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# Multiscale Modelling of Large-Amplitude Fluctuations in Tokamak Edge Plasmas

Master's thesis in Physics and Astronomy

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# Abstract

For the efficient and safe operation of magnetic confinement fusion reactors, reliable theoretical descriptions of the dynamic behaviour of the confined plasma are required. One proven, highly successful description is gyrokinetics, which describes small amplitude turbulence in the core plasma. However, this description is expected to break down in the plasma edge, in the presence of large fluctuations and an extremely steep pressure gradient.

Attempting to capture these edge conditions, in this thesis we use multiscale analysis to produce a new set of equations we collectively refer to as Toroidal Kinetic Reduced MHD (TKRMHD). These equations, suitable for describing ITG-like turbulence, are fully kinetic, applicable in general field confinement configurations, and capable of describing the collisional to weakly-collisional transition. As such this description constitute a first step towards a first-principle description of L-mode tokamak edge turbulence.

In this thesis we furthermore demonstrate that the TKRMHD equations can be smoothly matched onto gyrokinetics through an intermediary set of equations, derived from suitable subsidiary multiscale expansions. This is important because gyrokinetic tokamak transport simulations strongly depend upon density and temperature edge boundary conditions. This new matching may make it possible to choose these conditions reliably, instead of relying on *ad-hoc* assumptions.

Keywords: gyrokinetics, L-mode, tokamak edge, multiscale analysis, ITG

# Contents

# 1
# Introduction

Humanity faces a great challenge this century. Over the coming decades the human population is projected to increase by several billions, particularly in developing nations. Because there is a direct correlation between standard of living and energy consumption, in order to achieve an increasing standard of living in both developing and industrialised nations, our energy production necessarily must be increased. At the moment, fossil fuels provide approximately 70% of our electricity and the absolute majority of the energy used for transportation (both of people and goods). However, easily exploitable fossil fuel sources are limited and may by some estimates already be on the brink of depletion, with both oil and natural gas peak production occurring within the next 20 years [1]. Additionally, currently available fossil fuel reserves are predominantly located in politically unstable regions, making major energy disruptions that could threaten national security a distinct risk.

Beyond dwindling fossil fuel reserves there is another reason our reliance on fossil fuels is troubling: global warming caused by $CO_2$ emissions. The various consequences of this climate change are manifold and the possible feedback mechanisms involved make accurate predictions difficult. Nevertheless, the scientific consensus is that even a few degrees of global warming will prove catastrophic [2]. It is thus clear that our current fossil energy production is unsustainable, even without accounting for projected population growth or a future continuation of the economic growth that has made the industrial world so prosperous the last 100 years.

Currently, the major alternatives to fossil fuels in use are various renewable energy sources and nuclear fission. Renewable energy sources like wind, hydroelectric, and solar power are politically attractive and, being heavily subsidised, are undergoing rapidly expansion. However these sources are inadequate to fully cover our energy consumption at current levels because the energy per land area they produce is too low [1]. This leaves nuclear fission, an energy source that has many attractive properties. Already nuclear power is economically viable, if beset by a prohibitive up-front capital cost, and it produces hardly any $CO_2$ pollution during operation. Modern nuclear power plants are exceedingly safe and technological advances in recent years have made long-term nuclear waste disposal seem ever more achievable. Despite this, the possibility for nuclear expansion in the near future seems slim at best. The public perception of nuclear power in Europe is decisively negative while the risk of nuclear proliferation makes it politically non-viable to suggest such an expansion.

If none of the currently available alternatives to fossil fuels are adequate to solve the energy problem at hand, this paints a bleak picture of the near future. Thankfully, there is a potential solution that is within reach. Nuclear fusion power, though currently still in the experimental stages, possesses many attractive features that makes it superior to conventional nuclear fission. First, fusion power does not rely on neutron multiplication to sustain a chain reaction like fission power does. Therefore, there is no need to store several years worth of fissile material within the reactor to maintain the reaction. Indeed, a fusion reactor must be continually fuelled to sustain its reaction, so there is no possibility of an uncontrollable runaway reaction causing a meltdown. Furthermore, a fusion reactor only produces short-lived nuclear waste from the interaction of neutrons with the reactor vessel and buildings. This is a major advantage because such waste only needs to be safely handled for about a hundred years. Additionally, this waste can not be used in a fashion that would raise nuclear proliferation concerns. Finally and most importantly, while the limiting fusion fuel tritium is extremely rare on earth, it can be bred from lithium by neutron bombardment in large enough quantities to cover all of our energy consumption at current levels for over 20 000 years. As such, fusion could conceivably completely replace fossil fuels and single-handedly avert the coming energy crisis [1].

Despite its many attractive features, fusion possesses one crippling drawback. In fusion reactions, two positively charged nuclei have to be brought together to fuse. In order for this to happen the Coulomb barrier must be overcome which, for appreciable cross sections, requires a relative kinetic energy of the order of 10-100 keV. If this sort of energy is to be supplied in thermal form the temperature must be approximately $10^8$ K and so the fuel must be provided as a plasma. This is such a tremendous temperature, exceeding the temperature of the Sun's core, that if the plasma was in thermal contact with a material surface it would cause irreparable damage. Therefore, in order to use fusion to safely generate power on Earth, we have but two options: either generate the plasma in extremely brief pulses or suspend it in a magnetic field configuration to limit thermal contact with the reactor vessel. Both avenues have been pursued, but because magnetic confinement has demonstrated better results over a longer time period, a greater amount of focus has been spent on it [3].

