Extensions of Constant Proportion Portfolio Insurance using the Geometric Ornstein-Uhlenbeck process and the Chan-Karolyi-Longstaff-Sanders process

Master's thesis in Engineering Mathematics and Computational Science

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Abstract

We investigate performance of the Constant Proportion Portfolio Insurance (CPPI) strategy and compare it with two of its extensions: Time Invariant Portfolio Protection (TIPP) and Exponential Proportion Portfolio Insurance (EPPI).

In order to do this, we model a risky asset (a stock or an index) using a Geometric Ornstein-Uhlenbeck process, and estimate its parameters using the likelihood ratio method with historical price data. We model a non-risky asset (a zero-coupon bound) using a Chan-Karolyi-Longstaff-Sanders process and estimate its parameters using the maximum likelihood method where we approximate the transition probability density function using a Hermite expansion.

We find that both extensions of the CPPI improve performance in different ways. The resulting distribution of simulated portfolio outcomes for the TIPP strategy has a lighter tail compared to the CPPI case, and the risk of loss is lower (this is also true compared to the EPPI strategy, but to a smaller degree). The EPPI strategy translates the distribution of simulated portfolio outcomes to the right, so that EPPI performs better than CPPI (and TIPP) in terms of both mean and median.

Keywords: constant proportion portfolio insurance (CPPI), time invariant portfolio protection (TIPP), exponential proportion portfolio insurance (EPPI), geometric Ornstein-Uhlenbeck process (GOU process), Chan-Karolyi-Longstaff-Sanders process (CKLS process), likelihood ratio, Hermite expansion, Hermite approximation, maximum likelihood approximation.

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Jonathan Bengtsson, Gothenburg, December 2025

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1

Introduction

A Portfolio insurance strategy is a specific type of investment strategy, with the goal of limiting the risk of loss in downward moving markets while allowing for participation in upward moving markets [15]. These types of strategies were first introduced by Rubinstein and Leland (1976)[18] after a market downturn where pension funds were withdrawn, where they noted that a portfolio insurance strategy could have convinced investors to participate in the market, and thus better taking advantage of the rise that followed a couple of years later.

In this thesis, we will investigate a specific strategy called Constant Proportion Portfolio Insurance (CPPI). Under the assumption of a market consisting of one risky asset, typically a stock or index, and one non-risky asset, typically a zero-coupon bound, CPPI suggests a systematic approach to continuously re-balance the proportion that is invested in the risky asset and the proportion invested in the non-risky asset [15]. The key idea behind the strategy is that one decides on a proportion of the funds that one is willing to risk. The CPPI strategy aims to ensure that the investor never loses more than that predefined amount.

We also consider two modifications (extensions) of the CPPI strategy: Time Invariant Portfolio Protection (TIPP) and Exponential Proportion Portfolio Insurance (EPPI). The TIPP strategy differs from CPPI in that it employs a ratchet mechanism, so that in upward moving markets one resets the amount one is willing to risk, and thus "locks in" gains [15]. The EPPI strategy differs from the CPPI in that it takes into account recent market movements in determining the proportion of the portfolio that should be exposed to the risky asset [15].

These three strategies are explained in more detail in Chapter 3.

1.1 Stochastic processes for assessing portfolio strategy performance

Although direct use of historical price data, of course, is the most reliable source of market behaviour, the amount of available historical price data is very limited. In addition to that, market behaviour tends to change over time, making old price data obsolete to assess strategy performance in the current market. Therefore, by using stochastic processes to model the behaviour of asset prices (and interest rates) hopefully one can capture the essence of the market behaviour by choosing a suitable stochastic process that, by some measure, seems to mimic the market behaviour. If the stochastic process reliably mimics the market behaviour, we can simulate as many price data paths as we need to make a good evaluation of an investment strategy.

In this thesis, we use the Geometric Ornstein-Uhlenbeck process (GOU), with the three parameters α, β and σ, to model stock or index assets, while we use the Chan-Karolyi-Longstaff-Sanders process (CKLS), with the four parameters α, β, σ and γ, to model interest rates. We use historical price data to estimate appropriate values of the parameters of the stochastic processes, in the hope that this flexibility is enough to capture the essence of the market behaviour we need, in order to evaluate the portfolio insurance strategies.

1.2 Software used

To estimate parameters and simulate portfolios we use *R*, with the libraries *pracma, stats, nleqslv, optimx.* Historical price data are obtained from Yahoo! Finance.

1.3 Outline of the thesis

After this brief introduction in Chapter 1, we present the relevant theory of stochastic processes in Chapter 2, including methods for estimating parameters of the GOU process and the CKLS process, and schemes for simulating them.

In Chapter 3 we present the details of portfolio investment strategies and how to implement them in practice for a given data series of price data.

In Chapter 4 we estimate parameters of the stochastic processes, with a short discussion on the historical price data that we use for the estimation. Here we find that we may have redundancy in both the parametrization of the GOU process and the CKLS process. Two of the parameters α and β seem to model something similar. In addition, the parameter γ for the CKLS process is often estimated at values close to 1. For a fixed $\gamma = 1$ the CKLS is the same stochastic process as the simpler GOU process, which may indicate that the CKLS process is more complex than needed, at least for markets similar to the ones from which we obtained historical price data.

Then, in Chapter 5, we use these estimated parameters to simulate price data paths and apply the CPPI, TIPP and EPPI strategies to see how well they perform. We find that both the TIPP and EPPI types of extensions of the CPPI strategy seem to improve it in some sense, but in different ways.

The use of the TIPP strategy modifies the distribution of outcomes of the portfolio simulations such that it has lighter tails. The TIPP strategy has a lower risk of loss compared to the CPPI.

The use of the EPPI strategy modifies the distribution of outcomes of the portfolio simulations by translating it to the right, thus yielding larger expected returns in terms of both median and mean than the CPPI strategy. For many of the parameter values on the stochastic processes that we try, we also find that the EPPI strategy often yields a smaller risk of loss than the CPPI, although this improvement is not as large as for the TIPP strategy.

2

Stochastic differential equations

A **stochastic differential equation** (SDE) can be seen as an extension of an ordinary differential equation obtained by adding a stochastic term.

Definition 1 *We say that the process $X(t)$ is a strong solution to* (2.1) *if for all $t > 0$ the integral $\int_0^t \mu(X(s), s)ds$ and the Itô integral $\int_0^t \sigma(X(s), s)dB(s)$ exist*

$$X(t) = X(0) + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dB(s). \tag{2.1}$$

The SDE (2.1) is referred to as a **diffusion-type SDE**, $\mu(x, t)$ is called the drift coefficient, and $\sigma(x, t)$ is called the diffusion coefficient [13, p. 126].

It is often convenient to express (2.1) on differential form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t). \tag{2.2}$$

However, a Brownian motion is nowhere differentiable, so $\frac{dB(t)}{dt}$ does not exist in the sense that derivatives of differentiable functions do [13, p. 124] . Therefore, (2.2) is merely a different notation for the relation expressed in (2.1).

2.1 Volatility induced stationarity (VIS)

A simplified, although many times useful way, of describing the dynamics of an SDE is to split it into an ODE for the drift, and think of the diffusion part as a white noise perturbation of the ODE. However, this is not always correct, since the drift part and the diffusion part interact, and this interaction sometimes becomes significant in the sense that the mean reversion associated with stationarity may come from high volatility rather than drift. When studying the CKLS model, this leads Conley, Hansen, Luttmer and Scheinkman (1997) [7] to define the concept of VIS. The definition we give here is a more general definition proposed by Albin et al. (2006)[2].

Given an $x_0 \in I$, and $I = (l, r), -\infty \leq l < r \leq \infty$ an open interval, the scale function is given by

$$S(x) = \int_{x_0}^{x} \exp\left(-2\int_{x_0}^{y} \frac{\mu(z)}{\sigma(z)^2} dz\right) dy \text{ for } x \in I \tag{2.3}$$

and the speed measure

$$\frac{dm(x)}{dx} = \frac{2}{\sigma(x)^2 S'(x)} \text{ for } x \in I. \tag{2.4}$$

We make two assumptions

1. The drift μ is continuous and the volatility σ is (strictly) positive and locally Hölder continuous of order $\frac{1}{2}$.

2. We have $S(l^+) = -\infty, S(r^-) = \infty$ and $m(I) < \infty$.

Under these assumptions, we can state the definition of VIS.

Definition 2 (Volatility induced stationarity) *The stationary solution to the SDE 2.2 has VIS at $l(r)$, and we call $l(r)$ a VIS boundary if $S'(l^+ < \infty(S'(r^- < \infty)$. If $S'(l^+) > 0(S'(r^-) > 0)$ we call $l(r)$ a positive VIS boundary – otherwise it is a null VIS boundary.*

2.2 Geometric Ornstein-Uhlenbeck (GOU) process

The **Geometric Ornstein-Uhlenbeck process** is a stochastic process that is defined as the solution $X(t)$ to the SDE

$$dX(t) = (\alpha + \beta X(t))dt + \sigma X(t)dB(t), \tag{2.5}$$

with parameters α, β and σ, where we require that parameter $\sigma > 0$.

The necessary conditions for the existence and uniqueness of solutions can be found in, for instance, Klebaner (2012)[13, p. 134]. For the GOU process, they are easy to verify.

Given data in the form of a time series, we want to estimate the parameters α, β and σ. We consider two methods for doing this: 1. linear regression and 2. likelihood ratio for α and β, and observed quadratic variation for σ.

2.2.1 Linear regression

Using Euler-Maruyama discretization (see section 2.4 for details) we can rewrite (2.5) as

$$X(t) = X(t-1) + (\alpha + \beta X(t-1))\Delta t + \sigma X(t-1)\sqrt{\Delta t}\varepsilon(t) \tag{2.6}$$

where $\varepsilon(t) \sim \mathrm{N}(0,1)$. By rearranging terms in (2.6) we se that we can use linear regression to estimate α and β:

$$R(t) = c_1 + c_2 \frac{1}{X(t-1)} + e(t) \tag{2.7}$$

where $R(t) = \frac{X(t)-X(t-1)}{X(t-1)}$, $c_1 = \beta\Delta t$, $c_2 = \alpha\Delta t$ and $e(t) = \sigma\sqrt{\Delta t}\varepsilon(t)$.

2.2.2 Likelihood ratio

If the diffusion process $X(t)$ is a solution to (2.8) with a P-Brownian motion $B(t)$, and $X(t)$ is also a solution to (2.9) with a Q-Brownian motion $W(t)$

$$dX(t) = \mu_1(X(t), t)dt + \sigma(X(t), t)dB(t) \tag{2.8}$$

$$dX(t) = \mu_2(X(t), t)dt + \sigma(X(t), t)dW(t) \tag{2.9}$$

we can form the likelihood ratio [13, p. 285]

$$\begin{aligned}\Lambda = \frac{dQ}{dP} = \exp\biggl(&\int_0^T \frac{\mu_2(X(t),t) - \mu_1(X(t),t)}{\sigma^2(X(t),t)} dX(t) \\ &+ \frac{1}{2}\int_0^T \frac{\mu_2^2(X(t),t) - \mu_1^2(X(t),t)}{\sigma^2(X(t),t)} dt\biggr).\end{aligned}$$

Following an example provided by Klebaner [13, p. 286], let $P_{\alpha,\beta}$ correspond to $X(t)$ in (2.5) and $P_{0,0}$ correspond to $\sigma X(t)dB(t)$ (i.e., $\alpha = \beta = 0$)

$$\begin{aligned}\Lambda &= \frac{P_{\alpha,\beta}}{P_{0,0}} = \exp\left(\int_0^T \frac{\alpha + \beta X(t)}{\sigma^2 X^2(t)} dX(t) - \frac{1}{2}\int_0^t \frac{(\alpha + \beta X(t))^2}{\sigma^2 X^2(t)} dt\right) \\ &= \exp\biggl(\frac{1}{\sigma^2}\biggl(\alpha\int_0^T \frac{1}{X^2(t)} dX(t) + \beta\int_0^T \frac{1}{X(t)} dX(t) - \alpha^2\frac{1}{2}\int_0^T \frac{1}{X^2(t)} dt \\ &\quad - \alpha\beta\int_0^T \frac{1}{X(t)} dt - \beta^2\frac{T}{2}\biggr)\biggr).\end{aligned} \tag{2.10}$$

We let $l(\alpha,\beta) = \alpha\int_0^T \frac{1}{X^2(t)} dX(t) + \beta\int_0^T \frac{1}{X(t)} dX(t) - \alpha^2\frac{1}{2}\int_0^T \frac{1}{X^2(t)} dt - \alpha\beta\int_0^T \frac{1}{X(t)} dt - \beta^2\frac{T}{2}$ and we can find optimal values for example by solving

$$\begin{aligned}0 = \frac{dl(\alpha,\beta)}{d\alpha} &= \int_0^T \frac{1}{X^2(t)} dX(t) - \alpha\int_0^T \frac{1}{X^2(t)} dt - \beta\int_0^T \frac{1}{X(t)} dt, \\ 0 = \frac{dl(\alpha,\beta)}{d\beta} &= \int_0^T \frac{1}{X(t)} dX(t) - \alpha\int_0^T \frac{1}{X(t)} dt - \beta T.\end{aligned}$$

2.2.3 Estimating σ using observed quadratic variation

The method above can be used to estimate α and β of the Geometric Ornstein-Uhlenbeck SDE (2.5), but it is not suitable for estimating σ. However, σ can easily be estimated by using observed quadratic variation.

We write the Geometric Ornstein-Uhlenbeck process (2.5) on integral form

$$X(t) = X(0) + \int_0^t (\alpha + \beta X(s))ds + \int_0^t \sigma X(s)dB(s). \tag{2.11}$$

We observe that (2.11) is an Itô process and, therefore, [13, p. 110]

$$[X, X](t) = \sigma^2 \int_0^t X^2(s)ds. \tag{2.12}$$

Observed quadratic variation can also be computed directly (approximately) using its definition [13, p. 8]

$$[X, X](t) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} (X(t_i^n) - X(t_{i-1}^n))^2 \tag{2.13}$$

where the limit is taken over partitions: $0 = t_0^n < t_1^n < ... < t_n^n) = t$, with $\delta_n = \max_{1 \leq i \leq n}(t_i - t_{i-1}^n)$. Combining (2.13) and (2.12) we can approximate σ as

$$\sigma^2 = \frac{\lim_{\delta_n \to 0} \sum_{i=1}^{n} (X(t_i^n) - X(t_{i-1}^n))^2}{\int_0^t X^2(s)ds}, \tag{2.14}$$

where the limit in (2.14) and the resolution of the integral are made as fine as allowed by the empirical data $(X_i)_{i=0}^N$.

2.3 Chan–Karolyi–Longstaff–Sanders (CKLS) process

The **Chan–Karolyi–Longstaff–Sanders process** is a stochastic process that is defined as the solution $X(t)$ to the SDE

$$dX(t) = (\alpha + \beta X(t))dt + \sigma X^{\gamma}(t)dB(t), \tag{2.15}$$

with parameters α, β, σ and γ, where we require that parameter $\sigma > 0$.

The process takes its name from those who studied it in 1992 [6] as an extension of the CIR process, studied by Feller (1951)[11] and Cox-Ingersoll and Ross (1985)[9]

The CKLS process has VIS for certain parameter values[2]:

$$\begin{aligned} &\left\{\frac{1}{2} < \gamma < 1, \alpha > 0, \beta = 0\right\} \cup \left\{\gamma = 1, \alpha > 0, 0 \leq \beta < \frac{1}{2}\sigma^2\right\} \\ &\cup \left\{\gamma > 1, \alpha > 0\right\} \cup \left\{\gamma > 1, \alpha = 0, \beta > 0\right\}. \end{aligned} \tag{2.16}$$

A stochastic process with VIS can be difficult to simulate, since the standard requirements for convergence (see section 2.4.1 and Appendix A) – Lipschitz conditions and linear growth for drift and diffusion coefficients – are not satisfied [2]. Therefore, in the next section, we consider implicit methods.

2.3.1 Maximum likelihood using a Hermite approximation

Maximum likelihood estimation is a traditional way to estimate parameters for probability density distributions. Applied to estimating parameters of a CKLS process $X(t)$ we obtain the estimates

$$(\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\gamma}) = \underset{\alpha,\beta,\sigma,\gamma}{\arg\max}\ \Pi_{i=1}^{n} f_{X(t_i)|X(t_{i-1})}(x_i|x_{i-1})$$

where $f_{X(t_i)|X(t_{i-1})}$ is the transition probability density function of $X(t-1)$ to $X(t)$ (which depends on α, β, σ and γ), and $x_1, ..., x_n$ are observations of the process X at the corresponding times. Furthermore, $0 < t_1 < ... < t_n$ and $X(0) = x_0$.

