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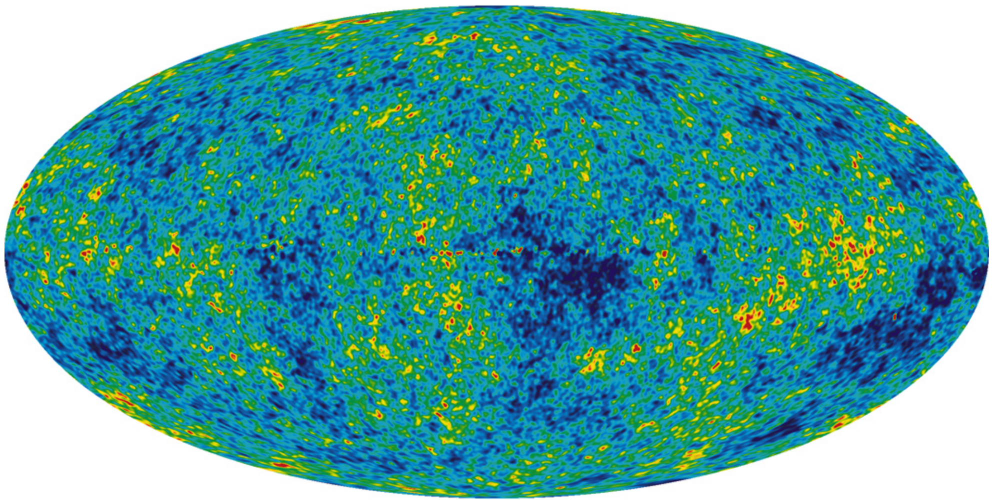

Big Bang Nucleosynthesis

A thesis for the degree Bachelor of Science

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Cover:
The cosmic microwave temperature fluctuations from the 5-year WMAP data seen over the full sky. The average temperature is 2.725 Kelvin, and the colors represent the tiny temperature fluctuations, as in a weather map. Red regions are warmer and blue regions are colder by about 0.0002 degrees.
Credit: NASA/WMAP Science Team

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SUMMARY
The fundamental physical processes that govern the Big Bang nucleosynthesis (BBN) have been studied. BBN refers to the production of predominantly light nuclei in the early Universe, which occurs on the time scale of a few minutes after the bang. An initial intensive literature study was carried out, followed by computer simulations with the scientific code NUC123.

The aim of the literature study was to build a theoretical basis from which observational support of BBN and key estimates of parameters could be understood, and in the case of the latter also reproduced. The emphasis has been placed on the time leading up to BBN, specifically the relation between time and temperature, the universal expansion and the baryon-to-photon ratio, in order to determine the onset of BBN.

Additionally, different simulations, based on models with varying degrees of complexity, have been performed in order to verify the theoretical work and the estimates of key parameters. By mass the most important abundances were found to be 75.2 % ^{1}H and 24.8 % ^{4}He with help of the NUC123 software.

These abundances were found to agree well with both observations and simulations referred to in literature. One important exception is ^{7}Li for which the calculated abundance differs significantly from the observational values. Even though the over all good agreement is a strong evidence for the standard models for both BBN and the Big Bang, this discrepancy points to shortcomings in the theory. Simply put, neither of these models can be completely wrong, though they do not paint the whole picture either.

Keywords: BBN, big bang nucleosynthesis, early Universe, nuc123, primordial nucleosynthesis.

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1 Introduction

Big Bang nucleosynthesis, often abbreviated BBN, refers to the network of nuclear reactions governing the formation of light elements, most significantly ^{2}H, ^{3}He, ^{4}He and ^{7}Li, in the early Universe [1]. More precisely, BBN is thought to begin 0.01 seconds after the big bang before coming to an end about 30 minutes thereafter [1]. It is also estimated that the rapidly expanding Universe, filled with a dense gas of particles and radiation, cooled from about 10^{11} to 10^9K during this time [1].

Remarkably, the primordial nucleosynthesis is one of the most easily simulated processes in the entire field of astrophysics [2]. As such, computational models of BBN yield results that are quite accurate compared to inherent errors in the observational and experimental data that are put into the equations [1, 2]. Many of the physical constants of importance for this process can be accurately measured in laboratories, because the relevant energy ranges are obtainable in a laboratory environment [1]. Consequently, modern BBN calculations for determining the abundances of light elements are carried out with only a single parameter, the baryon density [1].

These easily achievable precision calculations, under the assumption that the standard model of the Big Bang holds true, has and hopefully will help to shed light on both the preceding and following history of the Universe [1]. Indeed, "there are presently three observational evidences for the Big-Bang model: the universal expansion, the Cosmic Microwave Background (CMB) radiation and Primordial or Big-Bang Nucleosynthesis (BBN)" [3].

In 1929 Edwin Hubble and Milton Humason discovered that the velocity at which galaxies travel away from the earth is proportional to the distance between the earth and the galaxy. This means that the Universe is expanding, and it confirmed what Georges Lemaître had proposed two years earlier in his "hypothesis of the primeval atom" which later was termed the Big Bang theory. At present the Universe is large and cold, but because of the expansion we can extrapolate backwards to when the Universe was very hot and dense.

The idea of the primordial nucleosynthesis, that is the creation of nuclei before the galaxies were formed, first appeared in the 1940s in the work of Gamow and his collaborators [4]. Despite some errors with regards to the physics involved in the process, they were able to predict the existence of cosmic background radiation, which after it was discovered in 1965 gave essential evidence not only for BBN but the big bang model as a whole [4]. Since that time the subject has evolved significantly both with regard to the underlying theory and the computational models. During the last three decades BBN calculations has been able to determine the above mentioned baryon density with an unprecedented accuracy [1].

Further evidence of BBN, as a theory, comes from the fact that the ratio of ^{1}H and ^{4}He, predicted abundances of the light elements, ^{2}H, ^{3}He, ^{4}He and to a lesser extent ^{7}Li, agrees very well with observational measurements[1, 2]. This despite of the fact that these values spans nine orders of magnitude, since the ratio of the mass density of ^{7}Li to ^{4}He is in the order of 10^{-9} [1, 2].

Such comparisons, however, have relied heavily upon the contemporary understanding of the chemical evolution, that is the constant change in the chemical composition of matter because of nuclear transformation in for example stars [1]. This predicament stems from the fact that the abundance measurements can only be compared with the output from standard BBN calculations once they have been extrapolated to primordial abundances [1, 2, 3].

The situation has since changed entirely in light of new precise measurements of the CMB, which have been used to fix the baryon density [1, 3]. Thus, the last unknown in BBN calculations has been deduced, which in turn determines the primeval abundances of the light elements. Therefore, it is now possible to use these exact calculations to research the chemical evolution that has since taken place [1, 2]. Even so, it should be noted that the discrepancy between ^{7}Li abundances as calculated with BBN and the observed values remains quite large [2, 3]. While there exists many suggested explanations for this result, no clear solution to this problem has emerged as of yet [2, 3].

The success of the standard model for BBN has enabled it to be used as a tool for probing new physics, such as alternative theories of gravity or the existence of new light particle species [1, 2]. For instance, calculations on primordial nucleosynthesis used to give the best possible constraint on the number of neutrino flavors, before being overtaken by precise laboratory measurements in the late 1980's [1]. However, now that the baryon density has been fixed it would be possible, if the uncertainties in determination of the ^{4}He abundance can somehow be reduced, for BBN calculations to put a comparable limit on the number of neutrino flavors, thereby cooperating with laboratory experiments to put bounds on new physics [1]. This prospect serves to exemplify how the BBN theory will continue to nurture the bond that it had previously helped forge between cosmology and nuclear and particle physics [1].

With regards to the amount of time and resources that is put into researching the big bang and its implications, it is apparent that the interest for these events within the scientific community is quite substantial. Moreover, the diverse stories of creation that appears in scripture are a testimony to the fact that the origin of humanity has been an ever present subject within the minds of scholars and philosophers for thousands of years. With-

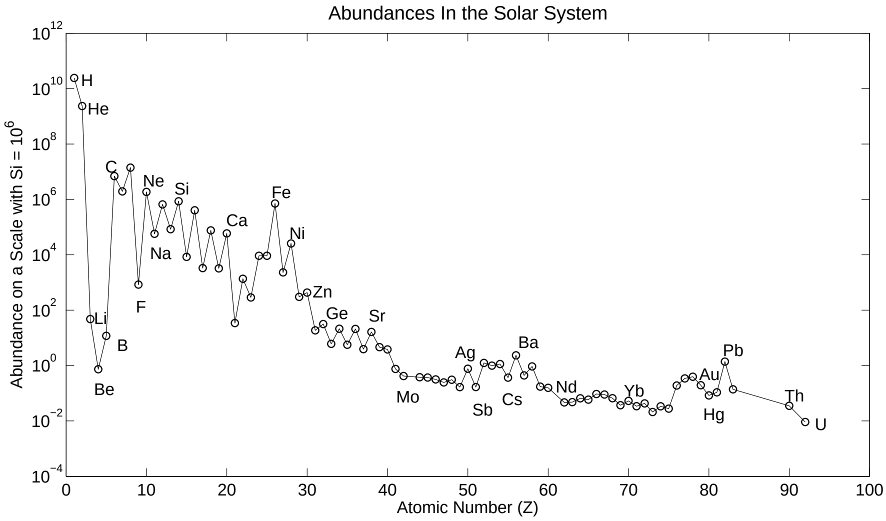

Figure 1: Abundances of elements in the solar system, data taken from [5].

out doubt, this will remain true at least for the foreseeable future and in so doing propel mankind to delve ever deeper into the story of the early Universe.

1.1 Specific Aims

The project aim is to study BBN and the formation of the light atomic nuclei, and consists of two main parts. The first part consists of a literature study to find the main observations that support the Big Bang theory in general and BBN in particular. The goal is also to find, understand and be able to reproduce the main parameters and conditions that describe the Universe prior to BBN. Mainly because these properties are essential for any effort to determine the outcome and the duration of BBN. Important aspects of these quantitative estimates is the time frame of BBN and the production of key isotopes.

The second part is to be based upon calculations with a computer model using the parameters and key estimates made from literature as input data. As the reaction networks that describe the BBN process are complex an available scientific code will be used to calculate the abundances. Hopefully, these calculations will help to explain the measured abundances of the elements. For instance, these ought to yield some clues to why the elemental abundances in the solar system have been observed to be distributed according to figure 1 and [5].

2 The Standard Model of Particle Physics

As explained in [6] the theory came out of advances made in physics in the 20^{th} century. Dirac combined quantum mechanics, electromagnetism and special relativity in his famous equation forming the first step towards quantum field theory. The first interaction to be successfully described within a field theory was that between the electron and the electromagnetic field.

According to the standard model there are four fundamental forces, or interactions, in nature [7]. These are gravity, the weak nuclear force, the electromagnetic force and the strong nuclear force. Each type of interaction has its own associated particles, called bosons, as outlined in table 1. Particles in a quantized interaction field will, in other words, interact by exchanging bosons. The members of this group of particles are characterized by having integer spins and that they obey Bose-Einstein statistics. Particles that have half integer spins are instead called fermions and obey Fermi-Dirac statistics.

Table 1: The four forces.

Force	Boson	Spin
gravity	gravitons (hypothetical)	2
weak nuclear force	W^+,W^-,Z	1
electromagnetic force	photons	1
strong nuclear force	gluons	1

Particles are divided into groups depending on which force that they can interact with. On the scale that concerns particles, gravity plays a minor role and it will not be dealt with any further. Charged particles, such as electrons, interact with the electromagnetic force, while The weak force interact with all particles. The strong force however, only interact with at particular set of different species. Specifically, particles that can interact with the strong force are called hadrons and those that do not are called leptons.

2.1 Hadrons

Hadrons are particles formed from quarks that interact with the strong force. The quarks are in turn elementary particles that can not exist freely and hence have to be combined. These are termed elementary since they can not be divided into smaller particles. There are six different types of quarks that all have corresponding anti particles, the properties of which are shown in table 2, as can be read in [7].

Table 2: Properties of quarks.

Quark	Symbol	Mass [MeV/c^2]	Charge [e]	B	Anti particle
up	u	5	$+2/3$	$+1/3$	$\bar{u}$
down	d	10	$-1/3$	$+1/3$	$\bar{d}$
charm	c	1500	$+2/3$	$+1/3$	$\bar{c}$
strange	s	200	$-1/3$	$+1/3$	$\bar{s}$
top	t	$1.7{\cdot}10^5$	$+2/3$	$+1/3$	$\bar{t}$
bottom	b	4300	$-1/3$	$+1/3$	$\bar{b}$

Quarks can be combined in two ways, either three quarks taken together or one quark and one anti quark. The former combination forms a group called baryons and the latter forms mesons. The most familial baryons, that is the proton and the neutron, both consists of up and down quarks, with the proton having (uud) and neutron (udd) [6]. Since a certain anti quark have the same mass as the corresponding quark but negative baryon number and charge, the baryon numbers of baryons and mesons are 1 and 0 respectively. This follows since, to our current knowledge, all reactions conserve the baryon number.

2.2 Leptons

The first elementary particles to be discovered was the electrons, which are part of the group of particles named leptons, as described in [6]. There are three families of leptons, each of which consists of a particle and an accompanying neutrino as well as the corresponding anti-particles. The properties of each member of the above mentioned families are shown in table 3. It shall also be noted that both particles in an particle-antiparticle pair have the same mass and spin, yet opposite charge.

As in the case of baryons, there exists a so called lepton number, which equals 1 for leptons, -1 for their corresponding antiparticles and 0 for non-leptons. Like the baryon number, both lepton number and electric charge are conserved in any reaction.

2.3 Bosons

As was mentioned earlier, each type of fundamental interaction in nature can be described as an exchange of bosons. The weak force is carried by W^+,W^- and Z bosons, the first two are charged and forms a particle-antiparticle

Table 3: Properties of leptons.

Particle	Symbol	Mass [MeV/c^2]	Charge [e]	Anti particle
Electron	e^-	0.511	-1	e^+
Electron neutrino	ν_e	$< 1 \cdot 10^{-7}$	0	$\bar{\nu}_e$
Muon	μ^-	105.7	-1	μ^+
Muon neutrino	ν_μ	$< 1 \cdot 10^{-7}$	0	$\bar{\nu}_\mu$
Tau	τ	1777	-1	$\bar{\tau}$
Tau neutrino	ν_τ	$< 1 \cdot 10^{-7}$	0	$\bar{\nu}_\tau$

pair while Z is uncharged [7]. As a result of the uncertainty principle, these particles, with masses between 80-90 GeV/c^2, are very short ranged [6].

On the other hand photons, like gluons, are massless and thus expected to have infinte range. The latter species are carriers of the strong force and are therefore responsible for making the quarks stick together as well as getting protons and neutrons to combine to form nuclei.

3 The Expansion

3.1 Hubble expansion

In 1929 Edwin Hubble noted that all distant galaxies in all directions seemed to be moving away from us [8], and even more remarkably, that their velocities were directly proportional to the intermediate distance. In short, the velocity was found to be described by Hubbles law (3.1.1) [8]:

$$v = HR \tag{3.1.1}$$

where H is the Hubble parameter and R is the relative distance between the two objects. Furthermore, the current value of H is often referred to as the Hubble constant, H_0, which in turn is sometimes expressed in terms of the dimensionless Hubble parameter, h, in accordance with (3.1.2) [9, 10].

$$H_0 = h \cdot 100km/(sMpc) \approx \frac{100h}{3.0857 \cdot 10^{19}}\,\mathrm{m/(sm)} \approx 3.241 \cdot 10^{-18} \cdot h\,\mathrm{m/(sm)} \tag{3.1.2}$$

as derived from the latest WMAP measurements, since $1\,\mathrm{pc} = 3.0857 \cdot 10^{16}\,\mathrm{m}$ [11, 12].

If distant objects seem to be moving away from Earth in all possible directions it might be assumed that the earth would be in the very center of the visible Universe[8]. Although this would undoubtedly be remarkable, the truth is even more so. Even though it may appear as if distant objects move away relative to the earth, it is in fact space itself that stretches between the earth and the objects [8]. This means that neither of them actually moves [8]. To illustrate this effect it is possible to paint spots on a half inflated balloon and watch how the spots appear to move away from each other as the balloon is filled with air. Alternatively, one can drink a shrinking potion, like Alice did in wonderland. As one shrinks together with Alice it may appear as if she is moving away, when in actuality both are standing still.

With this new view on the expansion it is now possible to regard R from (3.1.1) as a cosmic scale function [8]. Since (3.1.1) is linear, there is no reason, if neglecting gravitational effects, to think that the Hubble constant, and thus the expansion rate, has changed from the time of the early Universe[8]. If this assumption holds true it would be possible to find an upper limit on the age of the Universe(3.1.3)[8].

$$\begin{aligned} t &= R/v \\ &= R/H_0R \\ &= 1/H_0 \end{aligned} \tag{3.1.3}$$

With $H_0 = 71.0 \pm 2.5\,\mathrm{km/(sMpc)} \approx 71.0\,\mathrm{km/s/Mpc} \approx 71.0/3.0857 \cdot 10^{19}\,\mathrm{m/s/m} \approx 2.3009 \cdot 10^{-18}\,\mathrm{m/s/m}$, since $1\,\mathrm{pc} = 3.0857 \cdot 10^{16}\,\mathrm{m}$, one finds that $t \approx 13.78$ billion years [12].

3.2 Relativistic Model of the Expansion

As it turns out, the Universe does not behave as linearly as one would assume, for which reason it is necessary to involve general relativity[8]. In the following reasoning, taken from D.E. Neuenschwander[8], the main ideas of this approach are discussed. Most importantly, the model as defined must be able to predict the behaviour and the end of the Universe. In relativity one must thus define an invariant distance between points in space time, so that there exists a proper time between nearby events in space time. This distance, dt_p, is given by:

$$dt_p^2 = dt^2 - (dx^2 + dy^2 + dz^2) \tag{3.2.1}$$

where the speed of light, c, is set to unity. Furthermore, equation (3.2.1) can be written in spherical coordinates as:

$$dt_p^2 = dt^2 - (dr^2 + r^2 d\omega^2) \tag{3.2.2}$$

where $d\omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$.

By mixing the equations above with the scale function $R(r)$ and by allowing space to be non-Euclidian one arrives at:

$$dt_p^2 = dt^2 - R^2(r)\left[\left(\frac{dr^2}{1 - kr^2}\right) + r^2 d\omega^2\right] \tag{3.2.3}$$

Here k is the curvature parameter, which has three possible values.

- Case 1: $k = -1$ Space is hyperbolic.
- Case 2: $k = 0$ Space is Euclidean.
- Case 3: $k = 1$ Space is elliptic.

Case 1: Space will continue to expand forever with a non-vanishing velocity. This leads to what is called an "open Universe"

Case 2: The expansion velocity of space will decrease towards zero until equilibrium is reached with regards to the gravitational potential, at which point the Universe will have reached a fixed size.

Case 3: The gravitational potential is larger than the kinetic energy and will hence pull the Universe together again, resulting in what is usually called the "big crunch"

4 The Early Universe

4.1 The Very Early Universe

The time period lasting from the beginning of time, $t = 0$, until approximately one second after the bang, often referred to as the very early Universe, can roughly be broken down into the following epochs [13, 14, 15]:

- Planck epoch, $0\,\mathrm{s} < t < 10^{-43}\,\mathrm{s}$.
- Grand Unification, $10^{-43}\,\mathrm{s} < t < 10^{-35}\,\mathrm{s}$.
- Inflation & Baryon genesis, $10^{-35}\,\mathrm{s} < t < 10^{-33}\,\mathrm{s}$.
- Separation of the weak and electromagnetic forces, $10^{-33}\,\mathrm{s} < t < 10^{-5}\,\mathrm{s}$.
- Protons and neutrons are created, $10^{-5}\,\mathrm{s} < t < 1\,\mathrm{s}$.

The first of these eras, the Planck epoch, is framed by two fundamental points in time, specifically the birth of the Universe in the form of a singularity at $t = 0\,\mathrm{s}$ and the Planck instant at $t \approx 10^{-43}\,\mathrm{s}$. The latter marks the moment after which quantum effects no longer dictates all physical processes, which follows from the fact that the general theory of relativity breaks down during the Planck epoch. The physics of this era is largely unknown, partly because of the high temperature, $T \approx 10^{32}\,\mathrm{K}$.

Although equally difficult to imagine, the physics of the following era, The Grand Unification, is more in line with classical theory [14]. Yet, the temperature was still high enough, that is $\sim 10^{29}\,\mathrm{K}$, for all fundamental forces apart from gravity to be indistinguishable, which therefore is true for a number of particle species as well [14].

The Universe continues to grow and cool however, and eventually reaches a temperature just below $10^{28}\,\mathrm{K}$. As this occurs, the strong force begins to dominate over the other interactions, which in turn influences strongly on the nature of matter [14]. More to the point, the separation of forces shifts the equilibrium for the composition of matter, thereby provoking what can be described as a phase transition [14]. During this period $10^{-36}\,\mathrm{s} < t < 10^{-33}\,\mathrm{s}$ called the inflation the universal expansion takes place at an exponential rate [14]. Remarkably, by the end of this time period the Universe has expanded by a factor of approximately 10^{25} [14].

At $t \approx 10^{-9}\,\mathrm{s}$ the temperature in the Universe has dropped to about $10^{15}\,\mathrm{K}$ and the electromagnetic and weak forces start to separate, while simultaneously becoming significantly decoupled from the strong nuclear force

[14]. Though less potent than the inflation, this later shift of the fundamental forces results in a perturbation of the matter content by introducing a small, yet significant, asymmetry in the number of particles as compared to antiparticles [14].

