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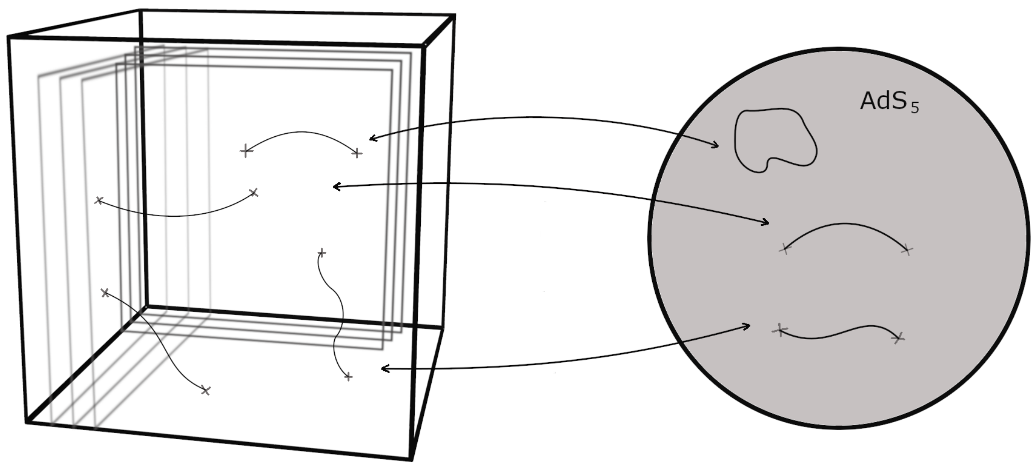


# A top-down model for layered holographic strange metals

## Analysis of electrical quantities dual to a D7 probe brane in a D3-D5 geometry

Master's thesis in Physics and astronomy

Olle Lexell

Department of Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2021

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Cover: Illustration of the different dualities between the strings in the model examined in this thesis.

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## Abstract

The AdS/CFT correspondence is a potentially powerful tool in describing condensed matter systems for which our current theoretical understanding is lacking. This is because it can be used to map a strongly coupled field theory to a weakly coupled gravitational theory. In this thesis I describe some aspects of the AdS/CFT correspondence and look at a top-down model. This model is built upon a large number of D3 and D5 branes with a D7 probe brane. The goal is to see whether this model can be used to describe aspects of the strange metal phase that is found in layered high-temperature superconductors. Starting from a weakly coupled string theory setting I derive the temperature-dependence of a DC current in different regimes, as well as dispersion relations for electromagnetic fluctuations. There is a possibility of obtaining resistivity that matches the linear in $T$ dependence for the resistivity in strange metals. This is done by adjusting the number of D5 branes depending on the temperature and the charge density, which is obtained holographically. With a particular choice of boundary conditions, plasmonic dispersion relations are found, as required. This is the first layered top-down model with this behaviour for the current available in its parameter-space. While it has not produced correct predictions without an adjustment of the parameter-space, the freedoms granted from the said parameter-space makes it possible that this model could describe other layered systems that lacks quasi-particles, and not just strange metals.

Keywords: AdS/CFT, Top-down, D3-D5 geometry, D7 probe brane, holographic strange metals, holographic plasmons, mixed boundary conditions, resistivity linear in temperature.

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Olle Lexell, Gothenburg, February 2021

# Contents

# 1

# Introduction

All theories in physics have their limits in which they apply. Newtonian mechanics apply for non-quantum objects at low velocity and weak gravitational field. At high velocity one needs special relativity. With a strong gravitational field, one needs general relativity. For the very small objects we need quantum mechanics and quantum field theory. In these different theories, it is often easy to describe the behaviour of a single particle, or a few particles interacting. Describing many interacting objects is always a challenge that requires approximations. One many-body system where there is currently a lack of theoretical understanding is in high temperature superconductors. This is part of what has led researchers to attempt to describe them using the Anti de-Sitter/Conformal field theory (AdS/CFT) correspondence [1].

## 1.1 String theory and AdS/CFT

String theory originated as a theory for hadrons, in an attempt to explain the strong interaction [1]. However, among the states that were found was a massless spin 2 particle - the graviton. This turned string theory from a theory of hadrons into a potential theory of everything. However, many years later, there is no observation that directly supports that string theory describes the fundamental nature of our world [2]. However, it still has its uses. One of them is via the AdS/CFT correspondence.

The correspondence was first proposed by Juan Maldacena in 1997, who conjectured that the compactification of a string theory on an Anti-deSitter (AdS) space-time is dual to a conformal field theory (CFT) in the large $N$ limit [3]. This conjecture is made by looking at a large stack of D3-branes from two different perspectives. We will take a closer look at the reasoning of this canonical example in chapter 4. What this conjecture essentially means is that this string theory and the conformal field theory will describe the same physics. The real strength in the correspondence is that it, in certain limits, is a weak-strong correspondence [2]. This means that we can have a strongly coupled field theory in Minkowski spacetime and examine its properties by looking at a weakly coupled gravitational theory. This is something very useful as many real-world systems are described by strongly coupled field theories [1]. In this thesis, we will concern ourselves with this form of the AdS/CFT correspondence. To note is that the opposite is also possible, that is, looking at a weakly coupled field theory can give us insights into strongly coupled string theories [4].

One way to view the correspondence is that the field theory lives on the boundary of the AdS spacetime in which the string-theory live. It is in-fact so that the conformal boundary of AdS spacetime is Minkowski spacetime with one fewer spatial dimensions [4]. We can therefore refer to these as the boundary and the bulk. While this viewpoint is useful, one must be careful, since we will add different terms to the action of the string-theory side of the correspondence that only live on the boundary of the

AdS spacetime. This will be done to either renormalise the action or change the boundary conditions of the fields in the bulk, both of which is explained in section 4.2.1. While adding the boundary-terms does change the physics that is modelled by our bulk-theory, we should not see this as adding the boundary-terms to the field theory that is said to live on the boundary. To avoid this confusion, we will mostly denote the two sides by the field-theory side and gravity/string theory side.

Moving on, when trying to use AdS/CFT there are two general methods, bottom-up and top-down. Bottom up is based on choosing the field-content of the gravity theory in order to tailor the behaviour of the field theory. Top-down is based on starting with a string theoretical system and deriving the behaviour of the field theory. The disadvantage of top-down is that this can become quite difficult. Additionally, one must perform a truncation of the Kaluza-Klein tower of fields that arise from the additional dimensions of string theory [1]. However, its advantage is that it guarantees that the Lagrangian of the boundary theory exists and that the bulk AdS theory is a consistent theory of quantum gravity. This is something bottom-up does not do. That said, bottom-up approaches have been the most successful in describing condensed matter physics, and so should not be overlooked [1]. In this thesis however, we will be using a top-down approach.

A key part of doing calculations with AdS/CFT is the Gubser-Klebanov-Polyakov-Witten (GKPW) rule. This rule is a statement that the partition functions of both sides are equal. This leads to a ”holographic dictionary” which can be used to translate quantities in one side to quantities on the other side [1]. For top-down models, the large $N$-limit important is very important in the application of this rule, since it will allow us to ignore the many different possible configurations present in quantum theories, and instead take things to be on-shell [1]. The explanation for this is the concern for chapter 2.

## 1.2 Strange metals

Let us look at a very quick description of strange metals, some of their properties and some argument as to why holography might be the right way to go. This is to give context and motivation to some of the steps in our analysis, mainly in chapter 6.

A strange metal is a phase often found in the copper oxide (cuprates) high-temperature superconductors. For which temperatures this behaviour emerges (if at all) depends on the doping of the superconductor. This phase is characterised by unconventional transport properties, which includes a resistivity linear in $T$. This temperature-dependence can persist to very low and/or high temperatures and is characterised by a lack of well-defined quasi-particles [5]. This lack of quasi-particles is one of the reasons holography might be a good tool [1].

The cuprates are layered systems and they show plasmonic dispersion relations in both the in- and off-plane directions [6]. However, the conductivity in the off-plane direction is not the same as for the in-plane direction, where they show insulator behaviour [1]. The conductivity behaviour varies on the doping, and also the in-plane direction can show insulator behaviour [1]. Usually, the focus is on the physics along the layers, but there is relevant information in regards to the dynamics in the orthogonal directions [6].

## 1.3 D3-D5-D7 model

As mentioned earlier, this thesis is concerned with a top-down model. These models make heavy use of the physics of Dirichlet-branes (D-branes), which is an important concept within string theory. They are $p$-dimensional hyper-surfaces on which open strings can end, giving them Dirichlet boundary conditions. The physics of D-branes will be explained further in chapter 3. As mentioned earlier, the canonical example of the correspondence is based on a large stack of D3 branes. If one wishes, one can add other D$p$-branes to this setup in order to model different physics.

In this thesis, I examine a top-down model in which there are many D3 and D5 branes as well as one D7 brane. The D3 and D5 branes curve the spacetime that they are in and give rise to an anisotrophy, where the physics along one of the non-compact directions behaves differently than in the other two. In fact, this direction becomes very small in comparison to the other two. This difference carries over to the field theory side, creating a system with different properties in the in-plane and off-plane directions [7]. The physics we are interested in modelling is dual to the physics of the one D7-brane, which uses the probe-approximation. This essentially means that the back-reaction of the brane on the surrounding geometry is ignored.

The fact that we get a system with different physics in different directions is an advantage since we are trying to model copper oxide high-temperature superconductors, which are layered systems. For this reason, it is of interest to develop a holographic model that captures both the in-plane and off-plane behaviour of these multi-layered systems. This is the motivation behind the model examined in this thesis.

## 1.4 Aim

The aim of this thesis is to study the AdS/CFT correspondence and the D3-D5-D7 model. The goal is to see whether it can reproduce experimental observations of the copper-oxide high temperature superconductors in the strange metal phase. One property examined regard the resistivity where I will see if resistivity linear in temperature can be obtained for a DC current. I will also look at how the conductivity of the model compares to that of strange metals at non-zero frequency. I will also see if it is possible to achieve plasmonic dispersion relations.

## 1.5 Outline

The outline for this thesis is as follows. In chapter 2 I will present key elements of field theory, important for understanding the rest of the thesis. Similarly, chapter 3 will cover important concepts of string theory, followed by an introduction to the AdS/CFT correspondence in chapter 4. This chapter also includes the concepts of holographic renormalisation, as well as a look at electromagnetic fields in the correspondence. The model I have examined will be presented in chapter 5. The main results of this thesis are then presented in chapter 6. Finally, conclusions and potential outlooks are located in chapter 7.

## 1.6 Notes on conventions

In this thesis, we will use several conventions, summarised here. For the signature of the metric, we use plus signs for the spatial coordinates. We always work in units of $c = \hbar = 1$ unless otherwise stated. Something that will be mentioned frequently is the bulk and boundary space. The usual convention that Greek indices include the time direction and lowercase Latin indices only includes spatial directions is also applied here. The same indices will be used for both the boundary and the bulk, and which coordinates these indices will range over will often be settled by the context. For example, a Greek index in the bulk might range over $t, x, y, z, r$ and on the boundary it would range over $t, x, y, z$. However, if there is a large possibility for confusion, there will be clarification. We will use $d$ to signify the number of spatial dimensions of the boundary, and $D = d + 1$ to signify the number of spacetime dimensions. This means that the bulk has $D + 1$ spacetime dimensions. This is something that varies a bit in the literature. Also note that the labels $t, x, y, z$ will be used as labels for non-compact but curved directions. These should therefore not be confused with the Cartesian coordinates. This will be the case in chapter 6.

# 2

# Elements of field theory

As mentioned in the introduction, a central item of the weak-strong duality part of the AdS/CFT correspondence, which is what we wish to utilise, is the large $N$-limit. Therefore, we will need some understanding of field-theories in the large $N$-limit as well as to look at their connection to condensed matter physics. Before that however, we must look at the generating functional of a QFT and how it is used to calculate correlation functions. This will also be the key for applying the GKWP rule to translate between bulk and boundary quantities. In the last part there will also be a brief introduction to non-linear electrodynamics.

## 2.1 The generating functional of a QFT

As mentioned earlier, in order to relate quantities on the two different sides of the correspondence we use the GKWP rule, which is to assume that the partition functions of the two sides are equal. For this we need the partition function of a quantum field theory, which, in the cases we consider, is the generating functional. The generating functional is attained via the Feynman path integrals and can be used to calculate correlation functions. In a scalar field theory, the generating functional of correlation functions is [8]

$$Z[J] = \int \mathcal{D}\phi \exp\left[i \int d^D x(\mathcal{L} + J\phi)\right], \tag{2.1}$$

where $\mathcal{L}$ is the field theory Lagrangian, and $J$ is a source. This clearly has a similar structure to an integral over all possible configurations of an exponential statistical weight, which is the partition function of statistical mechanics. For a more general field theory, we have

$$Z[J] = \int \mathcal{D}\phi \exp\left[i \int d^D x(\mathcal{L} + J_i\mathcal{O}_i)\right] \tag{2.2}$$

where the $\mathcal{O}_i$ are the field theory operators [4].

We can use the generating functional $Z$ to calculate our correlation functions by taking the functional derivative of $\ln Z$ with respect to the sources $J$ and taking the limit of vanishing sources,

$$\big\langle \mathcal{O}_i(x) \big\rangle = -i \frac{1}{Z[J=0]} \frac{\delta Z[J_i]}{\delta J_i(x)}\Big|_{J_i=0}. \tag{2.3}$$

The one-point correlation function is also referred to as the expectation value. The above equation can of course be generalised to an $n$-point correlation function

$$\big\langle \mathcal{O}_i 1(x_1) \mathcal{O}_i 2(x_2) ... \mathcal{O}_i n(x_n) \big\rangle = \frac{1}{Z[J_i=0]} \left[\prod_{j=1}^{n} \Big(-i\frac{\delta}{\delta J_i(x_j)}\Big) Z[J_i]\right]_{J_i=0}. \tag{2.4}$$

These equations are general and apply for both free and interacting field theories. However, in the case of an interacting field theory, $Z[J = 0]$ cannot be calculated explicitly. This is because, in a free field theory, the integral in eq. (2.2) is at most quadratic in the fields, making the integral Gaussian. In an interacting field theory however, this is no longer the case [8].

## 2.2 Large $N$ field theories

An important part of using the AdS/CFT correspondence is the large $N$-limit of matrix field theories, and we therefore need to understand what these field theories are, and their properties. This section will not cover detailed derivations, but rather the results needed to understand the calculations in later sections. If one wishes to know more, one can, for example, look at [9].

A good example of a matrix field theory is regular QCD, where the fields, represented as $3\times3$ matrices, transform in the adjoint of the symmetry group $SU(3)$. A generalised QCD with $N$ colours would have the symmetry group $SU(N)$. So a large $N$ field theory is, simply put, a field theory with a large $N$ symmetry group, for example, $U(N)$ or $SU(N)$. In condensed matter physics, large $N$ field theories are used, however mostly in the form of vector field theories [1]. In vector field theories the fields transform in the vector representation of some symmetry group rather than in the adjoint. There are cases in which matrix field theories are used for condensed matter physics, for example within quantum Hall physics [10].

Let us briefly look at some properties of vector field theories in the large $N$ limit. The most notable one is that the existence of a set of operators $\mathcal{O}_i$ that factorise up to $1/N$ corrections [11], which means that

$$\langle \mathcal{O}_{i1}\mathcal{O}_{i2}...\mathcal{O}_{in}\rangle = \langle \mathcal{O}_{i1}\rangle\langle \mathcal{O}_{i2}\rangle...\langle \mathcal{O}_{in}\rangle + \mathcal{O}(1/N). \tag{2.5}$$

Note that this also applies for matrix large $N$ theories. This operator factorisation follows from the fact that at large $N$, we arrive at a saddle-point description for the action. Despite solving saddle-point equations, the large $N$-limit still captures non-trivial quantum physics [1]. However, the vector field theory will be essentially free in the limit of $N \to \infty$ [11] and so is not what we are interested in, since we are looking for a large $N$-theory with strong coupling, which can be dual to our weakly coupled AdS supergravity.

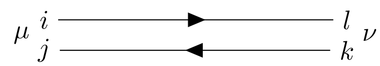
Moving on to matrix theories, we wish to understand their behaviour at $N \to \infty$. Just as in the vector case, there exists a mean-field description, but in this case, it is possible to have strong coupling when the saddle-point description is valid [11]. In the end, we will arrive at the result that the ultimate mean field description of a $U(N)$ matrix theory at large $N$ is a string theory [1]. That this mean field description had something to do with strings was discovered by 't Hooft using planar diagrams [12], however it would take until the discovery of the AdS/CFT correspondence to understand the exact nature of the this string-formulation [1]. We will go through the steps for arriving at this idea of a string description at a conceptual level. However, the step to the actual AdS/CFT correspondence will be taken by starting in string theory, and will be explained in section 4.1.

Let us begin by looking at an action for a $U(N)$ theory, in this example, a Yang-Mills action,

$$S = -\frac{1}{4g_{YM}^2}\int d^D x\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{2.6}$$

where $g_{YM}$ is the Yang-Mills coupling and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the field strength tensor. It should of course be noted that the components $A_\mu$ are $N \times N$ matrices. The key to understanding the large $N$ limit of this theory is to use that at large $N$, with $\lambda = g_{YM}^2 N$ (referred to as the 't Hooft coupling) kept fixed, planar diagrams dominate, as shown by 't Hooft in [12]. I will explain what planar diagrams are and why they dominate at large $N$ shortly. In this approach, the field $A_\mu$ is kept as a matrix. In contrast, the approach often taken is to expand the field in terms of the generators of the gauge-group.

In keeping the field as a matrix, the Feynman diagrams are written with a double line notation. Recall that the Feynman diagrams track the indices of the fields, and so now they track the matrix indices $i, j, k, l$ etc. as well as the vector indices $\mu, \nu, \sigma, \lambda$ etc. The propagator take the following form [1]




$$\left\langle A_\mu{}^i{}_j(p) A_\nu{}^k{}_l(k) \right\rangle = g^2 \delta^i{}_l \delta^k{}_j \frac{\eta_{\mu\nu}}{p^2} \delta^{(D)}(p+k) \tag{2.7}$$

The arrow on each line points from an upper to a lower matrix index. As can be observed, if there is a closed loop, the Kronecker deltas would give a factor $N$ for each loop. We can also see that using the $1/g^2_{YM}$-factor in the action can instead be written as $N/\lambda$. The prefactor clearly diverges with $N \to \infty$, as does the number of fields. To understand which diagrams are of leading order we must look at the number of loops, vertices and propagators. We have that vertices scale as $N/\lambda$, propagators scale as $\lambda/N$ [4], and as already mentioned, loops gives a factor $N$. By setting $V =$ the number of vertices, $P =$ the number of propagators and $L =$ the number of loops we have that

$$\text{Diagram}(V, P, L) \sim N^{V-P+L} \lambda^{P-V}. \tag{2.8}$$

We can then realise that the diagrams forms surfaces, with $P =$ the number of edges and $L =$ the number of faces. This means that we can utilise the the Euler characteristic, $\chi \equiv V - P + L = 2 - 2g$ (assuming the surface is orientable) where $g$ is the genus [4]. This means that the leading order diagrams have $g = 0$ and therefore scale as $N^2$. The diagrams that fulfil are called planar diagrams, since they can be mapped to a 2-sphere. Typical examples are the following two diagrams

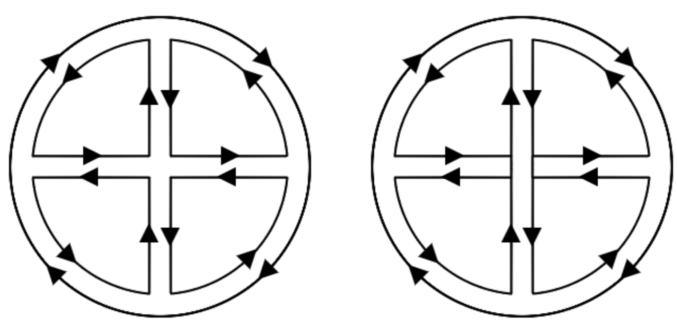


**Figure 2.1:** Two diagrams in the double line notation. The left one is planar with a genus of 0 and the right one is non-planar with a genus of 1. This can be verified by counting the number of propagators, vertices and loops.

In the large $N$ limit, these planar diagrams dominate, and this applies for scalar and fermionic fields as well, assuming they are matrix-valued [1]. This diagram scaling also leads to correlation function factorisation, when we consider the full correlations function that includes the disconnected diagrams [4]. This is because all diagrams have the same scaling, $N^2$, since we only consider planar diagrams. This means that the combination of two disconnected diagrams will scale as $N^4$ while a connected diagram scales as $N^2$, even though both cases contribute to the same (2-point in this example) correlation function. This means that the leading contribution to the full correlation functions are disconnected diagrams, and therefore we have that the correlation functions factorise.

$$\langle \mathcal{O}_1 \mathcal{O}_2 ... \mathcal{O}_n \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle ... \langle \mathcal{O}_n \rangle + ... \tag{2.9}$$

The difference compared to eq. (2.5) is that the next order contribution depends on $N$.

This factorisation mean that the variance $(\Delta\mathcal{O})^2 = \langle \mathcal{O}\mathcal{O} \rangle - \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle = 0$ in the large $N$ limit. What this means is that it is no longer necessary to average over different configurations of the fields, as done in the path integral. Rather, there must be a classical field that give us the values of the correlation functions [13].

Let us now take a look again at the planar Feynman diagrams. When these have many loops we can view them as a large tiled surface. When doing calculations with the planar diagrams, there is a need for a perturbation expansion since one does not wish to count infinite diagrams (just as in any weakly interacting QFT). At large $N$, this expansion is done in the effective coupling $\lambda$ instead. However, when $\lambda$ becomes large, these diagrams start to look like the world-sheet swept out by a dynamical string. The way to view this is that we need to look at diagrams with an essentially infinite amount of loops. If the total size of the diagram is kept fixed, and because of the tiled structure, the surface will start to look like that of string worldsheet. This was first observed by 't Hooft in [12]. One can also map the Feynman diagrams in the double line notation to string theory graphs of the same genus via a smooth topological transition [4]. As a note, that will become more relevant in section 4.1, the planar limit is the appropriate view when $\lambda = g^2 N \ll 1$ and the string view is appropriate when $\lambda = g^2 N \gg 1$ [1]. In section 4.1, we will look at these two limits from the string theory perspective and find a similar conclusion.

## 2.3 Non-linear electrodynamics

As will be seen later, non-linear electrodynamics will be important in the analysis of our model since it enters as a part of the D-brane action, seen in section 3.3. One of the issues of the Maxwell equations is that the self-energy of a point-charge is infinite, since there is no limit on the field strength. However, the electric field on a D-brane has a maximum value which can be shown using T-duality [2]. The lack of a maximal field strength is something that can be remedied by a non-linear description, that must also reduce to linear electrodynamics (Maxwellls equations) in the limit of weak fields.

The Maxwell equations in the presence of a material takes the form

$$\begin{aligned} &\nabla \vec{E} = -\partial_t \vec{B}, \quad \nabla \cdot \vec{D} = \rho \\ &\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{j} + \partial_t \vec{D} \end{aligned} \tag{2.10}$$

where we have used natural units ($c = 1$). Note that the $\vec{D}$ and $\vec{H}$ fields depend on the $\vec{E}$ and $\vec{B}$ fields through a non-trivial relation. Now, using the 4-potential $A_\mu = (\rho, \vec{A})$ we get the usual field strength tensor $F_{\mu\nu}$. The equations involving $\vec{E}$ and $\vec{B}$ are then reduced to the Bianchi identity $\partial_{[\gamma} F_{\mu\nu]} = 0$. We can now define a new antisymmetric tensor $G_{\mu\nu}$ where

$$G_{0i} = D_i, \quad G_{ij} = \epsilon_{ij}^{k} H_k \tag{2.11}$$

From the equations involving $\vec{D}$ and $\vec{H}$ in eq. (2.10) gives that

$$\partial_\nu G^{\mu\nu} = j^\mu. \tag{2.12}$$

Now we would like to examine the relation between $F_{\mu\nu}$ and $G_{\mu\nu}$, and confirm that eq. (2.12) follows from the variation of an action on the form

$$S = \int d^D x \Big( \mathcal{L}(F_{\mu\nu}) + A_\mu j^\mu \Big). \tag{2.13}$$

If one were to put in the Maxwell Lagrangian density $\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ one would get $\partial_\nu F^{\mu\nu} = j^\mu$ as the equation of motion. For simplicity, we will assume that the Lagrangian only depends on $F_{\mu\nu}$ and not, for example, its derivatives. Now we want to plug this into the Euler-Lagrange equations of motion and therefore we need to define the partial derivatives with respect to field strengths in order to get a consistent result. Due to the antisymmetry of $F_{\mu\nu}$ we define

$$\frac{\delta M}{\delta F_{\mu\nu}} = \frac{1}{2} \frac{\partial M}{\partial F_{\mu\nu}} \tag{2.14}$$

where $M$ is an arbitrary function of $F_{\mu\nu}$. We would also like to know how the chain rule behaves when $F_{\mu\nu}$ depends on some variable $U$, so we look at

$$\frac{\partial M}{\partial U} \equiv \frac{\delta M}{\delta U} = \frac{\delta M}{\delta F_{\mu\nu}} \frac{\delta F_{\mu\nu}}{\delta U} = \frac{1}{2} \frac{\partial M}{\partial F_{\mu\nu}} \frac{\partial F_{\mu\nu}}{\partial U}. \tag{2.15}$$

With this we can easily use the Euler Lagrange equations of motion on the integrand in eq. (2.13) by using its dependence on $A_\mu$. We arrive to

$$\frac{1}{2} \partial_\mu \Big( \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \Big) - j^\nu = 0. \tag{2.16}$$

Note that we don't actually need the anti-symmetazation in the above equation since we assumed $\mathcal{L}$ only depends on $F_{\mu\nu}$, but it has been written out for clarity. We note that with the chain rule we have that $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}$. We can see that eq. (2.16) matches eq. (2.12) if we make the identification

$$G^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \tag{2.17}$$

Thus we have a relation between $G_{\mu\nu}$ and $F_{\mu\nu}$ assuming the Lagrangian is known. In the case of $\mathcal{L}_{Maxwell}$ we find that $G^{\mu\nu} = F^{\mu\nu}$ and we therefore attain the Maxwell equations from eq. (2.12).

