_0 = \begin{pmatrix} -a \\ 1 \end{pmatrix} \; \bar{\varepsilon}_0 = (1,a), \tag{2.5}$$

where the parameter a corresponds to the gauge theory θ-angle. It was shown in [9] that for every choice of a there is a unique $\mathcal{U}$-invariant half-BPS boundary condition that preserves all of the gauge symmetry.

In order to be able to localize the action, we need to know the fermion fields of the theory. The fermion fields, λ, of $\mathcal{N} = 4$ super Yang-Mills theory are adjoint-valued fields and transforms under **16** of $SO(1,9)$, as do the supersymmetries. The boundary conditions for the λ's turn out to be, as derived in [9]:

$$\lambda \mid_{x^3=0} \in V_8 \otimes \vartheta \tag{2.6}$$

where $\vartheta \in V_2$ is

$$\vartheta = \begin{pmatrix} a \\ 1 \end{pmatrix}. \tag{2.7}$$

Actions

Let the scalar fields that transform under $SO(3)_{\vec{X}}$ be denoted $\vec{X}$. By considering the boundary conditions for them as well as the boundary conditions for the gauge fields at $x^3 = 0$ for general values of the parameter a (corresponding to the gauge theory θ-angle), as described in [4] it can be shown that the action for $\vec{X}$ and the gauge fields respectively can be written as:

$$\widehat{I_X} = \frac{1}{g_{YM}^2} \int_{x \geq 0} \mathrm{Tr}(D_\mu X_c D^\mu X^c) \, \mathrm{d}^4 x + \frac{2a}{3g_{YM}^2(1+a^2)} \int_{x^3=0} \epsilon^{cde} \, \mathrm{Tr}(X_c[X_d, X_e]).$$
$$\widehat{I_A} = \frac{1}{2g_{YM}^2} \int_{x^3>0} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) d^4 x - \frac{\theta}{32\pi^2} \int_{x^3 \geq 0} \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}) \mathrm{d}^4 x. \tag{2.8}$$

with

$$\frac{\theta}{2\pi} = \frac{2a}{1-a^2} \frac{4\pi}{g_{YM}^2}.$$

(2.9)

The second term in both these actions was added to satisfy the boundary conditions. In the action for the gauge fields, it is actually the "usual" topological term of four dimensional gauge theory. Note that for fixed values of θ and a, equation 2.9 gives us two possible values for g_{YM}. These correspond to half-BPS boundary conditions for the D3-NS5 and D3-$\overline{\text{NS5}}$ systems respectively.

One now wishes to leave the Lorentzian signature and go to Euclidean signature in order to approach topological field theory. A Wick rotation, $x^0 \to -ix^0$, is thus performed, which will reverse the sign of the second part of $\widehat{I_X}$ since this contains two factors of x^0, (one in $\mathrm{d}x^0$ and one in X_0) and multiplies the second term in $\widehat{I_A}$ by $-i$ since it only contains one factor of x^0 in the differential, giving us the boundary action

$$I^* = \frac{1}{g_{YM}^2} \int_{x^3=0} (\frac{-2a}{3(1+a^2)} \epsilon^{abc} \operatorname{Tr}(X_a[X_b,X_c]) + \frac{2ia}{1-a^2} \epsilon^{\mu\nu\lambda} \operatorname{Tr}(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda)) \mathrm{d}^3 x$$

(2.10)

2.2 Interpreting our theory in a topological field theory way

An attempt will now be made in order to understand equation 2.10 from the perspective of topological field theory. The basic idea is to construct a four-dimensional topological field theory by topological twisting of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Let the $\mathcal{N} = 4$ Yang-Mills theory be given by a system of D3-branes parameterized by x^0, x^1, x^2, x^3, where these coordinates are rotated by $SO(4)$, whereas the remaining coordinates, $x^4...x^9$ are rotated by $SO(6)$.

To obtain a topological field theory from this, we define $SO'(4)$ that acts on x^0, x^1, x^2, x^3 and simultaneously on x^4, x^5, x^6, x^7. Pick a parameter of the supersymmetry, ε, that is invariant under $SO'(4)$, that is, it satisfies

$$(\Gamma_{\mu\nu} + \Gamma_{\mu+4,\nu+4})\varepsilon = 0, \ \mu,\nu = 0,...,3.$$

(2.11)

According to [4], it can be shown that if we restrict ourselves to operators and states that are invariant under this supersymmetry, a four-dimensional topological field theory has been obtained.

Furthermore, $SO'(4)$ commutes with $SO(2) \simeq U(1)$, which is the R-symmetry group that acts on the two remaining coordinates. From the viewpoint of $SO'(4)$, four of the

adjoint scalar fields of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory are reinterpreted as an adjoint valued one-form $\phi = \phi_\mu \mathrm{d}x^\mu$. The other two combine into an adjoint valued complex scalar field σ which transforms under the $U(1)$. We can normalize the generator of this group to give σ charge 2. This will now give the boundary part of the action as:

$$I^* = \frac{1}{g_{YM}^2} \int_{x^3=0} \epsilon^{\mu\nu\lambda} \mathrm{Tr}(-\frac{4a}{3(1+a^2)}\phi_\mu\phi_\nu\phi_\lambda + i\frac{2a}{1-a^2}(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda))\mathrm{d}^3 x. \quad (2.12)$$

What happened to the boundary conditions? Do they preserve the symmetry of the topological field theory? Equation 2.11 has a two-dimensional space of solutions. We can pick a basis of solutions $\varepsilon_l, \varepsilon_r$ that are chiral in the four-dimensional sense:

$$\Gamma_{0123}\varepsilon_l = -\varepsilon_l$$
$$\Gamma_{0123}\varepsilon_r = \varepsilon_r. \quad (2.13)$$

They can be normalized so that they also satisfy:

$$\Gamma_{\mu,\mu+4}\varepsilon_l = -\varepsilon_r$$
$$\Gamma_{\mu,\mu+4}\varepsilon_r = \varepsilon_l. \quad (2.14)$$

When constructing our topological field theory, we can thus take our supersymmetry generator to be an arbitrary linear combination of these two. Since any scaling is uninteresting in this case, one can without loss of generality choose

$$\varepsilon = \varepsilon_l + t\varepsilon_r. \quad (2.15)$$

We here have a family of topological field theories that are parameterized by $t \in \mathbb{C}$. Now by (2.4) and (2.13), one can see that, after some basic gamma matrix algebra

$$B_0\varepsilon_l = i\varepsilon_l \quad (2.16)$$
$$B_0\varepsilon_r = -i\varepsilon_r.$$

Combining this with (2.11) and (2.14)

$$B_1\varepsilon_l = -\varepsilon_r \quad (2.17)$$
$$B_1\varepsilon_r = -\varepsilon_l.$$

From the above two relations, (2.16 and 2.17) is obtained that

$$\left(1 + i\frac{1-t^2}{1+t^2}B_0 + \frac{2t}{1+t^2}B_1\right)(\varepsilon_l + t\varepsilon_r) = 0. \quad (2.18)$$

However, by using the matrix expressions for the B-matrices, (2.4), we can see that the ε_0 as defined in (2.5) also satisfies

$$\left(1 + i\frac{1 - t^2}{1 + t^2}B_0 + \frac{2t}{1 + t^2}B_1\right)(\varepsilon_0) = 0. \tag{2.19}$$