The ratio of the number of baryons and leptons is conserved during the later stages of the Universe, as these amounts are thought to have been, almost, fixed during the baryon genesis, at $t \approx 10^{-34}$ s, and the electro-weak transition, at $t \approx 10^{-10}$ s, respectively [14]. Additionally, the number of leptons per baryon is related to the number of photons per baryon since photons were created as a result of the annihilation of leptons at high temperatures [14]. The latter quotient is in turn a measure of the entropy per particle [14].

As was mentioned in the previous section, hadrons are composed of quarks, held together by gluons associated with the strong nuclear force [14]. These two particles species did not begin to form into hadrons until $t \approx 10^{-5}$ s, though [15]. Previously, that is from the baryon genesis and forwards, the Universe is filled with a quark-gluon plasma that also contains electron-positron pairs, neutrinos and photons [14, 15]. During the phase transition that follows, bubbles of hadron gas forms and grows in what is best described as a sort of nucleation process. At $t \approx 10^{-4}$ s small droplets of gluons and quarks remain in the, at this point, dominating gas of hadrons and leptons [15]. When this period comes to an end, the protons and neutrons contained within the hadron gas are in thermal equilibrium [14].

4.2 The Early Universe

Before delving into the details of the final era of the very early Universe, that is the time period $0.01\,\mathrm{s} < t < 1.9\,\mathrm{s}$, it is helpful to present, as a reference, a list of events to be discussed, together with the approximate times at which they are thought to have begun [15].

- Neutrino oscillations are initiated, $t \approx 0.1$ s.
- The neutrinos decouple, $t \approx 1$ s.
- Simultaneously, the neutrons freeze-out, $t \approx 1$ s.

With reference to the first of these happenings, it is important to keep in mind that its occurrence is not predicted by the standard model for cosmology, since it includes the assumption that all neutrinos are massless [10]. In the present day model for particle physics however, no conflict exists[10]. Specifically, there are no theoretical restraints that compels the neutrino masses to be either zero or non-zero [10]. Given that the latter holds true, it would be possible for the weak eigenstates of the neutrinos to be formed from linear combinations of mass eigenstates, thereby providing a route for transitions between different neutrino flavors, often referred to as neutrino oscillations [10].

Neutrinos, after photons, are the most abundant particle species in the Universe [10]. Therefore it is not far fetched to assume that a non-zero neutrino mass, together with oscillations, would severely effect the cosmological evolution [10]. Indeed, the first of these deviations from the standard model would alone result in a profound contribution to the total energy density of the Universe [10]. Furthermore, neutrino oscillations is bound to have affected the universal expansion rate, neutrino densities and energy spectrum together with the asymmetry between neutrinos and anti-neutrinos as well as the neutrino dependent cosmological processes [10].

Before discussing further the implications of neutrino oscillations, one would benefit from having a rough estimate of the upper bound for the neutrino masses. This is possible, thanks to the requirement that the total mass density for all neutrino species should be less than equal to that of matter, ρ_m, as stated in (4.2.1).

$$\sum \rho_{\nu_f} \leq \rho_m. \tag{4.2.1}$$

For these calculations it will be assumed, in agreement with present day observations, that the non relativistic matter density in the Universe, ρ_m, is less than 30% of the so called critical mass density, ρ_c, defined by (4.2.3) [1, 3, 10].

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{4.2.2}$$

where G is Newtons gravitational constant and H_0 is the Hubble constant. Before continuing with this discussion it is convenient to introduce the property Ω_i, which represents the contribution of species i, by fraction, to the critical mass density [1, 3]. Thus, it is related to ρ_c by equation (4.2.4), where ρ_i is the mass density for species i [1, 3].

$$\Omega_i = \frac{\rho_i}{\rho_c} \tag{4.2.3}$$

By combining (4.2.2) and (4.2.3) one can easily derive the expression (4.2.4) for ρ_i [3].

$$
\begin{aligned}
\Omega_i &= \frac{\rho_i}{\rho_c} = \rho_i \frac{8\pi G}{3H_0^2} \\
\Leftrightarrow \quad \rho_i &= \frac{3H_0^2\Omega_i}{8\pi G}
\end{aligned}
\tag{4.2.4}
$$

By substituting ρ_m in (4.2.1) for (4.2.4) one thus arrives at the inequality in (4.2.5).

$$
\sum \rho_{\nu_f} \leq \Omega_m \cdot \frac{3H_0^2}{8\pi G} \tag{4.2.5}
$$

The procedure necessary to arrive at a precise limit for the sum of the neutrino masses, $\sum m_{\nu_f}$, is a bit to involved to be attempted here, as such only the result (4.2.6) shall be stated [10].

$$
\sum m_{v_f} \lesssim 94\,\mathrm{eV/c^2} \cdot \Omega_m h^2 \tag{4.2.6}
$$

As was stated above, it has been inferred that $\rho_m < 0.3 \cdot \rho_c$ or equally that $\Omega_m < 0.3$. Additionally, it will be assumed that the Hubble parameter $h = 0.7$, in agreement with section 3. Upon inserting the above values into (4.2.6) one finally arrives at the sought limit, (4.2.7) [10].

$$
\sum m_{v_f} \leq 15\,\mathrm{eV/c^2} \tag{4.2.7}
$$

Even so, there exists much more precise limits on the neutrino masses as obtained from observations, experiments and BBN calculations [10]. For example, some measurements indicate that massive neutrinos could be candidates for hot dark matter if $m_{v_f} \sim 5\,\mathrm{eV}$, which suggests that the usefulness of the estimate presented above is perhaps limited [10].

The kinetic decoupling of the neutrinos can be described as a decrease in thermal contact between these particles and the rest of the plasma. The process begins when $t \approx 0.12\,\mathrm{s}$ at a temperature of $T \approx 3 \cdot 10^{10}\,\mathrm{K}$ and then comes to a close $\sim 1.1\,\mathrm{s}$ after the bang [13]. Specifically, this means that the rates of the weak interactions, such as $e^+ + e^- \rightleftarrows \nu + \bar{\nu}$, whereby the neutrinos are kept in thermal equilibrium with the plasma drops below the expansion rate of the Universe [16, 17]. Afterward, the neutrinos only influence the cosmological evolution by their addition to the total mass-energy density of the Universe [13].

Lastly, it shall be noted that during the entirety of the time period that has been discussed the Universe is filled predominantly with photons, neutrinos and antineutrinos together with electron-positron pairs [13]. The neutrons, protons and electrons meanwhile are mixed into the primordial gas only in trace amounts [13]. Furthermore, the temperature, ranging from 10^{11} K at $t \approx 0.01$ s to $T \lesssim 10^{10}$ K once $t \lesssim 1.9$ s, is sufficiently high for $e^{\pm}$ pairs to be produced. As such the particles within the gas mixture are relativistic and the total behaviour of the fluid resembles more that of radiation than matter [13].

4.3 Freeze-out

Since the neutron number greatly influences the outcome of BBN, it is important to be able to calculate, at least approximately, the time for the neutron-to-proton freeze-out. Similarly to the neutrino decoupling, this freezeout is assumed to have occurred when the overall interconversion rate of protons and neutrons $\lambda_{n,p}$ fell below the universal expansion rate, due to decreasing temperature [3]. Specifically, it would seem likely that as the average time between collisions, that is reciprocal of the conversion rate, grows compared to the time scale in the Universe, as measured by $1/H$, these events will occur ever more seldom. One would thus expect the n-to-p interconversion to become more ineffective to sustain the equilibrium that had existed between the two species before the freeze-out [4]. It is therefore not unreasonable to assume an estimate temperature at the neutron freeze-out would correspond to the time at which the equality $\lambda_{n,p} = H$ was satisfied.

Before continuing with this discussion however, it is of the essence to describe the relationship between neutrons and protons at times when these were still kept in chemical equilibrium through the reactions in (4.3.1), (4.3.2) and (4.3.3),

$$\mathrm{n} + \mathrm{e}^{+} \rightleftharpoons \mathrm{p} + \bar{\nu}_e \tag{4.3.1}$$

$$\mathrm{n} + \nu_e \rightleftharpoons \mathrm{p} + \mathrm{e}^{-} \tag{4.3.2}$$

$$\mathrm{n} \rightarrow \mathrm{p} + \mathrm{e}^{-} + \bar{\nu}_e \tag{4.3.3}$$

The fact that the mass difference between the species, $Q = m_n c^2 - m_p c^2 \approx 1.293\,\mathrm{MeV}$, is greater than zero implies that there were fewer neutrons than protons in the early Universe, or equally that $n_n/n_p < 1$ [2]. Yet, the ratio of the neutron to proton number densities, is predicted to approach unity as the temperature goes to infinity, at least according to (4.3.4) [13].

$$\frac{n_n}{n_p} = \exp\left(\frac{-Q}{k_B T}\right) \tag{4.3.4}$$

In deriving equation (4.3.4), one proceed by first finding suitable expression for the neutron and proton number densities. As is shown in appendix

C, the number of particles of species i per unit volume and with momentum in the interval $[q,q+dq]$ is given by equation (4.3.5).

$$n_i(q)dq = \frac{4\pi g_i}{h^3}\frac{q^2dq}{\exp\left(\frac{E_i(p,q)-\mu_i}{k_BT}\right)\pm 1} \tag{4.3.5}$$

Both protons and neutrons have half integer spins and are thus fermions, for which reason the variant of (4.3.5) with a plus sign on the right hand side applies [11]. Furthermore, it can be assumed that both protons and neutrons were non-relativistic at the time of the n-to-p freeze-out, which shall later be shown to have occurred when $T \approx 10^{10}\,\mathrm{K}$ [13, 18, 19]. This assumption is justified by the fact that the electrons and positrons, with masses three orders of magnitude less than the nucleons, seized to be relativistic at similar temperatures [13, 19]. It follows that the energy of these particles, $E_i(p,q)$ in (4.3.5), can be written on the form (4.3.6) [11, 18].

$$E_j(q) = m_jc^2 + \frac{q^2}{2m_j} \tag{4.3.6}$$

where the subscript j has been included to distinguish the nucleons from the relativistic particles discussed in section C, with $j = n$ for neutrons and $j = p$ for protons. Most importantly, the chemical potentials, compared to those of positrons and electrons, do not vanish in this case. By proceeding in a manner identical to when deriving equation (C.1.29), $\mu_- + \mu_+ = 0$, in section C one ought to be able to prove that $\mu_p = \mu_n = \mu$. For example, one could substitute N_- for N_n and N_+ for N_p and then use the fact that the chemical potentials of each of the species e^-, e^+ and ν, which appear in reactions (4.3.1) to (4.3.3), are zero. In other words, what would be shown is that the chemical potentials are additively conserved in each of the named reactions, which in fact generally holds true [13]. Lastly, nucleons, being fermions with spin 1/2, have two spin degrees of freedom, $g_j = 2$.

With the above statements taken into account the expression (4.3.7), for the total number of particles j per unit volume, results when integrating (4.3.5) .

$$\begin{aligned} n_j &= \int_0^\infty n_j(q)dq = \int_0^\infty \frac{4\pi\cdot 2}{h^3}\frac{q^2dq}{\exp\left(\frac{m_jc^2+q^2/(2m_j)-\mu}{k_BT}\right)+1} \\ \Leftrightarrow \quad n_j &= \frac{8\pi}{h^3}\int_0^\infty \frac{q^2dq}{\exp\left(\frac{m_jc^2-\mu}{k_BT}\right)\exp\left(\frac{q^2}{2m_jk_BT}\right)+1} \end{aligned} \tag{4.3.7}$$

In order to ascertain an analytical solution to (4.3.7) it can be assumed that the exponential term in the denominator is much larger than unity, effectively inferring that the nucleons follow Maxwell-Boltzmann statistics. The assumption is the most critical for particles with small momentum q, for which $\exp[q^2/(2m_j k_B T)] \approx 1$. Therefore, ensurance that the magnitude of the factor $\exp[(m_j c^2 - \mu_j)/(k_B T)]$ is high enough for an appropriate temperature is sufficient evidence to validate this approximation. To show that this factor is indeed large compared to 1, without taking the chemical potential into account, one can determine the magnitude of the term $m_j c^2/(k_B T)$. For this purpose, the temperature can be taken to be 10^{10} K. With very simple estimates of the physical constants, one thus finds that $m_j c^2/(k_B T) \approx 10^{-27} \cdot 10^{17}/(10^{-23} \cdot 10^{10}) = 10^{-27+17+23-10} = 10^3$ [11]. As $\exp[m_j c^2/(k_B T)] \approx e^{1000} \gg 1$ unarguably, the stated assumption ought to be justified for all q, under which (4.3.7) will now be shown to reduce to the form (4.3.8). Note that the integral on the right hand side of (4.3.7) was evaluated with help of formula (D.1.3), derived in appendix D.1.

$$
\begin{aligned}
& n_j \approx \frac{8\pi}{h^3} \int_0^\infty \frac{q^2 dq}{\exp\left(\frac{m_j c^2 - \mu}{k_B T}\right) \exp\left(\frac{q^2}{2m_j k_B T}\right)} \\
= \quad & \frac{8\pi}{h^3} \exp\left(\frac{\mu - m_j c^2}{k_B T}\right) \int_0^\infty q^2 \exp\left(-\frac{q^2}{2m_j k_B T}\right) dq \\
= \quad & \left\{ x = \frac{q}{\sqrt{2m_j k_B T}} \Leftrightarrow \sqrt{2m_j k_B T} x = q \Rightarrow dq = \sqrt{2m_j k_B T} dx \right\} \\
= \quad & \frac{8\pi}{h^3} \exp\left(\frac{\mu - m_j c^2}{k_B T}\right) \int_0^\infty (2m_j k_B T) x^2 e^{-x^2} \sqrt{2m_j k_B T} dx \\
= \quad & \frac{8\pi}{h^3} \exp\left(\frac{\mu - m_j c^2}{k_B T}\right) (2m_j k_B T)^{3/2} \int_0^\infty x^2 e^{-x^2} dx \\
\Leftrightarrow \quad & n_j \approx \frac{8\pi}{h^3} \exp\left(\frac{\mu - m_j c^2}{k_B T}\right)(2m_j k_B T)\right)^{3/2} \frac{\sqrt{\pi}}{2} \\
\Leftrightarrow \quad & n_j \approx \frac{4(2\pi k_B T)^{3/2}}{h^3} \exp\left(\frac{\mu}{k_B T}\right) m_j^{3/2} \exp\left(\frac{-m_j c^2}{k_B T}\right) \qquad (4.3.8)
\end{aligned}
$$

The expression (4.3.9) is obtained by forming the ratio n_n/n_p and then introducing (4.3.8). Finally, (4.3.4) stated earlier follows from (4.3.9), by inferring that the smallness of the neutron to proton mass difference means that $m_n/m_p \approx 1$.

$$\frac{n_n}{n_p} \approx \left(\frac{m_n}{m_p}\right)^{3/2} \frac{\exp\left(\frac{-m_n c^2}{k_B T}\right)}{\exp\left(\frac{-m_p c^2}{k_B T}\right)}$$

$$\Rightarrow \quad \frac{n_n}{n_p} \approx \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(\frac{-Q}{k_B T}\right) \tag{4.3.9}$$

As mentioned though, equation (4.3.4) is only applicable at thermal equilibrium, times preceding the neutron to proton freeze-out. Thereafter, the number of neutrons decreases because of beta decay, according to the reaction in(4.3.10) [2].

$$n \rightarrow p + \mathrm{e}^- + \bar{\nu}_e \tag{4.3.10}$$

Given that the neutrons in a particular system is neither consumed nor created by any reaction except (4.3.10) one can calculate the number of neutrons at any time t, later than t_0, from (4.3.11).

$$N_n = N_{n,0} \exp\left(\frac{-(t - t_0)}{\tau_n}\right), \tag{4.3.11}$$

where $N_{n,0} = N_n(t_0)$ is the, known, number of neutrons at a particular time t_0, for example the time at the neutron to proton freeze-out, and $\tau_n \approx 885.7 \pm 0.8\,\mathrm{s}$ is the mean neutron life time [11, 19].

Figure 2 shows the preditcted evolution of the neutron-to-proton ratio for a decreasing temperature based on the previous discussion. Specifically, the temperature dependence of this ratio is governed by (4.3.9) for $T < T_{\text{freeze-out}}$ and by (4.3.11) for $T > T_{\text{freeze-out}}$.

From the statement above the simple relation (4.3.11) ceases to hold when the Universe has become cool enough for the nucleosynthesis to begin [2]. During this era most neutrons are fused into different nuclei, primarily ^{4}He [2]. Once the BBN process has come to an end however, the conditions for the neutrons return to those that persisted just before the onset and the remaining neutrons are thus comparably slowly converted into protons, through reaction (4.3.10), as time progresses [2].

With the above discussion in close mind, it is convenient to return the problem of calculating the temperature at the nucleon freeze-out. As was suggested earlier one ought to be able to estimate this temperature by solving the equation obtained by setting the Hubble parameter equal to the neutron to proton conversion rate. In order to achieve this however, one must first find an expression for the conversion rate and the Hubble parameter H as

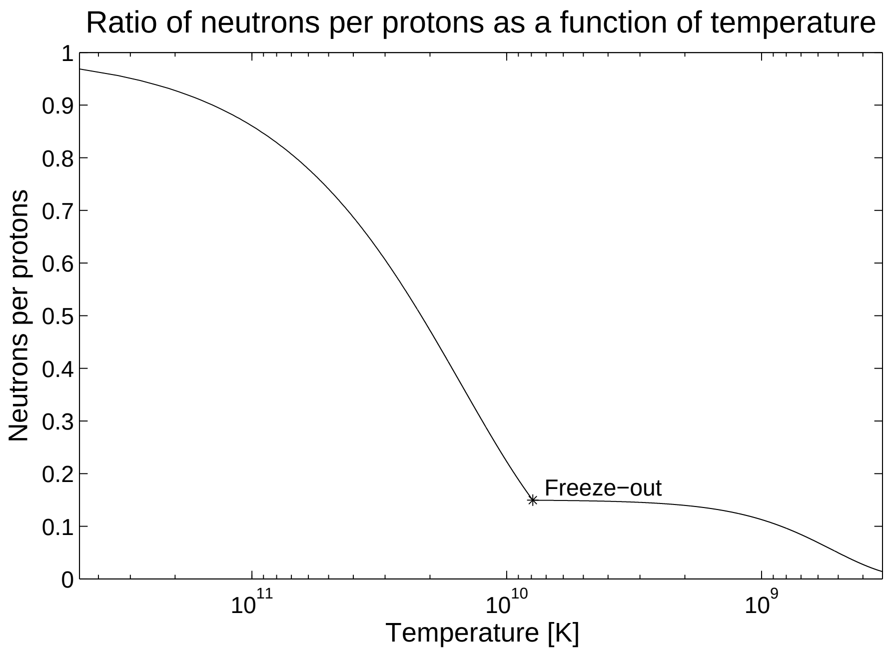

Figure 2: The evolution the neutron to proton ratio as a function of temperature before the Big Bang Nucleosynthesis, specifically (4.3.9) for $T < T_{\text{freeze-out}}$ and (4.3.11) for $T > T_{\text{freeze-out}}$. The asterisk, $*$, marks the point that corresponds to the n-to-p freeze-out, as calculated from equation (4.3.14). Before the freeze-out the ratio is just a function of the canonical ensemble and thereafter only of neutron decay.

functions of time. Because time and temperature of the early Universe are tightly linked, an almost equivalent approach would be to determine the temperature below which the protons and neutrons are no longer in equilibrium. Deducing equation (4.3.12), that shall be used for this comparison, is far beyond the scope of this text though, and as such it will be stated without proof [18].

$$\lambda_{n,p} = \frac{255}{\tau_n x^5}\left(12 + 6x + x^2\right), \quad x = \frac{Q}{k_B T} \tag{4.3.12}$$

Moving on to the Hubble parameter, one have by definition $H = \dot{R}/R$. With the help of expressions (C.1.6) and (C.1.38) derived in C, it is therefor possible to arrive at the formula (4.3.13) for $H(T)$. [1]

$$\frac{\dot{R}}{R} = \sqrt{\frac{8\pi G}{3c^2}\epsilon} \tag{C.1.6}$$

$$\epsilon \approx \frac{43}{8} a T^4 \tag{C.1.38}$$

$$\Rightarrow H = \frac{\dot{R}}{R} = \sqrt{\frac{8\pi G}{3c^2}\epsilon} \approx \left(\frac{8\pi G}{3c^2}\frac{43}{8} a T^4\right)^{1/2}$$

$$\Leftrightarrow H(T) \approx \left(\frac{43\pi a G}{3c^2}\right)^{1/2} T^2 \tag{4.3.13}$$

An estimate of the temperature at the nucleon freeze-out can be calculated solving the equation,(4.3.14) , by setting the Hubble parameter equal to the neutron to proton conversion rate.

$$\begin{aligned} H(T) &= \lambda_{n,p} \\ \Leftrightarrow \left(\frac{43\pi a G}{3c^2}\right)^{1/2} T^2 &= \frac{255}{\tau_n x^5}\left(12 + 6x + x^2\right), \quad x = \frac{Q}{k_B T} \end{aligned} \tag{4.3.14}$$

With numerical values for the physical constants appearing in (4.3.14), the temperature below which the neutrons and protons were no longer in equilibrium is calculated to be $T_{\text{freeze-out}} \approx 7.8965 \cdot 10^9\,\mathrm{K}$, as explained in section 8.1.