### 2.3.1 Born-Infeld electrodynamics

The Lagrangian of a theory of non-linear electrodynamics must be gauge and Lorentz invariant, and as in the previous section, for simplicity, we want it to depend only on $F_{\mu\nu}$ and none of its derivatives. We also require it to reduce to the Maxwell Lagrangian $\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ for small values of the field strengths. Since $F_{\mu\nu}$ is gauge invariant, we only need to look for Lorentz invariant terms. This means we need to look for terms with no free indices.The only two objects that we can form are [2]

$$\begin{aligned} s &= F_{\mu\nu} F^{\mu\nu} \quad \text{and} \\ p &= \tilde{F}^{\mu\nu} F_{\mu\nu} \end{aligned} \tag{2.18}$$

where $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \tilde{F}_{\rho\sigma}$. From these we can build the Born-Infeld (BI) Lagrangian [2]

$$\mathcal{L}_{BI} = -b^2 \sqrt{1 - \frac{2s}{b^2} - \frac{p^2}{b^4}} + b^2 = -b^2 \sqrt{- \det\Big( \eta_{\mu\nu} + \frac{1}{b} F_{\mu\nu} \Big)} + b^2 \tag{2.19}$$

where $b$ is the value of the critical electric field. The square root ensures that the electric field strength does not exceed a maximal value. A maximal electric field strength is required to ensure a finite self-energy for the point charge.

# 3

# Elements of string theory and supergravity

This chapter will serve as a short introduction to some string-theory concepts used in this thesis. This is to serve as a reminder for those already acquainted with string theory and a pointer for those that are not. We will also look at supergravity and D-branes, as they are important concepts in the derivation of the correspondence. We will look both at what supergravity contains, how D-branes behave from the supergravity point of view as well as the low-energy effective action for these D-branes. This action is central in the thesis, as it is what is used to examine what physics our model contains.

## 3.1 Quantised relativistic strings

The essential idea of string theory is to quantise relativistic strings. A way to write the action for the relativistic string is by the Nambu-Goto action [2]

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det(\partial_\mu X^\lambda \partial_\nu X^\gamma g_{\lambda\gamma})} \tag{3.1}$$

which is an area functional. Here $\alpha' = l_s^2$ where $l_s$ is the string length, $\sigma$ are the coordinates of the worldsheet traced by the string and $X^\mu$ is the embedding of the worldsheet into the target spacetime. In analogy to the point particle, this action will minimise the area of the worldsheet, just as the action of the point particle will minimise the length of the world-line. This action applies for both open and closed strings, but for open strings the boundary conditions needs to be specified as well.

With the action at hand the (simplified) quantisation procedure is to define a set of operators and impose reasonable commutation relations. Then the spectrum of the theory can be derived. Most notably the spectrum includes several massless states. These are gauge bosons (photon states), which come from open strings, excitations of an antisymmetric tensor field (the Kalb-Ramond field), a graviton and a dilaton (a scalar), which all come from the closed strings. We also obtain a restriction on the number of spacetime dimensions to 26, in order to preserve Lorentz invariance. There are also tachyons in the spectrum, which are states with a negative squared mass. These represent instabilities in the theory [2].

## 3.2 Superstrings

With the strings we just described we attain bosonic states. But as we know, we also require fermions if we wish to have something resembling the real world. To get fermions we add them to the worldsheet, by making the string action supersymmetric [4]. A brief description of supersymmetry is that it is a symmetry between fermions and bosons. It has a number of charges $Q_a$ that fulfils the following algebra [14]

$$\{Q_a, \bar{Q}_b\} = 2\gamma^{\mu}_{ab} P_{\mu} \tag{3.2}$$

where $P_\mu$ is the generator of momentum. The $Q_a$ are also the generators of the supersymmetry transformations, and how many there are can vary between different theories.

For adding fermions to the worldsheet via supersymmetry, we can use a different form of the string action, namely the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X^\mu \tag{3.3}$$

where $h_{\alpha\beta}$ is the induced metric on the worldsheet. Then adding fermions give us the action as [4]

$$S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} (\partial_\alpha X_\mu \partial_\beta X^\mu + i\bar{\psi}_\mu \gamma_\alpha \partial_\beta \psi^\mu). \tag{3.4}$$

When quantising the superstring(s) we now also obtain spacetime fermions and the dimensionality of spacetime is fixed to 10 [2]. Due to a freedom in the boundary conditions on the worldsheet fermions we get two different sectors, the Ramond (R) sector and Neveu-Schwarz (NS) sector. When quantised these sectors contain different states. These can then be truncated and combined in different ways for the closed strings. From these different combinations we arrive type at IIA and type IIB superstring theories (there are also 3 other types, but they are not important here). The important part to note is that these arise from closed strings and have slightly different massless fields. However in both cases we obtain a dilaton, a Kalb-Ramond field and a graviton [2]. In chapter 4 we will find the AdS/CFT correspondence by looking at D-branes from the open and closed string perspective and so let's look at bit more at open and closed superstrings.

The open superstrings are viewed in such a way that they end on D-branes, on which the string endpoint is fixed on. We will expand further on D-branes later. The most important part is that we get photon states, that is, a Maxwell field on the world-volume of the D-brane. We also get space-time fermionic states [2]. This is done by truncating the NS and R sectors in such a way that the truncated R-sector contributes the fermionic states and the truncated NS sector contribute the bosonic states. We end up with an equal number of bosonic and fermionic states at each mass level and we have gotten rid of the tachyonic state that the NS sector originally contained. The sectors that we have combined here are called the R- and the NS+ sectors.

For the closed superstrings we combine right-moving and left-moving sectors. In the case of type IIB superstrings, we combine the NS+ and R- sectors in the combinations [2]

(NS+,NS+), (NS+,R-), (R-,NS+), (R-,R-)

where in each parenthesis, there is a left- and right moving sector, ordered left to right. We attain fermionic states from the two combinations of R- and NS+, and bosonic states come from the (NS+,NS+) and (R-,R-). These states include a graviton, a dilaton and a Kalb-Ramond field, from (NS+,NS+). There is then another scalar and Kalb-Ramond field, as well a totally antisymmetric field with four indices, from (R-,R-) [2]. Note that we only care about massless fields, and skip a lot of details regarding the different sectors.

### 3.2.1 Supergravity

In short, supergravity can be obtained as the low-energy approximation of superstring theory [4]. It can also be obtained by imposing local supersymmetry [14], and then following a similar procedure as when deriving the equations of general relativity. The field content of supergravity is not set in stone but can vary. If derived from string theory, the supergravity action will contain the fields obtained from the massless closed string states [4]. And so, just as we have type IIA and IIB superstring theory, we also have typ IIA and IIB supergravity. Supergravity theories can also come in several other forms and in any dimensionality with $D \leq 11$. But here, type IIB (which lives in 10 dimensions) is the interesting one, as it will be what we use for the AdS/CFT correspondence.

Let us look a bit at the type IIB supergravity action. We will restrict ourselves to the bosonic part, ignoring the fermions. It takes the form [15]

$$S_{IIB} = \frac{1}{4\kappa_B^2}\int \left(\sqrt{g}e^{-2\phi}\left(2R_G + 8\partial_\mu\phi\partial^\mu\phi - |H_3|^2\right) - \sqrt{g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + C_4 \wedge H_3 \wedge F_3\right)$$

in the string frame. $R_G$ is the Ricci scalar and $\phi$ is the dilaton. The other components will be explained shortly. We should also mention that there is also a need for the self-dual constraint

$$*\tilde{F}_5 = \tilde{F}_5$$

which is compatible with the action but does not fall out naturally from it. Therefore, we need to tag it along to get the correct equations of motion for the fields. The action and constraint use differential forms as well as the wedge product $\wedge$ and Hodge dual $*$. Let us look at a quick explanation on these, as well as the different fields that we have, in order to get a feel for the content of the action.

Differential forms are essentially another way to write certain tensors. For example, $H_3$ can also be written as an antisymmetric tensor $H_{\mu\nu\rho}$, which is the field strength tensor for the Kalb-Ramond field. The relation between the two is normally written as $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, but in the formalism of differential forms, the exterior derivative d is used instead, and written as $H_3 = \mathrm{d}B_2$. Here, we have written the Kalb-Ramond field as a differential 2-form, and the exterior derivative on $B_2$ creates the differential 3-form. We then have the wedge product, which is essentially an antisymmetrised tensor product [4]. The Hodge dual is a bit more complicated, but the quick and dirty version for our $F_5$ is that [4]

$$(*\tilde{F})_{\mu_1\ldots\mu_5} = \frac{1}{5!}\epsilon^{\nu_1\ldots\nu_5}{}_{\mu_1\ldots\mu_5}\tilde{F}_{\nu_1\ldots\nu_5}. \tag{3.5}$$

Note that in the general case this is dependent on the number of spacetime dimensions.

With this out of the way, lets specify the different components of the action [15]

$$\begin{aligned}
F_1 &= \mathrm{d}C \\
F_3 &= \mathrm{d}C_2 \\
F_5 &= \mathrm{d}C_4 \\
H_3 &= \mathrm{d}B \\
\tilde{F}_3 &= F_3 - CH_3 \\
\tilde{F}_5 &= F_5\frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2F_3.
\end{aligned}$$

The $C$, $C_2$ and $C_4$ fields correspond to the scalar, Kalb-Ramond field and totally antisymmetric four-index field respectively that originates from the Ramond-Ramond sector of the closed type IIB superstrings. The $F_1$, $F_3$ and $F_5$ are their respective field strengths.

## 3.3 D-branes

D-branes are very important objects in string theory when dealing with open strings. When the equations of motion for these open strings are derived, it becomes important to ask what boundary conditions (b.c.) are imposed on the end points of the string. There are essentially two choices, Neumann or Dirichlet boundary conditions. Neumann b.c. means that the velocities of the endpoints are fixed while Dirichlet b.c. means the position of the endpoints are fixed. When solving the equations of motion in many dimensions (10 for superstrings) we can have Dirichlet b.c. in some directions and Neumann b.c. in the other directions, and these can also be different for the two endpoints. For example, we can have one endpoint which can move around on a plane in the $x, y$ directions (Neumann b.c.) but have a fixed $z$-value (Dirichlet b.c.), while the other endpoint is fixed in the $x$-direction and free to move in the other directions. In the case of the endpoint being free to move in the $x, y$ directions, one can view this as the string being attached to this $x, y$-plane. This plane is then called a D2-brane. The endpoint can move freely in the tangential directions to the brane but is fixed in the directions normal to the brane, which applies for D-branes of all dimensions. Note that the endpoint is always free to move in time. By convention, a $p$-dimensional D$p$ means it extends in $p$ spatial directions and always have time as a tangential direction.

### 3.3.1 The D-brane action

As mentioned earlier, we get photon states when quantising the open strings. These states can be shown to live only in the world-volume of the D$p$-brane [2]. One of the most important things we need is the low-energy effective action for D-branes since this is the basis for the AdS/CFT correspondence and will be used heavily in the model examined in this thesis. The action can be found by utilising T-duality [16] [2]. This is a duality in string theory which arises when there are compact dimensions and appears in both bosonic and superstring theory [17].

When we approach low energies, which means that we only concern ourselves with the massless string states, we can view the states on the D-brane as a field that lives on the brane. With this, a part of the low-energy effective action of a single D$p$-brane, denoted as the Dirac-Born-Infeld (DBI) action, is given by [18]

$$S_{DBI} = -\tau_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det(P[g]_{MN} + P[B]_{MN} + 2\pi\alpha' F_{MN})}. \tag{3.6}$$

Note the similarly to the string action, eq. (3.1). We here use $M, N$ as indices to specify that these are in the brane world-volume. $\tau_p$ is related to the tension of the brane $T_p$ via $\tau_p = T_p g_s$. The D$p$-brane tension is in turn related to $g_s$ (the string coupling) and $l_s$ (the string length) via

$$T_p = \left((2\pi)^p \alpha'^{(p+1)/2} g_s\right)^{-1}. \tag{3.7}$$

Remember that $\alpha' = l_s^2$ in natural units. Furthermore, $P[g]$ in eq. (3.6) denotes the pullback of the metric on the brane world-volume, meaning

$$P[g]_{MN} = \frac{\partial X^\mu}{\partial \xi^M} \frac{\partial X^\nu}{\partial \xi^N} g_{\mu\nu} \tag{3.8}$$

where $\xi^M$ denotes the coordinates of the brane world-volume. Similarly, $P[B]_{MN}$ denotes the pullback of the Kalb-Ramond field $B_{\mu\nu}$ on the brane. $F_{MN}$ is the field strength tensor for the gauge field $A_\mu$ living on the brane. If $P[B]_{MN} = 0$ then this action looks similar to the Born-infeld action (eq. (2.19)) of non-linear electrodynamics in a general metric. It is in fact the case that an electromagnetic field on a D$p$-brane is governed by Born-Infeld electrodynamics [2]. We can of-course rescale $F_{MN} \to \frac{F_{MN}}{2\pi\alpha'}$ in order to remove the $2\pi\alpha'$ prefactor for the gauge-field [19]. This will be done later on for convenience.

In a setup where the Kalb-Ramond field is 0, we can expand the action in eq. (3.6) to lowest non-trivial order in $F_{MN}$, which is done by using the relation

$$\det(M) = e^{\mathrm{Tr}(\ln(M))}.$$

Using matrix representation of the tensors, and letting $G$ denote the pullback of the metric, the square root in the Lagrangian density ends up becoming

$$\begin{aligned}&\sqrt{-\det(G+2\pi\alpha' F)} \approx \sqrt{-\det(G)}\sqrt{e^{\mathrm{Tr}\left(2\pi\alpha' G^{-1}F - \frac{1}{2}(2\pi\alpha')^2(g^{-1}F)^2\right)}} \approx \\ &\sqrt{-\det(G)}\sqrt{\left[1+2\pi\alpha' \,\mathrm{Tr}(G^{-1}F)+\frac{1}{2}(2\pi\alpha')^2\,\mathrm{Tr}(G^{-1}F)^2\right]\left[1-\frac{1}{2}(2\pi\alpha')^2\,\mathrm{Tr}((G^{-1}F)^2)\right]} \approx \\ &\sqrt{-\det(G)}\left[1+\frac{1}{2}2\pi\alpha'\,\mathrm{Tr}\big(G^{-1}F\big)-\frac{1}{4}(2\pi\alpha')^2\,\mathrm{Tr}\big((G^{-1}F)^2\big)+\frac{1}{8}(2\pi\alpha')^2\,\mathrm{Tr}\big(G^{-1}F\big)^2\right].\end{aligned} \tag{3.9}$$

Translating back to the index formalism we have that

$$\begin{aligned}&\mathrm{Tr}\big(G^{-1}F\big) = G^{MN}F_{NM} = 0 \\ &\mathrm{Tr}\big((G^{-1}F)^2\big) = G^{MN}F_{NK}G^{KL}F_{LM} = F^{ML}F_{LM} = -F^{MN}F_{MN}\end{aligned} \tag{3.10}$$

due to the antisymmetry in $F_{MN}$. In the end, our action then becomes

$$S_{DBI} \approx -\frac{(2\pi\alpha')^2\tau_p}{4}\int d^p\xi\sqrt{-G}e^{-\phi}F^{MN}F_{MN} \tag{3.11}$$

where we have now denoted $G = det(G_{MN})$. This we can recognise as the Yang-Mills action, with coupling constant

$$g_{YM}^2 = \frac{1}{(2\pi\alpha')^2\tau_p e^{-\phi}} = (2\pi)^{p-2}\alpha'^{(p-3)/2}e^{\phi}. \tag{3.12}$$

When there are $N$ coincident D-branes, the symmetry of the gauge field gets upgraded to $U(N)$ [20]. The DBI action then generalises to [16]

$$S_{DBI} = -\tau_p\int d^{p+1}\xi\,\mathrm{Tr}\left(e^{-\phi}\sqrt{-\det(P[g]_{MN}+P[B]_{MN}+2\pi\alpha' F_{MN})}\right).$$

As for the case of Yang-Mills theory, we can then obtain the 't Hooft coupling of $N$ D$p$-branes as

$$\lambda_{\mathrm{D}p} = Ng_{YM}^2 = N(2\pi)^{p-2}\alpha'^{(p-3)/2}e^{\phi}. \tag{3.13}$$

If we then have that $e^{\phi}\sim g_s$, the we have that $\lambda_{\mathrm{D}p} = N(2\pi)^{p-2}\alpha'^{(p-3)/2}g_s$. This is the case for a large number of D3-branes, as will be explained in the next section. This will however turn out not to be the case for our model [7]. The geometry created by a large number of D5-branes and D3 branes will cause the dilaton to have a non-trivial dependence on $r$.

Moving on from the DBI action, we should also mention that in superstring theory there is a second term to the bosonic part of the D$p$-brane action. It appears because D-branes carries Ramond-Ramond charges, and therefore couple to the Ramond-Ramond fields of closed strings. This extra term takes the Chern-Simons like form [18]

$$S_{CS} = \frac{(2\pi\alpha')^2}{2}\tau_p\int d^{p+1}\xi e^{F+P[B]}\wedge\sum_q P[C_{q+1}]. \tag{3.14}$$

This term is sometimes written as a Wess-Zumino term in the literature [21], and can also be deduced via T-duality [17].

**Einstein frame**

The action for the D-brane in eq. (3.6) has been given in the string-frame. However, in the later parts of this thesis we will instead work in the so-called Einstein frame. One of the reasons that the Einstein frame is often used is that it is the choice of frame for the canonically normalised action for string gravity [4]. The string frame and Einstein frame are related to each other by a conformal transformation. There is some debate as to which one is physical, which in this case means that the theory is consistent in that frame and that the physical observables can be, in principle, measured [22]. This is not something we will care about. We use it because the previous works on this model uses it [7]. The Einstein frame is related to the string frame via the transformation [23]

$$g^E_{\mu\nu} = e^{\phi/2} g^S_{\mu\nu} \tag{3.15}$$

where $g^E_{\mu\nu}$ and $g^S_{\mu\nu}$ denotes the metric in the Einstein and string frame respectively. This gives us that the DBI part of the D$_p$-brane action becomes

$$S_{DBI} = -T_p \int d^{p+1}\xi e^{\phi\frac{(p-3)}{4}} \sqrt{-\det\big(P[g]_{MN} + e^{-\phi/2}P[B]_{MN} + e^{-\phi/2}F_{MN}\big)} \tag{3.16}$$

Note that the Chern-Simons term is not affected by this transformation [23].

### 3.3.2 D-branes in supergravity

We can also view D-branes as solitonic solutions to the equations of motion for supergravity, that is, we view them as massive objects that curve the surrounding spacetime. The following ansatz for the metric solves the supergravity equations of motion in the case of a D$p$-brane [4],

$$ds^2 = h_p(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + h_p(r)^{1/2}dy^i dy^i. \tag{3.17}$$

The function $h_p(r)$ is

$$h_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p} \tag{3.18}$$

and $r$ is defined by $r^2 = y_i y_i$ where $i \in [p+1, 9]$. $\mu$ and $\nu$ span $0, 1, ..., p$. Specifying the metric is not quite enough though, as we also need an ansatz for the dilaton, the Kalb-Ramond field and the Ramond-Ramond fields from the closed strings. This ansatz looks like

$$\begin{aligned} e^\phi &= g_s h_p(r)^{(3-p)/4} \\ B_{MN} &= 0 \\ C_{p+1} &= (H_p(r)^{-1}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^p \end{aligned} \tag{3.19}$$

where the $M, N$ indices are used to make it clear that it applies for all bulk coordinates. We should remember that for type IIB string theory, only odd-numbered D-branes are stable. The constant $L_p$ can be obtained by integrating the Ramond-Ramond charge of the brane and can be found to be [4]

$$L_p^{7-p} = (4\pi)^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right) g_s N \alpha'^{(7-p)/2} \tag{3.20}$$

where the factor $N$ is the number of coincident D$p$-branes. The ansatz for the dilaton give us some more context for the discussion after eq. (3.13). For D3-branes, the dilaton will just be $e^\phi = g_s$. eq. (3.19) also gives us a hint as to why the dilaton in our model, that contains a large number of D3 and D5 branes, will not simply go as $e^\phi \sim g_s$. However, it does not paint the whole picture. The geometry of our model will be expanded on in chapter 5 but the full derivation of the geometry of our model is beyond the scope of this thesis.

# 4

# The AdS/CFT correspondence

Now that we have some understanding of both field-theory and string-theory aspects, we will finally introduce the AdS/CFT correspondence itself, starting from the string theory side. This will be done via Maldacena's canonical example of $AdS_5/CFT_4$, first proposed by Maldacena in [3]. The motivation is based on using an open and closed string perspective on a stack of D3-branes and comparing the results. This example will not contain all the detailed calculations but rather some qualitative arguments. We will then move on to important concepts for utilising the correspondence. These include the holographic dictionary which is used to translate between the different theories and holographic renormalisation, used to remove infinities. We will also see how we can change the boundary conditions of the gravitational theory, which is used to model different physics of the field theory. We will finish of by looking at electromagnetism in the correspondence and derive some useful quantities.

## 4.1 The canonical example

As already mentioned, in this setup we use $N$ parallel D3-branes in flat spacetime. We will label the directions by $x^i$ with $i = 0, 1, ..., 9$ since we have 10 spacetime dimensions. The D3-branes are set to extend in the first 3 space directions, $x^1, x^2, x^3$ (as well as the $x^0$ since it is required for a D-brane). For clarity, the following table shows which directions are tangential and which are normal to the branes

**Table 4.1:** Table showing the setup for the D3-branes. The $\times$ marks a direction tangential to the brane, the - marks directions normal to the branes.

| Direction: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D3-brane: | $\times$ | $\times$ | $\times$ | $\times$ | - | - | - | - | - | - |

These D3-branes are separated by a small distance $d_3$, which we will let go to 0. The content of our system is open strings stretching between the branes, representing excitations of the branes, and closed strings that represent excitations of the space-time. When we consider the system at low energies, that is energies much smaller than the inverse of the string length

$$E \ll \frac{1}{l_s}, \tag{4.1}$$

only the massless modes can be accessed [24]. This means that we are able to write down an effective action that involves the open strings (D3-brane DBI action), the closed strings and the interactions between the two,

$$S = S_{open} + S_{closed} + S_{int}. \tag{4.2}$$

$S_{closed}$ is essentially the action of type IIB supergravity, with some additional corrective terms [24]. As for $S_{int}$, in this low energy approximation, it is given by the Chern-Simons part of the D-brane action [4]. Before we can analyse this further, we must also look at $g_s N$. Depending on whether this quantity is $\ll 1$ or $\gg 1$, different perspectives will be appropriate to describe the physics [2]. We will later be taking the so-called Maldacena limit, where we let $\alpha' \to 0$ and as well as $d_3 \to 0$ while their ratio $u = \frac{d_3}{\alpha'}$ is kept fixed [3].

### 4.1.1 Open string perspective

For the case of $g_s \ll 1$, the appropriate way to describe the physics is from the perspective of open strings. The reason for this is because when we expand the interaction term $S_{int}$ in orders of $\alpha'$, we see that it vanishes in the Maldacena limit [4].

For $S_{open}$, the DBI action, eq. (3.6), for a single D3 brane is

$$S_{DBI} = -\frac{1}{(2\pi)^3 \alpha'^2} \int d^4\xi e^{-\phi} \sqrt{-\det(P[g]_{MN} + 2\pi\alpha' F_{MN})}, \tag{4.3}$$

where we have inserted the expression for $\tau_p$ and set the Kalb-Ramond field to 0 for simplicity. We have also seen this action expanded in orders of $\alpha'$ in eq. (3.11). Note that it was described as an expansion in orders of $F$, but the result is the same when seen as orders of $\alpha'$. Seeing it as an expansion in $\alpha'$ is more appropriate here, as we will be taking $\alpha' \to 0$. Note that if we also expand the induced metric in $\alpha'$, we find that [25]

$$P[g]_{MN} \approx \eta_{MN} + (2\pi\alpha')^2 \partial_M \phi^i \partial_N \phi^i \tag{4.4}$$

where the six scalars $\phi^i$ are the coordinates the parameterise the embedding of the brane. This means that the low-energy effective action, at lowest order in $\alpha'$ takes the form

$$-\frac{1}{2\pi g_s} \int d^4\xi \Big( \frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} \partial_M \phi^i \partial^M \phi^i + \mathcal{O}(\alpha') \Big) \tag{4.5}$$

where we have used that $e^{-\phi} \sim g_s$ in this case. As in the previous chapter, setting $g_{YM}^2 = 2\pi g_s$, we can identify this to a Yang-Mills theory when $\alpha' \to 0$. In fact, this is Super Yang-Mills theory with four supercharges ($\mathcal{N} = 4$) [25]. Taking the Maldacena limit, the D3-branes become coincident, and we the symmetry of the theory becomes $U(N)$ [24], which we also saw in the last chapter.

Regarding $S_{closed}$ we have noted as before that this is the action of type IIB supergravity with some higher order corrective terms. When we take the Maldacena limit, we arrive solely at type IIB supergravity in 10 dimensional Minkowski space [4]. This means that the end-result of the open-string perspective is that we have supergravity on $\mathbb{R}^{1,9}$ (10 dimensional Minkowski space) and an $\mathcal{N} = 4$ $U(N)$ SYM theory with coupling $g_{YM} = 2\pi g_s$ on $\mathbb{R}^{1,3}$, which is a conformal field theory [24].

### 4.1.2 Closed string perspective

For the case of $g_s N \gg 1$ gravitational effects become more important [2] and we must instead take the closed string perspective [4], where we view the D3-branes as massive charged objects. These will source the various fields of the string theory [24]. Despite the stronger string interaction we will see still see a decoupling between different strings, this time between closed strings in two different regions [2].

From the last chapter, we have the supergravity solution for a large amount of D$p$-branes. Applying this for our stack of D3-branes we have that our metric becomes

$$ds^2 = h(r)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + h(r)^{1/2} dy^i dy^i. \tag{4.6}$$

with

$$h(r) = 1 + \left(\frac{L}{r}\right)^4 \text{ and } L^4 = 4\pi g_s N \alpha'^2. \tag{4.7}$$

As before, the radial coordinate $r = y_i y_i$ for $i = 4, 5, ..., 9$. Looking closer at $h(r)$, we see that for $r \gg L$, $h(r) \approx 1$ and we get regular Minkowski spacetime. If we instead have $r \ll L$, called the near-horizon limit, we have that $h(r) \approx \frac{L^2}{r^2}$ and the metric takes the form of

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dy^i dy^i. \tag{4.8}$$

We can rewrite this metric using our radial coordinate, that is, we introduce spherical coordinates for our $y^i$. The resulting metric becomes

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 ds^2_{S^5} \tag{4.9}$$

where the $ds^2_{S^5}$ is the metric for the unit 5-sphere. We also recognise that the first part is the metric of AdS space, with at radius of $L$. We therefore get a region that is $\mathrm{AdS}_5 \times \mathrm{S}^5$ [24]. A further remark on the this region, which is referred to as the throat, is that the horizon at $r = 0$ (the end of the throat) is infinitely far away from any point in the plane [2].

