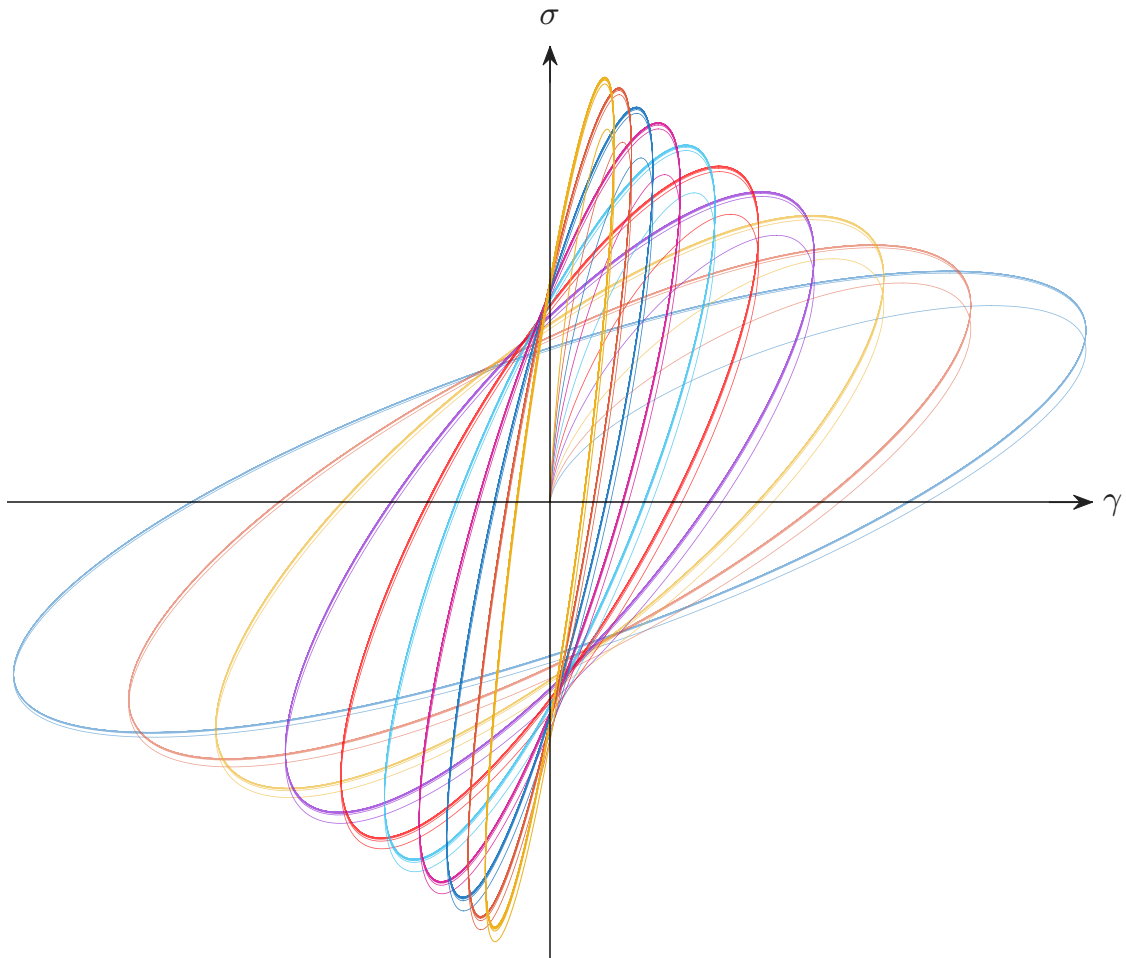




CHALMERS
UNIVERSITY OF TECHNOLOGY



Mechanical Behavior of Bitumen: In the Context of Bitumen Coated Piles

Master's thesis in the Master's Programme
Structural Engineering and Building Technology

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CHALMERS UNIVERSITY OF TECHNOLOGY
Master's thesis IMSX30
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Cover: Hysteresis loops for a specific bitumen at different frequencies, simulated in
MATLAB.

Department of Industrial and Materials Science
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ABSTRACT

Bitumen has long been used in geotechnical applications as a coating on piles to modify the interaction between the pile surface and the surrounding soil. Relative ground settlements induce negative skin friction, increasing axial loads and reducing structural capacity. A well-established method to mitigate this effect is to apply a thin layer of bitumen. However, despite bitumen's long-standing use in practice, the mechanisms governing the behavior of bitumen and its effectiveness in reducing pile–soil interaction remain insufficiently understood.

In this thesis, the fundamental mechanical behavior of bitumen is investigated to establish a basis for the numerical evaluation of adhesion strength in bitumen-coated pile design. A literature review is combined with the implementation of suitable constitutive material models, which are subsequently calibrated using available experimental data.

The results suggest significantly lower adhesion values than those commonly adopted in design. However, the essential viscoelastic behavior of bitumen can be adequately described within the calibrated range of the model. Nevertheless, due to limited experimental data for realistic loading conditions, significant extrapolation in both strain magnitude and strain rate was required, increasing uncertainty in the predicted response. Consequently, further experimental studies are required to characterize bitumen's behavior when subjected to the large deformations and low strain rates observed in practice. This will enable more reliable calibration of the material model for practical design applications.

Key words: Bitumen, Bitumen-coated piles, Negative skin friction, Pile-soil interaction, Viscoelasticity, Constitutive modeling

Bitumens mekaniska beteende:

I Kontexten av Bitumenbelagda Pålar

Examensarbete inom masterprogrammet Konstruktionsteknik och byggnadsteknologi

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SAMMANFATTNING

Bitumen har länge använts i geotekniska tillämpningar som ett yttre skikt på pålar för att minska skjuvspänningarna i glidytan mellan påle och omkringliggande jord. Dessa spänningar uppstår till följd av relativa marksättningar, vilka leder till ökade axiella laster och minskad bärförmåga hos pålen. Trots att bitumenbelagda pålar har använts i praktiken under lång tid är de bakomliggande mekanismerna för materialets beteende och dess effektivitet i att reducera interaktionen mellan påle och jord ännu inte fullständigt klarlagda.

I detta examensarbete undersöks bitumens mekaniska beteende i syfte att etablera ett underlag för numerisk utvärdering av vidhäftningsstyrka vid dimensionering av bitumenbelagda pålar. Arbetet bygger på en litteraturstudie med efterföljande implementering av lämpliga konstitutiva materialmodeller, vilka därefter kalibreras mot experimentella data.

Materialmodellen predikterar avsevärt lägre vidhäftningsstyrka än det som vanligtvis antas vid dimensionering. Dock kan bitumens viskoelastiska beteende beskrivas inom det kalibrerade modellområdet. För verkliga lastfall, utanför tillgängliga experimentella data, krävs betydande extrapoleringar av både töjningsnivåer och töjningshastigheter. Detta medför betydande osäkerhet i de predikterade spänningsnivåerna under realistiska belastningsförhållanden, för vilka experimentell verifiering i hög grad saknas. Följaktligen krävs ytterligare experimentella studier för att karakterisera bitumens beteende vid stora deformationer och låga töjningshastigheter. Detta skulle möjliggöra en mer tillförlitlig kalibrering av materialmodellen vid praktisk dimensionering.

Nyckelord: Bitumen, Bitumenbelagda pålar, påhängslaster, påle–jord-interaktion, viskoelasticitet, konstitutiv modellering

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Preface

In this study, the mechanical behavior of bitumen has been investigated in the context of bitumen-coated piles. The work was carried out between January 2026 and June 2026 at the Department of Mechanical Engineering, Chalmers University of Technology.

The project was conducted under the supervision of Per Kjetil and Helén Broo at the Bridge and Construction Department at Skanska in Gothenburg. We would like to express our sincere gratitude to Per Kjetil and Helén Broo for sharing their extensive expertise in mechanical and geotechnical applications, as well as for their valuable guidance throughout the project. We also wish to thank Skanska for providing us with this opportunity and for supplying the tools and resources necessary to complete this research.

Finally, we would like to extend our appreciation to Assistant Professor Knut Andreas Meyer at Chalmers for his experience within mechanical engineering, valuable insights, and constructive feedback.

Gothenburg, June 2026

Axel Isberg

Axel Jankav

Notations

List of acronyms

DSR	Dynamic shear rheometer
LVE	Linear viscoelastic
MW	Maxwell
TRS	Thermorheologically simple
TTS	Time-temperature superposition
WLF	William-Landel-Ferry

List of variables

Greek letters

α	Difference between upper and lower asymptote (sigmoidal model)
α_T	Shift factor
β	Shape constant (sigmoidal model)
δ	Phase angle
δ_s	Lower asymptote (sigmoidal model)
γ	Shear strain
$\dot{\gamma}$	Shear strain rate
γ_e	Elastic shear strain
γ_p	Prescribed shear strain
γ_v	Viscous shear strain
$\dot{\gamma}_s$	Shear strain rate in dashpot
η	Viscosity
η_0	Zero shear viscosity
η_∞	Infinite shear viscosity
η_s	Viscosity in the dashpot
$\hat{\eta}_s$	Cross-model viscosity
κ	Shear stress reduction coefficient
κ_{soft}	Softening coefficient
ν	Poisson's ratio
ω	Angular frequency
ω_r	Reduced angular frequency
σ	Shear stress
$\hat{\sigma}$	Steady-state shear stress
σ_{MW}	Stress in the Maxwell element
σ_s	Stress in the dashpot
τ	Relaxation time(s)
θ	Rotation angle

Roman letters

C_1	Material constant
C_2	Material constant
E	Young's modulus
E_a	Viscoelastic activation energy
G	Shear modulus
G_0	Instantaneous shear modulus
G_∞	Long-term shear modulus
G_i	Branch-specific shear modulus
g_i	Normalized branch-specific shear modulus
G^*	Complex shear modulus
G'	Storage modulus
G''	Loss modulus
h	Height
K	Material constant (Cross model)
m	Material constant (Cross model)
M	Torque
r	Radius
r_q	Radial position
R	Radius
S	Softening
S_{min}	Minimum softening factor
t	Time
Δt	Time step
T	Temperature
T_{ref}	Reference temperature

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1 Introduction

Pile foundations are widely used to transfer structural loads to deeper, more competent soil layers when near-surface conditions are insufficient. Bitumen is commonly used in geotechnical applications, including its use as a coating material on piles to modify the interaction between the pile surface and the surrounding soil. Bitumen-coating is primarily used to reduce negative skin friction and downdrag forces, which develop when the surrounding soil settles relative to the pile shaft. As the soil moves downwards, shear stresses are mobilized along the pile-soil interface, increasing axial loads in the pile and potentially affecting serviceability and structural safety. Despite the long-standing use of bitumen coated piles in practice, the mechanisms governing the performance of bitumen, as well as the influence of material properties, application methods and environmental conditions, are not yet fully understood.

In pile design, adhesion in the absence of bitumen is generally governed by the shear strength of the surrounding clay. However, Fellenius (1979) suggests, based on empirical field studies, that a properly applied bitumen coating can significantly reduce this adhesion to a conservative value of 10kPa (0.1kg/cm^2), as indicated by observed values within the range of (0 – 20kPa). While this suggestion is commonly adopted in current practice, the degree of its conservatism remains uncertain. Given the advancements in contemporary analytical tools and numerical methods, there is a clear opportunity to reassess this suggestion.

1.1 Purpose

The aim of this thesis is to enhance the understanding of bitumen and its mechanical behavior in the context of bitumen-coated piles. Furthermore, an appropriate material model will be identified and developed to describe the interaction between the bitumen coating and the surrounding soil.

1.2 Goals

The primary objective of this thesis is to develop a constitutive material model for bitumen based on 1D shear stress–strain relationships. The model is intended to describe the interaction between the bitumen coating and the surrounding soil under low loading rates and large deformations. Its numerical implementation is subsequently evaluated within a 2D finite element framework.

To achieve this objective, the following research questions have been formulated:

- How does bitumen respond to shear loading at low strain rates, and what characterizes its deformation behavior?
- How can a constitutive material model for bitumen be formulated to represent its behavior under realistic loading conditions?
- How well do the simulated results agree with available experimental data and reported field observations?

1.3 Limitations

In order to reduce the complexity of the problem and enable a focused analysis within the given time frame, a set of limitations has been imposed. These constraints define the scope of the study and should be considered when interpreting the results. The following limitations are applied in this study:

- No group pile effects are considered; the analysis is restricted to a single pile
- No creep effects of the bitumen are considered
- No aging effects of the bitumen are considered
- The numerical models are restricted to 1D and 2D frameworks

1.4 Simplified realistic loading condition

A simplified realistic loading case was defined to represent typical loading conditions. The case is characterized by a constant relative displacement between the soil and the pile of 5mm per year over a time horizon of 100 years. Assuming a bitumen coating thickness of 2mm, the corresponding shear strain rate for this scenario can be estimated as follows:

$$\dot{\gamma} = \frac{5\text{mm}}{(365 \cdot 24 \cdot 60 \cdot 60\text{s}) \cdot 2\text{mm}} \approx 8.0 \cdot 10^{-8} \text{ s}^{-1} \quad (1.1)$$

Accordingly, the relevant parameters governing the model response are a constant shear strain rate, $\dot{\gamma} = 8.0 \cdot 10^{-8} \text{ s}^{-1}$, with total deformation magnitudes of up to 500mm. Additionally, the ground temperature T is assumed to be in the range of $0 - 10^\circ\text{C}$, and the pile is assumed to be an end-bearing pile in clay.

1.5 Ecological, ethical and social aspects

As urban areas continue to expand, the demand for new construction is steadily increasing. This development commonly necessitates buildings in regions with suboptimal ground conditions thereby requiring the use of deep foundations. In cities such as Gothenburg, characterized by thick clay deposits, this presents significant challenges for pile capacity, particularly for end-bearing piles. At greater depths, piles are subjected to substantial loads from the surrounding clay, which in some cases may exceed the structural capacity of the pile itself. To mitigate this issue, bitumen has been employed as a coating layer at the pile–soil interface. However, the underlying mechanisms responsible for the effectiveness of bitumen remain insufficiently understood, and further research in this area could provide several advantages for sustainability, urban development, and the construction industry as a whole.

From an ecological perspective, a deeper understanding of the mechanical behavior of bitumen may enable a more efficient design of pile foundations, thereby reducing both the number of piles required and their dimensions. As concrete and steel are the most commonly used materials for piles, improved efficiency in their utilization can lead to a reduction in material consumption, thereby decreasing the environmental impact and minimizing the overall influence on the surrounding environment. For this reason,

the mechanical performance of bitumen is investigated in this study under a realistic loading scenario derived from representative field conditions (Section 1.4). However, to enable systematic analysis, the problem is approached using simplified assumptions, ensuring that the essential mechanical behavior is captured while maintaining a clear and controlled framework for investigation.

From an ethical perspective, it is important to maintain transparency regarding the limitations and assumptions of the study, as these directly influence the interpretation and applicability of the results. By clearly communicating uncertainties and possible sources of error, the risk of misinterpretation or overconfidence in the findings is reduced. This is particularly important in engineering applications, where decisions based on inaccurate conclusions could lead to potentially harmful design choices.

From a social perspective, the study contributes to an improved understanding of pile behavior and thereby supports more informed engineering decisions that benefit both current and future communities. In addition, a time horizon of 100 years is considered in order to capture long-term behavior and large deformations, thereby helping to ensure the performance of foundation systems with respect to public safety and long-term societal resilience.

2 Background and theoretical framework

As briefly introduced in the previous chapter, bitumen is commonly used in geotechnical applications, including its use as a coating material for piles. Its behavior under such applications has been examined in previous studies, and some conclusions have been drawn from empirical observations and numerical studies. However, the mechanical response of the material in this context remains insufficiently investigated. In contrast to geotechnical engineering, the mechanical behavior of bitumen has been extensively investigated within the pavement industry (Hettiarachchi & Mampearachchi, 2025). Nevertheless, the loading conditions and material parameters of interest differ significantly between these fields, with loading rates and deformation magnitudes representing the main distinctions.

To address these interdisciplinary differences, a more detailed examination of bitumen as a material, drawing on previous research within geotechnical applications as well as findings from the pavement discipline, is required. The following sections therefore present the relevant background on current knowledge of bitumen, along with the theoretical framework necessary for subsequent material modeling.

2.1 Bitumen

Bitumen is a byproduct of crude oil refining and belongs to a class of amorphous, dark-colored cementitious materials (Hettiarachchi & Mampearachchi, 2025). It primarily exhibits nonlinear viscous behavior and consists of high molecular weight hydrocarbons (Briaud, 1997). Due to its strong dependence on temperature and loading conditions, the mechanical behavior of bitumen is complex. Consequently, a more detailed introduction to its material properties and classification standards is required.

2.1.1 Properties of bitumen

Bitumen exists in different forms, and its behavior is influenced by its internal structure, in which the microstructure governs its material properties, including its adhesive characteristics (Porto et al., 2022). One of the most important properties of bitumen is its viscoelastic behavior, which governs its response to temperature and loading conditions (Pipintakos et al., 2024). Viscoelasticity reflects the combined viscous and elastic nature of the material. The viscous component is related to the material's resistance to flow, commonly described in terms of viscosity. A high viscosity corresponds to a slow-flowing material, whereas low viscosity corresponds to a readily flowing material. As the temperature increases, bitumen exhibits more viscous behavior, while at lower temperature it behaves more elastically (Behzadfar & Hatzikiriakos, 2013). At high loading rates, the material behaves more elastically, while at low loading rates it exhibits a more viscous response. Consequently, at low temperatures combined with rapid loading, bitumen becomes stiff and brittle. In contrast, at high temperatures and low loading rates, it behaves more like a viscous fluid and is prone to permanent deformation.

Bitumen is a thermoplastic material, meaning that its consistency and rheological behavior vary with temperature (Sybilski, 1993). Consequently, depending on the applied temperature and shear rate, bitumen may exhibit Newtonian behavior governed by Newton's linear law of viscosity, whereas under other conditions it demonstrates non-Newtonian behavior, in which shear stress varies nonlinearly with shear rate. At

service temperatures below approximately 50°C, the latter behavior has been observed in experimental studies by Wang et al. (2018) on bitumen used in pavement applications. The non-Newtonian response is generally categorized as either shear-thinning (i.e., viscosity decreases with increasing shear rate) or shear-thickening (i.e., viscosity increases with increasing shear rate) behavior. However, according to Sybilski (1993), non-Newtonian fluids predominantly exhibit shear-thinning characteristics.

2.1.2 Classification of bitumen

To determine the properties of bitumen, various standardized methods are employed. However, it is important to emphasize that the standards in Sweden today are explicitly focused and developed towards the pavement industry. Commonly used classification approaches include penetration grading using a penetrometer, determining the softening point by the Ring and Ball method, measuring the viscosity with a viscometer, and measure the elastic recovery. These methods, along with their classification criteria and preparatory procedures, are described in the standards SS-EN 12591(SIS, 2009), SS-EN 14023(SIS, 2010), SS-EN 14770(SIS, 2023) and SS-EN 12594(SIS, 2024).

In the literature regarding piles, it is customary to report the bitumen's penetration grade as a principal indicator of its consistency. Fellenius (2026) contends that, for locations not subject to exceptional climatic or temperature conditions, a penetration grade of 80/100 is generally appropriate. Here, the range represents the minimum and maximum permissible penetration values, expressed in tenths of a millimetre, determined at 25 degrees Celsius.

2.2 Bitumen in geotechnical application

To mitigate downdrag loads on piles, the use of bitumen-coated piles has proven to be an effective approach when applied under appropriate conditions. Bitumen is effective in reducing the shaft shear due to its viscous properties and inability to sustain shear forces and resist movement (Fellenius, 2006).

Fellenius (1979) argues that the critical issue is to apply the bitumen layer correctly to remain intact during storage and installation of the pile. In addition, the author indicated that there are robust literature that confirms that a thin layer of 1-2 mm is sufficient. To protect the coating, it is of importance that the bitumen layer does not flow too much during storage and that the layer does not shear off during installation, which is influenced by the temperate and load rate. Claessen and Horvat (1974) emphasize the importance of different temperatures at different stages. During application, the bitumen needs to be heated to a sufficiently high temperature to achieve near-liquid behavior, for the layer to adhere to the piles and enable smearing. At installation, the bitumen should be exposed to lower temperature to maintain elastic behavior during piling, which also is heavily enhanced at higher strain rates. A method to reduce the risk of damage to the layer is to apply a two-layer bitumen solution when needed, where the outer layer consists of a very stiff bitumen coating to cover the inner slip layer during installation (Fellenius, 2026).

Furthermore, as a result of differences in the horizontal soil pressure the slip layer must not be forced upwards or downwards. To prevent penetration of the slip layer, it is important to consider the grain-size of the soil layer (Claessen & Horvat, 1974). Something that, according to Fellenius (1999), only is relevant during driving to prevent the

layer from being scraped off. Claessen and Horvat (1974) further points out that if a pile is used as a deep foundation element, the lower end corresponding to about ten times the diameter or width of the pile should be uncoated to not reduce the point bearing capacity. Ideally, the pile should be coated only where negative skin friction is expected (caused by downward relative soil settlement inducing axial forces along the pile shaft), as the applied coating will equally well reduce the positive shaft resistance, resulting in a decreased load-bearing capacity (Fellenius, 2026). Therefore, a careful assessment is needed to determine if the use of bitumen is justified to mitigate negative skin friction. Fellenius (2026) argues that in most cases it is not necessary and points out that coating the piles is difficult, as well as expensive and time-consuming.

2.2.1 Designing for downdrag

Fellenius (2006) summarized some long-term measurements on instrumented piles during the period 1960-2000. The results showed that the load transition from the soil to the pile through negative skin friction, and from the pile back to the soil through positive shaft resistance, was governed by effective stresses. After the piles were installed, a force equilibrium developed, where the sustained loads on the pile head and the loads governed by downdrag were equal to the positive shaft resistance plus the pile toe resistance. Since the magnitude of the former parameters are depth dependent, the location of the force equilibrium occurred where the upward and downward forces were equal. This location is often referred to as the neutral plane, and is the position at which the relative movement between the pile and the soil is zero.

In design, there is no single factor governing the consideration of downdrag. However, according to Briaud (1997), there are some indicators gathered from experience. These include cases where the total settlement of the ground surface exceeds 100 mm, or where the post-installation settlement after driving or boring piles is greater than 10 mm. Additional attention is also warranted when the embankment height exceeds 2 m, when the soft compressible soil layer is thicker than 10 m, or when the groundwater table is expected to be lowered by more than 4 m. Furthermore, special design measures may be necessary if the pile length exceeds 25 m. Downdrag loads become increasingly significant with increasing pile length, as the axial load within the pile rises with depth due to the development of negative skin friction. This may ultimately lead to the pile capacity being exceeded (Sun & Yan, 2010).

In current design practice, Fellenius (1979) suggests, based on empirical field studies, that a correctly applied bitumen-coating can conservatively mitigate negative skin friction to approximately 10kPa (0.1 kg/cm^2), thereby providing a practical basis for estimating downdrag loads.

2.2.2 Previous field research

Bitumen was extensively studied during the mid and late 20th century. While many researchers agree that it can reduce drag load, their interpretations of its performance and application differ considerably across the literature. A comprehensive understanding therefore requires taking all these perspectives into account.

One of many studies was carried out at koninklijke/shell-laboratorium in Amsterdam in 1968 (Claessen & Horvat, 1974). A concrete pile ($0.32 \times 0.32 \times 14.00 \text{ m}$) with a slip layer of bitumen with a thickness of 10 mm was installed to a depth of 13 m

through the upper soil layers to reach the bearing sand. The bitumen coating was applied specifically to the upper 5 m of the pile to minimize negative skin friction from the settling sand fill. Three thin steel plates were placed at distances of 0, 4 and 8 mm from the pile shaft at an approximately depth of 4 m below ground level in order to measure the vertical displacement. The trials indicated that deformation occurs within the slip layer, increasing non-linearly with distance from the pile shaft, although with small deviation. The study indicates that it is possible to reduce number of piles required by 15-50% and decrease the negative skin friction to about 10% of that of a untreated pile. However, the achievable reduction of piles and negative skin friction relies on the soil, pile and bitumen properties and conditions. Further, it is stated that the external load from the settlements of the soil shall be limited to not generate significant negative friction. This interpretation was later questioned by Fellenius (1975), who pointed out a contradiction, since the fundamental mechanism of the slip layer is intended to reduce the drag loads. In general, the author suggested that a justification for the use of bitumen would have been appropriate. Fellenius further argued that a coating thickness of 10 mm does not improve the effect of the bitumen layer, but instead introduces unnecessary complications. Such a thickness would likely require a bitumen with higher viscosity to ensure a uniform coating, which is not desirable.

In Norway, full-scaled field experiments indicates similar results. Two separate studies of bitumen-coated piles to rock in soft clay at sites with ongoing settlements shows that negative skin friction on the pile can be reduced to 10% of a untreated pile (Bjerrum et al., 1969). This report likewise emphasizes the importance of protecting the bitumen layer during driving. In this study an enlarged point was used to protect the coating during driving with bentonite slurry between pile shaft and fill for stability. The report further explicates that the observations were conducted only on single piles, making conclusions regarding group behavior infeasible.

Björklund et al. (2025) specify that decreased settlements of the soil, caused by the interaction between the soil and the pile, reduce the downdrag forces acting on the pile. Furthermore, deformations in the pile, particularly creep deformations at high load levels and at the pile tip, as well as short-term loading at the pile top, have positive effects. On the other hand, the installation effects are shown to increase the downdrag forces. However, they emphasize the need for further comparison with measurement data to validate precise values.

2.3 Previous mechanical findings

In addition to field studies conducted within geotechnical applications, bitumen has long been investigated within other engineering disciplines. Selected mechanical findings relevant to this study are presented in the following paragraphs.

It is well established that the shear response of bitumen can be expressed as

$$\sigma = \eta_s \cdot \dot{\gamma} \quad (2.1)$$

where η_s is the secant viscosity, here after referred to as viscosity, and $\dot{\gamma}$ is the shear strain rate (Briaud, 1997). However, it is important to note that the viscous behavior of bitumen is not constant (Briaud & Tucker, 1997). The viscosity is strongly influenced by temperature and shear strain rate, making the material behavior highly dependent

on external conditions and thus more complex to understand and predict. As the shear strain rate increases, the shear stress increases accordingly. However, due to nonlinear shear-thinning behavior, the effective viscosity decreases. If the temperature is increased by a factor of two, the shear stress can decrease by a factor of ten. Consequently, bitumen exhibits a much lower viscosity with at higher temperatures.

Furthermore, Behzadfar and Hatzikiriakos (2013) investigated the viscoelastic behavior of bitumen and reported that the material response at low shear rates is primarily governed by viscous characteristics, whereas at higher shear rates the behavior is dominated by elastic properties.

During the late 20th century, Cheung and Cebon (1997) conducted a series of experimental studies on pure bitumen, analyzing its behavior in different temperatures for both shear and axial loading. The shear tests were conducted on two geometries, the first denoted *parallel-shear* and the second *coaxial-shear*. The coaxial test was performed by utilizing a cylinder of bitumen, located between two coaxial cylindrical surfaces. A constant relative rotation was then applied to the two coaxial surfaces, forcing the specimen to shear. The results from the coaxial shear test conducted at $T = 20^\circ\text{C}$ are presented for different strain rates ($0.01\text{s}^{-1} \leq \dot{\gamma} \leq 1\text{s}^{-1}$) in Figure 2.1.

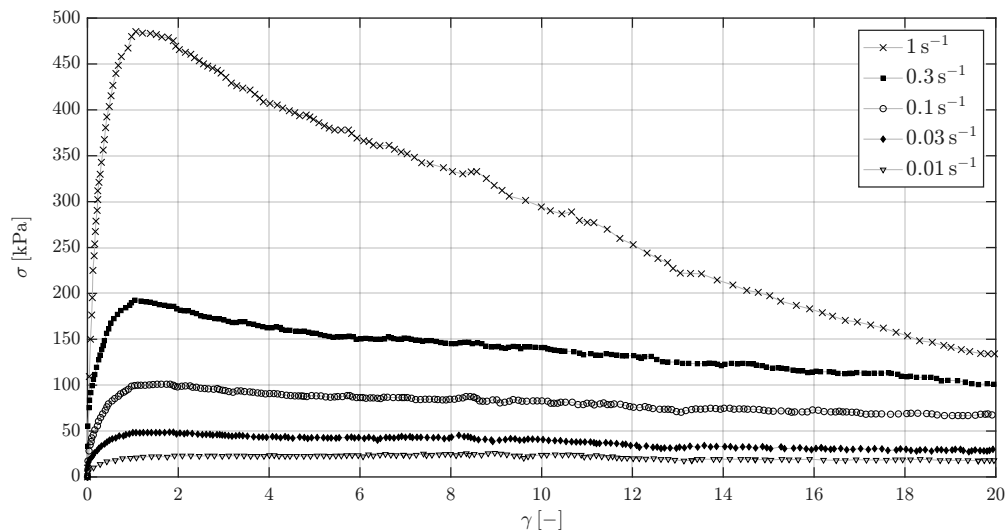


Figure 2.1: Reconstructed plot of manually digitized data from a coaxial shear test graph by Cheung and Cebon (1997), conducted at different strain rates.

The extracted data are the data referred to as "Corrected Data". The original test data are corrected due to adiabatic heating caused by the accumulation of plastic deformation. Furthermore, the corresponding properties of the tested bitumen specimen are summarized in Table 2.1.

Table 2.1: Material properties of the bitumen used in the coaxial shear tests reported by Cheung and Cebon (1997).

Batch No.	Penetration at 25°C [$\frac{\text{mm}}{10}$]	Softening point [°C]	Glass transition temperature [°C]
1	48	56	-30
2	48	53	-25

2.4 Fundamental models of viscoelastic materials

A body may undergo relative movement, i.e., deformation, when subjected to external loads (Shitikova & Krusser, 2022). The resulting deformation depends on the mechanical properties of the material and may be reversible, irreversible, or a combination of both. Reversible deformation, commonly referred to as elastic deformation, is a non-permanent process in which the material returns to its original configuration once the load is removed. Irreversible deformation, on the other hand, comprises viscous, plastic, or otherwise permanent changes that leave the body with residual deformation after unloading. Relations that describe the properties of a material, independent of the body's geometry, are known as rheological equations. The first constitutive relation for solids was formulated by Hooke (1678), describing the proportionality between the relative deformation of a spring and the applied load:

$$\sigma = G \cdot \gamma \quad (2.2)$$

where G denotes the shear modulus of elasticity, and γ the shear strain. In addition to Hooke's law, Isaac Newton (1687) introduced the governing equation for an ideal viscous fluid, establishing that the stress is proportional to the strain rate:

$$\sigma = \eta \cdot \dot{\gamma} \quad (2.3)$$

where η represents the viscosity of the dashpot. In addition to the constitutive relation for a solid, and the governing equation for an ideal viscous fluid, there are materials that cannot be described by only one of the two theories (Shitikova & Krusser, 2022). That is the case for *viscoelastic* materials. The behavior of a viscoelastic material is characterized by the deformation process in both an ideal elastic and an ideal viscous body, meaning that the behavior is a combination of Hooke's and Newton's constitutive relations. In theory, it can be difficult to visualize the work of viscoelastic materials. Therefore, it is appropriate to consider mechanical models that simulate the work. In such models, combinations of pure elastic and pure viscous elements are commonly used, i.e. combinations of springs and dashpots. The Maxwell elements and the Kelvin-Voigt elements are two frequently used models for describing viscoelastic solids. The former model utilizes a spring and dashpot connected in series, allowing the stress to be equal for the spring and the dashpot, with the total deformation as the sum of the deformations of the individual members. The Kelvin-Voigt element on the other hand, consist of a parallel connection, resulting in an equal deformation over the element and a total stress equal to the sum of the stresses in the stretched spring and dashpot. The Maxwell and Kelvin-Voigt elements are illustrated in Figure 2.2 below with their corresponding governing equations.

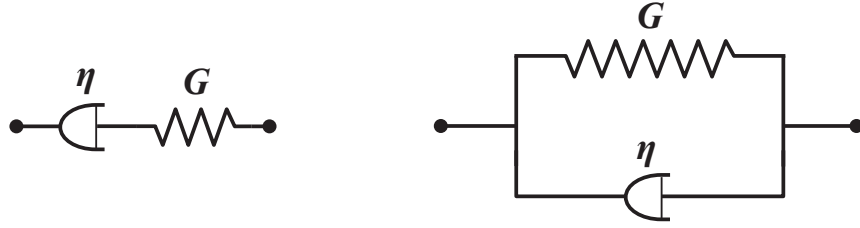


Figure 2.2: Schematic illustration of the Maxwell model (left) and the Voigt model (right).

Maxwell strain and stress

$$\gamma = \gamma_e + \gamma_v \quad (2.4)$$

$$\sigma = G(\gamma - \gamma_v) \quad (2.5)$$

Kelvin-Voigt stress

$$\sigma = G\gamma + \eta\dot{\gamma} \quad (2.6)$$

with γ_e as the elastic shear strain, and γ_v as the viscous shear strain.

2.4.1 The simplest viscoelastic model

One of the simplest viscoelastic representations, commonly referred to as a three-element elastic model, is obtained by connecting a spring in series with a Kelvin–Voigt element or by placing a spring in parallel with a Maxwell element (“Essentials of Linear Viscoelasticity”, n.d.). The American physicist Clarence Zener (1958) was the first to formulate a linear solid using the such mechanical arrangement, which subsequently led to the model becoming known as the *Zener model*. The model is a combination of the Kelvin-Voigt and Maxwell models, resulting in a more realistic description of a viscoelastic solid (Meyer, 2024). An illustration of the model is presented in Figure 2.3 below.

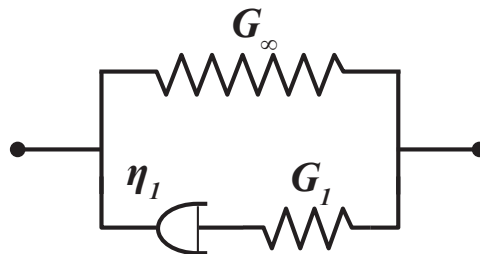


Figure 2.3: Schematic illustration of the Zener model.

Since the Zener model is a combination of the Maxwell and Kelvin-Voigt models, the governing equations for strain and stress are

Zener strain and stress

$$\gamma = \gamma_e + \gamma_v \quad (2.7)$$

$$\sigma = G_\infty \gamma + G_1(\gamma - \gamma_v) \quad (2.8)$$

2.4.2 The generalized Maxwell Model

The previously described Zener model is based on a mechanical system consisting of a spring arranged in parallel with a Maxwell element. However, when modeling materials whose behavior arises from a much larger number of internal mechanisms, this simple three-element representation is no longer as sufficient (Shitikova & Krusser, 2022). In such cases, more generalized models with extended series or parallel networks of springs and dashpots are required. The generalized Maxwell model originates from the Zener arrangement, but with additional Maxwell elements added into the system, see Figure 2.4.

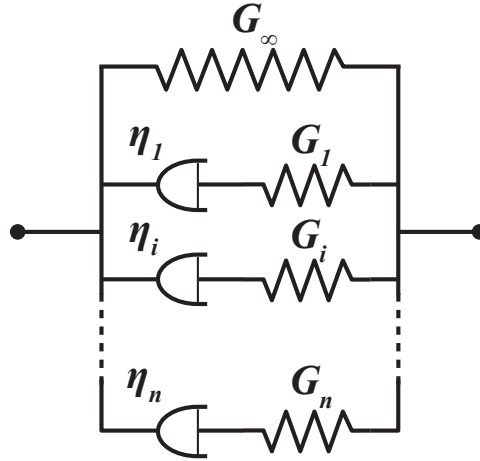


Figure 2.4: Schematic illustration of the Generalized Maxwell model

With additional Maxwell elements, the model accounts for the fact that relaxation does not occur at a single relaxation time (Roylance, 2001). Instead, the relaxation behavior is characterized by a distribution of relaxation times, resulting in a more realistic representation of the material's mechanical response compared to the simpler model.