The transition probability density function is not explicitly known for the CKLS process, therefore, we need to approximate it. To this end, we use Hermite polynomials.

Underlying ideas and assumptions

The approximation method we use here was first proposed by Aït-Sahalia (2002) [1]. The class of distributions for which an Hermite expansion, for a fixed step size Δ, converges as more terms are added to the expansion, is pretty limited. Doing a straight Hermite expansion of the transition density for a diffusion process almost certainly results in a non-converging series.

However, densities that are sufficiently close to the $N(0,1)$ density do converge. Therefore, we transform the SDE into an SDE with transition densities that are close to $N(0,1)$. We do this in two steps, 1. We transform X into an SDE Y with a diffusion coefficient equal to 1, and 2. Since the transition density p_Y of Y gets peaked around its conditional value y_0 when Δ is small (a behaviour similar to a Dirac mass), we make a second transformation from Y to Z. This second transformation is similar to a normalization. Although it is not a standardization in a strict sense, Aït-Sahalia (2002) [1] shows that it fulfills the role of making the distribution sufficiently close to $N(0,1)$ even for small Δ.

Aït-Sahalia (2002) [1] makes three assumptions regarding the drift coefficient and the diffusion coefficient, which in this section we denote $\tilde{\mu}$ and $\tilde{\sigma}$ respectively, in order to avoid confusion with the CKLS parameter σ. In this thesis we merely present an overview of this method and the necessary results to apply it, we do not consider convergence results etc. We do restate the assumptions made by Aït-Sahalia (2002) [1] since these are the prerequisites for using the method. The assumptions are:

Assumption 1 (Smoothness of the coefficients) *The functions* $\tilde{\mu}(x;\theta)$ *and* $\tilde{\sigma}(x;\theta)$ *are infinitely differentiable in* x*, and three times continuously differentiable in* θ*, for all* $x \in D_X$ *and* $\theta \in \Theta$*.*

Assumption 2 (Non-Degeneracy of the diffusion)

- *If* $D_X = (-\infty, \infty)$ *there exist a constant* c *such that* $0 < c < \tilde{\sigma}(x;\theta)$ *for all* $x \in D_X$ *and* $\theta \in \Theta$*.*
- *If* $D_X = (0, \infty), \lim_{x \to 0^+} \tilde{\sigma}(x;\theta) = 0$ *is possible, but there there exist constants* $\xi_0 > 0, \omega > 0, \rho \geq 0$ *such that* $\tilde{\sigma}(x;\theta) \geq \omega x^\rho$ *for all* $0 < x \leq \xi_0$ *and* $\theta \in \Theta$*. Whether or not* $\lim_{x \to 0^+} = 0$*,* $\tilde{\sigma}$ *is nondegenerate on* $(0, \infty)$*, that is for each* $\xi > 0$*, there exists a constant* c_ξ *such that* $\tilde{\sigma}(x;\theta) \geq c_\xi > 0$ *for all* $x \in [\xi, \infty]$ *and* $\theta \in \Theta$*.*

The third assumption considers the coefficients after the first transformation (2.17) from X to Y, see next page.

Assumption 3 (Boundary behavior) *For all* $\theta \in \Theta, \mu_Y(y;\theta)$ *and its derivatives w.r.t.* y *and* θ *have at most polynomial growth near the boundaries and* $\lim_{y \to \underline{y}^+ \text{ or } \overline{y}^-} \lambda_Y(y;\theta) < \infty$ *where* $-\lambda_Y$ *is the potential,* $\lambda_Y(y;\theta) \equiv -\frac{\mu_Y^2(y;\theta) + \partial \mu_Y(y;\theta)/\partial y}{2}$.

- *Lower boundary: If* $\underline{y} = 0$*, there exit constants* $\epsilon_0, \kappa, \alpha$ *such that for all* $0 < y \leq \epsilon_0$ *and* $\theta \in \Theta, \mu_Y(y;\theta) \geq \kappa y^{-\alpha}$ *where either* $\alpha > 1$ *or* $\kappa > 0$*, or* $\alpha = 1$ *and* $\kappa \geq 1$*. If* $\underline{y} = -\infty$*, there exist constants* $E_0 > 0$ *and* $K > 0$ *such that for all* $y \leq -E_0$ *and* $\theta \in \Theta, \mu_Y(y;\theta) \geq Ky$*.*
- *Upper boundary: If* $\overline{y} = +\infty$*, there exist constants* $E_0 > 0$ *and* $K > 0$ *such that for all* $y \geq E_0$ *and* $\theta \in \Theta, \mu_Y(y;\theta) \leq Ky$*. If* $\overline{y} = 0$*, there exist constants* $\epsilon_0, \kappa, \alpha$ *such that for all* $0 > y \geq -\epsilon_0$ *and* $\theta \in \Theta, \mu_Y(y;\theta) \leq -\kappa|y|^{-\alpha}$ *where either* $\alpha > 1$ *and* $\kappa > 0$ *or* $\alpha = 1$ *and* $\kappa \geq 1/2$*.*

Transforming the SDE to an SDE with normal density

A diffusion type SDE (2.2) can be transformed into an SDE with diffusion coefficient equal to 1 using

$$Y = \tilde{\gamma}(X;\theta) = \int^X \frac{du}{\tilde{\sigma}(u;\theta)}$$

where the integration constant is irrelevant – any primitive of $1/\tilde{\sigma}$ may be chosen, and by applying Itô's lemma

$$\begin{aligned} dY(t) &= \mu_Y(Y(t);\theta)dt + dB \\ \mu_Y(y;\theta) &= \frac{\tilde{\mu}(\tilde{\gamma}^{-1}(y;\theta);\theta)}{\tilde{\sigma}(\tilde{\gamma}^{-1}(y;\theta);\theta)} - \frac{1}{2}\tilde{\sigma}'(\tilde{\gamma}^{-1}(y;\theta);\theta). \end{aligned} \tag{2.17}$$

Here θ represents the SDE parameters, in the case of CKLS $\theta = (\alpha, \beta, \sigma, \gamma)$.

Secondly, we make the transformation

$$Z = \Delta^{-1/2}(Y - y_0).$$

Let $p_X(\Delta, x|x_0;\theta)$ denote the conditional density $X(t+\Delta)|X(t)$, and define analogously the conditional densities p_Y and p_Z for Y and Z.

Since we expand the density p_Z using Hermite polynomials but want to know p_X, we need to know the relationship between p_X and p_Z. This can be found by applying the Jacobian formula for the change of density:

$$p_Y(\Delta, y|y_0;\theta) = \Delta^{-1/2}p_Z(\Delta, \Delta^{-1/2}(y - y_0)|y_0;\theta) \tag{2.18}$$

and then

$$p_X(\Delta, x|x_0;\theta) = \tilde{\sigma}(x;\theta)^{-1} \cdot p_Y(\Delta, \tilde{\gamma}(x,\theta)|\tilde{\gamma}(x_0;\theta);\theta). \tag{2.19}$$

Approximating the transition probability density functions using Hermite polynomials

The classical Hermite polynomials are

$$H_j(w) = e^{w^2/2}\frac{d^j}{dw^j}[e^{-w^2/2}], j \geq 0.$$

Denote the $N(0,1)$ density function by $\phi(w) = e^{w^2/2}/\sqrt{2\pi}$ and define the Hermite expansion

$$p_Z^{(J)}(\Delta, z|y_0;\theta) = \phi(z)\sum_{j=0}^{J}\eta_Z^{(j)}(\Delta, y_0;\theta)H_j(z)$$

of the density function $z \mapsto p_Z(\Delta, z|y_0;\theta)$. The coefficients $\eta_Z^{(j)}$ are given by

$$\eta_Z^{(j)}(\Delta, y_0;\theta) = \frac{1}{j!}\int_{-\infty}^{\infty} H_j(z)p_Z(\Delta, z|y_0;\theta)dz, \tag{2.20}$$

since the Hermite polynomials divided by $\sqrt{j!}$ are orthogonal with respect to the $L^2(\phi)$ scalar product weighted by the Normal density. Analogously to (2.19) and (2.18), the sequence of approximations to p_Y and p_X are

$$\begin{aligned} p_Y^{(J)}(\Delta, y|y_0;\theta) &= \Delta^{-1/2}p_Z^{(J)}(\Delta, \Delta^{-1/2}(y-y_0)|y_0;\theta), \\ p_X^{(J)}(\Delta, x|x_0;\theta) &= \tilde{\sigma}(x;\theta)^{-1}p_Y^{(J)}(\Delta, \tilde{\gamma}(x,\theta)|\tilde{\gamma}(x_0;\theta);\theta). \end{aligned}$$

Taylor expansion of the Hermite polynomial coefficients

The coefficients of the Hermite expansion $\eta_Z^{(j)}$ can be written

$$\begin{aligned} \eta_Z^{(j)}(\Delta, y_0;\theta) &= \frac{1}{j!}\int_{-\infty}^{\infty} H_j(z)p_Z(\Delta, z|y_0;\theta)dz \\ &= \frac{1}{j!}\int_{-\infty}^{\infty} H_j(z)\Delta^{1/2}p_Y(\Delta, \Delta^{1/2}+y_0|y_0;\theta)dz \\ &= \frac{1}{j!}\int_{-\infty}^{\infty} H_j(\Delta^{-1/2}(y-y_0))\Delta^{1/2}p_Y(\Delta, y|y_0;\theta)dz \\ &= \frac{1}{j!}\mathrm{E}[H_j(\Delta^{-1/2}(Y(t+\Delta)-y_0))|Y(t)=y_0;\theta]. \end{aligned} \tag{2.21}$$

Next, we calculate a Taylor series expansion in Δ for the coefficients $\eta_Z^{(j)}$. If we apply Taylor's theorem to the function $s \mapsto \mathrm{E}[f(Y(t+s), y_0)|Y(t)=y_0]$, we get

$$\mathrm{E}[f(Y(t+\Delta), y_0)|Y(t)=y_0] = \sum_{k=0}^{K} A^k(\theta)\cdot f(y_0,y_0)\frac{\Delta^k}{k!} + \mathrm{E}[A^{K+1}(\theta)\cdot f(Y(t+\Delta),y_0)|Y(t)=y_0]\frac{\Delta^{K+1}}{(K+1)!}, \tag{2.22}$$

where the infinitesimal generator of the diffusion Y is defined as

$$A(\theta): f \mapsto \mu_Y(\cdot;\theta)\frac{\partial f}{\partial y} + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}.$$

We now let $p_Z^{(J,K)}$ denote the Taylor series (in Δ) up to order K of $p_Z^{(J)}$, and its coefficients (obtained by combining (2.21) and (2.22))

$$\eta_Z^{(j,k)}(\Delta, y_0;\theta) = \frac{1}{j!}A^k(\theta)\cdot H_j(0)\frac{\Delta^k}{k!}.$$

Collecting terms of the Taylor expansions

By arranging terms according to increasing powers of Δ (not the Hermite polynomials order), and denoting $\tilde{p}_\cdot^{(K)} \equiv p_\cdot^{(\infty,K)}$, Aït-Sahalia (2002) [1] derives the formula

$$\tilde{p}_Y^{(K)}(\Delta, y|y_0;\theta) = \frac{1}{\Delta^{1/2}}\phi\left(\frac{y-y_0}{\Delta^{1/2}}\right)\exp\left(\int_{y_0}^{y}\mu_Y(u;\theta)du\right)\sum_{k=0}^{K} c_k(y|y_0;\theta)\frac{\Delta^k}{k!}$$

where for $j=0$ we have $c_0(y|y_0;\theta)=1$ and for $j\geq 1$:

$$c_j(y|y_0;\theta) = j\frac{1}{(y-y_0)^j}\int_{y_0}^{y}(u-y_0)^{j-1}\left(\lambda_Y(u;\theta)c_{j-1}(u|y_0;\theta) + \frac{1}{2}\frac{\partial^2 c_{j-2}(u|y_0;\theta)}{\partial u^2}\right)du$$

with

$$\lambda_Y(y;\theta) = -\frac{1}{2}\left(\mu_Y^2(y;\theta) + \frac{\partial \mu_Y(y;\theta)}{\partial y}\right).$$

Expansion of order 2

Details of the following computations can be found in appendix B.

We decide to use an expansion of order $K = 2$ and note that

$$\tilde{p}_Y^{(2)}(\Delta, y|y_0;\theta) = \tilde{p}_Y^{(0)}(\Delta, y|y_0;\theta)\left(1 + c_1(y|y_0;\theta)\Delta + c_2(y|y_0;\theta)\frac{\Delta^2}{2}\right)$$

For a CKLS-type SDE for $\gamma = 1$, we get the GOU-type SDE for which an explicit formula for the transition density is known, (however, we treat this case in appendix B). For $\gamma \neq 1$ we have

$$\mu_Y(y;\theta) = \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} + \beta(1-\gamma)y - \frac{\gamma}{2(1-\gamma)y}$$

where $C = |\sigma(1-\gamma)|^{\frac{1}{1-\gamma}}$. Explicit computation, using the coefficients of an CKLS-type SDE, gives

$$\begin{aligned}\tilde{p}_Y^{(0)}(\Delta, y|y_0;\theta) =& \frac{1}{\sqrt{2\pi\Delta}}\exp\Bigg[-\frac{(y-y_0)^2}{2\Delta} + \frac{\alpha|\gamma-1|}{\sigma C^\gamma(2\gamma-1)}\left(|y_0|^{\frac{2\gamma-1}{\gamma-1}} - |y|^{\frac{2\gamma-1}{\gamma-1}}\right)\\ &+ \frac{\beta(1-\gamma)}{2}(y^2 - y_0^2) + \frac{\gamma}{2(1-\gamma)}\ln\left(\frac{y_0}{y}\right)\Bigg]\end{aligned}$$

It remains to evaluate the coefficients $c_1(...), c_2(...)$. We compute the primitive function $\Lambda_Y(y;\theta)$ of $\lambda_Y(y;\theta)$ that satisfies

$$\Lambda_Y(y;\theta) - \Lambda_Y(y_0;\theta) = \int_{y_0}^{y}\lambda(u;\theta)du.$$

Since μ_Y is known and relatively simple, we can relatively easily compute Λ_Y explicitly:

$$\begin{aligned}\Lambda_Y(y;\theta) =& \frac{\alpha^2|\gamma-1|}{2\sigma^2C^{2\gamma}(3\gamma-1)}|y|^{\frac{3\gamma-1}{\gamma-1}} - \frac{\beta^2(1-\gamma)^2y^3}{6} + \frac{\gamma(2-\gamma)}{8(1-\gamma)^2y}\\ &+ \frac{\alpha\beta(1-\gamma)^2}{\sigma C^\gamma(3\gamma-2)}|y|^{\frac{3\gamma-2}{\gamma-1}} - \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} - \frac{1}{2}\beta(1-2\gamma)y\end{aligned}$$

Using this, we can compute $c_1(...)$

$$
\begin{aligned}
c_1(y|y_0;\theta) &= 1 \cdot \frac{1}{y-y_0}\int_{y_0}^{y}(u-y_0)^0(\lambda_Y(u;\theta)\cdot 1 + \frac{1}{2}\cdot 0)du \\
&= \frac{1}{y-y_0}\int_{y_0}^{y}\lambda_Y(u;\theta)du = \frac{1}{y-y_0}(\Lambda_Y(y;\theta)-\Lambda_Y(y_0;\theta)).
\end{aligned}
$$

We now compute $c_2(...)$

$$
\begin{aligned}
c_2(y|y_0;\theta) &= \frac{2}{(y-y_0)^2}\int_{y_0}^{y}(u-y_0)(\lambda_Y(u;\theta)c_1(y|y_0;\theta)+0)du \\
&= \frac{2}{(y-y_0)^2}\int_{y_0}^{y}(u-y_0)\lambda_Y(u;\theta)\frac{1}{u-y_0}(\Lambda_Y(u;\theta)-\Lambda_Y(y_0;\theta))du \\
&= \frac{2}{(y-y_0)^2}\left(\int_{y_0}^{y}\lambda_Y(u;\theta)\Lambda_Y(u;\theta)du - \Lambda_Y(y_0;\theta)\int_{y_0}^{y}\lambda_Y(u;\theta)du\right) \\
&= \frac{2}{(y-y_0)^2}\left(\left[\frac{1}{2}\Lambda^2(u;\theta)\right]_{u=y_0}^{y} + \Lambda^2(y_0;\theta) - \Lambda(y;\theta)\Lambda(y_0;\theta)\right) \\
&= \left(\frac{\Lambda(y;\theta)-\Lambda(y_0;\theta)}{y-y_0}\right)^2.
\end{aligned}
$$