We then have closed strings that live on 10 dimensional Minkowski spacetime and closed strings that live in $\mathrm{AdS}_5 \times \mathrm{S}^5$. The most important part here is that the strings in these two different regions decouple in the low energy limit. To illustrate this, we imagine an observer that sits far away from the near horizon region, that is, in the flat region. This observer will observe the low energy modes in the flat region, which will have long wavelengths [24]. When these approach the near horizon region, the cross-section for them to be absorbed by this region goes as [26]

$$\sigma \sim E^3 L^8 \tag{4.10}$$

at low energies. The curvature $L$, does not diverge, and so the cross section will become very small for low energies. One can view this as that the wavelength of the mode becomes larger than the size of this region, which is of order $L$ [24].

Meanwhile, the modes that live in $\mathrm{AdS}_5 \times \mathrm{S}^5$ will be experience extreme red-shift from the point of view of the observer at infinity. The energy observed will be

$$E_{observed} = \sqrt{-g_{00}}E_{mode} = \frac{r}{L}E_{mode}. \tag{4.11}$$

When we let $\frac{r}{L}$ become very small, the observed energy becomes small even if $E_{mode}$ is large. This means that we should not disregard large $E_{mode}$ in the throat region, even when taking the low energy limit [4]. Essentially, the modes close to the horizon at r = 0 will find it very hard to climb out to the flat region [24]. Therefore, the closed strings of the two regions will decouple from each other. The result for this case of $g_s N \gg 1$ is that we have supergravity on $\mathbb{R}^{1,9}$ and supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Now we are ready to combine this with the open string perspective.

### 4.1.3 Combining the perspectives

With having looked at both the open and closed string perspectives in the Maldacena limit, for the two different cases of $g_s N \ll 1$ and $g_s N \gg 1$, we can now combine them. In both cases we have supergravity on $\mathbb{R}^{1,9}$. At $g_s N \ll 1$ we have $\mathcal{N} = 4$ $U(N)$ SYM theory on $\mathbb{R}^{1,3}$ and at $g_s N \gg 1$ we have supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Since we started from the same physics for both these descriptions, and both contain supergravity on $\mathbb{R}^{1,9}$, this suggest that the two theories should describe the same physics.

Since we have taken two different limits for these descriptions, the idea is that the supergravity theory describes the physics of the SYM theory when $g_s N = g_{YM}^2 N = \lambda \gg 1$ and $N \to \infty$. This is the weakest form of the AdS/CFT correspondence. Maldacena's conjecture was that the full type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is dual to $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1,3}$ [3], and that this is the case for any value of $N$ and $\lambda$. This is the strongest form of the correspondence. There is also a strong form where the 't Hooft limit of our CFT is taken to be dual to classical type IIB string theory (classical meaning we only take tree-level diagrams into account). The strength essentially is settled by what range of the parameters $N$, $\lambda$ (on the field theory side) and $g_s$, $\alpha'/L^2$ (on the string theory side) we say that the correspondence applies for. The three cases are divided in table 4.2 below.

**Table 4.2:** Table describing the different conjectured dualities in the canonical example, ordered by strength. On each row, the theories on each side is conjectured to be dual to each other. The term classical here means that we only take tree level diagrams into account.

| | **Field theory side**<br>with coupling $g_{YM} = 2\pi g_s$ | $\mathbf{AdS_5 \times S^5}$ **side**<br>with curvature $L^4 = 4\pi g_s N \alpha'^2$ |
|---|---|---|
| **Strongest from** | $\mathcal{N} = 4$ $U(N)$<br>super Yang-Mills (CFT)<br>Any $\lambda$ and $N$ | Type IIB string theory on<br>$g_s \neq 0$, $\alpha'/L^2 \neq 0$ |
| **Strong form** | 't Hooft limit of $\mathcal{N} = 4$ $U(N)$<br>$\lambda$ fixed and $N \to \infty$ | Classical type IIB string theory<br>$g_s \to 0$, $\alpha'/L^2 \neq 0$ |
| **Weak from** | Large 't Hooft limit of $\mathcal{N} = 4$ $U(N)$<br>$\lambda$ large and $N \to \infty$ | Classial supergravity<br>$g_s \to 0$, $\alpha'/L^2 \to 0$ |

Despite the weak form being very limited for when it can be applied, it is still very useful, and it is in fact the form that is the basis for the calculations done for our model, in chapter 5 and chapter 6. This is the strong-weak duality that makes the AdS/CFT correspondence so attractive, since we can describe a strongly coupled field theory with classical supergravity.

As a last note, we should mention that when the symmetries of the theories are examined, one finds that they coincide completely, including in the strongest form of the correspondence [4]. This concludes our look into the canonical example.

## 4.2 The holographic dictionary

Now that we have seen the string-theoretical origin of the correspondence, we must find a way to actually use it. We need to be able to translate between quantities on the gravity and field theory side. This, as mentioned in the introduction, can be done via the GKWP rule. This rule was discovered by Gubser, Klebanov, Polyakov [27] and Witten [28]. It is a statement that there is a one to one correspondence between the generating functional of the boundary theory and the partition function in the bulk. More precisely it identifies the QFT generating functional with sources $J$ with the partition function of the bulk gravitational theory where the bulk is asymptotically AdS and the boundary value of a fields $\phi$ in the bulk is equated with the sources $J$ [27] [28].

$$Z_{QFT} = \int \mathcal{D}\phi e^{iS_{bulk}(\phi(x,r))|_{\phi(x,r=\infty)=J(x)}} \tag{4.12}$$

Just as the AdS/CFT correspondence itself, this rule is still a conjecture [1]. The GKWP rule can be used together with eq. (2.4) to calculate correlation functions, and will be demonstrated in the example of a scalar field.

Let us already now mention that the large $N$ limit will be key to utilising this rule. Let's recall the form of the generating functional in eq. (2.2)

$$Z_{QFT}[J] = \int \mathcal{D}\phi \exp\left[ i \int d^D x (\mathcal{L}_{QFT} + J_i \mathcal{O}_i) \right].$$

If we recall the discussion after eq. (2.9), the path integral will collapse to a single (on-shell) configuration in the large $N$ limit. On the right hand side of eq. (4.12) we can take the action $S_{bulk}$ to be on-shell [1].

With the GKWP rule we can derive the relation between field-theory quantities and gravitational fields with relative ease. Generally these will follow a pattern where the leading term in the large $r$ expansion of the field is identified as the source (as essentially prescribed by the rule), and the sub-leading term is identified with the expectation value of the operator [1]. We will see that this is the case for the scalar field below. There is however a way to switch the roles of the leading and sub-leading term, but we will go through this when exploring the scalar field example. For the D3D5D7 model that we examine in this thesis, we are mostly interested in the electromagnetic current. We will wait with this derivation until section 4.3, where we look at the electromagnetic field in AdS spacetime a bit more thoroughly.

Two identifications between field- and bulk quantities that we do not look at in detail but should mention for later reference is listed in the following table.

**Table 4.3:** Two identifications between field- and gravity-theory quantities from [1].

| Field theory | Gravity theory |
|---|---|
| Energy momentum tensor | Metric tensor |
| Chemical potential | Boundary value of the electrostatic potential $A_t$ |

### 4.2.1 Scalar field example and holographic renormalisation

Let us now turn to the scalar field example and at the same time describe the method of holographic renormalisation. We will for this part throw away all knowledge of the string theoretical origin of the correspondence and look at a free scalar field in AdS$_{D+1}$ space. We will see that the on-shell action becomes divergent. This divergence must be removed in order for our calculation of quantities on the field theory side to make sense. This is done through the framework of holographic renormalisation.

It is in fact so that the on-shell action in the AdS bulk is often divergent and therefore in need of renormalisation. Most often, these divergences appear due to the infinite volume of the AdS space [29]. Let's now describe the method used to remove these divergences, as prescribed in [30]. First, we assume that we have solved the equations of motion for the field(s) asymptotically for large $r$. Then we introduce a cut-off to the action, that is, we regularise the integral by introducing a cut-off, $r = R$ where $R$ will later go to infinity. If we insert our asymptotic solution, we see the behaviour of the now on-shell action for $R \to \infty$. We then introduce boundary terms. These are terms that live on the $r = R$ boundary of the cut offed version of our AdS space. These are designed in such a way, with appropriate coefficients, to cancel the divergences when we let $R \to \infty$.

The need for this procedure might seem strange at first, but we conjecture that the theory in the AdS bulk is dual to a quantum field theory, which are also plagued by divergences that need to be removed. If fact, when renormalising a QFT, cancellation of UV divergences do not depend on the IR physics. With the procedure of adding boundary terms, removing the large $r$ divergences (which correspond to

UV divergences for the field theory) is independent of the physics for smaller $r$ (which correspond to the IR physics of the field theory) [29]. Again, similar to the renormalisation of a QFT, we can add finite boundary terms if we wish [30], which we will do later to change boundary conditions. Note that there is a deep connection between the radial coordinate $r$ and renormalisation group flow of the field theory [1], however that is not covered in this thesis.

Let us now turn to the actual calculations for our scalar field in AdS$_{D+1}$ space, and show the procedure of holographic renormalisation. First, we need to solve the equations of motion, which in this case is the Klein-Gordon equation. The Klein Gordon (K.G.) action is

$$S_{KG} = \int d^{D+1}x\sqrt{-g}\Big( -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\Big). \tag{4.13}$$

The K.G. equation in a general metric is then

$$\big( -\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu + m^2\big)\phi = 0 \tag{4.14}$$

where $g = \det g_{\mu\nu}$. To solve this we need the metric of our AdS$_{D+1}$ spacetime, and for this we use the Poincaré patch [31], where

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^i dx^i) + \frac{L^2}{r^2}dr^2. \tag{4.15}$$

Here we have one time-coordinate, $D$ spatial coordinates where one of these is the radial coordinate, $r$. $L$ is the AdS scale parameter. We should note the likeness of the metric to the one obtained from the supergravity solution of a stack of D3-branes in eq. (4.9). It is also prudent to mention that the Poincaré patch only covers half of the AdS space. However, since we only want to find the general behaviour for the field at large $r$, solving the K.G. equation for this part of our space is sufficient, since a change of coordinates cannot change the behaviour. Our metric is therefore

$$g_{tt} = -\frac{r^2}{L^2},\ g_{rr} = \frac{L^2}{r^2} \text{ and } g_{ij} = \frac{r^2}{L^2}\delta_{ij}, \text{ with } i,j \in [2, D] \implies \sqrt{-g} = \Big(\frac{r}{L}\Big)^{D-1}.$$

This means that

$$\frac{1}{\sqrt{-g}}\partial_r\sqrt{-g}g^{rr}\partial_r = \Big(\frac{L}{r}\Big)^{D-1}\partial_r\bigg[\Big(\frac{r}{L}\Big)^{D-1}\frac{r^2}{L^2}\partial_r\bigg] = \frac{(D+1)r}{L^2}\partial_r + \frac{r^2}{L^2}\partial_r^2.$$

Now our K.G. equation becomes

$$\Big(\frac{r^2}{L^2}\partial_t^2 - \frac{(D+1)r}{L^2}\partial_r - \frac{r^2}{L^2}\partial_r^2 - \frac{r^2}{L^2}\partial_i^2 + m^2\Big)\phi = 0.$$

We rewrite this to

$$\Big(\partial_t^2 - \frac{D+1}{r}\partial_r - \partial_r^2 - \partial_i^2 + \frac{m^2L^2}{r^2}\Big)\phi = 0$$

and use separation of variables, $\phi = G(r)F(t,x)$. Inserting this into the equation above give us, where we suppress the coordinate dependence of $L$ and $F$,

$$\frac{1}{G}\Big(\frac{D+1}{r}G' + G'' - \frac{m^2L^2}{r^2}G\Big) = \frac{\partial_t^2F - \partial_i^2F}{F} \tag{4.16}$$

and so, both the R.H. and L.H. sides must be constant. The form of the L.H. side suggests that we make the ansatz $G(r) = r^a$ where $a$ is a constant. This give us

$$a\frac{D+1}{r^2} + a(a-1)\frac{1}{r^2} - \frac{m^2L^2}{r^2} = constant. \tag{4.17}$$

Now we note that we must have the *constant* $= 0$, since the only $r$-dependence in eq. (4.17) is the $\frac{1}{r^2}$ factor. Then eq. (4.17) reduces to

$$(D+1)a + a^2 - a - m^2L^2 = 0 \implies a = -\frac{D}{2} \pm \sqrt{\frac{D^2}{4} + m^2L^2}.$$

Defining $\Delta_\pm := \frac{D}{2} \pm \sqrt{\frac{D^2}{4} + m^2L^2}$ we get our two solutions as $G(r) = \frac{1}{r^{\Delta_\pm}}$ barring any constant of proportionality or linear combination of the solutions. Now we note that the R.H. side of eq. (4.16), with the L.H. side $= 0$, simply becomes the wave equation. Since eq. (4.14) is second order we must have two independent solutions, and since it is linear, we can make a linear combination of the two solutions. We can then write our solution as

$$\phi = \frac{\alpha(t,x)L^{\Delta_-}}{r^{\Delta_-}} + \frac{\beta(t,x)L^{\Delta_+}}{r^{\Delta_+}} + ... \tag{4.18}$$

Now we look closer at our Klein-Gordon action. Following the prescription of holographic renormalisation, we regularise it, and use integration by parts to find that

$$S_{KG} = \int d^Dxdr\sqrt{-g}\phi\Big(\partial_\mu\partial^\mu\phi - m^2\phi\Big) - \frac{1}{2}\int_{r=R} d^Dx\sqrt{-\gamma}\phi n^\mu\partial_\mu\phi \tag{4.19}$$

The second term in eq. (4.19) is a boundary term, where $\gamma_{\mu\nu}$ is the surface metric and $n^\mu$ is a unit normal vector. We have that $\sqrt{-\gamma} = \left(\frac{r}{L}\right)^D$ and $n^r = \frac{r}{L}$ as the only non-zero component, which means that $n^\mu\partial_\mu = \frac{r}{L}\partial_r$. Inserting our solution eq. (4.18) means that the bulk term vanishes. The on-shell action (denoted by the *) then becomes

$$\begin{aligned} S^*_{KG} =& \frac{1}{2}\int_{r=R} d^Dx\left(\frac{R}{L}\right)^{D+1}\Big(\frac{\alpha L^{\Delta_-}}{R^{\Delta_-}} + \frac{\beta L^{\Delta_+}}{R^{\Delta_+}}\Big)\Big(\frac{\Delta_-\alpha L^{\Delta_-}}{R^{\Delta_-+1}} + \frac{\Delta_+\beta L^{\Delta_+}}{R^{\Delta_++1}}\Big) = \\ & \frac{1}{2L}\int_{r=R} d^Dx\left(\frac{R}{L}\right)^{D}\Big(\frac{\Delta_-\alpha^2L^{2\Delta_-}}{R^{2\Delta_-}} + \frac{D\alpha\beta L^D}{R^D} + \frac{\Delta_+\beta^2L^{2\Delta_+}}{R^{2\Delta_+}}\Big) = \\ & \frac{1}{2L}\int_{r=R} d^Dx\Big(\Delta_-R^{D-2\Delta_-}\alpha^2L^{2\Delta_--D} + D\alpha\beta + \frac{\Delta_+\beta^2L^{2\Delta_+-D}}{R^{2\Delta_+-D}}\Big). \end{aligned} \tag{4.20}$$

We note now that this is not finite when $R \to \infty$. We must therefore renormalise our action with a boundary term, which can be chosen as

$$\begin{aligned} S_{bdy} =& -\frac{\Delta_-}{2L}\int_{r=R} d^Dx\sqrt{-\gamma}\phi^2 = \\ & -\frac{\Delta_-}{2L}\int_{r=R} d^Dx\Big(R^{D-2\Delta_-}\alpha^2L^{2\Delta_--D} + 2\alpha\beta + \frac{\beta^2L^{2\Delta_+-D}}{R^{2\Delta_+-D}}\Big), \end{aligned}$$

where we have inserted our solution eq. (4.18) in the second line. Adding this boundary term to our action it give us

$$S^* = S^*_{KG} + S^*_{bdy} = \frac{(\Delta_+ - \Delta_-)}{2L}\int_{r=R} d^Dx\alpha\beta$$

when we let $R \to \infty$. The on-shell action has been made finite. We are now almost ready to calculate the correlation function. However, the keen reader will remember that the prescription of the GKWP rule, requires us to set the leading term of the field equal to the source in the generating functional of the QFT. Let us look at what this means below.

**Boundary conditions and correlation functions**

A familiar result from general relativity is that, in AdS space, a light-signal can travel an infinite distance in finite time. This means that what happens at $r \to \infty$ can affect the rest of the spacetime [4]. So we need to impose boundary conditions at $r = R$ even when we let $R \to \infty$ in order to make $\delta S = 0$. What this boundary condition is, is also closely related to the calculations of correlation functions of the field theory using the GKWP rule. Let us now look at the example of a scalar field, where we have just finished renormalising the action.

The first step we must take is to vary the action, starting from

$$S = S_{KG} + S_{bdy} = -\frac{1}{2}\int d^D x dr \sqrt{-g}\Big(\partial_\mu \phi \partial^\mu \phi + m^2\phi^2\Big) - \frac{\Delta_-}{4L}\int_{r=R} d^D x\sqrt{-\gamma}\phi^2. \tag{4.21}$$

Let us now vary $\phi$ around the solution to the e.o.m. (eq. (4.18)), by $\phi \to \phi + \delta\phi$ with

$$\delta\phi = \frac{\delta\alpha(t,x)L^{\Delta_-}}{r^{\Delta_-}} + \frac{\delta\beta(t,x)L^{\Delta_+}}{r^{\Delta_+}} + ... \tag{4.22}$$

The following calculations are very similar to when we renormalised the action, so we will be brief. The first term becomes

$$\begin{aligned}&\delta S^*_{KG} = \int d^D x dr\sqrt{-g}\Big(-\partial_\mu\phi\partial_\mu\delta\phi - m^2\phi\delta\phi\Big) = \\ &\int d^D x dr\sqrt{-g}\delta\phi\Big(\partial_\mu\partial^\mu\phi - m^2\phi\Big) - \int_{r=R} d^D x\sqrt{-\gamma}\delta\phi n^\mu\partial_\mu\phi = \\ &\frac{1}{L}\int d^D x\Big(\Delta_- R^{D-2\Delta_-}\delta\alpha\alpha L^{2\Delta_-} + \Delta_+\delta\alpha\beta L^D + \Delta_-\delta\beta\alpha L^D + \frac{\Delta_+\delta\beta\beta L^{2\Delta_+}}{R^{2\Delta_+-D}}\Big).\end{aligned} \tag{4.23}$$

On the third line, we inserted eq. (4.18) and eq. (4.22), and used that $r = R$. The second term becomes

$$\begin{aligned}&\delta S^*_{bdy} = -\frac{\Delta_-}{L}\int_{r=R} d^D x\left(\frac{R}{L}\right)^D\delta\phi\phi = \\ &-\frac{\Delta_-}{L}\int d^D x\Big(R^{D-2\Delta_-}\delta\alpha\alpha L^{2\Delta_-} + \delta\alpha\beta L^D + \delta\beta\alpha L^D + \frac{\delta\beta\beta L^{2\Delta_+}}{R^{2\Delta_+-D}}\Big).\end{aligned} \tag{4.24}$$

And so, we find that, ignoring all terms that go to 0 when $R \to \infty$,

$$\delta S^* = \frac{\Delta_+ - \Delta_-}{L}\int d^D x\delta\alpha\beta. \tag{4.25}$$

This means that we must have $\delta\alpha = 0$ at the boundary. Note that we could not have chosen the proportionality constant in $S_{bdy}$ so that we would require $\delta\beta = 0$, since we still needed to cancel a divergent term.

Now let's turn to the GKWP rule. If we set $\alpha = J(x,t)$ on the boundary then $\delta\alpha = 0$ as required. Setting $\alpha = J(x,t)$ also makes sense from the fact that, in the GKWP rule, we should set the boundary value of $\phi$ equal to the source $J$, and $\alpha L^{\Delta_-}/r^{\Delta_-}$ is the leading term at $r \to \infty$. Now we utilise the fact that we are in the large $N$-limit, meaning that the path integral collapses to a single (on-shell) configuration, as mention early in section 4.2. Letting $S^*$ denote the on-shell action, we then have that, for a 1-point correlation function,

$$\langle\mathcal{O}\rangle = -i\frac{1}{Z[J=0]}\frac{\delta Z[J]}{\delta J(x)}\Big|_{J=0} = -ie^{-iS^*}\frac{\delta e^{iS^*}}{\delta\alpha}\Big|_{\alpha=0} = \frac{\delta S^*}{\delta\alpha}\Big|_{\alpha=0} = \frac{\Delta_+ - \Delta_-}{L}\beta(t,x). \tag{4.26}$$

This means that the response $\langle \mathcal{O} \rangle$ to the source $J$ is proportional to the sub-leading term of $\phi$, when the source is proportional to the leading term. This follows the pattern that was mentioned early in section 4.2.

Now comes the question if this is the only choice of boundary condition and identifications for the response and source. The answer turns out to be no. We can in fact change these by utilising our freedom to add another boundary term. This is sometimes referred to as alternative quantisation [31]. The boundary term that needs to be added is

$$S_{bdy,2} = \int_{r=R} d^D x \sqrt{-\gamma} \phi n^\mu \partial_\mu \phi. \tag{4.27}$$

Varying this around our solution eq. (4.18) give us

$$\delta S_{bdy,2} = \int_{r=R} d^D x \sqrt{-\gamma} (\delta\phi n^\mu \partial_\mu \phi + \phi n^\mu \partial_\mu \delta\phi). \tag{4.28}$$

We immediately see that the first part is just $-\delta S_{KG}$. The second part becomes

$$-\frac{1}{L} \int d^D x \Big( \Delta_- R^{D-2\Delta_-} \delta\alpha\alpha L^{2\Delta_-} + \Delta_- \delta\alpha\beta L^D + \Delta_+ \delta\beta\alpha L^D + \frac{\Delta_+ \delta\beta\beta L^{2\Delta_+}}{R^{2\Delta_+ - D}} \Big). \tag{4.29}$$

If we also change the sign in front of $S_{bdy}$, the result is that

$$\delta S^*_{alt} = \delta S^*_{KG} - \delta S^*_{bdy} + \delta S^*_{bdy,2} = \frac{(\Delta_- - \Delta_+)}{L} \int d^D x \delta\beta\alpha. \tag{4.30}$$

Now we see that the roles of $\alpha$ and $\beta$ are reversed, since $\beta$ needs to be fixed at the boundary to make $\delta S = 0$. We should note however that this is only possible to do in a certain range of the value of $m^2 L^2$. It needs to be in the range [31]

$$-\frac{D^2}{4} < m^2 L^2 < -\frac{D^2}{4} + 1 \tag{4.31}$$

Notice that this implies a negative mass squared. In flat spacetime this would signify a tachyonic state. However in AdS spacetime, the curvature of the spacetime contributes to the effective potential of the field in such a way that the field remains stable even for negative mass squared down to $m^2 L^2 > -\frac{D^2}{4}$. This is called the Breitenlohner-Freedman bound. We can notice that for values of $m^2 L^2$ below this bound that the $\Delta_\pm$ would become complex. Since the scaling dimension of the operator is in this case become $\Delta_-$, the second restriction, $m^2 L^2 < \frac{D^2}{4} + 1$ comes from the fact that the operator needs to be unitary. This means that we require $\Delta_- < \frac{D}{2} - 1$ [1]. This is why the alternative quantisation scheme is only possible in the range given in eq. (4.31).

This concludes our examination of the scalar field example. The most important concepts to remember for further sections are the calculation of correlation functions as well as the method used to change the boundary conditions.

### 4.2.2 AdS/CFT at finite temperature

A very important step is to look at the duality at a finite temperature. This can be achieved by having a black hole with non-zero horizon radius $r_h$ in our AdS spacetime, transforming it to an AdS-Schwarzschild spacetime. We can in fact have a supergravity solution to our stack of D3-branes that incorporates a horizon. It takes the following form [4]

$$ds^2 = h(r)^{-1/2} \big( -f(r) dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \big) + \frac{(r)^{1/2}}{f(r)} dr^2 + L^2 ds^2_{S^5} \tag{4.32}$$

where $f(r) = 1 - \left(\frac{r_h}{r}\right)^4$ is the so-called emblackening factor. In the near horizon limit, $r \ll L$, we can see that this metric is asymptotically $\text{AdS}_5 \times \text{S}^5$ when $r \gg r_h$.

This black hole will have a Hawking temperature that is proportional to $r_h$, and this temperature will also be the temperature of the field theory [1]. It can be calculated by Wick-rotating to a Euclidean metric.

When using the real-time formalism, as we will be doing in the later chapters, it is also important to look at the boundary condition near the black hole horizon. These however cannot be changed by adding boundary terms. When solving the equations of motion near the horizon, one will find two solutions. One that falls into the horizon and one that comes out from the horizon. It can be shown that these correspond to retarded and advanced greens functions [11]. To maintain causality, we must therefore choose the in-falling solution, which is called the in-falling boundary condition. This is also consistent with the idea of a black hole. The in-falling boundary condition is simply that modes at the horizon must fall into the black hole, and not emerge from it.

## 4.3 Electrodynamics

One important calculation will be that of electrodynamical relations on the boundary by using the electrodynamics of the bulk theory. In this thesis we will mostly be concerned with dispersion relations and the electric current. This means we need a method to find relations between the electrodynamics of the bulk and boundary. We will do this in $D+1$-dimensional AdS space with a $D$-dimensional boundary. We will ignore the added complexity of the compact dimensions and resulting Kaluza-Klein tower of fields inferred in top-down constructions.

We begin by assuming a Lagrangian $\mathcal{L}$ that depends on $F_{\mu\nu}$ and none of its derivatives, just as in section 2.3. Note that it also includes the weight factor $\sqrt{-g}$ for the integral. We then vary the action by letting $A_\mu \to A_\mu + \epsilon\eta_\mu \implies F_{\mu\nu} \to F_{\mu\nu} + \epsilon\zeta_{\mu\nu}$ where $\zeta_{\mu\nu} = \partial_\mu\eta_\nu - \partial_\nu\eta_\mu$. We have that

$$
\begin{aligned}
&\frac{\partial S}{\partial \epsilon} = \int_M d^{D+1}x \frac{\partial \mathcal{L}}{\partial \epsilon} = \int_M d^{D+1}x \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\zeta_{\mu\nu} = \int_M d^{D+1}x \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\partial_\mu\eta_\nu = \\
&\int_M d^{D+1}x\Big( -\partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu + \partial_\mu\big(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu\big)\Big) = -\int_M d^{D+1}x \partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu + \int_{\partial M} d^D x \frac{\sqrt{-\gamma}}{\sqrt{-g}} n_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu
\end{aligned}
\tag{4.33}
$$

where $M$ is the AdS space, $\gamma$ is the determinant of the boundary metric and the factor $\frac{\sqrt{-\gamma}}{\sqrt{-g}}$ is there to compensate for the fact that $\mathcal{L}$ already includes $\sqrt{-g}$. The boundary normal $n_\mu$ only points in the radial direction of the AdS space. This means that the last component in the last equality simply becomes $\int_{\partial M} d^D x n_r \frac{\partial \mathcal{L}}{\partial F_{r\nu}}\eta_\nu$. From eq. (4.33) we can also obtain the equations of motion $\partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu = 0$, and boundary condition $\eta_\mu\big|_{\partial M} = 0$, which is a Dirichlet boundary condition.