The total stress σ transmitted by the model is equal to the stress in the isolated spring (with stiffness G_∞) and that in each of the Maxwell elements:

$$\sigma = G_\infty \gamma + \sum_i G_i(\gamma - \gamma_{i,v}), \quad (2.9)$$

with the viscous strain γ_v introduced for each element as

$$\gamma_v = \frac{\Delta t \cdot G_i \cdot \gamma + \eta_i \cdot {}^n\gamma_{i,v}}{G_i \cdot \Delta t + \eta_i}. \quad (2.10)$$

Here, ${}^n\gamma_{i,v}$ is the state variable, meaning that the current viscous strain is dependent on the previous state, and Δt as the time step.

2.4.3 Prony series

To evaluate the time-dependent stress–strain response of a linear viscoelastic material subjected to an arbitrary loading, it is necessary to account for the full deformation history (Chen, 2000). A common methodology for this is to utilize the hereditary integral (Flügge, 1975), expressed as

$$\sigma(t) = G_\infty \gamma(0) + \int_0^t G(t - \tau) \frac{d\gamma(\tau)}{d\tau} d\tau, \quad (2.11)$$

where G_∞ is the equilibrium modulus, $\gamma(0)$ is the initial strain, and $G(t)$ is the stress-relaxation modulus of the material. In the case of the previously presented Maxwell model, see Subsection 2.4, the total relaxation modulus $G(t, T)$ is obtained as the sum of the relaxation contributions G_i from each Maxwell element. A conventional representation of this is given by the so-called Prony series (Pacheco et al., 2015). A Prony series is a mathematical formulation that represents time-dependent relaxation as a summation of decaying exponential functions (Chen, 2000). In this manner, the modulus in the right-hand term of Equation 2.11 can be expressed as

$$G(t, T) = \sum_i G_i e^{-t/\tau_i(T)}, \quad (2.12)$$

where G_i is the relaxation contribution from each Maxwell element, t is the time and τ_i the relaxation time. In addition, Ferry (1980) notes that materials exhibiting solid-like behavior, in contrast to liquid-like materials, approach a finite equilibrium modulus G_∞ as $t \rightarrow \infty$. Consequently, the moduli in the hereditary integral can be expressed in its complete form using a Prony-series representation:

$$G(t, T) = G_\infty + \sum_i G_i e^{-t/\tau_i(T)}. \quad (2.13)$$

Alongside the long-term modulus, the initial modulus plays a fundamental role in characterizing a material's mechanical behavior. The instantaneous modulus G_0 represents the stiffness of a viscoelastic material at the moment of loading, i.e., at $t = 0$ (Pacheco et al., 2015). Hence, the instantaneous modulus G_0 , is given by

$$G_0 = G(0, T) = G_\infty + \sum_i G_i. \quad (2.14)$$

In some applications, it is convenient to express the Prony series terms in normalized form. Instead of working with the absolute moduli G_i , each term is expressed as a ratio relative to the instantaneous modulus (*Abaqus*, 2015; Chen, 2000; Kraus et al., 2017), such that

$$g_i = \frac{G_i}{G_0}, \quad (2.15)$$

with the conditions

$$0 \leq g_i \leq 1 \quad \sum_i g_i \leq 1. \quad (2.16)$$

2.5 Rheological testing

To extend the material model and enable a more data-driven formulation, Dynamic Shear Rheometer (DSR) test data was examined as a basis for describing and calibrating the material behavior, rather than directly fitting the model to experimental response curves. This approach facilitates a more transferable framework, allowing different types of bitumen to be modeled using the same methodology. In addition, DSR testing inherently accounts for temperature sensitivity, which is a critical aspect of bitumen behavior. Following a laboratory visit and further insight into DSR testing procedures, the suitability of the provided experimental data for calibrating a viscoelastic material model was investigated.

2.5.1 Dynamic shear rheometer

To measure the rheological properties of bitumen, a dynamic shear rheometer (DSR) is standard practice in the industry (**SSEN14770**). At given temperature and frequency, the rheometer calculates the complex shear modulus $|G^*|$ with corresponding storage modulus G' and loss modulus G'' as well as the phase shift angle δ when tested in oscillatory shear. The phase shift angle is the angular measure between two waveforms of the same frequency. In the context of DSR testing, the material response is evaluated to an ideally elastic material and represents the degree of viscous behavior. A phase angle of 0° corresponds to an ideal elastic response, whereas 90° corresponds to a fully viscous response. A non-ideal elastic material dissipates energy within its structure, which can be expressed through the damping factor $\tan(\delta)$. From which the phase angle δ can be derived according to Equation 2.17 (Tapia-Romero et al., 2020).

$$\delta = \arctan\left(\frac{G''}{G'}\right) \quad (2.17)$$

The test requires preparatory work to ensure reliable results. The geometry of the specimen is determined by the magnitude of the complex shear modulus, typically using plate diameters of 4 mm, 8 mm, or 25 mm with gap settings between 1 and 3 mm. Additionally, the specimen is sensitive to temperature, contamination, and geometric imperfections, and improper handling can easily invalidate the test. The rheometer is controlled either by a prescribed shear strain or a prescribed shear stress. When a specific shear strain is applied, the corresponding angular displacement and torque are measured to calculate the specimen shear strain and shear stress. If the shear stress controlled mode is used, the specimen shear stress is calculated from the applied torque, and the specimen shear strain is calculated from the measured angular displacement.

For the test to be valid, bitumen must remain within the linear viscoelastic (LVE) range. To ensure that an appropriate shear strain or shear stress must be selected. This is done by measuring G' and G'' at the lowest test temperature combined with the highest frequency, and at the highest temperature combined with the lowest frequency, and

verifying that these values deviate by no more than 5% from the fitted linear regression line. If the applied strain is too large, the internal structure responds non-linearly and the resulting stress is no longer sinusoidal (Anton Paar GmbH, 2026).

In the context of oscillatory loading, such as in DSR testing within the frequency domain, material properties such as storage modulus G' and loss modulus G'' are often used to characterize a material's viscoelastic behavior. Consistent with the theory introduced in Section 2.4, G' and G'' can also be represented using the Prony-series formulation (Tapia-Romero et al., 2020). In this formulation, the frequency-dependent storage and loss moduli are obtained by transforming the time-domain Prony terms into their corresponding expressions in the frequency domain:

$$G'(\omega, T) = G_\infty + \sum_i \frac{G_i (\omega \tau_i)^2}{1 + (\omega \tau_i)^2} \quad (2.18)$$

$$G''(\omega, T) = \sum_i \frac{G_i \omega \tau_i}{1 + (\omega \tau_i)^2}, \quad (2.19)$$

where ω is the angular frequency. Consequently, an additional viscoelastic parameter, the complex modulus G^* , can be introduced. The complex modulus quantifies a material's viscoelastic response under dynamic loading at a specified strain level by combining contributions its elastic stiffness (solid-like behavior) and viscous damping (liquid-like behavior) (Widyatmoko, 2016). In theory, this implies that G^* consists of a real part, originating from the stored energy G' , and an imaginary part, originating from the dissipated energy G'' (Greidanus et al., 2022; Ramesh et al., 2021), such that

$$G^* = G' + iG''. \quad (2.20)$$

By combining this with the mathematical definition of complex algebra, G^* can be obtained as

$$|G^*| = \sqrt{(G')^2 + (G'')^2}. \quad (2.21)$$

The interpretation of the complex modulus is often extended by examining the variation of $|G^*|$ relative to both temperature and loading frequency, as these parameters together determine the balance between elastic and viscous contributions in a viscoelastic material such as bitumen (Alhaddad, 2015). At high frequencies or low temperatures, bitumen exhibits predominantly elastic, glass-like behavior, whereas lower frequencies or elevated temperatures shift the response toward a more viscous regime. This frequency–temperature dependency forms the theoretical basis for constructing master curves, in which rheological data obtained at different temperatures are horizontally shifted to create a continuous representation of the material's viscoelastic behavior. The theoretical principles governing the construction of such master curves are further introduced in the following section.

2.5.2 Time-temperature superposition

From the standpoint of linear viscoelastic theory, high-frequency (short-time) behavior is representative of the material response at lower temperatures, whereas low-frequency (long-time) behavior reflects the response at higher temperatures (Ljubić et al., 2014). This relationship can be quantitatively exploited through the principle of time–temperature superposition (TTS). By applying this principle, extensive sets of isothermal oscillatory frequency data can be translated into a single master curve by using a horizontal shift factor α_T (Nickerson et al., 2004). This enables prediction and modeling of mechanical and rheological properties beyond experimentally accessible timescales (Ljubić et al., 2014).

According to the definition of TTS, each curve measured at a temperature different from the reference temperature can be shifted horizontally by α_T (Li, 2000). The magnitude of this shift depends on the chosen reference temperature as well as on the properties of the material under investigation. Throughout extensive research in the field, several formulations have been proposed to relate the shift factor to the chosen reference temperature. Two of the most widely recognized formulations are the Williams–Landel–Ferry (WLF) equation and the Arrhenius equation. It is generally accepted that the former is more reliable for temperatures above the glass transition temperature (as investigated in this study), whereas the latter is typically used for temperatures below or near this transition. The WLF equation is defined as

$$\log_{10} \alpha_T = \frac{-C_1(T - T_{ref})}{C_2 + (T - T_{ref})}, \quad (2.22)$$

where C_1 and C_2 are material specific constants, and T_{ref} and T corresponds to the reference temperature and the temperature of interest, respectively. For temperatures below the glass transition temperature the shift factor is more accurately described by the Arrhenius equation:

$$\ln \alpha_T = \frac{E_a}{R} \left(\frac{1}{T} - \frac{1}{T_{ref}} \right), \quad (2.23)$$

where E_a is the viscoelastic activation energy and R is the universal gas constant.

As previously described, the TTS principle can be utilized to extend mechanical and rheological properties beyond experimentally accessible data. In theory, this is achieved by transforming the relaxation times τ_i using the horizontal shift factor α_T , such that

$$\tau_i(T) = \alpha_T \tau_i(T_{ref}), \quad (2.24)$$

and thereby enabling the superposition of the material response measured at different temperatures (Aslan & Aktürk, 2024). When this relationship holds, several mechanical properties can be related to the superposition principle. Considering the relaxation modulus G , the relaxation times are incorporated according to

$$G(t, T) = \sum_i G_i e^{-t/\tau_i(T)}. \quad (2.25)$$

By combining this expression with Equation 2.24, the relaxation modulus can be superposed as

$$G(t, T) = \sum_i G_i e^{-t/(\alpha_T \tau_i(T_{ref}))} = G\left(\frac{t}{\alpha_T}, T_{ref}\right). \quad (2.26)$$

In addition to relaxation times, the TTS principle applies to other viscoelastic quantities as well (Ghobadi et al., 2015). The previously defined storage and loss moduli, Equations 2.18 and 2.19, can likewise be shifted using the horizontal shift factor α_T (Ferry, 1980), such that

$$G'(\omega, T) = G'(\alpha_T \omega, T_{ref}) \quad (2.27)$$

$$G''(\omega, T) = G''(\alpha_T \omega, T_{ref}). \quad (2.28)$$

In this way, slow loading conditions can be accessed by studying higher temperatures, as the shift factor extends the material response into the long-time regime while preserving its underlying viscoelastic nature (Oseli et al., 2019).

The principle of the horizontal shift factor α_T is commonly illustrated through master curves. The master curve is constructed by horizontally shifting experimental data measured at different temperatures onto a reduced frequency scale, resulting in a single, smooth representation of a material's behavior (Yusoff et al., 2013). In rheological applications, this approach is effectively demonstrated by fitting the shifted data to a so-called sigmoidal model, which is a widely used method in pavement engineering for characterizing material properties such as the complex modulus in asphalt mixtures and bituminous binders (Garcia & Thompson, 2007; Yusoff et al., 2013). Mathematically, the sigmoidal model is formulated as

$$\log_{10} |G^*| = \delta_s + \frac{\alpha}{1 + e^{\beta + \gamma(\log_{10} \omega_r)}}, \quad (2.29)$$

where the reduced frequency ω_r is derived from the previously introduced shift factor α_T and the logarithmic framework, such that

$$\omega_r = \alpha_T \omega \quad \leftrightarrow \quad \log_{10} \omega_r = \log_{10} \alpha_T + \log_{10} \omega. \quad (2.30)$$

By including additional material-specific parameters such as δ_s , α , β and γ , the model can more effectively capture the behavior of the material (Xu & Engquist, 2018). In practice, δ_s is the lower asymptote, α is the difference between the upper and lower asymptote, and β and γ define the shape around the curve around the inflection point (Yusoff et al., 2013). Typical master curves constructed using the sigmoidal function are presented in Figure 4.4 in Section 4.

3 Viscoelastic model based on literature data

The initial viscoelastic model was developed to capture the material behavior observed in the coaxial shear test in Figure 2.1, conducted by Cheung and Cebon (1997). Based on the identified material response, suitable constitutive models for bitumen were subsequently evaluated, with particular emphasis on formulations capable of capturing time-dependent deformation under large strains and low strain-rate conditions. The model was implemented in MATLAB, with a parallel analysis conducted in Abaqus to verify the numerical implementation and increase confidence in both the results and the underlying modeling assumptions.

To simplify the initial analysis the viscoelastic response of the bitumen was restricted to the shear response up to the peak stress for each strain rate, corresponding to strain levels within the range $\gamma \leq 1$. The extracted test data within this range are presented in Figure 3.1.

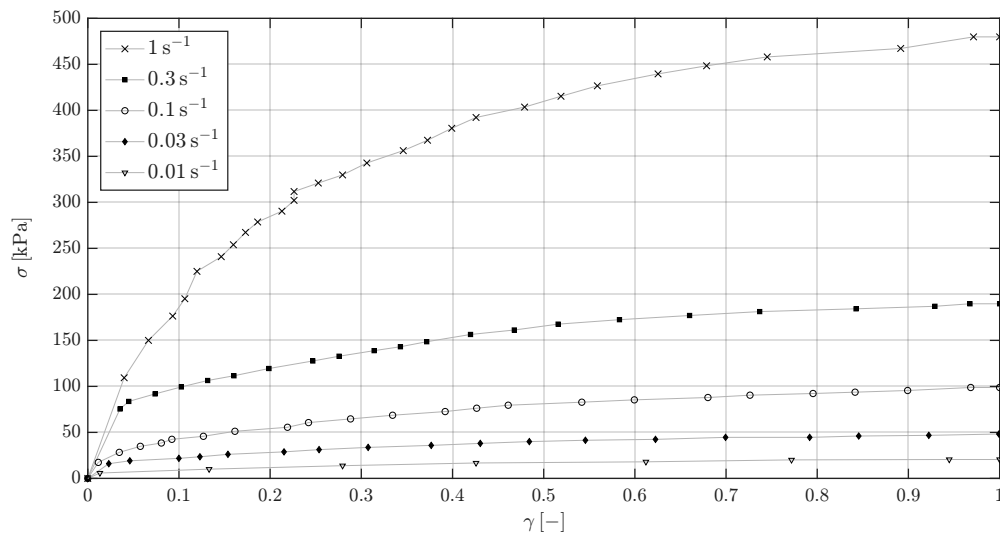


Figure 3.1: Reconstructed plot of manually digitized data for $\gamma \lesssim 1$ from a coaxial shear test graph by Cheung and Cebon (1997), conducted at different strain rates.

3.1 Model formulation

The initial viscoelastic implementation was carried out in MATLAB and was based on the theoretical background introduced in Section 2.4. Since viscous behavior is strongly influenced by contextual conditions such as temperature, loading rate, and time, particular attention was given to the more efficient Maxwell model with extended parallel networks, presented in Subsection 2.4. The system was created with one main spring coupled with three parallel Maxwell elements, see Figure 3.2 below.

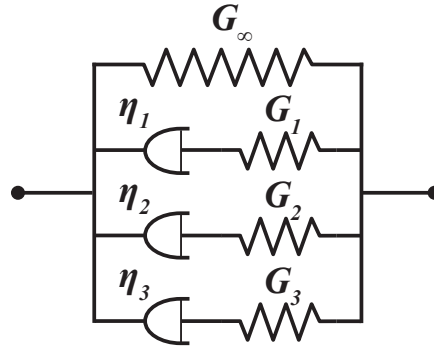


Figure 3.2: Schematic illustration of the Generalized Maxwell model with three branches.

The loading duration for each applied strain rate was derived from the relation $t = \frac{\gamma}{\dot{\gamma}}$, and is summarized in Table 3.1.

Table 3.1: Determined load durations for each strain rate.

$\dot{\gamma}$ [s ⁻¹]	γ [-]	Time [s]
1.0	1.0	1.0
0.30	1.0	3.33
0.10	1.0	10.0
0.030	1.0	33.3
0.010	1.0	100

Initial guesses for unknown variables such as, G_∞ , G_i and τ_i was then made in order to calculate the viscosity η and the viscous strain γ_v for the initial state. For each element, the viscosity η was calculated as

$$\eta_i = \tau_i \cdot G_i, \quad (3.1)$$

and the viscous strain γ_v was evaluated using the previously introduced relation in Equation 2.10. For the initial state, however, it was assumed that no viscous strain had yet developed, meaning that only elastic deformation occurred in the system. Consequently, the implicit numerical integration technique *Backward Euler method* was employed to obtain the viscous strain at each subsequent state, and thus calculate the stresses in the system.

Parallel to the implementation of the generalized Maxwell model, an Abaqus analysis was carried out using Abaqus/CAE 2024. Viscoelastic modeling in Abaqus is based on the generalized Maxwell framework described in Section 2.4, making it a reasonable way to verify the model implementation in MATLAB. The simulation aimed to reproduce the coaxial-shear test by simulating a simple shear test, see Figure 3.3. The bitumen in the coaxial shear test was assumed to be subjected to simple shear, such that only shear stresses were induced in the material.

In order to capture simple shear behavior in Abaqus, the vertical displacements of all boundaries were set to fixed, as well as the horizontal displacement of the bottom

boundary, according to Figure 3.3b. Shear stresses were subsequently introduced by prescribing a displacement to the top boundary, representing the rotating outer cylinder in the coaxial shear test, see Figure 3.3a.

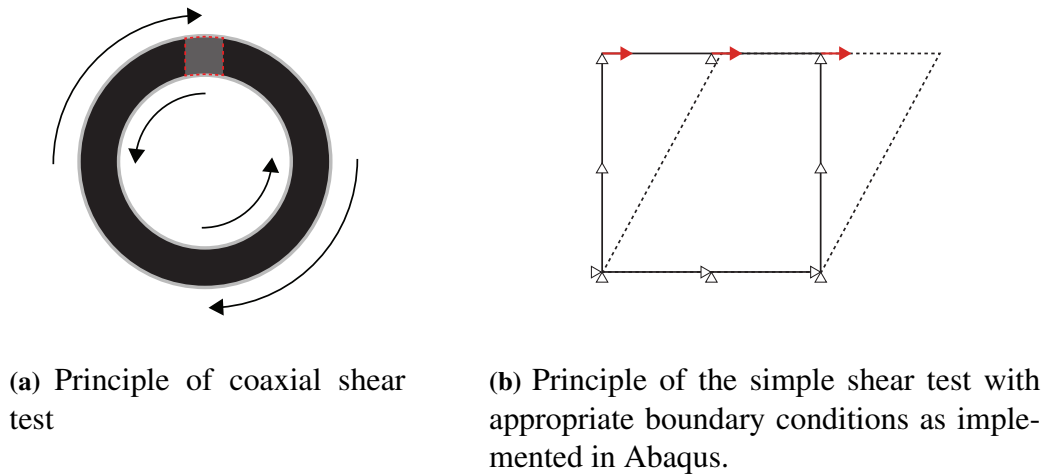


Figure 3.3: Interpretation of the bitumen response in a coaxial shear test and its translation to the simple shear test

With the boundary conditions set, an 2D implicit solid body with dimensions $0.01\text{m} \times 0.01\text{m}$ was introduced in Abaqus. The geometry was discretized using CP4ER elements (4-node bilinear plane strain quadrilateral, reduced integration, hourglass control). The prescribed displacement was set to $\gamma = 1$ in order to capture the behavior up to peak stress. Furthermore, the elastic contribution to the viscoelastic model was defined with Poisson's ratio and Young's modulus, with the later defined according to Equation 3.2 with G_0 obtained in Subsection 3.3.

$$E = 2G_0(1 + \nu) \quad (3.2)$$

3.2 Verification of model implementation

To ensure that the model implementation in MATLAB and the underlying assumptions in Abaqus were consistent with the theoretical framework, a verification procedure was conducted. This verification was carried out using a set of calibrated material parameters, with the procedure for obtaining these parameters described in Section 3.3. The initial verification consisted of several complementary assessments aimed at evaluating different aspects of the model response and increasing confidence in the predicted stress–strain behavior to be analyzed subsequently. First, the numerical implementation was examined in Abaqus by reproducing an ideal simple shear test, ensuring that the boundary conditions, deformation mode, and stress state were correctly represented. Thereafter, the constitutive response was verified in MATLAB by analyzing the material behavior at the theoretical limits. This was achieved by evaluating the instantaneous and long-term shear moduli through the model response at extreme strain rates, allowing direct comparison with the theoretical bounds defined by G_0 and G_∞ .

3.2.1 Simple shear behavior in Abaqus

The behavior of the Abaqus model was verified by examining the distribution of shear stress within the body under prescribed displacement, as previously introduced in Subsection 3.1. As a constant strain rate was applied, the displacement increased linearly along the height of the body, with material planes remaining parallel, as illustrated in Figure 3.4a. Consequently, the shear stress distribution was found to be homogeneous throughout the entire body (Figure 3.4b), indicating an ideal simple shear response. This behavior was observed consistently across the entire strain-rate range. By confirming an ideal simple shear response, the Abaqus model was subsequently used as a reference for comparison with the MATLAB results, further presented in Section 3.4.

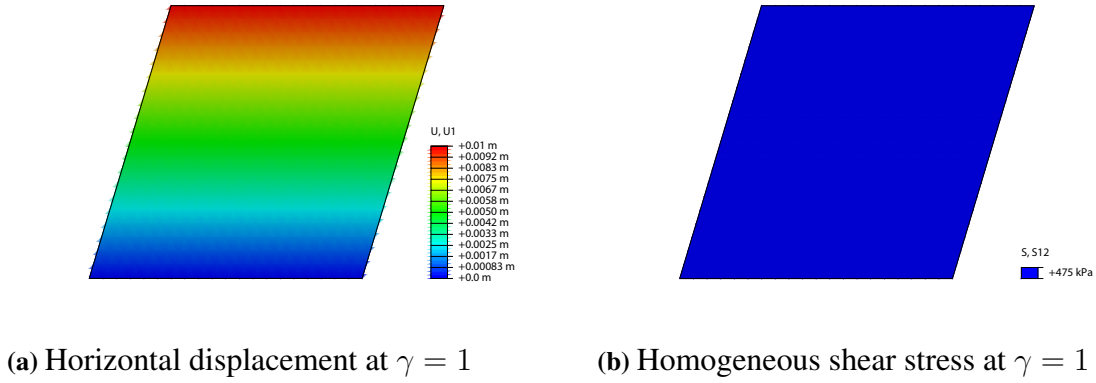


Figure 3.4: Abaqus results for $\dot{\gamma} = 1$.

3.2.2 Evolution of shear moduli

Similarly to Abaqus, the behavior of the Maxwell model implemented in MATLAB was verified. In this case, the evolution of the instantaneous shear modulus G_0 and the long-term modulus G_∞ was examined. To evaluate the model, calibrated material parameters reported in Table 3.2 were used as input, and a stress-strain analysis was conducted to obtain the model response.

As previously introduced in Equation 2.14, the instantaneous shear modulus G_0 can be determined from G_∞ and G_i . By employing the calibrated parameters presented in Table 3.2, this yields $G_0 \approx 3100\text{kPa}$. In theory, this represents the instantaneous slope of the stress-strain curve (see Figure 3.7), that is, the slope as $\dot{\gamma} \rightarrow \infty$. Consequently, the shear modulus evaluated at an arbitrary small strain should approach G_0 as the strain rate increases, with G_0 representing the theoretical upper limit of stiffness attainable by the model. By applying this approach, the shear modulus was extracted at a fixed strain level of 1% ($\gamma = 0.01$), defined as

$$G_{1\%} = \left. \frac{d\sigma}{d\gamma} \right|_{\gamma=1\%} \quad (3.3)$$

This procedure was applied across a range of strain rates, extending from those employed in the experimental tests to progressively higher strain rates, in order to investigate the evolution of $G_{1\%}$ as $\dot{\gamma} \rightarrow \infty$. The resulting variation of $G_{1\%}$ with increasing strain rate was subsequently plotted and is presented in Figure 3.5.

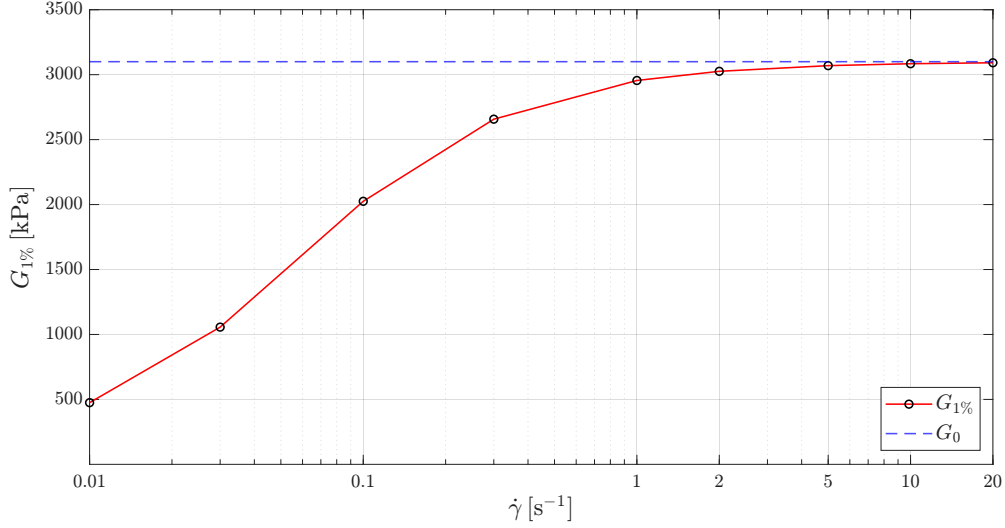


Figure 3.5: Variation of shear moduli G at strain level $\gamma = 1\%$ as $\dot{\gamma} \rightarrow \infty$.

As observed, the shear modulus $G_{1\%}$ increases with increasing strain rate and gradually approaches an asymptotic value. This trend is consistent with the theoretical expectation that the shear modulus evaluated at a small fixed strain converges toward the instantaneous modulus G_0 as $\dot{\gamma} \rightarrow \infty$, indicating that the model response with respect to G_0 appears physically reasonable.

Whereas the instantaneous shear modulus characterizes the initial response of the model, the long-term behavior asymptotically converges toward the equilibrium modulus G_∞ as $t \rightarrow \infty$, corresponding to loading durations $t \gg \tau_i$. At a fixed strain level, this behavior is equivalent to the limit $\dot{\gamma} \rightarrow 0$, implying that, at sufficiently large strains, the slope of the stress–strain curve approaches the equilibrium modulus, $\frac{d\sigma}{d\gamma} \rightarrow G_\infty$. To assess whether the material model reproduces this expected long-term behavior, the shear modulus was extracted at an arbitrary large strain level of 1000%, defined as

$$G_{1000\%} = \left. \frac{d\sigma}{d\gamma} \right|_{\gamma=1000\%} \quad (3.4)$$

This procedure was applied across a range of strain rates, spanning from those employed in the experimental tests to progressively lower strain rates, in order to investigate the evolution of $G_{1000\%}$ as $\dot{\gamma} \rightarrow 0$. The variation of $G_{1000\%}$ with decreasing strain rate is presented in Figure 3.6.

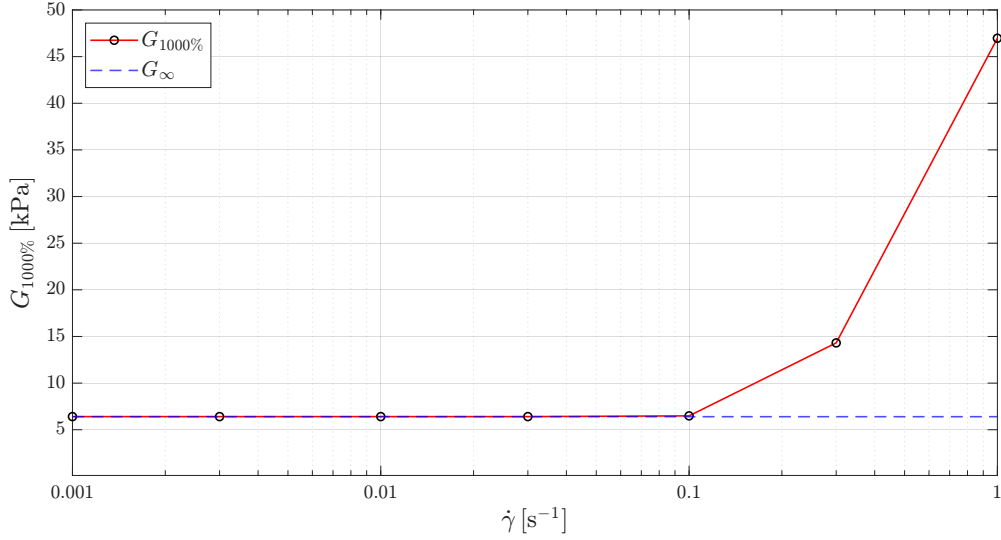


Figure 3.6: Variation of shear moduli G at strain level $\gamma = 1000\%$ as $\dot{\gamma} \rightarrow 0$.

Similarly to the instantaneous shear modulus, $G_{1000\%}$ gradually approaches an asymptotic value, however in this case the asymptote corresponds to the long-term equilibrium modulus $G_{\infty} = 6.41\text{kPa}$, as reported in Table 3.2. This observation indicates that the Maxwell model is also capable of capturing the long-term evolution of the shear modulus.

The verification of the instantaneous and long-term shear moduli indicates that the formulated material model in MATLAB performs as intended and is consistent with the theoretical framework introduced in Section 2.4. This demonstrates that the model is capable of capturing the evolution of the shear moduli across different strain rates and increasing strain, thereby enabling a more reliable prediction of the stress–strain response, which is further investigated in Section 3.4.

3.3 Model calibration

In order to calibrate the input parameters for the Maxwell model, a residual function was developed in parallel with the model formulation, defined as

$$error = \sigma_{measured} - \sigma_{model}. \quad (3.5)$$

The purpose of the residual function was to quantify the error between the initial input parameters and the extracted data from the coaxial shear tests conducted by Cheung and Cebon (1997). This residual formulation was subsequently integrated into the MATLAB optimization routine *lsqnonlin*. This gradient-based solver addresses non-linear least-square problems by initiating the search from an initial parameter estimate x_0 and iteratively refining the parameters to minimize the sum of squared residuals. The gradient is obtained numerically using finite differences, where the Jacobian of the residual function is approximated using central differences. The Jacobian is then used to compute the gradient ∇f of the objective function $f(x) = \|r(x)\|^2$, given by

$$\nabla f(x) = 2J(x)^T r(x), \quad (3.6)$$

which is subsequently employed within a trust-region-reflective algorithm to guide the iterative parameter updates. The optimization procedure continues until convergence is achieved, resulting in a minimized error between the Maxwell model and the experimental data. The optimized Maxwell parameters obtained from the optimization routine are summarized in Table 3.2.

Table 3.2: Optimized Maxwell-parameters in MATLAB.

No.	G_i [kPa]	τ_i [s]	G_∞ [kPa]
0	-	-	6.41
1	2597	0.090	
2	414.6	0.410	
3	82.40	12.7	

G_∞ represents the equilibrium modulus, that is, the long-term modulus, of the single spring in the Maxwell system. The parameters G_i form a vector containing three shear moduli, one associated with each Maxwell branch (see Figure 3.2), while τ_i denotes the corresponding vector of relaxation times for each branch.

Based on these parameters, the viscosity of each Maxwell branch can be computed according to Equation 3.1, and is presented in Table 3.3.

Table 3.3: Viscosity for each Maxwell-branch.

No.	η_i [kPa · s]
1	233.7
2	170.0
3	1047.3

The input data in Abaqus are derived from MATLAB according to theory presented in Section 2.4. The parameters used to define the material in Abaqus are presented in Table 3.4. A high value of Poisson's ratio is commonly suggested for bitumen, as its deformation is primarily governed by shear rather than volumetric strains. However, for numerical stability in Abaqus, Poisson's ratio must be strictly less than 0.5. Consequently, a value slightly below this upper limit was adopted.

Table 3.4: Input parameters used in Abaqus.

Elastic		Viscoelastic		
E [kPa]	ν [-]	No.	g_i [-]	τ_i [s]
9271	0.495	1	0.837	0.090
		2	0.134	0.410
		3	0.027	12.71

3.4 Results

The optimized parameters were subsequently implemented in the Maxwell model, and the system was subjected to a constant loading rate corresponding to the experimental data. The resulting shear stress-strain response indicates that the model with optimized

parameters yields a reasonable representation of the experimental data, as illustrated in Figure 3.7, with the corresponding shear stress at $\gamma = 1$ presented in Table 3.5. The largest deviation are observed at intermediate strain rates due to the interaction of multiple relaxation mechanism. However, the differences remains within approximately 5%, which enables the model to represent the experimental data.

Table 3.5: Maximum shear stress at $\gamma = 1$.

$\dot{\gamma}$ [s^{-1}]	Test data σ_{max} [kPa]	MATLAB σ_{max} [kPa]	Deviation [%]
1	479.9	474.4	-1.14
0.3	189.8	200.0	5.35
0.1	98.8	103.8	5.04
0.03	48.3	47.7	1.30
0.01	20.6	20.9	1.67

In addition, the response corresponds well to the relative values of the optimized moduli and relaxation times reported in Table 3.2. The pronounced early reduction in stress reflects the dominant magnitude of the initial shear modulus in G_i combined with its associated short relaxation time in τ_i . This behavior is also influenced by the prescribed strain levels and constant loading rates, which lead to differences in the corresponding analysis time scales. Consequently, at sufficiently high strain rates, the system is not influenced by all relaxation mechanisms, whereas at sufficiently low strain rates, the relaxation appears to occur instantaneously.

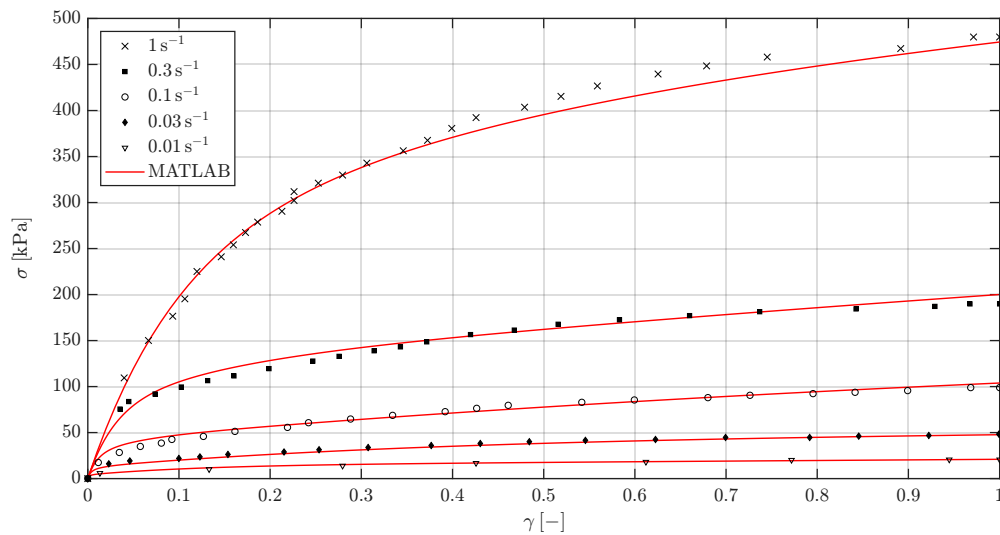


Figure 3.7: Comparison between digitized test data and MATLAB results for $\gamma \leq 1$.