Magnetic confinement works on the principle that electrically charged particles in a magnetic field will undergo gyromotion, i.e. be strongly confined to spiral around magnetic field lines. Therefore the particle flux across the magnetic field is greatly reduced so that, if the magnetic field topologically forms a closed surface, a plasma can be contained. Now, in order for there to exist any stable plasma equilibrium the magnetic field must topologically be toroidal, with both toroidal and poloidal components [4]. One configuration that satisfies these constraints, and which is the focus of this thesis, is the tokamak, in which the poloidal field is generated by the plasma itself by driving a toroidal current through it. Because of their simple axisymmetry, tokamaks have been able to be extensively developed and so constitute the cutting edge of current fusion reactors.

Of course, the goal of fusion power is not simply maintaining a steady-state plasma. Ultimately, a reactor must produce more energy than it consumes and ideally ignition, where no external power at all is needed, should be achieved. The condition for this to occur is the famous Lawson criterion which states that the triple product of the the plasma density, plasma temperature, and energy confinement time (the energy stored divided by the power loss) must exceed a critical value while the temperature simultaneously is maintained at a suitably high value [3]. This criterion has still not been met by any experimental reactor, and the reasons for this can be traced both to engineering inefficiencies, like the difficulty of creating large enough superconducting magnets, and to a lack of scientific understanding of the plasma within the reactor.

For obvious reasons, it is important to be able to predict the performance of a tokamak in advance of its construction. To fully achieve this, it is necessary to possess a first-principles description of plasma behaviour such that no possibly relevant phenomenon is neglected. Such a description already exists for the core in the form of a large-scale equilibrium, whose stability is determined from magnetohydrodynamics [5], in combination with turbulent small-scale fluctuations, described by gyrokinetics [6]. Simulations of this kind have proven to yield accurate results that have been repeatedly verified in experiments [7]. However, in the edge region there is an extremely steep pressure gradient that renders this theory inapplicable. Furthermore, there is mounting evidence that edge physics plays a vital role for overall tokamak performance.

Among the many different kinds of tokamak behaviours that seem to originate in edge physics, perhaps the most important and characteristic is the transition from low confinement L-mode to high confinement H-mode [8]. This transition occurs as a heating threshold is reached and causes the confinement time to rapidly increase by about a factor of 2. As a result the pressure gradient in the edge region rapidly steepens even further and becomes a pedestal upon which the shallower core gradient can rest, resulting in a significant core temperature increase. This is possible because the turbulent transport across the edge is quenched, which is an extremely desirable feature. However, in going to H-mode the plasma becomes unstable and exhibits short bursts of edge-localised modes (ELMs) that eject plasma into the scrape-off layer towards the reactor wall [9]. There are many different kinds of ELMs with different characteristics that range from moderately attractive to devastating. Thoroughly understanding these ELMs and the L-H transition could allow safe and efficient tokamak operation and is very desirable.

Now the focus of this thesis is not to study the L-H transition and indeed we will not be concerned with the more extreme H-mode physics at all. This transition is merely the most dramatic and noticeable edge physics effect, but there are sure to be many more, even for L-mode operation, that impact general tokamak performance and so also have to be described. To this end we will in this thesis develop a framework for describing some of these phenomena that should be suitable for common L-mode operation. Furthermore we will also investigate its compatibility with the proven gyrokinetics of the tokamak core.

# 2

# Analysing Toroidal Plasmas

Thoroughly describing plasma edge physics to a sufficient degree for the purpose of this thesis unsurprisingly requires a wide array of different mathematical techniques. These range in nature from those well-known by any physicist to the esoteric only applied within specific subsets of the field of plasma physics. As such, we will in this Chapter present these different tools for those that are unfamiliar. We will start by introducing the fundamental equations which form the basis for all subsequent work in this thesis. Then we will proceed by discussing multiscale analysis, the powerful tool which underpins both gyrokinetics and the new edge theory we shall develop. Its power comes from the large separation of scales present in a typical tokamak plasma, a fact we will next use when introducing straight field line motion to describe a few typical plasma phenomena of interest to us. Having done this, we will also find it suitable to introduce a set of coordinates, intimately linked to the magnetic field, which will prove convenient in describing these plasma phenomena. Logically we will then proceed to elaborate on the properties of the present fields and distributions in two ways. First we will highlight the splitting of fluctuating from non-fluctuating quantities and the mathematical implications of this splitting. Finally we will then investigate the large-scale properties of the magnetic field and the mathematical operations this allows us to introduce.

## 2.1 Fundamental Equations

A fusion reactor plasma is, at the most basic level, a collection of electrons and ions contained by a powerful magnetic field. Even though the plasma density, in practice, is so low that it can accurately be classified as a vacuum, the number density is still on the order of $10^{20}$ m$^{-3}$ [10]. Thus, it becomes completely infeasible to attempt to describe the system in its entirety with every particle's complete trajectory. This is both because the computational complexity would be too high and because it is impossible to know each particle's exact position and velocity at any given moment. Instead, we are immediately forced to use a statistical approach.