We now need to translate this to the electrodynamics at the field theory side. First we recognise that the source for the field theory current $\mathcal{J}^\mu$ is the field theory Maxwell field $\mathcal{A}_\mu$ due to the way we add the current to the electromagnetic action [32]. When trying to translate between the Maxwell field on the two sides of the correspondence, one consistent way to achieve this is to identify the field-strengths with each other, in the case of regular Maxwell electromagnetism in the bulk. This is because $F_{\mu\nu}$ is gauge-invariant. However, if we choose to work in the radial gauge, in which $A_r \equiv 0$, we can identify the vector potentials on the two sides with each other. That is, we set the boundary value of the vector potential equal to the source field $\mathcal{A}_\mu$, meaning $A_\mu(r \to \infty) = A_\mu^{(0)} = \mathcal{A}_\mu$ up to a constant. This is useful when applying the GKWP rule, which then give us that

$$\langle \mathcal{J}^{\mu} \rangle = \frac{\delta S^*}{\delta A_\mu}\bigg|_{r\to\infty} \tag{4.34}$$

where we keep in mind that the * denotes that the action is on-shell. In order to make the following calculation clear we must utilise the definition of the functional derivative. For the unfamiliar reader, [33] gives a good explanation. We Taylor-expand the Lagrangian and the bulk part becomes

$$\begin{aligned}
&\frac{\delta S}{\delta A_\lambda(y)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[\int_M d^{D+1}x\mathcal{L}(x, A_\nu(x) + \delta^\lambda_\nu \epsilon\delta^{(D)}(x-y), F_{\mu\nu}(x) + \partial_\mu\delta^\lambda_\nu\epsilon\delta^{(D)}(x-y) - \\
&\partial_\nu\delta^\lambda_\mu\epsilon\delta^{(D)}(x-y)) - \int_M d^{D+1}x\mathcal{L}(x, A_\nu(x), F_{\mu\nu}(x))\Bigg] = \\
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[\int_M d^{D+1}x\frac{\partial\mathcal{L}}{\partial A_\nu}\delta^\lambda_\nu\epsilon\delta^{(D)}(x-y) + \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\partial_\mu\left(\delta^\lambda_\nu\epsilon\delta^{(D)}(x-y)\right)\Bigg] = \\
&\int_M dr\left(\frac{\partial\mathcal{L}}{\partial A_\lambda} - \partial_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}\right) + \int_{\partial M} d^D x\frac{\sqrt{-\gamma}}{\sqrt{-g}}n_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}\delta^{(D)}(x-y) = \frac{\sqrt{-\gamma}}{\sqrt{-g}}n_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}\bigg|_{\partial M}
\end{aligned} \tag{4.35}$$

where we have used our definition for the functional derivative with respect to $F_{\mu\nu}$. We therefore have that

$$\langle \mathcal{J}^{\lambda} \rangle = \frac{\sqrt{-\gamma}}{\sqrt{-g}}n_r\frac{\partial\mathcal{L}}{\partial F_{r\lambda}}\bigg|_{\partial M}. \tag{4.36}$$

Note that we must insert a solution to the equations of motion for this apply, since we need it to be on-shell. We have also assumed that the on-shell action does not require renormalisation by use of boundary terms, which will turn out be the case when this analysis is used in chapter 6. This does not apply in general.

As a consistency check, we make sure that the field-theory current is conserved, by observing that

$$\nabla_\lambda\mathcal{J}^\lambda = \frac{\sqrt{-\gamma}}{\sqrt{-g}}n_r\partial_\lambda\frac{\partial\mathcal{L}}{\partial F_{r\lambda}} = 0 \tag{4.37}$$

by the equations of motion $\partial_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu = 0$. We have used that $\lambda$ do not range over $r$, and the calculation is assumed to take place on the boundary. This gives us some extra assurance that this identification is the correct one.

### 4.3.1 Changing boundary conditions

We now want to change our boundary conditions in the case of electromagnetism. The motivation for this is that we want to end up with Maxwell dynamics on the boundary [32]. As we want to model a condensed matter system that follow Maxwell dynamics, this is an important step. Why the change of boundary conditions leads to Maxwell dynamics will be made clear in the analysis below. As in section 4.2.1, changing the boundary conditions is done by adding boundary terms to the action. This analysis will be used later in section 6.2, where the effect of the different boundary conditions will be discussed.

When varying the action by $A_\mu \to A_\mu + \epsilon\eta_\mu$, as done in eq. (4.33), we find that in order to make it stationary, we must have the variation $\eta_\mu = 0$ at the boundary. To see what sort of boundary conditions we can have, we must look at what type of boundary terms we can have. The requirement on the boundary terms is that they are Lorentz invariant, and built from the boundary-values of the fields present in the bulk. There are then two types of boundary terms that we can add, a "$JA$"-term and an "$F^2$"-term. The "$JA$"-term is built from the boundary current and the gauge-field on the boundary. We should also remember that the addition of these boundary terms does not change the identification of the current. This will be explained at the end of this section.

Let us first look at the "$JA$"-term, which has the form

$$S_{JA} = \int_{\partial M} d^D x \frac{\sqrt{-\gamma}}{\sqrt{-g}} n_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} A_\nu \tag{4.38}$$

where $\mathcal{L}$ is the bulk Lagrangian. We then vary our boundary term by $A_\mu \to A_\mu + \epsilon\eta_\mu$ and find that

$$\begin{aligned}\frac{\partial S_{JA}}{\partial \epsilon} &= \int_{\partial M} d^D x \frac{\sqrt{-\gamma}}{\sqrt{-g}} n_\mu \frac{\partial}{\partial \epsilon}\Big(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\left(A_\nu + \epsilon\eta_\nu\right)\Big) = \\ &\int_{\partial M} d^D x \frac{\sqrt{-\gamma}}{\sqrt{-g}} n_\mu \left[\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\eta_\nu + \frac{\partial}{\partial \epsilon}\Big(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\Big) A_\nu\right].\end{aligned} \tag{4.39}$$

We note that the first term in the second line matches the last term in eq. (4.33). We can then let this terms cancel each other out in the variation by taking $S - S_{JA}$. With this configuration, we would have a boundary condition on the form

$$\frac{\sqrt{-\gamma}}{\sqrt{-g}} n_\mu \frac{\partial}{\partial \epsilon}\Big(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\Big) = 0 \tag{4.40}$$

since we would otherwise have to impose $A_\nu = 0$ at the boundary. This would not be wanted as it is related to the gauge-field on the field-theory side. We can note that if $\mathcal{L}$ would be the regular Maxwell Lagrangian (with the weight factor) $\mathcal{L} = -\frac{\sqrt{-g}}{4} F^{\mu\nu} F_{\mu\nu}$, then the boundary condition would look like

$$\sqrt{-\gamma} n_\mu \gamma^{\mu\sigma}\gamma^{\nu\lambda}\partial_{[\sigma}\eta_{\lambda]} = 0. \tag{4.41}$$

We know that $n_r$ is the only non-zero component of $n_\mu$. If we momentarily use our gauge freedom to set $A_r = 0$ (and therefore $\eta_r = 0$), which will be used in section 6.2, combined with the fact that $\gamma^{\mu\nu}$ is diagonal, the condition becomes

$$\sqrt{-\gamma} n_r \gamma^{rr}\gamma^{\nu\lambda}\partial_r \eta_\lambda = 0. \tag{4.42}$$

As we can see, this is the form of a Neumann boundary condition.

Lets now turn to the "$F^2$"-term, which simply looks like

$$S_{F^2} = \frac{\lambda^{-1}}{4}\int_{\partial M} d^D x \sqrt{-\gamma} F^{\mu\nu} F_{\mu\nu} \tag{4.43}$$

where $\lambda$ is a constant relating the electromagnetic coupling of the bulk and boundary. Varying this we find

$$\begin{aligned}\frac{\partial S_{F^2}}{\partial \epsilon} &= \frac{\lambda^{-1}}{2}\int_{\partial M} d^D x \sqrt{-\gamma} F^{\mu\nu}\partial_{[\mu}\eta_{\nu]} = \lambda^{-1}\int_{\partial M} d^D x \sqrt{-\gamma}\gamma^{\mu\sigma}\gamma^{\nu\lambda}\partial_\sigma A_\lambda \partial_{[\mu}\eta_{\nu]} = \\ &- \lambda^{-1}\int_{\partial M} d^D x \sqrt{-\gamma}\gamma^{\nu\lambda} A_\lambda \partial^\mu \partial_{[\mu}\eta_{\nu]}\end{aligned} \tag{4.44}$$

where we performed a partial integration in the last step. Since $\partial M$ itself do not have a boundary, there is no extra term from the partial integration. Putting it all together we find that

$$\frac{\partial}{\partial \epsilon}(S - S_{JA} + C_{F^2} S_{F^2}) = -\int_{\partial M} d^D x \sqrt{-\gamma} A_\mu \left[\frac{n_\sigma}{\sqrt{-g}}\frac{\partial}{\partial \epsilon}\Big(\frac{\partial \mathcal{L}}{\partial F_{\sigma\mu}}\Big) + \lambda^{-1} C_{F^2}\gamma^{\nu\mu}\partial^\sigma \partial_{[\sigma}\eta_{\nu]}\right]. \tag{4.45}$$

$C_{F^2}$ is a constant that we can choose as we want. Since we do not wish to impose further restrictions on the boundary value of $A_\mu$ the boundary condition must become

$$\sqrt{-\gamma}\left[\frac{n_\sigma}{\sqrt{-g}}\frac{\partial}{\partial\epsilon}\left(\frac{\partial\mathcal{L}}{\partial F_{\sigma\mu}}\right)+\lambda^{-1}C_{F^2}\gamma^{\nu\mu}\partial^\sigma\partial_{[\sigma}\eta_{\nu]}\right]=0. \tag{4.46}$$

This is what is called a mixed boundary condition since it is a ”mix” between Dirichlet and Neumann boundary conditions. The reason why this leads to Maxwell dynamics on the boundary should now be clear, as we have essentially imposed the Maxwell action on the boundary.

This mixed boundary condition is equivalent to a double trace deformation on the boundary field theory [34]. A quick explanation of double trace deformations is that they are related to the random phase approximation often used in condensed matter physics [1], where the response function takes the form

$$\big\langle\mathcal{O}(-k)\mathcal{O}(k)\big\rangle=\frac{G(k)}{1+fG(k)} \tag{4.47}$$

where $G$ is the non-RPA greens function.

In order to incorporate a double trace deformation, we must keep our current identification as in eq. (4.36) [34]. It can also be shown that the incorporation of this double trace deformation leads to plasmonic dispersion relations on the field theory side, as it is equivalent to eq. (4.46) [35] [32]. As mentioned in the introduction, plasmonic dispersion relations is something that is found in strange metals.

# 5

# D3-D5-D7 model

At last I will present the model that this thesis is about. When creating a top-down model, we use D-branes to which the open strings are attached. One well studied system is based upon a large number, $N_c$, of D3-branes, as presented in section 4.1. One can then add D7-branes in the so-called probe-approximation. This means that one assumes that the D7 branes do not affect the geometry of the system. On the field theory side this corresponds to ignoring quark loops, that is, we find ourselves in the quenched approximation [18]. In order to realise this, the number of D7-branes have to be small [4]. As by the name of our model, we will also add a D7-brane in the probe-approximation. However, we will first add a large number, $N_f$, of D5-branes. This will alter the geometry of our spacetime and subsequently change the physics of the boundary theory. An important aspect in this model is that there are a lot of different strings that stretch between the different D-branes, so let us first look at how we can deal with these strings in a known example, and what they mean in terms of the AdS/CFT correspondence.

The additional open strings introduced by the new D-branes will be part of a duality between the different open strings [4]. That is, we get a open-open string duality, compared to the open-closed string duality in section 4.1. Let us look at this in the example of a probe D7-brane added to the $N$ D3-branes in the canonical example. On the gravity side, we have strings that begin and end on the D7-brane which are dual to the strings that stretch between the D7-brane and the D3-branes on the field theory side. This means that, on the gravity side, we only consider strings that both begin and end on the D7-brane. The strings that stretch between the D7-brane and the D3-branes will be part of the field theory side of the correspondence [4].

The original use for introducing probe branes was to insert flavour into the duality [36]. This is because the degrees of freedom introduced by the addition of the D7-branes transform in the fundamental representation of the gauge group [4]. Recall that the degrees of freedom from the $N_c$ D3-branes transform in the adjoint representation. For our model, before adding the D7 probe branes we will add a large number $N_f$ of D5-branes. This will alter the geometry of our bulk spacetime, creating an ansisotrophy, as will be explained below. We will first look at the geometry of the spacetime, then look at what this means for the D7 probe-branes and lastly look at the contents of the model.

Let us also mention Kaluza-Klein reductions. Later on, we will assume that the fields on the D7 probe brane do not propagate in the compact directions. In the general case, there would be a infinite tower of fields. Since the directions that gives rise to this behaviour are compact, the momentum in these directions will be discretised. This in turn gives rise to an energy-gap. This means that, at least for fluctuations, we can ignore the compact direction [1]. However, we also want this to hold for the full (or linear response) equations of motion. For this we assume that the solutions to the truncated equations of motion (those that ignore the compact directions) also solves the equations of motion when the compact directions are included. If this holds, there is still a potential issue that the solution might not be a stable solution to the full equations [1]. This is not something that we will examine in detail, but rather we will simply hope that the solution to the truncated equations of motion solve the full version of the same, and that this solution is stable.

## 5.1 D3-D5 geometry

Let us now look at the geometry, which was first derived in [37]. The setup of the D3 and D5 branes are shown in table 5.1 below.

**Table 5.1:** The setup for the D3 and D5 branes. The $\times$ marks a directions tangential to the brane, the - marks directions normal to the branes.

| Direction: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D3: | $\times$ | $\times$ | $\times$ | $\times$ | - | - | - | - | - | - |
| D5: | $\times$ | $\times$ | $\times$ | - | $\times$ | $\times$ | $\times$ | - | - | - |

The smearing approximation is used for the D5 branes. This approximation means that the branes are not viewed as having 0 width in the direction(s) normal to the branes, but rather that they are smeared across the $x^3$ direction, and we consider a continuous distribution of branes. The smearing approximation is used because trying to find the resulting metric becomes very difficult otherwise, due to the fact that the mass-distribution will consist of delta-functions. This only works in the limit of large $N_f$, with $N_f/N_c \equiv \lambda_{fc}$ kept fixed, which is referred to as the Veneziano limit [37]. The resulting metric in the Einstein frame can be written as [7] [38]

$$\begin{aligned} ds_{10}^2 =& \frac{1}{\sqrt{h}}\Big[-B(dx^0)^2+(dx^1)^2+(dx^2)^2+e^{-2\phi}(dx^3)^2\Big]+\frac{\sqrt{h}}{B}dr^2+\\ &+\bar{R}^2\Big[d\chi^2+\frac{\cos^2\chi}{4}((\omega^1)^2+(\omega^2)^2)+\frac{\cos^2\chi\sin^2\chi}{4}(\omega^3)^2+\frac{1}{b}\Big(d\tau+\frac{1}{2}\cos^2\chi\omega^3\Big)^2\Big] \end{aligned} \tag{5.1}$$

where $h = \frac{R^4}{r^4}$ is the warp factor, $B = 1-\left(\frac{r_h}{r}\right)^{10/3}$ is the emblackening factor, $\bar{R}^2 = R^2/b$ and $b = \frac{8}{9}$. $R$ is now the curvature of the AdS space and is related to $N_c$ by [38]

$$R^4 = \frac{4096}{1215}\pi N_c \alpha'^2 g_s. \tag{5.2}$$

$r_h$ is the radius of the black hole horizon. $\phi$ is the string-theory dilaton and we have

$$e^{\phi} = \left(\frac{3r}{4Q_f}\right)^{2/3} g_s \text{ where } Q_f = \frac{4\pi N_f}{9\sqrt{3}} l_s. \tag{5.3}$$

$N_f$ is the number of D5-branes per unit length in the $x^3$-direction. I should mention that the $g_s$ factor for the dilaton lacks solid mathematical foundation. The sources for our geometry, [37] [38] [7] do not explicitly say this, as they work in $g_s = 1$. This can be seen as a bit strange as we would want $g_s \to 0$, as we are using the weak form of AdS/CFT. My argument for this $g_s$-factor is that it is present in the ansatz for the supergravity solution for a stack of $D_p$-branes, see section 3.3.2, eq. (3.19).

Moving on, for the angles, $\chi \in [0,\pi]$, $\tau \in [0, 2\pi]$ and the $\omega^i$ are left-invariant SU(2) one-forms,

$$\begin{aligned} \omega^1 &= \cos\psi d\theta + \sin\psi\sin\theta d\varphi \\ \omega^2 &= \sin\psi d\theta - \cos\psi\sin\theta d\varphi \\ \omega^3 &= d\psi + \cos\theta d\varphi \end{aligned} \tag{5.4}$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$ [7]. As we can observe there is a horizon at $r = r_h$ and as $r$ becomes very large, which means that $B \to 1$, the non-compact part of the metric in eq. (5.1) takes the form of an AdS metric, barring the peculiar $x^3$-direction.

Let us take a closer look at this $x^3$-direction. As we approach the boundary of the AdS space, this seem to vanish compared to the $x^0, x^1$ and $x^2$ directions, as it no longer contributes to $ds^2_{10}$. Essentially this direction is not being pushed open as much as the other, since the D5-branes do not extend in this direction. As can be viewed in fig. 5.1, the $x^3$-direction eventually collapses in comparison to the other non-compact directions. This collapse is what gives us our layered system.

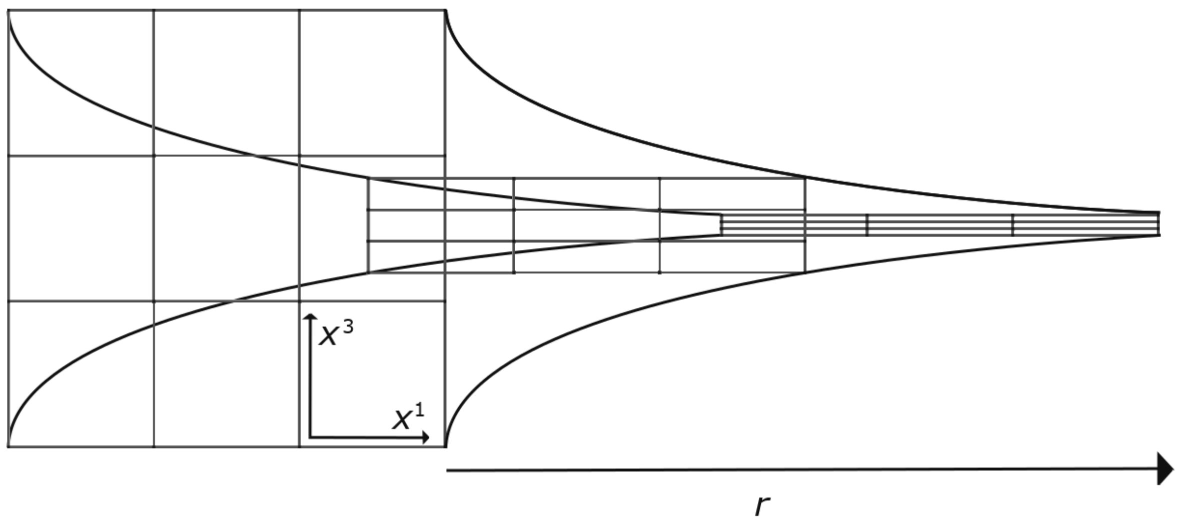



**Figure 5.1:** Visualisation of the different scaling for the $x^3$- and $x^1$ directions. As $r$ increases, we let the $x^1$-lines stay constant, but then the $x^3$-lines must move closer together due to the different metric components. As $r \to \infty$, the $x^3$-direction have no extension compared to the $x^1$-direction (as well as $x^0$ and $x^2$).

Let us also briefly compare eq. (5.1) to the metric obtained in case of only $N_c$ D3-branes

$$ds^2 = \frac{r^2}{R^2}(f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2$$

where $f(r) = 1 - \left(\frac{r_h}{r}\right)^4$ as the emblackening factor in this case. From this we can directly see that we can not just take the limit $N_f \to 0$ to arrive back at this metric [38]. This is understandable since the derivation of our metric eq. (5.1) required $N_f$ to be large.

As a last note, find the Hawking temperature $T$ of the black hole via the relation [7]

$$T = \frac{1}{2\pi}\left[\frac{1}{\sqrt{g_{rr}}}\frac{d}{dr}(\sqrt{-g_{x^0x^0}}\right]_{r=r_h} = \frac{5r_h}{6\pi R^2}. \tag{5.5}$$

As mentioned in section 4.2.2, this will be the temperature of our field theory.

## 5.2 D7 probe brane

Now we add a single D7-brane, in the probe approximation, to the geometry. The inclusion of a D7 probe brane was first explored in [7]. This brane is made to extend in the $x^0, x^1, x^2, x^3$- and $r$-directions as well as in the three sphere that correspond to the one-forms $\omega^i$ [7]. The setup looks like the following

**Table 5.2:** The setup for all the D-branes in our model. As before, the $\times$ marks a directions tangential to the brane, the - marks directions normal to the branes.

| Direction: | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $r$ | 5 | 6 | 7 | $\chi$ | $\tau$ |
|---|---|---|---|---|---|---|---|---|---|---|
| D3: | $\times$ | $\times$ | $\times$ | $\times$ | - | - | - | - | - | - |
| D5: | $\times$ | $\times$ | $\times$ | - | $\times$ | $\times$ | $\times$ | - | - | - |
| D7: | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | - | - |

The D7-brane do not extend in the $\chi$ or $\tau$-direction, and we will need to specify the brane's position in these directions, also referred to as the embedding of the D7-brane. We can, without loss of generality, set $\tau = const$ and consider the embedding $\chi$ as a function of $r$ [7]. This means that $d\chi^2 = \chi'^2 dr^2$. Now, the induced metric for the D7-brane world-volume, with the expressions for the one-forms inserted, is

$$\begin{aligned} ds_8^2 =& \frac{1}{\sqrt{h}}\Big[-B(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + e^{-2\phi}(dx^3)^2\Big] + \sqrt{h}\Big[\frac{1}{B} + \frac{r^2}{b}\chi'^2\Big]dr^2 + \\ &+ \frac{r^2\sqrt{h}}{4b}\cos^2\chi\Big[d\theta^2 + \big(\sin^2\theta + (1 - \frac{1-b}{b}\cos^2\chi)\cos^2\theta\big)d\varphi^2 + \\ &\Big(1 + \frac{1-b}{b}\cos^2\chi\Big)(d\psi^2 + 2\cos\theta d\psi d\varphi)\Big]. \end{aligned} \tag{5.6}$$

Note that this is given in the Einstein frame.

With the induced metric on the D7-brane at hand, let us next turn to the action of the brane. From eq. (3.16), the DBI part takes the general form of

$$S_{DBI} = -\tau_7 \int d^8\xi e^{\phi}\sqrt{-\det\big(P[g]_{\mu\nu} + e^{-\phi/2}P[B]_{\mu\nu} + e^{-\phi/2}F_{\mu\nu}\big)}$$

in the Einstein frame. The Wess-Zumino term gives no contribution to the action of the D7-brane. This is because of the pullback of the Ramond-Ramond seven form present in the background vanishes for our embedding [7]. Let us also look at the Kalb-Ramond field. In section 3.3.2 the Kalb-Ramond field was set to 0 in the ansatz for the supergravity-solution of a D$p$-brane. Since this applied for any (odd) $p$, we should be able to assume that it also holds for this system (note that this is not mentioned in [37], [38] or [7]). Assuming this holds, this means that the $P[B]_{\mu\nu}$ term in the DBI action disappears and we have that it ends up on the form

$$S_{\mathrm{D7}} = -\tau_7 \int d^8x e^{\phi}\sqrt{-\det\big(g_{\mu\nu} + e^{-\phi/2}F_{\mu\nu}\big)}. \tag{5.7}$$

As we can note, the Lagrangian density in this action contains the weight-factor $\sqrt{-g}$. For our later calculations it is therefore useful to know that it is

$$\sqrt{-g} = \frac{r^3 e^{-\phi}\sin(\theta)}{8b^2}\cos^3(\chi)\sqrt{\left[1 + \frac{1-b}{b}\cos^2(\chi(r))\right](b + Br^2\chi'(r))} \tag{5.8}$$

and in the case of $\chi \equiv 0$ (called the massless case) [7] we have

$$\sqrt{-g} = \frac{r^3 e^{-\phi} \sin(\theta)}{8b^2}. \tag{5.9}$$

Let us here take a brief look at what is meant by that $\chi \equiv 0$ is called the massless case, as well as a bit about what $\chi$ says about the embedding of the brane, and what it means on field-theory side. The coordinate $\chi(r)$ is one of the 5 angular coordinates of our metric and essentially tell us where on the deformed 5-sphere [38] the brane lies.

Depending on both the embedding $\chi$ and the horizon radius $r_h$, the D7-brane can either close of before the horizon, referred to as a Minkowski embedding, extend into the horizon, referred to as a black hole embedding or ”touch” the horizon at a singe point, referred to as a critical embedding [39]. The different embeddings are illustrated in fig. 5.2, as well as how the different embeddings are related to the branes position on the deformed 5-sphere. When we look at the embedding close to the boundary, we can generally attain an asymptotic solution. The leading order contribution of $\chi$ is identified with the mass of the quarks on the field-theory side [7] (note that here a quark is a generalised concept of those found in regular QCD). This of course means that setting $\chi \equiv 0$ means that the D7-branes position on the deformed 5-sphere is constant, and that the quarks are massless, which is why we refer to this embedding as the massless embedding.

