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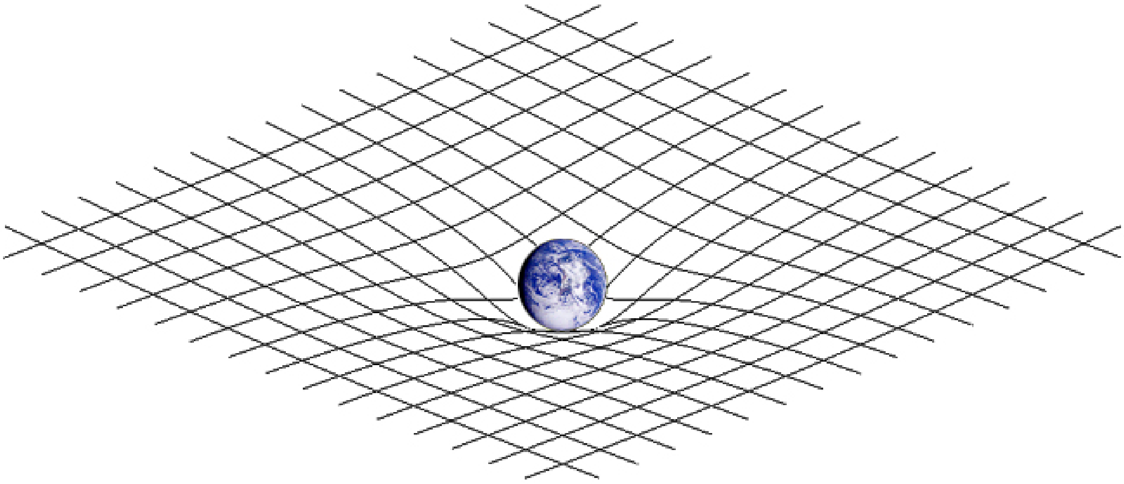

Black Holes and Solution Generating Symmetries in Gravity

Master's Thesis in Physics and Astronomy

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Department of Fundamental Physics
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Gothenburg, Sweden 2014

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Abstract

A gravity theory in D dimensions with a spacetime metric that admits n commuting Killing vectors can be dimensionally reduced to an effectively $(D-n)$-dimensional theory. The reduction is performed by using Kaluza-Klein techniques and, as it turns out, the lower dimensional theory reveals a hidden global symmetry, described by a group G, on the space of solutions. This symmetry is not seen in the original D-dimensional theory. When the solutions admit a sufficient number of Killing vectors we can dimensionally reduce down to $D = 3$ where we get a non-linear sigma-model. This sigma-model contains scalar fields which originate from the D-dimensional metric and whatever other possible D-dimensional fields the theory may contain. These scalar fields can also be described as parameters in a coset space G/H which depends on the particular D-dimensional theory we started from. If one rewrites the sigma-model in terms of coset representatives the hidden symmetry emerges and becomes manifest; the sigma-model is invariant under global G transformations. That is, given a solution to the equations of motion we can transform it to get a new solution. Thus, the technique utilizes the symmetries which become manifest upon dimensional reduction to generate new solutions.

For the case when four-dimensional pure gravity is reduced over the time dimension to three-dimensions the symmetry is described by $SL(2,\mathbb{R})$ and the coset space by $SL(2,\mathbb{R})/SO(1,1)$. We demonstrate how the Reissner-Nordström solution and the Schwarzschild solution are related by a $SO(1,1)$ transformation and identifiy the subgroup $SO(1,1)$ as the generator of electric charge. For the stationary axisymmetric solutions in $D = 4$ we can reduce down to two dimensions. The remarkable property of two-dimensional gravity is that the symmetry group G enlarges to an infinite-dimensional symmetry group. In terms of group theory this corresponds to the affine Kac-Moody group associated to the group G. In this thesis we explicitly show how $SL(2,\mathbb{R})$ enlarges to its affine extension $SL(2,\mathbb{R})^+$. The coset space G/H has to be extended to a coset space G^+/H^+ which requires an introduction of a spectral parameter and the so called monodromy matrix. This matrix encodes all the information about the spacetime metric and the key problem is to factorize this matrix. This amounts to a certain infinte-dimensional Riemann-Hilbert problem.

The main goal of this thesis has been to solve this for the case of minimal supergravity in five dimensions where the symmetry is given by $G_{2(2)}$ and the coset space by $G_{2(2)}/SO(2,2)$. As a result of this thesis, we have constructed the seed monodromy matrix for Schwarzschild and generated the five-dimensional Reissner-Nordström metric via a SO(2,2) transformation

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Contents

1

Introduction

1.1 Gravity, Black Holes and Symmetries

EINSTEIN'S field equations in $D = 4$ consist of ten non-linear coupled differential equations whose solutions constitute the spacetime metric g. The metric g is a symmetric $D \times D$ matrix and is the most central object in general relativity. In general relativity, we think of gravity as a manifestation of a curved spacetime. That is, gravity is not considered as a force such as electromagnetism but rather as a consequence of the geometry we live in. Knowing the curvature is the same as knowing the gravitational "force".

One way to visualize a curved geometry is to think of a sphere in a three-dimensional space. This surface is obviously not flat in the usual sense but is instead a two-dimensional curved space. This curved space has some considerable differences compared to a sheet, i.e. flat space. For example, we could easily construct a triangle whose angles do not add up to 180°. In a curved spacetime the concept of distance becomes a bit tricky and one cannot measure distances as we are used to, e.g. by using the Pythagorean theorem on the coordinates. The role of the metric g is thus to give us a prescription how this should be done. In other words, since the metric carries the information on how we should measure distance it also carries the information about the curvature of spacetime.

One of the most interesting features of Einstein's theory is that it allows for spacetimes with singularities, e.g. black hole solutions. Black holes are perhaps the most immoderate physical object we know about due to its extreme physics which really puts our physical intuition to the test. However, the number of analytically derived black hole-solutions as of yet are very limited because of the complicated set of equations that needs to be solved. A black hole is a remnant of a once very massive star, much heavier than the sun. At the end of a star's life it collapses under its own gravitational force and if the star is heavy enough, this implosion may result in an extremely dense object. As a result of the high mass density, the gravitational field outside the object becomes so large that not even light can escape, hence the name black hole. What is interesting is the so called event horizon, a border outside the black hole which marks the "point of no return". Once you have crossed the event horizon, the gravitational "force" becomes so strong that not even light can escape.

There is evidence that black holes exist and in particular, at the center of our own galaxy there exists a so called *supermassive black hole* [2]. The mass of a supermassive black hole is on the order of billions of solar masses. It is believed that almost every galaxy has a supermassive black hole at its center.

General relativity has an amazing structure and a lot of interesting physics and mathematics can be explored if one scratches the surface. For example, the presence of a symmetry is very appreciated since it simplifies and serves as a tool when we are trying to solve the equtions. Before we discuss symmetries in the context of gravity and how it will become useful in this thesis, we begin with some remarks about symmetries in general.

Perhaps the best way to define the concept of symmetries was given by Weyl: "a thing is symmetrical if there is something we can do to it so that after we have done it, it looks the same as it did before" [3]. We understand this definition perfectly well when we think of a geometrical object, like a circle or a sphere, whose symmetries are rotational symmetries. We also have symmetries in space and time, for example: it does not matter when and in which direction I throw my rock, its trajectory will be the same. The reason why we are interested in the symmetries of nature is not only because they are beautiful, they simplify and give us further support in matters. For instance, going back to the previous example about the trajectory of a rock, we know that nothing in the equations of motion should depend on the direction, or time, in which I throw my rock. Moreover, nothing in the solution should depend on the direction or time either. That is, the symmetries of a system can and should be used as an extra input when we are trying to solve a problem.

These are examples of symmetries acting on spacetime, *external symmetries*. If we study the microscopic world we find a lot of symmetries on the particle fields themselves which are called *internal symmetries*. The standard model is a so called *gauge theory* since it is invariant under gauge transformations. A gauge transformation is basically a transformation which depends on spacetime, *a local* *transformation*, in contrast to transformations that are constant, *global transformations*. In the standard model the gauge symmetries are responsible for the three fundamental interactions.

In special relativity we have coordinate invariance, Poincare invariance. If this is extended to local coordinate invariance we get general relativity and thereby gravity. For example, the measure d^4x changes to $\mathrm{d}^4x\sqrt{g}$ in order to be invariant under general coordinate transformations. Thus, gravity might be considered as a gauge theory of global spacetime transformations. In particle physics, beyond the standard model, it is believed that there exists a global, so called, supersymmetry which relates bosons to fermions. If we make this supersymmetry local as well we necessarily introduce gravity. Thus, supergravity might be considered as a gauge theory of global supersymmetry. In other words, supergravity, or local supersymmetry, is a combined theory of gravity and a supersymmetric particle theory. Just like the spin 1 gauge field A_μ is introcued when one implements local $U(1)$ invariance for a spin 1/2 field, we have to add the gravitino gauge field in order for the Lagrangian to be invariant under local supersymmetry. Thus, gravity is required for local supersymmetry.

There are different versions of supersymmetry depending on the number of supersymmetry generators. Minimal supergravity, or $N = 1$ supergravity, is a theory with only one supersymmetric generator, $N = 2$ has two supersymmetric generators which would for example give two gravitinos when acted upon a graviton and maximal supergravity, $N = 8$, has eight supersymmetric generators.

Another type of symmetry is a symmetry which acts on the space of solutions. This space may be considered to consist of functions. This does not mean that the functions themselves are unchanged under this symmetry operation. It means that given a function which is a solution we can find another function which is also a solution by acting with our symmetry operation. The notion of symmetry here is thus realized by the fact that we do not destroy the “solution property” of our functions.

1.2 Motivation to This Thesis

In this thesis we will make extensive use of symmetries connected to gravity. We will begin by assuming that the metric g does not depend on time. This means that the time-dimension is “superfluous” and that there is an effective lower dimensional theory. That is, our D-dimensional theory can be treated as a $(D-1)$-dimensional theory. If the metric is independent of more coordinates we can continue and formulate an even lower dimensional effective theory. As it turns out, when one reduces a gravitational theory to three dimensions a hidden symmetry emerges, a symmetry which was not seen from the higher dimensional theory. When we talk about a symmetry in this case we mean that there exists a set of transformations that we can apply to the fields in our Lagrangian which leaves it invariant. By the action principle this means that we have a symmetry on the space of solutions, just like we discussed above. By utilizing these newly found symmetries, solution generating methods can be developed.

Kaluza and Klein pioneered the idea of fomulating gravity in a lower dimension. They played with the idea of reducing five-dimensional pure gravity to four dimensions and discovered that the part of the metric polarized in the fifth dimension manifests itself as electromagnetism in four dimensions. This inspired others to reduce gravity and gravity-matter systems in various dimensions. The Einstein-Hilbert action in $D = 4$

$$S = \int \mathrm{d}^4 x \sqrt{g} R \tag{1.1}$$

has seemingly a relatively simple structure[1]. However, when one dimensionally reduces to lower dimensions hidden symmetries emerge.

As explained above, Einstein's field equations are very difficult to solve which is one of the main reasons why we are interested in finding symmetries. These symmetries allow us to take a "shortcut" in finding new solutions. For example, as we will see in this thesis, one can start from a spherically symmetric solution with mass m and generate a solution with electric charge added. Compared to if we would have solved Einstein's field equations with a stress tensor, we do not have to solve any equations.

Solutions and solution generating techniques to Einstein's four-dimensional field equations have been studied extensively over the years [4]. In the 1970's, Geroch observed that if one dimensionally reduces down to two dimensions one discovers an infinte-dimensional symmetry, an affine Kac-Moody group [5]. For the case of stationary, axisymmetric solutions in four dimensions this corresponds to the so called Geroch group. Julia was one of the first who tried to get a better group theoretical understanding of Geroch's result (see e.g. [6]). In [7] a linear system, a so called Lax-pair, was derived indicating that two-dimensional gravity is integrable giving it a connection to the "Inverse Scattering Method". The group theoretical aspects of it was further developed, among others, by Breitenlohner and Maison [8, 9]. The latter has been one of the most important sources of inspiration to this thesis. Their results include solution generating methods and uniqueness theorems for static black holes.

[1] R is the Ricci tensor and $g = |\det g_{\mu\nu}|$.

A recent development was made by Katsimpouri, Kleinschmidt and Virmani [10, 11], where the method by Breitenlohner and Maison is applied to the case of pure gravity and STU supergravity. The purpose of this thesis has been to follow and extend their approach. More specifically:

> *The goal has been to investigate whether previously known techniques can be used for the case when minimal supergravity in $D = 5$ is reduced down to two dimensions.*

In this case, the symmetry group is the affine extension of $G_{2(2)}$ and the coset space $G_{2(2)}/SO(2,2)$. We will not in this thesis discuss the nature of supergravity in any further extent. Instead we will focus on applying the solution generating technique discussed in the forthcoming chapters to this theory. For a more comprehensive discussion about supergravity, see [29].

1.3 Outline

In this thesis, the main focus has been to apply a solution generating technique to the $G_{2(2)}$ group and its affine extension. Consequently, a lot of effort has been put into finding explicit solutions and expressions. In the first half, symmetries in $D = 3$ are investigated and in the second half we move on to $D = 2$. Throughout the thesis, the required theory is presented prior to the calculations. However, many concepts and objects are not explained and assumed to be known to the reader. In particular, the reader is assumed to have knowledge within general relativity, tensor calculus, the action principle, basic differential geometry and group theory.

The group $SL(2,\mathbb{R})$ is the simplest example of a symmetry emerging from dimensional reduction and will therefore return occasionally for illustration purposes. In the first chapters of this thesis, this group will have a prominent role but in the last chapters we will instead focus on the $G_{2(2)}$ group.

The outline of the thesis is as follows. We begin in the second chapter by performing dimensional reduction of pure gravity from $D = 4$ to $D = 3$ and then dualize. Even though this is the simplest example of a reduction, most of the results are transferable to more complicated cases. The reason is because we have reduced the Ricci scalar which is by far the most difficult and tedious object. In chapter 3 we present the so called sigma-model and explain how and why there is a symmetry and how the solution generating process works. In chapter 4 we apply the method and show how the Schwarzschild and Reissner-Nordström metrics are related. Chapter 5 aims to give an understanding of how the infinite-dimensional symmetry emerges in two dimensions and a few important concepts are introduced. In chapter 6 we review some properties about the $G_{2(2)}$ group and the coset space $G_{2(2)}/SO(2,2)$. In the last chapter, the main results of this thesis are presented. Here we review and apply a solution generating method for the affine Kac-Moody group $G^+_{2(2)}$ by generating the five-dimensional Reissner-Nordström metric using Schwarzschild as seed solution.

2

Dimensional Reduction from $D = 4$ to $D = 3$

Under the assumption that there exists a coordinate on which the metric components do not depend, e.g. time, we can perform dimensional reduction of our theory. Dimensional reduction[1] is very close to the concept of compactification and we should say something about what separates them. Compactification is performed when one is trying to describe a theory in a spacetime in which some of the dimensions are compact. For example, a five-dimensional world where the fifth coordinate is compactified into a circle. If the radius of this circle is very small the theory can be considered as effectively four-dimensional. The mathematics behind this procedure was first developed by Kaluza and Klein (KK) and it works as follows: since the fifth coordinate parametrizes a circle we can Fourier expand all the fields of our theory in this coordinate. The different fields are constituents of the metric (more about this later) and other possible terms in the Lagrangian, e.g. a Maxwell term. As it turns out, the non-zero modes give rise to massive particles with mass proportional to the inverse radius of the compactified dimension. Since we assume that this radius is small these massive particles have masses which are far beyond our present energy scale. Thus, they can be ignored and what is left is all the zero modes which do not depend on the fifth-coordinate. Knowing this one can just assume that nothing in the theory depends on the fifth coordinate without having to do the Fourier expansion.

Now, consider a spacetime in which the fifth dimension is not compactified into a circle. Instead, we assume that there exists a symmetry such that nothing depends on the fifth coordinate. That is, we are restricting ourselfs to the special

[1]Henceforth referred to as just reduction.

case where the metric and the other fields in our theory only depend on the first four components. This makes our theory effectively four-dimensional and we can use the same technique described above to perform dimensional reduction. In particular, we can use the same ansatz for the metric.

To sum up: to perform dimensional reduction and compactification one uses the same mathematics. However, the physical motivation behind it differs.

In this chapter we will do the reduction of pure gravity in $D = 4$ to $D = 3$ by Kaluza-Klein techniques. First we will find a suitable parametrization of the four-dimensional metric in terms of the three-dimensional metric and some additional fields, then we will express the four-dimensional Einstein-Hilbert action in terms of these fields. This will result in a gravity matter system in three-dimensions where the matter term exhibits a global symmetry. The results of this chapter can be found in the literature, e.g. [12], but the derivations are often omitted. Thus, the purpose of this chapter is not only to introduce the concepts but also to present the explicit derivations. We will make use of a non-coordinate basis in this chapter, called vielbein. See appendix A for a brief introduction to the subject.

2.1 Parametrization of the Metric

We start from pure gravity in $D = 4$ which is described by the action

$$S = \int d^4x \sqrt{g^{(4)}} R^{(4)}. \tag{2.1}$$

We would like to find an appropriate[2] ansatz for the metric $g^{(4)}_{\hat{\mu}\hat{\nu}}$ which we will simply denote $g_{\hat{\mu}\hat{\nu}}$. We start by decomposing it as

$$g^{(4)}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & g_{\mu 3} \\ g_{3\nu} & g_{33} \end{pmatrix} \tag{2.2}$$

where $\hat{\mu}, \hat{\nu} = 0,1,2,3$ and $\mu, \nu = 0,1,2$. That is, four-dimensional objects are distinguished from three-dimensional objects by a hat. We will use this notation throughout unless clearly stated otherwise. We see from (2.2) that $\hat{\mu} = 3$ is the coordinate which the metric does not depend on and consequently, the dimension we can reduce. The signature of the metric depends on whether we reduce on a time like or a space like coordinate. If we reduce on a time like coordinate we have the signature $(1, 1, 1, -1)$ and if we reduce a space like coordinate we have the signature $(-1,1,1,1)$. If we were to insert our ansatz (2.2) into (2.1) the resulting action would look somewhat messy. For example, the three-dimensional Einstein Hilbert term $\sqrt{g}R^{(3)}$ will have some extra factors on it which will cause

[2] The precise meaning of this will become clear later on.

some unnecessary difficulties. This indicates that $g_{\mu\nu}$ is not the three-dimensional metric we are looking for. Moreover, $g_{\mu 3}$ will not turn out to be a useful parameter either. In order to find a better metric parameterization we must consider how the different parts in (2.2) transform under general coordinate transformations. Our four-dimensional metric transforms[3] as

$$g'_{\hat{\mu}\hat{\nu}}(x') = \frac{\partial x^{\hat{\rho}}}{\partial x'^{\hat{\mu}}} \frac{\partial x^{\hat{\sigma}}}{\partial x'^{\hat{\nu}'}} g_{\hat{\rho}\hat{\sigma}}(x) \tag{2.3}$$

which in infinitesimal form $x'^{\hat{\mu}} = x^{\hat{\mu}} + \epsilon\lambda^{\hat{\mu}}$ becomes

$$\delta g_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\nu}}\lambda^{\hat{\sigma}} g_{\hat{\mu}\hat{\sigma}} + \partial_{\hat{\mu}}\lambda^{\hat{\rho}} g_{\hat{\rho}\hat{\nu}} + \lambda^{\rho}\partial_{\rho} g_{\hat{\mu}\hat{\nu}} \tag{2.4}$$

Vectors and scalars transform as

$$\begin{aligned} \delta g_{\hat{\mu}} &= \partial_{\hat{\mu}}\lambda^{\hat{\sigma}} g_{\hat{\sigma}} + \lambda^{\rho}\partial_{\rho} g_{\hat{\mu}} \\ \delta g &= \lambda^{\rho}\partial_{\rho} g \end{aligned} \tag{2.5}$$

respectively. Let us now see how $g_{\mu\nu}, g_{3\nu}, g_{33}$ transform under the assumption that $g_{\hat{\mu}\hat{\nu}}$ transforms as a tensor under general coordinate transformations. First we consider *external* coordinate transformations, i.e. in the unreduced space which means that

$$\lambda^{\hat{\mu}} = \begin{cases} 0 & \hat{\mu} = 3 \\ \neq 0 & \hat{\mu} \neq 3 \end{cases}. \tag{2.6}$$

In this case we have that $\lambda^{\hat{\mu}}$ depends only on the external coordinates. If we put (2.6) into (2.4) we get for the different parts $g_{\mu\nu}, g_{3\nu}, g_{33}$

$$\begin{aligned} \delta g_{\mu\nu} &= \partial_{\nu}\lambda^{\hat{\sigma}} g_{\mu\hat{\sigma}} + \partial_{\mu}\lambda^{\hat{\rho}} g_{\hat{\rho}\nu} + \lambda^{\rho}\partial_{\rho} g_{\mu\nu}, \\ \\ \delta g_{\mu 3} &= \partial_3\lambda^{\hat{\sigma}} g_{\mu\hat{\sigma}} + \partial_{\mu}\lambda^{\hat{\sigma}} g_{\hat{\sigma}3} + \lambda^{\rho}\partial_{\rho} g_{\mu 3} \\ &= \partial_{\mu}\lambda^{\hat{\sigma}} g_{\hat{\sigma}3} + \lambda^{\rho}\partial_{\rho} g_{\mu 3}, \\ \\ \delta g_{33} &= \partial_3\lambda^{\hat{\sigma}} g_{3\hat{\sigma}} + \partial_3\lambda^{\hat{\rho}} g_{\hat{\rho}3} + \lambda^{\rho}\partial_{\rho} g_{33} \\ &= \lambda^{\rho}\partial_{\rho} g_{33}. \end{aligned} \tag{2.7}$$

We see that under external coordinate transformations $g_{\mu\nu}$ transforms as a 2-tensor, $g_{\mu 3}$ as a vector and g_{33} as a scalar. We continue by checking how they transform under *internal* coordinate transformations, i.e transformations of the reduced coordinates

$$\lambda^{\hat{\rho}} = \begin{cases} \neq 0 & \hat{\rho} = 3 \\ = 0 & \hat{\rho} = \rho \end{cases}. \tag{2.8}$$

[3] As we will discuss below, these are not the only symmetries our theory contains.

Contrary to the first case $\lambda^{\hat{\rho}}$ is allowed to depend on x^3 as will become apparent later. We insert (2.8) into (2.4)

$$
\begin{aligned}
\delta g_{\mu\nu} &= \partial_\nu \lambda^{\hat{\sigma}} g_{\mu\hat{\sigma}} + \partial_\mu \lambda^{\hat{\rho}} g_{\hat{\rho}\nu} + \lambda^\rho \partial_\rho g_{\mu\nu} \\
&= \partial_\nu \lambda^3 g_{\mu 3} + \partial_\mu \lambda^3 g_{3\nu} + \lambda^3 \partial_3 g_{\mu\nu} \\
&= \partial_\nu \lambda^3 g_{\mu 3} + \partial_\mu \lambda^3 g_{3\nu} \\
\\
\delta g_{\mu 3} &= \partial_3 \lambda^{\hat{\sigma}} g_{\mu\hat{\sigma}} + \partial_\mu \lambda^{\hat{\sigma}} g_{\hat{\sigma}3} + \lambda^\rho \partial_\rho g_{\mu 3} \\
&= \partial_\mu \lambda^3 g_{33} + \partial_3 \lambda^3 g_{3\mu} + \lambda^3 \partial_3 g_{3\nu} \\
&= \partial_\mu \lambda^3 g_{33} + \partial_3 \lambda^3 g_{3\mu} \\
\\
\delta g_{33} &= \partial_3 \lambda^{\hat{\sigma}} g_{3\hat{\sigma}} + \partial_3 \lambda^{\hat{\rho}} g_{\hat{\rho}3} + \lambda^\rho \partial_\rho g_{33} \\
&= 2\partial_3 \lambda^3 g_{33}.
\end{aligned}
\tag{2.9}
$$

Note here that $\partial_3 \lambda^3$ not neccesarily has to vanish but it should be independent of x^3 since otherwise g_{33} will depend on x^3. This suggests that

$$\lambda^3 = \tilde{\lambda}(x^\mu) + \Lambda x^3. \tag{2.10}$$

Inserting this into (2.9) yields

$$\delta g_{\mu\nu} = \partial_\nu \tilde{\lambda} g_{\mu 3} + \partial_\mu \tilde{\lambda} g_{3\nu} \tag{2.11a}$$
$$\delta g_{3\mu} = \partial_\mu \tilde{\lambda} g_{33} + \Lambda g_{3\mu} \tag{2.11b}$$
$$\delta g_{33} = 2\Lambda g_{33}. \tag{2.11c}$$

From this we see that $g_{\mu\nu}$ is not invariant, as we would like for the three-dimensional metric. The reason for this is that we want the metric to live only in this three dimensional spacetime and not depend on the fourth dimension at all. Consider the ansatz

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + G_{\mu\nu} \tag{2.12}$$

and the transformation properties

$$
\begin{aligned}
\delta \tilde{g}_{\mu\nu} &= 0 \\
\delta G_{\mu\nu} &= \partial_\nu \tilde{\lambda} g_{\mu 3} + \partial_\mu \tilde{\lambda} g_{3\nu}
\end{aligned}
\tag{2.13}
$$

according to (2.11a). We make this decomposition in order to find the three dimensional invariant metric which we now hope to be $\tilde{g}_{\mu\nu}$. For $\tilde{g}_{\mu\nu}$ to transform as a covariant 2 tensor under external coordinate transformations we need $G_{\mu\nu}$ to transform as a covariant 2 tensor as well[4]. A reasonable ansatz would be $G_{\mu\nu} \sim g_{\mu 3} g_{3\nu}$ since we have seen that $g_{\mu 3}$ is a vector. The transformation becomes

$$\begin{aligned}\delta G_{\mu\nu} &= \delta(g_{\mu 3} g_{3\nu}) = \delta g_{\mu 3} g_{3\nu} + g_{\mu 3} \delta g_{3\nu} \\ &= \left(\partial_\mu \tilde{\lambda} g_{33} + \Lambda g_{\mu 3}\right) g_{3\nu} + g_{\mu 3} \left(\partial_\nu \tilde{\lambda} g_{33} + \Lambda g_{3\nu}\right) \\ &= \partial_\mu \tilde{\lambda} g_{33} g_{3\nu} + \partial_\nu \tilde{\lambda} g_{33} g_{\mu 3} + 2\Lambda g_{\mu 3} g_{3\nu}.\end{aligned} \tag{2.14}$$

Obviously this is not right and we note the unwanted g_{33}. At this stage we have to consider if $g_{\mu 3}$ is really the vector we are looking for. Since g_{33} is the only scalar, we try $g_{\mu 3} = g_{33} \tilde{g}_\mu$ so that we do not change the transformation property under external transformations. We get

$$\delta(g_{33} \tilde{g}_\mu) = 2\Lambda g_{33} \tilde{g}_\mu + g_{33} \delta \tilde{g}_\mu = \partial_\mu \tilde{\lambda} g_{33} + \Lambda g_{\mu 3} \tag{2.15}$$

which means that

$$\delta \tilde{g}_\mu = \partial_\mu \tilde{\lambda} - \Lambda \tilde{g}_\mu. \tag{2.16}$$

As we can see, the part of the internal coordinate transformations that depends on x^μ manifests itself as a $U(1)$ gauge transformation together with a scaling $\Lambda \tilde{g}_\mu$. To remove the unwanted g_{33} from (2.14) we try the ansatz $G_{\mu\nu} = g_{33} \tilde{g}_\mu \tilde{g}_\nu$, i.e. a factor of g_{33} less than before, and see that

$$\begin{aligned}\delta G_{\mu\nu} &= \delta(g_{33} \tilde{g}_\mu \tilde{g}_\nu) = \delta g_{33} \tilde{g}_\nu \tilde{g}_\mu + g_{33} \delta \tilde{g}_\nu \tilde{g}_\mu + g_{33} \tilde{g}_\nu \delta \tilde{g}_\mu \\ &= 2\Lambda g_{33} g_\nu \tilde{g}_\mu + g_{33} (\partial_\nu \tilde{\lambda} - \Lambda \tilde{g}_\nu) \tilde{g}_\mu + g_{33} \tilde{g}_\nu (\partial_\mu \tilde{\lambda} - \Lambda \tilde{g}_\mu) \\ &= \partial_\nu \tilde{\lambda} g_{\mu 3} + \partial_\mu \tilde{\lambda} g_{3\nu}.\end{aligned} \tag{2.17}$$

Finally, $G_{\mu\nu}$ has the desired transformation property and we choose to parametrize our metric $g_{\hat{\mu}\hat{\nu}}$ as

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \tilde{g}_{\mu\nu} + g_{33} \tilde{g}_\mu \tilde{g}_\nu & g_{33} \tilde{g}_\mu \\ g_{33} \tilde{g}_\nu & g_{33} \end{pmatrix} \tag{2.18}$$

As it turns out when we perform the dimensional reduction we should make the replacement $\tilde{g}_{\mu\nu} \to g_{33}^{-1} \tilde{g}_{\mu\nu}$ in order to get the correct Einstein Hilbert term in the Lagrangian. Since g_{33} is a scalar we do not change any transformation properties of $\tilde{g}_{\mu\nu}$ under *external* coordinate transformations. More generally, one can do the replacement $\tilde{g}_{\mu\nu} \to \lambda \tilde{g}_{\mu\nu}$ and then find that $\lambda = g_{33}^{-1}$ is a suitable choice for the conformal factor. The final metric ansatz which we will use is therefore

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{33}^{-1} \tilde{g}_{\mu\nu} + g_{33} \tilde{g}_\mu \tilde{g}_\nu & g_{33} \tilde{g}_\mu \\ g_{33} \tilde{g}_\nu & g_{33} \end{pmatrix}. \tag{2.19}$$

[4] Since we have seen that $g_{\mu\nu}$ is a covariant 2 tensor.

To sum up:

- $\tilde{g}_{\mu\nu}$ is as 2 tensor under *external* coordinate transformations and is independent of *internal* coordinates. Under the *internal* coordinate transformation that depends on x^3 it transforms with a constant scaling factor Λ.
- $\tilde{g}_{\mu}$ is a vector under *external* coordinate transformations and transforms with a $U(1)$ gauge under *internal* coordinate transformations. Moreover, it also scales with the constant factor Λ in a way that makes the Lagrangian invariant.
- g_{33} is a scalar under *external* coordinate transformations and scales with a constant factor Λ under *internal* coordiante transformations.

It is a bit strange that our three-dimensional metric $\tilde{g}_{\mu\nu}$ transforms under internal coordinate transformations. To overcome this we use the fact that our equations of motions, i.e. Einstein's equations, have another symmetry. We can scale our metric with a constant k, $g_{\hat{\mu}\hat{\nu}} \to k^2 g_{\hat{\mu}\hat{\nu}}$. This is not a symmetry of the Lagrangian which picks up a constant factor but this does not effect the equations of motion. Now, if we consider infinitesimal transformations of this scaling symmetry, $k \approx (1 + a)$, we have $\delta g_{\hat{\mu}\hat{\nu}} = 2a g_{\hat{\mu}\hat{\nu}}$ where a is an infinitesimal constant parameter. The transformations under both this "new‘" scaling and the old one of the various components become

$$\begin{aligned} \delta g_{33} &= (2\Lambda + 2a) g_{33} \\ \delta \tilde{g}_{\mu\nu} &= 2a\tilde{g}_{\mu\nu} - 2\tilde{g}_{\mu\nu}(a + c) \end{aligned} \tag{2.20}$$

where c is the shift of the phase when one considers $g_{33} = e^{2\phi}$. The idea is to use the "k- transformation" to compensate for the transformation generated by the internal coordinate transformations in such a way that $\delta\tilde{g}_{\mu\nu} = 0$.

2.2 Reduction to $D = 3$

At this point it is a good idea to change our variables to make the notation a bit clearer. Let $g_{33} = \mathrm{e}^{\phi}$, $\tilde{g}_{\mu} = \mathcal{A}_\mu$ and $\tilde{g}_{\mu\nu} = g_{\mu\nu}$. That is

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \mathrm{e}^{-\phi} g_{\mu\nu} + \mathrm{e}^{\phi}\mathcal{A}_\mu\mathcal{A}_\nu & \mathrm{e}^{\phi}\mathcal{A}_\mu \\ \mathrm{e}^{\phi}\mathcal{A}_\nu & \mathrm{e}^{\phi} \end{pmatrix}. \tag{2.21}$$

To get the three-dimensional action we insert our metric ansatz (2.21) into (2.1). The effective three-dimensional theory is obtained by restricting our coordinate dependence as described above.