This means that the half-BPS boundary conditions preserve the supersymmetry of the twisted topological field theory as well as every supersymmetry with $\varepsilon = \eta \otimes \varepsilon_0$ with $\eta \in \mathbf{V}_8$ in the D3-NS5-system if the following relationship between the parameter t of the topological field theory and the parameter a describing the D3-NS5-system is satisfied:

$$a = i\frac{1 - it}{1 + it}. \tag{2.20}$$

Inserting this expression for a (2.20) into (2.9) and recalling the definition of τ, ($\tau_{YM} = \theta/2\pi + 4\pi i/g_{YM}^2$), one obtains

$$t^2 = \frac{\bar{\tau}}{\tau}. \tag{2.21}$$

It can be interesting to note that the operation $t \to -t$ corresponds to $a \to -1/a$ and the exchange of the D3-NS5-system to a D3-$\overline{\text{NS5}}$ system. Using the relationship between a and t (2.20), we can now rewrite the boundary couplings (2.12) in terms of t instead:

$$I^* = \frac{1}{g_{YM}^2}\int_{x^3=0}\epsilon^{\mu\nu\lambda}\text{Tr}(-\frac{t + t^{-1}}{3}\phi_\mu\phi_\nu\phi_\lambda + \frac{t + t^{-1}}{t - t^{-1}}(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda))\mathrm{d}^3x. \tag{2.22}$$

2.3 Wilson loops

$\mathcal{N} = 4$ super Yang-Mills theory in four dimensions admits 1/16-BPS Wilson loop operators. [10]. Below, a brief explanation of how these are constructed will be given.

The supersymmetry transformation law for bosonic fields in this theory is given by

$$\delta A_I = i\bar{\varepsilon}\Gamma_I\lambda = -i\bar{\lambda}\Gamma_I\varepsilon \tag{2.23}$$
$$I = 0,...,9$$

Above, a ten-dimensional notation has been used, thus for $I \leq 3$, A_I is a component of a gauge field and for $I \geq 4$ it is a scalar field. When we twisted, we converted four of those scalars into a one-form $\phi = \sum_{\mu=0}^3 A_{\mu+4}\mathrm{d}x^\mu$, similarly, we define another one-form $A = \sum_{\mu=0}^3 A_\mu\mathrm{d}x^\mu$. Recall that Greek letters are used to denote four-dimensional indices.

The Wilson operators will be on the form

$$\mathcal{W} = \text{Tr } P \exp(-\oint_K (\mathcal{A}))$$

$$\mathcal{A} = A + w\phi \tag{2.24}$$

giving us

$$\delta\mathcal{W} \propto (\delta A + w\delta\phi) = ... = -i\overline{\lambda}(\Gamma_\mu + w\Gamma_{\mu+4})\varepsilon. \tag{2.25}$$

In order for the Wilson operator to be invariant under the supersymmetry, we require that $\delta\mathcal{W} = 0$. However, this means that, for general values of w, there are no supersymmetric Wilson operators except at the boundary of V where one can use the boundary conditions obeyed by λ as well as the conditions obeyed by ε in order to make sure that (2.25) vanishes. Since we are at the boundary of V, only $\mu = 0,...,2$ is considered in (2.25). When this condition is satisfied, the Wilson operators in equation 2.24 indeed are supersymmetric for any knot K in the boundary of V.

Condition 2.25 is equivalent to $\overline{\lambda}\Gamma_\mu(1 + iwB_0B_1)\varepsilon = 0$ since $\Gamma_{\mu,\mu+4}\varepsilon = iB_0B_1\varepsilon$. Thus one considers the situation when

$$\overline{\lambda}\Gamma_\mu(1 + iwB_0B_1)\varepsilon = 0. \tag{2.26}$$

The fermion boundary condition of the D3-NS5-system says that λ, on the boundary of V is the tensor product of some vector in V_8 with $\vartheta \in V_2$ (ϑ was defined in 2.7), and similarly, the condition for the generator ε of an unbroken supersymmetry of the D3-NS5 boundary condition is that it must be the tensor product of some vector in V_8 and $\varepsilon_0 \in V_2$. Thus it is required that

$$\vartheta^T(1 + iwB_0B_1)\varepsilon_0 = 0 \tag{2.27}$$

in order for the Wilson operator in 2.24 to be invariant under the supersymmetry generated by ε.

2.4 Action of $\mathcal{N} = 4$ Super Yang-Mills Theory

Recalling the definitions of ϑ (2.7) and of ε_0 (2.5), as well as the expressions of the B-matrices (2.4), the condition 2.27 reduces to

$$w = i\frac{a^2 - 1}{a^2 + 1} = \frac{t - t^{-1}}{2}. \tag{2.28}$$

Now, by the definition of a, if $\theta, g_{YM} \in \mathbb{R}$, then $a \in \mathbb{R}$ which means that t, w must be purely imaginary. Furthermore, recalling that $t = \pm\frac{\overline{\tau}}{\tau}$ gives us that

$$w = \mp i \frac{Im(\tau)}{\tau} \tag{2.29}$$

with the corresponding signs.

The action of $\mathcal{N} = 4$ super Yang-Mills theory on a four mainfold V is the sum of a term proportional to $1/g_{YM}^2$ containing the kinetic energy for all fields, and a term proportional to θ:

$$I = \frac{1}{g_{YM}^2} \int_V \mathrm{d}^4x \sqrt{g} \mathcal{L}_{kin} + i \frac{\theta}{32\pi^2} \int_V \mathrm{d}^4x \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}). \tag{2.30}$$

In the case where V has no boundary, both these terms are Q-invariant. The θ-term is Q-invariant since it is, more generally, topologically invariant (unchanged under any continuous deformations). The first term is equivalent mod $\{Q,...\}$ to a multiple of the second term.

$$I = \{Q, ...\} + \frac{2\pi i \Psi}{32\pi^2} \int_V \mathrm{d}^4x \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}). \tag{2.31}$$

The parameter $\Psi = \frac{|\tau|^2}{Re(\tau)}$ is known as the canonical parameter and is always real for the D3-NS5-system.

Now, what happens when V has a boundary? (Since this is the case we have here!) Then the integral $\int_V \mathrm{d}^4x \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta})$ is no longer Q-invariant but varies with a boundary term. It is convenient to replace this integral by a multiple of the Chern-Simons function, defined below for any (possibly complex) connection $\mathcal{A} = A + w\phi$ as:

$$\mathrm{CS}(\mathcal{A}) = \frac{1}{4\pi} \int_{\partial V} \mathrm{d}^3x \epsilon^{\mu\nu\lambda} \mathrm{Tr}(\mathcal{A}_\mu \partial_\nu \mathcal{A}_\lambda + \frac{2}{3} \mathcal{A}_\mu \mathcal{A}_\nu \mathcal{A}_\lambda). \tag{2.32}$$

Thus we can write 2.31 as

$$I = \{Q, ...\} + i\Psi \mathrm{CS}(\mathcal{A}). \tag{2.33}$$

Note here that $\mathrm{CS}(\mathcal{A})$ is not quite gauge-invariant, so 2.33 must be treated with care. That the generalization of 2.31 in the presence of a boundary is exactly 2.33 is explained in [4].