[1] As was mentioned previously, (C.1.38) differs from equation 8.62 deduced by Islam since the contribution of the τ neutrino and its corresponding antiparticle has not been taken into account in the latter case [13]. Also, in deriving the same expression Islam has set the speed of light equal to unity, $c = 1$ [13]

5 Energy Density

5.1 The Baryon to Photon Ratio

Over the years accurate and independent experimental measurements have successively improved the estimates of the original input parameters to Big Bang Nucleosynthesis simulations. Eventually, these were pinned to within ranges that essentially promoted BBN to a model with a sole parameter, namely the baryon to photon ratio η [3].

Thanks to the Wilkinson Microwave Anisotropy Probe satellite, WMAP, this situation has recently changed rather dramatically [2]. After its launch by NASA in 2001, WMAP has mapped the cosmic microwave background, CMB, over the entire sky in great detail [20]. Specifically, multi-parameter expressions have been fitted to the observed anisotropy of the background radiation [21, 22]. The errors in the predicted values on these parameters has been further refined through comparison with other observational data [21, 22]. The baryon density was one of those chosen parameters and has, as such, been determined with an unprecedented accuracy [22].

As will be shown it is possible to deduce the photon number density given the black-body temperature that correspondence to the cosmic background radiation, 2.743 K [11]. This deduction will be based upon the assumption that radiation energy density of CBR follows Planck's radiation law, both in terms of frequencies (5.1.1) and wavelengths (5.1.2) [11]. Indeed, this is also what has been observed, mind the small fluctuations mentioned above [23].

$$du = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \tag{5.1.1}$$

$$du = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\exp\left(\frac{hc}{k_B T\lambda}\right) - 1} \tag{5.1.2}$$

The formulas (5.1.1) and (5.1.2) give the energy content per unit volume of black body radiation in the intervals $[\nu, \nu + d\nu]$ and $[\lambda, \lambda + d\lambda]$ respectively. Thus, the total energy density of the radiation emitted by a black body of temperature T can be deduced by integrating (5.1.1) over all frequencies ν according to (5.1.3).

$$u = \int_0^\infty \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

$$\Leftrightarrow \quad u = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \tag{5.1.3}$$

The relation (5.1.3) can be rearranged into the form (5.1.4), where $x = \frac{h\nu}{k_B T}$, that is more easily solvable.

$$u = \frac{8\pi h}{c^3} \int_0^\infty \left(\frac{k_B T}{h}\right)^3 \left(\frac{h\nu}{k_B T}\right)^3 \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \frac{k_B T}{h} \frac{h d\nu}{k_B T}$$

$$\Leftrightarrow \quad u = 8\pi \frac{(k_B T)^4}{(hc)^3} \int_0^\infty \frac{x^3}{e^x - 1} dx \tag{5.1.4}$$

The integral on the right hand side of (5.1.4) has, as shown in appendix D.2, the solution (5.1.5). This result can be inserted into (5.1.4) to yield the formula (5.1.6) for the CMB energy density [24].

$$(m-1)! \sum_{n=1}^{\infty} \frac{1}{n^m}\Big|_{m=4} = 6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15}$$

$$\Rightarrow \quad \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \tag{5.1.5}$$

$$\Rightarrow \quad u = \frac{8\pi^5}{15} \frac{(k_B T)^4}{(hc)^3} \tag{5.1.6}$$

Equally, (5.1.2) can be rewritten in terms of the number density of photons, N_γ, thus yielding the equation (5.1.7) since the photon energy equals $h\nu = hc/\lambda$ and $u = h\nu \cdot N_\gamma$.

$$dN_\gamma = \frac{du}{h\nu} = \frac{\lambda}{hc} du = \frac{\lambda}{hc} \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1}$$

$$\Leftrightarrow \quad dN_\gamma = \frac{8\pi}{\lambda^4} \frac{d\lambda}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} \tag{5.1.7}$$

Expression (5.1.8) for the total number density of photons follows from (5.1.7) by integrating both sides of the equation over all wavelengths.

$$
\begin{aligned}
& N_\gamma = \int_0^\infty \frac{8\pi}{\lambda^4} \frac{d\lambda}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} \\
\Leftrightarrow \quad & N_\gamma = 8\pi \int_0^\infty \frac{\lambda^{-4} d\lambda}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} \qquad (5.1.8)
\end{aligned}
$$

By the same token as (5.1.4), (5.1.9) represents a form of (5.1.8) that is more readily solvable, where $y = (hc)/(k_B T \lambda) \Rightarrow dy = -(hc)/(k_B T)\lambda^{-2} d\lambda$.

$$
\begin{aligned}
& N_\gamma = 8\pi \int_0^\infty \left(\frac{k_B T}{hc}\right)^2 \left(\frac{hc}{k_B T \lambda}\right)^2 \frac{1}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} \frac{k_B T}{hc} \frac{hc}{k_B T} \lambda^{-2} d\lambda \\
\Leftrightarrow \quad & N_\gamma = 8\pi \left(\frac{k_B T}{hc}\right)^3 \int_0^\infty \frac{y^2}{e^y - 1} dy \qquad (5.1.9)
\end{aligned}
$$

Though the integral on the right hand side of (5.1.9), compared to that in (5.1.4), cannot be obtained as an precise number, it can still be evaluated in the same manner as before. This results in the estimate (5.1.10), with which the final relation (5.1.11) is obtained [24].

$$
\begin{aligned}
& (m-1)! \sum_{n=1}^{\infty} \frac{1}{n^m} \bigg|_{m=3} \approx 2 \cdot 1.202 \approx 2.404 \\
\Rightarrow \quad & \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.404 \qquad (5.1.10) \\
\Rightarrow \quad & N_\gamma \approx 2.404 \cdot 8\pi \left(\frac{k_B T}{hc}\right)^3 \\
\Rightarrow \quad & N_\gamma \approx 60.42 \cdot \left(\frac{k_B T}{hc}\right)^3 \qquad (5.1.11)
\end{aligned}
$$

Hence, the photon number density in the cosmic background radiation is found, by evaluating (5.1.11) for $T = 2.743\,\mathrm{K}$ [11]. Combined with the WMAP data this yields the following result.

$$
\begin{aligned}
& N_\gamma \approx 60.42 \cdot \left(\frac{1.381 \cdot 10^{-23} \cdot 2.743}{6.626 \cdot 10^{-34} \cdot 2.998 \cdot 10^8}\right)^3 \\
\Rightarrow \quad & N_\gamma \approx 4.190 \cdot 10^8\,\mathrm{m}^{-3}
\end{aligned}
$$

Yet to find the sought baryon to photon ratio, one must first deduce the baryon number density n_b. Given the baryon mass density density ρ_b, the number density is most easily calculated by assuming the mass per baryon to be equal to that of a proton, $m_b \approx m_p \approx 1.6726216 \cdot 10^{-27}\,\mathrm{kg}$ [11, 13]. The main problem is therefore determining ρ_b. Fortunately, the baryon mass density can be calculated from the dimensionless number Ω_b that has been accurately fitted to the WMAP observations, as mentioned above. As was discussed in section 4.2 Ω_b is by definition the baryonic contribution, by fraction, to the so called critical mass density ρ_c, defined by (4.2.2) [1, 3]. Furthermore, this property conveniently appears in the expression (5.1.13) for the baryon mass density density, obtained simply by substituting the index i for b in (4.2.4) [3].

$$\Omega_b = \frac{\rho_b}{\rho_c} \tag{5.1.12}$$

$$\Rightarrow \quad \rho_b = \frac{3H_0^2\Omega_b}{8\pi G} \tag{5.1.13}$$

The gravitational constant will be taken as $G = 6.6726 \cdot 10^{-11}\,\mathrm{Nm}^2/\mathrm{kg}^2$ while the most up to date WMAP measurements give $\Omega_b h^2 = 0.02258^{+0.057}_{-0.056}$ [11, 12]. With H_0, given by (3.1.2), one can thus calculate the baryon number density from (5.1.14).

$$n_b = \frac{\rho_b}{m_b} = \frac{3H_0^2\Omega_b}{8\pi G m_b} \tag{5.1.14}$$

$$\Rightarrow \quad n_b = \frac{3 \cdot (3.241 \cdot 10^{-18} \cdot h)^2 \cdot 0.02258/h^2}{8\pi \cdot 6.6726 \cdot 10^{-11} \cdot 1.6726216 \cdot 10^{-27}}$$

$$\Rightarrow \quad n_b = \frac{3 \cdot (3.241 \cdot 10^{-18})^2 \cdot 0.02258}{8\pi \cdot 6.6726 \cdot 10^{-11} \cdot 1.6726216 \cdot 10^{-27}} \approx 0.2536\,\mathrm{m}^{-3}$$

The sought baryon to photon ratio, $\eta = n_b/n_\gamma$, is thus found to be

$$\eta = \frac{0.2536}{4.190 \cdot 10^8} \approx 6.1 \cdot 10^{-10}.$$

6 Relating Time and Temperature

6.1 Simple Model for Relating Time and Temperature

The Hubble parameter H_0 does not remain constant on large time scales and it is therefore necessary to find how $H(t)$, now without the subscript, varies with the expansion. A much more thorough derivation than what is below is found in appendix C. Intuitively a relation to a radius would be practical, but since the Universe lacks one the scale factor R will be used instead [25]. This parameter only depends on time and thus the distance between two point in space can be predicted given an expression for $R(t)$ and the magnitude of this distance at some arbitrarily time t_0 [26]. The relation between H and R is [25]

$$H = \frac{1}{R}\frac{dR}{dt}. \tag{6.1.1}$$

To obtain the relation for the time evolution of H the tensor equation of the theory of general relativity needs to be solved. The result is stated below [25].

$$H^2 = \frac{(dR/dt)^2}{R^2} = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{R^2} + \frac{\Lambda}{3}, \tag{6.1.2}$$

where G is the gravitational constant, $\rho(t)$ is the sum of the mean mass and the energy density of the Universe, k is the curvature parameter the value of which depends on whether the Universe is open, closed or flat, as was discussed in section 3. Lastly, Λ is the cosmological constant which will be ignored from this point onwards. Furthermore, the Universe will be assumed to be flat, which simplifies the calculations since the geometrical factor k is zero in this case.

In order to integrate (6.1.2) the dependence of ρ on R is needed. The early Universe was dominated by radiation as well as particles moving at relativistic speeds, for which reason it can be assumed that the radiation-like relationship $E = hc/\lambda$ was obeyed. Hence, one can use the radiant energy density ρ_R, which represents the energy content of the radiation per unit volume. In turn ρ_R is equal to the cross product of the energy per quantum and the number of quanta per unit volume [25]

$$\rho_R = \frac{\text{energy}}{\text{volume}} = \text{energy per quantum} \times \text{quanta per volume.}$$

The energy per quantum is proportional to $1/R$ and the quanta per unit volume is proportional to $1/R^3$ [25]. ρ in (6.1.2) will therefore be assumed to have the form $\rho_R = C/R^4$, with C a constant that will be shown to disappear in the calculations below. An approximate form for (6.1.2) is thus

$$
\begin{aligned}
H^2 = \left(\frac{1}{R}\frac{dR}{dt}\right)^2 &= \frac{8\pi G}{3}\frac{C}{R^4} \\
\Leftrightarrow \quad H = \frac{1}{R}\frac{dR}{dt} &= \sqrt{\frac{8\pi GC}{3}}\frac{1}{R^2},
\end{aligned} \tag{6.1.3}
$$

which can be integrated to yield

$$
t = \sqrt{\frac{3C}{32\pi G R^4}} = \sqrt{\frac{3}{32\pi G \rho_R}}. \tag{6.1.4}
$$

To arrive at the desired relation between time and temperature the temperature dependence of ρ_R is required. As was previously mentioned, the early Universe was dominated by radiation and relativistic particles. Therefore the energy density ρ_R can be taken as that of black body radiation for a radiating system, $u(T)$, at temperature T [25]

$$
u(T) = \sigma T^4, \tag{6.1.5}
$$

where σ is the Stefan-Boltzmann constant. The relation between temperature, in Kelvins, and the elapsed time since the big bang , in seconds, is

$$
T = \left(\frac{3}{32\pi G\sigma}\right)^{1/4} \cdot \frac{1}{t^{1/2}}, \tag{6.1.6}
$$

which reduces to (6.1.7) upon inserting numerical values for the physical constants.

$$
T \approx \frac{1.5 \cdot 10^{10}}{t^{1/2}}\,\mathrm{K} \cdot \mathrm{s}^{1/2}. \tag{6.1.7}
$$

The relation in (6.1.7) is shown in figure 3.

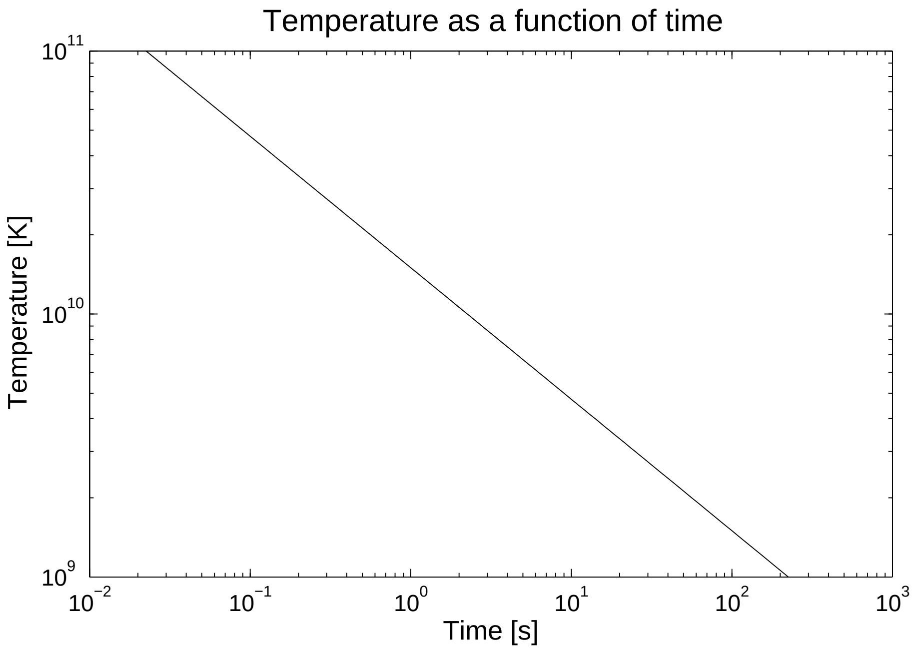

Figure 3: .Relation between time and temperature in the radiation dominated era.

7 Big Bang Nucleosynthesis

7.1 The Physical Process

The Big Bang Nucleosynthesis represents an era in the history of the Universe that is said to have lasted from about a second until thirty minutes after the Big Bang [1, 4]. During this process protons and neutrons were combined, as governed by a complex reaction network, to form a multitude of light nuclei [1, 4].

Before moving on to discuss the details of the primordial nucleosynthesis, it is convenient to discuss this and similar events in the early Universe in more general terms. Most importantly, the processes that have been or shall be described neither ceases nor are initialized at specific times, or temperatures. Indeed, regarding any changes as momentaneous, though helpful as a simplification for calculation and modeling purposes, is inherently flawed. Therefore, the specific times or temperatures related to these events, in many cases, represent points marking a shift in dominance of one physical property over another. For instance, the time for the neutron freeze-out is calculated by comparing the rate of the neutron to proton conversion and the expansion of the Universe respectively. This could falsely lead to the conclusion that no protons were converted into neutrons once the point of intersection had been reached. In reality though this process did occur, be it at an ever slower pace.

The value of the neutron to proton ratio at the beginning of BBN is one of the key factors that determines its outcome [9]. It is therefore convenient, given the previous example, to return to the era at hand. Nontheless, in order to fully appreciate the impact of the most crucial parameters, such as the n-to-p ratio, on the primordial nucleosynthesis a qualitative understanding of the underlying physics is required.

During the period at hand the Universe was still quite dense, with regards to the total energy content per unit volume, and the temperature correspondingly high [4, 13]. Furthermore, the Universe was predominantly inhabited by photons and the recently decoupled neutrinos, while the protons, neutrons and electrons were present only in trace amounts [4, 13]. It thus seems reasonable that the only nuclear reactions of interest would have involved interactions between, at most, two particles [4]. By the same token, it can be assumed that the more complex nuclei than D, could only have formed through a chain of two-body collisions [4]. This suggests that the formation of the lightest nuclei, the deuteron, could be regarded as the beginning of BBN. Indeed, the starting point for the primordial nucleosynthesis is often referred to as the time when the rate of deuterium nuclei formation, through (7.1.2), exceeded that of the reverse reaction [2].

Also, the right hand side of (7.1.2) suggest that the destruction of d was, at this stage, primarily due to photodissociation. The vast number of photons per baryon meant that this process was highly effective in hindering the deuterium nuclei to survive long enough for it to take part in other nuclear reactions [2]. This deadlock prevails long after the temperature, or rather $k_B \cdot T$, has dropped below the binding energy of the deuteron, $B_d \approx 2.23\,\mathrm{MeV}$. Specifically, the Big Bang Nucleosynthesis is estimated to have begun in earnest when $k_B \cdot T \approx 0.080\,\mathrm{MeV}$ [2].

The nucleosynthesis then proceeded through a intricate network of reactions, of which (7.1.1) to (7.1.11), shown in figure 4, represented the twelve of these with highest significance [2, 3].

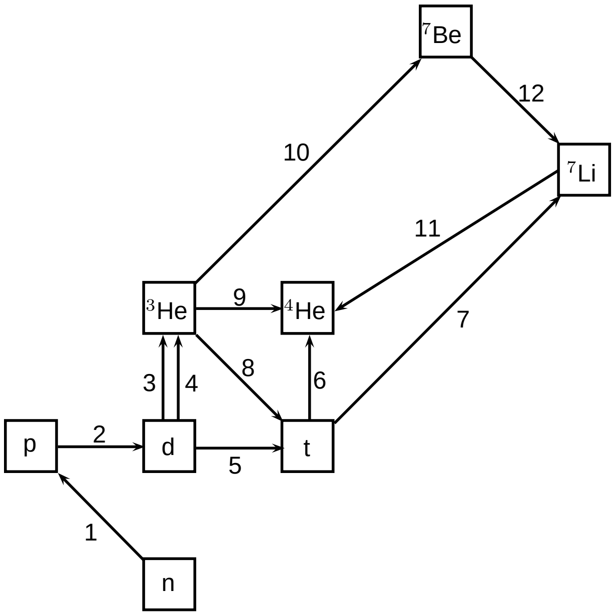

Figure 4: The twelve most important reactions in the reaction network that governs the BBN process. The reactions represented by numbers are reproduced in equations (7.1.1) – (7.1.11).

$$1. \quad n \rightarrow p + \mathrm{e}^- + \bar{\nu}_e \tag{7.1.1}$$
$$2. \quad p + n \rightarrow \mathrm{d} + \gamma \tag{7.1.2}$$
$$3. \quad \mathrm{d} + p \rightarrow^3 \mathrm{He} + \gamma \tag{7.1.3}$$
$$4. \quad \mathrm{d} + \mathrm{d} \rightarrow^3 \mathrm{He} + n \tag{7.1.4}$$
$$5. \quad \mathrm{d} + \mathrm{d} \rightarrow \mathrm{t} + p \tag{7.1.5}$$
$$6. \quad \mathrm{t} + \mathrm{d} \rightarrow^4 \mathrm{He} + n \tag{7.1.6}$$
$$7. \quad \mathrm{t} + {}^4\mathrm{He} \rightarrow^7 \mathrm{Li} + \gamma \tag{7.1.7}$$
$$8. \quad {}^3\mathrm{He} + n \rightarrow \mathrm{t} + p \tag{7.1.8}$$
$$9. \quad {}^3\mathrm{He} + \mathrm{d} \rightarrow^4 \mathrm{He} + p \tag{7.1.9}$$
$$10. \quad {}^3\mathrm{He} + {}^4\mathrm{He} \rightarrow^7 \mathrm{Be} + \gamma \tag{7.1.10}$$
$$11. \quad {}^7\mathrm{Li} + p \rightarrow^4 \mathrm{He} + {}^4\mathrm{He} \tag{7.1.11}$$
$$12. \quad {}^7\mathrm{Be} + n \rightarrow^7 \mathrm{Li} + p \tag{7.1.12}$$

In discussing nuclear reactions, it is sometimes useful to examine the nuclear binding energies, defined as the difference in mass between a nucleus and the sum of its individual nucleons. Specifically, the comparison of this property, evaluated for different nuclei, yields an important estimate of their relative stabilities. Given values for the binding energy per nucleon for the lightest elements it is thus possible to make qualitative predictions on BBN with regards to both the physical process and its outcome. Most importantly, in going from d through ^{3}He and T to ^{4}He the degree of binding increases, even though the tritium nuclei is in actuality unstable and later decays to ^{3}He [3, 11, 18].

By the same logic used to assess the starting point for BBN, it could be inferred that each nuclear species with a binding energy higher than the deuteron would in fact have been stable at temperatures higher than 0.080 MeV [13]. More precisely, in the sense that not enough energetic photons would have been available to photo fission them efficiently [13]. It is therefore helpful to think of the formation of d as a sort of bottleneck that delayed the nucleosynthesis. In addition, the tighter binding of the, slightly, more massive nuclides meant that these would be expected to have readily formed once this barrier had been breached [13].

It seems likely that most neutrons, which were outnumbered by the protons by a factor of at least 6, would have been incorporated into the most stable nuclei [1, 13]. Because of the mass gaps at $A = 5$ and $A = 8$, the most tightly bound nuclide produced during the primordial nucleosynthesis was ^{4}He [13]. Specifically, ^{4}He constitutes a local maximum for the binding energy per nucleon as a function of the nucleon number, A, as is seen in figure 5 [18, 27].