2.2.1 The Determinant

We start by expressing $\sqrt{g}$ in terms of $\det(g_{\mu\nu})$ which we will denote $g^{(3)}$. By using the fact that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C)$$

we get

$$\begin{aligned} \det(g_{\hat{\mu}\hat{\nu}}) &= \begin{pmatrix} \mathrm{e}^{-\phi} g_{\mu\nu} + \mathrm{e}^{\phi}\mathcal{A}_\mu\mathcal{A}_\nu & \mathrm{e}^{\phi}\mathcal{A}_\mu \\ \mathrm{e}^{\phi}\mathcal{A}_\mu & \mathrm{e}^{\phi} \end{pmatrix} \\ &= \det(\mathrm{e}^{\phi})\det(\mathrm{e}^{-\phi} g_{\mu\nu} + \mathrm{e}^{\phi}\mathcal{A}_\mu\mathcal{A}_\nu - \mathrm{e}^{\phi}\mathcal{A}_\mu \mathrm{e}^{-\phi}\mathrm{e}^{\phi}\mathcal{A}_\nu) \\ &= \mathrm{e}^{-2\phi}\det(g_{\mu\nu}). \end{aligned} \tag{2.22}$$

2.2.2 The Spin Connection

We continue the reduction by expressing $R^{(4)}$ in terms of $R, \mathcal{A}_\mu$ and e^{ϕ} where R is the three-dimensional Ricci scalar. This is preferably done in the vielbein basis using Cartan's equations. In the following calculations we distinguish the four-dimensional fields from the three-dimensional ones by a hat. In the vielbein basis we have that

$$d\hat{s}^2 = \hat{e}^{\hat{\alpha}}\hat{e}^{\hat{\beta}}\eta_{\hat{\alpha}\hat{\beta}} = \hat{e}^{\alpha}\hat{e}^{\beta}\eta_{\alpha\beta} + \hat{e}^3\hat{e}^3 \tag{2.23}$$

From our metric ansatz

$$d\hat{s}^2 = \mathrm{e}^{\alpha\phi}ds^2 + \mathrm{e}^{\beta\phi}(dz + \mathcal{A}_\mu dx^\mu)^2 = \mathrm{e}^{\alpha\phi}e^{\alpha}e^{\beta}\eta_{\alpha\beta} + \mathrm{e}^{\beta\phi}(dz + \mathcal{A}_\mu dx^\mu)^2 \tag{2.24}$$

we see that

$$\hat{e}^{\alpha} = \mathrm{e}^{\frac{\alpha\phi}{2}}e^{\alpha} \qquad\qquad \hat{e}^3 = \mathrm{e}^{\frac{\beta\phi}{2}}(dz + \mathcal{A}_\mu dx^\mu). \tag{2.25}$$

Here we consider a slightly more general ansatz than our previous one by allowing the dilaton field to have a prefactor of α and β. The reason for this is to make the dilaton kinetic term more general which means that one can choose different normalizations. We define the structure constants $\hat{C}^{\alpha}{}_{\beta\gamma}$

$$\mathrm{d}\hat{e}^{\alpha} = -\frac{1}{2}\hat{C}^{\alpha}{}_{\beta\gamma}\hat{e}^{\beta} \wedge \hat{e}^{\gamma}. \tag{2.26}$$

If we differentiate (2.25) we get

$$\begin{aligned}
\mathrm{d}\hat{e}^{\alpha} &= \mathrm{d}(\mathrm{e}^{\frac{\alpha\phi}{2}} e^{\alpha}) = \frac{\alpha}{2}\mathrm{e}^{\frac{\alpha\phi}{2}} d\phi \wedge e^{\alpha} + \mathrm{e}^{\frac{\alpha\phi}{2}} \mathrm{d}e^{\alpha} \\
&= \frac{\alpha}{2}\mathrm{e}^{\frac{\alpha\phi}{2}} \partial_{\gamma}\phi e^{\gamma} \wedge e^{\alpha} - \frac{1}{2}\mathrm{e}^{\frac{\alpha\phi}{2}} C^{\alpha}{}_{\beta\gamma} e^{\beta} \wedge e^{\gamma} \\
&= \frac{\alpha}{2}\partial_{\gamma}\phi e^{\gamma} \wedge \hat{e}^{\alpha} - \frac{1}{2} C^{\alpha}{}_{\beta\gamma} e^{\beta} \wedge \hat{e}^{\gamma} \\
&= -\frac{1}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}} \left(-\alpha\partial_{\beta}\phi\delta^{\alpha}_{\gamma} + C^{\alpha}{}_{\beta\gamma}\right) \hat{e}^{\beta} \wedge \hat{e}^{\gamma},
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
\mathrm{d}\hat{e}^{3} &= \mathrm{d}\left(\mathrm{e}^{\frac{\beta\phi}{2}} (dz + \mathcal{A}_{\mu} dx^{\mu})\right) \\
&= \frac{\beta}{2}\mathrm{e}^{\frac{\beta\phi}{2}} \left(\mathrm{d}\phi \wedge (dz + A_{\mu} dx^{\mu}\right) + \mathrm{e}^{\frac{\beta\alpha}{2}} \mathrm{d}(dz + \mathcal{A}_{\mu} dx^{\mu}) \\
&= \frac{\beta}{2}\mathrm{e}^{\frac{\beta\phi}{2}} \left(\partial_{\gamma}\phi e^{\gamma} \wedge (dz + A_{\mu} dx^{\mu}\right) + \mathrm{e}^{\frac{\beta\alpha}{2}} \left(\mathrm{d}(dz) + \mathrm{d}(\mathcal{A}_{\mu} dx^{\mu})\right) \\
&= \frac{\beta}{2}\partial_{\gamma}\phi e^{\gamma} \wedge \hat{e}^{3} + \mathrm{e}^{\frac{\beta\phi}{2}} \partial_{[\gamma}\mathcal{A}_{\beta]} e^{\gamma} \wedge e^{\beta} \\
&= \frac{\beta}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}} \partial_{\gamma}\phi \hat{e}^{\gamma} \wedge \hat{e}^{3} + \frac{1}{2}\mathrm{e}^{\left(\frac{\beta\phi}{2} - \alpha\phi\right)} \mathcal{F}_{\gamma\beta} \hat{e}^{\gamma} \wedge \hat{e}^{\beta}.
\end{aligned} \tag{2.28}$$

Here, $\partial_{\gamma}\phi$ should be interpreted as $e^{\mu}{}_{\gamma}\partial_{\mu}\phi$ and the field strength $\mathcal{F} = \mathrm{d}\mathcal{A}$. We continue by calculating the spin connections $\hat{\omega}^{\alpha}{}_{\beta}$ defined as

$$\nabla A^{\alpha} = \mathrm{d}A^{\alpha} + \hat{\omega}^{\alpha}{}_{\beta} \wedge A^{\beta}, \tag{2.29}$$

where A^{α} is a p-form. If we let $A^{\alpha} = \hat{e}^{\alpha}$ we get

$$\nabla \hat{e}^{\alpha} = \mathrm{d}\hat{e}^{\alpha} + \hat{\omega}^{\alpha}{}_{\beta} \wedge \hat{e}^{\beta}. \tag{2.30}$$

We can think of the spin connection ω as the Levi-Civita connection in the vielbein basis. The Levi-Civita connection is defined in such a way that the following two conditions are satisfied

$$\nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} = 0 \tag{2.31}$$

$$\Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\rho\nu} = 0. \tag{2.32}$$

One can express the spin connection in terms of the Levi-Civita connection as [13], p.246

$$\hat{\omega}^{\alpha}{}_{\beta\mu} = -\hat{e}_{\beta}{}^{\nu}(\partial_{\mu}\hat{e}^{\alpha}{}_{\nu} - \Gamma^{\lambda}_{\mu\nu}\hat{e}^{\alpha}{}_{\lambda}) \tag{2.33}$$

which yields

$$\Gamma^{\lambda}_{\mu\nu} = \hat{e}^{\beta}{}_{\nu}\hat{e}^{\lambda}{}_{\alpha}\hat{\omega}^{\alpha}{}_{\beta\mu} + \hat{e}^{\lambda}{}_{\alpha}\partial_{\mu}\hat{e}^{\alpha}{}_{\nu}. \tag{2.34}$$

The symmetry property (2.32) implies that $\Gamma^{\lambda}_{\mu\nu}dx^{\mu} \wedge dx^{\nu} = 0$. Consequently, by multiplying (2.34) by $dx^{\mu} \wedge dx^{\nu}$ we get

$$\begin{aligned}
\Gamma^{\lambda}_{\mu\nu}dx^{\mu} \wedge dx^{\nu} &= \left(\hat{e}^{\beta}{}_{\nu}\hat{e}^{\lambda}{}_{\alpha}\hat{\omega}^{\alpha}{}_{\beta\mu} + \hat{e}^{\lambda}{}_{\alpha}\partial_{\mu}\hat{e}^{\alpha}{}_{\nu}\right) dx^{\mu} \wedge dx^{\nu} \\
&= \hat{e}^{\lambda}{}_{\alpha}\left(\partial_{\mu}\hat{e}^{\alpha}{}_{\nu} + \hat{\omega}^{\alpha}{}_{\beta\mu}\hat{e}^{\beta}{}_{\nu}\right) dx^{\mu} \wedge dx^{\nu} \\
&= \hat{e}^{\lambda}{}_{\alpha}\nabla\hat{e}^{\alpha} = 0.
\end{aligned}$$

If we multiply by $\hat{e}^{\beta}{}_{\lambda}$, change our index back to α we get the so called *torsion free condition* $\nabla\hat{e}^{\alpha} = 0$ which, by using (2.30), can be expressed as

$$\mathrm{d}\hat{e}^{\alpha} = -\hat{\omega}^{\alpha}{}_{\beta} \wedge \hat{e}^{\beta}. \tag{2.35}$$

To read of the structure constants from (2.27) and (2.28) we have to remember to antisymmetrize in β and γ

$$\hat{C}^{\alpha}{}_{\beta\gamma} = \mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\alpha\partial_{[\gamma}\phi\delta^{\alpha}_{\beta]} + C^{\alpha}{}_{\beta\gamma}\right) \tag{2.36a}$$

$$\hat{C}^{3}{}_{3\gamma} = \frac{\beta}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial_{\gamma}\phi \tag{2.36b}$$

$$\hat{C}^{3}{}_{\beta\gamma} = -\mathrm{e}^{\frac{\beta\phi}{2}-\alpha\phi}\mathcal{F}_{\beta\gamma}. \tag{2.36c}$$

We can expand (2.35) as $\mathrm{d}\hat{e}^{\alpha} = -\hat{\omega}^{\alpha}{}_{\beta} \wedge \hat{e}^{\beta} = -\hat{\omega}_{\gamma}{}^{\alpha}{}_{\beta}\hat{e}^{\gamma} \wedge \hat{e}^{\beta}$ which means that

$$-\frac{1}{2}\hat{C}^{\alpha}{}_{\beta\gamma}\hat{e}^{\beta} \wedge \hat{e}^{\gamma} = \hat{\omega}_{\gamma}{}^{\alpha}{}_{\beta}\hat{e}^{\beta} \wedge \hat{e}^{\gamma}.$$

Since $\hat{C}^{\alpha}{}_{\beta\gamma}$ is antisymmetric in β and γ we get for $\alpha, \beta, \gamma = 0{,}1{,}2{,}3$

$$\hat{C}^{\alpha}{}_{\beta\gamma} = -\left(\hat{\omega}_{\gamma}{}^{\alpha}{}_{\beta} - \hat{\omega}_{\beta}{}^{\alpha}{}_{\gamma}\right) \tag{2.37}$$

and if we consider

$$\begin{aligned}
\frac{1}{2}\left(\hat{C}_{\alpha\beta\gamma} + \hat{C}_{\gamma\alpha\beta} - \hat{C}_{\beta\gamma\alpha}\right) &= \frac{1}{2}\left(\hat{\omega}_{\beta\alpha\gamma} - \hat{\omega}_{\gamma\alpha\beta} + \hat{\omega}_{\alpha\gamma\beta} - \hat{\omega}_{\beta\gamma\alpha} - \hat{\omega}_{\gamma\beta\alpha} + \hat{\omega}_{\alpha\beta\gamma}\right) \\
&= \hat{\omega}_{\beta\alpha\gamma}
\end{aligned}$$

we have the spin connection expressed in terms of the structure constants

$$\hat{\omega}_{\alpha\beta\gamma} = \frac{1}{2}\left(\hat{C}_{\alpha\beta\gamma} - \hat{C}_{\gamma\alpha\beta} - \hat{C}_{\beta\gamma\alpha}\right). \tag{2.38}$$

As indicated above, our indices run from 0 to 3, i.e. this is true in $D = 4$ as well. Now we can calculate the components of our spin connection

$$\begin{aligned}\hat{\omega}_{\alpha\beta\gamma} &= \frac{1}{2}\Big(\mathrm{e}^{-\frac{\alpha\phi}{2}}\Big(\alpha\partial_\alpha\phi\delta_{\alpha\beta} - \alpha\partial_\beta\delta_{\alpha\gamma} + C_{\alpha\beta\gamma}\Big)\\ &\quad - \mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\alpha\partial_\beta\phi\delta_{\alpha\gamma} - \alpha\partial_\alpha\delta_{\beta\gamma} + C_{\gamma\alpha\beta}\right)\\ &\quad - \mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\alpha\partial_\alpha\phi\delta_{\beta\gamma} - \alpha\partial_\gamma\delta_{\alpha\beta} + C_{\beta\gamma\alpha}\right)\Big)\\ &= \mathrm{e}^{-\frac{\alpha\phi}{2}}\omega_{\alpha\beta\gamma} + \alpha\mathrm{e}^{-\frac{\alpha\phi}{2}}(\partial_\gamma\phi\delta_{\alpha\beta} - \partial_\beta\phi\delta_{\alpha\gamma})\end{aligned} \tag{2.39}$$

$$\begin{aligned}\hat{\omega}_{3\alpha\beta} &= \frac{1}{2}\left(\hat{C}_{3\alpha\beta} - \hat{C}_{\beta 3\alpha} - \hat{C}_{\alpha\beta 3}\right) = \frac{1}{2}\hat{C}_{3\alpha\beta}\\ &= -\frac{1}{2}\mathrm{e}^{\frac{\beta\phi}{2}-\alpha\phi}\mathcal{F}_{\alpha\beta}\end{aligned} \tag{2.40}$$

$$\begin{aligned}\hat{\omega}_{\alpha 3\beta} &= \frac{1}{2}\left(\hat{C}_{\alpha 3\beta} - \hat{C}_{\beta\alpha 3} - \hat{C}_{3\beta\alpha}\right) = -\frac{1}{2}\hat{C}_{3\beta\alpha}\\ &= \frac{1}{2}\mathrm{e}^{\frac{\beta\phi}{2}-\alpha\phi}\mathcal{F}_{\beta\alpha}\end{aligned} \tag{2.41}$$

$$\begin{aligned}\hat{\omega}_{33\beta} &= \frac{1}{2}\left(\hat{C}_{33\beta} - \hat{C}_{\beta 33} - \hat{C}_{3\beta 3}\right) = \hat{C}_{33\beta}\\ &= \frac{\beta}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial_\beta\phi.\end{aligned} \tag{2.42}$$

Finally we get the spin connections $\hat{\omega}^\alpha{}_\beta = \hat{\omega}_\delta{}^\alpha{}_\beta\hat{e}^\delta + \hat{\omega}_3{}^\alpha{}_\beta\hat{e}^3$, $\alpha,\beta = 0, 1, 2, 3$

$$\hat{\omega}^\alpha{}_\beta = \omega^\alpha{}_\beta + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_\beta\phi\hat{e}^\alpha - \partial^\alpha\phi\eta_{\beta\gamma}\hat{e}^\gamma\right) - \frac{1}{2}\mathrm{e}^{\frac{\beta\phi}{2}-\alpha\phi}\mathcal{F}^\alpha_\beta\hat{e}^3 \tag{2.43}$$

$$\hat{\omega}^3{}_\alpha = \frac{\beta}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial^\alpha\phi\hat{e}^z + \frac{1}{2}\mathrm{e}^{\frac{\beta\phi}{2}-\alpha\phi}\mathcal{F}^\alpha{}_\beta\hat{e}^\beta \tag{2.44}$$

Other components are obtained from the antisymmetry property $\hat{\omega}^{\alpha\beta} = -\hat{\omega}^{\beta\alpha}$.

2.2.3 The Riemann Tensor

From now on we set $\alpha = -\beta$ to get the ansatz (2.24) to coincide with (2.21). To get the Riemann tensor $\hat{R}^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}}$ we use Cartan's equations

$$\frac{1}{2}\hat{R}^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}}\hat{e}^{\hat{\gamma}} \wedge \hat{e}^{\hat{\delta}} = \mathrm{d}\hat{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} + \hat{\omega}^{\hat{\alpha}}{}_{\hat{\kappa}} \wedge \hat{\omega}^{\hat{\kappa}}{}_{\hat{\beta}}. \tag{2.45}$$

The Ricci scalar $\hat{R}$ is given in terms of $\hat{R} = \hat{R}_{\beta\delta}\eta^{\beta\delta} + \hat{R}_{33}$ which in turn are given by

$$\begin{aligned}\hat{R}_{\beta\delta} &= \hat{R}^{\alpha}{}_{\beta\gamma\delta}\eta^{\gamma}{}_{\alpha} + \hat{R}^{3}{}_{\beta 3\delta}\\ \hat{R}_{33} &= \hat{R}^{\alpha}{}_{3\gamma 3}\eta^{\gamma}{}_{\alpha}.\end{aligned} \tag{2.46}$$

A useful identity is

$$\begin{aligned}\mathrm{d}(\hat{e}^{\kappa}{}_{\nu}) &= \partial_{\mu}(\hat{e}^{\kappa})dx^{\mu}\wedge dx^{\nu} = \partial_{\mu}(\mathrm{e}^{\frac{\alpha\phi}{2}}e^{\kappa}{}_{\nu})dx^{\mu}\wedge dx^{\nu}\\ &= \partial_{\mu}(\mathrm{e}^{\frac{\alpha\phi}{2}}e^{\kappa}{}_{\nu})e^{\mu}{}_{\gamma}e^{\nu}{}_{\delta}e^{\gamma}\wedge e^{\delta} = \frac{\alpha}{2}\partial_{\mu}\phi\mathrm{e}^{\frac{\alpha\phi}{2}}e^{\kappa}{}_{\nu}e^{\nu}{}_{\delta}e^{\mu}{}_{\gamma}e^{\gamma}\wedge e^{\delta}\\ &\quad + \mathrm{e}^{\frac{\alpha\phi}{2}}\partial_{\mu}(e^{\kappa}{}_{\nu})\hat{e}^{\nu}{}_{\delta}\hat{e}^{\mu}{}_{\gamma}\hat{e}^{\gamma}\wedge\hat{e}^{\delta} = \mathrm{e}^{\frac{\alpha\phi}{2}}\mathrm{d}(e^{\kappa}) + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial_{\gamma}\phi\eta^{\gamma}{}_{\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta}.\end{aligned} \tag{2.47}$$

We start with $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$

$$\frac{1}{2}\hat{R}^{\alpha}{}_{\beta\gamma\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta} = \mathrm{d}\hat{\omega}^{\alpha}{}_{\beta} + \hat{\omega}^{\alpha}{}_{\hat{\kappa}}\wedge\hat{\omega}^{\hat{\kappa}}{}_{\beta} = \underbrace{\mathrm{d}\hat{\omega}^{\alpha}{}_{\beta}}_{\mathrm{I}} + \underbrace{\hat{\omega}^{\alpha}{}_{\kappa}\wedge\hat{\omega}^{\kappa}{}_{\beta}}_{\mathrm{II}} + \underbrace{\hat{\omega}^{\alpha}{}_{3}\wedge\hat{\omega}^{3}{}_{\beta}}_{\mathrm{III}}. \tag{2.48}$$

In the following calculations we will drop all the $\hat{e}^{\alpha}\wedge\hat{e}^{3}$ terms since they cancel anyway. The different parts are given by:

I:

$$\begin{aligned}\mathrm{d}\hat{\omega}^{\alpha}{}_{\beta} &= \mathrm{d}\left(\omega^{\alpha}{}_{\beta} + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_{\beta}\phi\hat{e}^{\alpha} - \partial^{\alpha}\phi\eta_{\beta\gamma}\hat{e}^{\gamma}\right) - \frac{1}{2}\mathrm{e}^{\frac{\beta\phi}{2}-\alpha\phi}\mathcal{F}^{\alpha}_{\beta}\hat{e}^{3}\right)\\ &= \mathrm{d}\omega^{\alpha}_{\beta} - \frac{\alpha^{2}}{4}\partial_{\gamma}\phi\partial_{\beta}\phi\mathrm{e}^{-\alpha\phi}\delta^{\alpha}_{\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta} - \frac{\alpha^{2}}{4}\mathrm{e}^{-\alpha\phi}\partial_{\gamma}\phi\partial^{\alpha}\eta_{\beta\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta}\\ &\quad + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}(\partial_{\gamma}\partial_{\beta}\phi\delta^{\alpha}_{\delta} - \partial_{\gamma}\partial^{\alpha}\eta_{\beta\delta})\hat{e}^{\gamma}\wedge\hat{e}^{\delta} + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}(\partial_{\beta}\phi\delta^{\alpha}_{\kappa} - \partial^{\alpha}\phi\eta_{\beta\kappa})\mathrm{d}(\hat{e}^{\kappa})\\ &\quad - \frac{1}{2}\mathrm{e}^{-\frac{3\alpha\phi}{2}}\mathcal{F}^{\alpha}{}_{\beta}\mathrm{d}(\hat{e}^{3})\\ &= \mathrm{d}\omega^{\alpha}_{\beta} - \frac{\alpha^{2}}{4}\partial_{\gamma}\phi\partial_{\beta}\phi\mathrm{e}^{-\alpha\phi}\delta^{\alpha}_{\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta}\\ &\quad - \frac{\alpha^{2}}{4}\mathrm{e}^{-\alpha\phi}\partial_{\gamma}\phi\partial^{\alpha}\eta_{\beta\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta} + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}(\partial_{\gamma}\partial_{\beta}\phi\delta^{\alpha}_{\delta} - \partial_{\gamma}\partial^{\alpha}\eta_{\beta\delta})\hat{e}^{\gamma}\wedge\hat{e}^{\delta}\\ &\quad + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\left(\partial_{\beta}\phi\delta^{\alpha}_{\kappa} - \partial^{\alpha}\phi\eta_{\beta\kappa}\right)\left(\mathrm{e}^{\frac{\alpha\phi}{2}}\mathrm{d}(e^{\kappa}) + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial_{\gamma}\phi\eta^{\gamma}{}_{\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta}\right)\\ &\quad - \frac{1}{2}\mathrm{e}^{-\frac{3\alpha\phi}{2}}\mathcal{F}^{\alpha}{}_{\beta}\mathrm{d}(\hat{e}^{3})\\ &= \mathrm{d}\omega^{\alpha}_{\beta} + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}(\partial_{\gamma}\partial_{\beta}\phi\delta^{\alpha}_{\delta} - \partial_{\gamma}\partial^{\alpha}\eta_{\beta\delta})\hat{e}^{\gamma}\wedge\hat{e}^{\delta}\\ &\quad + \frac{\alpha}{2}\left(\partial_{\beta}\phi\delta^{\alpha}_{\kappa}\mathrm{d}e^{\kappa} - \partial^{\alpha}\phi\eta_{\beta\kappa}\mathrm{d}e^{\kappa}\right) - \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^{\alpha}{}_{\beta}\mathcal{F}_{\gamma\delta}\hat{e}^{\gamma}\wedge\hat{e}^{\delta}.\end{aligned} \tag{2.49}$$

II:

$$\begin{aligned}
\hat{\omega}^{\alpha}{}_{\kappa} \wedge \hat{\omega}^{\kappa}{}_{\beta} &= \Big(\omega^{\alpha}{}_{\kappa} + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_{\kappa}\phi\hat{e}^{\alpha} - \partial^{\alpha}\phi\eta_{\kappa\gamma}\hat{e}^{\gamma}\right)\Big) \wedge \Big(\omega^{\kappa}{}_{\beta} + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_{\beta}\phi\hat{e}^{\kappa} - \partial^{\kappa}\phi\eta_{\beta\gamma}\hat{e}^{\gamma}\right)\Big) \\
&= \omega^{\alpha}{}_{\kappa}\omega^{\kappa}{}_{\beta} + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_{\beta}\phi\omega^{\alpha}{}_{\kappa} \wedge \hat{e}^{\kappa} - \partial^{\kappa}\phi\eta_{\beta\delta}\omega^{\alpha}{}_{\kappa} \wedge \hat{e}^{\delta}\right) \\
&\quad + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_{\kappa}\phi\hat{e}^{\alpha} \wedge \omega^{\kappa}{}_{\beta} - \partial^{\alpha}\phi\eta_{\kappa\gamma}\hat{e}^{\gamma} \wedge \omega^{\kappa}{}_{\beta}\right) \\
&\quad + \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\Big(\partial_{\kappa}\phi\partial_{\beta}\phi\hat{e}^{\alpha} \wedge \hat{e}^{\kappa} - \partial_{\kappa}\phi\partial^{\kappa}\hat{e}^{\alpha} \wedge \hat{e}^{\delta}\eta_{\beta\delta} - \partial^{\alpha}\phi\partial_{\beta}\phi\eta_{\kappa\gamma}\hat{e}^{\gamma} \wedge \hat{e}^{\kappa} \\
&\quad + \partial^{\alpha}\phi\partial^{\kappa}\phi\eta_{\kappa\gamma}\eta_{\beta\delta}\hat{e}^{\gamma} \wedge \hat{e}^{\delta}\Big) \\
&= \omega^{\alpha}{}_{\kappa} \wedge \omega^{\kappa}{}_{\beta} + \frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_{\beta}\phi\omega^{\alpha}{}_{\kappa} \wedge \hat{e}^{\kappa} - \partial^{\alpha}\phi\eta_{\kappa\gamma}\hat{e}^{\gamma} \wedge \omega^{\kappa}{}_{\beta}\right) \\
&\quad + \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\Big(\partial_{\delta}\phi\partial_{\beta}\phi\eta^{\alpha}{}_{\gamma} - (\partial\phi)^2\eta_{\beta\delta}\eta^{\alpha}{}_{\gamma} - \partial^{\alpha}\phi\partial_{\beta}\phi\eta_{\delta\gamma} + \partial^{\alpha}\phi\partial_{\kappa}\phi\eta_{\beta\delta}\Big)\hat{e}^{\gamma} \wedge \hat{e}^{\delta} \\
&\quad + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\left(-\partial_{\kappa}\phi\omega_{\gamma}{}^{\kappa}{}_{\beta}\eta^{\alpha}{}_{\delta} - \partial^{\kappa}\phi\eta_{\beta\delta}\omega_{\gamma}{}^{\alpha}{}_{\kappa}\right)\hat{e}^{\gamma} \wedge \hat{e}^{\delta}.
\end{aligned} \tag{2.50}$$

Here we have defined $(\partial\phi)^2 = \partial_{\alpha}\phi\partial^{\alpha}\phi$.

III:

$$\begin{aligned}
\hat{\omega}^{\alpha}{}_{3} \wedge \hat{\omega}^{3}{}_{\beta} &= \left(-\frac{1}{2}\mathcal{F}^{\alpha}{}_{\gamma}\,\mathrm{e}^{-\frac{3\alpha\phi}{2}}\hat{e}^{\gamma}\right) \wedge \left(\frac{1}{2}\mathcal{F}_{\beta\delta}\,\mathrm{e}^{-\frac{3\alpha\phi}{2}}\hat{e}^{\delta}\right) \\
&= -\frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^{\alpha}{}_{\gamma}\mathcal{F}_{\beta\delta}\hat{e}^{\gamma} \wedge \hat{e}^{\delta}.
\end{aligned} \tag{2.51}$$

All together we get

$$\begin{aligned}
\mathrm{d}\hat{\omega}^{\alpha}{}_{\beta} + \hat{\omega}^{\alpha}{}_{\hat{\kappa}} \wedge \hat{\omega}^{\hat{\kappa}}{}_{\beta} &= \mathrm{d}\hat{\omega}^{\alpha}{}_{\beta} + \hat{\omega}^{\alpha}{}_{\kappa} \wedge \hat{\omega}^{\kappa}{}_{\beta} + \hat{\omega}^{\alpha}{}_{3} \wedge \hat{\omega}^{3}{}_{\beta} = \mathrm{d}\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\kappa} \wedge \omega^{\kappa}{}_{\beta} \\
&\quad + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\left(\partial_{\gamma}\partial_{\beta}\phi\delta^{\alpha}_{\delta} - \partial_{\gamma}\partial^{\alpha}\phi\eta_{\beta\delta} - \partial_{\kappa}\phi\omega_{\gamma}{}^{\kappa}{}_{\beta}\eta^{\alpha}{}_{\delta} - \partial^{\kappa}\phi\eta_{\beta\delta}\omega_{\gamma}{}^{\alpha}{}_{\kappa}\right)\hat{e}^{\gamma} \wedge \hat{e}^{\delta} \\
&\quad + \frac{\alpha}{2}\left(\partial_{\beta}\phi\eta^{\alpha}{}_{\kappa}\mathrm{d}e^{\kappa} - \partial^{\alpha}\eta_{\beta\kappa}\mathrm{d}e^{\kappa} + \partial_{\beta}\phi\omega^{\alpha}{}_{\kappa} \wedge e^{\kappa} - \partial^{\alpha}\phi\eta_{\kappa\gamma}e^{\gamma} \wedge \omega^{\kappa}{}_{\beta}\right) \\
&\quad + \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\Big(-\partial_{\gamma}\phi\partial_{\beta}\phi\eta^{\alpha}{}_{\delta} + \partial_{\gamma}\phi\partial^{\alpha}\phi\eta_{\beta\delta} + \partial_{\gamma}\phi\partial_{\beta}\phi\eta^{\alpha}{}_{\delta} \\
&\quad - \partial^{\alpha}\phi\partial_{\gamma}\phi\eta_{\delta\beta} + \partial_{\delta}\phi\partial_{\beta}\phi\eta^{\alpha}{}_{\gamma} - \partial^{\alpha}\phi\partial_{\beta}\phi\eta_{\delta\gamma} + \partial^{\alpha}\phi\partial_{\gamma}\phi\eta_{\beta\delta}\Big)\hat{e}^{\gamma} \wedge \hat{e}^{\delta} \\
&\quad - \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\left(\partial\phi\right)^2\eta_{\beta\delta}\eta^{\alpha}{}_{\gamma} - \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^{\alpha}{}_{\beta}\mathcal{F}_{\gamma\delta}\hat{e}^{\gamma} \wedge \hat{e}^{\delta} \\
&\quad - \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^{\alpha}{}_{\gamma}\mathcal{F}_{\beta\delta}\hat{e}^{\gamma} \wedge \hat{e}^{\delta} \\
&= \mathrm{d}\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\kappa} \wedge \omega^{\kappa}{}_{\beta} + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\left(\nabla_{\gamma}\partial_{\beta}\phi\eta^{\alpha}{}_{\delta} - \nabla_{\gamma}\partial^{\alpha}\phi\eta_{\beta\delta}\right)\hat{e}^{\gamma} \wedge \hat{e}^{\delta}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{2}\left(\partial_\beta\phi\nabla e^\alpha-\partial^\alpha\phi\eta_{\beta\kappa}\nabla e^\kappa\right)-\frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\left(\partial\phi\right)^2\eta^\alpha{}_\gamma\eta_{\beta_\delta}\hat{e}^\gamma\wedge\hat{e}^\delta\\
&+\frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\Big(\partial_\delta\phi\partial_\beta\phi\eta^\alpha{}_\gamma-\partial^\alpha\phi\partial_\beta\phi\eta_{\delta\gamma}+\partial^\alpha\phi\partial_\gamma\phi\eta_{\delta\beta}\Big)\hat{e}^\gamma\wedge\hat{e}^\delta\\
&-\frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^\alpha{}_\beta\mathcal{F}_{\gamma\delta}\hat{e}^\gamma\wedge\hat{e}^\delta-\frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^\alpha{}_\gamma\mathcal{F}_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta.
\end{aligned}
\tag{2.52}
$$

Now if we use $\nabla e^\alpha = 0$, antisymmetrize in γ and δ and multiply by a factor of 2 we get the Riemann tensor

$$
\begin{aligned}
\hat{R}^\alpha{}_{\beta\gamma\delta}=\;&\mathrm{e}^{-\alpha\phi}R^\alpha{}_{\beta\gamma\delta}+\frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\left(\nabla_\gamma\partial_\beta\phi\eta^\alpha{}_\delta-\nabla_\delta\partial_\beta\phi\eta^\alpha{}_\gamma-\nabla_\gamma\partial^\alpha\phi\eta_{\beta\delta}+\nabla_\delta\partial^\alpha\phi\eta_{\beta\gamma}\right)\\
&+\frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\Big(\partial_\delta\phi\partial_\beta\phi\eta^\alpha{}_\gamma-\partial_\gamma\phi\partial_\beta\phi\eta^\alpha{}_\delta+\partial^\alpha\phi\partial_\gamma\phi\eta_{\delta\beta}-\partial^\alpha\phi\partial_\delta\phi\eta_{\gamma\beta}\Big)\\
&-\frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\left(\partial\phi\right)^2\left(\eta^\alpha{}_\gamma\eta_{\beta_\delta}-\eta^\alpha{}_\delta\eta_{\beta_\gamma}\right)-\frac{1}{2}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^\alpha{}_\beta\mathcal{F}_{\gamma\delta}\\
&-\frac{1}{4}\mathrm{e}^{-3\alpha\phi}\left(\mathcal{F}^\alpha{}_\gamma\mathcal{F}_{\beta\delta}-\mathcal{F}^\alpha{}_\delta\mathcal{F}_{\beta\gamma}\right).
\end{aligned}
\tag{2.53}
$$

We continue by calculating $\hat{R}^3{}_{\beta3\delta}$.

$$
\frac{1}{2}\hat{R}^3{}_{\beta3\delta}\hat{e}^3\wedge\hat{e}^\delta=\underbrace{\mathrm{d}\hat{\omega}^3{}_\beta}_{\mathrm{I}}+\underbrace{\hat{\omega}^3{}_\kappa\wedge\hat{\omega}^\kappa{}_\beta}_{\mathrm{II}}
\tag{2.54}
$$

Just like above, we drop all the terms not containing $\hat{e}^3\wedge\hat{e}^\delta$ and get

I:

$$
\begin{aligned}
\mathrm{d}\hat{\omega}^3{}_\beta&=\mathrm{d}\left(\frac{-\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial_\beta\phi\hat{e}^3+\frac{1}{2}\mathrm{e}^{-\frac{3\alpha\phi}{2}}\mathcal{F}_{\beta\delta}\hat{e}^\delta\right)\\
&=\frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\partial_\delta\phi\partial_\beta\phi\hat{e}^\delta\wedge\hat{e}^3-\frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\partial_\delta\partial_\beta\phi\hat{e}^\delta\wedge\hat{e}^3-\frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\partial_\beta\phi\mathrm{d}(\hat{e}^3)\\
&\quad+\frac{1}{2}\mathrm{e}^{-\frac{3\alpha\phi}{2}}\mathcal{F}_{\beta\delta}\mathrm{d}(\hat{e}^\delta).
\end{aligned}
\tag{2.55}
$$

II:

$$
\begin{aligned}
\hat{\omega}^3{}_\kappa\wedge\hat{\omega}^\kappa{}_\beta=&\left(\frac{-\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\partial_\kappa\phi\hat{e}^3+\frac{1}{2}\mathrm{e}^{\frac{-3\alpha\phi}{2}}\mathcal{F}_{\kappa\delta}\hat{e}^\delta\right)\\
&\wedge\left(\omega^\kappa{}_\beta+\frac{\alpha}{2}\mathrm{e}^{-\frac{\alpha\phi}{2}}\left(\partial_\beta\phi\hat{e}^\kappa-\partial^\kappa\phi\eta_{\beta\delta}\hat{e}^\delta\right)-\frac{1}{2}\mathrm{e}^{-\frac{3\alpha\phi}{2}}\mathcal{F}^\kappa_\beta\hat{e}^3\right)\\
=&-\frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\left(\partial_\delta\phi\partial_\beta\hat{e}^3\wedge\hat{e}^\delta-(\partial\phi)^2\eta_{\beta\delta}\hat{e}^3\wedge\hat{e}^\delta\right)\\
&-\frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}_{\delta\kappa}\mathcal{F}^\kappa{}_\beta\hat{e}^3\wedge\hat{e}^\delta-\frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\partial_\kappa\phi\omega_\delta{}^\kappa{}_\beta\hat{e}^3\wedge\hat{e}^\delta.
\end{aligned}
\tag{2.56}
$$

All together we get

$$\begin{aligned}\mathrm{d}\hat{\omega}^3{}_\beta + \hat{\omega}^3{}_\kappa \wedge \hat{\omega}^\kappa{}_\beta = &-\frac{\alpha^2}{2}\mathrm{e}^{-\alpha\phi}\partial_\delta\phi\partial_\beta\phi\hat{e}^3 \wedge \hat{e}^\delta + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\nabla_\delta\partial_\beta\phi\hat{e}^3 \wedge \hat{e}^\delta \\ &+ \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}(\partial\phi)^2\eta_{\beta\delta}\hat{e}^3 \wedge \hat{e}^\delta - \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}_{\delta\kappa}\mathcal{F}^\kappa{}_\beta\hat{e}^3 \wedge \hat{e}^\delta \\ &- \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\partial_\beta\phi\partial_\delta\phi\hat{e}^3 \wedge \hat{e}^\delta. \end{aligned} \tag{2.57}$$

We have to divide by a factor of two to get the Riemann tensor component $\hat{R}^3{}_{\beta3\delta}$ to avoid overcounting but this factor cancels with factor of 2 from Cartan's equations.

$$\begin{aligned}\hat{R}^3{}_{\beta3\delta} = &-\frac{3\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\partial_\delta\phi\partial_\beta\phi + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\nabla_\delta\partial_\beta\phi + \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}(\partial\phi)^2\eta_{\beta\delta} \\ &+ \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}_{\kappa\delta}\mathcal{F}^\kappa{}_\beta. \end{aligned} \tag{2.58}$$

From (2.46) we get

$$\begin{aligned}\hat{R}_{\beta\delta} = &\,\mathrm{e}^{-\alpha\phi}R_{\beta\delta} + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\Big(\nabla_\delta\partial_\beta\phi - D\nabla_\delta\partial_\beta\phi - \Box\phi\eta_{\beta\delta} + \nabla_\delta\partial_\beta\phi + \nabla_\delta\partial_\beta\phi\Big) \\ &+ \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\Big(D\partial_\delta\phi\partial_\beta\phi - \partial_\delta\phi\partial_\beta\phi + (\partial\phi)\eta_{\delta\beta} - \partial_\delta\phi\partial_\beta - 3\partial_\delta\phi\partial_\beta\phi\Big) \\ &- \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}(\partial\phi)^2(D-2)\eta_{\beta\delta} + \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}_{\kappa\delta}\mathcal{F}^\kappa{}_\beta \\ &- \frac{1}{2}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^\alpha{}_\beta\mathcal{F}_{\gamma\delta}\eta^\gamma{}_\alpha - \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\Big(\mathcal{F}^\alpha{}_\gamma\mathcal{F}_{\beta\delta} - \mathcal{F}^\alpha{}_\delta\mathcal{F}_{\beta\gamma}\Big)\eta^\gamma{}_\alpha \\ = &\,\mathrm{e}^{-\alpha\phi}R_{\beta\delta} + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}(3-D)\nabla_\beta\partial_\beta\phi - \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\Box\phi\eta_{\beta\delta} \\ &+ \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}\partial_\delta\phi\partial_\beta\phi(D-5) - \frac{\alpha^2}{4}\mathrm{e}^{-\alpha\phi}(\partial\phi)^2(D-3)\eta_{\beta\delta} - \frac{1}{2}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^\alpha{}_\beta\mathcal{F}_{\alpha\delta}. \end{aligned} \tag{2.59}$$

In the above calculations we have used the covariant d'Alembertian $\Box = \nabla_\alpha\partial^\alpha$ and D to denote the dimension *after* reduction, i.e. $D = 3$ in our case[5]. Since $R_{\alpha\beta\gamma\delta} = R_{\beta\alpha\delta\gamma}$ we get that $\hat{R}^\alpha{}_{3\gamma3}$ is equal to $\hat{R}^3{}_{\beta3\delta}$, with the proper indices, and therefore

$$\hat{R}_{33} = \frac{\alpha^2}{2}\mathrm{e}^{-\alpha\phi}(\partial\phi)^2(D-3) + \frac{\alpha}{2}\mathrm{e}^{-\alpha\phi}\Box\phi + \frac{1}{4}\mathcal{F}^2\mathrm{e}^{-3\alpha\phi} \tag{2.60}$$

[5] Actually, our derivation does not become more general by writing D instead of 3 since we have already used that $\alpha = -\beta$ which is only true in $D = 3$. However, it makes it easier to follow the steps in the calculation if we use D instead of 3.

where $\mathcal{F}^2 = \mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}$. Finally we can put everything together to get $\hat{R}$ and at the same time put $D = 3$

$$\hat{R} = \mathrm{e}^{-\alpha\phi}R - \alpha\mathrm{e}^{-\alpha\phi}\Box\phi - \frac{\alpha^2}{2}\mathrm{e}^{-\alpha\phi}(\partial\phi)^2 - \frac{1}{4}\mathrm{e}^{-3\alpha\phi}\mathcal{F}^2. \tag{2.61}$$

To get the standard normalization for the dilaton kinetic term we choose $\alpha = -1$. If we now put this and (2.22) into (2.1) we obtain the $D = 3$ effective action

$$S = \int d^3x\sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\mathrm{e}^{2\phi}\mathcal{F}^2\right) \tag{2.62}$$

where the covariant d'Alembertian term has been dropped since it is a total derivative and the prefactor cancels with the factor from the determinant.

