Supermultiplets and Koszul Duality

Super-Yang-Mills and supergravity using pure spinors

Master's thesis in Physics

SIMON JONSSON

DEPARTMENT OF PHYSICS

CHALMERS UNIVERSITY OF TECHNOLOGY

Gothenburg, Sweden 2021

www.chalmers.se

MASTER'S THESIS 2021

Supermultiplets and Koszul Duality

Super-Yang-Mills and supergravity using pure spinors

SIMON JONSSON

Department of Physics
Division of Subatomic, High Energy and Plasma Physics
Mathematical Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2021

Supermultiplets and Koszul Duality
Super-Yang-Mills and supergravity using pure spinors
SIMON JONSSON

Supervisor: Martin Cederwall, Division of Subatomic, High Energy and Plasma Physics
Examiner: Martin Cederwall, Subatomic, High Energy and Plasma Physics

Master's Thesis 2021
Department of Physics
Division of Subatomic, High Energy and Plasma Physics
Mathematical Physics
Chalmers University of Technology
SE-412 96 Gothenburg
Telephone +46 31 772 1000

Printed by Chalmers Reproservice
Gothenburg, Sweden 2021

"If I have seen further it is by standing on the shoulders of Giants"

– Isaac Newton, *1675*

Abstract

There is a deep and general correspondence connecting constrained objects, superalgebras and supermultiplets. The most natural way of displaying this correspondence is using partition functions and BRST formalism. The partition functions of the constrained objects can be found to be dual to that of a Lie superalgebra and in some cases L_∞ algebras. The duality reveals itself through the partition functions being each others inverses.

We find the pure spinor partition functions in $D = 10$ to contain the supermultiplet for $D = 10$ linearised super-Yang-Mills. We find the dual algebra to be an extension of D_5 with an odd null root, defining an infinite dimensional graded Lie superalgebra called a Borcherds superalgebra. The algebra is proven to be freely generated by the super-Yang-Mills multiplet from order 3.

Further investigation concerns the case of $D = 11$ supergravity. The dual algebra is no longer just a Lie superalgebra. In addition to the Lie bracket structure there is also, at least, a 3- and a 4-bracket structure. It is conjectured that this algebra, from order 4, is freely generated, under the Lie bracket, by the $D = 11$ supergravity multiplet.

Keywords: Representation theory, pure spinors, supersymmetry, Lie superalgebras, partition functions, super-Yang-Mills, supergravity,

Acknowledgements

First and foremost, my most sincere gratitude goes out to my supervisor Martin Cederwall whom have been an amazing support throughout this project. Your incredible understanding of mathematics and physics never ceases to amaze me. It has been an honour and a privilege to have discussed the most astonishing concepts and formalisms within mathematical physics with you. The occasional anecdotes of what you have done during your research have been a pleasure to take part of.

A big thank you goes out to professor Bengt E.W. Nilsson whom have been an amazing lecturer and inspiration throughout most of my years of masters studies. You classes have been a true pleasure to attend and has provided me with the background knowledge to complete this thesis.

My love and gratitude goes out to my family and friends, your support during this thesis and throughout my studies have been truly appreciated. Especially I wish to thank my classmate Henric Ernbrink for all great discussions and collaborations during our years at Chalmers.

Simon Jonsson, Gothenburg, May 2021

Contents

Chapter 1

Introduction

To find a theory which unify all four fundamental forces in a single formalism have shown to be one of science hardest problems yet. For over 60 years there have really only been one candidate for this; String/M-Theory. With string theory we have been forced to completely change our intuition of reality; instead of picturing small pointlike particles bouncing around we now imagine the fundamental building block to be small vibrating strings dancing together, either in loops or connected to branes [1]. As if this wasn't enough we must also come to peace with the idea of more than three spatial dimensions, for bosonic string theory must be formulated in 26 dimensions. This is solved by having *compact dimensions* which means the excess dimensions are wrapped up in themselves. These compact dimensions must of course be very small in order to not be observable by us.

No theory of nature is complete without fermions, this is because the fermions are the building blocks of matter, i.e everything we see and feel. When introducing fermions to string theory we can reduce the amount of dimensions to "just" 10, i.e. 9 spatial dimensions. There are several ways to actually introducing fermions to string theory, all of which must be supersymmetric (for the readers who are not familiar to supersymmetry we refer to App. D for a review of the concept). One method uses so called *pure spinors* and was introduced by Berkovits [2, 3]. Pure spinors as mathematical objects were introduce by Cartan [4] as early as the 1930s. They are bosonic spinors in even dimensions constrained to reside in a minimal orbit. Our definition of pure spinors in the present thesis will coincide with the definition by Cartan in one of the cases we consider ($D = 10$). It was early realised that pure spinors may have a fundamental connection to maximally supersymmetric field theories [5, 6, 7]. In recent years a deep relation have been discovered between pure spinors and supersymmetric field theories [2, 3, 8, 9, 10], this provides a way to use pure spinors to write down manifestly supersymmetric, off-shell formulations of maximally supersymmetric field theories [11, 12, 13].

On another end, a deep and general correspondence have been discovered between constrained objects and superalgebras [14, 15, 16, 17]. This duality is observed using a BRST treatment of the constraint. The treatment involves introducing ghosts, and if the constraint is reducible ghosts for ghosts, in order to find the physical degrees of freedom.

The elements of the superalgebra are then identified with the ghosts. The realisation of the duality between pure spinors and superalgebras and the correspondence between pure spinors and supermultiplets is truly astonishing. The supermultiplets can actually be used to generate the superalgebras, and they furthermore may generate them freely, i.e. no relations are forced on the multiplication. This could tell us that there is something deep and fundamental with these quite simple constrained objects.

In this thesis we will first investigate pure spinors in 10 and 11 dimensions, we will investigate their relation to supermultiplets using cohomological algebra and partition functions. In Chapter 5 we will show that we can retrieve the supermultiplets of 10-dimensional super-Yang-Mills (SYM) and 11-dimensional supergravity (SG) from the partition functions of pure spinors. We will also further investigate the deep duality between pure spinors and superalgebras, we will investigate this in both 10 and 11 dimensions, and show that in the 10-dimensional case, the algebra is freely generated from order 3. In the 11 dimensional case we find, at least, an L_4 structure where the 2-bracket is conjectured to be freely generated from order 4 by the supergravity multiplet.

Chapter 2

Preliminaries

This chapter serves as a support for the vast sea of definitions, conventions, and notifications in mathematical physics. It will define some of the more common concepts to so that the thesis will be somewhat self-contained. The present chapter will in no way constitute a complete treatment of the concepts but, if needed, will allow one to go back to freshen up ones memory.

2.1 Conventions

We will work in the mostly + convention, meaning the Minkowski spacetime metric is $\eta^{ab} = diag(-1, +1, \ldots, +1)$. When working with curved spaces the metric is denoted g_{mn}.

Physical dimensions are denoted in powers of inverse length, i.e. if $[X] = L^x$ then L is omitted and we write $[X] = -x$.

2.1.1 Representations

Representations of Lie algebras are denoted either by their Dynkin labels or by a bold face number indicating the dimension of the representation. Bars over representations denote the dual representation. Elements in dual representations or dual vector spaces will be denoted by *. Following the physics terminology, we will refer to the module, i.e. the vector space a representation acts on, as representation as well. Tensor products of representations are denoted with $\otimes$, whilst direct sum are denoted by $\oplus$. We will in some cases use the symbol $\ominus$ this is a formal symbol used to "subtract" representations from expressions.

2.1.2 Indices

$a, b, c, \ldots$ represent Minkowski vector indices, and $i, j, k, \ldots$ are used for spatial parts. Greek lower case letters $\alpha, \beta, \delta, \ldots$ represent spinorial indices. When working with differential forms and curved spaces we will specify more clearly which part of the alphabet we take our indices from. Capital letters are often used to indicate two- or more different kinds of indices. This is also specified in the concerned sections.

2.2 Algebras and cohomology

An algebra, $\mathcal{A}$, is a vector space over a field K equipped with a bilinear (under addition), product $*$. That is to say, for $x,\ y,\ z \in \mathcal{A}$, and $a, b, c \in K$

$$ax * (by + cz) = (ab)x * y + (ac)x * z.$$

We will for the rest of the discussion refrain from writing out the $*$. Its presence will be obvious from context.

2.2.1 Superalgebras

A Superalgebra is an algebra, $\mathcal{A}$, which is $\mathbb{Z}_2$-graded commutative. That is, $\mathcal{A}$ is the direct sum

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1,$$

where, for $x, y \in \mathcal{A}$

$$xy = (-)^{\varepsilon_x \varepsilon_y} yx.$$

The number ε_x is called the parity of x and is 0 if $x \in \mathcal{A}_0$ and 1 if $x \in \mathcal{A}_1$. We will in most cases be a bit sloppy in the notation and omit the ε and simply write $(-)^{xy}$, instead of $(-)^{\varepsilon_x \varepsilon_y}$.

This notion of parity is often also called Grassmann parity and elements in $\mathcal{A}_0$ or $\mathcal{A}_1$ called Grassmann even or odd, respectively.

2.2.2 Lie superalgebras

The space of linear endomorphisms (functions from V to itself), $End(V)$, of a $\mathbb{Z}_2$-graded vector space V has a $\mathbb{Z}_2$-grading

$$End(V) = End(V)_0 \oplus End(V)_1,$$

that is induced from

$$\varepsilon_{Mx} = \varepsilon_M + \varepsilon_x \quad x \in V,\ M \in End(V).$$

We can define the graded commutator between two transforms $M_1, M_2 \in End(V)$ as

$$[M_1, M_2\} = M_1 M_2 - (-)^{\varepsilon_{M_1} \varepsilon_{M_2}} M_2 M_1. \tag{2.1}$$

The graded commutator (2.1) obeys the generalised Jacobi identity

$$\Big[[M_1, M_2\}, M_3\Big\} + (-)^{\varepsilon_{M_2}(\varepsilon_{M_1} + \varepsilon_{M_3})}\Big[[M_3, M_1\}, M_2\Big\} + (-)^{\varepsilon_{M_1}(\varepsilon_{M_2} + \varepsilon_{M_3})}\Big[[M_2, M_3\}, M_1\Big\},$$

and thus the space $\Big(End(V), +, [\cdot, \cdot\}\Big)$ defines a $\mathbb{Z}_2$ graded Lie algebra or Lie superalgebra. For deeper description of Lie algebras see App. A

2.2.3 Derivations, differentials and cohomology

A (graded) derivation, is a linear transform, D, on a $\mathbb{Z}_2$-graded vector space V that obeys the Leibniz rule

$$D(xy) = xD(y) + (-)^{\varepsilon_x \varepsilon_y} D(x)y.$$

The set $Der(V)$ is a Lie subalgebra of $End(V)$ [18].

One can define another notion of graded algebra, not necessarily related to the $\mathbb{Z}_2$ grading

$$\mathcal{A} = \bigoplus_n \mathcal{A}_n,$$

where n is an integer, one can also restrict to non negative integers as well. The degree of an element in $\mathcal{A}$ is determined by which subset of the partition it belongs.

$$deg(x) = n \Leftrightarrow x \in \mathcal{A}_n.$$

The multiplication of such an algebra is

$$\mathcal{A}_n \mathcal{A}_n \subset \mathcal{A}_{n+m}.$$

The grading of $\mathcal{A}$ induces a grading in $End(\mathcal{A})$ analogous to the aforementioned $\mathbb{Z}_2$-grading of $End(\mathcal{A})$.

A differential, d, is an odd, nilpotent derivation that acts on an graded algebra

$$d^2 \equiv \frac{1}{2}\{d, d\} = 0$$
$$\varepsilon(d) = 1.$$

A graded algebra together with a differential is called a *graded differential algebra*. If d acts on a graded algebra, $\mathcal{A}$, then d will have an induced degree since $d \in End(\mathcal{A})$. If we assume $\deg(d) = \pm 1$ we have that

$$d\mathcal{A}_n \subseteq \mathcal{A}_{n\pm 1}.$$

Consider a differential, d, with degree 1. Since $d^2 = 0$ we have that objects which are d-exact, i.e. can be written as d acting on something, are annihilated by d. Another way of saying this is that the Image of d is in the Kernel of d. This allows us to define the *cohomology* of d by

$$H^*(d) = \frac{Ker(d)}{Im(d)} = \bigoplus_n H^n(d),$$

i.e. the algebra of the elements in $\mathcal{A}$ annihilated by d (d-closed), modulo those that are d-exact ($x = dy \in \mathcal{A}$). The quotient algebra have inherited the same grading as the vector space or algebra on which d acts.

2.3 Exterior algebra and differential forms

We will here define the notion of differential forms and exterior algebra. Differential geometry is wildly used throughout almost all areas of physics. The reason why is because it is independent of coordinates. This simplifies things greatly. Coordinate systems are a human construct, it would indeed be strange if Nature it self was not coordinate independent, that there would be some kind of fundamental coordinate system that permeates the entire universe. This will not be a complete treatment of all the aspects of differential geometry, we will define some of the most frequent entities and try to motivate its convenience in calculations. For the interested reader we refer to [19] for a complete treatment.

We start by defining a coordinate system z^M over a D-dimensional smooth manifold $\mathcal{M}$. The tangent vectors of the coordinate lines at a point, p, on our manifold form a vector space, T_p. A natural basis for this vector space are the derivatives $\frac{\partial}{\partial z^M} \equiv \partial_M$, as they are the tangent vectors of the coordinate lines. We can define the *tangent bundle* $T_{\mathcal{M}}$ as the union of all tangent spaces on all points on the manifold. There exist a vector space, dual to T_p which we denote T_p^*, of functions called 1-forms. We define a basis for the 1-forms as

$$dz^M.$$

We then have a canonical product between these two spaces,

$$\langle\ ,\ \rangle : T_p^* \times T_p \to \mathbb{R},$$

such that,

$$\langle dz^N, \frac{\partial}{\partial z^M}\rangle \equiv \frac{\partial z^N}{\partial z^M} = \delta^N_M.$$

A general 1-form can be written $\omega = dz^M \omega_M$. Next we define the wedge product between two 1-forms as the antisymmetric tensor product

$$dz^M \wedge dz^N \equiv dz^M \otimes dz^N - dz^N \otimes dz^M.$$

The $\wedge$ product can be performed multiple times, a form consisting of p 1-forms wedged together is called a p-form. A general p-form can be written

$$\Omega = \frac{1}{p!} dz^{M_1} \wedge \cdots \wedge dz^{M_p} \omega_{M_1 \cdots M_p}(z).$$

To avoid clutter in calculations we will most of the time omit writing out the $\wedge$. The space of p-forms span a vector space, denoted Λ^p. The 1-forms, or the *covectors* span Λ^1, ordinary functions of spacetime span Λ^0. The direct sum of all these vector spaces form an algebra under the wedge product, the exterior algebra Λ,

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^D.$$

As the wedge product is antisymmetric the series will terminate after D. The dimension of the subspace consisting of p-forms is

$$dim(\Lambda^p) = \binom{D}{p}.$$

Which means the total dimension

$$dim(\Lambda) = \sum_{p=0}^{D} \binom{D}{p} = 2^D.$$

Note here also that $dim(\Lambda^p) = dim(\Lambda^{D-p})$ which suggests a duality between the vector spaces Λ^p and Λ^{D-p}. This duality is realised by the *Hodge star*.

$$\begin{aligned} \star : \Lambda^p &\to \Lambda^{D-p} \\ \omega_{M_1 \cdots M_p} &\mapsto \frac{1}{p!} g_{M_1 N_1} \cdots g_{M_{D-p} N_{D-p}} \epsilon^{N_1 \cdots N_D} \omega_{N_{D-p+1} \cdots N_D}, \end{aligned}$$

where ϵ is the $Levi-Civita$ tensor, i.e. it is normalised by the square root of the determinant of the metric [20].

We have the following properties for multiplication of forms

$$\begin{aligned} i) \quad & (c_1\Lambda_1 + c_2\Lambda_2)\Omega = c_1\Lambda_1\Omega + c_2\Lambda_2\Omega \\ ii) \quad & \Lambda\Omega = (-)^{pq}\Omega\Lambda \\ iii) \quad & \Lambda(\Xi\Omega) = (\Lambda\Xi)\Omega. \end{aligned}$$

For a q-form Ω, a p-form Λ, and an arbitrary form Ξ. We will in some cases, write only the component fields of the p-forms. Although the notation is sloppy we will sometimes refer to these as forms as well.

The exterior derivative, d, maps p-forms to $p+1$-forms. The action on a p-form is

$$\begin{aligned} \Omega &= \frac{1}{p!} dz^{M_1} \cdots dz^{M_p} \Omega_{M_1 \cdots M_p}(z) \mapsto \\ d\Omega &= \frac{1}{p!} dz^{M_1} \cdots dz^{M_p} dz^N \frac{\partial}{\partial z^N} \Omega_{M_1 \cdots M_p}(z). \end{aligned}$$

d is linear and enjoys the properties

$$\begin{aligned} d(\Omega\Sigma) &= \Omega d\Sigma + (-)^p d\Omega\Sigma \\ dd &= 0. \end{aligned}$$

From the first property we see that the exterior derivative acts from the right. The second property follows from the fact that the 1-forms and the partial derivatives have opposite symmetries.

We can, analogously to the exterior derivative, which is a function $d : \Lambda^p \to \Lambda^{p+1}$, define the interior derivative or the interior multiplication. The interior derivative is defined as

$$\begin{aligned} \iota_v :\ & \Lambda^p \to \Lambda^{p-1} \\ \iota_v \omega =& \frac{1}{(p-1)!} v^M \omega_{MN_2 \cdots N_p} dx^{N_2} \wedge \cdots \wedge dx^{N_p}, \end{aligned}$$

where v^m is a vector field We will not use the interior derivative throughout the paper, nonetheless we include it for completeness. We will use this briefly when defining spinors in App. E. For a rigorous introduction we refer the reader to [19].

2.3.1 Coordinate transformations

As we want a formulation that is completely covariant under general coordinate transformations we need to ensure that the forms satisfy this. The values of functions on spacetime need to be independent of the choice of coordinates.

An M-dimensional manifold, $\mathcal{M}$, is covered by a family of open sets, $\{U_i\}$, i.e $\bigcup_i U_i = \mathcal{M}$ [19]. To each open set, U_i, there is an associated homeomorphism (essentially smooth isomorphism), $\varphi_i : U_i \to U_i' \subset \mathbb{R}^M$. Depending of what kind of manifold $\mathcal{M}$ is, Euclidean, Minkwoskian, super-Euclidean, super-Minkowskian, the homeomorphism maps the subset to the corresponding flat space. φ_i can then be used to impose a coordinate system over $\mathcal{M}$.

Consider a point $p \in U_z \cap U_y$. Then p and the surrounding points in $U_z \cap U_y$ can be described by two coordinate systems[19],

$$\begin{aligned} z^M &= \varphi_z : \mathcal{M} \to \mathbb{R}^M \\ y^M &= \varphi_y : \mathcal{M} \to \mathbb{R}^M. \end{aligned}$$

We can relate these two coordinate systems by

$$y^M(z) = \varphi_y \circ \varphi_z^{-1}(z).$$

Now, for a function, $F : M \to \mathbb{R}$, over our manifold, we use our coordinate systems to realise the function. We thus have two different coordinate presentations for the function over our manifold

$$\begin{aligned} F_z(z) &= F \circ \varphi_z^{-1}(z) : \mathbb{R}^m \to \mathbb{R} \\ F_y(y) &= F \circ \varphi_y^{-1}(y) : \mathbb{R}^m \to \mathbb{R}. \end{aligned}$$

Remember that all of this is done in the subset $U_z \cap U_y$ around the point p in $\mathcal{M}$, but as p is arbitrary this logic is applicable to all point on our manifold. These two presentations must give the same value for the same point on $\mathcal{M}$, let's check this.

$$F_y(y(z)) = F \circ \varphi_y^{-1} \circ \varphi_y \circ \varphi_z^{-1}(z) = F_z(z).$$

Indeed we get that functions over $\mathcal{M}$ are independent of coordinates. This concept generalises to p-forms as

$$\begin{aligned}\Omega_y(y) =&\frac{1}{p!}dy^{M_1}\cdots dy^{M_p}(\Omega_y)_{M_1\cdots M_p}(y) = \\ =&\frac{1}{p!}dz^{N_1}\frac{\partial y^{M_1}}{\partial z^{N_1}}\cdots dz^{N_p}\frac{\partial y^{M_p}}{\partial z^{N_p}}\left(\Omega_y\right)_{M_1\cdots M_p}(y(z)) = \\ =&\frac{1}{p!}dz^{N_1}\cdots dz^{N_p}\left(\Omega_y\right)_{N_1\cdots N_p}(y(z)) = \\ =&\frac{1}{p!}dz^{N_1}\cdots dz^{N_p}(\Omega_z)_{N_1\cdots N_p}(z) = \\ =&\Omega_z(z).\end{aligned}$$

We see here that forms takes the same value at each point on our manifold regardless of our choice of coordinates.

The differential form formalism is a general covariant formalism completely independent of the choice coordinates.

2.4 Spinors

We will here present an introduction to maybe one of physics and mathematics most widely discussed and used objects; spinors.

2.4.1 The Lorentz Algebra

The Lorentz algebra $\mathfrak{so}(D)$ is frequently used throughout physics. It is a central cornerstone in physics ever since Einstein introduced special relativity in 1905 [21]. The Lorentz algebra satisfies the commutation relations

$$[J^{ab}, J^{cd}] = i(\eta^{bd}J^{ac} + \eta^{ac}J^{bd} - \eta^{ad}J^{bc} - \eta^{bc}J^{ad}). \tag{2.2}$$

Apart from the fundamental representation of the Lorentz algebra, i.e. vectors, there exist another representation which is of almost equal importance in physics; the *spinor representation*. The easiest way to understand this representation is using the Dirac- or Clifford algebra. The algebra is a set of matrices, γ^a, satisfying

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{2.3}$$

By finding matrices satisfying these relations we can construct the Lorentz generators by

$$J^{ab} = \frac{1}{4}\gamma^{[a}\gamma^{b]}. \tag{2.4}$$

These generators satisfy eq. (2.2). A γ-matrix have a index structure like $(\gamma^a)_\alpha{}^\beta$, where we use Greek letters to denote the so called *spinor indices* (the brackets separating the vector index from the spinor indices will sometimes be omitted). We will in this thesis treat spinors in 10 and 11 dimensions. We will cover the representations, both in Weyl/chiral- and Majorana basis.

2.4.2 Spinors in even dimensions

In $D = 2n$ dimensions we can redefine the γ-matrices like $a^a = \frac{1}{\sqrt{2}}\gamma^a$ to get

$$\{a^a, a^b\} = \eta^{ab}.$$

If we are in a signature (p, m) we can do a new redefinition $\alpha^a = (ia^1, \ldots ia^n, a^{p+1}, \ldots, ia^{p+m})$ to get

$$\{\alpha^a, \alpha^b\} = \delta^{ab}.$$

Thus in the complex algebra signature does not matter. We now redefine once again to

$$d_i = \frac{1}{\sqrt{2}}(\alpha^{2i-1} + i\alpha^{2i}), \qquad d_i^\dagger = \frac{1}{\sqrt{2}}(\alpha^{2i-1} - i\alpha^{2i}), \qquad i = 1, \ldots, n.$$

These objects now satisfy

$$\{d_i, d_j^\dagger\} = \delta_{ij},$$

i.e. an ordinary creation/annihilation algebra. The state space can then be defined from a ground state, $|0\rangle$, annihilated by all d_i. The state space will then be

$$\begin{aligned} d_{i_1}^\dagger \cdots d_{i_n}^\dagger |0\rangle &= |[i_1 \cdots i_n]\rangle \\ &\vdots \\ d_i^\dagger |0\rangle &= |i\rangle \\ &|0\rangle . \end{aligned}$$

The number of states at each level is $\binom{n}{k}$, where k is the number of $d^\dagger$ acting on $|0\rangle$. The total number of states is thus

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$

As the d:s are just a redefinition of the γ-matrices we thus have a 2^n dimensional representation of the Dirac algebra and thereby the Lorentz algebra. This representation is the spinor representation. Notice that the Lorentz generators (2.4) will be bilinear in d and hence Lorentz transformations will not mix states with odd and even number of $d^\dagger$:s. The representation will thus break down into two irreducible representations of dimension 2^{n-1}. We call these two representations *chiralities* and often refer to them as left or right chirality. A spinor which consist of both chiralities is called a *Dirac spinor*, the chiral ones are sometimes referred to as *Weyl spinors*. They are related as

$$\Psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

Notice that the dimension of the spinor representation is the same as that of the exterior algebra of n dimensions in sec. 2.3, namely 2^n. We can use forms in n dimensions to construct the spinor representation. We can order the forms in a 2^n dimensional vector on which the γ-matrices acts. We define the first n γ-matrices to act through the exterior derivative and the last n through the interior derivative. We order the forms as

$$\Omega = \begin{pmatrix} \omega^{(0)} \\ \omega^{(1)} \\ \vdots \\ \omega^{(n)} \end{pmatrix} = \omega^0 + \omega^{(1)}_m dx^m + \cdots + \frac{1}{n!}\omega^{(n)}_{m_1\cdots m_n} dx^{m_1} \wedge \cdots \wedge dx^{m_n},$$

Where m is here an index in a $\mathfrak{gl}(n)$ module. When constructing the Lorentz generators $\frac{1}{4}\gamma^{[ab]}$ they will raise and lower the form degree by 2 when acting on Ω, thus odd and even forms do not mix and we again see the chiralities.

With this definition of the spinor representation we can find a useful relation between different dimensions. If we want to construct a scalar out of two spinors Ω and Ω' we can use the Hodge star operation. We define the scalar product as

$$(\Omega, \Omega') = \star(\Omega \wedge \Omega') = \star(\omega^{(0)} \wedge \omega'^{(n)} + \omega^{(1)} \wedge \omega'^{(n-1)} + \cdots).$$

This gives an interesting insight in the relation between the chiralities; as we get scalars through contracting objects in the ordinary vector space and the dual we must for example have that $\omega^{(n)}$ is in the dual space to $\omega^{(0)}$. Denoting the two chiralities as S, and S' respectively we must have, as even and odd forms are different chiralities,

$$n = 2p \implies \text{Pairing even forms with even forms} \implies S = \overline{S} \text{ and } \overline{S}' = S'$$
$$n = 2p + 1 \implies \text{Pairing even forms with odd forms} \implies S = \overline{S}' \text{ and } \overline{S} = S'$$

The relation to the dimensions is summarised in table 2.1

Dimension	$\overline{S}$	$\overline{S}'$
$D = 4p$	S	S'
$D = 4p + 2$	S'	S

Table 2.1: *Summarise of the relations between the chiralities and the dual representations of even-dimensional spinors.*

In even dimensions we can construct a $\gamma^C \sim \gamma^1 \cdots \gamma^{2n}$. Because there are an even number of γ-matrices in γ^C this implies that $\{\gamma^C, \gamma^a\} = 0$. We can further more define the proportionality constant such that $(\gamma^C)^2 \pm \mathbf{1}_{2^n}$. We now note that γ^a and γ^C is now a set of $2n + 1$ anticommuting matrices. We can thus use this set to construct the spinor representation in odd dimensions. This matrix γ^C is often referred to as the chirality operator as it is used to construct projection operators onto the chiralities;

$$P_L = \frac{\mathbf{1}_{2^n} \pm \gamma^C}{2}, \qquad P_R = \frac{\mathbf{1}_{2^n} \mp \gamma^C}{2}.$$

Moving on, there exists six matrices, $A_\pm$, $B_\pm$, and $C_\pm$ such that

$$A_\pm \gamma^a A_\pm^{-1} = \pm(\gamma^a)^\dagger, \qquad B_\pm \gamma^a B_\pm^{-1} = \pm(\gamma^a)^*, \qquad C_\pm \gamma^a C_\pm^{-1} = \pm(\gamma^a)^\top.$$

Thus the transpose, the Hermitean conjugate, and the complex conjugate are all the same representations of the Clifford algebra, as they are linked via similarity-(or basis) transformations. The A, B, and C are used to move between these representations, they thus have both their indices up, C is used to raise indices of the spinors and γ-matrices.

$$(\gamma^a)^{\alpha\beta} = C^{\alpha\delta}(\gamma^a)_\delta{}^\beta = (C\gamma^a)^{\alpha\beta}$$
$$(\gamma^a)_{\alpha\beta} = (\gamma^a)_\alpha{}^\delta(C^{-1})_{\delta\beta} = (\gamma^a C^{-1})_{\alpha\beta}.$$

A, and B follow the same transformation, but the γs also gets Hermitean conjugated or complex conjugated, respectively.

2.4.2.1 Majorana spinors

C is used to raise indices on Dirac spinors, as this transform the γ-matrices to the dual representation. Thus for a Dirac spinor Ψ,

$$\Psi^\alpha = C^{\alpha\beta}\Psi_\beta$$

Here we again raise with C from the left, if we want to raise with C from the right we get the *Majorana conjugate* instead. The Majorana conjugate is defined as

$$\tilde{\Psi} \equiv \Psi^\top C.$$

We define the Dirac conjugate as

$$\bar{\Psi} \equiv \Psi \dagger A.$$

When working with Majorana spinors we have that the Majorana conjugate and the Dirac conjugate are equal;

$$\tilde{\Psi} = \bar{\Psi} \Leftrightarrow \Psi^\top C = \Psi^\dagger A.$$

The Majorana condition forces a Dirac spinor to, in principle become real, depending on which representation we are in we may still have complex valued components, however the degrees of freedom are those for a real spinor. In the Majorana representation we will have that $A = C$ and thus that Ψ is completely real.

2.4.2.2 Spinor bilinears

The γ-matrices are all invariant under Lorentz transformations. This supplies us with an opportunity; if we could find a basis for $2^{D/2} \times 2^{D/2}$-matrices using γ-matrices we would be able to express tensor products of spinors with the help of this basis. This is in fact possible. We consider $D = 2n$, we can define multi-indexed γ-matrices as

$$\gamma^{[n]} = \gamma^{a_1 \cdots a_n} \equiv \gamma^{[a_1} \ldots \gamma^{a_n]}.$$

These matrices, together with the identity $\mathbf{1}_{2^n}$ form a basis for $2^{D/2} \times 2^{D/2}$-matrices. To see this we must first confirm that the dimensions are correct. The number of matrices in $\{\gamma^{[i]}\}_{i=0}^{2n}$ is $\sum_{i=0}^{2n} \binom{2n}{i} = 2^{2n} = (2^{2n/2})^2$, and thus the dimensions agree. Next we need to prove linear independence. This is done using the trace. Consider first the case where an even number of γ-matrices are antisymmetrised.

$$Tr(\gamma^{[2p]}) = Tr(\gamma^{a_1} \cdots \gamma^{a_{2p}}) = \left\{ \begin{matrix} Tr(\gamma^{a_{2p}} \gamma^{a_1} \cdots \gamma^{a_{2p-1}}), & \text{Using cyclicity} \\ -Tr(\gamma^{a_{2p}} \gamma^{a_1} \cdots \gamma^{a_{2p-1}}), & \text{Using antisymmetry} \end{matrix} \right\} = 0$$

For an odd number of γ-matrices we cannot use this trick, but there is another;

$$Tr(\gamma^{[2p+1]}) = \pm Tr(\gamma^{[2p+1]}(\gamma^C)^2) = \mp Tr(\gamma^C \gamma^{[2p+1]} \gamma^C) = 0.$$

Where we moved γ^C past an odd number of γs and picked up a sign. This shows that all γ-matrices are traceless, except the identity. Independence follows from looking the at the trace $Tr(\gamma^{[n]}\gamma^{[m]})$. The product $\gamma^{[n]}\gamma^{[m]} \sim \gamma^{[m+n-2k]}$, where k is the number of γ-matrices that existed in both $\gamma^{[n]}$ and $\gamma^{[m]}$) and hence got canceled. Thus all $\gamma^{[n]}$ are orthogonal under the trace and thereby constitute a basis.

We can thus now use $\{\gamma^{[n]}\}_{n=0}^{2n}$ as a basis and expand a tensor product between two spinors as

$$\psi_\alpha \chi_\beta = \delta_{\alpha\beta} x^{(0)} + \gamma^a_{\alpha\beta} x^{(0)}_a + \cdots + \gamma^{[2n]}_{\alpha\beta} x^{(2n)}_{[2n]}, \tag{2.5}$$

where $x^{(i)}_{[2n]}$ is a tensor with $2n$ antisymmetrised indices. As the γ-matrices are invariant under Lorentz transformations we here see the real decomposition of the tensor product of spinors, each term in (2.5) transform independently.

The procedure of expanding spinor bilinears (or multilinears too for that matter) into the γ-basis is usually called *Fierzing*, and the different identities that may arise between various products of γ-matrices are referred to as *Fierz identities.*

Chapter 3

BRST and BV

When using path integral methods to perform calculations for gauge theories in quantum field theory one will get a lot of infinities. This is due to the fact that the infinite set of gauge variations yield equivalent contributions to the path integral which makes it diverge. To obtain any useful information when quantising one then have to gauge fix the theory. The gauge invariance is then, obviously, lost. The quantised theory has thus lost all its elegance that come from it being a gauge theory.

To create a procedure for which quantisation is possible without loosing the sense of the gauge symmetry is thus desired. This is precisely what the BRST treatment provides us with. It was developed by Becchi, Roet, Stora and Tyutin, [22, 23] and was originally formulated to deal with quantised systems, however it also showed having massive applicability to the classical theory. The general idea of the BRST theory is to change the gauge symmetry into a rigid fermionic symmetry over an extended phase space including so called ghost fields, where the ghost fields contain the gauge invariance. This new symmetry, the BRST symmetry, will completely capture the gauge invariance of the original theory [18].

3.1 Ghosts

The concept of ghosts and ghost fields have been introduced in the studies of physical systems. Ghosts were first introduced in quantum field theory as fields with wrong relation between spin and statistics. As mentioned above, we cannot directly perform path integrals over gauge theories. One must gauge fix the action such that the redundant gauge variables are removed. By doing this gauge invariance is of course lost. One can perform the path integrals but we loose track of the physics. The ghost fields, under the Fadeev-Popov formalism, were introduce precisely to remove the gauge degrees of freedom. BRST is instead a formalism constructed to be able to keep manifest gauge invariance without loosing the physical degrees of freedom.

The Fadeev-Popov method of dealing with gauge theories is sufficient when working with irreducible *closed* gauge theories, i.e. the gauge transformations generate the Lie algebra of the gauge group. The most common examples of this are ordinary Yang-Mills theories. The general procedure of the Fadeev-Popov method is that one introduces ghost fields i.e. virtual particles that are in the same representation as the gauge parameters but with opposite statistics. The new action, the Fadeev-Popov action, will consist of the original action and other terms containing the ghosts and potential interactions between the fields and the ghosts. This new action is no longer gauge invariant, due to the second term. That the ghost is of opposite statistics is what will make sure that nonphysical loop amplitude calculations will cancel. We will not discuss the Fadeev-Popov method any further. We will go on to the BRST formalism and then discuss shortly the Batalin-Vilkovisky formalism.

We will consider more general gauge theories, for these cases the Fadeev-Popov procedure is not enough. Such cases includes: Reducible gauge theories, i.e. where the gauge parameters has gauge freedom, and gauge theories which closes only on-shell. i.e. for two gauge transformations δ_ϵ and δ_η we would have

$$[\delta_\epsilon, \delta_\eta]\phi^i = \delta_{[\epsilon,\eta]}\phi^i + Field\ equations.$$

An example of this is the Wess-Zumino multiplet discussed in the beginning of App. D. Other examples with *open gauge algebras* include supergravity theories and the Green-Schwarz superstring. [24]

3.2 Gauge invariant functions

As previously mentioned, systems with gauge invariance require some care. We will now motivate how we can obtain the physical observables (gauge invariant entities). By considering the constraints of the system the gauge invariant functions can be obtained by a two-step reduction process of ordinary phase space functions;

1. Enforce the constraints by identifying arbitrary phase space functions which coincide on the surface where the constraints are realised.

2. Identify functions on the constraint functions which differs by a gauge transformation.

By considering functions subject to constraints we can in fact obtain the gauge invariant functions. The gauge invariance and constraints are actually closely connected as the constraint will generate the gauge transformations of the fields [18]. First we will consider the reduction of phase space functions by the constraints.

Consider a physical system subject to some constraints,

$$\phi^m(p,q) = 0,$$

on the phase space variables, where $m = 1, \ldots, M$. We denote by $\mathcal{P}$ the ordinary phase space, and $C^\infty(\mathcal{P})$ the algebra of smooth functions on $\mathcal{P}$. The physical system will be confined to be functions on the *constraint Surface*,

$$\Sigma := \{(p,q) \in \mathcal{P}, \quad \phi(p,q) = 0\}. \tag{3.1}$$

The algebra of functions on Σ is denoted $C^\infty(\Sigma)$. The constraints $\phi(p,q) = 0$ might be quite complicated to work with. We can instead identify functions in $C^\infty(\Sigma)$ with arbitrary functions in $C^\infty(\mathcal{P})$ modulo the constraint. If we identify functions in $C^\infty(\mathcal{P})$ that coincide on the constraint surface we can actually obtain $C^\infty(\Sigma)$. Define the equivalence relation

$$f(p,q) \sim g(p,q), \quad \Longleftrightarrow \quad f(p,q) - g(p,q) \in \mathcal{N}, \tag{3.2}$$

where $\mathcal{N} = \{n(p,q) \in C^\infty(\mathcal{P}),\ n(p,q) = 0,\ if\ (p,q) \in \Sigma\}$, i.e. the subalgebra of functions that vanish on the constraint surface. In general, an element in $\mathcal{N}$ can be written as

$$\mathcal{N} \ni F(p,q) = \phi^m(p,q)F_m(p,q).$$

$\mathcal{N}$ is in fact an ideal under multiplication as any product with an element in $\mathcal{N}$ will also vanish on the surface. We can thus define the quotient algebra

$$\frac{C^\infty(\mathcal{P})}{\mathcal{N}},$$

of equivalence classes of the relation (3.2). To see that the quotient algebra and the subalgebra $C^\infty(\Sigma)$ are algebraically equivalent we simply use the surjective homomorphism

$$\begin{aligned} \theta : C^\infty(\mathcal{P}) &\to C^\infty(\Sigma) \\ f(p,q) &\mapsto f(p,q). \end{aligned}$$

We find the kernel of θ to be, $Ker(\theta) = \{f(p,q),\ (p,q) \in \Sigma \implies f(p,q) = 0\} = \mathcal{N}$. Then by the *Fundamental homomorphism theorem*, [25], we find that the mapping

$$\begin{aligned} \rho : \frac{C^\infty(\mathcal{P})}{\mathcal{N}} &\to C^\infty(\Sigma) \\ f(p,q) + \mathcal{N} &\mapsto f(p,q) \end{aligned}$$

is an isomorphism and

$$\frac{C^\infty(\mathcal{P})}{\mathcal{N}} \simeq C^\infty(\Sigma).$$

The proof of the Fundamental homomorphism theorem is not very involved and can be found in [25].

We have now found that instead of examining functions of a set of complicated variables subject to a constraint we can just as well investigate functions of the general set of variables modulo the constraint. This is easier as we can now observe the constraints as a "gauge transformation" of the functions. It is important however to remember that these are not the actual gauge transformations of the fields but transformations to simplify the reduction of the phase space functions to the constraint surface. The ”transformations” of the fields can be written as.

$$\Delta f(p,q) = \phi^m(p,q)A_m(p,q).$$

The constraints may sometimes be reducible. There might exist some variation $\Delta A_m(p,q)$ such that it is not seen by the variation of f.

$$\Delta f(p,q) = \phi^m(p,q)A_m(p,q) + \phi^m(p,q)\Delta A_m(p,q) = \phi^m(p,q)A_m(p,q).$$

One has to find all reducibilities by searching for variations which is not seen by the higher level. With this procedure however one will end up with functions solely defined on Σ.

The aforementioned discussion does not treat the gauge invariance of the system. The gauge invariance can however be treated in an analogous manner. On Σ there are lines on which all functions on Σ are related by a gauge transformation. These lines are called *gauge orbits*. An illustration of the constraint functions and gauge orbits on it can be found if fig. 3.1.

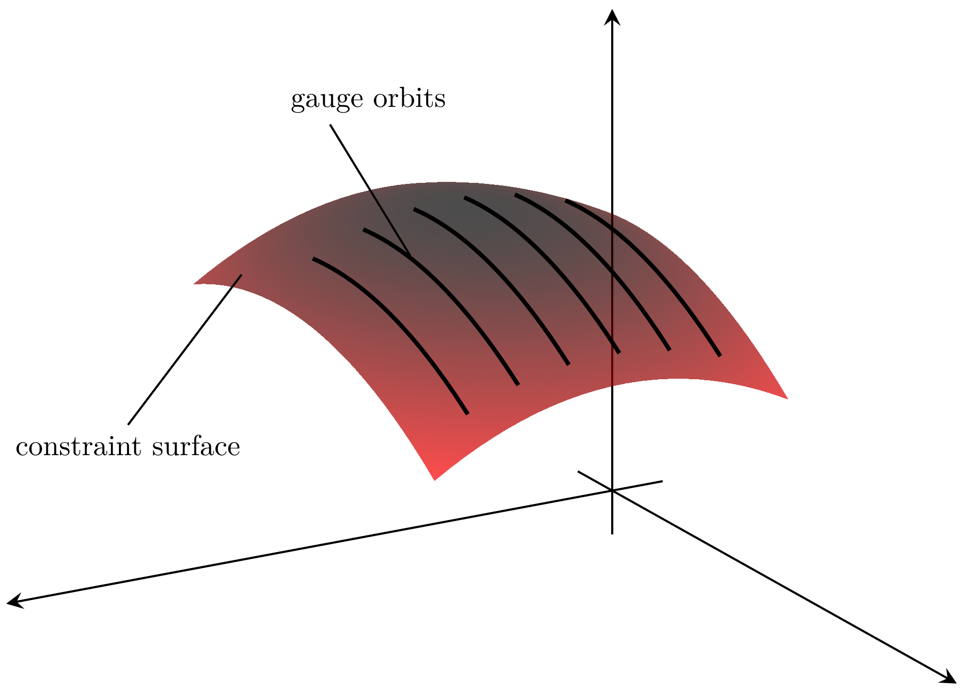

Figure 3.1: *Illustration of a constraint surface in phase space. Parallel lines are gauge orbits.*

These lines are the field lines of vector fields on Σ. To obtain the gauge invariant functions we must now identify functions which only differs by a gauge transformation on the constraint surface, i.e. making the gauge orbits to equivalence classes. The constraints will actually, under the Poisson bracket, generate the gauge transformations. Thus we use these to define the equivalence classes. For a function, F, on Σ we will have that

$$\delta F = \delta v_m[F, \phi^m]_{PB}.$$

Define now the subalgebra

$$\mathcal{I} = \{F \in C^{\infty}(\Sigma),\ F = v_m[f, \phi^m]_{PB}\}.$$

We can then define the equivalence relation

$$F \sim A \iff F - A \in \mathcal{I},$$

and apply the same argument as above and obtain the gauge invariant functions as

$$\{Gauge\ invariant\ functions\} \simeq \frac{C^{\infty}(\Sigma)}{\mathcal{I}}.$$

3.3 The BRST differential

How can we retrieve the gauge invariant functions in an easy way? The most used method is by using a BRST formalism. By defining a symmetry we can retrieve the gauge invariant objects. We will here describe, in short, the construction of the BRST symmetry. The most important property of the symmetry is that it is generated by a nilpotent odd derivation acting on an extended phase space. Thus, by the introduction to cohomological algebra in sec. 2.2.3 the BRST symmetry, s, is a differential. One can therefore examine its cohomology,

$$H^k(s) = \frac{Ker^k(s)}{Im^k(s)},$$

where k, is a grading called *ghost number*. We will define s, such that the observables (gauge invariant functions) will be retrieved at the zeroth order of the cohomology

$$H^0(s) = \{Gauge\ invariant\ functions\}.$$

The replacement of a gauge symmetry by a rigid, or global, one enables the opportunity to replace the gauge invariant action, yielding infinities when using path integral methods, by one that will be calculable. The ghosts are the ones compensating for the gauge degrees of freedom [18].

3.3.1 The BRST generator

The BRST operator s, can in fact be chosen to have a canonical action of the form

$$sx = [x, \Omega\}, \tag{3.3}$$

where Ω is the BRST generator, for classical systems the bracket in (3.3) is the graded Poisson bracket. The choice of a canonical action of the BRST operator in terms of (3.3) is what allows for quantisation.

3.3.2 Construction of the BRST generator

We will now provide a short introduction to the construction of the BRST differential. There are some details we do not consider in this explanation that is needed to, for example, really motivate why the observables can be retrieved in the cohomology. For a more detailed approach we recommend the reader to consult [18].

If we consider a set of constraints of the phase space variables, z^A,

$$G^{A_1}(z^A) = 0, \tag{3.4}$$

where A_1 is a set numbering the constraints, it can also be related to a representation of some gauge group. In that case the constraints will generate the gauge group as

$$[G^{A_1}, G^{B_1}] = f^{A_1 B_1}{}_{C1} G^{C_1}, \tag{3.5}$$

where $f^{A_1 B_1}{}_{C1}$ are the structure constants. Now, as we discussed above: The physical degrees of freedom are confined to be functions on the constraint surface,

$$C^\infty(\Sigma) \simeq \frac{C^\infty(P)}{\mathcal{N}},$$

where Σ is spanned by the phase space variables obeying the constraints (3.4). Here we must remember that we have the gauge invariance to take care of as well, this is one of the amazing parts of the BRST formalism, by construction the gauge invariance will be taken into consideration and we will receive the gauge invariant functions.

Consider a differential s, for which we assign the ghost numbers

$$gh(s) = 1, \quad gh(z) = 0$$

We want to retrieve the phase space functions $C^\infty(\Sigma)$ at the zeroth ghost number cohomology

$$H^0(s) = \frac{Ker^0(s)}{Im^0(s)} = C^\infty(\Sigma) \simeq \frac{C^\infty(P)}{\mathcal{N}}.$$

It is thus natural to identify

$$\begin{aligned} &i)\ Ker^0(s) = C^\infty(P), \\ &ii)\ Im^0(s) = \mathcal{N}. \end{aligned} \tag{3.6}$$

To achieve this we first assign that s annihilates all phase space variables $s(z) = 0$. This satisfies $i)$ in (3.6). Next, we know that if a function vanish on Σ it must be proportional to the constraints.

$$\mathcal{N} \ni f(z) \implies f(z) = f_{A_1} G^{A_1}.$$

We now extend our phase space by a new pair of variables $(C_1^{A_1}, B_{1A_1})$ such that

$$\{C_1^{A_1}, B_{1B_1}\} = \delta^{A_1}_{B_2}, \quad -gh(C_1^{A_1}) = gh(B_{1A_1}) = 1, \quad \text{and} \quad sC_1^{A_1} = \{C_1^{A_1}, \Omega\} = G^{A_1}.$$

A function that vanishes on Σ can thus be written as something s-exact and does not contribute to the cohomology.

$$\mathcal{N} \ni f(z) \implies f(z) = s(f_{A_1} C_1^{A_1}) \in Im^0(s).$$

We refer to C^{A_1} as *ghosts*. They are ghosts of the first generation, also denoted C-type ghosts. With this introduction of ghosts one can construct the BRST generator for a set of constraints satisfying eq. (3.5) as [18]

$$\Omega = B_{1A_1} G^{A_1} - \frac{1}{2} B_{1A_1} B_{1B_1} f^{A_1 B_1}{}_{C_1} C^{C_1}.$$

The nilpotency of Ω, $\{\Omega, \Omega\} = 0$ will follow from eq. (3.5) and its corresponding Jacobi identity.