Comparison of reactions (7.1.10) and (7.1.7) with those higher up in the list suggests that particles with higher positive charge, that is at least one with charge $+2e$ instead of $+e$, must have collided in order for the heavier nuclei, $A = 7$, to have formed. Since the radius of these light particles are in the order of a few fm, the interactions through which ^{7}Li and ^{7}Be were created would most definitely have involved significantly higher coloumb barriers [11, 13]. Additionally, the continuous expansion of the Universe rapidly drained away the energy that was needed to surmount these barriers. Therefore reactions (7.1.10) and (7.1.7) ought to have given birth to no more than trace amounts of $A = 7$ nuclei [1]. It would thus be expected that most neutrons were bound in ^{4}He once BBN had come to a close [1]. This nuclear freeze-out, occurred when $k_B T \approx 0.1$ MeV corresponding to $t \approx 30$ min, after which time no further nuclear reactions took place [1, 2]. Yet, both ^{7}Be and T as well as the neutron were unstable and continued to decay into ^{7}Li, ^{3}He and protons respectively [3]. Following the primordial nucleosynthesis the Universe ought to have contained, based on the previous discussion, primarily hydrogen-1 and helium-4 in addition to small remnants of unburned d, ^{3}He and ^{7}Li [1].

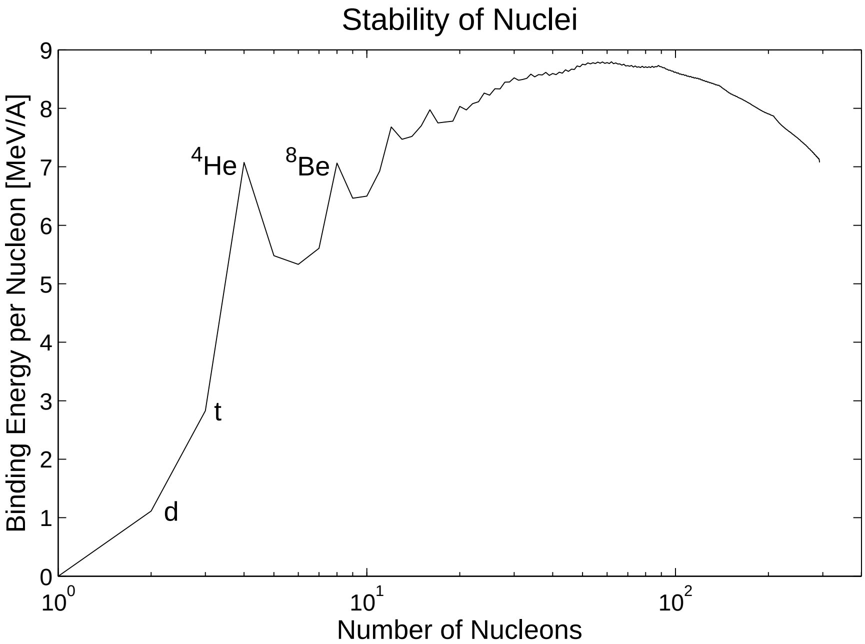

Figure 5: Maximum binding energies as a function of number of nucleons, A, [27]. Worth noting is that ^{8}Be has a very short half-life, and α-decays almost instantly into two ^{4}He.

7.2 The Impact of η

With the framework in place it is convenient to turn to the physical parameters that have the largest impact on the predicted result of the Big Bang Nucleosynthesis. Principally these are the nuclear cross sections that determine the nuclear reaction rates, the neutron lifetime, the baryon to photon ratio as well as the number of neutrino species [1]. Since the latter three have already been under scrutiny, in sections E.8, 5.1 and 4.2 respectively, while giving a meaningful introduction to cross sections is beyond the scope of this text, there shall be no effort to explain the underlying principles for these parameters.

The range of plausible values for η has, until very recently, been rather wide compared to those of the other physical properties, previously mentioned [3]. Indeed, BBN calculations, with the observed values for the primordial abundances used as input data instead of η, used to give the best, if not the only, estimates for the baryon density [2, 3]. It is therefore prudent, not only for historical reasons, to assess the dependence of the final number fractions of the light elements as well as the progression of the primordial synthesis on the baryon to photon ratio.

The photodissociation, responsible for delaying the formation of the deuteron and thus BBN as a whole, should in all likelihood have a rather sharp dependence on η [2, 9]. Although bold, this statement shall now be justified by analyzing the Boltzmann equation for the system of particles included in (7.1.2) [18].

As with most expressions involving the the cosmic scale factor $R(t)$, the complexity of the underlying theory regretfully means that it will be stated without proof. Furthermore, the form (7.2.1) used in the following derivation is approximate and only applicable for a system composed of particles 1, 2, 3 and 4 [18].

$$R^{-3}\frac{d}{dt}(n_1 R^3) = n_1^{(0)} n_2^{(0)} \langle\sigma\nu\rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \tag{7.2.1}$$

Here n_i is the number density of particle type i, while the superscript (0) indicates that the latter should be evaluated at equilibrium conditions [18]. Also, by stating this formula it has been inferred that the only reaction involving the species of interest, 1, is $1 + 2 \leftrightharpoons 3 + 4$ [18].

In (7.2.1) the nature of the interconversion process is taken into account by inclusion of the thermally averaged cross section $\langle\sigma\nu\rangle$ [18]. Though highly significant for modeling not only the system at hand but the Big Bang Nucleosynthesis as a whole, giving a qualitative description of these factors is beyond the scope of this text. However, by regarding the particle collisions in a macroscopic sense, at least a basic degree of understanding for the underlying physics can be achieved [28]. Classically, both the rates and likelihoods of such impacts are dependent on the sizes of the interacting particles, a property measured by for example their cross sectional areas [28]. Even so, the nuclear cross sections are immensely more intricate and depends on a far greater number of physical properties [18].

The term on the left hand side in (7.2.1) ought to represent the total rate of change in the number density for particle 1 [18]. From the definition of $R(t)$ given in the introductory chapter and the results in section C, this term can be seen to take into account the decrease of n_1 through the expansion of the Universe [18]. Moving on to the right hand side, $n_2^{(0)}\langle\sigma\nu\rangle$ should qualify as the rate of reaction [18]. The factor within the curly brackets, in turn, vanishes if the particles are in equilibrium, that is if $n_i = n_i^{(0)}\ \forall\, i$, and should hence give an estimate of the departure from equilibrium.

Since the Hubble parameter $H = \dot{R}/R$, or rather H^{-1}, is a measure of the cosmological time-scale, it is plausible that the magnitude of the term $\frac{\partial(n_1 R^3)}{\partial t}$ is in the order of $H_1 n_1$ [18]. Thus, if the rate of conversion $n_2^{(0)}\langle\sigma\nu\rangle$ is significantly greater than the expansion, the sum of the terms inside the curly brackets must be very small for the equality (7.2.1) to hold. One would in other words require the condition in (7.2.2), often referred to as the definition for chemical equilibrium, to be approximately true [18].

$$\frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} = \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} \tag{7.2.2}$$

Because the photons greatly outnumber the nucleons, $\eta \approx 10^{10}$, in the early Universe, it seems reasonable that the number density of the photons $n_\gamma \approx n_\gamma^{(0)}$ [18]. Equation (7.2.2) therefore takes on the form (7.2.3) for the system at hand [18].

$$\begin{aligned} \frac{n_D n_\gamma}{n_D^{(0)} n_\gamma^{(0)}} &= \frac{n_n n_p}{n_n^{(0)} n_p^{(0)}} \\ \Rightarrow \frac{n_D}{n_n n_p} &= \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}} \end{aligned} \tag{7.2.3}$$

In deriving formula (4.3.8) it was shown that the protons and neutrons could, with good approximation, be described by Boltzmann statistics. Moreover, this assumption was partly justified by comparing energies equivalent to the masses of these particles and the temperature respectively. For this reason, the same arguments should be applicable to the even more massive deuteron. Consequently, each of the number densities appearing in (7.2.3) could be replaced by (7.2.4), which is a more general form of formula (4.3.8) [18]. More precisely, the former follows from the latter by replacing a factor of 2, given by the number of spin degrees of freedom for the nucleons, with g_j and the nucleon chemical potential μ with μ_j.

$$n_j \approx \frac{2g_j(2\pi k_B T)^{3/2}}{h^3} \exp\left(\frac{\mu_j}{k_B T}\right) m_j^{3/2} \exp\left(\frac{-m_j c^2}{k_B T}\right) \qquad (7.2.4)$$

What is more, at equilibrium conditions the chemical potentials vanish and equation (7.2.4) takes on the form (7.2.5) for all relevant species.

$$n_j^{(0)} \approx 2\frac{(2\pi k_B T)^{3/2}}{h^3} \cdot g_j m_j^{3/2} \exp\left(\frac{-m_j c^2}{k_B T}\right) \qquad (7.2.5)$$

The approximate form (7.2.6) is obtained by introducing of (7.2.5) into the right hand side of (7.2.3).

$$\begin{aligned}
&\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}} \\
= \quad & \frac{2(2\pi k_B T)^{3/2}/h^3}{(2(2\pi k_B T)^{3/2}/h^3)^2} \frac{g_D m_D^{3/2} \exp\left(\frac{-m_D c^2}{k_B T}\right)}{g_n m_n^{3/2} \exp\left(\frac{-m_n c^2}{k_B T}\right) \cdot g_p m_p^{3/2} \exp\left(\frac{-m_p c^2}{k_B T}\right)} \\
= \quad & \frac{1}{2(2\pi k_B T)^{3/2}/h^3} \frac{g_D}{g_n g_p} \left(\frac{m_D}{m_n m_p}\right)^{3/2} \exp\left(\frac{-c^2(m_D - m_n - m_p)}{k_B T}\right) \\
\Leftrightarrow \quad & \frac{n_D}{n_n n_p} \\
& = \frac{g_D}{g_n g_p} \frac{h^3}{2} \left(\frac{m_D/m_n m_p}{2\pi k_B T}\right)^{3/2} \exp\left(\frac{c^2(-m_D + m_n + m_p)}{k_B T}\right) \qquad (7.2.6)
\end{aligned}$$

This equation can be further simplified, however. As was mentioned in section 4.3 both neutrons and protons are fermions and hence have spin 1/2 corresponding to $g_n = g_p = 2$ [18]. The deuteron, on the other hand, has $g_D = 3$ [18]. With these numbers, the ratio of the species spin degrees of freedom that appears on the right hand side of (7.2.6) is easily evaluated to be $g_D/(g_n g_p) = 3/4$. By inserting this result into the previous equation (7.2.6), one yields the expression (7.2.7) [18].

$$\frac{n_D}{n_n n_p} = \frac{3h^3}{8} \left(\frac{m_D}{2\pi m_n m_p k_B T}\right)^{3/2} \exp\left(\frac{c^2(-m_D + m_n + m_p)}{k_B T}\right) \qquad (7.2.7)$$

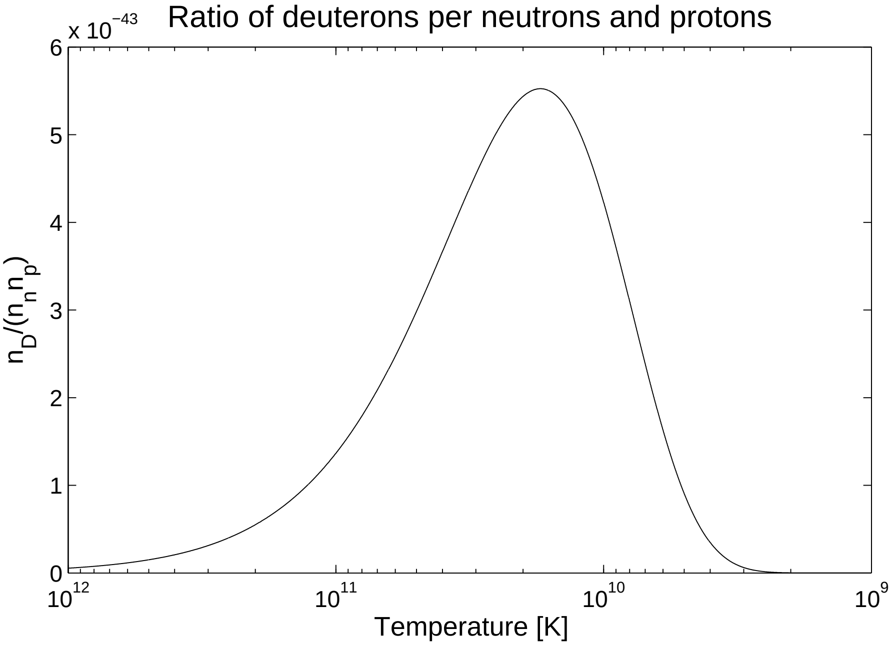

Figure 6: The graph shows the temperature dependence of the ratio $n_D/(n_n \cdot n_p)$.

It will also be assumed that the small differences in mass, between on one hand the proton and the neutron and on the other the deuteron and its constituents, are negligible, or equivalently that $m_D = m_n + m_p \approx 2m_n \approx 2m_p$, in the quotient on the right hand side of (7.2.7). In addition, the nominator in the exponential is substituted for the binding energy of the deuterium nuclei, $-Q = -m_D c^2 + (m_n c^2 + m_p c^2)$. As such, (7.2.7) is reduced to the form (7.2.8), which corresponds to the graph in figure 6.

$$\frac{n_D}{n_n n_p} = \frac{3h^3}{8}\left(\frac{2m_p}{2\pi m_p m_p k_B T}\right)^{3/2} \exp\left(\frac{-Q}{k_B T}\right)$$
$$\Rightarrow \quad \frac{n_D}{n_n n_p} = \frac{3h^3}{8}\left(\frac{1}{\pi m_p k_B T}\right)^{3/2} \exp\left(\frac{-Q}{k_B T}\right) \qquad (7.2.8)$$

Furthermore, nucleons are baryons by definition and it therefore seems likely that both the proton and neutron number densities are proportional to the number of baryons per unit volume, n_b [18]. Also, in section 5.1 an equation, (5.1.9), was derived, which shows that the the number density of the photons is proportional to the temperature cubed, $n_\gamma \propto (k_B T)^3$. By implication, these statements together with the above expression (7.2.8) lead to the proportionality in (7.2.9) [18].

$$
\begin{aligned}
\frac{n_D}{n_n n_p} &\propto \frac{n_D}{n_b n_b} \\
\Rightarrow \frac{n_D}{n_b} &\propto n_b h^3 \frac{(k_B T)^3}{(k_B T)^3} \left(\frac{1}{(m_p k_B T}\right)^{3/2} \exp\left(\frac{-Q}{k_B T}\right) \\
\Rightarrow \frac{n_D}{n_b} &\propto h^3 \frac{n_b}{n_\gamma} \left(\frac{k_B T}{m_p}\right)^{3/2} \exp\left(\frac{-Q}{k_B T}\right) \\
\Rightarrow \frac{n_D}{n_b} &\propto \eta h^3 \left(\frac{k_B T}{m_p}\right)^{3/2} \exp\left(\frac{-Q}{k_B T}\right) \qquad (7.2.9)
\end{aligned}
$$

Because of the smallness of the baryon to photon ratio, the prefactor will dominate over the exponential as long as the quotient $Q/(k_B T)$ is not very large [18]. For example, at $T = 10^{10}\,\mathrm{K}$ the exponent $Q/(k_B T) \approx 10^6\,\mathrm{eV}/(10^{-4} \cdot 10^{10}\,\mathrm{eV}) = 1$ [11]. In agreement with previous assessments, this result suggests that the numerosity of the photons hinders the deuterium nuclei from forming until the temperature, $k_B T$, is a few orders of magnitude lower than the binding energy of the deuteron, Q.

As mentioned the deuterium nuclei combined to form heavier nuclei shortly after having been created and it might thus be expected that these species would show a similar dependence on η. This is indeed true, at least to some extent. Still, the intricacy of the network of nuclear reactions governing the interactions results in a much more complex dependence. What shall also be mentioned, is that relation (7.2.8) only gives the equilibrium ratio $n_D/(n_n n_p)$, based soley on the reaction (7.1.2), and not the final abundance of D. Nonetheless, it is the dependence of the amounts of the different species produced on the baryon to photon ratio that shall henceforth be discussed. Also, to simplify the comparison with the results obtained from simulations, most importantly figures 9 and 10, the conclusions presented below are summarized in table 4.

Turning first to ^{4}He, it can be inferred that this species should be rather insensitive to η [4, 9]. This is due to the fact that the binding energies of the light nuclei were independent of the baryon to photon ratio, wherefore most neutrons would have been fused into ^{4}He, regardless of the precise value on η. The extent of the helium-4 production would accordingly have been determined, to a large extent, by the total content of neutrons in the Universe when the BBN process was initialized. In turn, this number depends on the temperature at which the neutron freeze-out occurs and thus on the competition between the rate of the n-to-p conversion and that of the expansion [9]. Still, for a large baryon to photon ratio the D-bottleneck ought to have been breached earlier [9]. This results in an increase in both the temperature as well as the total number neutrons at the onset of BBN, since fewer of them would have had time to decay in this case. One would thus expect that slightly more ^{4}He would have been produced for a greater value on the η[9]. However, the previous discussion suggests that the rate of increase, with respect to η, ought to be low. For similar conditions, this implies that lesser amounts would have remained of the species, specifically D and ^{3}He, that collided to form helium-4 nuclei for a higher baryon to photon ratio [4, 9].

The $A = 7$ nuclides, lastly, are of particular interest as there existed two dominant paths for ^{7}Li production [3, 9]. On one hand, these nuclei could have been created directly via reaction (7.1.7), a route that is assumed to have been favoured for a low baryon to photon ratio [3, 9]. This follows from the fact that a higher baryon density would have resulted in a greater number of protons able to destroy ^{7}Li nuclei through (7.1.11) [3, 9]. On the other hand, ^{7}Be have a higher binding energy per nucleon and should therefore be more stable with regards to such collisions [3, 9]. Consequently it ought to have been produced to a larger extent if η was high [3, 9]. These nuclei were unstable however and later decayed into ^{7}Li through the absorption of an electron [2, 3, 9].

The analysis given is made more complicated by (7.1.12), which suggests that the indirect route for lithium-7 synthesis ought to have been plausible even at a lower baryon to photon ratio [3]. Particularily, (7.1.12) was limited by the amount of neutrons present during the primordial nucleosynthesis, a number that would have been enhanced for lower values on η [3]. Yet, simulations in which the baryon to photon ratio is varied, such as figure 12 in section 8.4, reveals a minimum in the ^{7}Li production for intermediate η values [3]. This implies that there should indeed have existed two separate modes for lithium-7 creation in the early Universe and that the relative dominance of these ought to have dependended on the value of η [3]. Alternatively, one can infer, based on the previous discussion on the synthesis of $A = 7$ nuclei, that ^{7}Li would have been produced primarily via (7.1.7) for low η instead of (7.1.10) for a given temperature. This conclusion stems from the fact that the latter reaction is a two-body collision between particles with higher positive charge, that is one more particle with $q = +2e$ instead of $q = +e$.

Table 4: Prediction of relative abundances at different values of η.

η	Low	Intermediate	High
^{4}He	slightly lower	increases slightly	slightly higher
d	higher	decreases	lower
^{3}He	higher	decreases	lower
^{7}Li	higher	minimum	higher

7.3 Calculating the Fraction of High Energy Photons

Photons follow a black body spectrum, the number of photons per unit volume with energy between E and $E + dE$ is [25]

$$n(E)dE = \frac{8\pi E^2}{(hc)^3} \cdot \frac{1}{\exp(E/kT) - 1} dE. \tag{7.3.1}$$

where $n(E)$ is the fraction of photons of energy E. To get the number of photons with energy greater than $E_\circ$, (7.3.1) is integrated from said energy to infinity

$$\frac{8\pi}{(hc)^3} \int_{E_\circ}^{\infty} E^2 \exp(-E/kT)dE = n(E > E_\circ), \tag{7.3.2}$$

where the approximation

$$\frac{1}{\exp(E/kT) - 1} \approx \exp(-E/kT), \tag{7.3.3}$$

has been introduced, which is good for $E \gg kT$. The integral in (7.3.2) is solved through integration by parts

$$\begin{aligned}
&\frac{(hc)^3}{8\pi} n(E_\circ > E) = \int_{E_\circ}^{\infty} E^2 \exp(-E/kT)dE \\
&= -kTE^2 \exp(-E/kT)\Big|_{E_\circ}^{R\to\infty} + 2kT \int_{E_\circ}^{\infty} E \exp(-E/kT)dE \\
&= kTE_\circ^2 \exp(-E_\circ/kT) - 2(kT)^2 E \exp(-E/kT)\Big|_{E_\circ}^{R\to\infty} \\
&+2(kT)^2 \int_{E_\circ}^{\infty} \exp(-E/kT)dE \\
&= \exp(-E_\circ/kT)(kTE_\circ^2 + 2(kT)^2 E_\circ) - 2(kT)^3 \exp(-E/kT)\Big|_{E_\circ}^{\infty} \\
&= (kT)^3 \exp(-E_\circ/kT)\left[\left(\frac{E_\circ}{kT}\right)^2 + \frac{2E_\circ}{kT} + 2\right],
\end{aligned}$$

it follows that

$$n(E > E_\circ) = \frac{8\pi}{(hc)^3}(kT)^3 \exp(-E_\circ/kT)\left[\left(\frac{E_\circ}{kT}\right)^2 + \frac{2E_\circ}{kT} + 2\right]. \tag{7.3.4}$$

To calculate the fraction of photons with energy greater than $E_\circ$, $f(E > E_\circ)$, here done by dividing the result in (7.3.4) by the integral from zero to infinity in (7.3.1), which has been evaluated with MATLAB

$$\begin{aligned}\frac{8\pi}{(hc)^3}\int_0^\infty \frac{E^2}{\exp(E/kT)-1}dE &= \left[x = E/kT\right] \\ = \frac{8\pi}{(hc)^3}(kT)^3\int_0^\infty \frac{x^2}{\exp(x)-1}dx &\approx \frac{1}{0.42}\frac{8\pi}{(hc)^3}(kT)^3.\end{aligned} \tag{7.3.5}$$

Finally (7.3.4) is divided by the result in (7.3.5)

$$f(E > E_\circ) = 0.42\exp(-E_\circ/kT)\left[\left(\frac{E_\circ}{kT}\right)^2 + \frac{2E_\circ}{kT} + 2\right]. \tag{7.3.6}$$

8 Simulations

8.1 Calculation of $T_{\text{freeze-out}}$

Another estimate of interest is the temperature at the time of the neutron freeze-out. In section 4.3 equation (4.3.14), that is restated below for convenience, was derived based on the initial assumption that the chemical equilibrium between neutrons and protons ceases when the expansion rate of the Universe equals the n-to-p conversion rate.

$$
\begin{aligned}
H(T) &= \lambda_{n,p}(T) \\
\left(\frac{43\pi aG}{3c^2}\right)^{1/2} T^2 &= \frac{255}{\tau_n x^5}\left(12+6x+x^2\right), \quad x=\frac{Q}{k_B T}
\end{aligned}
$$

In order to solve the above algebraic equation the textscMatlab routine fsolve.m was applied to the stated problem. As a starting guess $T_{\text{guess}} = 10^{10}$K was chosen, in agreement with the most frequently occurring estimates of $T_{\text{freeze-out}}$ in literature [3, 4, 13]. These mathematical statements, together with numerical values on the relevant physical constants, are included in the textscMatlab function-file, 4.3 in Appendix E.