2.2.4 Dualization

In this section we will make a final adjustment of our three dimensional action by dualizing the field strength $\mathcal{F}$. This will result in a sigma-model on the coset space $SL(2,\mathbb{R})/SO(2)$ or $SL(2,\mathbb{R})/SO(1,1)$ depending on whether the three-dimensional space has Lorentzian or Euclidean signature. As it turns out, this sigma-model is invariant under $SL(2, \mathbb{R})$. We start by considering our three-dimensional Lagrangian from (2.62)

$$\mathcal{L} = \sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\mathrm{e}^{2\phi}\mathcal{F}^2\right). \tag{2.63}$$

It is importan to note here that R is independent of ϕ and $\mathcal{F}$. The dualization procedure of (2.63) works as follows: we have defined $\mathcal{F}$ as the field strength associated to the KK-vector $\mathcal{A}$. This means that the Bianci identity[6]and the field equations we get by varying $\mathcal{A}$ are

$$\begin{aligned} \nabla_\mu(\mathrm{e}^{2\phi}\mathcal{F}^{\mu\nu}) &= 0 \\ \nabla_\mu(\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}) &= 0. \end{aligned} \tag{2.64}$$

We denote the Levi-Civita *tensor* as ϵ and the Levi-Civita *symbol* as $\tilde{\epsilon}$. Now, let's treat $\mathcal{F}$ as in independent , i.e. independent of $\mathcal{A}$. However, we would not like to loose the information that $\mathcal{F}$ is a field strength so we have to add this separately into the Lagrangian by introducing a new field χ as

$$\mathcal{L} = \sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\mathrm{e}^{2\phi}\mathcal{F}^2 + \chi\frac{1}{2}\epsilon^{\mu\nu\rho}\partial_\mu\mathcal{F}_{\nu\rho}\right). \tag{2.65}$$

[6] $\mathcal{F}^{\mu\nu} = g^{\mu\rho}\mathcal{F}_{\rho\delta}g^{\nu\delta}$.

It might seem that the added term vanishes since $\epsilon^{\mu\nu\rho}\partial_\mu \mathcal{F}_{\nu\rho} = 0$ but now we have to remember that $\mathcal{F}$ is not considered as the field strength for $\mathcal{A}$. When we consider the Lagrangian (2.65) we should thus think of it as a function of the variables $g_{\mu\nu}, \mathcal{F}_{\mu\nu}, \phi, \chi$. By varying this Lagrangian with respect to the new field χ we get

$$\begin{aligned}\frac{1}{2}\sqrt{g}\epsilon^{\mu\nu\rho}\partial_\mu \mathcal{F}_{\nu\rho} = \frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho}\partial_\mu \mathcal{F}_{\nu\rho} &= \frac{1}{2}\partial_\mu(\tilde{\epsilon}^{\mu\nu\rho}\mathcal{F}_{\nu\rho}) \\ &= \frac{1}{2}\partial_\mu(\sqrt{g}\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}) = \frac{1}{2}\sqrt{g}\nabla_\mu(\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}) = 0\end{aligned}$$

i.e. $\nabla_\mu(\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho}) = 0$ which is nothing else than the Bianci identity for $\mathcal{F}$. Thus, by construction, variation with respect to the field χ gives the Bianci identity. One can say that the Lagrangians (2.63) and (2.65) contain the same amount of information but expressed in different ways. If we vary (2.65) with respect to $\mathcal{F}_{\mu\nu}$ we get

$$\begin{aligned}\delta\mathcal{L} &= -\frac{\sqrt{g}}{2}e^{2\phi}\delta\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\chi}{2}\tilde{\epsilon}^{\mu\nu\rho}\partial_\mu(\delta\mathcal{F}_{\nu\rho}) \\ &= -\frac{\sqrt{g}}{2}e^{2\phi}\mathcal{F}^{\mu\nu}\delta\mathcal{F}_{\mu\nu} + \partial_\mu\left(\frac{\chi}{2}\tilde{\epsilon}^{\mu\nu\rho}\delta\mathcal{F}_{\nu\rho}\right) - \delta\mathcal{F}_{\nu\rho}\partial_\mu\left(\frac{\chi}{2}\tilde{\epsilon}^{\mu\nu\rho}\right) \\ &= \left(-\frac{\sqrt{g}}{2}e^{2\phi}\mathcal{F}^{\mu\nu} - \partial_\rho\mathcal{F}\tilde{\epsilon}^{\rho\mu\nu}\right)\delta\mathcal{F}_{\mu\nu} + \partial_\mu\left(\frac{\chi}{2}\tilde{\epsilon}^{\mu\nu\rho}\right)\delta\mathcal{F}_{\nu\rho}.\end{aligned}$$

which gives us

$$\mathcal{F}^{\mu\nu} = -e^{-2\phi}\epsilon^{\rho\mu\nu}\partial_\rho\chi. \tag{2.66}$$

This tells us that our newly introduced field χ is nothing else than the dual of $\mathcal{F}_{\mu\nu}e^{2\phi}$. The dual of a two form[7] $\mathcal{F}' = e^{2\phi}\mathcal{F}$ is defined as

$$(\star\mathcal{F}')_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}(\mathcal{F}')^{\nu\rho}. \tag{2.67}$$

The inverse expression is obtained by first multiplying by $\epsilon^{\mu\sigma\lambda}$

$$(\mathcal{F}')^{\nu\rho}\epsilon_{\mu\nu\rho}\epsilon^{\mu\sigma\lambda} = 2(\star\mathcal{F}')_\mu\epsilon^{\mu\sigma\lambda} \tag{2.68}$$

and use that

$$\epsilon_{\mu\nu\rho}\epsilon^{\mu\sigma\lambda} = 2\sigma(\delta^\sigma_\nu\delta^\lambda_\rho - \delta^\lambda_\nu\delta^\sigma_\rho),$$

where $\sigma = +1$ for Euclidean signature and $\sigma = -1$ for Lorentizan,

$$(\mathcal{F}')^{\sigma\lambda} = \frac{1}{2\sigma}\epsilon^{\mu\sigma\lambda}(\star\mathcal{F}')_\mu.$$

[7] To simplify the notation we define $\mathcal{F}'$.

If we put this expression into (2.64) we get

$$\nabla_\mu\Big(\epsilon^{\rho\mu\nu}(\star\mathcal{F}')_\rho\Big) = 0,$$

i.e. the field equation for $\mathcal{F}'$ is the Bianci identity for $(\star\mathcal{F}')$ and vice versa. Since $(\star\mathcal{F}')$ satisfies a Bianci identity we can introduce a scalar F such that $(\star\mathcal{F}')_\mu = \partial_\mu F$ and finally arrive at

$$(\mathcal{F}')^{\sigma\lambda} = \frac{1}{2\sigma}\epsilon^{\mu\sigma\lambda}\partial_\mu\chi. \tag{2.69}$$

The last step is to insert (2.66) into the relevant part of (2.65)

$$\begin{aligned}
\mathcal{L}_\mathcal{F} &= \sqrt{g}\left(-\frac{1}{4}\mathrm{e}^{2\phi}\mathcal{F}^2 + \frac{\chi}{2}\epsilon^{\mu\nu\rho}\partial_\mu\mathcal{F}_{\nu\rho}\right)\\
&= -\frac{\sqrt{g}}{4}\mathrm{e}^{2\phi}\left(-\mathrm{e}^{-2\phi}\epsilon^{\rho\mu\nu}\partial_\rho\chi\right)\left(-\mathrm{e}^{-2\phi}\epsilon_{\lambda\mu\nu}\partial^\lambda\chi\right) + \frac{\chi}{2}\tilde{\epsilon}^{\mu\nu\rho}\partial_\mu(-\mathrm{e}^{-\phi}\epsilon_{\lambda\nu\rho}\partial^\lambda\chi)\\
&= -\frac{\sqrt{g}}{4}\mathrm{e}^{-2\phi}\epsilon^{\rho\mu\nu}\epsilon_{\lambda\mu\nu}\partial_\rho\chi\partial^\lambda\chi - \frac{\chi}{2}\tilde{\epsilon}^{\mu\nu\rho}\partial_\mu(\mathrm{e}^{-2\phi}\epsilon_{\lambda\nu\rho}\partial^\lambda\chi)\\
&= -\frac{\sqrt{g}}{4}\mathrm{e}^{-2\phi}\epsilon^{\rho\mu\nu}\epsilon_{\lambda\mu\nu}\partial_\rho\chi\partial^\lambda\chi + \frac{\sqrt{g}}{2}\partial_\mu\chi\mathrm{e}^{-2\phi}\tilde{\epsilon}^{\mu\nu\rho}\tilde{\epsilon}_{\lambda\nu\rho}\partial^\lambda\chi\\
&= -\frac{\sigma}{2}\sqrt{g}\mathrm{e}^{-2\phi}(\partial\chi)^2 + \sigma\sqrt{g}\mathrm{e}^{-2\phi}(\partial\chi)^2\\
&= \frac{\sigma}{2}\sqrt{g}\mathrm{e}^{-2\phi}(\partial\chi)^2.
\end{aligned} \tag{2.70}$$

The complete Lagrangian becomes

$$\mathcal{L} = \sqrt{g}\left(R - \frac{1}{2}\left((\partial\phi)^2 - \sigma\mathrm{e}^{-2\phi}(\partial\chi)^2\right)\right). \tag{2.71}$$

This describes a so called *non-linear sigma-model* coupled to gravity. A general sigma-model is written as

$$\mathcal{L} = \sqrt{g}\left(R - g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_j\gamma^{ij}(\phi)\right). \tag{2.72}$$

where $\gamma_{ij}(\phi)$ function as a metric on the coset space and ϕ_i as coordinates. The coset space is also called the *target space*. From (2.71) we end up with two cases

- If we reduce a spacelike dimension our remaining space has Lorentizan signature which means that $\sigma = -1$ and we get a sigma-model on $SL(2,\mathbb{R})/SO(2)$.

- If we reduce a timelike dimension our remaining space has Euclidean signature which means that $\sigma = +1$ and we get a sigma-model on $SL(2,\mathbb{R})/SO(1,1)$.

That (2.71) is invariant under $SL(2,\mathbb{R})$ is by no means obvious at this stage. In the next chapter we will show this by rewriting the sigma-model in a way which makes the symmetry become manifest.

3

Sigma-Models on Coset Spaces

In the previous chapter we saw that dimensional reduction and dualization of pure gravity from $D = 4$ to $D = 3$ gives rise to a sigma-model. This extends to higher dimensions and different gravity theories, e.g. Einstein-Maxwell and supergravity. That is, depending on the theory, the dimension we start with and the number of Killing vectors, we get different sigma-models. The purpose of this chapter is to investigate sigma-models with target space G/H and in particular show that they are invariant under the group G. At the end of this chapter we derive the three-dimensional field equations. Since $SL(2, \mathbb{R})$ is the symmetry group for the case of pure gravity we begin by reviewing some basic properties of $SL(2,\mathbb{R})$ and the coset space $SL(2, \mathbb{R})/SO(2)$.

3.1 The Coset Space $SL(2,\mathbb{R})/SO(2)$

We claimed in section 2.2.4 that dimensional reduction[1] of pure gravity leads to a sigma-model on the coset space $SL(2,\mathbb{R})/SO(2)$. This means that the field content in the sigma-model parametrize the coset space. Moreover, the sigma-model is invariant under $SL(2,\mathbb{R})$. In this section and the sequel we will mainly focus on this coset space but the results can be generalized for any sigma-model.

[1]Of a space-like dimension.

3.1.1 $SL(2,\mathbb{R})$

The group $SL(2,\mathbb{R})$ consists of all 2×2 real matrices with unit determinant. The corresponding three-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is generated by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

which defines the triangular decomposition of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}e \oplus \mathbb{R}h \oplus \mathbb{R}f. \tag{3.2}$$

The sum is a direct sum of vector spaces. The Lie algebra is defined by the commutation relations

$$\begin{aligned} [h,e] &= 2e \\ [h,f] &= -2f \\ [e,f] &= h. \end{aligned} \tag{3.3}$$

The group $SL(2,\mathbb{R})$ has $SO(2)$ as maximal compact subgroup with Lie algebra $(e-f)$. That is,

$$\mathfrak{so}(2) = \mathbb{R}(e-f). \tag{3.4}$$

3.1.2 The Cartan Involution

The *Cartan involution* θ on a Lie algebra $\mathfrak{g}$ is defined by its action on the Chevalley generators e_i, f_i, h_i as [14]

$$\theta(e_i) = -f_i \qquad \theta(f_i) = -e_i \qquad \theta(h_i) = -h_i. \tag{3.5}$$

We can define an invariant and an anti-invariant subspace with respect to θ as

$$\begin{aligned} \mathfrak{k} &= \{t \in \mathfrak{g} | \theta(t) = t\} \\ \mathfrak{p} &= \{t \in \mathfrak{g} | \theta(t) = -t\} \end{aligned} \tag{3.6}$$

which gives us a decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.7}$$

called the *Cartan decomposition*. The subspace $\mathfrak{k}$ defines the Lie subalgebra to the maximal compact subgroup to G. We can check this for the $\mathfrak{sl}(2,\mathbb{R})$ case. The Cartan involution becomes

$$\theta(e) = -f \qquad \theta(h) = -h \qquad \theta(f) = -e. \tag{3.8}$$

This can equivalently be described by

$$\theta : \qquad \theta(t) = -t^T, \qquad\qquad t \in \mathfrak{sl}(2,\mathbb{R}) \tag{3.9}$$

where $()^T$ denotes matrix transpose. From (3.8) we get that

$$\theta(e-f) = e-f \qquad \theta(e+f) = -(e+f) \qquad \theta(h) = -h \tag{3.10}$$

which means that

$$\begin{aligned} \mathfrak{k} &= \mathbb{R}(e-f) \\ \mathfrak{p} &= \mathbb{R}(e+f) \oplus \mathbb{R}h. \end{aligned} \tag{3.11}$$

We note here that $\mathfrak{p}$ is *not* a subalgebra since it does not close under the Lie bracket. Moreover, we also note that the subalgebra invariant under the Cartan involution is indeed the algebra $\mathfrak{so}(2)$. The Cartan decomposition (3.7) of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} = \mathbb{R}(e-f) \oplus \mathbb{R}h \oplus \mathbb{R}(e+f) = \underbrace{\mathfrak{so}(2)}_{\text{Invariant}} \oplus \underbrace{\mathbb{R}h \oplus \mathbb{R}(e+f)}_{\text{Anti-invariant}}. \tag{3.12}$$

The key point here is that θ is defined to leave the subalgebra $\mathfrak{so}(2)$ invariant, i.e. the Lie algebra which generates the maximal compact subgroup.

3.1.3 Iwasawa Decomposition

In general, the *triangular decomposition* of a Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_+ \tag{3.13}$$

where the sum is a direct sum of vector spaces. For a connected[2] semisimple Lie group G with Lie algebra $\mathfrak{g}$ there exists a another global decomposition called the *Iwasawa decomposition* [15]. The Iwasawa decomposition of the Lie algebra is given by [16],

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{a} \oplus \mathfrak{k} \tag{3.14}$$

where $\mathfrak{k}$ is the Lie algebra for the maximal compact subgroup, $\mathfrak{a} = \mathfrak{n}_0 \cap \mathfrak{p}$ is the subspace of non-compact Cartan generators. The sum is a direct sum of vector spaces. The Iwasawa decomposition of the Lie group G is given by [16],

$$G = NAK \tag{3.15}$$

[2] A connected space is a topological space that cannot be represented as the union of two or more open disjoint subsets.

where K, A and N are generated by $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}_+$ respectively. This can also be seen as a decomposition of a group element G into an orthogonal matrix K, a diagonal matrix A and an upper triangular matrix N. K is the maximal compact subgroup. From the Iwasawa decomposition we can easily construct a coset representative for the coset space G/K by choosing $K = \mathbb{1}$. That is, a coset representative for G/K is given by

$$G/K = NA. \tag{3.16}$$

From the Iwasawa decomposition (3.14) of the Lie algebra $\mathfrak{g}$ we see that this coset is generated by $\mathfrak{a} \oplus \mathfrak{n}_+$ which is called the *Borel subalgebra*. We can find the decomposition and coset representative explicitly for $SL(2,\mathbb{R})$. We have that $\mathfrak{a} = \mathbb{R}h, \mathfrak{n}_+ = \mathbb{R}e$ and $\mathfrak{k} = \mathbb{R}(e - f) = \mathfrak{so}(2)$ which means that (3.14) becomes

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}e \oplus \mathbb{R}h \oplus \mathfrak{so}(2). \tag{3.17}$$

Let $g \in SL(2,\mathbb{R})$ be an arbitrary element. By exponentiating with the parameters χ, ϕ, θ we get that (3.15) becomes

$$g = NAK = \exp(\chi e) \exp(-\phi/2h) \exp(\theta(e - f)) \tag{3.18}$$

which in matrix form is equal to

$$g = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi/2} & 0 \\ 0 & e^{\phi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{3.19}$$

The coset representative for $SL(2,\mathbb{R})/SO(2)$ is now given by (3.16). Let $V \in SL(2,\mathbb{R})/SO(2)$ be an arbitrary coset element. We have that

$$V = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi/2} & 0 \\ 0 & e^{\phi/2} \end{pmatrix}. \tag{3.20}$$

The choice $\theta = 0$ is of course not unique, although the most convenient one, as a coset representative. Since the sigma-model is parametrized on a coset space the choice of representative should not matter. As we will see, this is indeed the case and we may think of this as gauge freedom in our theory and the choice $\theta = 0$ as a choice of gauge.

In this section we have worked with coset spaces G/K where K is the maximal compact subgroup. For more general coset spaces where the denominator is not necessarily a compact subgroup, e.g. $SL(2,\mathbb{R})/SO(1,1)$ we will use the notation G/H. However, one has to be careful when H is a non-compact group since the results of the Iwasawa decomposition is not entirely transferable.

As we have seen the Cartan involution θ is defined to leave the Lie algebra $\mathfrak{k}$ to the maximal compact subgroup K of a Lie group G invariant. We can generalize this to an involution τ which is defined to leave a subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ invariant. The Cartan decomposition (3.7) generalizes to

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{3.21}$$

where $\mathfrak{m}$ is the anti-invariant subspace with respect to τ. This will become useful later on when we consider the coset space $SL(2,\mathbb{R})/SO(1,1)$.

We end this section by defining the transformation of a coset element V of a general coset space G/H. The coset element V transforms as

$$V \longrightarrow gVk(x). \tag{3.22}$$

g is a global element in G and k is a local element in the subgroup H depending on both g and V. The reason why $k(x)$ is spacetime dependent is because V is spacetime dependent. If we had just defined the transformation as $V \longrightarrow gV$ it is not guaranteed that we stay in the gauge. To ensure that the transformed element is in the gauge we have chosen, i.e. upper triangular matrices, we accompany the global element g with a local compensator which thus depends on both the coset element V and g.

3.1.4 The Matrix M

Consider the matrix M defined as

$$M = VV^{\mathcal{T}} \tag{3.23}$$

where $()^{\mathcal{T}}$ is the generalized transpose defined as $V^{\mathcal{T}} = \exp\Big(-\tau(t)\Big)$. Since elements in $\mathfrak{h}$ are invariant under τ, and therefore anti-invariant under $-\tau$, we get that

$$g^{\mathcal{T}} = g^{-1} \qquad\qquad g \in H. \tag{3.24}$$

By using this and (3.22) we get the following transformation property for M

$$M \longrightarrow (gVk)(gVk)^{\mathcal{T}} = gMg^{\mathcal{T}}. \tag{3.25}$$

This shows that M is independent of k. In general, it is quite difficult to determine the local compensator k and it is therefore impractical to work with the group element V to generate new solutions. Since the transformation of M does not exhibit this problem we will use this matrix when we do explicit calculations. The definition (3.23) is not unique and it is also common to define M as

$$M = V^{\mathcal{T}}V. \tag{3.26}$$

This change has no drastic consequences except that one has to change coset space from G/H to $H\backslash G$. That is, $V \in H\backslash G$ and transforms as $V \longrightarrow kVg$. Consequently, M transforms as

$$M \longrightarrow g^{\mathcal{T}} M g. \tag{3.27}$$

Actually, we will use the latter definition in this thesis with exception for the next section. The reason for this is that the latter definition is used more extensively in the literature.

3.2 Construction of an Invariant Sigma-Model

In this section we will show how one can rewrite a general sigma-model on a coset space G/H in a way which makes the symmetry G become manifest. At the end we will show this explicitly for the case $SL(2,\mathbb{R})$. We begin by defining the Lie algebra valued one form, called the *Maurer Cartan* form, ω_μ

$$\omega_\mu = V^{-1}(x)\partial_\mu V(x), \qquad \omega_\mu \in \mathfrak{g} \tag{3.28}$$

where $\mathfrak{g}$ is the Lie algebra of G and $V \in G/H$. Since ω is an element in the Lie algebra we can decompose it as follows

$$\omega_\mu = \mathcal{Q}_\mu + \mathcal{P}_\mu \tag{3.29}$$

where

$$\mathcal{Q}_\mu = \frac{1}{2}\Big(\omega_\mu + \tau(\omega_\mu)\Big) \tag{3.30}$$

$$\mathcal{P}_\mu = \frac{1}{2}\Big(\omega_\mu - \tau(\omega_\mu)\Big). \tag{3.31}$$

That is, an invariant and an anti-invariant subspace under the involution. Now, let's see how Q_μ and P_μ transform when $V \longrightarrow gVk$

$$\begin{aligned} \mathcal{Q}_\mu &\longrightarrow \frac{1}{2}\left(k^{-1}\omega_\mu k + k^{-1}\partial_\mu k + \tau(k^{-1}\omega_\mu k + k^{-1}\partial_\mu k)\right) \\ &= k^{-1}\partial_\mu k + \frac{1}{2}\left(k^{-1}\omega_\mu k + \tau(k^{-1}\omega_\mu k)\right) \end{aligned} \tag{3.32}$$

where we have used $k^{-1}\partial_\mu k \in \mathfrak{k}$. As a final step we would like to rewrite $\tau(k^{-1}\omega_\mu k)$ by using Baker Campbell Hausdorff formula

$$\exp(-X)Y\exp(X) = Y - [X,Y] + \frac{1}{2!}[X,[X,Y]] + ... \tag{3.33}$$

$$\begin{aligned}
\tau(k^{-1}\omega_\mu k) &= \tau(\exp(-t)\omega_\mu \exp(t)) \\
&= \tau\left(\omega_\mu - [t,\omega_\mu] + \frac{1}{2!}[t,[t,\omega_\mu]] + ...\right) \\
&= \tau(\omega_\mu) - [\tau(t),\tau(\omega_\mu)] + \frac{1}{2!}[\tau(t),[\tau(t),\tau(\omega_\mu)]] + ... \\
&= \tau(\omega_\mu) - [t,\tau(\omega_\mu)] + \frac{1}{2!}[t,[t,\tau(\omega_\mu)]] + ... \\
&= k^{-1}\tau(\omega_\mu)k.
\end{aligned} \tag{3.34}$$

Here we have used the definition of the involution τ, namely $\tau(t) = t$ when $t \in \mathfrak{k}$. If we put this back into (3.32) we get

$$\mathcal{Q}_\mu \longrightarrow k^{-1}\mathcal{Q}_\mu k + k^{-1}\partial_\mu k. \tag{3.35}$$

This looks very much like the transformation of a gauge potential. If we turn our attention to $\mathcal{P}_\mu$ we get by a similar calculation

$$\mathcal{P}_\mu \longrightarrow k^{-1}\mathcal{P}_\mu k \tag{3.36}$$

which tells us that $\mathcal{P}_\mu$ transforms as a field strength. If we let ourselves be inspired by how one usually constructs invariant terms in the Lagrangian of field strengths we consider

$$\langle \mathcal{P}_\mu | \mathcal{P}^\mu \rangle = \mathrm{Tr}(\mathcal{P}_\mu \mathcal{P}^\mu). \tag{3.37}$$

This is invariant under both global, G, and local transformations, H. Since $\mathcal{P}_\mu$ is invariant under global transformations by itself we only need to check local transformations

$$\mathrm{Tr}(\mathcal{P}_\mu P^\mu) \longrightarrow \mathrm{Tr}(k^{-1}\mathcal{P}_\mu k k^{-1}\mathcal{P}^\mu k) = \mathrm{Tr}(\mathcal{P}_\mu \mathcal{P}^\mu), \tag{3.38}$$

where the cyclic invariance of the trace has been used. To show that (3.37) is equivalent to a sigma-model one has to parametrize the coset element V in a particular way. Exactly how the coset element V should be parametrized differs from group to group. As announced, we will show this for the $SL(2,\mathbb{R})$ case. Instead of finding ω and then constructing $\mathcal{P}_\mu$ we use the matrix M

$$M = VV^{\mathcal{T}}. \tag{3.39}$$

Since we would like to relate this to $\mathcal{P}_\mu$, which we know how to construct invariants of, we use the Maurer Cartan form to get a Lie algebra valued one form $M^{-1}\partial_\mu M$. This can be rewritten as

$$M^{-1}\partial_\mu M = (VV^\mathcal{T})^{-1}\partial_\mu(VV^\mathcal{T}) = (V^{-1})^\mathcal{T}\left(\omega_\mu + \partial_\mu V^\mathcal{T}(V^{-1})^\mathcal{T}\right)V^\mathcal{T}. \tag{3.40}$$

The second term in the parenthesis can be rewritten as

$$(\partial_\mu V^\mathcal{T})(V^{-1})^\mathcal{T} = (V^{-1}\partial_\mu V)^\mathcal{T} = \omega_\mu^\mathcal{T} = -\tau(\omega_\mu). \tag{3.41}$$

If we put this back into (3.41) we obtain the wanted relation between M and $\mathcal{P}_\mu$ as

$$M^{-1}\partial_\mu M = (V^{-1})^\mathcal{T}\Big(\omega_\mu - \tau(\omega_\mu)\Big)V^\mathcal{T} = 2(V^{-1})^\mathcal{T}\mathcal{P}_\mu V^\mathcal{T}. \tag{3.42}$$

This implies that

$$\frac{1}{4}\langle M^{-1}\partial_\mu M|M^{-1}\partial^\mu M\rangle = \langle\mathcal{P}_\mu|\mathcal{P}^\mu\rangle \tag{3.43}$$

and therfore is $\langle M^{-1}\partial_\mu M|M^{-1}\partial^\mu M\rangle$ G invariant as well. Thus, a G invariant Lagrangian can be written in the following form

$$\mathcal{L} = \sqrt{g}\Big(R - \frac{g^{\mu\nu}}{4}\langle M^{-1}\partial_\mu M|M^{-1}\partial_\nu M\rangle\Big) \tag{3.44}$$

which we consider to be manifestly G-invariant. If one parametrizes the coset element V in the right way this is equivalent to a sigma-model. We emphasize here that (3.44) is a general result, i.e. this is the Lagrangian after dimensional reduction for a general gravity-matter system. The only thing that differs is the coset space G/H. If we choose to look at the special case when $G/K = SL(2,\mathbb{R})/SO(2)$, i.e. pure gravity, we have from the Iwasawa decomposition that

$$V = \begin{pmatrix} \mathrm{e}^{-\phi/2} & \chi\mathrm{e}^{\phi/2} \\ 0 & \mathrm{e}^{\phi/2} \end{pmatrix} \tag{3.45}$$

where $V \in G$, and consequently

$$M = \begin{pmatrix} \mathrm{e}^{-\phi} + \chi^2\mathrm{e}^{\phi} & \chi\mathrm{e}^{\phi} \\ \chi\mathrm{e}^{\phi} & \mathrm{e}^{\phi} \end{pmatrix}. \tag{3.46}$$

Finally, by using that $\langle\cdot|\cdot\rangle = \mathrm{Tr}(\cdot\cdot)$ we get that

$$\mathcal{L} = \sqrt{g}\left(R - \frac{g^{\mu\nu}}{2}\Big(\partial_\mu\phi\partial_\nu\phi + \mathrm{e}^{2\phi}\partial_\mu\chi\partial_\nu\chi\Big)\right). \tag{3.47}$$

We see that this is exactly the same Lagrangian as (2.71) if one reduces a spacelike dimension. This completes our derivation of showing the invariance of (2.71) under $SL(2,\mathbb{R})$.

3.2.1 Alternative Description

There is an alternative way of writing $d^3x\sqrt{g}\langle\mathcal{P}_\mu|\mathcal{P}^\mu\rangle$ using the language of differential geometry, which will become useful later on,

$$\mathrm{d}^3x\sqrt{g}\mathrm{Tr}(\mathcal{P}_\mu\mathcal{P}^\mu) = \mathrm{Tr}\left(\star\mathcal{P}\wedge\mathcal{P}\right). \tag{3.48}$$

To see this we just use the definition of the hodge dual of a one-form $\mathcal{P}$ in three dimensions

$$\star\mathcal{P} = \frac{1}{2}\epsilon^\rho{}_{\mu\nu}\mathcal{P}_\rho\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu. \tag{3.49}$$

This gives us

$$\begin{aligned}
\star\mathcal{P}\wedge\mathcal{P} &= \frac{1}{2}\epsilon^\rho{}_{\mu\nu}\mathcal{P}_\rho\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathcal{P}_\sigma\mathrm{d}x^\sigma\\
&= \frac{1}{2}\mathcal{P}_\rho\mathcal{P}_\sigma\epsilon^\rho{}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\sigma\\
&= \frac{1}{3!}\mathcal{P}^\rho\mathcal{P}_\rho\epsilon_{\sigma\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\sigma\\
&= \mathcal{P}^\rho\mathcal{P}_\rho\epsilon\\
&= \sqrt{g}\mathcal{P}^\rho\mathcal{P}_\rho\mathrm{d}^3x.
\end{aligned}$$

In the third equality we have used that $\rho=\sigma$ and the antisymmetry property of ϵ and the three-form.

3.3 Field Realization of $SL(2,\mathbb{R})$

Now after we have seen that we really have a $SL(2,\mathbb{R})$ symmetry it is natural to ask which symmetry transformations that actually give us something. We can take a look at how our fields ϕ and χ form a representation, i.e. a field realization of the symmetry group. There are three independent transformations under consideration, $g = n, a, k$ generated by e,h,$(e-f)$ respectively. For the elements in the Borel algebra we do not need the compensator k since their corresponding transformation preserve the upper triangular form.

3.3.1 $g = \exp(\xi e)$

We use the same parametrization for $V\in G$ as above

$$V' = \begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\phi/2} & \chi\mathrm{e}^{\phi/2}\\ 0 & \mathrm{e}^{\phi/2}\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-\phi/2} & \chi\mathrm{e}^{\phi/2}+\xi\mathrm{e}^{\phi/2}\\ 0 & \mathrm{e}^{\phi/2}\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-\phi'/2} & \chi'\mathrm{e}^{\phi'/2}\\ 0 & \mathrm{e}^{\phi'/2}\end{pmatrix}.$$

By comparing the different components we find that

$$\phi' = \phi \qquad\qquad \chi' = \chi + \xi. \tag{3.50}$$

From the dualization procedure (2.69) we see that

$$(\mathcal{F}')^{\sigma\lambda} = \frac{1}{2\sigma}\epsilon^{\mu\sigma\lambda}\partial_\mu\chi' = \frac{1}{2\sigma}\epsilon^{\mu\sigma\lambda}\partial_\mu\chi + \frac{1}{2\sigma}\epsilon^{\mu\sigma\lambda}\partial_\mu\chi = \mathcal{F}^{\sigma\lambda}$$

since ξ is spacetime independent. This tells us that a transformation generated by e does not have any effect in the four dimensional theory, i.e. our metric does not change.

3.3.2 $g = \exp(-\varphi/2h)$

In the same way as above we find that

$$\phi' = \phi + \varphi \qquad\qquad \chi' = \mathrm{e}^{-\varphi}\chi \tag{3.51}$$

From the previous section we have that $g_{33} = \mathrm{e}^{\phi}$ and therefore we get

$$\delta(\mathrm{e}^{\phi}) = \delta\phi\mathrm{e}^{\phi} = \varphi\mathrm{e}^{\phi} \tag{3.52}$$

which in finite form becomes $\mathrm{e}^{\phi'} = \mathrm{e}^{\varphi}\mathrm{e}^{\phi}$. From the dualization (2.69) we get $\delta\mathcal{F}^{\sigma\lambda} = -\varphi\mathcal{F}^{\sigma\lambda}$ and by using the definition of $\mathcal{F}$ we get

$$\delta\mathcal{A}^{\lambda} = -\varphi\mathcal{A}^{\lambda} \tag{3.53}$$

which in finite form becomes $(\mathcal{A}')^{\lambda} = \mathrm{e}^{-\varphi}\mathcal{A}^{\lambda}$. From (2.11c) we can identify $\varphi = 2\Lambda$ which tells us that this transformation is nothing else than a rescaling of the reduced coordinate. From (2.16) we also see that

$$\delta\mathcal{A}_{\lambda} = \partial_\lambda\tilde{\lambda} - \Lambda\mathcal{A}_{\mu} = -\varphi\mathcal{A}_{\lambda} = -2\Lambda\mathcal{A}_{\lambda} \tag{3.54}$$

which tells us that we also have $\partial_\lambda\tilde{\lambda} = -\Lambda\mathcal{A}_\lambda$, i.e. a coordinate transformation of the reduced coordinate that depends on the unreduced coordinates. So, what we have seen is that $g = n, a$ are transformations corresponding to scaling and gauge symmetry and therefore do not give us any new physical solutions. However, the non-linear action of $K = SO(2)$ will turn out to provide us with non-trivial transformations. It is believed that this result can be generalized to any sigma-model of a dimensionally reduced gravity theory, e.g. Maxwell-Einstein, five-dimensional minimal supergravity[3] and 11-dimensional supergravity.

[3] In [17] is is shown that this is indeed true for $G_2/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$.

3.4 Field Equations in $D = 3$

We end this chapter by deriving the field equations in $D = 3$. These will be solved in the next chapter for a spherical symmetric black hole. Our Lagrangian looks like

$$\begin{aligned}\mathcal{L} &= \sqrt{g}\Big(R - g^{\mu\nu}\langle\mathcal{P}_\mu|\mathcal{P}_\nu\rangle\Big) = \sqrt{g}\Big(R - \frac{g^{\mu\nu}}{4}\langle M^{-1}\partial_\mu M|M^{-1}\partial_\nu M\rangle\Big) \\ &= \sqrt{g}\left(R - \frac{g^{\mu\nu}}{4}\mathrm{Tr}(M^{-1}\partial_\mu M M^{-1}\partial_\nu M)\right).\end{aligned} \tag{3.55}$$

There are two different field equations to be derived. The first is the one we get from the sigma-model and the second one is Einstein's equations (obtained by varying $g_{\mu\nu}$) with matter content. Since there are basically three different ways of expressing the sigma-model we will get three equivalent field equations but with different physical interpretations. We know that the Lagrangian is invariant under a G transformation so we should expect a conserved current.