While producing a statistical description of a plasma we can thankfully utilise our knowledge of the microscopic forces that determine individual trajectories. We know that any non-relativistic plasma particle of species $s$ obeys the equations of motion

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}, \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \tag{2.1}$$

where we have used Gaussian units which will be used throughout this thesis. To use these deterministic equations to derive a statistical equation we now assign the particle a probability distribution function over $(\mathbf{r}, \mathbf{v})$-phase space so that at time $t$ it has probability $f_s \mathrm{d}^3 r \mathrm{d}^3 v$ to be within the infinitesimal phase-space volume $\mathrm{d}^3 r \mathrm{d}^3 v$ at $(\mathbf{r}, \mathbf{v})$. Then, by the equation of motion, it must at a later time have that same probability to be within the infinitesimal volume at the correspondingly later point on the trajectory. Thus the total derivative along the trajectory is zero, so [11, 12]

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0. \tag{2.2}$$

After making two important conceptual modifications to this equation by changing $f_s$ to be a distribution function over all particles of species $s$ and letting $\mathbf{E}$ and $\mathbf{B}$ be the ensemble averaged fields, neglecting microscopic self-generated fields, this equation becomes the Vlasov equation. It describes how the distribution function evolves over time, subject to given, self-consistent fields and in the absence of particle interactions.

The Vlasov equation is not sufficient to completely describe a plasma. Collisions, which intrinsically depend on the short-range self-generated fields, generally play an important role in the plasma dynamics and so must be accounted for. This is usually done by adding a collisional term to the right hand side of (2.2) to produce the Boltzmann equation

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = C[f_s]. \tag{2.3}$$

In general, the form of the collisional term is some complicated integral so the Boltzmann equation is an integro-differential equation which are notoriously difficult to handle. But since collisions in plasmas are predominantly small-angle, causing only cause small deflections, the integral form can be simplified into a manageable form known as the Fokker-Planck equation [13]. Before proceeding, it is worth stressing the natural property that the collisional operator's zeroth and energy velocity moments vanish, because collisions conserve particles and energy [14].

Even after the inclusion of collisions the Boltzmann equation alone is still not sufficient to fully describe the plasma. The electric and magnetic fields too have to be specified in some fashion. Specifically, they have to be determined in a self-consistent way that takes into account the self-generated macroscopic fields. This is naturally done through Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2.4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.5}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}, \tag{2.6}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{2.7}$$

where the charge density $\rho$ and current density $\mathbf{j}$ are given by moments of the distribution functions

$$\rho = \sum_s Z_s e \int \mathrm{d}^3 v f_s, \tag{2.8}$$

$$\mathbf{j} = \sum_s Z_s e \int \mathrm{d}^3 v \mathbf{v} f_s. \tag{2.9}$$

Equations (2.3)-(2.7) are sufficient to describe all non-relativistic plasma phenomena. Indeed, they describe the plasma too completely. This is because they describe everything from light waves, through fast gyromotion, to slow bulk transport. It is not possible to describe everything within this spectrum, either analytically or computationally, because of the huge scale separation involved. Much work has therefore been done over the last decades to reduce these equations in some fashion and restrict them to only describe a select subset of phenomena. Continuing in this tradition, we will proceed to do exactly this in the coming Chapters.

Because this thesis is focused on terrestrial fusion plasmas with negligible relativistic and Debye-scale effects, there are two natural restrictions we can immediately use. Neglecting light waves, we will assume that $v_{th,s}^2/c^2 \ll 1$ and $\rho_s^{-2}\lambda_{De}^2 \ll 1$ where $v_{th,s}$ is the thermal velocity of species $s$, $\rho_s$ the typical gyroradius, and $\lambda_{De}$ the Debye length. The first of these conditions allows us to ignore the displacement current in Ampère's law so it becomes

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}, \tag{2.10}$$

while the second leads us to replace Gauss' law with the quasineutrality condition[10]

$$\rho = 0. \tag{2.11}$$

Of course, these limitations are not enough to make our equation system tractable by themselves and we will have to restrict ourselves further. To this end, we will employ a mathematical tool which will be described in the next Section.

## 2.2 Multiscale Analysis

While there are many different ways one can go about using the Fokker-Planck equation to produce interesting physics, here we will describe a tool for this task that is of primary importance for us. This tool is known as multiscale analysis, or asymptotology [15], or perturbation methods [16], and can be found in many different mathematical branches. As the name suggests, it relies upon an inherent, large separation between different scales which allows one to treat them differently in the problem at hand. Formally, if a problem contains a small parameter $\epsilon \ll 1$ such as the ratio between two different scales then different terms can be expanded in orders of $\epsilon$.