In comparison with the Abaqus simulation, the shear stress–strain response shows a nearly identical trend to that obtained from the material model implemented in MATLAB, as illustrated in Figure 3.8. Since the material parameters used in Abaqus are derived from the MATLAB implementation, an exact agreement between the results is expected. Nevertheless, minor discrepancies are observed, with Abaqus predicting slightly higher stress levels over certain strain intervals.

These differences are primarily attributed to variations in the numerical time integration schemes. In the MATLAB implementation, a constant time step is used, whereas Abaqus employs an adaptive time stepping strategy, refining the increments in regions with higher nonlinearity or rapid changes in the response. This leads to small deviations in the local solution, although the overall agreement between the two approaches remains consistent.

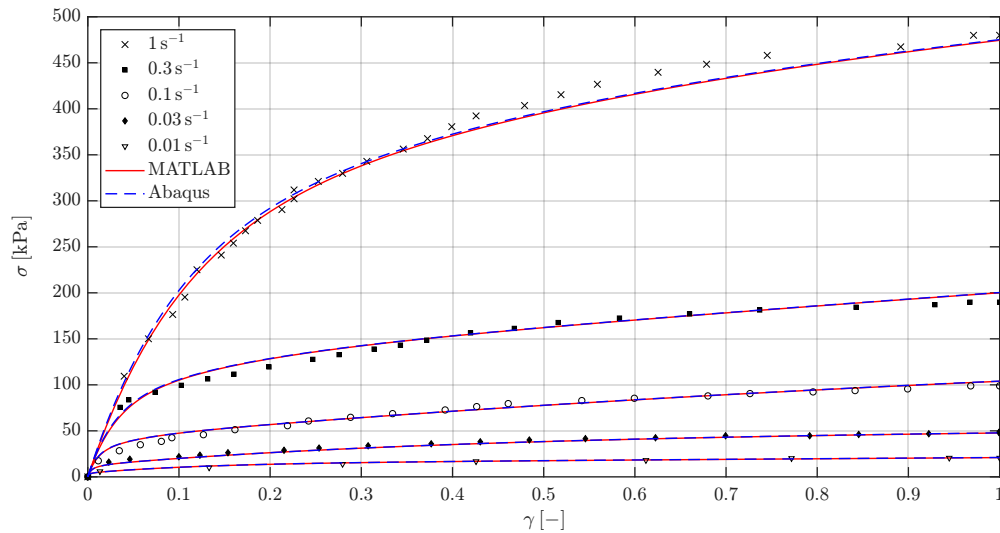


Figure 3.8: Comparison between digitized test data and results from MATLAB and Abaqus for $\gamma \leq 1$.

Due to the dominant viscoelastic contribution in the Prony series ($\sum g_i = 0.998$), see Table 3.4, almost the entire initial shear modulus relaxes within a short time. After the transient viscoelastic response has decayed, only a small equilibrium shear modulus remains. Consequently, the response is predominantly viscoelastic and the shear stress increases slowly with strain after full relaxation. However, it is important to note that the stress does not stagnate but continues to increase.

3.5 Model limitation

Based on the results obtained from the model and the corresponding verification procedures, the implemented approach appears to provide a reasonable representation of the material behavior. However the present formulation is limited to capturing the response up to the peak stress due to the remaining equilibrium shear modulus, which is consistent with both the underlying theoretical assumptions and the design of the model. In order to capture the post-peak behavior observed in Figure 2.1, the model must be extended to account for mechanisms governing stiffness degradation in the post-peak regime.

In addition, the underlying input data for the model are subject to considerable uncertainty. Based on the coaxial shear tests conducted by Cheung and Cebon (1997), the available information regarding specimen type and material properties is limited. Consequently, the model's reliability could only be assessed through comparison with the measured test responses, rather than against independently established material parameters.

Furthermore, the optimization routine is limited to minimizing the discrepancy between the experimental data and the model-predicted response, without explicitly accounting for the actual values of the material parameters G_∞ , G_i and τ_i . Consequently, the resulting parameter set is not necessarily physically representative, but rather tailored to achieve a close fit to the experimental response.

At low strain rates, the model loses the ability to capture the viscoelastic response, as the relaxation effectively occurs instantaneously. Consequently, the material response is dominated by the equilibrium modulus after the transient period. At high strain rates, however, the material does not have sufficient time to relax, and the viscoelastic mechanisms remain active, resulting in a stiffer response.

Overall, the model is capable of reproducing the experimental response within the calibrated range, but its applicability is limited by uncertainties in the input data, lack of physical constraints in the parameter identification, and its inability to capture post-peak behavior. Moreover, temperature-dependent behavior is not captured, as the model is based solely on data obtained at a single temperature. Further refinement is therefore required to extend its predictive capability. To address these limitations, the model is further developed and calibrated using improved material data. The associated methodology and results are presented in the following chapter.

4 Viscoelastic model based on experimental data

Whereas the previous viscoelastic model was calibrated using experimental co-axial shear test data, the model presented in this chapter is calibrated based on test data from a dynamic shear rheometer (DSR). This enables a more data-driven formulation and allows temperature-dependent material behavior to be considered.

To further improve the understanding of the bitumen's material behavior, the constitutive model was updated and subjected to a sinusoidal loading to replicate a DSR test, see Section 2.5. In addition, the cyclic loading enables the extraction of additional viscoelastic parameters in terms of the complex shear modulus and phase angle, which provide a measure of the viscoelastic response of the material. In parallel, simulations were performed in Abaqus to validate the implementation of the constructed material model.

The test data used to calibrate the material model were provided by one of Skanska's testing laboratories. The DSR tests were conducted in accordance with standards issued by the *Swedish Institute for Standards (SIS)*, with the testing methodology described in Section 2.5. The dataset corresponds to a bitumen specimen of penetration grade 50/70, tested over a temperature range from $-30\text{ }^{\circ}\text{C}$ to $80\text{ }^{\circ}\text{C}$ in increments of $10\text{ }^{\circ}\text{C}$. Furthermore, the specimen was tested at fixed shear strain amplitudes over an angular frequency range of $0.1 - 100\text{ s}^{-1}$. For illustrative purposes, data from one temperature is summarized in Table 4.1, with the complete dataset reported in Appendix 8.

Table 4.1: DSR-data for Bitumen 50/70 PP08.

Temperature [$^{\circ}\text{C}$]	Angular Frequency [1/s]	Shear Strain [%]	Shear Stress [Pa]	Complex Shear Modulus [Pa]	Phase Shift Angle [$^{\circ}$]	Storage Modulus [Pa]	Loss Modulus [Pa]
10	0.1	0.5020	25532	5.08E+06	35.39	4.14E+06	2.94E+06
10	0.215	0.3940	26976	6.84E+06	33.79	5.69E+06	3.81E+06
10	0.464	0.3130	28419	9.07E+06	32.45	7.65E+06	4.87E+06
10	1	0.2480	29496	1.19E+07	31.26	1.02E+07	6.18E+06
10	2.15	0.1960	30272	1.55E+07	30.22	1.34E+07	7.78E+06
10	4.64	0.1550	30815	1.99E+07	29.28	1.74E+07	9.74E+06
10	10	0.1220	31121	2.55E+07	28.44	2.24E+07	1.21E+07
10	21.5	0.0966	31221	3.23E+07	27.67	2.86E+07	1.50E+07
10	46.4	0.0763	31133	4.08E+07	26.95	3.64E+07	1.85E+07
10	100	0.0603	30842	5.11E+07	26.23	4.59E+07	2.26E+07

The tabulated quantities include the imposed angular frequency and shear strain, together with the measured shear stress and the derived viscoelastic parameters, namely the complex shear modulus, phase angle, storage modulus, and loss modulus. The experimental data were then utilized to characterize the rheological response of the material for each temperature increment, serving as the foundation for subsequent extensions of the viscoelastic material model.

4.1 Model formulation

A rheological characterization of the investigated bitumen was performed based on the experimentally measured data. The dataset was subsequently analyzed to determine the Prony parameters and the time–temperature superposition (TTS) parameters for the generalized Maxwell model, see Figure 2.4.

As oscillatory loading was implemented, the applied shear strain cannot be described as $\dot{\gamma} \cdot t$. Instead, to replicate the loading conditions of a DSR test, the shear strain at a given radial position r_q is expressed as

$$\gamma_q(t) = \gamma_p \frac{r_q}{R} \sin(\omega t) \quad (4.1)$$

Here, γ_p denotes the prescribed strain amplitude on the surface where $r_q = R$, and ω is the constant angular frequency defining the cyclic loading with time t .

Since the shear stress and shear strain vary along the specimen radius, Gaussian quadrature was introduced to numerically integrate the local stress contributions over the radius, with the torque given by:

$$M = \int_0^R 2\pi r^2 \sigma dr \quad (4.2)$$

This enables comparison between the measured and simulated torque response, and evaluation of the radial shear stress distribution. Although the strain varies linearly with the radius, the torque depends on a weighted radial integral of the stress, which motivates the use of numerical integration. The numerical integration approximates the integral by a weighted sum of function values within the integration domain according to Equation 4.3 (Davis & Rabinowitz, 2007).

$$\int_a^b f(x) dx \approx \sum_{i=1}^n w_i f(x_i). \quad (4.3)$$

Subsequently, Gauss quadrature was implemented with $N = 10$ points to approximate the integral, with the integrand defined as

$$f(x) = \sigma(r)r^2 \quad (4.4)$$

The strain for each Gaussian point was then calculated according to Equation 4.1. Consequently, the *Backward Euler method* was applied to update the viscoelastic strain ($\gamma_{v,i,q}$) in each Maxwell branch. The resulting stress in each Gaussian point was then calculated as:

$$\sigma(r_q) = G_0 \gamma_q + \sum_{i=1}^N G_i (\gamma_q - \gamma_{v,i,q}) \quad (4.5)$$

Finally, the resulting torque from the model was obtained by adding the contributions from all Gaussian points:

$$M_{\text{model}} = 2\pi \sum_{q=1}^{N_q} \sigma(r_q) r_q^2 w_q \quad (4.6)$$

The analytical solution, representing the ideal elastic response, can be calculated according to Equation 4.7. Where σ_p is the shear stress calculated from the measured torsional moment within the DSR machine.

$$M_{\text{analytic}} = \sigma_p \cdot \sin(\omega t) \cdot \pi \cdot \frac{R^3}{2} \quad (4.7)$$

By extracting the final cycle of the torque response and assuming a harmonic response, the viscoelastic response is characterized. The phase angle is evaluated as:

$$\delta = \arctan \left(\frac{\sum_k M_{\text{model}}(t_k) \cos(\omega t_k)}{\sum_k M_{\text{model}}(t_k) \sin(\omega t_k)} \right) \quad (4.8)$$

Where t_k are the discrete time points within the final cycle, and k is the corresponding index. This approach enables the phase angle to be determined directly from the simulated torque response, ensuring consistency with the DSR measurements. Visually, the phase angle corresponds to the horizontal phase lag between the model-predicted torque response relative to the ideal elastic response, as illustrated in Figure 4.1.

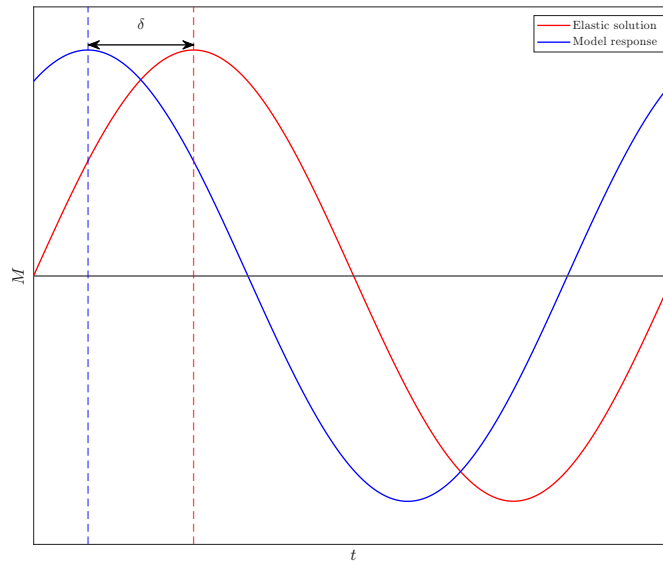


Figure 4.1: Phase angle δ illustrating the delayed stress response due to viscoelastic behavior.

4.1.1 Time-temperature superposition

In order to approximate the response beyond the frequency ranges accessible in laboratory testing, time-temperature superposition (TTS) was implemented according to Section 2.5. While the lower frequency in the DSR experiments was limited to $\omega = 0.1/\text{s}$, the strain rate associated with the considered field case (Equation 1.1) was several orders of magnitude lower. Consequently, TTS was employed to translate the experimentally

measured rheological response from the DSR data to equivalent angular frequencies corresponding to the realistic loading conditions considered in this study. However, in order to apply time-temperature superposition, the material has to be thermorheologically simple (TRS), meaning that a variation in temperature should not modify the form of the constitutive behavior of the material, but only shift the response along the time or frequency scale. In practice, this means that the rheological properties (such as phase angle δ , complex shear modulus G^* , storage modulus G' , and loss modulus G'') of the material exhibit similar behavior for different combinations of temperature and frequency. The verification of this requirement is further described in Subsection 4.2. Based on the previously introduced shift factor α_T (Equation 2.22) and the reduced angular frequency ω_r (Equation 2.30), it was possible to predict rheological properties between two equivalent frequency-temperature pairs. Subsequently, properties such as δ , G^* , G' and G'' were predicted in accordance to the relationship presented in Equation 2.27 and 2.28, such that

$$\begin{aligned}\delta(\omega, T) &\Leftrightarrow \delta(\omega_r, T_{ref}) \\ G^*(\omega, T) &\Leftrightarrow G^*(\omega_r, T_{ref}) \\ G'(\omega, T) &\Leftrightarrow G'(\omega_r, T_{ref}) \\ G''(\omega, T) &\Leftrightarrow G''(\omega_r, T_{ref})\end{aligned}\tag{4.9}$$

For the realistic loading condition ($\dot{\gamma} = 8 \cdot 10^{-8} \text{ s}^{-1}$ and $\gamma = 500$) a reference temperature of $T_{ref} = 0^\circ\text{C}$ was adopted and the angular frequency of interest was derived from the realistic loading condition according to

$$\omega = \frac{\dot{\gamma}}{\gamma} \quad \Rightarrow \quad \omega = 1.6 \cdot 10^{-10} \text{ s}^{-1}.\tag{4.10}$$

Moreover, T_{ref} and the realistic angular frequency were applied in Equation 4.16, in combination with an arbitrarily selected angular frequency $\omega = 0.1 \text{ s}^{-1}$ within the range of the DSR data, such that

$$\begin{aligned}\delta(0.1, T) &\Leftrightarrow \delta(1.6 \cdot 10^{-10}, 0) \\ G^*(0.1, T) &\Leftrightarrow G^*(1.6 \cdot 10^{-10}, 0) \\ G'(0.1, T) &\Leftrightarrow G'(1.6 \cdot 10^{-10}, 0) \\ G''(0.1, T) &\Leftrightarrow G''(1.6 \cdot 10^{-10}, 0)\end{aligned}\tag{4.11}$$

where T denoted the unknown temperature at which the rheological properties were equivalent to the frequency-temperature pair $(1.6 \cdot 10^{-10} \text{ s}^{-1}, 0^\circ\text{C})$. The unknown temperature T was subsequently solved from Equation 2.22 with α_T defined by 2.30, and C_1 and C_2 calibrated according to Section 4.3 for $T_{ref} = 0^\circ\text{C}$, such that

$$T = T_{ref} - \frac{\log_{10}(\alpha_T)C_2}{\log_{10}(\alpha_T) + C_1}.\tag{4.12}$$

With T solved, rheological properties were extracted from the obtained DSR data for the case $(0.1, T)$, which, by definition, are equivalent to those corresponding to the condition $(1.6 \cdot 10^{-10}, 0)$. From the determined material properties, Prony series were

extracted according to the methodology presented in Section 4.3, and further implemented into the extended material model introduced in Section 4.1. Subsequently, the stress-strain response was extracted and presented in Section 4.4.

4.1.2 Abaqus

Simultaneously as the MATLAB model was constructed, a two-dimensional axisymmetric solid model with the twist formulation was implemented in Abaqus. Subsequently, meshed with elements CGAX4 (4-node generalized bilinear axisymmetric quadrilateral, twist). This allows a three-dimensional response to be captured using a two-dimensional representation by permitting rotation about the axis of symmetry. The dimensions of the part are defined in accordance with industry standards, with a radius $R = 0.004\text{m}$ and a height $h = 0.001\text{m}$. Since DSR test theoretically only allows rotation of the specimen, all boundaries in the model are horizontally and vertically fixed, visualized in Figure 4.2. The bottom boundary represents the contact surface of the bitumen specimen and the underlying surface and is constrained against rotation. To introduce stresses in the model, a prescribed rotation about the axis of symmetry was applied, calculated as:

$$\theta(\gamma) = \gamma_p \frac{h}{R} \quad (4.13)$$

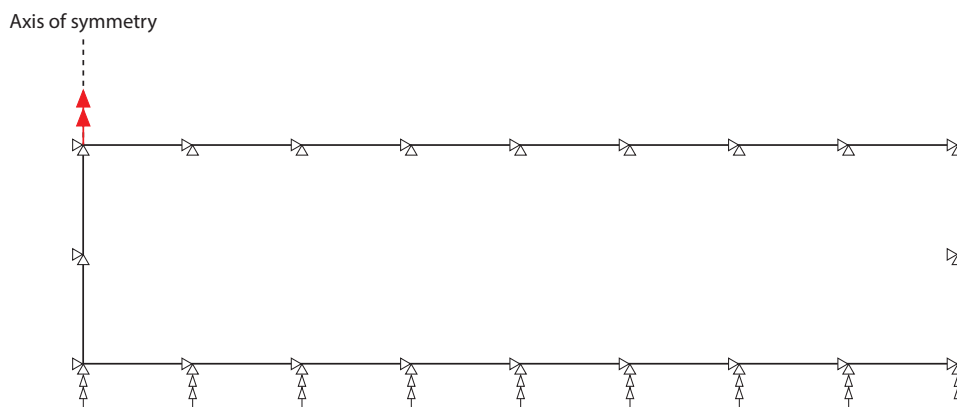


Figure 4.2: Boundary conditions for the axisymmetric model applied in Abaqus.

4.2 Verification of model implementation

This section aims to verify the results obtained in MATLAB where no TTS is applied. The Prony parameters are verified with storage and loss moduli, experiencing deviating behavior. The shear stress-strain response as well as the oscillatory predictions is compared with the provided DSR data. The implementation and assumptions of the model is verified with the results of Abaqus.

To verify the Prony parameters prior to their implementation in the Maxwell model, the storage and loss moduli were recomputed using the identified parameter set and subsequently compared with the experimentally measured data. The resulting comparison, serving as a qualitative validation of the fitted parameters, is presented in Figure 4.3 for $T = 10^\circ\text{C}$.

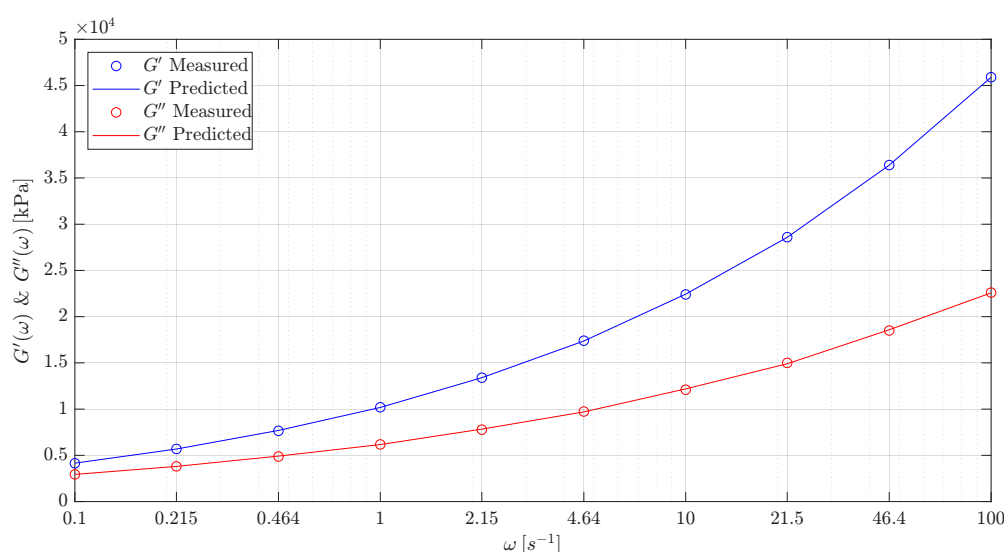
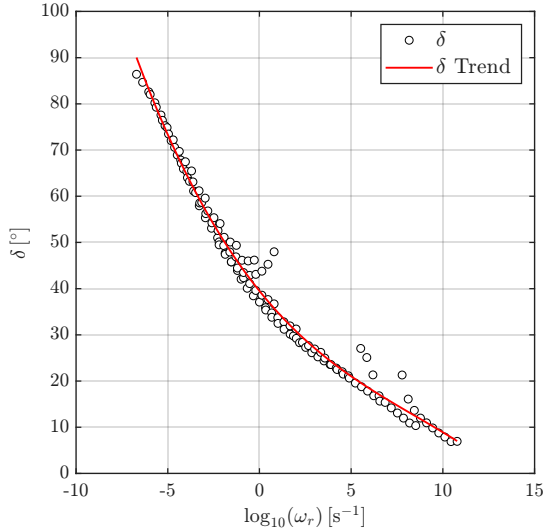


Figure 4.3: Prony fit of the Storage and Loss moduli at $T = 10^\circ\text{C}$, obtained from MATLAB.

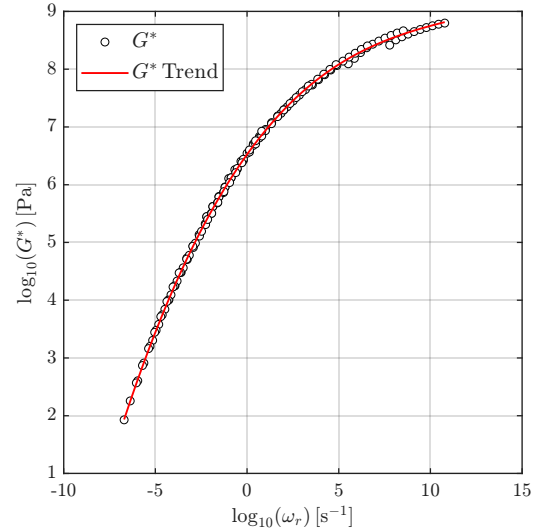
As can be observed, the model-predicted moduli show good agreement with the experimentally measured storage and loss moduli at $T = 10^\circ\text{C}$, with similar behavior observed across the remaining temperature range. This indicates a properly implemented model formulation for the extraction of the Prony parameters.

4.2.2 TTS parameters

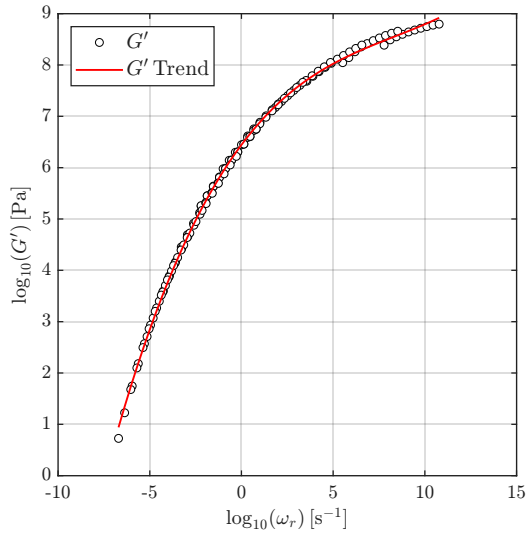
As previously described in Subsection 2.5, the objective of the time–temperature superposition (TTS) implementation was to enable the determination of rheological properties at strain rates lower than those accessible through the experimental measurements. One way to assess the assumption of TRS and verify the implementation of the shift factor a_T is to examine whether rheological properties at different temperatures can be shifted along the frequency axis to form a single master curve. This verification procedure was applied to the phase angle δ and the complex shear modulus G^* at a reference temperature of $T_{ref} = 20^\circ\text{C}$, using a set of calibrated material parameters. The procedure used to obtain these parameters is described in Section 4.3. The resulting master curves are presented in Figure 4.4.



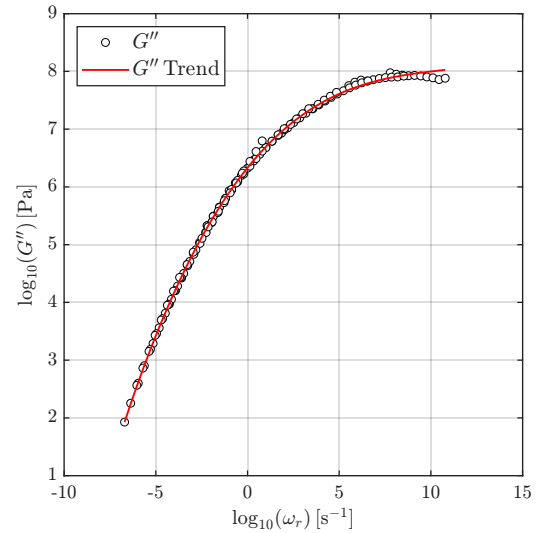
(a) Phase angle δ at $T_{ref} = 20^\circ\text{C}$



(b) Complex shear modulus G^* at $T_{ref} = 20^\circ\text{C}$



(c) Storage modulus G' at $T_{ref} = 20^\circ\text{C}$



(d) Loss modulus G'' at $T_{ref} = 20^\circ\text{C}$

Figure 4.4: Constructed master curves at $T_{ref} = 20^\circ\text{C}$, obtained from MATLAB.

As observed in Figure 4.4, both δ , G^* , G' and G'' can, with a reasonably high degree of confidence, be shifted along the frequency axis to form individual master curves, indicating thermorheologically simple behavior and a correctly implemented TTS formulation.

A further verification of this behavior was carried out by using the constructed master curves to estimate the rheological properties at a different temperature–frequency combination. Specifically, the material response at $T = 10^\circ\text{C}$ and $\omega = 4.64\text{s}^{-1}$ was approximated by applying the corresponding shift factor to the reference curve, in order to assess whether the predicted values of δ , G^* , G' and G'' are consistent with independent results at this temperature and frequency. The shift factor α_T was calculated (according to Equation 2.22) for $T_{ref} = 20^\circ\text{C}$ and $T = 10^\circ\text{C}$, with the calibrated parameters C_1

and C_2 reported in Table 4.6, yielding $\alpha_T \approx 22.0$. Next $\omega_r \approx 100s^{-1}$ was calculated from Equation 2.30 before the parameters δ , G^* , G' and G'' were predicted for the case ($T = 10^\circ\text{C}$, $\omega = 4.64 s^{-1}$) by means of the master curves presented in Figure 4.4, yielding the results in Table 4.2.

Table 4.2: Measured and predicted material properties of the analyzed bitumen at $T = 10^\circ\text{C}$, $\omega = 4.64 s^{-1}$.

	Measured	Predicted	Deviation
$\delta [^\circ]$	29.30	31.00	5.80%
$G^* [\text{MPa}]$	19.90	19.95	0.25%
$G' [\text{MPa}]$	17.40	17.34	-0.34%
$G'' [\text{MPa}]$	9.74	10.00	2.66%

As can be observed in Table 4.2 all the predicted properties show reasonable agreement with the corresponding DSR data. The phase angle and the loss modulus have a deviation of slightly below 6% and 3% respectively, while the deviation in the complex shear modulus and the storage modulus are insignificantly small, indicating that the implementation of the time-temperature superposition was applied correctly.

4.2.3 Oscillatory response

To verify the implemented model, Abaqus simulations were performed according to the procedure described in Section 4.1 using the material parameters presented in 4.4, derived from MATLAB. The results correspond to a temperature $T = 10^\circ\text{C}$, an angular frequency of $\omega = 100s^{-1}$, and a prescribed rotation of $\theta = 15.075 \cdot 10^{-4}$.

The Abaqus simulations produced a linear radial stress distribution, as shown in Figure 4.5, in agreement with the theoretical predictions. This indicates that the model formulation and its underlying assumptions have been correctly implemented.

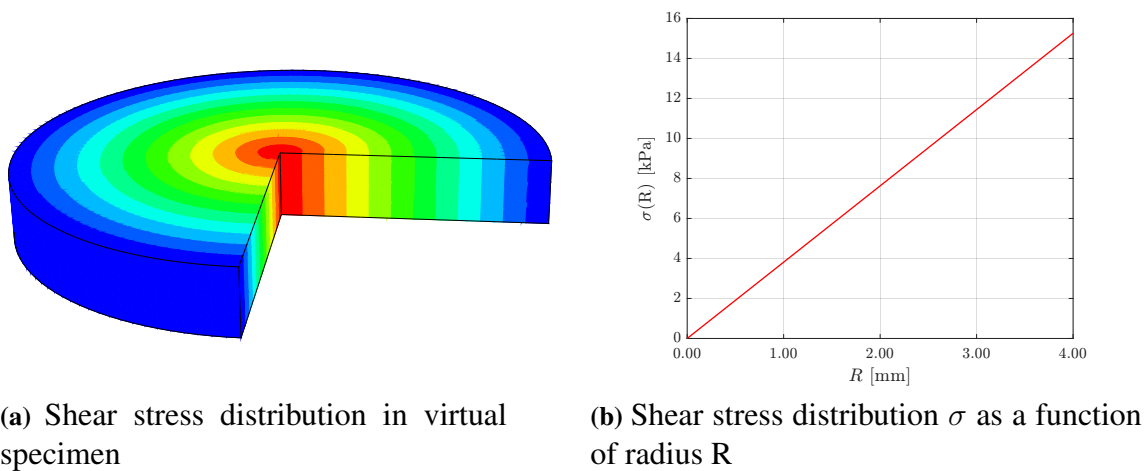


Figure 4.5: Linear increase of stress (σ) with radius (R) from Abaqus.

The resulting moment response exhibited a periodic steady-state behavior with a distinct horizontal phase lag relative to the elastic solution, see Figure 4.6. The presence of a phase shift is characteristic of viscoelastic materials and demonstrates that the model captures the expected time-dependent behavior. Furthermore, the absence of waveform distortion suggests that the response remains within the linear viscoelastic regime. The close agreement between the Abaqus and MATLAB results therefore provides strong support for the validity of the implementation

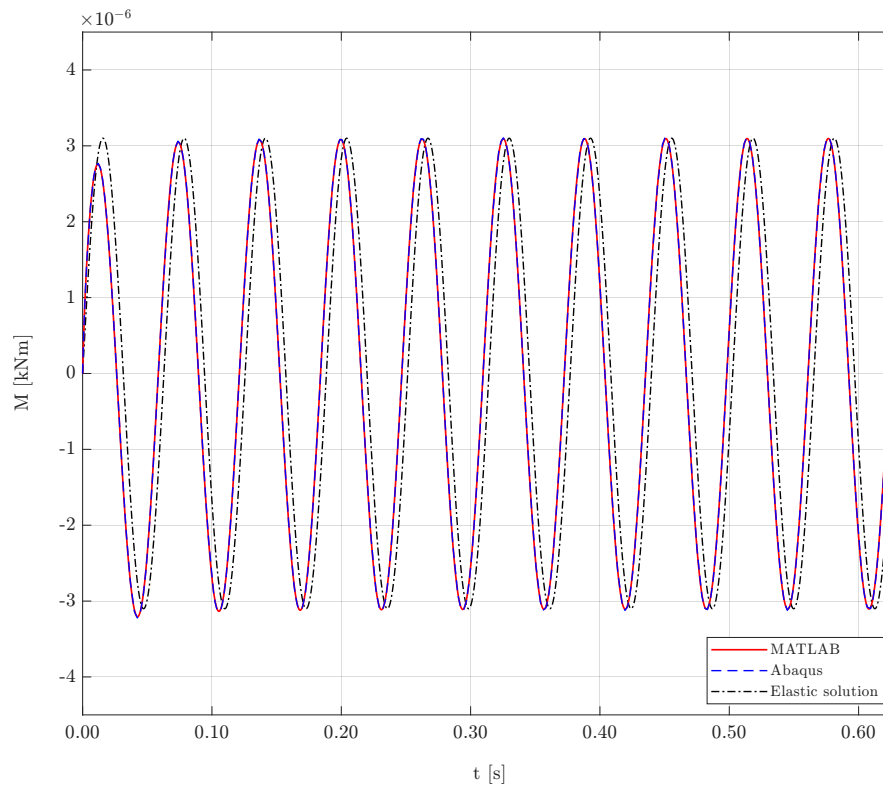


Figure 4.6: Comparison of the elastic solution and the moment response (M) obtained from MATLAB and Abaqus, as a function of time (t).

The shear stress-strain response obtained from Abaqus is presented in Figure 4.7a. Comparison with the corresponding MATLAB results reveals minor differences, which are quantified in Table 4.3 and illustrated in Figure 4.7b. The same reasoning as presented in Section 3.4 explains the discrepancies observed in these simulations. The Abaqus results generally predict slightly higher stress levels, with larger deviations at the local maxima, due to the numerical time integration schemes. Nevertheless, the small deviations indicate that the constitutive model has been implemented consistently and that the viscoelastic response is captured reliably.

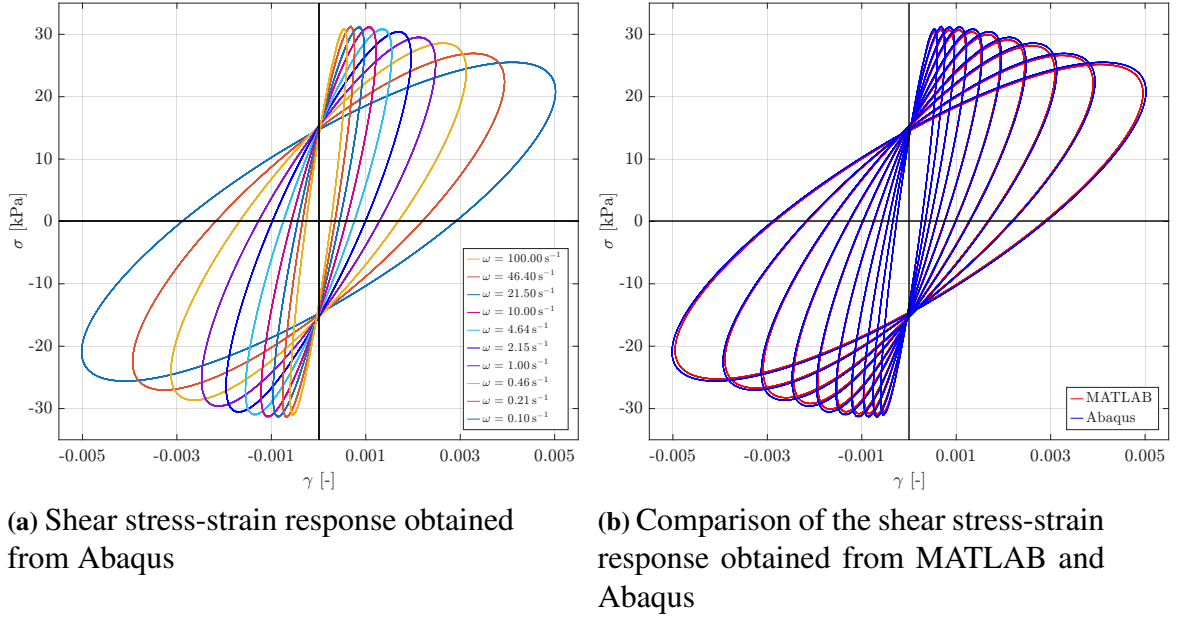


Figure 4.7: Shear stress-strain response at $T = 10^\circ\text{C}$, with comparison between Abaqus and MATLAB.

Table 4.3: Comparison between maximum shear stress (σ_{max}) obtained from MATLAB and Abaqus at $T = 10^\circ\text{C}$.