One can write $\mathrm{CS}(\mathcal{A})$ explicitly as a function of A, ϕ and w, giving us

$$\mathrm{CS}(\mathcal{A}) = \mathrm{CS}(A) + \frac{1}{4\pi} \int_{\partial V} \mathrm{d}^3x \epsilon^{\mu\nu\lambda} \mathrm{Tr}(w\phi_\mu F_{\nu\lambda} + w^2 \phi_\mu D_\nu \phi_\lambda + \frac{2w^3}{3} \phi_\mu \phi_\nu \phi_\lambda). \tag{2.34}$$

2.5   Localization of the path integral

As described in the introduction, this path integral can now be localized on configurations that satisfy $\{Q,\zeta\} = 0$ for all fermion fields ζ. The fermion fields in this theory are one self-dual two-form χ^+, one anti-self dual two form χ^- and a scalar η, all adjoint-valued. For these fields, it is true that

$$
\begin{aligned}
\{Q, \chi^+\} &= (F - \phi \wedge \phi + t\mathrm{d}_A\phi)^+ \\
\{Q, \chi^-\} &= (F - \phi \wedge \phi - t^{-1}\mathrm{d}_A\phi)^- \\
\{Q, \eta\} &= D_\mu\phi^\mu
\end{aligned}
\tag{2.35}
$$

The localization equations thus become

$$
\begin{aligned}
(F - \phi \wedge \phi + t\mathrm{d}_A\phi)^+ = (F - \phi \wedge \phi - t^{-1}\mathrm{d}_A\phi)^- = D_\mu\phi^\mu &= 0 \\
D_\mu\sigma = [\phi_\mu, \sigma] = [\sigma, \bar{\sigma}] &= 0
\end{aligned}
\tag{2.36}
$$

These localization equations are a set of elliptical differential equations on the manifold V such that $\partial V = W$. Under favorable conditions, the last row of the equations imply $\sigma \equiv 0$. [4].

However, instead of solving these equations here already, one now preforms an S-duality, which turns out to simplify them considerably.

2.6   S-duality

We now apply EM-duality to $\mathcal{N} = 4$ super Yang-Mills on $V = W \times \mathbb{R}^+$. The gauge group then transforms as $G \to G^V$, where G^V is the Langlands dual group. As before, this new G^V gauge theory will have a theta-angle and gauge coupling (denoted θ^V and g_{YM}^V), where we again can define:

$$
\tau^V = \frac{\theta^V}{2\pi} + \frac{4\pi i}{(g_{YM}^V)^2}.
\tag{2.37}
$$

The family of topological field theories that is relevant here is mapped to itself by electro-magnetic duality. The twisting parameter of the dual description, t^V is related to the twisting parameter in the original description by:

$$
t^V = \pm\frac{\tau}{|\tau|}t.
\tag{2.38}
$$

For the D3-NS5-system, we have $t^2 = \frac{\bar{\tau}}{\tau}$ which leads to

$$t^V = \pm 1. \tag{2.39}$$

This sign is however uninteresting. This means that the localization equations in G^V become surprisingly simple, giving us

$$F - \phi \wedge \phi + \star_4 \mathrm{d}_A \phi = 0 = \mathrm{d}_A \star_4 \phi. \tag{2.40}$$

These equations were treated for the case $W = S^3$ in [5]. It is worth noting that when we S-dualize, the Wilson operators becomes 't Hooft operators.

We now need to see how the boundary conditions away from the 't Hooft operators look in the dualized G^V gauge theory. After S-dualization, the D3-NS5-system goes into a D3-D5 system, and these are now the required boundary conditions. These are described by specifying the singular behavior of the fields near the boundary. They turn out to actually have a half-BPS-symmetry. On the boundary, the gauge fields A will be the connection induced by the metric (for example, if W is flat, A will vanish). The normal part of the one-form ϕ vanishes on the boundary. Because of rotational- and translation invariance, it is known that ϕ can only be a function of $x^3 = y$ on the boundary. By considering how 2.40 looks under these assumptions, it can be shown, together with requiring conformal invariance of the D3-D5 boundary condition that the remaining parts of ϕ, that is, the one form on W, which is here denoted $\vec{\phi}$ will satisfy:

$$\vec{\phi} = \frac{\vec{e}}{y} + \dots \tag{2.41}$$

where $\vec{e}$ is the vielbein. Thus $\vec{\phi}$ has a regular Nahm-pole at infinity.

We will herein only consider the solutions far away from the t'Hooft operators. [9], [4].

Chapter 3

Four-Dimensional Equations

In the last chapter it was shown that the localization equations of the four-dimensional S-dualized topologically twisted theory takes the form:

$$F - \phi \wedge \phi + \star_4 \mathrm{d}_A \phi = 0 = \mathrm{d}_A \star_4 \phi$$
$$D_\mu \sigma = [\phi_\mu, \sigma] = [\sigma, \bar{\sigma}] = 0 \tag{3.1}$$

The $\star_4$ here denotes the Hodge operator in the four-dimensional sense. The second set of equations forces σ to vanish everywhere under favorable conditions, and thus will henceforth not be considered. In [5], these were studied on the four-manifold $V = W \times I$ where I is an interval parametrized by the coordinate $x^3 = y$.

We now make the gauge choice $A_y = 0$. Equation 3.1 together with suitable boundary conditions gives us $\phi_y = 0$. This will simplify equation 3.1 to:

$$\partial_y A = \star_3 (\mathrm{d}_A \phi)$$
$$\partial_y \phi = \star_3 (F - \phi \wedge \phi)$$
$$\partial_A (\star_3 \phi) = 0. \tag{3.2}$$

The $\star_3$ here represents the Hodge operator in the three-dimensional sense. By aid of the Bianchi-identity together with the top two equations, it can be shown that the last one of these is identically satisfied if it is satisfied for some value of y. Thus we do not consider this equation any further.

In general, these equations have an infinite-dimensional space of solutions, but a boundary with half-BPS boundary conditions will define a "middle-dimensional" solution set. Here we have two boundaries, (one at $y = 0$ and one at either infinity or at finite distance) which means they each define a middle-dimensional solution set. The intersection of these sets will in general be given by a discrete set of solutions. This was first stated in [4] and then confirmed in [5].

3.1 Boundary Conditions at Infinite Distances

We now consider what kind of boundary conditions apply at infinite distances. In [5], the boundary condition that was considered was that $A + i\phi \to \rho$ as $y \to \infty$ where ρ is a flat connection on the complexification $E_{\mathbb{C}}$ of the gauge bundle E. Let $\rho = \rho_1 + i\rho_2$ where ρ_1, ρ_2 denotes the real respectively imaginary part of ρ. Expanding A and ϕ around these gives us:

$$A = \rho_1 + a$$
$$\phi = \rho_2 + \varphi \tag{3.3}$$

where

$$\begin{pmatrix} a \\ \varphi \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{as} \ \ y \to \infty. \tag{3.4}$$

3.2 Special properties of the case $W = S^3$

It is here important to know that the three-sphere also can be considered as a group. This can be done since S^3 is isomorfic to $SU(2)$, which will be shown below.

If we consider $SU(2) = \left\{ g \in \mathbb{C}^{2 \times 2} : \det(g) = 1, g^{\dagger} g = \mathbb{1} \right\}$, one can let $g = x^0 \mathbb{1} + i\vec{x} \cdot \vec{\sigma}$ where $\vec{x} = (x^1, x^2, x^3)$ and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. The condition

$$g^{\dagger} g = \mathbb{1} \tag{3.5}$$

then gives us that

$$|x^0|^2 + \vec{x}^{\dagger} \cdot \vec{x} = 1. \tag{3.6}$$

Furthermore, $\det(g) = 1$ gives us that

$$X^{\dagger} \cdot X = 1 \tag{3.7}$$

where $X = (x^0, x^1, x^2, x^3)$. In order for these two to be compatible, X must be real since otherwise the triangle inequality would not be satisfied ($|X^{\dagger} \cdot X| \leq |x^0|^2 + \vec{x}^{\dagger} \cdot \vec{x}$). However, S^3 can be parametrized by $X^{\dagger} \cdot X = 1$ when embedded in $\mathbb{R}^4$. Thus $SU(2) \simeq S^3$.