This actually concludes the case of irreducible constraints. For reducible constraints (such as the constraints we will be investigating) the introduction of a ghost transforming as the constraint may give rise to more unwanted cohomology. That is

$$s(C_1^{A_1} R_{A_1}{}^{A_2}) = 0, \tag{3.7}$$

Where $R_{A_1}{}^{A_2}$ is some function of phase space (at higher levels of reducibility we might also have extra terms consisting of ghosts of lower orders). This is resolved by introducing a new set of ghosts, or *ghosts of ghosts*, (C^{A_2}, B_{2A_2}) such that

$$s(C_2^{A_2}) = C_1^{A_1} R_{A_1}{}^{A_2}.$$

This then makes (3.7) s-exact and does not contribute to the cohomology. This procedure of finding unwanted cohomology and introducing ghosts, and ghosts for ghosts, continues as long as there is cohomology to kill. This construction also ensures the nilpotency of s and yields the cohomology that we want; only the physical fields. In the end the BRST generator will then take the schematic form

$$\Omega = \sum BCC \cdots C.$$

3.4 The Batalin-Vilkovisky method

In this section we will present a brief review of the Batalin-Vilkovisky method used to handle difficult gauge theories. The review is mostly based on [26, 27, 28]. For a full treatment of the method we refer to [18].

This will provide a meaning and intuition behind the ghosts and antifields we will find in the following chapters. The BV procedure can be viewed as trying to go from a Hamiltonian to Lagrangian formalism, such that we also preserve manifest Lorentz invariance. In general we are looking for an off-shell description of the theory i.e. an action, this is what the BV procedure provides us with.

We first introduce the ordinary fields of the theory, collectively denoted ϕ^i and a set of ghosts C^A. These can be ordered as

$$\Phi^I = (\phi^i, C^A).$$

The ghosts C^A are here the ghosts of the gauge parameters of their respective gauge symmetry or reducibility of such. We introduce for each field ϕ^i, and C^A a corresponding antifield; ϕ_i^* and C_A^*, collectively denoted

$$\Phi_I^* = (\phi_i^*, C_A^*).$$

Moreover we define the antifields to have opposite Grassmann parity i.e. Φ_I^* is fermionic if Φ^I is bosonic and vice versa. We also define the antifields to have ghost number

$$gh(\Phi_I^*) = -gh(\Phi^I) - 1.$$

We can now move on to defining the antibracket and soon the master equation. The antibracket is a bilinear form of functionals. It can be viewed as a generalisation of the Poisson bracket

$$(A, B) = \int \mathrm{d}x A \frac{\overleftarrow{\delta}}{\delta \Phi^I} \frac{\overrightarrow{\delta}}{\delta \Phi_I^*} B - A \frac{\overleftarrow{\delta}}{\delta \Phi_I^*} \frac{\overrightarrow{\delta}}{\delta \Phi^I} B,$$

Where the arrows indicate the direction of the variation of the functionals. These are related by [28]

$$\begin{aligned} \frac{\overrightarrow{\delta}}{\delta \Phi^I} A &= (-)^{(A-I)I} A \frac{\overleftarrow{\delta}}{\delta \Phi^I}, \\ \frac{\overrightarrow{\delta}}{\delta \Phi_I^*} A &= (-)^{(A-I)I+1} A \frac{\overleftarrow{\delta}}{\delta \Phi_I^*}. \end{aligned}$$

The antibracket have the symmetry

$$(A, B) = -(-)^{(A+1)(B+1)}(B, A),$$

i.e. it is symmetric if A, and B are bosonic and antisymmetric otherwise. Now the BRST transformation of any functional can be seen as

$$sF = (\mathcal{S}, F),$$

where $\mathcal{S}$ is the generator of the BRST symmetry, just in the same way as we defined it in sec 3.3.1, however now it will be expressed as an integral. The generator is sometimes called the master- or BV- action.

The nilpotence of the BRST-transformation is now equivalent to

$$(\mathcal{S}, \mathcal{S}) = 0. \tag{3.8}$$

Eq. (3.8) is referred to as the Master Equation. Note that the master equation can also be seen as the invariance of the action itself under the BRST-transformation.

We can use the relations between the left and right acting variations to rewrite (3.8) as

$$\frac{\overrightarrow{\delta}\mathcal{S}}{\delta\Phi^I}\frac{\overrightarrow{\delta}\mathcal{S}}{\delta\Phi^*_I} = 0.$$

The master action $\mathcal{S}$ satisfying this will contain all information of the gauge structure of the theory. It can be constructed in a sequential form with respect to the antifields

$$\mathcal{S} = S_0 + S_1 + \cdots ,$$

Where for example S_0 is just the ordinary action of the theory. Which is good since we want the master action to reduce to the ordinary action when removing the anti-fields. The solution $\mathcal{S}$ to the master action is determined up to canonical transformations

$$(\mathcal{S} + (\delta F, \mathcal{S}), \mathcal{S} + (\delta F, \mathcal{S}) = (\mathcal{S}, \mathcal{S}) - 2((\delta F, \mathcal{S}), \mathcal{S}) = 0,$$

due to the graded Jacobi identity satisfied by the antibracket, δF here needs to be an infinitesimal fermionic functional. The solution and the canonical transformations is the complete BV formalism. When determined we will have a complete off-shell description of the gauge theory. The antifields are introduced to actually put the theory off-shell, under the canonical symmetries they transform like the equations of motion of the physical fields. This is why BV is so powerful, there is no longer any difference between gauge symmetries and equations of motion, they are all part of one big symmetry.

Chapter 4

Pure spinor cohomology

We will now investigate more in depth the notion of pure spinors and how we can use them to explicitly find supermultiplets (see App. D for introduction to supersymmetry). We will first introduce the pure spinor constraint and then construct a nilpotent operator which can be interpreted as a BRST operator. By doing so we investigate the cohomology and find that the cohomology coincide with the superspace calculation of the supersymmetric field theories. We will investigate both 10 dimensional pure spinors, which will yield a super-Yang-Mills theory, and 11-dimensional "pure spinors" which yields $D = 11$ supergravity, the low energy limit of M-theory.

4.1 The pure spinor constraint and Q

We start our discussion in a general dimension D. The covariant derivative in superspace, with respect to supersymmetry transformations is defined as (see App. D for an introduction to supersymmetry)

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - (\gamma^a\theta)_\alpha\partial_a,$$

where θ^α are the fermionic superspace coordinates. The D_α satisfies the supersymmetry algebra

$$\{D_\alpha, D_\beta\} = -2\gamma^a\partial_a.$$

We can define the operator Q by

$$Q \equiv \lambda^\alpha D_\alpha.$$

Which will be nilpotent by

$$Q^2 = \{Q, Q\} = \{\lambda^\alpha D_\alpha, \lambda^\beta D_\beta\} = \lambda^\alpha\lambda^\beta\{D_\alpha, D_\beta\} = 2\lambda^\alpha\gamma^a_{\alpha\beta}\lambda^\beta\partial_a = 0,$$

if

$$\lambda\gamma^a\lambda = 0. \tag{4.1}$$

Eq. (4.1) is called the *pure spinor constraint* and a spinor satisfying (4.1) is called a *pure spinor*. This terminology is bit dangerous to use, as discussed in the Introduction, *pure spinors* were originally introduced by Cartan [4] as spinors in even dimensions which is in a minimal orbit. An orbit is the subspace of a representation module for which the group action is closed. A minimal orbit is defined as the smallest such subspace. In other words, the largest constraint you can enforce on an object in a module such that it is still in an orbit. Representation wise, consider an object in a representation, $R(\mu)$, characterised by a highest weight μ. For the object to be in a minimal orbit the only module left in a bilinear would be the representation characterised by $R(2\mu)$, i.e. the sum of the weights. The constraint (4.1) does not necessarily put a spinor in a minimal orbit, however we will still use the terminology pure spinors for objects satisfying eq. (4.1).

4.1.1 Pure spinor superfields

It is possible to observe λ^α as an additional coordinate, making us expand our notion of superspace to cover an extra set of coordinates. We can expand these *pure spinor superfields* in a power series as

$$\Psi(x,\theta,\lambda) = c^{(0)}(x,\theta) + \lambda^\alpha A^{(1)}_\alpha(x,\theta) + \lambda^\alpha\lambda^\beta A^{(2)}_{\alpha\beta}(x,\theta) + \cdots$$

The superscript on the component fields denote at which order of λ they are. It is also possible to expand the component fields in powers of θ as well. The expansion of a scalar pure spinor superfield is then

$$\begin{aligned}\Psi(x,\theta,\lambda) =& c^{(0,0)}(x) + \theta^\alpha c^{(0,1)}_\alpha(x) + \theta^\alpha\theta^\beta c^{(0,2)}_{\alpha\beta}(x) + \cdots \\ &+ \lambda^\alpha A^{(1,0)}_\alpha(x) + \lambda^\alpha\theta^\beta A^{(1,1)}_{\alpha\beta}(x) + \lambda^\alpha\theta^{\beta_1}\theta^{\beta_2} A^{(1,2)}_{\alpha\beta_1\beta_2}(x) + \cdots \\ &+ \lambda^{\alpha_1}\lambda^{\alpha_2} A^{(2,0)}_{\alpha_1\alpha_2}(x) + \lambda^{\alpha_1}\lambda^{\alpha_2}\theta^\beta A^{(2,1)}_{\alpha_1\alpha_2\beta}(x) + \cdots \\ &\vdots\end{aligned}$$

4.1.2 Action of Q

The space of pure spinor superfields, $\mathcal{V}$, on which Q acts has a natural bigrading in terms of orders of λ and θ

$$\mathcal{V} = \bigoplus_{i=0}^{\infty}\bigoplus_{j=0}^{2^{2/D}} \mathcal{V}^{(i,j)},$$

where

$$\mathcal{V}^{(i,j)} = \{\Psi(x,\theta,\lambda), \quad \Psi = \lambda^{\alpha_1}\cdots\lambda^{\alpha_k}\theta^{\beta_1}\cdots\theta^{\beta_l} A^{(i,j)}_{\alpha_1\cdots\alpha_i\beta_1\cdots\beta_j}(x)\}.$$

Observing the structure of Q we can observe that it acts on the subspaces as

$$Q : \mathcal{V}^{(i,j)} \to \mathcal{V}^{(i+1,j-1)} \oplus \mathcal{V}^{(i+1,j+1)}.$$

We will in the next section examine the cohomology of Q, it is therefore of interest to know what a Q-exact field looks like. A general Q-exact field can be written as

$$Q\omega = 0 + \lambda^\alpha D_\alpha \omega^{(0)} + \lambda^\alpha \lambda^\beta D_\beta \omega^{(1)}_\alpha + \cdots . \tag{4.2}$$

4.1.3 Cohomology of Q

We now introduce a grading, the ghost number $gh(\cdot)$, and define $(x^\mu, \theta^\beta, \lambda^\alpha)$ to have ghost numbers $(0, 0, 1)$ respectively. The operator $Q = \lambda D$ will then be a odd nilpotent derivation (i.e. a differential) with ghost number 1. It can thus be seen as a BRST operator. We should mention that this form of Q as a BRST operator is not stemming from a conventional construction as discussed in sec. 3.3.2. This definition is rather *ad hoc* but we will still see that it works beautifully. From what we learned in Chapter 3 it is interesting to examine the cohomology of Q, namely the space of functions annihilated by Q

$$Q\Psi = 0, \tag{4.3}$$

modulo the equivalence relation

$$\Psi \sim \Psi + Q\omega.$$

The nilpotency of Q is what implies that functions differing by something Q-exact will transform equally under Q. We refer to this ambiguity as a gauge freedom. Eq. (4.3) will be referred to as the equation of of motion. The cohomology of Q can be expressed by

$$H(Q) = \frac{Ker(Q)}{Im(Q)}.$$

Using eq. (4.2) we can do a gauge transformation of the pure spinor superfield

$$\Psi' = \Psi + Q\omega = c^{(0)} + \lambda^\alpha (A^{(1)}_\alpha + D_\alpha \omega^{(0)}) + \lambda^\alpha \lambda^\beta (A^{(2)}_{\alpha\beta} + D_\alpha \omega^{(1)}_\beta) + \cdots . \tag{4.4}$$

We see that it looks very similar to the gauge transformations of the connections in ordinary gauge theories. The variation, $\delta\Psi$, is just $Q\omega$.

One important thing to notice here is that $Q\Psi = 0$ is a linear equation on the field Ψ. Thus we cannot expect to obtain any theories with interactions when solving eq. (4.3). In the 10-dimensional case discussed below we will obtain linearised super-Yang-Mills i.e. abelian super-Yang-Mills. In the 11-dimensional case we will obtain linearised supergravity. The equation of motion, (4.3) will, for each order in λ and θ, take the general form

$$\lambda^{\alpha_1} \cdots \lambda^{\alpha_i} \theta^{\beta_1} \cdots \theta^{\beta_j} \left(a^{(i-1,j+1)}_{\alpha_1 \cdots [\alpha_i \beta_1 \cdots \beta_j]} - \gamma^a_{\alpha_i \beta_1} \partial_a a^{(i-1,j-1)}_{\alpha_1 \cdots \alpha_{i-1}[\beta_2 \cdots \beta_j]} \right) = 0. \tag{4.5}$$

The algorithm for solving equations of this form is pretty straight forward; expand $(\lambda)^i(\theta)^j$ and $a^{(i-1,j+1)}$ into irreducible representations using the γ-basis. The only non-zero contractions are those from contractions of dual representations. Thus by computing the irreducible representations present in a product of λ and θ with the ones present in

$a^{(i-1,j+1)}$ we find that if some modules coincide they must either be zero or coincide with a module in $\partial_a a^{(i-1,j-1)}$. We must also consider the possible gauge transformations from eq. (4.4). By evaluating what irreps are present in $\omega^{(i-1,j+1)}$ we find which ones in $a^{(i,j)}$ that can be gauged away. This procedure must be done carefully though. Due to the γ^a in the second term in (4.5), things are a bit complicated. When expanding $\partial_a a^{(i-1,j-1)}$ in γ-matrices and these are contracted with λ and θ some products will, due to θ, become fermionic, which forces some representations to vanish. This complicates the search for the representations in the second term. .

4.1.3.1 Zero mode cohomology

The process of finding the exact cohomology will become unbearable at higher orders due to the amount of γ-matrix algebra. One would quickly loose track of what one is doing. There is a remedy to this though. We can actually pick up the relevant information by only looking at the zero mode cohomology, that is, ignoring the x-dependence of the fields. This means removing the bosonic derivative term in Q, i.e. $Q \rightarrow Q' = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$. This makes the process of finding the zero mode cohomology a strictly algebraic problem; match the irreducible representations in $\lambda^i\theta^j$ with those in $a^{(i-1,j+1)}$, if some modules agree these cannot be part of the zero mode cohomology. One of course needs to take the gauge transformations into consideration as well. Using the software LiE, [29], we can easily perform tensor product decompositions and determine the zero mode cohomology, now at higher ghost numbers as well.

Now how can we retrieve the full cohomology? Let's assume that we have found some module $a^{(p,q)}$ in the zero mode cohomology at order $\lambda^p\theta^q$, that is

$$Q'(\lambda^p\theta^q a^{(p,q)}) \sim \lambda^{p+1}\theta^{q-1}a^{(p,q)} = 0.$$

We can now ask ourselves the question: What happens if we let $a^{(p,q)}$ become x-dependent? The only possibility for the cohomology to survive is for it to be in the same module as the zero mode but subject to some equation of motion. The equation of motion will in turn be found at order $\lambda^{p+1}\theta^{q+2n-1}$[27]. It will state that a certain representation of derivatives on the cohomology at $\lambda^p\theta^q$ will not be present in the cohomology. This representation will instead be found at order $\lambda^{p+1}\theta^{q+2n-1}$ in the cohomology. The zero mode cohomology at higher order will in a sense tell us the form of the equations of motion the cohomology must satisfy.

The zero mode cohomology tells us which representations that in a sense are the fundamental blocks of the full cohomology. Any other representation in the full cohomology will be, in one way or another, a derivative of the fields in the zero mode cohomology. We will also find which representations the equations of motion will sit in.

4.2 Pure spinors and super-Yang-Mills in $D = 10$

In 10 dimensions the spinor representation of $D_5 \simeq \mathfrak{so}(10)$ is 32 dimensional, which breaks up in two 16-dimensional chiralities. Consider now a bosonic spinor λ^α, of one chirality. A bilinear in λ can be written in terms of the γ-basis. Being a bosonic spinor we are only interested in the symmetric γ- matrices, we can then use table E.3 to find the appropriate ones. A symmetric spinor bilinear can be expanded as

$$\lambda^\alpha \lambda^\beta = \frac{1}{16}\gamma_a^{\alpha\beta}\lambda\gamma^a\lambda + \frac{1}{1920}\gamma_{abcde}^{\alpha\beta}\lambda\gamma^{abcde}\lambda.$$

By enforcing the pure spinor constraint (4.1) we thus eliminate the vector part of a symmetric spinor bilinear. This puts λ^α in a minimal orbit and thus in 10 dimensions our definition of pure spinors coincide with the definition by Cartan.

The constraint $\lambda\gamma^a\lambda = 0$ can be solved by rewriting the $SO(10)$ basis in a $SU(5)$ one and from that solve and conclude that a pure spinor has 11 degrees of freedom. This is done in [3] for example. This is however not a covariant procedure and it lies in our interest to investigate methods to determine the degrees of freedom in a more covariant fashion.

4.2.1 Super-Yang-Mills in $D = 10$

Gauge theories are a crucial part of theoretical physics. It is the central cornerstone of the standard model and used frequently throughout physics. Supersymmetric gauge theories are more or less ordinary gauge theories which are also supersymmetric, that is, gauge theories which are invariant under supersymmetry transformations and where the fermionic and bosonic degrees of freedom are equal.

We will here discuss 10-dimensional super-Yang-Mills. Which is pretty similar to ordinary Yang-Mills. It consist of a 2-form field strength, F_{ab}, constructed from a 1-form potential A, and a fermionic spinor χ^α. Super-Yang-Mills is covered in detail in App. D.8.3, but we will here do a quick review. We can obtain 10-dimensional super-Yang-Mills through introducing a gauge group G to superspace, we then have to introduce a gauge connection $A = dz^M (A_M)_a{}^b$ and a gauge-covariant derivative $\mathscr{D} \sim d + \wedge A$. By also introducing the field strength $F = \mathscr{D}A = dA + A \wedge A$ we obtain the Bianchi identities $\mathscr{D}F = 0$. Now normally how one would proceed is to derive the equations of motion from an action $S \sim \int Tr(F^2)$, where the trace is over the gauge group. In superspace however we take an entirely different approach via the Bianchi identities.

By enforcing *conventional constraints* we can relate A_α to A_a. This a redefinition of the gauge field as a shift by the vector part of $F_{\alpha\beta}$. It does not affect the supersymmetry [30]. We impose this constraint in order to remove unnecessary degrees of freedom. The constraint is equivalent to removing the vector part of the spinor components of the field strength i.e.

$$\gamma_a^{\alpha\beta} F_{\alpha\beta} = 0.$$

After this we impose the *dynamical constraints*, this is the constraint which will put the whole theory on-shell. The dynamical constraint is

$$(\gamma^{abcde})^{\alpha\beta} F_{\alpha\beta} = 0. \tag{4.6}$$

The dynamical and conventional constraint combined will force $F_{\alpha\beta} = 0$. Plugging this into the Bianchi identities $\mathscr{D}F = 0$ and simplifying will now have consequences of the constraints, they will no longer be identities but to be satisfied the equations of motion must hold.

The Poincaré lemma states that if the Bianchi identities are satisfied, i.e. $dF = 0$, then locally $F = dA$, i.e. we can write F in terms of a potential. This in turn mean that by enforcing constraints on F and still have it satisfy the Bianchi identities will yield the same set of equations as solving the differential equations for A that arise solely from the constraints. In the example of super-Yang-Mills this mean eq. (4.6) will also yield the equations of motion instead of solving the Bianchi identities.

4.2.2 Cohomology of Q in 10 dimensions

We will now go on to determining the cohomology of Q. As Q is seen as a BRST operator this means that the physical degrees of freedom will be retrieved at ghost number zero in the cohomology. We thus want to define the ghost number, $gh(\Psi)$, of the pure spinor superfield such that the equation of motion (4.3) at ghost number 0 coincide with the constraint we set on the fields in the superspace formalism of a supersymmetric theory. The dynamical constraints imposed in linearised super-Yang-Mills is that the 5-form of the spinorial part of the field strength vanish i.e.

$$\gamma^{[5]\alpha\beta} F_{\alpha\beta} = \gamma^{[5]\alpha\beta} D_{(\alpha} A_{\beta)} = 0. \tag{4.7}$$

Acting with Q on Ψ gives

$$Q\Psi = \cdots + \lambda^\alpha \lambda^\beta D_\beta A_\alpha + \cdots = 0.$$

To understand this better we can expand $\lambda^\alpha\lambda^\beta$ into the basis of γ-matrices. As they are bosonic this is a symmetric bilinear and we thus only need γ^a and $\gamma^{[5]}$

$$\begin{aligned} Q\Psi &= \Big(-\frac{1}{16} (\gamma^a)^{\alpha\beta} \lambda\gamma_a\lambda + \frac{1}{1920} (\gamma^{[5]})^{\alpha\beta} \lambda\gamma_{[5]}\lambda \Big) D_\beta A_\alpha = 0, \\ &\implies \lambda\gamma_{[5]}\lambda (\gamma^{[5]})^{\alpha\beta} D_\beta A_\alpha = 0 \end{aligned} \tag{4.8}$$

Eq. (4.8) implies that both the vector and 5-form part are 0, which is in agreement with eq. (4.7). Due to the motivation in the last section this will thus provide us with the equations of motion for super-Yang-Mills. Choosing $gh(\Psi) = 1$ we have that A_α in (4.8) have $gh(A_\alpha) = 0$. We also choose the dimension, $dim(\Psi)$, such that the superfields in the cohomology with ghost number 0 have the same dimensions as the superfields in the supersymmetric theory. λ^α have the same dimension as θ^α, namely -1/2, thus $dim(\Psi) = 0 \implies dim(A_\alpha) = 1/2$, which is what we want. Further more Ψ must be fermionic, as A_α in SYM is fermionic. These choices of ghost number and dimension will cause the full cohomology at ghost number 0 and the superspace theory to coincide. Meaning we will retrieve the supermultiplet from the cohomology.

We can now start examining the cohomology. Our first observation, is that at order λ^0 only constants are allowed. By acting with Q on the scalar superfield we get

$$\lambda^\alpha D_\alpha c^{(0)} = \lambda^\alpha (c_\alpha^{(0,1)} + \theta^\beta (2c_{\alpha\beta}^{(0,2)} - \gamma^a_{\alpha\beta}\partial_a c^{(0,0)}) + \cdots = 0 \tag{4.9}$$

This tells us that $c_\alpha^{(0,1)} = 0$ which effectively will force all component fields at uneven order in θ to zero. Moreover as the two terms at order θ in (4.9) have opposite symmetry they must both be zero by them selves. This forces all component fields at even order to zero as well, except for $c^{(0,0)}$ which only is required to have zero gradient. Thus we have found our first element in the cohomology, namely the constant "ghost" $c = c^{(0,0)}$. It is a ghost as $gh(c^{(0)} = 1$, it is actually the ghost of the gauge symmetry of A.

Let's move on to the first order in λ. This an important order as, by noticing Q as a BRST operator, the fields of ghost number 0 will be the physical fields. We omit the superscript indicating the λ-order as this is not important through these calculations. The field of interest is now $\lambda^\alpha A_\alpha(x,\theta)$ Once again we must take into consideration the a gauge freedom as

$$A_\alpha \to A_\alpha + \delta A_\alpha = A_\alpha + D_\alpha \omega.$$

Now the work of finding out the explicit relations between the different orders of θ includes a lot of γ-matrix algebra and quickly becomes complicated. We will instead here focus on simply the representation content at the different levels. As discussed in sec. 4.1.3 one has to be careful when doing this and cannot just blindly write down the representations. We use the software LiE, [29], to help us perform the tensor products of representations.

c We present the representations by their corresponding Dynkin labels. As we are working in 10 dimensions we use the Dynkin labels of the rank 5 algebra $D_5 \simeq \mathfrak{so}(10)$. Some frequently used Dynkin labels and their tensor structure can be found in table 4.1.

Dynkin label	Representation
(00000)	Singlet
(10000)	Vector
(01000)	2-Form
(00100)	3-form
(00010)	Cospinor
(00001)	Spinor
(00011)	4-Form
(00020),(00002)	5-Form (self-dual)
(20000)	Traceless Symmetric Tensor
(10001)	γ-Traceless Vectorspinor

Table 4.1: *List of frequently used Dynkin labels of D_5 and their tensor structure*

In table 4.2 we find what representations are present in the cohomology at ghost number 0 i.e. the physical fields. We have also indicated the implications that the fields must satisfy in order to be in the cohomology, i.e. the equations of motion.

Component	Dimension	Representation	"Implication"
$a^{(1,0)}$		$\bullet$	
$a^{(1,1)}$	1	(10000)	$a^{(1,1)} \sim A_a$
$a^{(1,2)}$	3/2	(00001)	$a^{(1,2)} \sim \chi^\alpha$
$a^{(1,3)} \propto \partial_a a^{(1,1)}$	2	(01000)	$F_{ab} \sim \partial_{[a} A_{b]}$
$a^{(1,4)} \propto \partial_a a^{(1,2)} \propto \partial_a \chi^\alpha$	5/2	(10001)	$(\partial\!\!\!/ \chi)_\beta = 0$
$a^{(1,5)} \propto \partial_a a^{(1,3)} \propto \partial_a F_{bc}$	3	(11000)	$\begin{cases} \partial_{[a} F_{bc]} = 0 \\ \partial_a F^{ab} = 0 \end{cases}$
$a^{(1,6)} \propto \partial_a a^{(1,4)} \propto \partial_a \partial_b \chi^\alpha a^{(1,2)}$	7/2	(20001)	
$a^{(1,7)} \propto \partial_a a^{(1,5)} \propto \partial_a \partial_b a^{(1,3)} \propto \partial_a F_{bc}$	4	(21000)	

Table 4.2: *Full cohomology, to $\mathcal{O}(\theta^7)$, of Q at ghost number 0.*

We see here that what we retrieve is the on-shell multiplet for the non-interactive (linearised) super-Yang-Mills theory (SYM). The gauge transformations generated by the Q-exact objects also correspond to the gauge transformations of SYM [26].

As was stated in sec. 4.1.3.1 we can also get out the necessary information from the zero mode cohomology, i.e letting $Q \to \lambda^\alpha \frac{\partial}{\partial \theta^\alpha}$. The different modules present in the zero mode cohomology are depicted in table 4.3

dim \ gh#	1	0	-1	-2	-3
0	(00000)				
1/2	•	•			
1	•	(10000)	•		
3/2	•	(00001)	•	•	
2	•	•	•	•	•
5/2	•	•	(00010)	•	•
3	•	•	(10000)	•	•
7/2	•	•	•	•	•
4	•	•	•	(00000)	•

Table 4.3: *Zero mode cohomology of Q for $D = 10$. The numbering in the vertical direction are the dimensions of the representations present in Ψ at the corresponding order in λ.*

What is interesting is that not only do we retrieve the super-Yang-Mills multiplet in the zero mode cohomology (the spinor and vector at ghost number 0 in table 4.3). The fields found at ghost number -1 are also the antifields in the Batalin-Vilkovisky sense [3]. They thus represent the equations of motion of the physical fields. They are found with the same dimension and in the same representation as the equations of motion for their respective physical field. Thus we see, already from the zero mode cohomology that we retrieve the on-shell super-Yang-Mills multiplet.

4.3 Pure Spinors and supergravity in $D = 11$

We will now proceed to pure spinors in 11 dimensions. This case will now provide us with the $D = 11$ linearised supergravity multiplet. The Dynkin diagram of $\mathfrak{so}(11)$ is B_5 and representations will again be described by their Dynkin labels. Representations can be understood as the geometrical tensors in table 4.4. As we now have left the 10 dimensional case there will be no ambiguity of the meaning of the Dynkin labels.

Dynkin label	Representation
(00000)	Singlet
(10000)	Vector
(20000)	Traceless Symmetric Tensor
(00001)	Spinor
(01000)	2-Form
(00100)	3-form
(00010)	4-form
(00002)	5-Form
(10001)	γ-Traceless Vectorspinor

Table 4.4: *List of frequently used Dynkin labels of B_5 and their tensor structure.*

In 11 dimensions we can see use table E.4 to find that a symmetric spinor bilinear can be written as

$$\lambda_{(\alpha}\lambda_{\beta)} = \frac{1}{32}\gamma^{a}_{\alpha\beta}\lambda\gamma_a\lambda - \frac{1}{64}\gamma^{ab}_{\alpha\beta}\lambda\gamma_{ab}\lambda + \frac{1}{3804}\gamma^{abcde}_{\alpha\beta}\lambda\gamma_{abcde}\lambda.$$

We here see that enforcing the pure spinor constraint (4.1) does not put λ^α in a minimal orbit. One might however assume that we should, i.e. that the pure spinor constraint in 11 dimensions instead should be

$$\lambda\gamma^a\lambda = 0, \quad \text{and} \quad \lambda\gamma^{ab}\lambda = 0. \tag{4.10}$$

This would once again put the pure spinor in a minimal orbit and make it a pure spinor in the Cartan sense. However it turns out that this constraint does not give us the supergravity multiplet. In principle the constraint (4.10) can be treated by chiral spinors in 12 dimensions. It is further discussed in [8]. We will still refer to 11-dimensional spinors satisfying eq. (4.1) as pure. This constraint is the most reasonable as if we consider our operator $Q = \lambda^\alpha D_\alpha$ we use the supersymmetry algebra to show that it is nilpotent. We would get a very strange supersymmetry algebra if we were to include $\lambda\gamma^{ab}\lambda$ in the commutation relations.

4.3.1 Cohomology of Q in 11 dimensions

We will now look for the cohomology of the nilpotent operator Q in $D = 11$. We will again define our pure spinor superfield such that the cohomology and the superspace calculation will coincide. But which superspace calculation should we choose?

The most traditional way to derive supergravity is to consider the (super-)vielbeins and torsion in superspace [31, 32], work out the Bianchi identities and their appropriate constraints and one will end up with the on-shell supergravity multiplet. This is reviewed in App. D.9 and for a full treatment we refer to [33, 10]. The constituent fields of $D = 11$ supergravity are the super-Riemann tensor R, the gravitino Ψ_a (superpartner to the graviton) with a field strength $S \sim \partial_{[a}\Psi_{b]}$, and a 4-form $H_{abcd} = 4\partial_{[a}C_{bcd]}$.

The gist of it all is that all physical degrees of freedom will be expressible by the component ${E_\mu}^a$ of the super-vielbein 1-forms $E^A = dz^M {E_M}^A$. If we flatten the μ index, by ${E_\mu}^a {E_\alpha}^\mu = {\phi_\alpha}^a$ we are left with an index structure that look strikingly similar to that of the first order component field of a vector valued pure spinor superfield

$$\Phi^a(x,\theta,\lambda) = \varphi^a + \lambda^\alpha {\phi_\alpha}^a + \cdots .$$

This method works and we can retrieve the linearised equations of motion by $Q\Phi^a = 0$, [12]. A problem with this formulation is that the 4-form field-strength appears in the super-vielbeins without the gauge connection 3-form C. This is a problem as in the supergravity action we have a Chern-Simons term $\sim C \wedge H \wedge H$. As C is not present in the super vielbeins this will make it difficult to create an action for supergravity based strictly on a vector valued pure spinor superfield.

There is however another way to formulate linearised 11-dimensional supergravity, perhaps more fundamental than the superspace formulation. This is based on the 3-form gauge field C_{ABC}, where the lowest dimensional components, $C_{\alpha\beta\gamma}$, contain all physical degrees of freedom. It is of dimension $\frac{-3}{2}$ and the only modules present are

$$C_{\alpha\beta\gamma} = (01001) + (00003),$$

This form fits perfectly in the third order of λ (as will be shown later) in a scalar pure spinor superfield

$$\Psi = \cdots + \lambda^\alpha\lambda^\beta\lambda^\gamma C_{\alpha\beta\gamma} + \cdots .$$

With this formulation we get a natural inclusion of the 3-form gauge field which will make it easier to write down an action. We will from now on focus on the scalar pure spinor superfield, we will return to Φ^a in Chapter 6.

The physical degrees of freedom should be retrieved at ghost number 0, if the 3-form gauge field is at λ^3 this would imply $gh(\Psi) = 3$. Further more as $[C_{\alpha\beta\gamma}] = \frac{-3}{2}$ and $[\lambda] = \frac{-1}{2}$ we must choose $[\Psi] = -3$. Moreover Ψ must be fermionic, as $C_{\alpha\beta\gamma}$ is fermionic. With these definitions we can determine the cohomology. We will not perform any calculations of the full cohomology as we know that the calculations will coincide with the derivation of supergravity. We will instead focus on the zero mode cohomology, which can be observed in table 4.5. We can there identify the supergravity multiplet at ghost number 0. We see the gravitino $(10001) \oplus (00001)$, the metric $(20000) \oplus (00000)$ and the 3-form (00100). At ghost number 3 and 2 we find the ghost for ghost for ghost, and the ghost for ghost, of the gauge symmetry of C, respectively. At ghost number 1 we see the ghost for the gauge symmetry of C; (01000), the ghost for general covariance (diffeomorphisms), (10000), and the ghost for supersymmetry, (00001). At negative ghost numbers we instead see the antifields and antighosts of the supergravity multiplet.

dim \ $gh\#$	3	2	1	0	-1	-2	-3	-4	-5
-3	(00000)								
$-5/2$	•	•							
-2	•	(10000)	•						
$-3/2$	•	•	•	•					
-1	•	•	(01000) (10000)	•	•				
$-1/2$	•	•	(00001)	•	•	•			
0	•	•	•	(00000) (00100) (20000)	•	•	•		
$1/2$	•	•	•	(00001) (10001)	•	•	•	•	
1	•	•	•	•	•	•	•	•	•
$3/2$	•	•	•	•	(00001) (10001)	•	•	•	•
2	•	•	•	•	(00000) (00100) (20000)	•	•	•	•
$5/2$	•	•	•	•	•	(00001)	•	•	•
3	•	•	•	•	•	(01000) (10000)	•	•	•
$7/2$	•	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	(10000)	•	•
$9/2$	•	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	(00000)	•
$11/2$	•	•	•	•	•	•	•	•	•

Table 4.5: *Zero mode cohomology of Q for $D = 11$. The numbering in the vertical direction are the dimensions of the representations present in Ψ at the corresponding order in λ.*

Chapter 5

Partition functions

There is another approach, equally rewarding, to investigate the cohomology of Q. By using partition functions, or generating functions, we can actually find the same zero mode cohomology as we did in Chapter 4. We will actually retrieve the full cohomology too. We will do this for both $D = 10$ super-Yang-Mills and $D = 11$ supergravity.

5.1 Partition functions overview

A partition function, or generating function, is a way to more compactly describe, for instance, the content of a spectrum or in the case of pure spinor superfields: the representation content at different orders of λ. One expresses the state content as a power series. As an example consider the spectrum of one quantum mechanical harmonic oscillator: at each level there is only one state and thus a generating function is of the form (subscript b indicating bosonic)

$$Z_{HO_B}(t) = 1 + t + t^2 + \cdots = \frac{1}{1-t} \tag{5.1}$$

If we consider n independent harmonic oscillators, for example a bosonic unconstrained spinor ($n = 16$ in D=10), the partition function can be presented as

$$Z_{S_B}(t) = \frac{1}{(1-t)^n}.$$

If one also include fermionic states we can differentiate the fermionic from the bosonic by putting a $-$ sign at each power containing a fermionic state. Because fermions are anticommuting their power series terminate after just one level. If we again consider the spectrum of the harmonic oscillator but now with anticommutators instead of commutators we only have two states for each oscillator and thus the partition function is of the form

$$(Z_{HO_F})^n = (1-t)^n. \tag{5.2}$$

Note that the bosonic (5.1) and the fermionic (5.2) partition functions are each others inverses

$$(Z_{HO_F})^n (Z_{HO_B})^n = 1.$$

5.1.1 Partition functions of representations

The previous section discussed partition functions from a superficial viewpoint. We considered the spectrum of several harmonic oscillators. We can however construct more refined partition functions by considering actual representations of a symmetry group or algebra. If we know which representations there are at each level we can create a partition function of the form

$$\mathcal{Z}(t) = \bigoplus_{k=1}^{n} R_k t^k.$$

Partition functions like this are now a sum over representations, where addition now is the direct sum of representations. A fermionic state in a representation will now be identified by $\ominus$; a formal definition which makes it possible to handle partition functions of representations similar to those that simply count the number of states.

When considering refined partition functions we can define partition functions of functions of individual objects in some representation, R. This is useful as we are interested in constructing functions of an object in some representation, for example the pure spinor superfields. We then consider power series of that object and hence want to find what representations are present at each power.

For fermionic or bosonic objects the partition function will be, respectively

$$\begin{aligned} \mathcal{Z}_F &= \bigoplus_{k=0}^{dim(R)} \wedge^k R(-t)^k, \\ \mathcal{Z}_B &= \bigoplus_{k=0}^{\infty} \vee^k R t^k, \end{aligned} \tag{5.3}$$

where $\wedge$ and $\vee$ represents antisymmetric and symmetric tensor products, respectively. Notice here that we once again used minus signs to describe fermionic states. The two partition functions in eq. (5.3) are actually each others inverses [16].

$$\mathcal{Z}_F \otimes \mathcal{Z}_B = 1.$$

Which is in agreement with the case for harmonic oscillators. We can thus take on a formal definition of the two partition functions as

$$\begin{aligned} \mathcal{Z}_F &= (1-t)^R, \\ \mathcal{Z}_B &= \frac{1}{(1-t)^R} = (1-t)^{-R}. \end{aligned} \tag{5.4}$$

To finish off we introduce two more partition functions. The first one deals with objects which are maximally constrained. Meaning that any bilinear vanish. It is similar to the case of a single fermion but here we admit both kinds of statistics, we introduce σ which is -1 for fermions and 1 for bosons. The partition function for a maximally constrained object is then

$$\mathcal{Z}_{R^2=0}(\sigma,t) = 1 + \sigma R t. \tag{5.5}$$

The second partition function deals with the opposite, namely ”free” objects, not admitting any statistic. This means that both symmetric and antisymmetric products are allowed. The object can still be thought of as bosonic or fermionic and we therefore denote fermionic states with a minus sign. The partition function for such an object is

$$\mathcal{Z}_O(\sigma,t) = \bigoplus_{k=0}^{\infty} \sigma^k \otimes^k Rt^k = \mathbf{1} \oplus \sigma R t \oplus \sigma^2 \otimes^2 Rt^2 \oplus \cdots = \frac{1}{1-\sigma R t}. \tag{5.6}$$

An object with a partition function of this form is somewhat fictional, but still useful. Note here also that the two partition functions, (5.5) and (5.6), are each others inverses for opposite choices of σ.

5.2 Pure spinor partition function in $D = 10$

We will now determine the partition function of the pure spinor λ^α in 10 dimensions. We want to find the representation content at different orders of λ, but we want to make sure that it is consistent with the pure spinor constraint

$$\lambda\gamma^a\lambda = 0. \tag{5.7}$$

We will investigate this from two perspectives that must give the same result. The first thing we can do is try to determine what representations are present at an arbitrary power of λ^α subject to the pure spinor constraint. We can see that λ^α subject to the constraint (5.7) is in a minimal orbit. This implies that the only representation present in a tensor product of n pure spinors is in fact $(0000n)$. To see this consider a symmetric spinor multilinear $\lambda^{(\alpha_1}\cdots\lambda^{\alpha_n)}$, it can be expanded as (not taking into account possible coefficients)

$$\lambda^{(\alpha_1}\cdots\lambda^{\alpha_n)} \sim \tilde{\lambda}^{\alpha_1\cdots\alpha_n} + \gamma_a^{(\alpha_1\alpha_2}\lambda\gamma^{|a|}\lambda\tilde{\lambda}^{\alpha_3\cdots\alpha_n)} + \gamma_a^{(\alpha_1\alpha_2}\gamma_b^{\alpha_3\alpha_4}\lambda\gamma^{|a|}\lambda\lambda\gamma^{|b|}\lambda\tilde{\lambda}^{\alpha_5\cdots\alpha_n)} + \cdots$$

where $\tilde{X}$ denotes γ-tracelessness i.e. $\gamma^a_{\alpha\beta}\tilde{X}_a{}^\beta = 0$. All terms but the first one will be zero due to the pure spinor constraint. The term $\tilde{\lambda}^{\alpha_1\cdots\alpha_n}$ is then completely γ-traceless and is in the representation $(0000n)$.

5.2.1 $D = 10$ pure spinor partition function in summation form

When we now want to write down the partition function of a pure spinor we must make a choice whether we want to write the partition function as the representations present in tensor products of λ or in the coefficients of a function of λ, in other words, are we interested in the basis elements or the coefficients. This choice is of no essential difference but alas we need to be precise. We will choose to write it in terms of the fields/coefficients $A^{(n)}_{\alpha_1\cdots\alpha_n}$ as these will contain the physics.

Only the $(0000n)$ module is present in the tensor product of pure spinors. Thus the fields $A^{(n)}_{\alpha_1 \cdots \alpha_n}$ then only belongs to the module $(000n0)$ and a partition function of the pure spinor will be

$$\mathcal{Z}_{\lambda_{10}} = \bigoplus_{k=1}^{\infty} (000k0) t^k. \tag{5.8}$$

A less refined partition function, only considering the dimensions of the representations, is

$$\begin{aligned} Z_\lambda(t) =& \sum_{n=0}^{\infty} dim((000k0)) t^k = \sum_{n=0}^{\infty} \frac{1}{10} \binom{7+n}{n} \binom{5+n}{3} t^n = \\ =& \frac{(1+t)(1+4t+t)}{(1-t)^{11}} = \frac{1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8}{(1-t)^{16}} = \\ =& Z_{S_B}(t) \times \Big(1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8\Big). \end{aligned} \tag{5.9}$$

The calculation of $dim((000n0))$ can be found in App. B. In (5.9) we see something quite remarkable; the unrefined partition function of the constrained spinor is equal to the partition function of an unconstrained spinor times a polynomial. The more exciting fact about this form is that it is the same as the zero mode cohomology we found in sec. 4.2.2. Comparing with table 4.3 we see that the dimensions of the representations are the same albeit with a different sign, this is because Ψ is fermionic. The physical dimensions also agree but with a factor of 2 in the power of t.

The refined partition function of the pure spinor can, in agreement with (5.9), be shown ([8]) to equal

$$\begin{aligned} \mathcal{Z}_{\lambda_{10}}(t) =& \bigoplus_{k=1}^{\infty} (000k0) t^k = \frac{1}{(1-t)^{(00010)}} \otimes \mathcal{P}_0(t), \quad \text{where} \\ \mathcal{P}_0(t) =& (00000) \ominus (10000)t^2 \oplus (00001)t^3 \ominus (00010)t^5 \oplus (10000)t^6 \ominus (00000)t^8. \end{aligned} \tag{5.10}$$

5.2.2 $D = 10$ pure spinor partition function in product form

An alternative, but just as justified, method of obtaining the partition function for the pure spinor is by using a BRST formalism. By observing that the pure spinor constraint $\lambda\gamma^a\lambda = 0$ is reducible we can achieve the same result as in the previous subsection. Using the ghost for ghost procedure discussed in Chapter 3 we can construct a BRST operator for the pure spinor constraint. We can then use the ghosts, and ghost for ghosts to define the partition function for the pure spinor. For an explicit calculation up to order 4 we refer the reader to [34].

Using this method we get an infinite sequence of ghosts that compensate for each other. As all these ghosts have no constraints on them their individual partition functions are simple to conclude. To write down the partition function for the pure spinor we simply multiply the partition functions of all the ghosts with that of an unconstrained spinor. Using only their degrees of freedom (dimensions of representations) we end up with an unrefined partition function

$$Z_{\lambda_{10}}(t) = \underbrace{(1-t)^{-16}}_{Z_{S_B}}(1-t^2)^{10}(1-t^3)^{-16}(1-t^4)^{45}\cdots = \prod_{n=1}^{\infty}(1-t^n)^{-A_n}, \tag{5.11}$$

where A_n is the dimension of the representation of the ghost of generation n (note that as we assigned $gh(\lambda) = 1$ this is also considered a ghost). Observe that $A_1 = 16$ refers to an unconstrained bosonic spinor, 00i.e. λ. The alternating sign in the powers indicate whether the ghost is fermionic or bosonic. The exponent for t indicate the ghost number.

There is a closed formula for A_n derived in [8]. We will not use is here, instead we list a few of the first dimensions of the sequence A_n,

$$A_n = \{16,\ -10,\ 16,\ -45,\ 144,\ -456,\ 1440,\ \cdots\}.$$

Interesting to note here is that, since $1-t^n = (1-t)(1+t+\cdots+t^{n-1})$, eq. (5.11) will have a factor

$$(1-t)^{-16+10-16+45-\cdots}.$$

What the BRST procedure does is remove the degrees of freedom that aren't there due to the constraints. Thus comparing eqs. (5.11) and the first equality on the second line of (5.9) we must actually have that the strength of the pole at 1 indicates the degrees of freedom. This agrees with what was said in the beginning of sec. 4.2. We must then have the sum

$$16-10+16-45+\cdots = 11,$$

which is proved rigorously in [8].

By using the formal way of writing refined partition functions in eq. (5.4) we can replace A_n with the the actual representations R_n and replace the product with a tensor product to retrieve the refined partition function $\mathcal{Z}_{\lambda_{10}}(t)$. The Dynkin labels of the first 7 levels are then [34]

$$\begin{aligned} R_n =\{&(00010),\ \ominus(10000),\ (00001),\ \ominus(01000),\ (10001),\\ &\ominus(11000)\ominus(10000)\ominus(00002),\\ &(20001)\oplus(01001)\oplus(10010)\oplus(00001),\cdots\}. \end{aligned}$$

5.3 Pure spinor partition function in $D = 11$

We will here present the partition for the pure spinor in $D = 11$ and show that it, analogous to the 10 dimensional case, contains the zero mode cohomology found in Chapter 4. The pure spinor constraint is still

$$\lambda\gamma^a\lambda = 0,$$

with now $a = 0, \ldots, 10$. The formal partition function of a pure spinor in 11 dimensions is a bit different as we now have two modules in a spinor bilinear (01000) and (00002), keep in mind that the Dynkin labels are now for B_5 and not D_5. Because of this an 11-dimensional pure spinor is no longer in a minimal orbit. But we can still determine which representations are present at any power of λ. Consider a symmetric spinor multilinear $\lambda^{(\alpha_1} \cdots \lambda^{\alpha_n)}$, just as in the 10 dimensional case we can write this as a γ-traceless part and a sequence of traces.

$$\lambda^{(\alpha_1} \cdots \lambda^{\alpha_n)} \sim \tilde{\lambda}^{(\alpha_1 \cdots \alpha_n)} + \gamma_a^{(\alpha_1\alpha_2}\lambda\gamma^a\lambda\tilde{\lambda}^{\alpha_3\cdots\alpha_n)} + \gamma_a^{(\alpha_1\alpha_2}\gamma_b^{\alpha_3\alpha_4}\lambda\gamma^{(a}\lambda\lambda\gamma^{b}\lambda\tilde{\lambda}^{\alpha_5\cdots\alpha_n)} + \cdots .$$

$\tilde{\lambda}^{\alpha_1\cdots\alpha_n}$ is as usual here a symmetric multispinor, traceless only under γ^a. This is the only representation that does not vanish due to the pure spinor constraint. We can expand this further into traceless objects under γ^{ab}

$$\tilde{\lambda}^{\alpha_1\cdots\alpha_n} \sim \hat{\lambda}^{\alpha_1\cdots\alpha_n} + \gamma_{ab}^{(\alpha_1\alpha_2}\lambda\gamma^{|ab|}\lambda\hat{\lambda}^{\alpha_3\cdots\alpha_n)} + \gamma_{ab}^{(\alpha_1\alpha_2}\gamma_{cd}^{\alpha_3\alpha_4}\lambda\gamma^{|ab}\lambda\lambda\gamma^{cd|}\lambda\hat{\lambda}^{\alpha_5\cdots\alpha_n)} + \cdots$$

where $\hat{X}$ now represent tracelessness under γ^{ab}. We can now identify the irreducible representations present. The first term is in $(0000n)$, just as in the 10-dimensional case. The following terms are in $(0p00(n-2p))$ as we for each term remove 2 spinor indices and add 2 antisymmetric vector indices.