ω [1/s]	Predicted (MATLAB) σ_{max} [kPa]	Predicted (Abaqus) σ_{max} [kPa]	Deviation [%]
0.1	25.2	25.5	1.19
0.215	26.6	27.0	1.50
0.464	28.2	28.6	1.42
1	29.1	29.6	1.72
2.15	30.0	30.5	1.67
4.64	30.4	30.9	1.64
10	30.7	31.2	1.63
21.5	30.7	31.2	1.63
46.4	30.7	31.3	1.95
100	30.4	30.9	1.64

4.3 Model Calibration

In order to further develop the initial viscoelastic model in both MATLAB and Abaqus, the experimental data were reformulated into a set of input parameters compatible with the respective numerical implementations. Consequently, the data were translated into the parameter set of the Generalized Maxwell model, expressed through a Prony-series representation consisting of the equilibrium modulus G_∞ , branch-specific shear moduli G_i , and the corresponding relaxation times τ_i . A Prony series comprising $i = 1, \dots, 10$ Maxwell branches was adopted, resulting in ten pairs of shear moduli and relaxation times to be determined through parameter identification. Following the formulation presented in Section 2.5, the Prony-series representation was applied in the frequency domain, allowing the storage and loss moduli to be expressed in terms of the General-

ized Maxwell parameters. The Prony coefficients were subsequently identified by fitting the experimentally measured storage and loss moduli to the corresponding theoretical expressions given by Equation 2.18 and 2.19 respectively. For this purpose, a residual function was formulated, and analogous to the optimization procedure employed in the previous chapter, the nonlinear least-square solver *lsqnonlin* was used to minimize the error between the experimentally measured and model-predicted moduli. The residual vector minimized by the solver was defined as

$$error = [G'_{measured} - G'_{predicted}, G''_{measured} - G''_{predicted}]. \quad (4.14)$$

After convergence was achieved, the experimentally measured and model-predicted storage and loss moduli were evaluated as functions of angular frequency for each temperature increment (Figure 4.3). This evaluation served as a qualitative assessment of the model's predictive capability.

As the model formulation was verified at $T = 10^\circ\text{C}$ (Subsection 4.2), the calibrated Prony parameters for this temperature are reported in Table 4.4. By applying the conversion described by Equation 3.2, these parameters were transformed into the corresponding Abaqus input parameters, which are presented in Table 4.5.

Table 4.4: Optimized Prony parameters for $T = 10^\circ\text{C}$ in MATLAB.

No.	G_i [kPa]	τ_i [s]	G_∞ [kPa]
-	-	-	7.77E+02
1	5.43E+04	0.5020	
2	2.52E+04	0.3940	
3	1.65E+04	0.3130	
4	1.08E+04	0.2480	
5	0.69E+04	0.1960	
6	0.43E+04	0.1550	
7	0.27E+04	0.1220	
8	0.08E+04	0.0966	
9	0.10E+04	0.0763	
10	0.23E+01	0.0603	

Table 4.5: Input parameters used in Abaqus for $T = 10^\circ\text{C}$.

No.	g_i [-]	τ_i [s]	E [kPa]	ν [-]
-	-	-	3.69E+5	0.495
1	4.40E-1	0.00100		
2	2.04E-1	0.00464		
3	1.34E-1	0.0215		
4	8.78E-2	0.100		
5	5.60E-2	0.464		
6	3.47E-2	2.15		
7	2.20E-2	10.0		
8	6.78E-3	46.4		
9	8.24E-3	215		
10	1.85E-5	1000		

To subsequently calibrate the TTS parameters, the measured complex shear modulus G^* was fitted to the corresponding model-predicted modulus using the sigmoidal master curve formulation defined in Equation 2.29. The error between the measured and predicted complex modulus was subsequently minimized using the solver *lsqnonlin*, with the corresponding residual vector defined as

$$error = \log_{10}(G^*_{measured}) - \log_{10}(G^*_{predicted}). \quad (4.15)$$

The parameters C_1 , C_2 , α , β , γ , and δ_s were subsequently identified through the fitting procedure, and the reduced frequency $\log_{10} \omega_r$ was computed in accordance with Equation 2.22 and 2.30. The resulting TTS parameters corresponding to $T_{ref} = 20^\circ\text{C}$ are reported in Table 4.6.

Table 4.6: Optimized TTS-parameters for $T_{ref} = 20$ in MATLAB.

$C_1[-]$	$C_2 [^\circ\text{C}]$	$\alpha [-]$	$\beta [-]$	$\gamma [-]$	$\delta_s [-]$
22.80	179.86	22.57	-2.00	-0.19	-13.36

4.4 Results

The resulting shear stress-strain response is extracted for $T = 10^\circ\text{C}$ and $T = 30^\circ\text{C}$, as illustrated in Figures 4.8 and 4.9. The prescribed strain γ_p from the DSR data was applied at different temperatures with corresponding strain rates. As relevant data can only be obtained under steady-state conditions, the transient response was excluded. The resulting steady-state response forms a hysteresis loop for each loading condition, representing the phase lag between stress and strain and reflecting the energy dissipation and the viscoelastic nature of the material.

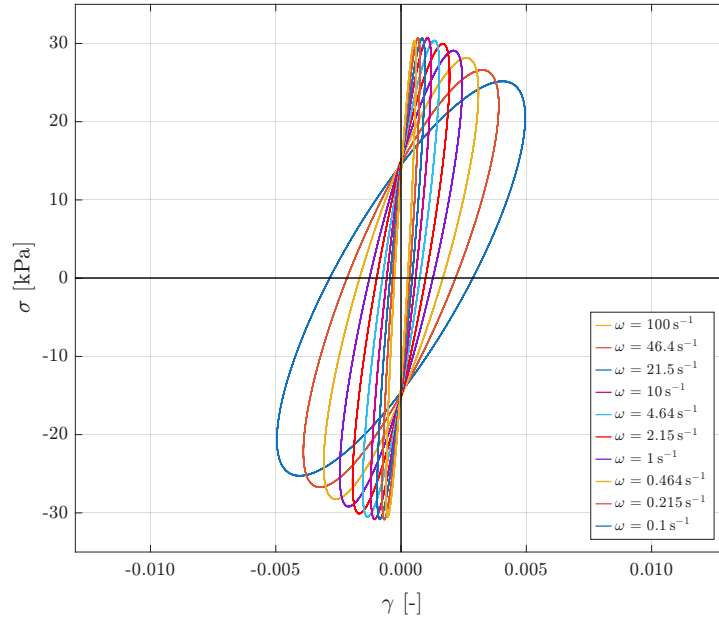


Figure 4.8: Hysteresis loop obtained from material model in MATLAB at $T=10^\circ\text{C}$ for different angular frequencies ω .

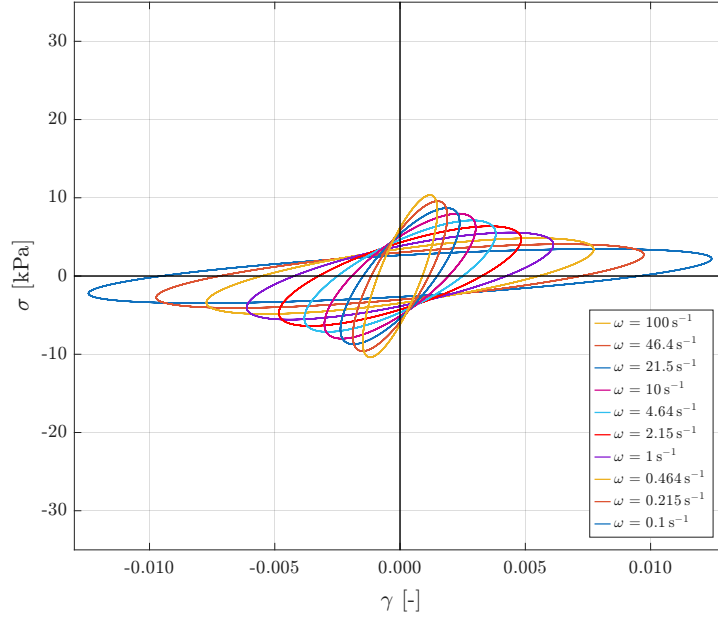


Figure 4.9: Hysteresis loop obtained from material model in MATLAB at $T=30^{\circ}\text{C}$ for different angular frequencies ω .

The model response was in good agreement with the provided DSR data. The model response at 10°C and frequency 2.15s^{-1} is highlighted in Figure 4.10, where the marker represents the maximum measured shear stress and the prescribed maximum shear strain applied by the DSR. The model shows consistent agreement across all temperatures and frequencies, with similar error levels observed. The deviation at $T = 10^{\circ}\text{C}$ for the shear stress (σ_{max}), and the phase angle (δ) are presented in Tables 4.7 and 4.8 respectively.

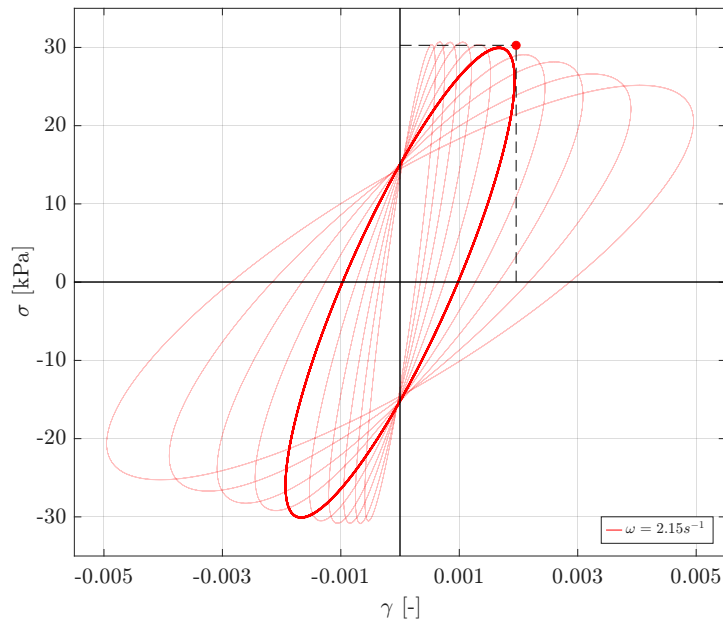


Figure 4.10: Hysteresis loop obtained from material model in MATLAB at $T=10^{\circ}\text{C}$, highlighting the response at $\omega = 2.15\text{s}^{-1}$ with the corresponding measured maximum shear stress and shear strain from DSR.

Table 4.7: Measured and predicted maximum shear stress (σ_{max}) at $T=10^{\circ}C$.

ω [s ⁻¹]	Measured (DSR) σ_{max} [kPa]	Predicted (MATLAB) σ_{max} [kPa]	Deviation [%]
0.1	25.5	25.2	-1.18
0.215	27.0	26.6	-1.48
0.464	28.4	28.2	-0.70
1	29.5	29.1	-1.36
2.15	30.3	30.0	-0.99
4.64	30.8	30.4	-1.30
10	31.1	30.7	-1.29
21.5	31.2	30.7	-1.60
46.4	31.1	30.7	-1.29
100	30.8	30.4	-1.30

Table 4.8: Measured and predicted phase angle (δ) at $T=10^{\circ}C$.

ω [s ⁻¹]	Measured (DSR) δ [°]	Predicted (MATLAB) δ [°]	Deviation [%]
0.1	35.39	35.25	-0.40
0.215	33.79	33.79	0.00
0.464	32.45	32.58	0.40
1	31.26	31.20	-0.19
2.15	30.22	30.32	0.33
4.64	29.28	29.18	-0.34
10	28.44	28.50	0.21
21.5	27.67	27.54	-0.47
46.4	26.95	27.04	0.33
100	26.23	26.19	-0.15

The phase angle is directly reflected in the shape of the hysteresis loop. Increasingly viscous behavior results in a more circular response with a phase angle approaching 90° , while increasingly elastic behavior results in a more linear response with a phase angle approaching 0° . It is important to note that the phase angle is not influenced by the magnitude of the applied strain, provided the material remains within the linear viscoelastic range. To facilitate a clearer interpretation of the phase angle, the curves in Figures 4.11 and 4.12 were normalized, highlighting the decrease in phase angle with increasing frequency, and vice versa.

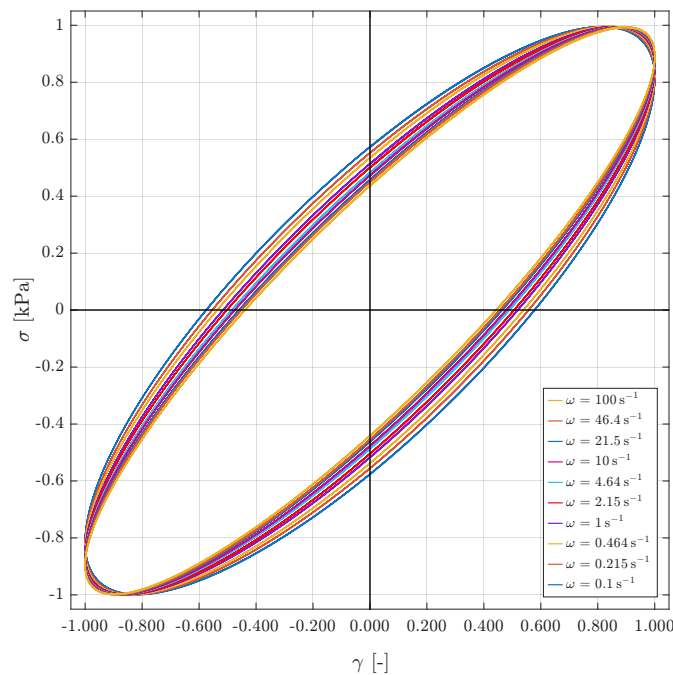


Figure 4.11: Normalized hysteresis loop obtained from material model in MATLAB at $T=10^{\circ}C$ for different angular frequencies ω .

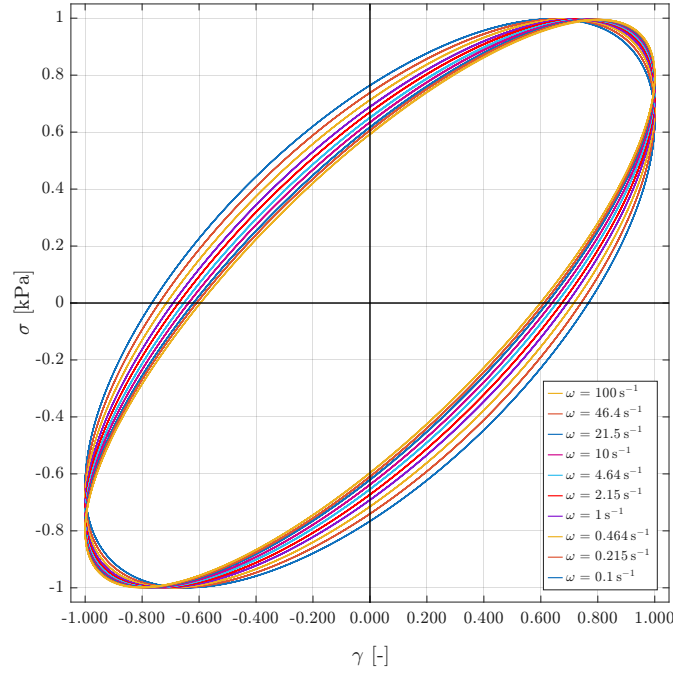


Figure 4.12: Normalized hysteresis loop obtained from material model in MATLAB at $T=30^{\circ}\text{C}$ for different angular frequencies ω .

An increase in temperature from $T=10^{\circ}\text{C}$ to $T=30^{\circ}\text{C}$ results in more pronounced viscoelastic material behavior, visualized by a more circular hysteresis loop, corresponding to an increased phase angle. Accordingly, the viscoelastic response is influenced by both temperature and frequency, as illustrated in Figure 4.13. This reflects the reduced resistance to deformation at higher temperatures and demonstrated that the model captures the temperature-dependent viscoelastic response.

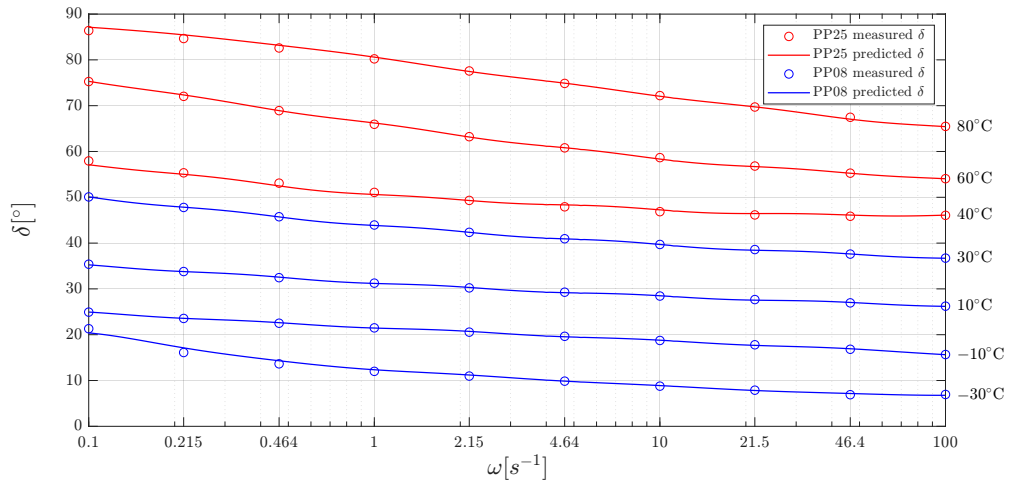


Figure 4.13: Comparison between the measured phase angle (δ) and the model prediction obtained from MATLAB as a function of angular frequency (ω) at different temperatures.

Consequently, the model is valid within the calibrated frequency-temperature range. However, as the frequency decreases beyond the testing interval, the phase angle de-

viates and approaches zero, as shown in Figure 4.14. This behavior is caused by the limited range of relaxation times included in the model, since the Prony parameters are calibrated for specific temperatures. At low frequencies, the viscoelastic branches fully relax, leaving the response dominated by the remaining elastic contribution and tending toward ideal elastic behavior. This highlights a limitation of the model, as it is unable to accurately represent the viscoelastic behavior outside the calibrated range.

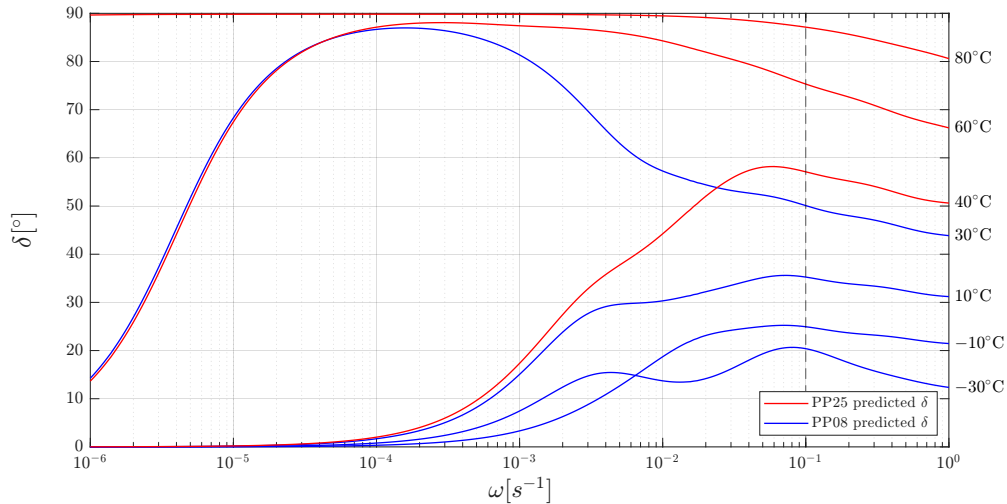


Figure 4.14: The phase angle (δ) as a function of frequency (ω) at different temperatures obtained from MATLAB beyond the measured DSR-range.

Nevertheless, with the implementation of the time-temperature superposition (formulated in Section 4.1), it was possible to predict the material behavior outside of the obtained frequency range. By solving Equation 4.12 for the unknown temperature T , using the calibrate parameters reported in Table 4.6, a temperature of $T = 83.4^\circ\text{C}$ was obtained. Consequently, the rheological properties were equivalent between the following two frequency-temperature pairs:

$$\begin{aligned} \delta(0.1, 83.4) &\Leftrightarrow \delta(1.6 \cdot 10^{-10}, 0) \\ G^*(0.1, 83.4) &\Leftrightarrow G^*(1.6 \cdot 10^{-10}, 0) \\ G'(0.1, 83.4) &\Leftrightarrow G'(1.6 \cdot 10^{-10}, 0) \\ G''(0.1, 83.4) &\Leftrightarrow G''(1.6 \cdot 10^{-10}, 0). \end{aligned} \quad (4.16)$$

However, since the temperature range for the obtained DSR data were limited to $-30 \leq T \leq 80$, rheological properties were extracted for the case $\omega = 0.1 \text{ s}^{-1}$ and $T = 80^\circ\text{C}$, resulting in the following material parameters under the realistic loading condition, summarized in Table 4.9.

Table 4.9: Resulting rheological properties for the realistic loading condition at $T_{ref} = 0^\circ\text{C}$.

δ [°]	G^* [MPa]	G' [MPa]	G'' [MPa]
86.40	84.8E-06	5.3E-06	84.6E-06

Subsequently, Prony series were calibrated in accordance with Section 4.3 and were implemented into the extended material model formulated in Section 4.1. The resulting stress-strain response for $\omega = 0.1 \text{ s}^{-1}$ at $T = 80^\circ\text{C}$ (equivalent to $\omega = 1.6 \cdot 10^{-10} \text{ s}^{-1}$ at $T = 0^\circ\text{C}$), is presented in Figure 4.15.

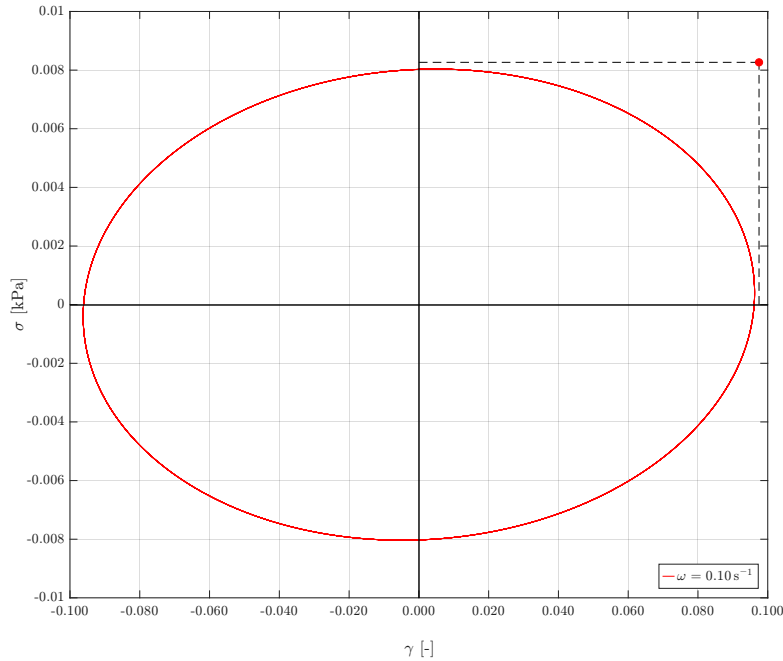


Figure 4.15: Shear stress-strain response obtained with TTS implementation in MATLAB, for an equivalent frequency-temperature pair $(\omega = 0.1 \text{ s}^{-1}, T = 80^\circ\text{C}) \Leftrightarrow (\omega = 1.6 \cdot 10^{-10} \text{ s}^{-1}, T = 0^\circ\text{C})$.

As observed in Figure 4.15, the shape of the stress–strain response is approximately circular, indicating that $\delta \rightarrow 90^\circ$, which is consistent with the resulting phase angle extracted from the model ($\delta \approx 87^\circ$). Moreover, the maximum stress reaches approximately $\sigma_{max} \approx 8.0 \text{ Pa}$ at a strain level of $\gamma = 0.0975$, which is in good agreement with the stress–strain response measured in the DSR tests ($\sigma_{max} = 8.2 \text{ Pa}$).

4.5 Model limitation

As the Prony parameters are calibrated separately for each temperature, the model response is in good agreement with the provided DSR data, yielding reliable and physically meaningful parameters. However, the model are restricted to the measured temperatures. Furthermore, when the frequency range is extended beyond the calibrated domain, the model exhibits deviating viscoelastic behavior. This effect is particularly noticeable at lower frequencies and can be explained by the relationship between the relaxation times and the duration of the analysis, resulting in an ideal elastic response once the Maxwell branches have fully relaxed. In theory, introducing additional Maxwell branches with longer relaxation times could improve the model response. However, this approach may lead to overfitting and reduce the physical reliability of the model.

With the implementation of TTS, the relaxation spectrum can be shifted to extend the applicability of the model. Specifically, selecting a higher reference temperature allows

the relaxation spectrum to shift toward significantly lower frequencies, thereby enabling representation over longer timescales. Consequently, material properties for frequencies beyond the available test data can be estimated.

Nevertheless, DSR measurements are conducted within the LVE regime, meaning that the model is inherently restricted to the same range of material behavior. At higher strain levels, the internal structure of the bitumen undergoes changes, resulting in non-linear and strain-dependent behavior. Consequently, the present model is only valid within the LVE regime and cannot capture the material response beyond the peak shear stress as the model does not include any mechanism to further reduce the stress after complete relaxation of the Maxwell element. Thus, while the model provides a consistent description of the viscoelastic response within the LVE regime, its predictive capability for larger deformations remains limited.

To overcome this limitation, an additional dashpot can be connected in series with the generalized Maxwell model. This modification enables the model to capture viscous behavior even after the Maxwell branches have fully relaxed, thereby preventing convergence to an ideal elastic response. This approach is implemented and discussed in Chapter 5.

5 Final viscoelastic model

As previously concluded in Section 3.5, the initial viscoelastic model was unable to capture the post-peak behavior observed in the coaxial shear test shown in Figure 2.1. For shear strains in the target case, with a displacement of 500 mm over a period of 100 years, see Section 1.4, the post-peak response is crucial for predicting large deformations under slow loading conditions. To capture this behavior, mechanisms governing stiffness degradation must therefore be incorporated into the model. At the same time, it was concluded from the results of the DSR implementation (Section 4.5), that the current model exhibits predominantly elastic behavior as $\dot{\gamma}$ approaches zero. This contradicts observations reported in previous studies, such as those by Fellenius (1970), which indicate that low loading rates are associated with more pronounced viscous behavior.

5.1 Model formulation

To account for both the post-peak response and the viscous dominance at slow loading rates, the model was subsequently extended by introducing a softening-controlled dashpot in series with the Maxwell element, illustrated in Figure 5.1.

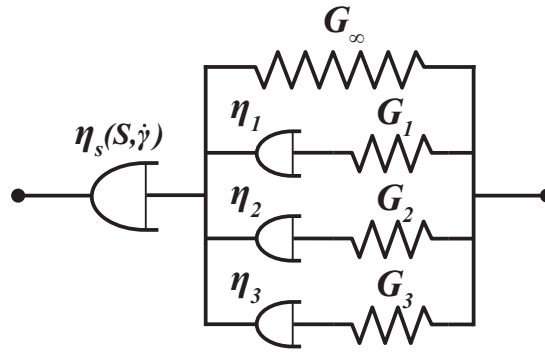


Figure 5.1: Schematic illustration of the Generalized Maxwell model in series with a softening-controlled dashpot.

For a system connected in series, the equilibrium conditions differ from those governing the parallel arrangement within the generalized Maxwell element. In a series configuration, the stress is uniform across all elements, while the total strain is given as the sum of the individual strains:

$$\sigma = \sigma_s = \sigma_{MW} \quad \gamma = \gamma_s + \gamma_{MW}. \quad (5.1)$$

Consequently, the stress response of the system is governed by the uniform stress shared among the individual components, enabling the dashpot and the Maxwell element to dominate the stress and strain responses at different stages of the deformation process. In practical terms, the generalized Maxwell element primarily governs the stress magnitude up to a peak value, after which the dashpot contribution becomes increasingly significant, allowing for continued viscous deformation and a redistribution of strain within the system.

To describe the viscous behavior of the dashpot, the constitutive relation proposed by Cross (1979) was adopted, such as

$$\hat{\eta}_s(\dot{\gamma}_s) = \eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + (K\dot{\gamma}_s)^m}, \quad (5.2)$$

where η_∞ denotes the limiting viscosity at high shear rates, η_0 the limiting viscosity at low shear rates, $\dot{\gamma}_s$ the current shear rate in the dashpot, and K and m are material-specific constants. The viscosity of the dashpot was evaluated based on the optimized steady-state parameters in the Cross model, in combination with the softening coefficient S_{min} and the derived relation given in Equation 5.12, such that

$$\eta_s(\dot{\gamma}_s) = \frac{1}{S_{min}} \hat{\eta}_s \quad \rightarrow \quad \eta_s(\dot{\gamma}_s) = \frac{1}{S_{min}} \left[\eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + (K\dot{\gamma}_s)^m} \right]. \quad (5.3)$$

To subsequently regulate the stress response of the system and account for the stress reduction observed in the coaxial shear tests, an additional softening parameter S was introduced into the dashpot formulation. The magnitude of the softening was defined using the softening function

$$S(\gamma) = 1 + (S_{min} - 1)(1 - e^{-k_{soft} \cdot \gamma}), \quad (5.4)$$

where the evolution of S was governed by the current total strain (γ) in the system, in combination with the softening coefficient k_{soft} , and the previously introduced parameter S_{min} . This assumption regarding the total strain in the system was adopted based on the observed shear stress reduction at a strain level of $\gamma \approx 1$ for all strain rates in the coaxial shear test (Figure 2.1). The softening function was constructed to incorporate both a maximum and a minimum plateau, with an exponentially decreasing response governed by the total strain describing the transition between these limits. The lower asymptotic limit, corresponding to the steady-state condition, was defined by the magnitude of S_{min} , such that

$$S(\gamma \rightarrow \infty) = 1 + (S_{min} - 1)(1 - e^{-k_{soft} \cdot \gamma}) \Rightarrow S = S_{min}. \quad (5.5)$$

For the upper asymptotic limit, $\gamma \rightarrow 0$, the magnitude of S was given by

$$S(\gamma \rightarrow 0) = 1 + (S_{min} - 1)(1 - e^{-k_{soft} \cdot \gamma}) \Rightarrow S = 1, \quad (5.6)$$

indicating that no softening was applied to the dashpot in the initial state. The evolution of softening behavior between the two asymptotic limits was governed by the parameters k_{soft} and S_{min} , which were calibrated based on the slope of the experimentally measured response, further described in Section 5.3.

With the introduction of the nonlinear dashpot and its associated softening function, the constitutive model was fully established. The viscoelastic response of the material

arises from the combined action of the generalized Maxwell elements and the dashpot, as formulated in Equation 5.1. The Maxwell branches primarily govern the initial stress relaxation, whereas the dashpot controls the rate-dependent viscous response and the progressive stress reduction associated with material degradation.

Since the resulting set of constitutive equations is nonlinear, an iterative solution procedure was required. Accordingly, the updated governing equations of the series-coupled system, defined in Equation 5.1, were solved using the *Newton–Raphson method*. The Newton iteration was governed by the strain in the Maxwell element, which was selected as the primary internal variable. Owing to the equilibrium conditions for a system in series, the stress must be uniform across all components, allowing the entire constitutive update to be expressed as a nonlinear equation in the Maxwell strain. Consequently, the remaining variables became implicitly dependent on the Maxwell strain, which therefore served as the controlling degree of freedom in the Newton–Raphson iteration. The residual for the iterations was defined as

$$R(\gamma_{MW}) = \sigma_{MW} - \sigma_s, \quad (5.7)$$

with convergence declared when

$$|R(\gamma_{MW})| < 1 \times 10^{-6} \cdot \max(|\sigma_{MW,i}|). \quad (5.8)$$

Here, $\sigma_{MW,i}$ denotes the stress in the Maxwell element at iteration i . Once convergence was achieved, all state variables were updated, including the strains and stresses of the individual components, as well as the corresponding viscosity and softening parameters. The strain in the dashpot was subsequently determined from the total strain relation given in Equation 5.1, and the stress in the system was obtained from

$$\sigma_s = S(\gamma) \cdot \eta_s(\dot{\gamma}_s) \cdot \dot{\gamma}_s, \quad (5.9)$$

where the strain rate in the dashpot, $\dot{\gamma}_s$, was defined as

$$\dot{\gamma}_s = \frac{\gamma_s^n - \gamma_s^{n-1}}{dt}. \quad (5.10)$$

The constitutive update procedure described above was implemented within a time-stepping framework, where the stress response was evaluated iteratively at each increment. To verify that the model formulation was correctly implemented and ensure consistency with the underlying theoretical formulation, a model verification study was conducted. The verification procedure and its results are presented in the following section.

5.2 Verification of model implementation

To verify the implementation of the model, a set of material parameters obtained by calibration in Section 5.3 was employed. To ensure that the desired behavior was achieved, the strain distribution within the system, as well as the response of the softening-controlled dashpot, were evaluated for the highest ($\dot{\gamma} = 1 \text{ s}^{-1}$) and lowest ($\dot{\gamma} = 0.01 \text{ s}^{-1}$) strain rates obtained from the coaxial shear test, as intermediate strain rates did not exhibit any significant deviations.

The strain distribution within the system, as illustrated in Figure 5.2, indicates that the maxwell element dominates the strain response at small deformation levels, after which a progressive redistribution of strain to the series-connected dashpot occurs. This behavior aligns with the theoretical framework presented in Section 5.1, confirming that the strain within the model has been formulated correctly.

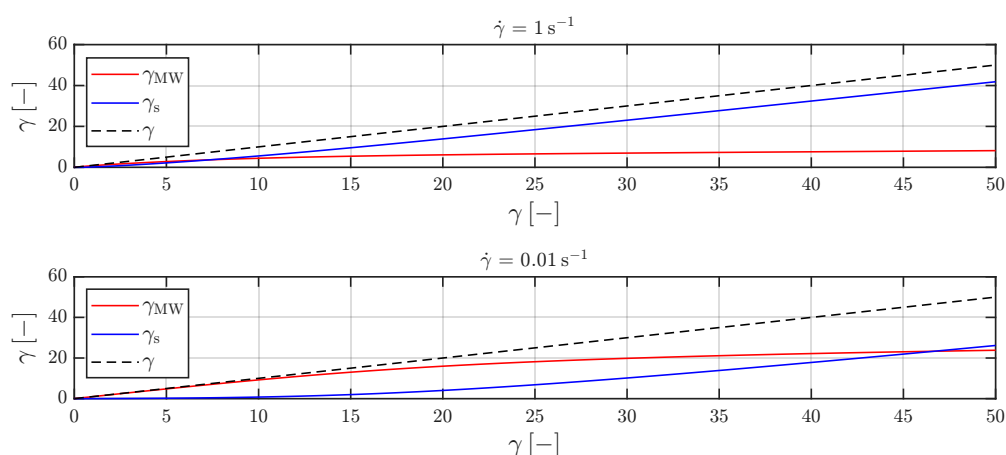


Figure 5.2: Shear strain distribution within the system obtained from MATLAB for $\dot{\gamma} = 1 \text{ s}^{-1}$ and $\dot{\gamma} = 0.01 \text{ s}^{-1}$, with the black dashed line representing the total shear strain.

To further verify the model response shown in Figure 5.2 against the total strain relationship introduced in Equation 5.1, the correspondence between the total strain and its distributed components can be examined. As illustrated in Figure 5.2, the total strain is consistently represented as the sum of the contributions from the Maxwell element and the dashpot across the entire deformation range. This observation provides additional confirmation of the consistency of the implementation with the underlying theory.

Furthermore, the Cross-model viscosity was evaluated at the maximum and minimum strain rate, see Figure 5.3, to assess its variation across the loading conditions and to further verify the viscous response of the model formulated in Equation 5.12.