If we endow S^3 with the standard round metric, we can define the Maurer-Cartan form

$$e = g^{-1} \mathrm{d}g. \tag{3.8}$$

This can be seen as a one-form on S^3 with values in the Lie algebra of $SU(2)$, and hence as relating one direction in space to one direction in the Lie algebra. It will then satisfy

$$de = -e \wedge e$$
$$\omega = \frac{1}{2}e$$
$$\star_3(e \wedge e) = e \tag{3.9}$$

where ω is the connection on the sphere induced by the metric.

3.3   Boundary Conditions at Finite Distance

At finite distance, we instead have the boundary conditions

$$A = w + a$$
$$\phi = y^{-1}e + \varphi \tag{3.10}$$

where $a, \varphi \in \Omega^1(W, \mathrm{ad}(E))$. These are shown to give a middle-dimensional solution space in [5], if the boundary conditions at $y = 0$ are as below.

$$a = \mathcal{O}(y)$$
$$\varphi = \mathcal{O}(y) \tag{3.11}$$

3.4   Spherically Symmetric Solutions

We can now make the maximally symmetric ansatz

$$A = \frac{1}{2}(1 + u)e$$
$$\phi = se \tag{3.12}$$

where u, s are functions of only y. Thus equations 3.2 reduce to differential equations for u and s as follows:

$$\partial_y u = 2su$$
$$\partial_y s = \frac{1}{4}u^2 - s^2 - \frac{1}{4}. \tag{3.13}$$

We have two critical points in the $u - s$ plane, namely $s = 0, u = \pm 1$, which correspond to the trivial configurations $\phi = 0, A = 0$ and $\phi = 0, A = e$ respectively. In [5], it is stated that these are related to each other by a gauge transformation, thus being gauge
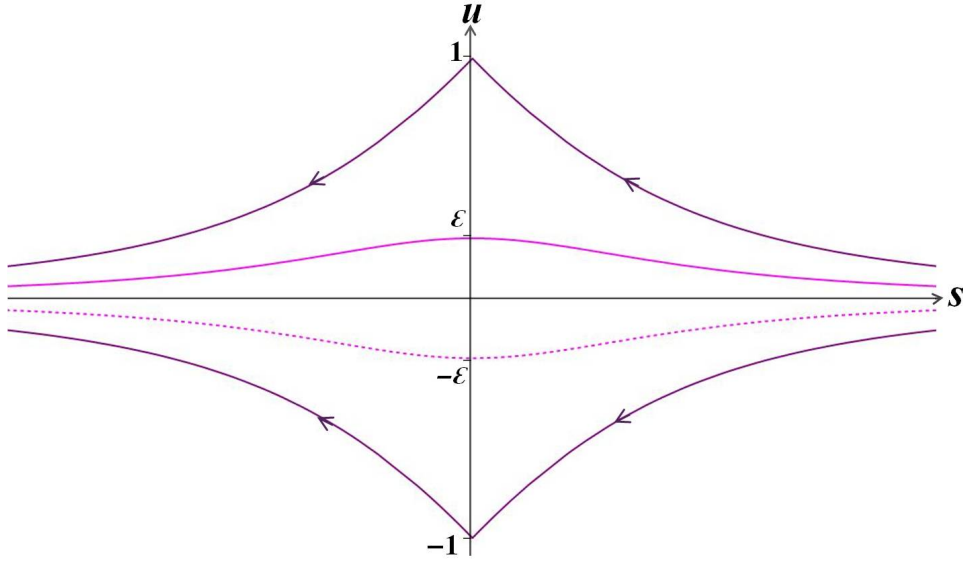
Figure 3.1: The flow in the $s - u$-plane, where arrows denote direction of increasing y. Only the interior of the space defined by the solutions flowing to or from the critical points $s = 0$, $u = \pm 1$ will satisfy our boundary conditions. Solutions here will appear pairwise by a solution parametrized by ε and its reflection through the s-axis.

equivalent. These fullfill the boundary conditions at $y \to \pm\infty$. These points can be viewed in the figure 3.1.

However, we cannot find solutions explicitly for all configurations of $V = I \times S^3$. Let Δy denote the interval length. This will then be related to a parameter denoted by ε (note that this has nothing whatsoever to do with the supersymmetry parameters of the theory) by defining ε as the value of u at $s = 0$. This is shown in figure 3.1. In the interior of the region bordered by the solutions flowing to and from the critical points $s = 0$, $u = \pm 1$, we have solutions that satisfy our boundary conditions. These will be uniquely determined by ε, and will all have a similar bell-shaped form as in figure 3.1. For a fixed value of the interval length, we will have a pair of solutions, which will be related by reflection through the s-axis. This can also be viewed as letting $\varepsilon \to -\varepsilon$. As ε approaches 1, the solutions will approach the constant solution at the critical point $s = 0, u = 1$, and this is the solution that corresponds to $I = \mathbb{R}^+$. The solution given by

$$u = 0$$
$$s = -\frac{1}{2} \tan \frac{y}{2} \tag{3.14}$$

corresponds to the critical interval length $\Delta y = \Delta y_{crit} = 2\pi$.

In [5], it was also there stated that for $\Delta y > \Delta y_{crit}$, solutions will appear pairwise related by a reflection in the u-axis, but these solutions are not gauge equivalent. This

poses an interesting question as to why this is the case, especially since they disappear at $\Delta y < \Delta y_{crit}$. Thus we expect them to be connected by a tunneling instanton solution of a set of five-dimensional localization equations obtained by T-dualizing the four dimensional theory. How to obtain these equations has been described in [4]. This will later in this text be shown to actually be the case, thus explaining the occurrence of pairwise, non-gauge equivalent solutions.

However, first it is interesting to see what the solutions for $\Delta y > \Delta y_{crit}$ look like. In this thesis, we will obtain this by a series expansion around the exact solution in 3.14 in the parameter ε. That parameter will later on be related to the interval length.

One can thus make the ansatz that

$$
\begin{aligned}
u &= 0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3) \\
s &= s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \mathcal{O}(\varepsilon^3)
\end{aligned}
\tag{3.15}
$$

where u_0, s_0 are given by 3.14.

One finds the functions u_1, u_2, s_1, s_2 etc by solving (3.13) order by order in ε. It is easy to see that u must be a odd function of ε and similarly s must be an even function of ε. Thus reducing our ansatz to

$$
\begin{aligned}
u &= 0 + \varepsilon u_1 + \varepsilon^3 u_3 + \mathcal{O}(\varepsilon^5) \\
s &= s_0 + \varepsilon^2 s_2 + \varepsilon^4 s_4 + \mathcal{O}(\varepsilon^6).
\end{aligned}
\tag{3.16}
$$

Herein, u, s has been found up to order 4 in powers of ε. To exemplify the approach, the calculations required to find u_1 will be performed below, but for the others only the result will be given since the calculations are straight-forward and follows the same pattern as for u_1.

To find u_1, we consider the $\partial_y u$-equation of 3.13 to first order in ε, thus giving us

$$
\partial_y u_1 = 2 s_0 u_1 = -\tan \frac{y}{2} u_1,
$$

which leads to

$$
u_1(y) = \cos^2 \frac{y}{2} u_1(0),
\tag{3.17}
$$

where $u_1(0)$ is a constant of integration which must be 1 by requiring that $u(y) = \varepsilon$ when $s(y) = 0$. The differential equations do become more complicated for higher powers of ε, but will always be ordinary differential equations with only one unknown and can thus always be solved by the method of integrating factor, at least numerically.