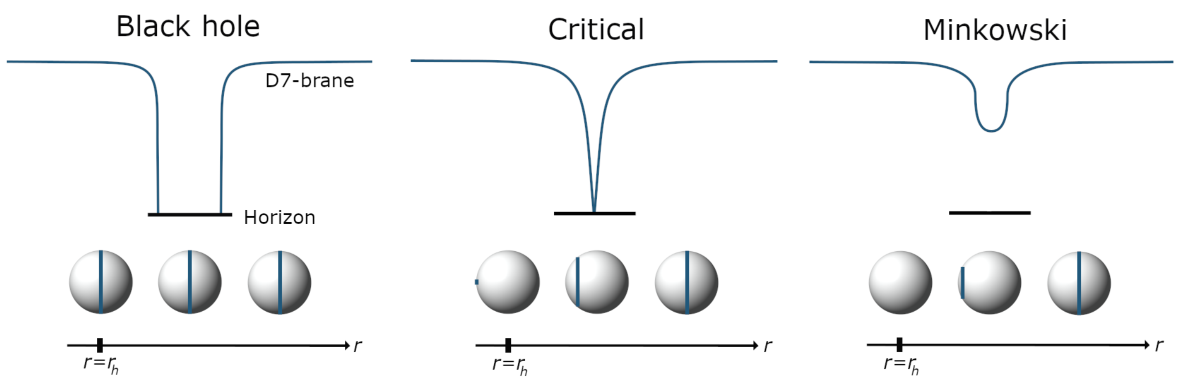



**Figure 5.2:** Illustration of the different types of embeddings for the D7-brane. In the upper part, we show the where the brane lies in relation to the horizon. In the black hole embedding (left) the brane extends into the horizon. For the critical embedding (middle) the brane touches the horizon in a single point and for the Minkowski embedding (right) the brane closes of before the horizon. In the lower part of the figure, the D7-branes position on the deformed 5-sphere for different $r$ is illustrated. The sphere represents the deformed 5-sphere and the line represents the D7-brane. In the black hole embedding, the position is the same down to the horizon $r = r_h$. In the critical embedding the brane ”falls of” the sphere at the horizon and in the Minkowski embedding the brane is no longer present on the sphere as we approach the horizon.

Since we are going to make use of boundary-terms to change our boundary conditions in chapter 6, we also wish to know the weight-factor $\sqrt{-\gamma}$ in the massless case, which is

$$\sqrt{-\gamma} = \frac{\sqrt{B} r^4 e^{-\phi} \sin(\theta)}{8b^2 R} = \frac{r^4 e^{-\phi} \sin(\theta)}{8b^2 R} \tag{5.10}$$

since we are at $r \to \infty$ at the boundary. Note that the boundary to our space-time has 9 dimensions, 4 non-compact and 5 compact ones, due to the compact dimensions in the bulk. However, the $\sqrt{-\gamma}$ just presented is only for 7 of the boundary dimensions. That is because when we add boundary terms, they will be to change the boundary condition of the fields present in the action for the D7-brane. Adding boundary terms in this case means adding them to the brane action, and so these terms must also live on the D7-brane.

## 5.3 Content

Lets us now describe the contents of this model, how we view the content and some loose motivations for why parts of it can be ignored. As mentioned earlier, in the example of $N_c \gg 1$ D3 branes and one D7 probe brane, we essentially have that the open strings between the D3 and D7 brane are dual to strings that begin and end on the D7-brane. In our model we still have that the 3-3 strings are dual to the closed strings and the 3-7 strings will again be dual to 7-7 strings. The 3-5 strings, that will give rise to flavour degrees of freedom, should be dual to the 5-5 strings on the D5-branes. These flavour degrees of freedom will follow a $U(1)^{N_f}$ symmetry, since the D5 branes are not coincident [37]. Lastly the 5-7 strings are ignored. The different dualities between the strings are illustrated in fig. 5.3. Note that it is not made clear in previous works on this model [7] exactly how the mathematics behind this works.

There is however, an argument regarding the coupling strengths that motivates some of the what we have just described. By looking at the 't Hooft coupling eq. (3.13) for the different strings, we see for example that

$$\frac{\lambda_{\mathrm{D7}}}{\lambda_{\mathrm{D3}}} = \frac{1}{N_c}(2\pi)^4\alpha'^2 \tag{5.11}$$

vanishes, both due to the Maldacena limit $\alpha' \to 0$ and because $N_c \to \infty$. Therefore, the 7-7 strings decouple from the 3-3 strings [4]. By similar arguments, the 7-7 strings decouple from the 3-7 strings [18], as well as from the 5-5 and 5-7 strings. Because the 7-7 strings decouple from the 5-5 strings, on the field theory side, the 3-7 and 3-5 strings also decouple from each-other. This means that the flavor degrees of freedom described by the 3-7 strings do not interact with the flavor degrees of freedom from the 3-5 strings. Similarly, the arguments regarding the coupling strength should make it so that we can disregard the 5-7 strings completely.

With these arguments in mind, the reason for why we only look at the 7-7 strings is because they model the physics that we are interested in. The physics of the 7-7 strings has been explored in [7], and our investigation is partly built on this previous work. Some analysis of what physics the D5 brane action give on the field theory side have been done in [37] and [38]. So in the next chapter, we will look at some of the physics the 7-7 strings give us, by use of the D7-brane action.

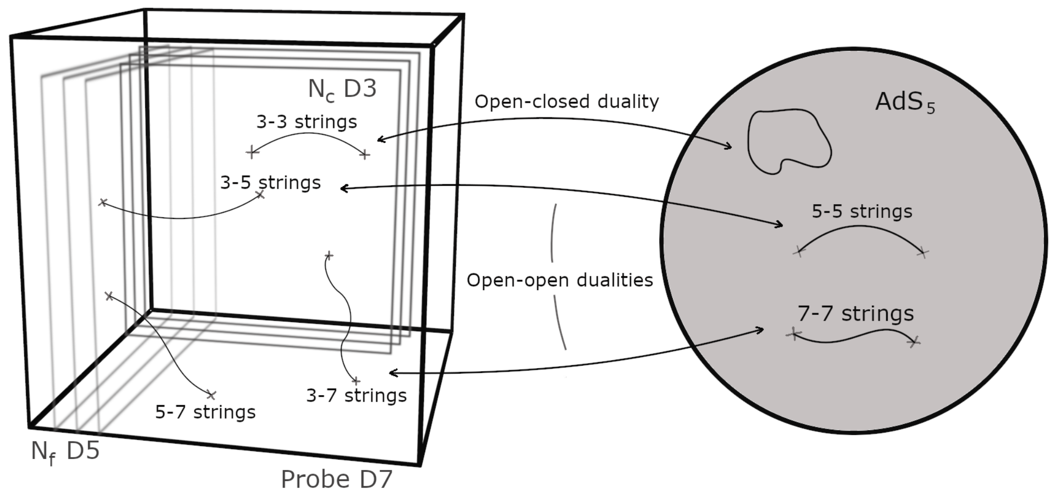



**Figure 5.3:** An illustrative sketch of the dualities at play in our model. The circle represents our AdS space on the supergravity side. Note that the $S^5$ is not drawn, for simplicity. On the field theory side we have the probe D7 brane represented by the cube, and the $N_c$ D3 and $N_f$ D5 branes represented by the stacks of planes inside the cube. On each side of the duality we have the strings of interest. Figure adapted from [1].

# 6

# Properties of the D3-D5-D7 model

In this chapter we will look at some properties of the D3-D5-D7 model, particularly the current resulting from an electric field. This will first be done with the full non-linear equations of motion, in a very simple DC case, that is the electric field is constant in only one direction. Then we will turn to linear response in order to look at what happens from small electric fields but with non-zero $k$ and $\omega$. We will look both at the current as well as dispersion relations. Throughout the first two sections of this chapter we will set $l_s = g_s = 1$.

## 6.1 Non-linear DC current

Let us begin with the DC current in case of the full e.o.m. If we want to model a strange metal, we would want the boundary current $\mathcal{J}^x$ to go as $1/T$ for some range of parameters. The tricky part lies in calculating the current in the holographic approach, in terms of a constant electric field, for which we will follow the methodology of [40]. Their procedure is to set the gauge potential to something very simple, which allows them to identify constants of motion. These constants of motion can be identified as being proportional to the boundary current and charge density. By then looking at the on-shell action they find a relation between the constants of motion, the temperature, and the electric field, which means that this relation will apply for the boundary charge density and current as well. Let us begin by demonstrating this method by calculating the in-plane current, that is, in the case of the electric field pointing in a spatial direction that is shared between the D5 and the D7 brane.

### 6.1.1 In-plane current

For this approach the gauge-field on the world-volume of the D7-brane will have a potential $A_\mu$ where $A_t = A_t(r)$, $A_x = -Et + a(r)$ and the rest are zero. Note that we use $t$ as a label for $x^0$ and $x$ as a label for $x^1$ here, and that $E$ is a constant. Note also that since this gauge-field lives on the word volume of the D7-brane, it has 8 components. The resulting field strength has non-zero, independent components are $F_{rt} = A_t'$, $F_{rx} = a'$ and $F_{tx} = -E$. Requiring that the components of the gauge-field makes the action eq. (5.7) stationary , we get the usual Euler-Lagrange equations for the gauge components

$$\nabla_\mu \Big( \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \Big) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0. \tag{6.1}$$

Note that $\nabla_\mu$ denotes the covariant derivative. We have that $\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ and

$$\nabla_t \frac{\partial \mathcal{L}}{\partial(\partial_t A_x)} = \nabla_t \frac{\partial \mathcal{L}}{\partial E} = 0 \tag{6.2}$$

since there is no time dependent component in $\mathcal{L}$. Remember that $\mathcal{L}$ depends on $F_{tx} = -E$ and not directly on $A_x$. From eq. (6.1) we then get two equations that can be integrated once to give

$$\frac{\partial \mathcal{L}}{\partial(\partial_r A_x)} = \frac{\partial \mathcal{L}}{\partial a'} = j^x \text{ and } \frac{\partial \mathcal{L}}{\partial(\partial_r A_t)} = \frac{\partial \mathcal{L}}{\partial A_t'} = d \tag{6.3}$$

where $j^x$ and $d$ are constants of motion.

Recalling eq. (5.7), the Lagrangian for our system is given by

$$\begin{aligned} \mathcal{L}_{D7} = &-\tau_7 e^{\phi}\sqrt{-\det\Big(g^8_{\mu\nu} + e^{-\frac{\phi}{2}}F_{\mu\nu}\Big)} = \\ &-\tau_7 e^{-\phi/2}\frac{\sin\theta}{8b^{3/2}}r^3\cos^3\chi\Big[\big(1+\frac{1-b}{b}\cos^2\chi\big)\Big(\big(\frac{1}{B}+\frac{r^2}{b}\chi'^2\big)\big(e^{\phi}B - E^2h\big) + a'^2B - A_t'^2\Big)\Big]^{1/2} \end{aligned} \tag{6.4}$$

where $g^8_{\mu\nu}$ denotes the metric for the world-volume of our D7-brane. After taking performing the derivatives in eq. (6.3) we arrive to

$$\begin{aligned} j^x &= \frac{-e^{-\phi/2}r^3\cos^3\chi\sqrt{b+(1-b)\cos^2\chi}\,a'B}{\sqrt{\big(\frac{1}{B}+\frac{r^2}{b}\chi'^2\big)\big(e^{\phi}B - E^2h\big) + a'^2B - A_t'^2}} \\ d &= \frac{e^{-\phi/2}r^3\cos^3\chi\sqrt{b+(1-b)\cos^2\chi}\,A_t'}{\sqrt{\big(\frac{1}{B}+\frac{r^2}{b}\chi'^2\big)\big(e^{\phi}B - E^2h\big) + a'^2B - A_t'^2}}. \end{aligned} \tag{6.5}$$

From this we can immediately see that

$$A_t' j^x = -a'Bd. \tag{6.6}$$

Before we proceed, we will define some functions to help us do the algebra. We define

$$\begin{aligned} &f_1 \equiv b + (1-b)\cos^2\chi \ , \ f_2 \equiv -E^2h\big(\frac{1}{B}+\frac{r^2}{b}\chi'^2\big), \\ &f_3 \equiv 1 + \frac{r^2}{b}\chi'^2B \ , \ f_4 \equiv r^3\cos^3\chi. \end{aligned} \tag{6.7}$$

Using these we find that

$$d^2 = \frac{e^{-\phi}f_4^2 f_1 A_t'^2}{f_2 - A_t'^2 + e^{\phi}f_3 + a'^2B} \implies A_t'^2 = \frac{d^2(f_2 + f_3e^{\phi} + a'^2B)}{e^{-\phi}f_4^2f_1 + d^2} \tag{6.8}$$

and then eq. (6.6) can be used to find that

$$a'^2B^2d^2 = (j^x)^2\frac{d^2(f_2+f_3e^{\phi}+a'^2B)}{e^{-\phi}f_4^2f_1+d^2} \implies a'^2 = \frac{(f_2+e^{\phi}f_3)(j^x)^2}{B^2(e^{-\phi}f_4^2f_1+d^2)-(j^x)^2B}. \tag{6.9}$$

Then using eq. (6.6) again we get that

$$A_t'^2 = \frac{(f_2+e^{\phi}f_3)Bd^2}{B(e^{-\phi}f_4^2f_1+d^2)-(j^x)^2}. \tag{6.10}$$

We can then put $a'^2$ and $A_t'^2$ into our Lagrangian from eq. (6.4)

$$\begin{aligned} \mathcal{L}_{D7} \propto &-e^{-\phi/2}f_4\big[f_1(f_2+a'^2B-A_t'^2+e^{\phi}f_3)\big]^{1/2} = \\ &-e^{-\phi/2}f_4\Big[f_1(f_2+e^{\phi}f_3)\Big(1+\frac{(j^x)^2-Bd^2}{B(f_1e^{-\phi}f_4^2+d^2)-(j^x)^2}\Big)\Big]^{1/2} \equiv -e^{-\phi/2}f_4\sqrt{f_1H_{||1}H_{||2}} \end{aligned} \tag{6.11}$$

where we have introduced the functions $H_{||1}$ and $H_{||2}$ to again shorten our notation. Now we need to analyse the functions $f_1$, $H_{||1}$ and $H_{||2}$. We note that the expression within the square root in eq. (6.11) needs to be positive or 0. We first note that $f_1 > 0 \forall \chi \in \mathbb{R}$. Looking at $H_{||1}$ we see that it is negative when $r \to r_h$ and positive when $r \to \infty$. The same goes for $H_{||2}$. Therefore there must be a point $r_s$ at which both $H_{||1}$ and $H_{||2}$ switches sign, in order for $\mathcal{L}_{D7}$ to be real for all $r \geq r_h$. A numerical investigation (at $\chi \equiv 0$) reveals that $H_{||1}$ goes to 0 for this $r \to r_s$ and $H_{||2}$ diverges when $r \to r_s$, and switches sign at this point. We therefore get two equations from eq. (6.11),

$$\Big(\frac{B_s r_s^{2/3}}{\alpha} - E^2 \frac{R^4}{r_s^4}\Big)\Big(\frac{1}{B_s} + \frac{r_s^2 \chi_s'^2}{b}\Big) = 0 \tag{6.12}$$

$$(j^x)^2 = B_s\bigg(\Big(b + (1-b)\cos^2\chi_s\Big)\frac{\alpha}{r_s^{2/3}} r_s^6 \cos^6\chi_s + d^2\bigg) \tag{6.13}$$

where $B_s = B(r = r_s)$, $\chi_s = \chi(r = r_s)$, $\chi_s' = \chi'(r = r_s)$ and

$$\alpha \equiv \Big(\frac{4Q_f}{3}\Big)^{2/3}. \tag{6.14}$$

Assuming that $\chi_s'^2$ is non-singular, we can note that $\big(\frac{1}{B_s} + \frac{r_s^2\chi_s'^2}{b}\big) > 0$ and finite, which means that the equation for $r_s$ reduces to

$$\frac{B_s r_s^{2/3}}{\alpha} - E^2 \frac{R^4}{r_s^4} = 0 \implies r_s^{14/3} - r_h^{10/3} r_s^{4/3} - \alpha E^2 R^4 = 0. \tag{6.15}$$

To make the calculation a bit simpler we will assume massless embedding of the brane, that is $\chi \equiv 0$. This means that

$$(j^x)^2 = B_s\big(r_s^{16/3}\alpha + d^2\big). \tag{6.16}$$

**Renormalisation and current identification**

We now want to make sure that our action is finite, as well as find the identification of the boundary current. In the massless case, the on-shell action is

$$S^{on-shell} \sim -\int_M d^8x e^{-\phi/2} r^3 \sqrt{\frac{-E^2 h}{B} + a'^2 B - A_t'^2 + e^{\phi}}. \tag{6.17}$$

where $a'$ and $A_t'$ are as in eq. (6.9) and eq. (6.10) respectively. Since the integrand only depends on $r$, we can write

$$S^{on-shell} \sim -\int_{r_h}^{\infty} dr e^{-\phi/2} r^3 \sqrt{\frac{-E^2 h}{B} + a'^2 B - A_t'^2 + e^{\phi}}. \tag{6.18}$$

At large $r$, we can series expand the integrand to find that it goes as

$$-r^3 + \mathcal{O}\bigg(\frac{1}{r^{5/3}}\bigg). \tag{6.19}$$

The first term will cause a divergence in the action, and so must be removed. However, we do not need to do this via a boundary-term. Instead we can see that, at large $r$, this divergence originates from the $e^{\phi}$-term in the square root in eq. (6.17) which in terms comes only from the weight-factor multiplied with $e^{\phi}$, since it does not contain $E$, $a'$ or $A_t'$. Therefore we can remove this simply by subtracting $-\sqrt{-g}e^{\phi} \sim -r^3$, up to a constant (see eq. (5.9)). This is okay since it does not change the equations of motion.

To make sure that there is no divergence for $r \sim r_h$, we series expand in this region and find that the integrand goes as $\mathcal{O}(1)$, and so does not give rise to a divergence in the integration. Of course, we also require that $r$ in the region between $r_h$ and $\infty$ does not give rise to a divergence in the integral as well. The only potential problem is close to the point $r_s$. When inserting the solutions eq. (6.15) and eq. (6.16) into our Lagrangian and series-expanding around $r_s$, we find that there is no pole. Our regularised action is therefore

$$S_{reg} \sim \int_M d^8 x r^3 \left[ 1 - e^{-\phi/2} r^3 \sqrt{\frac{-E^2 h}{B} + a'^2 B - A_t'^2 e^{-2\phi} + e^{\phi}} \right]. \tag{6.20}$$

Now we can use eq. (4.36). By recognising that $\frac{\sqrt{-\gamma}}{\sqrt{-g}} n_r$ is a constant, we find that we do in-fact have

$$\left\langle \mathcal{J}^t \right\rangle \sim d \;\&\; \left\langle \mathcal{J}^x \right\rangle \sim j^x. \tag{6.21}$$

**Result**

Despite the simplification of working with massless embedding, we still need to employ numerical methods to solve for $j^x$. For this numerical calculation, both $N_c$ and $N_f$ were set to 1000. The instantaneous exponent for the $T$-dependence was calculated, and the result is presented in fig. 6.1. Looking at different orders of magnitude for $N_c$ and $N_f$ yielded similar results.

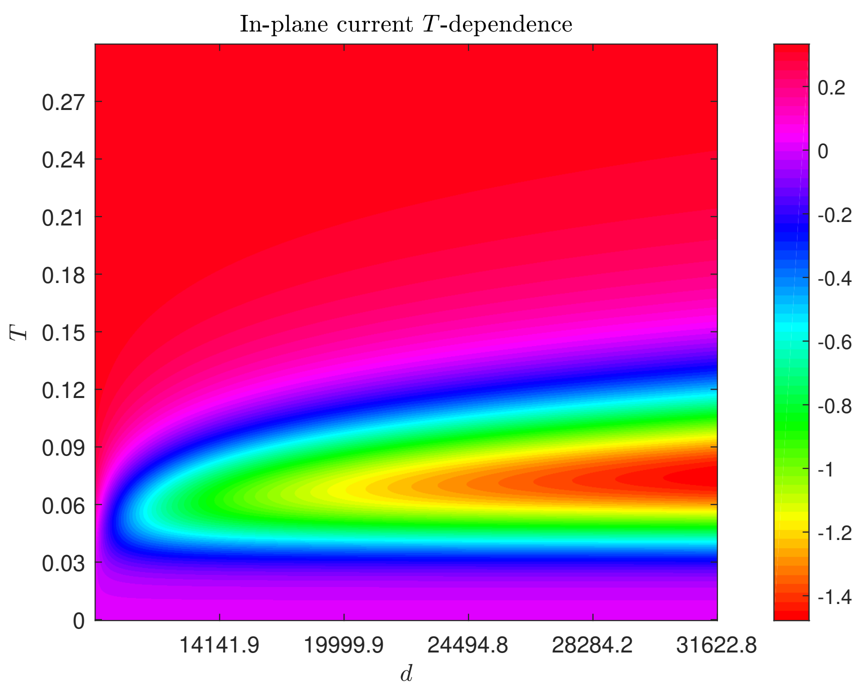



**Figure 6.1:** Exponent of $T$-dependence for $j^x$. The big red region for higher $T$ gives an exponent for the $T$-dependence of $\frac{1}{3}$.

The big region with red colour represents a $T^{1/3}$ dependence. This can also be found analytically.

First we note that, by using eq. (6.15) $B_s = \frac{\alpha E^2 R^4}{r_s^{14/3}}$. Inserted into eq. (6.16), we find that

$$(j^x)^2 = \alpha E^2 R^4 \big(\alpha r_s^{2/3} + \frac{d^2}{r_s^{14/3}}\big). \tag{6.22}$$

Then we note that $r_s \geq r_h$ and so when $r_h^{10/3} * r_s^{4/3} \gg \alpha E^2 R^4$ we have that $r_s \approx r_h$. Then, if $\frac{d^2}{r_s^{16/3}} \ll \alpha$ we find that $Dx \sim r_s^{1/3} \sim T^{1/3}$. For the other parts in fig. 6.1 it is clear that the exponent do not take a fixed value for any larger region.

An interesting thing to note is that we have many different possible behaviours. If we examine the result and allow ourselves to fix for example $\alpha$ in terms of other parameters, we can for find a resistivity linear in $r_h \propto T$ for small electric fields. We can of course find other behaviours, but this is the one of most interest. We start by Taylor-expanding the solution to eq. (6.15) for small $E$ to order $E^2$. The result is

$$r_s \approx r_h + \frac{3}{10 r_h^{11/3}} \alpha R^4 E^2. \tag{6.23}$$

Inserting this into eq. (6.22) and do another Taylor-expansion for small $E$ we arrive at

$$(j^x)^2 \approx \frac{\alpha R^4 E^2 \left(d^2 + \alpha r_h^{16/3}\right)}{r_h^{14/3}}. \tag{6.24}$$

If we now wish to find $\alpha$ so that $\frac{j^x}{E} = \frac{C}{r_h}$ where $C$ is some constant, we find that

$$\alpha = \frac{1}{2 r_h^{16/3}} \left( \sqrt{\frac{4C^2 r_h^8}{R^4} + d^4} - d^2 \right). \tag{6.25}$$

We can therefore find a resistivity linear in temperature (see eq. (5.5)) for small $E$ using this expression for $\alpha$. This resistivity is the resistivity found in the field theory, since $E$ will be the electric field also on field theory side. This makes the relation very interesting, as we are looking for resistivity linear in $T$. The question then becomes if $\alpha$ having this dependence is okay, but this will be done later on.

### 6.1.2 Off-plane DC current

We now turn to off-plane case. Here we set $A_z = -Et + a(r)$ and keep our $A_t = A_t(r)$, where $z$ is used as a label for $x^3$. As in the in-plane case we now we obtain constants of motion that we name $d$ and $j^z$. Our Lagrangian becomes

$$\mathcal{L}_{D7} = -\tau_7 e^{\phi/2} \frac{\sin\theta}{8 b^{3/2}} r^3 \cos^3\chi \Big[\big(1 + \frac{1-b}{b}\cos^2\chi\big)\Big(\big(\frac{1}{B} + \frac{r^2}{b}\chi'^2\big)\big(e^{-\phi}B - E^2 h\big) + a'^2 B - A_t'^2 e^{-2\phi}\Big)\Big]^{1/2}. \tag{6.26}$$

Using this our constants of motion become

$$\begin{aligned} j^z &= \frac{-e^{\phi/2} r^3 \cos^2\chi \sqrt{b + (1-b)\cos^2\chi}\, a' B}{\sqrt{\big(\frac{1}{B} + \frac{r^2}{b}\chi'^2\big)\big(e^{-\phi}B - E^2 h\big) + a'^2 B - A_t'^2 e^{-2\phi}}} \\ d &= \frac{e^{\phi/2} r^3 \cos^2\chi \sqrt{b + (1-b)\cos^2\chi}\, A_t' e^{-2\phi}}{\sqrt{\big(\frac{1}{B} + \frac{r^2}{b}\chi'^2\big)\big(e^{-\phi}B - E^2 h\big) + a'^2 B - A_t'^2 e^{-2\phi}}}. \end{aligned} \tag{6.27}$$

Just as in the in-plane case we solve for $A_t'^2$ and $a'^2$. We have that

$$j^z A_t' e^{-2\phi} = -B a' d \tag{6.28}$$

and so, we arrive at

$$A_t'^2 = \frac{d^2 e^{2\phi}(f_2 + e^{-\phi} f_3)B}{B(d^2 + e^{-\phi} f_4^2 f_1) - e^{-2\phi}(j^z)^2} \text{ and } a'^2 = \frac{(j^z)^2 e^{-2\phi}(f_2 + e^{-\phi} f_3)B}{B(d^2 + e^{-\phi} f_4^2 f_1) - e^{-2\phi}(j^z)^2} \tag{6.29}$$

where have used the $f_1, f_2, f_3$ and $f_4$ from eq. (6.7). We then insert these into our Lagrangian. We can write it as

$$\begin{aligned}\mathcal{L}_{D7} &\propto -e^{\phi/2} f_4 \big[f_1(f_2 + a'^2 B - A_t'^2 e^{-2\phi} + e^{-\phi} f_3)\big]^{1/2} = \\ &- e^{\phi/2} f_4 \left[f_1(f_2 + e^{-\phi} f_3)\Big(1 + \frac{(j^z)^2 e^{-2\phi} - Bd^2}{B(f_1 e^{-\phi} f_4^2 + d^2) - e^{-2\phi}(j^z)^2}\Big)\right]^{1/2} \equiv -e^{\phi/2} f_4 \sqrt{f_1 H_{\perp 1} H_{\perp 2}}.\end{aligned} \tag{6.30}$$

Examining the signs and behaviours of $H_{\perp 1}$ and $H_{\perp 1}$ we find that, just as before, there must be a point $r_s$ which satisfies

$$\frac{\alpha B_s}{r_s^{2/3}} - E^2 \frac{R^4}{r_s^4} = 0 \tag{6.31}$$

$$(j^z)^2 = \frac{B_s r_s^{4/3}}{\alpha^2} \left( \big(b + (1-b)\cos^2 \chi_s\big) \frac{\alpha}{r_s^{2/3}} r_s^6 \cos^6 \chi_s + d^2 \right). \tag{6.32}$$

We note that unlike the in-plane case, it is possible to find a closed form expression for $r_s$ from eq. (6.31)

$$r_s = \big(\frac{E^2 R^4}{\alpha} + r_h^{10/3}\big)^{3/10}. \tag{6.33}$$

This solution fulfils the requirements that $r_s > r_h$ and is real. Just as for the in-plane current we now turn to the massless case to make the examination of the current easier. We use that we can write $B_s = \frac{E^2 R^4}{\alpha r_s^{10/3}}$ and insert it into eq. (6.32)

$$(j^z)^2 = \frac{E^2 R^4}{\alpha^3 r_s^2} \Big( \frac{1}{\alpha b} r_s^{16/3} + d^2 \Big). \tag{6.34}$$

**Renormalisation and current identification**

Just as in the in-plane case, we now want to make sure that our action is finite, as well as find the identification of the boundary current. In the massless case, the on-shell action is

$$S^{on-shell} \sim - \int_M d^8 x e^{\phi/2} r^3 \sqrt{\frac{-E^2 h}{B} + a'^2 B - A_t'^2 e^{-2\phi} + e^{-\phi}}. \tag{6.35}$$

where $a'$ and $A_t'$ are as in eq. (6.29). Since the integrand only depends on $r$, we can write

$$S^{on-shell} \sim - \int_{r_h}^{\infty} dr e^{\phi/2} r^3 \sqrt{\frac{-E^2 h}{B} + a'^2 B - A_t'^2 e^{-2\phi} + e^{-\phi}}. \tag{6.36}$$

Following the same procedure as last time, we series-expand the integrand at large $r$ to find that it too goes as

$$-r^3 + \mathcal{O}\left(\frac{1}{r^{5/3}}\right).$$

By the exact same argument as for the in-plane case, we can remove this divergence by adding the term $\sqrt{-g} e^{\phi}$, up to a constant, to the action. To make sure that there is no divergence for $r \sim r_h$, we series expand in this region as well and find that the integrand goes as $\mathcal{O}(1)$, and so does not give rise to a divergence in the integration. Just as for the in-plane case, we also series expand close to the point $r_s$ and we find that there is once again no pole. Our regularised action is therefore

$$S_{reg} \sim \int_M d^{D+1}xr^3\left[1 - e^{\phi/2}r^3\sqrt{\frac{-E^2h}{B} + a'^2B - A_t'^2e^{-2\phi} + e^{-\phi}}\right]. \tag{6.37}$$

Now we can again use eq. (4.36) and find that we do in-fact have

$$\left\langle \mathcal{J}^t \right\rangle \sim d \;\&\; \left\langle \mathcal{J}^z \right\rangle \sim j^z. \tag{6.38}$$

**Results**

We can clearly tell from eq. (6.34), that when $r_h \gg (E^2R^4/\alpha)^{3/10}$ we get $r_s \approx r_h$. And then, when $r_s^{16/3}/\alpha \gg d^2$ we see that $j^z \sim E^2T^{5/3}$ since $r_h$ is linear in $T$. As can be observed, eq. (6.34) does seem to make it possible for $j^z \sim 1/T$. Looking at fig. 6.2, this is indeed possible but only in a thin band. This figure was created in the same way as for $T$-dependence of $j^x$. We can also observe the region where the exponent becomes $\frac{5}{3}$ as predicted analytically.