3.4.1 Conserved Current Equation

Due to the global symmetry G of our Lagrangian $\mathcal{L}$ we should expect a conserved current. In this section we derive the field equation for M and its variants, one of which describes the conservation of a current. Variation with respect to M gives the following:

$$\delta\mathcal{L} = \sqrt{g}g^{\mu\nu}\mathrm{Tr}\Big(\partial_\mu\delta M\partial_\nu M^{-1} + \partial_\mu M\partial_\nu\delta(M^{-1})\Big).$$

Here we need to express $\delta(M^{-1})$ in terms of δM. In general, for two invertible matrices A and B we have, [18],

$$(A + B)^{-1} = A^{-1} - A^{-1}(\mathbb{1} + BA^{-1})^{-1}BA^{-1}. \tag{3.56}$$

In our case $A = M$ and $B = \delta M$ and since the latter is considered infinitesimal we get to the first order that

$$(M + \delta M)^{-1} = M^{-1} - M^{-1}\delta M M^{-1}. \tag{3.57}$$

That is, $\delta(M^{-1}) = M^{-1}\delta M M^{-1}$. If we put this back into the variation of the Lagrangian we get

$$\begin{aligned}
\delta\mathcal{L} &= \sqrt{g}g^{\mu\nu}\mathrm{Tr}\left(\partial_\mu\delta M\partial_\nu M^{-1} - \partial_\mu M\partial_\nu(M^{-1}\delta M M^{-1})\right)\\
&= \mathrm{Tr}\Big(\partial_\mu(\sqrt{g}g^{\mu\nu}\delta M\partial_\nu M^{-1}) - \delta M\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu M^{-1})\\
&\quad - \sqrt{g}g^{\mu\nu}\partial_\mu M(\partial_\nu M^{-1}\delta M M^{-1} + M^{-1}\partial_\nu\delta M M^{-1} + M^{-1}\delta M\partial_\nu M^{-1})\Big)\\
&= \partial_\mu\mathrm{Tr}\Big(\sqrt{g}g^{\mu\nu}\delta M\partial_\nu M^{-1}\Big) - \mathrm{Tr}\Big(\delta M\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu M^{-1})\\
&\quad - \sqrt{g}g^{\mu\nu}\delta M M^{-1}\partial_\mu M\partial_\nu M^{-1} - \sqrt{g}g^{\mu\nu}\partial_\nu\delta M M^{-1}\partial_\mu M M^{-1}\\
&\quad - \sqrt{g}g_{\mu\nu}\delta M\partial_\nu M^{-1}\partial_\mu M M^{-1}\Big)\\
&= \mathrm{Tr}\Big(- \delta M\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu M^{-1}) + \delta M\partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M M^{-1})\\
&\quad - \sqrt{g}g^{\mu\nu}2\delta M\partial_\mu M^{-1}\partial_\nu M M^{-1}\Big).
\end{aligned}$$

We have dropped divergent terms along the way. A final simplification gives

$$\begin{aligned}
\delta\mathcal{L} &= \mathrm{Tr}\Big(- \delta M(\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu M^{-1}) - \partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M M^{-1})\\
&\quad + 2\partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M M^{-1}) - 2M^{-1}\partial_\nu(\sqrt{g}g^{\mu\nu}\partial_\mu M M^{-1}))\Big)\\
&= \mathrm{Tr}\left(-\delta M(-2M^{-1}\partial_\nu(\sqrt{g}g^{\mu\nu}\partial_\mu M M^{-1}))\right)\\
&= \mathrm{Tr}\left(2M^{-1}\partial_\nu(\sqrt{g}g^{\mu\nu}\partial_\mu M M^{-1})\delta M\right)\\
&= \mathrm{Tr}\Big(\partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M M^{-1})\delta M - \sqrt{g}g^{\mu\nu}\partial_\nu M^{-1}\partial_\mu M M^{-1}\delta M\Big)\\
&= \mathrm{Tr}\Big(\partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M)M^{-1}\delta M\Big) + \mathrm{Tr}\Big(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M\partial_\nu M^{-1}\delta M\Big)\\
&\quad - \mathrm{Tr}\Big(\sqrt{g}g^{\mu\nu}\partial_\nu M^{-1}\partial_\mu M M^{-1}\delta M\Big)\\
&= \mathrm{Tr}\Big(\partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M)M^{-1}\delta M\Big) = 0.
\end{aligned}$$

Since δM is arbitrary we have that

$$\partial_\nu(\sqrt{g}g^{\mu\nu}M^{-1}\partial_\mu M) = 0 \tag{3.58}$$

since the trace is invariant under cyclic permutations. By using the covariant derivative[4] ∇_ν this can be written as

$$\nabla_\mu(M^{-1}\partial^\mu M) = 0. \tag{3.59}$$

From this we can define a conserved current $j^\mu = M^{-1}\partial^\mu M$ which is due to our global symmetry. (3.58) can be rewritten in terms of $\mathcal{P}_\mu$ by using the relation

$$M^{-1}\partial_\mu M = 2(V^{-1})^{\mathcal{T}}\mathcal{P}_\mu V^{\mathcal{T}}. \tag{3.60}$$

[4] $\nabla_\mu(v^\mu) = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}v^\mu)$.

We get

$$\begin{aligned}\partial_\nu(\sqrt{g}g^{\mu\nu}(V^{-1})^{\mathcal{T}}\mathcal{P}_\mu V^{\mathcal{T}}) &= \partial_\nu(\sqrt{g}g^{\mu\nu}(V^{-1})^{\mathcal{T}}\mathcal{P}_\mu)V^{\mathcal{T}} + \sqrt{g}g^{\mu\nu}(V^{-1})^{\mathcal{T}}\mathcal{P}_\mu\partial_\nu V^{\mathcal{T}}\\ &= (V^{-1})^{\mathcal{T}}\partial_\nu(\sqrt{g}g^{\mu\nu}\mathcal{P}_\mu V^{\mathcal{T}} + \sqrt{g}g^{\mu\nu}\partial_\nu(V^{-1})\mathcal{P}_\mu V^{\mathcal{T}}\\ &\quad + \sqrt{g}g^{\mu\nu}(V^{-1})^{\mu}\mathcal{P}_\mu\partial_\nu V^{\mathcal{T}} = 0.\end{aligned} \tag{3.61}$$

Here we can use the different versions of $V^{-1}\partial_\nu V = \mathcal{P}_\nu + \mathcal{Q}_\nu$

$$\begin{aligned}\partial_\nu(V^{-1})^{\mathcal{T}} &= (V^{-1})^{\mathcal{T}}(-\mathcal{P}_\nu + \mathcal{Q}_\mu)\\ \partial_\nu V^{\mathcal{T}} &= (\mathcal{P}_\nu - \mathcal{Q}_\nu)V^{\mathcal{T}}\end{aligned} \tag{3.62}$$

to get

$$\begin{aligned}\partial_\nu(\sqrt{g}g_{\mu\nu}(V^{-1})^{\mathcal{T}}\mathcal{P}_\mu V^{\mathcal{T}}) &= (V^{-1})^{\mathcal{T}}\partial_\nu(\sqrt{g}g^{\mu\nu}\mathcal{P}_\mu) + \sqrt{g}g^{\mu\nu}(V^{-1})^{\mathcal{T}}(-\mathcal{P}_\nu\\ &\quad + \mathcal{Q}_\nu)\mathcal{P}_\mu V^{\mathcal{T}} + \sqrt{g}g^{\mu\nu}(V^{-1})^{\mathcal{T}}\mathcal{P}_\mu(\mathcal{P}_\nu - \mathcal{Q}_\nu)V^{\mathcal{T}}\\ &= (V^{-1})^{\mathcal{T}}\left(\partial_\nu(\sqrt{g}g^{\mu\nu}\mathcal{P}_\mu) + \sqrt{g}g^{\mu\nu}[\mathcal{Q}_\nu,\mathcal{P}_\mu]\right)V^{\mathcal{T}} = 0.\end{aligned} \tag{3.63}$$

So, an equivalent field equation to (3.58) is

$$\partial_\nu(\sqrt{g}g^{\mu\nu}\mathcal{P}_\mu) + \sqrt{g}g^{\mu\nu}[\mathcal{Q}_\nu,\mathcal{P}_\mu] = 0. \tag{3.64}$$

Here it is convenient to introduce a covariant derivative on the coset space G/K

$$D_\mu = \partial_\mu + [\mathcal{Q}_\mu,\cdot] \tag{3.65}$$

which brings our field eqution into the form

$$D_\mu(\sqrt{g}\mathcal{P}^\mu) = 0. \tag{3.66}$$

There is actually a third equation equivalent to (3.58) expressed in the coset parameters ϕ_i. To derive these equations we have to rewrite our Lagrangian using

$$\frac{1}{4}\mathrm{Tr}(M^{-1}\partial_\mu M M^{-1}\partial^\mu M) = \gamma^{ij}\partial_\mu\phi_i\partial^\mu\phi_j \tag{3.67}$$

to get

$$\mathcal{L} = \sqrt{g}g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\gamma_{ij}(\phi). \tag{3.68}$$

Variation with respect to ϕ^i gives

$$\begin{aligned}\delta\mathcal{L} = \sqrt{g}\Big(&g^{\mu\nu}\partial_\mu\delta\phi^i\partial_\nu\phi^j\gamma_{ij}(\phi) + g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\delta\phi^j\gamma_{ij}(\phi)\\ &+ g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\frac{\partial\gamma_{ij}(\phi)}{\partial\phi^k}\delta\phi^k\Big) = \partial_\mu(\sqrt{g}g^{\mu\nu}\delta\phi^i\partial_\nu\phi^j\gamma_{ij}(\phi))\\ &+ \partial_\nu(\sqrt{g}g^{\mu\nu}\partial_\mu\phi^i\delta\phi^j\gamma_{ij}(\phi)) - \delta\phi^i\partial_\nu(\sqrt{g}g^{\mu\nu}\partial_\mu\phi^j\gamma_{ij}(\phi))\\ &- \delta\phi^j\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\mu\phi^i\gamma_{ij}(\phi)) + \sqrt{g}g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\frac{\partial\gamma_{ij}}{\partial\phi^k}\delta\phi^k.\end{aligned}$$

We drop the divergent terms and conclude that

$$2\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j\gamma_{ij}) - \sqrt{g}g^{\mu\nu}\partial_\mu\phi^k\partial_\nu\phi^j\frac{\partial\gamma_{kj}}{\partial\phi^i} = 0 \tag{3.69}$$

If we use that

$$\frac{\partial\gamma_{kj}}{\partial\phi^i} = \Gamma^m_{ik}\gamma_{mj} + \Gamma^m_{ij}\gamma_{mk}$$

the first terms becomes

$$\begin{aligned}2\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j\gamma_{ij}) &= 2\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j)\gamma_{ij} + 2\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j\partial_\mu\phi^k\frac{\partial\gamma_{ij}}{\partial\phi^k}\\ &= 2\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j)\gamma_{ij} + 2\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j\partial_\mu\phi^k(\Gamma^m_{ik}\gamma_{mj} + \Gamma^m_{ij}\gamma_{mk})\end{aligned}$$

and the last term becomes

$$\sqrt{g}\partial_\mu\phi^k\partial^\mu\phi^j(\Gamma^m_{ik}\gamma_{mj} + \Gamma^m_{ij}\gamma_{mk}) = 2\sqrt{g}\Gamma^m_{ik}\partial_\mu\phi^k\partial^\mu\phi^k\gamma_{mj}.$$

which means that our field equations (3.69) become

$$\begin{aligned}&2\partial_\mu(\sqrt{g}\partial^\mu\phi^j)\gamma_{ij} + 2\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j\partial_\mu\phi^k(\Gamma^m_{ik}\gamma_{mj} + \Gamma^m_{kj}\gamma_{mi} - \Gamma^m_{ik}\gamma_{mj})\\ &\quad= 2\gamma_{mi}(\partial_\mu(\sqrt{g}\partial^\mu\phi^m) + 2\sqrt{g}g^{\mu\nu}\partial_\nu\phi^j\partial_\mu\phi^k\Gamma^m_{kj})\\ &\quad= 2\sqrt{g}\gamma_{mi}\left(\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial^\mu\phi^m) + \Gamma^m_{kj}\partial_\mu\phi^j\partial^\mu\phi^k\right) = 0.\end{aligned}$$

Thus, our field equations are

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial^\mu\phi^i) + \Gamma^i_{kj}\partial_\mu\phi^j\partial^\mu\phi^k = 0. \tag{3.70}$$

As it turns out, this equation is more convenient to work with since it leads to a geodesic equation on the coset space G/H when spherical symmetry is assumed. We note that Γ^i_{kj} is a connection on the coset space, not spacetime.

3.4.2 Gravity-Matter Equation

To find the field equation for $g_{\mu\nu}$ we will rewrite Einstein's equations. First we use the following result,

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{3.71}$$

Einstein's equations can be written in a form where the Ricci tensor $R_{\mu\nu}$ only appears once. Consider Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}. \tag{3.72}$$

Multiply both sides by $g^{\mu\nu}$ and use that $g^{\lambda}{}_{\lambda} = D$,

$$\begin{aligned} g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) &= g^{\mu\nu}T_{\mu\nu} \\ R - \frac{1}{2}g^{\lambda}{}_{\lambda}R &= T^{\lambda}{}_{\lambda} \\ R - \frac{1}{2}DR &= T^{\lambda}{}_{\lambda} \\ R &= \frac{T^{\lambda}{}_{\lambda}}{1-\frac{D}{2}}. \end{aligned}$$

By inserting this expression back into Einstein's equations we get

$$R_{\mu\nu} = T_{\mu\nu} + g_{\mu\nu}\frac{T^{\lambda}{}_{\lambda}}{2-D}. \tag{3.73}$$

We begin by finding the stress tensor $T_{\mu\nu}$ which is defined as

$$T_{\mu\nu} = -\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} \tag{3.74}$$

where $\mathcal{L}_{\text{matter}}$ consists of everything in $\mathcal{L}$ except R. If we vary with respect to $g^{\mu\nu}$ we get that

$$\begin{aligned} \delta\mathcal{L}_{\text{matter}} &= -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}(-\langle P_\rho|P_\sigma\rangle g^{\rho\sigma}) - \sqrt{g}\langle P_\rho|P_\sigma\rangle g^{\rho\sigma}\delta g^{\mu\nu} \\ &= \sqrt{g}\left(\frac{1}{2}g_{\mu\nu}\langle P_\rho|P_\sigma\rangle g^{\rho\sigma} - \langle P_\rho|P_\sigma\rangle\right)\delta g^{\mu\nu} \end{aligned}$$

and thus, (3.74) becomes

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\langle P_\rho|P^\rho\rangle + \langle P_\mu|P_\nu\rangle. \tag{3.75}$$

For $D = 3$ we get

$$\begin{aligned} R_{\mu\nu} &= T_{\mu\nu} - g_{\mu\nu} T^{\lambda}{}_{\lambda} \\ &= -\frac{1}{2} g_{\mu\nu} \langle P_\rho | P^\rho \rangle + \langle P_\mu | P_\nu \rangle - g_{\mu\nu} \left(-\frac{1}{2} g^{\lambda}{}_{\lambda} \langle P_\rho | P^\rho \rangle + \langle P_\rho | P_\rho \rangle \right) \\ &= -\frac{1}{2} g_{\mu\nu} \langle P_\rho | P^\rho \rangle + \langle P_\mu | P_\nu \rangle + \frac{3}{2} g_{\mu\nu} \langle P_\rho | P^\rho \rangle - g_{\mu\nu} \langle P_\rho | P^\rho \rangle \\ &= \langle P_\mu | P_\nu \rangle. \end{aligned} \tag{3.76}$$

That is,

$$R_{\mu\nu} = \langle P_\mu | P_\nu \rangle. \tag{3.77}$$

We note here that the right hand side is invariant under G and consequently, the three dimensional metric g is also invariant under g. We will see this explicitly in section 7.3.

4

Black Hole Solutions on $SL(2,\mathbb{R})/SO(1,1)$

In the last chapter we showed that the Lagrangian in three dimensions exhibits a symmetry described by a group G. The purpose of this chapter is to use this symmetry as part of a solution generating technique. We will begin this chapter with a section describing how this symmetry can be used in order to generate new solutions. In section 2.2.4 we saw that if we reduce the timelike dimension we get a sigma-model on the coset space $SL(2,\mathbb{R})/SO(1,1)$. As it turns out, this is also the coset space for Einstein-Maxwell theory when we consider static, electrically charged solutions. In this chapter we will see how dimensional reduction of Einstein-Maxwell theory in four dimensions gives rise to this sigma-model. Then we will solve our field equations for a spherically symmetric black hole with electric charge q and mass m. By using the $SL(2,\mathbb{R})$ symmetry we can relate the Schwarzschild solution to the Reissner-Nordström solution by identifying $SO(1,1)$ as the generator of electric charge.

4.1 Solution Generating Technique

In this section we will give an overview of the solution generating technique that follows from the symmetry property of our Lagrangian. We have seen in the previous sections that dimensional reduction gives us an effective lower dimensional theory which exhibits hidden symmetries. These symmetries are manifested in a sigma-model whose field content parametrize a coset space G/H. The idea is to exploit these symmetries to generate new solutions from a given solution, called a *seed solution*. The procedure works as follows:

(i) Start with a seed solution metric g_0 and extract the field parameters according to (2.19)

(ii) Dualize the three-dimensional fields to scalars

(iii) Construct the matrix M_0

(iv) Transform M_0 by a suitable group element g as $M' = gMg^{\mathcal{T}}$

(v) Extract the new fields from M'

(vi) Do the inverse dualization procedure to get the metric components and physical fields.

A "suitable group element" means a group element that actually gives us new physical solutions. We will elaborate on this more in the next section. Moreover, not every solution can serve as a seed solution which we will see later on. It should be mentioned that when one is actually performing calculations the expressions tend to become very complicated. For $SL(2,\mathbb{R})$ this is manageable but as the groups become more complicated, e.g. $SL(3,\mathbb{R})$, the matrix M becomes very messy. Another issue one should have in mind is that the inverse dualization often leads to a very difficult integration.

4.2 $SL(2,\mathbb{R})/SO(1,1)$

Since $SL(2,\mathbb{R})/SO(1,1)$ is the coset space of interest in the following example we will in this section discuss some of its properties. The main difference between this coset space and $SL(2,\mathbb{R})/SO(2)$ is that $SO(1,1)$ is not compact, in contrast to $SO(2)$ which makes the Iwasawa decomposition non-global. That is, not all the group elements can be decomposed by a non-global decomposition. As usual we have

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}e \oplus \mathbb{R}h \oplus \mathbb{R}f. \tag{4.1}$$

We define an involution by

$$\tau: \quad \tau(V) = \eta^{-1}(V^{-1})^T\eta, \qquad V \in SL(2,\mathbb{R}) \tag{4.2}$$

and on the Lie algebra

$$\tau: \quad \tau(t) = -\eta^{-1}t^T\eta, \qquad t \in \mathfrak{sl}(2,\mathbb{R}) \tag{4.3}$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.4}$$

If we replace $\eta \to \delta$ we get the Cartan involution. We define an involution on the premises that the space H in the coset space[1] G/H should be invariant. From this we see that

$$\tau(e) = f \qquad \tau(f) = e \qquad \tau(h) = -h \tag{4.5}$$

which means that we can form

$$\tau(e+f) = e+f \qquad \tau(e-f) = -(e-f) \tag{4.6}$$

For the case $SL(2,\mathbb{R})/SO(2)$ we decomposed the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ in an invariant and an anti-invariant subspace, denoted $\mathfrak{k}, \mathfrak{p}$ respectively. Since $H = SO(1,1)$ is not a compact subgroup we use $\mathfrak{h}$ and $\mathfrak{m}$ instead to avoid confusion,

$$\begin{aligned} \mathfrak{h} &= \{t \in \mathfrak{sl}(2,\mathbb{R}) | \tau(t) = t\} \\ \mathfrak{m} &= \{t \in \mathfrak{sl}(2,\mathbb{R}) | \tau(t) = -t\}. \end{aligned} \tag{4.7}$$

From this we get a decomposition of $\mathfrak{sl}(2,\mathbb{R})$ into an invariant and an anti-invariant subspace according to

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{m} \tag{4.8}$$

where $\mathfrak{h}$ and $\mathfrak{m}$ are given by (4.7) and (4.5). That is,

$$\begin{aligned} \mathfrak{h} &= \mathbb{R}(e+f) \\ \mathfrak{m} &= \mathbb{R}(e-f) \oplus \mathbb{R}h \end{aligned} \tag{4.9}$$

i.e. an invariant and an anti-invariant *subspace*.[2] The subgroup $SO(1,1)$ is generated[3] by

$$e+f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.10}$$

As for the $SL(2,\mathbb{R})/SO(2)$ case we have another decomposition

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}e \oplus \mathbb{R}h \oplus \mathbb{R}(e+f). \tag{4.11}$$

[1] Note that we denote the coset space G/H rather than G/K here since K denotes the maximal compact subgroup.

[2] $\mathfrak{m}$ is *not* a subalgebra since is does not close under the Lie bracket.

[3] It is not difficult to show this. Just use the definition of SO(1,1) and expand in the parameter.

This decomposition, contrary to the one above, splits $\mathfrak{sl}(2,\mathbb{R})$ into two *subalgebras*. When this is exponentiated we get the following decomposition

$$SL(2,\mathbb{R}) = NAH \tag{4.12}$$

where $H = SO(1,1)$ and N, A as before. Explicitly, we have

$$SL(2,\mathbb{R}) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi/2} & 0 \\ 0 & e^{\phi/2} \end{pmatrix} \begin{pmatrix} \cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & \cosh(\varphi) \end{pmatrix}. \tag{4.13}$$

We can now choose the $\varphi = 0$ coset element as a representative for $SL(2, \mathbb{R})/SO(1,1)$. This composition is called "non-global" in contrast to the Iwasawa decomposition which is a global decomposition, [15]. Thus, (4.13) is not a global equality. This is due to the non-compact group $SO(1,1)$. However, the decomposition is valid on a "sufficiently large" part of the group as we will see later on. We would now like to define the matrix M for this coset. It would of course be perfectly right to construct M according to (3.23) but we will make a small adjustment in order to get M symmetric. The important thing is that M becomes invariant under $SO(1,1)$. Consider

$$M = VV^{\mathcal{T}}\eta \tag{4.14}$$

where the generalized transpose is defined as

$$V^{\mathcal{T}} = \eta^{-1}V^T\eta \tag{4.15}$$

or equivalently as

$$V^{\mathcal{T}} = \exp\Big(-\tau(t)\Big). \tag{4.16}$$

We can simplify the expression for M by first rewriting (4.2) as

$$\Big(\tau(V)\Big)^{-1} = \eta(V^T)\eta^{-1}.$$

If we use this in the definition (4.14) for M we get

$$M = VV^{\mathcal{T}}\eta = V\eta V^T. \tag{4.17}$$

and it is straightforward to check the it transforms in the right way

$$M \longrightarrow (gVk)\eta(gVk)^T = gV\underbrace{k\eta k^T}_{=\eta}V^Tg^T = gMg^T.$$

From (4.13) we get that

$$M = \begin{pmatrix} e^{-\phi} - \xi^2 e^{\phi} & -\xi e^{\phi} \\ -\xi e^{\phi} & -e^{\phi} \end{pmatrix} \tag{4.18}$$

which yields

$$\mathrm{Tr}(\partial_\mu M^{-1} \partial^\mu M) = -2\Big(\partial_\mu \phi \partial^\mu \phi - e^{2\phi} \partial_\mu \xi \partial^\mu \xi\Big). \tag{4.19}$$

Note the minus sign and that the additional factor η in the definition of M vanishes. We can define $\Delta = e^{-2\phi}$ to get

$$\mathrm{Tr}(M^{-1}\partial_\mu M M^{-1} \partial^\mu M) = \Big(\frac{1}{2\Delta^2}\partial_\mu \Delta \partial^\mu \Delta - \frac{2}{\Delta}\partial_\mu \xi \partial^\mu \xi\Big). \tag{4.20}$$

The conclusion of this is that when we go from $SL(2,\mathbb{R})/SO(2)$ to $SL(2,\mathbb{R})/SO(1,1)$ we change the sign in front of the $(\partial\xi)^2$ term. This is known in the physics litterature as a Wick-rotation $\xi \longrightarrow i\xi$.

4.3 Dimensional Reduction of Einstein-Maxwell Theory in $D = 4$

In this section we will solve our equations for a spherical symmetric black hole. The aim is to find a non-rotating electrically charged black hole, related to Schwarzschild by a group element. In [19], the same example is discussed but with most of the calculations omitted. Therefore, we will present a fairly thorough calculation in this section. The Einstein-Maxwell Lagrangian is given by

$$\mathcal{L} = \sqrt{g}(R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}). \tag{4.21}$$

We use (2.19) as an ansatz for the four-dimensional metric and find

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \Delta^{-1} g_{\mu\nu} & 0 \\ 0 & \Delta \end{pmatrix}. \tag{4.22}$$

From the assumption of spherical symmetry, no cross terms exists. Moreover, we make the following ansatz for the three-dimensional metric

$$ds_3^2 = -dr^2 - f(r)^2 d\Omega^2. \tag{4.23}$$

Since we are interested in the Schwarzschild solution and the electrically charged Schwarzschild we have that $\Delta = \Delta(r)$ which allows us to reduce the time dimension. The Einstein-Hilbert term is the same as before with the KK-vectors set to zero. The Maxwell part, however, needs to be decomposed. Let

$$A^{\hat{\mu}}(r) = \begin{cases} A(r) & \hat{\mu} = 3 \\ 0 & \hat{\mu} \neq 3. \end{cases} \tag{4.24}$$

The only non zero components of $F_{\mu\nu}$ are $F_{tr} = -F_{rt} = -\partial_r A$ which give

$$F_{\mu\nu}F^{\mu\nu} = F_{tr}F^{tr} + F_{rt}^{F}rt = 2F_{tr}F_{\hat{\mu}\hat{\nu}}g^{\hat{\mu}t}g^{\hat{\nu}r} = -2(\partial_r A)^2.$$

The three-dimensional Lagrangian is

$$\begin{aligned} \mathcal{L} &= \sqrt{g}\left(R^{(3)} - \frac{1}{2\Delta^2}\partial_r\Delta\partial^r\Delta + \frac{1}{2\Delta}\partial_r A\partial^r A\right) \\ &= \sqrt{g}\left(R^{(3)} - \frac{1}{4}\left(\frac{2}{\Delta^2}\partial_r\Delta\partial^r\Delta - \frac{2}{\Delta}\partial_r A\partial^r A\right)\right). \end{aligned} \tag{4.25}$$

The Δ^{-1} factor in the second term comes from the determinant of the metric, see (2.22). We can redefine the electric potential $A(r) \longrightarrow 2A(r)$ to get

$$\mathcal{L} = \sqrt{g}\left(R^{(3)} - \left(\frac{1}{2\Delta^2}\partial_r\Delta\partial^r\Delta - \frac{2}{\Delta}\partial_r A\partial^r A\right)\right). \tag{4.26}$$

This is the sigma-model (4.20) on the coset space $SL(2,\mathbb{R})/SO(1,1)$.

4.4 Solutions of the Field Equations

Our field equations (3.70) can be rewritten by a change of variables into a geodesic equation. Let

$$\begin{aligned} \tau &= -\int_r^\infty \frac{1}{f^2(s)}ds \\ \frac{d\tau}{dr} &= \frac{1}{f^2(r)}. \end{aligned} \tag{4.27}$$

From this we get

$$\begin{aligned} &\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial^\mu\phi^i) + \Gamma^i_{mk}\partial_\mu\phi^m\partial^\mu\phi^k \\ &\qquad = \frac{1}{f^2(r)}\frac{d\tau}{dr}\frac{d}{d\tau}\left(f^2(r)\frac{d\phi^i}{d\tau}\frac{d\tau}{dr}\right) + \Gamma^i_{jk}\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau}\left(\frac{d\tau}{dr}\right)^2 \\ &\qquad = \frac{d^2\phi^i}{d\tau^2} + \Gamma^i_{jk}\frac{d\phi^j}{d\tau}\frac{d\phi^k}{d\tau} = 0. \end{aligned} \tag{4.28}$$

This result is quite remarkable; stationary black hole solutions are described by geodesics on the target space G/H ($SL(2,\mathbb{R})/SO(1,1)$ in this case), see for example [20]. From our ansatz of the three-dimensional metric, the gravity matter equations (3.77) become, [19]

$$R_{rr} = -2f^{-1}(r)\frac{d^2f(r)}{dr^2} = \gamma_{ij}\frac{d\phi^i}{dr}\frac{d\phi^j}{dr} \tag{4.29}$$

$$R_{\varphi\varphi} = R_{\theta\theta} = f^{-2}(r)\left(\frac{d}{dr}\left(f(r)\frac{df(r)}{dr}\right) - 1\right) = 0 \tag{4.30}$$

Now we can express our field equations (4.28) in terms of Δ and A. From (4.26) we find that

$$\gamma_{ij} = \begin{pmatrix} \frac{1}{2\Delta^2} & 0 \\ 0 & \frac{-2}{\Delta} \end{pmatrix} \tag{4.31}$$

which means that the non-vanishing components of Γ^i_{jk} are

$$\Gamma^0_{00} = \frac{-1}{\Delta} \qquad\qquad \Gamma^0_{11} = -2 \qquad\qquad \Gamma^1_{10} = \frac{-1}{2\Delta}. \tag{4.32}$$

So, our field equations (4.28) become

$$\ddot{\Delta} - \frac{\dot{\Delta}}{\Delta} - 2\dot{A} = 0 \tag{4.33}$$

$$\ddot{A} - \frac{\dot{\Delta}}{\Delta}\dot{A} = 0 \tag{4.34}$$

where $\dot{\Delta} = \frac{d\Delta}{d\tau}$. Note that our fields are functions of τ. Since ϕ^i describe a geodesics we have

$$\gamma_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} = 2v^2 \tag{4.35}$$

where v is a properly chosen constant. Here we can make an observation: since the left hand side is invariant under $SL(2,\mathbb{R})$ we can only transform one black hole solution to another if they have the same v^2. The solution of (4.30) is given by

$$\begin{aligned} \frac{d}{dr}\left(f(r)\frac{df(r)}{dr}\right) - 1 = 0 \\ f(r)\frac{df(r)}{dr} &= r - r_0 \\ \int f df &= \int (r - r_0)dr \\ f^2 &= r^2 + 2rr_0 - r_0^2 + c \\ f^2(r) &= (r - r_0)^2 + c. \end{aligned} \tag{4.36}$$

If we put this into (4.29) we find $c = -v^2$. Our second geodesic equation (4.34) is easily solved in terms of Δ

$$\dot{A} = B\Delta(\tau) \tag{4.37}$$

and the first one (4.33) by

$$\Delta(\tau) = \frac{C}{4B^2}\left(-1 + \tanh^2\left[\frac{1}{2}(\tau\sqrt{C} + \sqrt{D})\right]\right). \tag{4.38}$$

C, B and D are constants of integration. Δ is the gravitational potential and A is the electric potential and therefore we impose the following boundary conditions at $\tau = 0$ $(r = \infty)$

$$\Delta(\tau)\Big|_{\tau=0} = 1 \tag{4.39}$$

$$A(\tau)\Big|_{\tau=0} = 0. \tag{4.40}$$

From these boundary conditions we might guess that B is related to the charge since for $B = 0$ we get that $A = 0$. We impose the following boundary conditions as well

$$\begin{aligned} \Delta(r) &\xrightarrow{r\to\infty} 1 - \frac{2m}{r} \\ A(r) &\xrightarrow{r\to\infty} \frac{q}{r}. \end{aligned} \tag{4.41}$$

Now we can find the constants C, B and D. We begin with $\Delta(0) = 1$,

$$\begin{aligned} \Delta(0) &= \frac{1}{4B^2}C\left(-1 + \tanh^2\left[\frac{1}{2}(\sqrt{D})\right]\right) = 1 \\ C &= \frac{4B^2}{-1 + \tanh^2\left[\frac{1}{2}(\sqrt{D})\right]}. \end{aligned}$$

To simplify the notation we define

$$k = \frac{2B}{\sqrt{\tanh^2\left(\frac{1}{2}\sqrt{D}\right) - 1}}. \tag{4.42}$$

Our solution is now given by

$$\Delta(\tau) = \frac{k^2}{4B^2}\left(\tanh^2\left[\frac{1}{2}\tau k + \frac{\sqrt{D}}{2}\right] - 1\right). \tag{4.43}$$

Now we can find A from (4.37)

$$A(\tau) = \frac{-k}{2B}\tanh\left[\frac{1}{2}\left(\tau k + \sqrt{D}\right)\right] + M \tag{4.44}$$

and find M by imposing $A(0) = 0$

$$M = \frac{k}{2B}\tanh(\sqrt{D}/2)$$

So, to sum up

$$A(\tau) = \frac{-k}{2B}\tanh\left[\frac{1}{2}\left(\tau k + \sqrt{D}\right)\right] + \frac{k}{2B}\tanh(\sqrt{D}/2) \tag{4.45}$$

$$\Delta(\tau) = \frac{k^2}{4B^2}\left(\tanh^2\left[\frac{1}{2}\tau k + \frac{\sqrt{D}}{2}\right] - 1\right). \tag{4.46}$$

Where B and D are the remaining constants of integration. In order to continue we need to find $\tau(r)$ from (4.27) given our solution for f^2. We assume that $v^2 > 0$

$$\tau(r) = -\int_r^\infty \frac{1}{(s-r_0)^2 - v^2} ds = -\int_r^\infty \frac{1}{v^2\left(\frac{(s-r_0)^2}{v^2} - 1\right)} ds$$

$$= \frac{1}{2v}\left[\ln\left|\frac{1+\left(\frac{2-r_0}{v}\right)}{1-\left(\frac{2-r_0}{v}\right)}\right|\right]_r^\infty = \frac{1}{2v}\ln\left(\frac{r-r_0-v}{r-r_0+v}\right) \tag{4.47}$$

In the last step we have chosen $r > r_0 + v$. If we now insert $\tau(r)$ into $A(\tau)$, define $x = \frac{1}{r}$ and expand around $x = 0$ we get

$$A(x) \approx \frac{k^2}{4B}\left(1 - \tanh^2\left(\frac{\sqrt{D}}{2}\right)\right)x = -Bx \tag{4.48}$$

i.e. we see that $B = -q$. Here we have used (4.41). From (4.37) we see that $\dot{A}(0) = -q$. We do the same for Δ

$$\Delta(x) \approx 1 + k\tanh\left(\frac{\sqrt{D}}{2}\right)x \tag{4.49}$$

and find that

$$k\tanh\left(\frac{\sqrt{D}}{2}\right) = -2m. \tag{4.50}$$

For simplicity we define

$$F = \tanh\left(\frac{\sqrt{D}}{2}\right) \tag{4.51}$$

so that (4.68) squared becomes

$$\frac{q^2}{F^2-1}F^2 = m^2. \tag{4.52}$$

If we now replace k with F and use some identities for hyperbolic functions we get

$$\Delta(\tau) = \frac{4m^2}{4F^2q^2}\left[\frac{F^2-1}{\left(\cosh\left(\frac{-m}{F}\tau\right)+F\sinh\left(\frac{-m}{F}\tau\right)\right)^2}\right]. \tag{4.53}$$

The cosh and sinh parts can be expressed in terms of r as

$$\begin{aligned}
\cosh\left(\frac{-m}{F}\tau\right) &= \frac{1+\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{Fv}}}{2\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{2Fv}}}\\
\sinh\left(\frac{-m}{F}\tau\right) &= \frac{-1+\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{Fv}}}{2\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{2Fv}}}.
\end{aligned}$$

Finally we can get an expression for $\Delta(r)$

$$\begin{aligned}
\Delta(r) &= \frac{2m^2}{4F^2q^2}\left[\frac{F^2-1}{\left(\cosh\left(\frac{-m}{F}\tau\right)+F\sinh\left(\frac{-m}{F}\tau\right)\right)^2}\right]\\
&= \frac{2m^2}{4F^2q^2}\frac{F^2-1}{\left(\frac{1+\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{Fv}}}{2\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{2Fv}}}+F\left(\frac{-1+\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{Fv}}}{2\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{2Fv}}}\right)\right)^2}\\
&= \frac{4\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{Fv}}}{\left(1-F+(1+F)\left(\frac{r-r_0-v}{r-r_0+v}\right)^{\frac{-m}{Fv}}\right)^2}\\
&= \frac{4\left(1-\frac{2v}{r-r_0+v}\right)^{\frac{-m}{Fv}}}{\left(1-F+(1+F)\left(1-\frac{2v}{r-r_0+v}\right)^{\frac{-m}{Fv}}\right)^2}.
\end{aligned} \tag{4.54}$$

In order to continue we need to find out what v is. To do this we first note that

$$\begin{aligned}\dot{\Delta}(\tau)\Big|_{\tau=0} &= \frac{k^3}{4q^2}\tanh\left(\frac{k\tau}{2}+\frac{\sqrt{D}}{2}\right)\frac{1}{\cosh^2\left(\frac{k\tau}{2}+\frac{\sqrt{D}}{2}\right)}\Big|_{\tau=0} \\ &= \frac{k^3}{4q^2}\tanh\left(\frac{\sqrt{D}}{2}\right)\frac{1}{\cosh^2\left(\frac{\sqrt{D}}{2}\right)} = \frac{k^3}{4B^2}F(1-F^2) \\ &= \frac{-8m^3}{4B^2}\frac{(1-F^2)}{F^2} = 2m.\end{aligned} \tag{4.55}$$

Since (4.35) is valid for every τ we can set $\tau = 0$ to get

$$\begin{aligned}2v^2 &= \frac{\dot{\Delta}^2(0)}{2} - 2\dot{A}^2(0) = 2m^2 - 2q^2 \\ v^2 &= m^2 - q^2.\end{aligned} \tag{4.56}$$

Now consider the exponent in (4.54)

$$\frac{m^2}{F^2v^2} = \frac{m^2}{(m^2-q^2)}\frac{m^2-q^2}{m^2} = 1. \tag{4.57}$$

Here we have used (4.51). (4.54) now becomes

$$\begin{aligned}\Delta(r) &= \frac{\left(1-\frac{2v}{r-r_0+v}\right)}{\left(1-\left(1-\frac{m}{v}\right)\frac{v}{r-r_0+v}\right)^2} \\ &= \frac{(r-r_0-v)(r-r_0+v)}{(-m-r-r_0)^2}.\end{aligned} \tag{4.58}$$

Redefine $r - r_0 \longrightarrow r$

$$\begin{aligned}\Delta(r) &= \frac{(r-v)(r+v)}{(r+m)^2} = \frac{r^2-v^2}{(r+m)^2} = \frac{r^2-m^2+q^2}{(r+m)^2} \\ &= \frac{(r-m)(r+m)+q^2}{(r+m)^2} = \frac{r-m}{r+m} + \frac{q^2}{(r+m)^2} \\ &= 1 - \frac{2m}{r+m} + \frac{q^2}{(r+m)^2}.\end{aligned} \tag{4.59}$$

As a las step, $r + m \longrightarrow r$

$$\Delta(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \tag{4.60}$$

The only thing that remains to get the full metric is to calculate $f^2(r)$. Consider (4.36), where r is the "old r" used up until (4.58),

$$\begin{aligned} f^2(r) &= (r-r_0)^2 - v^2 = (r-r_0)^2 - m^2 + q^2 = \left[\ r - r_0 \longrightarrow r\ \right] \\ &= r^2 - m^2 + q^2 = (r-m)(r+m) + q^2 = [r + m \longrightarrow r] \\ &= (r-2m)r + q^2 = r^2\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right). \end{aligned} \tag{4.61}$$

Thus, our spherical symmetric Einstein-Maxwell metric, called the *Reissner Nordström metric*, is given by

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{4.62}$$

where q is the charge and m is the mass.