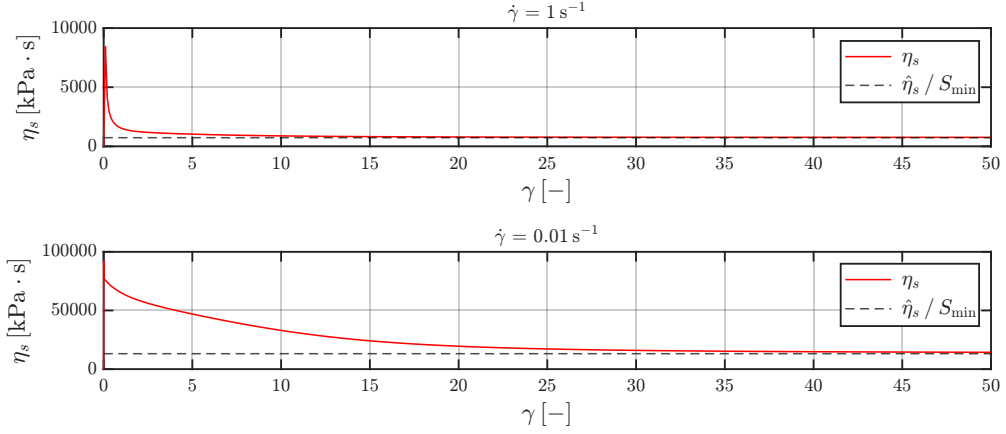


Figure 5.3: Cross-model viscosity in the dashpot obtained from MATLAB for $\dot{\gamma} = 1 \text{ s}^{-1}$ and $\dot{\gamma} = 0.01 \text{ s}^{-1}$, with the black dashed line representing the steady-state viscosity.

As demonstrated in Figure 5.3, the viscous response of the model, η_s , approaches the limiting value $\frac{1}{S_{min}}\hat{\eta}_s$ with increasing deformations. Accordingly, the modeled behavior is consistent with the theoretical formulation introduced in Equation 5.3, confirming a correct implementation and calibration of the Cross model.

To assess the evolution of the softening behavior during the simulated shear test, the magnitude of the softening function was extracted for the maximum and minimum strain rate, presented in Figure 5.4.

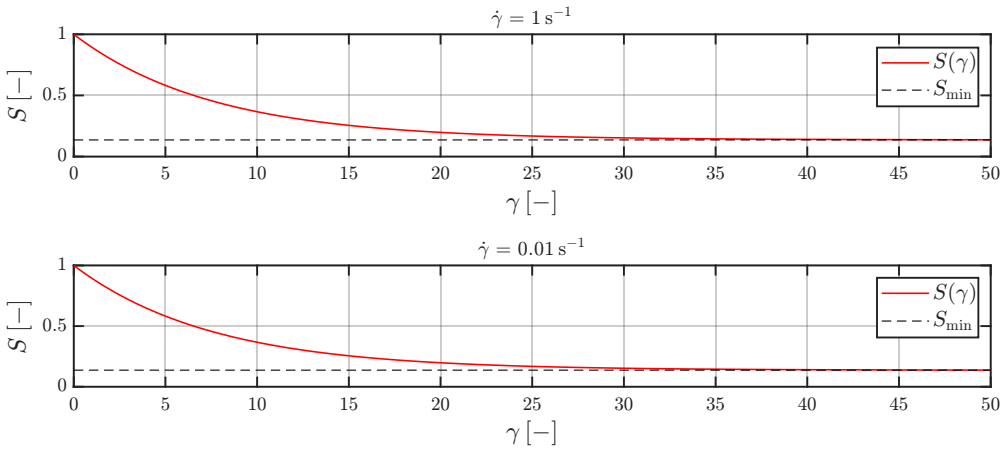


Figure 5.4: Response of the softening function from MATLAB for $\dot{\gamma} = 1 \text{ s}^{-1}$ and $\dot{\gamma} = 0.01 \text{ s}^{-1}$, with the black dashed line representing the minimum softening factor.

As observed in Figure 5.4, the softening function behaves in accordance with the theoretical formulations introduced in Equations 5.5 and 5.6, satisfying the condition $S_{min} \leq S \leq 1.0$. In addition, since the magnitude of the softening parameter is governed by the total strain in the system, γ , the function evolves consistently over the two strain rates. This behavior reflects the underlying modeling assumption that the onset of softening occurs at approximately the same strain level, $\gamma \approx 1$ for all strain rates, followed by a gradual transition towards a steady-state plateau. Collectively, these observations con-

firm that the softening behavior of the dashpot operates as intended within the model framework.

Furthermore, the model's global steady-state behavior was also verified for the maximum and minimum strain rate. With the updated model formulation, the global response approaches the asymptotic steady-state shear stress at large deformations, as illustrated in Figure 5.5. This confirms that the formulated model captures the expected long-term behavior, further described in the following section.

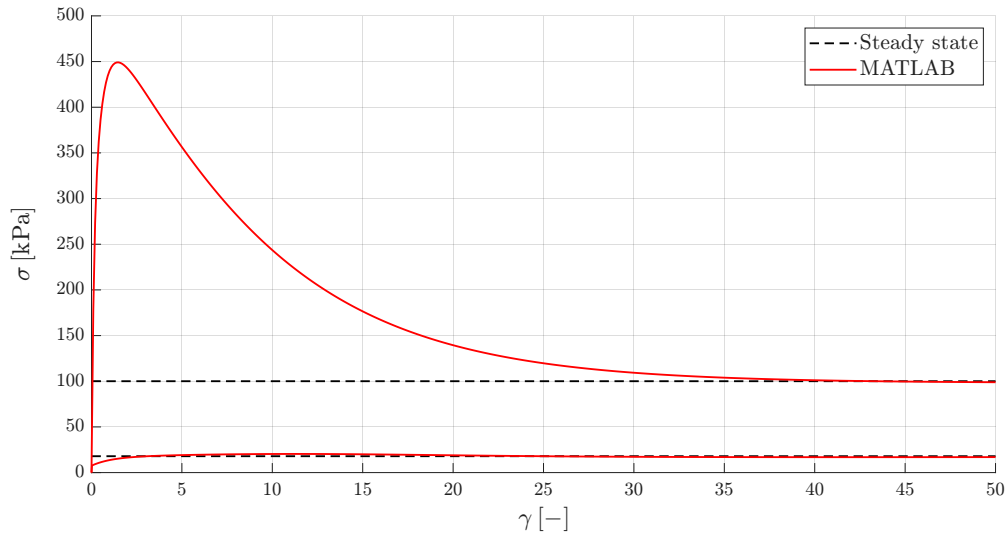


Figure 5.5: Shear stress (σ) as a function of shear strain (γ) obtained from MATLAB for $\dot{\gamma} = 1s^{-1}$ and $\dot{\gamma} = 0.01s^{-1}$, with the black dashed line representing the steady-state shear stress ($\hat{\sigma}$).

5.3 Model calibration

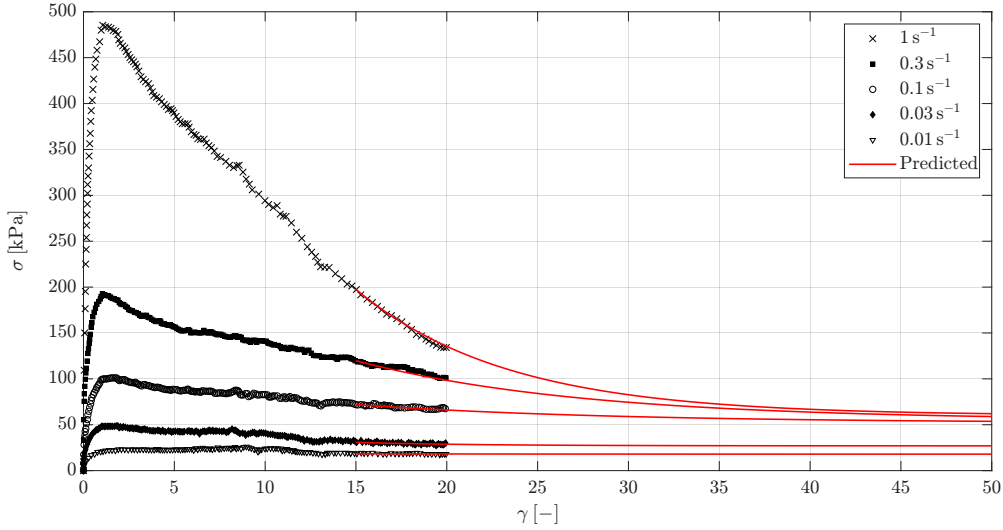
To describe the viscous behavior of the additional dashpot using the Cross model, the parameters introduced in Equation 5.2 were calibrated. For each strain rate, the asymptotic response beyond peak stress was matched by visually fitting the relation defined in Equation 5.11 to the experimental data. This procedure enabled the determination of the steady-state shear stress, $\hat{\sigma}$, which was subsequently used in the calibration of the Cross model.

$$\sigma(\gamma) = \sigma_0 + (\hat{\sigma} - \sigma_0)(1 - e^{-\kappa \cdot \gamma}) \quad (5.11)$$

Here, σ_0 represents the shear stress at the selected strain level, and κ the rate at which the shear stress decreased with increasing strain. The manually fitted variables were extracted from MATLAB and are summarized in Table 5.1, with the corresponding response shown in Figure 5.6.

Table 5.1: Input parameters for manually predicted steady state function.

$\dot{\gamma}$ [s ⁻¹]	κ [-]	σ_0 [kPa]	$\hat{\sigma}$ [kPa]
1.0	0.12	197	60
0.3	0.08	120	55
0.1	0.07	72	52
0.03	0.15	31	27
0.01	0.18	18	18

**Figure 5.6:** Reconstructed plot of manually digitized data from a coaxial shear test graph by Cheung and Cebon (1997), with manually predicted response beyond the measured range.

Once each strain rate curve was sufficiently represented, the steady state shear stress $\hat{\sigma}$ was extracted and divided by the corresponding strain rate $\dot{\gamma}$ to obtain the steady state viscosity $\hat{\eta}_s$, summarized in Table 5.2.

Table 5.2: Calculated steady-state viscosity ($\hat{\eta}_s$) from predicted steady-state shear stress ($\hat{\sigma}$) for all strain rates ($\dot{\gamma}$).

$\dot{\gamma}$ [s ⁻¹]	$\hat{\sigma}$ [kPa]	$\hat{\eta}_s$ [kPa · s]
1.0	60	60
0.3	55	183
0.1	52	520
0.03	27	900
0.01	18	1800

The Cross model $\eta_s(\dot{\gamma})$, Equation 5.2, was then multiplied by the minimum softening factor S_{min} (the strain-softening applied to the dashpot, defining the lowest viscosity attainable in the system, i.e. the viscosity at steady state) and subsequently fitted to the steady state viscosity, using the *lsqnonlin* routine in MATLAB, such as

$$\hat{\eta}_s = S_{min} \cdot \eta_s(\dot{\gamma}_s), \quad (5.12)$$

where the residual error was defined as

$$error = \hat{\eta}_{s,predicted} - \hat{\eta}_{s,Cross}. \quad (5.13)$$

Subsequently, the Cross model parameters η_0 , η_∞ , K and m were calibrated, summarized in Table 5.3. It should be noted that η_∞ approaches zero during calibration. Liao and Chen (2011) observed a similar model response, which was attributed to the assumption that an increasing shear rate reduces viscosity, resulting in no distinct plateau at high shear rates.

Table 5.3: Optimized Cross-model parameters for steady state prediction.

η_∞ [kPa · s]	η_0 [kPa · s]	K [-]	m [-]
2.98E-17	1.40E+04	1.87E+03	0.655

The optimized cross model deviates compared to the steady-state viscosity predicted using the help function, as shown in Figure 5.7. The corresponding values are reported in Table 5.4. Moreover, the residual was defined without any weighting of the error, leading to a bias toward lower strain rates.

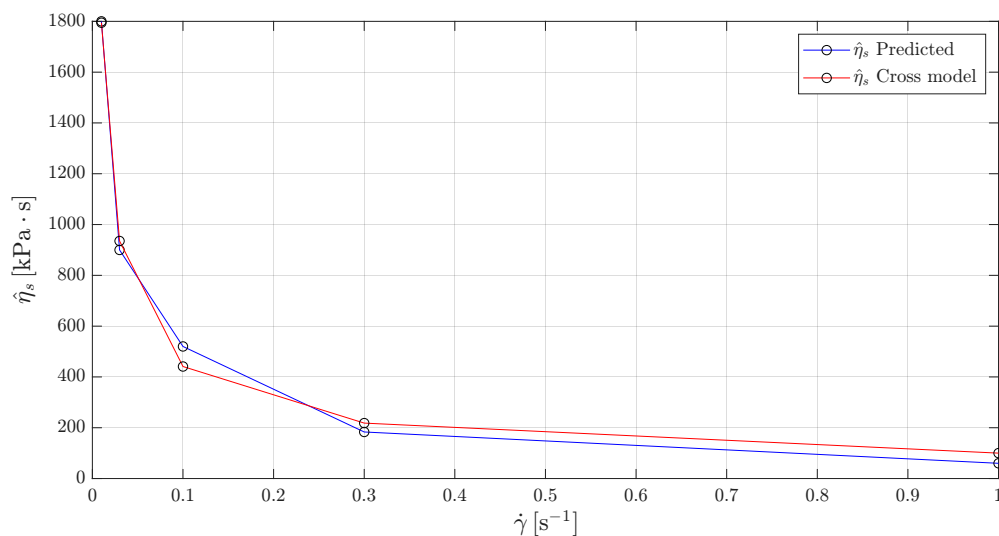


Figure 5.7: Comparison of the steady-state viscosity ($\hat{\eta}_s$) obtained from the optimized Cross model and the predicted viscosity.

Table 5.4: Deviation in steady-state viscosity ($\hat{\eta}_s$) between the prediction and the Cross model.

$\dot{\gamma}$ [s ⁻¹]	Predicted $\hat{\eta}_s$ [kPa · s]	Cross model $\hat{\eta}_s$ [kPa · s]	Error [%]
1.0	60	100	40.0
0.3	183	218	16.0
0.1	520	441	-18.0
0.03	900	935	4.0
0.01	1800	1795	- 0.3

Since the post-peak behavior was derived from the steady-state parameters of the Cross model, the stress response at increasing strain levels was expected to approach the steady-state shear stress $\hat{\sigma}$, and could thus be approximated as

$$\hat{\sigma} = \hat{\eta}_s \cdot \dot{\gamma}_s = S_{min} \cdot \eta_s(\dot{\gamma}_s) \cdot \dot{\gamma}_s. \quad (5.14)$$

With the calibrated dashpot incorporated into the Maxwell system, the Generalized Maxwell parameters G_∞ , G_i and τ_i together with the softening parameters k_{soft} and S_{min} were subsequently optimized against the full experimental strain range ($0 \leq \gamma \leq 20$). The optimization was performed using the nonlinear solver *lsqnonlin* in MATLAB, with the objective of minimizing the error between the experimentally measured data from the coaxial shear tests and the model-predicted stress–strain response of the Maxwell system, such that

$$error = \sigma_{measured} - \sigma_{predicted}. \quad (5.15)$$

The resulting parameters obtained from the optimization procedure are summarized in Table 5.5.

Table 5.5: Optimized Prony and softening parameters, obtained from MATLAB.

No.	G_i [kPa]	τ_i [s]	G_∞ [kPa]	S_{min} [-]	k_{soft} [-]
0	-	-	0.610	0.136	0.132
1	2.89E+03	0.08			
2	0.61E+03	0.88			
3	0.87E+01	113.02			

5.4 Results

This section presents the results obtained using the final model in order to evaluate its ability to reproduce the observed material behavior. The predicted response of the complete material model is subsequently presented and verified against the experimentally obtained test data from Cheung and Cebon (1997). Finally, the loading rate derived in Equation 1.1 is applied to the model, and a coaxial shear test is simulated, with the resulting response presented and discussed.

With all input parameters calibrated and the additional dashpot incorporated into the Maxwell system, following the procedure derived in Section 5.3, the material model yields the response shown in Figure 5.8. As can be observed, the model is capable of simulating the general trend of the coaxial shear test. The main deviation is observed in the transition phase, where the material degradation (activation of the softening mechanism) is not fully captured, illustrated by slightly different shear stress magnitudes compared to the test data. Furthermore, the model predicts the shear stress fairly well at the end of the calibrated range ($\gamma = 20$), with small deviations observed over all strain rates. At increasing shear strain levels beyond the calibrated range, the model response slowly approaches the manually fitted steady-state plateau.

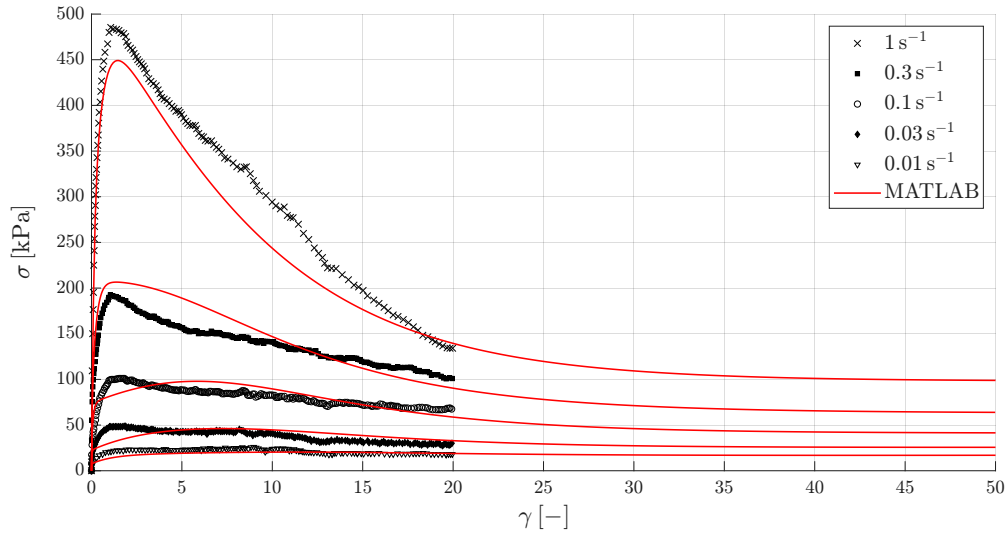


Figure 5.8: Comparison of the shear stress-strain response of the final model obtained from MATLAB and the manually digitized data.

Based on these observations, the model was subsequently applied under realistic loading conditions to facilitate a more direct interpretation of the resulting response in the context of design practice.

5.4.1 Realistic loading case

In the final stage of the study, the realistic loading conditions ($\dot{\gamma} = 8 \cdot 10^{-8} \text{ s}^{-1}$, $\gamma = 500$), derived in Section 1.4, were applied to the model. The predicted shear stress - shear strain response of the model is presented in Figure 5.9.

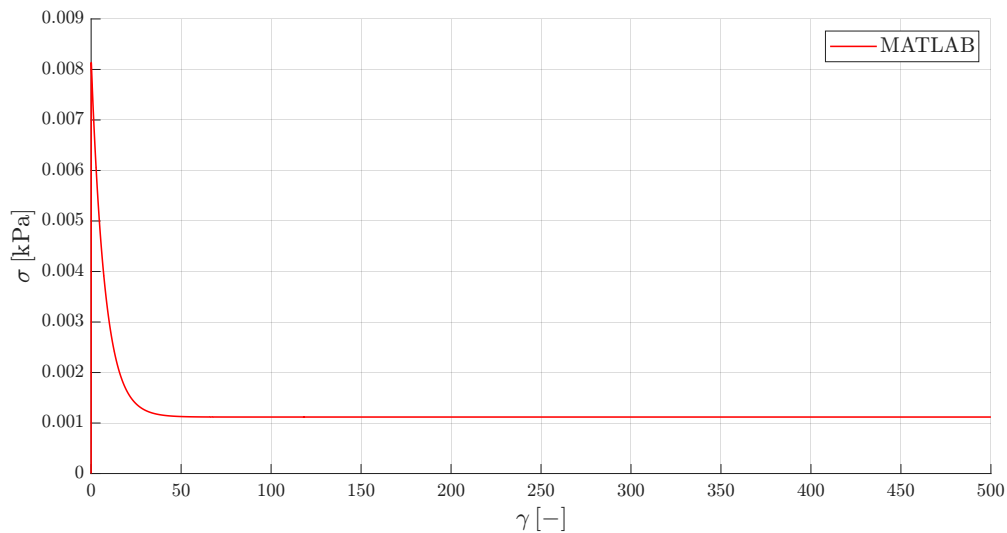


Figure 5.9: Shear stress-strain response for final model under realistic loading conditions, obtained from MATLAB.

As illustrated in Figure 5.9, the shear stress obtained at the realistic loading rate is of significantly lower magnitude compared to the shear stress levels corresponding to the loading rates applied in the coaxial shear tests (see Figure 5.8). The peak stress occurs at approximately $\sigma_{max} \approx 8.0\text{Pa}$, followed by a steady-state plateau slightly above $\hat{\sigma} \approx 1.0\text{Pa}$.

The resulting model response, was further analyzed by assessing the shear strain distribution within the system, presented in Figure 5.10.

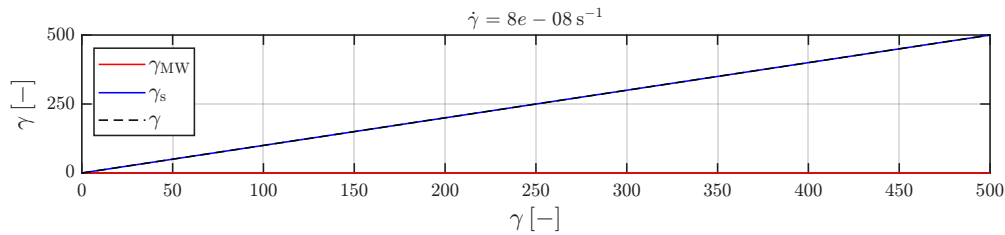


Figure 5.10: Shear strain distribution in system under realistic loading conditions, obtained from MATLAB.

In contrast to the strain distribution observed at higher loading rates (Figure 5.2), Figure 5.10 clearly demonstrates that the dashpot governs the total shear strain of the system under realistic loading conditions. This behavior persists throughout the entire simulation and indicates a viscosity-dominated response at very low strain rates, consistent with findings reported in the literature.

In addition to the strain distribution, the evolution of the Cross-model viscosity in the dashpot was also examined and is presented in Figure 5.11.

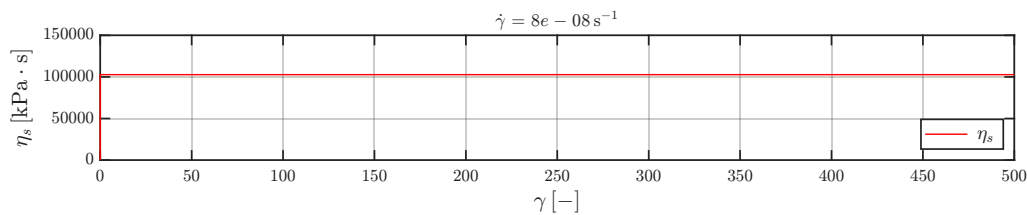


Figure 5.11: Cross-model viscosity in the dashpot under realistic loading conditions, obtained from MATLAB.

As observed in Figure 5.11, the Cross-model viscosity η_s increases rapidly until reaching a maximum plateau of approximately $\eta_{s,max} \approx 100\,000\text{kPa} \cdot \text{s}$, after which the viscosity remains constant with further deformation. Since the realistic loading rate is significantly lower than the strain rate from the coaxial shear test ($0.01\text{ s}^{-1} \leq \dot{\gamma} \leq 1\text{ s}^{-1}$), the viscosity plateau can be derived from Equation 5.3 directly, by approaching the limit $\dot{\gamma} \rightarrow 0$, such that

$$\lim_{\dot{\gamma} \rightarrow 0} \eta_s(\dot{\gamma}_s) = \frac{1}{S_{min}} \left[\eta_{\infty} + \frac{\eta_0 - \eta_{\infty}}{1 + (K\dot{\gamma}_s)^m} \right] = \frac{1}{S_{min}} \eta_0 \quad (5.16)$$

Consequently, the viscosity of the dashpot converges toward an asymptotic upper bound defined by $\frac{1}{S_{min}}\eta_0$ corresponding to the steady-state viscous response under long-term, low strain-rate loading conditions. By applying the resulting values for η_0 and S_{min} reported in Table 5.3 and 5.5 respectively, the upper bound viscosity can be approximated as

$$\eta_s = \frac{1}{S_{min}}\eta_0 \approx 103\,000 \text{ kPa} \cdot \text{s}, \quad (5.17)$$

which is in accordance with the resulting maximum viscosity observed in Figure 5.11. Consequently, with the current model formulation the shear stress under realistic loading rates is governed by the zero-shear viscosity, η_0 . This indicates that the model's shear stress response is highly dependent on the calibrated parameters within the Cross model, in combination with the applied softening.

The softening response under realistic loading conditions was identical to the behavior observed in Figure 5.4. However, in this case, the strain in the dashpot governs the total strain in the system (Figure 5.10). Despite this, the softening function evolves in a comparable manner due to its dependence on the total strain, starting from an initial value of 1.0 and gradually decreasing toward the calibrated asymptotic value $S_{min} = 0.136$ as the strain level increases.

Overall, the final model generally captures the intended qualitative behavior and overall trends across both laboratory-scale and realistic loading conditions. In particular, the transition toward a viscous-dominated response at low strain rates and the strain-controlled evolution of the softening function are consistently observed. However, deviations remain in the predicted magnitude of the shear stress under realistic conditions, with the largest discrepancies arising relative to the currently adopted design value of 10kPa. These differences motivate a more detailed interpretation of the current results, their practical implications, and potential directions for future improvements, which are addressed in the following discussion.

6 Discussion

The initial modeling approach based on literature data, relies solely on least square fitting of coaxial test curves to describe the behavior of the material, which introduces uncertainty in the physical interpretation of the parameters. However, as the model is extended and calibrated using DSR data, relevant constitutive relations can be extracted. Within the linear viscoelastic (LVE) regime, the phase angle, complex shear modulus, and storage and loss moduli are well represented. Accordingly, the model response shows good agreement with the DSR data in this regime, indicating consistent material behavior and a well-formulated calibration procedure. Although, the data covers a wide range of temperatures and frequencies. DSR tests are standardized for the pavement industry, consequently meaning significantly different frequencies of interest. For material behavior beyond the experimentally obtained data range, the model fail to capture the response and cannot reproduce the viscous behavior previously reported in the literature. To address this limitation, time-temperature superposition (TTS) was adopted as a suitable approach to study lower loading rates. With TTS implemented, the frequency range was shifted and an equivalent frequency-temperature pair was utilized to evaluate the rheological properties under realistic condition ($\omega = 1.6 \cdot 10^{-10} \text{ s}^{-1}$ at $T = 0^\circ\text{C}$). The resulting hysteresis response (Figure 4.15) indicates that the TTS implementation was capable of capturing viscosity-dominated behavior at low loading rates, characterized by $\delta \rightarrow 90^\circ$. However, this response was interpreted with caution since the frequency axis was shifted over several orders of magnitude, introducing significant uncertainty. Furthermore, since the model remained based on linear viscoelastic theory, it was not capable of capturing the characteristic stress reduction observed in the coaxial shear test (Figure 2.1). In addition, the model was limited to smaller strain levels within the LVE regime, several magnitudes smaller compared to long-term pile application under realistic conditions. Moreover, as previously noted, the model was unable to capture viscosity-dominated behavior at low loading rates without the use of the inherently uncertain TTS implementation. Consequently, rather than continuing with the DSR analysis, the initial generalized Maxwell model was extended by introducing an additional dashpot connected in series.

The underlying concept of the extended model was to allow the Maxwell element to govern the response at low strain levels, while the additional dashpot progressively dominated the response as the strain increased. To describe the behavior of the dashpot, the non-linear Cross model was employed to characterize the viscosity as a function of the strain rate, in combination with a softening function. The Cross model was calibrated against a predicted steady-state condition, whereas the generalized Maxwell element was calibrated against the stress-strain response for ($0 \leq \gamma \leq 20$) observed in the coaxial shear test. This implementation resulted in a maximum shear stress of 8.0Pa and a steady-state response at 1.0Pa.

The extended model suggests a response at larger deformations, however, this relies partly on manually defined adjustments to describe the approach toward steady-state conditions. Additionally, the limited dataset introduces further uncertainty, and validation against additional experimental data is required to confirm the observed trends.

Given the uncertainties discussed above, the following sections expand the discussion by examining the results from different perspectives. The practical implications are first

considered, followed by a more detailed analysis of the mechanisms behind the model response. The reliability of the results and the relevance of the reference value of 10kPa are then assessed. Finally, material properties beneficial for pile applications (ranging from installation to long-term behavior) are examined, followed by a discussion of directions for future research.

Practical implications

Under realistic loading conditions, the material model resulted in a maximum shear stress of 8.0Pa and a steady-state shear stress of approximately $\sigma \approx 1.0\text{Pa}$, representing a substantial reduction compared to the proposed design value of 10kPa, and placing the results at the very end of the reported range of 0 – 20kPa. In practice, these results suggest that the bitumen coating significantly reduces the shear resistance at the pile–soil interface, resulting in negligible downdrag loads on the pile. At the same time, with little to no shaft friction at the interface, the surrounding soil is not supported by the pile, leading to increased settlements. However, since the primary purpose of the bitumen coating is to reduce downdrag loads, these increased settlements may be considered acceptable within the design context. From a theoretical perspective, a very low adhesion strength is beneficial, although such a condition may not be entirely realistic in practice.

In pile design, negligible adhesion strength implies negligible downdrag loads, resulting in a significant reduction in the total axial load acting on the pile. Consequently, the required load-bearing capacity can be reduced. This may allow for a reduced number of piles or increased spacing between them, which may be advantageous from both an economic and an environmental standpoint. However, as discussed in the previous paragraph, such reduction in the adhesion strength is not as reasonable in practice, and should be interpreted with caution.

Underlying causes of the low shear stress response

The coaxial shear tests conducted by Cheung and Cebon (1997) were performed at a temperature of approximately 20°C, whereas the conditions considered in the realistic case fall within the range of 0 – 10°C. As observed in the DSR-data, all shear moduli decrease with increasing temperature (Appendix 8), indicating a strong correlation with temperature. Since both the material parameters and the softening behavior were derived at 20°C, this temperature discrepancy may have a significant influence on the material response, and consequently on the obtained results. However, under realistic loading conditions, the dashpot was observed to govern the total strain in the system (Figure 5.10), meaning that the total stress at steady state is primarily limited by the zero-shear viscosity reported in Equation 5.3 and the applied strain rate ($\dot{\sigma} = 14\,000\text{kPa} \cdot \text{s} \times 8 \cdot 10^{-8}\text{s}^{-1} \approx 1.0\text{Pa}$). Consequently, the temperature-dependent shear moduli have no direct influence on the steady-state shear stress under these conditions. The governing factors behind the discrepancy in the resulting long-term behavior, compared to the reference value of 10kPa, instead lie in the calibration of the Cross model, and in particular the zero-shear viscosity η_0 . It should be noted, however, that the calibration of the Cross-model parameters is itself also based on the coaxial shear tests conducted at $T = 20^\circ\text{C}$. This implies that the resulting stress response from the model may be underestimated, as higher temperatures correspond to a less stiff material response compared to the expected behavior at 0 – 10°C. While this effect may partly

explain the low predicted shear stress values, it cannot, on its own, account for the large deviation from the reference value of 10kPa.

In addition, the manually fitted help function used for calibrating the long-term behavior may have been inadequately estimated, with an overestimation of the decay rate leading to excessively low steady-state values. Furthermore, the extensive extrapolation required introduces significant uncertainty, making it difficult to obtain reliable predictions of the steady-state response.

Reliability of the results

The predicted shear stress of approximately 1.0Pa should be interpreted with caution, as it lies outside the range for which the model has been calibrated. Although the quantitative value remains uncertain, the model consistently captures a transition toward viscous-dominated behavior at low strain rates, consistent with trends reported in previous studies. In addition, the current model formulation does not account for several factors present in real conditions, such as creep effects, dynamic loading, and elastic recovery. These mechanisms may influence both the short-term and long-term response of the bitumen layer and could lead to deviations from the predicted material behavior. However, for the considered long-term, low strain-rate conditions, viscous effects are expected to dominate, which suggests that the neglect of certain short-term mechanisms may have a limited influence on the overall response.

Furthermore, the model based on experimental data (Section 4) showed good agreement with the obtained DSR data, as well as with the time-temperature superposition within the LVE regime, capturing the evolution of the phase angle across varying temperatures and loading rates. However, for deformations beyond the LVE regime, the results become difficult to assess and validate against experimental measurements. Consequently, the results of the DSR analysis remain uncertain, however, the model once again was capable of capturing a viscosity-dominated response at low loading rates.

A further source of uncertainty arises from the calibration of the final material model itself. As discussed in previous sections, the model parameters are primarily obtained through curve fitting, rather than being directly linked to measurable physical properties. Consequently, while the model is able to reproduce the observed experimental behavior within the calibrated range, its predictive capability outside this domain remains uncertain.

Despite these limitations, consistent trends can be identified across the different modeling approaches. In particular, the final model formulation indicate a transition toward a viscous-dominated response at low strain rates, resulting in a near-constant shear stress at large deformations. This qualitative behavior is therefore considered to be a robust outcome of the study. However, the exact magnitude of the steady-state shear stress is highly sensitive to the calibration of the viscosity in the Cross model, and in particular the zero-shear viscosity η_0 . As this parameter is derived indirectly from experimental data obtained at significantly higher strain rates and temperatures, its extrapolation to realistic conditions introduces considerable uncertainty.

Overall, while the final model provides a consistent qualitative description of the expected material behavior, the quantitative prediction of the shear stress should be interpreted with caution. The results should therefore not be relied upon for quantitative assessment, but rather serve to highlight general trends.

Interpretations of the reference value 10kPa

As observed in the stress–strain responses presented in Section 5.4, the stress increases significantly with strain up to approximately $\gamma \approx 1$, beyond which the magnitude decreases. Since the shear stress corresponds to the adhesion strength at the pile–soil interface, this implies that the pile is subjected to a downdrag load, whose magnitude depends on the strain level. Consequently, according to the results, the maximum downdrag load on the pile does not occur under steady-state conditions, but rather at the strain level corresponding to the peak shear stress. Therefore, it is essential that the pile is capable of withstanding the additional loads that develop at relatively low deformation levels, as these will govern the design (up to an cohesion strength equal to the shear capacity of the clay). With this in mind, it is pertinent to assess whether the shear stress corresponding to the reference value of 10kPa occurs at lower strain levels or at steady state. In the original source of this value, this distinction is not specified in detail, it is merely stated that an assumption of 10kPa is of a conservative nature.

On the other hand, it is difficult, based on the results obtained in this study, to assess to what extent the model response reflects the actual behavior of bitumen. It is possible that the final model formulation introduces an exaggerated peak prior to the steady-state plateau, thereby reducing the relative importance of the steady-state response in the design considerations, as discussed in the previous paragraph. If the actual material behavior is more realistically described by a gradual increase in stress toward a maximum plateau at steady-state, it is instead the long-term response that governs the design. Accordingly, the viscous properties control the actual material response, and consequently the adhesion strength at the interface.

Pile installation and varying shear rates

In practical pile engineering applications, not only the long-term behavior of the bitumen is of interest, but also its instantaneous response during pile installation. At this stage, high loading rates are combined with large forces, leading to significantly increased strength demands within the bitumen layer, to prevent failure. Consequently, a sufficiently high material stiffness is required at higher loading rates to prevent detachment of the layer, while still permitting viscous behavior to dominate at lower loading rates, as indicated in this study. In accordance with viscoelastic theory, this corresponds to a phase angle δ evolving from relatively small values at high loading rates to larger values at lower loading rates. As observed in the normalized hysteresis responses, Figure 4.11 and 4.12, this implies that the loop approaches a more linear shape at higher rates and a more circular shape at lower loading rates. In the context of coating design, this behavior can be controlled through the selection of the bitumen grade, as well as through potential modifications using suitable additives. However, a more detailed investigation of these aspects is beyond the scope of the present study and should be addressed in future work.