On a final note, the solution $T_{\text{freeze-out}} \approx 7.8965 \cdot 10^9$ K, presented in section 4.3 was used in the program `partphotons.m`, described in section 8.2, to calculate the number of neutrons remaining at the time of the n-to-p freeze-out.

8.2 Simple Predictions

In the program `partphotons.m`, to be used with MATLAB, the abundance of ^{4}He and ^{1}H are calculated according to a very simple model. In this model it is assumed that at freezeout, when neutrons and protons no longer are in thermal equilibrium, the only reaction mechanism at work is the beta decay of the neutrons, see equation (4.3.3) in section E.8 . The program thus only needs to calculate the elapsed time from freezout to the start of BBN in order to determine the number of neutrons that have had time to decay.

For BBN to start deuterium nucleus must not only have begun to form, but accumulate as well. This is only possible when the number of photons with energy equal to, or greater than, the binding energy of deuterium becomes less numerous than the number of deuterium nuclei that are being formed. This follows because in the formation of a deuteron a photon is created according to (8.2.1). Another photon can reverse the reaction, however,

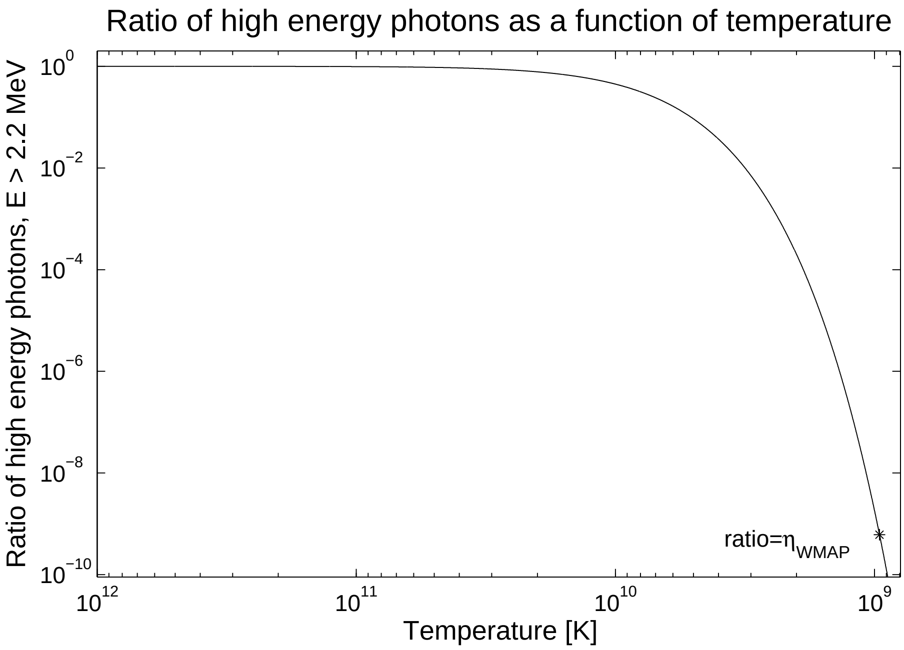

Figure 7: The graph shows the temperature dependence of the fraction of photons with energy greater than the binding energy of the deuteron. The asterisk,$*$, marks the point at which this fraction is equal to η.

and the onset for BBN is therefor approximately taken as the point in time when the fraction of photons, with enough energy, is equal to the baryon-to-photon ratio. In the model at hand it is also inferred that all neutrons and an equal number of protons combine to form ^{4}He, while the remaining protons end up as ^{1}H.

$$\mathrm{p} + \mathrm{p} \rightleftharpoons \mathrm{d} + \gamma \tag{8.2.1}$$

Given a value on the baryon-to-photon ratio, the program also calculates the time and temperature at the onset of BBN. This is done in the following manner. First, the fraction of photons with energy greater than the binding energy of deuterium for a particular temperature, $f(E > E_\circ)$, is calculated, the time evolution of which is depicted in figure 7. More precisely, the program evaluates (7.3.2), without the approximation in (7.3.3). To find $f(E > E_\circ)$ for a certain temperature, this result is then divided by the integral of (7.3.1), taken from zero to infinity. Lastly, the textscMatlab routine `fzero.m` is used to solve for which temperature $f(E > E_\circ)$ equals the baryon-to-photon ration, corresponding to the point marked with an asterisk, $*$, in figure 7.

In order to convert the temperature at the onset of BBN to a corresponding time, equation (C.2.14) is implemented. Thereafter, the time span from freeze-out to BBN, t_{span}, can be used to obtain the number ratio of remaining neutrons to protons, given the initial value, $N_{n,0}$, for this quotient and the mean life time of the neutron, τ, as was discussed in section 4.3.

$$N_n = N_{n,0} \exp(-t_{span}/\tau). \tag{8.2.2}$$

In the program $N_{n,0}$ is calculated, beforehand, with help of equation (4.3.4). Since the mass of the neutron and proton are almost equal, the mass percent of ^{4}He is taken as two times N_n, while that of ^{1}H is assumed to be given by the remainder, $N_{n,0} - N_n$.

The most probable values for the onset of BBN and the abundances of ^{4}He and H ought to be obtained with the value on the baryon to photon ratio presented in section 5.1, mind the simplicity of this pedagogical model. As was pointed out in section 7, however there is much insight to be gained from analyzing the results of such simulations for a range of different values on η. In this case the code described above can easily be used to perform such calculations, for example by evaluating the function in a loop where the baryon to photon ratio is set to a new value at the start of each new iteration. The result of such a procedure, specifically the time for the onset of BBN as a function of η, is represented by the thick line in figure 14.

8.3 Big Bang Nucleosynthesis Using NUC123

It is difficult to find good experimental data to support the previously presented estimations. Therefore, the results are compared to the output of the state-of-the-art software NUC123, written by Lawrence Kawano[29], although the majority of the code stems from the work of R.V. Wagoner.

NUC123 solves the coupled ordinary differential equations related to the nucleosynthesis reaction network using the Runge-Kutta approximation, which is the same method as MATLAB uses to solve ordinary differential equations. All of the reactions within this network that NUC123 takes into account are included in figure 8[29].

Within the program, specifically with help of the user interface, it is also possible to change the model parameters of the standard model, specifically:

- Newton's gravitational constant
- the neutron lifetime
- the number of neutrino species

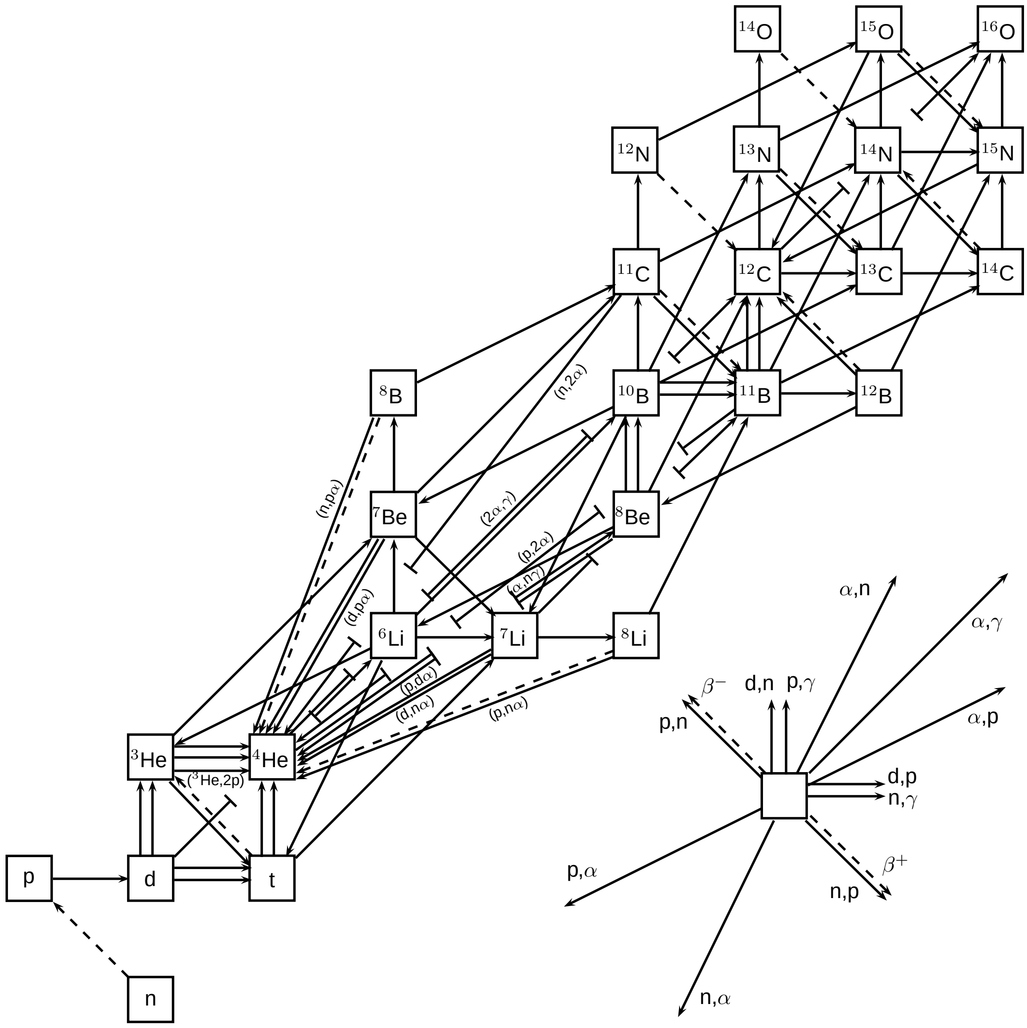

Figure 8: All of the reactions in the reaction network that NUC123 simulates, that is 26 nuclides and 88 reactions.

- the final baryon to photon ratio
- the cosmological constant
- the possibility for neutrino degeneracy

Most of these are either accurately known today or very hard to motivate changes to based on the present accepted theories. For this reason the only parameter to be varied is the different baryon to photon ratio. More precisely, the program will be evaluated for a range of different values on the former, so as to determine its impact on the final abundances.

These calculations are simplified by the fact that modern personal computers are many times more powerful than the computers that NUC123 was originally written for. This enables the use of wrapper software and data analysis using MATLAB. Specifically, the wrapper script runs NUC123 like any normal user, that is by giving certain commands in the user interface, and varies the final baryon to photon ratio in controlled steps. All available data is written onto the disc and finally analyzed using MATLAB. These scripts can be found in appendix E.

All of the data concerning the light elements generated by NUC123 are presented in number densities relative to the number density of hydrogen, except for ^{4}He and p which are presented in mass percentage. This means that these abundances have to be converted to mass percentage as well. Even though this conversion cannot be done exactly, a good approximation is that all nuclei, created in the physical process, is included in the output of the NUC123 program.

8.4 Simulation of the Time Evolution of BBN

Figure 9 presents the results of a run of BBN123 using $\eta = 6.1 \cdot 10^{-10}$[12], while all other parameters were set to their standard values. In figure 11 the final abundances relative to hydrogen can be seen and in figure 10 the change in the abundance of hydrogen is visible. As is indicated by these figures, all simulations stop 28 days after the big bang. Even though there is no reason, with regards to the physical process, for the evaluations to seize when $t \geq 28$ days, this choice does make it possible to illustrate the abundance of free neutrons before they finally decayed. Furthermore, the values obtained, except ^{7}Li, agree well with observed values [3].

The data previously shown is only valid just after the Big Bang Nucleosynthesis has ended, however some elements like tritium or free neutrons are unstable. For comparison, a list with final abundances after a majority of the synthesized neutrons and tritium have decayed are presented in table 5.

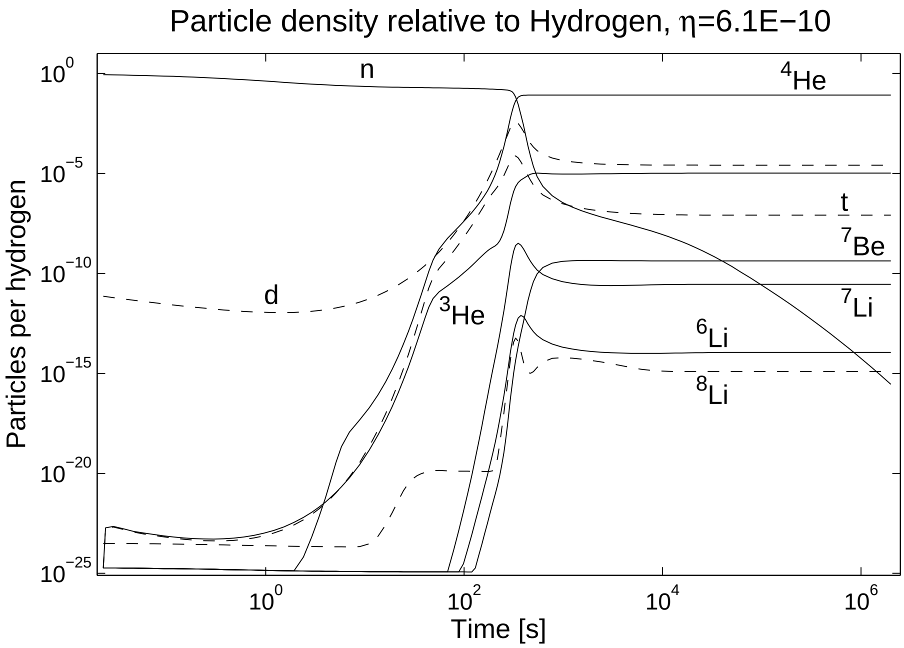

Figure 9: Abundances relative to hydrogen as a function of time.

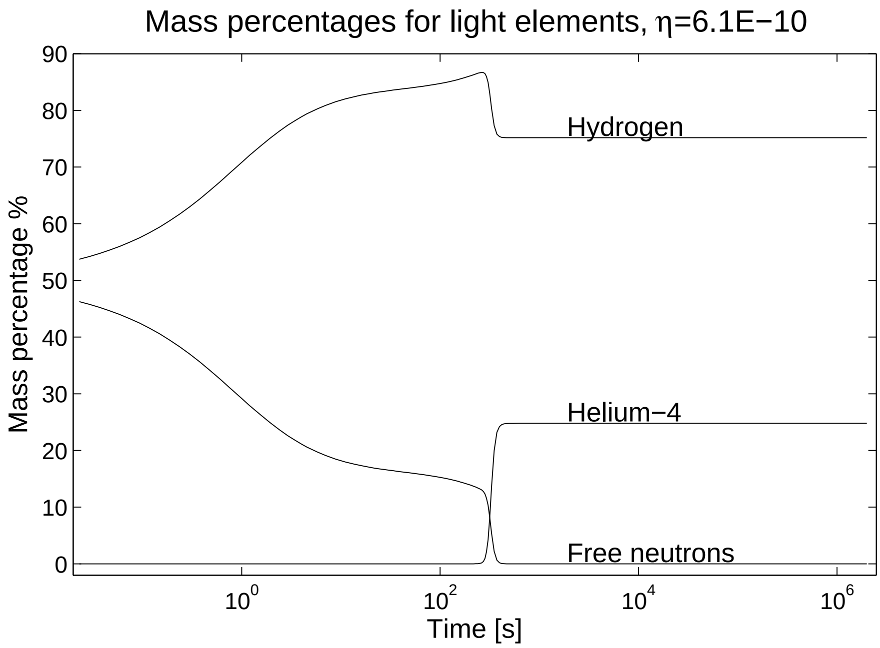

Figure 10: Abundances in mass percentage as a function of time.

Table 5: Table of final abundances after decay, $\eta = 6.1 \cdot 10^{-10}$

Element	Mass percentage	Particles per hydrogen
H	75.2	1
n	0	0
d	3.90E-03	2.58E-05
t	0	0
^{3}He	2.40E-03	1.04E-05
^{4}He	24.8	0.0825
^{6}Li	5.06E-12	1.12E-14
^{7}Li	1.50E-08	2.85E-11
^{8}Li	7.40E-13	1.23E-15
^{7}Be	2.21E-07	4.20E-10

These results have been obtained from a single run of the NUC123 software by only taking into account the predicted values on the elemental abundances after the last time step. Also, the final time has been chosen sufficiently large for the above mentioned species to have decayed though less than ~ 300000 years. This upper limit follows from the fact that the ^{7}Be nuclei becomes unstable once formed into atoms, which in turn is assumed to have occurred around this point in time [9, 14].

Estimates, comparable to those in table 5, can also be obtained based on the calculations presented in 8.2. Specifically, the values in table 6 from part of the output from `partphotons.m`, described in appendix E.7.

Table 6: Simple estimates of light element abundances by mass.

^{1}H	81.80 %
^{4}He	18.20 %

8.5 BBN Calculations for a Range of Value on η

Since the baryon to photon ratio,η, is well determined[12], it might seem unnecessary to regard it as a variable parameter. Even so, the final elemental abundances for a sweep over η is presented in figure 11. While on the subject, the graph in figure 11 corresponding to ^{7}Li is displayed by itself in figure 12 to ease the comparison with the discussion in section 7. Since the abundances

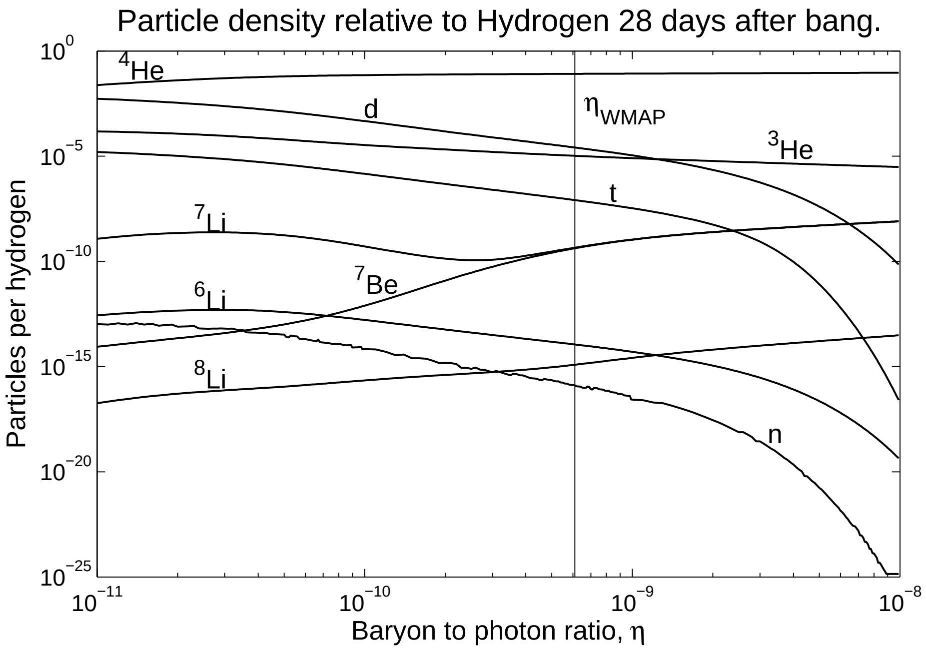

Figure 11: Abundances relative to Hydrogen as a function of baryon to photon ratio.

of ^{4}He and H are significantly greater than those of the other elements, the results for these species are also shown separately in figure 13. The latter diagram also includes graphs, that is the dashed lines, corresponding to the abundances of ^{4}He and H respectively calculated with help of `partphotons.m` for a similar range of values on η.

As became apparent in section 7.3, the nucleosynthesis should start at an earlier time for a greater value on baryon to photon ratio. Remarkably, if the onset of nucleosynthesis is defined as the point in time where there is the highest quantity of deuterium this trend becomes apparent even though there is a large oscillation around the onset time, see figure 14. The thick line in figure 14 represent the corresponding onset times given by the simple estimations done in `partphotons.m`, see section 8.2.