In multiscale analysis one is more interested in how different quantities scale with $\epsilon$ rather than what the actual value of $\epsilon$ is [16]. In a physics context, this allows one to see how the dynamics of different orders, or equivalently scales, and the interactions between them behave. Usually the lowest order dynamics alone is that which is studied, neglecting other phenomena as higher order contributions. Naturally, the lower order dynamics are expected to dominate when $\epsilon$ is small. Indeed, in the not uncommon case where the multiscale expansion fails to converge for finite $\epsilon$-values the lowest order dynamics may still prove useful. This is because the dynamics of different orders correspond to separate kinds of physics, interacting only appreciably for larger $\epsilon$-values.

Using multiscale analysis properly is not a straightforward matter, especially in large, complex problems such as the one of interest to us. Choosing how to order various terms to encapsulate the desired phenomena is almost an art. Failing to make a proper choice will at best mean that the resulting theory will describe some other set of physics than that of interest. In addition to this delicate choice, there are also technical difficulties that have to be dealt with. Typically, in conjunction with the ordering choice, one introduces various quantities to study. Careless choices at this stage could prove disastrous as the resulting equation system for these quantities could be either under- or overdetermined. In the former case the lowest order evolution of some quantities can only be determined by going to higher order, which in turn introduces other quantities whose evolution must be determined at even higher order, *ad infinitum* [15]. Such an ordering choice fails to simplify the problem because it retains the full information of the original equation system and all its dynamics. Thankfully for the purposes of this thesis, there is a remedy to underdetermination of which we will make liberal use. That remedy is annihilation operators, and for our purposes specifically the gyroaverage, the flux-surface average, and the turbulence average which will be introduced in Sections 2.4-2.6. Applying these operators will remove higher order terms and allow our equations to be closed at a finite order.

So far, this Section has been an abstract description of multiscale analysis. Returning to the purpose of this thesis, we will now narrow the scope. Terrestrial, magnetically contained fusion plasmas are invariably strongly magnetised. That is, the magnetic field is strong enough that the typical gyroradius $\rho_s$ of plasma particles is much smaller than any other important length scale. It is therefore only natural to use this scale separation to apply multiscale analysis with the ratio between $\rho_s$ and some other typical length scale as the expansion parameter. Indeed, this is a natural basis for gyrokinetics [10] and will also be the basis for the theory to be developed in the next Chapter. Of course, this is not the only kind of scale separation present and selecting how to order these scale separations separates different theories. We will go into greater detail of these orderings and their justifications for both gyrokinetics and the new set of equations we will introduce in the following Chapters. However, we will study one such scale separation further in the next Section, that between the fast gyromotion and other slow particle motion.

## 2.3 Straight Field Line Motion

Generally, in order to solve the equations of motion (2.1) analytically, the field configuration has to be in some kind of tractable form. Even without accounting for short-range, self-generated fields, this is not the case in a tokamak so some approximation has to be employed. The starting point for this is that the plasma is suspended in a strong magnetic field. Now, if the magnetic field is uniform in the direction of $\mathbf{b}$, it is obvious that particles are constrained in the perpendicular direction to gyrate around a guiding centre $\mathbf{R}_s$ with gyrofrequency

$$\Omega_s = \frac{q_s B}{m_s c}, \tag{2.12}$$

and gyroradius

$$\rho_s = \frac{v_\perp}{\Omega_s}, \tag{2.13}$$

like

$$\mathbf{v} = v_\parallel \mathbf{b} + v_\perp(\cos\vartheta \mathbf{e}_1 - \sin\vartheta \mathbf{e}_2), \tag{2.14}$$

where we have introduced orthogonal unit vectors satisfying

$$\mathbf{b} = \mathbf{e}_1 \times \mathbf{e}_2. \tag{2.15}$$

Of course, in a tokamak the magnetic field is not uniform. However, because the magnetic field is so strong, the gyroradius is much smaller than the typical variation length $a$ of the magnetic field. Therefore, to lowest order in $\rho_s/a$ and in the absence of collisions, gyration around the field lines is still the dominant particle motion in tokamaks. Nevertheless, it is known that collisional transport is insufficient to account for the bulk of tokamak transport so that the full transverse motion has to be specified further.

In studying particle motion beyond mere gyration, the first thing to note is that in the presence of a spatially constant force $\mathbf{F}$, a superposition of the aforementioned gyromotion and a transverse drift

$$\frac{c}{q_s B^2}\mathbf{F} \times \mathbf{B}, \tag{2.16}$$

clearly satisfies the equation of motion (2.1) [17]. The first such drift of importance for this thesis is that arising from the electric field which we will denote by

$$\mathbf{u}_E = \frac{c}{B^2}\mathbf{E} \times \mathbf{B}. \tag{2.17}$$

In this thesis, this drift will prove to be of utmost importance, since its corresponding term in the Fokker-Planck equation (2.3) will give rise to nonlinearity and produce the dominant transport.

The second type of transverse drift is that arising from the non-uniformity of the magnetic field which we denote by

$$\mathbf{V}_{Ds} = \frac{\mathbf{b}}{\Omega_s} \times \left( v_\parallel^2 \mathbf{b} \cdot \nabla \mathbf{b} + \frac{1}{2} v_\perp^2 \nabla \ln B \right). \tag{2.18}$$

In the first of these term, the force in question is the centripetal force that keeps the guiding centre moving along the curved magnetic field lines. The second term in contrast corresponds to a fictitious force, accounting for how a non-uniform magnetic field strength causes the gyroradius to vary, as is obtained by Taylor-expanding the magnetic field around the guiding centre and averaging the Lorentz force over an assumed gyration [17].