Further studies

To further develop the material model, additional experimental investigations are required. Particular attention should be given to determining whether a peak in cohesion strength followed by a reduction toward a steady-state plateau accurately reflects the real material behavior, or whether a gradual increase in strength toward a steady-state

maximum provides a more realistic description. Such clarification would also contribute to a better understanding of the relationship between the somewhat unspecified reference value of 10kPa (as proposed in the literature during the 20th century) and today's advancements in numerical modelling. In addition to improving the understanding of the actual mechanical behavior, well-defined material data are also required, particularly at significantly lower strain rates and larger deformations than those achieved in the coaxial shear test and in the DSR application. Based on such data, more reliable predictions of the long-term behavior and the non-Newtonian viscosity can be obtained, leading to results that are more firmly grounded in observed material behavior rather than empirical fitting and extrapolation.

7 Conclusion

By applying constitutive material models based on viscoelastic theory, the mechanical response of bitumen can be characterized under varying loading conditions. This is supported by the findings of the present study, which demonstrate that the adopted modeling framework is capable of capturing the mechanical behavior within the range of available experimental data. However, to represent realistic loading conditions for bitumen-coated piles, the model must be applied far beyond the range of the available data, which introduces significant uncertainty in the predicted response. Therefore, this work primarily provides a modeling approach that enables future calibration using more representative experimental data, ultimately allowing for more reliable estimation of the adhesion strength of bitumen.

The final material model predicts an adhesion strength of approximately 8.0 Pa at peak and 1.0 Pa at steady state. These values are several orders of magnitude lower than the commonly adopted reference value of 10 kPa used in practice. However, previous field studies have indicated adhesion values within the range of 0 – 20 kPa.

According to the model response, the reference value of 10kPa appears to correspond to a peak stress at low strain levels rather than the steady-state plateau. From a theoretical standpoint, this suggests that accurately predicting the peak response may be sufficient for determining a representative design value, provided that material parameters are calibrated using well-defined experimental data. In contrast, if the model does not accurately capture the true material behavior, the steady-state response may instead govern the design value, requiring a formulation capable of more accurately describing the long-term viscous behavior.

The DSR analysis provides a good understanding of the material response within the linear viscoelastic (LVE) regime, and the model shows good agreement with experimental data. However, at larger deformations, the reliability of the material properties becomes uncertain and is therefore insufficient as a basis for calibration. Time-temperature superposition (TTS) can be employed to evaluate loading rates beyond the experimentally measured range, however superposition over several orders of magnitude should be interpreted with caution.

Regarding the coaxial shear test conducted by Cheung and Cebon (1997), the strain rates were limited to values significantly higher than the strain rate representing a realistic loading condition, reducing their suitability for calibration. Moreover, for steady-state behavior, test data at strain levels $\gamma > 20$ are required to better predict large-deformation response without relying on significant extrapolation.

Further studies are necessary to improve the understanding of the mechanical behavior of bitumen in coated pile applications, particularly whether a peak in cohesion strength followed by a reduction to a steady-state plateau accurately reflects material behavior, or a gradual increase toward a steady-state maximum provides a more realistic response. Moreover, experimental investigations of larger deformations and lower strain rates must be conducted to better reflect realistic field scenarios. Consequently, this would enable the establishment of a less uncertain design value, ultimately improving the reliability and efficiency of design guidance for bitumen-coated piles.

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Appendix A: MATLAB code

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VISCOELASTIC MODEL BASED ON LITERATURE DATA

Written by: Axel Isberg & Axel Jankavs

Publication date: 12-06-2026

```
%-----

clc
clear all
format short
close all

% --- MATERIAL INPUTS/INITIAL GUESS --- %
G_inf_init = 8.3e3;
G_init = [36e6 12e3 3.3e4];
tau_init = [0.02 0.2 2];
d_g = [1 0.3 0.1 0.03 0.01];

% --- MATRIX & LIST FORMATTING --- %
x0 = [G_inf_init, G_init, tau_init];
log_x0 = log(x0);

% --- EXTERNAL TEST-DATA --- %
run('DATA_SHEAR_TEST_1s.m');

d_g_test = {1, 0.3, 0.1, 0.03, 0.01};
g_tot_test = {x_data_1, x_data_03, x_data_01, x_data_003, x_data_001};
stress_test = {y_data_1, y_data_03, y_data_01, y_data_003, y_data_001};

% --- COST-FUNCTION --- %
cost = @(logx) GMM_residual(logx, d_g_test, g_tot_test, stress_test);

% --- CALIBRATIONS --- %
options = optimoptions('lsqnonlin', 'Display', 'iter',...
    'FunctionTolerance', 1e-12, 'StepTolerance', 1e-12,...
    'OptimalityTolerance', 1e-12, 'MaxIterations', 5e3,...
    'MaxFunctionEvaluations', 2e5, 'FiniteDifferenceType', 'central',...
    'UseParallel', false);
```

```

logx_opti = lsqnonlin(cost, log_x0, [], [], options);

% --- CALIBRATED PARAMETERS --- %
N = numel(G_init);
Ginf_opti = exp(logx_opti(1));
Gi_opti = exp(logx_opti(2:1+N));
tau_opti = exp(logx_opti(2+N:end));

% --- PLOT MODEL CURVES --- %

% Preperation for result analysis
stress_analysis = zeros(length(d_g),1);
g_analysis      = zeros(length(d_g),1);
stress_max = zeros(size(d_g));

figure; hold on;

for i = 1:length(d_g)
    [g_tot, stress, n] = Generalized_Maxwell_model_func(Ginf_opti, Gi_opti,
    tau_opti, d_g(i), 0);

    stress_max(i) = max(stress);

    % Model plot
    if i == 1
        h_model = plot(g_tot, stress/1e3, 'r-', ...
            'LineWidth', 0.9, ...
            'DisplayName', '$\sigma(\gamma)$, \mathrm{MATLAB}$');
    else
        plot(g_tot, stress/1e3, 'r-', ...
            'LineWidth', 0.9, ...
            'HandleVisibility','off');
    end
end

% ---- TEST DATA (COAXIAL SHEAR TEST CHEUNG & CEBOM 1997) ---- %
h_leg = zeros(1,5);

h_leg(1) = plot(x_data_1, y_data_1/1e3, 'x', ...
    'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k', ...
    'MarkerSize',6, 'LineWidth',0.6);

h_leg(2) = plot(x_data_03, y_data_03/1e3, 's', ...
    'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k', 'MarkerFaceColor','k', ...
    'MarkerSize',3.5, 'LineWidth',0.6);

h_leg(3) = plot(x_data_01, y_data_01/1e3, 'o', ...
    'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k', ...
    'MarkerSize',4, 'LineWidth',0.6);

```

```

h_leg(4) = plot(x_data_003, y_data_003/1e3, 'd', ...
    'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k', 'MarkerFaceColor','k', ...
    'MarkerSize',3, 'LineWidth',0.6);

h_leg(5) = plot(x_data_001, y_data_001/1e3, 'v', ...
    'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k', ...
    'MarkerSize',3, 'LineWidth',0.6);

stress_max_test = [max(y_data_1) max(y_data_03) max(y_data_01)
max(y_data_003) max(y_data_001)];

grid on

% ---- PLOT-SETTINGS ---- %
xlabel('$\gamma \, [-]$', ...
    'Interpreter','latex', ...
    'Units','normalized', ...
    'Position', [0.5 -0.07]);

ylabel('$\sigma \, [\mathrm{kPa}]$', ...
    'Interpreter','latex', ...
    'Units','normalized', ...
    'Position', [-0.06 0.5]);

ax = gca;

ax.Box = 'on';
ax.TickLabelInterpreter = 'latex';
ax.YAxis.Exponent = 0;

ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;

ax.LineWidth = 0.8;
ax.FontSize = 12;

lgd = legend([h_leg, h_model], ...
    {'$1\, \mathrm{s}^{-1}$', ...
    '$0.3\, \mathrm{s}^{-1}$', ...
    '$0.1\, \mathrm{s}^{-1}$', ...
    '$0.03\, \mathrm{s}^{-1}$', ...
    '$0.01\, \mathrm{s}^{-1}$', ...
    '$\mathrm{MATLAB}$'}, ...
    'Location','northwest', ...
    'Interpreter','latex');

lgd.FontSize = 12;
lgd.ItemTokenSize = [25, 10];
lgd.Box = 'on';

set(gcf, 'Position', [100 100 1000 480])

```

```
% --- PRINT CALIBRATED PARAMETERS --- %
disp('Optimized Ginf [kPa]:'); disp(Ginf_opti'/1e3)
disp('Optimized G-values [kPa]:'); disp(Gi_opti'/1e3)
disp('Optimized tau-values [s]:'); disp(tau_opti')

% --- PRONY SERIES FOR ABAQUS --- %
[g_i, tau_i] = Create_Prony(Ginf_opti, Gi_opti, tau_opti);

disp('Prony series Abaqus:')
disp('g_i [-]'); disp(g_i)
disp('tau_i [s]'); disp(tau_i)
disp('Sigma_max [kPa]'); disp(stress_max'/1e3)
disp('Sigma_max (Test) [kPa]'); disp(stress_max_test'/1e3)

E = E_mod_inst(Ginf_opti, Gi_opti, 0.495);
disp('E [kPa]'); disp(E/1e3)

%
%
%
```

Functions

```
% RESIDUAL FUNCTION FOR MODEL CALIBRATION
function r = GMM_residual(logx, d_g_test, g_tot_test, stress_test)

% INPUT: logx    = Convert parameters to log-format
%          d_g    = List of strain rates
%          g_tot  = Total strain data
%          stress = Stress data
%
% OUTPUT: r      = Residual
%
% Written by Axel Isberg & Axel Jankavs
%-----

N = (numel(logx) - 1) / 2;

log_G0 = logx(1);
log_G = logx(2 : 1 + N);
log_tau = logx(2 + N : end);

G0 = exp(log_G0);
G = exp(log_G);
tau = exp(log_tau);

% Constraints for stability
G0 = min(max(G0, 1e0), 1e9);
G = min(max(G, 1e0), 1e9);
```

```

r = [];

for i = 1:length(d_g_test)

    d_g_list = d_g_test{i}(:);
    g_tot_list = g_tot_test{i}(:);
    stress_list = stress_test{i}(:);

    [g_tot, stress] = Generalized_Maxwell_model_func(G0, G, tau,
d_g_list, 0);
    g_tot = g_tot(:) ;
    stress = stress(:);

    stress_interp = interp1(g_tot, stress, g_tot_list, 'linear',
'extrap');
    stress_interp = stress_interp(:);

    res = (stress_list - stress_interp);
    r = [r; res];

end
end

% GENERALIZED MAXWELL FUNCTION
function [g_tot, stress, n] = Generalized_Maxwell_model_func(G_inf, G, tau,
d_g, p)
%
%
%
%          G1
%  -----xxxxxxxxx-----
%  I                      I
%  I                      I
%  O---I                  I---O
%  I          G2          n          I
%  I-----xxxxxxx-----[-----I
%
%
%
%
%  INPUT: G_inf = Long-term modulus          [Pa]
%         G     = Shear modulus for dashpot branches [Pa]
%         tau   = Relaxation time              [s]
%         d_g   = strain rates                 [/s]
%         plot  = if == 1 -> plot, if == 0 no plot
%
%
%  OUTPUT: g_tot = Total shear strain          [-]
%         stress = Total shear stress         [Pa]
%         n      = Viscosity                  [Pas]
%

```

```

%
% Written by Axel Isberg & Axel Jankavs
%-----

% --- LOOP OVER SEVERAL STRAIN RATES --- %

% --- TIME STEP --- %
t0 = 0;
t_final = 1/d_g;
dt = 0.01;
t = t0:dt:t_final;

% --- MATRIX FORMATTING --- %
g_v = zeros(length(t),length(tau));
n = zeros(size(tau));

% --- INITIAL CONDITION --- %
g_v(1,:) = 0;

% --- TOTAL STRAIN --- %
g_tot = d_g * t(:);

% --- BACKWARD EULER (ZENER MODEL) --- %
for j = 1:length(tau)

    n(j) = tau(j) * G(j);

    for i = 2:length(t)

        g_v(i,j) = (dt * G(j) * g_tot(i) + n(j) * g_v(i-1,j)) / (G(j) *
dt + n(j));

    end
end

% --- STRESS SUMMATION (ZENER MODEL) --- %
stress_sec = zeros((length(t)),length(tau));

for k = 1:length(tau)

    stress_sec(:,k) = G(k) * (g_tot - g_v(:,k));
end

stress = G_inf * g_tot + sum(stress_sec,2);

% --- PLOT --- %

```

```

    if p == 1
        plot(g_tot, stress, 'Color', [0.5843 0.8157 0.9882]);
        grid on
        hold on

    elseif p == 0
        % No plot!
    end

end

```

```

% FUNCTION FOR PRONY SERIES FOR ABAQUS
function [g_i, tau_i] = Create_Prony(G_inf, G, tau)
%
%
%   INPUT: G_inf = Long-term Shear mod.          [Pa]
%           G     = Shear mod. for each branch    [Pa]
%           tau    = Relaxation time              [s]
%
%   OUTPUT: g_i   = Normalized prony series        [-]
%           tau_i  = Relaxation time              [s]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% --- TOTAL SHEAR MOD. --- %
G_tot = G_inf + sum(G);

% --- CALCULATING g_i --- %
g_i = zeros(length(G),1);

    for i = 1:length(G)

        g_i(i) = G(i) / G_tot;
    end

tau_i = tau';
end

```

```

% FUNCTION FOR CALCULATING INSTANT E-MODULUS FOR ABAQUS
function E = E_mod_inst(G_inf, G, v)
%
%
%   INPUT: G_inf = Long-term Shear mod.          [Pa]
%           G     = Shear mod. for each branch    [Pa]
%           v     = Poisson's ratio              [-]
%
%   OUTPUT: E     = Young's modulus              [Pa]
%
%

```

```
% Written by Axel Isberg & Axel Jankavs
```

```
%-----
```

```
    E = 2 * (G_inf + sum(G)) * (1 + v);
```

```
end
```

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VISCOELASTIC MODEL BASED ON EXPERIMENTAL DATA

Written by Axel Isberg & Axel Jankavs

Publication date: 12-06-2026

```
clear all
close all
clc
format long

% NOTE!
% ---- DSR-DATA (Start by running this section) ---- %
run('DSR_DATA_50_70.m')
%
%
%
```

Material parameters

```
% NOTE!
% ---- SELECT TEMPERATURE FOR DSR DATA ---- %
% i = 1 -> T = -30°C PP08
% i = 2 -> T = -20°C PP08
% i = 3 -> T = -10°C PP08
% i = 4 -> T = 0°C PP08
% i = 5 -> T = 10°C PP08
% i = 6 -> T = 20°C PP08
% i = 7 -> T = 30°C PP08
% i = 8 -> T = 30°C PP25
% i = 9 -> T = 40°C PP25
% i = 10 -> T = 50°C PP25
% i = 11 -> T = 60°C PP25
% i = 12 -> T = 70°C PP25
% i = 13 -> T = 80°C PP25

i = 5;
```

```

% ---- SETUP FOR SYSTEM ---- %
% Number of Maxwell-branches
N = 10;

% Storage & Loss moduls
G_si = G_s(:,i);
G_li = G_l(:,i);

% Tau
tau = logspace(log10(1/(10*max(w))), log10(100/(min(w))), N).';

% Initial guess for G_i
Gi = ones(N, 1) * max(G_si);

% Initial guess of long-term modulus
G_inf = 0;

% Vector with all initial guesses
x0 = [Gi; G_inf];

% ---- OPTIMIZATION REGIME ---- %
% Optimization of parameters
options = optimoptions('lsqnonlin', 'MaxFunctionEvaluations', 1e6,
'MaxIterations', 5e4);

% Bounderies for optimazation
lb = zeros(N+1,1); % Gi and G_inf > 0
ub = []; % No upper boundary limit

x = lsqnonlin(@(x) Prony_error(x, w, G_si, G_li, tau), x0, lb, ub, options);

% Extract solutions
Gi = x(1:N); % Prony-coefficients
G_inf = x(N+1); % Glass-modulus

% ---- RESULTING MODULUS ---- %
[G_s_calc, G_l_calc] = DSR_calc(w, Gi, tau, G_inf);

% ---- CREATION OF PRONY SERIES ---- %
[g_i, tau_i] = Create_Prony(G_inf, Gi, tau);

% ---- PLOTS & PRONY-RESULTS ---- %

```

```

figure(1)
semilogx(w, G_si/1e3, 'bo', w, G_s_calc/1e3, 'b-');
hold on
semilogx(w, G_li/1e3, 'ro', w, G_l_calc/1e3, 'r-');
grid on

xlabel('$\omega$, [s-1]', 'Interpreter','latex')
ylabel('$G'(\omega)\ \&\ G'(\omega)$, [\mathrm{kPa}]]$',
'Interpreter','latex')

lgd = legend( ...
    "$G'$ Measured", ...
    "$G'$ Predicted", ...
    "$G'' Measured", ...
    "$G'' Predicted", ...
    'Location','northwest', ...
    'Interpreter','latex');

ax = gca;
ax.XScale = 'log';
ax.XTick = w;
ax.XTickLabel = compose('%.3g$', w);
ax.TickLabelInterpreter = 'latex';
ax.XMinorTick = 'off';

xlim([min(w) max(w)]);

ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;
ax.LineWidth = 0.6;
ax.FontSize = 12;

lgd.FontSize = 12;
lgd.ItemTokenSize = [30, 10];
lgd.Box = 'on';

set(gcf, 'Position', [100 100 1000 480])
hold off

disp('Prony series:')
disp('g_i [-]')
disp(g_i)
disp('tau_i [s]')
disp(tau_i')

% ----- E-MODULUS FOR ABAQUS ----- %
v = 0.495;
E = E_mod_inst(Gi, G_inf, v);

disp('Instant E-modulus [Pa]:')

```

```
disp(E)
```

```
% ---- SHEAR MODS. ---- %
disp('Gi [Pa]:')
disp(Gi)
disp('G_inf [Pa]:')
disp(G_inf)

Gi = Gi';
tau = tau';
%
%
%
%
```

WLF - Constants

```
% Reference temperature
T_ref = T(1,i);

% ---- INITIAL GUESSES ---- %
C1 = 11.51;
C2 = 108.5;
alpha = 11.9;
beta = -0.97;
gamma = 0.34;
delta = -2.9;

x0 = [C1; C2; alpha; beta; gamma; delta];

% ---- SETTINGS & BOUNDARIES ---- %
lb = [10 100 -inf -inf -5 -Inf];
ub = [50 300 inf inf 5 inf];

options = optimoptions('lsqnonlin', ...
    'Algorithm','trust-region-reflective', ...
    'Display','iter', ...
    'TypicalX', x0, ...
    'StepTolerance',1e-8, ...
    'FunctionTolerance',1e-8, ...
    'MaxIterations',2000);

x = lsqnonlin(@(x) WLF_cost(x, T, w, G_comp, T_ref), x0, lb, ub, options);

% ---- RESULTS ---- %
fprintf('\nOptimized WLF parameters:\n');
fprintf('C1 = %.2f\n', x(1));
```

```

fprintf('C2      = %.2f\n', x(2));
fprintf('α       = %.2f\n', x(3));
fprintf('β       = %.2f\n', x(4));
fprintf('γ       = %.2f\n', x(5));
fprintf('δ       = %.2f\n', x(6));

C1      = x(1);
C2      = x(2);
alpha   = x(3);
beta    = x(4);
gamma   = x(5);
delta   = x(6);

% ---- PLOT ---- %
% Extract data from optimization
[log_aT, log_wr, G_comp_mes, G_comp_calc] = WLF_plot(T, w, G_comp, T_ref,
C1, C2, alpha, beta, gamma, delta);

aT = 10.^(log_aT);

% Convert matrices to vectors
log_wr_vec      = log_wr(:);
G_mes_log_vec   = G_comp_mes(:);
G_calc_log_vec  = G_comp_calc(:);
aT_vec          = aT(:);

% Plot: Measured data
figure(4)
plot(log_wr_vec, G_mes_log_vec, 'ko', 'Markersize', 5,
'MarkerFaceColor','w', 'DisplayName','G^* Measured');
hold on

% Plot: Calculated data
[log_wr_sorted, idx] = sort(log_wr_vec);
G_calc_sorted = G_calc_log_vec(idx);

p = polyfit(log_wr_sorted, G_calc_sorted, 3);
G_trend = polyval(p, log_wr_sorted);

plot(log_wr_sorted, G_trend, 'r-', 'DisplayName','G^* Predicted');

xlabel('log_{10}(\omega_r) [1/s]', 'FontSize', 14);
ylabel('log_{10}(G^*) [Pa]', 'FontSize', 14);
legend('Location','northwest', 'FontSize', 14);
grid on

ax = gca;
ax.XMinorTick = 'off';

ax.XLabel.FontSize = 16;

```

```

ax.YLabel.FontSize = 16;
ax.FontSize = 12;

lgd.FontSize = 12;
lgd.ItemTokenSize = [30, 10];
lgd.Box = 'on';

set(gcf, 'Position', [100 100 1000 480])
hold off

%
%
%
%
```

Modified Maxwell model

```

% ---- Sample dimensions ---- %
R = 0.004;

% ---- SELECT ANGULAR FREQUENCY FOR DSR DATA ---- %
% j = 1 -> w = 0.1      [/s]
% j = 2 -> w = 0.215    [/s]
% j = 3 -> w = 0.464    [/s]
% j = 4 -> w = 1        [/s]
% j = 5 -> w = 2.15     [/s]
% j = 6 -> w = 4.64     [/s]
% j = 7 -> w = 10       [/s]
% j = 8 -> w = 21.5     [/s]
% j = 9 -> w = 46.4     [/s]
% j = 10 -> w = 100     [/s]
j = 5;

% Prescribed inputs
w_int = w(j);
g_DSR = g(j,i);
stress_DSR = stress(j,i);

% Number of cycles
num_cycles = 5;

% DSR-Simulation
[M_all, M_analytic, t, delta_deg, stress_max, stress_store1, g_store1] =
Modified_Maxwell(R, G_inf, Gi, tau, w_int, g_DSR, stress_DSR, num_cycles, 0);

% --- RESULTS --- %
% Torsion
figure(5)
plot(t, M_analytic/1e3, 'rx', 'LineWidth', 1.3);
hold on
```

```

plot(t, M_all/1e3, 'b-', 'LineWidth', 1.3);
xlabel('Tid [s]');
ylabel('Moment [kNm]')
hold off

% Shear stress
figure(6)
plot(g_store1(N,:), stress_store1(N,:)/1e3, 'b-');
xlabel('Shear strain', 'FontSize', 14);
ylabel('Shear stress [kPa]', 'FontSize', 14);
title('Shear stress-strain curve', 'FontSize', 18, 'FontWeight', 'bold');
legend('Shear response', 'Location', 'northwest', 'FontSize', 14);
grid on

%
%
%
%
%
%

```

Functions

```

% RESIDUAL FUNCTION FOR MODEL CALIBRATION FROM DSR
function error = Prony_error(x, w, G_s, G_l, tau)
%
% INPUT: x      = Calibrated parameters      [-]
%          w      = Angular frequency          [1/s]
%          G_s     = Measured Storage modulus  [Pa]
%          G_l     = Measured Loss modulus     [Pa]
%          tau     = Relaxation time           [s]
%
% OUTPUT: error  = Residual                   [-]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% Unpack parameters
N = length(tau);
G = x(1:N);
G_inf = x(N+1);

% Conversion to storage & loss moduli
[G_s_calc, G_l_calc] = DSR_calc(w, G, tau, G_inf);

% Residual
error = [G_s - G_s_calc; G_l - G_l_calc];

end

```

```

% FUNCTION CONVERTING PRONY PARAMETERS TO STORAGE & LOSS MODULI
function [G_s_calc, G_l_calc] = DSR_calc(w, G, tau, G_inf)
%
%   INPUT: G_inf  = Long-term Shear mod.          [Pa]
%           G      = Shear mod. for each branch    [Pa]
%           tau    = Relaxation time                [s]
%           w      = Angular frequency             [1/s]
%
%   OUTPUT: G_s_calc = Calculated storage modulus [Pa]
%           G_l_calc = Calculated loss modulus    [Pa]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% Reduced angular frequency
wt = w .* tau';

% Calculated storage and loss moduli
G_s_calc = G_inf + sum((G'.*(wt.^2)) ./ (1 + wt.^2), 2);
G_l_calc = sum((G'.*wt) ./ (1 + wt.^2), 2);

end

% FUNCTION FOR PRONY SERIES
function [g_i, tau_i] = Create_Prony(G_inf, G, tau)
%
%
%   INPUT: G_inf  = Long-term Shear mod.          [Pa]
%           G      = Shear mod. for each branch    [Pa]
%           tau    = Relaxation time                [s]
%
%   OUTPUT: g_i    = Normalized prony series        [-]
%           tau_i   = Relaxation time                [s]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% --- TOTAL SHEAR MOD. --- %
G_tot = G_inf + sum(G);

% --- CALCULATING g_i --- %
g_i = zeros(length(G),1);

    for i = 1:length(G)

        g_i(i) = G(i) / G_tot;
    end

tau_i = tau';

```

end

% FUNCTION FOR CALCULATING INSTANT E-MODULUS FOR ABAQUS

function E = E_mod_inst(G_inf, G, v)

%

%

% INPUT: G_inf = Long-term Shear mod. [Pa]

% G = Shear mod. for each branch [Pa]

% v = Poisson's ratio [-]

%

% OUTPUT: E = Young's modulus [Pa]

%

%

% Written by Axel Isberg & Axel Jankavs

%-----

E = 2 * (G_inf + sum(G)) * (1 + v);

end

% FUNCTION FOR TTS-CALIBRATION

function res = WLF_cost(x0, T, w, G_comp, T_ref)

%

% INPUT: x = Calibrated parameters [-]

% T = Temperature [C]

% w = Angular frequency [1/s]

% G_comp = Measured Complex shear modulus [Pa]

% T_ref = Reference temperature [C]

%

% OUTPUT: res = Residual [-]

%

%

% Written by Axel Isberg & Axel Jankavs

%-----

% Parameters to be calibrated

C1 = x0(1);

C2 = x0(2);

alpha = x0(3);

beta = x0(4);

gamma = x0(5);

delta = x0(6);

% Measured data

log_G_comp_mes = log10(G_comp);

log_w = log10(w);

% Matrix formatting

```

[ row, col ] = size(T);
log_G_comp_calc = zeros(size(T));

% Pre. residual
res = [];

% WLF - Equations
for h = 1:col
    for i = 1:row

        log_aT = (-C1 * (T(i,h) - T_ref)) / ((T(i,h) - T_ref) + C2);
        log_wr = log_w(i) + log_aT;

        log_G_comp_calc(i,h) = delta + (alpha / (1 + exp(beta + gamma *
(log_wr)))));

    end
end

% Sum of square errors
rk = log_G_comp_mes - log_G_comp_calc;
res = [res; rk];

end

% FUNCTION FOR TTS IMPLEMENTATION
function [log_aT, log_wr, G_comp_mes, G_comp_calc] = WLF_plot(T, w, G_comp,
T_ref, C1, C2, alpha, beta, gamma, delta)
%
% INPUT: T          = Temperature          [C]
%          w         = Angular frequency    [1/s]
%          G_comp     = Measured Complex shear modulus [Pa]
%          T_ref      = Reference temperature [C]
%          C1         = WLF constant        [-]
%          C2         = WLF constant        [C]
%          alpha      = Constant for sigmoidal model [-]
%          beta       = Constant for sigmoidal model [-]
%          gamma      = Constant for sigmoidal model [-]
%          delta      = Constant for sigmoidal model [-]
%
% OUTPUT: log_aT      = Shift factor (log)          [-]
%          log_wr      = Reduced angular frequency (log) [1/s]
%          G_comp_mes  = Measured Complex shear modulus [Pa]
%          G_comp_calc = Calculated Complex shear modulus [Pa]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% Measured data

```

```

G_comp_mes = log10(G_comp);
log_w = log10(w);

% Matrix formatting
[row, col] = size(T);
log_wr = zeros(row,col);
log_aT = zeros(row,col);
G_comp_calc = zeros(row,col);

% WLF - Equations
for h = 1:col
    for i = 1:row

        log_aT(i,h) = (-C1 * (T(i,h) - T_ref)) / ((T(i,h) - T_ref) + C2);
        log_wr(i,h) = log_w(i) + log_aT(i,h);

        log_G_comp = delta + (alpha / (1 + exp(beta + gamma * (log_wr(i,h)))));
        G_comp_calc(i,h) = log_G_comp;

    end
end

end

% FUNCTION FOR MODIFIED MAXWELL IMPLEMENTATION (DSR)
function [M_all, M_analytic, t, delta_deg, stress_max, stress_store,
g_store] = Modified_Maxwell(R, G_inf, Gi, tau, w, g_DSR, stress_DSR,
num_cycles, p)
%
%
% Function recreating a DSR-test (10 cyclic loadings) based on a Generalized
Maxwell model
%
%
% INPUT:      R      = Radius of sample                      [m]
%             G_inf   = Long-term shear modulus              [Pa]
%             Gi      = Sher modulus for Maxwell elements    [Pa]
%             tau     = Relaxation time                      [s]
%             w       = Angular frequency                   [1/s]
%             g_DSR   = Shear strain from DSR test           [-]
%             stress_DSR = Shear stress from DSR-test        [Pa]
%             num_cycles = Number of test cycles             [-]
%             p       = Sanity check (if p == 1), no check (P == 0)
%
%
% OUTPUT:     M_all      = All torque moments for each time-step      [Nm]
%             M_analytic = Analytical torque moments for each time-step [Nm]
%             t          = Total time of test                        [s]
%             delta_deg   = Phase angle                              [deg]
%             stress_max  = Maximal calculated shear stress          [Pa]
%

```

```

%-----
% ---- DATA PREPERATION ---- %
eta = Gi .* tau;
nBranches = numel(Gi);

% ---- TIME DESCRETIZATION ---- %
t0 = 0;
tf = num_cycles * (2*pi/w);
dt = max(0.001, 0.05*min(tau));    %Bytt från min
t = t0:dt:tf;

% ---- GAUSSIAN POINTS [-1,1] ---- %
N = 10;

xi = [ ...
    -0.9739065285171717
    -0.8650633666889845
    -0.6794095682990244
    -0.4333953941292472
    -0.1488743389816312
     0.1488743389816312
     0.4333953941292472
     0.6794095682990244
     0.8650633666889845
     0.9739065285171717];

wi = [ ...
    0.0666713443086881
    0.1494513491505806
    0.2190863625159820
    0.2692667193099963
    0.2955242247147529
    0.2955242247147529
    0.2692667193099963
    0.2190863625159820
    0.1494513491505806
    0.0666713443086881];

% Mapping [0, R]
r_pts = 0.5 * R * (xi + 1);
w_pts = 0.5 * R * wi;

% --- Preallocation ---
g_v = zeros(nBranches, N);
M_all = zeros(size(t));

g_store = zeros(N, length(t));
stress_store = zeros(N, length(t));

```

```

% ----- BACKWARD EULER LOOP ----- %
for n = 1:length(t)

    M = 0;

    for q = 1:N
        r_q = r_pts(q);
        w_q = w_pts(q);

        % Strain for each Gaussian point
        g_q = g_DSR * (r_q / R) * sin(w * t(n));

        % Update Euler step
        for j = 1:nBranches
            g_v(j,q) = (dt * Gi(j) * g_q + eta(j) * g_v(j,q)) / (Gi(j) * dt
+ eta(j));
        end

        % Shear stress for each Gaussian point
        stress_q = G_inf * g_q + sum(Gi .* (g_q - g_v(:,q)'));

        % Torque moment
        M = M + 2*pi * stress_q * r_q^2 * w_q;

        g_store(q, n) = g_q;
        stress_store(q, n) = stress_q;
    end

    M_all(n) = M;
end

% ----- RESULTS ----- %

% Calculated torque vs test-data
stress_analytic = stress_DSR * sin(w*t);
M_analytic = (stress_analytic * pi * R^3)/2;

% Phase angle
Tper = 2*pi/w;
t_start = t(end) - Tper;           % cut to last cycle
idx = t >= t_start;

s = sin(w*t(idx));
c = cos(w*t(idx));
Mall = M_all(idx);

A = sum(Mall .* s);
B = sum(Mall .* c);

```

```

delta_rad = atan2(B, A);
delta_deg = rad2deg(delta_rad);

% Wrap 0..180°
delta_deg = mod(delta_deg, 360);
if delta_deg > 180
    delta_deg = 360 - delta_deg;
end

disp(' ')
disp('SHEAR RESPONSE NO WLF:')
fprintf("Phase angle  $\delta$  = %.2f° \n", delta_deg);

stress_max = max(stress_store(N,:));
fprintf("Maximum shear stress  $\tau$  = %.2f kPa \n", stress_max/1e3)

error = ((max(stress_store(N,:))-stress_DSR)/stress_DSR)*1e2;
fprintf("Error (Test vs model) = %.2f %% \n", error)
disp(' ')

% ----- COMPARISON ----- %
if p == 1
    figure(1)
    plot(t, M_analytic/1e3, 'rx', 'LineWidth', 1.3);
    hold on
    plot(t, M_all/1e3, 'b-', 'LineWidth', 1.3);
    xlabel('Tid [s]');
    ylabel('Moment [kNm]')
    hold off

elseif p == 0

end

end

```

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VISCOELASTIC MODEL BASED ON EXPERIMENTAL DATA (MASTERCURVES)

Script creating TTS-Mastercurves from DSR data

Written by Axel Isberg & Axel Jankavs

Publication date: 12-06-2026

```
-----

clc
clear all
close all

% NOTE!
% ----- DSR-DATA (Start by running this section) ----- %
run('DSR_DATA_50_70.m')

% NOTE!
% ----- SELECT TEMPERATURE FOR DSR DATA ----- %
% i = 1 -> T = -30°C PP08
% i = 2 -> T = -20°C PP08
% i = 3 -> T = -10°C PP08
% i = 4 -> T = 0°C PP08
% i = 5 -> T = 10°C PP08
% i = 6 -> T = 20°C PP08
% i = 7 -> T = 30°C PP08
% i = 8 -> T = 30°C PP25
% i = 9 -> T = 40°C PP25
% i = 10 -> T = 50°C PP25
% i = 11 -> T = 60°C PP25
% i = 12 -> T = 70°C PP25
% i = 13 -> T = 80°C PP25

i = 6;

% Reference temperature
T_ref = T(1,i);

% ----- INITIAL GUESSES ----- %
C1 = 11.51;
C2 = 108.5;
alpha = 11.9;
```

```

beta = -0.97;
gamma = 0.34;
delta = -2.9;

x0 = [C1; C2; alpha; beta; gamma; delta];

% ----- SETTINGS & BOUNDERIES ----- %
lb = [10 100 -inf -inf -5 -Inf];
ub = [50 300 inf inf 5 inf];

options = optimoptions('lsqnonlin', ...
    'Algorithm','trust-region-reflective', ...
    'Display','iter', ...
    'TypicalX', x0, ...
    'StepTolerance',1e-8, ...
    'FunctionTolerance',1e-8, ...
    'MaxIterations',2000);

x = lsqnonlin(@(x) WLF_cost(x, T, w, G_comp, T_ref), x0, lb, ub, options);

% ----- RESULTS ----- %
fprintf('\nOptimized WLF parameters:\n');
fprintf('C1      = %.2f\n', x(1));
fprintf('C2      = %.2f\n', x(2));
fprintf('α       = %.2f\n', x(3));
fprintf('β       = %.2f\n', x(4));
fprintf('γ       = %.2f\n', x(5));
fprintf('δ       = %.2f\n', x(6));

C1      = x(1);
C2      = x(2);
alpha   = x(3);
beta    = x(4);
gamma   = x(5);
delta   = x(6);

[log_aT, log_wr, G_comp_mes, G_comp_calc] = WLF_plot(T, w, G_comp, T_ref,
C1, C2, alpha, beta, gamma, delta);

% Storage modulus
alpha = 11.9;
beta = -0.97;
gamma = 0.34;
delta = -2.9;

x0 = [alpha; beta; gamma; delta];

% ----- SETTINGS & BOUNDERIES ----- %