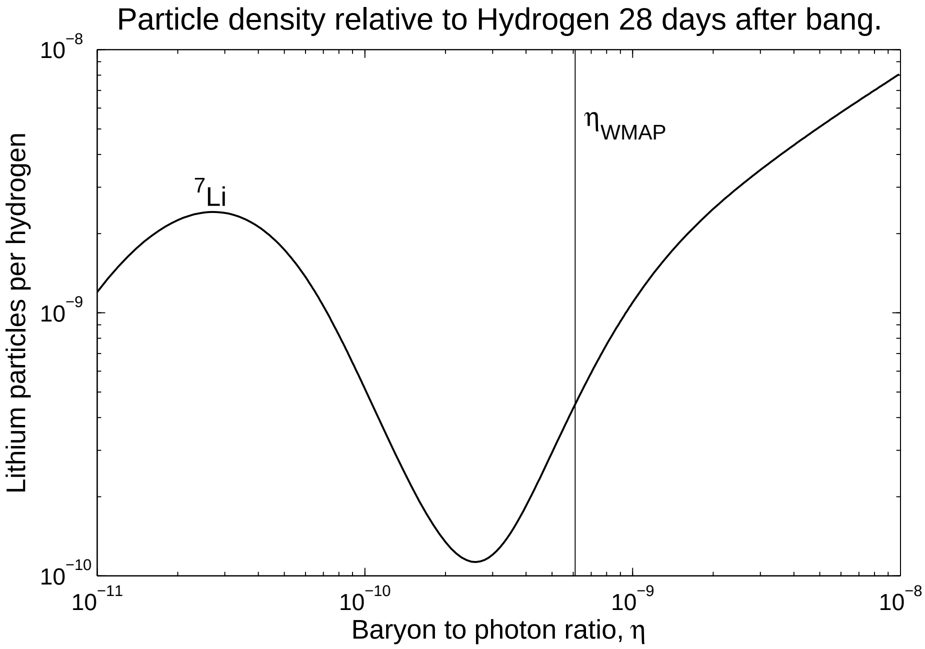

Figure 12: Abundance of ^{7}Li relative to Hydrogen as a function of baryon to photon ratio.

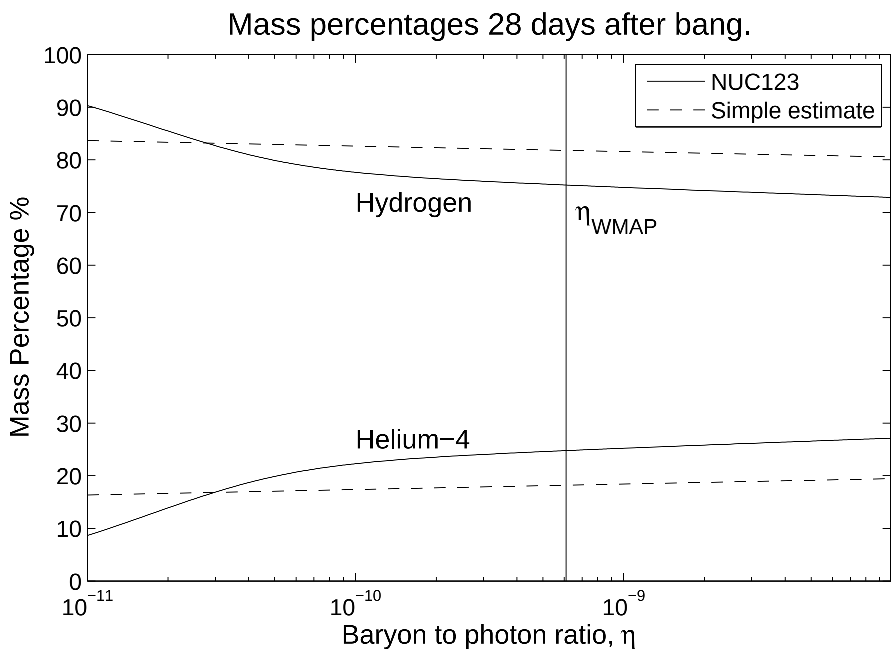

Figure 13: Abundances of ^{4}He and H in mass percentage as a function of baryon to photon ratio, where the dashed and solid lines represent the outputs from `partphotons.m` and NUC123 respectively.

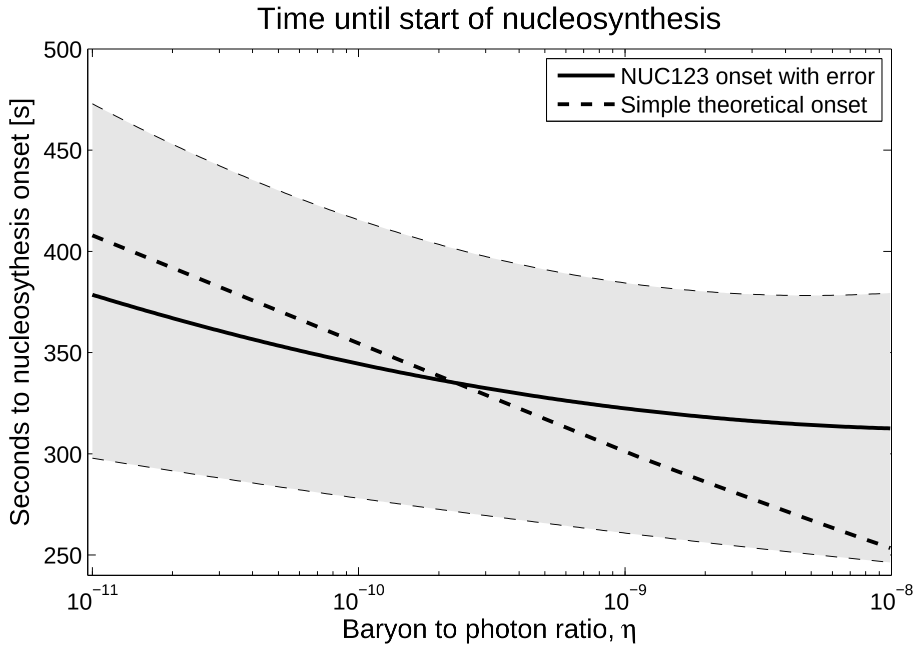

Figure 14: The onset time of BBN as a function of baryon to photon ratio. The thick line is the result of the simple calculation from `partphotons.m` (see appendix E.7).

9 Discussion

Generally, any efforts to describe and make predictions on physical phenomena, assumptions are necessary both in order to get a qualitative understanding of the process and to be able to perform relevant calculations. Obviously, this is also the case for this report. However, to assess the validity for a given simplification one is required to have a deeper understanding of the process or to possess a significant quantity of data, measured with a precision that makes it possible to make valid comparisons with the simulations. For the model at hand, there exists some evidence, namely the ^{7}Li discrepancy discussed below, to suggest that some of the simplifications might be flawed. However, the degree of understanding of the authors of this report is confined to the physics that has been presented so far. It is therefore difficult to point towards particular simplifications that would be less likely to hold if put under scrutiny. Still, an effort shall be made to indicate which assumptions was found to be the most astonishing.

With regards to the primordial nucleosynthesis, the most profound assumption that is probably that all particles, that have not yet decoupled, are in thermal equilibrium. Additionally, no effort has been made to take into account possible concentration gradients and thus the entire Universe has, in fact, been inferred to act as a perfectly stirred reactor. These assumptions seem plausible however given the high kinetic energies of the particles involved in the process and the smallness of the anisotropy in the cosmic background radiation models as well as the fact that none of the alternative models presented so far significantly improves on the standard model predictions [4]. As of yet though, it is not possible to entirely disregard the plausibility that some of these simplifications are inherently flawed.

Another interesting simplification is the assumption that the baryonic content of the Universe only consists of protons and neutron. What is more, this is inferred without giving any compelling reason for why it ought to hold. The same is true for leptons, since the only species under consideration in addition to electrons are the neutrinos and anti-neutrinos.

Moving on to the presented results, what is the most surprising is the accuracy with which the amounts of most light elements created in the nucleosynthesis can be calculated. The degree of correspondence between the very simple calculations, described in sections 8.2, and the more accurate simulations performed by the NUC123 software is also remarkably high, given the roughness of the simplifications in the former. Specifically, a comparison of tables 6 and 5 in section 8.4 reveals that the error in the estimates of ^{4}He and H are in the order of a few mass percent. This is not surprising since the NUC123 simulations, presented in figure 9 and table 5, shows that elements other than H and ^{4}He were only synthesized to a very small extent. In addition, the graphs in figures 13 and 14, in 8.5, that represent calculations with NUC123 and `partphotons.m` respectively are in good agreement as well, perhaps to a less degree with regards to the latter diagram. It is also important to note that the predicted mass percent of ^{4}He at the end of the BBN process is underestimated in the simplified calculations, even though the opposite might be expected given the assumption that all neutrons are fused into ^{4}He nuclei. One plausible explanation for this result could be that texttttfreezeout.m predicts the n-to-p freeze-out to have occurred at a later time compared to NUC123, in which case fewer neutrons survive long enough to take part in the nucleosynthesis process.

While on the subject, one shall not fail to mention that the conclusions drawn regarding the dependencies of the elemental abundances in η discussed in section 7, particularly with respect to table 4, also agrees with the behaviors shown in figure 9 and 12.

In table 7 the output from the NUC123 software are presented together with the primordial abundances given by observations and recent calculations with the latest data on nuclear reaction rates, more precisely cross section measurements [30].

Table 7: Observed and calculated abundances.

	Coc et al. 2010	Calculated	Observed	Factor
^{4}He	0.2486 ± 0.0002	.248	0.232-0.258	$\times 10^0$
D/H	2.49 ± 0.17	2.58	$2.82^{+0.20}_{-0.19}$	$\times 10^{-5}$
^{3}He/H	1.00 ± 0.07	1.04	0.9-1.3	$\times 10^{-5}$
^{7}Li/H	$5.24^{+0.71}_{-0.67}$	0.285	1.1 ± 0.1	$\times 10^{-10}$

When the columns in the above table are compared, the most noticeable discrepancies are seen to concern ^{7}Li. Most importantly, the abundance of ^{7}Li obtained as an output to NUC123 lies one order of magnitude lower than both the results presented by Coc and Vangioni, as well as the observational averages [30]. Yet, it is also obvious that even these most recent simulations, with the baryon-to-photon ratio taken from the WMAP observations, still fail to comply with the observed ^{7}Li abundances. This oddity is very important as it may point to errors in NUC123 or even the standard model itself. The poor agreement between the two simulations however, could perhaps be explained by the fact that the results given in the second column are based on more recent estimates of the relevant physical parameters, most importantly the nuclear cross sections [30]. Additionally, the system of differential equation was solved using a more sophisticated computer program, specifically with the help of Monte-Carlo calculations [30].

On a final not, it might appear as if the standard model for the Big Bang, as in the case of classical physics in the 19$^{\text{th}}$ century, gives an almost complete description of the underlying physics of the processes it describes. It is furthermore often indicated that the few flaws that do exist are only minor in nature. Still, one is compelled to draw a comparison between the ^{7}Li deficiency and the problem with the black body radiation in classical physics. Thus, it seems likely that the present theory for the standard Big Bang model in general and BBN in particular will change rather dramatically sometime in the not so distant future.

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A Glossary

BBN	Big Bang nucleosynthesis (*also primordial n.*), refers to the synthesis of light elements a few minutes after the Big Bang.
CBR	Cosmic background radiation, a cosmic electromagnetic radiation that is found throughout the Universe.
CMB	Cosmic microwave background, the CBR at its present temperature.
Helmholtz energy	The amount of useful work obtainable from a closed system.
Planck epoch	The very earliest time period of the Universe lasting from the bang to 10^{-43} s.
Planck instant	When $t = 10^{-43}$ s, that is when the Planck epoch was over.
QCD	Quantum chromodynamics, a theory for the strong interaction (*or force*) that describes the interactions of quarks and gluons.
annihilation	When matter and anti matter react to form photons.
antimatter	Matter that has the same properties as regular matter but when it comes in contact with regular matter both annihilates to photons.
baryon	Hadrons that are made from three quarks.
baryon density	The density of baryons in the Universe, usually protons and neutrons.
baryon genesis/baryogenesis	The process through which an asymmetry between baryons and anti-baryons was created to favor regular matter that make up the universe today.
baryon number	A quantum number, the quarks have baryons number 1/3 and anti quarks have -1/3, the baryon number is conserved in all reactions.
boson	Particles that carry the four forces, have whole integer spin and follow Bose-Einstein statistics.

chemical evolution	The evolution of the composition of matter in the Universe through nucleosynthesis, both in the early Universe and in stars.
cosmological principle	A principle which states that the universe is homogeneous and uniform, and that an observer sees the same object regardless of vantage point and that the physical principles hold true in the whole Universe.
decoupling	A process during which a type of particle that can interact ceases to do so because the rate of the interaction drops below that of the universal expansion.
electromagnetic force	*Also the electromagnetic interaction*. It is a fundamental force through which charged particle interact by exchanging photons in electromagnetic fields.
electron-positron pair	A pair that consists of an electron and its anti particle, the positron.
energy density	The amount of energy of a system divided by its volume.
exotics	Physics that is not included in any theory generally accepted by the scientific community.
fermion	A particle with half integer spin and that follows Fermi-Dirac statistics.
fundamental forces	A force that can not be described in terms of other forces.
fundamental particle	Particle that can not be divided further into smaller particles.
gluon	Boson for the strong interaction.
graviton	Hypothetical boson for gravity.
gravity	A fundamental force that acts on all massive particles.
hadron	Particles that can interact with the strong force.
hadron gas	A gas of hadrons, that is baryons and mesons.
kinetic energy	The energy of an object related to its speed.

latent heat	The energy change that a chemical substance emits or absorbs when going through an isothermal phase transition.
lepton	Particles that can not interact with the strong force.
light elements	Usually refers to hydrogen, helium and lithium.
mesons	Hadrons that are made from one quark and one anti quark.
muon	An unstable massive lepton that is not as unstable as the tauon and can replace electrons in ordinary matter.
neutrino	Fundamental particle, each family of leptons has a neutrino species, or flavour, associated with it.
neutrino flavors/species	In the standard model the leptons have three families of particles, each with an associated neutrino. In other non-standard models the number of families, and thereby the number different kinds of neutrinos, may be different.
nuclear reactions	A reaction between nuclei or other subatomic particles.
particle	Refers to elementary particles such as photons, electrons, protons, neutrons, etc.
phase transition	When matter goes from one phase to another such as ice melting to water.
physical constant	A constant, such as Planck's constant or the speed of light, which are thought to be universal and non-transient.
primeval abundances	The abundances of elements created in BBN.
primordial nucleosynthesis	see BBN
quantum effects	Phenomena at the subatomic level explained only by quantum mechanics, such as tunneling.
quantum gravity	An effort to try to unite quantum mechanics and general relativity in consistent theory.

quark	Fundamental particles that are the building blocks of hadrons.
quark-gluon plasma	A phase in QCD that existed at very high temperatures in which quarks and gluons could exist freely without having to form baryons and mesons..
radiation	The transportation of energy with particles such as photons, alpha particles, electrons, neutrons, etc.
relativistic particles	A particle whose speed is or is close to that of light.
scale factor	A transient function that scales distances in the universe at different times with respect to to distances in the present.
strong force	The strong force (*also strong interaction or strong nuclear force*) is a fundamental force that holds protons and neutrons together in a nucleus as well as the protons and neutrons themselves. At sufficiently low energies it is the strongest force.
tauon	A very massive lepton that is unstable. Also, the only lepton that can decay into hadrons.
transient scalar field	A scalar field that changes with time. A scalar field is a field that associates a scalar value to every point in a predefined domain, such temperature or pressure.
universal expansion	The expansion of the Universe, that is the expansion of space itself.
W^+,W^-,Z	Bosons for the weak interaction.
weak force	*Also the weak interaction or the weak nuclear force.* A fundamental force that, among other things, is responsible for beta decay.

B List of Symbols

Latin symbols

a	Stefan's constant
A	nucleon number
c	speed of light in vacuum
E	energy
F	Helmholtz energy
g	number of degrees of freedom
G	gravitational constant
h	dimensionless Hubble constant, Planck's constant
H	Hubble parameter
H_0	Hubble constant
k	curvature parameter
k_B	Boltzmann's constant
m_b	mass per baryon
m_n	mass of neutron
m_p	proton mass
m_{v_f}	mass of neutrino
n_b	baryon number density
n_i	number density of species i
$\bar{n}_i$	mean occupation number of species i
N_γ	number density of photons
p	pressure
q	particle momentum
Q	massdifference between the proton and neutron in eV
R	scale factor
s	entropy density
S	entropy
t	time
T	temperature
u	energy density of photons
U	internal energy
V	volume

Greek symbols

ϵ	energy density of the Universe
η	baryon-to-photon ratio
λ	wavelength, conversion rate
Λ	cosmological constant
μ	chemical potential
ν	frequency
ρ	sum of the mean mass and energy density of the Universe
ρ_b	baryon mass density
ρ_c	critical mass density
ρ_R	energy content of radiation per unit volume
σ	Stefan-Boltzmann constant
τ	mean lifetime
Ω_m	critical density

C Elaborate Deduction of $t(T)$

C.1 $t(T)$ for Temperatures $10^{12}\,\mathrm{K} > T > 5.5 \cdot 10^{9}\,\mathrm{K}$

According to the cosmological principle the Universe can be regarded, on sufficiently large scales, as a perfect fluid [13, 31, 32]. Specifically, the latter is defined as a distribution of matter in which a co moving observer, at rest with respect to the overall motion of the fluid, would regard each direction drawn from its position as equivalent [13, 31, 32]. Under this assumption, it is possible to derive a relation, (C.1.1), between the energy density, pressure and expansion rate of the Universe from Einsteins equations [13].

$$\dot{\epsilon} + 3(p+\epsilon)\frac{\dot{R}}{R} = 0, \tag{C.1.1}$$

where ϵ and p are the energy density and pressure of the Universe respectively, whereas R is the so called scale factor [13]. Even though the nature of the latter is still unknown, one can describe it as the time dependent part of the distance between any two points of the Universe [13]. In other words, one can define the distance between any two points A and B moving with the expanding Universe as $f_{AB}R(t)$ where f_{AB} is independent of time [13].

It is possible to rewrite (C.1.1) into (C.1.3) for times when the energy density of the Universe was dominated by the contributions from radiation and relativistic particles, since (C.1.2) holds true under such condition [13]. [2]

$$p = \frac{1}{3}\epsilon \tag{C.1.2}$$

$$\begin{aligned} 0 &= \dot{\epsilon} + 3(p+\epsilon)\frac{\dot{R}}{R} = \dot{\epsilon} + 3\left(\frac{1}{3}\epsilon + \epsilon\right)\frac{\dot{R}}{R} \\ &\Leftrightarrow \dot{\epsilon} + 4\epsilon\frac{\dot{R}}{R} = 0 \\ &\Leftrightarrow \frac{\dot{R}}{R} = -\frac{1}{4}\frac{\dot{\epsilon}}{\epsilon}. \end{aligned} \tag{C.1.3}$$

Deducing (C.1.1), (C.1.2) and (C.1.4) requires the application of Einstein's equations within the Friedmann models, a procedure that is far beyond the scope of this text and as such shall not be attempted. Even so, it is

[2] In actuality, (C.1.2) follows from a more general equation of state $p(\epsilon)$ such that $p(\epsilon) \to \frac{1}{3}\epsilon$ as $R \to 0$ [13].

of importance that the final steps of the derivation rests on the assumption that R small, which in turn infers that $t \approx 0$ [13]. In other words, (C.1.4) holds true only in the early Universe [13]. What is more, the terms multiplied by the factor k, that defines the curvature of space, drops out of the original expressions if $R \to 0$ and as such does not appear in (C.1.4) [13]. Thus, the expansion in the early Universe is independent of whether the Universe is open or closed [13].

Next, given (C.1.4) one can replace the left hand side of (C.1.3) by (C.1.6) and thus ascertain (C.1.7) [13].

$$\dot{R}^2 = \frac{8\pi G}{c^2}\epsilon R^2 \tag{C.1.4}$$

$$\Leftrightarrow \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{c^2}\epsilon \tag{C.1.5}$$

$$\Leftrightarrow \frac{\dot{R}}{R} = \sqrt{\frac{8\pi G}{3c^2}\epsilon}, \tag{C.1.6}$$

$$-\frac{1}{4}\frac{\dot{\epsilon}}{\epsilon} = \frac{\dot{R}}{R} = \sqrt{\frac{8\pi G}{3c^2}\epsilon}$$

$$\Leftrightarrow \dot{\epsilon} = -4\left(\frac{8\pi G}{3c^2}\epsilon\right)^{1/2}\epsilon$$

$$\Leftrightarrow \dot{\epsilon} = -4\left(\frac{8\pi G}{3c^2}\right)^{1/2}\epsilon^{3/2}. \tag{C.1.7}$$

where, $G \approx 6.6726 \cdot 10^{-11}\,\mathrm{Nm^2/kg^2}$ is Newtons gravitational constant and $c = 299792458\,\mathrm{m/s}$ the velocity of light, in vacuum [11, 13]. Also, notice that (C.1.6) is taken to be the non negative solution of (C.1.5). This is the only plausible choice given the above mentioned definition of R and the assumption that the Universe is constantly expanding, whereby we must have $R > 0$ and $\dot{R} = \frac{\partial R}{\partial t} > 0$.