Summary of explicit expressions

For convenience, we summarize our explicit expression of $\tilde{p}_Y^{(2)}$ for the CKLS-type of SDE:

$$\begin{aligned}
\tilde{p}_Y^{(2)}(\Delta, y|y_0;\theta) =& \tilde{p}_Y^{(0)}(\Delta, y|y_0;\theta)\left(1 + c_1(y|y_0;\theta)\Delta + c_2(y|y_0;\theta)\frac{\Delta^2}{2}\right) \\
\tilde{p}_Y^{(0)}(\Delta, y|y_0;\theta) =& \frac{1}{\sqrt{2\pi\Delta}} \exp\Bigg[-\frac{(y-y_0)^2}{2\Delta} + \frac{\alpha|\gamma-1|}{\sigma C^\gamma(2\gamma-1)}\left(|y_0|^{\frac{2\gamma-1}{\gamma-1}} - |y|^{\frac{2\gamma-1}{\gamma-1}}\right) \\
&+ \frac{\beta(1-\gamma)}{2}(y^2 - y_0^2) + \frac{\gamma}{2(1-\gamma)}\ln\left(\frac{y_0}{y}\right)\Bigg] \\
c_1(y|y_0;\theta) =& \frac{1}{y-y_0}(\Lambda_Y(y;\theta) - \Lambda_Y(y_0;\theta)) \\
c_2(y|y_0;\theta) =& \left(\frac{\Lambda(y;\theta) - \Lambda(y_0;\theta)}{y-y_0}\right)^2
\end{aligned}$$

with

$$\begin{aligned}
\Lambda_Y(y;\theta) =& \frac{\alpha^2|\gamma-1|}{2\sigma^2 C^{2\gamma}(3\gamma-1)}|y|^{\frac{3\gamma-1}{\gamma-1}} - \frac{\beta^2(1-\gamma)^2 y^3}{6} + \frac{\gamma(2-\gamma)}{8(1-\gamma)^2 y} \\
&+ \frac{\alpha\beta(1-\gamma)^2}{\sigma C^\gamma(3\gamma-2)}|y|^{\frac{3\gamma-2}{\gamma-1}} - \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} - \frac{1}{2}\beta(1-2\gamma)y \\
C =& |\sigma(1-\gamma)|^{\frac{1}{1-\gamma}}
\end{aligned}$$

which are expressions that are explicit enough to use in computer simulations.

In order to obtain the $(J=\infty, K)$ approximation of p_X, we use the analogue of (2.19), i.e.

$$\tilde{p}_X^{(K)}(\Delta, x|x_0;\theta) = \tilde{\sigma}(x;\theta)^{-1} \cdot \tilde{p}_Y^{(K)}(\Delta, \tilde{\gamma}(x,\theta)|\tilde{\gamma}(x_0;\theta);\theta),$$

and, as before,

$$\begin{aligned}
\theta &= (\alpha, \beta, \sigma, \gamma), \\
\tilde{\sigma}(w;\theta) &= \sigma w^\gamma, \\
\tilde{\gamma}(w;\theta) &= \frac{1}{\sigma(1-\gamma)} w^{1-\gamma}.
\end{aligned}$$

2.4 Discretization schemes

In order to simulate a stochastic differential equation

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t). \tag{2.23}$$

we need a method to discretize it. Here we present the Euler-Maruyama scheme and a class of implicit discretization schemes including the fully implicit Euler scheme.

2.4.1 Euler-Maruyama scheme

We use methods based on the Euler scheme for ordinary differential equations, extended for usage with stochastic differential equations. [12, p. 164]

Definition 3 (Euler-Maruyama scheme) *The Euler-Maruyama scheme $X^{(h)}$ associated with the stochastic differential equation 2.23 and with a time step h is defined by*

$$\begin{aligned} X_0^{(h)} &= x \qquad (2.24) \\ X_t^{(h)} &= X_{ih}^{(h)} + \mu(X_{ih}^{(h)}, ih)(t - ih) + \sigma(X_{ih}^{(h)}, ih)(B_t - B_{ih}), \\ &\quad \text{for } i \geq 0,\ t \in (ih, (i+1)h]. \end{aligned}$$

For a discussion on statistical error, discretization error, and convergence, see appendix A.

2.4.2 Implicit discretization schemes

Simulating an SDE with VIS, may be difficult since the requirements for convergence are not satisfied [2]. We therefore consider implicit discretization schemes

$$
\begin{aligned}
X_0^{(h)} &= x \\
X_t^{(h)} &= X_{ih}^{(h)} + (\theta_\mu \bar{\mu}(X_{(i+1)h}^{(h)}, (i+1)h) + (1-\theta_\mu)\bar{\mu}(X_{ih}^{(h)}, ih))(t-ih) \\
&\quad + (\theta_\sigma \sigma((X_{(i+1)h}^{(h)}, (i+1)h) + (1-\theta_\sigma)\sigma(X_{ih}^{(h)}, ih))(B_t - B_{ih}) \\
&\quad \text{for } i \geq 0,\ t \in (ih, (i+1)h]
\end{aligned}
$$

where $\bar{\mu} = \mu - \theta_\sigma \sigma\sigma'$ is a correction term, in order to ensure convergence to an Itô integral.

The explicit Euler-Maruyama scheme (2.24) is obtained by setting $\theta_\mu = \theta_\sigma = 0$. If we set $\theta_\mu = \theta_\sigma = 1$, we obtain the **fully implicit Euler scheme**:

$$
\begin{aligned}
X_0^{(h)} &= x \\
X_t^{(h)} &= X_{ih}^{(h)} + \bar{\mu}(X_{(i+1)h}^{(h)}, (i+1)h)(t-ih) \\
&\quad + \sigma((X_{(i+1)h}^{(h)}, (i+1)h)(B_t - B_{ih}) \qquad (2.25) \\
&\quad \text{for } i \geq 0,\ t \in (ih, (i+1)h].
\end{aligned}
$$

Fully implicit Euler scheme for the CKLS process

Simulating taking steps of size h, we get that at simulation step i, the next step is at $t = (i+1)h$ so that $X_t^{(h)} = X_{(i+1)h}^{(h)}$. Since the unknown $X_{(i+1)h}^{(h)}$ appears on both sides in (2.25), for each step in the implicit Euler scheme we need to solve for $X_{(i+1)h}^{(h)}$. For the CKLS process this means that we need to solve the equation

$$f(x) = x - (\alpha + \beta x + \gamma\sigma^2 x^{2\gamma-1})h - \sigma x^\gamma (B_{(i+1)h} - B_{ih}) = X_{ih}^{(h)} \qquad (2.26)$$

for x, in order to use this solution as the next simulation step $X^{(h)}_{(i+1)h} = x$.

It is not sure whether (2.26) has a unique solution, whether this is the case or not depends on the size of $\Delta B_i = B_{(i+1)h} - B_{ih}$ [2].

Whether this constitutes a problem or not depends on if we have VIS – and if we do, how much VIS is present. If we only have little or no VIS, it may be sufficient to pick either one of the solutions.

For cases where this is a problem, we refer to Albin et al. (2006) [2] for a more complete treatment.

3

Portfolio strategies

3.1 Constant Proportion Portfolio Insurance (CPPI)

The idea behind the CPPI strategy, popularized by Black and Jones (1987)[4], Perold (1986)[17] and, Black and Perold (1992)[5], is to ensure that the value of the portfolio at maturity time is greater than a guaranteed amount, while also being exposed to the potential upside of a risky asset. We start by defining a set of concepts:

- $S(t)$, value of an **active asset** at time t. This is a risky asset with a potential upside that one wants to be exposed to. In our case, the active asset is modeled as a GOU process.
- $R(t)$, value of an **reserve asset** at time t. This is a risk-free (or low-risk) asset. In our case, the reserve asset is modeled as a CKLS process.
- $V(t)$, the value of the **total wealth** (portfolio value) at time t.
- The **floor** of the portfolio, $F(t)$, is defined to be equal to the value of a fixed number of shares of the reserve asset at time t, i.e. $F(t)$ is proportional to $R(t)$. As parameter we choose a staring value of the floor as a proportion of the starting value of the portfolio $f = F(t_0)/V(t_0)$ so, assuming the fixed number of shares of the reserve asset can be non-integer, this fixed number of shares of the reserve asset is equal to $f\frac{V(t_0)}{R(t_0)}$.

- The **cushion** $C(t)$, is defined as the excess of wealth above the floor $C(t) = V(t) - F(t)$.
- The **exposure** $E(t)$, is the investment in the active asset $S(t)$.

The CPPI strategy is defined by its decision rule, that is, keeping the exposure $E(t)$ a constant multiple $m > 1$ (referred to as **multiplier**) of the cushion $C(t)$, as long as the portfolio value is greater than the floor $F(t)$:

$$E(t) = \min\{m \cdot C(t), V(t)\}. \tag{3.1}$$

This leads to the dynamic

$$dV(t) = E(t)\frac{dS(t)}{S(t)} + (V(t) - E(t))\frac{dR(t)}{R(t)} \tag{3.2}$$

of the portfolio value. In theory, the portfolio should be rebalanced continuously, so for numerical implementations we re-balance the portfolio at every time step. By discretizing (3.2) and (3.1) we can implement the strategy numerically

$$\begin{aligned} E(t_i) &= \min\{m(V(t_{i-1}) - F(t_{i-1})), V(t_{i-1})\} \\ V(t_i) &= E(t_i)\frac{S(t_i)}{S(t_{i-1})} + (V(t_{i-1}) - E(t_i))\frac{R(t_i)}{R(t_{i-1})}. \end{aligned}$$

Some typical values for the multiplier m are 3-5 [3][8].

3.2 Time Invariant Portfolio Protection (TIPP)

This portfolio insurance strategy is a modification of the CPPI strategy. It differs in how the floor $F(t)$ is chosen. TIPP (proposed by Estep and Kritzman (1988) [10]) has a ratchet mechanism, so that a proportion of the upside performance is locked in. This is achieved by defining the floor as the maximum between the CPPI floor $F_{\text{CPPI}}(t)$ (defined in the same way as for the

CPPI strategy) and a percentage of the historical maximum portfolio value

$$F_{\text{TIPP}}(t) = \max\{F_{\text{CPPI}}(t), PL \cdot \sup_{s \le t} V_s\}, \forall t \in [0, T]$$

with PL is a parameter – the **protection level**.

3.3 Exponential Proportion Portfolio Insurance (EPPI)

The Exponential Proportion Portfolio Insurance is also a modification of the CPPI, more specifically a specific strategy within the larger category of strategies called Variable Proportion Portfolio Insurance (VPPI)[15]. The idea behind VPPI strategies is to modify the CPPI by not using a constant multiplier m, but varying it depending on the portfolio dynamics. The EPPI strategy, proposed by Lee et al. (2008)[14] uses multiplier values defined as

$$m(t) = \nu + \exp\left\{a \ln\left(\frac{S(t)}{S(t-1)}\right)\right\},$$

where $\nu > 1$ and $a > 1$. Here, $e^{a \ln \frac{S(t)}{S(t-1}}$ is called the **dynamic multiple adjustment factor** (DMAF). The parameter a is a magnification factor of the DMAF.

4

Estimating parameters of the GOU process and the CKLS process

4.1 Empirical data

Historical data of stock quotes and interest rates are obtained from Yahoo! Finance.

Since the U.S. foreign policies during 2025 regarding import taxation have affected the market in an atypical way, which our models are not suited for, we avoid using data from this period of time. In an industrial application this should be accounted for using more complex models suited for rare case events. However, this is beyond the scope of this thesis.

4.2 Asset prices for the GOU process

We use $\Delta t = 0.001$ (corresponding to a change of units for time), and we normalize price data by dividing by the mean price during the observed period (change of units for price). The reason we do this is because the GOU process behaves differently depending on the value of X, whereas a we mostly do not expect stock prices or indices to behave differently depending on this – we expect percentage changes of the prices to be similar independently of the prices.

For one year observations, we use the calendar year 2024, for two years observations, we also include 2023, and for three years observations, we use the years 2022-2024. We avoid using data for 2025 since this is heavily influenced by the very unusual event of very atypical U. S. import tax policies, which is not the type of event our methods aim to model.

To estimate the parameters of the GOU process, we use six different data series. We use three indicies: S&P500, NIKKEI255, DAX P and three stocks: GOOG (Alphabet C), GM (General Motors) and PG (Procter & Gamble). We use one quote per day (since this is the smallest resolution available for free) – closing prices adjusted for dividends.

After some initial attempts of re-estimating parameters of a simulated GOU process with known parameters using both linear regression (see section 2.2.1) and the likelihood ratio method (see sections 2.2.2. and 2.2.3), we decide to use the likelihood ratio method, since this seems to perform somewhat better.

The estimated parameters for the GOU process (using the likelihood ratio method) can be seen in Table 4.1.

The GOU process may take values below zero, which is not desirable, since we aim to model stock (or index) prices, which cannot be negative. Since our estimates, seen in Table 4.1, have $\alpha < 0$ for all cases, we decide to modify our model by only accepting $\alpha \geq 0$ when optimizing (2.10) to find estimates of α and β. These estimates can be seen in Table 4.2.

4.3 Interest rates for the CKLS process

Just as when we estimate parameters for the GOU process, we use $\Delta t = 0.001$ (change of units for time) and normalize the price data by dividing by the mean price during the observed period (change of units for price), for the same reason as for the GOU process.

The data we use are taken from the same time periods as the stock quotes; see the previous section.

	Estimated parameters		
Asset (mean price)	α	β	σ
Index			
S&P500			
1 year (5428.235)	-2.23224	3.066528	0.2531142
2 years (4858.262)	-1.228202	2.075539	0.2542669
3 years (4605.013)	-5.884279	6.21008	0.326606
NIKKEI255			
1 year (38396.74)	-68.1813	68.85964	0.4967195
2 years (34548.83)	-1.690682	2.519377	0.4364588
3 years (32128.4)	-5.350938	5.805436	0.4287106
DAX P			
1 year (18394.24)	-10.52214	11.21454	0.2407312
2 years (17032.53)	-2.945481	3.630804	0.2464684
3 years (15974.48)	-6.062389	6.396048	0.3130737
Stock			
GOOG			
1 year (164.3488)	-15.18489	16.41284	0.5533988
2 years (141.6646)	-1.590099	2.996481	0.5664853
3 years (132.5947)	-7.035187	7.505376	0.6184116
GM			
1 year (45.13083)	-7.976477	9.493649	0.6460155
2 years (39.51398)	-8.242768	9.259269	0.6532054
3 years (39.31967)	-7.606858	7.428501	0.731241
PG			
1 year (160.4914)	-7.21457	7.758196	0.2982301
2 years (150.6896)	-8.065115	8.369887	0.2976791
3 years (145.7396)	-9.496059	9.64626	0.342728

Table 4.1: Estimated parameters for the GOU process using the likelihood ratio method with historical data for indices S&P500, NIKKEI255, DAX P, and stocks GOOG (Alphabet C), GM (General Motors), PG (Procter & Gamble). Historical data from before Jan 1, 2025 is used for periods of 1, 2 and 3 years. For NIKKEI255 the currency is yen, for DAX P the currency is Euro, otherwise the currency is the U.S. dollar.