```

```

lb = [ -inf -inf -5 -Inf];
ub = [  inf  inf  5  inf];

options = optimoptions('lsqnonlin', ...
    'Algorithm','trust-region-reflective', ...
    'Display','iter', ...
    'TypicalX', x0, ...
    'StepTolerance',1e-8, ...
    'FunctionTolerance',1e-8, ...
    'MaxIterations',2000);

x = lsqnonlin(@(x) WLF_cost_s_1(x, C1, C2, T, w, G_s, T_ref), x0, lb, ub,
options);

% ---- RESULTS ---- %
fprintf('\nOptimized WLF parameters:\n');
fprintf('α      = %.2f\n', x(1));
fprintf('β      = %.2f\n', x(2));
fprintf('γ      = %.2f\n', x(3));
fprintf('δ      = %.2f\n', x(4));

alpha = x(1);
beta  = x(2);
gamma = x(3);
delta = x(4);

[~, ~, G_s_mes, G_s_calc] = WLF_plot(T, w, G_s, T_ref, C1, C2, alpha, beta,
gamma, delta);

% Loss modulus
alpha = 11.9;
beta  = -0.97;
gamma = 0.34;
delta = -2.9;

x0 = [alpha; beta; gamma; delta];

% ---- SETTINGS & BOUNDERIES ---- %
lb = [ -inf -inf -5 -Inf];
ub = [  inf  inf  5  inf];

options = optimoptions('lsqnonlin', ...
    'Algorithm','trust-region-reflective', ...
    'Display','iter', ...
    'TypicalX', x0, ...
    'StepTolerance',1e-8, ...
    'FunctionTolerance',1e-8, ...
    'MaxIterations',2000);

```

```

x = lsqnonlin(@(x) WLF_cost_s_l(x, C1, C2, T, w, G_l, T_ref), x0, lb, ub,
options);

% ---- RESULTS ---- %
fprintf('\nOptimized WLF parameters:\n');
fprintf('α      = %.2f\n', x(1));
fprintf('β      = %.2f\n', x(2));
fprintf('γ      = %.2f\n', x(3));
fprintf('δ      = %.2f\n', x(4));

alpha = x(1);
beta  = x(2);
gamma = x(3);
delta = x(4);

[~, ~, G_l_mes, G_l_calc] = WLF_plot(T, w, G_l, T_ref, C1, C2, alpha, beta,
gamma, delta);

% Master curves
aT = 10.^(log_aT);

% Convert matrices to vectors
log_wr_vec      = log_wr(:);
G_mes_log_vec_comp = G_comp_mes(:);
G_mes_log_vec_s  = G_s_mes(:);
G_mes_log_vec_l  = G_l_mes(:);

G_calc_log_vec_comp = G_comp_calc(:);
G_calc_log_vec_s    = G_s_calc(:);
G_calc_log_vec_l    = G_l_calc(:);
aT_vec = aT(:);

% Plot: Measured data
figure(4)
plot(log_wr_vec, G_mes_log_vec_comp, 'ko', 'Markersize', 5,
'MarkerFaceColor','w', 'DisplayName','G^* Measured');
hold on

[log_wr_sorted, idx] = sort(log_wr_vec);
G_calc_sorted_comp = G_calc_log_vec_comp(idx);

p = polyfit(log_wr_sorted, G_calc_sorted_comp, 3);
G_trend_comp = polyval(p, log_wr_sorted);

% Plot trend
plot(log_wr_sorted, G_trend_comp, 'r-', 'DisplayName','G^* Predicted');

```

```

xlabel('log_{10}(\omega_r) [1/s]', 'FontSize', 14);
ylabel('log_{10}(G^*) [Pa]', 'FontSize', 14);
legend('Location','northwest', 'FontSize', 14);
grid on
hold off

% Storage modulus
figure(1)
plot(log_wr_vec, G_mes_log_vec_s, 'ko', 'Markersize', 5,
'MarkerFaceColor','w', 'DisplayName', '$G^{\prime}$');
hold on

G_calc_sorted_s = G_calc_log_vec_s(idx);

p = polyfit(log_wr_sorted, G_calc_sorted_s, 3);
G_trend_s = polyval(p, log_wr_sorted);

% Plot trend
plot(log_wr_sorted, G_trend_s, 'r-', ...
'LineWidth', 1.2, ...
'DisplayName', '$G^{\prime}\backslash, \mathrm{Trend}$');

xlabel('$\log_{10}(\omega_r)\backslash, [\mathrm{s}^{-1}]$', ...
'Interpreter','latex', ...
'FontSize',14);

ylabel('$\log_{10}(G^{\prime})\backslash, [\mathrm{Pa}]$', ...
'Interpreter','latex', ...
'FontSize',14);

lgd = legend('Location','northwest', 'Interpreter','latex');
lgd.FontSize = 14;
lgd.Box = 'on';
hold off
ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 0.8;
grid on

set(gcf, 'Position', [100 100 500 480])

% Loss modulus
figure(2)
plot(log_wr_vec, G_mes_log_vec_l, 'ko', 'Markersize', 5,
'MarkerFaceColor','w', 'DisplayName', '$G^{\prime\prime}$');
hold on

G_calc_sorted_l = G_calc_log_vec_l(idx);

p = polyfit(log_wr_sorted, G_calc_sorted_l, 3);

```

```

G_trend_l = polyval(p, log_wr_sorted);

% Plot trend
plot(log_wr_sorted, G_trend_l, 'r-', ...
     'LineWidth', 1.2, ...
     'DisplayName', '$G^{\prime\prime}\backslash,\mathrm{Trend}$');

xlabel('$\log_{10}(\omega_r)\backslash,[\mathrm{s}^{-1}]$', ...
     'Interpreter','latex', ...
     'FontSize', 14);

ylabel('$\log_{10}(G^{\prime\prime}\backslash,[\mathrm{Pa}]$', ...
     'Interpreter','latex', ...
     'FontSize', 14);

ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 0.8;

lgd = legend('Location','northwest', 'Interpreter','latex');
lgd.FontSize = 14;
lgd.Box = 'on';

grid on
hold off

set(gcf, 'Position', [100 100 500 480])

% Phase angle
delta_rad = zeros(size(G_trend_l));
delta_rad_mes = zeros(size(G_trend_l));

G_mes_sorted_s = G_mes_log_vec_s(idx);
G_mes_sorted_l = G_mes_log_vec_l(idx);

for j = 1:length(delta_rad)

    delta_rad_mes(j) = atan(10^(G_mes_sorted_l(j)) ./
10^(G_mes_sorted_s(j)));
end

delta_deg_mes = rad2deg(delta_rad_mes);

p = polyfit(log_wr_sorted, delta_deg_mes, 3);
delta_deg_trend = polyval(p, log_wr_sorted);

figure(3)

plot(log_wr_sorted, delta_deg_mes, 'ko', ...
     'MarkerSize',5, ...

```

```

        'MarkerFaceColor','w', ...
        'DisplayName','$\delta$');
hold on

plot(log_wr_sorted, delta_deg_trend, 'r-', ...
     'LineWidth',1.2, ...
     'DisplayName','$\delta \ \mathrm{Trend}$');

xlabel('$\log_{10}(\omega_r)\,,[\mathrm{s}^{-1}]$', ...
      'Interpreter','latex', ...
      'FontSize',14);

ylabel('$\delta\,,[^{\circ}]$', ...
      'Interpreter','latex', ...
      'FontSize',14);

ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 0.8;

lgd = legend('Location','northeast','Interpreter','latex');
lgd.FontSize = 14;
lgd.Box = 'on';

grid on
hold off

set(gcf, 'Position', [100 100 500 480])

% Complex modulus
figure(4)

plot(log_wr_vec, G_mes_log_vec_comp, 'ko', ...
     'MarkerSize',5, ...
     'MarkerFaceColor','w', ...
     'DisplayName','$G^{*}\, \mathrm{}$');
hold on

[log_wr_sorted, idx] = sort(log_wr_vec);
G_calc_sorted_comp = G_calc_log_vec_comp(idx);

p = polyfit(log_wr_sorted, G_calc_sorted_comp, 3);
G_trend_comp = polyval(p, log_wr_sorted);

plot(log_wr_sorted, G_trend_comp, 'r-', ...
     'LineWidth',1.2, ...
     'DisplayName','$G^{*}\, \mathrm{Trend}$');

```

```

xlabel('$\log_{10}(\omega_r)\backslash, [\mathrm{s}^{-1}]$', ...
      'Interpreter','latex', ...
      'FontSize',14);

ylabel('$\log_{10}(G^{*})\backslash, [\mathrm{Pa}]$', ...
      'Interpreter','latex', ...
      'FontSize',14);

ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 0.8;

lgd = legend('Location','northwest', ...
            'Interpreter','latex');
lgd.FontSize = 14;
lgd.Box = 'on';

grid on
hold off

set(gcf, 'Position', [100 100 500 480])
%
%
%
%
```

Functions

```

% FUNCTION FOR TTS-CALIBRATION (Storage & Loss moduli)
function res = WLF_cost_s_l(x0, C1, C2, T, w, G_comp, T_ref)
%
%   INPUT: x      = Calibrated parameters          [-]
%           C1     = WLF constant                   [-]
%           C2     = WLF constant                   [C]
%           T      = Temperature                    [C]
%           w      = Angular frequency              [1/s]
%           G_comp = Measured Complex shear modulus [Pa]
%           T_ref  = Reference temperature          [C]
%
%   OUTPUT: res    = Residual                      [-]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% Parameters to be calibrated
alpha = x0(1);
beta  = x0(2);
gamma = x0(3);
```

```

delta = x0(4);

% Measured data
log_G_comp_mes = log10(G_comp);
log_w = log10(w);

% Matrix formatting
[row, col] = size(T);
log_G_comp_calc = zeros(size(T));

% Pre. residual
res = [];

% WLF - Equations
for h = 1:col
    for i = 1:row

        log_aT = (-C1 * (T(i,h) - T_ref)) / ((T(i,h) - T_ref) + C2);
        log_wr = log_w(i) + log_aT;

        log_G_comp_calc(i,h) = delta + (alpha / (1 + exp(beta + gamma *
(log_wr)))));
    end
end

% Sum of square errors
rk = log_G_comp_mes - log_G_comp_calc;
res = [res; rk];
end

% FUNCTION FOR TTS-CALIBRATION
function res = WLF_cost(x0, T, w, G_comp, T_ref)
%
%   INPUT: x      = Calibrated parameters      [-]
%           T      = Temperature                [C]
%           w      = Angular frequency          [1/s]
%           G_comp = Measured Complex shear modulus [Pa]
%           T_ref  = Reference temperature      [C]
%
%   OUTPUT: res   = Residual                    [-]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

```

```

% Parameters to be calibrated
C1    = x0(1);
C2    = x0(2);
alpha = x0(3);
beta  = x0(4);
gamma = x0(5);
delta = x0(6);

% Measured data
log_G_comp_mes = log10(G_comp);
log_w = log10(w);

% Matrix formatting
[row, col] = size(T);
log_G_comp_calc = zeros(size(T));

% Pre. residual
res = [];

% WLF - Equations
for h = 1:col
    for i = 1:row

        log_aT = (-C1 * (T(i,h) - T_ref)) / ((T(i,h) - T_ref) + C2);
        log_wr = log_w(i) + log_aT;

        log_G_comp_calc(i,h) = delta + (alpha / (1 + exp(beta + gamma *
(log_wr))));

    end
end

% Sum of square errors
rk = log_G_comp_mes - log_G_comp_calc;
res = [res; rk];
end

% FUNCTION FOR TTS IMPLEMENTATION
function [log_aT, log_wr, G_comp_mes, G_comp_calc] = WLF_plot(T, w, G_comp,
T_ref, C1, C2, alpha, beta, gamma, delta)
%
%     INPUT: T           = Temperature           [C]
%            w           = Angular frequency     [1/s]
%            G_comp      = Measured Complex shear modulus [Pa]
%            T_ref       = Reference temperature  [C]
%            C1          = WLF constant          [-]
%            C2          = WLF constant          [C]
%            alpha       = Constant for sigmoidal model [-]

```

```

%          beta    = Constant for sigmoidal model    [-]
%          gamma   = Constant for sigmoidal model    [-]
%          delta   = Constant for sigmoidal model    [-]
%
%  OUTPUT: log_aT    = Shift factor (log)            [-]
%          log_wr    = Reduced angular frequency (log) [1/s]
%          G_comp_mes = Measured Complex shear modulus [Pa]
%          G_comp_calc = Calculated Complex shear modulus [Pa]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% Measured data
G_comp_mes = log10(G_comp);
log_w = log10(w);

% Matrix formatting
[row, col] = size(T);
log_wr = zeros(row,col);
log_aT = zeros(row,col);
G_comp_calc = zeros(row,col);

% WLF - Equations
for h = 1:col
    for i = 1:row

        log_aT(i,h) = (-C1 * (T(i,h) - T_ref)) / ((T(i,h) - T_ref) + C2);
        log_wr(i,h) = log_w(i) + log_aT(i,h);

        log_G_comp = delta + (alpha / (1 + exp(beta + gamma * (log_wr(i,h)))));
        G_comp_calc(i,h) = log_G_comp;

    end
end

end

```

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CALIBRATION OF NONLINEAR VISCOSITY (CROSS-MODEL) FOR DASHPOT

Written by Axel Isberg & Axel Jankavš

Publication date: 12-06-2026

```
-----

clc
clear all
close all

% ---- EXTRACTED TEST-DATA ---- %
run('Approximerad_Corrected_Data.m');

figure(1)

% ---- DATA ---- %

% 1 /s
plot(x_1, y_1/1e3, 'x', ...
     'Color',[0.7 0.7 0.7], ...
     'MarkerEdgeColor','k', ...
     'MarkerSize',6, 'LineWidth',0.6, ...
     'DisplayName',' $1\,\mathrm{s}^{-1}$ ');
hold on

% 0.3 /s
plot(x_03, y_03/1e3, 's', ...
     'Color',[0.7 0.7 0.7], ...
     'MarkerEdgeColor','k', ...
     'MarkerFaceColor','k', ...
     'MarkerSize',3.5, 'LineWidth',0.6, ...
     'DisplayName',' $0.3\,\mathrm{s}^{-1}$ ');

% 0.1 /s
plot(x_01, y_01/1e3, 'o', ...
     'Color',[0.7 0.7 0.7], ...
     'MarkerEdgeColor','k', ...
     'MarkerSize',4, 'LineWidth',0.6, ...
     'DisplayName',' $0.1\,\mathrm{s}^{-1}$ ');

% 0.03 /s
```

```

plot(x_003, y_003/1e3, 'd', ...
     'Color',[0.7 0.7 0.7], ...
     'MarkerEdgeColor','k', ...
     'MarkerFaceColor','k', ...
     'MarkerSize',3, 'LineWidth',0.6, ...
     'DisplayName',' $0.03 \mathrm{s}^{-1}$ ');

% 0.01 /s
plot(x_001, y_001/1e3, 'v', ...
     'Color',[0.7 0.7 0.7], ...
     'MarkerEdgeColor','k', ...
     'MarkerSize',3, 'LineWidth',0.6, ...
     'DisplayName',' $0.01 \mathrm{s}^{-1}$ ');

grid on

xlabel('$\gamma \backslash, [-]$', 'Interpreter','latex', ...
      'Units','normalized', 'Position',[0.5 -0.07]);

ylabel('$\sigma \backslash, [\mathrm{kPa}]$', 'Interpreter','latex', ...
      'Units','normalized', 'Position',[-0.06 0.5]);

xlim([0 50])

% ----- FIGURE SETTINGS ----- %
ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;
ax.LineWidth = 0.6;
ax.FontSize = 12;

set(gcf, 'Position', [100 100 1000 480])

% ----- EXTRAPOLATION ----- %
g_end = 50;
g_0 = 15;
g = linspace(g_0, g_end);

% ----- 1/S ----- %
tau_inf = 60e3;
[~, idx] = min(abs(x_1 - g_0));
k = 0.12;
tau_0 = y_1(idx);
f = Help_function(tau_0, tau_inf, k, g);
plot(g, f/1e3, 'r-', 'DisplayName',' $\sigma(\gamma)$ ');
tau_inf_1 = tau_inf;

% ----- 0.3/S ----- %
tau_inf = 55e3;

```

```

[~, idx] = min(abs(x_03 - g_0));
k = 0.08;
tau_0 = y_03(idx);
f = Help_function(tau_0, tau_inf, k, g);
plot(g, f/1e3, 'r-', 'HandleVisibility','off')
tau_inf_03 = tau_inf;

% ----- 0.1/S ----- %
tau_inf = 52e3;
[~, idx] = min(abs(x_01 - g_0));
k = 0.07;
tau_0 = y_01(idx);
f = Help_function(tau_0, tau_inf, k, g);
plot(g, f/1e3, 'r-', 'HandleVisibility','off')
tau_inf_01 = tau_inf;

% ----- 0.03/S ----- %
tau_inf = 27e3;
[~, idx] = min(abs(x_003 - g_0));
k = 0.15;
tau_0 = y_003(idx);
f = Help_function(tau_0, tau_inf, k, g);
plot(g, f/1e3, 'r-', 'HandleVisibility','off')
tau_inf_003 = tau_inf;

% ----- 0.01/S ----- %
tau_inf = 18e3;
[~, idx] = min(abs(x_001 - g_0));
k = 0.18;
tau_0 = y_001(idx);
f = Help_function(tau_0, tau_inf, k, g);
plot(g, f/1e3, 'r-', 'HandleVisibility','off')
tau_inf_001 = tau_inf;

hold off

% ----- LEGEND ----- %
lgd = legend('Location','northeast', 'Interpreter','latex');
lgd.FontSize = 12;
lgd.Box = 'on';

% ----- CONSTRUCTION OF STRESS VS STRAIN RATE PLOT ----- %

% Extracted stress values from previous plot
tau_inf_all = [tau_inf_1 tau_inf_03 tau_inf_01 tau_inf_003 tau_inf_001];

% Corresponding strain rates
d_g = [1 0.3 0.1 0.03 0.01];

```

```

figure(2)
plot(d_g, tau_inf_all/1e3, 'b-', 'Marker', 'o', 'MarkerEdgeColor', 'k')
grid on

xlabel('d\gamma [1/s]', 'Units','normalized', 'Position', [0.5 -0.07]);
ylabel('\sigma_{inf} [kPa]', 'Units','normalized', 'Position', [-0.06 0.5]);

set(gca,'FontSize',12,'LineWidth',0.6)
set(get(gca,'XLabel'),'FontSize',16)
set(get(gca,'YLabel'),'FontSize',16)

set(gcf, 'Position', [100 100 1000 480])
hold off

% ----- CONSTRUCTION OF VISCOSITY VS STRAIN RATE PLOT ----- %

eta_inf_test = zeros(1, length(d_g));

for i = 1:length(d_g)

    eta_inf_test(i) = tau_inf_all(i) ./ d_g(i);

end

figure(3)

plot(d_g, eta_inf_test/1e3, 'b-o', 'MarkerEdgeColor','k', 'DisplayName','$\hat{\eta}_s$');

grid on

xlabel('$\dot{\gamma} \backslash, [\mathrm{s}^{-1}]$', 'Interpreter','latex',
'Units','normalized', 'Position', [0.5 -0.07]);
ylabel('$\hat{\eta}_s$ [kPa]', 'Interpreter', 'latex', 'Units','normalized',
'Position', [-0.06 0.5]);
legend('Interpreter','latex','Location','best')
%lgd.FontName = 'Times New Roman';

set(gca,'FontSize',12,'LineWidth',0.6)
set(get(gca,'XLabel'),'FontSize',16)
set(get(gca,'YLabel'),'FontSize',16)
set(gca,'TickLabelInterpreter','latex')

set(gcf, 'Position', [100 100 1000 480])
hold off

```

Calibration of Cross model

```
% Initial values
eta_inf = eta_inf_test(1);
eta_0 = eta_inf_test(end);
K = 100; % Time constant [s]
m = 1; % Cross exponent [-]

x0 = [eta_inf, eta_0, K, m];

% ----- SETTINGS & BOUNDARIES ----- %
lb = [0 0 0 0];
ub = [inf inf inf inf];

options = optimoptions('lsqnonlin', ...
    'Algorithm','trust-region-reflective', ...
    'Display','iter', ...
    'TypicalX', x0, ...
    'StepTolerance',1e-10, ...
    'FunctionTolerance',1e-10, ...
    'MaxIterations',2000);

x = lsqnonlin(@(x) eta_opti_new(x, eta_inf_test, d_g), x0, lb, ub, options);

eta_inf = x(1);
eta_0 = x(2);
K = x(3);
m = x(4);

% Eta_s with optimized parameters
eta_s_hat = eta_s_func(eta_0, eta_inf, K, m, d_g);

figure(4)

plot(d_g, eta_inf_test/1e3, 'b-o', ...
    'MarkerEdgeColor','k', ...
    'DisplayName',' $\hat{\eta}_s$   $\mathrm{Predicted}$ ');

hold on

plot(d_g, eta_s_hat/1e3, 'r-o', ...
    'MarkerEdgeColor','k', ...
    'DisplayName',' $\hat{\eta}_s$   $\mathrm{Cross}$   $\mathrm{model}$ ');
```

```

grid on

xlabel('$\dot{\gamma} \backslash, [\mathrm{s}^{-1}]$', ...
      'Interpreter','latex', ...
      'Units','normalized', ...
      'Position', [0.5 -0.07]);

ylabel('$\hat{\eta}_s \backslash, [\mathrm{kPa} \cdot \mathrm{s}]$', ...
      'Interpreter','latex', ...
      'Units','normalized', ...
      'Position', [-0.06 0.5]);

ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 0.6;

ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;

% ---- LEGEND (LaTeX) ---- %
lgd = legend('Location','northeast','Interpreter','latex');
lgd.FontSize = 12;
lgd.Box = 'on';

set(gcf, 'Position', [100 100 1000 480])

hold off

disp('eta_inf')
disp(eta_inf)

disp('eta_0')
disp(eta_0)

disp('K')
disp(K)

disp('m')
disp(m)
%
%
%
%
%
%
%
%
%
%
```

Functions

```
% HELP-FUNCTION FOR STEADY-STATE PREDICTION
function f = Help_function(tau_0, tau_inf, k, g)
%
% INPUT:  tau_0    = Initial shear stress
%         tau_inf  = Steady-state shear stress
%         k        = Constant
%         g        = Shear strain
%
%
% OUTPUT: f        = Function value
%
% Written by Axel Isberg & Axel Jankavs
% -----

    f = tau_0 + (tau_inf - tau_0) * (1 - exp(-k * (g - g(1))));

end

% RESIDUAL FOR CALIBRATION OF CROSS MODEL
function res = eta_opti_new(x, eta_inf_test, d_g)
%
% INPUT:  x          = Initial shear stress          [-]
%         eta_inf_test = Predicted Steady-state viscosity [Pas]
%         d_g        = Strain rate                    [1/s]
%
%
% OUTPUT: res        = Residual [-]
%
% Written by Axel Isberg & Axel Jankavs
% -----

    % Unpack parameters
    eta_inf = x(1);
    eta_0   = x(2);
    K       = x(3);
    m       = x(4);

    res = [];

    eta_s = zeros(size(d_g));

    for i = 1:length(d_g)

        eta_s(i) = eta_inf + ((eta_0 - eta_inf) ./ (1 + (K * d_g(i)).^m));
    end

    % Error
    rk = (eta_inf_test) - (eta_s);
    res = [res; rk];
```

end

```
% FUNCTION CALCULATING CROSS-MODEL VISCOSITY
function eta_s = eta_s_func(eta_0, eta_inf, K, m, d_g)
%
% INPUT:  eta_0    = Zero shear viscosity          [Pas]
%          eta_inf  = Infinite shear viscosity     [Pas]
%          K        = Cross-model constant         [-]
%          m        = Cross-model constant         [-]
%          d_g      = Strain rate                   [1/s]
%
%
% OUTPUT: eta_s    = Cross-model viscosity [Pas]
%
% Written by Axel Isberg & Axel Jankavs
% -----

    eta_s = zeros(1,length(d_g));

    for i = 1:length(d_g)

        eta_s(i) = eta_inf + ((eta_0 - eta_inf) ./ (1 + (K * d_g(i)).^m));
    end
end
```

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FINAL VISCOELASTIC MODEL

Written by Axel Isberg & Axel Jankavs

Publication date: 12-06-2026

```
-----  
  
clc  
clear all  
close all  
  
run('Cross_extrapolation_final.m');  
  
% Initial guesses  
G_inf = 2e5;  
Gi     = [2.6e3  2.3e4  1.28e6];  
tau    = [0.1    0.4    12];  
k_soft = 5;  
Smin   = 0.5;  
  
% Loading  
d_g     = [1 0.3 0.1 0.03 0.01]; % strain rate [1/s]  
  
% Parameter optimization  
x0 = [G_inf Gi tau k_soft Smin];  
nB = numel(Gi);  
  
stress_test = {y_1, y_03, y_01, y_003, y_001};  
g_test      = {x_1, x_03, x_01, x_003, x_001};  
  
% Bounderies and settings for optimization  
lb = [0 0 0 0 0 0 0 0 0];  
ub = [inf inf inf inf inf inf inf inf 200 1];  
  
options = optimoptions('lsqnonlin', ...  
    'Algorithm','trust-region-reflective', ...  
    'Display','iter', ...  
    'TypicalX', x0, ...  
    'StepTolerance',1e-8, ...
```

```

        'FunctionTolerance',1e-8, ...
        'MaxIterations',2000);

x = lsqnonlin(@(x) MW_damper_opti(x, eta_0, eta_inf, K, m, d_g, g_test,
stress_test), x0, lb, ub, options);

% Optimized input parameters
idx = 1;
G_inf = x(idx);

idx = idx + 1;
Gi = x(idx : idx + nB - 1);

idx = idx + nB;
tau = x(idx : idx + nB -1);

idx = idx + nB;
k_soft = x(idx);

idx = idx + 1;
Smin = x(idx);

% Display results
fprintf('\nOptimized Cross-model parameters:\n');

disp('eta_0 [Pas] =');
disp(eta_0/Smin);

disp('eta_inf [Pas] =');
disp(eta_inf/Smin);

disp('K =');
disp(K);

disp('m =');
disp(m);
disp('-----')

fprintf('\nOptimized Maxwell-model parameters:\n')

disp('G_inf [Pa] =');
disp(G_inf);

disp('Gi [Pa] =');
disp(Gi);

disp('tau [s] =');
disp(tau);

disp('k_soft [-] =');
disp(k_soft);

```

```
disp('Smin [-] =');
disp(Smin);
```

Result-plots with calibrated parameters

```
% Total strain in system
g_final = 50;

% Strain rates
d_g = [1 0.3 0.1 0.03 0.01];
%d_g = [1 0.01];

% Loop preperation
n = numel(d_g);

% Creating cells
g_tot      = cell(n,1);
stress_tot = cell(n,1);
stress_s   = cell(n,1);
stress_MW  = cell(n,1);
g_MW       = cell(n,1);
g_s        = cell(n,1);
eta_s      = cell(n,1);
S          = cell(n,1);

for k = 1:n
    [g_tot{k}, stress_tot{k}, stress_s{k}, stress_MW{k}, ~, g_MW{k}, g_s{k},
    eta_s{k}, S{k}] = MW_damper_func(G_inf, Gi, tau, eta_0, eta_inf, K, m,
    k_soft, Smin, d_g(k), g_final);
end

% Component strains vs total strain in system
figure(5)
tiledlayout(n,1)

for k = 1:n
    nexttile

    plot(g_tot{k}, g_MW{k}, 'r-', ...
        'LineWidth',1.0, ...
        'DisplayName', '$\gamma_{\mathrm{MW}}$');
    hold on

    plot(g_tot{k}, g_s{k}, 'b-', ...
        'LineWidth',1.0, ...
        'DisplayName', '$\gamma_{\mathrm{s}}$');
    hold on
```

```

plot(g_tot{k}, (g_MW{k} + g_s{k}), 'k--', ...
     'LineWidth',1.0, ...
     'DisplayName',' $\gamma$ ');

grid on

title(sprintf(' $\dot{\gamma} = %.3g\backslash\backslash\mathrm{s}^{-1}$ ',
d_g(k)), 'Interpreter', 'latex')

xlabel(' $\gamma \backslash[-]$ ', ...
      'Interpreter', 'latex')

ylabel(' $\gamma \backslash[-]$ ', ...
      'Interpreter', 'latex')

ax = gca;
ax.TickLabelInterpreter = 'latex';
ax.XAxis.TickLabelInterpreter = 'latex';

ax.YAxis.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 1;

ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;

ax.Toolbar.Visible = 'off';

lgd = legend('Interpreter','latex','Location','northwest');
lgd.FontSize = 14;
lgd.Box = 'on';

set(gcf, 'Position', [100 100 1000 1200])

end

%set(gcf,'Renderer','painters')
%exportgraphics(gcf,'figure5.pdf','ContentType','vector')

% Viscosity in damper vs total strain in system
figure(6)
tiledlayout(n,1)

for k = 1:n
    nexttile

```

```

% ----- PLOT ----- %

plot(g_tot{k}, eta_s{k}/1e3, 'r-', ...
     'LineWidth',1.0, ...
     'DisplayName',' $\eta_s$ ');

yline((eta_s_hat(k)/1e3)/Smin, 'k--', ...
     'LineWidth',1.0, ...
     'DisplayName',' $\hat{\eta}_s / S_{\min}$ ');

grid on

title(sprintf(' $\dot{\gamma} = %.3g \backslash \backslash \mathrm{s}^{-1}$ ', d_g(k)), ...
      'Interpreter','latex')

xlabel(' $\gamma \backslash [-]$ ', 'Interpreter','latex')

ylabel(' $\eta_s \backslash [\mathrm{kPa} \cdot \mathrm{s}]$ ', ...
      'Interpreter','latex')

ax = gca;
ax.YAxis.Exponent = 0;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 1.2;

ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;

ax.GridAlpha = 0.25;
ax.GridColor = [0.2 0.2 0.2];

ax.Toolbar.Visible = 'off';

lgd = legend('Interpreter','latex','Location','northeast');
lgd.FontSize = 14;
lgd.Box = 'on';

end

set(gcf, 'Position', [100 100 1000 1200])

%set(gcf,'Renderer','painters')
%exportgraphics(gcf,'figure5.pdf','ContentType','vector')

hold off

```

```

% Softening in damper vs total strain in system
figure(7)
tiledlayout(n,1)

for k = 1:n
    nexttile

    plot(g_tot{k}, S{k}, 'r-', ...
        'LineWidth', 1.0, ...
        'DisplayName', '$S(\gamma)$')
    hold on

    yline(Smin, 'k--', ...
        'LineWidth', 1.0, ...
        'DisplayName', '$S_{\min}$')

    grid on

    % ---- TITEL ---- %
    title(sprintf('$\dot{\gamma} = %.3g\backslash,\backslash\mathrm{s}^{-1}$', d_g(k)), ...
        'Interpreter', 'latex')

    % ---- AXLAR ---- %
    xlabel('$\gamma \backslash,[-]$', ...
        'Interpreter', 'latex')

    ylabel('$S \backslash,[-]$', ...
        'Interpreter', 'latex')

    ax = gca;
    ax.TickLabelInterpreter = 'latex';
    ax.FontSize = 12;
    ax.LineWidth = 1.2;

    ax.XLabel.FontSize = 16;
    ax.YLabel.FontSize = 16;

    ax.GridAlpha = 0.25;
    ax.GridColor = [0.2 0.2 0.2];

    ax.Toolbar.Visible = 'off';

    lgd = legend('Interpreter','latex','Location','northeast');
    lgd.FontSize = 14;
    lgd.Box = 'on';
end

set(gcf, 'Position', [100 100 1000 1200])

%set(gcf,'Renderer','painters')
%exportgraphics(gcf,'figure5.pdf','ContentType','vector')

```

```

hold off

% Stress vs strain for system
figure(8)
hold on

% Test data
plot(x_1, y_1/1e3, 'x', 'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k',
'MarkerSize',6, 'LineWidth',0.6, 'DisplayName',' $1\,\mathrm{s}^{-1}$ ');

plot(x_03, y_03/1e3, 's', 'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k',
'MarkerFaceColor','k', 'MarkerSize',3.5, 'LineWidth',0.6,
'DisplayName',' $0.3\,\mathrm{s}^{-1}$ ');

plot(x_01, y_01/1e3, 'o', 'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k',
'MarkerSize',4, 'LineWidth',0.6, 'DisplayName',' $0.1\,\mathrm{s}^{-1}$ ');

plot(x_003, y_003/1e3, 'd', 'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k',
'MarkerFaceColor','k', 'MarkerSize',3, 'LineWidth',0.6,
'DisplayName',' $0.03\,\mathrm{s}^{-1}$ ');

plot(x_001, y_001/1e3, 'v', 'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k',
'MarkerSize',3, 'LineWidth',0.6, 'DisplayName',' $0.01\,\mathrm{s}^{-1}$ ');

% Steady-state shear stress (Comparison)
x_ss_plot = linspace(0,g_final,50);
sigma_ss = zeros(size(d_g));
%eta_s_hat = [eta_s_hat(1) eta_s_hat(end)];

for i = 1:length(d_g)

    sigma_ss(i) = eta_s_hat(i) * d_g(i);

end

%sigma_ss_1 = linspace(sigma_ss(1), sigma_ss(1), 50);
%sigma_ss_03 = linspace(sigma_ss(2), sigma_ss(2), 50);
%sigma_ss_01 = linspace(sigma_ss(3), sigma_ss(3), 50);
%sigma_ss_003 = linspace(sigma_ss(4), sigma_ss(4), 50);
%sigma_ss_001 = linspace(sigma_ss(2), sigma_ss(2), 50);

%plot(x_ss_plot, sigma_ss_1/1e3, 'k--', 'LineWidth', 1.2, 'DisplayName',' $\mathrm{Steady}\,\mathrm{s}\,\mathrm{state}$ ');
%plot(x_ss_plot, sigma_ss_03/1e3, 's',
'Color',[0.7 0.7 0.7], 'MarkerEdgeColor','k',
'MarkerFaceColor','k', 'MarkerSize',3, 'LineWidth',0.6,
'DisplayName',' $0.3\,\mathrm{s}^{-1}$ ');

```

```

%plot(x_ss_plot, sigma_ss_01/1e3, 'o', 'Color',[0.7
0.7 0.7],'MarkerEdgeColor','k', 'MarkerSize',4,'LineWidth',0.6,
'DisplayName','$0.1\,\mathrm{s}^{-1}$');
%plot(x_ss_plot, sigma_ss_003/1e3, 'd',
'Color',[0.7 0.7 0.7],'MarkerEdgeColor','k',
'MarkerFaceColor','k','MarkerSize',3,'LineWidth',0.6,
'DisplayName','$0.03\,\mathrm{s}^{-1}$');
%plot(x_ss_plot, sigma_ss_001/1e3, 'k--','LineWidth', 1.2,
'HandleVisibility','off');
hold on

for k = 1:n
    if k == 1

        plot(g_tot{k}, stress_tot{k}/1e3, 'r','LineWidth', 1.2, ...
            'DisplayName','$\mathrm{MATLAB}$');

    else

        plot(g_tot{k}, stress_tot{k}/1e3, 'r','LineWidth', 1.2, ...
            'HandleVisibility','off');

    end
end

hold off

grid on

xlabel('$\gamma \, [-]$', ...
    'Interpreter','latex', ...
    'Units','normalized', ...
    'Position',[0.5 -0.07]);

ylabel('$\sigma \, [\mathrm{kPa}]$', ...
    'Interpreter','latex', ...
    'Units','normalized', ...
    'Position',[-0.06 0.5]);

ylim([0 500])

ax = gca;
ax.YAxis.Exponent = 0;
ax.TickLabelInterpreter = 'latex';
ax.FontSize = 12;
ax.LineWidth = 0.6;

ax.XLabel.FontSize = 16;
ax.YLabel.FontSize = 16;

% ----- LEGEND ----- %
lgd = legend('Location','northeast','Interpreter','latex');

```

```

lgd.FontSize = 14;
lgd.Box = 'on';

set(gcf, 'Position', [100 100 1000 480])

%set(gcf, 'Renderer', 'painters')
%exportgraphics(gcf, 'figure5.pdf', 'ContentType', 'vector')
%
%
%
%
%
%
%
```

Functions

```

% RESIDUAL FUNCTION FOR CALIBRATION OF FINAL MODEL
function res = MW_damper_opti(x, eta_0, eta_inf, K, m, d_g, x_test, y_test)
%
% INPUT:  x          = Calibrated parameters                [-]
%          eta_0      = Zero shear viscosity                [Pas]
%          eta_inf    = Infinite shear viscosity            [Pas]
%          K          = Cross-model constant                [-]
%          m          = Cross-model constant                [-]
%          d_g        = Strain rate                         [1/s]
%          x_test     = Measured shear strain (Coaxial shera test) [-]
%          y_test     = Measured shear stress (Coaxial shera test) [Pa]
%
%
% OUTPUT: res        = Residual [-]
%
% Written by Axel Isberg & Axel Jankavs
% -----

% Unpack parameters
G_inf = x(1);
Gi     = x(2:4);
tau    = x(5:7);
k_soft = x(8);
Smin   = x(9);

% Pre. residual
res = [];

% Stress & Strain calulation
for k = 1:length(d_g)

    [g_tot, stress_tot, ~, ~, ~, ~, ~, ~] = MW_damper_func(G_inf, Gi,
tau, eta_0, eta_inf, K, m, k_soft, Smin, d_g(k), 20);

    % Test-data

```

```

%          g_MW      = Shear strain in MW-block      [-]
%          g_s_tot_i = Shear strain in dashpot      [-]
%          eta_s      = Cross-model viscosity        [Pas]
%          S          = Softening magnitude          [-]
%
%
% Written by Axel Isberg & Axel Jankavs
%-----

% Time discretization
t0 = 0;
t_final = g_tfinal/d_g;
dt = 0.1;
t = (t0:dt:t_final)';
n = length(t);

% Allocate variables
nB = length(Gi);
g_MWv = zeros(n, nB);
g_MW = zeros(n,1);
g_tot = d_g * t;
stress_tot = zeros(n,1);
stress_s = zeros(n,1);
stress_MW = zeros(n,1);
g_s_tot_i = zeros(n,1);
g_s_tot = 0;
eta_s = zeros(n,1);
S = ones(n,1);

%
%-----

for i = 2:n

    g_MW_i = g_MW(i-1);

    % Newton iteration
    for iter = 1:20

        % Maxwell branches
        for j = 1:nB
            g_MWv(i,j) = (dt * Gi(j) * g_MW_i + tau(j) * Gi(j) *
g_MWv(i-1,j)) ./ (Gi(j) * dt + tau(j) * Gi(j));
        end

        % MW stress
        stress_MW_i = G_inf * g_MW_i + sum(Gi .* (g_MW_i - g_MWv(i,:)));

        % Damper strain & rate
        g_s_i = g_tot(i) - g_MW_i;
        g_s_old = g_tot(i-1) - g_MW(i-1);
        dg_s = (g_s_i - g_s_old) / dt;

        % Softening

```

```

g_tot_i = g_tot(i);
S_i = 1 + (Smin - 1) * (1 - exp(-k_soft * g_tot_i));

% Damper viscosity (CROSS)
eta_s_i = (1/Smin) * (eta_inf + ((eta_0 - eta_inf) ./ (1 +
(K*abs(dg_s))^m)));

% Stress in Damper
stress_s_i = (S_i * eta_s_i) * dg_s;

% Residual
R = stress_MW_i - stress_s_i;
dR = (G_inf + sum(Gi)) + (S_i * eta_s_i)/dt;

% Newton solution
g_MW_i = g_MW_i - R/dR;

if abs(R) < 1e-6 * max(abs(stress_MW_i),1)
    break
end
end

% Update parameters
g_MW(i)      = g_MW_i;
g_s_tot_i(i) = g_s_i;
g_s_tot      = g_s_tot + abs(dg_s) * dt;
eta_s(i)     = eta_s_i;
S(i)         = S_i;

% Final stress (and verification of stress equilibrium in system)
stress_s(i)  = stress_s_i; % Total stress in damper
stress_tot(i) = stress_s(i); % Total stress in system
stress_MW(i) = stress_MW_i; % Stress in MW-block

end

end

```

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Appendix B: Manually digitized data from coaxial shear test

Digitally extracted data points from a coaxial shear test graph reported by Cheung and Cebon (1997).