5.3.1 $D = 11$ pure spinor partition function in summation form

As we are in odd dimensions we do not have chiral spinors which means that every representation of $\mathfrak{so}(11)$ is its own dual. Thus we do not have to worry about if we should choose basis elements or coefficients to display the partition function. The refined partition function for a pure spinor in 11 dimensions in a sum form can be written as

$$\mathcal{Z}_{\lambda_{11}}(t) = \bigoplus_{n=0}^{\infty} \oplus_{i=0}^{\frac{n}{2}} \Big(0i00(n-2i)\Big)t^n, \tag{5.12}$$

where for odd n the second sum goes up to $\frac{n-1}{2}$. This can once again be rewritten as the partition function for a free spinor modulo the constraint. The partition function then takes the form [8]

$$\begin{aligned}
\mathcal{Z}_{\lambda_{11}} =& \frac{1}{(1-t)^{(00001)}} \otimes \mathcal{G}_0(t), \quad \text{where} \\
\mathcal{G}_0(t) =& (00000) \ominus (10000)t^2 \oplus \Big((10000) \oplus (01000)\Big)t^4 \\
& \ominus (00001)t^5 \ominus \Big((00000) \oplus (00100) \oplus (20000)\Big)t^6 \\
& \oplus \Big((00001) \oplus (10001)\Big)t^7 \ominus \Big((00001) \oplus (10001)\Big)t^9 \oplus \Big((00000) \oplus (00100) \oplus (20000)\Big)t^{10} \\
& \oplus (00001)t^{11} \ominus \Big((10000) \oplus (01000)\Big)t^{12} \\
& \oplus (10000)t^{14} \ominus (00000)t^{16}.
\end{aligned} \tag{5.13}$$

Note here again the agreement of the numerator with the zero mode cohomology of table 4.5. Here we once again see all representations present in the zero mode cohomology, albeit with a shifted statistics. This is again due to Ψ being fermionic. The dimensions are also consistent with the powers of t. However now the shift is $t^{2([a]+3)}$, where $[a]$ is the dimension of the zero-mode cohomology. The 3 comes from the fact that $[\Psi] = -3$.

5.3.2 $D = 11$ pure spinor partition function in product form

Just as in sec. 5.2.2 we can write the partition function of the pure spinor in 11 dimensions on a product form

$$Z_{\lambda_1 1}(t) = \prod_{n=1}^{\infty} (1 - t^n)^{R_n}, \tag{5.14}$$

where the representations can be both positive and negative referring to fermions and bosons respectively.

There are at least two ways to obtain the representations R_n, the first one is to simply match the expansion of (5.14) with that of eq. (5.12). This can be done level by level and the first 12 levels are [17]

$$\begin{aligned}
R_1 &= -(00001) \\
R_2 &= (10000) \\
R_3 &= 0 \\
R_4 &= -(10000) \\
R_5 &= (00001) \\
R_6 &= -(01000) \\
R_7 &= 0 \\
R_8 &= (00000) + (20000) + (00010) \\
R_9 &= -(00001) - (01001) - (10001) \\
R_{10} &= (00000) + (00010) + 2(00100) + (02000) + (10000) + (10010) + (11000) \\
R_{11} &= -(00001) - (01001) - (10001) - (11001) \\
R_{12} &= (00000) + (00010) + (02000) + (12000) + (20000) + (20010) \\
&\quad - (00002) - (00100) - (10000) - (10010) - (11000)
\end{aligned} \tag{5.15}$$

Notice here that the signs for the representation refers to the sign in the exponent of the partition function $(1 - t^n)^{R_n}$. This means negative signs refer to bosons and positive to fermions.

Another way of obtaining the representations is through a BRST treatment of the pure spinor constraint. This is done to level 8 in sec. 9.3. We can notice here that due to the holes at levels 3 and 7 we will have an inconsistent grading for levels 4, 5, and 6 (usually even levels are bosons while odd ones are fermions). This will imply some complications later on.

5.4 The on-shell supermultiplets

We have now motivated the partition function for the pure spinor from two ways, what can these two tell us?

We will here demonstrate that not only does the partition function of the pure spinor contain the zero mode cohomology of Q, which in turn contain a supermulitplet, it also carry information of the full cohomology, and by that the full on-shell supermultiplet. We will show this explicitly for both super-Yang-Mills and supergravity.

5.4.1 Super-Yang-Mills

Using the Dynkin labels to present the representations of the ghosts we can combine eq. (5.10) with the product form and write the partition function of the pure spinor as

$$\begin{aligned}\mathcal{Z}_{\lambda_{10}}(t) &= \bigoplus_{k=1}^{\infty}(000k0)t^k = \bigotimes_{n=1}^{\infty}(1-t^n)^{R_n} = \\ &= (1-t)^{-(00010)} \otimes \mathcal{P}_0(t) = \\ &= (1-t)^{-(00010)} \otimes (1-t^2)^{(10000)} \otimes (1-t^3)^{-(00001)} \otimes (1-t^4)^{(01000)} \cdots\end{aligned} \tag{5.16}$$

As we see from (5.16) we can remove the partition function of the unconstrained bosonic spinor and find the equality

$$\mathcal{P}_0(t) = (1-t^2)^{(10000)} \otimes (1-t^3)^{-(00010)} \otimes (1-t^4)^{(01000)} \cdots$$

We have seen that the zero-mode cohomology is contained in the partition function for a pure spinor, the zero-mode cohomology in turn demonstrates what exists in the θ expansion of the superfield. We first only considered functions of just λ, eq (5.8) but if we ”pull out” the partition function of a bosonic spinor (the inverse to a fermionic one) we get eq. (5.10) where $\mathcal{P}_0(t)$ are the representations present in the θ expansion. Thus in a way the factoring out of a bosonic spinor is like introducing θ dependence to the pure spinor superfield.

We can in fact do the exact same with a fermionic vector and introduce x-dependence as well, what we then retrieve will be the complete on-shell supermultiplet. This is all demonstrated in the following theorem.

Theorem 1. *The on-shell linearised super-Yang-Mills multiplet is contained in the partition function for a pure spinor in 10 dimensions. Explicitly,*

$$\mathcal{Z}_{\lambda_{10}}(t) = ((00000) \ominus \mathcal{Z}_{SYM}(t)) \otimes (1-t)^{-(00010)} \otimes (1-t^2)^{(10000)}, \tag{5.17}$$

where

$$\mathcal{Z}_{SYM}(t) = t^4 \bigoplus_{k=0}^{\infty}(k1000)t^{2k} \ominus t^3 \bigoplus_{k=0}^{\infty}(k0001)t^{2k} \tag{5.18}$$

is the partition function for super-Yang-Mills.

Proof. To prove this, note that

$$\mathcal{Z}_{\lambda_{10}}(t) = \mathcal{P}_0(t) \otimes (1-t)^{-(00010)},$$

with $\mathcal{P}_0(t)$ as in eq. (5.10). Comparing with (5.17) we see we must have

$$\mathcal{P}_0(t) \otimes (1-t^2)^{-(10000)} = (00000) \ominus \mathcal{Z}_{SYM}(t). \tag{5.19}$$

The first 8 orders in t of eq. (5.19) can be done explicitly and are

$$\begin{aligned} \mathcal{P}_0(t) \otimes (1-t^2)^{-(10000)} &= \bigoplus_{k=1}^{\infty} \vee^k(10000)t^{2k} \otimes \mathcal{P}_0(t) = \\ =&(00000) \oplus (00001)t^3 \ominus (01000)t^4 \oplus (10001)t^5 \ominus (11000)t^6 \oplus (20001)t^7 \ominus (21000)t^8 \cdots \end{aligned} \tag{5.20}$$

The explicit calculation for all orders is somewhat tedious and not particularly insightful, the proof is therefore relocated to App. C. In the end we get exactly the right hand side of eq. (5.18).

What is now left to prove is that the partition function for the SYM multiplet agrees with (5.18). The fields are the spinor, χ, and the 2-form field strength F_{ab} with dimensions 3/2, and 2, respectively. We can write their partition functions, not as a power series showing the representations present in tensor products of them, but as their derivative expansion in a power series in x. We count the levels as the dimensions of the fields plus the numbers of derivatives. For example, an arbitrary term in the Taylor expansion of χ is $(\partial)^n\chi$ which thus has dimension $3/2+n$. To avoid awkward powers in t we grade the dimension with an additional factor two, so that $\partial^n\chi$ will be on order t^{3+2n}, which is consistent with how we have graded dimensions previously. Forcing χ to be on-shell we see that no representations of the form (k0010) $k \leq n$ can be present in $\partial^n\chi$, as these correspond to derivatives of the Dirac equation. Furthermore, representations like $(k0001)$ $k<n$ will not be present as these will correspond to the contraction of two or more derivatives acting on χ. If the spinor is on-shell i.e. satisfying the Dirac equation it also satisfies the Klein-Gordon equation. Thus in a term $(\partial)^n\chi$ only the representation (n0001) is present. Thus the partition function for the spinor is

$$\mathcal{Z}_\chi(t) = \ominus \bigoplus_{n=0} (n0001)t^{3+2n},$$

where the minus is because the spinor is a fermion.

The partition function for the field strength can be found through analogous arguments. It is in the representation (01000) and an arbitrary term in its Taylor expansion will have dimension $[(\partial)^n F_{ab}] = 2+n$. Due to the Bianchi identity $\partial_{[a}F_{bc]} = 0$ we will not have any antisymmetric parts at an arbitrary level of derivatives. Further more, $\partial^a\partial_{[a}F_{bc]} = \Box F_{bc} + \partial_c\partial^a F_{ab} + \partial_b\partial^a F_{ca} = 0 \implies \Box F_{bc} = 0$, as F is on-shell. The partition function for the field strength is then

$$\mathcal{Z}_F(t) = \bigoplus_{n=0}(n1000)t^{4+2n}.$$

Thus

$$\mathcal{Z}_{SYM}(t) = \mathcal{Z}_\chi(t) \oplus \mathcal{Z}_F(t) = t^4 \bigoplus_{k=0}^{\infty}(k1000)t^{2k} \ominus t^3 \bigoplus_{k=0}^{\infty}(k0001)t^{2k}$$

which completes the proof $\Box$

The implication of Theorem 1 is rather extraordinary. We can observe that the partition function of the pure spinor, actually contains all the information about the linearised on-shell super Yang-Mills multiplet. Using eq. (5.19) it is clear that what the vector and spinor does is in fact introducing x- and θ-dependence, respectively. When we multiply $\mathcal{Z}_{\lambda_{10}}$ with the partition function for a fermionic spinor we found $\mathcal{P}_0(t)$, which contain the representations of the ghost, the constituent fields (χ^α, A_a), their respective anti-fields, and the antighost for the linearised super Yang-Mills theory. This was equivalent to introducing θ-dependence to the pure spinor superfield. The presence of the antifields furthermore tells us that the vector and spinor are on-shell (the introduction of antifields is related to putting an on-shell gauge theory off-shell as was discussed briefly in sec. 3.4, they will transform as the equations of motion). Furthermore, through eq. (5.19) it is now clear that

$$\mathcal{Z}_{\lambda_{10}}(t) \otimes (1-t)^{(00010)} \otimes (1-t^2)^{-(10000)} = (00000) \ominus \mathcal{Z}_{SYM}(t),$$

and we can thus see the multiplication of the bosonic vector as introducing x-dependence. When introducing x-dependence the fields and antifields will start to interact in such a way that the fields will be put on shell. It is remarkable that the partition function of the pure spinor actually contain information about the complete super-Yang-Mills Theory.

5.4.2 Supergravity

We will now investigate the partition function of 11-dimensional pure spinors and show that it also contain full information of the on-shell supergravity multiplet. This is a bit more involved but follows in the same principle. We already know (eq. (5.13)) that the zero mode cohomology is contained in the partition function for 11-dimensional pure spinors. That the full on-shell supergravity multiplet can be retrieved from the pure spinor partition function is described in the following theorem.

Theorem 2. *The on-shell linearised D=11 supergravity multiplet is contained in the partition function for a pure spinor in 11 dimensions. Explicitly,*

$$\mathcal{Z}_{\lambda_{11}} = \Big((00000) \ominus \mathcal{Z}_{\overline{SG}}(t)\Big) \otimes (1-t)^{-(00001)} \otimes (1-t^2)^{(10000)}, \tag{5.21}$$

where

$$\mathcal{Z}_{\overline{SG}} = \ominus(10000)t^4 \oplus (00001)t^5 \ominus (01000)t^6 \oplus t^6 \mathcal{Z}_{SG}(t), \tag{5.22}$$

and

$$\mathcal{Z}_{SG}(t) = \bigoplus_{n=0}^{\infty}(n0010)t^{2n+2} \ominus \bigoplus_{n=0}^{\infty}(k1001)t^{2n+3} \oplus \bigoplus_{n=0}^{\infty}(k2000)t^{2n+4}. \tag{5.23}$$

is the partition function for supergravity.

Proof. The partition function for 11-dimensional pure spinors is

$$\mathcal{Z}_{\lambda_{11}} =(1-t)^{-(00001)} \otimes \mathcal{G}_0(t), \tag{5.24}$$

where

$$\begin{aligned}\mathcal{G}_0(t) =&(00000) \ominus (10000)t^2 \oplus \Big((10000) \oplus (01000)\Big)t^4 \\ &\ominus (00001)t^5 \ominus t^6\Big((00100) \oplus (20000)\Big) \\ &\oplus (10001)t^7 \ominus (10001)t^9 + t^{10}\Big((00100) \oplus (20000)\Big) \\ &\oplus (00001)t^{11} \ominus t^{12}\Big((10000) \oplus (01000)\Big) \\ &\oplus t^{14}(10000) - (00000)t^{16}.\end{aligned}$$

Comparing equations (5.21), (5.22), and (5.24) we see that we must show that

$$\mathcal{G}_0(t) \otimes (1-t^2)^{-(10000)} = ((00000) \oplus (10000)t^4 \ominus (00001)t^5 \oplus (01000)t^6 \ominus t^6 \mathcal{Z}_{SG}(t)). \tag{5.25}$$

We also must show that the partition function for supergravity takes the form of eq. (5.23).

That we multiply $\mathcal{G}_0$ with a bosonic vector can and should once again be interpreted as introducing x-dependence. The first 16 orders are done explicitly and can be found in (5.26).

$$\begin{aligned}\mathcal{G}_0(t) \otimes \bigoplus_{n=0}^{\infty} \vee^k(10000)t^{2k} =&(00000) \oplus (10000)t^4 \ominus (00001)t^5 \oplus (01000)t^6 \ominus (00010)t^8 \\ &\oplus (01001)t^9 \ominus \Big((02000) \oplus (10010)\Big)t^{10} \\ &\oplus (11001)t^{11} \ominus \Big((12000) \oplus (20010)\Big)t^{12} \\ &\oplus (21001)t^{13} \ominus \Big((22000) \oplus (30010)\Big)t^{14} \\ &\oplus (31001)t^{15} \ominus \Big((32000) \oplus (40010)\Big)t^{16} \oplus \cdots .\end{aligned} \tag{5.26}$$

The proof to all orders is similar to the case of the SYM multiplet. It is even more involved and will not supply us with any more insight, it is therefore moved to App. C.

What still needs to be done is to explain the form of the partition function for the supergravity multiplet

$$\mathcal{Z}_{SG}(t) = \bigoplus_{n=0}^{\infty}(n0010)t^{2n+2} \ominus \bigoplus_{n=0}^{\infty}(k1001)t^{2n+3} \oplus \bigoplus_{n=0}^{\infty}(k2000)t^{2n+4}. \tag{5.27}$$

To find the partition function for the supergravity multiplet we need the covariant objects, and to determine which representations there are when acting with n derivatives. We think of the partition function of a multiplet as the representations present in a Taylor expansion of the fields around some point. The covariant objects in 11 dimensional supergravity are the curvature R_{abcd}, the field-strength for the 3-form gauge field, $H_{abcd} \sim \partial_{[a}C_{bcd]}$, and the field strength for the gravitino, $S \sim \partial_{[a}\psi_{b]}$.

We start with the 4-form H, it is in the representation (00010) and have dimension 1. We count levels, just as before, as 2 times the dimension. We want to find the representations in a term $(\partial)^n H$. All antisymmetrisations are zero due to the Bianchi identity $\partial_{[a}H_{bcde]}$, and the commutativity of the derivatives. Moreover, the equation of motion is $\partial^a H_{abcd} = 0$ which implies, analogously to the proof for the 2-form field strength in Thm. 1, that if we contract the Bianchi identity with ∂^a we will get that H satisfies Klein-Gordon. This leaves only the representation $(n0010)$ in a term $(\partial)^n H$.

The field strength for the gravitino, $S_{ab} \sim \partial_{[a}\psi_{b]}$, is in the representation (01001) with dimension 3/2. As the field strength already is proportional to a derivative we cannot have any antisymmetric representations other than the antisymmetry we already have for S_{ab}. Thus only symmetric products will remain. The gauge symmetry for the gravitino is the local supersymmetry and we can then gauge fix such that the gravitino, and consequently also the field strength, satisfy the Klein-Gordon equation. Thus only $(n1001)$ will be present in a term $(\partial)^n S_{ab}$.

Moving on to the curvature tensor R. It is of dimension 2 and breaks down [20] into the curvature scalar, the Ricci tensor, and the Weyl tensor. As we are in linearised on-shell supergravity we work around a flat Minkowski space and thus both the curvature scalar and the Ricci tensor are zero on-shell. The only thing left which are not present in the equations of motion is the Weyl-tensor which is in the module (02000). The same arguments apply here. Due to the Bianchi identities, $\partial_{[a}R_{bc]de} = 0$, we cannot have any antisymmetrisations with derivatives. Further more, contracting with η^{ad} yields that $\partial^a R_{bcae} = 0$ which analogously to the H case implies that R satisfies Klein-Gordon. This implies that only $(n2000)$ is present in a term $(\partial)^n R$.

With all this we can now write the partition function for 11-dimensional supergravity as eq. (5.27). This together with the calculation in App. C concludes the proof. □

Theorem 2 once again shows that pure spinors hide the most mysterious abilities. The pure spinor, at first glance, knows in principle nothing about either supersymmetry or the cohomology. But still it contains information about the zero mode cohomology of Q and when factoring out a vector i.e. introducing x-dependence the zero modes in the cohomology start to interact such that the multiplet is put on-shell.

When observing the small polynomial at order 4, 5 and 6 in eq. (5.22) we can interpret these as the zero modes (zero modes as we only have the first representations in a Taylor series) of the ghosts of the symmetries we have, as only the zero modes are present these are ghosts of global symmetries. They have the wrong statistics as they are the ghosts and not the symmetries themselves. The representations (10000) and (01000) are the ghosts for diffeomorphisms, these are the Killing vectors of Minkowski space. (00001) is the ghost for global supersymmetry transformations.

The extra factor of $t^{2\times 3}$ multiplying the partition function $\mathcal{Z}_{SG}$ in eq. (5.22) is due to the fact that we choose the dimension of the pure spinor superfield Ψ in Chapter 4 to have dimension -3. As the partition function for the pure spinor contains what representations are presents at each order in λ there is a close relation between $\mathcal{Z}_{\lambda_{11}}(t)$ and $\Psi(\lambda)$. To get the physical dimensions from the partition function we must remove t^6 from all terms, including the ghosts.

Chapter 6

Interactions and integration in pure spinor space

So far we have only discussed the linearised cases of $D = 10$ super-Yang-Mills and $D = 11$ supergravity, no interactions are considered. This is because the equation $Q\Psi = 0$ is linear in the field Ψ and hence no nonlinear terms can arise. Before we dive into one of the main parts of this thesis; the duality between pure spinors and superalgebras, we will do a short review covering the concepts and challenges there are if one wants to include interactions in the pure spinor formalism. In the end what we are looking for is a possibility to write down an action which is manifestly supersymmetric and Lorentz invariant. To do this we must first discuss how the equation of motion, $Q\Psi = 0$, changes to cover interactions. Then we will go on to discuss off-shell formulations in the pure spinor formalism, i.e. finding an appropriate action. To formulate an action one needs to define what integration with respect to a pure spinor implies, this we will discuss in the end of the chapter.

6.1 The action principle

As we discussed in Chapter 4 the equation

$$Q\Psi = 0, \tag{6.1}$$

will be equivalent, given appropriate dimension and ghost number, to the superspace calculation of a linearised supersymmetric theory. In our case super-Yang-Mills or supergravity. An action corresponding to an equation of motion of this form would be

$$\mathcal{S} \sim \int \Psi Q \Psi.$$

Varying this with respect to Ψ we get back $Q\Psi = 0$, assuming partial integration with respect to Q is possible and does not produce boundary terms.

In the SYM case the modification needed to incorporate interactions in the equation of motion is pretty straight forward; the equation of motion for interacting super-Yang-Mills is equivalent to (see App. D) $F_{\alpha\beta} = D_{(\alpha}A_{\beta)} + \gamma^a_{\alpha\beta}A_a - A_\alpha A_\beta = 0$, in particular the $\gamma^{[5]}$ part will yield the equations of motion. This tells us we need to add a Ψ^2 term to (6.1)

$$Q\Psi - \Psi\Psi = 0.$$

At the λ^2 order we will then have

$$\lambda^\alpha\lambda^\beta(D_\alpha A_\beta + \gamma^a_{\alpha\beta}A_a - A_\alpha A_\beta) = \frac{1}{1920}\lambda\gamma_{[5]}\lambda\gamma^{[5]}(D_\alpha A_\beta - A_\alpha A_\beta) = 0$$

which is exactly what we are looking for. This modification will not change the field content of the cohomology, only extra terms will appear corresponding to interaction terms. This would be equivalent to introducing a Ψ^3 term in the action.

We are now interested in formulating an action. With an action formulation we get an off-shell description of the theory. Furthermore we would like the action to manifestly display the symmetries that we have; Lorentz invariance (or general covariance for supergravity) and supersymmetry. Note also that if we formulate our action using the pure spinor superfield we will automatically end up in a Batalin-Vilkovisky formulation of the theory, as the ghost, antifields and anti-ghosts all are present in Ψ.

6.2 Integration in pure spinor space

Before moving on to concretely formulating the actions of SYM and supergravity we need to discuss integration in pure spinor space. Due to the pure spinor constraint this is not a trivial question. We need to define what we mean by integration with respect to pure spinors. That means we need to define a measure that is non-degenerate and with the appropriate dimensions and ghost numbers. By non-degenerate we mean

$$\int [d\lambda] f \cdot g = 0 \;\forall g \implies f = 0.$$

Let's start with determining the dimension and ghost number the measure need to have for super-Yang-Mills and supergravity. We let $[dZ]$ denote the measure over all spaces, superspace and pure spinor space, $[d\lambda]$ we use do denote the measure for pure spinor space. For super-Yang-Mills, the measure over superspace will be $\frac{1}{g^2}\int d^{10}x d^{16}\theta$ which will be of dimension $-4 + \frac{1}{2} \times 16 = 4$, where we defined g such that the integration over x will have dimension -4 to agree with the ordinary dimensions of a Lagrangian. Moreover, the dimension of the term $\Psi Q\Psi$ is 0 meaning that the dimension of the measure with respect to pure spinors must be -4 for the action to be dimensionless. Moreover as $gh(\Psi Q\Psi) = 3$ we must have that $gh(\int[d\lambda]) = -3$. In supergravity the measure over superspace is instead $\frac{1}{\kappa^2}\int d^{11}x d^{32}\theta$, with dimension $-2 + \frac{1}{2} \times 32 = 14$, where κ is defined analogously to g, however in this case we define it such that $[\frac{1}{\kappa^2}d^{11}x] = -2$ because for gravity the Einstein-Hilbert term $\sqrt{g}R$ is of dimension 2. Moreover the term $\Psi Q\Psi$, now of dimension -6, implies the dimension of the measure with respect to pure spinors must be -8. Moreover as $gh(\Psi Q\Psi) = 7$ we must have that $gh(\int[d\lambda]) = -7$.

Given these parameters there seems to be a natural candidate for pure spinor integration. Looking at the zero mode cohomologies in tables 4.3 and 4.5 we see we have scalars at order $\lambda^3\theta^5$ and $\lambda^7\theta^9$ for SYM and supergravity, respectively. These components will have the appropriate dimensions and ghost numbers for $d\lambda$. Thus defining an "integration" that only returns this component would satisfy the necessary dimensions and ghost numbers. Integration with respect to λ would then be, for 10 and 11 dimensions respectively,

$$\int [d\lambda]\Psi \sim \theta^5 A^{(3,5)}, \qquad \int [d\lambda]\Psi \sim \theta^9 A^{(7,9)}$$

This definition is unfortunately degenerate, as we only have positive powers of λ in a pure spinor superfield. Any function $f = \lambda^4 A + \cdots$, in 10-dimensions will yield zero when integrated with any other function as all terms in the integration will be of higher order than 3 in λ.

The solution for this predicament is by introducing so called *non-minimal variables* in the theory, [11]. By doing this we can effectively treat negative powers of λ such that our measure no longer is degenerate. The minimal variables we introduce consist of an another pure spinor $\bar{\lambda}_\alpha$ which can be seen as the complex conjugate to λ^α. We also introduce a fermionic spinor, r_α which is pure relative $\bar{\lambda}$ i.e $(\bar{\lambda}\gamma^a r) = 0$. The fermionic spinor r_α can be thougth of as the differential $d\bar{\lambda}_\alpha$. With the introduction of these new variables we must modify our BRST operator in order to make sure we retrieve the same cohomology as before. This modification is

$$Q = \lambda D + r\frac{\partial}{\partial\bar{\lambda}} = Q_0 + d\bar{\lambda}\frac{\partial}{\partial\bar{\lambda}} = Q_0 + \bar{\partial},$$

where $\bar{\partial}$, is called the *Dolbeault* operator, it is the exterior derivative with respect to the complex conjugate of λ^α.

Once these non-minimal variables are introduced in a consistent way we are able to define a measure and integration which is non-degenerate and yields what we are after. One important feature we need is that $\int[dZ]Q\Psi = 0$ such that we do not get any "boundary" terms when performing partial integration.

The precise treatment of defining the measure and integration is delicate and tricky, we will not treat this formulation in this paper but refer the reader to [11, 12, 13, 27].

6.3 Manifestly supersymmetric actions of $D = 10$ SYM and $D = 11$ SG

Once the integration is properly defined we can now present the manifestly supersymmetric action for 10-dimensional super-Yang-Mills and 11-dimensional supergravity

$$\mathcal{S}_{SYM} = \int [dZ]\frac{1}{2}\Psi Q\Psi + \frac{1}{3}\Psi^3,$$
$$\mathcal{S}_{SG} = \int [dZ]\Psi Q\Psi + \frac{1}{6}\lambda\gamma_{ab}\lambda(1\frac{2}{3}T\Psi)\Psi R^a\Psi R^b\Psi.$$

The SYM action is easy and leads exactly to the equation of motion we want. The supergravity action is however a bit more involved. First off we have introduced an operator R^a of ghost number -2 and dimension 2. This operator is needed to relate the two fields Ψ and Φ^a discussed in Chapter 4,

$$R^a\Psi = \Phi^a.$$

The term $\lambda\gamma_{ab}\lambda\Psi R^a\Psi R^b\Psi$ is what generates the Chern-Simons term $C \wedge H \wedge H$. A nilpotent operator T and a 4-point coupling must also be introduced in order to satisfy the master equation, $(\mathcal{S}, \mathcal{S}) = 0$, introduced in Chapter 3.

Chapter 7

Koszul duality of constrained objects

We will in this chapter motivate a duality between constrained objects and algebras. This correspondence is motivated by looking at the BRST treatment of a constrained object and explicitly at the unwanted cohomology that arises when introducing ghosts. These ghost can in turn be interpreted as elements in the coalgebra (defined below) of an algebra whose partition function is the inverse of the partition function for the constrained object. A duality displayed as two partition functions of algebraic structures being each others inverses can be referred to as a *Koszul duality*.

We will first consider this duality for Lie superalgebras and we will then move on to consider more general algebras with higher order brackets.

7.1 The coalgebra and the duality

To appreciate the duality we will first define the concept of the *coalgebra*. Given a Lie algebra $\mathfrak{g}$, the coalgebra of $\mathfrak{g}$ is defined as the dual vector space, $\mathfrak{g}^*$, of $\mathfrak{g}$ equipped with the map

$$d : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*,$$

called the coproduct, such that for a basis $E_a \in \mathfrak{g}$ and its dual $E^{*a} \in \mathfrak{g}^*$

$$[E_a, E_b] = f_{ab}{}^c E_c \iff dE^{*c} = f_{ab}{}^c E^{*a} \wedge E^{*b}.$$

The definition can easily be extended to Lie superalgebras by admitting graded commutators and wedge products, i.e if the objects E_a and E_b are fermionic we will as usual have a anti-commutator and the corresponding coproduct will be the symmetric product $E^{*a} \vee E^{*b}$. We will in in most cases let $\wedge$ denote graded wedge product.

The definition of the coalgebra implies, since it uses the same structure constants, that it contains all information of the actual algebra. So it does not really matter if we work with the coalgebra or the ordinary algebra. An interesting fact is that the Jacobi identity is equivalent to the nilpotency of the coproduct d. Choosing a basis E^M for the coalgebra of a Lie superalgebra we get

$$\begin{aligned} ddE^M = d\Big(f_{AB}{}^M E^A \wedge E^B\Big) =& f_{AB}{}^M f_{CD}{}^B E^A \wedge E^C \wedge E^D - f_{AB}{}^M f_{CD}{}^A E^C \wedge E^D \wedge E^B = \\ =& f_{AB}{}^M f_{CD}{}^B \Big(1 - (-)(-)^{AB}(-)^2(-)^{A(C+D)}\Big) E^A \wedge E^C \wedge E^D = \\ =& f_{AB}{}^M f_{CD}{}^B (1 + (-)^{A(B+C+D)} E^A \wedge E^C \wedge E^D = \\ =& 2 f_{AB}{}^M f_{CD}{}^B E^A \wedge E^C \wedge E^D = 0 \end{aligned}$$

where we used that $C + D \equiv B \mod 2$ as we deal with a superalgebra, i.e. if C and D are both fermionic then B must be bosonic.

Consider now a $\mathbb{Z}$-graded Lie superalgebra, $\mathscr{B} = \bigoplus_{n\in\mathbb{Z}} \mathscr{B}_n$, where odd levels are fermionic and even ones bosonic. Grade is conserved under the Lie-bracket,

$$[\mathscr{B}_p, \mathscr{B}_q\} \subseteq \mathscr{B}_{p+q}.$$

In particular this tells us that all $\mathscr{B}_n$ are representations of level zero $\mathscr{B}_0$. We define the algebra to be generated from level 1 such that

$$\underbrace{[\mathscr{B}_1, [\mathscr{B}_1, \cdots, \{\mathscr{B}_1 \mathscr{B}_1\}}_{n} \cdots\} = \mathscr{B}_n.$$

We let R_n denote the representation at level n. Define a basis, E^M, for level 1, where M is the appropriate index for the representation R_1. Algebras with this structure are called Borcherds superalgebras, we will discuss these further in Chapter 8.

Before continuing to write down the partition function of an algebra with this structure we need to define the *universal enveloping algebra*, $\mathcal{U}$, of a Lie algebra or Lie superalgebra $\mathfrak{g}$. We do this because we will actually consider the partition functions of the universal enveloping algebra rather than the algebras themselves. The Universal enveloping algebra can be thought of as the tensor algebra of $\mathfrak{g}$

$$T(\mathfrak{g}) = 1 \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \oplus \cdots,$$

modulo the equivalence relation

$$x \sim y \iff x \otimes y \pm y \otimes x = [x, y\}, \tag{7.1}$$

for $x,\ y \in \mathfrak{g}$. The equivalence relation, $\sim$ extends the graded Lie bracket to tensor products of the Lie algebra $\mathfrak{g}$. Here we can see the similarity to the partition functions of the pure spinors. The universal enveloping algebra is sort of a power series in the Lie algebra $\mathfrak{g}$, we thus look as "functions" of $\mathfrak{g}$ rather than $\mathfrak{g}$ itself, just as for pure spinors.

For a freely generated algebra (freely generated algebras will be defined in Chapter 8, but they are exactly what one would assume; no Serre relations at all) this definition becomes even simpler. Consider an $\mathbb{N}$-graded algebra $\mathscr{B}_+$ freely generated from level one. As there are no relations enforced on the bracket the equivalence relation (7.1) will be trivial and the universal enveloping algebra will then coincide with the tensor product space (or tensor algebra) of $\mathscr{B}_1$. If $\mathscr{B}_1$ is bosonic the partition function will then simply be that of a bosonic object of indefinite statistics in eq. (5.6), with $\sigma = 1$.

$$\mathcal{Z}_{U(\mathscr{B}_+)} = 1 \oplus R_1 t \oplus \otimes^2 R_1 t^2 \oplus \cdots .$$

This can also be observed by expanding the function

$$\begin{aligned} \mathcal{Z}_{U(\mathscr{B}_+)}(t) =& (1-t)^{-R_1}(1-t^2)^{-R_2}(1-t^3)^{-R_3}\cdots = \\ =& \mathbf{1} \oplus R_1 t \oplus (\vee^2 R_1 \oplus R_2)t^2 \oplus (\vee^3 R_1 \oplus R_1 \otimes R_2 \oplus R_3)t^3 \oplus \cdots \end{aligned} \tag{7.2}$$

As the algebra is generated from level 1 we can rewrite all higher order representations in terms of R_1

$$R_2 = \wedge^2 R_1, \quad R_3 = R_1 \otimes R_2 = \underbrace{R_1 \wedge R_1 \wedge R_1}_{Jacobi \implies =0} \oplus R_1 \vee R_1 \wedge R_1, \tag{7.3}$$

substituting (7.3) in (7.2) we get

$$\begin{aligned} \mathcal{Z}_{U(\mathscr{B}_+)}(t) =& \mathbf{1} \oplus \sigma R_1 t \oplus (\vee^2 R_1 \oplus \wedge^2 R_1)t^2 \oplus (\vee^3 R_1 \oplus R_1 \otimes \wedge^2 R_1 \oplus R_1 \vee R_1 \wedge R_1)t^3 \oplus \cdots = \\ =& \mathbf{1} \oplus R_1 t \oplus \otimes^2 R_1 t^2 \oplus \otimes^3 R_1 t^3 \cdots \end{aligned}$$

Thus we see that the partition function for the universal enveloping algebra of a freely generated algebra is

$$\mathcal{Z}_{U(\mathscr{B}_+)}(t) = \bigotimes_{n=1} (1-t^n)^{(-)^{n+1} R_n}, \tag{7.4}$$

Where we have generalised to superalgebras by alternating the sign in the exponent, as odd levels are fermionic.

This same methodology works just as well if we have some Serre relations that kill certain representations in the tensor products of R_1. We then simply subtract these representations in eq. (7.3) and we can still write the partition function for the universal enveloping algebra as in (7.4). We will sometimes be a bit sloppy in the notation and simply say that (7.4), is the partition function for the algebra.

The partition function of the algebra $\mathscr{B}$ mentioned above will take the form (7.4). Consider the subalgebra $\mathscr{B}_+ = \bigoplus_{n>0} \mathscr{B}_n$ of positive levels. The coproduct for the first few levels of the coalgebra will be

$$\begin{aligned} dE^{*M} &= 0, \\ dE^{*MN} &= E^{*M} \vee E^{*N}\Big|_{\overline{R}_2}, \\ dE^{*MNP} &= E^{*M} \wedge E^{*NP}\Big|_{\overline{R}_3}, \end{aligned}$$

where E^{*MNP} must satisfy the Jacobi identity. The first line is a because, as $\mathscr{B}_+$ is generated by $\mathscr{B}_1$, we cannot commute anything in $\mathscr{B}_+$ to get something in $\mathscr{B}_1$. The bar denotes the dual representation. If we want to write down the relation for any level we notice that, as the coproduct is dual to the commutator, the action of d on $E^{*(n)}$ tells us what we need to commute to get $E_{(n)}$, where (n) denotes the number of indices and also the level in $\mathscr{B}_+$. This is simply all possible two part partitions of the integer n. Thus for any level we can write

$$dE^{*(n)} = \sum_{k=1}^{n-1} E^{*(k)} \wedge E^{*(n-k)}. \tag{7.5}$$

We can now see the striking resemblance to the BRST operator discussed in Chapter 3. Consider a bosonic object $\overline{\lambda}$ in the representation $\overline{R_1}$ subject to a constraint $\overline{\lambda}^2|_{\overline{R}_2} = 0$ such that $\overline{\lambda}$ is in a minimal orbit. The BRST treatment of this constraint will yield a BRST generator of the form [16, 34]

$$\Omega = \sum_{n=1}^{|G|} \sum_{k=1}^{n} B_{n+1} C^{n-k+1} C^k,$$

where $|G|$ is the highest order of reducibility of the constraint. Note here that we have switched the notation $\overline{\lambda} \to C^{(1)}$. The first term of the BRST generator is always a B-type ghost times the constraint which in this case is exactly $C^{(1)}C^{(1)}|_{\overline{R}_2}$. The action of the BRST operator on the ghosts will be

$$sC^{(p)} = [C^{(p)}, \Omega\} = \sum_{k=1}^{p-1} C^{(p-k)} C^{(k)}.$$

This is exactly on the same form as (7.5). There is thus no real difference between the construction of the coalgebra of a Borcherds superalgebra and the BRST formalism of a constrained object in a minimal orbit. It is more or less a matter of notation, the coalgebra generators $E^{*(n)}$ correspond to the ghosts C^n and the action of the coproduct is simply just the BRST operator acting on the ghosts

$$dE^{*(n)} \leftrightarrow sC^n = [C^n, \Omega\}.$$

The only difference between these two pictures are the shift in statistics, odd levels in the algebra are fermionic whilst odd generations of the ghosts are bosonic. We will refer to the algebra side of this duality as the *Algebra picture* and the ghost side as the *Ghost picture*.

As the coalgebra contain all information of the algebra we can consider this duality in the context of the original algebra and a bosonic object $\lambda \in R_1$ subject to a constraint $\lambda^2|_{R_2} = 0$.

The partition function for $\lambda \in R_1$ will be, as discussed in Chapter 5,

$$\mathscr{Z}_\lambda(t) = (1-t)^{-R_1}(1-t^2)^{R_2}(1-t^3)^{-R_3}\cdots .$$

This form of the partition function is exactly the same as the one for the algebra eq. (7.4), albeit with a sign difference in the exponent due to the change in statistics of $E^{*(n)} \leftrightarrow C^n$. The representations at each level in the algebra will coincide with the representations of the ghosts of the constraint $\lambda^2|_{R_2}$. We thus have

$$\mathscr{Z}_\lambda(t) \otimes \mathscr{Z}_{U(\mathscr{B}_+)}(t) = \bigotimes_{n=1}(1-t^n)^{(-1)^n R_n} \otimes \bigotimes_{n=1}(1-t^n)^{-(-1)^n R_n} = \mathbf{1}$$

This duality between the BRST formalism and a Lie superalgebra is what could be considered a Koszul duality. It has furthermore been proven that the dual algebras for objects in minimal orbits are always Borcherds superalgebras [16].

7.2 Examples

We will here display some examples of the duality. We will, in Chapters 8 and 9, return to the dual algebras of pure spinors in 10 and 11 dimensions.

7.2.1 Extreme Cases

Two easy examples for demonstrating the duality is by considering extreme cases, namely objects either subject to no constraints or constraints such that all bilinears vanish. We start with the first case. A bosonic object not subject to any constraints have a partition function

$$\mathscr{Z}(t) = (1-t)^{-R}.$$

This is the inverse of a partition function of fermion in R. Or in the case of the duality: the inverse of an algebra where the Serre relations cover everything i.e. $\{E_M, E_N\} = 0$ for all representations. Here the statistics can be interchanged meaning that the constrained object could be a fermion and the algebra element a boson.

The other extreme case is when we consider an algebra freely generated by $\mathscr{B}_1$. Then we do not have any Serre relations which means that all products of level one will be allowed. The partition function of the algebra will then take the form of eq. (5.6), which is the inverse of a maximally constrained object, i.e. eq. (5.5). Also here the statistics of the object and $\mathscr{B}_1$ are interchangeable.

7.2.2 $SO(8)$

As a final example we will consider the case of pure spinors in 8 dimensions. The Dynkin diagram is now $D_4 \simeq \mathfrak{so}(2 \cdot 4)$ and the spinor representation has the Dynkin label (0001). The table E.2 can help us find an appropriate form of the pure spinor constraint. As a symmetric spinor bilinear in 8 dimensions can be written

$$\lambda^\alpha \lambda^\beta = \frac{1}{8} C^{\alpha\beta} \lambda\lambda + \frac{1}{384} \gamma^{abcd} \lambda \gamma_{abcd} \lambda$$

we must choose the constraint $\lambda^\alpha \lambda_\alpha = 0$, if we choose $\lambda\gamma^{abcd}\lambda = 0$ that would force $\lambda = 0$.

The BRST treatment of a constraint on this form is rather easy. As the constraint is a scalar it is irreducible, we thus only require one set of ghosts, C, to take care of the constraint. The partition function of the pure spinor will then be

$$\mathcal{Z}_{\lambda_8}(t) = (1-t)^{(0001)}(1-t^2) \tag{7.6}$$

The dual algebra to 8-dimensional pure spinors will actually be the Lie superalgebra obtained by attaching an odd null root in the spinor node on the Dynkin diagram of D_4, this is depicted in fig. 7.1. What this extension mean will be described in Chapter 8.

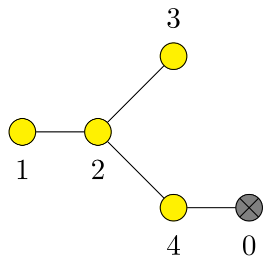

Figure 7.1: *The extension of D_4 by an odd null root. The odd root is indicated by a grey crossed out node*

For this algebra the first level will contain fermions in the representation (0001). The second level will only contain 1 boson. As scalars commute with everything we have nothing at level 3. The algebra ends at level 2 and will have partition function

$$\mathcal{Z}_{\mathscr{B}}(t) = (1-t)^{(0001)}(1-t^2)^{-1} = \left(\mathcal{Z}_{\lambda_8}(t)\right)^{-1}.$$

Which is the inverse of eq. (7.6). The duality between superalgebras and BRST is quite remarkable.

7.3 Generalisations

In the previous sections we discussed constrained objects in minimal orbits. i.e objects with a BRST generator on the form $\Omega \sim \Sigma BCC$. But what happens if an object will produce a BRST operator with say $BCCC$ or $BCCCC$ terms? What would be the dual "algebra" then?

The coalgebra defined earlier is only defined for Lie algebras, or, with minor modifications, for Lie superalgebras. But as argued above we identify the coproduct on the coalgebra elements as the BRST action on the ghosts. If we now have higher order terms i.e. more than two C:s we need to extend our algebra to include higher order brackets as well. By introducing higher order coproducts

$$d_n E^{*M} = f_{A_1 \cdots A_n}{}^M E^{*A_1} \wedge \cdots \wedge E^{*A_n},$$

we can match the higher order terms in the BRST operator with higher order coproducts. This will in the end produce a L_∞ structure, a generalisation of Lie algebras which contain multilinear brackets. This will be explained more in Chapter 9.

Chapter 8

Borcherds superalgebras and $D = 10$ super-Yang-Mills

We have so far covered the ghost picture of the duality between pure spinors and superalgebras. We will now dive in to the algebra picture of it. In this chapter we will discuss Borcherds superalgebras, which were mentioned in the previous chapter. There we said that objects in minimal orbits has a dual algebra which is a Borcherds superalgebra. We will further more see that the super-Yang-Mills multiplet plays a crucial role in the generation of the algebra.

8.1 Extensions of root systems

A semisimple Lie algebra, $\mathfrak{g}$, is defined, up to isomorphisms, by its root system, or equivalently by the Cartan matrix

$$A_{ij} = 2\frac{(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)},$$

where $(\cdot, \cdot)$ is the inner product of the root space $\mathfrak{h}^*$ defined from the canonical bilinear form between the Cartan algebra and its dual $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to K$, where K is the field over the vector space $\mathfrak{g}$.

Consider now a semisimple Lie algebra, $\mathfrak{g}$ of rank r with simple roots α_i. We can extend $\mathfrak{g}$ by adding another root, β_0, to our simple roots. By defining β_0 to be an odd null root

$$(\beta_0, \beta_0) = 0,$$

we obtain a new rather interesting algebra, $\mathscr{B}$. The complete set of simple roots is now $\beta_I = (\beta_0, \beta_i)$, where $\beta_i = \alpha_i$. The Cartan matrix for $\mathscr{B}$ is obtained by extending the Cartan matrix for $\mathfrak{g}$. We add an extra column

$$B_{i0} = 2\frac{(\beta_0, \beta_i)}{(\beta_i, \beta_i)},$$

and an extra row $B_{0I} = (\beta_I, \beta_0)$, note the diagonal entry $B_{00} = (\beta_0, \beta_0) = 0$. An example of an extension of this sort can be seen in fig 8.1, this extension will be used later on. The extension of a semisimple finite dimensional Lie algebra with an odd root β_0 in this way defines a Borcherds superalgebra.

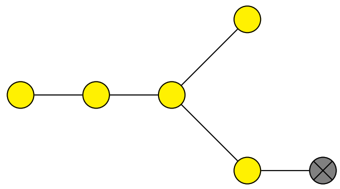

Figure 8.1: *D_5 diagram extended with an odd null root. The odd root is indicated by a grey crossed out node*

Before continuing our investigation of these algebras we need yet another definition. A Lie (super)algebra is called *freely generated* or *free* if there are no imposed relations on the Lie bracket, other than (graded) antisymmetry and the (graded) Jacobi identity. One implication of the definition of a free Lie algebra is that it is always infinite dimensional, with the exception when there is only one generator.

8.2 Decomposition

The generators of the Borcherds superalgebra, $\mathscr{B}$, are the Chevalley generators, h_i, e_i, f_i, associated to the roots β_i for the underlying algebra, $\mathfrak{g}$, together with three new generators, h_0, e_0, f_0, of which e_0 and f_0 obey fermionic statistics. The new generators we associate to the new root β_0. We combine these generators as h_I, e_J, f_J. The algebra is then generated by h_I, e_J, f_J modulo the Chevalley relations

$$[h_I, e_J] = B_{IJ}, \quad [h_I, f_J] = -B_{IJ}, \quad [e_I, f_J\} = \delta_{IJ} h_J, \tag{8.1}$$

and the Serre relations

$$(ad_{e_I})^{1-B_{IJ}} e_J = (ad_{f_I})^{1-B_{IJ}} e_J = 0, \quad I \neq J.$$

We can use the graded Jacobi identity to find that

$$\{e_0, [e_0, e_j]\} = 0, \ \forall j.$$

Thus e_0 can only act once, which is expected as it is fermionic. This tells us B_{i0} is either 0 or -1.