In addition to these two perpendicular drifts there is one more kind of perpendicular motion which will be of interest to us. Unlike the previous drifts, this motion is not a guiding centre drift. Instead, it is a phenomenon which arises from the collective motion of many particles, all undergoing gyromotion. When determining the total fluid velocity of all these particles, if they are not uniformly distributed there will be an imbalance in the amount of particles crossing surfaces during gyromotion from either direction. As a result there is a net fluid velocity

$$\frac{c}{q_s n_s B^2}\mathbf{B} \times \nabla p_s, \tag{2.19}$$

called the diamagnetic drift [18].

Before we proceed further there is one final topic of particle gyromotion we must discuss, the so called first adiabatic invariant. It is a commonly derived result that the magnetic moment

$$\mu_s = \frac{m_s v_\perp^2}{2B}, \tag{2.20}$$

of a particle is to lowest order conserved in slowly varying, compared to the gyrofrequency, electric and magnetic fields like those of interest to us [17]. Indeed, it can be shown to be the lowest order contribution to an exact invariant corresponding to the gyromotion [18, 19]. In the next Section, we will see why this is an important feature for a convenient mathematical description of gyromotion.

## 2.4 Catto-Transformed variables

For containment purposes rapid gyromotion is of little interest compared to slow drifts across the containment vessel. Hence, when one wishes to describe transport phenomena like in gyrokinetics or our proposed TKRMHD this rapid motion should be averaged away. To this end, we return to the fundamental Fokker-Planck equation (2.3) which is expressed in conventional $(\mathbf{r}, \mathbf{v})$-variables. Though suitable for comparison with experiments, these make it difficult to separate the two different kinds of motion for analytical purposes. Instead we will make use of so called Catto-transformed variables [20]. These are given by $(\mathbf{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma)$, where $\mathbf{R}_s$ is the guiding centre position, $\varepsilon_s$ the particle energy, $\mu_s$ the exact first adiabatic invariant, $\vartheta$ the gyroangle, and $\sigma \in \{-1, 1\}$ indicates the direction of motion along the field line[21],

$$\mathbf{R}_s = \mathbf{r} - \frac{1}{\Omega_s}\mathbf{b} \times \mathbf{v}, \tag{2.21}$$

$$\varepsilon_s = \frac{1}{2}m_s v^2 + Z_s e\varphi, \tag{2.22}$$

$$\mu_s = \frac{m_s v_\perp^2}{2B} + \mathcal{O}\left(\epsilon\frac{T}{B}\right). \tag{2.23}$$

Because $\dot{\mu}_s = 0$ in the absence of collisions, in Catto-transformed variables the Fokker-Planck equation becomes

$$\left(\left.\frac{\partial}{\partial t}\right|_{\mathbf{R}_s,\varepsilon_s,\mu_s,\vartheta} + \dot{\mathbf{R}}_s \cdot \left.\frac{\partial}{\partial \mathbf{R}_s}\right|_{t,\varepsilon_s,\mu_s,\vartheta} + \dot{\varepsilon}_s \left.\frac{\partial}{\partial \varepsilon_s}\right|_{t,\mathbf{R}_s,\mu_s,\vartheta} + \dot{\vartheta}_s \left.\frac{\partial}{\partial \vartheta}\right|_{t,\mathbf{R}_s,\varepsilon_s,\mu_s}\right) f_s = C[f_s]. \tag{2.24}$$

where all partial derivatives have explicitly been taken while holding remaining variables fixed. As useful as this form will be for us in deriving TKRMHD and manipulating its equations the original $(\mathbf{r}, \mathbf{v})$-coordinates are more intuitive and useful for comparison with experiments. We will therefore find it convenient to switch between these two systems, both by switching derivatives with the multivariable chain rule and by Taylor expanding to switch between evaluating quantities at $\mathbf{R}_s$ or $\mathbf{r}$. Because we for convenience's sake will neglect specifying what quantities are kept fixed in these partial derivatives one simple rule should be kept in mind to avoid confusion regarding the time-derivative: if no partial $\mathbf{R}_s$-derivative is present in the equation the time-derivatives that appear are taken with $(\mathbf{r}, \mathbf{v})$ fixed.

Having introduced Catto-transformed variables there are now two more properties of these variables that we will later find useful. The first is the explicit form of complete velocity space integration in these coordinates which is given by

$$\int \mathrm{d}^3 v a = \sum_\sigma \int_{Z_s e\varphi}^{\infty} \mathrm{d}\varepsilon_s \int_0^{\varepsilon_s - Z_s e\varphi} \mathrm{d}\mu_s \int_0^{2\pi} \frac{B \mathrm{d}\vartheta}{m_s^2 |v_\parallel|} a, \tag{2.25}$$

where $a$ is an arbitrary function.