$\dot{\gamma} = 1$ [s ⁻¹]		$\dot{\gamma} = 0.3$ [s ⁻¹]		$\dot{\gamma} = 0.1$ [s ⁻¹]		$\dot{\gamma} = 0.03$ [s ⁻¹]		$\dot{\gamma} = 0.01$ [s ⁻¹]	
γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	9.64	0.01	5.46	0.01	8.36	0.00	3.64	0.01	5.88
0.04	109.39	0.01	33.35	0.01	17.44	0.01	8.73	0.13	10.08
0.07	150.04	0.01	55.49	0.03	28.34	0.01	12.86	0.28	13.85
0.09	176.45	0.04	75.50	0.06	34.88	0.02	16.01	0.43	16.79
0.11	195.31	0.04	83.69	0.08	38.52	0.05	19.17	0.61	18.05
0.12	225.06	0.07	91.87	0.09	42.51	0.10	21.83	0.77	20.15
0.15	240.99	0.10	99.45	0.13	45.78	0.12	23.53	0.94	20.57
0.16	253.98	0.13	106.43	0.16	51.24	0.15	26.20	1.08	20.99
0.17	267.39	0.16	111.58	0.22	55.60	0.22	28.87	1.22	21.83
0.19	278.71	0.20	119.47	0.24	60.68	0.25	31.30	1.41	22.25
0.21	290.44	0.25	127.65	0.29	64.68	0.31	33.72	1.57	22.25
0.23	302.18	0.28	132.81	0.33	68.68	0.38	35.90	1.76	21.83
0.23	311.82	0.31	138.87	0.39	72.67	0.43	38.09	1.92	23.09
0.25	321.04	0.34	143.12	0.43	76.31	0.48	40.03	2.09	23.09
0.28	329.84	0.37	148.57	0.46	79.58	0.55	41.48	2.28	23.09
0.31	342.83	0.42	156.46	0.54	82.85	0.62	42.46	2.46	22.67
0.35	356.24	0.47	161.31	0.60	85.39	0.70	44.64	2.65	23.09
0.37	367.56	0.52	167.68	0.68	87.94	0.79	44.64	2.79	23.09
0.40	380.55	0.58	172.53	0.73	90.48	0.85	46.09	2.97	22.67
0.43	392.29	0.66	177.08	0.80	92.30	0.92	46.82	3.11	23.09
0.48	403.60	0.74	181.32	0.84	93.75	1.00	48.28	3.29	23.09
0.52	415.34	0.84	184.35	0.90	95.57	1.07	48.28	3.45	23.09
0.56	426.66	0.93	187.08	0.97	98.84	1.15	48.03	3.59	22.67
0.63	439.65	0.97	189.81	1.04	99.56	1.23	48.28	3.74	22.67
0.68	448.45	1.04	192.54	1.11	99.93	1.32	48.28	3.89	22.67
0.75	458.09	1.18	191.63	1.18	99.93	1.41	48.28	4.01	23.51
0.89	467.31	1.29	190.72	1.27	99.93	1.51	48.28	4.11	23.51
0.97	479.88	1.38	190.42	1.36	100.29	1.59	48.28	4.22	22.67
1.06	485.33	1.49	189.21	1.44	100.65	1.70	48.28	4.35	22.67
1.21	483.65	1.58	187.69	1.53	101.02	1.81	48.76	4.46	22.67
1.38	483.24	1.70	187.08	1.63	101.02	1.89	48.76	4.60	22.67
1.52	481.98	1.79	186.17	1.74	101.02	1.96	46.82	4.76	22.67
1.64	479.46	1.90	185.87	1.82	101.02	2.04	47.79	4.91	22.67
1.78	478.63	1.97	183.14	1.89	99.93	2.13	47.06	5.06	23.09
1.89	475.27	2.04	181.63	1.93	98.84	2.20	47.79	5.20	23.09
1.94	469.40	2.13	181.02	1.98	98.11	2.31	47.31	5.34	22.67
2.02	465.63	2.22	181.02	2.07	98.84	2.43	46.58	5.47	23.09
2.16	462.70	2.31	179.20	2.16	99.20	2.53	46.09	5.59	23.09
2.28	460.60	2.40	177.99	2.26	99.20	2.61	46.34	5.71	23.51
2.36	456.83	2.50	175.56	2.32	97.75	2.70	46.34	5.83	23.51
2.46	453.48	2.57	174.65	2.40	97.75	2.76	45.61	5.96	23.51
2.55	450.13	2.68	174.04	2.49	96.66	2.85	46.09	6.07	22.25
2.65	447.61	2.80	172.53	2.58	96.29	2.91	46.09	6.19	24.35

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γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]
2.75	445.52	2.90	172.23	2.66	95.93	2.97	45.61	6.29	23.09
2.86	443.00	3.00	171.01	2.78	95.57	3.03	45.37	6.44	23.93
2.94	440.07	3.10	169.50	2.86	96.29	3.11	44.88	6.56	24.77
3.05	435.46	3.19	168.28	2.90	95.20	3.18	44.40	6.68	23.93
3.17	429.17	3.27	168.28	2.97	94.48	3.27	44.88	6.80	23.51
3.29	426.24	3.36	169.19	3.03	94.11	3.34	45.37	6.96	23.93
3.42	423.72	3.47	169.19	3.11	93.02	3.42	44.88	6.96	23.93
3.58	421.63	3.55	167.07	3.20	92.30	3.52	44.40	7.09	23.09
3.69	417.02	3.66	166.46	3.30	93.75	3.66	43.67	7.24	23.93
3.79	412.82	3.75	164.95	3.38	93.75	3.77	43.43	7.40	25.19
3.87	408.63	3.86	163.74	3.46	93.02	3.86	43.43	7.47	23.51
4.02	406.96	3.96	162.52	3.57	92.66	3.94	43.18	7.58	23.93
4.18	405.28	4.06	162.52	3.64	92.30	4.07	43.91	7.74	24.35
4.30	401.93	4.16	163.43	3.75	91.57	4.16	43.67	7.88	24.77
4.44	398.99	4.24	162.83	3.83	91.21	4.25	42.70	8.00	24.77
4.56	396.90	4.28	160.70	3.92	90.48	4.35	43.91	8.12	23.51
4.67	393.55	4.37	159.79	4.01	90.84	4.42	42.94	8.22	24.77
4.82	394.38	4.46	159.79	4.10	90.84	4.52	42.21	8.34	25.61
4.90	392.29	4.59	158.88	4.18	90.48	4.67	43.43	8.41	23.93
4.98	389.77	4.68	158.58	4.28	89.75	4.77	42.94	8.52	24.77
5.10	386.00	4.77	158.88	4.38	89.03	4.85	41.97	8.68	23.93
5.22	382.65	4.87	157.67	4.47	88.66	4.93	42.70	8.82	25.61
5.32	380.13	4.95	156.76	4.56	88.30	5.06	42.94	8.97	26.03
5.44	378.04	5.04	156.46	4.65	88.30	5.20	42.70	9.10	25.19
5.59	378.04	5.13	154.94	4.71	88.30	5.29	42.46	9.23	24.35
5.75	378.04	5.22	154.03	4.76	87.94	5.37	42.46	9.39	23.09
5.82	374.27	5.32	153.12	4.83	87.57	5.47	42.21	9.51	23.09
5.93	369.24	5.41	152.21	4.93	88.66	5.54	42.70	9.61	20.57
6.08	366.30	5.50	152.52	5.04	88.30	5.72	42.94	9.70	22.25
6.20	365.47	5.58	152.52	5.14	88.30	5.81	43.43	9.81	23.09
6.29	361.69	5.67	152.82	5.24	87.21	5.88	42.46	9.99	23.93
6.44	360.85	5.76	153.43	5.35	86.85	5.97	42.94	10.11	23.93
6.64	361.27	5.85	152.82	5.44	87.21	6.10	41.97	10.30	24.35
6.76	357.08	5.92	150.09	5.51	87.57	6.20	43.18	10.46	23.93
6.91	354.15	6.02	150.39	5.61	87.57	6.30	42.21	10.59	24.35
7.04	352.05	6.13	151.00	5.72	88.66	6.39	40.03	10.73	23.93
7.20	348.28	6.23	150.39	5.78	88.66	6.45	41.97	10.87	23.93
7.36	342.41	6.35	150.09	5.83	87.21	6.57	42.94	10.94	21.83
7.54	341.16	6.45	150.09	5.93	86.48	6.66	42.94	11.07	22.25
7.84	336.97	6.54	150.39	6.01	86.48	6.75	42.46	11.20	23.51
8.04	332.77	6.63	152.82	6.06	86.48	6.85	43.18	11.32	22.25
8.26	330.26	6.69	151.61	6.13	85.76	6.92	44.15	11.47	21.41
8.45	332.36	6.74	150.70	6.22	86.12	7.00	43.18	11.56	22.25
8.57	332.77	6.84	150.09	6.29	86.12	7.10	43.91	11.66	20.99
8.78	325.23	6.92	150.09	6.39	85.76	7.16	42.94	11.76	20.57
8.96	317.69	6.97	151.30	6.46	85.39	7.26	43.18	11.88	20.57
9.13	312.24	7.06	151.00	6.52	86.85	7.34	43.18	12.02	20.15
9.29	305.95	7.13	150.09	6.61	86.85	7.48	42.94	12.15	20.15
9.65	301.34	7.22	149.18	6.73	86.48	7.62	42.70	12.30	19.73
9.99	294.22	7.31	148.88	6.85	86.48	7.72	43.18	12.47	20.15
10.19	290.03	7.38	147.97	6.93	86.48	7.86	41.73	12.63	19.31
10.45	286.67	7.46	148.57	6.97	86.48	7.96	42.21	12.79	19.31
10.66	288.77	7.57	148.57	7.04	86.48	8.02	43.43	12.93	18.47
10.83	279.55	7.67	147.67	7.14	85.76	8.18	45.37	13.13	17.63

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γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]
11.00	277.45	7.78	146.45	7.23	84.30	8.29	43.91	13.31	18.05
11.15	277.03	7.88	145.85	7.33	84.30	8.38	43.91	13.31	18.05
11.44	269.91	8.01	145.24	7.44	84.67	8.44	42.94	13.43	19.31
11.72	259.85	8.11	145.24	7.53	84.67	8.52	41.73	13.53	19.31
12.02	253.14	8.20	146.15	7.64	84.67	8.61	40.51	13.79	19.31
12.32	243.92	8.31	146.76	7.71	84.30	8.75	41.00	13.92	19.31
12.56	238.06	8.40	146.76	7.82	83.94	8.89	40.27	14.15	19.31
12.77	233.45	8.47	145.24	7.90	83.94	8.92	38.57	14.38	19.31
12.89	227.16	8.59	146.15	7.98	84.67	9.07	40.03	14.54	19.31
13.05	222.13	8.68	145.24	8.08	84.67	9.20	40.76	14.70	18.89
13.28	221.71	8.77	142.51	8.17	85.76	9.32	41.73	14.70	18.89
13.52	221.29	8.90	141.60	8.27	86.12	9.42	41.48	14.92	18.89
13.87	214.59	8.98	141.90	8.33	87.57	9.52	40.03	15.05	18.47
14.21	209.14	9.05	142.21	8.43	87.57	9.62	40.27	15.29	19.31
14.52	203.27	9.12	140.69	8.51	86.85	9.70	41.24	15.45	18.89
14.77	201.17	9.20	140.08	8.51	86.85	9.80	41.24	15.65	18.89
15.02	197.40	9.27	141.30	8.58	85.39	9.92	41.00	15.90	19.31
15.26	191.53	9.38	142.21	8.63	82.85	10.03	40.76	16.21	18.47
15.61	186.92	9.48	141.90	8.70	82.49	10.15	39.79	16.49	18.05
15.90	183.15	9.54	139.78	8.80	82.49	10.29	39.54	16.69	18.05
16.18	178.96	9.63	141.60	8.92	83.94	10.42	39.54	16.93	18.89
16.41	175.19	9.74	141.60	9.01	83.94	10.54	38.57	17.15	18.47
16.69	170.58	9.86	141.30	9.08	80.67	10.66	38.33	17.37	18.47
16.95	168.90	9.99	140.99	9.19	81.40	10.76	38.09	17.59	18.47
17.25	165.55	10.07	140.99	9.29	82.49	10.85	38.09	17.79	18.89
17.52	162.20	10.13	139.48	9.37	84.30	10.95	38.09	17.95	18.89
17.80	157.59	10.20	138.87	9.49	83.58	11.08	38.09	18.12	18.89
18.06	153.81	10.28	138.27	9.57	82.49	11.15	37.36	18.34	19.31
18.32	148.37	10.38	137.66	9.67	82.49	11.21	36.39	18.54	18.89
18.60	146.69	10.44	137.05	9.81	82.85	11.28	36.63	18.78	18.05
18.86	143.34	10.51	137.05	9.90	82.85	11.41	37.36	18.96	18.05
19.02	141.24	10.71	136.45	10.01	82.85	11.52	36.88	19.17	18.05
19.24	138.73	10.84	134.93	10.13	82.49	11.65	36.39	19.39	18.05
19.43	136.63	10.92	133.11	10.20	80.31	11.78	35.66	19.61	17.63
19.67	134.53	11.09	134.32	10.27	80.31	11.85	35.66	19.61	17.63
19.83	134.12	11.19	132.81	10.35	80.31	11.96	34.21	19.75	18.05
19.97	134.12	11.30	132.50	10.42	81.40	12.09	34.45	19.87	18.05
		11.42	134.32	10.52	81.40	12.15	32.27	19.96	18.05
		11.54	133.41	10.67	81.40	12.31	33.72		
		11.65	132.81	10.80	78.85	12.45	32.27		
		11.78	131.90	10.92	78.85	12.54	32.51		
		11.86	131.90	11.00	80.31	12.63	31.05		
		12.03	132.20	11.10	80.31	12.71	31.54		
		12.14	130.69	11.16	78.49	12.83	31.54		
		12.19	129.17	11.29	79.58	12.97	31.30		
		12.27	130.99	11.37	80.67	13.04	32.51		
		12.41	131.29	11.45	78.85	13.14	31.78		
		12.54	127.96	11.53	78.85	13.27	33.48		
		12.66	125.53	11.60	79.58	13.44	32.51		
		12.78	125.23	11.73	79.22	13.58	34.21		
		12.98	125.23	11.85	77.76	13.72	32.75		
		13.10	123.71	11.94	76.67	13.85	33.24		
		13.24	123.71	12.02	75.94	14.04	32.99		
		13.43	123.41	12.16	75.58	14.16	32.99		

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γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]
		13.59	123.41	12.28	76.67	14.27	31.54		
		13.69	124.01	12.36	75.58	14.43	32.75		
		13.80	123.71	12.45	74.49	14.57	32.75		
		13.87	122.80	12.60	73.76	14.71	31.54		
		13.96	121.29	12.70	73.76	14.87	32.99		
		14.04	122.80	12.78	73.76	14.96	31.78		
		14.18	123.71	12.86	73.40	15.04	30.81		
		14.32	123.10	12.97	71.22	15.27	32.75		
		14.43	123.71	13.07	70.49	15.35	30.81		
		14.56	123.41	13.18	72.31	15.55	31.30		
		14.66	123.10	13.30	73.40	15.69	31.78		
		14.73	121.89	13.39	74.13	15.83	30.57		
		14.85	121.29	13.50	73.76	15.94	29.60		
		14.95	119.77	13.60	74.13	16.04	31.05		
		15.08	118.56	13.72	74.85	16.16	31.78		
		15.21	118.86	13.83	74.49	16.16	31.78		
		15.28	118.56	13.94	73.76	16.25	30.57		
		15.40	117.65	14.02	74.85	16.36	30.08		
		15.54	116.43	14.14	74.85	16.44	30.57		
		15.65	115.52	14.24	73.76	16.59	31.05		
		15.75	115.52	14.35	74.13	16.72	29.60		
		15.86	113.71	14.46	74.13	16.86	30.08		
		15.96	114.31	14.54	73.76	17.03	30.81		
		16.04	115.22	14.66	73.76	17.19	29.84		
		16.14	114.61	14.79	73.40	17.33	30.81		
		16.33	114.61	14.89	72.67	17.49	30.57		
		16.43	114.31	15.00	71.95	17.62	28.87		
		16.53	112.80	15.10	71.58	17.81	29.35		
		16.68	112.49	15.27	71.95	17.90	30.57		
		16.78	112.80	15.44	71.58	18.02	30.08		
		16.78	112.80	15.50	70.49	18.13	28.87		
		16.84	112.80	15.57	70.86	18.26	28.14		
		16.94	113.71	15.73	70.86	18.33	29.35		
		17.10	113.10	15.85	70.86	18.45	29.60		
		17.19	112.80	15.97	73.04	18.59	28.14		
		17.32	113.10	16.09	72.67	18.71	29.84		
		17.48	113.10	16.22	71.22	18.80	29.60		
		17.64	113.40	16.37	70.86	18.91	28.87		
		17.70	112.80	16.47	70.86	19.04	29.35		
		17.84	111.89	16.55	71.58	19.11	30.08		
		17.92	109.76	16.63	70.86	19.22	30.08		
		18.04	109.16	16.71	70.86	19.29	28.63		
		18.12	109.76	16.83	70.49	19.41	27.41		
		18.25	108.55	16.93	70.49	19.47	28.38		
		18.43	108.55	17.03	69.40	19.55	29.35		
		18.57	108.55	17.14	68.31	19.65	28.63		
		18.70	107.34	17.26	68.68	19.74	28.38		
		18.80	104.61	17.36	68.68	19.81	28.14		
		18.95	105.22	17.49	68.68	19.86	28.87		
		19.16	104.91	17.68	68.31	19.94	30.08		
		19.33	103.40	17.86	68.68				
		19.48	101.58	18.04	69.04				
		19.62	100.67	18.13	69.40				
		19.81	101.58	18.28	69.40				

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γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]	γ [-]	σ [kPa]
		19.96	100.97	18.48	69.04				
				18.59	68.68				
				18.71	67.59				
				18.87	66.86				
				19.04	66.86				
				19.17	66.86				
				19.27	66.86				
				19.39	66.86				
				19.55	67.22				
				19.67	68.31				
				19.77	68.31				
				19.83	68.68				
				19.92	67.59				

Appendix C: DSR data

DSR-data for Bitumen 50/70 PP08

Temperature [°C]	Angular Frequency [1/s]	Shear Strain [%]	Shear Stress [Pa]	Complex Shear Modulus [Pa]	Phase Shift Angle [°]	Storage Modulus [Pa]	Loss Modulus [Pa]
-30	0.1	0.0652	1.70E+05	2.61E+08	21.31	2.43E+08	9.48E+07
-30	0.215	0.0512	1.65E+05	3.22E+08	16.11	3.09E+08	8.92E+07
-30	0.464	0.0406	1.48E+05	3.64E+08	13.64	3.54E+08	8.60E+07
-30	1	0.0321	1.30E+05	4.05E+08	12.00	3.96E+08	8.41E+07
-30	2.15	0.0253	1.13E+05	4.48E+08	10.97	4.40E+08	8.52E+07
-30	4.64	0.0200	97390	4.87E+08	9.86	4.80E+08	8.34E+07
-30	10	0.0158	83229	5.26E+08	8.77	5.20E+08	8.02E+07
-30	21.5	0.0125	70552	5.65E+08	7.89	5.59E+08	7.74E+07
-30	46.4	0.00987	59353	6.01E+08	6.90	5.97E+08	7.22E+07
-30	100	0.00792	49837	6.29E+08	6.95	6.25E+08	7.62E+07
-20	0.1	0.1120	1.39E+05	1.24E+08	27.10	1.11E+08	5.67E+07
-20	0.215	0.0876	1.35E+05	1.54E+08	25.08	1.39E+08	6.51E+07
-20	0.464	0.0695	1.35E+05	1.94E+08	21.37	1.81E+08	7.07E+07
-20	1	0.0550	1.29E+05	2.35E+08	16.79	2.25E+08	6.80E+07
-20	2.15	0.0434	1.18E+05	2.71E+08	15.37	2.61E+08	7.18E+07
-20	4.64	0.0343	1.05E+05	3.07E+08	14.22	2.98E+08	7.54E+07
-20	10	0.0271	93520	3.45E+08	13.09	3.36E+08	7.81E+07
-20	21.5	0.0214	82174	3.84E+08	11.98	3.75E+08	7.96E+07
-20	46.4	0.0169	71518	4.23E+08	10.93	4.15E+08	8.01E+07
-20	100	0.0134	62093	4.62E+08	10.35	4.54E+08	8.30E+07
-10	0.1	0.1880	1.00E+05	5.33E+07	24.93	4.83E+07	2.25E+07
-10	0.215	0.1480	97109	6.58E+07	23.55	6.03E+07	2.63E+07
-10	0.464	0.1170	93970	8.02E+07	22.51	7.41E+07	3.07E+07
-10	1	0.0926	89715	9.68E+07	21.49	9.01E+07	3.55E+07
-10	2.15	0.0732	84727	1.16E+08	20.56	1.08E+08	4.07E+07
-10	4.64	0.0578	79508	1.38E+08	19.65	1.30E+08	4.62E+07
-10	10	0.0457	73993	1.62E+08	18.74	1.53E+08	5.20E+07
-10	21.5	0.0361	68320	1.89E+08	17.80	1.80E+08	5.79E+07
-10	46.4	0.0285	62557	2.20E+08	16.82	2.10E+08	6.35E+07
-10	100	0.0224	56830	2.53E+08	15.67	2.44E+08	6.84E+07

DSR-data for Bitumen 50/70 PP08

Temperature [°C]	Angular Frequency [1/s]	Shear Strain [%]	Shear Stress [Pa]	Complex Shear Modulus [Pa]	Phase Shift Angle [°]	Storage Modulus [Pa]	Loss Modulus [Pa]
0	0.1	0.3110	53384	1.72E+07	29.72	1.49E+07	8.52E+06
0	0.215	0.2440	54140	2.22E+07	28.35	1.95E+07	1.05E+07
0	0.464	0.1930	54841	2.83E+07	27.21	2.52E+07	1.30E+07
0	1	0.1530	54526	3.56E+07	26.18	3.20E+07	1.57E+07
0	2.15	0.1210	53637	4.44E+07	25.26	4.01E+07	1.89E+07
0	4.64	0.0955	52340	5.48E+07	24.40	4.99E+07	2.26E+07
0	10	0.0754	50736	6.72E+07	23.60	6.16E+07	2.69E+07
0	21.5	0.0596	48855	8.20E+07	22.81	7.56E+07	3.18E+07
0	46.4	0.0471	46733	9.93E+07	22.02	9.20E+07	3.72E+07
0	100	0.0372	44452	1.20E+08	21.22	1.12E+08	4.33E+07
10	0.1	0.5020	25532	5.08E+06	35.39	4.14E+06	2.94E+06
10	0.215	0.3940	26976	6.84E+06	33.79	5.69E+06	3.81E+06
10	0.464	0.3130	28419	9.07E+06	32.45	7.65E+06	4.87E+06
10	1	0.2480	29496	1.19E+07	31.26	1.02E+07	6.18E+06
10	2.15	0.1960	30272	1.55E+07	30.22	1.34E+07	7.78E+06
10	4.64	0.1550	30815	1.99E+07	29.28	1.74E+07	9.74E+06
10	10	0.1220	31121	2.55E+07	28.44	2.24E+07	1.21E+07
10	21.5	0.0966	31221	3.23E+07	27.67	2.86E+07	1.50E+07
10	46.4	0.0763	31133	4.08E+07	26.95	3.64E+07	1.85E+07
10	100	0.0603	30842	5.11E+07	26.23	4.59E+07	2.26E+07
20	0.1	0.8010	10245	1.28E+06	42.08	9.50E+05	8.57E+05
20	0.215	0.6290	11434	1.82E+06	40.21	1.39E+06	1.17E+06
20	0.464	0.4990	12677	2.54E+06	38.58	1.99E+06	1.58E+06
20	1	0.3950	13833	3.50E+06	37.14	2.79E+06	2.11E+06
20	2.15	0.3120	14893	4.77E+06	35.88	3.87E+06	2.80E+06
20	4.64	0.2470	15880	6.44E+06	34.73	5.29E+06	3.67E+06
20	10	0.1950	16780	8.61E+06	33.72	7.16E+06	4.78E+06
20	21.5	0.1540	17591	1.14E+07	32.81	9.60E+06	6.19E+06
20	46.4	0.1220	18311	1.51E+07	32.01	1.28E+07	7.98E+06
20	100	0.0961	18933	1.97E+07	31.30	1.68E+07	1.02E+07
30	0.1	1.2600	3484.9	2.77E+05	50.08	1.78E+05	2.13E+05
30	0.215	0.9880	4152.1	4.20E+05	47.78	2.83E+05	3.11E+05
30	0.464	0.7830	4894.3	6.25E+05	45.75	4.36E+05	4.48E+05
30	1	0.6200	5661.2	9.14E+05	43.95	6.58E+05	6.34E+05
30	2.15	0.4900	6446.3	1.32E+06	42.35	9.73E+05	8.87E+05
30	4.64	0.3870	7245.1	1.87E+06	40.95	1.41E+06	1.23E+06
30	10	0.3060	8056.8	2.64E+06	39.70	2.03E+06	1.68E+06
30	21.5	0.2410	8874.5	3.68E+06	38.59	2.87E+06	2.29E+06
30	46.4	0.1910	9691.6	5.08E+06	37.60	4.02E+06	3.10E+06
30	100	0.1510	10500	6.97E+06	36.72	5.58E+06	4.16E+06

DSR-data for Bitumen 50/70 PP25

Temperature [°C]	Angular Frequency [1/s]	Shear Strain [%]	Shear Stress [Pa]	Complex Shear Modulus [Pa]	Phase Shift Angle [°]	Storage Modulus [Pa]	Loss Modulus [Pa]
30	0.1	1.2600	3523.0	2.81E+05	49.51	1.82E+05	2.13E+05
30	0.215	0.9880	4115.3	4.17E+05	47.49	2.82E+05	3.07E+05
30	0.464	0.7830	4797.9	6.13E+05	45.81	4.27E+05	4.39E+05
30	1	0.6200	5581.5	9.01E+05	44.44	6.43E+05	6.31E+05
30	2.15	0.4900	6445.2	1.32E+06	43.51	9.55E+05	9.06E+05
30	4.64	0.3870	7397.3	1.91E+06	43.02	1.40E+06	1.30E+06
30	10	0.3060	8449.8	2.76E+06	43.04	2.02E+06	1.89E+06
30	21.5	0.2410	9636.2	3.99E+06	43.70	2.88E+06	2.76E+06
30	46.4	0.1910	11014	5.77E+06	45.24	4.06E+06	4.10E+06
30	100	0.1510	12683	8.42E+06	47.98	5.64E+06	6.26E+06
40	0.1	1.9400	1041.8	5.36E+04	57.94	2.84E+04	4.54E+04
40	0.215	1.5300	1323.4	8.66E+04	55.34	4.92E+04	7.12E+04
40	0.464	1.2100	1651.4	1.36E+05	53.09	8.19E+04	1.09E+05
40	1	0.9580	2008.2	2.10E+05	51.07	1.32E+05	1.63E+05
40	2.15	0.7570	2418.5	3.19E+05	49.32	2.08E+05	2.42E+05
40	4.64	0.5980	2901.9	4.85E+05	47.92	3.25E+05	3.60E+05
40	10	0.4730	3456.3	7.31E+05	46.86	5.00E+05	5.34E+05
40	21.5	0.3730	4084.1	1.09E+06	46.15	7.58E+05	7.89E+05
40	46.4	0.2950	4791.0	1.62E+06	45.86	1.13E+06	1.17E+06
40	100	0.2330	5590.4	2.40E+06	46.06	1.66E+06	1.73E+06
50	0.1	2.9600	301.5	1.01E+04	67.14	3.95E+03	9.38E+03
50	0.215	2.3300	414.95	1.78E+04	64.02	7.80E+03	1.60E+04
50	0.464	1.8500	558.81	3.02E+04	61.14	1.46E+04	2.65E+04
50	1	1.4600	740.5	5.06E+04	58.54	2.64E+04	4.32E+04
50	2.15	1.1500	945.94	8.19E+04	56.24	4.55E+04	6.81E+04
50	4.64	0.9120	1179.9	1.29E+05	54.24	7.55E+04	1.05E+05
50	10	0.7210	1466.6	2.04E+05	52.54	1.24E+05	1.62E+05
50	21.5	0.5690	1813.9	3.19E+05	51.16	2.00E+05	2.48E+05
50	46.4	0.4500	2224.2	4.94E+05	50.09	3.17E+05	3.79E+05
50	100	0.3550	2700.7	7.60E+05	49.40	4.95E+05	5.77E+05
60	0.1	4.4600	89.796	2.01E+03	75.27	5.12E+02	1.94E+03
60	0.215	3.5100	133.73	3.81E+03	72.04	1.17E+03	3.62E+03
60	0.464	2.7800	193.28	6.95E+03	68.92	2.50E+03	6.48E+03
60	1	2.2000	272.45	1.23E+04	65.94	5.05E+03	1.13E+04
60	2.15	1.7400	373.05	2.14E+04	63.23	9.66E+03	1.91E+04
60	4.64	1.3700	500.02	3.64E+04	60.79	1.77E+04	3.17E+04
60	10	1.0800	650.85	5.99E+04	58.66	3.12E+04	5.12E+04
60	21.5	0.8570	832.96	9.71E+04	56.82	5.31E+04	8.13E+04
60	46.4	0.6770	1059.7	1.56E+05	55.27	8.91E+04	1.29E+05
60	100	0.5350	1338.9	2.50E+05	54.06	1.47E+05	2.03E+05

DSR-data for Bitumen 50/70 PP25

Temperature [°C]	Angular Frequency [1/s]	Shear Strain [%]	Shear Stress [Pa]	Complex Shear Modulus [Pa]	Phase Shift Angle [°]	Storage Modulus [Pa]	Loss Modulus [Pa]
70	0.1	6.6300	26.672	4.02E+02	82.03	5.57E+01	3.98E+02
70	0.215	5.2200	42.582	8.16E+02	79.27	1.52E+02	8.02E+02
70	0.464	4.1300	65.505	1.58E+03	76.43	3.71E+02	1.54E+03
70	1	3.2700	98.394	3.00E+03	73.47	8.55E+02	2.88E+03
70	2.15	2.5800	143.67	5.55E+03	70.59	1.84E+03	5.24E+03
70	4.64	2.0400	204.95	1.00E+04	67.84	3.78E+03	9.29E+03
70	10	1.6100	284.33	1.76E+04	65.34	7.35E+03	1.60E+04
70	21.5	1.2700	386.84	3.03E+04	63.11	1.37E+04	2.70E+04
70	46.4	1.0100	519.32	5.15E+04	61.19	2.48E+04	4.51E+04
70	100	0.7960	684.06	8.59E+04	59.60	4.34E+04	7.41E+04
80	0.1	9.7500	8.2651	8.47E+01	86.40	5.32E+00	8.46E+01
80	0.215	7.6700	13.831	1.80E+02	84.66	1.67E+01	1.79E+02
80	0.464	6.0800	22.497	3.70E+02	82.60	4.76E+01	3.67E+02
80	1	4.8100	35.565	7.39E+02	80.22	1.25E+02	7.28E+02
80	2.15	3.8000	55.213	1.45E+03	77.58	3.12E+02	1.41E+03
80	4.64	3.0000	83.547	2.78E+03	74.86	7.26E+02	2.68E+03
80	10	2.3700	123.30	5.19E+03	72.20	1.58E+03	4.94E+03
80	21.5	1.8700	177.88	9.48E+03	69.71	3.29E+03	8.89E+03
80	46.4	1.4800	251.24	1.69E+04	67.47	6.49E+03	1.56E+04
80	100	1.1700	348.16	2.97E+04	65.48	1.23E+04	2.70E+04