The extension of an odd null root induces a $\mathbb{Z}$-grading of the Borcherds superalgebra. $\mathscr{B}$ can then be decomposed as

$$\mathscr{B} = \bigoplus_{p \in \mathbb{Z}} \mathscr{B}_p,$$

where

$$\mathscr{B}_p = \{b \in \mathscr{B},\ [h, b] = \langle p\beta_0 + \alpha, b\rangle b,\ \alpha \in \mathfrak{h}^* \text{ and } \forall h \in \mathscr{H}\}, \tag{8.2}$$

and $\mathscr{H}$, is the Cartan subalgebra $\{h_0, \mathfrak{h}\}$ Observe here that (8.2) implies $\mathfrak{g} \subset \mathscr{B}_0$, as α is any root in the dual vector space $\mathfrak{h}^*$. The integer p here denotes, for $p > 0$, the number of e_0 present in an element in $\mathscr{B}_p$. For $p < 0$ it denotes the number of f_0 present. Thus for odd p the subspace $\mathscr{B}_p$ consist of fermionic elements and bosonic for even p.

The decomposition implies

$$[\mathscr{B}_p, \mathscr{B}_q\} \subseteq \mathscr{B}_{p+q},$$

and in particular

$$[\mathscr{B}_0, \mathscr{B}_p\} \subseteq \mathscr{B}_p. \tag{8.3}$$

Equation (8.3) tells us that $\mathscr{B}_p$ defines a representation module of $\mathscr{B}_0$ under the adjoint action. As $\mathfrak{g} \subset \mathscr{B}_0$ we have that the $\mathbb{Z}$-grading defines a sequence of representations of $\mathfrak{g}$. We denote the representation at level p with R_p. The representations R_p and R_{-p} are conjugate to each other as elements in the two representations will have eigenvalues with opposite signs under the action of h_i.

Some interesting observations can now be made. We can see from eq. (8.1) that e_0 defines a lowest weight state as it is annihilated by all f_i. The representation, R_1, built from e_0 can be used to construct the representation R_2. This is not hard to see as $\{\mathscr{B}_1, \mathscr{B}_1\} \subseteq \mathscr{B}_2$. By defining a basis E_M of $\mathscr{B}_1$, where the index M agrees with the dimension of the representation R_1 we can write an element in $\mathscr{B}_2$ as

$$\{E_M, E_N\} = E_{MN} \in \mathscr{B}_2. \tag{8.4}$$

However, we must be careful here. As $\{e_0, e_0\} = 0$, this means that not all representations from the symmetric product $\vee^2 R_1$ of (8.4) will survive. As the largest irreducible representation in a tensor product is the one defined by the sum of the lowest (or highest) weights there is quite a lot that is not present at level 2. The representation generated by $\{e_0, e_0\}$ is denoted $R_2^{\perp}$ and is the complementary representation to the actual representation R_2 in the symmetric product

$$\vee^2 R_1 = R_2 \oplus R_2^{\perp}.$$

Thus we can rewrite the Serre relations in a more covariant way as

$$\{E_M, E_N\}\Big|_{R_2^{\perp}} = 0,$$

where we have indicated that we only evaluate the representation $R_2^{\perp}$.

8.2.1 Partition function for a Borcherds superalgebra

How can we write down the partition function for a Borcherds superalgebra? First we must define what we mean by the partition function. Following the discussion in Chapter 7 we will write the partition function as that of the universal enveloping algebra. We know that at each level, p, we have a set of objects in a representation R_p. For odd p these are fermionic (as they contain an odd number of e_0) and the opposite for even p. The partition functions for level p is therefore

$$\mathcal{Z}_{U(\mathscr{B}_p)}(t) = (1 - t^p)^{-(-)^p R_p}.$$

We will now focus on the subalgebra spanned by the positive levels $\mathscr{B}_+ = \bigoplus_{n=1} \mathscr{B}_n$. Just as in Chapter 7, the full partition function for $\mathscr{B}_+$ will be the product of the partition functions at each level.

$$\mathcal{Z}_{U(\mathscr{B}_+)}(t) = \bigotimes_{p=1}^{\infty} (1 - t^p)^{-(-1)^p R_p}.$$

8.3 Duality to $D = 10$ pure spinors

Consider the extension of the D_5 Dynkin diagram with an odd null root on the spinorial node. The diagram is depicted in fig. 8.2.

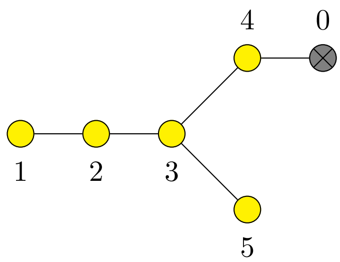

Figure 8.2: *Dynkin diagram of D_5 extended with an odd null root in the split tail, and with the nodes numbered. The odd root is indicated by the cross over the node*

This extension will define a Borcherds superalgebra which is infinite dimensional and has the decomposition shown in fig. 8.3. The odd levels are fermionic and are therefore accompanied by a minus sign.

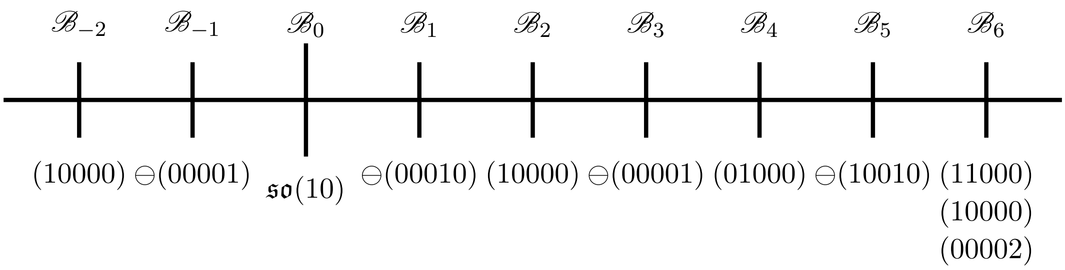

Figure 8.3: *Representations of the first levels in the Borcherds superalgebra constructed from $D_5 \simeq \mathfrak{so}(10)$*

The sequence of representations in fig. 8.3 for positive levels is exactly the sequence for the representations for the ghost for ghosts used to construct the partition function for the pure spinor in sec. 5.2.2, but with the statistics shifted. From the discussion in Chapter 7 we have hence found the algebra dual to pure spinors in 10-dimensions. Each level in the Borcherds superalgebra 8.2 correspond to the ghosts of the 10-dimensional pure spinor constraint but with statistics changed. The partition functions are each others inverses.

$$\mathcal{Z}_{\lambda_{10}}(t) \otimes \mathcal{Z}_{U(\mathscr{B}_+)}(t) = \mathbf{1}. \tag{8.5}$$

A significant implication of (8.5) is presented in the following theorem.

Theorem 3. *The subalgebra $\mathscr{B}_{n\geq 3} = \bigoplus_{n=3}^{\infty} \mathscr{B}_n \subset \mathscr{B}_+$ is freely generated by the super-Yang-Mills multiplet.*

Proof. Taking a closer look at (8.5) and writing it out in product form we have

$$\begin{aligned} \mathcal{Z}_{\lambda_{10}}(t) \otimes \mathcal{Z}_{U(\mathscr{B}_+)}(t) =& (1-t)^{-(00010)}(1-t^2)^{(10000)}(1-t^3)^{-(00001)} \cdots \\ & \otimes (1-t)^{(00010)}(1-t^2)^{-(10000)}(1-t^3)^{(00001)} \cdots = \mathbf{1}. \end{aligned} \tag{8.6}$$

By canceling the partition functions of the two spinors and using eq. (5.10) we can rewrite (8.6) as

$$\mathbf{1} = \mathcal{P}_0(t) \otimes (1-t^2)^{-(10000)} \otimes \mathcal{Z}_{U(\mathscr{B}_{n\geq 3})}(t).$$

Using Theorem 1 we can write this as

$$\mathbf{1} = (1 - \mathcal{P}_{SYM}(t)) \otimes \mathcal{Z}_{U(\mathscr{B}_{n\geq 3})}(t).$$

Which means that the partition function for the Borcherds superalgebra from level 3 must take the form

$$\mathcal{Z}_{U(\mathscr{B}_{n\geq 3})}(t) = \frac{1}{1-\mathcal{P}_{SYM}} = \mathbf{1} \oplus \mathcal{P}_{SYM} \oplus \otimes^2 \mathcal{P}_{SYM} \oplus \cdots \tag{8.7}$$

Which, as described in Chapter 7 is the form of an algebra freely generated by the super-Yang-Mills multiplet. Before we can conclude that the algebra actually is freely generated we must clarify something. Because we have a consistent grading of the partition function of the super-Yang-Mills multiplet there will be no risk of cancellations between representations in eq. (8.7). Because of this all representations will be present in $\mathcal{Z}_{U(\mathscr{B}_{n\geq 3}}(t)$ which concludes the proof. □

Theorem 3 demonstrates the remarkable connection we have between supermultiplets, pure spinors and superalgebras. Not only are the pure spinors dual to a Lie superalgebra but the super-Yang-Mills multiplet in some sense connect the two.

Chapter 9

L_∞ superalgebras and $D = 11$ supergravity

We will in this chapter investigate the dual algebra to the BRST treatment of the pure spinor constraint in 11-dimensions. We found in Chapter 5 that the partition function for a pure spinor in 11 dimensions contain the on-shell linearised supergravity multiplet. But we also noticed that the set of ghosts contain holes, we have no cohomology at ghost number 3 and 7. In Chapter 8 we found that the algebra dual to the pure spinor constraint in 10 dimensions was freely generated from level 3 by the super-Yang-Mills multiplet. We will here find that the corresponding algebra no longer is a Lie superalgebra but in fact have an L_∞ structure, which is characterised by having brackets of more than two elements.

9.1 L_∞ algebras

Before proceeding we will make a short introduction to L_∞ algebras and their relation to the BRST formalism. In fact L_∞ algebras can be defined entirely from a BRST perspective.

Consider a set $\mathcal{C}$ containing all the ghosts of a BRST formulation, including ghosts for ghosts and arbitrary functions of them. We can write an arbitrary element as $C = C(C_1, \cdots)$ where C_i is a ghosts with ghost number i. An example of possible unwanted cohomologies killed off by the ghosts are

$$\begin{aligned}
[\Omega, C_1\} =& 0, \\
[\Omega, C_2\} =& C_1 C_1, \\
[\Omega, C_4\} =& C_2 C_1 C_1, \\
[\Omega, C_5\} =& C_4 C_1 + C_2 C_2 C_1, \\
[\Omega, C_6\} =& C_5 C_1 + C_4 C_2, \\
[\Omega, C_8\} =& C_6 C_1 C_1 + C_5 C_2 C_1 + C_4 C_4 + C_2 C_2 C_2 C_2 \\
& \vdots
\end{aligned} \tag{9.1}$$

The general form of the BRST action on any object $C \in \mathcal{C}$ can then be written as

$$[\Omega, C] = \sum_{n=1} [\![C^n]\!], \tag{9.2}$$

where we introduced the bracket $[\![\ \cdot\ ]\!]$ which is a graded symmetric bracket. That is, bosons commute with bosons and fermions, whilst fermions and fermions anticommute, i.e. the same relations as ordinary multiplication. The form in (9.2) is a short hand notation for

$$[\![C^n]\!] = [\![\underbrace{C, C, \ldots, C}_{n}]\!].$$

That we have killed off cohomology which is not just quadratic in the ghosts implies that the dual algebra can no longer be just a Lie superalgebra. The higher order terms in the ghost picture implies higher order multilinear brackets in the algebra picture. Algebras of this kind are called L_∞ algebras. If we for example only have killed cohomology at most quartic in the ghosts we would have an L_4 algebra.

The bracket $[\![\ \cdot\ ]\!]$ should roughly be thought of as the bracket structure of the dual L_∞ algebra. Interpreting the ghosts as elements in a coalgebra as discussed in Chapter 7 we see that we must extend the coproduct to not only include bilinear forms but higher multilinear ones as well. In the example above we will for example have a 3-bracket and a 4-bracket structure. The coproducts will then be on the form

$$\begin{aligned} d_2 D^{*2} &= D^{*1} \wedge D^{*1} \\ (d_2 + d_3) D^{*5} &= D^{*4} \wedge D^{*1} + D^{*2} \wedge D^{*2} \wedge D^{*1} \\ d_4 D^{*8} &= D^{*2} \wedge D^{*2} \wedge D^{*2} \wedge D^{*2}. \end{aligned}$$

This introduction of L_∞ algebras is somewhat superficial, but for our intents and purposes this definition, of algebras with multilinear bracket structures, is sufficient. There is however one complication we need to address, which comes from the change in statistics when going from the ghost picture to the algebra picture. Consider the symmetry properties of the ghost picture bracket

$$[\![B, B']\!] = [\![B', B]\!], \quad [\![F, B]\!] = [\![B, F]\!], \quad [\![F, F']\!] = -[\![F', F]\!],$$

where F is a fermion and B a boson. When moving over to the algebra picture we must shift statistic. When we then try to construct the brackets on that side, which are graded antisymmetric we run into an inconsistency. The symmetry properties of a bracket which is graded antiymmetric will be

$$[B, B'] = -[B', B], \quad [F, B] = -[B, F], \quad \{F, F'\} = \{F', F\}$$

The transition from the ghost picture to the algebra picture is consistent for the brackets with 2 fermions or 2 bosons. But for the brackets with one of each we have a predicament

$$\overbrace{[\![F, B]\!] = [\![B, F]\!]}^{\textit{Ghost Picture}} \leftrightarrow \overbrace{[B, F] = -[F, B]}^{\textit{Algebra Picture}}.$$

When we transition from ghost picture to the algebra picture, how will we know in which order we define the brackets of the algebra? We cannot just take any cohomology term and go to the algebra picture without treating this sign ambiguity consistently. There is a remedy to this problem and is treated in detail in [35].

This concludes our superficial definition of L_∞ algebras. Let us now apply this to our 11-dimensional pure spinors and supergravity.

9.2 The dual algebra of $D = 11$ pure spinors

We have laid the ground work for trying to interpret the dual algebra of 11-dimensional pure spinors. We will denote this algebra $\mathscr{A}$. We know from Thm. 2 that the partition function of a pure spinor can be written as

$$\mathcal{Z}_{\lambda_{11}}(t) = \Big((00000) \ominus \mathcal{P}_{\overline{SG}}(t)\Big)(1-t)^{-(00001)}(1-t^2)^{(10000)} \tag{9.3}$$

The dual algebra will be the algebra which has a partition function inverse to (9.3),

$$\mathcal{Z}_{\lambda_{11}}(t) \otimes \mathcal{Z}_{U(\mathscr{A})} = \mathbf{1}.$$

Before diving further into this we must determine the set of ghosts needed in order kill of all cohomology in the BRST treatment of the pure spinor constraint.

9.3 Ghost picture of $D = 11$ pure spinors

We will here construct the BRST operator for the pure spinor constraint up to order 8. The procedure is simple,

- Find the unwanted cohomology
- Introduce a ghost which kills that cohomology
- Repeat until no more unwanted cohomology can be found.

This is discussed in a bit more detail in Chapter 3. We invoke a different grading here however. As previously we have that $gh(\lambda) = 1$, We also set $gh(s) = 0$). Note that this is not consistent with the ghost and antighost grading of [18]. This does not matter as the gradings are just numbers that can be chosen somewhat arbitrarily. Below we list the transformations of the ghosts up to order 8. The right hand side of the equations symbolises the unwanted cohomology which we kill off.

$$
\begin{aligned}
s\Big(C_2^a\Big) &= \lambda\gamma^a\lambda,\\
s\Big(C_{4a}\Big) &= C_2^b\lambda\gamma_{ba}\lambda,\\
s\Big(C_{5\alpha}\Big) &= C_{4a}(\gamma^a\lambda)_\alpha + \frac{1}{2}C_2^aC_2^b(\gamma_{ab}\lambda)_\alpha,\\
s\Big(C_6^{[ab]}\Big) &= C_{5\alpha}(\lambda\gamma^{ab})^\alpha + 2C_4^{[a}C_2^{b]},\\
s\Big(C_{8_1}\Big) &= C_6^{[ab]}\lambda\gamma_{ab}\lambda + 10C_{5\alpha}C_2^a(\lambda\gamma_a)^\alpha + 6C_4^aC_{4a},\\
s\Big(C_{8_2}^{\tilde{(ab)}}\Big) &= C_6^{c(a}\lambda\gamma^{b)}{}_c\lambda + \frac{\eta^{ab}}{11}C_6^{cd}\lambda\gamma_{cd}\lambda + C_{5\alpha}C_2^{(a}(\lambda\gamma^{b)})^\alpha\\
&\quad - \frac{\eta^{ab}}{11}C_{5\alpha}C_2^c(\lambda\gamma_c)^\alpha - \frac{1}{2}C_4^{(a}C_4^{b)} + \frac{\eta^{ab}}{22}C_4^cC_{4c},\\
s\Big(C_{8_3}^{[abcd]}\Big) &= C_6^{[ab}\lambda\gamma^{bc]}\lambda + \frac{1}{24}C_6^{ef}\lambda\gamma^{abcd}{}_{ef}\lambda - \frac{2}{3}C_{5\alpha}C_2^{[a}(\lambda\gamma^{bcd]})^\alpha\\
&\quad - \frac{1}{12}C_{5\alpha}C_2^e(\lambda\gamma^{abcd}{}_e)^\alpha + \frac{1}{2}C_2^{[a}C_2^bC_2^cC_2^{d]}.
\end{aligned}
\tag{9.4}
$$

The coefficients between terms were in most cases determined by two sets of equations; one linear and one involving γ-matrices. The solutions for these must agree and thus Fierz identities for the γ-equations was required. For this the software *GAMMA*, [36] was useful. The identities were checked with tracing by γ-matrices. The identities used were

$$
\begin{aligned}
\lambda\gamma^a\lambda\lambda\gamma_{ab}\lambda =&0,\\
\lambda\gamma_a\lambda(\lambda\gamma^{ab})^\alpha =&-\lambda\gamma_{ab}\lambda(\lambda\gamma^a)^\alpha,\\
(\lambda\gamma^{c(a})^\alpha\lambda\gamma_c{}^{b)}\lambda - \lambda\gamma^{(a}\lambda(\lambda\gamma^{b)})^\alpha =&+\frac{\eta^{ab}}{11}\Big((\lambda\gamma^{cd})^\alpha\lambda\gamma_{cd}\lambda - \lambda\gamma^c\lambda(\lambda\gamma_c)^\alpha\Big),\\
\lambda\gamma^{[ab}\lambda(\lambda\gamma^{cd]})^\alpha + \frac{1}{24}\lambda\gamma^{abcd}{}_{ef}\lambda(\lambda\gamma^{ef})^\alpha =&\frac{2}{3}\lambda\gamma^{[a}\lambda(\lambda\gamma^{bcd]})^\alpha + \frac{1}{12}\lambda\gamma^e\lambda(\lambda\gamma^{abcd}{}_e)^\alpha.
\end{aligned}
$$

Note that we do not have any cohomology at order 3 or 7, which is in agreement with the list of ghost in eq. (5.15). We can write the BRST operator in a schematic form by changing the notation for λ to C_1 (as it has ghost number 1) and ignore constants and γ-matrices. Notice that by writing it schematically we get exactly the cohomology from the example in eq. (9.1). There are now two distinctions that differ from the case of 10-dimensional pure spinors. The first one is the holes at ghost number 3 and 7, this does not happen in the 10-dimensional case. There we have cohomology at every level. The second distinction is that we have terms with more than 2 C:s We have quite a lot of CCC-terms and even one $CCCC$-term in the BRST action on the ghosts in eq. (9.4). The implication of the terms with more than two C:s is that the dual algebra now will be at least an L_4 algebra i.e. an algebra with at least a 3- and a 4-bracket. As the BRST action on the ghosts could be identified with the coproduct of a Lie coalgebra acting on the coalgebra elements,

$$sC = CC \leftrightarrow dE = E \wedge E,$$

we must have that the terms with more than two C:s are identified with a more general coproduct $d_n : \mathscr{A} \to \underbrace{\mathscr{A} \times \cdots \times \mathscr{A}}_{n}$. This was also discussed in Chapter 7.

9.4 Algebra picture of $D = 11$ pure spinors

We now investigate the algebra picture of this duality. We will determine the bracket structure of the algebra $\mathscr{A}$ in a schematic way up to level 8. We first present the coalgebra and coproduct structure that we have. This will essentially just be a reformulation of eq. (9.1). We denote the algebra element at level i by D_{iM}, where M is an appropriate index for the $\mathfrak{so}(11)$ representation. The coalgebra structure is

$$\begin{aligned}
d_2 D_1^{*\alpha} =& 0 \\
d_2 D_2^{*a} =& f_{\alpha\beta}{}^{a} D_1^{*\alpha} \wedge D_1^{*\beta} \\
d_3 D_4^{*a} =& f_{bc\alpha}{}^{a} D_2^{*b} \wedge D_2^{*c} \wedge D_1^{*\alpha} \\
(d_2 + d_3) D_5^{*\alpha} =& f_{a\beta}{}^{\alpha} D_4^{*a} \wedge D_1^{*\beta} + f_{ab\beta}{}^{\alpha} D_2^{*a} \wedge D_2^{*b} \wedge D_1^{*\beta} \\
d_2 D_6^{*[ab]} =& f_{\alpha\beta}{}^{[ab]} D_5^{*\alpha} D_1^{*\beta} + f_{cd}{}^{[ab]} D_4^{*c} \wedge D_2^{*d} \\
(d_2 + d_3) D_8^{*(ab)} =& f_{[cd]\alpha\beta}{}^{(ab)} D_6^{*[cd]} \wedge D_1^{*\alpha} \wedge D_1^{*\beta} + f_{\alpha c\beta}{}^{(ab)} D_5^{*\alpha} \wedge D_2^{*c} \wedge D_1^{*\beta} + f_{cd}{}^{(ab)} D_4^{*c} D_4^{*d} \\
(d_3 + d_4) D_8^{*[abcd]} =& f_{[ef]\alpha\beta}{}^{[abcd]} D_6^{*[ef]} \wedge D_1^{*\alpha} \wedge D_1^{*\beta} + f_{\alpha e\beta}{}^{[abcd]} D_5^{*\alpha} \wedge D_2^{*e} \wedge D_1^{*\beta} + \\
& + f_{efgh}{}^{[abcd]} D_2^{*e} \wedge D_2^{*f} \wedge D_2^{*g} \wedge D_2^{*h} \\
& \vdots
\end{aligned} \tag{9.5}$$

We have not written out the exact structure constants due to the sign discrepancy described in sec. 9.1. The wedges are to be thought of as graded antisymmetric. From eq. (9.5) we can now schematically write down the bracket structure of $\mathscr{A}$

$$\begin{aligned}
&\{D_1, D_1\} \sim D_2 \\
&[D_2, D_1, D_1] \sim D_4 \\
&[D_4, D_1] \sim [D_2, D_2, D_1] \sim D_5 \\
&[D_5, D_1] \sim [D_4, D_2] \sim D_6 \\
&[D_6, D_1, D_1] \sim [D_5, D_2, D_1] \sim \{D_4, D_4\} \sim [D_2, D_2, D_2, D_2] \sim D_8
\end{aligned} \tag{9.6}$$

We see here that we have a 2-bracket, 3-bracket and 4-bracket structure in $\mathscr{A}$. There may be higher order bracket present higher up in the algebra. However, the higher order brackets were introduced to be able to "jump over" the holes in the algebra. There will be no more holes higher up in the algebra and it is therefore unexpected that higher order brackets will arise.

9.4.1 Jacobi identities

As we now have an algebra with higher order brackets we can ask ourselves if there are any restrictions on these brackets. Are there any kind of generalised Jacobi identities?

Well yes, there must be. Due to the nilpotency of the BRST operator and by that the coproduct we must have Jacobi identities. In Chapter 7 we showed that the nilpotency of the coproduct is equivalent to the Jacobi identity for Lie superalgebras. We must now generalise this. If we consider an arbitrary BRST operator like the one in eq. (9.2) the coproduct equivalent to this will be a sum of coproducts of all orders

$$\tilde{d} = \sum_{n=1} d_n.$$

and the Jacobi identity can be written as

$$\tilde{d}\tilde{d} = d_1 d_1 + d_1 d_2 + d_2 d_1 + d_2 d_2 + \cdots = 0.$$

As $d_n : \mathscr{A} \to \underbrace{\mathscr{A} \times \cdots \times \mathscr{A}}_{n}$ we must have that terms $d_m d_n$ where $m + n = p$ all must be zero together. In our case we do not have any 1-brackets corresponding to d_1, the Jacobi identities then takes the form

$$\begin{aligned}
&(d_2 d_2)D = 0 \\
&(d_2 d_3 + d_3 d_2)D = 0 \\
&(d_3 d_3 + d_2 d_4 + d_4 d_2)D = 0 \\
&\qquad\qquad \vdots
\end{aligned} \tag{9.7}$$

We can determine the explicit formulation in terms of the structure constants by writing out $d_m d_n D^M$ for appropriate combinations of m and n. This is a pretty straight forward calculations however there are a few subtleties one need to keep in mind. The first one is that d_m is fermionic/odd with respect to form degree when m is even. That is

$$d_m(D^A \wedge D^B) = D^A \wedge d_m D^B - (-)^m d_m D^A \wedge D^B.$$

This is because d_m has form degree $m - 1$. Further more, when performing the calculation one will find signs (coming from moving indices) accompanied by structure constants of the form

$$f_{ABCD}{}^E (-)^{M(A+B+C+D+E)}.$$

Because of the structure constant we must have that $A + B + C + D \equiv E \mod 2$ as they must obey the boson/fermion multiplication rules i.e. $Fermion \times Fermion = Boson$ etc. This then implies that $A + B + C + D + E \equiv 0 \mod 2$. Dealing with these subtleties one will get the generalised Jacobi identities

$$\begin{aligned}
&f_{[AB}{}^M f_{CD)}{}^B = 0, \\
&3 f_{[ABC}{}^M f_{DE)}{}^C + 2 f_{[AB}{}^M f_{CDE)}{}^B = 0, \\
&4 f_{[ABCD}{}^M f_{EF)}{}^D + 3 f_{[ABC}{}^M f_{DEF)}{}^C + 2 f_{[AB}{}^M f_{CDEF)}{}^B = 0.
\end{aligned}$$

9.4.2 Interpreting the bracket structure

What can we now say about the structure of $\mathscr{A}$? The first thing we can determine is that level 1 and 2 form a small subalgebra under the 2-bracket. This subalgebra is the supersymmetry algebra. We have from the first line in (9.4) that level 1 are spinors and 2 are vectors. From eqs. (9.5) and (9.4) we can determine the structure constant ${f_{\alpha\beta}}^a$ and we get

$$d_2 D^{*a} = \gamma^a_{\alpha\beta} D^{*\alpha} \vee D^{*\beta} \iff \{D_\alpha, D_\beta\} = \gamma^a_{\alpha\beta} D_a.$$

Thus level 1 and 2 generate the 11-dimensional supersymmetry algebra.

To get higher up in the algebra we must introduce the 3-bracket to get to level 4. Thus the 3-bracket glues together the supersymmetry algebra with the rest. At level 4, 5, and 6 we have a small global supermultiplet, consisting of the ghost corresponding to translations, supersymmetry, and rotations. It is a supermultiplet as from eq. (9.6) we see it is a representation of supersymmetry under the 2-bracket. Level 8 and upwards will consist of tensor products of levels 4, 5 and 6 which also will be representations of supersymmetry. Moreover the physical fields in the supergravity multiplet will also be present in the algebra, these will also form a representation of supersymmetry.

To get past the hole at level 7 we must use the 3-bracket and in one case even a 4-bracket. Note that the 4-bracket is introduced to cancel cohomology in the (00010) representation which is the same as the 4-form field strength. Analogous to the SYM case we will have that the supergravity multiplet will be in the algebra. This will be shown explicitly in the next section. The 4-bracket is thus needed to enter the supergravity multiplet. Which also will be a representation of the supersymmetry algebra under the 2-bracket. The purpose of the higher order brackets are now clear; they are needed to jump over the holes, but they are also needed enter or ”kick-start” the two supermultiplets.

9.4.3 Freely generated by supergravity

The previous sections discussed the bracket structure of the algebra. Here we will now focus on the generators of the algebra. We will see that it may be freely generated by the supergravity multiplet. We will do the same manipulation as we did for Thm. 3 in Chapter 8, where we found that the Borcherds superalgebra $\mathscr{B}_+$ was freely generated from level 3 by the super-Yang-Mills multiplet.

The partition function for our 11-dimensional pure spinor is the inverse of the partition function for the dual algebra $\mathscr{A} = \bigoplus_{n=1}^{\infty} \mathscr{A}_n$.

$$\begin{aligned} \mathcal{Z}_{\lambda_{11}}(t) \otimes \mathcal{Z}_{U(\mathscr{A})}(t) =& (1-t)^{-(00001)}(1-t^2)^{(10000)}(1-t^4)^{-(10000)} \cdots \\ & \otimes (1-t)^{(00001)}(1-t^2)^{-(10000)}(1-t^4)^{(10000)} \cdots = \mathbf{1}. \end{aligned}$$

By canceling the partition functions of the two spinors and using eq. (5.13) we get

$$\mathbf{1} = \mathcal{G}_0(t) \otimes (1 - t^2)^{-(10000)} \otimes \mathcal{Z}_{U(\mathscr{A}_{n\geq 4})}(t).$$

Using Thm. 2 we can write this as

$$\mathbf{1} = \Big(\mathbf{1} \ominus \mathcal{P}_{\overline{SG}}(t)\Big) \otimes \mathcal{Z}_{U(\mathscr{A}_{n\geq 4})}(t).$$

Which mean that the partition function for the the algebra from level 4 must take the form

$$\mathcal{Z}_{U(\mathscr{A}_{n\geq 4})}(t) = \frac{1}{1 - \mathcal{P}_{\overline{SG}}(t)} = \mathbf{1} \oplus \mathcal{P}_{\overline{SG}} \oplus \otimes^2 \mathcal{P}_{\overline{SG}} \oplus \cdots . \tag{9.8}$$

Here we now see explicitly that the supergravity multiplet resides in the algebra $\mathscr{A}$. Does eq. (9.8) mean, analogous to the SYM case, that the algebra $\mathscr{A}$ is freely generated from level 4 by the supergravity multiplet, including the ghosts? This question is not as easy to answer as in the SYM case. This is due to the ghost multiplet with "wrong" statistics at level 4, 5 and 6. As they have wrong statistics, i.e. are fermionic at even orders in t, it is possible that tensor products of these or tensor products of these with the physical fields could produce representations which will coincide with representations at the same order in t but with opposite statistics. They would then cancel and not show up in the partition function for the algebra. If the algebra would be freely generated by the supergravity multiplet it would mean that all tensor products of the multiplet, including the ghosts, are present in the algebra. Thus we could in principle have both fermionic and bosonic states in the same representation in the algebra. But we would not be able to see this from the partition function.

Note that the form of eq. (9.8) is restricted to a partition function for a freely generated algebra under a bilinear product, eq. (5.6) is defined from arbitrary tensor products of an object. Thus it is only the 2-bracket which would be freely generated. That the 2-bracket is freely generated by the supergravity multiplet would make the case pretty similar to the super-Yang-Mills. The 3- and 4-brackets are then simply needed to tie everything together.

We have not been able to prove this to all orders. However, we can show that to the linear order in the physical fields the ghosts cannot cancel them. This is because the representations (10000), (00001) and (01000) are small relative the levels they are on. To see this let's say we want to cancel any of the physical fields with tensor products of the ghosts, the tensor products at lowest possible order in t that could cancel the physical fields are

$$\begin{aligned}
&1)\ (n0010) \leftrightarrow \otimes^{n-2}(10000) \otimes^3 (01000),\ n \geq 2\\
&2)\ (n1001) \leftrightarrow \otimes^{n}(10000) \otimes (01000) \otimes (00001)\\
&3)\ (n2000) \leftrightarrow \otimes^{n}(10000) \otimes^2 (01000).
\end{aligned}$$

These are determined simply by looking at the various tensor products, and, using LIE [29], deduced by which tensor product is at the lowest order in t. The representations ($n0010$), ($n1001$), and ($n2000$) are at order t^{8+2n}, t^{9+2n} and t^{10+2n}, respectively. The tensor products of the ghosts are respectively at

1) $t^{4(n-2)+18}$
2) t^{4n+11}
3) t^{4n+12}

Which are at higher orders in t than their respective physical representation they try to cancel. We could still have cancellations at linear orders in the physical fields by say creating $((n+1)1001)$ by $(10000)\otimes(n1001)$, however this product would be at t^{2n+9+4} which is higher than $t^{2(n+1)+9}$. This is once again a consequence of level 4, 5 and 6 being to small. Thus at linear order in the physical fields we cannot have any cancellations. This also proves that there can be no holes higher up in the algebra as in order to have holes in particular we must have that representations linear in the physical fields are canceled.

Unfortunately we did not have time to finish formulating a proof of this, we instead conclude by postulating a conjecture.

Conjecture. *The algebra $\mathscr{A}_{n\geq 4} = \bigoplus_{n=4}^{\infty} \subset \mathscr{A}$ is freely generated by the supergravity multiplet, including the global ghosts for translations, rotations, and supersymmetry under the 2-bracket.*

Chapter 10

Discussion and outlook

We have investigated the correspondence connecting pure spinors, superalgebras and supermultiplets. Our focus has been on 10- and 11-dimensional pure spinors, where the corresponding supermultiplets were linearised $D = 10$ super-Yang-Mills and $D = 11$ supergravity, respectively. We found that the dual algebra in the 10 dimensional case was a Borcherds superalgebra obtained by extending the D_5 diagram with an odd null root on the spinorial node. We also showed that this algebra is freely generated from level 3 by the super-Yang-Mills multiplet. In the 11-dimensional case things were a bit more complicated. We found that the algebra was no longer just a Lie superalgebra but had also at least an L_4 structure.

Can we summarise the algebraic structure of $\mathscr{A}$? Let us first discuss the Lie superalgebra structure, i.e. the 2-bracket. Levels 1 and 2 generate the supersymmetry algebra, and assuming the conjecture 9.4.3 is accurate we have that levels 4 and up are freely generated by the supergravity multiplet, including the ghost zero modes. We also found that level 4, 5, and 6 comprise of a small finite dimensional supermulitplet, i.e it is a representation of supersymmetry. Level 8 and upwards will also consist of a representation of supersymmetry.

Now what about the 3- and 4-brackets? As discussed previously the 3-bracket is needed to jump over the holes at levels 3 and 7, it is in some way the glue of the algebra itself. It keeps the supersymmetry algebra connected with the small global ghost multiplet at level 4, 5 and 6. But it is also needed, together with the 4-bracket, to enter into the physical supergravity multiplet at level 8. The 4-bracket is explicitly needed to enter into the physical supergravity multiplet, but it is also needed to skip over the hole at level 7. This calls for an interesting thought; ordinary gauge theories such as (super-)Yang-Mills, Einstein gravity etc. are all ”generated” by commuting covariant derivatives. But here in the supergravity case we have that the 4-form is generated by ”commuting” four elements of level 2, which we know are covariant derivatives in a flat basis. Does this mean that supergravity is fundamentally different from ordinary gauge theories? In order to further investigate this one would need to calculate higher orders of the cohomology. This should supply us with insight of how the field strength for the gravitino and the curvature tensor are generated.

Another question one could ask is if the 3- and 4-brackets branch off and form their own structures. From the Jacobi identities in eq. (9.7) we see that the 3- and 4-bracket must together satisfy an identity, but it could just as well be that the 3-bracket will satisfy a generalised Jacobi by itself. Further more, if we assume that no higher brackets will arise we must have that the 4-bracket satisfy a Jacobi identity by itself. This means that the physical supergravity multiplet would branch of and disconnect from the tensor products of the ghosts.

Areas of development are firstly to prove or disprove the conjecture 9.4.3, and find out whether or not the algebra in Chapter 9 is freely generated by the supergravity multiplet. Secondly we need to determine the exact algebraic structure, i.e. are there higher brackets or does it stop at 4? Also determining, explicitly the true functions of the 3- and 4-brackets is of importance.

It could also be interesting to investigate what distinguishes pure spinors from arbitrary objects under constraints. What is the explanation behind why we find supermultiplets, and in particular why $D = 10$ SYM and $D = 11$ SG? Is it possible to find other supermultiplets for example by examining the exceptional groups?

Appendix A

Lie Groups and Lie Algebras

This section will cover the most essential parts of one of the most used mathematical objects in physics, Lie algebras. We will define what Lie algebras are and how they are classified.

A.1 Lie groups

A *group* $(G, *)$ is a set equipped with a binary operation $*$ which obeys the following axioms

- (Closed) If $a, b \in G$ then $a * b \in G$.
- (Associativity) $(a * b) * c = a * (b * c)$.
- (Identity) There exist an element $e \in G$ such that $\forall a \in G$ $e * a = a * e = a$.
- (Inverses) For each $a \in G$ there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

Examples of groups are the set of integers, $(\mathbb{Z}, +)$, under addition, the set of real numbers $(\mathbb{R}, \times)$, under multiplication. There are infinitely many more examples.

A *Lie group* is a group that is also a smooth manifold. Examples of Lie groups are

- $GL(n)$; The group of invertible linear transformations in n dimensions
- $SO(n)$; The group of rotations in n dimensions
- $U(n)$; The group of unitary transformations in n dimensions.

The conditions for being a Lie group are often nonlinear and awkward to work with. For example consider $SO(n)$, we can represent this with $n \times n$ matrices. The condition on these matrices for being in $SO(n)$ will then be $RR^{\top} = I$. This is an nonlinear constraint. How we can treat this is instead by going to the corresponding Lie algebra, $\mathfrak{g}$, of the group. The Lie algebra is the tangent space to the Lie group at the identity element.

What are the conditions on the objects in the Lie algebra? To answer this we write a group element as $R = I + \epsilon\mathcal{J}$, where ϵ is an infinitesimal parameter and $\mathcal{J} \in \mathfrak{g}$ is referred to as the generator of the Lie group. As the tangent space is a vector space addition is perfectly justified. The constraint on elements in the Lie algebra will then be, to the linear order

$$(I + A)(I + A)^\top = I + A^\top + A = I \implies A = -A^\top.$$

I.e. matrices in the Lie algebra $\mathfrak{so}(n)$ are antisymmetric. This condition is much easier to work with. This demonstrates the convenience to working in the Lie algebra instead.

One might now however ask: This only deals with infinitesimal group transformations what about global ones? Well, here is what the true beauty of Lie theory comes in. Suppose you want to perform a rotation θ degrees around some axis. There is no difference between rotating the complete θ degrees in one go or to split it up and rotate $\frac{\theta}{2}$ twice. Thus what if we split up the rotation $R(\theta)$ in N pieces and performed them successively? Further more what if we let the number of splits go to infinity? If we do this we can rewrite the rotation $R(\theta)$ as

$$R(\theta) = \lim_{N\to\infty} \Big(R(\frac{\theta}{N})\Big)^N = \lim_{N\to\infty} \Big(I + \frac{\theta\mathcal{J}}{N}\Big)^N = e^{\theta\mathcal{J}}.$$

This shows the elegance of the Lie algebra. We can obtain a global transformation from an infinitesimal one by using the exponential function. This is truly remarkable. We will now leave Lie groups and move our focus to Lie algebras.

A.2 Lie algebras

A lie algebra, $\mathfrak{g}$, is a vector space equipped with a antisymmetric bilinear product, $[\cdot, \cdot]$, called the Lie bracket. The Lie bracket must also satisfy the Jacobi identity, that is

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0 \quad \forall A, B, C \in \mathfrak{g}.$$

A Lie algebra, $\mathfrak{g}$ is called *simple* if it is nonabelian and contains no nonzero proper ideals. A Lie algebra is called *semisimple* if it is a direct sum of simple Lie algebras.

A representation of a Lie algebra is a homomorphism from the algebra to the space of linear operators, $\mathcal{L}(V)$, acting on some vector space V. The homomorphism preserves the Lie bracket and takes it to the commutator of those linear operators.

$$\rho : \mathfrak{g} \to \mathcal{L}(V)$$
$$\rho([g, f]) = [\rho(g), \rho(f)].$$

The dimension of the representation is the dimension of the module (the vector space) on which the elements of the algebra acts. A special representation is the *Adjoint representation*, where the module is the algebra itself. It is therefore a $\dim(\mathfrak{g})$-dimensional representation.

The action under the adjoint representation is realised by the Lie-bracket.

$$\begin{aligned} ad : \mathfrak{g} &\to End(\mathfrak{g}) \\ (ad\ g)\ f = ad_g f &= [g, f]. \end{aligned}$$

Given a semisimple Lie algebra, $\mathfrak{g}$, it is possible to write the generators in a so called Chevalley-Serre basis, such that

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, E_j] &= A_{ij} E_j, \\ [H_i, F_j] &= -A_{ij} F_j, \\ [E_i, F_j] &= \delta_{ij} H_i, \end{aligned} \tag{A.1}$$

Where A_{ij} is the elements of the *Cartan Matrix* and $i = 1, \cdots, r$. The H_i:s together form the *Cartan subalgebra*, $\mathfrak{h}$; an algebra of mutually commuting objects. The dimension of the Cartan algebra is called the rank, r, of $\mathfrak{g}$. The remaining generators are constructed by successive commutation of the E:s and F:s subject to the *Serre-relations*

$$\begin{aligned} (ad\ E_i)^{1-A_{ij}} E_j &= 0, \\ (ad\ F_i)^{1-A_{ij}} F_j &= 0. \end{aligned}$$

Consider a representation of $\mathfrak{g}$. To not make the calculations to messy we will use the same symbols for the matrices as for the elements of the algebra. As the elements in $\mathfrak{h}$, all commute they are simultaneously diagonalisable and there exist a common eigenbasis for all $h \in \mathfrak{h}$. We can decompose the module, V, into the eigenvectors of $\mathfrak{h}$ such that

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda, \tag{A.2}$$

Where Λ is the set of eigenvalues and $H_i V_\lambda = \lambda_i V_\lambda$. The eigenvalues are also called weights. E_i and F_i raises and lowers the eigenvalues, respectively, by the elements in the Cartan matrix

$$\begin{aligned} H_i(E_j V_\lambda) &= E_j H_i V_\lambda + [H_i, E_j] V_\lambda = (\lambda_i + A_{ij}) V_\lambda \\ H_i(F_j V_\lambda) &= F_j H_i V_\lambda + [H_i, F_j] V_\lambda = (\lambda_i - A_{ij}) V_\lambda. \end{aligned}$$

Thus for a finite dimensional representation we can start from a highest weight state, $|\lambda\rangle$ annihilated by all raising operators

$$E_i \, |\lambda\rangle = 0 \quad \forall i,$$

and create the entire representation module by acting with the lowering operators.

A.2.1 Tensor products of representations

The tensor product of two representations of a Lie algebra is also a representation. Consider a representation $R(\mu)$ of an semisimple Lie algebra, characterised by its highest weight μ. A tensor product of this representation with itself will always contain the irreducible representation characterised by 2μ. Consider a highest weight state $|\lambda\rangle$ with highest weight μ. Then we have for the generators of the Cartan algebra that

$$h_i |\lambda\rangle = \mu_i |\lambda\rangle .$$

The representation formed by the tensor product

$$R(\mu) \otimes R(\mu), \qquad h_i \to h_i \otimes I + I \otimes h_i$$

will now have a highest weight state, $|\lambda\rangle \otimes |\lambda\rangle$, with eigenvalues

$$(h_i \otimes I + I \otimes h_i) |\lambda\rangle \otimes |\lambda\rangle = \mu_i |\lambda\rangle \otimes |\lambda\rangle + |\lambda\rangle \otimes \mu_i |\lambda\rangle = 2\mu_i |\lambda\rangle \otimes |\lambda\rangle .$$

Thus the sum of the weights will always be present in the tensor product of multiple representations. This is also the biggest irreducible representation of the tensor product.

A.2.2 Roots and the dual Cartan algebra

In the adjoint representation, the eigenvalues, or weights are called roots. The roots are elements of the dual Cartan algebra, $\mathfrak{h}^*$. To see this consider a basis for $\mathfrak{h}$, $\{H_i\}$, and a basis for $\mathfrak{h}^*$, $\{a^i\}$. The canonical product

$$\langle \cdot \, , \, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to K,$$

to a field K, is such that for $h = h^i H_i$ and $\alpha = \alpha_i a^i$ then $\langle h, \alpha \rangle = h^i \alpha_i$. Similarly for an element, $E^\alpha \in \mathfrak{g}$, such that

$$[H_i, E^\alpha] = \alpha_i E^\alpha,$$

We have that

$$[h, E^\alpha] = h^i \alpha_i E^\alpha = \langle h, \alpha \rangle E^\alpha. \tag{A.3}$$

Thus indeed the roots are elements of the dual space $\mathfrak{h}^*$. In particular the roots associated with the E_i are called simple roots, these form a basis for $\mathfrak{h}^*$. Thus comparing the Chevally relations in eq. (A.1) with eq. (A.3) we see the elements in the Cartan matrix are the components of the simple roots. The full Lie algebra thus, analogously to eq. (A.2), has a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \langle h, \alpha \rangle g \ \forall h \in \mathfrak{h}\}$. We can also define a metric on the cartan algebra using the adjoint representation. The Cartan-Killing form

$$K(H_i, H_j) = Tr(ad\ H_i \cdot ad\ H_j) = K_{ij}$$

can be used to raise and lower indices (relate elements in $\mathfrak{h}^*$ to elements in $\mathfrak{h}$). We denote H^γ the element in $\mathfrak{h}$ associated with the root $\gamma \in \mathfrak{h}^*$. The canonical product between a root, $\gamma \in \mathfrak{h}^*$ and an element, $h \in \mathfrak{h}$, becomes

$$\langle h, \gamma \rangle = K(h, H^\gamma).$$

The roots can thus be realised as $\gamma = K(H^\gamma, \cdot)$. The scalar product between two elements in $\mathfrak{h}$ can now be written

$$(h, f) = K(h, f) = h^i f^j K(H_i, H_j) = h^i f^j K_{ij}.$$

Reversely, the inverse of K_{ij} can be used as the metric on $\mathfrak{h}^*$. Using the Cartan-Killing-form, the elements in the Cartan Matrix can be written as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

We can now normalise the roots and define the *coroot*

$$\alpha^\vee \equiv \frac{2\alpha}{(\alpha, \alpha)}.$$

This simplifies the Cartan matrix to

$$A_{ij} = (\alpha_i, \alpha_j^\vee).$$

The simple coroots, associated with the simple roots, are of course a basis over $\mathfrak{h}^*$. Moreover the elements in $\mathfrak{h}$ associated with the coroots are the Cartan generators,

$$\alpha_i^\vee = K(H_i, \cdot).$$

A.2.3 Fundamental weights, Dynkin labels and Dynkin diagrams

With the Cartan matrix we can easily classify and characterise the simple Lie Algebras by so called *Dynkin diagrams*. By associating each simple root with a node in a diagram and then drawing $|A_{ij}|$ lines between the two nodes i, and j. If the roots have different lengths an arrow, pointing towards the shorter root, will indicates this. The classification of the finite-dimensional simple Lie algebras then boils down to the Dynkin diagram shown in fig. A.1

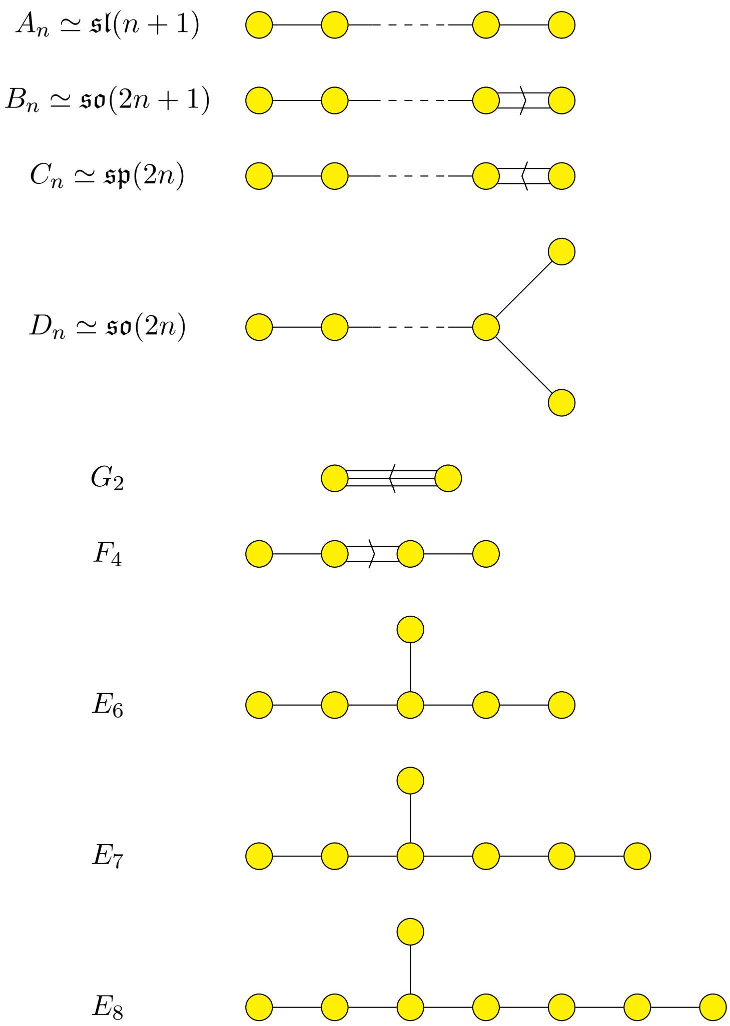

Figure A.1: *Dynkin diagram for all simple Lie algebras*

As stated previously, the representation module and by that the representation of the Lie algebra can be constructed from a highest (or lowest) weight state, similar to the spectrum of the harmonic oscillator in quantum mechanics.