From (C.1.7) one can deduce (C.1.8) by first separating variables and then integrating both sides of the equation.

$$\begin{aligned}
& \dot{\epsilon} = \frac{\partial \epsilon}{\partial t} = -4\left(\frac{8\pi G}{3c^2}\right)^{1/2} \epsilon^{3/2} \\
\Leftrightarrow \quad & \epsilon^{-3/2}\frac{\partial \epsilon}{\partial t} = -4\left(\frac{8\pi G}{3c^2}\right)^{1/2} \\
\Rightarrow \quad & \int \epsilon^{-3/2} d\epsilon = \int -4\left(\frac{8\pi G}{3c^2}\right)^{1/2} \\
\Leftrightarrow \quad & \frac{\epsilon^{-1/2}}{-1/2} + \text{const.} = -4\left(\frac{8\pi G}{3c^2}\right)^{1/2} t \\
\Leftrightarrow \quad & t = \left(\frac{3}{32\pi G}\right)^{1/2} c\epsilon^{-1/2} + \text{const.}. \qquad \text{(C.1.8)}
\end{aligned}$$

Though (C.1.8) is an explicit expression for the time t after the big bang, the temperature dependence is as of yet hidden inside the energy density $\epsilon(T)$. As such, one now seek to replace ϵ with an equivalent function of T.

During the time period when $10^{12}\,\mathrm{K} > T > 5.5 \cdot 10^{9}\,\mathrm{K}$ the greatest contribution to the total energy density came from relativistic species [13]. Specifically, the plasma consisted mainly of photons, electron-positron pairs as well as electron- and muon-neutrinos together with their antiparticles, denoted by γ, e^-, e^+, ν_e, ν_μ, ν_τ, $\bar{\nu_e}$, $\bar{\nu_\mu}$ and $\bar{\nu_\tau}$ respectively [13]. Fortunately, only negligible quantities of protons, neutrons and electrons, lacking positron partners, were mixed into the relativistic gas [13]. Therefore, these latter three need not be taken into account when calculating the energy density of the Universe, obtained by summing the individual contributions (C.1.9) for each species i [13].

$$\epsilon_i = \int_0^\infty E_i(q) n_i(q) dq. \qquad \text{(C.1.9)}$$

where the integral is taken over the entire momentum space [13]. This requires however that the number densities $n_i(q)$ for each type of relevant particle is known for an arbitrary particle momentum, q, and temperature respectively. [3]

By definition $n_i(q)dq$ equals the total number of particles i per unit volume with a momentum in the interval $[q, q + dq]$. Additionally, if the gas can be regarded as an ideal, that is if the particle interactions can be neglected, each particles species can be described as a grand canonical ensemble. That

[3] Note that the derivation that follows has been based solely on fundamental equations in statistical physics and furthermore has , as of yet, not been verified in its entirety.

is, a system with a non-constant number of particles held at a particular temperature T, where the mean number of particles is approximately given by (C.1.10) [11].

$$\bar{N}_i = \sum \bar{n}_{i,j} \approx \frac{g}{h^3} \iint \bar{n}_i dq dp, \tag{C.1.10}$$

where the integrals are taken over the generalized one-particle momentum and spatial coordinates q and p respectively, while g is the number of degrees of freedom in the system [11].

The assumption of gas ideality allows one to predict the distribution of states for fermions and bosons by Fermi-Dirac and Bohr-Einstein statistics respectively [11]. Specifically, the mean occupation numbers, that is the average number of particles in a state j defined by a certain one-particle energy ϵ_j, at a specific temperature T are given by (C.1.11) for bosons and (C.1.12) for fermions. Here μ is the chemical potential of the species and k_B the Boltzmann constant.

$$\bar{n}_{i,j} = \frac{1}{\exp\left(\dfrac{E_{i,j} - \mu_i}{k_B T}\right) - 1}, \tag{C.1.11}$$

$$\bar{n}_{i,j} = \frac{1}{\exp\left(\dfrac{E_{i,j} - \mu_i}{k_B T}\right) + 1}. \tag{C.1.12}$$

Although the energy levels, E_i,j, are discrete, these shall be substituted for a continuous distribution $E_i(p,q)$. Admittedly, this approach is somewhat flawed, but in return greatly simplifies the calculations since all particles of interest behave relativistically and as such have energies given by (C.1.13) [13].

$$E_i(p,q) = c(q^2 + m_i^2 c^2)^{1/2}. \tag{C.1.13}$$

It is convenient to introduce an approximate and continuously distributed mean occupation number, (C.1.14), as a function the generalized coordinates, p, and momentum, q.

$$\bar{n}_i(p,q) = \frac{1}{\exp\left(\dfrac{E_i(p,q) - \mu_i}{k_B T}\right) \pm 1}. \tag{C.1.14}$$

where the plus sign is for fermions and the minus sign for bosons. Since no particular restraints has been put on the system in defining the problem at hand, the system can be described in terms of Cartesian coordinates [33].

Accordingly (C.1.10) can be written as (C.1.15), where the $dV_{\mathbf{r}}$ and $dV_{\mathbf{q}}$ are infinitesimal volume elements in the spatial coordinate space and momentum space respectively.

$$\bar{N}_i \approx \frac{g}{h^3} \iint \bar{n}_i dV_{\mathbf{r}} dV_{\mathbf{q}}. \tag{C.1.15}$$

The total energy in (C.1.13) is independent of the spatial coordinate, $\mathbf{r}$, and depends only on the magnitude of the momentum, $q = |\mathbf{q}|$. Therefore the mean number density distribution should, by (C.1.14), show a similar dependence. , Formula (C.1.15) can hence be rewritten into the form (C.1.16), where the system volume $V_{\mathbf{r}} = \int dV_{\mathbf{r}}$ has been introduced.

$$\begin{aligned}
& \bar{N}_i \approx \frac{g}{h^3} \iint \bar{n}_i(q) dV_{\mathbf{r}} dV_{\mathbf{q}} \\
\Leftrightarrow \quad & \bar{N}_i = \frac{g}{h^3} \int dV_{\mathbf{r}} \int \bar{n}_i(q) dV_{\mathbf{q}} \\
\Leftrightarrow \quad & \bar{N}_i = \frac{g}{h^3} V_{\mathbf{r}} \int \bar{n}_i(q) dV_{\mathbf{q}} \\
\Leftrightarrow \quad & \frac{\bar{N}_i}{V_{\mathbf{r}}} = \frac{g}{h^3} \int \bar{n}_i(q) dV_{\mathbf{q}},
\end{aligned} \tag{C.1.16}$$

The volume element, with regards to the integral over momentum space, is given by $dV_{\mathbf{q}} = dq_x dq_y dq_z$ in Cartesian coordinates, that is if $\mathbf{q} = q_x \hat{\mathbf{e}}_x + q_y \hat{\mathbf{e}}_y + q_z \hat{\mathbf{e}}_z$. Nevertheless, it is convenient to transform the problem into spherical coordinates. In so doing, the the volume element must be substitute for its spherical equivalent, $dq_x dq_y dq_z = q'^2 sin(\theta) dq' d\theta d\phi$. As mentioned, $\bar{n}_i(q)$ is independent of the angular coordinates, θ and ϕ, for which reason (C.1.17) follows from (C.1.16) almost by default.

$$\begin{aligned}
& \frac{\bar{N}_i(q)}{V_{\mathbf{r}}} = \frac{g_i}{h^3} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{q'=0}^{q} \bar{n}_i(q') q'^2 sin(\theta) dq' d\theta d\phi \\
\Leftrightarrow \quad & \frac{\bar{N}_i(q)}{V_{\mathbf{r}}} = \frac{g_i}{h^3} \int_{\phi=0}^{2\pi} d\phi \cdot \int_{\theta=0}^{\pi} sin(\theta) d\theta \cdot \int_{q'=0}^{q} \bar{n}_i(q') q'^2 dq' \\
\Leftrightarrow \quad & \frac{\bar{N}_i(q)}{V_{\mathbf{r}}} = \frac{g_i}{h^3} 2\pi \cdot 2 \cdot \int_{q'=0}^{q} \bar{n}_i(q') q'^2 dq' \\
\Leftrightarrow \quad & \frac{\bar{N}_i(q)}{V_{\mathbf{r}}} = \frac{4\pi g_i}{h^3} \int_{q'=0}^{q} \bar{n}_i(q') q'^2 dq'.
\end{aligned} \tag{C.1.17}$$

From the expression for the average number of particles in the Universe per unit volume, (C.1.17), one finds $n_i(q)dq$ by differentiating (C.1.17) with respect to q, finally arriving at (C.1.18) after substituting $\bar{n}_i(q)$ for (C.1.14).[4]

$$\begin{aligned} n_i(q)dq &= \frac{\partial}{\partial q}\frac{\bar{N}_i(q)}{V_{\mathbf{r}}}dq = \frac{4\pi g_i}{h^3}\bar{n}_i(q)q^2dq \\ \Leftrightarrow \quad n_i(q)dq &= \frac{4\pi g_i}{h^3}\frac{q^2dq}{\exp\left(\dfrac{E_i(p,q)-\mu_i}{k_BT}\right)\pm 1}. \end{aligned} \tag{C.1.18}$$

The formula (C.1.19) for ϵ_i is then obtained by introducing (C.1.18) in (C.1.10), together with (C.1.13).

$$\epsilon_i = \int_0^\infty \frac{c(q+m_ic^2)^{1/2}4\pi g_i q^2 dq}{h^3\exp\left(\dfrac{c(q+m_ic^2)^{1/2}-\mu_i}{k_BT}\right)\pm 1}. \tag{C.1.19}$$

The most immediate concern at this point, with regards to the evaluation of (C.1.19), is to find suitable expressions fort the chemical potentials. Firstly, both the photon and the different neutrino flavours have zero chemical potential, under the assumption that the standard model holds and that these species are in thermal equilibrium [13, 34]. This follows from the fact that there are no conservation laws that governs the number of such particles in a particular system, unlike for instance electrons and positrons [13, 34, 35]. In other words, the reactions through which these species are created, or destroyed, take place simply if there is sufficient energy available for such transitions. These processes are for example not constrained by a requirement that charge be conserved [34, 35].

In proving this statement one can analyze a system of volume V held at a temperature T containing a total of N particles. Without any conversion laws, N would take on a value that minimizes the free, Helmholtz, energy F of the system and thus (C.1.20) must hold true [34].

$$\left(\frac{\partial F}{\partial N_\gamma}\right)_{T,V} = 0. \tag{C.1.20}$$

By definition, the chemical potential is the amount by which the internal energy changes when a particle is added to the system. As a result, the

[4] Note that Islam presents an equation, 8.35, that , in comparison to 8.52a and 8.52b derived therefrom and most critically (C.1.18) in this text, lacks an exponent -1 on the right hand side [13].

second and third laws of thermodynamics takes on the form (C.1.21) when the number of particles N in the system is non constant [34].

$$dU = TdS - pdV + \mu dN. \tag{C.1.21}$$

From the definition of the Helmholtz energy, $F = U - TS$, and (C.1.21) one finds that a differential change, dF, of the same can be written as (C.1.22) [34].

$$dF = -SdT - pdV + \mu dN. \tag{C.1.22}$$

Hence, the expression (C.1.23) for μ, for a system with fixed volume and temperature, can be deduced [34].

$$\mu = \left(\frac{\partial F}{\partial N_\gamma}\right)_{T,V}. \tag{C.1.23}$$

According to (C.1.20) and (C.1.23), particle species not governed by any conservation laws must have zero chemical potential, $\mu_\gamma = 0$ [34]. This does not hold true for electrons and positrons, however. Instead, it will be shown that it is reasonable to assume that the chemical potentials of both species vanish [13].

Fundamentally, these particles are being annihilated and created through the reaction (C.1.24) [34, 35].

$$e^- + e^+ \rightleftarrows \gamma \tag{C.1.24}$$

This conversion is further constrained by the conservation of charge $N_+ - N_- = N_0$, where N_0 is constant, while N_+ and N_- are the number of positrons and electrons respectively [34, 35].

In turn, the free energy of the Universe depends on T and V, both of which are held constant. Furthermore, this energy can be chosen to depend explicitely on any pairwise combination of N_0, N_- and N_+, since these are not independent variables [35]. As such one can state the conservation law on the form (C.1.25) [35].

$$F(T,V,N_0,N_+) \equiv F(T,V,N_0,N_-) \equiv F(T,V,N_-,N_+ = N_- - N_0). \tag{C.1.25}$$

If the system is at equilibrium, the Helmholtz energy must be minimal with respect to N_-, or equally N_+, as stated in (C.1.26) [34, 35].

$$\left(\frac{\partial F}{\partial N_-}\right)_{T,V,N_0} = 0. \tag{C.1.26}$$

In addition, F ought to be a sum of two terms corresponding to the positron and electron contributions to the free energy respectively, $F(T,V,N_-,N_+) = F_+(T,V,N_+) + F_-(T,V,N_-)$ [35]. Furthermore, this equality implies that (C.1.27) holds true, since both $F_+ = F_+(T,V,N_+ = N_0 - N_-)$ and $F_- = F_-(T,V,N_- = N_0 - N_+$ only depend on N_0 implicitely.[5]

$$\begin{aligned}\left(\frac{\partial F}{\partial N_-}\right)_{T,V,N_+} &= \left(\frac{\partial F_-}{\partial N_-}\right)_{T,V,N_+} = \left(\frac{\partial F_-}{\partial N_-}\right)_{T,V,N_0},\\ \left(\frac{\partial F}{\partial N_+}\right)_{T,V,N_-} &= \left(\frac{\partial F_+}{\partial N_+}\right)_{T,V,N_-} = \left(\frac{\partial F_+}{\partial N_+}\right)_{T,V,N_0}.\end{aligned} \tag{C.1.27}$$

With (C.1.27) and the fact that $N_+ = N_- - N_0$, one can then deduce (C.1.28) from (C.1.26) [34, 35].

$$\begin{aligned}& 0 = \left(\frac{\partial F}{\partial N_-}\right)_{T,V,N_0}\\ \Leftrightarrow\quad & 0 = \left(\frac{\partial F_-}{\partial N_-}\right)_{T,V,N_0} + \left(\frac{\partial F_+}{\partial N_-}\right)_{T,V,N_0}\\ \Leftrightarrow\quad & 0 = \left(\frac{\partial F_-}{\partial N_-}\right)_{T,V,N_0} + \left(\frac{\partial F_+}{\partial N_+}\right)_{T,V,N_0}\frac{dN_+}{dN_-}\\ \Leftrightarrow\quad & 0 = \left(\frac{\partial F_-}{\partial N_-}\right)_{T,V,N+} + \left(\frac{\partial F_+}{\partial N_+}\right)_{T,V,N_-}\frac{dN_+}{dN_-},\\ & \{N_+ = N_- - N_0 \Rightarrow \frac{dN_+}{dN_-} = 1\},\\ \Leftrightarrow\quad & \left(\frac{\partial F}{\partial N_-}\right)_{T,V,N+} + \left(\frac{\partial F}{\partial N_+}\right)_{T,V,N_-} = 0.\end{aligned} \tag{C.1.28}$$

Next, (C.1.28) is reduced to (C.1.29) by applying (C.1.23) [34, 35].

$$\mu_- + \mu_+ = 0. \tag{C.1.29}$$

where μ_+ and μ_- are the chemical potential of positrons and electrons respectively.

[5] Note that neither (C.1.27) nor (C.1.28), which is shown to follow from the former by implication, has been adopted from any reference. Instead, these merely state that F should be a sum of two terms, one for the electron and positron respectively, and that (C.1.28) must therefore be correct [34, 35]

There must have been a excess of electrons compared to positrons, since otherwise no atoms would have been able to form after the e^+-e^- pair annihilation. However, as mentioned earlier such electrons were very rare at the relevant temperatures [13]. As such, it is not far fetched to assume that the chemical potentials of both species vanish [13]. Specifically, this would mean that the number densities of positrons and electrons were approximately equal. This statement in turn can be shown to imply that $\mu_- \approx \mu_+ \approx 0$. Specifically, one first form the equality $n_+ \approx n_-$, then substitute n for (C.1.18) before introducing (C.1.29).

$$
\begin{aligned}
& n_+ dq = n_- dq \\
\Rightarrow \quad & \frac{4\pi g_+}{h^3} \frac{q^2}{\exp\left(\dfrac{E_+(p,q)-\mu_+}{k_B T}\right)+1} = \frac{4\pi g_-}{h^3} \frac{q^2}{\exp\left(\dfrac{E_-(p,q)-\mu_-}{k_B T}\right)+1}, \\
& \{g_+ = g_-\}, \\
\Rightarrow \quad & \frac{1}{\exp\left(\dfrac{E_+(p,q)-\mu_+}{k_B T}\right)+1} = \frac{1}{\exp\left(\dfrac{E_-(p,q)-\mu_-}{k_B T}\right)+1} \\
\Rightarrow \quad & \exp\left(\frac{E_+(p,q)-\mu_+}{k_B T}\right)+1 = \exp\left(\frac{E_-(p,q)-\mu_-}{k_B T}\right)+1 \\
\Rightarrow \quad & \exp\left(\frac{E_+(p,q)}{k_B T}\right)\exp\left(\frac{-\mu_+}{k_B T}\right) = \exp\left(\frac{E_-(p,q)}{k_B T}\right)\exp\left(\frac{-\mu_-}{k_B T}\right), \\
& \{E_-(p,q) = E_+(p,q)\}, \\
\Rightarrow \quad & \exp\left(\frac{-\mu_+}{k_B T}\right) = \exp\left(\frac{-\mu_-}{k_B T}\right), \\
& \{\mu_+ = -\mu_-\}, \\
\Rightarrow \quad & \mu_+ = \mu_- = 0.
\end{aligned}
$$

Additionally, the highly relativistic nature of these particles at the temperature at hand means that they can be regarded as being approximately massless [13]. More to the point, the magnitudes of the momentas of both e^+ and e^- are very high, since particle speeds are close to that of light, during the times considered. Consequently, the term mc^2 is negligible compared to q^2 on the right hand side of equation (C.1.13) for the relativistic energy. The same holds true for photons and neutrinos, at least according to the standard particle model, though not by approximation /citeIslam.

For the reasons given above, (C.1.19) reduces to (C.1.30) for each particle species so far considered.

$$
\begin{aligned}
\epsilon_i &= \frac{4\pi c g_i}{h^3} \int_0^\infty \frac{q^2(q^2 + m_i^2 c^2)^{1/2} dq}{\exp(c(q^2 + m_i^2 c^2)^{1/2} - \mu_i)/k_B T) \pm 1} \\
\Rightarrow \quad \epsilon_i &= \frac{4\pi c g_i}{h^3} \int_0^\infty \frac{q^3 dq}{\exp(cq/k_B T) \pm 1}. \qquad \text{(C.1.30)}
\end{aligned}
$$

In further rewriting (C.1.30), one obtains a more easily evaluated expression for the energy densities ϵ_i, specifically (C.1.31) where $x = cq/k_B T \Rightarrow dx = cdq/k_B T$.

$$
\begin{aligned}
\epsilon_i &= \frac{4\pi c g_i}{h^3} \int_0^\infty \frac{(k_B T/c)^3 (cq/k_B T)^3 (k_B T/c)(cdq/k_B T)}{\exp(cq/k_B T) \pm 1} \\
\Leftrightarrow \quad \epsilon_i &= \frac{4\pi c g_i}{h^3} \left(\frac{k_B T}{c}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x \pm 1} \\
\Leftrightarrow \quad \epsilon_i &= \frac{4\pi g_i k_B^4}{(ch)^3} T^4 \int_0^\infty \frac{x^3 dx}{e^x \pm 1}. \qquad \text{(C.1.31)}
\end{aligned}
$$

The integral on the right hand side of (C.1.31) can be evaluated by applying (D.2.3) and (D.2.4), deduced in appendix D.2, with $m - 1 = 3$ for fermions and bosons respectively. With help of, for example, a suitable table of sums, this results in the final expressions (C.1.32) for bosons and (C.1.33) for fermions [11].

$$
\begin{aligned}
\sum_{n=1}^\infty \frac{1}{n^m}\bigg|_{m=4} &= \frac{\pi^4}{90} \\
\Rightarrow \quad m! \sum_{n=1}^\infty \frac{1}{n^m}\bigg|_{m=4} &= 6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15} \\
\Rightarrow \quad \int_0^\infty \frac{x^3 dx}{e^x - 1} &= \frac{\pi^4}{15} \qquad \text{(C.1.32)} \\
\Rightarrow \quad \int_0^\infty \frac{x^3 dx}{e^x + 1} &= \frac{7}{8}\frac{\pi^4}{15}. \qquad \text{(C.1.33)}
\end{aligned}
$$

Lastly, one must take into account the number of spin degrees of freedom, g_i, for the relevant species. Reasonably, $g_i = 2$ for both e^- and e^+, since both have spin 1/2 [13]. Conversely, neutrinos although fermions have only one spin degree of freedom, $g_\nu = 1$ [13]. By convention the term helicity is used instead of spin for massless particles. Furthermore, neutrinos with helicity $-1/2$ are termed “left handed” whereas the corresponding “right handed” antineutrinos have helicity +1/2 [16]. Photons on the other hand, have $g_\gamma = 2$. This is one degree less than what would be predicted given only that $s = 1$, since a helicity of zero is not allowed for massless particles [36].