To estimate the parameters of the CKLS process we use three different data series consisting of historical data for the bounds 13 WEEK TREASURY BILL, Treasury Yield 5 Years and CBOE Interest Rate 10 Year T No. We use one quote per day (since this is the smallest resolution available for free). The 13 WEEK TREASURY BILL is a zero-coupon bond, while the others make periodic interest payments every six months. Since the model we use

	Estimated parameters		
Asset (mean price)	$\alpha(\geq 0.0)$	β	σ
Index			
S&P500			
1 year (5428.235)	0.0	0.8251456	0.2531142
2 years (4858.262)	0.0	0.8257708	0.2542669
3 years (4605.013)	0.0	0.2103623	0.3266060
NIKKEI255			
1 year (38396.74)	0.0	0.6027456	0.4967195
2 years (34548.83)	0.0	0.8011638	0.4364588
3 years (32128.4)	0.0	0.3296174	0.4287106
DAX P			
1 year (18394.24)	0.0	0.6694667	0.2407312
2 years (17032.53)	0.0	0.6621352	0.2464684
3 years (15974.48)	0.0	0.2346381	0.3130737
Stock			
GOOG			
1 year (164.3488)	0.0	1.1004640	0.5533988
2 years (141.6646)	0.0	1.3400540	0.5664853
3 years (132.5947)	0.0	0.1567386	0.6184116
GM			
1 year (45.13083)	0.0	1.4018160	0.6460155
2 years (39.51398)	0.0	0.7387244	0.6532054
3 years (39.31967)	0.0	-0.4189908	0.7312410
PG			
1 year (160.4914)	0.0	0.53041600	0.2982301
2 years (150.6896)	0.0	0.25598160	0.2976791
3 years (145.7396)	0.0	0.07517357	0.3427280

Table 4.2: Estimated parameters for the GOU process using the likelihood ratio method with historical data for indicies S&P500, NIKKEI255, DAX P, and stocks GOOG (Alphabet C), GM (General Motors), PG (Procter & Gamble) with the constraint $\alpha \geq 0$. Historical data from before Jan 1, 2025 is used for periods of 1, 2 and 3 years. For NIKKEI255 the currency is yen, for DAX P the currency is Euro, otherwise the currency is the U.S. dollar.

is made to suit zero-coupon bonds, using data from bonds paying interest periodically is not ideal, but since this is the data available to us, we decide to try it. However, for the portfolio simulation in the next section, we only use parameters estimated using the 13 WEEK TREASURY BILL data.

The estimated parameters for the CKLS process (using the Hermite approximation for the maximum likelihood method presented in section 2.3.1) can be seen in Table 4.3, where we also indicate whether the estimated parameters are in any of the ranges (2.16) where the CKLS process has VIS.

	Estimated parameters				
Interest rate (mean price)	α	β	σ	γ	**VIS**
13 WEEK TREASURY BILL					
1 year (4.95425)	22.2363746	-32.2812932	0.0832809	1.0187323	Yes
2 years (4.999524)	46.8454693	-6.2390386	0.4674551	1.0144053	Yes
3 years (3.998299)	1.8201677	1.3201811	0.8562640	0.9874594	
Treasury Yield 5 Years					
1 year (4.12706)	1511.8249268	-887.8772009	0.2421946	-1.7523361	
2 years (4.097671)	12.2641687	-31.2347013	0.5543384	1.0140400	Yes
3 years (3.735108)	413.253721	-380.324826	0.541392	1.729732	Yes
CBOE Interest Rate 10 Year T No					
1 year (4.207135)	10.7150415	10.7395117	0.5040791	1.0142179	Yes
2 years (4.086568)	22.9870116	13.4937768	0.6059842	1.0137887	Yes
3 years (3.709991)	3.1851056	7.9371117	0.6742577	1.0135347	Yes

Table 4.3: Estimated parameters for the CKLS process using Maximum-likelihood with a Hermite approximation with historical data for the bounds 13 WEEK TREASURY BILL, Treasury Yield 5 Years and CBOE Interest Rate 10 Year T No. Historical data from before Jan 1, 2025 is used for periods of 1, 2 and 3 years. We indicate whether the CKLS process with the estimated parameters has VIS, i.e. if the estimated parameters are in any of the ranges in (2.16).

We note that the estimated parameter α in Table 4.3 is positive for all input data, preventing the process from reaching values below zero when simulating the process.

5

Portfolio simulation results and analysis

We decide to simulate pathways consisting of 500 trading days, which roughly corresponds to two years. As starting point for the simulations we use the last price in the empirical series that were used to estimate the parameters of the stochastic processes.

We set all initial portfolio values to 1.0, as this makes the simulation results easy to interpret.

Since we do not have access to over-night interest rates, we use the 13 week Treasury bill as an approximation for over-night interest rate, since hopefully a short-term interest rate better approximates this than a five-year bond (Treasury Yield 5 years) or a ten-year bond (CBOE Interest Rate 10 Year T No), which are the data that are (freely) available to us. Also, the 13 week Treasury bill is the only bond that is a zero-coupon bond, which is what our model is adapted to.

Simulation methods

As when estimating parameters, we use $\Delta t = 0.001$ (change of units for time) and simulate normalized price data (divided by the mean price during the observed period used to estimate the stochastic processes parameters).

For the GOU process, we use the standard Euler-Maruyama method with $h = \Delta t = 0.001$, presented in section 2.4.1. Since the CKLS process may have

VIS, we use the fully implicit Euler method with $h = \Delta t = 0.001$, presented in section 2.4.2. Since we simulate normalized price data, we obtain the final paths by multiplying with the mean price during the observed period used to estimate the stochastic processes parameters (as presented in Tables 4.1, 4.2 and 4.3 along with the parameter values of the processes).

Choice of parameters for trading strategies

For CPPI we use $m = 3$, $f = 0.7$. For TIPP we use $m = 3$, $f = 0.7$, and PL $= 0.7$. For EPPI we use $f = 0.7$, $\nu = 2.0$, and $a = 3$ (so that when the value of the asset $S(t) = S(t-1)$ then $m(t) = 3$). The choices of parameters are in the range of common values in applications [3] [8] and we try to make parameter choices such that they correspond to a similar risk level for the different trading strategies, in order to highlight the differences between the strategies.

5.1 Convergence of portfolio simulations

In order to get a grasp of the number of simulations (simulated pathways of the GOU process and the CKLS process that we apply the trading strategies to) necessary to get stable results, we decide to start by investigating convergence for one set of parameters for the GOU and CKLS processes. Here we mean convergence of the distribution of portfolio outcomes in the sense that performing more simulations would not affect the distribution of the portfolio outcomes. We measure this by plotting the 10th, 25th, 50th, 75th, and 90th percentiles of the distribution of portfolio outcomes as a function of the number of simulated portfolios.

We use parameters for the GOU and CKLS processes that were estimated from 3 years of observations of the S&P500-index (with no restrictions on α during the estimation) and the 13 week treasury bill.

We perform this analysis for all three trading strategies (CPPI, TIPP and EPPI).

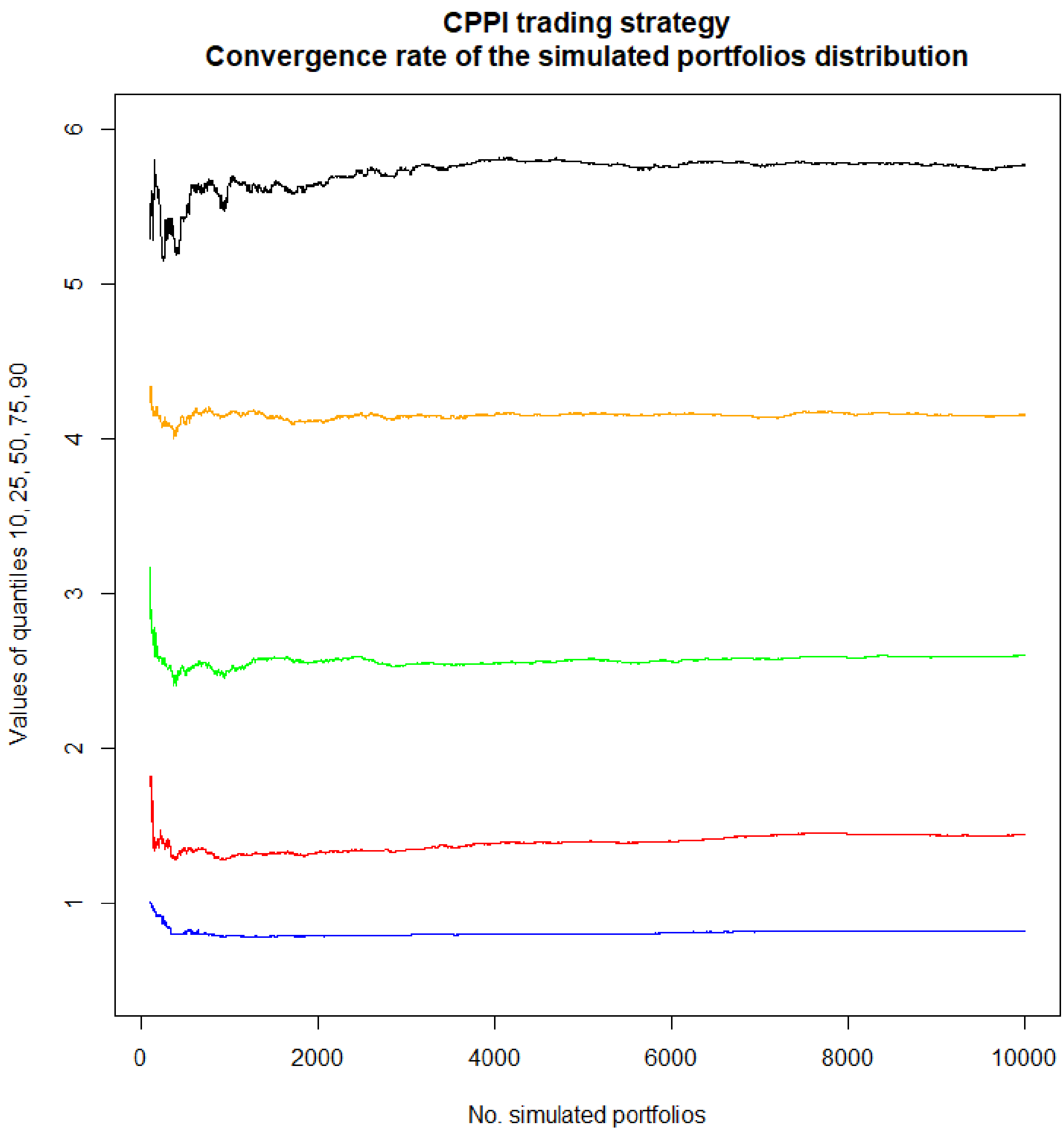

Figure 5.1: CPPI strategy. Parameters $m = 3$, $f = 0.7$. Convergence of 10th, 25th, 50th, 75th, and 90th quantiles of the simulated portfolio distribution using three different trading strategies. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see Tables 4.1 and 4.3. For each path 500 trading days were simulated.

For all three strategies, we see in Figures 5.1, 5.2 and 5.3 that 10 000 simulations is clearly enough, the distributions seem to stabilize even at 5000 simulations. Since 10 000 simulations can be performed in a relatively short time, we decide to continue to use this number of simulations when using

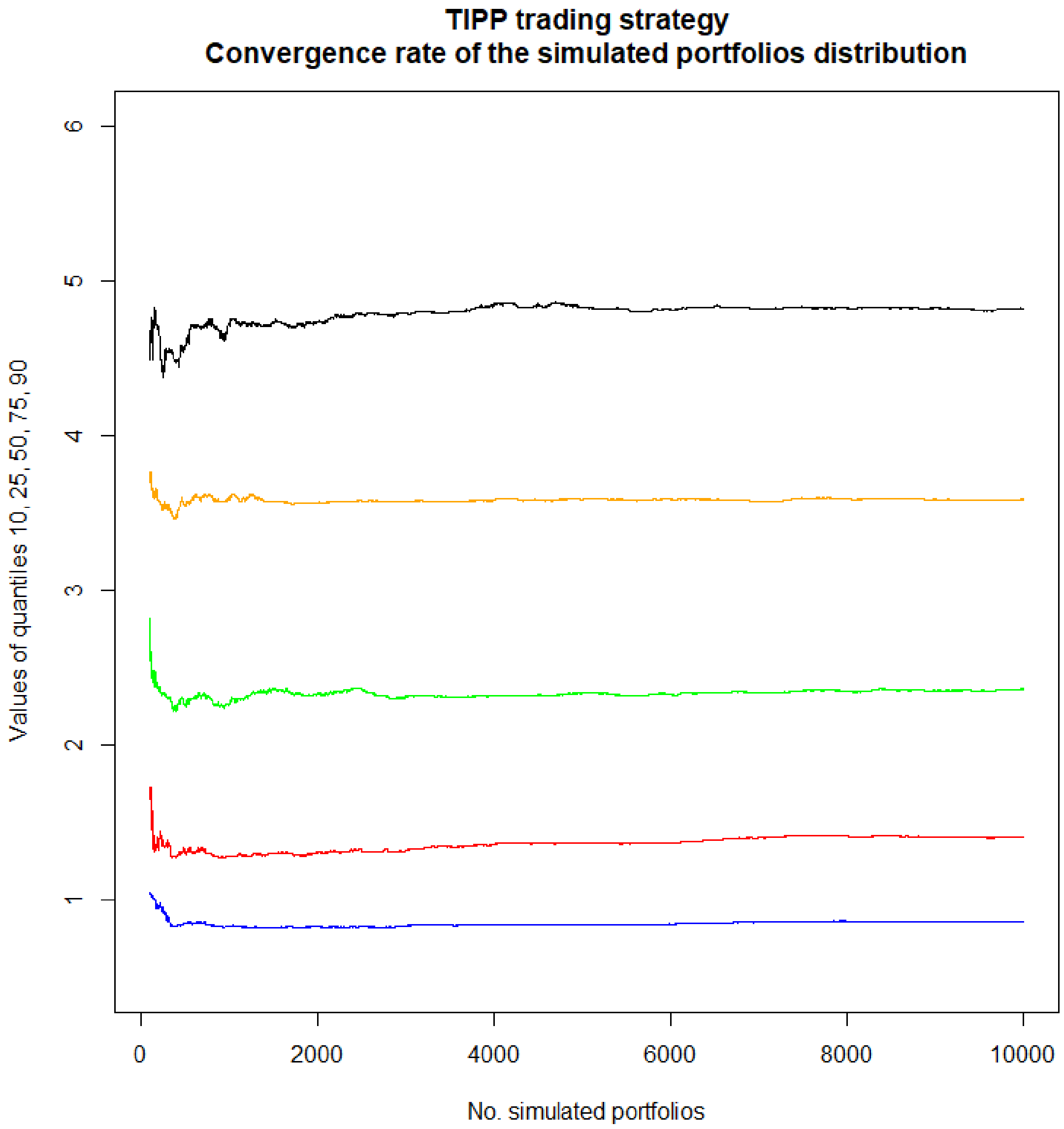

Figure 5.2: TIPP strategy. Paramters $m = 3$, $f = 0.7$, PL $= 0.7$. Convergence of 10th, 25th, 50th, 75th, and 90th quantiles of the simulated portfolio distribution using three different trading strategies. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see Tables 4.1 and 4.3. For each path 500 trading days were simulated.

other data in subsequent sections.

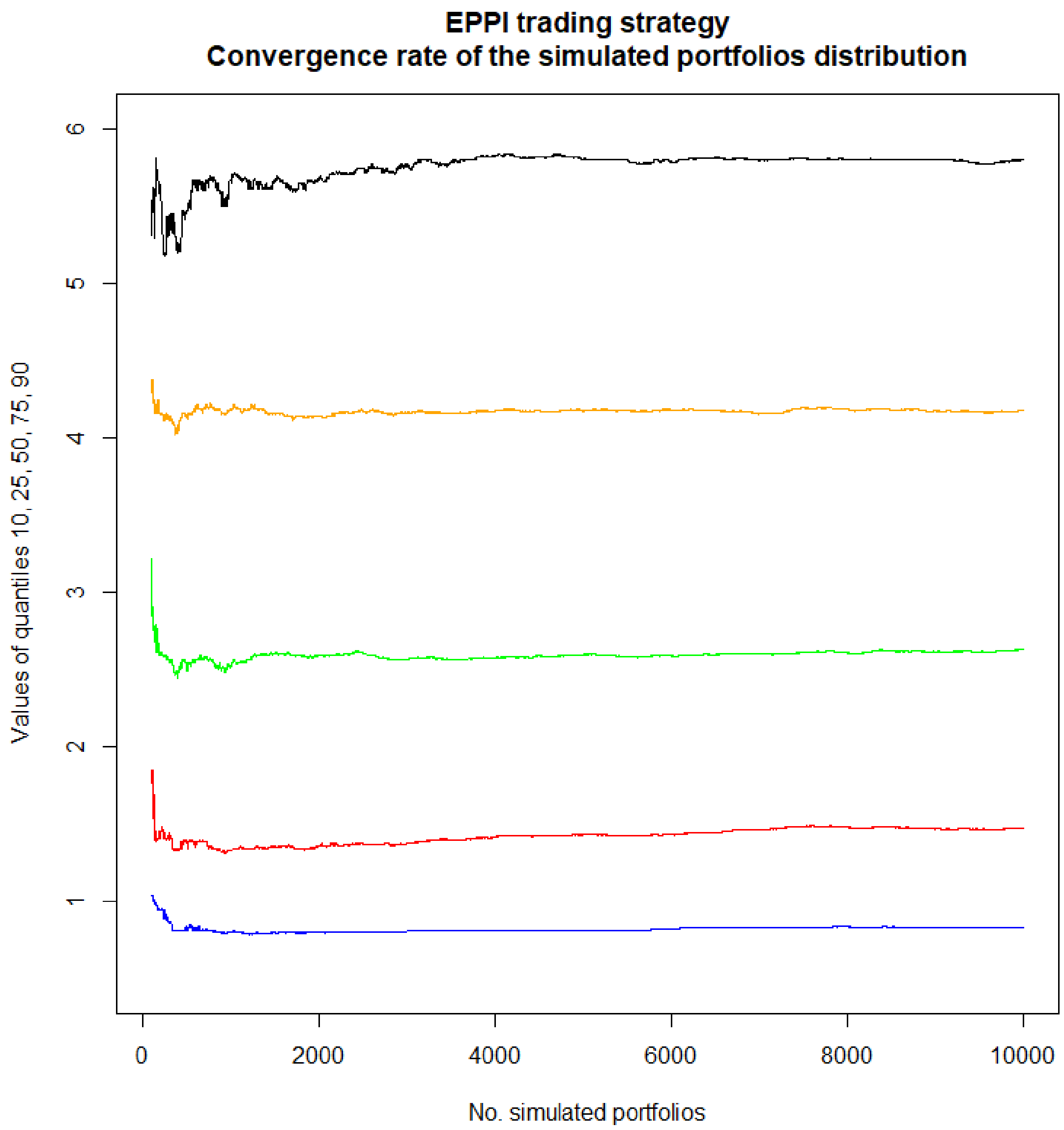

Figure 5.3: EPPI strategy. Paramters $f = 0.7$, $\nu = 2.0$, $a = 3$. Convergence of 10th, 25th, 50th, 75th, and 90th quantiles of the simulated portfolio distribution using three different trading strategies. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see Tables 4.1 and 4.3. For each path 500 trading days were simulated.