We can define a new basis for $\mathfrak{h}^*$, dual to the simple coroots, by

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}.$$

The ω_j are called the *fundamental weights*. With this new basis the expansion coefficients of a weight, $\lambda = \sum_1^r \lambda_i \omega_j \in \mathfrak{h}^*$ become integers and are called the *Dynkin labels* of the representation. A representation, $R(\lambda)$ is often defined from its highest weight, λ in terms if its Dynkin labels

$$R(\lambda) \leftrightarrow \lambda = (\lambda_1 \lambda_2 \cdots \lambda_r).$$

Consider a highest weight state $|\lambda\rangle$ such that

$$\begin{aligned}
&E_i\,|\lambda\rangle = 0 \ \ \forall i \\
&H_i\,|\lambda\rangle = \langle H_i, \lambda\rangle\,|\lambda\rangle = \lambda^j(\alpha_i^\vee, \omega_j)\,|\lambda\rangle = \lambda_i\,|\lambda\rangle
\end{aligned}$$

To now construct the representation based on this weight state we successively act with the lowering operators F_i to create our module. However we must make sure that λ is in fact the highest weight. This is done by only acting with the lowering operator corresponding to the simple root α_i parallel to the fundamental weight which have a positive Dynkin label [37]. For clarification, let the highest weight be represented by the Dynkin labels

$$\lambda = (\lambda_1 \cdots \lambda_r).$$

If, say $\lambda_i > 0$ then we create the sequence of states $(F_i)^j\,|\lambda\rangle$, for $j = 1, \cdots, \lambda_i$ which have the corresponding weights (if there are more than one $\lambda_i > 0$ the same procedure is carried out for each of them)

$$\begin{aligned}
|\lambda\rangle &\leftrightarrow \lambda \\
F_i\,|\lambda\rangle &\leftrightarrow \lambda - \alpha_i \\
(F_i)^2\,|\lambda\rangle &\leftrightarrow \lambda - 2\alpha_i \\
&\vdots \\
(F_i)^{\lambda_i}\,|\lambda\rangle &\leftrightarrow \lambda - \lambda_i\alpha_i.
\end{aligned}$$

This procedure is carried out until no weights with positive Dynkin labels appear. If one wishes to create the module from a lowest weight state then the method is reversed. The lowest weights are simply the negative of the highest weights. One then then builds the module through the exact same procedure but instead of acting with lowering operators one acts with raising operators on the states of which the weight components are negative instead of positive. An example in the case of $D_5 \simeq \mathfrak{so}(10)$ will be carried out in the following section.

A.2.3.1 Constructing the spinor representation of $\mathfrak{so}(10)$

We will construct the chiral spinor representation of $\mathfrak{so}(10) \simeq D_5$. We will do this from the lowest weight instead of the highest. We follow the procedure described in the end of the last section. The result is depicted in figure A.2, we see here that we get 16 states. Which is expected as the spinor representation in even dimension is reducible and breaks down into two chiralities. Dynkin labels are often presented from the highest weights and not the lowest. In table A.1 we find some other Dynkin labels of frequently used representations of $\mathfrak{so}(10)$

Dynkin label	Representation
(00000)	Singlet
(10000)	Vector
(01000)	2-Form
(00100)	3-form
(00010)	Cospinor
(000010)	Spinor
(00011)	4-Form
(00020),(00002)	5-Form (self-dual)
(20000)	Traceless Symmetric Tensor
(10001)	γ-Traceless Vectorspinor

Table A.1: *List of usual Dynkin labels and their index structure*

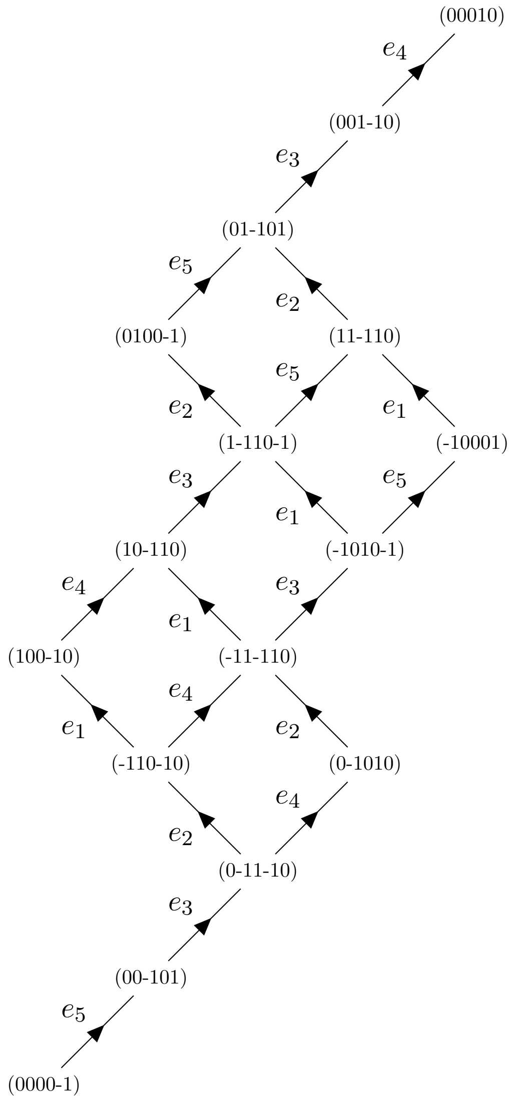

Figure A.2: *Construction of the spinor representation of* $\mathfrak{so}(10)$ *from the lowest weight* (0000–1).

Appendix B

Calculation of the dimension of $(000n0)$ in D_5

We will here determine $dim((000n0))$, for D_5. From Chapter 3 we found that we can write functions of constrained objects as functions of the free objects modulo the constraints.

We now apply this method on pure spinors in 10-dimensions to find their partition function. The constraints in (3.1) are now

$$\phi(\lambda) = \lambda\gamma^a\lambda = 0.$$

Now, instead of examining functions depending on pure spinors we can investigate arbitrary functions of spinors modulo the constraint. Thus two functions are considered equivalent if and only if

$$c'(\lambda) - c(\lambda) = \lambda\gamma^a\lambda a_a(\lambda).$$

Thus the variation of a function

$$\Delta c(\lambda) = \lambda\gamma^a\lambda a_a(\lambda),$$

does not yield any new information. Expanding $c(\lambda)$ in a power series it is of course of interest to find what irreducible representations are allowed at each order of λ. By determining the degrees of freedom at $\mathcal{O}(\lambda^n)$ we get an idea of what kind of irreducible representations are allowed. To determine this we we must however subtract the degrees of freedom of $a_a(\lambda)$ at $\mathcal{O}\Big(\lambda^{n-2}\Big)$.

We are, unfortunately, not done here though. Because of the Fierz identity in 10 dimensions

$$\gamma_{a(\alpha\beta}\gamma^a_{\gamma)\rho} \equiv 0$$

we find that the transformation

$$\Delta a_a(\lambda) = \lambda\gamma_a\chi(\lambda)$$

of the gauge field $a_a(\lambda)$ is unseen by $\Delta c(\lambda)$. There is thus a reducibility in the transformation. We must compensate for this by adding the degrees of freedom χ^β at $\mathcal{O}\Big(\lambda^{n-3}\Big)$. With this same procedure we find multiple levels of reducibility, all the transformations are summarised in eq. (B.1)

$$
\begin{aligned}
\Delta c(\lambda) &= \lambda\gamma^a\lambda a_a(\lambda) \\
\Delta a_a(\lambda) &= \lambda\gamma_a\chi(\lambda) \\
\Delta\chi^\beta(\lambda) &= \lambda^\beta\lambda\tilde{\chi}(\lambda) - \frac{1}{2}\lambda\gamma^b\lambda(\gamma_b\tilde{\chi})^\beta \\
\Delta\tilde{\chi}_\beta(\lambda) &= (\lambda\gamma^a)_\beta\tilde{a}_a(\lambda) \\
\Delta\tilde{a}_a(\lambda) &= \lambda\gamma_a\lambda\tilde{c}(\lambda)
\end{aligned}
\tag{B.1}
$$

All transformations are obtained to make sure that the transformation of a field will not be seen by the transformation of a field higher up in the list (B.1) (or in simpler terms, making sure that $\Delta^2 = 0$).

Now back to determine the degrees of freedom. As previously discussed, to get back the degrees of freedom at $\mathcal{O}(\lambda^n)$ we must subtract the gauge freedom generated by the constraint. But then we found that the constraint had reducibility and we then have to add something to compensate. The tensor product of n bosonic spinors is a completely symmetric tensor with n indices, which have

$$\binom{15+n}{n}$$

degrees of freedom. For each λ in the transformation in eq. (B.1) we must reduce the symmetric tensor by 1 index. But we must also keep in mind all free indices in the transformations, e.g. a_a will contribute with a tensor with one vector index and $n-2$ symmetric spinor indices. Performing this calculation we get that the degrees of freedom at $\mathcal{O}(\lambda^n)$ is

$$
\begin{aligned}
dim((000n0)) =& \\
=&\binom{15+n}{n} - 10\binom{15+n-2}{n-2} + 16\binom{15+n-3}{n-3} - \\
&-16\binom{15+n-5}{n-5} + 10\binom{15+n-6}{n-6} - \binom{15+n-8}{n-8} = \\
=&\frac{(1+n)(2+n)(3+n)^2(4+n)^2(5+n)^2(6+n)(7+n)}{302400} = \\
=&\frac{1}{10}\frac{(7+n)!}{7!n!}\frac{(5+n)!}{(n+2)!3!} = \frac{1}{10}\binom{7+n}{n}\binom{5+n}{3}.
\end{aligned}
$$

Appendix C

Complement of proofs of theorems 1 and 2

We will here present the explicit calculations for the proofs of Theorems 1 and 2. The calculations both involve calculating

$$\mathcal{P}(t) \otimes \bigoplus_{n=1}^{\infty} \vee^k (10000) t^{2k},$$

for some representation polynomial $\mathcal{P}$. It is thus convenient to calculate $\vee^k(10000)$. The following lemma is valid for Dynkin labels of both D_5 and B_5.

Lemma 1. *The symmetric product of vectors,*

$$\vee^k (10000) = \begin{cases} \bigoplus_{i=0}^{\frac{k}{2}} (2i0000), \ k = 2n \\ \bigoplus_{i=0}^{\frac{k-1}{2}} ((2i+1)0000), \ k = 2n+1 \end{cases}$$

Proof. As eigenvalues add under tensor products we know that the representation $(k0000)$, must be present. This is a traceless symmetric tensor with k indices (traceless in the sense of contraction with η^{ab}). Thus the symmetric tensor product can be decomposed as ($\sim$ denoting tracelessness)

$$M_{(a_1 \cdots a_k)} = \tilde{M}_{(a_1 \cdots a_k)} + \eta_{a_1 a_2} M_{(a_3 \cdots a_k)},$$

where now $M_{(a_3 \cdots a_k)}$ is a symmetric tensor with $k-2$ indices, which can be decomposed analogously. This continues until we reach a scalar for even k and a vector for odd k. $\square$

C.1 Calculation of Theorem 1

Before proving Thm. 1 we can simplify our calculations by presenting the following Lemma.

Lemma 2. *The tensor products ($k \geq 1$)*

$$(k0000) \otimes (00010) = (k0010) \oplus ((k-1)0001)$$
$$(k0000) \otimes (00001) = (k0001) \oplus ((k-1)0010)$$

Proof. Consider the tensor-spinor $M_{(a_1 \cdots a_k)}{}^{\alpha}$, traceless in its k vector indices. The only way we can decompose this is by using the γ-matrices. These are the only invariant objects connecting the spinor indices with the vector indices. Thus we can create a γ-traceless tensor-spinor by $\gamma^{a_1}{}_{\alpha\beta} M_{(a_1 \cdots a_k)}{}^{\beta} = 0$. Thus $M_{(a_1 \cdots a_k}{}^{\alpha}$ decomposes as ($\sim$ now denoting γ-tracelessness)

$$M_{(a_1 \cdots a_k}{}^{\alpha} = \tilde{M}_{(a_1 \cdots a_k)}{}^{\alpha} + \gamma_{a_1}{}^{\alpha\beta} M_{(a_1 \cdots a_k)\beta}$$

□

We can now go on to prove (5.19). The first levels levels up to t^8 have been done explicitly in eq. (5.20). The higher orders, $t^{n>8}$, when all terms in $\mathcal{P}_0(t)$ are present, can be calculated order by order. We prove this for odd and even powers of t separately. For odd orders, t^{2n+1} we can break the calculation for when n is even or odd. If n is odd we have using Lemmas 1 and 2

$$\begin{aligned} t^{2n+1}\Big(&\vee^{n-1} (10000) \otimes (00001) \ominus \vee^{n-2}(10000) \otimes (00010)\Big) = \\ &= \bigoplus_{i=0}^{\frac{n-1}{2}} ((2i)0000) \otimes (00001) \ominus \bigoplus_{i=0}^{\frac{n-3}{2}} ((2i+1)0000) \otimes (00010) = \\ &= \bigoplus_{i=0}^{\frac{n-1}{2}} ((2i)0001) \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} ((2i-1)0010) \ominus \bigoplus_{i=0}^{\frac{n-3}{2}} ((2i)0001) \ominus \bigoplus_{i=1}^{\frac{n-3}{2}} ((2i+1)0010) = \\ &= ((n-1)0001) \end{aligned}$$

The case for even n is similar. For even powers of t we instead have

$$t^{2n}\Big(\vee^{n} (10000) \ominus \vee^{n-1} \otimes (10000) \oplus \vee^{n-3}(10000) \otimes (10000) \ominus \vee^{n-4}(10000)\Big). \quad \text{(C.1)}$$

As the symmetric product of vectors decompose as in lemma 1 we have that the first and last term in eq. (C.1) will cancel, leaving the representations

$$(n0000) \oplus ((n-2)0000). \quad \text{(C.2)}$$

The same is true for the second and third term. But here we also have an extra tensor product with a vector. As the tensor product

$$(k0000) \otimes (10000) = ((k+1)0000) \oplus ((k-1)10000) \oplus ((k-1)0000),$$

we have that the only surviving terms from the second and third term in eq. (C.1) are

$$\ominus (n0000) \ominus ((n-2)1000) \ominus ((n-2)0000)$$

Which together with (C.2) leaves $\ominus((n-2)1000)$. Combining now even and odd powers in t and the terms in eq. (5.20) We can write

$$\mathcal{P}_0(t) \otimes (1-t^2)^{-(10000)} = (00000) \oplus t^3 \bigoplus_{k=0}^{\infty}(k0001)t^{2k} \ominus t^4 \bigoplus_{k=0}^{\infty}(k1000)t^{2k}.$$

Which is precisely on the form of eq. (5.18).

C.2 Calculation of Theorem 2

First off, Lemma 1 is valid for Dynkin labels of B_5. To calculate (5.26) to all orders we must first calculate the tensor products

$$\begin{aligned}
(k0000) \otimes (10000) =& ((k+1)0000) \oplus ((k-1)1000) \oplus ((k-1)0000) \\
(k0000) \otimes (01000) =& ((k-2)1000) \oplus ((k-1)0100) \oplus (k0000) \oplus (k1000) \\
(k0000) \otimes (20000) =& ((k+2)0000) \oplus (k1000) \oplus (k0000) \oplus \\
& ((k-2)0000) \oplus ((k-2)1000) \oplus ((k-2)2000) \\
(k0000) \otimes (00100) =& ((k-2)0100) \oplus ((k-1)0010) \oplus ((k-1)1000) \oplus (k0100) \\
(k0000) \otimes (00001) =& ((k-1)0001) \oplus (k0001) \\
(k0000) \otimes (10001) =& ((k-2)0001) \oplus ((k-2)1001) \oplus ((k-1)0001) \oplus ((k-1)1001) \oplus \\
& \oplus (k0001) \oplus ((k+1)0001)
\end{aligned} \tag{C.3}$$

These are all proved similar to Lemmas 1 and 2 and will not be presented here.

We can now go on to prove (5.25). The first levels levels up to t^{16} have been done explicitly in eq. (5.26). For orders, $t^{m\geq 16}$, all terms in $\mathcal{G}_0(t)$ will be present and we can do the calculation for odd and even orders of t separately. We start by odd orders, at t^{2n+1} and $n \geq 8$ we will have

$$\begin{aligned}
odd =& \ominus \vee^{n-2}(10000) \otimes (00001) \oplus \vee^{n-3}(10000) \otimes \Big((10001) \oplus (00001)\Big) \\
& \ominus \vee^{n-4}(10000) \otimes \Big((10001) \oplus (00001)\Big) \oplus \vee^{n-5}(10000) \otimes (00001).
\end{aligned}$$

We only do the case where n is odd here, the case for n even is identical albeit with minor corrections to the sums limits.. We have, at t^{2n+1}

$$\underbrace{\Big(\ominus\bigoplus_{i=0}^{\frac{n-3}{2}}((2i+1)0000\oplus\bigoplus_{i=0}^{\frac{n-3}{2}}((2i)0000\ominus\bigoplus_{i=0}^{\frac{n-5}{2}}((2i+1)0000)\oplus\bigoplus_{i=0}^{\frac{n-5}{2}}((2i)0000)\Big)\otimes(00001)}_{=\boldsymbol{I}}$$
$$\oplus\underbrace{\Big(\bigoplus_{i=0}^{\frac{n-3}{2}}((2i)0000\ominus\bigoplus_{i=0}^{\frac{n-5}{2}}((2i+1)0000)\Big)\otimes(10001)}_{=\boldsymbol{II}}.$$

We now look closer to the two terms $\boldsymbol{I}$ and $\boldsymbol{II}$. In $\boldsymbol{I}$ we can combine the first and second sums into one. The same can be done for the third and fourth sums. In $\boldsymbol{II}$ we can also combine the sums into one.

$$\begin{aligned}
\boldsymbol{I} =&\Big(\bigoplus_{k=0}^{n-2}(-1)^k(k0000)\oplus\bigoplus_{k=0}^{n-4}(-1)^k(k0000)\Big)\otimes(00001)=\\
=&2\times(00001)\oplus\bigoplus_{k=1}^{n-2}(-1)^k\Big(((k-1)0000)\oplus(k0001)\Big)\\
&\oplus\bigoplus_{k=1}^{n-4}(-1)^k\Big(((k-1)0000)\oplus(k0001)\Big)=\\
=&\bigoplus_{k=1}^{n-3}(-1)^{k+1}(k0001)\oplus\bigoplus_{k=1}^{n-2}(-1)^k(k0001)\\
&\oplus\bigoplus_{k=1}^{n-5}(-1)^{k+1}(k0001)\oplus\bigoplus_{k=1}^{n-4}(-1)^k(k0001)=\\
=&\ominus((n-2)0001)\ominus((n-4)0001)
\end{aligned}\tag{C.4}$$

For $\boldsymbol{II}$ we instead have

$$\begin{aligned}
\boldsymbol{II} =&\bigoplus_{k=0}^{n-3}(-1)^k(k0000)\otimes(10001)=\\
=&(10001)\ominus(00001)\ominus(01001)\ominus(10001)\ominus(20001)\\
&\oplus\bigoplus_{k=2}^{n-3}(-1)^k\Big(((k-2)0001)\oplus((k-2)1001)\oplus((k-1)0001)\\
&\qquad\oplus((k-1)1001)\oplus(k0001)\oplus((k+1)0001)\Big)=\\
=&(10001)\ominus(00001)\ominus(01001)\ominus(10001)\ominus(20001)\\
&\oplus\bigoplus_{k=0}^{n-5}(-1)^k\Big((k0001)\oplus(k1001)\Big)\oplus\bigoplus_{k=1}^{n-4}(-1)^{k+1}\Big((k0001)\oplus(k1001)\Big)\\
&\oplus\bigoplus_{k=2}^{n-3}(-1)^k(k0001)\oplus\bigoplus_{k=3}^{n-2}(-1)^{k+1}(k0001)=\\
=&\ominus(20001)\oplus((n-4)0001)\oplus((n-4)1001)\oplus(20001)\oplus((n-2)0001).
\end{aligned}\tag{C.5}$$

Combining eqs. (C.4) and (C.5) we are only left with

$$t^{2n+1}((n-4)1001).$$

Which is equivalent to

$$t^{2n+9}(n1001).$$

Moving on to even powers in t. At t^{2n}, and $n \geq 8$ we have

$$\begin{aligned}
even = &\underbrace{\vee^{n}(10000) \ominus \vee^{n-3}(10000) \oplus \vee^{n-5}(10000) \ominus \vee^{n-8}(10000)}_{=\boldsymbol{III}} \\
&\oplus \underbrace{\Big(\ominus \vee^{n-1}(10000) \oplus \vee^{n-2}(10000) \ominus \vee^{n-6}(10000) \oplus \vee^{n-7}(10000)\Big)}_{=\boldsymbol{IV}} \otimes (10000) \\
&\oplus \underbrace{\Big(\vee^{n-2}\,(10000) \ominus \vee^{n-6}(10000)\Big)}_{=\boldsymbol{V}} \otimes (01000) \\
&\oplus \underbrace{\Big(\ominus \vee^{n-3}(10000) \oplus \vee^{n-5}(10000)\Big)}_{=\boldsymbol{VI}} \otimes \Big((20000) \oplus (00100)\Big).
\end{aligned} \tag{C.6}$$

We again do it for even n, the odd n will be equivalent. Using Lemma 1 we can expand each term by itself and simplify.

$$\boldsymbol{III} = \bigoplus_{i=0}^{\frac{n}{2}}((2i)0000) \ominus \bigoplus_{i=0}^{\frac{n-4}{2}}((2i+1)0000) \oplus \bigoplus_{i=0}^{\frac{n-6}{2}}((2i+1)0000) \ominus \bigoplus_{i=0}^{\frac{n-8}{2}}((2i)0000) =$$

$$= \bigoplus_{i=0}^{3}((n-2i)0000) \ominus ((n-3)0000)$$

$$\boldsymbol{IV} = \ominus \bigoplus_{i=0}^{\frac{n-2}{2}}((2i+1)0000) \oplus \bigoplus_{i=0}^{\frac{n-2}{2}}((2i)0000) \ominus \bigoplus_{i=0}^{\frac{n-6}{2}}((2i)0000) \oplus \bigoplus_{i=0}^{\frac{n-8}{2}}((2i+1)0000) =$$

$$= \bigoplus_{k=1}^{5}(-1)^{k}((n-k)0000)$$

$$\boldsymbol{V} = \bigoplus_{i=0}^{\frac{n-2}{2}}((2i)0000) \ominus \bigoplus_{i=0}^{\frac{n-6}{2}}((2i)0000) = ((n-4)0000) \oplus ((n-2)0000)$$

$$\boldsymbol{VI} = \ominus \bigoplus_{i=0}^{\frac{n-4}{2}}((2i+1)0000) \oplus \bigoplus_{i=0}^{\frac{n-6}{2}}((2i+1)0000) = \ominus((n-3)0000)$$

We now need to multiply these terms with the corresponding modules in eq. (C.6). We first observe the term

$$
\begin{aligned}
&\boldsymbol{V} \otimes (01000) \oplus \boldsymbol{VI} \otimes \Big((20000) \oplus (00100)\Big) = \\
&\oplus((n-4)1000) \oplus ((n-3)0100) \oplus ((n-2)0000) \oplus ((n-2)1000) \\
&\oplus((n-6)1000) \oplus ((n-5)0100) \oplus ((n-4)0000) \oplus ((n-4)1000) \\
&\ominus\Big(((n-1)0000) \oplus ((n-3)1000) \oplus ((n-3)0000) \oplus ((n-5)0000) \oplus ((n-5)1000) \\
&\oplus((n-5)2000) \oplus ((n-5)0100) \oplus ((n-4)0010) \oplus ((n-4)1000) \oplus ((n-3)0100)\Big) = \\
&= \bigoplus_{k=2}^{6} (-1)^{k+1}((n-k+1)0000) \\
&\quad \oplus \bigoplus_{k=1}^{5} (-1)^{k-1}((n-k-1)1000) \ominus ((n-5)2000) \ominus ((n-4)0010).
\end{aligned}
$$

We used eq. (C.3) to compute the tensor products. We also have

$$
\begin{aligned}
\boldsymbol{IV} \otimes (10000) =& \bigoplus_{k=1}^{5} (-1)^k((n-k)0000) \otimes (10000) = \\
=& \bigoplus_{k=1}^{5} (-1)^k \Big((n-k+1)0000) \oplus ((n-k-1)1000) \oplus ((n-k-1)0000)\Big).
\end{aligned}
$$

Now, using eq. (C.3) again we can substitute everything, into eq. (C.6).

$$
\begin{aligned}
even =& \bigoplus_{i=0}^{3} ((n-2i)0000) \ominus ((n-3)0000) \\
& \oplus \bigoplus_{k=1}^{5} (-1)^k \Big((n-k+1)0000) \oplus ((n-k-1)1000) \oplus ((n-k-1)0000)\Big) \\
& \oplus \bigoplus_{k=2}^{6} (-1)^{k+1}((n-k+1)0000) \\
& \oplus \bigoplus_{k=1}^{5} (-1)^{k-1}((n-k-1)1000) \ominus ((n-5)2000) \ominus ((n-4)0010) = \\
=&(n0000) \oplus ((n-2)0000) \oplus ((n-4)0000) \oplus ((n-6)0000) \ominus ((n-3)0000) \\
& \oplus (n0000) \ominus ((n-5)0000) \oplus \bigoplus_{k=1}^{5} (-1)^k((n-k-1)0000) \ominus ((n-5)2000) \ominus ((n-4)0010) = \\
=& \ominus ((n-5)2000) \ominus ((n-4)0010)
\end{aligned}
$$

Thus at order t^{2n} we have the representations $\ominus((n-5)2000) \ominus ((n-4)0010)$, which can also be stated as $\ominus(n2000)$ is at order t^{10+2n} and $(n0010)$ is at order t^{8+2n}

To write everything together we combine the terms in eq. (5.26) with what we got to even an odd orders and end up at

$$\begin{aligned}\mathcal{G}_0(t) \otimes (1-t^2)^{-(10000)} =&(00000) \ominus (10000)t^4 \oplus (00001)t^5 \ominus (01000)t^6 \\ &\ominus \bigoplus_{n=0}^{\infty}(n0010)t^{2n+8} \oplus \bigoplus_{n=0}^{\infty}(k1001)t^{2n+9} \ominus \bigoplus_{n=0}^{\infty}(k2000)t^{2n+10}.\end{aligned}$$

Which is in perfect agreement with eqs. (5.22) and (5.23).

Appendix D

Introduction to supersymmetry and supergravity

This appendix was written as part of a project course in supersymmetry (SUSY) taken in the fall of 2020. It is aimed to supply the average master's level student of theoretical physics with the necessary understanding to what supersymmetry is and how it is used. The present appendix was written with the intent of being self-contained. There might hence be some overlap with the rest of the thesis, together with some slight disagreement in conventions. This will however not imply any difficult conversion.

We will begin with a short motivation and discussion on what SUSY is and why it is of relevance to study it. Then we will go on introducing the first known example of a supersymmetric field theory; the Wess-Zumino Model. We then move on to the representation theory of the supersymmetry algebra. We see that the SUSY algebra is in fact a creation/annihilation algebra, just like the one used for the harmonic oscillator of quantum mechanics. We then go on introducing the notion of superspace; an extension of spacetime which simplifies calculations and provides new insights in the theory of supersymmetry. To proceed we will expand the concept of differential forms to cover superspace and go on to study gauge theories in a covariant way. When the complete formalism is properly introduced and understood we go on to the concrete superspace derivation of $D = 10$ super-Yang-Mills (SYM). Lastly we will discuss the theory of 11-dimensional supergravity (SUGRA).

The structure and theory is based first and foremost on Appendix C in Aspman [33]. Further more Wess & Bagger [38] was also used extensively.

D.1 What is supersymmetry?

Supersymmetry is a spacetime symmetry between two kinds of particles; bosons and fermions. It is a conjectured relationship that has not yet been verified by experiments. The symmetry, if applicable to nature, relates every boson in the standard model to a so called fermion superpartner and vice versa.

The supersymmetry generators, often denoted Q, changes the spin of a particle but keeps all other quantum numbers fixed. A field theory which is invariant under supersymmetric transformations is called a supersymmetric field theory. Such a theory has an equal number of bosonic and fermionic degrees of freedom. Moreover, a supersymmetric theory, which is generally covariant under coordinate transformations is called supergravity (SUGRA). It is in fact so, which is not easy to show, that supergravity is synonymous to having local supersymmetry. In this sense supersymmetry and gravity is tightly related.

D.2 Why study supersymmetry?

Supersymmetry, except for being an intriguing theory suitable to study just to get a broader understanding of symmetries and the innovative thinking of extending theories, provides us with a lot of solutions to current problems not explained by the Standard Model. For starters supersymmetry provides an elegant solution to the naturalness problem, the question of why the Higgs mass is not in the order of the Planck scale. Current measurements states the Higgs mass at about $m_h \sim 125GeV$. This agrees with the theory at tree level. But once radiative corrections are calculated the mass diverges and at the UV cut-off the Standard Model breaks down. With a supersymmetric Standard Model, all radiative corrections (loop calculations) from fermions will cancel out the correction from the corresponding bosonic superpartners and vice versa. This offers us a way out, [39].

Further more, supersymmetry is a necessity in order formulate superstring/M-theory, the currently leading candidate for a theory of everything. This means detection of superpartners gives an experimental indication that we might be on the right track with string theory. Unfortunately the Large Hadron Collider (LHC) have not yet detected any trace of supersymmetry. Do not be disheartened by this, the lack of detection of superpartners of equal mass implies that if in fact supersymmetry is realised in Nature it must be spontaneously broken at some higher scale.

D.3 The Wess-Zumino model

We will here introduce one of the first supersymmetric models formulated; the Wess-Zumino model. The Lagrangian for the Wess-Zumino model is

$$\mathscr{L} = -\frac{1}{2}\partial_a S\partial^a S - \frac{1}{2}\partial_a P\partial^a P + \frac{i}{2}\bar{\psi}\gamma^a\partial_a\psi.$$

Where S, and P are real scalar fields, and ψ is a Dirac spinor (We will see that ψ needs to be Majorana) [33].

We let roman letters denote Lorentz vector indices whilst greek letter are used for spinors. We will work in the mostly plus convention, meaning $\eta_{ab} = diag(-1, 1, 1, 1)$. The transformations that leave this Lagrangian invariant are

$$\begin{aligned}
\delta S &= -i\bar{\varepsilon}\psi \\
\delta P &= -\bar{\varepsilon}\gamma^5\psi \\
\delta\psi &= \not{\partial}(S - i\gamma^5 P)\varepsilon \\
\delta\bar{\psi} &= -\bar{\varepsilon}\partial_a(S - i\gamma^5)\gamma^a
\end{aligned}$$

Note here that the scalars transform into the spinor and vice versa. To show that $\mathscr{L}$ is in fact invariant under these transformations we first must show that the transformations for S and P are real.

$$(\bar{\varepsilon}\psi)^\dagger = (\varepsilon^\dagger A\psi)^\dagger = \psi^\dagger A^\dagger\varepsilon = -\bar{\psi}\varepsilon.$$

As $A = -\gamma^0$ is antihermitean (see App. E.1). This does not provide us with much, because we have put no restrictions on ψ and ε yet. If we assume that they are Majorana spinors however we can do a Majorana flip, i.e. $\bar{\psi}\varepsilon = \bar{\varepsilon}\psi$, see sec. E.1.2.1 and thus see that the variation for S is real. For P we do a similar check, and use that γ^5 is hermitean. Thus the complex conjugate of the variation of P is

$$(\delta P)^* = -(\bar{\varepsilon}\gamma^5\psi)^\dagger = -\psi^\dagger\gamma^5 A^\dagger\varepsilon = -\bar{\psi}\gamma^5\varepsilon.$$

Once again if we assume Majorana spinors we can do a flip and we see that it is completely real.

The variation for $\bar{\psi}$ can be derived by

$$\begin{aligned}
\delta\bar{\psi} = \delta\psi^\dagger A = \left(\gamma^a\partial_a(S - i\gamma^5 P)\varepsilon\right)^\dagger A &= \varepsilon^\dagger\partial_a(S + i\gamma^5)\gamma^{a\dagger}A \\
= -\varepsilon^\dagger\partial_a(S + i\gamma^5)(-\gamma^0)\gamma^a &= -\bar{\varepsilon}\partial_a(S - i\gamma^5)\gamma^a.
\end{aligned}$$

Now we need to know if these transformations actually leave the Lagrangian invariant,

$$\begin{aligned}\delta\mathscr{L} =& -\partial_a S\partial^a(\delta S) - \partial_a P\partial^a(\delta P) + \frac{i}{2}\Big(\delta\bar{\psi}\partial\!\!\!/\psi + \bar{\psi}\partial\!\!\!/\delta\psi\Big) = \\ =& \Box S\delta S + \Box P\delta P - \frac{i}{2}\Big(\bar{\varepsilon}\partial_a(S - i\gamma^5 P)\gamma^a\Big)\partial\!\!\!/\psi + \frac{i}{2}\Big(\bar{\psi}\partial\!\!\!/(\partial\!\!\!/(S - i\gamma^5 P)\varepsilon)\Big) = \\ =& -i\bar{\varepsilon}\psi\Box S - \bar{\varepsilon}\gamma^5\psi\Box P + \frac{i}{2}\Big(\bar{\varepsilon}\gamma^a\gamma^b\partial_a\partial_b(S - i\gamma^5 P)\psi + \frac{i}{2}\Big(\bar{\psi}\gamma^a\gamma^b\partial_a\partial_b(S - i\gamma^5 P)\varepsilon\Big) = \\ =& i\Box S\Big(\frac{1}{2}\bar{\varepsilon}\psi + \frac{1}{2}\bar{\psi}\varepsilon - \bar{\varepsilon}\psi\Big) + \Box P\Big(\frac{1}{2}\bar{\varepsilon}\gamma^5\psi + \frac{1}{2}\bar{\psi}\gamma^5\varepsilon - \bar{\varepsilon}\gamma^5\psi\Big).\end{aligned} \tag{D.1}$$

We used here partial integration in both the second and third step. Further more we used that $\gamma^a\gamma^b = \gamma^{ab} + \eta^{ab}$. The fact that γ^{ab} is antisymmetric in a, and b, and $\partial_a\partial_b$ is symmetric implies that terms where these two multiply are zero. We can thus conclude from eq. (D.1) that if we impose the Majorana condition on our spinors and perform a Majorana flip, we get that the Lagrangian is invariant.

D.3.1 Commuting the transformations

Let us now find what kind of algebra these transformations generate,

$$\begin{aligned}[\delta_1, \delta_2]S =& \delta_1(-i\bar{\varepsilon}_2\psi) - \delta_2(-i\bar{\varepsilon}_1\psi) = \\ =& -i\bar{\varepsilon}_2\partial\!\!\!/(S - i\gamma^5 P)\varepsilon_1 + i\bar{\varepsilon}_1\partial\!\!\!/(S - i\gamma^5 P)\varepsilon_2 = 2i\bar{\varepsilon}_1\gamma^a\varepsilon_2\partial_a S.\end{aligned} \tag{D.2}$$

Where we once again used a Majorana flip, to flip the ε:s. The ε is to be thought of as parameters of the transformation. It is then natural to define, [33],

$$\delta \equiv -i\bar{\varepsilon}Q = -i\bar{Q}\varepsilon = -i\bar{\varepsilon}^\alpha Q_\alpha = -i\varepsilon_\beta C^{\beta\alpha}Q_\alpha = -i\varepsilon_\beta Q^\beta = iQ^\alpha\varepsilon_\alpha,$$

where we see Q as a generator of the symmetry. Using this definition we see that

$$\begin{aligned}(\delta_1\delta_2 - \delta_2\delta_1)S =& (-i)^2(\bar{\varepsilon}_1 Q\bar{\varepsilon}_2 Q - \bar{\varepsilon}_2 Q\bar{\varepsilon}_1 Q)S = (\bar{Q}^\alpha\varepsilon_{2\beta}\bar{\varepsilon}_1^\alpha Q_\beta - \bar{\varepsilon}_1^\alpha Q_\alpha\bar{Q}^\beta\varepsilon_{2\beta})S = \\ =& -(\bar{\varepsilon}_1^\beta\varepsilon_{2\alpha}\bar{Q}^\alpha Q_\beta + \bar{\varepsilon}_1^\alpha\varepsilon_{2\beta}Q_\alpha\bar{Q}^\beta)S = -\bar{\varepsilon}_1^\alpha\varepsilon_{2\beta}\{Q_\alpha, \bar{Q}^\beta\}S\end{aligned} \tag{D.3}$$

Comparing eqs. (D.2) and (D.3) we arrive at

$$\{Q_{\bar{\alpha}}, \bar{Q}^{\bar{\beta}}\}S = -2i(\gamma^a)_{\bar{\alpha}}{}^{\bar{\beta}}\partial_a S \tag{D.4}$$

Where the bars on the indices are to clarify that these are Dirac indices. We can thus as in App. E.1.1 go to the chiralities. This is done simply by looking at the block structure of the γ-matrices and keeping track of indices. We denote the Dirac spinor Q as $Q_{\bar{\alpha}} = \binom{q_\alpha}{\bar{q}^{\dot{\alpha}}}$. Equation (D.4) then gives

$$\begin{gathered}\{Q_{\bar{\alpha}}, \bar{Q}^{\bar{\beta}}\} = -2i(\gamma^a)_{\bar{\alpha}}{}^{\bar{\beta}}\partial_a = \\ i\begin{pmatrix}\{q_\alpha, q^\beta\} & \{q_\alpha, \bar{q}_{\dot{\beta}}\} \\ \{\bar{q}^{\dot{\alpha}}, q^\beta\} & \{\bar{q}^{\dot{\alpha}}, \bar{q}_{\dot{\beta}}\}\end{pmatrix} = -2i\begin{pmatrix}0_\alpha{}^\beta & i\sigma^a_{\alpha\dot{\beta}} \\ i(\bar{\sigma}^a)^{\dot{\alpha}\beta} & 0^{\dot{\alpha}}{}_{\dot{\beta}}\end{pmatrix}\partial_a,\end{gathered}$$

where the extra i on the left hand side comes from the Dirac conjugate $Q^{\bar{\beta}}$, as can be observed from eq. (E.3). We thus get, renaming $q_\alpha = Q_\alpha$,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = -2i\sigma^a_{\alpha\dot{\beta}}\partial_a,$$

where the indices are now Weyl two-valued indices. This is the SUSY algebra defined in [38]. This calculation was carried out with S as a test function. The calculation for P is identical. Consider now what happens when acting on the spinor ψ.

$$\begin{aligned}[\delta_1, \delta_2]\psi_\alpha =&\delta_1(\gamma^a\partial_a(S - i\gamma^5 P)\varepsilon_2)_\alpha - \delta_2(\gamma^a\partial_a(S - i\gamma^5 P)\varepsilon_1)_\alpha = \\ =&\not{\partial}\Big(- i\bar{\varepsilon}_1\psi + i\gamma^5\bar{\varepsilon}_1\gamma^5\psi)\varepsilon_2\Big)_\alpha - 1 \leftrightarrow 2 = \\ =& - i\Big[(\bar{\varepsilon}_1\partial_a\psi)\gamma^a\varepsilon_2 - (\bar{\varepsilon}_1\gamma^5\partial_a\psi)\gamma^a\gamma^5\varepsilon_2\Big]_\alpha - 1 \leftrightarrow 2.\end{aligned} \tag{D.5}$$

Nothing interesting yet, have faith. We now take a look at the first and third term. They can be written as

$$\varepsilon_{1\beta}\partial_a\psi^\beta(\gamma^a)_a{}^\sigma\varepsilon_{2\sigma} - \varepsilon_{2\beta}\partial_a\psi^\beta(\gamma^a)_a{}^\sigma\varepsilon_{1\sigma} = -2\partial_a\psi^\beta(\gamma^a)_\alpha{}^\sigma\varepsilon_{1(\beta}\varepsilon_{2\sigma)}. \tag{D.6}$$

We can now use an application of the Fierz identity App. E.1.3 and rewrite (D.6) as

$$-\frac{1}{2}\bar{\varepsilon}_1\gamma_b\varepsilon_2(\gamma^a\gamma^b\partial_a\psi)_\alpha + \frac{1}{4}\bar{\varepsilon}_1\gamma_{bc}\varepsilon_2(\gamma^a\gamma^{bc}\partial_a\psi)_\alpha.$$

We now take a look at the second and fourth term of eq. (D.5),

$$\begin{aligned}-(\bar{\varepsilon}_1\gamma^5\partial_a\psi\gamma^a\gamma^5\varepsilon_2)_\alpha - 1 \leftrightarrow 2 =&\varepsilon_{2\beta}(\gamma^5)^{\beta\sigma}\partial_a\psi_\sigma(\gamma^a\gamma^5)_\alpha{}^\delta\varepsilon_{1\delta} - 1 \leftrightarrow 2 = \\ =& - \varepsilon_{2\beta}\varepsilon_{1\delta}(\gamma^5)^{\beta\sigma}\partial_a\psi_\sigma(\gamma^a\gamma^5)_\alpha{}^\delta - 1 \leftrightarrow 2 = \\ =&2\varepsilon_{1(\beta}\varepsilon_{2\delta)}(\gamma^5)^{\beta\sigma}\partial_a\psi_\sigma(\gamma^a\gamma^5)_\alpha{}^\delta = \\ =&\frac{-1}{2}\gamma^a\gamma^c\partial_a\psi\bar{\varepsilon}_1\gamma_a\varepsilon_2 - \frac{1}{4}\gamma^a\gamma^{bc}\partial_a\psi\bar{\varepsilon}_1\gamma_{bc}\varepsilon_2.\end{aligned}$$

Eq.(D.5) now becomes

$$\begin{aligned}[\delta_1, \delta_2]\psi_\alpha = -i\Big[&- \frac{1}{2}\bar{\varepsilon}_1\gamma_b\varepsilon_2(\gamma^a\gamma^b\partial_a\psi)_\alpha + \frac{1}{4}\bar{\varepsilon}_1\gamma_{bc}\varepsilon_2(\gamma^a\gamma^{bc}\partial_a\psi)_\alpha \\ &- \frac{1}{2}\bar{\varepsilon}_1\gamma_b\varepsilon_2(\gamma^a\gamma^b\partial_a\psi)_\alpha - \frac{1}{4}\bar{\varepsilon}_1\gamma_{bc}\varepsilon_2(\gamma^a\gamma^{bc}\partial_a\psi)_\alpha\Big] = \\ =&i\gamma^a\gamma^b\partial_a\psi_\alpha\bar{\varepsilon}_1\gamma_b\varepsilon_2 = 2i\bar{\varepsilon}_1\gamma^a\varepsilon_2\partial_a\psi_\alpha - i\bar{\varepsilon}_1\gamma_b\varepsilon_2\gamma^b\not{\partial}\psi_\alpha.\end{aligned}$$

Using eq. (D.3) we see that

$$[\delta_1, \delta_2]\psi_\alpha = -\bar{\varepsilon}_1^\beta\varepsilon_{2\sigma}\{Q_\beta, Qb^\sigma\}\psi_\alpha = 2i\bar{\varepsilon}_1^\beta\varepsilon_{2\sigma}(\gamma^a)_\beta{}^\sigma\partial_a\psi_\alpha - i\bar{\varepsilon}_1^\beta\varepsilon_{2\sigma}(\gamma_b)_\beta{}^\sigma(\gamma^b\not{\partial}\psi)_\alpha.$$

We can remove the ε and write it as

$$\{Q_\alpha, \bar{Q}^\beta\}\psi_\sigma = -2i(\gamma^a)_\alpha{}^\beta\partial_a\psi_\sigma + i(\gamma^b)_\alpha{}^\beta(\gamma_b\not{\partial}\psi)_\sigma. \tag{D.7}$$

This is rather peculiar. We get the SUSY algebra back but with an extra term in shape of the massles Dirac equation for ψ. We know that the Lagrangian is invariant under these transformations. But it does not seem to be the SUSY algebra generating these transformations, unless ψ is on-shell. Does this mean that our model actually is on-shell? We shall see later, when we talk about superspace and superfields, that this is the case.

D.4 Representation theory of the supersymmetry algebra

We will now look at the different representations of the SUSY algebra. Representation is, in this context, referred to the module of the realisation of the algebra. We can generalise the SUSY algebra (D.2) to $\mathcal{N}$ supersymmetries

$$\{Q_{\alpha i}, \bar{Q}^j_{\dot{\beta}}\} = 2\sigma^a_{\alpha\dot{\beta}} P_a \delta^j_i.$$

Where $i, j = 1, \ldots, \mathcal{N}$. We have here written the algebra in momentum basis. We will investigate both the massive and the massless case.