4.5 The Transformation of the Reissner-Nordström Metric

In this section we will see how one can obtain the Schwarzschild solution by using the $SL(2,\mathbb{R})$ symmetry. Of course, we could just set $q = 0$ but for the purpose of illustrating the "solution generating technique" using symmetries we will do this fairly thoroughly following the steps presented in section 4.1. Our seed solution is the Reissner-Nordström metric and our coset element V is thus given by

$$V = \begin{pmatrix} \Delta^{1/4} & A\Delta^{-1/4} \\ 0 & \Delta^{-1/4} \end{pmatrix} \tag{4.63}$$

which we recall transforms as

$$V \longrightarrow gVk. \tag{4.64}$$

Consider an infinitesimal transformation where g is an element in $SO(1,1)$

$$(f+e)V + cV(f+e) = \begin{pmatrix} 0 & \Delta^{-1/4} \\ \Delta^{1/4} & A\Delta^{-1/4} \end{pmatrix} + c\begin{pmatrix} A\Delta^{-1/4} & \Delta^{1/4} \\ \Delta^{-1/4} & 0 \end{pmatrix}. \tag{4.65}$$

If we choose $c = -\Delta^{1/2}$ we restore the upper triangular form

$$\delta(V)_{SO(1,1)} = \begin{pmatrix} -A\Delta^{1/4} & \Delta^{-1/4} - \Delta^{3/4} \\ 0 & A\Delta^{-1/4} \end{pmatrix}. \tag{4.66}$$

We compare this with the matrix we get when we consider infinitesimal transformations of the fields Δ and A

$$\delta(V)_{SO(1,1)} = \begin{pmatrix} \frac{1}{4\Delta^{3/4}}\delta\Delta & \delta A\Delta^{-1/4} - \frac{A}{4}\frac{1}{\Delta^{5/4}}\delta\Delta \\ 0 & \frac{-1}{4\Delta^{5/4}}\delta\Delta \end{pmatrix} \tag{4.67}$$

If we compare the components we get

$$\begin{aligned} \frac{1}{4\Delta^{3/4}}\delta\Delta &= -A\Delta^{1/4} \\ \delta\Delta &= -4A\Delta \end{aligned}$$

$$\begin{aligned} \delta A\Delta^{-1/4} - \frac{A}{4}\frac{1}{\Delta^{5/4}}\delta\Delta &= \Delta^{-1/4} - \Delta^{3/4} \\ \delta A &= 1 - \Delta - A^2. \end{aligned}$$

If we insert $\tau = 0$ and use $\Delta(\tau = 0) = 1, A(\tau = 0) = 0$ we see that both $\Delta(0)$ and $A(0)$ are invariant which imply that $SO(1{,}1)$ might be the correct transformation in order to get the Schwarzschild metric. Consider the seed matrix $M_0 = V\eta V^T$,

$$M_0 = \begin{pmatrix} \Delta^{1/2} - \frac{A^2}{\Delta^{1/2}} & -\frac{A}{\Delta^{1/2}} \\ -\frac{A}{\Delta^{1/2}} & -\frac{1}{\Delta^{1/2}} \end{pmatrix}. \tag{4.68}$$

Now we transform $M \longrightarrow gMg^T = M'$ with $g \in SO(1{,}1)$

$$M' = \begin{pmatrix} \frac{\cosh^2(\xi)\Delta-(A\cosh(\xi)+\sinh(\xi))^2}{\Delta^{1/2}} & \frac{\sinh(2\xi)\left(-A^2+\Delta-1\right)-2A\cosh(2\xi)}{2\Delta^{1/2}} \\ \frac{\sinh(2\xi)\left(-A^2+\Delta-1\right)-2A(x)\cosh(2\xi)}{2\Delta^{1/2}} & \frac{\sinh^2(\xi)\Delta-(A\sinh(\xi)+\cosh(\xi))^2}{\Delta^{1/2}} \end{pmatrix}. \tag{4.69}$$

If we differentiate with respect to τ and compare the M_{22} components we get

$$\begin{aligned} \frac{dM'_{22}}{d\tau} &= \frac{\left(\Delta\sinh^2(\xi) + (\cosh(\xi) + A\sinh(\xi))^2\right)\dot{\Delta} - 4\sinh(\xi)(\cosh(\xi) + A\sinh(\xi))\Delta\dot{A}}{2\Delta^{3/2}} \\ &= \frac{\dot{\Delta}'}{2(\Delta')^{3/2}}. \end{aligned}$$

In the last step we have differentiated the M' version of (4.68). Let $\tau = 0$

$$\dot{\Delta}'\Big|_{\tau=0} = 2m\Big(\sinh^2(\xi) + \cosh^2(\xi)\Big) + 2q\sinh(2\xi).$$

Recall that $\dot{\Delta}(\tau = 0) = 2m$ and $\dot{A}(\tau = 0) = -q$. If we do the same for the M_{12} component we get

$$\dot{A}'\Big|_{\tau=0} = -m\sinh(2\xi) - q\cosh(2\xi). \tag{4.70}$$

If we would like to transform the Reissner Nordström metric into the Schwarzschild metric we should choose a ξ such that $q' = 0$. We find that

$$\tanh(2\xi) = -\frac{q}{m}. \tag{4.71}$$

It is not difficult to show that A' vanishes for this choice of ξ. We can check that $(v')^2 = (m')^2 - (q')^2$ has not changed, i.e. $v' = v$, by first rewriting $\dot{\Delta}'(0)$

$$\begin{aligned}\dot{\Delta}'(0) &= 2m\Big(\sinh^2(\xi) + \cosh^2(\xi)\Big) + 2q\sinh(2\xi) = 2m\left(\frac{\cosh(2\xi)-1}{2} + \frac{\cosh(2\xi)+1}{2}\right) \\ &\quad + 2q\sinh(2\xi) = 2m\cosh(2\xi) + 2q\sinh(2\xi) = 2m\sqrt{1-\frac{q^2}{m^2}},\end{aligned}$$

where we have used

$$\cosh 2\xi = \frac{1}{\sqrt{1-\frac{q^2}{m^2}}} \qquad\qquad \sinh 2\xi = \frac{-\frac{q}{m}}{\sqrt{1-\frac{q^2}{m^2}}},$$

and put into $(v')^2 = (m')^2$,

$$(v')^2 = m^2\left(1 - \frac{q^2}{m^2}\right) = m^2 - q^2 = v^2. \tag{4.72}$$

So, we have transformed A and Δ in such a way that the new boundary conditions become

$$\begin{aligned} \Delta'\Big|_{\tau=0} &= 1 & \qquad A'\Big|_{\tau=0} &= 0 \\ \dot{\Delta}'\Big|_{\tau=0} &= 2m' & \qquad \dot{A}'\Big|_{\tau=0} &= 0. \end{aligned}$$

We get the explicit expression for Δ from the first line of (4.59)

$$\begin{aligned}\Delta(r) &= \frac{(r-v)(r+v)}{(r+m)^2} = \frac{r^2-v^2}{(r+m')^2} = \frac{r^2-(m')^2}{(r+m')^2} \\ &= \frac{(r-m')(r+m')}{(r+m')^2}.\end{aligned}$$

Just like before, we redefine $r + m' \longrightarrow r$ to get

$$\Delta(r) = 1 - \frac{2m'}{r}. \tag{4.73}$$

A completely analogous calculation to (4.61) gives that $f^2 = r^2\left(1 - \frac{2m'}{r}\right)$. From what we have seen in this section we can conclude that the one-dimensional group

$SO(1,1)$ generates electric charge. This result will be recognized for larger groups G where each generator in H will generate[4] a corresponding charge, e.g. electric, magnetic, angular momentum and NUT-charge. Perhaps it would be more natural to generate the Reissner-Nordström metric from the Schwarzschild instead of the opposite. In fact, in [19] it is shown that any static black hole solution in $D = 4$ with non-degenerate horizon can be generated from the Schwartzschild solution by a suitable group transformation.

4.6 The Charge Matrix Q

We saw in the previous chapter that there exists a conserved current $j_\mu = M^{-1}\partial_\mu M$ and consequently conserved charges. This makes it natural to define a so called charge matrix which encodes the conserved charges. The purpose of this chapter is to show how the charge matrix enters as a coefficient in the expansion of M in the radial coordinate. Moreover, the expansion reveals a lot of information on how the group acts on a solution M. The charge matrix Q is defined in [9] as

$$Q = \frac{1}{4\pi}\int_{r=\infty} j_\mu \mathrm{d}s^\mu, \tag{4.74}$$

where $\mathrm{d}s^\mu$ is the infinitesimal vector area. We continue to follow [9] by requiring M to have a converging power series in $1/r$. That is,

$$M = M_0 + \frac{1}{r}M_1 + \mathcal{O}(\frac{1}{r^2}). \tag{4.75}$$

By inserting the definition of j^μ into (4.74) we get

$$\begin{aligned} Q &= \frac{1}{4\pi}\int_{r=\infty} M^{-1}\partial_\mu M \mathrm{d}s^\mu \\ &= \frac{1}{4\pi}\int_{r=\infty} \left(\frac{-1}{r^2}\right) M_0^{-1}M_1 r^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi \\ &= -2M_0^{-1}M_1. \end{aligned} \tag{4.76}$$

For the Reissner-Nordström case we get, by expanding (4.68) in the redefined coordinate $r - r_0 \longrightarrow r$ that

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -m & -q \\ -q & -m \end{pmatrix}\frac{1}{r} + \mathcal{O}(\frac{1}{r^2}). \tag{4.77}$$

[4] H in G/H.

which means that $M_0 = \eta$, M_1 equals the coefficient for $1/r$ and

$$Q = -2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-m & -q\\ -q & -m\end{pmatrix} = 2\begin{pmatrix}m & q\\ -q & -m\end{pmatrix}. \tag{4.78}$$

To find out how Q transforms under $SO(1,1)$ we first note that

$$M_i \longrightarrow gM_ig^T \qquad\qquad i = 0, 1, 2, ... \tag{4.79}$$

since g is spacetime independent. From (4.76) we get the following transformation property for Q,

$$Q \longrightarrow g^{-1}Qg, \tag{4.80}$$

where $g \in SO(1,1)$. We see from the expansion (4.75) that M_0 encodes the boundary condition at $r = \infty$ and that M_1 encodes the conserved charges. M_0 is flat space in our case and by definition, $SO(1,1)$ does not change this matrix. The charge matrix for Reissner-Nordström is given by

$$Q_{RN} = 2\begin{pmatrix}m & q\\ -q & -m\end{pmatrix} \tag{4.81}$$

and according to (4.80) we can construct a new solution with charge matrix Q' given by

$$Q' = \begin{pmatrix} 2(m\cosh(2\xi) + q\sinh(2\xi)) & 2(q\cosh(2\xi) + m\sinh(2\xi)) \\ -2(q\cosh(2\xi) + m\sinh(2\xi)) & -2(m\cosh(2\xi) + q\sinh(2\xi)) \end{pmatrix} \tag{4.82}$$

and by letting $q' = 0$ we get the same ξ as before. The fact that M_0 is invariant under $SO(1,1)$ means that, in $D = 3$, Minkowski spacetime cannot be used as seed solution and that the boundary condition is preserved under $SO(1,1)$. For pure gravity in $D = 5$ with symmetry group $SL(3,\mathbb{R})$ this is no longer the case, i.e. the boundary condition may not be invariant. If this is desired, one has to find the subgroup of $SL(3,\mathbb{R})$ which leaves the asymptotic behaviour invariant. A very interesting consequence of this is that solutions with $\mathbb{R}^{4,1}$ asymptotics can be transformed into a solution with $\mathbb{R}^{3,1} \times S^1$ [21].

The charge matrix Q can be used to find solutions to the geodesic equation (4.28) which can be written as, [9],

$$\frac{d}{d\tau}\left(M(\tau)\frac{d}{d\tau}M(\tau)\right) = 0. \tag{4.83}$$

The quantity inside the parentheses is independent of τ and turns out to be equal to the charge matrix. It is well known that the solutions to this equation is given by $M(\tau) \propto \exp(\tau Q)$. See [21] or [22] for a more detailed discussion.

5

Dimensional Reduction to $D = 2$

Now that we have explored the symmetries in $D = 3$ we can continue and see what happens if we reduce our theory further to $D = 2$. In particular, we will see what happens to the case of pure gravity. This requires another Killing vector in addition to the first one. That is, if we start in four dimensions we need two commuting Killing vectors. The reduction down to two dimensions can basically be performed i two different ways: the Ehlers or the Matzner-Misner reduction. In the Ehlers reduction one reduces the theory in steps, i.e. first down to three dimensions and then down to two dimensions. In the Matzner-Misner reduction one reduces directly down to two dimensions. Obviously, the two ways of reducing the four-dimensional theory should be physically equivalent which means, as we will see, that there exists a duality in the two-dimensional theory.

Since we have made the reduction down to three-dimensions already we will do the Ehlers reduction. The set up here is a dimensionally reduced gravity theory in $D = 3$ which means that we have a sigma-model on a general coset space G/H. The matrix M might therefore be parametrized by fields in addition to the metric components, e.g. scalars from a Maxwell term. The reduction to two dimensions is of interest when one considers stationary axisymmetric solutions in $D = 4$, e.g. the Schwarzschild and Kerr metrics. In the first section we will parametrize the three-dimensional metric and present the two-dimensional Lagrangian for a general coset space. For illustrating purposes we will then dualize the two-dimensional theory explicitly for the $SL(2, \mathbb{R})/SO(2)$ case.

5.1 Parametrization of the Metric in $D = 3$

We make the following ansatz for the three-dimensional metric, [10],

$$g_{\mu\nu} = \begin{pmatrix} f^2 g^{(2)}_{ab} + \rho^2 \mathcal{B}_a \mathcal{B}_b & \rho^2 \mathcal{B}_a \\ \rho^2 \mathcal{B}_b & \rho^2 \end{pmatrix}$$

where $a, b = 1{,}2$. In D dimensions our Kaluza-Klein vectors $\mathcal{B}_a$ have $D-2$ degrees of freedom due to gauge invariance, hence $\mathcal{B}_a = 0$ in $D = 2$. With this taken into account we simply have

$$g_{\mu\nu} = \begin{pmatrix} f^2 g^{(2)}_{ab} & 0 \\ 0 & \rho^2 \end{pmatrix} . \tag{5.1}$$

The Lagrangian in $D = 2$ is now obtained by a similar calculation as for the $D = 3$ case and the result is [19]

$$\mathcal{L} = \rho \sqrt{g^{(2)}} \left(R^{(2)} - \frac{1}{4} \mathrm{Tr}(M^{-1} \partial_a M M^{-1} \partial^a M) + 2 f^{-1} \partial_a f \rho^{-1} \partial^a \rho \right) . \tag{5.2}$$

We note here that we cannot choose the conformal factor f in a suitable way to remove the prefactor of the Einstein-Hilbert term, in contrast to the $D = 3$ case. The field equations are easily obtained: variation of M gives the same equation as (3.66) but with an additional factor of ρ, the gravity matter equations follow from variation of $g^{(2)}$ and the equation for ρ follows from a simple variation, [19],

$$\begin{aligned} R^{(2)}_{ab} - \frac{1}{2} g^{(2)}_{ab} R^{(2)} &= \frac{1}{4} \mathrm{Tr}(M^{-1} \partial_a M M^{-1} \partial_b M) - 2 f^{-1} \partial_{(a} f \rho^{-1} \partial_{b)} \rho \\ &\quad - \frac{1}{2} g^{(2)}_{ab} \left(\frac{1}{4} \mathrm{Tr}(M^{-1} \partial_k M M^{-1} \partial^k M) - 2 f^{-1} \partial_k f \rho^{-1} \partial^k \rho \right) \\ D_a(\rho \mathcal{P}^a) &= 0 \\ \nabla_a \partial^a \rho &= 0. \end{aligned}$$

In $D = 2$, any metric is conformally flat and we can therefore take $g^{(2)}_{ab} = \delta_{ab}$ by absorbing the conformal factor into f. This simplifies our field equations considerably since $R^{(2)}_{ab} = 0$ and the covariant derivatives become covariant derivatives in flat space. We follow [10, 23] by choosing ρ and its conjugate variable z, defined[1] as $\mathrm{d}\rho = \star_2 \mathrm{d}z$, as coordinates on the two-dimensional space. Since ρ and z are harmonic coordinates[2] the metric will become equal to the unit matrix when

[1] $\star_2$ is the Hodge dual in two dimensions.
[2] Since they are harmonic functions.

expressed in these coordinates [24, 25]. The equation for ρ in the new coordinates $x^m = \rho, z$ becomes

$$\partial_m \partial^m \rho = 0 \tag{5.3}$$

which is trivially satisfied. It is easy to confirm that these new coordinates, called *Weyl canonical coordinates*, define an orthogonal coordinate system since $\partial_\mu \rho \partial^\mu z = 0$. In these coordinates our field equations reduce to

$$\begin{aligned}
\partial_\rho(\log f) &= \frac{\rho}{8}\Big(\mathrm{Tr}(M^{-1}\partial_\rho M M^{-1}\partial_\rho M - \mathrm{Tr}(M^{-1}\partial_z M M^{-1}\partial_z M\Big) \\
\partial_z(\log f) &= \frac{\rho}{4}\Big(\mathrm{Tr}(M^{-1}\partial_\rho M M^{-1}\partial_z M\Big)
\end{aligned} \tag{5.4}$$

$$D_a(\rho \mathcal{P}^a) = 0 \tag{5.5}$$

Once we have solved (5.5), (5.4) are solved by integration which means that (5.5) is the one we should focus on.

5.2 Dualization

In this section we will dualize the $SL(2,\mathbb{R})/SO(2)$ Ehler's model in two dimensions, i.e. the sigma-model of pure gravity. This will result in a dual theory with fields related to the Ehler's fields by dual transformations. We follow the procedure from 2.2.4. In terms of the Ehler's fields, the two-dimensional Lagrangian (5.2) is given by[3]

$$\mathcal{L} = \rho\left(R^{(2)} - \frac{\delta^{ab}}{2}\Big(\partial_a\phi\partial_b\phi + \mathrm{e}^{-2\phi}\partial_a\chi\partial_b\chi\Big) + 2f^{-1}\partial_a f \rho^{-1}\partial^a\rho\right). \tag{5.6}$$

Let $C_a = \partial_a\chi$ and add the Lagrange multiplier $\tilde{\chi}\partial_a(\epsilon^{ab}C_b)$

$$\mathcal{L}' = \rho\left(R^{(2)} - \frac{\delta^{ab}}{2}\Big(\partial_a\phi\partial_b\phi + \mathrm{e}^{-2\phi}C_aC_b\Big) + 2f^{-1}\partial_a f \rho^{-1}\partial^a\rho\right) + \tilde{\chi}\partial_a(\epsilon^{ab}C_b). \tag{5.7}$$

Variation of $\tilde{\chi}$ gives the Bianchi identity for C_a. If we vary with respect to C_a

$$C_a = \frac{\mathrm{e}^{2\phi}}{\rho}\delta_{ac}\epsilon^{bc}\partial_b\tilde{\chi} \tag{5.8}$$

and put the result back into (5.7) we get

$$\mathcal{L}' = \rho\left(R^{(2)} - \frac{\delta^{ab}}{2}\left(\partial_a\phi\partial_b\phi - \frac{\mathrm{e}^{2\phi}}{\rho^2}\partial_a\tilde{\chi}\partial_b\tilde{\chi}\right) + 2f^{-1}\partial_a f \rho^{-1}\partial^a\rho\right). \tag{5.9}$$

[3]Note that $\sqrt{g^{(2)}} = 1$.

We note that this looks quite similar to (5.6) and we have a first glimpse of the duality. In fact, if we define

$$\begin{aligned} e^{-2\tilde{\phi}} &= \frac{e^{2\phi}}{\rho^2} \\ \tilde{\rho} &= \rho \\ \tilde{f} &= f\rho^{1/4}e^{\phi/2} \end{aligned} \tag{5.10}$$

we get

$$\tilde{\mathcal{L}} = \tilde{\rho}\left(R^{(2)} - \frac{\delta^{ab}}{2}\left(\partial_a\tilde{\phi}\partial_b\tilde{\phi} - e^{-2\tilde{\phi}}\partial_a\tilde{\chi}\partial_b\tilde{\chi}\right) + 2\tilde{f}^{-1}\partial_a\tilde{f}\tilde{\rho}^{-1}\partial^a\tilde{\rho}\right). \tag{5.11}$$

This Lagrangian is identical to (5.6) except for the different sign in front of the $\tilde{\chi}$ term and they are connected through the duality relations (5.8) and (5.10). These duality relations are more commonly known as the Kramer-Neugebauer mappings which have generalizations to other theories. As mentioned earlier, this Lagrangian is also obtained if one reduces the four-dimensional theory directly to two dimensions, called the *Matzner-Misner* reduction. That is, the Kramer-Neugebauer mapping takes us from the Ehler's Lagrangian to the Matzner-Misner Lagrangian and vice versa. The change in sign is due to the signature of the unreduced two-dimensional space. In our case we have Euclidean signature which means that we have reduced the time dimension. We recall from section 4.2 that this change in sign implies a change of the coset space from $SL(2,\mathbb{R})/SO(2)$ to $SL(2,\mathbb{R})/SO(1,1)$. Each of these sigma-models has a $SL(2,\mathbb{R})$ symmetry and because of the duality relations these two $SL(2,\mathbb{R})$'s will not commute, i.e. it is not the same $SL(2, \mathbb{R})$ in both cases. In the next section we will see that this will enlarge our symmetry group to an affine Kac-Moody group.

5.3 Infinite Dimensional Symmetry

In this section we will see what consequences the duality relations derived in the previous section have. As already mentioned, the duality gives rise to an infinite dimensional symmetry, called an *affine Kac-Moody algebra*. We will show this for the case $SL(2, \mathbb{R})$ but the results are quite general. That is, when we reduce to two dimensions our symmetry group enlarges to an infinite-dimensional symmetry group. For a brief introduction to affine Kac-Moody algebras, see appendix B. For a detailed discussion about the loop-group and the central extension, see [8] and [26].

5.3.1 Field Representation

The Ehler's and the Matzner-Misner $\mathfrak{sl}(2,\mathbb{R})$'s will be denoted

$$\begin{aligned} \mathfrak{g} &= \mathbb{R}e \oplus \mathbb{R}h \oplus \mathbb{R}f \\ \tilde{\mathfrak{g}} &= \mathbb{R}\tilde{e} \oplus \mathbb{R}\tilde{h} \oplus \mathbb{R}\tilde{f} \end{aligned} \tag{5.12}$$

respectively, where e, h, f and $\tilde{e}, \tilde{h}, \tilde{f}$ are the usual generators of $SL(2,\mathbb{R})$. Under an infinitesimal transformation our group element V transforms as $V \longrightarrow \delta g V + V \delta k$. The coset representative is given by

$$V = \begin{pmatrix} \mathrm{e}^{\phi/2} & \chi \mathrm{e}^{-\phi/2} \\ 0 & \mathrm{e}^{-\phi/2} \end{pmatrix}. \tag{5.13}$$

Note here that we have only made the change $\phi \to -\phi$ to the usual coset representative[4] to get (5.6). In our case we have three independent transformations $(-e, -h, -f)$ and we would like to find a field representation with module consisting of the fields $\Delta = \mathrm{e}^{\phi}, \chi$. From the transformation of V we can read of the transformation properties of Δ and χ. When we use the f transformation we have to recall to use a compensating factor k to restore the upper triangular gauge. We find

$$\begin{aligned} &\delta_e \Delta = 0 && \delta_e \chi = -1 \\ &\delta_h \Delta = -2\Delta && \delta_h \chi = -2\chi \\ &\delta_f \Delta = 2\chi\Delta && \delta_f \chi = \chi^2 - \Delta^2. \end{aligned} \tag{5.14}$$

For the Matzner-Misner case, we get the exact same transformations, that is

$$\begin{aligned} &\delta_{\tilde{e}} \tilde{\Delta} = 0 && \delta_{\tilde{e}} \tilde{\chi} = -1 \\ &\delta_{\tilde{h}} \tilde{\Delta} = -2\tilde{\Delta} && \delta_{\tilde{h}} \tilde{\chi} = -2\tilde{\chi} \\ &\delta_{\tilde{f}} \tilde{\Delta} = 2\tilde{\chi}\tilde{\Delta} && \delta_{\tilde{f}} \tilde{\chi} = \tilde{\chi}^2 - \tilde{\Delta}^2. \end{aligned} \tag{5.15}$$

This may be considered as field representation of the $SL(2,\mathbb{R})$ groups. Due to the duality relations (5.10) we will get induced transformations on the Ehlers field when we transform the Matzner-Misner fields and vice versa. Consider ϕ and χ as the module for the affine Kac-Moody group with Chevalley generators $e, \tilde{e}, f, \tilde{f}$. We would now like to find the Cartan matrix associated with this symmetry group. To do this we need to calculate the commutation relations and we begin by ensuring that the two $\mathfrak{sl}(2,\mathbb{R})$ have the right commutation relations, then we intertwine

[4] The change in sign only affects the factor $\mathrm{e}^{2\phi} \to \mathrm{e}^{-2\phi}$.

them. Consider

$$\begin{aligned}\delta_{[h,e]}\Delta &= (he - eh)\Delta = \delta_h\delta_e\Delta - \delta_e\delta_h\Delta = 2\delta_e\Delta\\ \delta_{[h,f]}\Delta &= (hf - fh)\Delta = \delta_h\delta_f\Delta - \delta_f\delta_h\Delta = -2\delta_f\Delta\\ \delta_{[e,f]}\Delta &= (\delta_e\delta_f - \delta_f\delta_e)\Delta = \delta_h\Delta.\end{aligned} \tag{5.16}$$

This shows that $\delta_e, \delta_h, \delta_f$ really serve as a $\mathfrak{sl}(2,\mathbb{R})$ representation[5]. To get the intertwined commutation relations we need to use the duality relations given in the previous section. Consider

$$\begin{aligned}\delta_{[\tilde{h},e]}\Delta &= (\delta_{\tilde{h}}\delta_e - \delta_e\delta_{\tilde{h}})\Delta = -\delta_e\delta_{\tilde{h}}\left(\frac{\rho}{\tilde{\Delta}}\right) = -2\delta_e\Delta\\ \delta_{[h,\tilde{e}]}\tilde{\Delta} &= (\delta_h\delta_{\tilde{e}} - \delta_{\tilde{e}}\delta_h)\tilde{\Delta} = -\delta_{\tilde{e}}\delta_h\left(\frac{\rho}{\Delta}\right) = -2\delta_{\tilde{e}}\tilde{\Delta}.\end{aligned} \tag{5.17}$$

By a completely analogous computation we find $[\tilde{h}, f] = 2f$ and $[h, \tilde{f}] = 2\tilde{f}$. Thus the Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{5.18}$$

This is the Cartan matrix for the infinte-dimensional affine Kac-Moody algebra $A_1^+ = SL(2,\mathbb{R})^+$. The implementation of the Matzner-Misner $SL(2,\mathbb{R})$ on χ is not as straightforward as for Δ. Consider[6] the $\delta_{\tilde{f}}$ transformation of χ

$$\begin{aligned}\delta_{\tilde{f}}\partial_a\chi &= \delta_{\tilde{f}}\left(\frac{\Delta^2}{\rho}\delta_{ac}\epsilon^{bc}\partial_b\tilde{\chi}\right)\\ &= \frac{2\Delta}{\rho}\delta_{\tilde{f}}\left(\frac{\rho}{\tilde{\Delta}}\right)\delta_{ac}\epsilon^{bc}\partial_b\tilde{\chi} + \frac{\Delta^2}{\rho}\delta_{ac}\epsilon^{bc}\partial_b\left(\tilde{\chi}^2 - \tilde{\Delta}^2\right)\\ &= 2\epsilon^{bc}\left(\delta_{ac}\frac{\rho}{\Delta}\partial_b\Delta - \frac{\Delta^2}{\rho}\delta_{ac}\tilde{\chi}\partial_b\tilde{\chi}\right) = 2\partial_a\varphi.\end{aligned}$$

That is, $\delta_{\tilde{f}}\partial_a\chi = 2\partial_a\varphi$ and if we use that $\delta_{\tilde{f}}$ and ∂ commute we get

$$\delta_{\tilde{f}}\chi = 2\varphi \tag{5.19}$$

where φ is a new field not contained in V. If we apply the Matzner-Misner $SL(2,\mathbb{R})$ on this field we will get another field, not contained in V, φ. This never stops and an infinite "chain" of new fields is generated [8]. Moreover, these new fields

[5] Of course, the same is true for the Matzner-Misner $SL(2,\mathbb{R})$.

[6] It is a bit misfortune that δ has two different meanings here but it should be clear from the indices which one is which.

depend on Δ, χ. Before we end this section we should make some observations regarding the conformal factor f and the central extension. Due to the invariant right hand side of (5.4), f does not transform under Ehler's $SL(2,\mathbb{R})$. However, it does transform under the Matzner-Misner $SL(2,\mathbb{R})$ because of (5.10) which suggests that we should include f in the set of Ehler's fields when we consider the intertwined set of symmetries. The central extension c of the loop algebra is given by

$$c = h + \tilde{h} \tag{5.20}$$

and it is straightforward to check that $[c, g] = 0$ for all $g \in \mathfrak{sl}(2,\mathbb{R})$. The conformal factor f transform under this element according to

$$\delta_c f = -f. \tag{5.21}$$

5.3.2 BM Linear System

As we have seen in the previous section, in two dimensions our symmetry groups intertwine and enlarges to an infinite-dimensional affine Kac-Moody group. By implementing the Matzner-Misner $SL(2,\mathbb{R})$ on the Ehler's fields Δ, χ and vice versa we were able to construct the Cartan matrix A. If one explores the implementation of the Matzner-Misner group on the Ehler's fields a bit further one finds that the field content Δ, χ is not enough; one has to introduce more fields. That is, the transformation of Δ, χ by the Matzner-Misner group generates an infinte sequence of new fields [8]. However, these fields are not independent and can be expressed in terms of the original fields Δ, χ through a recursive process [27]. In [8], a practical way of dealing with these fields is presented. A generating function $\mathcal{V}(t,x)$ is introduced and is given by

$$\mathcal{V}(t,x) = V_0 + tV_1 + t^2 V_2 + ... \tag{5.22}$$

where t is a complex spectral parameter, $V_0 \in G/H$ is the usual coset representative and x is a collective spacetime parameter. The motivation for extending V_0 is to be able to act with the infinte dimensional symmetry group. The higher order terms V_i contain the infinte sequence of new potentials generated by the Matzner-Misner group when implemented on the Ehler's fields. This object is an element of the loop extension of a group G, [23], and the triangular gauge is generalized to the requirement $\mathcal{V}(t,x) \to V_0$ as $t \to 0$.