The second is the gyroaverage which is of vital importance for this thesis. This is because the gyroaverage is the first of the annihilation operators we will use to produce a closed equation system as described in Section 2.2. Additionally, it is also the reason Catto-transformed variables lend themselves so well to discarding uninteresting gyromotion. There are two different forms this operation can take, either

$$\langle a \rangle_{\mathbf{R}_s} = \frac{1}{2\pi} \oint \mathrm{d}\vartheta a(\mathbf{R}_s, \varepsilon_s, \mu_s, \vartheta, \sigma), \tag{2.26}$$

or

$$\langle a \rangle_{\mathbf{r}} = \frac{1}{2\pi} \oint \mathrm{d}\vartheta a(\mathbf{R}_s(\mathbf{r}, v_\perp, \vartheta), \varepsilon_s(\mathbf{r}, v_\parallel, v_\perp, \vartheta), \mu_s(\mathbf{r}, v_\parallel, v_\perp, \vartheta), \sigma), \tag{2.27}$$

where $(\varepsilon_s, \mu_s, \sigma, \mathbf{R}_s)$ or $(\mathbf{r}, v_\parallel, v_\perp)$ are held constant, respectively [10]. The former of these corresponds to averaging over a particle's gyromotion, while the second corresponds to averaging over all possible guiding centres at fixed distance from $\mathbf{r}$.

## 2.5 Field Structure

Having described how to eliminate uninteresting gyromotion using the gyroaverage in the previous Section we now turn to the quantities to which we will apply this operation: the distribution functions $f_s$. It is commonly seen in fusion plasmas that a comparatively quiescent equilibrium can be distinguished from turbulent fluctuations. In the core, the equilibrium constitutes the bulk of the plasma whose transport properties we obviously wish to accurately describe so as to be able create a sustainable fusion reaction. However, we also know that the fluctuations give rise to the transport which presently prevents sustainable fusion. Both must therefore be included in any useful theory, including ours.

To distinguish the quiescent equilibrium and turbulent fluctuations from both each other and the total quantities we will be using two different notational conventions. For scalar quantities like the distribution functions we will use a $\delta$ to signify the fluctuating piece while the mean part, when discernible, will be represented by a capital letter, like so

$$f_s = F_s + \delta f_s. \tag{2.28}$$

The only scalar quantity which will diverge from this pattern is the temperature $T_s$, which will everywhere denote the full temperature.

Now for the electric and magnetic fields (and their magnitudes) we will use a slightly different convention because we wish to keep the interpretation that lower case vectors correspond to unit vectors. Thus we will keep the $\delta$ to signify fluctuations but change the total fields to be denoted by a tilde, so that for the magnetic field we have

$$\tilde{\mathbf{B}} = \mathbf{B} + \delta\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \delta\mathbf{B}. \tag{2.29}$$

Here we have additionally introduced a decomposition of $\mathbf{B}$ into the time-independent confinement magnetic field $\mathbf{B}_0$, which is to be specified independently, and the plasma-generated non-fluctuating magnetic field $\mathbf{B}_1$. The notation here is suggestive, as the former in this thesis will be larger than the latter in our multiscale orderings. Therefore we will frequently make use of the lowest order expression

$$|\tilde{\mathbf{B}}| = \tilde{B} = |\mathbf{B}_0| = B, \tag{2.30}$$

and define

$$\mathbf{b} = \frac{\mathbf{B}_0}{B}, \quad \tilde{\mathbf{b}} = \frac{\tilde{\mathbf{B}}}{B}. \tag{2.31}$$

Having introduced the decomposition (2.29) we will find it suitable to introduce the potentials $\tilde{\mathbf{A}}$ and $\varphi$. We specify that the vector potential $\tilde{\mathbf{A}}$ satisfies the Coulomb gauge

$$\nabla \cdot \tilde{\mathbf{A}} = 0, \tag{2.32}$$

and that it only determines the plasma-generated magnetic field

$$\mathbf{B}_1 + \delta\mathbf{B} = \nabla \times \tilde{\mathbf{A}}, \tag{2.33}$$

so that the fields in terms of potentials are given by

$$\tilde{\mathbf{B}} = \mathbf{B}_0 + \nabla \times \tilde{\mathbf{A}} \tag{2.34}$$

and

$$\tilde{\mathbf{E}} = -\nabla\varphi - \frac{1}{c}\frac{\partial \tilde{\mathbf{A}}}{\partial t}. \tag{2.35}$$

For the scalar potential $\varphi$ we will commonly find that it is composed of only a fluctuating part,

$$\varphi = \delta\varphi. \tag{2.36}$$

This is because of an important and severe limitation that we will make in this thesis. We will assume that the bulk plasma is non-rotating. Then Debye-screening will rapidly diminish the electric field strength inside the plasma and no large electric field can be set up [22]. Of course substantial bulk rotation is present in all tokamaks for stability purposes and should be accounted for. However, its inclusion was omitted to limit the scope of this thesis. Although it would have given rise to a plethora of new terms in the equations of the following Chapters the logic of the arguments presented would not substantially change.