D.4.1 Massive case

First observation to be done is that $P^2 = -m^2$ commutes with all supersymmetry generators, it is thus a Casimir operator. This means that all state generated from the SUSY algebra will have the same mass. Now if we boost to the rest frame $P_a = (-m, 0, 0, 0)$ the SUSY algebra reads

$$\{Q_{\alpha i}, \bar{Q}^j_{\beta}\} = 2m\delta_{\alpha\beta}\delta^j_i,$$

where the dotted indices are discarded due to the fact that since we have fixed the momentum to be zero in the spatial components we have lowered our symmetry to the *Little group*, which in this case is $SO(3)$. In 3 dimensions we have no chiral representations. We now see that the SUSY is really a creation/annihilation algebra, we just have to normalise the operators. Defining

$$a_{\alpha i} \equiv \frac{1}{\sqrt{2m}} Q_{\alpha i} \qquad (a^\dagger)^i_\alpha \equiv \frac{1}{\sqrt{2m}} \bar{Q}^i_\alpha.$$

These satisfy the normalised creation/annihilation algebra.

$$\{a_{\alpha i}, a^{\dagger j}_{\beta}\} = \delta_{\alpha\beta}\delta^j_i$$

This means that we can create our basis in a Hilbert space by acting with raising operators on a ground state. We first define a ground state $|\Omega\rangle$ which is annihilated by all lowering operators

$$a_{\alpha i}|\Omega\rangle = 0 \ \ \forall \alpha, i.$$

We can add an index the the ground state to indicate what spin it has from the beginning, if the ground state is a scalar at first we write $|\Omega\rangle_0$ and so on. The $a^{\dagger j}_1$ and $a^{\dagger j}_2$ will raise and lower the magnetic quantum number (i.e. the spin) with $\frac{1}{2}$, respectively. This gives $2\mathcal{N}$

raising operators (even though $a_2^{\dagger j}$ lowers the magnetic quantum number it is still a raising operator). We can now start creating our states. We will start by creating our states on the scalar ground state, to extend to different ground states one uses the rules of angular momentum addition. The rules of angular momentum addition when multiplying spin $\boldsymbol{\frac{1}{2}}$ representations are

$$\overbrace{\boldsymbol{\frac{1}{2}} \otimes \cdots \otimes \boldsymbol{\frac{1}{2}}}^{n \text{ times}} = \boldsymbol{\frac{n}{2}} + \boldsymbol{\frac{n-2}{2}} \oplus \cdots ,$$

which terminate with $\mathbf{0}$ if n is even and $\boldsymbol{\frac{1}{2}}$ if n is odd.

A typical representation of supersymmetry is depicted below.

$$\begin{aligned}
(a_\alpha^{\dagger j})^{2\mathcal{N}} &\left|\Omega\right\rangle_0 \\
&\vdots \\
(a_\alpha^{\dagger j})^{n} &\left|\Omega\right\rangle_0 \\
&\vdots \\
(a_\alpha^{\dagger j})^{2} &\left|\Omega\right\rangle_0 \\
a_\alpha^{\dagger j} &\left|\Omega\right\rangle_0 \\
&\left|\Omega\right\rangle_0 .
\end{aligned}$$

In each class of states (classes defined by the number of $a^\dagger$ acting on the ground state) there are $\binom{2\mathcal{N}}{n}$ states. The total number of states are then

$$\sum_{i=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{i} = 2^{2\mathcal{N}}.$$

Now lets do this for $\mathcal{N} = 1$, then we can discard of the j index. The four different states we have is thus

$$\begin{aligned}
&\left|\Omega\right\rangle_0 \\
a_1^\dagger &\left|\Omega\right\rangle_0 \\
a_2^\dagger &\left|\Omega\right\rangle_0 \\
a_1^\dagger a_2^\dagger &\left|\Omega\right\rangle_0 .
\end{aligned}$$

We thus have two spin 0 and one spin $\frac{1}{2}$ representations i.e. two scalars and one spinor. The complete spinor representation consist of the two states $a_1^\dagger \left|\Omega\right\rangle_0$ and $a_2^\dagger \left|\Omega\right\rangle_0$. Thus a representation module of supersymmetry is a vector of representations of $\mathfrak{so}$. The different states for one supersymmetry is summarised in table D.1.

Spin	$\|\Omega\rangle_0$	$\|\Omega\rangle_{\frac{1}{2}}$	$\|\Omega\rangle_1$	$\|\Omega\rangle_{\frac{3}{2}}$
0	2	1		
$\frac{1}{2}$	1	2	1	
1		1	2	1
$\frac{3}{2}$			1	2
2				1

Table D.1: *Table over the different states in the massive supersymmetric multiplet for $\mathcal{N} = 1$ supersymmetries, the last three columns are created from the first by "angular momentum addition"*

Next let's do the case of $\mathcal{N} = 4$, We will now omit the spinorial index, and write the states in a more schematic kind of manor. The different states are written out in eq. (D.8).

$$\begin{aligned}
|\Omega\rangle_0 &= \binom{\mathcal{N}}{0} \times 0 \\
a^{\dagger j}\,|\Omega\rangle_0 &= \binom{\mathcal{N}}{1}\binom{\mathcal{N}}{0} \times \pm\frac{1}{2} \\
(a^{\dagger j})^2\,|\Omega\rangle_0 &= \binom{\mathcal{N}}{2}\binom{\mathcal{N}}{0} \times \pm 1 + \binom{\mathcal{N}}{1}\binom{\mathcal{N}}{1} \times 0 \\
(a^{\dagger j})^3\,|\Omega\rangle_0 &= \binom{\mathcal{N}}{3}\binom{\mathcal{N}}{0} \times \pm\frac{3}{2} + \binom{\mathcal{N}}{2}\binom{\mathcal{N}}{1} \times \pm\frac{1}{2} \\
(a^{\dagger j})^4\,|\Omega\rangle_0 &= \binom{\mathcal{N}}{4}\binom{\mathcal{N}}{0} \times \pm 2 + \binom{\mathcal{N}}{3}\binom{\mathcal{N}}{1} \times \pm 1 + \binom{\mathcal{N}}{2}\binom{\mathcal{N}}{2} \times 0.
\end{aligned} \tag{D.8}$$

The left hand sides of (D.8) represents all possible combinations of raising operators to the written power on the ground state, i.e. all combinations of $i = 1, \ldots, 4$ and all combinations of $\alpha = 1, 2$. On the right hand sides we write the number of states times the spin quantum number for a state on that form. When reading the expression with a + the first binomial factor is the multiplicity of $a_1^{\dagger j}$ the second binomial factor is that of $a_2^{\dagger j}$ (as these raises and lowers the spin quantum number, respectively). The two binomial factors changes place when reading the the expression with the - instead. When we have created all states we must collect them into spin-representations, just as in the case for $\mathcal{N} = 1$. To do this we must know which spin quantum numbers are present in each representation. The spin quantum numbers for the different spin representations are

$$\begin{aligned}
&\mathbf{0} : 0, \\
&\frac{\mathbf{1}}{\mathbf{2}} : \frac{1}{2},\ \frac{-1}{2}, \\
&\mathbf{1} : 1,\ 0,\ -1, \\
&\frac{\mathbf{3}}{\mathbf{2}} : \frac{3}{2},\ \frac{1}{2},\ 0,\ \frac{-1}{2},\ \frac{-3}{2}, \\
&\mathbf{2} : 2,\ 1,\ 0,\ -1,\ -2.
\end{aligned}$$

To get the complete spin-representations we just start filling them up, for each class of states, starting with the highest spin first.

$\mathcal{N} = 4$ is in fact the highest number of supersymmetries we can have as this generates spin 2 states. There are no states with higher than spin 2 in Nature. The content of the multiplet for $\mathcal{N} = 4$ is presented in table D.2

Spin	$\lvert\Omega\rangle_0$
0	42
$\frac{1}{2}$	48
1	27
$\frac{3}{2}$	8
2	1

Table D.2: *Table over the different states in the massive supersymmetric multiplet for $\mathcal{N} = 4$ supersymmetries.*

D.4.2 Massless case

In the massless case we can boost to a light like reference frame $P_a = (-E, 0, 0, E)$, here the Little group is now $SO(2)$. In this frame the algebra becomes

$$\{Q_{\alpha j}, \bar{Q}^i_{\dot{\beta}}\} = 2\Big[\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \Big]_{\alpha\beta} \delta^i_j .$$

This tells us that we only have one kind of raising and lowering operator per supersymmetry, as the $\alpha = 2$ index anticommutes with all combinations of Q. We can once again normalise the algebra to satisfy the correct equations for the creation/annihilation algebra. Let

$$a_j \equiv \frac{1}{2\sqrt{E}} Q_{1j}, \quad \text{and} \quad a^{\dagger i} \equiv \frac{1}{2\sqrt{E}} \bar{Q}^i_{\dot{1}} .$$

Now we can create our basis for our hilbert space. We do it in the same manor as in the massive case, but now we do it in terms of helicity. Define a ground state with determined minimal helicity, $\Omega_{\bar{\lambda}}$, so that it gets annihilated by all the a_is.

Now we can just start building. We start from the lowest helicity and climb our way up, for each time we add a $a^\dagger$ the helicity increases by $\frac{1}{2}$. The multiplicity of the state $(a^{\dagger i})^n \Omega_{\bar{\lambda}}$ is $\binom{\mathcal{N}}{n}$. To create a supermultiplet we need that the representations are symmetrical around zero helicity, we cannot make a parity, charge, and time (PCT) invariant theory with a state with say just helicity, $\lambda = -1$, we need $\lambda = 1$ as well. We will thus add the PCT conjugate in the tables to make the representations symmetrical around zero. We present the states for $\mathcal{N} = 1$, 4, and 8 in tables D.3, D.4, and D.5, respectively.

	Minmal Helicity $\bar{\lambda}$							
Helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
2	1							1
$\frac{3}{2}$	1	1					1	1
1		1	1			1	1	
$\frac{1}{2}$			1	1	1	1		
0				2	2			
$-\frac{1}{2}$			1	1	1	1		
-1		1	1			1	1	
$-\frac{3}{2}$	1	1					1	1
-2	1							1

Table D.3: *Table over the multiplicities, for different values of minimal helicity in the massless supersymmetric multiplet for $\mathcal{N} = 1$ supersymmetries.*

	Minmal Helicity $\bar{\lambda}$				
Helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
2	1				1
$\frac{3}{2}$	4	1		1	4
1	6	4	1	4	6
$\frac{1}{2}$	4	7	4	7	4
0	2	8	6	8	2
$-\frac{1}{2}$	4	7	4	7	4
-1	6	4	1	4	6
$-\frac{3}{2}$	4	1		1	4
-2	1				1

Table D.4: *Table over the multiplicities, for different values of minimal helicity in the massless supersymmetric multiplet for $\mathcal{N} = 4$ supersymmetries.*

The most interesting number of supersymmetries are $\mathcal{N} = 8$ Because if we start from a ground state with helicity $\bar{\lambda} = -2$, we will cover the complete spectrum of helicities from -2 to 2. And as there are no particles with helicity higher than 2 this is the maximal number of supersymmetries we can have.

Helicity	Multiplicity
2	1
$\frac{3}{2}$	8
1	28
$\frac{1}{2}$	56
0	70
$-\frac{1}{2}$	56
-1	28
$-\frac{3}{2}$	8
-2	1

Table D.5: *Table over the multiplicities, for different values of minimal helicity in the massless supersymmetric multiplet for $\mathcal{N} = 8$ supersymmetries.*

D.5 Superspace and superfields in four dimensions

We can extend Minkowski space to superspace by adding two new coordinates θ^α and $\bar{\theta}_{\dot{\alpha}}$. A general coordinate for superspace can now be written

$$z^A = (x^a, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}).$$

But θ and $\bar{\theta}$ are no ordinary coordinates, they are odd Grassmann numbers, i.e anticommuting coordinates. When we, from now on, refer to odd or even quantities we refer to Grassmann odd or even, i.e anticommuting- or commuting quantities, respectively. Our underlying manifold is what we call a *graded manifold*, it is expressed in odd and even coordinates, the even being the ordinary spacetime coordinates x^a. It may be a bit counter intuitive to think of coordinates as fermionic but we will see it works out. Fields depending on the superspace variables are called *superfields*, and are often denoted with a Φ.

Can and should we construct a metric for our superspace? I.e should we find a composite tensor that we can raise and lower complete superspace indices with? There are a few ways of to think about this; if we would impose a metric on superspace it would be impossible to raise and lower indices of different grade independently, which would be kind of inconvenient convenient, we also do not have a super-Lorentz group, i.e a Lorentz group that mixes the odd and even indices. It will act block wise on the different kind of indices. We will therefore not introduce a metric over superspace, and keep the different coordinates separate. We thus raise and lower spacetime indices with η_{ab}. Indices on θ and $\bar{\theta}$ are raised and lowered by ϵ.

The coordinates of superspace can be seen as parameters of the SUSY algebra[38]. We can define a group element

$$G(x, \theta, \bar{\theta}) = e^{i(x^a P_a + \theta Q + \bar{\theta} Q)}, \tag{D.9}$$

Just as in the case of ordinary Lie group theory. This group element induces translations in the coordinate space as

$$G(y, \varepsilon, \bar{\epsilon})(x, \theta, \bar{\theta}) = (x^a + y^a + i\theta\sigma^a\bar{\varepsilon} - i\varepsilon\sigma^a\bar{\theta}, \theta + \varepsilon, \bar{\theta} + \bar{\varepsilon}).$$

Next we define our derivative rules with respect to these funny coordinates, θ, and $\bar{\theta}$. We define derivation as

$$\begin{aligned}
\partial_\alpha \theta^\beta \equiv \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha^\beta \qquad & \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \\
\partial^\alpha \theta_\beta \equiv \frac{\partial}{\partial\theta_\alpha}\theta_\beta = \delta_\beta^\alpha \qquad & \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} \equiv \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\bar{\theta}_{\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}} \\
\frac{\partial}{\partial\theta^\alpha}\theta^\beta\theta^\sigma = \delta_\alpha^\beta\theta^\sigma & - \theta^\beta\delta_\alpha^\sigma.
\end{aligned}$$

One needs to be careful when dealing with the symbols ∂_α, as these are a bit ambiguous when it comes to raising and lowering. Note for example, if we raise ∂_α from the left we get,

$$\partial_\alpha \theta^\beta = \epsilon_{\alpha\sigma}\epsilon^{\beta\gamma}\partial^\sigma\theta_\gamma = \epsilon_{\alpha\sigma}\epsilon^{\beta\gamma}\delta^\sigma_\gamma = -\epsilon^{\beta\sigma}\epsilon_{\sigma\alpha} = -\delta^\beta_\alpha.$$

Other useful identities are

$$\begin{aligned}\theta^2 =& \theta^\alpha\theta_\alpha = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta \implies \frac{-1}{2}\theta^2\epsilon^{\alpha\beta} = \theta^\alpha\theta^\beta \\ \bar\theta^2 =& \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} \implies \frac{1}{2}\bar\theta^2\epsilon^{\dot\alpha\dot\beta} = \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}.\end{aligned}$$

We now realise the supersymmetry algebra on the coordinates [33].

$$\begin{aligned}Q_\alpha &= \partial_\alpha - i\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_a \\ \bar Q_{\dot\alpha} &= -\bar\partial_{\dot\alpha} + i\theta^\beta\sigma^a_{\beta\dot\alpha}\partial_a.\end{aligned}$$

This we need to check satisfies the algebra,

$$\begin{aligned}\{Q_\alpha, \bar Q_{\dot\beta}\} =& -\underbrace{\{\partial_\alpha, \bar\partial_{\dot\beta}\}}_{=0} + \{\partial_\alpha, i\theta^\beta\sigma^a_{\beta\dot\beta}\partial_a\} + \{i\sigma^a_{\alpha\dot\sigma}\bar\theta^{\dot\sigma}\partial_a, \bar\partial_{\dot\beta}\} + \underbrace{\{\sigma^a_{\alpha\dot\sigma}\bar\theta^{\dot\sigma}\partial_a, \theta^\beta\sigma^b_{\beta\dot\alpha}\partial_b\}}_{=\sigma^a_{\alpha\dot\sigma}\partial_a\sigma^b_{\beta\dot\alpha}\partial_b\{\bar\theta^{\dot\sigma},\theta^\beta\}=0} = \\ =& 2i\sigma^a_{\alpha\dot\beta}\partial_a.\end{aligned}$$

The other combination of indices are trivial. What we see here is that it satisfies the algebra but with the opposite sign from what we got in the Wess-Zumino Model. This is because Q_α are realised on the coordinates and not the fields. The Q_α generate transformations on the coordinate space, not on the fields themselves.

We can in a similar manor define two new operators

$$\begin{aligned}D_\alpha &= \partial_\alpha + i\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_a \\ \bar D_{\dot\alpha} &= -\bar\partial_{\dot\alpha} - i\theta^\beta\sigma^a_{\beta\dot\alpha}\bar\theta^{\dot\beta}\partial_a.\end{aligned} \tag{D.10}$$

These two satisfy the algebra with the correct sign, $\{D_\alpha, \bar D_{\dot\beta}\} = -2i\sigma^a_{\alpha\dot\beta}\partial_a$. Thus the D:s can be seen as realisation of the algebra onto the superfields. Further more the D:s anti commute with the Q:s with all combinations of indices. We can thus see the D:s as covariant derivatives with respect to the supersymmetric transformations.

We can expand a superfield $\Phi(x, \theta, \bar\theta)$ in a power series in θ and $\bar\theta$. Noting that the complete expansion is finite as all higher powers than two of the Grassmann coordinates is zero we can write a general scalar superfield as

$$\begin{aligned}\Phi = \varphi + \theta^\alpha\psi_\alpha + \bar\theta_{\dot\alpha}\chi^{\dot\alpha} + \theta^\alpha\theta^\beta A_{\alpha\beta} + \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}B_{\dot\alpha\dot\beta} + \theta^\alpha\bar\theta^{\dot\beta}C_{\alpha\dot\beta} \\ + \theta^\alpha\theta^\beta\psi'_{\alpha\beta\dot\gamma}\bar\theta^{\dot\gamma} + \theta^\alpha X'_{\alpha\dot\beta\dot\gamma}\bar\theta^{\dot\beta}\bar\theta^{\dot\gamma} + \theta^\alpha\theta^\beta D_{\alpha\beta\dot\sigma\dot\gamma}\bar\theta^{\dot\sigma}\bar\theta^{\dot\gamma}.\end{aligned} \tag{D.11}$$

We can rewrite eq. (D.11) to a more comapct form by observing that A, and B are totally antisymmetric in their indices and therefore proportional to ϵ. This is also the case for ψ' and X', they are totally antisymmetric in their two undotted and dotted indices, respectively. Further more C can be written as $C_{\alpha\dot{\beta}} = \sigma^a_{\alpha\dot{\beta}} c_a$. D is also totally antisymmetric in both its dotted and undotted indices and thus proportional to $\epsilon_{\alpha\beta}\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}$. We can thus write a superfield as

$$\begin{aligned}\Phi = \varphi + \theta\psi + \bar{\theta}\chi + \theta^2 a + \bar{\theta}^2 b + \theta\sigma^a\bar{\theta}c_a \\ + \theta^2\psi'\bar{\theta} + \theta\chi'\bar{\theta}^2 + \theta^2\bar{\theta}^2 d.\end{aligned}$$

We see here that we have a total of 4 scalars, 4 fermions, and one vector. But for one supersymmetry in 1+3 dimensions we know, from table D.1, that our supermultiplet has either three or four degrees of freedom, depending on what ground state we start at. This means that Φ is a reducible representation of the SUSY algebra. We thus need further constraints on the fields to lower the degrees of freedom. One such example is the notion of chiral superfields, which have the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{D.12}$$

This equation is fairly difficult to solve. However we can find a transformation, T, that reduces the equation to

$$-\bar{\partial}_{\dot{\alpha}}\tilde{\Phi} = 0, \tag{D.13}$$

where $\tilde{\Phi} = T\Phi$. We transform eq. (D.12) to

$$T\bar{D}_{\dot{\alpha}}\Phi = T\bar{D}_{\dot{\alpha}}T^{-1}T\Phi = -\bar{\partial}_{\dot{\alpha}}\tilde{\Phi}.$$

We now need to find T such that $-T^{-1}\bar{\partial}_{\dot{\alpha}}T = \bar{D}_{\dot{\alpha}}$. The T satisfying this is $T = e^{-i\theta\sigma^a\bar{\theta}\partial_a}$ [33]. It can be checked by

$$-T^{-1}\bar{\partial}_{\dot{\alpha}}T = -T^{-1}(\bar{\partial}_{\dot{\alpha}}T) - T^{-1}T\bar{\partial}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + \bar{\partial}_{\dot{\alpha}}(i\theta^\beta\sigma^a_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_a) = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^a_{\beta\dot{\alpha}}\partial_a = \bar{D}_{\dot{\alpha}}.$$

Now our job reduces to finding a $\tilde{\Phi}$ satisfying eq. (D.13). But that is just a superfield independent of $\bar{\theta}$,

$$\tilde{\Phi} = \varphi + \theta\psi + \theta^2 F.$$

Now we can simply go back to our old Φ by

$$\begin{aligned}\Phi = T^{-1}\tilde{\Phi} &= (1 + i\theta\sigma^a\bar{\theta}\partial_a + \frac{1}{4}\theta^2\bar{\theta}^2\Box)(\varphi + \theta\psi + \theta^2 F) \\ &= \varphi + \theta\psi + \theta^2 F + i\theta\sigma^a\bar{\theta}\partial_a\varphi - \frac{i}{2}\theta^2\partial_a\psi\sigma^a\bar{\theta} + \frac{1}{4}\Box\varphi.\end{aligned} \tag{D.14}$$

Where we used the expansion of $e^{i\theta\sigma^a\bar{\theta}\partial_a}$. The expansion is finite here as well. The fact that the expansions terminate is rather remarkable.

Now we have our chiral superfield. We call φ, ψ, and F component fields. We see here is that we now have the correct number of degrees of freedom for a spin-0 ground state, we have 2 scalars, φ and F and one fermion ψ. This representation is thus irreducible.

D.5.1 Transformations in superspace

We will now investigate deeper how supersymmetry transformations act on the superfields. We begin by observing that we can fish out the component fields from the superfield via the covariant derivative.

$$\begin{aligned} \varphi &= \Phi\Big|_{\theta=\bar{\theta}=0} \\ \psi_\alpha &= D_\alpha \Phi\Big|_{\theta=\bar{\theta}=0} \\ F &= \frac{1}{4}\epsilon^{\alpha\beta} D_\alpha D_\beta\Big|_{\theta=\bar{\theta}=0} = \frac{-1}{4} D^2\Big|_{\theta=\bar{\theta}=0}. \end{aligned}$$

We now define a supersymmetric transformation of a superfield by

$$\delta_\epsilon \Phi = (\epsilon Q + \bar{\epsilon}\bar{Q})\Phi$$

where $(\epsilon Q + \bar{\epsilon}\bar{Q})$ comes from the odd part of the algebra in eq.(D.9). This means that in order to do a field variation we actually go to the parameter realisation of the SUSY algebra and then back to the field realisation. We can look at it as we find out how the field have to change in order to match the transformations of the parameters.

We can now see how the component fields transform, starting with φ

$$\delta_\varepsilon \varphi = \delta_\varepsilon \Phi\Big|_{\theta=\bar{\theta}=0} = (\varepsilon Q + \bar{\varepsilon}\bar{Q})\Phi\Big|_0 = (\varepsilon D + \bar{\varepsilon}\bar{D})\Phi\Big|_0 = \varepsilon D \Phi\Big|_0 = \varepsilon\psi.$$

Where we used that at $\theta = \bar{\theta} = 0$, Q and D are just partial derivatives, we also used that Φ is chiral. This is the same structure we had for the transformations for the scalar fields (remember that S, and P are real and φ is complex) in the Wess-Zumino multiplet. The following calculations will be evaluated at $\theta = \bar{\theta} = 0$, however we omit writing this out to avoid clutter.

We move on now to the transformation of ψ

$$\delta_\varepsilon \psi_\alpha = (\varepsilon D + \bar{\varepsilon}\bar{D}) D_\alpha \Phi = \varepsilon^\beta(-2\epsilon_{\beta\alpha}F) + \bar{\varepsilon}_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\delta}}\{D_\alpha, \bar{D}_{\dot{\delta}}\}\Phi = 2\varepsilon_\alpha F + 2i\bar{\varepsilon}^{\dot{\alpha}}\sigma^a_{\alpha\dot{\alpha}}\partial_a\varphi$$

We see here that ψ transforms like in the Wess-Zumino model, except for the term involving P and with an extra term F.

Finally the transformation of F

$$\begin{aligned}\delta_\varepsilon F =&\frac{1}{4}\epsilon^{\alpha\beta}\delta_\varepsilon D_\alpha D_\beta\Phi = \frac{\epsilon^{\alpha\beta}}{4}\Big(\underbrace{\varepsilon^\sigma D_\sigma D_\alpha D_\beta\Phi}_{=0\text{ as no }\theta^3\text{ terms}} -\bar\varepsilon^{\dot\alpha}\bar D_{\dot\alpha} D_\alpha D_\beta\Phi\Big) = \\ =&\frac{-1}{4}\epsilon^{\alpha\beta}\bar\varepsilon^{\dot\beta}[\bar D_{\dot\beta}, D_\alpha D_\beta]\Phi = \frac{-1}{4}\epsilon^{\alpha\beta}\bar\varepsilon^{\dot\beta}\Big(\{\bar D_{\dot\beta}, D_\alpha\}D_\beta - D_\alpha\{\bar D_{\dot\beta}, D_\beta\}\Big)\Phi = \\ =&\frac{-1}{4}\epsilon^{\alpha\beta}\bar\varepsilon^{\dot\beta}\Big(-2i\sigma^a_{\alpha\dot\beta}\partial_a D_\beta + 2i\sigma^a_{\beta\dot\beta}\partial_a D_\alpha\Big)\Phi = i\partial_a\psi\sigma^a\bar\varepsilon = i\bar\varepsilon\bar\sigma^a\partial_a\psi\end{aligned} \tag{D.15}$$

We see here that F transforms as the Dirac equation.. These two is actually what eliminates the worrisome Dirac term we had in the Wess-Zumino model.

D.5.1.1 Resolving the Wess-Zumino model

Let us now test our transformations and see in fact that the troublesome Dirac equation in eq. (D.7) is not there when we take our F in consideration.

$$\begin{aligned}[\delta_1,\delta_2]\psi_\alpha =&\delta_1(2\varepsilon_{2\alpha}F + 2i\bar\varepsilon_2^{\dot\beta}\sigma^a_{\alpha\dot\beta}\partial\varphi) - 1\leftrightarrow 2 = \\ =&2i(\varepsilon_{2\alpha}\bar\varepsilon_1^{\dot\beta}\sigma^a_{\beta\dot\beta}\partial_a\psi^\beta + \bar\varepsilon_2^{\dot\beta}\varepsilon_1^\beta\sigma^a_{\alpha\dot\beta}\partial_a\psi_\beta) - 1\leftrightarrow 2 = \\ =&2i\Big(\sigma^a_{\alpha\dot\beta}\partial_a\psi_\beta + \epsilon_{\alpha\beta}\sigma^a_{\sigma\dot\beta}\partial_a\psi^\sigma\Big)\Big(\bar\varepsilon_2^{\dot\beta}\varepsilon_1^\beta + \varepsilon_2^\beta\bar\varepsilon_1^{\dot\beta}\Big).\end{aligned} \tag{D.16}$$

Now let's look at the commutator of two general supersymmetric transformations

$$\begin{aligned}\delta_1 =&\varepsilon_1 Q + \bar\varepsilon_1\bar Q \implies [\delta_1\delta_2]\psi_\alpha = (\varepsilon_1 Q + \bar\varepsilon_1\bar Q)(\varepsilon_2 Q + \bar\varepsilon_2\bar Q)\psi_\alpha - 1\leftrightarrow 2 = \\ =&\Big(-\{Q_\beta,\bar Q_{\dot\beta}\}(\varepsilon_2^\beta\bar\varepsilon_1^{\dot\beta} + \bar\varepsilon_2^{\dot\beta}\varepsilon_1^\beta) - \varepsilon_2^\alpha\varepsilon_1^\beta\{Q_\alpha,Q_\beta\} - \bar\varepsilon_2^{\dot\alpha}\bar\varepsilon_1^{\dot\beta}\{\bar Q_{\dot\alpha},\bar Q_{\dot\beta}\}\Big)\psi_\alpha.\end{aligned} \tag{D.17}$$

Now, comparing terms in eq.(D.16) and (D.17) we first see that the anticommutators $\{Q,Q\}$ and $\{\bar Q,\bar Q\}$ are zero. Secondly we see that,

$$\begin{aligned}\{Q_\beta,\bar Q_{\dot\beta}\}\psi_\alpha =&-2i(\sigma^a_{\alpha\dot\beta}\partial_a\psi_\beta + \epsilon_{\alpha\beta}\sigma^a_{\sigma\dot\beta}\partial_a\psi^\sigma) = \\ =&-2i(\sigma^a_{\alpha\dot\beta}\partial_a\psi_\beta + \epsilon_{\alpha\beta}\epsilon^{\sigma\gamma}\sigma^a_{\sigma\beta}\partial_a\psi_\gamma + \epsilon_{[\alpha\beta}\partial_{|a|}\psi_{\gamma]}\sigma^a_{\sigma\beta}\epsilon^{\sigma\gamma}) = \\ =&-2i(\sigma^a_{\beta\dot\beta}\partial_a\psi_\alpha).\end{aligned}$$

Where we in the third step used the fact that antisymmetrising over three two-valued indices is zero. We get back the SUSY algebra, now with the correct sign as the algebra is realised on the fields and not on the coordinates. Further more the Dirac equation is no longer there.

D.5.2 Action and Lagrangian in superspace

We now know how the field transforms under a supertransformation. What is left now is to construct our Lagrangian and our action.

To further understand how a potential action would look we first need to know how we integrate over the Grassmann variables. We define our integrations to be

$$\int d\theta^\alpha \theta_\beta \equiv \delta^\alpha_\beta \qquad \int d\theta_\alpha \theta^\beta = \delta^\beta_\alpha$$
$$\int d\bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \equiv \delta^{\dot{\alpha}}_{\dot{\beta}} \qquad \int d\bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}.$$

Further more, we need a means of normalisation when integrating over both Grassmann coordinates. We normalise as follows,

$$1 = a \int d^2\theta\theta^2 \int d\theta^\alpha d\theta^\beta \theta_\gamma \theta_\sigma \epsilon_{\alpha\beta}\epsilon^{\sigma\gamma} = 2a\epsilon_{\alpha\beta}\epsilon^{\sigma\gamma}\delta^{[\alpha}_\sigma \delta^{\beta]}_\gamma = 2a\epsilon^{\alpha\beta}\epsilon_{\alpha\beta} = -4a \implies a = \frac{-1}{4}.$$

Now instead of having a outside the integral we can simply redefine

$$d^2\theta = \frac{-1}{4}\epsilon_{\alpha\beta} d\theta^\alpha d\theta^\beta.$$

Notice here that integration and differentiation works in the same way for Grassmann variables.

$$\int d\theta^\alpha = \frac{\partial}{\partial\theta_\alpha} = D^\alpha\Big|_0$$
$$\int d\bar{\theta}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} = -\bar{D}^{\dot{\alpha}}\Big|_0$$
$$\int d^2\theta = \frac{-1}{4}D^2\Big|_0$$
$$\int d^2\bar{\theta} = \frac{-1}{4}\bar{D}^2\Big|_0.$$

Lets have a quick dimensional interlude. We know that $[\varphi] = L^{-1}$ as it is a scalar field. This means that also Φ has dimension L^{-1}. Furthermore as $[\psi] = L^{-3/2}$ this means that $[\theta] = [\bar{\theta}] = L^{+1/2}$. A potential action will be on the form

$$S \sim \int d^4x \mathscr{L} = \int d^4x d^2\theta d^2\bar{\theta}\mathcal{L} = \frac{1}{16}\int d^4x D^2\bar{D}^2\mathcal{L}\Big|_{\theta=\bar{\theta}=0}.$$

Here we have separated the notion of the Lagrangian over spacetime, $\mathscr{L}(x) = \int d^2\theta d^2\bar{\theta}\mathcal{L}(x,\theta,\bar{\theta})$, and the Lagrangian over superspace, $\mathcal{L}(x,\theta,\bar{\theta})$. Now we need the allowed superspace Lagrangians. First of, as $[\mathscr{L}] = L^{-4}$ and $[\int d^2\theta d^2\bar{\theta}] = L^{-2}$ (from the fact that it works as derivatives), the allowed dimensions for the superspace Lagrangian are $[\mathcal{L}] = L^{-2}$. The only real term satisfying this is $\Phi^*\Phi$. This term represent the free Lagrangian. Now, this is not the only terms allowed. We can also construct spacetime Lagrangians where we have just integrated over θ. This is because of the structure of the chiral superfield in eq. (D.14). All terms in Φ containing $\bar{\theta}$ also contain a free spacetime derivative over the fields. This means the spacetime integral will be zero for these terms as we assume that the fields die out at spacetime infinities. We can thus just omit $\bar{\theta}$ and consider spacetime Lagrangians of the form

$$\mathscr{L} = \int d^2\theta\Big(\lambda\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3\Big) + \text{complex conjugate.} \tag{D.18}$$

These will be our interaction terms.

Now let's take a closer look at the Lagrangian containing $\Phi^*\Phi$. We denote it $\mathscr{L}_{free}$.

$$\begin{aligned}
\mathscr{L}_{free} &= \int d^2\theta d^2\bar{\theta}\Phi^*\Phi = \frac{1}{16}D^2\bar{D}^2\Phi^*\Phi\Big|_0 = \\
&= \frac{1}{16}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}D_\alpha D_\beta\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}(\Phi^*\Phi) = \\
&= \frac{1}{16}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\Big(\overbrace{\underbrace{(D_\alpha D_\beta\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\Phi^*)}_{=D_\alpha[D_\beta,\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}]\Phi^*}\Phi}^{\mathbf{I}} - \overbrace{\underbrace{(D_\beta\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\Phi^*)}_{=[D_\beta,\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}]}(D_\alpha\Phi)}^{\mathbf{II}} \\
&\qquad + \overbrace{\underbrace{(D_\alpha\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\Phi^*)}_{=[D_\alpha,\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}]}(D_\beta\Phi)}^{\mathbf{III}} + \overbrace{(\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\Phi^*)(D_\alpha D_\beta\Phi)}^{\mathbf{IV}}\Big).
\end{aligned}$$

Here we used Leibniz rule of differentiation. We also used that Φ is chiral to get the commutators. Observe that it is implied that we evaluate this at $\theta = \bar{\theta} = 0$. Now we rewrite the commutators in terms of anticommutators as we did in eq. (D.15). We do the calculation term by term. Observe that the factor $\frac{1}{16}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}$ is included in these calculations. The first term is

$$\begin{aligned}
\mathbf{I} =&\Phi\frac{i}{8}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}D_\alpha\Big(\sigma^a_{\beta\dot{\beta}}\partial_a\bar{D}_{\dot{\alpha}} - \sigma^a_{\beta\dot{\alpha}}\partial_a\bar{D}_{\dot{\beta}}\Big)\Phi^* = \\
=&\Phi\frac{1}{4}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\Big(\sigma^a_{\beta\dot{\beta}}\sigma^b_{\alpha\dot{\alpha}}\partial_a\partial_b - \sigma^a_{\beta\dot{\alpha}}\sigma^b_{\alpha\dot{\beta}}\partial_a\partial_d\Big)\Phi^* = \\
=&\varphi\Box\varphi^* = -\partial_a\varphi\partial^a\varphi^*,
\end{aligned}$$

where we in the third step used $Tr(\sigma^a\bar{\sigma}^b) = -2\eta^{ab}$. In the last step we used integration by parts as we know we we are under an action.

Observing, **II**, and **III** we see that they are the same but with α and β interchanged. These are however just dummy indices so we can match them, pick up an extra minus sign in **III** from the switch in the ϵ tensor, and simply add them

$$\begin{aligned}
\mathbf{II} + \mathbf{III} =&\frac{-i}{4}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\Big(\sigma^a_{\beta\dot{\beta}}\partial_a\bar{D}_{\dot{\alpha}} - \sigma^a_{\beta\dot{\alpha}}\partial_a\bar{D}_{\dot{\beta}}\Big)\Phi^*D_\alpha\Phi = \\
=&\frac{i}{2}(\bar{\sigma}^{a\dot{\alpha}\alpha}\partial_a\bar{D}_{\dot{\alpha}}\Phi^*)D_\alpha\Phi = \\
=&\frac{i}{2}\partial_a\bar{\psi}\bar{\sigma}^a\psi = \frac{-i}{2}\bar{\psi}\bar{\sigma}^a\partial_a\psi.
\end{aligned}$$

The fourth term is quite simple,

$$\mathbf{IV} = \frac{1}{16}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\Phi^*D_\alpha D_\beta\Phi = \frac{1}{16}\Big((-4F^*)(-4F)\Big) = |F|^2.$$

Our free Lagrangian will now become

$$\mathscr{L}_{free} = -\partial_a\varphi^*\partial^a\varphi - \frac{i}{2}\bar{\psi}\bar{\sigma}^a\partial_a\psi + |F|^2.$$

This is just the kinetic Lagrangian we had in the Wess-Zumino Model excluding the F field. But the equation of motion for F here is $F = 0$ which means that it was on-shell in the Wess-Zumino model. But if one equation of motion is satisfied that means all fields are on-shell. Thus the troublesome Dirac equation term in eq. (D.7) is not there because ψ is on-shell. Our symmetry transformations introduced in sec. D.3.1 thus generate the SUSY algebra.

D.5.3 Supersymmetric interactions

We still have left the other terms in eq. (D.18). These are the interaction terms, we evaluate them separately.

$$\begin{aligned}
\mathscr{L}_\lambda =&\lambda\int d^2\theta\Phi + c.c = \frac{-\lambda}{4}D^2\Phi\Big|_0 + c.c = \lambda F + \lambda^* F^* \\
\mathscr{L}_m =&\frac{m}{2}\int d^2\theta\Phi^2 + c.c = \\
=&\frac{-m}{8}D^\alpha(2\Phi D_\alpha\phi) + c.c = \frac{-m}{4}(D^\alpha\Phi D_\alpha\Phi + \Phi D^\alpha D_\alpha\Phi) + c.c = \\
=&\frac{-m}{4}\psi^2 + m\varphi F + \frac{-m^*}{4}\bar{\psi}^2 + m^*\varphi^* F^* \\
\mathscr{L}_g =&\frac{-g}{12}D^2\Phi^3 + c.c = \frac{-g}{4}D^\alpha(\Phi^2 D_\alpha\Phi) = \frac{-g}{4}(2\Phi D^\alpha\Phi D_\alpha\Phi + \Phi^2 D^2\Phi) + c.c = \\
=&-\frac{g}{2}\varphi\psi^2 + g\varphi^2 F - \frac{g^*}{2}\varphi^*\bar{\psi}^2 + g^*(\varphi^*)^2 F^*.
\end{aligned}$$

Here we note that in the total Lagrangian we do not have any derivatives acting on F. The field has no dynamics whatsoever, it sole purpose to the supersymmetric Lagrangian is to make sure that the fermion field transform properly under supersymmetry. This we saw in section D.5.1.1 where the extra transformation of F cancel out the worrisome Dirac term. We call fields like F, *auxiliary fields*.

D.6 Differential forms in superspace

We will now develop the notion of differential forms in superspace. Doing this will help us on our way to formulate supergravity. The supersymmetry transformations are a part of the general coordinate transformations of superspace as can be seen from eq. (D.9). Thus extending the notion of differential forms to superspace can help us define supergravity in a covariant way under the coordinate transformations of superspace.

We start by defining a coordinate system z^M over superspace. We think of superspace as a supermanifold, i.e a manifold that is described by both fermionic (odd grade) and bosonic (even grade) coordinates. The coordinate system is

$$z^M = (z^m, \theta^\mu, \bar{\theta}_{\dot{\mu}}).$$

Note that this is the same coordinate system setup as when we first introduced superspace. We have however changed the indices to M, N, L instead of A, B, C. The fermionic and bosonic properties arise when we multiply two elements of superspace,

$$z^M z^N = (-)^{MN} z^N z^M,$$

where the indices M and, N in $(-)^{MN}$ are the grade of the components z^M and z^N.

A general super-1-form can be written $\omega = dz^M \omega_M$. We will from now on omit writing out super on cases when it is clear from the context. Next we define the wedge product between two 1-forms as

$$\begin{aligned} dz^M \wedge dz^N &\equiv -(-)^{MN} dz^N \wedge dz^M \\ dz^M z^N &\equiv (-)^{MN} z^N dz^M. \end{aligned}$$

This is in complete analogy to ordinary space. A general p-form can be written

$$\Omega = \frac{1}{p!} dz^{M_1} \wedge \cdots \wedge dz^{M_p} \Omega_{M_p \cdots M_1}(z).$$

Observe here that the indices is summed in such a way that there is always an even object between two contracted indices. This makes calculations easy as we do not have to worry about sign shifts. To avoid clutter in expressions and calculations we will refrain from writing out the wedge product in places where their presence is clear from the context. We can just as for ordinary space create the exterior superalgebra, Λ, with the wedge product as the bilinear operation. The exterior algebra in superspace, Λ, is decomposed as

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots$$

In contrast to ordinary space the exterior algebra of superspace is not compact. As the wedge product of two 1-forms with odd indices is symmetric there is no limit to how high order of forms we can have. However the number of spacetime 1-forms we can have are still the restricted to the dimension which we are working in.

Superforms obey the same multiplication as ordinary forms (see sec. 2.3).

$$\begin{aligned} i) &\quad (c_1\Lambda_1 + c_2\Lambda_2)\Omega = c_1\Lambda_1\Omega + c_2\Lambda_2\Omega \\ ii) &\quad \Lambda\Omega = (-)^{pq}\Omega\Lambda \\ iii) &\quad \Lambda(\Xi\Omega) = (\Lambda\Xi)\Omega. \end{aligned}$$

For a q-form Ω, a p-form Λ, and an arbitrary form Ξ. In order to ensure the permutation symmetry of the indices in a super form we introduce the graded commutator

$$[\,\cdot\,,\,\cdot\,\},$$

which is a anticommutator if both arguments are odd, otherwise a commutator. This notion is applicable to the indices as well. We introduce a graded symmetrisation of indices $[\cdot,\cdot)$ analogous to the graded commutator but with the appropriate combinatorial factor in front to properly normalise.

This allows us to, in some cases, write only the component fields of the p-forms. Let's give an example in the case of a 2-form

$$A = \frac{1}{2} dz^M dz^N A_{[NM)} = \frac{1}{2}\frac{1}{2} dz^M dz^N (A_{NM} - (-)^{MN} A_{MN}).$$

Here we see that if M and N are both odd we have symmetrised the indices and other wise they are antisymmetrised.

We can now define the exterior derivative of superspace, d, this maps p-forms to $p+1$-forms. The exterior derivative on a p-form Ω is

$$d\Omega = \frac{1}{p!} dz^{M_1} \cdots dz^{M_p} dz^N \frac{\partial}{\partial z^N} \Omega_{M_p \cdots M_1}(z).$$

We define d to have the same algebraic properties as the exterior derivative of ordinary space, see sec. 2.3.

D.6.1 Super-vielbeins and torsion

We assume here that we work in flat superspace. Next we need to clarify the different indices. Upper case letters in the beginning of the alphabet represent Flat or Lorentz indices. Upper case letters from the middle of the alphabet represent curved indices, i.e indices that transform under general coordinate transformations, we are still in flat space. All uppercase indices, as before, has a bosonic (even) and a fermionic (odd) part.

Now, as we know we are in flat space the first thing we can do is to make a change of basis so that we can write our basis 1-forms and other quantities i Lorentz indices instead of curved ones, this is done using super-vielbeins. Further more we want to make a use of our covariant derivatives with respect to supersymmetric transformations,

$$\begin{aligned} D_a &= \frac{\partial}{\partial z^a} \\ D_\alpha &= \partial_\alpha + i\sigma^a_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_a \\ \bar{D}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^a_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_a. \end{aligned} \tag{D.19}$$

These derivatives were defined previously in eq. (D.10). Because the normal $\frac{\partial}{\partial z^M}$ does not commute with our SUSY generators we want to use the derivatives in eq. (D.19) in our formulation of the exterior derivative. We therefore define a new set of basis 1-forms, as

$$e^A(z) = dz^M {e_M}^A(z).$$

The matrix ${e_M}^A$ is called a *super-vielbein*. The exterior derivative can now be written in terms of the supersymmetric covariant derivatives

$$d = dz^M \frac{\partial}{\partial z^M} = e^A D_A = dz^M {e_M}^A {e_A}^N \frac{\partial}{\partial z^N}.$$

We thus have $D_A = {e_A}^M \partial_M$. The matrices follows from eq. (D.19)

$$\begin{aligned} {e_A}^M &= \begin{pmatrix} {e_a}^m = {\delta_a}^m & {e_a}^\mu = 0 & {e_a}^{\dot\mu} = 0 \\ {e_\alpha}^m = i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} & {e_\alpha}^\mu = {\delta_\alpha}^\mu & {e_\alpha}^{\dot\mu} = 0 \\ e^{\dot\alpha m} = i\theta^\alpha \sigma^m_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\alpha} & e^{\dot\alpha\mu} = 0 & {e^{\dot\alpha}}_{\dot\mu} = {\delta^{\dot\alpha}}_{\dot\mu} \end{pmatrix} \\ {e_M}^A &= \begin{pmatrix} {e_m}^a = {\delta_m}^a & {e_a}^\mu = 0 & {e_a}^{\dot\mu} = 0 \\ {e_\mu}^\alpha = -i\sigma^a_{\mu\dot\mu}\bar\theta^{\dot\mu} & {e_\mu}^\alpha = {\delta_\mu}^\alpha & {e_\mu}^{\dot\alpha} = 0 \\ e^{\dot\mu a} = -i\theta^\rho \sigma^a_{\rho\dot\nu}\epsilon^{\dot\nu\dot\mu} & e^{\dot\mu\alpha} = 0 & {e^{\dot\mu}}_{\dot\alpha} = {\delta^{\dot\mu}}_{\dot\alpha} \end{pmatrix}. \end{aligned} \tag{D.20}$$

They are each others inverses, i.e

$${e_M}^A {e_A}^N = {\delta_M}^N \qquad {e_A}^M {e_M}^B = {\delta_A}^B.$$

Using these two matrices we can convert any curved index into a Lorentz index. Later we will define super-vielbeins over curved space. Changing the basis from dz^M to e^A is called to choose a flat basis, or Lorentz basis. The super-vielbeins can be used to "flatten" the indices of forms written in the coordinate basis (dz^M)

Important to note is that the exterior derivative on the new basis 1-forms (also called super-vielbeins) does not vanish for all components

$$\begin{aligned} de^A &= dz^M dz^N \partial_N {e_M}^A \\ de^a &= -2ie^\alpha \sigma^a_{\alpha\dot\alpha} e^{\dot\alpha} \\ de^\alpha &= 0 \\ de_{\dot\alpha} &= 0. \end{aligned}$$

We can define the *torsion* as

$$T^A \equiv de^A. \tag{D.21}$$

This is however not the full definition of the torsion as we are working with flat superspace. In sec. D.7.1 we will consider curved superspace which will force us to include another term in the definition of the torsion. For now we notice that we only have torsion on the even part of superspace.

D.7 Gauge theory in superspace

We will now introduce the action of a local Lie structure group, G. I.e a gauge group. The module of a representation of the structure group is spanned of the super-forms. We introduce the gauge group with the right action

$$\Omega'^a = \Omega^b X_b{}^a(z).$$

For $X \in G$. Here the indices a, and b are not coordinate indices nor Lorentz indices, they are a new set of indices that transform solely via elements of the structure group. When we later do Cartan gravity this gauge group will be the Lorentz group. Right now we take it to be any Lie group.