5.2 Distribution of portfolio simulations

Continuing our investigation from the previous section, we plot histograms of the simulated portfolio outcomes for 10 000 simulations.

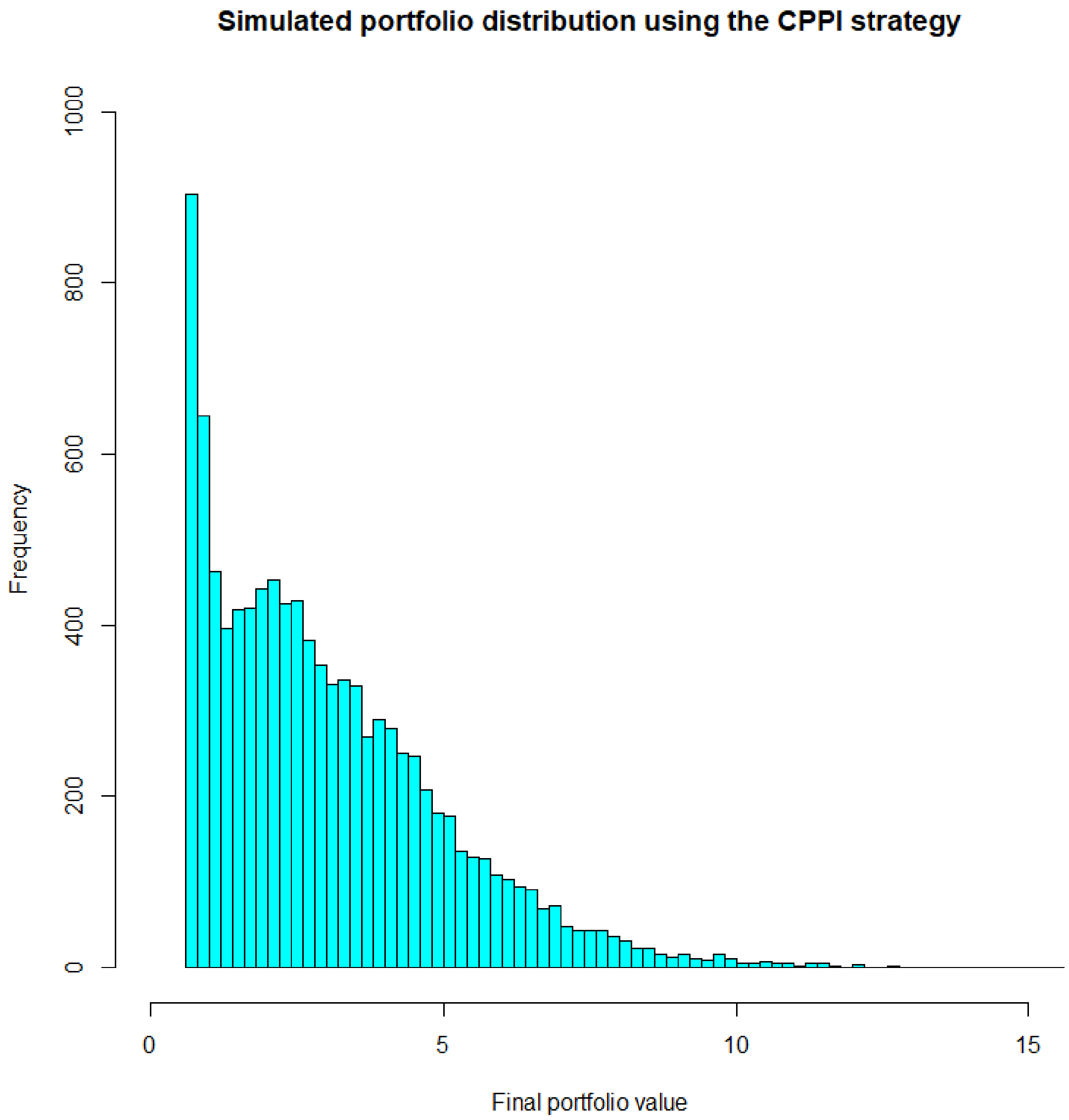

Figure 5.4: CPPI strategy. Parameters $m = 3$, $f = 0.7$. Distributions of the simulated portfolios using 10 000 simulations. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see tables 4.1 and 4.3. For each path 500 trading days were simulated.

In Figures 5.4, 5.5 and 5.6, we see that none of the portfolios outcomes seems to be normally distributed. The tails of the distributions seem to be significantly heavier. Using QQ-plots and comparing some common distributions (normal, lognormal, Pareto, exponential, and Weibull) we find that the portfolios seem to be Weibull distributed; see Figures 5.7, 5.8 and 5.9.

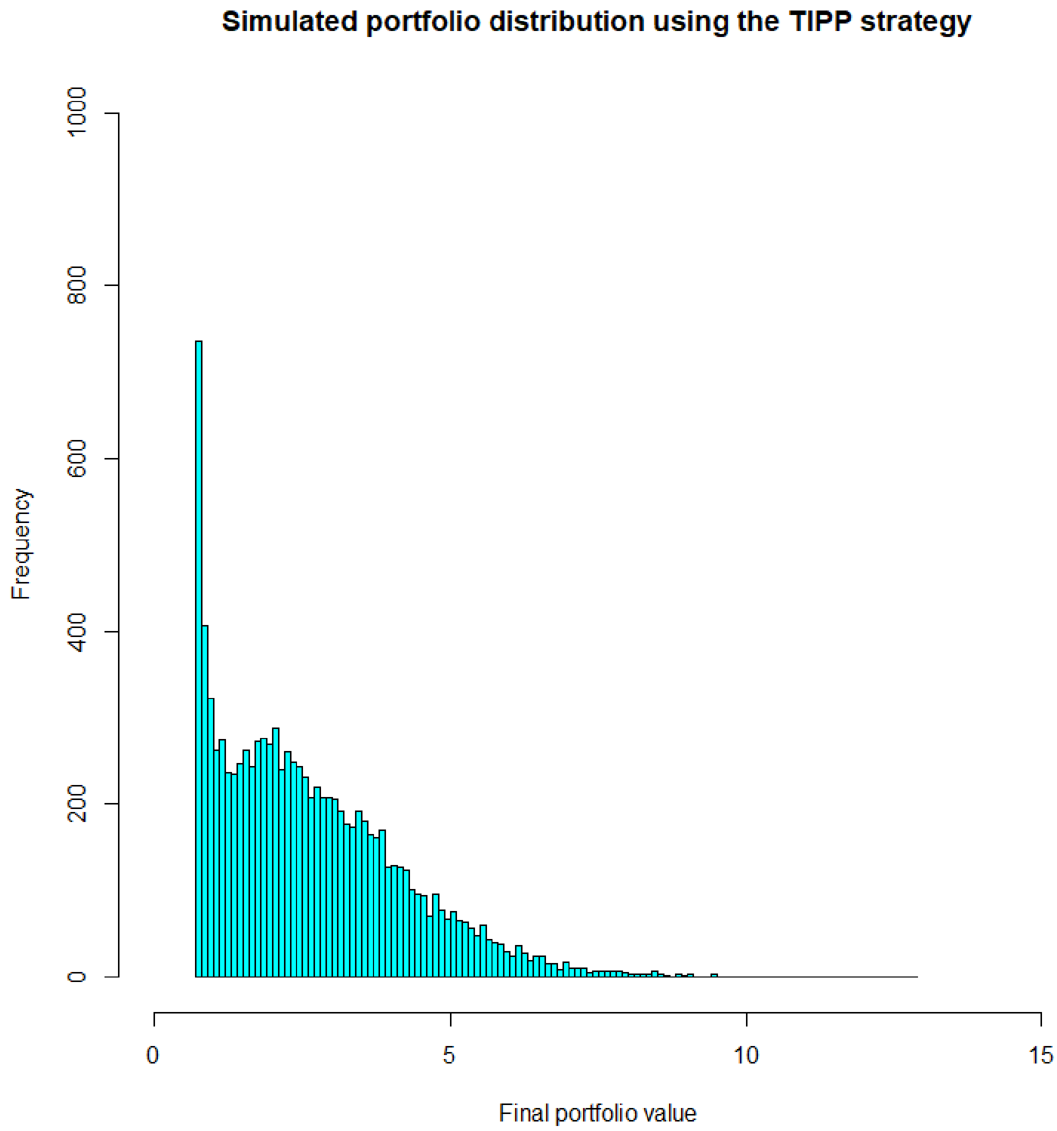

Figure 5.5: TIPP strategy. Parameters $m = 3$, $f = 0.7$, PL $= 0.7$. Distributions of the simulated portfolios using 10 000 simulations. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see tables 4.1 and 4.3. For each path 500 trading days were simulated.

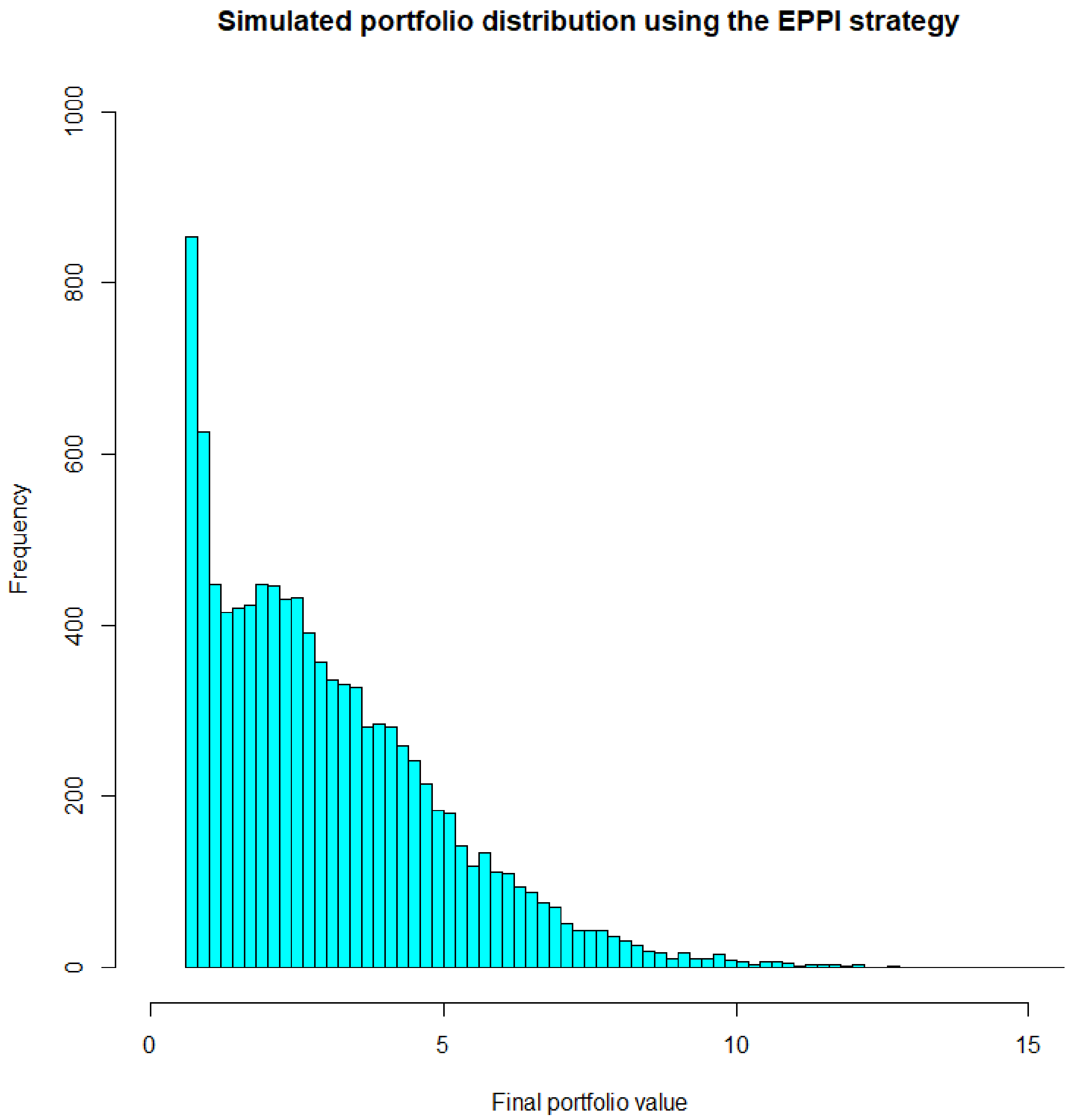

Figure 5.6: EPPI strategy. Parameters $f = 0.7$, $\nu = 2.0$, $a = 3$. Distributions of the simulated portfolios using 10 000 simulations. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see tables 4.1 and 4.3. For each path 500 trading days were simulated.