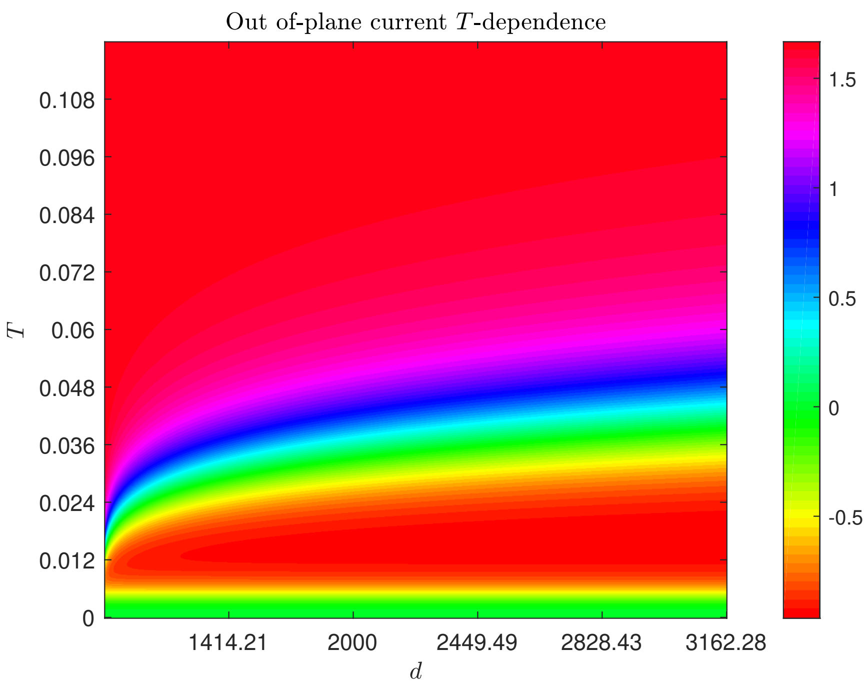



**Figure 6.2:** Exponent of $T$-dependence for $j^z$. The big red region for higher $T$ gives an exponent for the $T$-dependence of $\frac{5}{3}$.

Let us briefly mention that the fact that we get a finite current is not without its questions, especially later in the next section where we examine the linear response. There we will find that we cannot have a finite boundary current in the off-plane direction. Looking at small $E$, we have that

$$(j^z)^2 \approx \frac{R^4E^2\left(d^2 + \alpha r_h^{16/3}\right)}{\alpha^3 r_h^2}, \tag{6.39}$$

which is very similar to the in-plane result eq. (6.24). Let us keep this result in mind for when we look at the off-plane current in linear response.

## 6.2 Linear response

In order to further analyse the behaviour of the model for weak electric fields, we can use linear response theory. This is needed in order to investigate what happens at non-zero $k_\mu$. The following derivation of the linear response equations of motion follows the steps in [7]. We will assume a small perturbation $a_\mu$ of a background solution $A_\mu^{(0)}$. This background solution is chosen to be the solution to the full e.o.m. when $A_t = A_t(r)$ is the only non-zero component. We will also assume massless embedding of the D7 branes, which means that the background solution can be found via eq. (6.5) by setting $\chi \equiv 0$, $E = 0$ and $a(r) \equiv 0$ (which should not be confused with our perturbation $a_\mu$). The result is

$$d = \frac{e^{-\phi/2} r^3 A_t'}{\sqrt{b}\sqrt{e^\phi - A_t'^2}} \implies A_t' = \frac{d e^\phi}{\sqrt{d^2 e^\phi + r^6}} \implies A_t = d \int \frac{e^\phi}{\sqrt{d^2 e^\phi + r^6}} dr. \tag{6.40}$$

A side note is that $A_t$ is proportional to the chemical potential of the field theory, according to the holographic dictionary [1]. Now we let $A_\mu \to A_\mu^{(0)} + a_\mu \implies F_{\mu\nu} \to F_{\mu\nu}^{(0)} + f_{\mu\nu}$. We can then rewrite

$$g_{\mu\nu} + e^{-\phi/2} F_{\mu\nu} = (g_{\mu\lambda} + e^{-\phi/2} F_{\mu\lambda}^{(0)})(\delta^\lambda{}_\nu + X^\lambda{}_\nu).$$

In order to make the following calculation a bit simpler (and consistent) we will move to the matrix representation of these tensors and drop the indices. Then we write $X = (g + e^{-\phi/2} F^{(0)})^{-1} e^{-\phi/2} f$ and can then use the relation

$$\det M = e^{\mathrm{Tr}(\ln(M))}. \tag{6.41}$$

By following the same procedure as in eq. (3.9), we get the Taylor expansion of the DBI determinant as

$$\sqrt{-\det\big(g + e^{-\phi/2} F\big)} \approx \sqrt{-\det\big(g + e^{-\phi/2} F^{(0)}\big)} \left[1 + \frac{\mathrm{Tr}(X)}{2} - \frac{\mathrm{Tr}\big(X^2\big)}{4} + \frac{\mathrm{Tr}(X)^2}{8} + \mathcal{O}\big(X^3\big)\right]. \tag{6.42}$$

Then we use that fact that $(g + e^{-\phi/2} F^{(0)})^{-1}$ contains a symmetric and an anti-symmetric part to write it as

$$(g + e^{-\phi/2} F^{(0)})^{-1} = G + \mathcal{J} \tag{6.43}$$

where $G$ is symmetric and $\mathcal{J}$ is anti-symmetric. And since the only symetric part of $(g + e^{-\phi/2} F^{(0)})^{-1}$ will be diagonal, we can then write, now using index notation,

$$X^\mu{}_\lambda = (G^{\mu\nu} + J^{\mu\nu}) e^{-\phi/2} f_{\nu\lambda}.$$

With this, and $\mathrm{Tr}(X^\mu{}_\lambda) = X^\sigma{}_\sigma$, the Taylor-expanded part of the DBI determinant in eq. (6.42) becomes

$$\begin{aligned}
&\left[1 + \frac{X^\mu{}_\mu}{2} - \frac{X^\mu{}_\nu X^\nu{}_\mu}{4} + \frac{X^\mu{}_\mu X^\nu{}_\nu}{8} + \mathcal{O}\big(X^3\big)\right] = \\
&\left[1 + \frac{1}{2} e^{-\phi/2} \mathcal{J}^{\mu\nu} f_{\nu\mu} - \frac{1}{4}\big(-G^{\mu\lambda} G^{\sigma\nu} + \mathcal{J}^{\mu\lambda}\mathcal{J}^{\sigma\nu}\big) e^{-\phi} f_{\lambda\nu} f_{\mu\sigma} + \frac{1}{8} e^{-\phi} \mathcal{J}^{\mu\nu} f_{\nu\mu} \mathcal{J}^{\lambda\sigma} f_{\sigma\lambda}\right] = \\
&e^{-\phi}\left[e^\phi + \frac{1}{2} e^{\phi/2} \mathcal{J}^{\mu\nu} f_{\nu\mu} + \frac{1}{4}\big(G^{\mu\lambda} G^{\sigma\nu} - \mathcal{J}^{\mu\lambda}\mathcal{J}^{\sigma\nu} + \frac{1}{2}\mathcal{J}^{\lambda\nu}\mathcal{J}^{\mu\sigma}\big) f_{\lambda\nu} f_{\mu\sigma}\right].
\end{aligned} \tag{6.44}$$

Looking at $\sqrt{-\det\big(g + e^{-\phi/2} F^{(0)}\big)}$ we see that it becomes

$$\begin{aligned}
&\sqrt{-\det\big(g + e^{-\phi/2} F^{(0)}\big)} = \frac{e^{-3\phi/2} r^3 \sin(\theta)}{8b^2} \sqrt{e^\phi - (A_t')^2} = \frac{e^{-3\phi/2} r^6 \sin(\theta)}{8b^2 \sqrt{d^2 + e^{-\phi} r^6}} = \\
&P \frac{H e^{-\phi/2}}{\sqrt{d^2 + H}} = P \mathcal{L}_*
\end{aligned} \tag{6.45}$$

where $H \equiv r^6 e^{-\phi} = \alpha r^{16/3}$, $P \equiv \frac{\sin(\theta)}{b^2}$ and $\mathcal{L}_* \equiv \frac{He^{-\phi/2}}{\sqrt{d^2+H}}$. Inserting all of this into the expression for the Lagrangian (eq. (6.4)), we see that its expansion around $A_\mu^{(0)}$ becomes

$$\mathcal{L} \approx -\tau_7 P \mathcal{L}_* \left[ e^{\phi} + \frac{1}{2} e^{\phi/2} \mathcal{J}^{\mu\nu} f_{\nu\mu} + \frac{1}{4} \left( G^{\mu\lambda} G^{\sigma\nu} - \mathcal{J}^{\mu\lambda} \mathcal{J}^{\sigma\nu} + \frac{1}{2} \mathcal{J}^{\lambda\nu} \mathcal{J}^{\mu\sigma} \right) f_{\lambda\nu} f_{\mu\sigma} \right]. \tag{6.46}$$

Now we wish to derive the e.o.m. for $a_\nu$. We note that the Lagrangian only depends on the derivatives of $a_\nu$ and therefore, the e.o.m. will be $\partial_\mu \frac{\partial \mathcal{L}}{\partial f_{\mu\nu}} = 0$. The resulting equations are

$$\partial_\mu \left( \mathcal{L}_* \left[ e^{\phi/2} \mathcal{J}^{\nu\mu} + \left( G^{\sigma\nu} G^{\lambda\mu} - \mathcal{J}^{\sigma\nu} \mathcal{J}^{\lambda\mu} + \frac{1}{2} \mathcal{J}^{\mu\nu} \mathcal{J}^{\sigma\lambda} \right) f_{\sigma\lambda} \right] \right) = 0. \tag{6.47}$$

In order to get further we need to find the non-zero components of $G^{\mu\nu}$ and $\mathcal{J}^{\mu\nu}$ from eq. (6.43). We can see that, in the matrix representation,

$$(g+e^{-\phi/2}F^{(0)})^{-1} = \begin{bmatrix} \frac{g_{\rm rr}}{g_{\rm rr}g_{\rm tt}+e^{-\phi}A_t'^2} & \frac{e^{-\phi/2}A_t'}{g_{\rm rr}g_{\rm tt}+e^{-\phi}A_t'^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{e^{-\phi/2}A_t'}{g_{\rm rr}g_{\rm tt}+e^{-\phi}A_t'^2} & \frac{g_{\rm tt}}{g_{\rm rr}g_{\rm tt}+e^{-\phi}A_t'^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{g_{\rm xx}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{g_{\rm xx}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{g_{\rm zz}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{g_{55}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{g_{77}}{-g_{67}^2+g_{66}g_{77}} & -\frac{g_{67}}{-g_{67}^2+g_{66}g_{77}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{g_{67}}{-g_{67}^2+g_{66}g_{77}} & \frac{g_{66}}{-g_{67}^2+g_{66}g_{77}} \end{bmatrix}$$

where we can immediately see the symmetric and anti-symmetric part. We note here that we assume that the $a_\mu$ have no component in the $\theta$, $\psi$ or $\phi$-directions and do not depend on these coordinates. Therefore, we can safely ignore these components of $G$. We find that the non-zero components are

$$\begin{aligned} G^{tt} &= -\frac{h^{1/2}(d^2+H)}{HB} & G^{rr} &= \frac{Bh^{-1/2}(d^2+H)}{H} \\ G^{xx} &= G^{yy} = h^{1/2} & G^{zz} &= h^{1/2}e^{2\phi} \end{aligned}$$

$$\mathcal{J}^{tr} = -\mathcal{J}^{rt} = -d\frac{\sqrt{d^2+H}}{H}.$$

We note here that $\partial_\mu \mathcal{L}_* e^{\phi/2} \mathcal{J}^{\nu\mu} = 0$ since $e^{\phi/2}\mathcal{L}_* \mathcal{J}^{tr} = -d$, which doesn't depend on either $t$ or $r$.

We can now write the equations for $a_\mu$ explicitly. But first we use our gauge-invariance to choose $a_r = 0$ (radial gauge). The e.o.m. for $a_r$ then results in the following constraint,

$$G^{tt}\partial_t f_{tr} + G^{xx}\partial_x f_{xr} + G^{xx}\partial_y f_{yr} + G^{zz}\partial_z f_{zr} = 0. \tag{6.48}$$

For the other components, the equations are

$$\begin{aligned} &\partial_r \left[ \mathcal{L}_* G^{tt} G^{rr} f_{rt} \right] + \mathcal{L}_* G^{tt} \left[ G^{xx}\partial_x f_{xt} + G^{xx}\partial_y f_{yt} + G^{zz}\partial_z f_{zt} \right] = 0 \\ &\partial_r \left[ \mathcal{L}_* G^{xx} G^{rr} f_{rx} \right] + \mathcal{L}_* G^{xx} \left[ G^{tt}\partial_t f_{tx} + G^{xx}\partial_y f_{yx} + G^{zz}\partial_z f_{zx} \right] = 0 \\ &\partial_r \left[ \mathcal{L}_* G^{xx} G^{rr} f_{ry} \right] + \mathcal{L}_* G^{xx} \left[ G^{tt}\partial_t f_{ty} + G^{xx}\partial_x f_{xy} + G^{zz}\partial_z f_{zy} \right] = 0 \\ &\partial_r \left[ \mathcal{L}_* G^{zz} G^{rr} f_{rz} \right] + \mathcal{L}_* G^{zz} \left[ G^{tt}\partial_t f_{tz} + G^{xx}\partial_x f_{xz} + G^{xx}\partial_y f_{yz} \right] = 0. \end{aligned} \tag{6.49}$$

### 6.2.1 Linear response renormalisation and boundary conditions

Among the quantities that are of interest in this model is plasmon dispersion relations. However, in order to obtain these, we need to change to a mixed boundary condition [35]. This can be done by adding boundary terms to the action, similarly to how holographic renormalisation is done. In this section we will both make sure that the action is properly renormalised and introduce boundary terms to change our b.c.

**Solving for $a_\mu$**

First of all, we must solve our equations of motion (eq. (6.48) and eq. (6.49)) and plug it into our linear response action to see whether it diverges. The solution will be done as a power expansion in $r$ near the boundary at $r \to \infty$. First, we Fourier-transform our equations in regards to $t, x, y, z$. Since $\mathcal{L}_*$ and $G^{\mu\nu}$ only depend on $r$, we only need to worry about the derivatives, using that $\partial_\mu \to ik_\mu$ (where $\mu \neq r$) under the Fourier-transform and $k_\mu = (-\omega, k_x, k_y, k_z)$. After this, the differential equations now consist of (the Fourier-transformed) $a_\mu$ and their first and second derivatives in $r$, which we will denote simply by $a'_\mu$ and $a''_\mu$ respectively. We also note that the there is no difference between the $x$ and $y$-direction, and so we can always align our coordinate system so that $k_y \equiv 0$. We also make a change of variable $u = \frac{1}{r^{1/3}}$. This of course means that

$$\begin{aligned} a'_\mu(r) &= \frac{du}{dr}\frac{d}{du}a_\mu(u) = -\frac{1}{3r^{4/3}}a'_\mu(u) = -\frac{x^4}{3}a'_\mu(u) \\ a''_\mu(r) &= -\frac{x^4}{3}\Big(-\frac{4x^3}{3}a'_\mu(u) - \frac{x^4}{3}a''_\mu(u)\Big) = \frac{x^7}{9}\big(4a'_\mu(u) + xa''_\mu(u)\big) \end{aligned} \tag{6.50}$$

Inserting this change of variables into our equations, and writing them in matrix-form, we can use row-reduction to simplify them a bit. The resulting equations are

$$
\begin{aligned}
0 =& \frac{9R^4\left(\alpha^2u^4k_x^2+k_z^2\right)}{\alpha\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_t(u)+\frac{9\alpha R^4u^4\omega k_x}{\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_x(u)+\\
&\frac{9R^4\omega k_z}{\alpha\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_z(u)+\frac{\left(21d^2u^{16}-3\alpha\right)}{d^2u^{17}+\alpha u}a_t'(u)+a_t''(u)\\
0 =& \frac{9R^4\left(\frac{k_z^2\left(u^{10}r_h^{10/3}-1\right)}{\alpha\left(\alpha+d^2u^{16}\right)}+u^4\omega^2\right)}{\left(u^{10}r_h^{10/3}-1\right)^2}a_x(u)-\frac{9R^4k_xk_z}{\alpha\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_z(u)+\\
&\frac{\left(5d^2u^{16}\left(3u^{10}r_h^{10/3}-1\right)+\alpha\left(7u^{10}r_h^{10/3}+3\right)\right)}{u\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_x'(u)+\frac{9R^4u^4\omega k_x}{\left(u^{10}r_h^{10/3}-1\right)^2}a_t(u)+a_x''(u)\\
0 =& \frac{9R^4\left(\alpha u^4\left(\omega^2\left(\alpha+d^2u^{16}\right)+\alpha k_x^2\left(u^{10}r_h^{10/3}-1\right)\right)+k_z^2\left(u^{10}r_h^{10/3}-1\right)\right)}{\alpha\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)^2}a_y(u)+\\
&\frac{\left(5d^2u^{16}\left(3u^{10}r_h^{10/3}-1\right)+\alpha\left(7u^{10}r_h^{10/3}+3\right)\right)}{u\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_y'(u)+a_y''(u)\\
0 =& \frac{9R^4u^4\left(\frac{\alpha k_x^2\left(u^{10}r_h^{10/3}-1\right)}{\alpha+d^2u^{16}}+\omega^2\right)}{\left(u^{10}r_h^{10/3}-1\right)^2}a_z(u)-\frac{9\alpha R^4u^4k_xk_z}{\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_x(u)+\\
&\frac{9R^4u^4\omega k_z}{\left(u^{10}r_h^{10/3}-1\right)^2}a_t(u)+\frac{\left(d^2u^{16}\left(11u^{10}r_h^{10/3}-1\right)+\alpha\left(3u^{10}r_h^{10/3}+7\right)\right)}{u\left(\alpha+d^2u^{16}\right)\left(u^{10}r_h^{10/3}-1\right)}a_z'(u)+a_z''(u)
\end{aligned}
$$

As can be observed, the equation for $a_y$ has decoupled and can be solved separately. All these equations must be solved using a series solution. That suits us well since that is the form we wish to put the solution in.

The series solution that we assume follows (note that Einstein's summation convention is not used here) as

$$a_\mu(u) = \mathcal{A}_\mu\Big[u^{p_\mu}+\sum_{n=1}\mathcal{A}_{\mu n}u^{n+p_\mu}\Big] \tag{6.51}$$

where we only look at $\mu = t, x, z$ for the moment. Plugging this into our equation and series expanding it near $u = 0$, we first find equations for the $p_\mu$. The solutions are

$$
\begin{aligned}
p_t &= 0 \text{ or } p_t = 4\\
p_x &= 0 \text{ or } p_x = 4\\
p_z &= 0 \text{ or } p_z = 8
\end{aligned}
$$

We then look at solutions on the form

$$a_\mu(u) = \mathcal{A}_\mu\Big[1+\sum_{n=1}\mathcal{A}_{\mu n}u^n\Big]+\mathcal{B}_\mu u^4\Big[1+\sum_{n=1}\mathcal{B}_{\mu n}u^{n+4}\Big]+C_\mu\ln u\Big[C_{\mu 0}+\sum_{n=1}C_{\mu n}u^n\Big] \tag{6.52}$$

for $\mu = t, x, z$. All $\mathcal{A}, \mathcal{B}, C$ are independent of $p_\mu$ but can of course depend on $k_\mu$ and other parameters. By inserting our series ansatz into the series-expanded equations for $a_\mu$ ($\mu = t, x, z$) we can get equations for the constants. After changing our variable back to $r$, the solutions end up being

$$\begin{aligned}
&a_t(r) = \mathcal{A}_t - \frac{9R^4 k_z\left(\mathcal{A}_t k_z + \omega\mathcal{A}_z\right)}{4\alpha^2 r^{2/3}} + \frac{\mathcal{B}_t}{r^{4/3}} + \frac{27R^8 k_z^3 \ln(r)(\mathcal{A}_t k_z + \omega\mathcal{A}_z)}{16\alpha^4 r^{4/3}} + \frac{3R^4\mathcal{B}_t k_z^2}{4\alpha^2 r^2} + \\
&\frac{81R^{12}k_z^5 \ln(r)(k_z\mathcal{A}_t + \omega\mathcal{A}_z)}{64\alpha^6 r^2} + \frac{3R^4 k_x\left(\mathcal{A}_t k_x + \omega\mathcal{A}_x\right)}{4r^2} + \frac{81R^{12}k_z^5\left(\mathcal{A}_t k_z + \omega\mathcal{A}_z\right)}{32\alpha^6 r^2} \\
&a_x(r) = \mathcal{A}_x + \frac{9R^4 k_z(\mathcal{A}_z k_x - \mathcal{A}_x k_z)}{4\alpha^2 r^{2/3}} + \frac{\mathcal{B}_x}{r^{4/3}} + \frac{27R^8 k_z^3 \ln(r)(\mathcal{A}_x k_z - \mathcal{A}_z k_x)}{16\alpha^4 r^{4/3}} + \frac{3R^4\mathcal{B}_x k_z^2}{4\alpha^2 r^2} + \\
&\frac{81R^{12}k_z^5 \ln(r)(\mathcal{A}_x k_z - \mathcal{A}_z k_x)}{64\alpha^6 r^2} - \frac{3R^4\omega(\mathcal{A}_t k_x + \omega\mathcal{A}_x)}{4r^2} + \frac{81R^{12}k_z^5(\mathcal{A}_x k_z - \mathcal{A}_z k_x)}{32\alpha^6 r^2} \\
&a_z(r) = \mathcal{A}_z + \frac{3R^4\left(k_z(\mathcal{A}_x k_x + \omega\mathcal{A}_t) + \mathcal{A}_z(\omega^2 - k_x^2)\right)}{4r^2} + \frac{\mathcal{B}_z}{r^{8/3}} - \\
&\frac{27R^8 k_z^2 \ln(r)\left(\omega\mathcal{A}_t k_z + \mathcal{A}_x k_x k_z + (\omega^2 - k_x^2)\mathcal{A}_z\right)}{32\alpha^2 r^{8/3}}.
\end{aligned} \tag{6.53}$$

These solutions directly satisfies the first-order constraint as well, as is required if the radial gauge is a consistent gauge choice.