What we first observe is that the exterior derivative does not transform covariantly under these transformations,

$$d\Omega'^a = \Omega^b dX_b{}^a + d\Omega^b X_b{}^a.$$

We can thus introduce a 1-form connection or a gauge field, ϕ, such that

$$\phi_a{}^b = dz^M(\phi_M)_a{}^b = dz^M \phi^r_M i(T^r)_a{}^b = e^A(\phi_A)_a{}^b.$$

Where T^r are the generators of the group. We define, [38], ϕ to transforms as

$$\phi' = X^{-1}\phi X - X^{-1}dX.$$

We can then define a covariant derivative with respect to gauge transformations as

$$\begin{aligned}\mathscr{D}\omega^a &= d\omega^a + \omega^b\phi_b{}^a \\ \mathscr{D}\omega_a &= d\omega_a - (-)^p\phi_a{}^b\omega_b.\end{aligned} \tag{D.22}$$

Eq. (D.22) implies that the covariant derivative of a matrix is

$$\mathscr{D}\Omega_a{}^b = d\Omega_a{}^b + ([\Omega, \phi\}_p)_a{}^b.$$

The graded commutator is now with respect to the form-degree, p, of Ω, i.e if p is even it is a commutator and an anticommutator if p is odd. We can write this in component form as

$$\mathscr{D}_A\Omega_B = D_A\Omega_B - [\phi_A, \Omega_B\}, \tag{D.23}$$

Where B denotes any combination of Lorentz indices. It is straight forward to show that $(\mathscr{D}\Omega^a)' = (\mathscr{D}\Omega^b)X_b{}^a$. Observe also that the super-vielbein forms are already covariant under the gauge group, i.e they do not transform. This implies that the the action of the connection on the super-vielbein forms are zero. Thus the covariant derivative on a super-vielbein 1 form is just the exterior derivative, and thus the torsion,

$$\mathscr{D}E^B = dE^B = T^B.$$

With this we can construct a new tensor with respect to the gauge transformations. This is the curvature tensor, or field strength tensor, F,

$$F = \mathscr{D}\phi = d\phi + \phi \wedge \phi.$$

F is a 2-form and transforms as

$$\begin{aligned}
F' &= d\phi' + (\phi\phi)' = \\
&= d(X^{-1}\phi X - X^{-1}dX) + (X^{-1}\phi X - X^{-1}dX)^2 = \\
&= X^{-1}\phi dX + X^{-1}d\phi X - dX^{-1}\phi X + dX^{-1}dX + \\
&+ X^{-1}\phi\phi X - X^{-1}\phi dX - X^{-1}dXX^{-1}\phi X + X^{-1}dXX^{-1}dX = \\
&= X^{-1}FX - dX^{-1}(\phi X - dX) - X^{-1}dXX^{-1}(\phi X - dX) = \\
&= X^{-1}FX - (dX^{-1} + X^{-1}dXX^{-1})(\phi X - dx) = \\
&= X^{-1}FX,
\end{aligned}$$

where we, in the last step used, $dX^{-1} = d(X^{-1}XX^{-1}) = dX^{-1} + X^{-1}dXX^{-1} + dX^{-1}$. So the field strength is a tensor with respect to gauge transformations.

Further more we have the Bianchi identity

$$\begin{aligned}
\mathscr{D}F = dF + F\phi - \phi F &= d(\phi\phi) + d\phi\phi + \phi^3 - \phi d\phi - \phi^3 = \\
&= \phi d\phi - d\phi\phi + d\phi\phi - \phi d\phi = 0.
\end{aligned} \tag{D.24}$$

In general, [38], the field strength and the covariant derivative are the only tensorial object we can make for the structure group, as $dd = 0$. We find for example that

$$\begin{aligned}
\mathscr{D}\mathscr{D}\Omega^a =& d(d\Omega^a + \Omega^b\phi_b{}^a) + (d\Omega^c + \Omega^b\phi_b{}^c)\phi_c{}^a = \\
=& \Omega^b d\phi_b{}^a - d\Omega^b\phi_b{}^a + d\Omega^b\phi_b{}^a + \Omega^c\phi_c{}^b\phi_b{}^a = \Omega^b F_b{}^a.
\end{aligned} \tag{D.25}$$

Similarly we also find

$$\begin{aligned}
\mathscr{D}\mathscr{D}\Omega_a =& d\Big(d\Omega_a - (-)^p\phi_a{}^b\Omega_b\Big) - (-)^{p+1}\phi_a{}^b\Big(d\Omega_b - (-)^p\phi_b{}^c\Omega_c\Big) = \\
=& (-)^{p+1}\Big(\phi_a{}^b d\Omega_b + (-)^p(d\phi)_a{}^b\Omega_b\Big) \\
&+ (-)^{p+2}\phi_a{}^b d\Omega_b + (-)^{2p+1}\phi_a{}^b\phi_b{}^c\Omega_c = \\
=& (-)^{2p+1}\Big((d\phi)_a{}^b\Omega_b + \phi_a{}^b\phi_b{}^c\Omega_c\Big) = -F_a{}^b\Omega_b.
\end{aligned} \tag{D.26}$$

Equations (D.25) and (D.26) implies that

$$\mathscr{D}\mathscr{D}\Omega_a{}^b = \Omega_a{}^c F_c{}^b - F_a{}^c\Omega_c{}^b = [\Omega, F]_a{}^b,$$

as when acting on a matrix the covariant derivative acts from both left and right.

In particular we have for a one form Ω_A

$$\mathscr{D}\mathscr{D}\Omega_A = \frac{1}{2}E^B E^C\Big((-)^{A(B+C)}\Omega_A F_{CB} - F_{CB}\Omega_A\Big).$$

But we also have

$$\begin{aligned}
\mathscr{D}\mathscr{D}\Omega_A =&\mathscr{D}(E^B\mathscr{D}_B\Omega_A) = E^B\mathscr{D}\mathscr{D}_B\Omega_A + \underbrace{\mathscr{D}E^B}_{=T^B}\mathscr{D}_B\Omega_A = \\
=&E^B E^C\mathscr{D}_C\mathscr{D}_B\Omega_A + T^D\mathscr{D}_D\Omega_A = \\
=&\frac{1}{2}E^B E^C[\mathscr{D}_C,\mathscr{D}_B\}\Omega_A + \frac{1}{2}E^B E^C T^D{}_{CB}\,\mathscr{D}_D\Omega_A \\
\Longrightarrow& [\mathscr{D}_C,\mathscr{D}_B\}\Omega_A = (-)^{A(B+C)}\Omega_A F_{CB} - F_{CB}\Omega_A - T^D{}_{CB}\,\mathscr{D}_D\Omega_A.
\end{aligned} \tag{D.27}$$

The identities above are called the Bianchi identities. These we will use later on to obtain the equations of motion for a super-Yang-Mills theory. With this we end our current discussion, we will now go on to consider the concept of curved space.

D.7.1 Cartan formulation of gravity

After introducing the gauge theory in flat space we will now do the same over an arbitrary curved manifold. We still do not have a complete metric over our superspace and thus the components are raised and lowered independently. Now we could debate what dimension we should do this in. However we do not really use the dimension in our calculations here. The calculations are done assuming chiralities of the spinor representation, which implies even dimensions. We have also assumed that they are not Majorana spinors. Thus if we want to restrict ourselves to Majorana spinors or odd dimensions we simply remove the objects with dotted indices from the calculations.

We will now introduce the super-vielbein, $E_M{}^A$ and $E_A{}^M$, in the same way as we did in flat space. They are a way to go between the curved indices of superspace to the flat Lorentz indices in the tangent space. This is just as we did in the flat space case but in that case the tangent space and the complete space coincide.

For zero curvature in four spacetime dimension these matrices coincide with the matrices $e_A{}^M$ introduced in eq. (D.20). We can use the vielbeins to define a new basis for our tangent space and cotangent space.

$$\begin{aligned}
E_A &= E_A{}^M\partial_M \\
E^A &= dz^M E_M{}^A.
\end{aligned}$$

Once again the indices with letters from the beginning of the alphabet are Lorentz indices whilst indices from the middle of the alphabet are curved or Einstein indices. The vielbeins connect the two kinds of indices

$$V_M = E_M{}^A V_A \qquad V_A = E_A{}^M V_M$$
$$V^M = V^A E_A{}^M \qquad V^A = V^M E_M{}^A$$

Thus, explicitly for a 1-form

$$dz^M V_M = dz^M E_M{}^A V_A = E^A V_A.$$

The next thing we do is to introduce our gauge group. We will choose it to be the Lorentz group. An element in the gauge group is now a local Lorentz transformation, $\Lambda(x)_A{}^B$, that transform each tangent space in superspace. The gauge group thus act on anything which has Lorentz indices. Note that, as we do not have a super Lorentz group the transformation will not mix the odd and even parts. The new basis 1-forms will transform as

$$E^A = (E^a, E^\alpha, E_{\dot\alpha}) \to E^B \Lambda_B{}^A = (E^b \Lambda_b{}^a, E^\beta \Lambda_\beta{}^\alpha, E_{\dot\beta} (\Lambda^*)^{\dot\beta}{}_{\dot\alpha}).$$

Now as the Lorentz transformations are gauged we need to introduce a connection, just as in regular gauge theory. We denote the connection

$$\omega_A{}^B = (\omega_a{}^b, \omega_\alpha{}^\beta, \omega^{\dot\alpha}{}_{\dot\beta}).$$

ω is more frequently called the spin-connection. ω is a 1-form just as before

$$\omega_A{}^B = dz^M (\omega_M)_A{}^B = E^C (\omega_C)_A{}^B = E^C \omega_C^r i (L^r)_A{}^B,$$

where L are now the generators of the Lorentz group, note that depending of the indices there are different spin representations of the Lorentz group.

Once again we can construct covariant derivatives, now with respect to Lorentz transformations,

$$\mathscr{D} V^A = dV^A + V^B \omega_B{}^A.$$

Which transform covariantly under local Lorentz transformations. With the spin-connection and the covariant derivative introduced we can now define torsion as

$$T^A \equiv \mathscr{D} e^A. \tag{D.28}$$

This is the full definition of the torsion. Important to notice is that for flat space the tangent spaces all coincide, therefore the Lorentz transformations will be the same at each point and therefore global. Thus the exterior derivative will now be a covariant derivative. Equivalently the connection is no longer needed to construct covariant derivatives. Therefore in flat space

$$\omega = 0.$$

In flat space the torsion then reduces to $T^a = de^a$, as in agreement with eq. (D.21). We can now just as before construct our curvature tensor, here denoted R,

$$R = d\omega + \omega\omega.$$

Or in index notation

$$R_A{}^B = \frac{1}{2} dz^M dz^N (R_{NM})_A{}^B = \frac{1}{2} E^C E^D (R^r_{DC}) i (L^r)_A{}^B$$

Now just as before we have the identity eq. (D.24),

$$\mathscr{D}R = 0.$$

The identity eq. (D.25) we can write as

$$\mathscr{D}\mathscr{D}V^A = V^B R_B{}^A. \tag{D.29}$$

What follows from (D.29) is

$$\mathscr{D}T^A = \mathscr{D}\mathscr{D}E^A = E^B R_B{}^A.$$

What is a bit more interesting is if we write out the identity (D.29) in its flat indices

$$\begin{aligned}
\mathscr{D}\mathscr{D}V^A =& \mathscr{D}E^B \mathscr{D}_B V^A = E^B \mathscr{D}\mathscr{D}_B V^A + \underbrace{\mathscr{D}(E^B)}_{=T^B} \mathscr{D}_B V^A = \\
=& E^B E^C \mathscr{D}_{[C} \mathscr{D}_{B)} V^A + T^B \mathscr{D}_B V^A = \\
=& \frac{1}{2} E^B E^C [\mathscr{D}_C, \mathscr{D}_B\} V^A + \frac{1}{2} E^B E^C T_{CB}{}^D \mathscr{D}_D V^A \\
\Longrightarrow & \frac{1}{2} E^B E^C [\mathscr{D}_C, \mathscr{D}_B\} V^A = V^D \frac{1}{2} E^B E^C R_{CBD}{}^A - \frac{1}{2} E^B E^C T_{CB}{}^D \mathscr{D}_D V^A.
\end{aligned}$$

Or written without the forms

$$[\mathscr{D}_C, \mathscr{D}_B\} V^A = (-)^{D(C+B)} V^D R_{CBD}{}^A - T_{CB}{}^D \mathscr{D}_D V^A. \tag{D.30}$$

Here we see the Torsion T introduced in eq. (D.28) as the covariant derivative on the basis forms. The extra factor $(-)^{D(C+B)}$ comes from moving V^D past the forms E^B and E^C.

Now, from eq. (D.30) an theory of gravity can be derived. It is the theory formulated by Élie Cartan. It is in some sense a more general formulation of Einstein gravity, as this formulation treats torsion, which is linked to the spin of particles. It formulates gravity on a

Riemann-Cartan (RC) spacetime, as apposed to general relativity (GR), which is formulated on a Riemannian Geometry. The fundamental differences between these two geometries is that Riemann-Cartan holds the property of incorporating a gauged Lorentz symmetry, which a Riemannian geometry does not. Further more a RC geometry has an affine connection, or when working with spinors a spin connection, that is independent of the metric, it is expressed in terms of the vielbeins. The ability to work with a gauged Lorentz symmetry allows us to couple gravity to fermions, which are the particles making up matter. The torsion we have defined in eq. (D.21) is the antisymmetric part of the affine connection over a RC spacetime [40]. If the torsion is non-zero this implies that the Ricci tensor is not symmetrical and consequently, when deriving the field equations, neither is the Energy-Momentum Tensor. When deriving the field equations we vary the action with respect to the metric and the affine connection, independently. This yields two sets of equations, the first being the modified Einsteins equations now with extra terms and extra equations containing the torsion and the intrinsic angular momentum of the matter [41].

The second set of equations are the so called Cartan equations which is a set of algebraic equations coupling spin to the torsion. Analogously to the energy-momentum tensor being the source of curvature, torsion is generated by the intrinsic angular momentum of the matter fields. These effects are however quite small, in most cases one is perfectly fine with normal GR. The only cases where this formulation might contribute with significant effects would be areas with very high particle density, e.g in the early Universe or around Black Holes [40].

To retrieve GR from the Cartan formulation we set the torsion to zero. This yields no dependence on the spin angular momentum of the matter and we retrieve the symmetrical field equations.

D.8 Supersymmetric 10-dimensional Yang-Mills

The formulation of supergravity in 11 dimensions is a complex and long process. However the methodology is fairly similar to that of Yang-Mills in 10 dimensions but with somewhat kinder calculations. We will thus do the full derivation for Yang-Mills and then we will motivate the corresponding calculations for supergravity.

We start of by defining our coordinates. We now work in 9+1 dimensions which means that we can have both Weyl- and Majorana spinors. Our coordinates are

$$Z^M = (x^m, \theta^\mu)$$

Where $m = 0, \dots, 9$ are the usual, curvelinear coordinates, of flat 10 dimensional spacetime. $\mu = 1, \dots, 16$ are the odd coordinates. We would have $2^{\frac{10}{2}} = 32$ odd indices but as we have both Majorana and Weyl this breaks down to 16. Next we define our super-vielbeins in the same way we did in 4 dimensions. However now we must take use of App. E.2.2 to work correctly with the Γ-matrices. Note that, as we work with Weyl spinors, we will use the left projected block of the total 32 dimensional representation. We want to define our super-vielbeins such that the exterior derivative can be written as the contraction of the basis 1-forms with the supersymmetric covariant derivatives. First of, SUSY generators are now

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\Gamma^a_{\alpha\sigma}\theta^\sigma\partial_a.$$

These generators satisfy the 10 dimensional SUSY algebra

$$\{Q_\alpha, Q_\beta\} = -2i\Gamma^a_{\alpha\beta}\partial_a,$$

where the Γ^a, are the gamma matrices for $SO(1,9)$ in App. E.2.2. Next, the new covariant derivatives are

$$\begin{aligned} D_a &= \frac{\partial}{\partial x^a} \\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\Gamma_{\alpha\sigma}\theta^\sigma\partial_a. \end{aligned}$$

With these definitions the super vielbeins are

$$\begin{aligned} e_A{}^M &= \begin{pmatrix} e_a{}^m = \delta_a{}^m & e_a{}^\mu = 0 \\ e_\alpha{}^m = i\Gamma^m_{\alpha\beta}\theta^\beta & e_\alpha{}^\mu = \delta_\alpha{}^\mu \end{pmatrix} \\ e_M{}^A &= \begin{pmatrix} e_m{}^a = \delta_m{}^a & e_m{}^\alpha = 0 \\ e_\mu{}^a = -i\Gamma^a_{\mu\nu}\theta^\nu & e_\mu{}^\alpha = \delta_\mu{}^\alpha \end{pmatrix}. \end{aligned}$$

D.8.1 The Bianchi identities

We introduce a gauge group G with generators T^r. With this we also define the connection or gauge field

$$A = dz^M A_M = E^A A_A.$$

We also have the field strength,

$$F = \frac{1}{2}dz^M dz^N F_{NM} = \frac{1}{2}E^A E^B F_{BA}, \tag{D.31}$$

which can be written in terms of the connection and torsion as

$$\begin{aligned} F =& dA + AA = d(E^A A_A) + E^A A_A E^B A_B = \\ =& E^A dA_A - (dE^A)A_A + (-)^{AB} E^A E^B A_A A_B = \\ =& E^A E^B D_B A_A - T^A A_A - E^A E^B A_B A_A = \\ =& \frac{1}{2}E^A E^B \Big(D_B A_A - (-)^{AB} D_A A_B - T^C_{BA} A_C - (A_B A_A - (-)^{AB} A_A A_B) \Big) \end{aligned} \tag{D.32}$$

Comparing (D.31) and (D.32) we see that the components of F can be written as

$$F_{AB} = D_A A_B - (-)^{AB} D_B A_A - T^C_{AB} A_C - (A_A A_B - (-)^{AB} A_B A_A). \tag{D.33}$$

We know for flat space the only non-zero components of the torsion is

$$\begin{aligned}T^D =& dE^D = dz^M dz^N \partial_N E_M{}^D = dz^\mu dz^\nu(-i\Gamma^d_{\mu\nu}) = \\ =& E^A E_A{}^\mu E^B E_B{}^\nu (-i\Gamma^d_{\mu\nu}) = \\ =& \frac{1}{2} E^\beta E^\alpha (2i\Gamma^d_{\alpha\beta}) \implies T^d_{\alpha\beta} = 2i\Gamma^d_{\alpha\beta}.\end{aligned}$$

Thus (D.33) decomposes into its Lorentz components as

$$\begin{aligned}F_{ab} &= \partial_{[a}A_{b]} - [A_a, A_b] \\ F_{a\beta} &= \partial_a A_\beta - D_\beta A_a - [A_a, A_\beta] \\ F_{\alpha\beta} &= D_{(\alpha}A_{\beta)} + 2i\Gamma^a A_a - \{A_\alpha, A_\beta\},\end{aligned}$$

We have seen in the previous sections that $\mathscr{D}F = 0$. Using this we get

$$\begin{aligned}&\frac{1}{2}\mathscr{D}(E^B E^A F_{AB}) = 0 \\ &\implies E^B \wedge E^A \underbrace{\mathscr{D}}_{=E^C\mathscr{D}_C} F_{AB} + E^B \underbrace{(\mathscr{D}E^A)}_{=T^A} - \underbrace{(\mathscr{D}E^B)}_{=T^B} E^A F_{AB} = \\ &\quad =E^A \wedge E^B \wedge E^C \mathscr{D}_C F_{BA} + \frac{1}{2} E^A E^B E^C T^D_{CB} F_{DA} - \frac{1}{2} E^B E^C T^D_{CB} E^A F_{AD} = \\ &\quad =E^A \wedge E^B \wedge E^C (\mathscr{D}_C F_{BA} + \frac{1}{2} T^D_{CB} - (-)^{A(C+B+D)}(-)^{A(C+B)}(-)(-)^{AD} \\ &\implies \mathscr{D}_{[A}F_{BC)} + T^D_{[AB}F_{|D|C)} = 0.\end{aligned}$$

These are the Bianchi identities for super-Yang-Mills. We can write them out in their corresponding Lorentz indices. The Bianchi identities becomes

$$\begin{aligned}&(1)\mathscr{D}_{[a}F_{bc]} = 0 \\ &(2)\mathscr{D}_{[a}F_{b\gamma)} = 2\mathscr{D}_{[a}F_{b]\gamma} + \mathscr{D}_\gamma F_{ab} = 0 \\ &(3)\mathscr{D}_{[a}F_{\beta\gamma)} + T^D_{[a\beta}F_{|D|\gamma)} = \mathscr{D}_a F_{\beta\gamma} + 2\mathscr{D}_{(\beta}F_{\gamma)a} + 2i\Gamma^d_{\beta\gamma}F_{da} = 0 \\ &(4)\mathscr{D}_{(\alpha}F_{\beta\gamma)} + T^D_{(\alpha\beta}F_{|D|\gamma)} = \mathscr{D}_{(\alpha}F_{\beta\gamma)} + 2i\Gamma^d_{(\alpha\beta}F_{|d|\gamma)} = 0.\end{aligned} \tag{D.34}$$

These identities are equations that any field strength will satisfy. Thus if we impose external constraints on the field strength we can solve these equations to find the degrees of freedom for the constrained theory. This will be done in the following section.

D.8.2 Solving the identities

The components of F_{AB} does not mix, F_{ab}, $F_{a\beta}$, and $F_{\alpha\beta}$ all transform differently under Lorentz transformations. This is a consequence of the fact that we do not have a super-lorentz group. Thus by constraining one of the components we do not break Lorentz symmetry for the other. We first enforce the *Conventional Constraint*

$$\gamma^{a\alpha\beta}F_{\alpha\beta} = 0 \implies \gamma^{\alpha\beta}_a(D_{(\alpha}A_{\beta)} - \{A_\alpha, A_\beta\}) = -32iA_a,$$

which tells us that A_a is now completely expressible through A_α, we enforce this constraint in order to remove superfluous degrees of freedom at order θ in $F_{\alpha\beta}$. This is however not sufficient, the conventional constraint does not imply anything in terms of the Bianchi identities. They will still be manifestly satisfied. We also need the *Dynamical Constraint*

$$\gamma^{[5]\alpha\beta}F_{\alpha\beta} = 0,$$

which together with the conventional constraint now forces

$$F_{\alpha\beta} = 0.$$

With these constraints the Bianchi identities (D.34) become

$$\begin{aligned}
&(1)\ \mathscr{D}_{[a}F_{bc]} = 0 \\
&(2)\ 2\mathscr{D}_{[a}F_{b]\gamma} + \mathscr{D}_\gamma F_{ab} = 0 \\
&(3)\ 2\mathscr{D}_{(\beta}F_{\gamma)a} + 2i\Gamma^d_{\beta\gamma}F_{da} = 0 \\
&(4)\ \Gamma^d_{(\alpha\beta}F_{|d|\gamma)} = 0.
\end{aligned} \tag{D.35}$$

Now we can start solving these for the unconstrained field strengths F_{ab}, and $F_{a\beta}$. The last equation in eq. (D.35) could imply that $F_{a\beta} = 0$, but this would make the whole theory trivial. We make no assumptions and see where it gets us. First we decompose $F_{a\beta}$ into irreducible components of $SO(1,9)$ this we do by observing that $F_{a\beta}$ is a product of a spin-1-representation and a spin-$\frac{1}{2}$-representation. We use the rule

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}.$$

We can thus write

$$F_{a\beta} = \tilde{F}_{a\beta} + (\Gamma_a)_{\beta\sigma}\chi^\sigma.$$

Here $\tilde{F}_{a\beta}$ is still a $1 \otimes \frac{1}{2}$ tensor. We have to put constrains so that the spinorial part of $\tilde{F}_{a\beta}$ vanishes. This is done by $(\Gamma^a)^\beta{}_\gamma F_{a\gamma} = 0$. Important to notice here is that as F_{AB} is a matrix with respect to the gauge group, so is χ. With this constraint equation (4) in (D.35) becomes

$$\Gamma^a_{(\alpha\beta}\tilde{F}_{|a|\gamma)} + (\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta}\chi^\delta = 0. \tag{D.36}$$

Now let us take a closer look at the second term. It is symmetric in α, β. Further more, we can expand any spinor bilinear in the gamma basis we developed in App. E.2.2. This is due to the Fierz identity. We also know that when contracting a bilinear with an element of the gamma basis all terms expect the term with the same element vanish, this is due to the basis elements being "orthogonal" under the trace.

As we want to expand eq. (D.36) in $\alpha\beta$ we only need the symmetric basis elements. A look in table E.3 we see the only ones are $\Gamma^a_{\alpha\beta}$, and $\Gamma^{[5]}_{\alpha\beta}$. Thus we can write

$$(\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta}\chi^\delta = \Gamma^a_{(\alpha\beta}v_{\gamma)a} + \Gamma^{[5]}_{\alpha\beta}v_{[5]\gamma}. \tag{D.37}$$

Now lets contract with Γ^a

$$\begin{aligned}(\Gamma^b)^{\alpha\beta}(\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta}\chi^\delta =& \frac{1}{3}(\Gamma^b)^{\alpha\beta}\Big(\Gamma^a_{\alpha\beta}(\Gamma_a)_{\gamma\delta} + \Gamma^a_{\beta\gamma}(\Gamma_a)_{\alpha\delta} + \Gamma^a_{\gamma\alpha}(\Gamma_a)_{\beta\delta}\Big)\chi^\delta = \\ =& \frac{1}{3}\left(Tr(\Gamma^b\Gamma^a)(\Gamma_a)_{\gamma\delta} + 2(\Gamma^a\Gamma^b\Gamma_a)_{\gamma\delta}\chi^\delta\right) = \\ =& \frac{1}{3}\left(16\eta^{ab}(\Gamma_a)_{\gamma\delta} - 2\cdot 8(\Gamma^b)_{\gamma\delta}\right)\chi^\delta = 0.\end{aligned}$$

In the last step we used that $\Gamma^a\Gamma^b\Gamma_a = 2\eta^{ab}\Gamma_a - \Gamma^b\Gamma^a\Gamma_a = 2\Gamma^b - 10\Gamma^b = -8\Gamma^b$. This means that $v_{a\gamma}$ in eq. (D.37) is zero. Similarly when contracting with $(\Gamma^{[5]})^{\alpha\beta}$

$$\begin{aligned}\frac{1}{3}(\Gamma^{[5]})^{\alpha\beta}\Big(\Gamma^a_{\alpha\beta}(\Gamma_a)_{\gamma\delta} + 2\Gamma^a_{\beta\gamma}(\Gamma_a)_{\alpha\delta}\Big)\chi^\delta &= \\ = \frac{2}{3}(\Gamma^a\Gamma^{[5]}\Gamma_a)_{\gamma\delta}\chi\delta.&\end{aligned} \tag{D.38}$$

Now we argue that $\Gamma^a\Gamma^{[b_1\cdots b_5]}\Gamma_a = 0$. It is zero because five terms in the sum will have an $a \in \{b_1, \dots, b_5\}$ yielding $\Gamma^{[5]}$ whilst five terms will have an $a \notin \{b_1, \dots, b_2\}$ yielding $-\Gamma^{[5]}$ Thus $\Gamma^a\Gamma^{[5]}\Gamma_a = (5\Gamma^{[5]} - 5\Gamma^{[5]}) = 0$.

We thus have that eq. (D.38) is zero. This implies that $v_{[5]\gamma} = 0$ which makes

$$(\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta} \equiv 0.$$

Equation (4) in (D.35) now becomes

$$\Gamma^a_{(\alpha\beta}\tilde{F}_{|a|\gamma)} = 0.$$

Contracting this with $(\Gamma^b)^{\alpha\beta}$ again gives

$$\begin{aligned}(\Gamma^b)^{\alpha\beta}(\Gamma^a)_{(\alpha\beta}\tilde{F}_{|a|\gamma)} &= 16\eta^{ab}\tilde{F}_{a\gamma} + 2(\Gamma^a\Gamma^b)_\gamma{}^\beta\tilde{F}_{a\beta} = \\ &= 16\eta^{ab}\tilde{F}_{a\gamma} + 2(2\eta^{ab}\delta_\gamma{}^\beta\tilde{F}_{a\beta} - \Gamma^b\underbrace{(\Gamma^a)_\gamma{}^\beta\tilde{F}_{a\beta}}_{=0}) = 20\eta^{ab}\tilde{F}_{a\gamma} = 0 \\ &\implies \tilde{F}_{a\beta} = 0.\end{aligned}$$

This means hat $F_{a\beta} = (\Gamma_a)_{\beta\sigma}\chi^\sigma$. The Bianchi identities now read

$$\begin{aligned}&(1')\ \mathscr{D}_{[a}F_{bc]} = 0 \\ &(2')\ 2\mathscr{D}_{[a}(\Gamma_{b]})_{\gamma\sigma}\chi^\sigma + \mathscr{D}_\gamma F_{ab} = 0 \\ &(3')\ \mathscr{D}_{(\beta}(\Gamma_a)_{\gamma)\sigma}\chi^\sigma = i\Gamma^d{}_{\beta\gamma}F_{da} \\ &(4')\ 0 = 0.\end{aligned} \tag{D.39}$$

Contracting (3′), with $(\Gamma_c)^{\beta\gamma}$ we get on the right hand side

$$(\Gamma_c)^{\beta\gamma} i(\Gamma^b)_{\beta\gamma} F_{ba} = iTr(\Gamma_c\Gamma^b)F_{ba} = 16iF_{ca}.$$

On the left hand side we get

$$(\Gamma_c)^{\beta\gamma} \mathscr{D}_{(\beta}(\Gamma_a)_{\gamma)\delta}\chi^\delta = \mathscr{D}_\beta(\Gamma_c\Gamma_a)^\beta{}_\delta\chi^\delta = (\Gamma_{ca})^\beta{}_\delta\mathscr{D}_\beta\chi^\delta + \eta_{ca}\mathscr{D}_\beta\chi^\beta. \tag{D.40}$$

As F_{ab} is antisymmetric eq. (D.40) tells us

$$\begin{aligned} &i)\ \mathscr{D}_\beta\chi^\beta = 0 \\ &ii)\ F_{ab} = \frac{-i}{16}(\Gamma_{ab})^\alpha{}_\beta\mathscr{D}_\alpha\chi^\beta. \end{aligned} \tag{D.41}$$

Next we expand $\mathscr{D}_\alpha\chi^\beta$ in the gamma basis. The basis matrices with an index structure $()_\alpha{}^\beta$ are $\delta_\alpha{}^\beta$, $(\Gamma^{[2]})_\alpha{}^\beta$ and, $(\Gamma^{[4]})_\alpha{}^\beta$. We have

$$\mathscr{D}_\alpha\chi^\beta = \delta_\alpha{}^\beta v + (\Gamma^{ab})_\alpha{}^\beta v_{ab} + (\Gamma^{abcd})_\alpha{}^\beta v_{abcd}. \tag{D.42}$$

Eq. (D.41) tells us, as all gamma matrices in the basis are traceless, that v in eq. (D.42) is zero. The next thing we can do is contract (3′) with $(\Gamma_{[5]})^{\beta\gamma}$ this will make the right hand side zero. The left hand side becomes

$$\begin{aligned} (\Gamma_{[5]})^{\beta\gamma}\mathscr{D}_{(\beta}(\Gamma_a)_{\gamma)\sigma}\chi^\sigma =& 2(\Gamma_{[5]})^{\beta\gamma}(\Gamma_a)^\gamma{}_\sigma\mathscr{D}_\beta\chi^\sigma = \\ =& 2(\Gamma_{[5]})^{\beta\gamma}(\Gamma_a)_{\gamma\sigma}\left((\Gamma^{b_1b_2})_\beta{}^\sigma v_{b_1b_2} + (\Gamma^{c_1\cdots c_4})_\beta{}^\sigma v_{c_1\cdots c_4}\right) = \\ =& 2\underbrace{Tr(\Gamma^{b_1b_2}\gamma_a\Gamma_{[5]})}_{=0} v_{b_1b_2} + 2Tr(P_L\Gamma^{c_1\cdots c_4}\Gamma_a\Gamma_{[5]})v_{c_1\cdots c_4} = \\ =& 2\eta_{ab}Tr\left(P_L(\Gamma^{c_1\cdots c_4b} + 4\Gamma^{[c_1\cdots c_3}\eta^{c_4]b})\Gamma_{[5]}\right) = \\ =& 2\eta_{ab}Tr\left(P_L\Gamma^{c1\cdots c_4b}\Gamma_{a_1\cdots a_5}\right)v_{c_1\cdots c_4} \end{aligned} \tag{D.43}$$

Where the first term in the third line is zero as it can be written as an expansion of the gamma basis with the lowest term being a $\Gamma^{[2]}$, and all basis elements have trace zero except the identity. The same is true for the second term in the fourth line. In the second term in the third line we inserted the projection operator $P_L = \frac{1}{2}(\mathbf{1} + \Gamma^{11})$ to imply that we are working in the block form of the matrices. Thus the trace in the last 3 lines is now over the complete 32-dimensional matrix, but with the projection .

Now how can we better understand $\Gamma^{c_1\cdots c_4b}\Gamma_{a_1\cdots a_5}$? As we know we will be taking the trace of the result and that all basis elements in the gamma basis except the identity are traceless we should expand it in the gamma basis, this is done in eq. (D.44). This will require some combinatorics. For each term in the expansion we assume that a number of the indices are the same thus squaring the corresponding gamma matrices to 1. But as we cannot know which ones are equal we must antisymmetrise over both the upper and lower indices. We also need that each term in the expansion normalise to just the corresponding gamma matrix.

Thus we need to counteract the antisymmetrisation, hence the $(5!)^2$. The other factors comes all the terms in the antisymmetrisation which due to the commutation of the deltas or the anticommutation of the $\Gamma^{[n]}$ adds up to the same. The expansion is thus

$$\begin{aligned}\Gamma^{c_1\cdots c_4 b}\Gamma_{a_1\cdots a_5} = & \overbrace{\Gamma^{c_1\cdots c_4 b}{}_{a_1\cdots a_5}}^{=\epsilon^{c_1c_2c_3c_4b}{}_{a_1a_2a_3a_4a_5}\Gamma^{11}} + \Big(\frac{5!}{4!}\Big)^2 \delta^{[c_1}_{[a_1}\Gamma^{c_2c_3c_4b]}{}_{a_2\cdots a_5]} \\ & - \frac{1}{2!}\Big(\frac{5!}{3!}\Big)^2 \delta^{[c_1}_{[a_1}\delta^{c_2}_{a_2}\Gamma^{c_3c_4b]}{}_{a_3a_4a_5]} - \frac{1}{3!}\Big(\frac{5!}{2!}\Big)^2 \delta^{[c_1}_{[a_1}\delta^{c_2}_{a_2}\delta^{c_3}_{a_3}\Gamma^{c_4b]}{}_{a_4a_5]} + \\ & \frac{1}{4!}\Big(\frac{5!}{1!}\Big)^2 \delta^{[c_1}_{[a_1}\delta^{c_2}_{a_2}\delta^{c_3}_{a_3}\delta^{c_4}_{a_4}\Gamma^{b]}{}_{a_5]} + \frac{5!^2}{5!}\underbrace{\delta^{[c_1}_{[a_1}\delta^{c_2}_{a_2}\delta^{c_3}_{a_3}\delta^{c_4}_{a_4}\delta^{b]}_{a_5]}}_{=\frac{1}{5!}\delta^{c_1c_2c_3c_4b}_{a_1a_2a_3a_4a_5}}.\end{aligned} \tag{D.44}$$

Now we go back to eq. (D.43). As all $\Gamma^{[n]}$ are traceless only the last term in the expansion should survive. However, we must take in consideration that we are now working in the left projected blocks of the matrices. This could mean that the traces are not zero. But as we know from our construction of the $SO(1,9)$ representation in App. E.2.2 the matrices are block diagonal with the gamma matrices from the $SO(8)$ representation, which are also traceless. So all is well. Although $\Gamma^{11} = \sigma^3 \otimes \mathbf{1}$ and thus not traceless in its block. Equation (D.43) now becomes

$$\begin{aligned}&(\Gamma_{a_1\cdots a_5})^{\beta\gamma}\mathscr{D}_{(\beta}(\Gamma_a)_{\gamma)\sigma}\chi^\sigma = \\ =&2\eta_{ab}Tr\left(P_L\Gamma^{c_1c_2c_3c_4b}\Gamma_{a_1a_2a_3a_4a_5}\right)v_{c_1c_2c_3c_4} = \\ =&2\eta_{ab}\delta^{c_1c_2c_3c_4b}_{a_1a_2a_3a_4a_5}Tr\Big(\frac{1}{2}(\mathbf{1}+\Gamma^{11})\Big)v_{c_1c_2c_3c_4} + \\ &+2\eta_{ab}\epsilon^{c_1c_2c_3c_4b}{}_{a_1a_2a_3a_4a_5}Tr\Big(\frac{1}{2}(\mathbf{1}+\Gamma^{11})\Gamma^{11}\Big)v_{c_1c_2c_3c_4} = \\ =&2\eta_{ab}\delta^{c_1c_2c_3c_4b}_{a_1a_2a_3a_4a_5}\frac{32}{2}v_{c_1c_2c_3c_4} + 2\eta_{ab}\epsilon^{c_1c_2c_3c_4b}{}_{a_1a_2a_3a_4a_5}\frac{32}{2}v_{c_1c_2c_3c_4}\end{aligned}$$

Then as a is a free index we can choose $a = a_5$ which puts the second term to zero. This implies that $v_{c_1\cdots c_4} = 0$. The expansion of the covariant derivative is now

$$\mathscr{D}_\alpha\chi^\beta = (\Gamma^{b_1b_2})_\alpha{}^\beta v_{b_1b_2}.$$

Using this in $ii)$ in eq. (D.41) we get

$$F_{ab} = \frac{-i}{16}(\Gamma_{ab})^\alpha{}_\beta\mathscr{D}_\alpha\chi^\beta = \frac{i}{16}Tr(\Gamma_{ab}\Gamma^{b_1b_2})v_{b_1b_2} = -2i\delta^{b_1b_2}_{ab}v_{b_1b_2} = -2iv_{ab}.$$

Using this we have

$$\mathscr{D}_\alpha\chi^\beta = \frac{i}{2}(\Gamma^{ab})_\alpha{}^\beta F_{ab}$$

We see now that we only have two fields present, F_{ab} and χ^α. To get here we have used up all information in $(3')$ and $(4')$. But we still have $(2')$ and $(1')$ left. Remarkably $(2')$ helps us in a somewhat unusual manner. We write down the Dirac equation for χ

$$\begin{aligned}(\not{\mathscr{D}}\chi)_\alpha =&(\Gamma^a)_{\alpha\beta}\mathscr{D}_a\chi^\beta = \frac{-1}{2i}\{\mathscr{D}_\alpha,\mathscr{D}_\beta\}\chi^\beta = \frac{i}{2}\mathscr{D}_\beta\left(\frac{-i}{2}(\Gamma^{ab})_\alpha{}^\beta F_{ab}\right) = \\ =&\frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta\mathscr{D}_\beta F_{ab} = \frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta\left(-2\mathscr{D}_{[a}(\Gamma_{b]})_{\beta\sigma}\chi^\sigma\right) = \\ =&-\frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta\left(\mathscr{D}_a(\Gamma_b)_{\beta\sigma}-\mathscr{D}_b(\Gamma_a)_{\beta\sigma}\right)\chi^\sigma = \\ =&-\frac{1}{2}(\Gamma^{ab}\Gamma_b)_{\alpha\sigma}\mathscr{D}_a\chi^\sigma = \frac{1}{4}(\Gamma^a\Gamma^b\Gamma_b-\Gamma^b\Gamma^a\Gamma_b)_{\alpha\sigma}\mathscr{D}_a\chi^\sigma = \\ =&-\frac{1}{4}(18\Gamma^a)_{\alpha\sigma}\mathscr{D}_a\chi^\sigma = \frac{-9}{2}(\not{\mathscr{D}}\chi)_\alpha.\end{aligned}$$

Where we, in the second line used (2′). We also used eq. (D.27) and that $F_{\alpha\beta} = 0$ to get $\{\mathscr{D}_\alpha,\mathscr{D}_\beta\}\chi^\sigma = -T^D_{\alpha\beta}\mathscr{D}_D\chi^\sigma = -2i(\Gamma^d)\mathscr{D}_d\chi^\sigma$. We thus get $(\not{\mathscr{D}}\chi)_\alpha = 0$, which is just the source free Dirac equation. Notice that the Bianchi identities helped us find the equation of motion for χ. Next we want to find the equation of motion for the field strength. We can use (2′) in eq. (D.39) to get there. By contracting the second term with $\mathscr{D}_\beta(\Gamma^a)^{\beta\gamma}$ we get

$$\mathscr{D}_\beta(\Gamma^a)^{\beta\gamma}\mathscr{D}_\gamma F_{ab} = \frac{1}{2}(\Gamma^a)^{\beta\gamma}\{\mathscr{D}_\beta,\mathscr{D}_\gamma\}F_{ab} = \frac{1}{2}(\Gamma^a)^{\beta\gamma}(-2i(\Gamma^c)_{\beta\gamma})\mathscr{D}_c F_{ab} = -16i\mathscr{D}^a F_{ab}.$$

For the first term we get

$$\begin{aligned}\mathscr{D}_\beta(\Gamma^a)^{\beta\gamma}2\mathscr{D}_{[a}(\Gamma_{b]})_{\gamma\sigma}\chi^\sigma =&\mathscr{D}_\beta(\Gamma^a)^{\beta\gamma}\Bigg((\Gamma_b)_{\gamma\sigma}\mathscr{D}_a-(\Gamma_a)_{\gamma\sigma}\mathscr{D}_b\Bigg)\chi^\sigma = \\ = \mathscr{D}_\beta\left((\Gamma^a\Gamma_b)^\beta{}_\sigma\mathscr{D}_a - 10\delta^\beta{}_\sigma\mathscr{D}_b\right)\chi^\sigma =&\mathscr{D}_\beta\Bigg(2\delta^a{}_b\delta^\beta{}_\sigma\mathscr{D}_a\chi^\sigma-(\Gamma_b)^{\beta\gamma}\underbrace{(\not{\mathscr{D}}\chi)_\gamma}_{=0}-10\mathscr{D}_b\chi^\beta\Bigg) = \\ = \mathscr{D}_\beta\Big(2\mathscr{D}_b\chi^\beta-10\mathscr{D}_b\chi^\beta\Big) = -8\mathscr{D}_\beta\mathscr{D}_b\chi^\beta =&-8[\mathscr{D}_\beta,\mathscr{D}_b]\chi^\beta = -8(-\chi^\beta F_{\beta b}-F_{\beta b}\chi^\beta) = \\ =&8\Big(-\chi^\beta(\Gamma_b)_{\beta\alpha}\chi^\alpha-(\Gamma_b)_{\beta\alpha}\chi^\alpha\chi^\beta\Big) = \\ =&-8(\Gamma_b)_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}.\end{aligned}$$

Now, (2') tells us

$$-16i\mathscr{D}^a F_{ab} = 8(\Gamma_b)_{\alpha\beta}\{\chi^\alpha,\chi^\beta\} \Leftrightarrow \mathscr{D}^a F_{ab} = \frac{i}{2}(\Gamma_b)_{\alpha\beta}\{\chi^\alpha,\chi^\beta\}. \tag{D.45}$$

And so we get the Yang-Mills equation of motion with a source term.

Let's take a quick break and think over what we just did. We started of with the Bianchi identities for supersymmetric Yang-Mills in 10 dimensions. We then put restrictions on the spinorial part of the field strength and found that the theory consisted of only the field strength and a spin 1/2 particle. But not only this, after some tedious algebra we also obtained the equations of motion. It is pretty remarkable that identities of a symmetry can provide us with not only the constituent fields of a field theory but the complete dynamics of said theory. Now all that is left is to make sure we have done the calculations correct.

D.8.3 SUSY transformations in $D = 10$ super-Yang-Mills

When we now have the constituent fields of our theory and their corresponding equations of motion we need to construct a Lagrangian invariant under supertransformations and which gives the correct dynamics.

We look first at the supersymmetry transformations of the theory. The transformations are as before

$$\delta_\varepsilon \chi^\alpha \Big|_{\theta=0} \equiv -i\bar{\varepsilon} Q \chi^\alpha \Big|_{\theta=0}.$$

We do not have the complex conjugate here as we are working with Weyl-Majorana spinors. Further more we evaluate at $\theta = 0$ because we are interested in the physical component field in the expansion of a superfield. Now just as when we did the supertransformations in sec. D.5.1 we want to exchange the Q with a covariant derivative. At $\theta = 0$ we have from eq. (D.23)

$$Q_\alpha \chi^\beta = D_\alpha \chi^\beta = \mathscr{D}_\alpha \chi^\beta + \{\chi^\beta, \phi_\alpha\}.$$

Thus for a supertransformations we have

$$\delta_\varepsilon \chi^\beta \Big|_{\theta=0} \equiv -i\bar{\varepsilon} Q \chi^\beta \Big|_{\theta=0} = -i\bar{\varepsilon}^\alpha \mathscr{D}_\alpha \chi^\beta - i\bar{\varepsilon}^\alpha (\chi^\beta \phi_\alpha + \phi_\alpha \chi^\beta).$$

We can make this entirely dependent on the covariant derivative by performing a gauge transformation δ_Λ with our supertransformation. When working with infinitesimal transformations we write the group action in first order. Thus

$$\chi' = X^{-1} \chi X \simeq (\mathbf{1} - \Lambda)\chi(\mathbf{1} + \Lambda) \implies \delta_\Lambda \chi = \chi\Lambda - \Lambda\chi.$$

By choosing $\Lambda = -i\bar{\varepsilon}^\beta \phi_\beta$ and combining this with the supertransformation we get, evaluation at $\theta = 0$ implied,

$$\begin{aligned} \delta_s \chi^\alpha \equiv \delta_\varepsilon \chi^\alpha + \delta_\Lambda \chi^\alpha = -i\bar{\varepsilon}^\beta \mathscr{D}_\beta \chi^\alpha - i\bar{\varepsilon}^\beta (\chi^\alpha \phi_\beta + \phi_\beta \chi^\alpha) - \chi^\alpha (i\bar{\varepsilon}^\beta \phi_\beta - i\bar{\varepsilon}^\beta \phi_\beta \chi^\alpha) = \\ -i\bar{\varepsilon}^\beta \mathscr{D}_\beta \chi^\alpha = \frac{\bar{\varepsilon}^\beta}{2} (\Gamma^{ab})_\beta{}^\alpha F_{ab}. \end{aligned}$$

Similarly we get the variation of the field strength as

$$\delta_s F_{ab} = -i\bar{\varepsilon}^\beta \mathscr{D}_\beta F_{ab} = 2i\bar{\varepsilon}^\beta \mathscr{D}_{[a} (\Gamma_{b]})_{\beta\sigma} \chi^\sigma$$

We do not need to look at $F_{a\beta}$ as it only contains χ as a field.