5.2.1 The Weibull distribution and heavy-tailed vs. light-tailed distributions

We saw that the simulated portfolios seem to be Weibull distributed

$$f(x) = c\tau x^{\tau-1} e^{-cx^{\tau}}, x \geq 0, c > 0, \tau > 0.$$

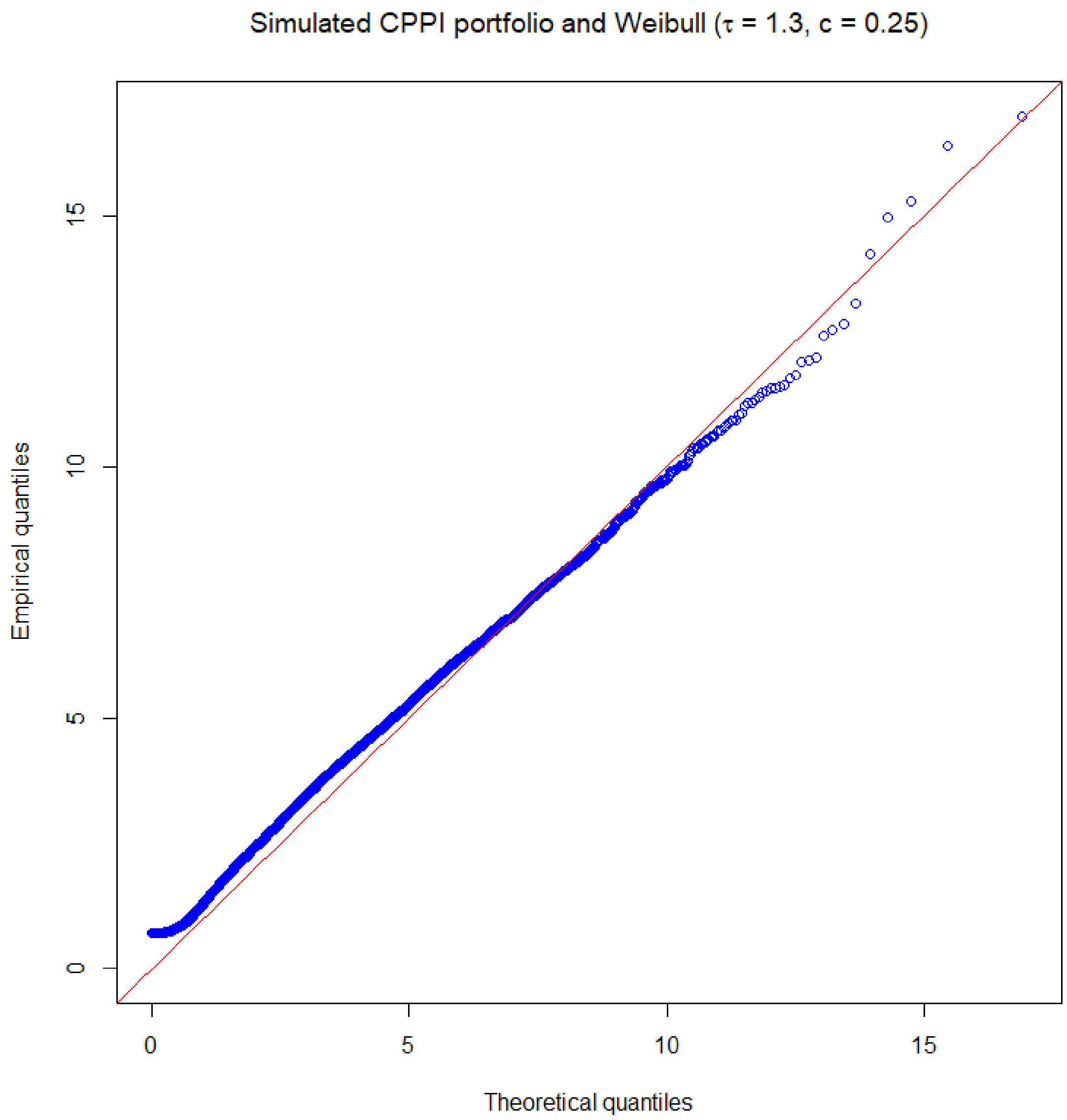

Figure 5.7: CPPI strategy. Parameters $m = 3$, $f = 0.7$. Comparing quantiles of distributions of the simulated portfolios using 10 000 simulations to theoretical quantiles of the Weibull distribution. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see Tables 4.1 and 4.3. For each path 500 trading days were simulated.

In order to characterize a distribution as heavy-tailed, one often uses the more precise concept of subexponential distributions [16, p. 103], see appendix C for a definition of subexponential distributions and a definition of the tail of a distribution.

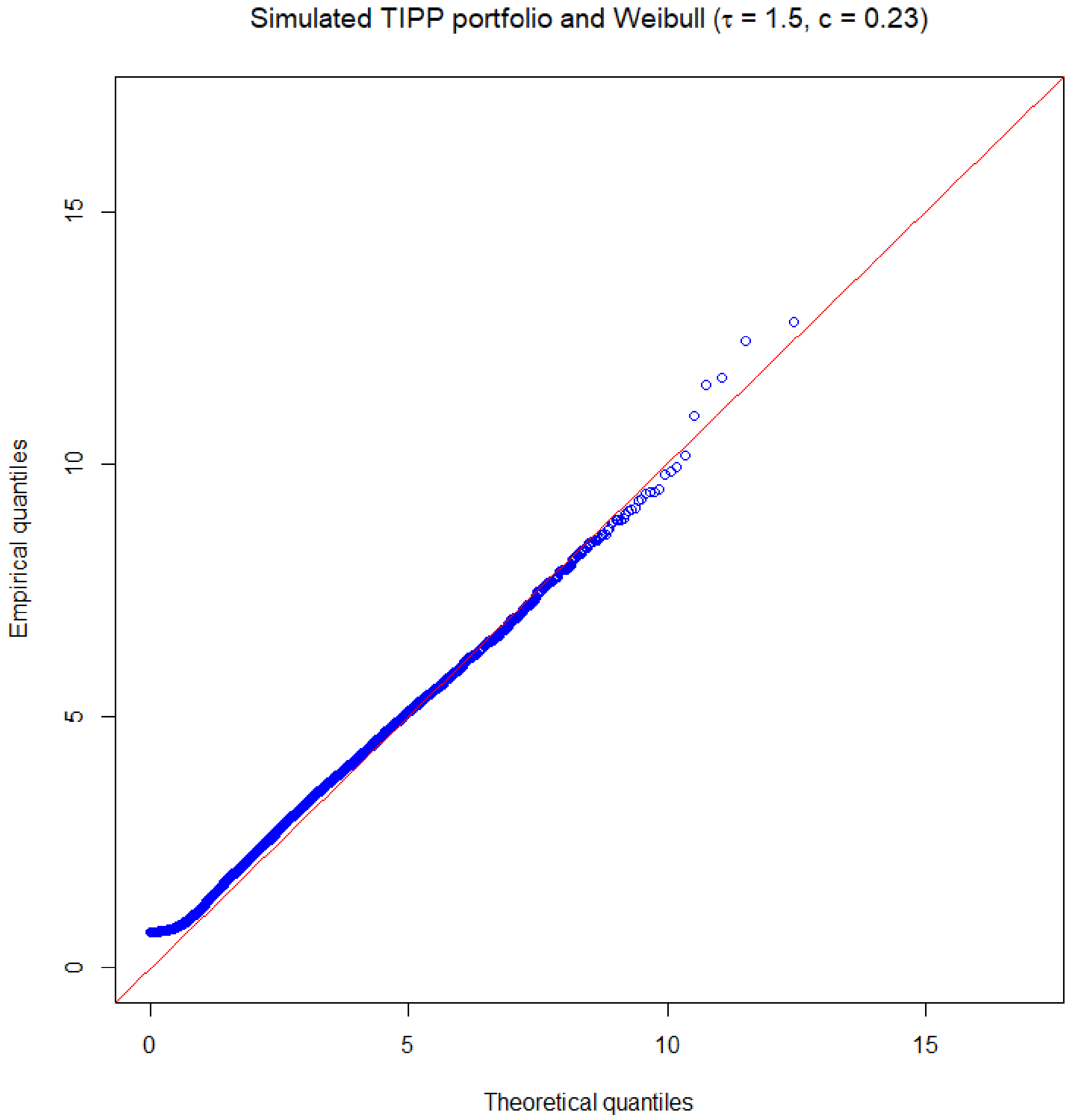

Figure 5.8: TIPP strategy. Paramters $m = 3$, $f = 0.7$, PL $= 0.7$. Comparing quantiles of distributions of the simulated portfolios using 10 000 simulations to theoretical quantiles of the Weibull distribution. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see Tables 4.1 and 4.3. For each path 500 trading days were simulated.

The Weibul distribution has the tail

$$\overline{F}(x) = e^{-cx^{\tau}}, x \geq 0$$

and the Weibull distribution is subexponential, and thus heavy-tailed for $\tau \in$

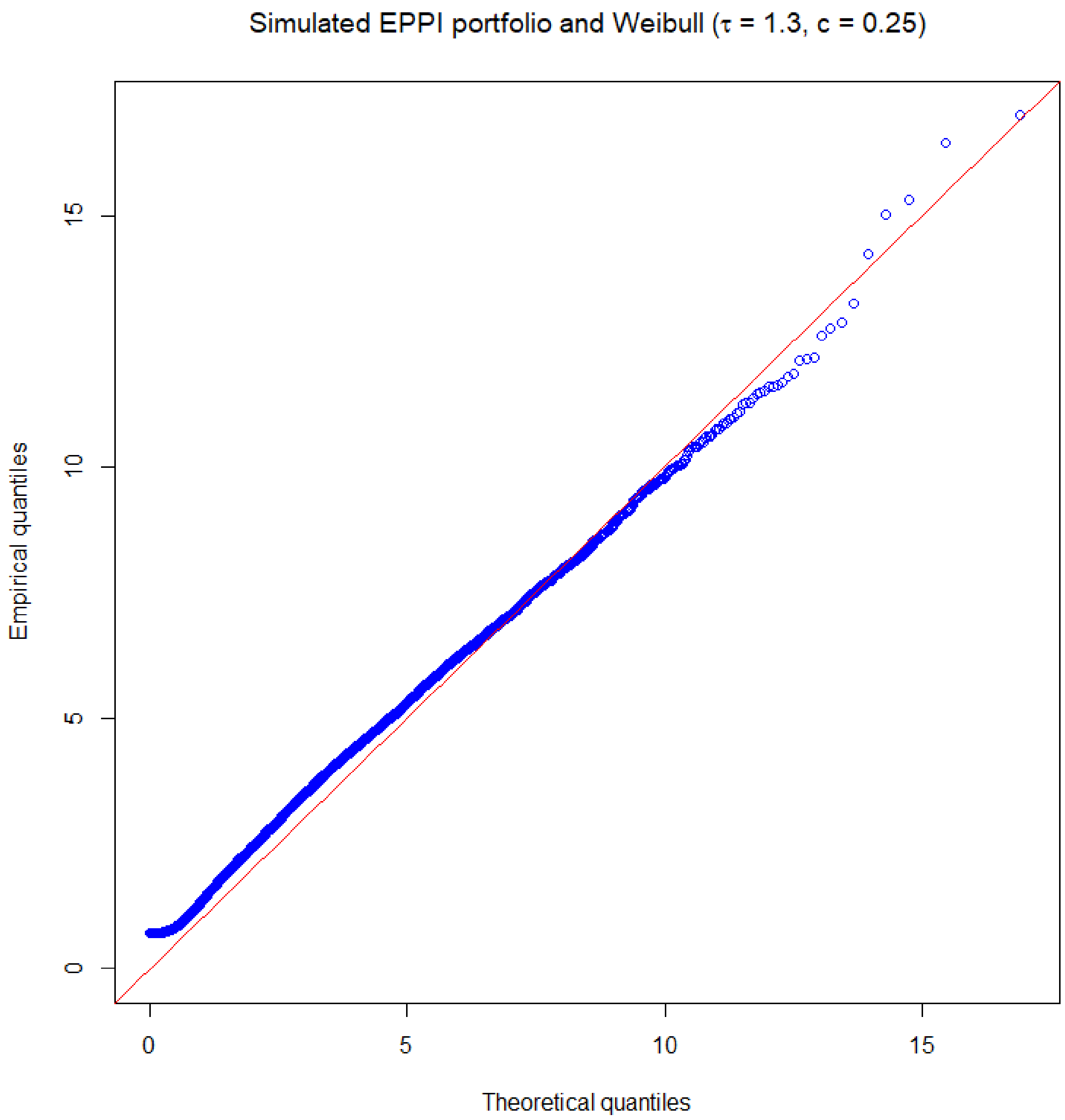

Figure 5.9: EPPI strategy. Paramters $f = 0.7$, $\nu = 2.0$, $a = 3$. Comparing quantiles of distributions of the simulated portfolios using 10 000 simulations to theoretical quantiles of the Weibull distribution. Parameters for the GOU and CKLS processes were estimated from 3 years of observations of the S&P500-index and the 13 week treasury bill (see Tables 4.1 and 4.3. For each path 500 trading days were simulated.

$(0, 1)$ and light-tailed for $\tau \geq 1$ [16, p. 88, 105].

In Figures 5.7, 5.8 and 5.9, we see no clear difference in distributions when using the CPPI strategy and the EPPI strategy, both simulated portfolio distributions seem to follow a Weibull distribution with $\tau = 1.3$ and $c = 0.25$ (to be exact, $2.9^{-1.3}$ since the software we use uses a different parameterization). However, when we apply the TIPP strategy we see a clear difference between distributions, the simulated TIPP portfolio distribution seems to follow a Weibull distribution with $\tau = 1.5$ and $c = 0.23$ ($2.7^{-1.5}$). Although none of these distributions are subexponential, the TIPP portfolio distribution seems to have a lighter tail than the CPPI and EPPI portfolio distributions.

We note that if we use subexponentiallity as the definition of heavy-tailness, none of the portfolio distributions is heavy-tailed, although the values $\tau = 1.3$ and $\tau = 1.5$ are not very far from $\tau < 1$ where the Weibull distribution is heavy-tailed.

5.3 Results of portfolio simulations

In Table 5.1 we present the simulated portfolios using GOU parameters (modeling the asset) estimated from S&P500 (without restrictions on α while estimating the parameters) and the CKLS parameters (modeling the interest rate) estimated using the 13 week treasury bill. The three different portfolio strategies are applied to the same set of estimated paths.

For parameter estimates (Table 4.1) using NIKKEI225 and DAX P data, the simulated GOU paths take values below zero very often. For this reason, we decide to continue to use the estimated parameters obtained under the constraint $\alpha \geq 0$ (Table 4.2). The result of these simulated portfolios can be seen in Tables 5.2, 5.3 and 5.4.

Asset	**Strategy**	**Portfolio value at maturity**						
		Percentile					Mean	Risk of loss
		10th	25th	50th	75th	90th		
S&P500								
1 year								
	CPPI	0.963	1.205	1.526	1.890	2.265	1.584	0.1216
	TIPP	0.979	1.185	1.455	1.760	2.072	1.501	0.1128
	EPPI	0.985	1.223	1.538	1.898	2.273	1.597	0.1093
2 years								
	CPPI	1.241	1.430	1.668	1.942	2.230	1.710	0.0088
	TIPP	1.288	1.459	1.667	1.905	2.149	1.699	0.0027
	EPPI	1.253	1.438	1.676	1.951	2.237	1.719	0.0067
3 years								
	CPPI	0.822	1.442	2.602	4.154	5.764	3.031	0.1547
	TIPP	0.860	1.410	2.363	3.588	4.822	2.663	0.1464
	EPPI	0.836	1.477	2.630	4.176	5.796	3.054	0.1479

Table 5.1: Results for portfolios using 10 000 simulations for each setting, with GOU parameters (modeling the asset) estimated from S&P500. The CKLS parameters (modeling the interest rate) were estimated using the 13 week treasury bill. For the CPPI strategy we use $m = 3$ and floor $F(t) = 0.7 \cdot R(t)$. For the TIPP strategy we use protection level PL $= 0.7$, $f = 0.7$, and $m = 3$. For the EPPI we use the $\nu = 2$ (so that when $S(t) = S(t-1)$ then $m(t) = 3$) and $a = 1$.

After simulating the portfolios we take a glance at the distribution by plotting them in histograms, but we do not present them here, since it would take up much space without adding very much information in addition to what we

Asset	**Strategy**	**Portfolio value at maturity**						
		Percentile					Mean	Risk of
($\alpha \geq 0$)		10th	25th	50th	75th	90th		loss
S&P500								
1 year								
	CPPI	1.169	1.305	1.477	1.672	1.861	1.501	0.0161
	TIPP	1.156	1.269	1.412	1.577	1.734	1.433	0.0108
	EPPI	1.183	1.316	1.486	1.680	1.866	1.511	0.0113
2 years								
	CPPI	1.184	1.317	1.485	1.671	1.859	1.509	0.0068
	TIPP	1.244	1.360	1.507	1.670	1.833	1.527	0.0013
	EPPI	1.196	1.326	1.492	1.679	1.867	1.517	0.0049
3 years								
	CPPI	0.877	0.943	1.068	1.252	1.439	1.121	0.3781
	TIPP	0.910	0.987	1.100	1.251	1.400	1.134	0.2782
	EPPI	0.896	0.969	1.097	1.275	1.458	1.144	0.3136
NIKKEI255								
1 year								
	CPPI	0.835	0.944	1.223	1.570	1.961	1.323	0.3097
	TIPP	0.885	0.997	1.194	1.446	1.738	1.263	0.2537
	EPPI	0.881	1.015	1.280	1.615	1.995	1.375	0.2334
2 years								
	CPPI	0.991	1.162	1.415	1.736	2.078	1.495	0.1081
	TIPP	1.099	1.254	1.462	1.728	2.006	1.523	0.0350
	EPPI	1.029	1.199	1.445	1.762	2.103	1.526	0.0758
3 years								
	CPPI	0.846	0.925	1.097	1.363	1.641	1.183	0.3755
	TIPP	0.891	0.986	1.133	1.334	1.553	1.188	0.2756
	EPPI	0.874	0.967	1.145	1.395	1.671	1.221	0.2994

Table 5.2: Results for portfolios using 10 000 simulations for each setting, with GOU parameters (modeling the asset) estimated from S&P500 and NIKKEI255 data under the constraint $\alpha \geq 0$. The CKLS parameters (modeling the interest rate) is estimated using the 13 week treasury bill. For the CPPI strategy we use $m = 3$ and floor $F(t) = 0.7 \cdot R(t)$. For the TIPP strategy we use protection level PL $= 0.7$, $f = 0.7$, and $m = 3$. For the EPPI we use the $\nu = 2$ (so that when $S(t) = S(t-1)$ then $m(t) = 3$) and $a = 1$.

present in the tables. However, we note that for some parameters of the GOU process, the simulated portfolios seem to be similar to normally distributed, and for other parameters, portfolios seem to be similar to exponentially distributed, which may not be evident if we just look at percentiles. This is consistent with our conclusion in the previous section that the distribution of portfolio outcomes may be well modeled using a Weibull distribution.