Solving these gives us:

$$u_1(y) = \cos^2\frac{y}{2}$$

$$u_3(y) = \frac{1}{96}\sin y(15y + (8 + \cos y)\sin y)$$

$$s_2(y) = \frac{5}{32}\left(\frac{y}{2}\cos^{-2}\frac{y}{2} + \sin\frac{y}{2}\cos^{-1}\frac{y}{2}\right) + \frac{1}{192}\left(20\cos\frac{y}{2}\sin\frac{y}{2} + 16\cos^3\frac{y}{2}\sin\frac{y}{2}\right)$$

$$s_4(y) = \frac{1}{18432}\left(427\sin y - 3\sin 2y - 24\sin 3y - \sin 4y + 645\tan\frac{y}{2} - \right.$$

$$\left. \frac{15y\left(31 + 32\cos y + 22\cos 2y + 4\cos 3y + 30y\tan\frac{y}{2}\right)}{1 + \cos y}\right). \tag{3.18}$$

These have later been used in order to find the solution to the time-dependent, five dimensional equations in the next chapter.

3.5 The Relationship Between ε and the Interval Length

Here, the expansion parameter ε will be related to the interval length Δy. This can be done, again to any desired order in ε, but has here only been done to the order ε^2.

$$\Delta y = -\int dy = -\int_{-\infty}^{\infty}(\partial_y s)^{-1}\,ds = \int_{-\infty}^{\infty}\frac{ds}{\frac{1}{4} + s^2 - \frac{1}{4}u^2} \tag{3.19}$$

where we have used the expression for $\partial_y s$ in 3.13. However, one now wishes to express u in terms of s. To do this, we first recall that

$$u(y) = \varepsilon\cos^2\frac{y}{2} + \mathcal{O}(\varepsilon^3)$$

$$s(y) = -\frac{1}{2}\tan\frac{y}{2} + \mathcal{O}(\varepsilon^2), \tag{3.20}$$

giving

$$y(s) = -2\arctan(2s + \mathcal{O}(\varepsilon^2)) = -2\arctan(2s) + \mathcal{O}(\varepsilon^2). \tag{3.21}$$

Inserted in the expression for $u(y)$ one then obtains

$$u(s) = \frac{\varepsilon}{1 + 4s^2} + \mathcal{O}(\varepsilon^3), \tag{3.22}$$

resulting in

$$\Delta y = \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{\frac{1}{4} + s^2 - \frac{1}{4}(\frac{\varepsilon}{1+4s^2})^2 + \mathcal{O}(\varepsilon^4)} = 2\pi + \varepsilon^2 \frac{5\pi}{8} + \mathcal{O}(\varepsilon^4). \qquad (3.23)$$

It should be noted that this interval length will now be considered "fixed" for a certain value of ε and that it will not be changed by the time-dependent solutions that will be considered in section 4.

Chapter 4

Five-Dimensional Equations

We now wish to consider the problem from a five-dimensional viewpoint instead. This can be done in several ways.

One way is to simply perform a lift from four to five dimensions, that is, claiming that the four-dimensional theory is the theory obtained from considering a five-dimensional maximally supersymmetric Yang-Mills theory compactified on a circle. This new coordinate introduced here can be thought of as a time-coordinate. In this way, the boundary conditions of the theory can as well be lifted in a straight forward way, giving us that three of the scalar fields have the singular behavior at the boundary earlier described. This can be interpreted as T-dualizing the theory from a D-brane point of view.

However, the maximally supersymmetric five-dimensional Yang-Mills theory is not ultraviolet complete. It does, on the other hand, have a canonical completion in six dimensions in the form of the (0,2)-theory. Thus this formulation will be more convenient for our purposes.

The basic idea here is to obtain a five-dimensional version of our four-dimensional theory by twisting of the six-dimensional (0,2) superconformal field theory associated to a simple and simply-laced Lie group G. More specifically, one wishes to identify the localization equations of the five-dimensional topological field theory on the five-manifold $\mathbb{R} \times V$, (i.e. the equations describing a supersymmetric field configuration). By formulating the five-dimensional maximally symmetric Yang-Mills theory in terms of a dimensionally reduced theory from ten dimensions it was shown in [4] that one arrives at the five-dimensional localization equations:

$$-F^+ + \frac{1}{4}B \times B - \frac{1}{2}D_y B = 0$$
$$F_{y\mu} + D^\nu B_{\nu\mu} = 0 \tag{4.1}$$

together with

$$\sigma = 0. \tag{4.2}$$

B is a self-dual two-form related to ϕ by

$$B_{0i} = \phi_i, \ B_{ij} = \epsilon_{ijk}\phi_k, \ i,j,k = 1,...3. \tag{4.3}$$

This can be done since ϕ here will be a self-dual two form on V with values in the adjoint bundle ad(E) (derived from the G^V bundle $E \to \mathbb{R} \times W \times I$). Hence we can use this to define an anti-symmetric tensor field B as above (equation 4.3). We can also define a cross-product operation on B:

$$(B \times B)_{\mu\nu} = \sum_{\tau} [B_{\mu\tau}, B_{\nu\tau}]. \tag{4.4}$$

The right hand side is here the commutator in the Lie algebra and is self-dual in B if B is, thus $B \times B$ is also a self-dual two form with values in ad(E). Furthermore, F^+ is here the selfdual projection of the two-form F (on $\mathbb{R} \times W$), defined by $F^+ = (1 + \star_4)F/2$, with the Hodge star defined in the four-dimensional sense $(\star(\mathrm{d}x^0 \wedge \mathrm{d}x^1) = \mathrm{d}x^2 \wedge \mathrm{d}x^3)$.

These are now considered on the manifold $\mathbb{R} \times W \times I$. In equation 4.1, one can make the gauge choice $A_0 = 0$. In this notation, the time component of the gauge field is also equal to the component along the interval I of the one form ϕ, so this as well vanishes in this gauge. We denote the remaining fields as $A_{1...3}, \phi_{1...3}, A_y = A_4 = \chi$. The indices 1...3 are denoted by Latin letters. Indexes running from 0...3 are denoted with Greek letters. The $0^t h$ direction is a time-like dimension, and will thus be denoted by t.

First, let us now consider the 0,i component of the first one of the equations in 4.1:

$$\frac{1}{2}(F_{0i} + \epsilon_{ijk}F_{ij}) - \frac{1}{4}(B \times B)_{0i} - \frac{1}{2}(\partial_y B_{0i} + [A_y, B_{0i}]) = 0 \tag{4.5}$$

where we have

$$(B \times B)_{0i} = \sum_{\tau}[B_{0,\tau}, B_{i\tau}] = \sum_{j}[\phi_j, \phi_k]\epsilon_{ijk} = \star_3(\phi \wedge \phi). \tag{4.6}$$

Thus this results in that 4.5 can be written as

$$\partial_t A = \star_3(F - \phi \wedge \phi) + \mathrm{D}_{\mu}\phi^{\mu}. \tag{4.7}$$

Now consider the $\mu = 0$-part of the second of the equations 4.1.