As discussed in chapter 5, (5.5) is the equation we need to solve and due to the integrability of this system there exists a so called *Lax-pair* with a compatibility condition. This basically means that one can solve a set of linear equations (Lax-pair) instead of the non-linear equation (5.5). For a more detailed discussion on the matter, see [8]. The linear system is given by, [11],

$$\partial_\mu \mathcal{V}\mathcal{V}^{-1} = \mathcal{Q}_\mu + \frac{1+t^2}{1-t^2}\mathcal{P}_\mu + \frac{2t}{1-t^2}\epsilon_{\mu\nu}\mathcal{P}^\nu \tag{5.23}$$

and the compatibility condition by

$$\partial_\mu(\partial_\nu \mathcal{V}\mathcal{V}^{-1}) - \partial_\nu(\partial_\mu \mathcal{V}\mathcal{V}^{-1}) + [\partial_\mu \mathcal{V}\mathcal{V}^{-1}, \partial_\nu \mathcal{V}\mathcal{V}^{-1}] = 0. \tag{5.24}$$

The latter is simply an identity. We see that by letting $t \to 0$ in (5.23) we get back

$$\partial_\mu V V^{-1} = \mathcal{Q}_\mu + \mathcal{P}_\mu. \tag{5.25}$$

A solution to (5.23) has the form of (5.22). By inserting (5.23) into (5.22) and comparing order by order in t we can solve for V_i. One can show that if we use (5.23) in (5.24) we get the original non-linear equation (5.5) *if* the spectral parameter t satisfies the condition

$$\partial_\mu t = -\frac{1}{2}\epsilon_{\mu\nu}\partial^\nu \left(\rho\left(t + \frac{1}{t}\right)\right). \tag{5.26}$$

That is, given a solution to (5.23), (5.5) is solved as well. The solutions to (5.26) is given by, [11],

$$t_\pm = \frac{1}{\rho}\left((z-w) \pm \sqrt{(z-w)^2 + \rho^2}\right). \tag{5.27}$$

w is a constant spectral parameter and $\pm$ refers to two different types of solutions. We note here that $t_+ = -1/t_-$. We will follow [11] and use t_+ which we will simply denote t from now on. The transformation of $\mathcal{V}(t,x)$ generalizes to, [11],

$$\mathcal{V}(t,x) \longrightarrow k(t,x)\mathcal{V}(t,x)g(w) \tag{5.28}$$

where $g(w)$ is an element in the loop group associated to G and $k(t,x)$ is defined below. Just like for the finite-group case, $k(t,x)$ is needed to preserve the gauge choice, i.e. it makes sure that $\mathcal{V}$ has the form of (5.22). The problem of finding the correct element $k(t,x)$ is still an issue which leads us to define a generalization of M. To do this we first have to generalize the involution τ as

$$\tau^\infty : \ \tau^\infty\Big(\mathcal{V}(t,x)\Big) = \tau\left(\mathcal{V}\Big(-1/t,x\Big)\right). \tag{5.29}$$

The last action should be interpreted as an action on each of V_i. The generating function $\mathcal{V}(t,x)$ can be seen as an element in the generalized coset space $H^\infty \backslash G^\infty$ [8], where H^∞ is the subgroup invariant under τ^∞ and G^∞ is the loop group associated to G. Just like the upper triangular matrices defined the Borel gauge for G/H, the regularity condition $\mathcal{V}(t,x) \to V_0$ as $t \to 0$ defines a "Borel gauge" for $H^\infty \backslash G^\infty$. $k(x,t)$ is an element in the subgroup H^∞, i.e.

$$k(t,x) = \tau\left(k(-1/t,x)\right). \tag{5.30}$$

Two elements $\mathcal{V}_1$ and $\mathcal{V}_2$ in the coset space $H^\infty \backslash G^\infty$ will thus be considered gauge equivalent if there exists an element $k \in H^\infty$ such that

$$\mathcal{V}_1 = k\mathcal{V}_2. \tag{5.31}$$

Consider the so called *monodromy matrix* $\mathcal{M}(w)$

$$\mathcal{M}(w) = \left(\mathcal{V}(t)\right)^{\mathcal{T}^\infty} \mathcal{V}(t) = \mathcal{V}^{\mathcal{T}}(-1/t)\mathcal{V}(t). \tag{5.32}$$

We will show below that $\partial_\mu \mathcal{M}(t,x) = 0$ which means that $\mathcal{M}$ depends only on the constant spectral parameter w. This result is quite unexpected; the spacetime independent matrix $\mathcal{M}(w)$ encodes information about the spacetime dependent metric $g(x)$. The transformation property of this matrix follows from (5.28)

$$\mathcal{M}(w) \longrightarrow g^{\mathcal{T}}(w)\mathcal{M}(w)g(w). \tag{5.33}$$

That is, we do not need $k(t,x)$ which again makes it more advantageous to use $\mathcal{M}(w)$. However, there is one important issue; due to (5.32) we cannot simply let $t \to 0$ to get M after we have made the transformation, as we would like to be able to identify the physical fields. Instead, we have to factorize $\mathcal{M}(w)$, called a *Riemann-Hilbert problem*, which is a non-trivial task and we will elaborate on this in chapter 7. The solution generating process works as before and once a seed solution is obtained, new solutions can relatively easy[7] be obtained. Figure 5.1 shows an overview of the process of generating new solutions. The possibilities arising from the infinite-dimensional symmetry are of course a significant difference to the three-dimensional case. For example, it allows Minkowski spacetime to serve as a seed solution.

[7] Given that we know how to factorize $\mathcal{M}$.

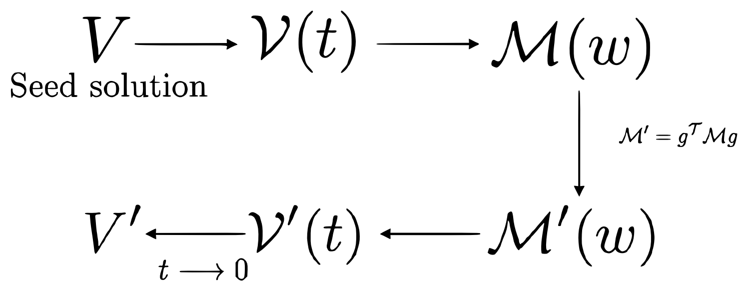

Figure 5.1: Summary of the solution generating method in $D = 2$.

The generalized coset space $H^\infty \backslash G^\infty$ is still the space of solutions to (5.5) but from the fact that there exists a central element acting on the conformal factor f we should include the conformal factor f into the enlarged coset space of solutions $H^\infty \backslash G^\infty_{ce}$. Thus, the coset space $H^\infty \backslash G^\infty_{ce}$, with elements $(\mathcal{V}, f)$, denotes the space of solution to both (5.4) and (5.5). See [8] for further details.

We end this section with some comments on the linear system (5.23). We note that the right hand side is invariant under τ^∞,

$$\begin{aligned} &\tau^\infty \left(\mathcal{Q}_\mu + \frac{1+t^2}{1-t^2}\mathcal{P}_\mu + \frac{2t}{1-t^2}\epsilon_{\mu\nu}\mathcal{P}^\nu \right) \\ &= \mathcal{Q}_\mu - \frac{1+1/t^2}{1-1/t^2}\mathcal{P}_\mu + \frac{2/t}{1-1/t^2}\epsilon_{\mu\nu}\mathcal{P}^\nu \\ &= \mathcal{Q}_\mu + \frac{1+t^2}{1-t^2}\mathcal{P}_\mu + \frac{2t}{1-t^2}\epsilon_{\mu\nu}\mathcal{P}^\nu . \end{aligned} \tag{5.34}$$

From (5.23) we see that, on shell, $\partial_\mu \mathcal{V}\mathcal{V}^{-1}$ is invariant under τ^∞ as well which means that $\partial_\mu \mathcal{V}\mathcal{V}^{-1} \in \mathrm{Lie}(H^\infty)$. This justifies the transformation (5.28) since

$$\partial_\mu \mathcal{V}\mathcal{V}^{-1} \longrightarrow \partial_\mu (k\mathcal{V}g)(k\mathcal{V}g)^{-1} = \partial_\mu k k^{-1} + k\partial_\mu \mathcal{V}\mathcal{V}^{-1}k^{-1}. \tag{5.35}$$

We recall that this is exactly how the τ invariant $\mathcal{Q}$ transforms. That is, $\partial_\mu \mathcal{V}\mathcal{V}^{-1}$ can be seen as an element in some generalized τ^∞ invariant space $\mathcal{Q}^\infty$. A final important observation is that the t-dependence on the right hand side of (5.23) is invariant under the transformation (5.28) [28]. By using that $\tau^\infty(\partial_\mu \mathcal{V}\mathcal{V}) = \partial_\mu \mathcal{V}\mathcal{V}$

we can show that $\mathcal{M}$ is independent of x as announced above. Consider

$$\begin{aligned}\partial_\mu \mathcal{M} &= \partial_\mu \left(\mathcal{V}^{\tau^\infty}\mathcal{V}\right) = \partial_\mu \left(\tau^\infty(\mathcal{V}^{-1})\mathcal{V}\right)\\ &= \partial_\mu \left(\tau^\infty(\mathcal{V}^{-1})\right)\mathcal{V} + \tau^\infty(\mathcal{V}^{-1})\partial_\mu\mathcal{V}\\ &= \tau^\infty\mathcal{V}^{-1}\left(\partial_\mu\mathcal{V}\mathcal{V}^{-1} + (\tau^\infty\mathcal{V}^{-1})^{-1}\partial_\mu(\tau^\infty\mathcal{V}^{-1})\right)\mathcal{V}\\ &= \tau^\infty\mathcal{V}^{-1}\left(\partial_\mu\mathcal{V}\mathcal{V}^{-1} - \tau^\infty(\partial_\mu\mathcal{V}\mathcal{V})\right)\mathcal{V} = 0.\end{aligned} \tag{5.36}$$

Here we have used that $\tau^\infty(\partial_\mu\mathcal{V}\mathcal{V}) = -\tau^\infty(\mathcal{V})\partial_\mu(\tau^\infty(\mathcal{V}^{-1}))$ and that $(\tau^\infty(\mathcal{V}))^{-1} = \tau^\infty(\mathcal{V}^{-1})$.

5.3.3 Relation Between $\mathcal{M}$ and M

There exists a very useful relation between the monodromy matrix $\mathcal{M}$ and M presented in [9]. We will use this relation later on when we construct the monodromy matrix $\mathcal{M}$ for the Schwarzschild metric. First we note that the functions $t_\pm$ have branch points at $\rho = \pm\mathrm{Im}(w)$ and $z = \mathrm{Re}(w)$ with values, [9],

$$t_\pm\Big|_{\substack{\rho=\mathrm{Im}(w)\\ z=\mathrm{Re}(w)}} = -i \qquad\qquad t_\pm\Big|_{\substack{\rho=-\mathrm{Im}(w)\\ z=\mathrm{Re}(w)}} = i. \tag{5.37}$$

From (5.27) we see that

$$\begin{cases} t_+ \longrightarrow 0 & \text{as } \rho \to 0, z < \mathrm{Re}(w)\\ t_- \longrightarrow \infty & \text{as } \rho \to 0, z < \mathrm{Re}(w)\end{cases}. \tag{5.38}$$

We define the following,

$$\begin{aligned}\mathcal{V}_+(w,\rho,z) &= \mathcal{V}(t_+(w,\rho,z),\rho,z)\\ \mathcal{V}_-(w,\rho,z) &= \mathcal{V}(t_-(w,\rho,z),\rho,z)\end{aligned} \tag{5.39}$$

and find that the Lax-equations (5.23) become in the limit $\rho \to 0, z < \mathrm{Re}(w)$

$$\begin{aligned}\partial_\mu\mathcal{V}_+\mathcal{V}_+^{-1} &= \mathcal{Q}_\mu + \mathcal{P}_\mu = \partial_\mu V V^{-1}\\ \partial_\mu\mathcal{V}_-\mathcal{V}_-^{-1} &= \mathcal{Q}_\mu - \mathcal{P}_\mu = -(\partial_\mu V V^{-1})^{\mathcal{T}}.\end{aligned} \tag{5.40}$$

The solutions to these equations are given by

$$\begin{aligned}\mathcal{V}_+(w,0,z) &= V(0,z)D(w)\\ \mathcal{V}_-(w,0,z) &= (V^{\mathcal{T}}(0,z))^{-1}C(w)\end{aligned} \tag{5.41}$$

where $C(w)$ is a constant matrix. We see that $D(w) = 1$ in order for the requirement $\mathcal{V}(t_+ = 0, \rho, z) = V(\rho, z)$ to be satisfied. By using the fact that $t_\pm$ are equal for each branch point we get

$$\mathcal{V}_+(w, \rho, z)\Big|_{\substack{\rho=\mathrm{Im}(w)\\ z=\mathrm{Re}(w)}} = \mathcal{V}_-(w, \rho, z)\Big|_{\substack{\rho=\mathrm{Im}(w)\\ z=\mathrm{Re}(w)}} \tag{5.42}$$

and by letting $\mathrm{Im}(w) \longrightarrow 0$, i.e. $\rho = 0$ and $z = w$, we get

$$\mathcal{V}_+(w, 0, w) = \mathcal{V}_-(w, 0, w) \tag{5.43}$$

which is equivalent to

$$V(0, z) = (V^{\mathcal{T}}(0, z))^{-1} C(w). \tag{5.44}$$

In the last step we have used (5.41). We recall that $M(0, z) = V^{\mathcal{T}}(0, z) V(0, z)$, thus giving us that $C(w) = M(0, w)$ and consequently, $\mathcal{V}_-(w, 0, w) = V(0, w)$. Finally, we get by using the fact that $\mathcal{M}(w)$ is spacetime independent that

$$\begin{aligned}
\mathcal{M}(w) &= (\mathcal{V}(t_+, \rho, z))^{\mathcal{T}^\infty} \mathcal{V}(t_+, \rho, z) \\
&= \mathcal{V}^{\mathcal{T}}(-1/t_+, \rho, z) \mathcal{V}(t_+, \rho, z) = \mathcal{V}^{\mathcal{T}}(t_-, \rho, z) \mathcal{V}(t_+, \rho, z) \\
&= \mathcal{V}_-^{\mathcal{T}}(w, \rho, z) \mathcal{V}_+(w, \rho, z) = \mathcal{V}_-^{\mathcal{T}}(w, 0, w) \mathcal{V}_+(w, 0, w) \\
&= V^{\mathcal{T}}(0, w) V(0, w) = M(0, w).
\end{aligned} \tag{5.45}$$

6

$D = 5$ Minimal Supergravity

In this chapter we will continue to analyze how symmetries can be used to generate new solutions to Einstein's equations. The next step is to consider five-dimensional supergravity with symmetry group $G_{2(2)}$ and coset space $G_{2(2)}/SO(2,2)$. In particular, we will consider stationary axisymmetric solutions in five-dimensions which depend on only two coordinates, thus rendering the theory to an effectively two-dimensional theory. Much has been written about symmetries in this theory, e.g. [17, 23], but the group theoretical approach taken by Breitenlohner and Maison in two dimensions, has yet not been applied in any further extent. Their method has been proven successful in the case of pure gravity and STU supergravity, see [10, 11], and it is therefore natural to move on and see if this works for more complex theories. Thus, the purpose of this and the forthcoming chapters is to investigate if the method of Breitenlohner and Maison can be applied to this theory in order to generate new solutions. The Lagrangian[1] for $D = 5$ minimal supergravity is given[2] by

$$\mathcal{L} = R^5 \star 1 - \frac{1}{2} \star F^5_{(2)} \wedge F^5_{(2)} - \frac{1}{3\sqrt{3}} F^5_{(2)} \wedge F^5_{(2)} \wedge A^5_{(1)} \tag{6.1}$$

where $F^5_{(2)} = \mathrm{d}A^5_{(1)}$. Indices inside parentheses denotes the type of p-form.

[1]This Lagrangian includes both the integrand and the measure in the action and should therefore be considered as a D-form.

[2]$\star 1 = \epsilon = \epsilon_{\mu_1 \ldots \mu_5} \mathrm{d}x^{\mu_1} \ldots \otimes \mathrm{d}x^{\mu_5} = 1/5! \epsilon_{\mu_1 \ldots \mu_5} \mathrm{d}x^{\mu_1} \ldots \wedge \mathrm{d}x^{\mu_5} = \sqrt{g} \mathrm{d}x^1 \ldots \wedge \mathrm{d}x^5 = \sqrt{g} \mathrm{d}^5 x$

6.1 Dimensional Reduction to $D = 3$

As an intermediate step, we reduce to three dimensions just like we have done before a la Ehlers. The dimensional reduction procedure is very similar to pure gravity, except that we now have a vector field $A^5_{(1)}$ in the five-dimensional theory. We will not do the reduction again but instead follow [23] throughout this section. We use the coordinates r, θ, ϕ, ψ, t in our five-dimensional spacetime and by restricting the coordinate dependence of our fields to r, θ, ϕ and reduce the ψ and t dimension we obtain a three-dimensional theory. If we instead reduce over only spacelike coordinates we get $G_{2(2)}/SO(4)$ rather than $G_{2(2)}/SO(2,2)$. As an ansatz for the three-dimensional metric we take

$$ds_5^2 = \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1+\phi_2} ds_3^2 + \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1-\phi_2}(\mathrm{d}\psi + \mathcal{A}^2_{(1)})^2 - \mathrm{e}^{\frac{-2}{\sqrt{3}}\phi_1}(\mathrm{d}t + \chi_1 \mathrm{d}\psi + \mathcal{A}^1_{(1)})^2 \tag{6.2}$$

where $\mathcal{A}^{1,2}_{(1)}$ are the two Kaluza-Klein one-forms. If we write this in matrix form we get

$$g^5 = \begin{pmatrix} \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1+\phi_2} g^{(3)}_{\mu\nu} + \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1-\phi_2}\mathcal{A}^2_\mu\mathcal{A}^2_\nu - \frac{\mathcal{A}^1_\mu\mathcal{A}^1_\nu}{\mathrm{e}^{\frac{2}{\sqrt{3}}\phi_1}} & \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1-\phi_2}\mathcal{A}^2_\nu - \frac{\chi_1}{\mathrm{e}^{\frac{2}{\sqrt{3}}\phi_1}}\mathcal{A}^1_\nu & \frac{\mathcal{A}^1_\nu}{\mathrm{e}^{\frac{2}{\sqrt{3}}\phi_1}} \\ & \mathrm{e}^{\frac{1}{\sqrt{3}}\phi_1-\phi_2} - \frac{\chi_1^2}{\mathrm{e}^{-\frac{2}{\sqrt{3}}\phi_1}} & -\mathrm{e}^{-\frac{2}{\sqrt{3}}\phi_1}\chi_1 \\ & & -\mathrm{e}^{-\frac{2}{\sqrt{3}}\phi_1} \end{pmatrix}. \tag{6.3}$$

Since this is a metric it is symmetric and therefore we only present the upper triangular part. As usual, we have to dualize to get the sigma-model. We start by defining the following field strengths

$$\mathcal{F}_{(1)} = \mathrm{d}\chi_1 \tag{6.4a}$$
$$\mathcal{F}^1_{(2)} = \mathrm{d}\mathcal{A}^1_{(1)} + \mathcal{A}^2_{(1)} \wedge \mathrm{d}\chi_1 \tag{6.4b}$$
$$\mathcal{F}^2_{(2)} = \mathrm{d}\mathcal{A}^2_{(1)} \tag{6.4c}$$

The vector field $A^5_{(1)}$ decomposes as

$$A^5_{(1)} = A_{(1)} + \chi_3 \mathrm{d}\psi + \chi_2 \mathrm{d}t. \tag{6.5}$$

χ_2 and χ_3 are two scalar fields with the corresponding field strengths

$$F^1_{(1)} = \mathrm{d}\chi_2 \tag{6.6a}$$
$$F^2_{(1)} = \mathrm{d}\chi_3 - \chi_1 \mathrm{d}\chi_2 \tag{6.6b}$$
$$F_{(2)} = \mathrm{d}A_{(1)} - \mathrm{d}\chi_2 \wedge (\mathcal{A}^1_{(1)} - \chi_1\mathcal{A}^2_{(1)}) - \mathrm{d}\chi_3 \wedge \mathcal{A}^2_{(1)} \tag{6.6c}$$

We can now dualize our field strengths[3] as

$$G_{(1)4} = \mathrm{e}^{\sqrt{3}\phi_1-\phi_2} \star F_{(2)} = \mathrm{d}\chi_4 + \frac{1}{\sqrt{3}}(\chi_2\mathrm{d}\chi_3 - \chi_3\mathrm{d}\chi_2) \tag{6.7a}$$

$$G_{(1)5} = -\mathrm{e}^{-\sqrt{3}\phi_1-\phi_2} \star \mathcal{F}^1_{(2)} = \mathrm{d}\chi_5 - \chi_2\mathrm{d}\chi_4 + \frac{\chi_2}{3\sqrt{3}}(\chi_3\mathrm{d}\chi_2 - \chi_2\mathrm{d}\chi_3) \tag{6.7b}$$

$$\begin{aligned} G_{(1)6} = \mathrm{e}^{-2\phi_2} \star \mathcal{F}^2_{(2)} &= \mathrm{d}\chi_6 - \chi_1\mathrm{d}\chi_5 + (\chi_1\chi_2 - \chi_3)\mathrm{d}\chi_4 \\ &+ \frac{1}{3\sqrt{3}}(\chi_3 - \chi_1\chi_2)(\chi_3\mathrm{d}\chi_2 - \chi_2\mathrm{d}\chi_3) \end{aligned} \tag{6.7c}$$

The Lagrangian (6.1) can now be expressed in terms of the scalar fields $\phi_1, \phi_2, \chi_1, \ldots, \chi_6$ as

$$\begin{aligned} \mathcal{L} = R \star 1 &- \frac{1}{2} \star \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_1 - \frac{1}{2} \star \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_2 + \frac{1}{2}\mathrm{e}^{-\sqrt{3}\phi_1+\phi_2} \star \mathrm{d}\chi_1 \wedge \mathrm{d}\chi_1 \\ &+ \frac{1}{2}\mathrm{e}^{\frac{2}{\sqrt{3}}\phi_1} \star \mathrm{d}\chi_2 \wedge \mathrm{d}\chi_2 - \frac{1}{2}\mathrm{e}^{-\frac{1}{\sqrt{3}}\phi_1+\phi_2} \star (\mathrm{d}\chi_3 - \chi_1\mathrm{d}\chi_2) \wedge (\mathrm{d}\chi_3 - \chi_1\mathrm{d}\chi_2) \\ &+ \frac{1}{2}\mathrm{e}^{\frac{1}{\sqrt{3}\phi_1}+\phi_2} \star G_{(1)4} \wedge G_{(1)4} - \frac{1}{2}\mathrm{e}^{\sqrt{3}\phi_1+\phi_2} \star G_{(1)5} \wedge G_{(1)5} \\ &+ \frac{1}{2}\mathrm{e}^{2\phi_2} \star G_{(1)6} \wedge G_{(1)6}. \end{aligned} \tag{6.8}$$

All the fields, including the three-dimensional Ricci scalar R, depend only on r, θ, ϕ. Although it is not that obvious, this is a sigma-model coupled to gravity on the coset space $G_{2(2)}/SO(2,2)$.

6.2 The Coset Space $G_{2(2)}/SO(2,2)$

Since $G_{2(2)}/SO(2,2)$ is the coset space for the sigma-model in three dimensions we will in this section briefly discuss its properties, define the involution, define the generalized transpose, define the matrix M and discuss how the coset space should be parametrized.

6.2.1 $G_{2(2)}$

Here we will follow [30]. The Lie algebra $\mathfrak{g}_{2(2)}$ is the split real form of $\mathfrak{g}_2$. It is a 14 dimensional algebra summarized in the Cartan matrix $A(G_{2(2)})$

$$A(G_{2(2)}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{6.9}$$

[3] $G_{(1)4,5,6}$ should not be confused with the *group* $G_{2(2)}$.

The Cartan matrix is of rank 2 which means that we have two triples of Chevalley generators h_i, e_i, f_i, $(i = 1, 2)$. From the Serre relations we get that we can form

$$\begin{aligned} e_3 &= [e_1, e_2] & f_3 &= [f_2, f_1] \\ e_4 &= [e_3, e_2] & f_4 &= [f_2, f_3] \\ e_5 &= [e_4, e_2] & f_5 &= [f_2, f_4] \\ e_6 &= [e_1, e_5] & f_6 &= [f_5, f_1] \end{aligned}$$

which together with our two Chevalley triples give us a total of 14 generators. The real span gives us the split real form $\mathfrak{g}_{2(2)}$ with triangular decomposition

$$\mathfrak{g}_{2(2)} = \mathfrak{n}_- \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_+ \tag{6.10}$$

where

$$\begin{aligned} \mathfrak{n}_- &= \mathrm{Span}_{\mathbb{R}}\{e_1, \ldots, e_6\} \\ \mathfrak{n}_0 &= \mathrm{Span}_{\mathbb{R}}\{h_1, h_2\} \\ \mathfrak{n}_+ &= \mathrm{Span}_{\mathbb{R}}\{f_1, \ldots, f_6\}. \end{aligned} \tag{6.11}$$

We define another basis for $\mathfrak{g}_{2(2)}$ as, [23],

$$\begin{aligned} h'_1 &= \frac{1}{\sqrt{3}} h_2 & h'_2 &= h_2 + 2h_1 \\ e'_1 &= e_1 & e'_2 &= \frac{1}{\sqrt{3}} e_2 \\ e'_3 &= \frac{1}{\sqrt{3}} e_3 & e'_4 &= \frac{1}{\sqrt{12}} e_4 \\ e'_5 &= \frac{1}{6} e_5 & e'_6 &= \frac{1}{6} e_6. \end{aligned}$$

This basis will be used when we create the coset representative.

6.2.2 Involution and Generalized Transpose

We define the involution τ as

$$\tau : \ \tau(t) = -(\eta^{-1})(t)^T \eta, \qquad t \in \mathfrak{g}_{2(2)} \tag{6.12}$$

η is defined below, which gives us

$$\begin{aligned} \tau(e_1) &= f_1 & \tau(e_2) &= f_2 \\ \tau(e_3) &= -f_3 & \tau(e_4) &= f_4 \\ \tau(e_5) &= -f_5 & \tau(e_6) &= f_6 \\ \tau(h_1) &= -h_1 & \tau(h_2) &= -h_2 \end{aligned} \tag{6.13}$$

The set of all generators invariant under the involution is given by

$$\mathfrak{h} = \{k = t + \tau(t) : t \in \mathfrak{g}_{2(2)}\}. \tag{6.14}$$

Explicitly, the set consists of

$$\begin{aligned} k_1 &= e_1 + f_1 & k_2 &= e_2 + f_2 & k_3 &= e_3 - f_3 \\ k_4 &= e_4 + f_4 & k_5 &= e_5 - f_5 & k_6 &= e_6 + f_6. \end{aligned}$$

These six generators define two $\mathfrak{sl}(2,\mathbb{R})$'s, easily seen by calculating the commutation relations. k_1, k_2, k_3 define one $SL(2,\mathbb{R})$ and k_4, k_5, k_6 another. These are of course not the same $SL(2,\mathbb{R})$'s that appear in the Chevalley triples. Since $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2,2)$ we have that $k_1,\dots,k_6$ generate $SO(2,2)$. Even though they are denoted with a k they do not form a compact subalgebra. We end this section by defining the generalized transpose as

$$V^{\mathcal{T}} = \eta^{-1} V^T \eta \tag{6.15}$$

where

$$\eta = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{6.16}$$

By a straightforward calculation one can check that $V^{\mathcal{T}} = V^{-1}$ for all $V \in SO(2,2)$.

6.2.3 The Matrix M

From now on we will use

$$M = V^{\mathcal{T}} V \tag{6.17}$$

as the definition of M. This does not change any result we have obtained so far but the coset space changes to H/G and the relationship between M and $\mathcal{P}_\mu$ changes to

$$M^{-1}\partial_\mu M = 2^{-1} V^{-1} \mathcal{P}_\mu V \tag{6.18}$$

obtained by a completely analogous calculation. Even though $H\backslash G$ is the correct coset space there is a trend in the litterature to keep denoting the coset space as G/H. We will also do this to stay in line with the common notation. The transformation of M is given by (3.27). Defined in [23] we can also define a $\hat{M}$ as

$$\hat{M} = S^T \eta V^T V S \tag{6.19}$$

where

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.20}$$

The advantage of using the latter definition is that it makes the vacuum truncation more connected to the $SL(3,\mathbb{R})/SO(2,1)$ coset space, see section 6.3. The downside of using it is that M is not an element in $G_{2(2)}$ and this will cause difficulties in the factorization, see below. In the following chapters we will use (6.17) unless clearly stated otherwise.

6.2.4 Parametrization of the Coset Space

By exponentiating the Borel subalgebra we get a coset representative much like we got for the $SL(2,\mathbb{R})/SO(2)$ case. The Borel subalgebra is generated by $h_1', h_2', e_1',\ldots, e_6'$ which gives ut the coset element V as, [23],

$$V = \exp\left(\frac{1}{2}\phi_1 h_1' + \frac{1}{2}\phi_2 h_2'\right) \exp(\chi_1 e_1') \exp(-\chi_2 e_2' + \chi_3 e_3') \exp(\chi_6 e_6') \exp(\chi_4 e_4' - \chi 5 e_5'). \tag{6.21}$$

We stress once more that this is actually an element in the right coset space $H\backslash G$ rather than the left G/H. With this parametrization one can show, by a straightforward calculation, that

$$\mathcal{L} = R \star 1 - \frac{1}{8}\mathrm{Tr}\Big(\star (M^{-1}\mathrm{d}M) \wedge (M^{-1}\mathrm{d}M)\Big). \tag{6.22}$$

The choice of coset representative is not unique and we might as well choose to exponentiate the sum of all generators. However, that choice is not very convenient since it does not allow us to identify $\phi_1, \phi_2, \chi_1, \ldots, \chi_6$ with the fields in the Lagrangian. We might say that if we want to parametrize our coset space with the same field as in the Lagrangian, (6.21) is unique up to a constant matrix as in (6.19).

6.3 Vacuum Truncation

If we would like to study vacuum solutions in five dimensions we could simply truncate our theory by letting $\chi_2 = \chi_3 = \chi_4 = 0$, i.e. no vector potential $A^5_{(1)}$. This should of course be equivalent to the case of pure gravity in five dimensions with the corresponding coset space $SL(3,\mathbb{R})/SO(2,1)$. Thus, if the truncation is consistent we should have that

$$\frac{SL(3,\mathbb{R})}{SO(2,1)} \subset \frac{G_{2(2)}}{SO(2,2)}. \tag{6.23}$$

However, even though we are effectively working in the $SL(3,\mathbb{R})/SO(2,1)$ coset space we have to remember that this is an embedding in the $G_{2(2)}/SO(2,2)$ space which means that we will use seven-dimensional representations. Our Lagrangian reduces to

$$\begin{aligned}\mathcal{L} = R \star 1 - \frac{1}{2} \star \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_1 - \frac{1}{2} \star \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_2 + \frac{1}{2} \mathrm{e}^{-\sqrt{3}\phi_1 + \phi_2} \star \mathrm{d}\chi_1 \wedge \mathrm{d}\chi_1 \\ - \frac{1}{2} \mathrm{e}^{\sqrt{3}\phi_1 + \phi_2} \star G_{(1)5} \wedge G_{(1)5} + \frac{1}{2} \mathrm{e}^{2\phi_2} \star G_{(1)6} \wedge G_{(1)6}\end{aligned} \tag{6.24}$$

where

$$G_{(1)5} = -\mathrm{e}^{-\sqrt{3}\phi_1 - \phi_2} \star \mathcal{F}^1_{(2)} = \mathrm{d}\chi_5 \tag{6.25a}$$

$$G_{(1)6} = \mathrm{e}^{-2\phi_2} \star \mathcal{F}^2_{(2)} = \mathrm{d}\chi_6 - \chi_1 \mathrm{d}\chi_5. \tag{6.25b}$$

The embedding (6.23) can be made more apparent if we consider the subalgebra spanned by h_1, h_2, e_1, e_5, e_6. If we make a change of basis of the Cartan algebra as

$$\begin{aligned}\tilde{h}_1 &= h_1 + h_2 \\ \tilde{h}_2 &= h_2\end{aligned} \tag{6.26}$$

we get, by using the commutation relations given by (6.9), that $\tilde{h}_1, \tilde{h}'_2, e_1, e_5, e_6$ satisfy the $\mathfrak{sl}(3,\mathbb{R})$ algebra. Thus, $SL(3,\mathbb{R})$ is a subgroup of $G_{2(2)}$. Our coset representative reduces to

$$V = \exp\left(\frac{1}{2}\phi_1 h'_1 + \frac{1}{2}\phi_2 h'_2\right) \exp(\chi_1 e'_1) \exp(\chi_6 e'_6) \exp(-\chi 5 e'_5) \tag{6.27}$$

We conclude this section by explicitly calculating the matrix $\hat{M}$ given by the definition (6.19). By using the representation given in appendix C we get that

$$\hat{M} = \begin{pmatrix} M^{-1}_{3\times 3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M_{3\times 3} \end{pmatrix} \tag{6.28}$$

where

$$M_{3\times 3} = \begin{pmatrix} -\frac{\left(e^{\sqrt{3}(\phi_1+\phi_2)}\chi_5{}^2+1\right)}{e^{\frac{2(\phi_1+\phi_2)}{\sqrt{3}}}} & -\frac{\left(\chi_1+e^{\sqrt{3}(\phi_1+\phi_2)}\chi_5\chi_6\right)}{e^{\frac{2(\phi_1+\phi_2)}{\sqrt{3}}}} & -e^{\frac{\phi_1+\phi_2}{\sqrt{3}}}\chi_5 \\ -\frac{\left(\chi_1+e^{\sqrt{3}(\phi_1+\phi_2)}\chi_5\chi_6\right)}{e^{\frac{2(\phi_1+\phi_2)}{\sqrt{3}}}} & \frac{\left(-\chi_1{}^2-e^{\sqrt{3}(\phi_1+\phi_2)}\left(\chi_6{}^2-1\right)\right)}{e^{\frac{2(\phi_1+\phi_2)}{\sqrt{3}}}} & -e^{\frac{\phi_1+\phi_2}{\sqrt{3}}}\chi_6 \\ -\frac{\chi_5}{e^{\frac{-\phi_1-\phi_2}{\sqrt{3}}}} & -e^{\frac{\phi_1+\phi_2}{\sqrt{3}}}\chi_6 & -e^{\frac{\phi_1+\phi_2}{\sqrt{3}}} \end{pmatrix} \tag{6.29}$$

As it turns out, $M_{3\times 3}$ is a representation for $SL(3,\mathbb{R})$ [23]. Thus, when the symmetries of vacuum gravity is embedded into the coset space $G_{2(2)}/SO(2,2)$ it appears twice.

6.4 The Reissner-Nordström Metric in $D = 5$

In our efforts to investigate the solution generating properties of $G_{2(2)}$ we will consider an explicit example, the five-dimensional Reissner-Nordström metric. This metric is the electrically charged five dimensional Schwarzschild solution parametrized by the mass m and charge q. The components are given explicitly in [23] but we will reproduce them here for convenience and to establish the notation. In the next chapter this solution will be generated. In the coordinates used above we have

$$\begin{aligned} g^{(5)}_{tt} &= -\frac{r^2(r^2-2m)}{(r^2+2m\sinh^2(\delta))^2} \\ g^{(5)}_{\phi\phi} &= \sin^2(\theta)(r^2+2m\sinh^2(\delta)) \\ g^{(5)}_{\psi\psi} &= \cos^2(\theta)(r^2+2m\sinh^2(\delta)) \\ g^{(5)}_{rr} &= \frac{r^2+2m\sinh^2(\delta)}{r^2-2m} \\ g^{(5)}_{\theta\theta} &= r^2+2m\sinh^2(\delta) \\ A_t &= \frac{2\sqrt{3}m\sinh(\delta)\cosh(\delta)}{r^2+2m\sinh^2(\delta)} \end{aligned} \tag{6.30}$$

where δ is the parameter which regulates the electric charge. To get the coset representative one needs to dualize, which involves integration. However, for more complicated examples this integration can become very difficult. To overcome this problem one can follow the procedure from [21]. First we make the following observation for M. If we calculate the matrix valued one-form product $\star M^{-1}\mathrm{d}M$

we find that the following matrix components become

$$\begin{aligned}
\left(\star M^{-1}\mathrm{d}M\right)_{61} &= \frac{\mathrm{e}^{2\phi_2}}{2}(\chi_1 \star \mathrm{d}\chi_5 - \star\mathrm{d}\chi_6) \\
\left(\star M^{-1}\mathrm{d}M\right)_{51} &= \frac{\mathrm{e}^{\sqrt{3}\phi_1+\phi_2}}{2} \star \mathrm{d}\chi_5 - \frac{\mathrm{e}^{2\phi_2}}{2}\chi_1(\chi_1 \star \mathrm{d}\chi_5 - \star\mathrm{d}\chi_6).
\end{aligned}$$

By a straightforward calculation we find, by using (6.25a), (6.25b), (6.4b) and (6.4c), that

$$\begin{aligned}
\left(\star M^{-1}\mathrm{d}M\right)_{61} &= -\frac{1}{2}\mathrm{d}\mathcal{A}^2_{(1)} \\
\left(\star M^{-1}\mathrm{d}M\right)_{51} &= -\frac{1}{2}\mathrm{d}(\mathcal{A}^1_{(1)} - \chi_1\mathcal{A}^2_{(1)}).
\end{aligned}$$

This leads us to define[4] a matrix valued one-form N

$$\mathrm{d}N = \star M^{-1}\mathrm{d}M \tag{6.31}$$

and from the result above we get

$$\begin{aligned}
N_{23} &= -\frac{1}{2}\mathcal{A}^2_{(1)} \\
N_{13} &= -\frac{1}{2}(\mathcal{A}^1_{(1)} - \chi_1\mathcal{A}^2_{(1)}).
\end{aligned} \tag{6.32}$$

From the defining relation of N we get an induced transformation from M as

$$N \longrightarrow g^{-1}Ng. \tag{6.33}$$

Thus, the matrix N allows us to extract the transformed $\mathcal{A}^{1,2}_{(1)}$ directly without performing any dualization.