For the $a_y(u)$-equation, we use a series solution as well, however, the fact that it is uncoupled from the other equations makes it significantly easier to solve. The solution is

$$\begin{aligned}
&a_y(r) = \frac{\mathcal{B}_y}{r^{4/3}}\left(1 + \frac{3R^4 k_z^2}{4\alpha^2 r^{2/3}} + \frac{27R^8 k_z^4}{128\alpha^4 r^{4/3}} + \frac{3R^4}{20r^2}(k_x^2 - \omega^2) + \frac{81R^{12}k_z^6}{2560\alpha^6 r^2}\right) + \\
&\mathcal{A}_y\left(1 - \frac{9R^4 k_z^2}{4\alpha^2 r^{2/3}} + \frac{R^8 k_z^4(81 + 108\ln(r))}{64\alpha^4 r^{4/3}} + \frac{3R^4(k_x^2 - \omega^2)}{4r^2} + \frac{81R^{12}k_z^6(11 + 4\ln(r))}{256\alpha^6 r^2}\right).
\end{aligned} \tag{6.54}$$

**Checking for and removing divergences**

With the e.o.m. solved asymptotically near the boundary we can plug these into the action to check whether it is finite. The first step is to write the solution to $a_\mu$ on a form that lets us check this more easily. We write them as

$$a_\mu(r) = a_\mu^{(0)} + \frac{a_\mu^{(1)}}{r^{2/3}} + \frac{a_\mu^{(2)}}{r^{4/3}} + \frac{a_\mu^{(3)}\ln(r)}{r^{4/3}} + \frac{a_\mu^{(4)}}{r^2} + \frac{a_\mu^{(5)}\ln(r)}{r^2} \tag{6.55}$$

for $\mu = t, x, y$ and

$$a_z(r) = a_z^{(0)} + \frac{a_z^{(1)}}{r^2} + \frac{a_z^{(2)}}{r^{8/3}} + \frac{a_z^{(3)}\ln(r)}{r^{8/3}} \tag{6.56}$$

On the boundary, we still have some gauge-freedom, and we use that to set $a_t^{(0)} = 0$. Now we look at our action, which we need to massage a bit before we insert our solutions. The Lagrangian comes from eq. (6.46).

$$S = -\tau_7 P\int_M dr d^7x \mathcal{L}_*\left[e^\phi + \frac{1}{2}e^{\phi/2}\mathcal{J}^{\mu\nu}f_{\nu\mu} + \frac{1}{4}\left(G^{\mu\lambda}G^{\sigma\nu} - \mathcal{J}^{\mu\lambda}\mathcal{J}^{\sigma\nu} + \frac{1}{2}\mathcal{J}^{\lambda\nu}\mathcal{J}^{\mu\sigma}\right)f_{\lambda\nu}f_{\mu\sigma}\right]. \tag{6.57}$$

We immediately notice that the $e^\phi$-term in the action will cause a divergence in the integral, since $\mathcal{L}_* e^\phi \sim r^3$ at large $r$. However, this term can be safely removed without changing the equations of motion. The term is removed by adding a term $\sqrt{-g}e^{\phi} = Pr^3$ (up to a constant) to the action. This is also what is done in eq. (6.20) and eq. (6.37). For the other parts of the action, we have

$$\begin{aligned}
&\int_M drd^7x\mathcal{L}_*\left[\frac{1}{2}e^{\phi/2}\mathcal{J}^{\mu\nu}f_{\nu\mu} + \frac{1}{4}\big(G^{\mu\lambda}G^{\sigma\nu} - \mathcal{J}^{\mu\lambda}\mathcal{J}^{\sigma\nu} + \frac{1}{2}\mathcal{J}^{\lambda\nu}\mathcal{J}^{\mu\sigma}\big)f_{\lambda\nu}f_{\mu\sigma}\right] = \\
&\int_M drd^7x\left[\frac{1}{2}\partial_\sigma\left(\mathcal{L}_*\big(G^{\mu\lambda}G^{\sigma\nu} - \mathcal{J}^{\mu\lambda}\mathcal{J}^{\sigma\nu} + \frac{1}{2}\mathcal{J}^{\lambda\nu}\mathcal{J}^{\mu\sigma}\big)f_{\lambda\nu}\right)a_\mu\right] + \\
&\int_{\partial M} d^7x\frac{\sqrt{-\gamma}}{\sqrt{-g}}n_r\mathcal{L}_*\left[e^{\phi/2}J^{\sigma r} + \frac{1}{2}\big(G^{r\lambda}G^{\sigma\nu} - \mathcal{J}^{r\lambda}\mathcal{J}^{\sigma\nu} + \frac{1}{2}\mathcal{J}^{\lambda\nu}\mathcal{J}^{r\sigma}\big)f_{\lambda\nu}\right]a_\sigma
\end{aligned} \tag{6.58}$$

where we again have used that $\partial_\mu\mathcal{L}_*e^{\phi/2}\mathcal{J}^{\mu\nu} = \partial_r(-d) = 0$. After the partial integration, the first integral in eq. (6.58) is zero when we insert our solution to the equation of motion, so we are only left with the boundary integral, which we write in momentum space, for compatibility with our $a_\mu$, below.

$$\int_{r=\Lambda}\frac{d^7k}{(2\pi)^7}\frac{\sqrt{-\gamma}}{\sqrt{-g}}n_r\mathcal{L}_*\left[e^{\phi/2}J^{\sigma r} + \frac{1}{2}\big(G^{r\lambda}G^{\sigma\nu} - \mathcal{J}^{r\lambda}\mathcal{J}^{\sigma\nu} + \frac{1}{2}\mathcal{J}^{\lambda\nu}\mathcal{J}^{r\sigma}\big)f_{\lambda\nu}\right]a_\sigma \tag{6.59}$$

where $f_{\mu\nu} = ik_\mu a_\nu - ik_\nu a_\mu$ and $\Lambda$ is the UV-regulator for $r$, which will be taken to infinity in the end. Now we wish to look at the behaviour at large $r$. First of all

$$\frac{\sqrt{-\gamma}}{\sqrt{-g}}n_r = 1 \tag{6.60}$$

at large $r$. Note that both sides of the above equation is a scalar as we only take the $r$-component of the normal vector $n_\mu$. We then have that

$$\mathcal{L}_*e^{\phi/2}J^{\sigma r}a_\sigma = \mathcal{L}_*e^{\phi/2}J^{tr}a_t \sim \frac{a_t^{(1)}}{r^{2/3}} + \mathcal{O}\left(\frac{1}{r^{4/3}}\right) \tag{6.61}$$

$$\begin{aligned}
\mathcal{L}_*G^{r\lambda}G^{\sigma\nu}f_{\lambda\nu}a_\sigma &= \mathcal{L}_*G^{rr}\big(G^{tt}a_t'a_t + G^{xx}a_x'a_x + G^{xx}a_y'a_y + G^{zz}a_z'a_z\big) \sim \\
&-a_t^{(0)}a_t^{(1)}r^{2/3} + a_x^{(0)}a_x^{(1)}r^{2/3} + a_y^{(0)}a_y^{(1)}r^{2/3} + a_z^{(0)}a_z^{(1)}r^{2/3} + \mathcal{O}(1).
\end{aligned} \tag{6.62}$$

The last two terms in eq. (6.59) are 0 since we work in $a_r = 0$ gauge and $\mathcal{J}^{tr} = -\mathcal{J}^{rt}$ is the only non-zero element of $\mathcal{J}^{\mu\nu}$. As we can see from the expression above, the on-shell action is divergent for $r \to \infty$.

Now we need to remove these divergences from the on-shell action. The way to do this is to re-scale $k_z$ and the components of $a_z$ by an essentially infinitesimal factor. This idea was conceived by Marcus Tornsö. For $k_z$ we re-scale it by

$$k_z \to e^{-\phi_{UV}}k_z = \frac{\alpha}{\Lambda^{2/3}}k_z. \tag{6.63}$$

The argument behind this is due to how the $z$-direction is singled out in the metric. If one looks at the metric in eq. (5.1) we see that the $e^{-2\phi}$-factor makes it so that the $z$-direction gives no contribution, comparatively to the other directions, to $ds^2$ at very large $r$. One can essentially see this as the $z$-direction having collapsed, no longer having an extension. This is visualised in fig. 5.1. It is only natural then that a wave travelling in this direction gives rise to problems, in this case divergences in the action. Similarly, we argue that also all the components in $a_z$ also should be rescaled by the same factor. Later on, when dealing with boundary conditions, we will also rescale components with a free $z$-index by the same factor. All in all, the rescaling of $k_z$ and $a_z$ means that all the divergences in eq. (6.62) disappear without any addition of boundary terms.

### Boundary conditions

We now wish to utilise the analysis done in section 4.2.1 to change our boundary conditions. The boundary condition we wish to utilise is the one written in eq. (4.46). Before we can do that however, there is a subtlety we must take into account, which is that we must add a dilaton-factor into the $F^2$-term. This term now looks like

$$S_{F^2} = \frac{\lambda^{-1}}{4} \int_{\partial M} d^D x \sqrt{-\gamma} e^{\phi} F^{\mu\nu} F_{\mu\nu}. \tag{6.64}$$

This dilaton-factor simply carries over into the boundary condition, that now take the following form

$$\sqrt{-\gamma} \left[ \frac{n_r}{\sqrt{-g}} \frac{\partial}{\partial \epsilon} \Big( \frac{\partial \mathcal{L}}{\partial F_{r\mu}} \Big) + e^{\phi} \lambda^{-1} C_{F^2} \gamma^{\nu\mu} \partial^{\sigma} \partial_{[\sigma} \eta_{\nu]} \right] = 0. \tag{6.65}$$

The reason we want to use this mixed b.c. is because this type of b.c. has given rise to plasmonic dispersion relations in earlier work [35]. The idea is to see whether the same is true is this case.

We will make use of the re-scaling of $k_z, \mathcal{A}_z$ and $\mathcal{B}_z$ done to remove the divergences, but we need to be careful using it. We will therefore start without this re-scaling and add it later. Let us now look at this boundary condition, starting with the first term. For this we need the explicit form of the Lagrangian which is a very large expression, and no-one will be made happy by writing it out. Let us instead skip to the part where the expression is manageable. We take the derivative of the Lagrangian with respect to $A'_\mu$, let $A_\mu \to A_\mu + \epsilon \eta_\mu$ and take the derivative with respect to $\epsilon$. Let us here note that we put restrictions on $A_\mu$ beyond the radial gauge, namely that $A_{x,y,z} \equiv 0$ (according to our background solution) and that the components in the direction of the 3-sphere are 0 as well. This is because in later steps we wish to set $a_\mu = \eta_\mu$. With these restrictions, letting $\epsilon \to 0$ yields

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \Big( \frac{\partial \mathcal{L}}{\partial F_{rt}} \Big) \bigg|_{\epsilon \to 0} = &\frac{\tau_7 \sqrt{B} e^{\phi/2} \sqrt{-g}}{\left( B\left(e^{\phi} - A_t'^2\right) + h A_t^2 k_x^2 + h e^{2\phi} A_t^2 k_z^2 \right)^{3/2}} \Big[ e^{\phi} \big( - h\omega e^{\phi} A_t k_z \eta_z A'_t + \\ & h e^{\phi} A_t k_z^2 \left( A_t \eta'_t - \eta_t A'_t \right) + B \eta'_t \big) + h\omega \left(-A_t\right) k_x \eta_x A'_t + h A_t k_x^2 \left( A_t \eta'_t - \eta_t A'_t \right) \Big] \end{aligned} \tag{6.66}$$

for the $t$-component. The result for $x, y, z$ follows similarly. Remembering eq. (6.60) and the background solution for $A_t$ in eq. (6.40) we can look at the leading order in $r$, and find that

$$\sqrt{-\gamma} \frac{n_r}{\sqrt{-g}} \frac{\partial}{\partial \epsilon} \Big( \frac{\partial \mathcal{L}}{\partial F_{r\mu}} \Big) \bigg|_{\epsilon \to 0} = \begin{cases} \tau_7 P \alpha r^{7/3} \eta'_t & \mu = t \\ -\tau_7 P \alpha r^{7/3} \eta'_x & \mu = x \\ -\tau_7 P \alpha r^{7/3} \eta'_y & \mu = y \\ -\tau_7 P r^3 e^{\phi} \eta'_z & \mu = z \end{cases} \tag{6.67}$$

where we remember that $P = \frac{\sin(\theta)}{8b^2}$.

Let us now turn to the second term. Ignoring the constants $\lambda$ and $C_{F^2}$ for the moment, we get that, at leading order in $r$, it becomes

$$\sqrt{-\gamma} \gamma^{\nu\mu} \partial^{\sigma} \partial_{[\sigma} \eta_{\nu]} = \begin{cases} R^3 P \left( k_x^2 \eta_t + \omega k_x \eta_x + e^{2\phi} (k_z^2 \eta_t + \omega k_z \eta_z) \right) & \mu = t \\ -R^3 P \left( \omega k_x \eta_t + \omega^2 \eta_x + e^{2\phi} (-k_z^2 \eta_x + k_x k_z \eta_z) \right) & \mu = x \\ -R^3 P \left( -k_x^2 \eta_y + \omega^2 \eta_y - e^{2\phi} k_z^2 \eta_y \right) & \mu = y \\ -R^3 P e^{2\phi} \left( \omega k_z \eta_t + k_x k_z \eta_x - k_x^2 \eta_z + \omega^2 \eta_z \right) & \mu = z. \end{cases} \tag{6.68}$$

To get further, we must realise that we can set $\eta_\mu = a_\mu$. Since we are looking at boundary conditions, we set $r = \Lambda$. Performing the re-scaling and setting $C_{F^2} = \frac{4}{3}R\tau_7$ we find the following set of boundary conditions

$$\begin{aligned}
&-\lambda\alpha\mathcal{B}_t + R^4\big((k_x^2 + k_z^2)\mathcal{A}_t + \omega(k_x\mathcal{A}_x + k_z\mathcal{A}_z)\big) = 0 \\
&-\lambda\alpha\mathcal{B}_x - R^4\big((\omega^2 - k_z^2)\mathcal{A}_x + k_x(\omega\mathcal{A}_t + k_z\mathcal{A}_z)\big) = 0 \\
&-\lambda\alpha\mathcal{B}_y - R^4\mathcal{A}_y\big(\omega^2 - k_x^2 - k_z^2\big) = 0
\end{aligned} \tag{6.69}$$

Note that for the $z$-component it is a bit trickier due to the re-scaling. Re-scaling $k_z$, all components in $a_z$ as well as rescaling the $z$-component of the boundary condition (because it has a free $z$-index) we find that the boundary condition becomes

$$k_z(\omega\mathcal{A}_t + k_x\mathcal{A}_x) + \mathcal{A}_z(\omega^2 - k_x^2) = 0 \tag{6.70}$$

for the $z$-component.

As is apparent, in order to find a dispersion relations, we need a relation between the different $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$. This essentially mean that we must solve the e.o.m. in the entire bulk (and use in-falling boundary conditions at the horizon). However, this cannot be done analytically. Instead we must use numerical methods. Then, we can read off the values of $a_\mu$ and $a'_\mu$ at the boundary, and since we know that the leading orders for those are proportional to $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ respectively, we can find $\frac{\mathcal{A}_\mu}{\mathcal{B}_\nu}$. The result will of course be dependent on $k_\mu$, and so we must choose a $k_\mu$, solve the e.o.m. (eq. (6.49)) and then see if it fulfils the b.c. (eq. (6.69) and eq. (6.70)).

The first step is to change our variables, by setting

$$u = \frac{r_h}{r}.$$

Because of the horizon at $r_h$, we only solve the equations in the range $r \in (r_h, \infty)$, we will have that $u \in (0, 1)$. We then solve the equations asymptotically near the horizon (at $u = 1$) and use the in-falling boundary condition. We can then solve the equations of motion numerically from $u \to 1$ to $u \to 0$, for a given $k_\mu$. We then check whether the boundary conditions are fulfilled. If not, we choose a new $k_\mu$ and redo the process. Eventually we will find a dispersion relation for $\omega$ and $k_i$. We also define new quantities, $\hat{k}_\mu \equiv \frac{R^2 k_\mu}{r_h}$, $\hat{\alpha} \equiv \alpha r_h^{-2/3}$ and $\hat{d} \equiv \frac{d}{r_h^{8/3}\sqrt{\alpha}}$ to make the solution simpler. Implementing the solution was done by Marcus Tornsö. One of the modes can be seen in fig. 6.3. As can be observed from its gapped nature, the mode follows that of a plasmonic dispersion relation.

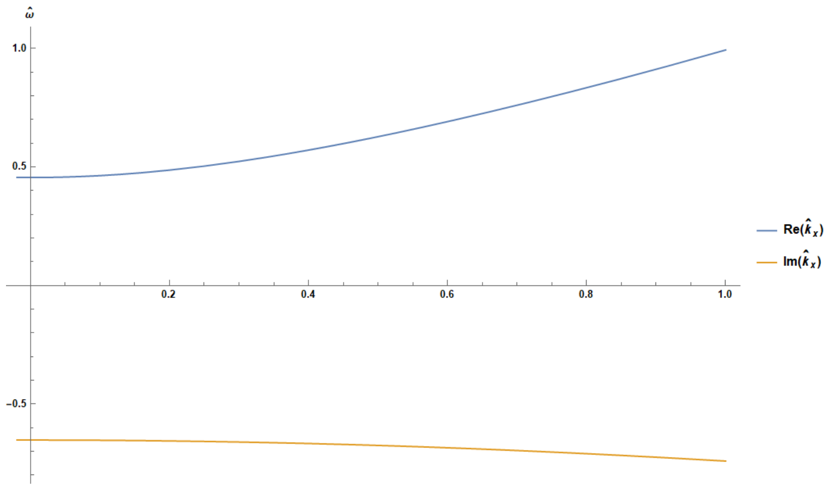



**Figure 6.3:** Dispersion relation $\hat{k}_x(\hat{\omega})$ for a plasmonic mode.

### 6.2.2 Linear response conductivity at $k = 0$

We will now calculate the behaviour for $\sigma(\omega, k = 0)$ from the linear response expansion of our Lagrangian. Let us first look at the field theory current with our Lagrangian, which we calculate by eq. (4.36)

$$\langle \mathcal{J}^{\mu} \rangle = \frac{\sqrt{-\gamma}}{\sqrt{-g}} n_r \frac{\partial \mathcal{L}}{\partial F_{r\mu}}\bigg|_{\partial M} = \begin{cases} \tau_7 P \alpha r^{7/3} a_t' & \mu = t \\ -\tau_7 P \alpha r^{7/3} a_x' & \mu = x \\ -\tau_7 P \alpha r^{7/3} a_y' & \mu = y \\ -\tau_7 P r^3 e^{\phi} a_z' & \mu = z \end{cases} \tag{6.71}$$

This result follows from a calculation that is nearly identical to the one done to arrive at eq. (6.67). The only real difference is that this one uses slightly different formalism. For eq. (6.67) we vary the gauge-field by $A_\mu \to A_\mu + \epsilon \eta_\mu$ and for eq. (6.71) we vary the gauge field by $A_\mu \to A_\mu + a_\mu$. We should keep in mind that the linear response is done around a background solution. Then remembering that the leading order of the vector potential of the field theory is proportional to the vector potential of the gravity theory, which means that we have

$$\mathcal{E}_i \sim -i(\omega a_i^{(0)} + k_i a_t^{(0)}). \tag{6.72}$$

Even though we have used our gauge freedom in the bulk, we should still have gauge-freedom on the field theory side. The field theory does not know anything about its gravitational dual, and so its gauge-freedom should not be used up when we choose our radial gauge. Therefore, we choose to use this gauge freedom to set $a_t^{(0)} = 0$ at the boundary. This means that we have

$$\mathcal{E}_i \sim -i\omega a_i^{(0)}. \tag{6.73}$$

Next, the fact that $k = 0$ means that all the equations of motion decouple from each other. This means that the equation of motion for $a_x$ and $a_y$ are now identical, and so we only need to consider one of them. It also means that there will only be diagonal components in the conductivity matrix. We can therefore calculate it by

$$\sigma^{xx} = \frac{\mathcal{J}^x}{\mathcal{E}_x} \sim -\frac{i\alpha r^{7/3} a_x'(r)}{\omega a_x^{(0)}} \text{ and } \sigma^{zz} = \frac{\mathcal{J}^z}{\mathcal{E}_z} \sim -\frac{i r^{11/3} a_z'(r)}{\alpha \omega a_z^{(0)}}. \tag{6.74}$$

Note that we will wait with any re-scaling of $a_z$ to later on. From this rquation we find that we need to know $a_x^{(1)}/a_x^{(0)}$ and $a_z^{(1)}/a_z^{(0)}$ at the boundary. But this cannot be found by simply expanding the e.o.m. at the boundary and using the boundary conditions. Instead we must, again, solve the e.o.m. near the horizon, use in-falling boundary conditions and then solve for $a_x$ and $a_z$ out to the boundary.

**Conductivity for $\omega = 0$**

Let us begin by looking at the case where $\omega = 0$ as well. Or rather, we consider $\omega$ to be small and let it go to 0 at the end. This case can actually be solved analytically. Let us start with the in-plane case, that is, solving for $a_x(r)$. The equation of motion is

$$\frac{r^{8/3}R^4\omega^2}{\left(r^{10/3}-r_h^{10/3}\right)^2}a_x(r)+\frac{\alpha r^{16/3}\left(3r_h^{10/3}+7r^{10/3}\right)-d^2\left(r^{10/3}-11r_h^{10/3}\right)}{3r\left(r^{10/3}-r_h^{10/3}\right)\left(d^2+\alpha r^{16/3}\right)}a_x'(r)+a_x''(r)=0. \quad (6.75)$$

This equation can be written on the form

$$q(r)a_x(r)+p(r)a_x'(r)+a_x''(r)=0 \quad (6.76)$$

where we see that $r = r_h$ is a regular singular point for our equation. Due to the form of $q$ and $p$, we can use Frobenius method. We make a series ansatz $a_x = \sum_{n=0} a_n(r-r_h)^{n+b}$ where $a_n$ are the coefficients. Series expanding $q$ and $p$, we find that

$$q(r)\approx\frac{9R^4\omega^2}{100r_h^2\left(r-r_h\right)^2} \text{ and } p(r)\approx\frac{1}{r-r_h} \quad (6.77)$$

to leading order in $r - r_h$. We can now solve for $b$,

$$b(b-1)+b+\frac{9R^4\omega^2}{100r_h^2\left(r-r_h\right)^2}=0 \implies b=\pm i\frac{3R^2\omega}{10r_h}. \quad (6.78)$$

As we approach $r = r_h$, the factor $(r - r_h)^b$ will oscillate rapidly. Now we make use of the in-falling boundary condition, which says that we must choose $b = -i\frac{3R^2\omega}{10r_h}$. This can be seen by restoring the time dependence. Remember that we Fourier-transformed our $a_x$ earlier. Near the horizon, we can write our $a_x(k, r)$ as

$$a_x(k,r)\approx(r-r_h)^{\pm i\frac{3R^2}{10r_h}\omega}=\exp\left(\pm i\ln(r-r_h)\frac{3R^2}{10r_h}\omega\right) \quad (6.79)$$

changing variables to $\tilde{r} \equiv \ln(r - r_h)\frac{3R^2}{10r_h}$ we have that the inverse Fourier transform of this is $\delta(t \pm \tilde{r})$. For the plus-sign this is a wave moving out from the horizon, and for the minus sign it is moving towards the horizon. As stated earlier in section 4.2.2, we must choose the wave moving towards the horizon to keep our field theory causal.

With this, we rewrite our equation by factoring out this oscillating factor. We define

$$\tilde{a}_x(r)\equiv(1-\frac{r_h}{r})^{i\frac{3\hat{\omega}}{10}}a_x(r) \quad (6.80)$$

where we have defined

$$\hat{\omega}\equiv\frac{R^2\omega}{r_h}. \quad (6.81)$$

We also define

$$\hat{d}\equiv\frac{d}{r_h^{8/3}\sqrt{\alpha}}. \quad (6.82)$$

We use these to make the following expressions a bit easier to handle. We then multiply our equation eq. (6.75) by $(1-\frac{r_h}{r})^{i\frac{3\hat{\omega}}{10}}$. After some algebra we arrive at the following equation

$$\left(-\frac{10i\left(\alpha\hat{d}^2r_h^{16/3}\left(4r^{10/3}r_h+17rr_h^{10/3}-14r_h^{13/3}-7r^{13/3}\right)+\alpha r^{16/3}\left(-4r^{10/3}r_h+9rr_h^{10/3}-6r_h^{13/3}+r^{13/3}\right)\right)}{(r-r_h)^2\left(r^{10/3}-r_h^{10/3}\right)\left(\alpha\hat{d}^2r_h^{16/3}+\alpha r^{16/3}\right)}+\right.$$
$$\left.\frac{100r^{14/3}\hat{\omega}r_h}{\left(r^{10/3}-r_h^{10/3}\right)^2}-\frac{9\hat{\omega}r_h}{(r-r_h)^2}\right)\frac{\hat{\omega}r_h}{100r^2}\tilde{a}_x(r)+$$
$$\left(\frac{\hat{d}^2\left(11r_h^{26/3}-r^{10/3}r_h^{16/3}\right)+3r^{16/3}r_h^{10/3}+7r^{26/3}}{3r\left(r^{10/3}-r_h^{10/3}\right)\left(\hat{d}^2r_h^{16/3}+r^{16/3}\right)}-\frac{3i\hat{\omega}r_h}{5r^2-5rr_h}\right)\tilde{a}'_x(r)+\tilde{a}''_x(r)=0.$$

Now we assume that $\omega$ (and therefore $\hat{\omega}$) is small. So we make an ansatz for the solution $\tilde{a}_x(r)=1+i\hat{\omega}f(r)$ where $f$ is a function of $r$. With this ansatz inserted into our equation, we can see that to first order in $\hat{\omega}$, there will not be any $f(r)$-term. So we define $g(r)\equiv f'(r)$. Only keeping the terms that are at most first order in $\hat{\omega}$, we can solve the equation analytically (where we of course use Mathematica). The solution for $g(r)$ is

$$g(r)=\frac{3r_h}{10(r^2-rr_h)}+\frac{C_1r^{11/3}}{\left(r^{10/3}-r_h^{10/3}\right)\sqrt{\hat{d}^2r_h^{16/3}+r^{16/3}}}$$

where $C_1$ is a constant. In order to make sure that the solution is regular at the horizon, we series expand the solution at $r-r_h$ and find that it goes as

$$\frac{3}{10}\left(1+\frac{C_1}{\sqrt{1+\hat{d}^2r_h^{4/3}}}\right)\frac{1}{r-r_h}+\mathcal{O}(1) \tag{6.83}$$

We require that the solutions is regular at the horizon, and we must therefore choose

$$C_1=-r_h^{4/3}\sqrt{\hat{d}^2+1} \tag{6.84}$$

to remove the $(r-r_h)^{-1}$ behaviour. Now let's turn this back into our $a_x(r)$. Since we want to know what $a_x(r)$ and $a'_x(r)$ is on the boundary, we expand our solution at $r\to\infty$, take the primitive function to get $f(r)$, and use that $\tilde{a}_x(r)=1+i\hat{\omega}f(r)$. Multiplying by $(1-\frac{r_h}{r})^{-i\frac{3\hat{\omega}}{10}}$ we arrive at our $a_x(r)$. We have, at $r\to\infty$

$$a_x(r)\approx 1+\frac{3i\sqrt{\hat{d}^2+1}\hat{\omega}r_h^{4/3}}{4r^{4/3}}. \tag{6.85}$$

Inserting this solution into eq. (6.74) we find that, in the $\omega\to 0$ limit,

$$\sigma^{xx}\sim\frac{R^2}{r_h^{7/3}}\sqrt{\alpha\left(d^2+\alpha r_h^{16/3}\right)} \tag{6.86}$$

Just like in the non-linear case in section 6.1.1, if we want to have $\sigma\sim\frac{C}{r_h}$, we can solve for $\alpha$ and find

$$\alpha=\frac{1}{2r_h^{16/3}}\left(\sqrt{\frac{4C^2r_h^8}{R^4}+d^4}-d^2\right). \tag{6.87}$$

Both eq. (6.86) and the expression for $\alpha$ matach exactly what we found when we required the same behaviour in the small $E$ expansion of the non-linear current in section 6.1.1, see eq. (6.24) and eq. (6.25). Let us now take the analysis a step further. This expression for $\alpha$ gives us linear in $T$ resistivity for all temperatures. If we wish, we can restrict ourselves to large or small temperatures by series-expanding $\alpha$ and $\sigma$. We have that

$$\alpha = \frac{C}{R^2 r_h^{4/3}} + \mathcal{O}(r_h^{-16/3}) \implies \sigma^{xx} = \frac{C}{r_h} + \mathcal{O}(r_h^{-5}) \tag{6.88}$$

for $r_h \gg \frac{d^4R^4}{4C^2}$ and

$$\alpha = \frac{C^2 r_h^{8/3}}{d^2R^4} + \mathcal{O}(r_h^{32/3}) \implies \sigma^{xx} = \frac{C}{r_h} + \mathcal{O}(r_h^{7}) \tag{6.89}$$

for $r_h \ll \frac{d^4R^4}{4C^2}$. We could of course expand around other values, but these are the most interesting ones that have been examined.