Asset	Strategy	Portfolio value at maturity						
		Percentile					Mean	Risk of
($\alpha \geq 0$)		10th	25th	50th	75th	90th		loss
DAX P								
1 year								
	CPPI	1.089	1.215	1.369	1.535	1.699	1.386	0.0361
	TIPP	1.092	1.193	1.323	1.463	1.602	1.338	0.0264
	EPPI	1.105	1.226	1.378	1.542	1.706	1.396	0.0266
2 years								
	CPPI	1.108	1.220	1.364	1.536	1.708	1.391	0.0215
	TIPP	1.172	1.274	1.402	1.551	1.701	1.423	0.0052
	EPPI	1.119	1.231	1.373	1.542	1.714	1.400	0.0161
3 years								
	CPPI	0.886	0.956	1.083	1.257	1.441	1.130	0.3476
	TIPP	0.919	0.998	1.113	1.254	1.403	1.142	0.2540
	EPPI	0.905	0.982	1.110	1.276	1.455	1.151	0.2872
GOOG								
1 year								
	CPPI	0.914	1.169	1.560	2.067	2.624	1.694	0.1522
	TIPP	0.971	1.152	1.427	1.791	2.219	1.527	0.1199
	EPPI	0.993	1.246	1.617	2.114	2.678	1.754	0.1031
2 years								
	CPPI	1.079	1.358	1.770	2.317	2.965	1.927	0.0673
	TIPP	1.208	1.430	1.757	2.182	2.679	1.870	0.0176
	EPPI	1.144	1.413	1.823	2.355	3.009	1.977	0.0395
3 years								
	CPPI	0.774	0.824	0.946	1.270	1.669	1.111	0.5703
	TIPP	0.823	0.899	1.046	1.276	1.557	1.133	0.4303
	EPPI	0.800	0.872	1.038	1.354	1.740	1.179	0.4546

Table 5.3: Results for portfolios using 10 000 simulations for each setting, with GOU parameters (modeling the asset) estimated from DAX P and GOOG data under the constraint $\alpha \geq 0$. The CKLS parameters (modeling the interest rate) is estimated using the 13 week treasury bill. For the CPPI strategy we use $m = 3$ and floor $F(t) = 0.7 \cdot R(t)$. For the TIPP strategy we use protection level PL $= 0.7$, $f = 0.7$, and $m = 3$. For the EPPI we use the $\nu = 2$ (so that when $S(t) = S(t-1)$ then $m(t) = 3$) and $a = 1$.

Asset	Strategy	Portfolio value at maturity						
		Percentile					Mean	Risk of
($\alpha \geq 0$)		10th	25th	50th	75th	90th		loss
GM								
1 year								
	CPPI	0.915	1.248	1.751	2.439	3.224	1.962	0.1417
	TIPP	0.990	1.205	1.537	2.023	2.572	1.691	0.1059
	EPPI	1.029	1.354	1.830	2.510	3.303	2.046	0.0894
2 years								
	CPPI	0.869	0.979	1.268	1.721	2.289	1.455	0.2740
	TIPP	0.988	1.138	1.381	1.734	2.153	1.498	0.1105
	EPPI	0.916	1.057	1.351	1.798	2.350	1.526	0.1927
3 years								
	CPPI	0.738	0.761	0.809	0.938	1.285	0.929	0.7989
	TIPP	0.773	0.821	0.917	1.081	1.313	0.994	0.6498
	EPPI	0.753	0.787	0.864	1.069	1.413	0.995	0.7003
PG								
1 year								
	CPPI	0.959	1.084	1.257	1.453	1.647	1.286	0.1466
	TIPP	0.984	1.087	1.224	1.385	1.547	1.250	0.1181
	EPPI	0.985	1.109	1.275	1.465	1.660	1.305	0.1151
2 years								
	CPPI	0.938	1.005	1.123	1.277	1.444	1.163	0.2380
	TIPP	0.996	1.079	1.194	1.333	1.479	1.220	0.1074
	EPPI	0.953	1.024	1.140	1.293	1.457	1.179	0.1947
3 years								
	CPPI	0.845	0.898	0.995	1.164	1.352	1.057	0.5085
	TIPP	0.877	0.941	1.042	1.178	1.331	1.080	0.4024
	EPPI	0.863	0.923	1.028	1.190	1.373	1.082	0.4410

Table 5.4: Results for portfolios using 10 000 simulations for each setting, with GOU parameters (modeling the asset) estimated from GM and PG data under the constraint $\alpha \geq 0$. The CKLS parameters (modeling the interest rate) is estimated using the 13 week treasury bill. For the CPPI strategy we use $m = 3$ and floor $F(t) = 0.7 \cdot R(t)$. For the TIPP strategy we use protection level PL $= 0.7$, $f = 0.7$, and $m = 3$. For the EPPI we use the $\nu = 2$ (so that when $S(t) = S(t-1)$ then $m(t) = 3$) and $a = 1$.

6

Discussion

Parameter estimation

When estimating parameters for the GOU process, we observe in Tables 4.1 and 4.2 that the estimates of the drift parameters α and β seem to indicate a general upward trend in the market. When estimating parameters under the condition $\alpha \geq 0$ this is especially clear since then estimates of α are equal to zero, for all input data with only one exception. All data are taken from the same period, the years $2022-2024$, so it is reasonable that the trend is similar for different indices and stocks, since market events in different parts of the world depend on each other to some extent. However, this has the disadvantage that when evaluating portfolio strategy performance, we mainly get information on how the strategies perform in markets during upward trends.

The estimates $\hat{\alpha}$ and $\hat{\beta}$ in Table 4.1 (no constraint on α) always have a different sign; $\hat{\alpha}$ is negative and $\hat{\beta}$ is positive, but $|\hat{\alpha}|$ and $|\hat{\beta}|$ are mostly fairly close, although $|\hat{\beta}|$ is always somewhat larger than $|\hat{\alpha}|$. The absolute values of $\hat{\alpha}$ range from 1.23 to 68.18 and similarly for $\hat{\beta}$ from 2.08 to 68.86. This indicates that there may be a redundancy for the parameterization, the value of the sum $\alpha + \beta$ seems to hold most of the important information about the drift. This seems reasonable because the drift parameter of the GOU $\alpha + \beta X(t)$ is equal to $\alpha + \beta$ when $X = 1$, which is the case for our input data, since we divided the input data by the mean, so the parameters are estimated on the input data close to 1.

When we optimize under the constraint $\alpha \geq 0$, we always get estimates of α very close to zero, but β is also affected in a way consistent with our previous reasoning about $\alpha+\beta$ (or some similar expression) is what holds the most important information about the drift. The parameters β are estimated to have values much smaller than before, and varying significantly less, with one exception the estimates vary between 0.08 and 1.40 (the exception being one case where the drift is negative $\hat{\beta} = -0.42$.

Something similar can be said for the estimates of the drift parameters α and β for the CKLS process. Here, the values vary even more, from 1.82 to 1511.82 for $\hat{\alpha}$ and between -887.88 and 13.49 for $\hat{\beta}$. However, the difference $|\alpha - \beta|$ also seems to vary more than for GOU. For example, for some input data, α and β are both positive. Under the assumption that we have a redundancy of parameters for the drift, the fact that we do not use the same methods for parameter estimation for GOU and CKLS may affect the range of the absolute values of estimates of α and β, i.e. this does not necessarily depend (only) on the difference of type of input data.

The diffusion parameter σ is estimated to have relatively small values for both GOU and CKLS, between 0.25 and 0.73 for the GOU process, and between 0.24 and 0.86 for the CKLS process, indicating low variance and thus low risk and high predictability of the stock market and interest rate.

For the CKLS process the parameter γ is estimated mostly to values around 1. For 7 of 9 estimations, the parameter is estimated to have values between 1.014 and 1.019. The other two estimates -1.75 and 1.73 are also not very far away.

The negative estimate -1.75, seems particularly strange, since this would make the diffusion term of the process larger for small values of X and smaller for large values of X, hence randomness plays a greater role for small values and a smaller role for large values . For the same data α is estimated to be 1511.82 and β to -887.88 which seem to have a destabilizing effect, in the sense that small changes in X affect dX to a large extent. The entire set of estimated parameters except maybe for $\hat{\sigma} = 0.24$ seems strange, and we may suspect that the algorithm for estimating the parameters does not work very well in this case. On the other hand, we may have atypical data, so that the parameters we found are actually the best (in a maximum likelihood sense) if the data were a path of a CKLS process, but the CKLS process may not be suitable for these atypical data.

However, in most cases γ is estimated to have values close to 1, which tells us, since α is positive for all estimates, that VIS is present (except when $\hat{\gamma} = 0.99$), but the effect from VIS is not very large. Since the CKLS process is equal to a GOU process if $\gamma = 1$, it is possible that we could simplify our model to using just a GOU process for the interest rate as well. Large values of γ, which would indicate a larger effect of VIS, would cause sudden peaks of the interest rate, which then rapidly returns to the value that the process had before the peak. This type of behaviour is not what one would expect from interest rate price data, so the fact that γ is not very large seems reasonable.

Simulated portfolios

When comparing investment strategies, see Tables 5.1, 5.2, 5.3 and 5.4, for different estimated parameter sets of the GOU process and the CKLS process, we see that EPPI performs similar to CPPI, but somewhat better in the sense that the entire distribution of outcomes is slightly translated to the right. TIPP on the other hand, modifies the shape of the distribution. This is especially clear when studying the distribution of outcomes in detail for the S&P500 data, see Figures 5.4, 5.5 and 5.6 (histograms of distributions), and Figures 5.7, 5.8 and 5.9 (QQ-plots).

In terms of mean and median, EPPI generally performs best, but in the sense of minimizing the risk of loss, the TIPP strategy performs best. EPPI has mostly a lower risk of loss than CPPI, which is expected if the distribution of outcomes is translated to the right.

For the one case with negative β (PG 3 years) – the only case we have with a negative market trend, which is hence of particular interest – TIPP has a higher median than both EPPI and CPPI (EPPI has a higher median than CPPI). In terms of mean, EPPI performs the best, but here TIPP outperforms

CPPI. Although the risk of loss is significantly higher compared to cases with positive market trends, TIPP has the lowest risk of loss, EPPI the second lowest, and CPPI the highest risk of loss. Hence, the effect of the TIPP and EPPI extensions of the CPPI trading strategy seems to be the same for both positive and negative market trends.

As for the distribution of outcomes in the QQ-plots 5.7, 5.8 and 5.9 we see that for using the parameters estimated using S&P500 data, it seems that we can use the Weibull distribution as a model. We see that both CPPI and EPPI seem to be similarly distributed, a Weibull distribution with parameters $\tau = 1.3$ and $c = 0.25$, while the TIPP portfolio outcome distribution seems to follow a Weibull distribution with parameters $\tau = 1.5$ and $c = 0.23$. A higher value on τ indicates a lighter tail – this is the type of behaviour one would expect for a more conservative strategy such as TIPP with its ratchet mechanism.

The Weibull distribution is flexible in the sense that it for some parameters mimics the normal distribution and for other parameters mimics the exponential distribution. As mentioned in section 5.3, taking a quick look at the distribution of portfolio outcomes using other parameter sets for the simulation, we see both more normal-like distributions and more exponential like distributions, which indicates that Weibull seems like a good choice to model this.

The question whether a light tail or a heavy tail is preferable is not so clear since the low probability events of earning large amounts are positive events for the investor. However, as noted in the result section, using subexponentiality as a marker of heavy-tailedness, none of the investment strategies gives rise to a distribution of portfolio outcomes that is heavy tailed. TIPP has the advantage of minimizing the risk of loss, whereas EPPI has the advantage of maximizing the expected gain, both in terms of the median and mean. The strategy one chooses depends on whether one prefers a safe bet or a larger potential upside.

We also notice, as can be seen in Figures 5.1, 5.2 and 5.3, the distributions of portfolio outcomes seem to converge using relatively few simulations, 5000 seems to be enough. Even 2000 could perhaps be used if one wants to prioritize computational speed to the price of a small loss of accuracy, which could be relevant in a high-frequency trading application.

Modeling choices and quality assessment

The three trading strategies that we investigate in this thesis have different sets of parameters (multiplier m, starting level of the floor f, etc). The parameters we use were chosen with the goal of having a similar risk level. However, both TIPP and EPPI are more flexible than CPPI, so the potential advantages (or disadvantages) of these strategies could be different for different parameter values, which could affect what strategy an investor would prefer.

The number of trading days, for our portfolio simulations was 500, roughly corresponding to two calendar years. This was chosen in attempt to make a compromise between having enough trading days for the effects of the strategies become apparent, and having a time period short enough so that the market behaviour used to estimate parameters would not become obsolete. We have not investigated what effect a different number of trading days would have.

In this thesis we have not considered the choice of the GOU process and CKLS process to model assets and interest rates, these are merely taken as prerequisites for our investigation. While estimating parameters, we obtain estimates that are optimal (in a likelihood ratio sense for GOU and maximum likelihood sense for CKLS) conditioned on the fact that we use the GOU process and the CKLS process, not to what degree these processes themselves are optimal to use. As discussed previously, we may have found that these models are even redundant α and β seem to model something similar at least for GOU, and for CKLS the γ parameters were often estimated to have values close to 1, indicating that we could have used the much simpler GOU process for those cases.

This assumption on the suitability of these two processes while investigating the performance of portfolio strategies is something that limits the degree of certainty with which we can draw conclusions of our experiments. Investigating this suitability is beyond the scope of this thesis. In order to draw conclusions about the portfolio strategies, it is reasonable to investigate to what degree the GOU process and the CKLS process actually are good models of assets and interest rates. It is also reasonable to do similar experiments for the same portfolio strategies using different stochastic processes.

We also note that we have worked with data with a resolution of one quote per day, thus there is a lot of information in the trading history within days that we have not accounted for (we did not do this since this data is not available for free). Since market behaviour changes, one would not like to work with data that is too old. On the other hand, since we only use one quote per day, this severely limits the number of data points we can use to estimate the parameters. We chose time series of 1, 2, and 3 years of data, in an attempt to make a compromise between these two goals, but we have not investigated very deeply the effect of this. One way of handling this could be to weight the price data, since prices closer to the present should be more relevant. If one for example uses a maximum likelihood estimation based method for parameter estimation, this could easily be implemented.

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A

Statistical error, discretization error and convergence for the Euler-Maruyama scheme

A.1 Statistical error and discretization error

Generally when we simulate stochastic processes we want to repeat the process several times and use information from all simulated paths, often by computing the expectation of the process (or of a functional of the process). Assume that we generate M independent paths by using the Euler-Maruyama scheme 2.24, and let $(X_{ih}^{(h,m)})_{0\leq i\leq N}, 1 \leq m \leq M$ denote the m-th simulation of a stochastic process X defined by an SDE 2.23. We can then approximate the expectation by

$$\mathrm{E}[X] \approx \frac{1}{M}\sum_{m=1}^{M} X^{(h,m)}. \tag{A.1}$$

The error of A.1 can be decomposed into a superposition of two errors with a different nature [12, p. 168-170]:

$$\begin{aligned}&\frac{1}{M}\sum_{m=1}^{M} X^{(h,m)} - \mathrm{E}[X] \\ &= \underbrace{\frac{1}{M}\sum_{m=1}^{M} X^{(h,m)} - \mathrm{E}[X^{(h,m)}]}_{\text{statistical error}} + \underbrace{\mathrm{E}[X^{(h,m)}] - \mathrm{E}[X]}_{\text{discretization error}}.\end{aligned} \tag{A.2}$$

The statistical error occurs because of the finite number of simulations. By the law of large numbers, it converges to 0 as $M \to \infty$. This error depends also on h but it has only a minor impact.