6.5 Dimensional Reduction to $D = 2$

The reduction to two dimensions is performed in the same way as in the previous chapter and we choose (ρ, z) as coordinates on the flat two-dimensional space. The ansatz

$$g^{(3)} = \begin{pmatrix} f^2 g^{(2)} & 0 \\ 0 & \rho^2 \end{pmatrix} \tag{6.34}$$

[4] Actually, the result above does not provide enough evidence for us to make this assumption since we need to check if all components can be written like this. However, on shell one can show that this assumption is valid [17].

gives the two-dimensional Lagrangian, [23]

$$\mathcal{L} = \rho R \star 1 - \frac{1}{8}\rho \operatorname{Tr}\Big(\star (M^{-1}\mathrm{d}M) \wedge (M^{-1}\mathrm{d}M)\Big) + 2f^{-1} \star \mathrm{d}\rho \wedge \mathrm{d}f. \qquad (6.35)$$

Since there are no structural differences in this Lagrangian compared to the one discussed in chapter 5, the field equations look the same. As pointed out at end of chapter 5, the conformal factor f^2 is obtained by integration once M is known. In [10], a useful and neat expression for calculating f is presented. It requires some concepts which we will present in the following chapter.

7

Solution to the Riemann-Hilbert Problem

When we generated solutions in three dimensions we transformed the matrix M and then read off the different fields directly. This procedure is rather straightforward, putting aside any issues with the inverse dualization. In two dimensions we cannot simply read of the fields after we have made the transformation. This is due to the nature of the monodromy matrix $\mathcal{M}$, whose connection to the physical fields is far from obvious. In order to find the physical fields after a transformation we have to find the corresponding matrix M. To do this one have to factorize $\mathcal{M} = \mathcal{V}^T\mathcal{V}$, which in general is quite difficult, called a *Riemann-Hilbert problem.* As it turns out, this problem becomes manageable if one considers special types of meromorphic matrices $\mathcal{M}$ with single poles. Breitenlohner and Maison worked out a factorization procedure for this kind of matrices, presented in unpublished notes[1], and in [10, 11] this method is applied to pure gravity and STU supergravity. As explained in the previous chapter, the natural way to proceed is to check if this method works for other groups which emerge from more complicated gravity theories and the next in line would be the $G_{2(2)}$ group. There seem to be no *a priori* indication that the method cannot be applied to this group but when it comes to actually performing calculations some practical difficulties may arise since this is a larger group in the sense of matrix representation. Since the purpose of the forthcoming chapters is to apply this to the $G_{2(2)}$ group, we will reproduce the method here, adapted to the $G_{2(2)}$ group. The presentation closely follows [11]. At the end of this chapter we will apply the method to factorize the Schwarzschild metric which will be one of the main results of this thesis. This will function not only as an illuminating example but also as a seed solution to generate the Reissner-Nordström metric.

[1] I am grateful to Axel Kleinschmidt for providing these notes.

7.1 Riemann-Hilbert Problem for $G_{2(2)}$

Due to the fact that $\mathcal{M} = \mathcal{V}^{\mathcal{T}}(-1/t)\mathcal{V}(t)$ we cannot simply let $t \to 0$ in order to get M. We have to obtain the corresponding matrix $\mathcal{V}(t)$ which amounts to solve the Riemann-Hilbert problem. In this section we will assume that the monodromy matrix $\mathcal{M}$ is meromorphic[2] in the spectral paramter w. $G_{2(2)}$ is a subgroup of $SO(3^+,4^-)$ and should therefore preserve a quadratic form. Let η' denote the quadratic form preserved by an arbitrary coset element

$$\eta' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.1}$$

The matrix $\mathcal{M}(t,x) \in G_{2(2)}$ is parametrized as

$$\mathcal{M}(t,x) = Y + \frac{A_1}{w-c} + \frac{A_2}{w+c} \tag{7.2}$$

where Y is given by the boundary condition. This follows from (5.45) and the assumption that M has an expansion in $1/r$, i.e. $M = Y + M_1/r + \mathcal{O}(1/r^2)$. A_k is parametrized as

$$A_k = \alpha_k a_k a_k^{\mathcal{T}} + \beta_k b_k b_k^{\mathcal{T}} \tag{7.3}$$

where the generalized transpose on a column vector is defined[3] as $a_k^{\mathcal{T}} = a_k^T \eta$ and on a row vector as $(a_k^T)^{\mathcal{T}} = \eta^{-1} a_k$. The prefactors α_k and β_k are added for convenience and can be used to tune $\det \mathcal{M} = 1$. We assume that the matrices A_1 and A_2 are rank 2 matrices but the generalization to higher rank should be straightforward [11]. From the relation (5.27)

$$t = \frac{1}{\rho}\left(z - w + \sqrt{(z-w)^2 + \rho^2}\right) \tag{7.4}$$

we can express $\mathcal{M}$ as

$$\mathcal{M}(t,x) = Y + \sum_{k=1}^{2} \frac{\nu_k t_k A_k}{t - t_k} + \sum_{k=1}^{2} \frac{\nu_k A_k}{1 + t t_k} \tag{7.5}$$

[2]A meromorphic function is an analytical function on an open set except for a set of isolated points.

[3]This definition is obtained by looking at $A_k^{\mathcal{T}}$.

where

$$\nu_k = -\frac{2}{\rho\left(t_k + \frac{1}{t_k}\right)} \tag{7.6}$$

and t_k are the poles at $\pm c$ given by (7.4). We need to find the restrictions on the vectors a_k and b_k which ensure that $\mathcal{M}$ is an element in $G_{2(2)}/SO(2,2)$. To do this we use the fact that a coset element should preserve the matrix η'. Consider

$$\begin{aligned}
\mathcal{M}(t,x)\mathcal{M}^{-1}(t,x) &= \mathcal{M}(t,x)(\eta')^{-1}\mathcal{M}^T(t,x)\\
&= \left(Y + \nu_-\left(\frac{t_1}{t-t_1} + \frac{1}{1+tt_1}\right)A_1 + \nu_2\left(\frac{t_2}{t-t_2} + \frac{1}{1+tt_2}\right)A_2\right)\\
&\quad(\eta')^{-1}\left(Y^T + \nu_1\left(\frac{t_1}{t-t_1} + \frac{1}{1+tt_1}\right)A_1^T + \nu_2\left(\frac{t_2}{t-t_2} + \frac{1}{1+tt_2}\right)A_2^T\right)\\
&= Y(\eta')^{-1}Y^T + \frac{\nu_1^2 A_1(\eta')^{-1}A_1^T}{(1+tt_1)^2} + \frac{\nu_+^2 A_2(\eta')^{-1}A_2^T}{(1+tt_2)^2}\\
&\quad + \frac{A_1(\eta')^{-1}\nu_1}{1+tt_1}\left(Y^T + \nu_1\frac{t_1}{t-t_1}A_1^T + \nu_2\left(\frac{t_2}{t-t_2} + \frac{1}{1+tt_2}\right)A_2^T\right)\\
&\quad + \left(Y + \nu_1\left(\frac{t_1}{t-t_1}\right)A_1 + \nu_2\left(\frac{t_2}{t-t_2} + \frac{1}{1+tt_2}\right)A_2\right)\frac{(\eta')^{-1}\nu_1 A_1^T}{1+tt_1}\ldots
\end{aligned}$$

Since $\mathcal{M}(t,x)\mathcal{M}^{-1}(t,x) = (\eta')^{-1}$ the left hand side should not contain any poles. In particular, the absence of double pole at $t = -1/t_\pm$ gives us

$$A_k(\eta')^{-1}A_k^T = 0. \tag{7.7}$$

Using the parametrization of A_k, $k = 1,2$, we get

$$\begin{aligned}
A_k(\eta')^{-1}A_k^T &= (\alpha_k a_k a_k^T + \beta_k b_k b_k^T)(\eta')^{-1}(\alpha_k a_k a_k^T + \beta_k b_k b_k^T)^T\\
&= \alpha^2 a_k a_k^T \eta(\eta')^{-1}\eta^T a_k a_k^T + \beta_k^2 b_k b_k^T \eta(\eta')^{-1}\eta^T b_k b_k^T\\
&\quad + \alpha_k\beta_k a_k a_k^T \eta(\eta')^{-1}\eta^T b_k b_k^T + \alpha_k\beta_k b_k b_k^T \eta(\eta')^{-1}\eta^T a_k a_k^T\\
&= \alpha^2 a_k a_k^T \eta' a_k a_k^T + \beta_k^2 b_k b_k^T \eta' b_k b_k^T\\
&\quad + \alpha_k\beta_k a_k a_k^T \eta' b_k b_k^T + \alpha_k\beta_k b_k b_k^T \eta' a_k a_k^T.
\end{aligned}$$

Here we have used $\eta(\eta')^{-1}\eta^T = \eta'$. From this we get that

$$\begin{aligned}
a_k^T\eta' a_k &= 0\\
b_k^T\eta' b_k &= 0\\
a_k^T\eta' b_k &= 0,
\end{aligned} \tag{7.8}$$

where $k = 1, 2$. The absence of single poles at $t = -1/t_k$ gives

$$A_k(\eta')^{-1}\mathcal{A}_k^T + \mathcal{A}_k(\eta')^{-1}A_k^T = 0 \tag{7.9}$$

where

$$\mathcal{A}_k = \left(\mathcal{M}(t,x) - \frac{\nu_k A_k}{1+tt_k}\right)\Big|_{t\to -\frac{1}{t_k}}. \tag{7.10}$$

Here, $\mathcal{A}_k$ should not be confused with the Kaluza-Klein vectors. If we insert the expression for A_k we get

$$\alpha_k a_k a_k^T \eta'\eta^{-1}\mathcal{A}_k^T + \beta_k b_k b_k^T \eta'\eta^{-1}\mathcal{A}_k^T = -\alpha_k \mathcal{A}_k \eta^{-1}\eta' a_k a_k^T - \beta_k \mathcal{A}_k \eta^{-1}\eta' b_k b_k^T. \tag{7.11}$$

If there exist numbers γ_k the solution is given by

$$\begin{aligned} \mathcal{A}_k\eta^{-1}\eta' a_k &= -\nu_k\beta_k\gamma_k b_k \\ b_k^T\eta'\eta^{-1}\mathcal{A}_k^T &= \nu_k\alpha_k\gamma_k a_k^T. \end{aligned} \tag{7.12}$$

Note that we cannot simply calculate γ_k from one of the equations above since both of them need to be satisfied. (7.12) should be considered as a further restriction on the vectors a_k and b_k in order to have a solution of (7.11).

7.1.1 Ansatz for Factorization

We continue to follow [11] and factorize $\mathcal{M}$ into the form

$$\mathcal{M}(t,x) = A_-^{\mathcal{T}}(t,x)M(x)A_+(t,x). \tag{7.13}$$

Due to the fact that $\mathcal{M}^{\mathcal{T}} = \mathcal{M}$ we have that

$$A_-(t,x) = A_+\left(-\frac{1}{t},x\right). \tag{7.14}$$

Moreover, since $\mathcal{M}(t,x) = \mathcal{V}^{\mathcal{T}}\left(-\frac{1}{t}\right)\mathcal{V}(t)$ and $M(x) = V^{\mathcal{T}}(x)V(x)$ we have that

$$\mathcal{V}(t,x) = V(x)A_+(t,x). \tag{7.15}$$

The poles of $\mathcal{M}$ are obviously distributed between A_+ and A_- and due to the fact that $\mathcal{V}(t,x)$ needs to be regular at $t = 0$ we have that the poles at $t = -1/t_k$ belong to A_+. Thus, we make the following ansatz for A_+

$$A_+(t,x) = \mathbb{1} - \frac{tC_1}{1+tt_1} - \frac{tC_2}{1+tt_2}. \tag{7.16}$$

where C_k is a matrix to be determined. From (7.14) we get that

$$A_-(t,x) = \mathbb{1} + \frac{C_1}{t-t_1} + \frac{C_2}{t-t_2}. \tag{7.17}$$

In a similar way as we did before we study the product $A_+(t,x)\eta'\mathcal{M}^T(t,x)$

$$\begin{aligned} A_+(t,x)\eta'\mathcal{M}^T(t,x) &= A_+(t,x)\eta' A_+^T(t,x) M^T(x)\Big(A_-^T(t,x)\Big)^T \\ &= \eta' M^T(x)\Big(A_-^T(t,x)\Big)^T. \end{aligned} \tag{7.18}$$

This tells us that $A_+(t,x)\eta'\mathcal{M}^T(t,x)$ does not have any poles at $t = -1/t_k$ and by expanding $\mathcal{M}(t,x)$ in terms of A_k we get

$$\begin{aligned} A_+(t,x)\eta'\mathcal{M}^T(t,x) = &\left(\mathbb{1} - \frac{tC_1}{1+tt_1} - \frac{tC_2}{1+tt_2}\right)\eta' \\ &\left(Y^T + \nu_1\left(\frac{t_1}{t-t_1} + \frac{1}{1+tt_1}\right)A_1^T + \nu_2\left(\frac{t_2}{t-t_2} + \frac{1}{1+tt_2}\right)A_2^T\right) \\ &= \eta' Y^T + \eta'\frac{\nu_1 t_1 A_1^T}{t-t_1} + \eta'\frac{\nu_1 A_1^T}{1+tt_1} + \eta'\frac{\nu_2 t_1 A_2^T}{t-t_2} + \eta'\frac{\nu_2 A_2^T}{1+tt_2} \\ &- \frac{tC_1}{1+tt_1}\eta' Y^T - \frac{tC_2}{1+tt_2}\eta' Y^T - \frac{tC_1\eta'\nu_1 A_1^T}{(1+tt_1)^2} - \frac{tC_2\eta'\nu_2 A_2^T}{(1+tt_2)^2} + \ldots \end{aligned}$$

The absence of double poles at $t = -1/t_k$ give us

$$C_k\eta' A_k^T = 0. \tag{7.19}$$

Now, if we recall (7.8) we make the following ansatz for C_k

$$C_k = c_k a_k^T \eta - d_k b_k^T \eta \tag{7.20}$$

where c_k and d_k are constant vectors. The reason why there is a minus sign in front of the d_k term will become apparent below. Given that (7.8) are satisfied, (7.19) is satisfied as well. The absence of single poles at $t = -1/t_k$ give us

$$\frac{1}{t_k}C_k\eta'\mathcal{A}_k^T + \left(A_+(t,x) + \frac{tC_k}{1+tt_k}\right)\Bigg|_{t=-\frac{1}{t_k}}\eta'\nu_k A_k^T = 0. \tag{7.21}$$

If we insert the expression for A_k, A_+ and use (7.12) we get

$$\begin{aligned} &\frac{1}{t_k}(-\nu_k\alpha_k\gamma_k d_k a_k^T - \nu_k\beta_k\gamma_k c_k b_k^T) + \alpha_k\nu_k\eta'\eta^T a_k a_k^T + \beta_k\nu_k\eta'\eta^T b_k b_k^T \\ &+ \frac{1}{t_k - t_l}\Big(\alpha_k\nu_k c_l a_l^T\eta' a_k a_k^T + \beta_k\nu_k c_l a_l^T\eta' b_k b_k^T - \alpha_k\nu_k d_l b_l^T\eta' a_k a_k^T \\ &- \beta_k\nu_k d_l b_l^T\eta' b_k b_k^T\Big) = 0, \end{aligned} \tag{7.22}$$

where $l \neq k$. Here we have used $\eta\eta'\eta = \eta'$. This expression can be simplified if we assume that

$$\begin{aligned} a_l^T \eta' a_k &= 0 \\ b_l^T \eta' b_k &= 0 \end{aligned} \tag{7.23}$$

for $l \neq k$, i.e. a generalization of (7.8). In addition to the vanishing of the two terms in the parenthesis we can use the orthogonality of a_k and b_k to get two equations that need to be satisfied. By multiplying from the right with $\eta' a_l$ and $\eta' b_l$ we get that (7.22) is solved if

$$\begin{aligned} \left(-\frac{1}{t_k}\nu_k\alpha_k\gamma_k d_k + \alpha_k\nu_k\eta'\eta a_k + \frac{\alpha_k\nu_k}{t_k - t_l}\left(c_l a_l^T\eta' a_k - d_l b_l^T \eta' a_k\right)\right) a_k^T\eta' b_l = 0 \\ \left(-\frac{1}{t_k}\nu_k\beta_k\gamma_k c_k + \beta_k\nu_k\eta'\eta b_k + \frac{\beta_k\nu_k}{t_k - t_l}\left(c_l a_l^T\eta' b_k - d_l b_l^T \eta' b_k\right)\right) b_k^T\eta' a_l = 0. \end{aligned} \tag{7.24}$$

Thus, (7.24) becomes

$$\begin{aligned} \tilde{\eta} a_k &= \frac{1}{t_k}\gamma_k d_k + \frac{1}{t_k - t_l} d_l a_k^T \eta' b_l \\ \tilde{\eta} b_k &= \frac{1}{t_k}\gamma_k c_k - \frac{1}{t_k - t_l} c_l a_l^T \eta' b_k \end{aligned} \tag{7.25}$$

where we have defined $\tilde{\eta} = \eta'\eta$. We note here that both α_k and β_k have disappeared. This means that we have a freedom in choosing the vectors a_k and b_k which can be utilized to satisfy the constraint $\det \mathcal{M} = 1$. If we define a and b as matrices with a_k and b_k as column vectors, respectively, (7.25) can be written as matrix equations. If we define Γ as

$$\Gamma_{kl} = \begin{cases} \frac{\gamma_k}{t_k} & \text{for} \quad k = l \\ \frac{a_k^T \eta' b_l}{t_k - t_l} & \text{for} \quad k \neq l \end{cases} \tag{7.26}$$

we get that (7.25) is equivalent to

$$\begin{aligned} \tilde{\eta} a &= d\Gamma^T \\ \tilde{\eta} b &= c\Gamma. \end{aligned} \tag{7.27}$$

This result agrees with [11]. Here we have also defined the matrices c and d completely analogous to a and b. The reason why we had a minus sign in front of d_k becomes clear here as it allows us to use Γ and Γ^T. The linear system (7.27) should be considered as equations for d and c. Once they are known, we can form A_+. To get $M(x)$ we use the fact that $\mathcal{M}(t,x) \longrightarrow Y$ and $A_-(t,x) \longrightarrow \mathbb{1}$ as $t \longrightarrow \infty$

$$M(x) = Y A_+^{-1}(t \to \infty). \tag{7.28}$$

To summarize: we have a method to factorize monodromy matrices with single poles in w. The residues of the poles are in general of arbitrary rank but for Schwarzschild they are of rank 2. These residues are in turn parametrized by vectors which have to satisfy a set of constraints in order for $\mathcal{M}$ to be in the coset space. As a result, the only non-trivial part in the factorization method is to solve a set of linear equations where the vectors are assumed to be known. Thus, when we want to factorize the monodromy matrix $\mathcal{M}$ we need the vectors. We end this section by referring to an important result in [8]: the factorization of $\mathcal{M}$ is *unique*.

7.2 The Monodromy Matrix $\mathcal{M}$ for Schwarzschild

In this section we will apply the factorization method described above to the Schwarzschild metric. Once the monodromy matrix $\mathcal{M}$ is obtained it can serve as a seed solution to generate other solutions. At least to the best of the author's knowledge, the Schwarzschild monodromy matrix $\mathcal{M}$ for the case of G_2 has not been constructed elsewhere. One way to construct $\mathcal{M}$ is to solve the Lax-pair (5.23) to get $\mathcal{V}$. This approach is not recommended for two reasons: (i) the calculations tend to get very cumbersome. (ii) To use the method described above we need the vectors a_k and b_k which means that knowing $\mathcal{M}$ is not sufficient.

7.2.1 The Schwarzschild Metric in New Coordinates

As it turns out, the coordinates we used in 6.4 are not very practical when it comes to actually performing calculations. The main problem is that the matrix Y, introduced in the previous section, is not well defined since it diverges. In these coordinates the Schwarzschild metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{\mu}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{\mu}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2 + \cos^2(\theta)\mathrm{d}\psi^2\right) \tag{7.29}$$

were $\mu = 2m$. We would like to find a set of coordinates in which the limit $r \longrightarrow \infty$ yields a constant invertible matrix Y. If we make the following coordinate transformations, [21],

$$\begin{aligned} \phi_- &= \psi - \phi \\ \phi_+ &= \frac{\sqrt{\mu}}{2\sqrt{2}}(\psi + \phi), \end{aligned} \tag{7.30}$$

where $\mu \neq 0$, the Schwarzschild metric is given in terms of the coordinates $r, \theta, \phi_-, \phi_+, t$ as

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{\mu}{r^2}\right)\mathrm{d}t^2+\left(1-\frac{\mu}{r^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\theta^2+\frac{2r^2}{\mu}\mathrm{d}\phi_+^2+\frac{r^2}{4}\mathrm{d}\phi_-^2 \\ &+\sqrt{\frac{2}{\mu}}r^2\cos(2\theta)\mathrm{d}\phi_+\mathrm{d}\phi_-. \end{aligned} \tag{7.31}$$

We see that in these coordinates the Schwarzschild metric is not diagonal and consequently, not the monodromy matrix $\mathcal{M}$ either. However, this is not a big issue. Moreover, in these new coordinates Minkowski spacetime is not represented since it is not allowed to put $\mu = 0$. To find the coset element we start by identifying[4] the non-vanishing fields $\phi_1, \phi_2, ..., \mathcal{A}^2_\mu$ from 6.3

$$\begin{aligned} \mathrm{e}^{\frac{-2}{\sqrt{3}}\phi_1} &= 1-\frac{\mu}{r^2} \\ \mathrm{e}^{-\phi_2} &= \frac{2}{\mu}\sqrt{r^4-\mu r^2} \\ \mathcal{A}^2_{\mu=2} &= \frac{\sqrt{\mu}}{2\sqrt{2}}\cos(2\theta). \end{aligned} \tag{7.32}$$

We also need the three-dimensional metric $\mathrm{d}s_3^2$

$$\mathrm{d}s_3^2 = \frac{2}{\mu}\left(r^2-\mu\right)\left(\frac{r^2}{4}\sin^2(2\theta)\mathrm{d}\phi_-^2+\left(1-\frac{\mu}{r^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\theta^2\right). \tag{7.33}$$

From the dualization procedure we get that the only non-trivial equation is

$$\mathrm{e}^{-2\phi_2} \star_3 \mathrm{d}\mathcal{A}^2_{(1)} = \mathrm{d}\chi_6 \tag{7.34}$$

which in component form becomes

$$\mathrm{e}^{-2\phi_2}\sqrt{g}g^{\nu\sigma}g^{\rho\lambda}\tilde{\epsilon}_{\mu\sigma\lambda}\partial_{[\nu}\mathcal{A}^2_{\rho]} = \partial_\mu\chi_6. \tag{7.35}$$

The solution to this equation is given by

$$\chi_6 = -\frac{2r^2}{\mu}+b \tag{7.36}$$

[4]We reduce the t and ξ dimensions.

where b is a constant of integration which we set $b = 1$. It is straightforward to identify that $\chi_1 = \chi_5 = 0$. In these new coordinates the matrix M takes the form

$$M_S = \begin{pmatrix} \frac{\mu}{2r^2-2\mu} & 0 & 0 & 0 & 0 & \frac{\mu-2r^2}{r^2-\mu} & 0 \\ 0 & \frac{\mu}{2r^2} & 0 & 0 & 0 & 0 & \frac{\mu}{r^2}-2 \\ 0 & 0 & \frac{1}{1-\frac{\mu}{r^2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{\mu}{r^2} & 0 & 0 \\ \frac{2r^2-\mu}{4r^2-4\mu} & 0 & 0 & 0 & 0 & -\frac{\mu}{2r^2-2\mu} & 0 \\ 0 & \frac{1}{2}-\frac{\mu}{4r^2} & 0 & 0 & 0 & 0 & -\frac{\mu}{2r^2} \end{pmatrix}. \tag{7.37}$$

By expanding in $1/r$ we get that

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.38}$$

We will consider this matrix as the M matrix for the Minskowski solution $\mathbb{R}^{4,1}$ even though we do not have a metric representing flat space. Thus, every solution M, in these coordinates, with asymptotic Minkowski spacetime should satisfy

$$\lim_{r\to\infty} M = Y \tag{7.39}$$

for Y given above. In the next section we will generate the Reissner-Nordström metric from Schwarzschild. Since the Schwarzschild solution is now expressed in the $r, \theta, \phi_-, \phi_+, t$ coordinates all solutions generated from this seed solution will also be expressed in theses coordinates. Thus, we have to make the same coordinate transformation to the Reissner-Nordström solution presented in 6.4 in order to recognize it. In particular, we need the non-vanishing coset scalars. By comparing it with (6.3) we find

$$\begin{aligned} \mathrm{e}^{-\frac{2}{\sqrt{3}}\phi_1} &= \frac{r^2(r^2-2m)}{(r^2+2m\sinh^2(\delta))^2} \\ \mathrm{e}^{-\phi_2} &= \frac{r\sqrt{r^2-2m}}{m} \\ \chi_2 &= \frac{2\sqrt{3}m\sinh(\delta)\cosh(\delta)}{r^2+2m\sinh^2(\delta)} \\ \mathcal{A}^2_{\mu=2} &= \frac{\sqrt{m}\cos(2\theta)}{2}. \end{aligned} \tag{7.40}$$

7.2.2 Parametrization of $\mathcal{M}$

We start with the ansatz (7.2)

$$\mathcal{M}(w) = Y + \frac{A_1}{w - \mu/4} + \frac{A_2}{w + \mu/4}. \tag{7.41}$$

From the derived constraints (7.8) and (7.23), the results from section 5.3.3 and the embedding of the $SL(3,\mathbb{R})$ group, i.e. the vacuum truncation, it turns out that the vectors a_k and b_k should be

$$\begin{aligned} a_1 &= (1/2, 0, 1/\sqrt{2}, 0, 0, 1/4, 0)^T \\ a_2 &= (0, 1/2, 0, 0, 1/\sqrt{8}, 0, -1/4)^T \\ b_1 &= (-1/2, 0, 1/\sqrt{2}, 0, 0, -1/4, 0)^T \\ b_2 &= (0, -1/2, 0, 0, 1/\sqrt{8}, 0, 1/4)^T. \end{aligned} \tag{7.42}$$

In order to tune $\det \mathcal{M} = 1$ we choose

$$\begin{aligned} \alpha_1 &= \beta_1 = \mu \\ \alpha_2 &= \beta_2 = -\mu. \end{aligned} \tag{7.43}$$

This gives us the Schwarzschild monodromy matrix

$$\mathcal{M}_S = \begin{pmatrix} \frac{-\mu/4}{\left(w+\frac{\mu}{4}\right)} & 0 & 0 & 0 & 0 & \frac{\mu/2}{\left(w+\frac{\mu}{4}\right)} - 2 & 0 \\ 0 & \frac{-\mu/4}{\left(w-\frac{\mu}{4}\right)} & 0 & 0 & 0 & 0 & \frac{-\mu/2}{\left(w-\frac{\mu}{4}\right)} - 2 \\ 0 & 0 & 1 - \frac{\mu/2}{\left(w+\frac{\mu}{4}\right)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mu/2}{\left(w-\frac{\mu}{4}\right)} + 1 & 0 & 0 \\ \frac{1}{2} - \frac{\mu/8}{\left(w+\frac{\mu}{4}\right)} & 0 & 0 & 0 & 0 & \frac{\mu/4}{\left(w+\frac{\mu}{4}\right)} & 0 \\ 0 & \frac{\mu/8}{\left(w-\frac{\mu}{4}\right)} + \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\mu/4}{\left(w-\frac{\mu}{4}\right)} \end{pmatrix}. \tag{7.44}$$

To show that this matrix factorizes to (7.37) we apply the method described above, which should be straightforward once the vectors a_k and b_k are determined. Basically, the only non-trivial part in the calculation is to solve the linear system (7.27). The different constituents can be found in appendix D. The result is given by

$$M_S = \begin{pmatrix} A & 0 & 0 & 0 & 0 & D & 0 \\ 0 & B & 0 & 0 & 0 & 0 & E \\ 0 & 0 & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C^{-1} & 0 & 0 \\ -D/4 & 0 & 0 & 0 & 0 & -A & 0 \\ 0 & -E/4 & 0 & 0 & 0 & 0 & -B \end{pmatrix} \tag{7.45}$$

where

$$\begin{aligned}
A &= \frac{2\mu}{-2\mu + \sqrt{16\rho^2 + (\mu - 4z)^2} + \sqrt{16\rho^2 + (\mu + 4z)^2}} \\
B &= \frac{2\mu}{2\mu + \sqrt{16\rho^2 + (\mu - 4z)^2} + \sqrt{16\rho^2 + (\mu + 4z)^2}} \\
C &= \frac{\mu + \sqrt{16\rho^2 + (\mu + 4z)^2} + 4z}{-\mu + \sqrt{16\rho^2 + (\mu - 4z)^2} + 4z} \\
D &= -\frac{16z}{\sqrt{16\rho^2 + (\mu - 4z)^2} - \sqrt{16\rho^2 + (\mu + 4z)^2} + 8z} \\
E &= -\frac{\sqrt{16\rho^2 + (\mu - 4z)^2} + \sqrt{16\rho^2 + (\mu + 4z)^2} + 8z}{\mu + \sqrt{16\rho^2 + (\mu + 4z)^2} + 4z}.
\end{aligned} \tag{7.46}$$

In order to recognize this as the Schwarzschild solution we have to change coordinates to r and θ. From the canonical parametrization of the three-dimensional metric

$$g^{(3)} = \begin{pmatrix} f^2 g^{(2)} & 0 \\ 0 & \rho^2 \end{pmatrix} \tag{7.47}$$

we see, by comparing with (7.33), that

$$\rho(r, \theta) = \frac{r}{\sqrt{2\mu}} \sqrt{r^2 - \mu} \sin(2\theta). \tag{7.48}$$

To find $z(r, \theta)$ we use the two versions of the two-dimensional metric i.e. $\mathrm{d}s_2^2 = \mathrm{d}\rho^2 + \mathrm{d}z^2$ and g^2. These two metrics are obviously related through a coordinate transformation $(\rho, z) \leftrightarrow (r, \theta)$. Since $f^2 g^2$ is diagonal, g^2 is diagonal as well and from this observation $z(r, \theta)$ can be found. We have

$$\begin{aligned}
\mathrm{d}\rho^2 + \mathrm{d}z^2 = &\left(\frac{\partial z}{\partial r}\right)^2 \mathrm{d}r^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \mathrm{d}\theta^2 + 2\frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}\mathrm{d}\theta\mathrm{d}r + 2\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial \theta}\mathrm{d}\theta\mathrm{d}\mathrm{d}r \\
&+ \left(\frac{\partial \rho}{\partial r}\right)^2 \mathrm{d}r^2 + \left(\frac{\partial \rho}{\partial \theta}\right)^2 \mathrm{d}\theta^2
\end{aligned} \tag{7.49}$$

and the requirement that no cross terms exist implies

$$\frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = -\frac{\partial \rho}{\partial r}\frac{\partial \rho}{\partial \theta}. \tag{7.50}$$

The solution to this differential equation is given by

$$z(r,\theta) = \frac{1}{2\sqrt{2\mu}}\left(\frac{2r^2}{\mu} - 1\right)\cos(2\theta). \tag{7.51}$$

If we insert this into (7.45) we regain the matrix M_S. However, this step is not as straightforward as it might seem due to (7.48) and (7.51) . To circumvent this problem one can introduce the (x,y) coordinates as an intermediate step, [23],

$$x = \frac{2r^2}{\mu} - 1 \qquad \rho = \frac{1}{2\sqrt{2\mu}}\sqrt{(x^2-1)(1-y^2)} \tag{7.52}$$

$$y = \cos(2\theta) \qquad z = \frac{1}{2\sqrt{2\mu}}xy. \tag{7.53}$$

The θ dependence on both ρ and z implies the restriction $\rho \geq 0$. Otherwise we get that $\theta = 0$ and $\theta = \pi$ for a fixed r has the same ρ, z coordinate. Moreover, we note that the horizon $r = \sqrt{\mu}$ corresponds to $\rho = 0$.

7.3 Adding Electric Charge

Now that we have the monodromy matrix $\mathcal{M}_S$ for Schwarzschild we can use it as a seed solution. In this section we will demonstrate how to add electric charge to the Schwarzschild solution and thereby get the five-dimensional Reissner-Nordström solution. The monodromy matrix $\mathcal{M}$ transforms as

$$\mathcal{M}(w) \longrightarrow g^{\mathcal{T}}(w)\mathcal{M}(w)g(w) \tag{7.54}$$

where $g(w)$ is an element in G_2^+. If we restrict ourselves to the case when $g(w) \in SO(2,2)$ is independent of w we get the following transformation properties for the vectors a_k and b_k

$$\begin{aligned} a_k &\longrightarrow g^{-1}a_k \\ b_k &\longrightarrow g^{-1}b_k \\ a_k^{\mathcal{T}} &\longrightarrow a_k^{\mathcal{T}}g \\ b_k^{\mathcal{T}} &\longrightarrow b_k^{\mathcal{T}}g. \end{aligned} \tag{7.55}$$

These are obtained from the definitions of a_k and b_k. The next step is to face the question which group element g we should use to add electric charge. From previous analysis we know that this group element lies in $SO(2,2)$, i.e. H. Since the Reissner-Nordström metric has the same asymptotic behaviour as Schwarzschild we also require that

$$g^{-1}Yg = Y. \tag{7.56}$$

It should be emphasized that this is a requirement on the particular generator we are looking after. It is not a requirement on $g(w)$ in general. In [17], an analysis of the action of the generators in $\mathfrak{h}$ was made and as it turned out, k_2 generates electric charge. This means that

$$g = \exp(\sqrt{3}\delta k_2) \tag{7.57}$$

and one can confirm that this group element indeed satisfies (7.56). The prefactor is added for convenience. Using this group element the new vectors a'_k and b'_k become

$$a'_1 = \left(\frac{\cosh(\delta)}{2}, -\frac{\sinh(\delta)}{2}, \frac{\cosh^2(\delta)}{\sqrt{2}}, -\frac{\sinh(\delta)\cosh(\delta)}{\sqrt{2}}, \frac{\sinh^2(\delta)}{2\sqrt{2}}, \frac{\cosh(\delta)}{4}, -\frac{\sinh(\delta)}{4}\right)$$
$$a'_2 = \left(-\frac{\sinh(\delta)}{2}, \frac{\cosh(\delta)}{2}, \frac{\sinh^2(\delta)}{\sqrt{2}}, -\frac{\sinh(\delta)\cosh(\delta)}{\sqrt{2}}, \frac{\cosh^2(\delta)}{2\sqrt{2}}, \frac{\sinh(\delta)}{4}, -\frac{\cosh(\delta)}{4}\right)$$
$$b'_1 = \left(-\frac{\cosh(\delta)}{2}, \frac{\sinh(\delta)}{2}, \frac{\cosh^2(\delta)}{\sqrt{2}}, -\frac{\sinh(\delta)\cosh(\delta)}{\sqrt{2}}, \frac{\sinh^2(\delta)}{2\sqrt{2}}, -\frac{\cosh(\delta)}{4}, \frac{\sinh(\delta)}{4}\right)$$
$$b'_2 = \left(\frac{\sinh(\delta)}{2}, -\frac{\cosh(\delta)}{2}, \frac{\sinh^2(\delta)}{\sqrt{2}}, -\frac{\sinh(\delta)\cosh(\delta)}{\sqrt{2}}, \frac{\cosh^2(\delta)}{2\sqrt{2}}, -\frac{\sinh(\delta)}{4}, \frac{\cosh(\delta)}{4}\right)$$

and accordingly the monodromy matrix $\mathcal{M}_{RN}$

$$\mathcal{M}_{RN} = \begin{pmatrix}
\frac{\mu c_-}{\mu^2-16w^2} & -\frac{\mu^2 s(2\delta)}{\mu^2-16w^2} & 0 & 0 & 0 & \frac{-8wc_-}{16w^2-\mu^2} & \frac{8w\mu s(2\delta)}{16w^2-\mu^2} \\
\frac{\mu^2 s(2\delta)}{\mu^2-16w^2} & \frac{\mu c_+}{\mu^2-16w^2} & 0 & 0 & 0 & -\frac{8w\mu s(2\delta)}{16w^2-\mu^2} & -\frac{8wc_+}{16w^2-\mu^2} \\
0 & 0 & \frac{(c_-)^2}{16w^2-\mu^2} & \frac{\mu s_-}{\mu^2-16w^2} & -\frac{2\mu^2 s^2(2\delta)}{\mu^2-16w^2} & 0 & 0 \\
0 & 0 & \frac{-\mu c_- s(2\delta)}{\mu^2-16w^2} & \frac{16w^2-\mu^2 c(4\delta)}{16w^2-\mu^2} & \frac{\mu s_+}{\mu^2-16w^2} & 0 & 0 \\
0 & 0 & \frac{\mu^2 s^2(2\delta)}{32w^2-2\mu^2} & -\frac{\mu c_+ s(2\delta)}{\mu^2-16w^2} & \frac{(c_+)^2}{16w^2-\mu^2} & 0 & 0 \\
\frac{2wc_-}{16w^2-\mu^2} & -\frac{2w\mu s(2\delta)}{16w^2-\mu^2} & 0 & 0 & 0 & \frac{-\mu c_-}{\mu^2-16w^2} & \frac{\mu^2 s(2\delta)}{\mu^2-16w^2} \\
\frac{2w\mu s(2\delta)}{16w^2-\mu^2} & \frac{2wc_+}{16w^2-\mu^2} & 0 & 0 & 0 & -\frac{\mu^2 s(2\delta)}{\mu^2-16w^2} & -\frac{\mu c_+}{\mu^2-16w^2}
\end{pmatrix}$$

where $s(2\delta) = \sinh(2\delta)$ and

$$c_- = 4w - \mu\cosh(2\delta) \qquad s_- = 8w\sinh(2\delta) - \mu\sinh(4\delta)$$
$$c_+ = 4w + \mu\cosh(2\delta) \qquad s_+ = 8w\sinh(2\delta) + \mu\sinh(4\delta).$$

The constants α_k and β_k remain the same. The factorization works as before and one finds that the M_{RN} matrix becomes

$$M_{RN} = \begin{pmatrix} \frac{\mu(r^2+s^2\mu)}{2r^2(r^2-\mu)} & -\frac{cs\mu^2}{2r^4-2r^2\mu} & 0 & 0 & 0 \\ \frac{cs\mu^2}{2r^4-2r^2\mu} & \frac{\mu a}{2r^2(r^2-\mu)(r^2+s^2\mu)} & 0 & 0 & 0 \\ 0 & 0 & \frac{(r^2+s^2\mu)^2}{r^4-r^2\mu} & -\frac{2cs\mu(r^2+s^2\mu)}{r^4-r^2\mu} & \frac{2c^2s^2\mu^2}{r^4-r^2\mu} \\ 0 & 0 & \frac{cs\mu(r^2+s^2\mu)}{r^4-r^2\mu} & 1-\frac{2c^2s^2\mu^2}{r^4-r^2\mu} & \frac{-2cs\mu a}{r^2(r^2-\mu)(r^2+s^2\mu)} \\ 0 & 0 & \frac{c^2s^2\mu^2}{2r^4-2r^2\mu} & \frac{cs\mu a}{r^2(r^2-\mu)(r^2+s^2\mu)} & \frac{a^2}{(r^2-\mu)(r^3+s^2\mu r)^2} \\ \frac{(2r^2-\mu)(r^2+s^2\mu)}{4r^2(r^2-\mu)} & -\frac{cs(2r^2-\mu)\mu}{4r^2(r^2-\mu)} & 0 & 0 & 0 \\ \frac{cs(2r^2-\mu)\mu}{4r^2(r^2-\mu)} & \frac{(2r^2-\mu)a}{4r^2(r^2-\mu)(r^2+s^2\mu)} & 0 & 0 & 0 \end{pmatrix}$$

where

$$a = r^4 - \mu r^2 - c^2s^2\mu^2$$

and $s = \sinh(\delta)$ and $c = \cosh(\delta)$. From this matrix we can read off the ϕ_1, ϕ_2, χ_2 and χ_6 components by comparing it with (D.1). The $\mathcal{A}^2_{(1)}$ component is obtained by inverse dualization, (7.35), which in general requires that we know the three-dimensional metric $\mathrm{d}s_3^2$. This metric is obtained once we have solved the equation for the conformal factor, (5.4). However, in this case we know from section 3.4.2 that the three-dimensional metric is the same for Reissner-Nordström as for Schwarzschild since we have transformed with the finite group G. Since χ_6 is the same as for the Schwarzschild solution, $\mathcal{A}^2_{(1)}$ is the same as well. In this example and the previous one we have reintroduced the r, θ coordinates. The reason for this is to make it easier to recognize the solutions as the Schwarzschild and Reissner-Nordström metric. However, when we generate new, previously unknown, solutions this is of course not possible since it requires that we know the explicit metric in the r, θ coordinates.