Let us move on to the off-plane conductivity. The steps taken are essentially the same, and the equations and solutions look very similar, mainly differing by $e^{\phi}$-factors. We will therefore omit to writing down most of the expressions. We begin by using Frobenius method to find that the oscillating factor, after using in-falling boundary conditions, is

$$(r - r_h)^{-i\frac{3R^2}{10r_h}\omega}. \tag{6.90}$$

Factoring this factor out from our $a_z(r)$, multiplying the equation by $(1-\frac{r_h}{r})^{i\frac{3R^2}{10r_h}\omega}$, changing to the hatted quantities and making the ansatz $\tilde{a}_z(r) = 1 + i\hat{\omega}f(r)$ we again find that the $f(r)$-term vanishes from the equation when we expand in small $\omega$. Then we set $g(r) \equiv f'(r)$ and find the solution to be

$$g(r) = \frac{r^{7/3}}{10\left(r^{10/3} - r_h^{10/3}\right)}\left[3r_h^{1/3}\left(\frac{r_h^3}{r^{10/3}} + \frac{r_h^2}{r^{7/3}} + \frac{r_h}{r^{4/3}} - \frac{r_h^{1/3} + r^{1/3}}{(rr_h)^{1/3} + r_h^{2/3} + r^{2/3}} + \frac{1}{r^{1/3}}\right) + \frac{10C_2}{\sqrt{\hat{d}^2 r_h^{16/3} + r^{16/3}}}\right].$$

Demanding the solution to be regular at the horizon we find that

$$C_2 = -\sqrt{\hat{d}^2 + 1} - r_h^{8/3}. \tag{6.91}$$

Then series-expanding the solution and taking us back to $a_z(r)$, it becomes

$$a_z(r) = 1 + \frac{3i\sqrt{\hat{d}^2+1}\hat{\omega} r_h^{8/3}}{8r^{8/3}} + \frac{9\hat{\omega}^2 r_h^2}{200r^2}. \tag{6.92}$$

Note that the third term here does not quite match the numerical factor when compared to the more general solutions given in eq. (6.53), in the $k_i \to 0$ limit. This should be due to the fact that we have only solved the equation of motion to first order in $\omega$, while this is a second order term. Transforming back from our hatted quantities, we find that the conductivity is, in the $\omega \to 0$ limit,

$$\sigma^{zz} \sim \frac{R^2\sqrt{d^2 + \alpha r_h^{16/3}}}{\alpha^{3/2} r_h} \tag{6.93}$$

which is the same expression as the one found when we took the small $E$ limit of the non-linear current calculation, see eq. (6.39). One interesting thing to note here is that we have not flattened our $a_z$ and received a finite current. One might object with that our action should be divergent then, but as both $k_i$ and $\omega$ are 0, all the divergent terms disappear, as can be found by looking at the solutions for $a_\mu$ (eq. (6.53) and eq. (6.54)) and the search for divergences in the on-shell action.

**Conductivity for $\omega \neq 0$**

Moving on to the case where $\omega \neq 0$, we need to solve the equations of motion numerically. This will only be done for the in-plane case and so the equation we wish to solve is eq. (6.75), however we must first change variables to $u \equiv \frac{r_h}{r}$. Also changing to the hatted quantities, the equation takes the following (rather compact) form

$$\frac{\hat{\omega}^2}{\left(u^{10/3}-1\right)^2} a_x(u) + \frac{\left(\hat{d}^2\left(17u^{10/3}-7\right)u^{16/3}+9u^{10/3}+1\right)}{3u\left(u^{10/3}-1\right)\left(\hat{d}^2u^{16/3}+1\right)} a_x'(u) + a_x''(u) = 0. \tag{6.94}$$

This equation can be solved numerically, provided with proper boundary conditions. Just as for eq. (6.75), we solve our new equation near $u = 1$ to leading order to find the leading behaviour as

$$a_x(u) \approx (u-1)^{-\frac{1}{10}(3i\hat{\omega})} \tag{6.95}$$

where we have used our in-falling boundary condition. We use this in our numerical solution to fixate $a_x(u=\epsilon)$ and $a_x'(u=1-\epsilon)$, where $\epsilon$ is a small number. Given numerical values for $\hat{d}$ and $\hat{\omega}$. The plots below show $a_x(u)$ for $\hat{d} = \hat{\omega} = 1$ and for $\hat{d} = \hat{\omega} = 1$. As can be observed, increased $\hat{\omega}$ increases the oscillatory behaviour of $a_x(u)$. Changes in $\hat{d}$ do not produce large changes in the general behaviour.

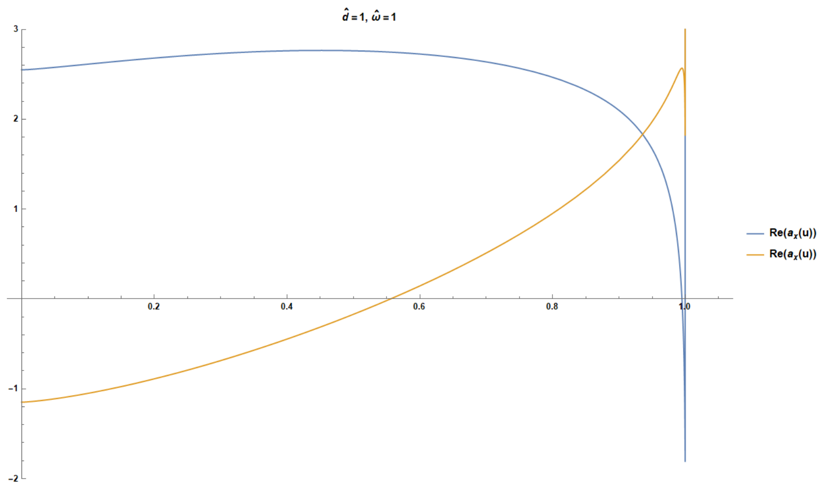



**Figure 6.4:** A solution of eq. (6.94) for $a_x(u)$ for $\hat{d} = \hat{\omega} = 1$. We have $u$ in the range $0.0001 \leq u \geq 0.9999$.

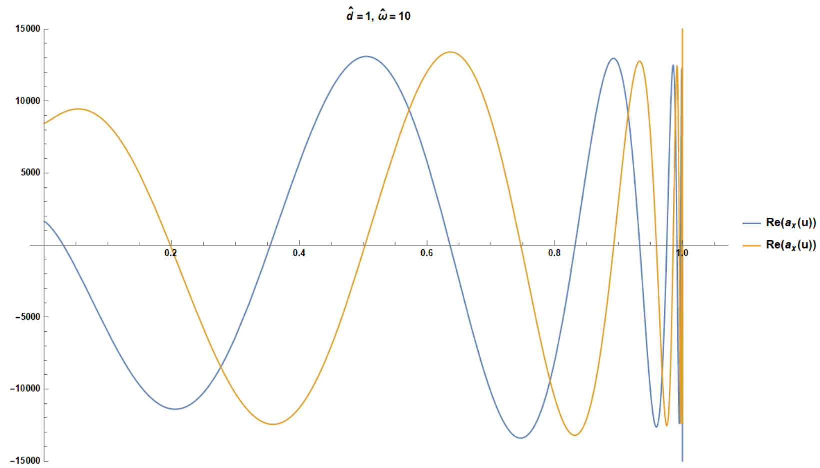



**Figure 6.5:** A solution of eq. (6.94) for $a_x(u)$ for $\hat{d} = 1$ and $\hat{\omega} = 10$. We have $u$ in the range $0.0001 \leq u \geq 0.9999$.

From these solutions we can read of the values of $a_x(u = \epsilon)$ and $a_x'(u = \epsilon)$. It should be noted however that $a_x'(u) \to 0$ as $u \to 0$, and so we should be careful with reading of this value.

The next step is to calculate the conductivity for different $\hat{\omega}$, $\hat{d}$ and $r_h$. This is done via equation eq. (6.74). We will have to choose a value for $\alpha$, and we use that expressions we found that give linear in $T$ resistivity for all temperatures. In fact, by using $\hat{d}$, we have that the expression for $\alpha$ becomes

$$\alpha = \frac{1}{2}\left(\frac{\sqrt{4C^2 + \alpha^2\hat{d}^4R^4r_h^{8/3}}}{R^2r_h^{4/3}} - \alpha\hat{d}^2\right) \implies \alpha = \frac{1}{R^2r_h^{4/3}\sqrt{1+\hat{d}^2}}. \tag{6.96}$$

As can be noticed, there is no need for Taylor-expansions for the different magnitudes of $r_h$. Numerically, we set $R = 1$, as this is the length-scale we should compare everything to. This is okay, despite the fact that we have earlier set $\alpha' = 1$, since we plot dimensionless quantities. When using our hatted quantities, the conductivity becomes

$$\sigma^{xx} = -\frac{iCr^{7/3}a_x'(r)}{r_h^{4/3}\sqrt{1+\hat{d}^2}r_h\hat{\omega}a_x(r)} = \frac{iCu^2a_x'(u)}{u^{7/3}r_h\sqrt{1+\hat{d}^2}\hat{\omega}a_x(u)} = \frac{iCa_x'(u)}{u^{1/3}r_h\sqrt{1+\hat{d}^2}\hat{\omega}a_x(u)} \tag{6.97}$$

when using that $r = r_h/u$. This means that our conductivity always has a $r_h^{-1}$-dependence when using these quantities. Of course, the actual temperature-dependence might look different, but it is hidden within $\hat{d}$ and $\hat{\omega}$. Since $C$ and $r_h$ mealy changes the order of magnitude for $|\sigma(\hat{\omega})|$, we set them both to 1 for the numerical calculation.

We now plot $|\sigma(\hat{\omega})|R$ and $\arg(\sigma(\hat{\omega}))$, starting at $\hat{\omega} = 0.001$ increasing by factors of 1.5 until reaching $\hat{\omega} = 16586$. We plot the results logarithmically since we want to capture many orders of magnitude. In the case of $|\sigma(\hat{\omega})|$, it is the exponent of the $\hat{\omega}$-dependence that is interesting (which is the same as the exponent of the $\omega$-dependence). The result is found in fig. 6.6.

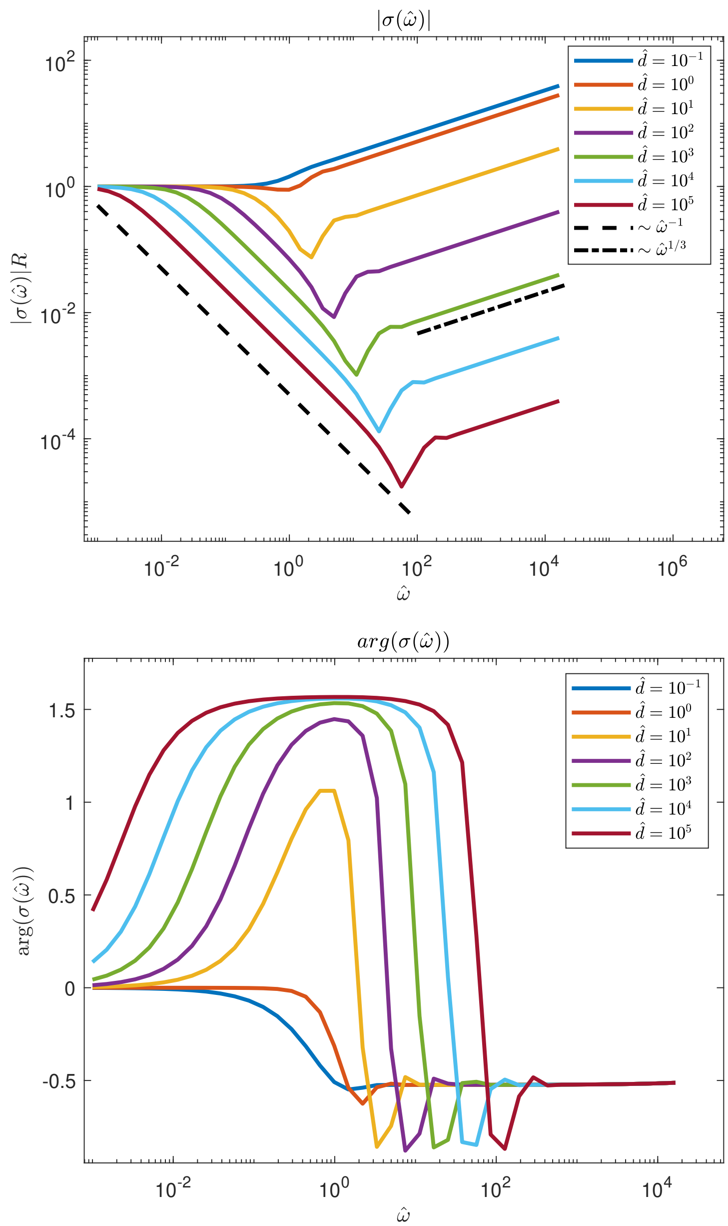



**Figure 6.6:** Plot of $|\sigma(\hat{\omega})|R$ (upper) and $\arg(\sigma(\hat{\omega}))$ (lower) for different values of $\hat{d}$. The dashed line in the left figure shows a $\hat{\omega}^{-1}$-dependence and the dash-dot line shows $\hat{\omega}^{1/3}$-dependence.

Starting with $|\sigma(\hat{\omega})|$, there is a range of $\hat{\omega}$ where $|\sigma(\hat{\omega})| \sim \hat{\omega}^{-1}$ for $\hat{d} \geq 10$. However, for smaller $\hat{d}$ this is not the case. Additionally, for all the $\hat{d}$-values, the $|\sigma(\hat{d})| \sim \hat{\omega}^{1/3}$ for large enough $\hat{\omega}$. The reason for stopping at $\hat{d} = 0.1$ is that every values of $\hat{d}$ below this produce the same $|\sigma(\hat{\omega})|$. This is easily understood since $\sqrt{1+\hat{d}} \approx 1$ when $\hat{d}$ becomes small. For the argument, all the different cases tend to 0 as $\hat{\omega} \to 0$, however this happens outside the plotted range for the larger $\hat{d}$-cases. For the larger $\hat{d}$, the argument reaches $\pi/2$ and stays there for a while until it drops down. For all the different (examined) values of $r_h$ and $\hat{d}$, $\arg(\sigma) \approx -0.52 = -30°$ for large enough $\hat{\omega}$.

## 6.3 Possible expressions for $\alpha$ for linear in $T$ resistivity

In several instances in this chapter we have given an expression for $\alpha$ in order to get resistivity linear in temperature. However, we must ask ourselves whether we can really set $\alpha$ to depend on the other quantities in this way. We must remember from section 5.1 that $\alpha$ is already defined via other quantities. Looking at eq. (5.3)

$$e^{\phi} = \left(\frac{3r}{4Q_f}\right)^{2/3} g_s \text{ where } Q_f = \frac{4\pi N_f}{9\sqrt{3}} l_s$$

and the definition of $\alpha$ eq. (6.14)

$$\alpha \equiv \left(\frac{4Q_f}{3}\right)^{2/3} \frac{1}{g_s} = \left(\frac{16\pi N_f}{27\sqrt{3}} l_s\right)^{2/3} \frac{1}{g_s}$$

(where we have now included $g_s$) we see that to determine the behaviour of $\alpha$, we must ask ourselves how exactly $\frac{(N_f l_s)^{2/3}}{g_s}$ behaves. Remember that we require $l_s/R \ll 1$ to avoid quantum gravitational contributions and $N_f \gg 1$ for the smearing approximation [37]. We can also get some clues from the length scale $R$. From eq. (5.2) we have that

$$R^4 = \frac{4096}{1215} \pi N_c \alpha'^2 g_s$$

which means that $N_c \gg 1$ in order for $R \gg l_s$. $g_s$ should also be small to avoid stringy corrections. Since $N_f/N_c \equiv \lambda_{fc}$ must be kept fixed and finite, we can write

$$\frac{(N_f l_s)^{2/3}}{g_s} \sim \frac{(N_c l_s)^{2/3}}{g_s} = \frac{(N_c l_s^4 g_s)^{2/3}}{l_s^2 g_s^{5/3}} \sim \frac{R^2}{l_s^2} \frac{1}{g_s^{5/3}} R^{2/3} \implies \alpha \gg R^{2/3} \tag{6.98}$$

So $\alpha$ must become very large in comparison to $R$, even with a finite $g_s$. A quick note on units is that in order to restore the length-units of the different quantities, we only need to re-insert factors of $\alpha'$. This will be used below.

So, does the requirement that $\alpha$ be very large mean that also $C$, the proportionality constant that determines the slope of the linear in $T$ resistivity also need to large? This would make the resistivity essentially 0, as it goes as

$$\rho \sim \frac{r_h}{C} \tag{6.99}$$

At first glance, it would appear that the answer is yes. With the units restored, the expression for $\alpha$, eq. (6.87), becomes

$$\alpha = \frac{\alpha'^2}{2r_h^{16/3}} \left(\sqrt{\frac{4C^2 r_h^8}{R^4} + d^4\alpha'^8} - d^2\alpha'^4\right). \tag{6.100}$$

This mean that, assuming that $r_h/R \sim 1$ and $dR^3 \sim 1$,

$$\frac{\alpha}{R^{2/3}} \sim \frac{\alpha'^2}{2R^6} \left(\sqrt{4C^2R^4 + \frac{\alpha'^8}{R^{-12}}} - \frac{\alpha'^4}{R^6}\right) \sim \frac{\alpha'^2}{R^4}\sqrt{C} \tag{6.101}$$

which becomes very small if we want $l_s \ll R$, unless $C \sim R^8/\alpha'^4 \gg 1$. Even if $r_h \sim l_s$ $d \sim l_s^{-3}$ we would have

$$\frac{\alpha}{R^{2/3}} \sim \frac{1}{2l_s^{4/3} R^{2/3}} \left(\sqrt{\frac{4C^2 l_s^8}{R^4} + l_s^4} - l_s^2\right) = \frac{l_s^{2/3}}{2R^{2/3}} \left(\sqrt{\frac{4C^2 l_s^4}{R^4} + 1} - 1\right) \tag{6.102}$$

which is at least $< 1$ with $l_s/R \ll 1$, unless $C \sim R^2/l_s^2 \gg 1$. It is clear that this behaviour for $\alpha$ is not amenable with eq. (6.98). So from this analysis it would appear that $C$ would need to be very large indeed. However, there are arguments to why this need not be true, but that is more appropriate for the next chapter.

## 6.4 Summary

Let us in this section quickly sum up the most important results, so that they can be more easily over-viewed. First, the conductivity at zero wave-vector and zero frequency in the in-plane direction is

$$\sigma^{xx} \sim \frac{R^2}{r_h^{7/3}} \sqrt{\alpha \left(d^2 + \alpha r_h^{16/3}\right)}.$$

We want it to go as $\sigma \sim C/r_h$, as that give us resistivity linear in temperature, remembering that the temperature of the field theory is proportional to $r_h$. This behaviour can be achieved if we set

$$\alpha = \frac{1}{2r_h^{16/3}} \left( \sqrt{\frac{4C^2 r_h^8}{R^4} + d^4} - d^2 \right).$$

Some potential issues with this choice of $\alpha$ is described in the previous section (section 6.3). With this relation for $\alpha$ we can find $\sigma(\omega)$. Using the hatted quantities $\hat{\omega}$ and $\hat{d}$ from eq. (6.81) and eq. (6.82) respectively, we can numerically find $|\sigma(\hat{\omega})|$ and $\arg(\sigma(\hat{\omega})$ for different $\hat{d}$ as in fig. 6.6. We find a region where $|\sigma(\hat{\omega})| \sim \hat{\omega}^{-1}$ and one where $|\sigma(\hat{\omega})| \sim \hat{\omega}^{1/3}$. For the argument we find a constant angle of 90° for large $\hat{d}$ in a range of $\hat{\omega}$ and a constant angle of $\approx -30°$ for larger $\hat{\omega}$ in all the $\hat{d}$-cases. For the off-plane current, we find a finite current in the case of zero wave-vector and zero frequency. We have not looked at any frequency dependence for the off-plane conductivity because the current will be zero due to the required re-scaling done to $a_z$. This re-scaling is required to keep the on-shell action finite.

For our dispersion relations, by adding boundary terms to the action we get mixed boundary conditions. Due to these, we find that we arrive at plasmonic dispersion relations as in fig. 6.3.

# 7

# Discussion and conclusions

In this thesis I have examined a top-down model and looked at some possible behaviour for the conductivity of the field theory that it is dual to the 7-7 strings. As can be expected from the model, as it is derived from string theory, it has a large phase space of possible behaviour. We have used this and chosen a particular configuration that give us resistivity that is linear in temperature, as this is something found in the strange metal phase of the cuprates. When examining the validity of this choice, we have however found an issue with regards to the order of magnitude of $N_c$ and $N_f$. When the geometry was derived these were assumed to be large [37], but this means that $\alpha \gg R^{2/3}$ from eq. (6.98). This has not taken into account the strength of the correspondence we work in, which might set additional limits on for example $\alpha'$ and $g_s$. However, our expression for $\alpha$ implies that $\alpha \ll R^{2/3}$ unless $C$ is very large. But, as mentioned in section 6.3 there are arguments that we do not need $N_c$ and $N_f$ large, nor $\alpha'/R \to 0$ or $g_s \to 0$ (as required in the weak form of the correspondence), and still might be able to work in the supergravity approximation, at least for IR physics (which we are interested in).

So, what we essentially want is to be able to use our supergravity solution (including the probe-limit) as an approximation for the IR (near horizon) physics in the strongest form of the correspondence. Let us first look at the case of AdS/QCD. This has been a rather successful application of the correspondence and it used $N_c = 3$, as expected for QCD [4]. So $N_c$ might not need to be as large as one might first expect. This we can argue should extend to $N_f$. Additionally, for the IR physics the supergravity approximation to string theory works better as the quantum corrections from a finite $\alpha'$ and $g_s$ are still negligible for this region [41]. Since it is the IR-physics we wish to model, this means that we might be able to use our supergravity solution even in the strongest form of the correspondence. We would of course not have a UV complete theory however, but this is not something we require.

We should also ask whether it is okay for $\alpha$ to have these dependencies from a physical perspective, disregarding the issue of limits. What we are essentially doing is saying that the number of D5-branes must depend on the radius of the black hole and $d$, which is the charge density of the field theory. What is happening then is we slightly alter the geometry of the model when we change the temperature and charge density. The geometry would change even if $\alpha$ was independent of $r_h$ as a higher $r_h$ means a larger black hole. An $r_h$ and $d$ dependent $\alpha$ would introduce another change in the geometry. We know that the metric tensor of the gravity theory is dual to the energy momentum tensor of the field theory [1]. Changing $T$ and $d$ should change the energy-momentum tensor and we should therefore also see a change in the metric tensor (which we do, since we alter $e^{\phi}$). We can therefore assume that the change of the metric tensor created by a change in $d$ and $r_h$ would correspond to the same change in the energy-momentum tensor by the same change in $d$ and $T$. This would be an interesting check of our model.

If we allow $\alpha$ to have the expression eq. (6.87), there is some interesting experiments to compare our results to, which can be found in [42]. The most interesting is that they find a power-law behaviour for $|\sigma(\omega)| \sim \omega^{-2/3}$ for a range of $\omega$ and that $\arg((\sigma(\omega)) \approx 60°$ for a different range of $\omega$. Although our power-law behaviours and angles do not match those in the experiment, it is somewhat encouraging that we can find power-law behaviours and constant angles. Also regarding matching against experiments, we find plasmonic dispersion relations with our boundary conditions, as one could expect from previous work on holographic electrodynamics [32] [35] [43], and as wanted for a model of strange metals.

Next, let us mention the fact that we can find a finite current in the off-plane direction when $k_\mu = 0$, but not for $k_\mu \neq 0$ without causing the on-shell action to diverge is quite interesting. But as mentioned in chapter 6, a wave propagating in this direction is problematic as the direction collapses as $r \to \infty$. But when there is no propagation, this issue goes away, and we can obtain a finite current as well as a finite on-shell action.

In the end, while there is not overwhelming support for this model to be able to describe strange metals, it certainly shows some prospects. Assuming the arguments regarding the values of $N_c, N_f$ etc. hold, then further analysis could tell whether there is a possibility for a description of strange metals in the parameter space. This would require some work in finding the (if any) freedoms that we have left. This might include a different embedding of the D7-brane. We have restricted ourselves to the massless case, and so it would be interesting to see what happens with a different embedding. This is most likely a very difficult problem. It is worth mentioning that this is the first layered model that have the linear in $T$ resistivity in its parameter space. There is also no demand that this model need only be able to describe strange metals. There could be other choices to make in the parameter space that allows for descriptions other layered systems that lack quasi-particles.

To summarise my suggestions for future studies of this model, I would suggest making sure that it is possible to trust our solutions even in the strongest form of the correspondence. In that case, it would be interesting to see whether further adjustments of the parameter-space could result in better agreement with experimental data from strange metals. It would of course also be interesting to see if it truly is possible to model other systems by a different choice of parameters. All in all, there is much in this model to discover and examine further, with a potential to give new insights into using top-down models to model condensed matter systems.

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