Now we need to construct our Lagrangian so that it is invariant under these transformations and that it gives the correct dynamics. As both the spinor field and the field strength are Lie algebra valued, meaning they are a linear combination of the generators, we should take the trace over the square of the fields. In fact the trace will help us in a few of the calculations so the use of the trace to make the Lagrangian a scalar is justified on many levels. The

Lagrangian can be written as

$$\mathscr{L} = BTr(F_{ab}F^{ab}) + CTr\Big(\chi \not{\mathscr{D}} \chi\Big),$$

where B and C are just constants we need to determine to get the correct dynamics. For that we vary the Lagrangian with respect to the gauge field.

$$\begin{aligned}
\delta_\phi \mathscr{L} =& Tr\Big(2BF^{ab}\delta_\phi F_{ab} + C(\Gamma^b)_{\alpha\beta}\delta_\phi(\chi^\alpha D_b\chi^\beta - \chi^\alpha[\phi_b, \chi^\beta]))\Big) = \\
=& Tr\Big(4BF^{ab}\mathscr{D}_a\delta\phi_b + C(\Gamma^b)_{\alpha\beta}(\chi^\alpha\chi^\beta\delta\phi_b - \chi^\alpha\delta\phi_b\chi^\beta)\Big) = \\
=& Tr\Big(- 4B\mathscr{D}_a F^{ab}\delta\phi_b + C(\Gamma^b)_{\alpha\beta}\{\chi^\alpha, \chi^\beta\}\delta\phi_b)\Big).
\end{aligned} \tag{D.46}$$

Here we used a few tricks to get to the last line. In the second line we used that

$$\begin{aligned}
\delta_\phi F_{ab} =& \delta_\phi(\partial_a\phi_b - \partial_b\phi_a - [\phi_a, \phi_b]) = \partial_a\delta\phi_b - \partial_b\delta\phi_b - [\delta\phi_a, \phi_b] - [\phi_a, \delta\phi_b] = \\
=& \partial_a\delta\phi_b + [\delta\phi_b, \phi_a] - \partial_b\delta\phi_a - [\delta\phi_a, \phi_b] = 2\mathscr{D}_{[a}\delta\phi_{b]}.
\end{aligned}$$

We then used the cyclicity of the trace on the last term on the second line to move δA to the right, this added a minus sign when χ^β passed χ^α. Lastly we performed a partial integration to move the covariant derivative to F. Partial integration with respect to $\mathscr{D}$ is valid as

$$\begin{aligned}
Tr\Big(F^{ab}\mathscr{D}_a\delta\phi_b\Big) =& Tr\Big(F^{ab}D_a\delta\phi_b - F^{ab}\phi_a\delta\phi_b + F^{ab}\delta\phi_b A_a\Big) = \\
=& Tr\Big(- D_a F^{ab}\delta\phi_b + [\phi_a, F^{ab}]\delta\phi_b\Big) = -Tr\Big(\mathscr{D}_a F^{ab}\delta\phi_b\Big).
\end{aligned}$$

So we see from (D.46) that putting $B = \frac{-1}{4}$ and $C = \frac{-i}{2}$ we retrieve eq. (D.45). The Lagrangian is thus

$$\mathscr{L} = \frac{-1}{4}Tr(F_{ab}F^{ab}) - \frac{i}{2}Tr\Big(\chi \not{\mathscr{D}} \chi\Big).$$

The only thing left to check is invariance under supertransformations.

$$\begin{aligned}
\delta_s\mathscr{L} =& - Tr\left[\frac{2}{4}F^{ab}\delta_s F_{ab} + \frac{i}{2}\Big(\delta_s\chi\not{\mathscr{D}}\chi + \chi\not{\mathscr{D}}\delta_s\chi\Big)\right] = \\
=& Tr\left[i\bar{\varepsilon}^\beta F^{ab}\mathscr{D}_a(\Gamma_b)_{\alpha\beta}\chi^\alpha + \frac{i}{2}\Gamma^c{}_{\alpha\beta}\delta_s\chi^\alpha\mathscr{D}_c\chi^\beta + (-)^2\frac{i}{2}\Gamma^c{}_{\alpha\beta}\delta_s\chi^\beta\mathscr{D}_c\chi^\alpha)\right] \\
=& - Tr\left[i\bar{\varepsilon}^\beta F^{ab}\mathscr{D}_a(\Gamma_b)_{\alpha\beta}\chi^\alpha + \frac{i}{2}\Gamma^c{}_{\alpha\beta}\frac{\bar{\varepsilon}^\gamma}{2}(\Gamma^{ab})_\gamma{}^\alpha F_{ab}\mathscr{D}_c\chi^\beta + \frac{i}{2}\Gamma^c{}_{\alpha\gamma}\frac{\bar{\varepsilon}^\beta}{2}(\Gamma^{ab})_\beta{}^\alpha F_{ab}\mathscr{D}_c\chi^\gamma))\right] = \\
=& - Tr\left[i\bar{\varepsilon}^\beta F^{ab}\mathscr{D}_a(\Gamma_b)_{\alpha\beta}\chi^\alpha + \frac{i\bar{\varepsilon}^\gamma}{2}F^{ab}(\Gamma_{ab}\Gamma^c)_{(\gamma\beta)}\mathscr{D}_c\chi^\beta\right] = \\
=& - Tr\left[i\bar{\varepsilon}^\beta F^{ab}\mathscr{D}_a(\Gamma_b)_{\alpha\beta}\chi^\alpha + \frac{i\bar{\varepsilon}^\gamma}{2}F^{ab}\Big(\underbrace{(\Gamma_{ab}{}^c)_{(\gamma\beta)}}_{=0} + 2(\Gamma_{[a}\delta^c_{b]})_{(\gamma\beta)}\Big)\mathscr{D}_c\chi^\beta\right] = \\
=& - Tr\left[i\bar{\varepsilon}^\beta F^{ab}\mathscr{D}_a(\Gamma_b)_{\alpha\beta}\chi^\alpha + i\bar{\varepsilon}^\beta F^{ab}(\Gamma_a)_{\beta\alpha}\mathscr{D}_b\chi^\alpha\right] = 0.
\end{aligned}$$

With this we are now done with our discussion on 10 dimensional super-Yang-Mills. We have shown that our theory is supersymmetric and we have found the constituent physical fields. We willl now continue on with supergravity in 11 dimensions.

D.9 Supergravity in $D = 11$

In this section we will do a brief review of the superspace derivation of supergravity in 11-dimensions[31, 32]. The logic will follow pretty much the same flow as for the super-Yang-Mills. We will start from the Bianchi Identities and with some restrictions on the components we will solve these to end up at the equations of motion. After this we will compactify to 4-dimensions. For a full treatment of $D = 11$ supergravity we refer to [33], [42], and [10].

D.9.1 Why supergravity?

We have now formulated our framework of supersymmetry. The transformations can be interpreted as coordinate transformations in superspace. Then why not gauge this symmetry, i.e making the transformations local? We know that when introducing local symmetries the theory will require a gauge field in order to make the theory invariant. The field needed when we gauge SUSY is in fact a spin 3/2 field, called the Gravitino. Furthermore the theory will require a massless spin 2 field. This field can be related to the metric, i.e the graviton. This implies a natural connection between local SUSY and gravity. Perhaps more important, local SUSY provides a unification between particle physics and gravity [43].

There are some additional, more subtle reasons for why we should analyse 11 dimensional supergravity. Eleven dimensions is the highest dimension allowing SUSY, given we do not consider particles with spin higher than 2. 11-dimensional supergravity is also the low energy limit of our current leading candidate for a theory of everything; M-Theory.

Einsteins theory of gravity is nonrenormalisable in its formulation. The quantisation of gravity is one of the biggest problems in modern physics. In supergravity the UV divergences when doing quantum gravity calculations become milder. The use of SUGRA might thus be a step in the right direction in order to formulate a quantum theory of gravity.

D.9.2 Introduction and the Bianchi identities

We will start of our discussion in superspace. After that we will, just as before evaluate at $\theta = 0$ to get our physical fields. We will use the same convention of indices as above, when working in superspace.

Let's first consider the symmetries we need in order to formulate 11-dimensional supergravity. The continuous symmetries are: 11-dimensional general covariance, one local supersymmetry, local $SO(1, 10)$ Lorentz invariance, and abelian gauge invariance [42]. When considering superspace the general covariance and the local supersymmetry combine to general covariance in superspace. The local Lorentz invariance means we are building on what we did in sec.

D.7.1. This is due to the fact that our theory, being supersymmetric, contains fermions. Fermions can only be defined in flat space, i.e in the tangent space of a curved manifold. In the tangent space we reduce the general covariance to Lorentz invariance and since our tangent spaces do not generally coincide we must have a local Lorentz invariance. The abelian gauge invariance is similar to the Maxwell theory it is needed in order to complete our theory of supergravity [42].

In our theory we have three fields present. We have the graviton; a spin 2 field ${e_m}^a$. We also have the superpartner of the graviton; the Gravitino, a spin $\frac{3}{2}$ field, ${\Psi_m}^\alpha$. The Gravitino is the gauge field of the local supersymmetry transformations. Lastly we have a three-form C_{mnp}, the gauge field of our abelian gauge theory. Further more, just as in sec. D.7.1, we need a gauge field for the Lorentz symmetry. This is the spin-connection ${\omega_A}^B$. The spin-connection will thus help us define covariant derivatives with respect to local Lorentz transformations.

$$\mathscr{D}_A(\omega)V_B = D_A V_B - {\omega_{AB}}^C V_C,$$
$$\mathscr{D}_A(\omega)^V B = D_A V^B - {\omega_{AC}}^B V^C.$$

As we are working in ten dimensions the spin connection takes the form

$${\omega_A}^B = ({\omega_a}^b,\ {\omega_\alpha}^\beta) = ({\omega_a}^b,\ \omega^{ab}\frac{1}{4}{(\Gamma_{ab})_\alpha}^\beta).$$

The Γ-matrix in the spinor part of the spin connection is simply because $\frac{1}{4}\Gamma^{ab}$ are the generators of the Lorentz group in the spinor representation.

The gravitational fields are in the spacetime vielbein. It carries the same physical information as the metric does. It is the space part of the super-vielbein. The super-vielbein are on the form, at $\theta = 0$,

$${E_M}^A = \begin{pmatrix} {e_m}^a(x) & {\Psi_m}^\alpha(x) \\ 0 & {\delta_\mu}^\alpha \end{pmatrix}, \qquad {E_A}^M = \begin{pmatrix} {e_a}^m(x) & -{\Psi_a}^\mu(x) \\ 0 & {\delta_\alpha}^\mu \end{pmatrix}.$$

We see here that we let the Gravitino be one component in the vielbein. There is no proper argument for this other than the fact it has the same index structure and the correct dimension to fit in there. We simply just try it out to put it there and we see that it works out.

The field strength for the 3-form is $H = 4dC$, or in component form $H_{ABCD} = 4\partial_{[A}C_{BCD]}$. The field strength of the spin connection is the curvature tensor or the super-Riemann tensor $R = d\omega + \omega \wedge \omega$.

The Bianchi identities we have are [33, 42]

$$\begin{aligned} \mathscr{D}T^A =& E^B R_B{}^A \\ \mathscr{D}R_A{}^B =& 0 \\ dH =& 0. \end{aligned} \tag{D.47}$$

These are the ones used when formulating supergravity. Now we need to impose the appropriate constraints in order to advance. First, there is a theorem by Dragon which states the covariant derivative on the curvature will automatically vanish if the first equation is satisfied [33]. We will thus only impose constraints on the torsion and gauge field strength.

D.9.3 Dimensional analysis and constraints

In order to validate the constraints we will investigate the dimensions of the different components. The dimension of the fields are

$$\begin{gathered} [e_m{}^a] = L^0 \\ [\Psi_m{}^\alpha] = L^{\frac{-1}{2}} \\ [H_{abcd}] = L^{-1} \implies [H] = L^3. \end{gathered}$$

Further more we need the dimensions of the derivatives and corresponding 1-forms

$$\begin{gathered} [D_a] = L^{-1}, \quad [E^a] = L^1 \\ [D_\alpha] = L^{-1/2}, \quad [E^\alpha] = L^{\frac{1}{2}}. \end{gathered}$$

We can now list the dimensions of the different components of the torsion and field strength.

Torsion	Dimension (L^{-1})
$T_{\alpha\beta}{}^\gamma$	1/2
$T_{\alpha\beta}{}^c$	0
$T_{\alpha b}{}^\gamma$	1
$T_{\alpha b}{}^c$	1/2
$T_{ab}{}^\gamma$	3/2
$T_{ab}{}^c$	1

Field Strenth	Dimension (L^{-1})
$H_{\alpha\beta\gamma\delta}$	-1
$H_{\alpha\beta\gamma d}$	$-1/2$
$H_{\alpha\beta cd}$	0
$H_{\alpha bcd}$	1/2
H_{abcd}	1

Now just as in the Yang-Mills case we try to find appropriate constraints on these components. It turns out that we can set

$$T_{\alpha\beta}{}^\gamma = T_{\alpha b}{}^c = T_{ab}{}^c = 0 \qquad H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma d} = H_{\alpha bcd} = 0.$$

This is arguments based on dimensional analysis. As we do not have any fields of dimension 1/2 we can set the corresponding torsion and field strength components to zero. Further more for the components of dimension 0 we can set these equal to gamma matrices

$$T_{\alpha\beta}{}^{c} = 2i(\Gamma^c)_{\alpha\beta} \qquad H_{ab\gamma\delta} = 2i(\Gamma_{ab})_{\gamma\delta}.$$

The Γ-matrices are now the ones from the 10+1 dimensional representation of the Dirac algebra, App. (E.2.3).

D.9.4 Solving the Bianchi identities

The procedure in which we solve the identities in eq. (D.47) is similar to that of the Yang-Mills case. We write out our Bianchi Identities in component form and insert our restrictions on the torsion and H and start solving. We use pretty much the same techniques. We expand in the gamma basis and contract with symmetric or antisymmetric parts. We also break down $T_{ab}{}^{\gamma}$ in its irreducible parts, as we did for $F_{a\beta}$ in Yang-Mills. This will in fact yield that $T_{ab}{}^{\gamma}$ is the field strength for the Gravitino. The explicit calculations can be found in [33] and [10]

Now what falls out from solving these equations and evaluating at $\theta = 0$ are, remarkably, Einstein's equations and the equation of motion for the Gravitino and the field strength. They are on the form [42]

$$\begin{aligned}
&(i)\ R_{mn}(\tilde{\omega}) - \frac{1}{2}g_{mn}R(\tilde{\omega}) = \frac{1}{3}\tilde{H}_{mpqr}\tilde{H}_{n}{}^{pqr} - \frac{1}{24}g_{mn}\tilde{H}_{pqrs}\tilde{H}^{pqrs},\\
&(ii)\ \hat{\Gamma}^{mnp}\tilde{D}_n(\tilde{\omega})\Psi_p = 0,\\
&(iii)\ \nabla_m \tilde{H}^{mpqr} = \frac{-1}{576}\varepsilon^{pqrm_1\cdots m_8}\tilde{H}_{m_1\cdots m_4}\tilde{H}_{m_5\cdots m_8},
\end{aligned}$$

where ε is the Levi-Cevita symbol and [42]

$$\begin{aligned}
\tilde{D}(\tilde{\omega})\Psi_n &= \mathscr{D}_m(\tilde{\omega})\Psi_n + T_n{}^{pqrs}\tilde{H}_{pqrs}\Psi_n\\
T^{smnpq} &= \frac{-i}{144}(\hat{\Gamma}^{smnpq} - 8\hat{\Gamma}^{[mnp}g^{q]s}),\\
\nabla_m(\omega)A_n{}^a &= \partial_m A_n{}^a - A_n{}^b\omega_{mb}{}^a - \Gamma_{mn}{}^p A_p{}^a,\\
\omega_{mab} &= \frac{1}{2}(-\Omega_{mab} + \Omega_{abm} - \Omega_{bma}) + K_{mab}\\
K_{mab} &= \frac{i}{4}\Big(- \bar{\Psi}_n\hat{\Gamma}_{mab}{}^{np}\Psi_p + 2(\bar{\Psi}_m\hat{\Gamma}_b\Psi_a - \bar{\Psi_m}\hat{\Gamma}_a\Psi_b + \bar{\Psi}_b\hat{\Gamma}_m\Psi_a)\Big),\\
\Omega_{mn}{}^a &= 2\partial_{[n}e_{m]}{}^a,\\
\tilde{\omega}_{mab} &= \omega_{mab} + \frac{i}{4}\bar{\Psi}_n\hat{\Gamma}_{mab}{}^{np}\Psi_p,\\
\tilde{H}_{mnpq} &= H_{mnpq} - 3\bar{\Psi}_{[m}\hat{\Gamma}_{np}\Psi_{q]},
\end{aligned}$$

where $\Gamma_{mn}{}^p = \Gamma_{nm}{}^p$ is here the affine connection. The hats on the $\hat{\Gamma}$-matrices are because they satisfy the Dirac algebra with opposite signature, i.e.

$$\{\hat{\Gamma^A}, \hat{\Gamma^b}\} = -2\eta^{ab}.$$

The $\sim$ on the different fields and derivatives are due to so called torsion and contorsion terms. We will not go deeper in on their meaning and implications their presence imply that we no longer work with ordinary gravity. Now when we have all equations of motion we can go a head and present the Lagrangian for 11-dimensional supergravity [44],[42] .

$$\begin{aligned}\mathscr{L} = e\Bigg({e_b}^n {e_a}^m {R_{mn}}^{ab}(\omega) - \frac{1}{2\cdot 4!}F_{mnpq}F^{mnpq} - \frac{i}{2}\bar{\Psi}_m\Gamma^{mnp}\mathscr{D}_n\Big(\frac{1}{2}(\omega+\tilde{\omega})\Big)\Psi_p\Bigg)+\\ -\frac{2}{(12)^4}\epsilon^{m_1\cdots m_{11}}F_{m_1\cdots m_4}F_{m_5\cdots m_8}C_{m_9m_{10}m_{11}}+\\ \frac{3}{4(12)^2}e\Big(\bar{\Psi}_m\hat{\Gamma}^{mnpqrs}\Psi_n + 12\bar{\Psi}^p\hat{\Gamma}^{qr}\Psi^s\Big)\Big(H_{pqrs}+\tilde{H}_{pqrs}\Big),\end{aligned}$$

where $e = \det({e_m}^a)$, i.e. the square root of the metric determinant, $\bar{\Psi} = \Psi^\dagger\hat{\Gamma}_0$ and ϵ is here the Levi-Civita tensor in 11-dimensions.

D.9.5 Compactification of supergravity

We will now resort to compactifying our theory of gravity. Why do we want to do this and what does it mean? As far as experiments go we have found no indication that reality is anything else than 4 dimensional. Thus to make sense of theories formulated in higher dimensions we need to make the extra dimensions undetectable by current experiments. We do this with the concept of compactification. This means that the extra dimensions are closed within themselves with a very small range. Imagine for example a line dancer at a circus. As far as her perception of the rope goes it is one dimensional, she can move forwards and backwards. However for an ant living on the rope it sees in fact that the rope is a two dimensional cylinder, we here have an extra compact dimension.

We have a theory over an 11-dimensional manifold $\mathcal{M}_{11}$. We want to decompose this into a product manifold $\mathcal{M}_{11} = \mathcal{M}_4 \times \mathcal{M}_7$, where $\mathcal{M}_7$ is a compact manifold, usually the 7-torus or 7-sphere. The compact 7-dimensional manifold is sometimes referred to as the *internal* space, whilst the *external* space is the observable 3+1 dimensional spacetime. A compactification like this would imply that at every point in spacetime there is either a 7-torus or 7-sphere. We will not perform any compactification here but refer the interested reader to [42] for an extensive analysis of compactification. For an intuitive introduction to compact dimensions and compactification we recommend [1].

Appendix E

Spinors in 4, 8, 10, and 11 dimensions

E.1 Spinors in $D = 1 + 3$

In 1+3 dimensions we represent the Dirac algebra with 4x4 matrices. With our choice of signature, $(- + ++)$, we will have $(\gamma^0)^2 = -\mathbf{1}_4$, $(\gamma^i)^2 = \mathbf{1}_4$. We can here also define a fifth gamma matrix, γ^5

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{E.1}$$

It is easy to show that $\{\gamma^a, \gamma^5\} = 0$.

E.1.1 Weyl/Chiral representation

We know that the spin representation of $\mathfrak{so}(1,3)$ breaks up into two chiralities; left, and right. We will here write down the Dirac algebra in the Weyl/chiral basis. We first define

$$\sigma^a_{\alpha\dot{\alpha}} = (-\mathbf{1}, \sigma^i)_{\alpha\dot{\alpha}}.$$

Where σ^i are the Pauli matrices. Here α and $\dot{\alpha}$ are 2-valued indices, the undotted are for the Left chiral spinors and the dotted ones are for the Right chiral ones. We raise and lower chiral indices with the $\epsilon^{\alpha\beta}$-tensor which is completely antisymmetric, numerically the dotted and undotted ϵ are identical. Next we can define

$$\bar{\sigma}^{a\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^a_{\beta\dot{\beta}} = (-\mathbf{1}, -\sigma^i).$$

We can use these to represent the γ matrices, namely let

$$(\gamma^a)_{\bar{\beta}}{}^{\bar{\alpha}} = \begin{pmatrix} 0_\beta{}^\alpha & i\sigma^a_{\beta\dot{\alpha}} \\ i(\bar{\sigma}^a)^{\dot{\beta}\alpha} & 0^{\dot{\beta}}{}_{\dot{\alpha}} \end{pmatrix}.$$

Here we clarify that $\bar{\alpha}$ is a four valued index $()^{\bar{\alpha}} = ()^{\alpha}{}_{\dot{\alpha}}$ whilst $()_{\bar{\alpha}} = ()_{\alpha}{}^{\dot{\alpha}}$. These matrices satisfy the Clifford algebra (2.3). We also have the defining equations for A, B, and C,

$$A_{\pm}\gamma^a A_{\pm}^{-1} = \pm(\gamma^a)^{\dagger}, \qquad B_{\pm}\gamma^a B_{\pm}^{-1} = \pm(\gamma^a)^{*}, \qquad C_{\pm}\gamma^a C_{\pm}^{-1} = \pm(\gamma^a)^{\top}. \tag{E.2}$$

Using (E.2) we get $A = -\gamma^0$, the extra minus sign is purely conventional. With this convention γ^5 in eq. (E.1) will take the form

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A Dirac spinor can, in the chiral basis, be written as

$$\Psi_{D\bar{\alpha}} = \begin{pmatrix} \xi_{\alpha} \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}.$$

We can now define the Dirac conjugate as

$$\bar{\Psi} = -\Psi^{\dagger}\gamma^0 = -\begin{pmatrix} \xi^{\dagger\beta} & \bar{\psi}^{\dagger}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} 0_{\beta}{}^{\alpha} & -i_{\beta\dot{\alpha}} \\ -i^{\dot{\beta}\alpha} & 0^{\dot{\beta}}{}_{\dot{\alpha}} \end{pmatrix} = i\begin{pmatrix} \psi^{\alpha} & \bar{\xi}_{\dot{\alpha}} \end{pmatrix}. \tag{E.3}$$

Where we have defined $\xi^{\dagger}_{\alpha} = \bar{\xi}_{\dot{\alpha}} \Leftrightarrow (\bar{\psi}^{\dot{\alpha}})^{\dagger} = \psi^{\alpha}$.

Using γ^5 we can create projection operators on to the different chiralities.

$$P_L = \frac{\mathbf{1}_4 - \gamma^5}{2}, \qquad P_R = \frac{\mathbf{1}_4 + \gamma^5}{2}.$$

E.1.2 Majorana representation

Supersymmetry requires Majorana spinors, this we can see in the Wess-Zumino theory in App. D.3.1. We will now use a different representation of the Dirac algebra, namely the Majorana representation, in this representation all matrices are real, as a Majorana spinor correspond to a real Dirac spinor. We will construct the γ-matrices in a systematic way by the tensor product. We will use the real matrices σ^1, σ^3, $\epsilon = i\sigma^2$, and $\mathbf{1}$. For $\mathfrak{so}(1,3)$ we can construct the γ-matrices as

$$\begin{aligned} \gamma^0 &= \epsilon \otimes \mathbf{1} \qquad (\gamma^0)^2 = -1 \\ \gamma^1 &= \sigma^1 \otimes \sigma^1 \qquad (\gamma^1)^2 = 1 \\ \gamma^2 &= \sigma^1 \otimes \sigma^3 \qquad (\gamma^2)^2 = 1 \\ \gamma^3 &= \sigma^3 \otimes \mathbf{1} \qquad (\gamma^3)^2 = 1. \end{aligned}$$

Now using the defining equations for A, B, and C, eq. (E.2) and the fact that all γ^a are real and equal to their transpose, except for $\gamma^0 = -(\gamma^0)^\top$, we get that

$$A_+ = C_+ = \gamma^1\gamma^2\gamma^3 \qquad A_- = C_- = \gamma^0. \tag{E.4}$$

E.1.2.1 Majorana flips

In supersymmetry Majorana spinors are used frequently, one particular thing used in calculations are the "Majorana flips", we use the symmetric or antisymmetric properties of the "all index up" matrices in the γ-basis to flip the order of various contracted spinors. C in the following equations is here C_- defined in eq. (E.4).

$$\begin{aligned}
\tilde{\psi}\chi &= \tilde{\psi}^\alpha\chi_\alpha = \psi_\beta C^{\beta\alpha}\chi_\alpha = -\chi_\alpha C^{\beta\alpha}\psi_\beta = \tilde{\chi}^\alpha\psi_\alpha = \tilde{\chi}\psi\\
\tilde{\psi}\gamma^a\chi &= \tilde{\psi}^\alpha(\gamma^a)_\alpha{}^\beta\chi_\beta = \psi_\delta C^{\delta\alpha}(\gamma^a)_\alpha{}^\beta\chi_\beta = -\chi_\beta C^{\beta\alpha}(\gamma^a)_\alpha{}^\delta\psi_\delta = -\tilde{\chi}\gamma^a\psi\\
\tilde{\psi}\gamma^{ab}\chi &= \psi_\delta C^{\delta\alpha}(\gamma^{ab})_\alpha{}^\beta\chi_\beta = -\chi_\beta C^{\beta\delta}(\gamma^{ab})_\delta{}^\delta\psi_\delta = -\tilde{\chi}\gamma^{ab}\psi\\
\tilde{\psi}\gamma^a\gamma^5\chi &= \psi_\delta C^{\delta\alpha}(\gamma^a\gamma^5)_\alpha{}^\beta\chi_\beta = \chi_\beta C^{\beta\delta}(\gamma^a\gamma^5)_\delta{}^\delta\psi_\delta = \tilde{\chi}\gamma^a\gamma^5\psi\\
\tilde{\psi}\gamma^5\chi &= \psi_\delta C^{\delta\alpha}(\gamma^5)_\alpha{}^\beta\chi_\beta = \chi_\beta C^{\beta\delta}(\gamma^5)_\delta{}^\delta\psi_\delta = \tilde{\chi}\gamma^5\psi.
\end{aligned}$$

E.1.3 The γ-basis and its symmetries

The basis for 4x4 matrices using the γ-matrices is

$$\Big\{\mathbf{1}, \gamma^a,\ \gamma^{ab},\ \gamma^{abc},\ \gamma^{abcd}\Big\}.$$

Now, it will be important for us to know the different symmetries of the basis elements of the γ basis, we will look at the basis with both indices up, i.e with multiplication of $C_\pm$ from the left. Let's look at the transpose of $C_\pm\gamma^{[n]}$.

$$(C_\pm\gamma^{[n]})^\top = (\gamma^{[n]})^\top C_\pm^\top = (-1)^\mu(-1)^{\frac{n(n-1)}{2}}(\gamma^\top)^{[n]}C_\pm.$$

Where the factor $(-1)^\mu$ comes from $C_\pm$ being either symmetric ($\mu = 0$) or antisymmetric ($\mu = 1$). The second factor comes from the fact that when we do the transpose of a product with n gamma matrices, we need to anticommute them past each other to come in the correct order. We then use eq. (E.2) again on $(\gamma^\top)^{[n]}C_\pm$. Then for each time moving $C_\pm$ to the left of γ-matrix we will pick up a factor (± 1). We thus end up with

$$(C_\pm\gamma^{[n]})^\top = (\pm 1)^n(-1)^\mu(-1)^{\frac{n(n-1)}{2}}C_\pm\gamma^{[n]}.$$

This can be used to construct a table for the different symmetries of the gamma basis, this is table E.1. Notice that for the case of $C_\pm$ being symmetric we do not generate a proper basis, as we need 10 symmetric and 6 antisymmetric matrices to form a basis. We will see, for example in the Wess-Zumino model (App. D.3.1), that we need $C\gamma^a$ symmetric and $C\gamma^{[abcd]}$

antisymmetric to uphold supersymmetry. This means that we need to use the antisymmetric $C = C_-$.

	$C_- = -C_-^\top$		$C_- = C_-^\top$		$C_+ = -C_+^\top$		$C_+ = C_+^\top$	
	Anti-symmetric	Symmetric	Anti-symmetric	Symmetric	Anti-symmetric	Symmetric	Anti-symmetric	Symmetric
$C\mathbf{1}$	1			1	1			1
$C\gamma^a$		4	4		4			4
$C\gamma^{ab}$		6	6			6	6	
$C\gamma^{[abc]}$	4			4		4	4	
$C\gamma^{[abcd]}$	1			1	1			1

Table E.1: *Table over the symmetries of the γ-basis of $\mathfrak{so}(1,3)$ with different choices of the symmetry of the C Matrix*

As discussed in sec. 2.4.2 we can use the γ-basis to expand spinor bilinears in their irreducible representations.

$$\psi_\alpha \chi_\beta = \frac{-1}{4} C_{\alpha\beta} \bar{\psi}\chi + \frac{1}{4}(\gamma^a)_{\alpha\beta}\bar{\psi}\gamma_a\chi - \frac{1}{8}(\gamma^{ab})_{\alpha\beta}\bar{\psi}\gamma_{ab}\chi + \frac{1}{4}(\gamma^a\gamma^5)_{\alpha\beta}\bar{\psi}\gamma_a\gamma_5\chi - \frac{1}{4}\gamma^5\bar{\psi}\gamma^5\chi.$$

The proportionality constants are determined using the trace. Another useful identity we get if we symmetrise over α and β

$$\psi_2^{(\alpha}\psi_1^{\beta)} = \frac{1}{2}(\psi_2^\alpha\psi_1^\beta + \psi_2^\beta\psi_1^\alpha) = \frac{1}{4}\Big((\gamma^a)^{\alpha\beta}\bar{\psi}_2\gamma_a\psi_1 - \frac{1}{2}(\gamma^{ab})^{\alpha\beta}\bar{\psi}_2\gamma_{ab}\psi_1\Big).$$

E.2 Majorana representation in higher dimensions

We will now discuss spinors in 8, 10, and 11 dimensions. We will present the concrete construction of the Majorana representation for each dimension.

E.2.1 Majorana representation in $D = 8$

We will construct the Majorana representation of $SO(8)$. We can as usually go from the Clifford algebra, and using linear combinations we can then create the generators for $SO(8)$. We will only look at the Clifford algebra though. In Euclidean signature the Clifford algebra now reads

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}.$$

We are looking for 8 anticommuting matrices of dimension $16 = 2^{8/2}$. All we need to do is to create 8 anticommuting matrices, we do it is the same way as we did for $\mathfrak{so}(1,3)$.

$$\begin{aligned}
\gamma^1 &= \sigma^1 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \\
\gamma^2 &= \epsilon \otimes \epsilon \otimes \sigma^1 \otimes \sigma^3 \\
\gamma^3 &= \epsilon \otimes \epsilon \otimes \sigma^1 \otimes \sigma^1 \\
\gamma^4 &= \epsilon \otimes \epsilon \otimes \sigma^3 \otimes \mathbf{1} \\
\gamma^5 &= \epsilon \otimes \sigma^1 \otimes \sigma^3 \otimes \epsilon \\
\gamma^6 &= \epsilon \otimes \sigma^3 \otimes \sigma^3 \otimes \epsilon \\
\gamma^7 &= \epsilon \otimes \mathbf{1} \otimes \epsilon \otimes \mathbf{1} \\
\gamma^8 &= \epsilon \otimes \mathbf{1} \otimes \sigma^1 \otimes \epsilon.
\end{aligned}$$

We can now define

$$\gamma^9 \equiv \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8 = \sigma^3 \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}.$$

All matrices are real which mean that $A = C$. Further more, all the matrices are equal to their transpose, thus the defining equation (E.2) for $C_\pm$ becomes

$$C_\pm \gamma^a C_\pm^{-1} = \pm\gamma^a \Leftrightarrow C_\pm \gamma^a = \pm\gamma^a C_\pm.$$

This means that

$$[C_+, \gamma^a] = 0 \qquad \{C_-, \gamma^a\} = 0 \implies C_+ = \mathbf{1} \text{ and } C_- = \gamma^9.$$

So both C_+ and C_- are symmetric. We will now summarise the symmetries of the element of a basis for all 16×16matrices. The results is presented in table E.2. Since we are in a even dimension we know the representation splits up in two 8 dimensional chiralities. This is also clear as we see that all matrices are in block off-diagonal form, indicating that we are in both the Majorana- and Weyl representation in 8 dimensions. In the table we will also indicate the index structure of the left projected chirality of the matrices. Note that the symmetries of the block matrices are the same as the symmetries of the full matrices and we will thus not have any problem with this two part visualisation of the gamma basis. Observing table E.2 we see that we have, for both choices of C, a complete basis with 136 symmetrical and 120 antisymmetrical matrices. For the left projected block we need $8^2 = 64$ matrices, where 36 are symmetric and 28 antisymmetric. We get 28 from $C\gamma^{[2]}$ and we get 36 from half of the $C\gamma^{[4]}$ as this is self dual, meaning, for the blocks $\epsilon^{abcdefgh}\gamma_{efgh} \propto \gamma^{abcd}$. The identity supplies us with the final symmetric matrix.

	$C = C_+ = \mathbf{1}$		$C = C_- = \gamma^9$	
	Anti-symmetric	Symmetric	Anti-symmetric	Symmetric
$C\mathbf{1}_{\alpha\beta}$		1		1
$C\gamma^{[1]}{}_{\alpha}{}^{\dot\beta}$		8	8	
$C\gamma^{[2]}_{\alpha\beta}$	28		28	
$C\gamma^{[3]}{}_{\alpha}{}^{\dot\beta}$	56			56
$C\gamma^{[4]}_{\alpha\beta}$		70		70
$C\gamma^{[5]}{}_{\alpha}{}^{\dot\beta}$		56	56	
$C\gamma^{[6]}_{\alpha\beta}$	28		28	
$C\gamma^{[7]}{}_{\alpha}{}^{\dot\beta}$	8			8
$C\gamma^{[8]}_{\alpha\beta}$		1		1

Table E.2: *Table over the symmetries of the γ-basis of $\mathfrak{so}(8)$ with different choices of the symmetry of the C Matrix*

E.2.2 Majorana representation in $D = 1 + 9$

In 1+9 dimensions the spinor representation is $2^{\frac{10}{2}} = 32$ dimensional. We can use the γ matrices used for $\mathfrak{so}(8)$ to create our 10 γ-matrices for $\mathfrak{so}(1,9)$. The Clifford algebra is now

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \tag{E.5}$$

Where η is diagonal with $\eta^{00} = -1$ and the rest ones. We construct our gamma matrices as

$$\begin{aligned}\Gamma^0 &= \epsilon \otimes \mathbf{1}_{16} \\ \Gamma^a &= \sigma^1 \otimes \gamma^a \qquad a = 1, \ldots, 9\end{aligned}$$

These satisfy eq. (E.5). Once again we can form the chirality operator $\Gamma^{11} = \Gamma^0 \cdots \Gamma^9 = \sigma^3 \otimes \mathbf{1}$. We see here that this choice of representation is also the Weyl/chiral basis, i.e. all matrices are on block form.

Using the defining equation for $C_\pm$ we get

$$\{C_-, \Gamma^i\} = 0 = [C_+, \Gamma^i], \qquad [C_-, \Gamma^0] = 0 = \{C_+, \Gamma^0\}.$$

This is satisfied if $C_+ = \sigma^1 \otimes \mathbf{1}$ and $C_- = \Gamma^0$. Here C_+ is symmetric whilst C_- is antisymmetric. We can summarise the symmetries of a 32 dimensional basis. Note however also that the matrices all have a block diagonal or block off-diagonal structure. We will, just as in the situation for $SO(8)$, in the table indicate which Weyl index structure the matrices has. We will do it for the left projected chirality. Observe that the notation in table E.3 could be somewhat misleading as the first column is referring to the left projected 16 dimensional nonzero block of the 32 dimensional matrix. But the symmetry properties in the other columns refer to the full 32 dimensional matrix. However the symmetry properties of the block are, just as in the eight dimensional case, the same as for the full matrix.

	$C = C_- = \Gamma^0$		$C = C_+ = \sigma^1 \otimes \mathbf{1}$	
	Anti-symmetric	Symmetric	Anti-symmetric	Symmetric
$(C\mathbf{1})^{\alpha}{}_{\beta}$	1			1
$(C\Gamma^{[1]})^{\alpha\beta}$		10		10
$(C\Gamma^{[2]})^{\alpha}{}_{\beta}$		45	45	
$(C\Gamma^{[3]})^{\alpha\beta}$	120		120	
$(C\Gamma^{[4]})^{\alpha}{}_{\beta}$	210			210
$(C\Gamma^{[5]})^{\alpha\beta}$		252		252
$(C\Gamma^{[6]})^{\alpha}{}_{\beta}$		210	210	
$(C\Gamma^{[7]})^{\alpha\beta}$	120		120	
$(C\Gamma^{[8]})^{\alpha}{}_{\beta}$	45			45
$(C\Gamma^{[9]})^{\alpha\beta}$		10		10
$(C\Gamma^{11})^{\alpha}{}_{\beta}$		1	1	

Table E.3: *Table over the symmetries of the gamma basis for $\mathfrak{so}(1,9)$ with different choices of the symmetry of the C Matrix*

Now, since we now have written our matrices in block form it is possible to find a basis for matrices projected on the left (right) chirality. I.e we want now to find a basis for 16 dimensional matrices. We then need 120 anti-symmetrical and 136 symmetrical. We do this in the upper left block of the $C\Gamma$ matrices, i.e with index structure $()^{\alpha\beta}$. This can be done by acting with the left projection, $P_L = \frac{1}{2}(\mathbf{1} + \Gamma^{11})$ operator from right and left.

Now we simply just pick out from table E.3. $\Gamma^{[3]}$ will supply us with 120 anti-symmetrical matrices which is just what we need. What about $\Gamma^{[7]}$? It is related to $\gamma^{[3]}$ via Γ^{11} by

$$\Gamma^{11}\Gamma^{abcdefg} = \frac{-1}{3!}\epsilon^{abcdefghij}\Gamma_{hij}. \tag{E.6}$$

Equation (E.6) can be altered to fit any combination of antisymmetrised Γ matrices, as long as the sum of the antisymmetrised indices are 10. As Γ^{11} is block diagonal with $\mathbf{1}$ on the top left, when projecting this equation of the left chirality Γ^{11} will act only as an identity. This means that on a block level $\Gamma^{[3]} = \Gamma^{[7]}$. Thus $\Gamma^{[7]}$ will not supply any more independent matrices.

Now for the symmetrical ones, we get 10 from $\Gamma^{[1]}$ and by the same argument as above we do not get any new from $\Gamma^{[9]}$. What is left is $\Gamma^{[5]}$ which are 256 symmetrical ones. However only half of these will provide independent matrices in the left chirality block. This is because it is in fact self dual, meaning

$$\Gamma^{[5]} \propto \epsilon^{[5][5']}\Gamma_{[5']}.$$

We thus have $10 + \frac{256}{2} = 136$ which is what we are looking for.

E.2.3 Majorana representation in $D = 1 + 10$

To create the representation for the Clifford algebra in odd dimensions the simplest thing is to add the chirality operator from the dimension under to the set of matrices. As the chirality operator anticommutes with all the Γ matrices in even dimension they all will satisfy the Clifford algebra in one dimension higher, given one increases the dimension in the correct signature; + if the chirality operator squares to one and - if it squares to -1.

Here we once again have a 32 dimensional representation, we just add Γ^{11} from our representation for $\mathfrak{so}(1,9)$ and rename it Γ^{10}. Our matrices are

$$\begin{aligned}\Gamma^0 &= \epsilon \otimes \mathbf{1}_{16} \\ \Gamma^a &= \sigma^1 \otimes \gamma^a \qquad a = 1, \ldots, 9 \\ \Gamma^{10} = \Gamma^{11} &= \sigma^3 \otimes \mathbf{1}_{16}.\end{aligned}$$

These 11 Matrices satisfy the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$$

for $\eta = diag(-1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. We could try, for fun, to make a chirality operator

$$\Gamma^{12} = \Gamma^0 \cdots \Gamma^{10} = (\epsilon \otimes \mathbf{1})((\sigma^1)^8 \otimes \gamma^9)(\sigma^1 \otimes \gamma^9)(\sigma^3 \otimes \mathbf{1}) = \mathbf{1}_{32}.$$

We get the same commutation relations for $C_\pm$ here as for $\mathfrak{so}(1,9)$

$$\begin{aligned}\{C_-, \Gamma^i\} =& 0 = [C_+, \Gamma^i] \\ [C_-, \Gamma^0] =& 0 = \{C_+, \Gamma^0\}\end{aligned}$$

Which implies that

$$C_- = \Gamma^0.$$

We cannot find a matrix satisfying the equations for C_+. The only ones we have are C_- which are matrices proportional to Γ^0. We can in the same way create a table over the symmetries of the basis elements.

	$C = C_- = \Gamma^0$	
	Anti-symmetric	Symmetric
$C\mathbf{1}$	1	
$C\Gamma^{[1]}$		11
$C\Gamma^{[2]}$		55
$C\Gamma^{[3]}$	165	
$C\Gamma^{[4]}$	330	
$C\Gamma^{[5]}$		462

Table E.4: *Table over the symmetries of the gamma basis for* $\mathfrak{so}(1,10)$

How come not the higher antisymmetrised gamma matrices are included? That is because the higher does not contribute any more independent matrices, as we can see from table E.4 we have already a complete basis of 528 symmetric matrices and 496 anti-symmetric. This is because the relation $\Gamma^{[n]} \propto \Gamma^{12}\Gamma^{[11-n]}$, analogous to eq. (E.6), implies that the higher order antisymmetrised matrices are linearly dependent on the lower ones, as $\Gamma^{12} = \mathbf{1}$.

Bibliography

[1] B. Zwiebach, *A First Course in String Theory.* Cambridge University Press, Cambridge, 2009.

[2] N. Berkovits, *Super Poincare covariant quantization of the superstring*, JHEP **04**, 018 (2000) [`hep-th/0001035`].

[3] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, JHEP **2001** (9, 2001) [`hep-th/0105050`].

[4] Élie Cartan, *Theory of Spinors.* Dover Publications, New York, 1937.

[5] B. E. W. Nilsson, *Pure Spinors as Auxiliary Fields in the Ten-dimensional Supersymmetric Yang-Mills Theory*, Class. Quant. Grav. **3**, L41 (1986).

[6] P. S. Howe, *Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions*, Phys. Lett. B **273**, 90–94 (1991).

[7] P. S. Howe, *Pure spinors lines in superspace and ten-dimensional supersymmetric theories*, Phys. Lett. B **258**, 141–144 (1991). [Addendum: Phys.Lett.B 259, 511 (1991)].

[8] N. Berkovits and N. A. Nekrasov, *The Character of Pure Spinors*, Lett. Math. Phys. **74** (10, 2005) [`hep-th/0503075`].

[9] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, *Spinorial cohomology and maximally supersymmetric theories*, JHEP **02**, 009 (2002) [`hep-th/0110069`].

[10] M. Cederwall, U. Gran, B. E. Nilsson and D. Tsimpis, *Supersymmetric corrections to eleven-dimensional supergravity*, JHEP **2005**, 1477–1533 (5, 2005) [`hep-th/0409107`].

[11] N. Berkovits, *Pure spinor formalism as an $N = 2$ topological string*, JHEP **2005** (10, 2005) [`hep-th/0509120`].

[12] M. Cederwall, *D=11 supergravity with manifest supersymmetry*, Mod. Phys. Lett. A **25**, 3201–3212 (2010) [1001.0112].

[13] M. Cederwall, *Towards a manifestly supersymmetric action for 11-dimensional supergravity*, JHEP **01**, 117 (2010) [0912.1814].

[14] M. Movshev and A. S. Schwarz, *On maximally supersymmetric Yang-Mills theories*, Nucl. Phys. B **681**, 324–350 (2004) [hep-th/0311132].

[15] M. Movshev and A. Schwarz, *Supersymmetric Deformations of Maximally Supersymmetric Gauge Theories*, JHEP **09**, 136 (2012) [0910.0620].

[16] M. Cederwall and J. Palmkvist, *Superalgebras, constraints and partition functions*, JHEP **2015** (8, 2015) [1503.06215].

[17] M. Cederwall, J. Palmkvist and I. Saberi, *Work in progress,*.

[18] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, 1994.

[19] M. Nakahara, *Geometry, Topology and Physics.* CRC Press, 10, 2018.

[20] S. Weinberg, *Gravitation And Cosmology.* John Wiley & Sons, inc., 1972.

[21] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Annalen der Physik **322** (1905).

[22] C. Becchi, A. Rouet and R. Stora, *Renormalization of Gauge Theories*, Annals Phys. **98**, 287–321 (1976).

[23] I. V. Tyutin, *Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism*, 0812.0580.

[24] M. B. Green and J. H. Schwarz, *Covariant description of superstrings*, Phys. Lett. B **136** (3, 1984).

[25] J. R. Durbin, *Modern Algebra An Introduction.* John Wiley & Sons, inc., 6 ed., 2008.

[26] Eliasson, Fredrik, *Super-Yang-Mills Theory using Pure Spinors*, Master's thesis, Gothenburg, 2006.

[27] M. Cederwall, *Pure spinor superfields – an overview*, Springer Proc. Phys. **153**, 61–93 (2014) [1307.1762].

[28] A. Fuster, M. Henneaux and A. Maas, *BRST-antifield quantization: A short review*,

International Journal of Geometric Methods in Modern Physics **02** (10, 2005) [hep-th/0506098].

[29] A.M. Cohen, B. Lisser and M.A.A. van Leeuwen, *LiE*, 2000.

[30] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, *The Structure of maximally supersymmetric Yang-Mills theory: Constraining higher order corrections*, JHEP **06**, 034 (2001) [hep-th/0102009].

[31] E. Cremmer and S. Ferrara, *Formulation of Eleven-Dimensional Supergravity in Superspace*, Phys. Lett. B **91**, 61–66 (1980).

[32] L. Brink and P. S. Howe, *Eleven-Dimensional Supergravity on the Mass-Shell in Superspace*, Phys. Lett. B **91**, 384–386 (1980).

[33] J. Aspman, *On the predictability of string theory*, Master's thesis, 2018.

[34] Y. Aisaka, E. A. Arroyo, N. Berkovits and N. Nekrasov, *Pure Spinor Partition Function and the Massive Superstring Spectrum*, JHEP **08**, 050 (2008) [0806.0584].

[35] M. Cederwall and J. Palmkvist, *L_∞ Algebras for Extended Geometry from Borcherds Superalgebras*, Comm. in Math. Phys. **369** (4, 2019) [1804.04377].

[36] U. Gran, *GAMMA: A Mathematica package for performing gamma-matrix algebra and Fierz transformations in arbitrary dimensions*, hep-th/0105086.

[37] P. Di Francesco, P. Mathieu and D. Sénéchal, *Simple Lie Algebras*, in *Conformal Field Theory*, pp. 489–555. Springer New York, New York, NY, 1997.

[38] J. Wess and J. Bagger, *Supersymmetry and Supergravity.* Princeton University Press, 1, 1983.

[39] Matteo Bertolini, *Lectures on Supersymmetry*, SSISA (2020).

[40] F. Cabral, F. S. N. Lobo and D. Rubiera-Garcia, *Einstein–Cartan–Dirac gravity with $U(1)$ symmetry breaking*, Eur. Phys. J. C **79**, 1023 (2019) [1902.02222].

[41] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, *General relativity with spin and torsion: Foundations and prospects*, Reviews of Modern Physics **48**, 393–416 (7, 1976).

[42] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein Supergravity*, Phys. Rept. **130**, 1–142 (1986).

[43] R. N. Mohapatra, *Local Supersymmetry (N = 1)*, pp. 290–315. 1992.

[44] E. Cremmer, B. Julia and J. Scherk, *Supergravity in theory in 11 dimensions*, Phys. Lett. B **76** (1978).

DEPARTMENT PHYSICS
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden
www.chalmers.se

CHALMERS
UNIVERSITY OF TECHNOLOGY