Although we have introduced a division into fluctuating and mean quantities we have so far not specified precisely how this done. Proceeding to do just this, we now introduce the second annihilation operator hinted at in Section 2.2: the turbulence average. Typically, turbulent fluctuations occur at a perpendicular length scale $l_\perp$ and frequency $\omega$ much smaller and higher than those associated with the system, which are $a$ and $\tau_E^{-1}$ respectively. The turbulence average is then defined by picking intermediate scales $\lambda_\perp$ and $T^{-1}$ and averaging over a perpendicular patch $p$ of length scale $\lambda_\perp$ and an interval of $T$ in the following way:

$$\langle a \rangle_{turb} = \frac{1}{T \int_p \mathrm{d}^2 r_\perp} \int_{t-T/2}^{t+T/2} \mathrm{d}t' \int_p \mathrm{d}^2 r_\perp a(\mathbf{r}_\perp, l, \mathbf{v}, t). \tag{2.37}$$

Here $l, \mathbf{v}, t$, where $l$ is the distance along a field line, are kept constant [10]. This operation now allows us to uniquely determine $\delta f_s$ by requiring its turbulence average to vanish.

## 2.6 Field Geometry

In the previous Section we dealt with the difference between fluctuating and mean quantities. Now we will proceed by elaborating on the structure of the mean magnetic field. Since we are interested in describing tokamaks the first obvious point to note is the toroidal topology of the containment field. Specifically, inside the core there exists a set of nested toroidal surfaces called flux surfaces whose normal vector is everywhere tangent to the magnetic field. The innermost of these is a simple circle called the magnetic axis and the outermost, beyond which the field lines become "open" and divert to the tokamak boundary, is intuitively called the last closed flux surface (LCFS) [4]. Even though the focus in this thesis is the plasma edge we will not at any point attempt to venture from the closed-field line region beyond the LCFS to the scrape off layer.

The reason for limiting ourselves to the closed field region is because by remaining there we gain access to the final annihilation operator we need to close our equation system: the flux surface average. This operator is most readily employed by switching to Clebsch coordinates, a set of spatial coordinates that encapsulate the magnetic field structure [23]. We will introduce two such sets of coordinates, $(\tilde{\psi}, \tilde{\alpha}, \tilde{l})$ for the exact field and $(\psi, \alpha, l)$ for the equilibrium field. In what follows we will use the exact coordinates, but entirely analogous relations will hold for the equilibrium coordinates.

The defining relation for our Clebsch coordinates is

$$\tilde{\mathbf{B}} = \nabla\tilde{\psi} \times \nabla\tilde{\alpha} = \tilde{B}\nabla\tilde{l}. \tag{2.38}$$

Furthermore, $\tilde{\psi}$ is chosen to be the poloidal flux function which is constant on every flux surface and defined by

$$\tilde{\psi} = \int_{S_P} \mathrm{d}\mathbf{S} \cdot \tilde{\mathbf{B}}, \tag{2.39}$$

where $S_P$ is the surface of constant poloidal angle between the flux surface and the magnetic axis [4]. After then choosing a suitable cut-off line on the flux surface so that this system becomes single-valued over the (exact) flux surface $\tilde{l}$ becomes the length from the branch cut along magnetic field lines and $\tilde{\alpha}$ a index coordinate for different magnetic field lines.

Given the intimate link between Clebsch coordinates and the magnetic field structure as outlined above it is intuitive why they simplify flux surface averaging so much. Because the poloidal flux $\tilde{\psi}$ is constant over flux surfaces we can choose to define the flux surface average as

$$\langle a\rangle_{\tilde{\psi}} = \lim_{\Delta\tilde{\psi}\to 0}\left(\int_{\Delta(\tilde{\psi})} \mathrm{d}^3\mathbf{r}\, a \bigg/ \int_{\Delta(\tilde{\psi})} \mathrm{d}^3\mathbf{r}\right), \tag{2.40}$$

where $\Delta(\tilde{\psi})$ is the volume enclosed between the flux surfaces labelled by $\tilde{\psi}$ and $\tilde{\psi}+\Delta\tilde{\psi}$. This can be shown to be equivalent to

$$\langle a\rangle_{\tilde{\psi}} = \left(\frac{\partial V}{\partial\tilde{\psi}}\right)^{-1}\int_{S_{\tilde{\psi}}} \frac{\mathrm{d}S}{|\nabla\tilde{\psi}|} a = \frac{1}{V'}\int_{S_{\tilde{\psi}}} \frac{\mathrm{d}S}{|\nabla\tilde{\psi}|} a, \tag{2.41}$$

where $S_{\tilde{\psi}}$ is the flux surface labelled by $\tilde{\psi}$ and $V$ the volume it encloses [10]. By then using the defining relation (2.38) to find the Jacobian $1/\tilde{B}$ for our Clebsch coordinates we arrive at the final, simple expression

$$\langle a\rangle_{\tilde{\psi}} = \frac{1}{V'}\int \frac{\mathrm{d}\tilde{l}\mathrm{d}\tilde{\alpha}}{\tilde{B}} a \tag{2.42}$$

for the flux surface average [24].

Equipped with the final expression for the flux surface average we can now immediately deduce that applying it to any term of the form

$$\tilde{B}\frac{\partial}{\partial\tilde{\alpha}}\bigg|_{\tilde{\psi},\tilde{l}} a, \quad \text{or} \quad \tilde{B}\frac{\partial}{\partial\tilde{l}}\bigg|_{\tilde{\psi},\tilde{\alpha}} a \tag{2.43}$$

will yield a vanishing result because $\tilde{l}$ and $\tilde{\alpha}$ by toroidal geometry are periodic coordinates. We will make use of this extensively in the subsequent Chapters to remove three particular classes of terms.