$$F_{y0} + D^{\nu}B_{\nu 0} = 0$$

which leads to

$$\partial_t \chi = -\partial^i \phi_i + [A_i, \phi_i]. \tag{4.8}$$

Similarly, the $\mu = i$ part can be written as:

$$F_{yi} + D^{\nu}B_{\nu i} = 0$$

giving

$$\partial_y A_i - \partial_i \chi + [\chi, A_i] + \mathrm{D}^0 B_{0i} + \mathrm{D}^j B_{ji} = 0$$

which leads to

$$\partial_y A_i - \mathrm{d}_A \chi + \partial_t \phi_i + \mathrm{d}_j \epsilon_{ijk} \phi_k = 0$$

and then one finally obtains

$$\partial_t \phi = \star_3 \mathrm{d}_A \phi - F_y. \tag{4.9}$$

Hence in the $A_0 = 0$-gauge, the five-dimensional localization equations can be written as

$$\begin{aligned}
\partial_t A &= - \star_3 (F - \phi \wedge \phi) + d_\chi \phi \\
\partial_t \phi &= \star_3 d_A \phi - F_y \\
\partial_t \chi &= - (\star_3 (d_A \star_3 \phi)) .
\end{aligned} \tag{4.10}$$

This is the general appearance of these equations, since we in five dimensions do not have a vanishing theorem that sets $\chi \equiv 0$ However, it is here worth noting that χ is a 0-form with values in the Lie algebra. This will aid when specializing to the spherically symmetric case.

4.1 The special case $W = S^3$

We will now consider spherically symmetric solutions of the equations 4.10 in the case where $W = S^3$. This is interesting because it might shed some light on the phenomenon that pairwise, non-gauge equivalent solutions occur in the four-dimensional equations. Thus our goal here is to find a solution in five dimensions which interpolates between the solution pairs in four dimensions. However, there is no spherically symmetric, Lie algebra valued non-vanishing one-form so this means that we in this case will have $\chi \equiv 0$. This will simplify the equations significantly, giving us:

$$\begin{aligned}
\partial_t A &= - \star_3 (F - \phi \wedge \phi) + \partial_y \phi \\
\partial_t \phi &= \star_3 d_A \phi - \partial_y A \\
\chi &\equiv 0.
\end{aligned} \tag{4.11}$$

Note that the right hand side in these equations is exactly the four-dimensional equations. Thus for time-independent solutions, the four-dimensional equations will be satisfied.

In the special case of spherical symmetry, we have, as in four dimensions

$$A = \frac{1}{2}(1+u)e$$
$$\phi = se \tag{4.12}$$

where u and s are functions of y,t, and e is the Maurer-Cartan form satisfying equation 3.9. By inserting this into 4.11 we obtain

$$\partial_t u = 2\partial_y s - \frac{1}{2}u^2 + 2s^2 + \frac{1}{2}$$
$$\partial_t s = -\frac{1}{2}\partial_y u + su. \tag{4.13}$$

In the four dimensional case, we saw that $s(y)$ only contains even powers of ε, and similarly, the function $u(y)$ only contains odd powers of ε. It stands to reason that this should be the case even in the time-dependant situation, which means that, in the five-dimensional case we cannot simply compare the equations order by order. Thus we here must make an important variable transformation, that is

$$t \to \tau = \varepsilon t \tag{4.14}$$

where ε is the parameter of the expansion around the exact solution for critical interval length. This will then effect the time-derivative terms in 4.15, and will give us the following equations for s and u as functions of y and τ.

$$\varepsilon\partial_\tau u = 2\partial_y s - \frac{1}{2}u^2 + 2s^2 + \frac{1}{2}$$
$$\varepsilon\partial_\tau s = -\frac{1}{2}\partial_y u + su. \tag{4.15}$$

We can now solve the equations order by order in ε .

We search for solutions on the form

$$s(y,\tau) = s_0(y) + \varepsilon^2(\tilde{s}_2(y) + d_2(y,\tau)) + \varepsilon^4(\tilde{s}_4(y) + d_4(y,\tau)) + \mathcal{O}(\varepsilon^6)$$
$$u(y,\tau) = \varepsilon u_1(y,\tau) + \varepsilon^3 u_3(y,\tau) + \mathcal{O}(\varepsilon^5) \tag{4.16}$$

where $\tilde{s}(y)$ denotes the stationary part (the four dimensional solution), and $d(y,\tau)$ the dynamic part.

The Boundary Conditions

We now need to consider what boundary conditions will be imposed on u and s respectively. When we go from four- to five dimensions, the boundary conditions in the y-direction does not change. This means that we still require the $s \xrightarrow[y\to y_0]{} \infty$ and $s \xrightarrow[y\to y_1]{} -\infty$, together with the fact that $u = 0$ at the interval ends. y_0, y_1 here denote the interval endpoints.

In the time-direction though, our boundary conditions can be stated as requiring the five-dimensional solutions to approach the four-dimensional as $t \to \pm\infty$. This is satisfied by requiring that the dynamic part vanishes as $t \to \pm\infty$. Furthermore, the dynamic part of the solution cannot have any poles since any occurrence of poles herein would result in a distortion of the interval length, which must be fixed for fixed value of ε.

Moreover, the appearance of the four-dimensional solutions leads us to believe it is reasonable to search for solutions satisfying $s(y,\tau) = -s(-y,\tau)$ and $u(y,\tau) = u(-y,\tau)$.

Solving the Equations Order by Order

Thus if we consider the equations in order ε we obtain

$$0 = s_0(y)u_1(y,\tau) - \partial_y u_1(y,\tau).$$

which if we recall that $s_0(y) = -\frac{1}{2}\tan(\frac{y}{2})$ gives us

$$u_1(y,\tau) = \cos^2(\frac{y}{2})\mathrm{T}_1(\tau). \tag{4.17}$$

Here, T_1 is an integration constant that is allowed to depend on the transformed time, τ. In order to find this time-dependence, we go to order ε^2:

$$\cos^2(\frac{y}{2})\partial_\tau \mathrm{T}_1 = 2\partial_y(\tilde{s}_2 + d_2) - \frac{1}{2}\cos^4(\frac{y}{2})\mathrm{T}_1^2 + 4(s_0(\tilde{s}_2 + d_2)). \tag{4.18}$$

By using the fact known from the four dimensional equations, namely that $2(\partial_y \tilde{s}_2 - \tan(\frac{y}{2})\tilde{s}_2) = \frac{1}{2}\tilde{u}_1^2$, where once again the $\tilde{\ }$ denotes the four dimensional solution (stationary solution) we obtain a non-linear partial differential equation. However, if this PDE is considered for a fixed value of τ we obtain an ordinary differential equation in y, and we can thus solve it in the standard manner of multiplication with integrating factor.

$$\partial_y d_2 - \tan(\frac{y}{2})d_2 = \frac{1}{2}\cos^2(\frac{y}{2})(\partial_\tau \mathrm{T}_1 + \frac{1}{2}\cos^2(\frac{y}{2})(\mathrm{T}_1^2 - 1))$$

which leads to

$$d_2 = \frac{1}{2}\left(\partial_\tau(\mathrm{T}_1) + \frac{5}{12}(\mathrm{T}_1^2 - 1)\right)\cos^{-2}(\frac{y}{2})\int \cos^4(\frac{y}{2})dy + \frac{1}{12}\sin(\frac{y}{2})\cos^3(\frac{y}{2}). \tag{4.19}$$