The discretization error occurs because of the time discretization only (independent of M). Here we expect that the accuracy should be better as $h \to 0$.

A.2 Strong convergence of the discretization

The Euler-Maruyama approximation of the corresponding SDE can be shown to converge (under certain conditions) in the sense of closeness of trajectories of two processes measured using L^p-norm [12, p. 170-171]. This is called **strong convergence**.

Theorem 1 (Strong convergence at order $\frac{1}{2}$) *Suppose that the coefficients (μ, σ) from 2.23 for X satisfy*

1. $|\mu(x,t) - (y,s)| + |\sigma(x,t) - (y,s)| \leq C_{\mu,\sigma}(|x-y| + |t-s|[\frac{1}{2})$ *for all* $(t,s,x,y) \in [0,T] \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,
2. $|\mu(0,0)| + |\sigma(0,0)| \leq C_{\mu,\sigma}$,

for a certain finite constant $C_{\mu,\sigma}$.

For any $p > 0$, there exists a constant C (depending on $T, x, C_{\mu,\sigma}$ and p) such that

$$\mathrm{E}\left[\underset{0\leq t\leq T}{sup} |X_t^{(h)} - X_t|^p\right] \leq Ch^{\frac{p}{2}}. \tag{A.3}$$

Since the L^p-norm is of magnitude $h^{\frac{1}{2}}$ we say that the convergence is of order

$\frac{1}{2}$.

A.3 Weak convergence of the discretization

If we want to evaluate $\mathrm{E}[X_T]$ using the Monte-Carlo method (see A.1) and the Euler-Maruyama scheme, the discretization error $\mathrm{E}[X_T^{(h)}] - \mathrm{E}[X_T]$ is called the **weak error** because it is the error between the distributions of $X_T^{(h)}$ and X_T, not between their trajectories (this is the strong error). Under certain conditions, one can show that the convergence rate is h (weak convergence at order 1) [12, p. 173].

A.4 Discretization error vs. statistical error

By examining the asymptotic convergence rate Gobet (2016) [12, p. 192-194], discusses how to best allocate the effort between the discretization ($h \to 0$) and independent simulations ($M \to \infty$), given a computational cost $\mathcal{C}_{\text{cost}}$. The squared error can be calculated:

$$\mathrm{E}(\mathrm{Error}_{h,M}^2) = \frac{\mathrm{Var}(X^{(h)})}{M} + [\mathrm{E}(X^{(h)}) - \mathrm{E}(X)]^2 \approx \frac{\mathrm{Var}(X^{(h)})}{M} + [Ch]^2,$$

with C being a constant. He arrives at three asymptotic cases:

1. $M >> h^{-2}$: Here the statistical error becomes negligible – the discretization error dominates. $\mathrm{Error}_{h,M}^2 \approx Ch$.
2. $M << h^{-2}$: Here the discretization error becomes negligible. The distribution of the total error converges to the distribution of a Gaussian random variable with variance $\mathrm{Var}(X)$ which can be asymptotically calculated using a sample of M simulations. This means that it is possible to form confidence intervals.
3. $M\ h^{-2}$: Here the discretization error and the statistical error are of same magnitude. It is possible to write confidence intervals, but they are not asymptotically centered in X, as they are in the previous case.

Gobet (2016) [12, p. 194] concludes that the second case seems to be preferable, since 1. for a given computational cost, it (almost) achieves the best accuracy, and 2. it gives asymptotically centered confidence intervals.

B

Computing the coefficients of the Taylor expansion of the Hermite approximation

For the CKLS SDE

$$dX = \tilde{\mu}(X;\theta)dt + \tilde{\sigma}(X;\theta)dB(t)$$

we have

$$\begin{aligned} \theta &= (\alpha, \beta, \sigma, \gamma), \sigma > 0 \\ \tilde{\mu}(x;\theta) &= \alpha + \beta x \\ \tilde{\sigma}(x;\theta) &= \sigma x^{\gamma}. \end{aligned}$$

Change of variables

$$y = \tilde{\gamma}(x;\theta) = \int^x \frac{1}{\tilde{\sigma}} du = \int^x \frac{1}{\sigma} u^{-\gamma} du = \begin{cases} \frac{1}{\sigma}\ln(x) & \text{if } \gamma = 1 \\ \frac{1}{\sigma(1-\gamma)} x^{1-\gamma} & \text{if } \gamma \neq 1 \end{cases}$$

and thus

$$x = \tilde{\gamma}^{-1}(y;\theta) = \begin{cases} e^{\sigma y} & \text{if } \gamma = 1 \\ (\sigma(1-\gamma)y)^{\frac{1}{1-\gamma}} = C|y|^{\frac{1}{1-\gamma}} & \text{if } \gamma \neq 1 \end{cases}$$

where $C = |\sigma(1-\gamma)|^{\frac{1}{1-\gamma}}$. For $\gamma = 1$ the SDE is equivalent to a GOU process – we ignore this case for now. The drift coefficient of the proposed variable change

$$\mu_Y(y;\theta) = \frac{\tilde{\mu}(\tilde{\gamma}^{-1}(y;\theta);\theta)}{\tilde{\sigma}(\tilde{\gamma}^{-1}(y;\theta);\theta)} - \frac{1}{2}\frac{\partial \tilde{\sigma}}{\partial x}(\tilde{\gamma}^{-1}(y;\theta);\theta)$$

is (for $\gamma \neq 1$) given by

$$\begin{aligned} \mu_Y(y;\theta) &= \frac{\alpha + \beta C|y|^{\frac{1}{1-\gamma}}}{\sigma C^\gamma |y|^{\frac{\gamma}{1-\gamma}}} - \frac{1}{2}\sigma\gamma C^{\gamma-1}|y|^{\frac{\gamma-1}{1-\gamma}} \\ &= \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} + \frac{\beta C^{1-\gamma}}{\sigma}|y| - \frac{\sigma\gamma C^{\gamma-1}}{2|y|} \\ &= \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} + \beta|1-\gamma||y| - \frac{\gamma}{2|1-\gamma||y|} \\ &= \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} + \beta(1-\gamma)y - \frac{\gamma}{2(1-\gamma)y} \end{aligned}$$

since $(1-\gamma)y > 0 \implies |1-\gamma||y| = (1-\gamma)y$. We have defined

$$\lambda_Y(y;\theta) = -\frac{1}{2}\left(\mu_Y^2(y;\theta) + \frac{\partial \mu_Y}{\partial y}(y;\theta)\right).$$

We compute

$$
\begin{aligned}
\int_{y_0}^{y} \mu_Y^2(w;\theta)dw &= \int_{y_0}^{y} \left(\frac{\alpha}{\sigma C^\gamma}|w|^{\frac{\gamma}{\gamma-1}} + \beta(1-\gamma)w - \frac{\gamma}{2(1-\gamma)w} \right)^2 dw \\
&= \int_{y_0}^{y} \frac{\alpha^2}{\sigma^2 C^{2\gamma}}|w|^{\frac{2\gamma}{\gamma-1}} + \beta^2(1-\gamma)^2 w^2 + \frac{\gamma^2}{4(1-\gamma)^2 w^2} \\
&\quad + \frac{2\alpha\beta|1-\gamma|}{\sigma C^\gamma}|w|^{\frac{2\gamma-1}{\gamma-1}} - \frac{\alpha\gamma}{\sigma C^\gamma |1-\gamma|}|w|^{\frac{1}{\gamma-1}} - \beta\gamma dw \\
&= \Bigg[-\frac{\alpha^2|\gamma-1|}{\sigma^2 C^{2\gamma}(3\gamma-1)}|w|^{\frac{3\gamma-1}{\gamma-1}} + \frac{\beta^2(1-\gamma)^2 w^3}{3} - \frac{\gamma^2}{4(1-\gamma)^2 w} \\
&\quad - \frac{2\alpha\beta(1-\gamma)^2}{\sigma C^\gamma(3\gamma-2)}|w|^{\frac{3\gamma-2}{\gamma-1}} + \frac{\alpha}{\sigma C^\gamma}|w|^{\frac{\gamma}{\gamma-1}} - \beta\gamma w \Bigg]_{w=y_0}^{y},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\int_{y_0}^{y} \lambda_Y(w;\theta)dw =& -\frac{1}{2}\Bigg[-\frac{\alpha^2|\gamma-1|}{\sigma^2 C^{2\gamma}(3\gamma-1)}|w|^{\frac{3\gamma-1}{\gamma-1}} + \frac{\beta^2(1-\gamma)^2 w^3}{3} - \frac{\gamma^2}{4(1-\gamma)^2 w} \\
& - \frac{2\alpha\beta(1-\gamma)^2}{\sigma C^\gamma(3\gamma-2)}|w|^{\frac{3\gamma-2}{\gamma-1}} + \frac{\alpha}{\sigma C^\gamma}|w|^{\frac{\gamma}{\gamma-1}} - \beta\gamma w \\
& + \frac{\alpha}{\sigma C^\gamma}|w|^{\frac{\gamma}{\gamma-1}} + \beta(1-\gamma)w - \frac{\gamma}{2(1-\gamma)w} \Bigg]_{w=y_0}^{y} \\
=& -\frac{1}{2}\Bigg[-\frac{\alpha^2|\gamma-1|}{\sigma^2 C^{2\gamma}(3\gamma-1)}|w|^{\frac{3\gamma-1}{\gamma-1}} + \frac{\beta^2(1-\gamma)^2 w^3}{3} - \frac{\gamma(2-\gamma)}{4(1-\gamma)^2 w} \\
& - \frac{2\alpha\beta(1-\gamma)^2}{\sigma C^\gamma(3\gamma-2)}|w|^{\frac{3\gamma-2}{\gamma-1}} + \frac{2\alpha}{\sigma C^\gamma}|w|^{\frac{\gamma}{\gamma-1}} + \beta(1-2\gamma)w \Bigg]_{w=y_0}^{y}.
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\Lambda_Y(y;\theta) =& \frac{\alpha^2|\gamma-1|}{2\sigma^2 C^{2\gamma}(3\gamma-1)}|y|^{\frac{3\gamma-1}{\gamma-1}} - \frac{\beta^2(1-\gamma)^2 y^3}{6} + \frac{\gamma(2-\gamma)}{8(1-\gamma)^2 y} \\
& + \frac{\alpha\beta(1-\gamma)^2}{\sigma C^\gamma(3\gamma-2)}|y|^{\frac{3\gamma-2}{\gamma-1}} - \frac{\alpha}{\sigma C^\gamma}|y|^{\frac{\gamma}{\gamma-1}} - \frac{1}{2}\beta(1-2\gamma)y.
\end{aligned}
$$

We now compute the integral of $\mu_Y(y;\theta)$, necessary to obtain an explicit expression for $p_Y^{(0)}$.

$$\begin{aligned}\int_{y_0}^{y} \mu_Y(w;\theta)dw &= \int_{y_0}^{y} \frac{\alpha}{\sigma C^\gamma}|w|^{\frac{\gamma}{\gamma-1}} + \beta(1-\gamma)w - \frac{\gamma}{2(1-\gamma)w}dw \\ &= \left[-\frac{\alpha|\gamma-1|}{\sigma C^\gamma(2\gamma-1)}|w|^{\frac{2\gamma-1}{\gamma-1}} + \frac{\beta(1-\gamma)}{2}w^2 - \frac{\gamma}{2(1-\gamma)}\ln(w)\right]_{w=y_0}^{y} \\ &= \frac{\alpha|\gamma-1|}{\sigma C^\gamma(2\gamma-1)}\left(|y_0|^{\frac{2\gamma-1}{\gamma-1}} - |y|^{\frac{2\gamma-1}{\gamma-1}}\right) + \frac{\beta(1-\gamma)}{2}(y^2-y_0^2) \\ &\quad + \frac{\gamma}{2(1-\gamma)}\ln\left(\frac{y_0}{y}\right)\end{aligned}$$

and hence

$$\begin{aligned}p_Y^{(0)}(\Delta, y|y_0;\theta) =& \frac{1}{\sqrt{2\pi\Delta}}\exp\left[-\frac{(y-y_0)^2}{2\Delta} + \frac{\alpha|\gamma-1|}{\sigma C^\gamma(2\gamma-1)}\left(|y_0|^{\frac{2\gamma-1}{\gamma-1}} - |y|^{\frac{2\gamma-1}{\gamma-1}}\right)\right. \\ &\left. + \frac{\beta(1-\gamma)}{2}(y^2-y_0^2) + \frac{\gamma}{2(1-\gamma)}\ln\left(\frac{y_0}{y}\right)\right].\end{aligned}$$

The case $\gamma = 1$

For $\gamma = 1$ we have $x = \tilde{\gamma}^{-1}(y;\theta) = e^{\sigma y}$, so computations become much simpler.

$$\begin{aligned}\mu_Y(y;\theta) &= \frac{\alpha}{\sigma}e^{-\sigma y} + \frac{\beta}{\sigma} - \frac{\sigma}{2} \\ \int_{y_0}^{y}\mu_Y(w;\theta)dw &= \frac{\alpha}{\sigma^2}(e^{-\sigma y_0} - e^{-\sigma y}) + \left(\frac{\beta}{\sigma} - \frac{\sigma}{2}\right)(y-y_0) \\ \int_{y_0}^{y}\mu_Y^2(w;\theta)dw &= \left[-\frac{\alpha}{2\sigma^3}e^{-2\sigma w} + \left(\frac{\beta}{\sigma}-\frac{\sigma}{2}\right)^2 w + \frac{1}{\sigma}\left(\alpha - \frac{2\alpha\beta}{\sigma^2}\right)e^{-\sigma w}\right]_{w=y_0}^{y}\end{aligned}$$

From these results we can compute

$$\begin{aligned}-2\int_{y_0}^{y}\lambda_Y(w;\theta)dw &= \int_{y_0}^{y}\mu_Y^2(w;\theta) + \frac{\partial\mu_Y}{\partial w}(w;\theta)dw \\ &= \left[-\frac{\alpha}{2\sigma^3}e^{-2\sigma w} + \left(\frac{\beta}{\sigma}-\frac{\sigma}{2}\right)^2 w + \frac{1}{\sigma}\left(\alpha - \frac{2\alpha\beta}{\sigma^2}\right)e^{-\sigma w} + \frac{\alpha}{\sigma}e^{-\sigma w}\right]_{w=y_0}^{y} \\ &= \left[-\frac{\alpha}{2\sigma^3}e^{-2\sigma w} + \left(\frac{\beta}{\sigma}-\frac{\sigma}{2}\right)^2 w + \frac{2\alpha}{\sigma}\left(1 - \frac{\beta}{\sigma^2}\right)e^{-\sigma w}\right]_{w=y_0}^{y},\end{aligned}$$

so we obtain

$$\Lambda_Y(y;\theta) = \frac{\alpha}{4\sigma^3}e^{-2\sigma w} + \left(\frac{\sigma}{4} - \frac{\beta}{2\sigma}\right)^2 w + \frac{\alpha}{\sigma}\left(\frac{\beta}{\sigma^2} - 1\right)e^{-\sigma w}$$

and

$$p_Y^{(0)}(\Delta, y|y_0;\theta) = \frac{1}{\sqrt{2\pi\Delta}}\exp\left[-\frac{(y-y_0)^2}{2\Delta} + \frac{\alpha}{\sigma^2}(e^{-\sigma y_0} - e^{-\sigma y}) + \left(\frac{\beta}{\sigma} - \frac{\sigma}{2}\right)(y-y_0)\right].$$

C

Subexponential distributions

In order to characterize a distribution as heavy-tailed, one often use the more precise concept of subexponential distributions [16, p. 103].

Definition 4 (Subexponential distribution) *The positive random variable X with unbounded support and its distribution are said to be Subexponential if for a sequence $(X_i)_{i=1}^n$ of iid random variables with the same distribution as X the following relation holds:*

$$\textit{For all } n \geq 2: \\ P(X_1 + ... + X_n > x) = P(\max_{i=1,...,n} X_i > x)(1 + o(1)), \ \textit{as} \ \rightarrow \infty. \quad \text{(C.1)}$$

Defining the **tail** $\overline{F}(x) = 1 - F(x)$ of the distribution F, we can [16, p. 103] write (C.1) as

$$\text{For all } n \geq 2: \lim_{x\rightarrow\infty} \frac{P(X_1 + ... + X_n > x)}{\overline{F}(x)} = n.$$

One can show [16, p. 104] that the tail of a subsexponential distribution $\overline{F}(x)$ decays to 0 slower than any exponential function $e^{-\epsilon x}$, for $\epsilon > 0$, which justifies its name.

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