It is really worth noticing how relatively easy[5] this solution was obtained in the sense of the actual calculation but also compared to if we would have started with the Einstein-Maxwell action and solved the equations. This shows the entire purpose of using symmetries; almost no equations need to be solved.

[5] Once the seed solution is known.

8

Conclusions

In this thesis, solution generating techniques have been analysed and applied to various examples for the case of minimal supergravity in $D = 5$ and the group $G_{2(2)}$. The motivation for this has been to broaden the uses of previously known techniques which have been proven successful. We have seen that the techniques and methods are quite general and easily applied to many groups. However, the method relies on representation theory which causes some practical difficulties. The key challenges we have faced are:

- How should the metric be parametrized?
- What is the symmetry group of the theory?
- How should we parametrize the coset space?
- What should we use as seed solution and how do we find the explicit expressions?
- What is the correct group element to use in the transformation?
- How do we extract the new fields after a transformation?

Many of these questions have been taken care of by previous authors. Nonetheless, we have been quite cautious since every group requires some adjustments. In $D = 2$ we have seen that it is not that easy to find the seed solution. One should be able to start with Minkowski spacetime, however, we chose to start with Schwarzschild. The reason for this was because it made it easier to generate the Reissner-Nordström metric. We also chose Schwarzschild as seed solution to illustrate the factorization procedure of $\mathcal{M}$. From our analysis of the factorization we found one problematic step; how do we find a_k and b_k? For the case of pure gravity when $SL(2,\mathbb{R})/SO(2)$ is the coset space one can start with arbitrary a_k and b_k. Since these vectors only have two components each one can, by tuning them, satisfy the coset constraints. When the number of components increases, e.g. for $G_{2(2)}$, this is no longer practicable.

A lot of effort has been put into trying to understand the group theoretical nature of these vectors. One approach has been to embed the vacuum truncation in the group and see which further restrictions we get on the coset element and consequently the vectors a_k, b_k. Moreover, we have used some results in [9] about the analytical properties of $\mathcal{M}$ on the z-axis, i.e. for $\rho = 0$, for further guidance. Unfortunately, a completely satisfying understanding of the nature of these vectors is yet to be found.

In future studies, one can perhaps exploit the gauge symmetries related to the generators of the Borel algebra and reduce the number of degrees of freedom in a_k and b_k. As we have seen, these transformations do not generate new physical solutions but instead act as gauge transformations. We used that $G_{2(2)}$ should preserve a matrix η' since it is a subgroup of $SO(3,4)$. However, this does not completely ensure that we get an element in $G_{2(2)}$. If one can find further restrictions which guarantees that we get an element in $G_{2(2)}$, it would probably simplify the process considerably. The most promising approach would be to use that elements in $G_{2(2)}$ also preserve a three-form[1] which would give us further restrictions on a_k and b_k, [23]. That being said, one of the main results of this thesis is that the monodromy matrix $\mathcal{M}$ for Schwarzschild has been constructed and can thus be used as a seed solution.

Another difficulty which arises is the inverse dualization procedure. Even for the most simple solutions, e.g. Myers-Perry in $D = 5$ we get integrals which are very difficult to solve. It was desired to find the monodromy matrix for Myers-Perry but due to this difficulty and the time limit of this thesis, the Reissner-Nordström monodromy matrix was constructed instead. Probably, the solution to this amounts to finding appropriate coordinates. In [31], the Myers-Perry monodromy matrix was obtained but in the setting of five-dimensional pure gravity in which $SL(3,\mathbb{R})/SO(2,1)$ is the coset space. This coset space is much simpler to work with for two reasons; (i) the matrices are only 3×3 (ii) the residues are only of rank 1.

When we added electric charge to the Schwarzschild solution we only needed the "usual" finite group element $g \in SO(2,2)$. A natural continuation would be to con-

[1] I am grateful to Amitabh Virmani for sharing his ideas with me.

sider transformation from the full affine Kac-Moody group. This would allow one to start from Minkowski spacetime and add poles to get for example Schwarzschild.

A remarkable property of five-dimensional gravity is the existence of *black ring solutions* [32]. These are black holes but with a non-spherical horizon. Another interesting continuation of this thesis would be to generate the six-parameter black ring solution [33].

For $N = 1$ supergravity in $D = 11$ the symmetry group in three-dimensions is $E_{8(8)}$ and the coset space is $E_{8(8)}/SO(16)$ [9]. In this thesis a small step towards this "final theory" has been taken and will hopefully be a source of inspiration for future studies.

A

The Vielbein Formalism

In this appendix we present a brief introduction to vielbeins following [34]. Consider a manifold with tangent space T_p at a point p. A natural basis, $\hat{e}_\mu$, for this tangent space is given by the partial derivatives of the coordinates we are using. That is, $\hat{e}_\mu = \partial_\mu$. For the cotangent space T_p^* the basis is given by the gradient, $\hat{e}^\mu = \mathrm{d}x^\mu$.

As always, we have the freedom to choose any basis we like, although some are far more appropriate than others. Consider a new set of basis vectors $\hat{e}_a$ at a spacetime point p which satisfy the following condition

$$g(\hat{e}_a, \hat{e}_b) = \eta_{ab} \tag{A.1}$$

where g is the spacetime metric and η_{ab} is the metric representing the signature of the spacetime. That is η_{ab} is Minkowski for Lorentzian spacetime and the identity for Euclidean. The old basis $\hat{e}_\mu$ can be expressed in the new basis $\hat{e}_a$ as

$$\hat{e}_\mu = e_\mu{}^a \hat{e}_a. \tag{A.2}$$

The object $e_\mu{}^a$, which is the components of the vector $\hat{e}_\mu$ in the basis $\hat{e}_a$, is called *vielbein*. We define their inverse $e^\mu{}_a$ as

$$\hat{e}_a = e^\mu{}_a \hat{e}_\mu \tag{A.3}$$

which obviously satisfies

$$e^\mu{}_a e_\nu{}^a = \delta^\mu{}_\nu. \tag{A.4}$$

(A.1) can thus be written as

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab} \tag{A.5}$$

which tells us that η_{ab} is the metric $g_{\mu\nu}$ expressed in the $\hat{e}_a$ basis. By multiplying with $e_\sigma{}^a e_\lambda{}^b$ we get

$$g_{\sigma\lambda} = e_\sigma{}^a e_\lambda{}^b \eta_{ab}. \tag{A.6}$$

The vielbeins are used to switch from curved indices $\mu, \nu, ...$ to flat indices $a, b, ...$. For example, a tensor V can be expressed as

$$V = V^{ab} \hat{e}_a \otimes \hat{e}_b = V^{ab} e_\mu{}^a e_\nu{}^b \hat{e}^\mu \otimes \hat{e}^\nu = V^{\mu\nu} \hat{e}^\mu \otimes \hat{e}^\nu. \tag{A.7}$$

When we use ∂_μ as basis we get an indued transformation of the components of a tensor which we use to define the tensor. However, the $\hat{e}_a$ basis is independent of the coordinates and can therefore be transformed without having to do a coordinate transformation. Since (A.1) defines the vielbeins we restrict ourselfs to transformations which preserve this condition, i.e. Lorentz or Euclidean transformations depending on the signature of η. These transformations are local[1] which leads us to define the transformation of $\hat{e}_a$ as

$$\hat{e}'_a = \Lambda(x)^b{}_a \hat{e}_b. \tag{A.8}$$

A tensor V can be expressed both in the vielbein basis and the coordinate basis as

$$V = V^{a\mu} \hat{e}_a \otimes \hat{e}_\mu \tag{A.9}$$

and transforms by a mixed transformation as

$$(V')^{a\mu} = \Lambda(x)^a{}_b \frac{\partial (x')^\mu}{\partial x^\nu} V^{b\nu}. \tag{A.10}$$

A tensor expressed in the coordinate basis has a covariant derivative acting on the components with an ordinary partial derivative plus correction terms. For example,

$$\nabla_\mu V^{\nu\sigma} = \partial_\mu V^{\nu\sigma} + \Gamma^\nu{}_{\mu\rho} V^{\rho\sigma} + \Gamma^\sigma{}_{\mu\rho} V^{\rho\nu}. \tag{A.11}$$

For a tensor expressed in the vielbein basis we get something similar. For example, for V^{ab} the covariant derivative becomes

$$\nabla_\mu V^{ab} = \partial_\mu V^{ab} + \omega_\mu{}^b{}_c V^{ac} + \omega_\mu{}^a{}_c V^{cb}. \tag{A.12}$$

That is, $\omega_\mu{}^a{}_c$ serves as connections in the vielbein basis. This object is called *the spin connection*. By calculating ∇V both in the coordinate and the vielbein basis and then equate we get an expression relating the ordinary connection Γ with ω

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a. \tag{A.13}$$

From the first term, our conception of ω as Γ expressed in the vielbein basis gets further support. However, we have to keep in mind that these objects are not tensor and do not transform accordingly.

[1] Since there is a set of basis vectors $\hat{e}_a$ defined at every spacetime point p.

B

Kac-Moody Algebras

The following appendix is a brief introduction to Kac-Moody algebras and in particular, affine Kac-Moody algebras. The content is based on [14, 26, 35].

B.1 Chevalley-Serre Presentation

We take (e_i, h_i, f_i) to be a triple of $\mathfrak{sl}(2,\mathbb{R})$ generators , i.e.

$$[h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i, \qquad [e_i, f_i] = h_i. \tag{B.1}$$

Consider the complex *Kac-Moody algebra* $\mathfrak{g}$ constructed from r copies of the $\mathfrak{sl}(2,\mathbb{R})$ tripple, i.e. $i = 1, 2, ..., r$, subjected to the following conditions:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, \\ [h_i, e_j] &= A_{ji} e_j, \\ [h_i, f_j] &= -A_{ji} f_j, \\ [h_i, h_j] &= 0. \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} \mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) &= [e_i, [e_i, ...[e_i, e_j]...]] = 0 \\ \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) &= [f_i, [f_i, ...[f_i, f_j]...]] = 0. \end{aligned} \tag{B.3}$$

The matrix A_{ij} is called the generalized *Cartan matrix*. This matrix defines the algebra which suggests that it is more appropriate to denote the algebra $\mathfrak{g}$ as $\mathfrak{g}(A)$. Given the *Chevalley generators* e_i, h_i, f_i we can construct more generators by commutation. The two conditions (B.3) are called the *Serre relations* which are added to ensure that no ideal are present in the algebra $\mathfrak{g}(A)$ thus making it a simple algebra. It might seem that the Serre relations "cut the chain" of commutations and restricting the algebra to be finite dimensional. This is not true and the algebra $\mathfrak{g}(A)$ might as well be finite as infinite, given the information above. What determines whether the algebra is of infinite or finite dimension is the generalized Cartan matrix A so let us discuss it a bit further. A generalized Cartan matrix A_{ij} satisfies the following conditions, [26],

$$A_{ii} = 2, \quad i = 1, 2, ..., r \tag{B.4}$$
$$A_{ij} = 0 \Leftrightarrow A_{ji} = 0 \tag{B.5}$$
$$A_{ij} \in \mathbb{Z}_- \quad (i \neq j). \tag{B.6}$$

Furthermore, we will also assume that A is indecomposable. This means that we cannot divide the index set $(1,2, ..., r)$ into two non-empty sets $\mathcal{I}$ and $\mathcal{J}$ such that $A_{ij} = 0$ for $i \in \mathcal{I}$ and $j \in \mathcal{J}$ [14]. The crucial propery of A that determines whether the algebra $\mathfrak{g}(A)$ is finite or infinite-dimensional is the sign of the determinant. There are three different classes:

(i) $\mathfrak{g}(A)$ is finite-dimensional if A is positive definite, i.e. only positive eigenvalues. Examples of algebras of this type are the well-known A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7 and E_8 algebras.

(ii) $\mathfrak{g}(A)$ is infinite-dimensional if A is positive-semidefinite. That is, $\det A \geq 0$ and has only one zero eigenvalue. These algebras are called[1] *affine Kac-Moody algebras*.

(iii) $\mathfrak{g}(A)$ is called an indefinite Kac-Moody algebra if none of the above conditions on A is satisfied.

In the continuation of this appendix we will focus on the second case, i.e. affine Kac-Moody algebras. There exists a very natural decomposition of the affine Kac-Moody algebra $\mathfrak{g}(A)$ called *the triangular decomposition* given by

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_+ \tag{B.7}$$

where

$$\begin{aligned} \mathfrak{n}_- &= \mathrm{Span}_{\mathbb{R}}\{f_1, f_2, ...\}, \\ \mathfrak{n}_0 &= \mathrm{Span}_{\mathbb{R}}\{h_1, h_2, ..., h_r\}, \\ \mathfrak{n}_+ &= \mathrm{Span}_{\mathbb{R}}\{e_1, e_2, ...\}. \end{aligned} \tag{B.8}$$

[1]Actually, this is only the *derived* algebra, not the full affine Kac-Moody algebra. We will elaborate on this in the next sections.

where the sum is a direct sum of vector spaces. The term "triangular" becomes obvious here since e_i are upper triangular matrices and f_i are lower triangular matrices. The $\pm$ signs will be explained in the next section. The subalgebra $\mathfrak{n}_0$ is called the *Cartan subalgebra* which forms an abelian subalgebra of $\mathfrak{g}(A)$.

B.2 The Root Space

In this section we will introduce the concept of *root* and *root space*. Consider the adjoint action of a Cartan element h on the Chevalley generators e_i and f_i,

$$\begin{aligned} \mathrm{ad}_h(e_i) &= [h, e_i] = \alpha_i(h)e_i, \\ \mathrm{ad}_h(f_i) &= [h, f_i] = -\alpha_i(h)f_i, \end{aligned} \tag{B.9}$$

for $i = 1, 2,...,r$. The eigenvalue $\alpha_i(h)$ is considered as the value of a linear map, [14],

$$\alpha_i : \mathfrak{n}_0 \ni h \mapsto \alpha_i(h) \in \mathbb{R} \tag{B.10}$$

Furthermore, the functional α_i is called a *simple root* and belongs to the dual space of $\mathfrak{n}_0$, denoted $\mathfrak{n}_0^\star$. We emphesize here that the simple roots α_i corresponds to the Chevalley generators $e_1, e_2,...,e_r$. The Cartan elements h_i can also be referred to as *coroots*, denoted $\alpha_i^\vee$. The eigenvalue relations (B.9) can then be expressed as

$$A_{ij} = \langle \alpha_i | \alpha_j^\vee \rangle = \alpha_i(\alpha_j^\vee). \tag{B.11}$$

For the multiple commutator elements $[e_i, e_j]$ it is straightforward to verify that

$$[h, [e_i, e_j]] = (\alpha_i + \alpha_j)[e_i, e_j]. \tag{B.12}$$

That is, $[e_i, e_j]$ is also an eigenvector to h with eigenvalue $(\alpha_i + \alpha_j)$. If the Serre-relations allow $[e_i, e_j] \neq 0$, this is an element in $\mathfrak{n}_+$. Let $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ be the set of simple roots. This set forms a basis in the dual space $\mathfrak{n}_0$ since every root can be expressed as a linear combination of these. That is,

$$\mathfrak{n}_0^\star = \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \alpha_2, ..., \alpha_r\}. \tag{B.13}$$

The complete set of roots, *the root system*, Φ can be divided into a *positive* and a *negative* part,

$$\Phi = \Phi_+ \cup \Phi_-. \tag{B.14}$$

A root vector α,

$$\alpha = \sum_{i=1}^{r} m_i \alpha_i. \tag{B.15}$$

is called a *positive root* if $m_i \geq 0$ for all i and a *negative root* if $m_i \leq 0$ for all i. Thus, a positive root belongs to Φ_+ and a negative root to Φ_-. We can now decompose $\mathfrak{g}(A)$ into a so called *root space decomposition* given by

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{B.16}$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) : [h, x] = \alpha(h)x, \forall h \in \mathfrak{n}_0\}. \tag{B.17}$$

B.3 The Bilinear Form

In this section we will introduce an invariant bilinear form which we will see is an important object in the construction of affine Kac-Moody algebras. Let us assume that the Cartan matrix A is *non-degenerate* and *symmetrizable*. A non-degenerate Cartan matrix A has det $\neq 0$ and is indecomposable. This restriction will be abandoned later on. A symmetrizable matrix can be factorized as

$$A = DS \tag{B.18}$$

where S is a symmetric $r \times r$ matrix and $D = \mathrm{diag}(\epsilon_1, ..., \epsilon_r)$ with all $\epsilon_i \geq 0$. By using the matrix S we can define a bilinear form $(\cdot|\cdot)$ on $\mathfrak{n}_0^\star$ as , [14],

$$(\alpha_i|\alpha_j) = S_{ij}. \tag{B.19}$$

We note here that $\alpha_i, \alpha_j \in \Pi$. The next step is to find a bilinear form on the Kac-Moody algebra. Let the mapping μ

$$\mu : \mathfrak{n}_0^\star \longrightarrow \mathfrak{n}_0 \tag{B.20}$$

be defined as, [26],

$$\langle \alpha, \mu(\beta) \rangle = (\alpha|\beta), \qquad \beta, \alpha \in \mathfrak{n}_0^\star, \mu(\beta) \in \mathfrak{n}_0. \tag{B.21}$$

The inverse map μ^{-1} can then be used to define a bilinear form on the Cartan subalgebra $\mathfrak{n}_0$ as

$$(\alpha^\vee|\beta^\vee) = \langle \mu^{-1}(\alpha^\vee), \beta^\vee \rangle \qquad \alpha^\vee, \beta^\vee \in \mathfrak{n}_0, \mu(\alpha^\vee) \in \mathfrak{n}_0^\star. \tag{B.22}$$

One can derive an explicit relation between the bilinear form on $\mathfrak{n}_0$ and $\mathfrak{n}_0^\star$, e.g. [14]. The result is given by

$$(\alpha_i^\vee|\beta_j^\vee) = \epsilon_i \epsilon_j (\alpha_i|\alpha_j). \tag{B.23}$$

This non-degenerate bilinear form $(\cdot|\cdot)$ can be extended the entire algebra $\mathfrak{g}(A)$ with the following properties, [26]:

(i) $(\cdot|\cdot)$ is invariant, i.e. $(x|[y,z]) = ([x,y]|z)$ for all $x, y, z \in \mathfrak{g}(A)$

(ii) $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$

By using the invariance property one can show that $(e_i|f_j) = \epsilon_i \delta_{ij}$.

B.4 Affine Kac-Moody Algebras

We discussed in section B.1 how different classes of Kac-Moody algebras depends on the Cartan matrix A. We recall that an *affine Kac-Moody algebra* was characterized by the fact that det $(A) = 0$ and the existence of one zero eigenvalue. Because of these conditions, an affine Kac-Moody algebras has some interesting properties not shared with the other classes. Let us define the *center* of a Kac-Moody algebra. The center Z of a Kac-Moody algebra $\mathfrak{g}(A)$ is defined as

$$Z = \{x \in \mathfrak{g}(A) | [x, y] = 0, \forall y \in \mathfrak{g}(A)\}. \tag{B.24}$$

For the class of Kac-Moody algebras where the Cartan matrix is positive definite, the center $Z = 0$. Otherwise it would not be a simple algebra. Actually, $Z \neq 0$ if and only if det $A = 0$ [26]. For affine Kac-Moody algebras where we only have one zero eigenvalue we have that the Cartan matrix has rank $r - 1$. According to [14], we have that

$$\dim Z = r - n \tag{B.25}$$

where n is the rank of the Cartan matrix. Thus for an affine Kac-Moody algebra we have that dim $Z = 1$ suggesting that there only exists one central element $c \in Z$,

$$Z = \mathbb{R}c. \tag{B.26}$$

It can be shown that the central element c is in the Cartan subalgebra $\mathfrak{n}_0$, [14]. The fact that there exists a non-trivial center Z for affine implies that affine Kac-Moody algebras are not simple. Before we state a proper definition of an affine Kac-Moody algebra we have to introduce the *derivation* d. The reason for this is that the algebra we have constructed from the Cartan matrix A is only the *derived* algebra. Let us continue to call the affine Kac-Moody algebra $\mathfrak{g}(A)$ and the derived algebra $\mathfrak{g}'(A)$,

$$\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]. \tag{B.27}$$

That is, we only get the derived algebra when we commute elements in the affine Kac-Moody algebra. Since the Cartan matrix A is degenerate the bilinear form is also degenerate. As we will see, the derivation d is introduced to fix this. One may think of a rank $r = k+1$ affine Kac-Moody algebra $\mathfrak{g}(A)$ as an extension of a finite-dimensional rank k algebra $\bar{\mathfrak{g}}$ [14]. The extension amounts to adding a node to the Dynkin diagram. Let us denote the set of simple roots for $\bar{\mathfrak{g}}$ as

$$\bar{\Pi} = \{\alpha_1, \alpha_2, ..., \alpha_k\} \tag{B.28}$$

and the set of simple roots for $\mathfrak{g}(A)$ as

$$\Pi = \{\alpha_0, \alpha_1, ..., \alpha_k\}. \tag{B.29}$$

The root α_0 is called the *affine root* and is always of the form, [14],

$$\alpha_0 = \delta - \theta. \tag{B.30}$$

Here, θ is the highest root of $\bar{\mathfrak{g}}$ and δ is a so called null root since it satisfies $(\delta|\delta) = 0$. The fact that there exists a null root is a consequence of $\det A = 0$. We simply extend our algebra $\mathfrak{g}'(A)$ by adding the derivation d as

$$\langle \alpha_i, d\rangle = \delta_{i0}, \qquad i = 1, 2, ..., k \tag{B.31}$$

The derivation d is added as an element in the Cartan subalgebra $\mathfrak{n}_0$, thus making $\mathfrak{n}_0$ a $(k+2)$-dimensional subalgebra. That is,

$$\Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, ..., \alpha_k^\vee, d\}. \tag{B.32}$$

Moreover, we now have the following results, [14],

$$(c|\alpha_i^\vee) = 0 \qquad\qquad (c|c) = 0 \qquad\qquad (c|d) = 1. \tag{B.33}$$

We see that last result ensures that the bilinear form becomes non-degenerate. Due to (B.31) we also find that d will never appear on the right hand side of any commutator [14]. This explains (B.27). We end this section by summarizing the affine Kac-Moody algebra $\mathfrak{g}(A)$ as

$$\mathfrak{g}(A) = \mathfrak{g}' \oplus \mathbb{R}d. \tag{B.34}$$

B.5 Loop-Extension

This section is based on [35]. Affine Kac-Moody algebras can be realized through *loop algebras* which are of great interest in this thesis. Let us begin by defining a loop algebra: a loop algebra $L\bar{\mathfrak{g}}$ is the set of analytical maps from S^1 to a simple

Lie algebra $\bar{\mathfrak{g}}$. Let $\{T^a|a = 1, 2,...,d\}$ be a basis for the rank r algebra $\bar{\mathfrak{g}}$. The elements in the loop algebra $L\bar{\mathfrak{g}}$ are then given by

$$T^a_n = T^a \otimes z^m \tag{B.35}$$

where z is the complex coordinate on S^1 and m an integer. The Lie bracket on this space is naturally generalized to

$$[T^a_m, T^b_n] = [T^a, T^b] \otimes z^{m+n}, \tag{B.36}$$

or in terms of the structure constants $[T^a, T^b] = f^{ab}_c T^c$

$$[T^a_m, T^b_n] = f^{ab}_c T^c_{m+n}. \tag{B.37}$$

This algebra may be extended by introducing an additional element c, the central element. The commutator becomes

$$\begin{aligned} [T^a_m, T^b_n] &= f^{ab}_c T^c_{m+n} + m\delta^{m+n,0} B(T^a, T^b)c \\ [T^a_m, c] &= 0, \end{aligned} \tag{B.38}$$

where B is the symmetric invariant bilinear form on $\bar{\mathfrak{g}}$. This procedure is often refered to as *central extension*. We continue by enlarging the algebra even further by introducing the derivation d satisfying the following commutation relation:

$$\begin{aligned} [d, T^a_m] &= mT^a_m \\ [d, c] &= 0 \end{aligned} \tag{B.39}$$

for all T^a_m. We see that d can be regarded as a "counting operator" with respect to the mode number m. The final Lie algebra, denoted $\mathfrak{g}$, will be an untwisted affine Kac-Moody algebra. We summarize,

$$\mathfrak{g} = \mathbb{C}c \oplus \mathbb{C}d \bigoplus_{m=-\infty}^{\infty} \mathbb{C}(z^m \otimes \bar{\mathfrak{g}}). \tag{B.40}$$

For our new algebra $\mathfrak{g}$ we can find a maximal commuting set,

$$\{c, d, \bar{h}_i \otimes z^0 | i = 1,2,...r\}, \tag{B.41}$$

where $\bar{h}_i$ is a Cartan element in $\bar{\mathfrak{g}}$. Thus, the extended Cartan subalgebra for $\mathfrak{g}$ is $(r + 2)$-dimensional. We note that $\bar{h}_i \otimes z^k$ for $k \neq 0$ is not an element in the Cartan subalgebra due to the commutating relations (B.38),

$$[\bar{h}_i \otimes z^k, \bar{h}_j \otimes z^l] = k\delta^{k+l,0} B(\bar{h}_i, \bar{h}_j)c \tag{B.42}$$

which in general is non-vanishing. Let us denote this new Cartan subalgebra $\mathfrak{n}_0$,

$$\mathfrak{n}_0 = \mathbb{C}c \oplus \mathbb{C}d \bigoplus_{i=1}^{r} \mathbb{C}(\bar{h}_i \otimes z^0). \tag{B.43}$$

The roots $\bar{\alpha}_k \in \bar{\Pi}$ associated to $\bar{\mathfrak{g}}$ can be extended to become linear operators α on $\mathfrak{n}_0$, i.e. elements in $\mathfrak{n}_0^\star$, by the definitions:

$$\begin{aligned} \alpha(\bar{h}_i \otimes z^0) &= \bar{\alpha}(\bar{h}_i), i = 1,2,...,r \\ \alpha(c) &= \alpha(d) = 0. \end{aligned} \tag{B.44}$$

Now, let us introduce a new root δ on $\mathfrak{n}_0^\star$,

$$\begin{aligned} \delta(\bar{h}_i \otimes z^0) &= 0 \\ \delta(c) &= 0 \\ \delta(d) &= 1. \end{aligned} \tag{B.45}$$

By a straightforward calculation we find,

$$\begin{aligned} {[h, \bar{e}_j \otimes z^m]} &= \bar{\alpha}_j(\bar{h}_i)\bar{e}_j \otimes z^m = \alpha_j(h_i)\bar{e}_j \otimes z^m \\ {[d, \bar{e}_j \otimes z^m]} &= m(\bar{e}_j \otimes z^m), \end{aligned} \tag{B.46}$$

where $e_j \in \bar{\mathfrak{n}}_+$. These commutation relations can be combined as

$$[h, \bar{e}_j \otimes z^m] = \Big(m\delta(h) + \alpha(h)\Big)\bar{e}_j \otimes z^m. \tag{B.47}$$

This tells us that the root $m\delta + \alpha$ corresponds to the element $\bar{e}_j \otimes z^m$. Moreover,

$$[h, \bar{h}_i \otimes z^m] = m\delta(h)\bar{h}_i \otimes z^m. \tag{B.48}$$

From this we see that $\bar{h}_i \otimes z^m$ is not a Cartan element but rather a generator with the associate root $m\delta$. Here we note that there is a degeneracy. Since there are r linearly independent $\bar{h}_i$ we have a finite degeneracy, i.e. the root space corresponding to $m\delta$ is r-dimensional.

B.6 Examples

In this section we present two examples of affine Kac-Moody algebras that appear in this thesis. For more examples, see [35].

B.6.1 A_1^+

We have already encountered this affine Kac-Moody group when we intertwined the Ehler's $SL(2,\mathbb{R})$ with the Matzner-Misner $SL(2,\mathbb{R})$ but we will present it here again with some further details. For A_1^+ the corresponding simple Lie algebra $\bar{\mathfrak{g}}$ is the three-dimensional $\mathfrak{sl}(2,\mathbb{R})$, also known as A_1. The Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{B.49}$$

and the δ root is given by

$$\delta = \alpha_0 + \alpha_1 \tag{B.50}$$

where α_1 is the highest root of A_1. The Cartan subalgebra of A_1^+ is three-dimensional.

B.6.2 $G_{2(2)}^+$

The Cartan matrix for $G_{2(2)}^+$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \tag{B.51}$$

and the δ root by

$$\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 \tag{B.52}$$

where $2\alpha_1 + 3\alpha_2$ is the highest root for $G_{2(2)}$. The corresponding simple Lie algebra is $G_{2(2)}$ which is 14-dimensional and of rank 2 thus making the Cartan subalgebra of $G_{2(2)}^+$ 4-dimensional.

C

Representation of $G_{2(2)}$

We use the following representation of $G_{2(2)}$, also used in [17],

$$h_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

D

Details on the Calculation of $\mathcal{M}$

In this appendix we present some details of the calculation of $\mathcal{M}$ from section 7.2.2 for the interested reader.

D.1 Schwarzschild

$$\begin{aligned}\gamma_k &= 0\\ \Gamma &= \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}\end{aligned}$$

where

$$A = \frac{\sqrt{\frac{2}{\mu}} r \sqrt{r^2-\mu}\, s(2\theta)}{\left(2\mu + \sqrt{\frac{\left(\mu^{3/2}+\sqrt{2}(2r^2-\mu)c(2\theta)\right)^2}{\mu} + \frac{8r^2(r^2-\mu)s^2(2\theta)}{\mu}} - \sqrt{\frac{\left(\mu^{3/2}+\sqrt{2}(\mu-2r^2)c(2\theta)\right)^2}{\mu} + \frac{8r^2(r^2-\mu)s^2(2\theta)}{\mu}}\right)}$$

and $s = \sin, c = \cos$.

D.2 Reissner-Nordström

Both γ_k and Γ are the same for Reissner-Nordström as for Schwarzschild. If we make the assumption that the Reissner-Nordström solution is diagonal in the r, θ, ϕ, ψ, t coordinates and that the only non-vanishing component of A is the time component we get that the coset representative in the $r, \theta, \phi_-, \phi_+, t$ coordinates is parametrized by $\phi_1, \phi_2, \chi_2, \chi_6$. We do not need the entire matrix so we present only the left upper 5×5 matrix needed to solve for $\phi_1, \phi_2, \chi_2, \chi_6$

$$M_{RN} = \begin{pmatrix} e^{\frac{\phi_1}{\sqrt{3}}+\phi_2} & -\frac{e^{\frac{\phi_1}{\sqrt{3}}+\phi_2}\chi_2}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{e^{\frac{\phi_1}{\sqrt{3}}+\phi_2}\chi_2}{\sqrt{3}} & e^{\phi_2-\frac{\phi_1}{\sqrt{3}}} - \frac{1}{3}e^{\frac{\phi_1}{\sqrt{3}}+\phi_2}\chi_2^2 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{2\phi_1}{\sqrt{3}}} & -\frac{2e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2}{\sqrt{3}} & \frac{2}{3}e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2 \\ 0 & 0 & \frac{e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2}{\sqrt{3}} & 1-\frac{2}{3}e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2 & \frac{2\chi_2\left(e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2-3\right)}{3\sqrt{3}} \\ 0 & 0 & \frac{1}{6}e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2 & -\frac{\chi_2\left(e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2-3\right)}{3\sqrt{3}} & \frac{1}{9}e^{-\frac{2\phi_1}{\sqrt{3}}}\left(e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2-3\right)^2 \\ -\frac{1}{2}e^{\frac{\phi_1}{\sqrt{3}}+\phi_2}\chi_6 & \frac{e^{\frac{\phi_1}{\sqrt{3}}+\phi_2}\chi_2\chi_6}{2\sqrt{3}} & 0 & 0 & 0 \\ -\frac{e^{\frac{\phi_1}{\sqrt{3}}+\phi_2}\chi_2\chi_6}{2\sqrt{3}} & \frac{1}{6}e^{\phi_2-\frac{\phi_1}{\sqrt{3}}}\left(e^{\frac{2\phi_1}{\sqrt{3}}}\chi_2^2-3\right)\chi_6 & 0 & 0 & 0 \end{pmatrix} \tag{D.1}$$

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