The singular terms in d_2 will be a result of the integral above, which will give rise to poles of first- as well as second order. Thus in order to fulfill the regularity assumption of d_2, we must require that, for all values of τ we have

$$\partial_\tau(\mathrm{T}_1) + \frac{5}{12}(\mathrm{T}_1^2 - 1) = 0$$

giving

$$T_1 = \frac{e^{\frac{5\tau}{6}} - C}{e^{\frac{5\tau}{6}} + C}. \tag{4.20}$$

However, we can here choose C to be 1 without loss of generality since it is simply a consequence of translations in τ. This gives us:

$$T_1 = \tanh(\frac{5\tau}{12})$$
$$u_1 = \cos^2(\frac{y}{2})T_1(\tau) \qquad\qquad = \cos^2(\frac{y}{2})\tanh(\frac{5\tau}{12}). \tag{4.21}$$

If we insert this into 4.19 we obtain:

$$T_2 = \cosh^{-2}(\frac{5\tau}{12})$$
$$d_2(y,\tau) = -\frac{1}{12}\cosh^{-2}(\frac{5\tau}{12})\sin(\frac{y}{2})\cos^3(\frac{y}{2}). \tag{4.22}$$

It is clear here that this time-dependent solution interpolates between the two four-dimensional solutions with parameter $\pm\varepsilon$, since we see that

$$u_1(y,\tau) \xrightarrow{\tau\to\pm\infty} \pm\tilde{u}_1(y) \tag{4.23}$$
$$s_2(y,\tau) = \tilde{s}_2(y) + d_2(y,\tau) \xrightarrow{\tau\to\pm\infty} \tilde{s}_2(y).$$

Thus this is the desired time-dependent solution to lowest order in ε. This is the general process for obtaining the time-dependent s and u functions to any order: By first finding the structure of the u_{2n-1} by using the equation for order ε^{2n-1}, and then inserting this solution into the equation for order ε^{2n} together with the knowledge that the stationary parts of s must fulfill the four dimensional equations, one will obtain an ordinary differential equation for d_{2n} for all values of τ. However, in general, this will have double poles, which needs to be canceled by the regularity condition of d_{2n}. Imposing this will result in an ordinary differential equation for t_{2n-1}, which when solved will give both the full expression for u_{2n-1} and s_{2n}. During this process, several integration constants will appear, but in cases where these must not vanish by the parity of u and s, they turn out to simply be the result of the possibility of translation in τ and thus can be chosen to 1. In this manner, it is possible to find the time-dependence recursively to any desired order in ε.

In this work, the explicit form of the ε^3 and ε^4-terms have also been determined. This was done by the same procedure.

In order to facilitate the calculations, a change of variable from y to x has been performed. This will have no profound consequences but is simply a matter of convenience. They are related by:

$$x = \sin\frac{y}{2}. \tag{4.24}$$

By solving the ε^3-part of 4.15, one obtains

$$u_3(y,\tau) = \left(1 - x^2\right)$$
$$\left(\frac{1}{72}\mathrm{T}_1 x \left((27 - 22\mathrm{T}_3)x + 6(-1 + \mathrm{T}_2)x^3 + \frac{45\arcsin x}{\sqrt{1 - x^2}}\right) + \mathrm{T}_3(\tau)\right). \qquad (4.25)$$

As before, T_1 and T_2 simply denote the τ-dependent factors in u_1 and s_2 respectively. The τ-dependence of T_3 is explicitly shown here in order to emphasize that that factor is still an unknown as opposed to T_1 or T_2.

By then solving the ε^4-part of 4.15 as a ordinary differential equation of y for a constant value of τ, one finds the general expression of the time-dependent part of $s_2(x, \tau)$ as

$$d_4(x,\tau) = \frac{1}{207360(1 - x^2)}D_4(x, \tau), \qquad (4.26)$$

where D_4 is an expression containing factors of $\sqrt{1 - x^2}$ and $\arcsin x$ multiplying polynomials in x where the coefficients contain $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ and $\partial_\tau \mathrm{T}_3$. This has here been written below in order to facilitate for the reader as much as possible without loosing any information.

$$D_4(x, \tau) = x\sqrt{1 - x^2}\left(c_0 - c_2 x^2 + c_4 x^4 - c_6 x^6 + c_8 x^8\right) +$$
$$15\left(\tilde{c}_0 + \tilde{c}_2 x^2 - \tilde{c}_4 x^4 + \tilde{c}_6 x^6\right)\arcsin(x) +$$
$$4320\left(\mathrm{T}_1\left(x\sqrt{1 - x^2}\left(33 - 26x^2 + 8x^4\right) + 15\arcsin(x)\right)\mathrm{T}_3(\tau) +$$
$$6\left(x\left(5 - 2x^2\right)\sqrt{1 - x^2} + 3\arcsin(x)\right)\mathrm{T}_3'(\tau)\right) \qquad (4.27)$$

where

$$c_0 = -15\left(801 + \mathrm{T}_1^2(-801 + 46\mathrm{T}_2) - \mathrm{T}_2(477 + 104\mathrm{T}_2)\right)$$
$$c_2 = 10\left(1233 + \mathrm{T}_1^2(-1233 + 398\mathrm{T}_2) + \mathrm{T}_2(-1269 + 872\mathrm{T}_2)\right)$$
$$c_4 = 8\left(3411 + 2\mathrm{T}_2(-891 + 458\mathrm{T}_2) + \mathrm{T}_1^2(-3411 + 1786\mathrm{T}_2)\right)$$
$$c_6 = 144\left(99 + 14(-2 + \mathrm{T}_2)\mathrm{T}_2 + \mathrm{T}_1^2(-99 + 74\mathrm{T}_2)\right)$$
$$c_8 = 288\left(6 + 6\mathrm{T}_1^2(-1 + \mathrm{T}_2) + (-2 + \mathrm{T}_2)\mathrm{T}_2\right) \qquad (4.28)$$

and

$$\tilde{c}_0 = 801 - 801\mathrm{T}_1^2 - 477\mathrm{T}_2 + 46\mathrm{T}_1^2\mathrm{T}_2 - 104\mathrm{T}_2^2$$
$$\tilde{c}_2 = 2160\left(-2 + 2\mathrm{T}_1^2 + \mathrm{T}_2\right)$$
$$\tilde{c}_4 = 1080\left(-4 + 4\mathrm{T}_1^2 + \mathrm{T}_2\right)$$
$$\tilde{c}_6 = 1440\left(-1 + \mathrm{T}_1^2\right). \qquad (4.29)$$

As can be seen above the general structure of D_4 is first one term with a factor of $x\sqrt{x}$ multiplying a polynomial of degree 8 in x with coefficients completely specified by the known time-dependence. Then we have one term which is polynomial in degree 6 in x (again with coefficients totally specified by known time-dependence) multiplied by an $\arcsin(x)$-factor. Lastly we have the terms involving the unknown time-dependence, one consisting of a factor of $x\sqrt{x}$ times a polynomial of degree 4 in x, plus a term of $\arcsin(x)$, all multiplying the unknown time-dependence $T_3(\tau)$, and one consisting of a factor of $x\sqrt{x}$ times a polynomial of degree 2 in x, plus a term of $\arcsin(x)$ multiplying the τ-derivative of $T_3(\tau)$. This will always be the general structure of the $d_{2n}(x,\tau)$-piece of the solution, though the degrees of the polynomials will increase.

We can now impose the condition that this expression must be free of singularities, that is, $D4(x,\tau)$ must have a root at $x = \pm 1$ in order to prevent the possible poles that will arise there due to the denominator.

It is worth noting that we in the previous calculation, that is in order ε^2 had to worry about a potential double pole also appearing. However, the fact that we here instead have two single poles is a fact only due to that we have changed variables. The general solutions does thus not become less singular for each order, so this method is applicable to all orders of ε.