It is now possible to deduce a formula for the energy density in the early Universe as a function of temperature only. More precisely, ϵ is obtained by summing the contributions (C.1.35) through (C.1.37) for the relativistic particles present at those times, γ, ν_e, $\bar{\nu}_e$, ν_μ, $\bar{\nu}_\mu$, ν_τ, $\bar{\nu}_\tau$ as well as e^- and e^+ [13].[6]. However, it is convenient to first introduce Stefan's constant, as defined by (C.1.34), in the energy density expressions. This yields the formula (C.1.35) for ϵ_γ that is more easily recognized as Stefan-Boltzmann law [11].

$$a = \frac{4}{c}\sigma = \frac{8\pi^5 k_B^4}{15(hc)^3}, \tag{C.1.34}$$

$$\begin{aligned}
& \epsilon_\gamma = \frac{4\pi g_\gamma k_B^4}{(ch)^3} T^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \\
\Leftrightarrow \quad & \epsilon_\gamma = \frac{8\pi k_B^4}{(ch)^3} \frac{\pi^4}{15} T^4 \\
\Leftrightarrow \quad & \epsilon_\gamma = aT^4, && \text{(C.1.35)} \\
& \epsilon_{\bar{\nu}} = \epsilon_\nu = \frac{4\pi g_\nu k_B^4}{(ch)^3} T^4 \int_0^\infty \frac{x^3 dx}{e^x + 1} \\
\Leftrightarrow \quad & \epsilon_{\bar{\nu}} = \epsilon_\nu = \frac{4\pi k_B^4}{(ch)^3} \frac{7}{8} \frac{\pi^4}{15} T^4 \\
\Leftrightarrow \quad & \epsilon_{\bar{\nu}} = \epsilon_\nu = \frac{7}{16} aT^4, && \text{(C.1.36)} \\
& \epsilon_{e^+} \approx \epsilon_{e^-} \approx \frac{4\pi g_{e^-} k_B^4}{(ch)^3} T^4 \int_0^\infty \frac{x^3 dx}{e^x + 1} \\
\Leftrightarrow \quad & \epsilon_{e^+} \approx \epsilon_{e^-} \approx \frac{8\pi k_B^4}{(ch)^3} \frac{7}{8} \frac{\pi^4}{15} T^4 \\
\Rightarrow \quad & \epsilon_{e^+} \approx \epsilon_{e^-} \approx \frac{7}{8} aT^4, && \text{(C.1.37)} \\
& \epsilon = \epsilon_\gamma + \epsilon_{\nu_e} + \epsilon_{\bar{\nu}_e} + \epsilon_{\nu_\mu} + \epsilon_{\bar{\nu}_\mu} + \epsilon_{\nu_\tau} + \epsilon_{\bar{\nu}_\tau} + \epsilon_{e^-} + \epsilon_{e^+}, \\
\Rightarrow \quad & \epsilon \approx (1 + 6\frac{7}{16} + 2\frac{7}{8}) aT^4 \\
\Leftrightarrow \quad & \epsilon \approx \frac{43}{8} aT^4. && \text{(C.1.38)}
\end{aligned}$$

[6]Notice the difference between (C.1.38) and equation 8.40 presented by Islam, where the contribution of the tau-neutrino, and its anti-particle, has not been included [13]. The main reason for this is presumably that at the time of writing, that particular neutrino flavour had just been discovered [13]

Finally, by combining (C.1.38) with equation (C.1.8), one yields the explicit relation (C.1.39) between age of the Universe t and the temperature T [13].

$$t = \left(\frac{3}{32\pi G}\right)^{1/2} c\epsilon^{-1/2} + \text{const.} \approx \left(\frac{3}{32\pi G}\right)^{1/2} c\left(\frac{43}{8}aT^4\right)^{-1/2} + \text{const.}$$
$$\Rightarrow \quad t \approx \left(\frac{3}{172\pi aG}\right)^{1/2} cT^{-2} + \text{const.},\ 10^{12} < T < 5.5 \cdot 10^9. \qquad \text{(C.1.39)}$$

For clarity, it should be noted that (C.1.39) is not valid for temperatures above 10^{12} K since it is only at later times that the matter content of the Universe exists mainly in the form of non-relativistic hadrons [15]. By the same token, equation (C.1.39) seizes to hold once electron-positron pairs have annihilated, an event that is thought to have occurred around a temperature of $5 \cdot 10^9$ K [13].

C.2 $t(T)$ for Temperatures $5.5 \cdot 10^9\,\mathrm{K} > T > 10^9\,\mathrm{K}$

For later times, one must also take into account the effects of the neutrino decoupling. It is therefore convenient to assign the neutrinos a temperature T_ν that evolves differently from that of the Universe T [13]. Obviously, in order to deduce an expression similar to (C.1.39) one must first find the relation between T_ν and T. This task is most easily achieved by considering the entropy density in the Universe. As of yet though, one lacks an expression for the latter.

Since it has been assumed that all relativistic particles species in the Universe can be described as a grand canonical ensembles, (C.2.1) applies [11]. From this equation the entropy in the Universe is found to be given by (C.2.2).

$$U - TS - \mu\bar{N} = -PV \qquad \text{(C.2.1)}$$
$$\Leftrightarrow \quad S = \frac{U}{T} + \frac{pV}{T} - \frac{\mu\bar{N}}{T}. \qquad \text{(C.2.2)}$$

In accordance with the arguments presented above, it is reasonable to assume that the term containing the chemical potential, which represents the contributions from each particle species, is negligible. The approximate expression (C.2.3) for the entropy density s can thus follows by dividing (C.2.2) with the volume V and omitting the above mentioned term [18].

$$s \approx \frac{\epsilon + p}{T}. \tag{C.2.3}$$

However, it is already known that the pressure is simply related to the energy density, (C.1.3), at times when the latter was dominated by contributions from radiation and relativistic particles. Consequently, (C.2.3) can be rewritten in terms of the energy density only. When combined with a formula for $\epsilon(T)$, similar to (C.1.38), what results is a relation between the entropy and the temperature on the form $s \propto T^3$. Still, in deducing the neutrino temperature will prove to be fruitful to take the ratio of the temperature before and after the annihilation of $e^{\pm}$-pairs. As such, the exact nature of the formula for $s(T)$ is not of importance.

At this point it is convenient to introduce a property $g_{eff,i} = g_{1,i}g_{2,i}g_{3,i}$, defined as the contribution from particle species i to the total effective number of spin degrees of freedom g_{eff} [13]. Here, $g_{1,i}$ counts the number of spin oriententION available to the species i. In turn $g_{2,i}$ takes into account the contribution of a eventual antiparticle, equaling two if such a particle exists and unity otherwise. Lastly, $g_{3,i}$ is a statistic mechanical factor that takes on the value 7/8 for fermions and unity for bosons [13]. Note that each of these factors are included in the prefactor on the right side of (C.1.39). In this case $g_{1,i}$ is identical to g_i, and therefore $g_{1,i} \cdot g_{2,i}$ represents the sum of the equal contributions of particle i and its antiparticle $\bar{i}$. In other words $g_{1,i} \cdot g_{2,i} = g_i + g_{\bar{i}}$. The factor $g_{3,i}$, meanwhile, can be seen to have arisen naturally in the calculations by comparing (C.1.32) and (C.1.33) [13]. From this definition, together with (C.1.3), (C.1.34), (C.1.38) and (C.2.3), it seems reasonable that the only factors in the resulting formula for s that change with time are g_{eff} and the temperature T. Nonetheless, the entropy averaged over the entire Universe must remain constant over time [13, 18]. Indeed, it is possible to show that s is scaled by a factor $R(t)^{-3}$, why the product $sR(t)^3$ should be time independent [18].

In deducing the sought relation (C.2.7), one proceed by first rewriting (C.1.3) into the form (C.2.4), from which (C.2.5) follows by implication.

$$
\begin{aligned}
& \frac{\dot{R}}{R} = -\frac{1}{4}\frac{\dot{\epsilon}}{\epsilon} \\
\Leftrightarrow \quad & \dot{R}\epsilon = -\frac{1}{4}\dot{\epsilon}R \\
\Leftrightarrow \quad & -4\dot{R}\epsilon \cdot R^3 = \dot{\epsilon}R \cdot R^3 = \\
\Leftrightarrow \quad & 0 = 4R^3\dot{R}\epsilon + R^4\dot{\epsilon} = \left(\frac{\partial}{\partial t}R^4\epsilon\right) && \text{(C.2.4)} \\
\Rightarrow \quad & R^4\epsilon = \text{const.} \Leftrightarrow \epsilon \propto R^{-4}. && \text{(C.2.5)}
\end{aligned}
$$

When combined with (C.1.38), (C.2.5) leads to (C.2.6).

$$
\begin{aligned}
& T^4 \propto \epsilon \propto R^{-4} \Rightarrow (TR)^4 = \text{const.} \\
\Rightarrow \quad & TR = \text{const.} \Rightarrow T \propto R^{-1}. && \text{(C.2.6)}
\end{aligned}
$$

Since, as stated above, $s \propto \epsilon/T$, one finds (C.2.7) from (C.2.5) and (C.2.6).

$$
\begin{aligned}
& s \propto \frac{\epsilon}{T} \propto \frac{R^{-4}}{R^{-1}} = R^{-3} \\
\Rightarrow \quad & sR^3 = \text{const.}. && \text{(C.2.7)}
\end{aligned}
$$

Adding all results of the above discussion together, one finally arrives at the relation (C.2.8) [13].

$$
g_{eff}T^3R^3 = \text{const.}. \qquad \text{(C.2.8)}
$$

Since the neutrinos went out of thermal equilibrium before the annihilation of electron-positron pairs, g_{eff} must be given by (C.2.9) at times just before, t_1, and after, t_2, this event respectively [13].

$$
\begin{aligned}
g_{eff}(t_1) &= g_{eff,\gamma} + g_{eff,e^-} = 2 \cdot 2 \cdot \frac{7}{8} + 2 = \frac{11}{2}, \\
g_{eff}(t_2) &= g_{eff,\gamma} = 2. && \text{(C.2.9)}
\end{aligned}
$$

By taking the ratio of the left hand side of (C.2.8), evaluated at t_1 and t_2 respectively, one finds (C.2.10) once (C.2.9) has been taken into account.

$$
\begin{aligned}
1 = \frac{(g_{eff}T^3R^3)|_{t_1}}{(g_{eff}T^3R(t)^3)|_{t_2}} &= \frac{(11/2)T(t_1)^3R(t_1)^3}{2T(t_2)^3R(t_2)^3}, \\
\left(\frac{T(t_1)R(t_1)}{T(t_2)R(t_2)}\right)^3 &= \frac{4}{11} \\
\Leftrightarrow \frac{T(t_1)R(t_1)}{T(t_2)R(t_2)} &= \left(\frac{4}{11}\right)^{1/3}. \qquad \text{(C.2.10)}
\end{aligned}
$$

The increase in temperature implied by (C.2.10) was due to the energy released during the pair annihilations that reheated the Universe [4, 13]. However, none of this energy was absorbed by the neutrinos since these were kinetically decoupled from the other particles in the Universe at this time [4, 13]. Therefore, it ought to be possible to regard the neutrinos as a separate closed system with a temperature T_ν, scaled by R^{-1} [13]. Also, T_ν should be equal to T at times preceeding the annihilation. Taken together these assumption suggest that (C.2.11) holds true [13].

$$
T_\nu(t_2)R(t_2) = T_\nu(t_1)R(t_1) = T(t_1)R(t_1). \qquad \text{(C.2.11)}
$$

By combining (C.2.10) and (C.2.11) one can deduce (C.2.12) that relates the two temperatures of interest.

$$
\begin{aligned}
&\frac{T(t_2)}{T_\nu(t_2)} = \frac{T(t_2)R(t_2)}{T_\nu(t_2)R(t_2)} = \frac{T(t_2)R(t_2)}{T(t_1)R(t_1)} = \left(\frac{4}{11}\right)^{-1/3} \\
\Leftrightarrow \quad &\frac{T(t_2)}{T_\nu(t_2)} = \left(\frac{11}{4}\right)^{1/3} \Leftrightarrow T_\nu(t_2) = \left(\frac{4}{11}\right)^{1/3} T(t_2). \qquad \text{(C.2.12)}
\end{aligned}
$$

After the annihilation of $e^{\pm}$-pairs, but before the shift from radiation to matter dominance, the only significant contributions to the energy content of the Universe came from photons and neutrinos[13]. Similarily to earlier times, the total energy density in the Universe can be calculated by summing the contributions of every particle species present. In this case though, one must also take the difference in temperature between the photons and the neutrinos into account. Proceeding as when deducing (C.1.39), and with the additional help of (C.2.12), as well as (C.1.35) and (C.1.36), one can derive (C.2.11).

$$\begin{aligned}
&\epsilon \approx \epsilon_\gamma + \epsilon_{\nu_e} + \epsilon_{\bar{\nu}_e} + \epsilon_{\nu_\mu} + \epsilon_{\bar{\nu}_\mu} + \epsilon_{\nu_\tau} + \epsilon_{\bar{\nu}_\tau} \\
\Rightarrow \quad &\epsilon \approx \left(aT^4 + 6\frac{7}{16} aT_\nu^4 \right) = \left(aT^4 + \frac{21}{8} a (\frac{4}{11})^{4/3} T^4 \right) \\
\Leftrightarrow \quad &\epsilon \approx \left(1 + \frac{21}{8} \left(\frac{4}{11} \right)^{4/3} \right) aT^4. \qquad \text{(C.2.13)}
\end{aligned}$$

Lastly, the relation between time and temperature in (C.2.14), by the same token as when deriving (C.1.38), is deduced by substituting ϵ in (C.1.8) for (C.2.13).[7]

$$\begin{aligned}
t &= \left(\frac{3}{32\pi G} \right)^{1/2} c\epsilon^{-1/2} + \text{const.} \\
&\approx \left(\frac{3}{32\pi G} \right)^{1/2} c \left[\left(1 + \frac{21}{8} \left(\frac{4}{11} \right)^{4/3} \right) aT^4 \right]^{-1/2} + \text{const.} \\
&\Rightarrow \\
t &\approx \left(1 + \frac{21}{8} \left(\frac{4}{11} \right)^{4/3} \right)^{-1/2} \left(\frac{3}{32\pi aG} \right)^{1/2} cT^{-2} + \text{const.}, T < 10^9\,\text{K} \qquad \text{(C.2.14)}
\end{aligned}$$

As of yet, one lacks a formula for $t(T)$, comparable to (C.2.14) and (C.1.38), that is applicable for $5.5 \cdot 10^9 > T > 10^9\,\text{K}$, during which era all e^+ and most e^- underwent pairwise annihilations. The main difficulty in deducing such an equation stems from the fact that the electrons and positrons are non-relatistic during at least some part of this time period [13]. The formula (C.1.38) for the energy density in the Universe is thus incorrect, for which reason there will be no effort to derive a relation between time and temperature for this period [13].

[7]Notice that, similarly to (C.1.38), equation (C.2.13) derived in this text differs from 8.43, presented by Islam, since the contribution of the τ-neutrino has not been taken into account in the latter case [13].

D Evaluation of Important Integrals

D.1 $\int_0^\infty x^2 e^{-x^2} dx$

The integral expression (D.1.1), that appears in the derivation of (4.3.8) in section 4.3, will be shown to reduce to (D.1.3).

$$\int_0^\infty x^2 e^{-x^2} dx \tag{D.1.1}$$

Firstly, an expression, (D.1.2), for $x^2 e^{-x^2}$ must be deduced.

$$\begin{aligned}
& \frac{d}{dx} e^{-x^2} = -2x e^{-x^2} \\
\Rightarrow \quad & \frac{d^2}{dx^2} e^{-x^2} = -2e^{-x^2} + 4x^2 e^{-x^2} \\
\Leftrightarrow \quad & x^2 e^{-x^2} = \frac{1}{4}\frac{d^2}{dx^2} e^{-x^2} + \frac{1}{2} e^{-x^2}
\end{aligned} \tag{D.1.2}$$

By applying (D.1.2) one can then evaluate (D.1.1) and thereby ascertain the sought formula (D.1.3).

$$\begin{aligned}
\Rightarrow \quad & \int_0^\infty x^2 e^{-x^2} dx = \int_0^\infty \left(\frac{1}{4}\frac{d^2}{dx^2} e^{-x^2} + \frac{1}{2} e^{-x^2} \right) dx \\
\Leftrightarrow \quad & \int_0^\infty x^2 e^{-x^2} dx = \frac{1}{4} \int_0^\infty \frac{d^2}{dx^2} e^{-x^2} dx + \frac{1}{2} \int_0^\infty e^{-x^2} dx \\
\Leftrightarrow \quad & \int_0^\infty x^2 e^{-x^2} dx = \frac{1}{4} \left[\frac{d}{dx} e^{-x^2} \right]_0^\infty + \frac{1}{2}\sqrt{\pi} \\
\Leftrightarrow \quad & \int_0^\infty x^2 e^{-x^2} dx = \frac{1}{4} \left[-2x e^{-x^2} \right]_0^\infty + \frac{\sqrt{\pi}}{2} \\
\Leftrightarrow \quad & \int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}
\end{aligned} \tag{D.1.3}$$

D.2 $\int_0^\infty \frac{x^{m-1}dx}{e^x \pm 1}$

Equations (D.2.1) and (D.2.2) represent more general forms of the most frequently occurring integral expressions in the theory sections, that is (D.2.1) and (D.2.2) with either $m-1=2$, $m-1=3$ or $m-1=4$.

$$\int_0^\infty \frac{x^{m-1}dx}{e^x+1} \tag{D.2.1}$$

$$\int_0^\infty \frac{x^{m-1}dx}{e^x-1} \tag{D.2.2}$$

$$
\begin{aligned}
& \int_0^\infty \frac{x^m dx}{e^x-1} = \int_0^\infty \frac{x^m}{e^x-1}\frac{e^{-x}}{e^{-x}dx} = \int_0^\infty x^m e^{-x}\frac{1}{1-e^{-x}}dx \\
= \quad & \left\{\frac{1}{1-\xi} = (1+\xi+\xi^2+...+\xi^n+...),\ |\xi|<1\right\} \\
= \quad & \int_0^\infty x^m e^{-x}(1+e^{-x}+e^{-2x}+...+e^{-nx}+...)dx \\
= \quad & \int_0^\infty x^m e^{-x}dx + \int_0^\infty x^m e^{-2x}dx + ... + \int_0^\infty x^m e^{-(n+1)x}dx + ... \\
\Leftrightarrow \quad & \Bigg\{\text{integration by parts} \\
& \int_0^\infty x^m e^{-(n+1)x}dx = \frac{x^m e^{-(n+1)x}}{-(n+1)}\Bigg|_0^\infty - \int_0^\infty \frac{mx^{m-1}e^{-(n+1)x}}{-(n+1)} \\
& 0 + \frac{m}{n+1}\int_0^\infty x^{m-1}e^{-(n+1)x} = ... \\
= \quad & \frac{m!}{(n+1)^m}\int_0^\infty e^{-(n+1)x} = \frac{m!}{(n+1)^{m+1}}\Bigg\} \\
\Leftrightarrow \quad & \int_0^\infty \frac{x^m dx}{e^x-1} = m!\sum_{n=0}^{\infty}\frac{1}{(n+1)^{m+1}} \\
\Rightarrow \quad & \int_0^\infty \frac{x^m dx}{e^x-1} - \int_0^\infty \frac{x^m dx}{e^x+1} = \int_0^\infty (\frac{x^m}{e^x-1} - \frac{x^m}{e^x+1})dx \\
= \quad & \int_0^\infty \frac{x^m(e^x+1)-x^m(e^x-1)}{(e^x-1)(e^x+1)}dx = \int_0^\infty \frac{x^m}{e^{2x}}2dx \\
= \quad & \{\chi = 2x \Leftrightarrow x = \chi/2 \Rightarrow d\chi = 2dx\} \\
= \quad & \int_0^\infty \frac{1}{2^m}\frac{\chi^m}{e^\chi}d\chi = \frac{1}{2^m}\int_0^\infty \frac{x^m}{e^x}dx \\
\Leftrightarrow \quad & \int_0^\infty \frac{x^m dx}{e^x+1} = \frac{2^m-1}{2^m}\int_0^\infty \frac{x^m dx}{e^x-1},
\end{aligned}
$$

Thus if one substitutes m for $m-1$, the sought expressions for (D.2.1) and (D.2.2) are obtained.

$$\int_0^\infty \frac{x^{m-1}dx}{e^x-1} = (m-1)!\sum_{n=1}^{\infty}\frac{1}{n^m}, \tag{D.2.3}$$

$$\int_0^\infty \frac{x^{m-1}dx}{e^x+1} = \frac{2^{m-1}-1}{2^{m-1}}(m-1)!\sum_{n=1}^{\infty}\frac{1}{n^m}. \tag{D.2.4}$$

E Programs

All files are available upon request.

E.1 `bbn.f`

The NUC123 fortran source code. Needs to be compiled on every workstation.

E.2 `manyruns.sh`

This script will run NUC123 with standard settings only varying eta, with exponential spacing and store outputted data file in data/ and name set to the current eta value. manyruns.sh requires the subdir data to exist, and preferably be empty.

E.3 `analyzedata.m`

Will read and analyze data output from manyruns.sh, which runs NUC123 many times .

E.4 `analyzematdata.m`

labelanalyzematdata Will read and analyze matlab data files from analyzedata.m and produces nice graphics.

E.5 `analyzeautodata.m`

Will read and analyze data output from manyruns.sh, which runs NUC123 many times. The contents of data/ is analyzed and stored as a matlab data environment file, 'autodata.mat', to be plotted with analyzeautomatdata.m.

E.6 `analyzeautomatdata.m`

Will read and analyze data output from analyzeautodata.m and produce nice graphics.

E.7 `partphotons.m`

Plots the abundance of helium and hydrogen with all parameters based on previously presented calculations.

E.8 `freezeout.m`

Calculates the freeze-out temperature.

E.9 `TempofTime.m`

Plots the temperature as a function of time.

E.10 `Blackbody.m`

Shows the ratio of high energy photons, $E > 2.2\,\mathrm{MeV}$, as a function of temperature or time.

E.11 `canon.m`

Plots the behaviour of the canonical distribution before freeze-out and neutron decay thereafter.