The non-singularity-requirement of d_4 gives us

$$\frac{15}{2}\pi\left(-639 + 639T_1^2 + 603T_2 + 46T_1^2T_2 - 104T_2^2 + 4320T_1T_3(\tau) + 5184T_3'(\tau)\right) = 0 \quad (4.30)$$

This condition makes sure the vanishing of both poles. It gives us:

$$T_3(\tau) = \cosh^{-2}\frac{5\tau}{12}\left(-\frac{5\tau}{2592} + \frac{5}{72}\tanh\frac{5\tau}{12}\right) \quad (4.31)$$

Giving us

$$
\begin{aligned}
u_3(x,\tau) = \left(1 - x^2\right)\Bigg(& -\frac{\text{sech}^2\frac{5\tau}{12}\left(5\tau + 36\left(-5 + 22x^2\right)\tanh\frac{5\tau}{12}\right)}{2592} \\
& + \frac{1}{24}x\tanh\frac{5\tau}{12}\left(9x + \frac{15\arcsin x}{\sqrt{1-x^2}} - 2x^3\tanh^2\frac{5\tau}{12}\right)\Bigg)
\end{aligned}
\quad (4.32)
$$

$$
\begin{aligned}
d_4(x,\tau) = -\frac{1}{31104}\left(-1 + x^2\right)\text{sech}^4\frac{5\tau}{12}\Bigg(& 3\left(x\sqrt{1-x^2}\left(-85 - 378x^2 + 72x^4\right)\right. \\
& - 135\left(-1 + 4x^2\right)\arcsin x\Big)\cosh\frac{5\tau}{6} - 5\left(3x\sqrt{1-x^2}\left(-19 + 18x^2\right)\right. \\
& + 81\left(-1 + 4x^2\right)\arcsin x + x\sqrt{1-x^2}\tau\sinh\frac{5\tau}{6}\Big)\Bigg) \quad .
\end{aligned}
$$

These can of course be expressed in terms of y again simply by using 4.24. This will then give us
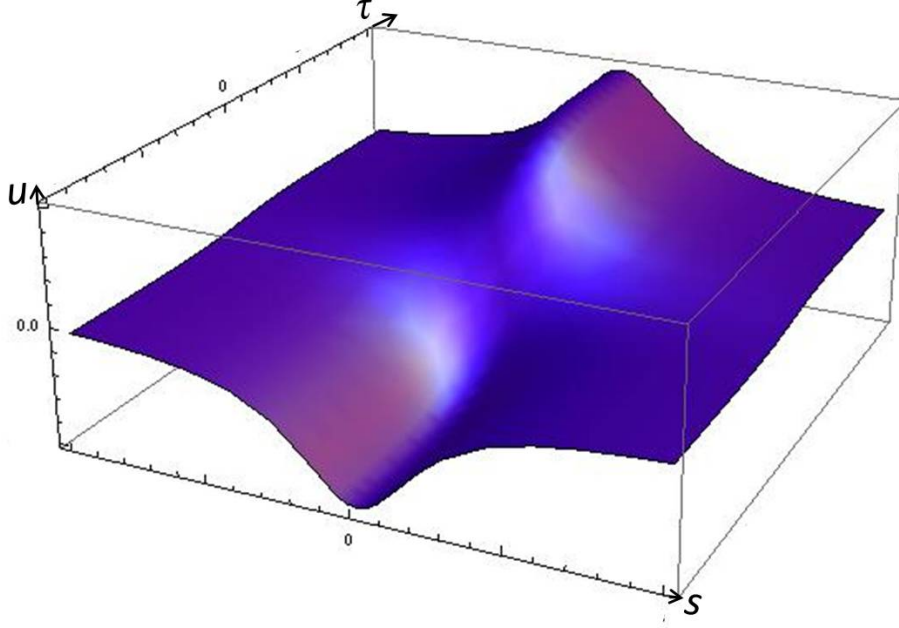
Figure 4.1: The time-dependent solution that interpolates between a stationary solution corresponding to $-\varepsilon$ and the stationary solution corresponding to ε. The direction inwards in the paper is the direction of increasing τ. The time-dependent solution can here clearly be seen as interpolating between the four-dimensional solutions.

$$
\begin{aligned}
u_3(y,\tau) = & -\frac{1}{2592}\cos^2\frac{y}{2}\left(\operatorname{sech}^2\frac{5\tau}{12}\left(5\tau - 36(-6 + 11\cos y)\,\tanh\frac{5\tau}{12}\right) + \right.\\
& \left. 54\tan\frac{y}{2}\,\tanh\frac{5\tau}{12}\left(-15y - 9\sin y + 2\sqrt{2}\sqrt{1+\cos y}\,\sin^3\frac{y}{2}\,\tanh^2\frac{5\tau}{12}\right)\right) \\
d_4(y,\tau) = & -\frac{1}{62208}\cos^2\frac{y}{2}\operatorname{sech}^4\frac{5\tau}{12}\left(3\cosh\frac{5\tau}{6}\left(-135y + 540y\sin^2\frac{y}{2}+\right.\right.\\
& 378\sqrt{2}\sqrt{1+\cos y}\,\sin^3\frac{y}{2} - 72\sqrt{2}\sqrt{1+\cos y}\,\sin^5\frac{y}{2}+ \\
& 85\sin y) + 5\left(-81y + 324y\sin^2\frac{y}{2} + 54\sqrt{2}\sqrt{1+\cos y}\,\sin^3\frac{y}{2} - 57\sin y + \tau\sin y\,\sinh\frac{5\tau}{6}\right)\bigg)\bigg)
\end{aligned}
$$

$$\tag{4.33}$$

These do satisfy the conditions that this time-dependent solution is an instanton tunneling between the two (non-gauge-equivalent) solutions in one "solution pair" as described in chapter 3 since we have

$$u_3(y,\tau) \xrightarrow{\ \tau \to \pm\infty\ } \pm \tilde{u}_3(y) \tag{4.34}$$

$$s_4(y,\tau) = \tilde{s}_4(y) + d_4(y\tau) \xrightarrow{\ \tau \to \pm\infty\ } \tilde{s}_4(y)$$

This can also be seen in figure 4.1. We can there see how our five-dimensional solution interpolates between the solutions in four dimensions. Thus we have now found an explicit form of:

$$u(y,\tau) = \varepsilon u_1(y, \tau) + \varepsilon^3 u_3(y, \tau) + \mathcal{O}(\varepsilon^5)$$
$$s(y,\tau) = s_0(y) + \varepsilon^2(\tilde{s}_2(y) + d_2(y, \tau)) + \varepsilon^4(\tilde{s}_4(y) + d_4(y, \tau)) + \mathcal{O}(\varepsilon^6) \tag{4.35}$$

where $\tilde{s}_2(y)$ and $\tilde{s}_4(y)$ are the time-independent functions as obtained in 3.18. $u_1(y, \tau)$ and $d_2(y, \tau)$ are given in 4.23, and $u_3(y, \tau)$ and $d_4(y, \tau)$ are the functions just obtained in 4.33

This method can be used to find higher ε-dependence as well, and can be done order by order to any desired order of ε. The variable y will be inconvenient for obtaining higher order ε-dependence. As seen here, a variable transformation to x was performed (4.24), but there may be other variable changes that could facilitate calculations further.

4.2   Outlook

A general feature here is that the maximally symmetric solutions disappear when the interval length is short enough. This is a feature of the equations that may be interesting to investigate for general three-manifolds W, since it may be a general property. Furthermore, the isolation of the solutions can also be investigated.

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