The first class of term that vanishes under the flux surface average is

$$\tilde{\mathbf{B}}\cdot\nabla a, \tag{2.44}$$

which is easily seen to be precisely the second term in (2.43). The second kind is the Poisson bracket

$$\{b,c\} = \tilde{\mathbf{b}}\cdot\nabla b\times\nabla c = \tilde{B}\left(\frac{\partial b}{\partial\tilde{\psi}}\frac{\partial c}{\partial\tilde{\alpha}} - \frac{\partial c}{\partial\tilde{\psi}}\frac{\partial b}{\partial\tilde{\alpha}}\right), \tag{2.45}$$

where the final equality holds by virtue of the defining relation (2.38). Now if $b$ is a function of $\tilde{\psi}$ alone then the second of these terms vanishes while in the first term $b$ can be pulled through the $\tilde{\alpha}$-derivative and vice versa for $c$. In either case the Poisson bracket reduces to the first term in (2.43) with $a = bc$ and so vanishes under flux-surface averaging.

The third and final class of terms which vanishes by flux surface averaging have the structure

$$\nabla \cdot \left( a \nabla \times \tilde{\mathbf{b}} \right) = \nabla \cdot \left( \tilde{\mathbf{b}} \times \nabla a \right). \tag{2.46}$$

To see that this vanishes under flux surface integration we use the specific form of the divergence in Clebsch coordinates which, because $|\nabla \tilde{l}| = 1$ and the Jacobian is $\tilde{B}^{-1}$, is given by

$$\nabla \cdot \mathbf{C} = \tilde{B} \frac{\partial}{\partial \tilde{\psi}} \left( |\nabla \tilde{\alpha}| \mathbf{C} \cdot \nabla \tilde{\psi} \right) + \tilde{B} \frac{\partial}{\partial \tilde{\alpha}} \left( |\nabla \tilde{\psi}| \mathbf{C} \cdot \nabla \tilde{\alpha} \right) + \tilde{B} \frac{\partial}{\partial \tilde{l}} \left( \frac{1}{\tilde{B}} \mathbf{C} \cdot \nabla \tilde{l} \right). \tag{2.47}$$

Here the last two terms clearly vanish under flux surface averaging and so, by also using the Poisson bracket (2.45), we find that

$$\left\langle \nabla \cdot \left( a \nabla \times \tilde{\mathbf{b}} \right) \right\rangle_{\tilde{\psi}} = \frac{1}{V'} \int \mathrm{d}\tilde{l} \mathrm{d}\tilde{\alpha} \frac{\partial}{\partial \tilde{\psi}} \left( |\nabla \tilde{\alpha}| \tilde{\mathbf{b}} \times \nabla a \cdot \nabla \tilde{\psi} \right) = \frac{1}{V'} \int \mathrm{d}\tilde{l} \mathrm{d}\tilde{\alpha} \frac{\partial}{\partial \tilde{\psi}} \left( |\nabla \tilde{\alpha}| \tilde{B} \frac{\partial a}{\partial \tilde{\alpha}} \right). \tag{2.48}$$

Invoking axisymmetry the $\tilde{\alpha}$-derivative can be moved through the $\tilde{\psi}$-derivative, at which point it is clear that this expression vanishes [24].

# 3

# Toroidal Kinetic Reduced Magnetohydrodynamics

Equipped with the mathematical tools described in the previous Chapter we are now in a position to broach the main topic of this thesis: Toroidal Kinetic Reduced MHD (TKRMHD). The specific form of this theory which will be developed in this Chapter describes strong ion scale turbulence in the presence of sharp spatial gradients. It is very similar to the recently proposed theory for describing the edge of a tokamak operating in H-mode from Abel and Hallenbert [25], differing in principle only by having slightly shallower spatial gradients. Thus, in tokamaks it may be applicable in an intermediate region away from the gyrokinetic core toward the edge region of the LCFS. Its suggestive name, and indeed its structure too, owes its origin to Kulsrud's Kinetic MHD [26, 27] and Strauss' Reduced MHD [28], to whom it shares similarities. Like the former, it neglects the high-collisionality assumption of MHD by introducing a kinetic equation for the evolution of the non-Maxwellian distributions. Like the latter, it relies heavily on the spatial anisotropy introduced by the strong magnetic field for closure in the multiscale expansion. Nevertheless, it is distinct from both, which the additional descriptor "toroidal", referring to its inclusion of magnetic curvature effects, further clarifies.

In this Chapter we will derive the equations that constitute TKRMHD, starting by introducing and justifying a set of fundamental orderings which we will apply to the Fokker-Planck equation (2.24). By focusing on ion scale phenomena we will have to retain the full ion kinetic equation but will be able to solve the lowest order electron equation. In so doing we will reduce the full electron kinetic equation into two equations describing the evolution of the electron density $n_e$ and temperature $T_e$, yielding a hybrid kinetic/fluid theory. However, manipulating the Fokker-Planck equation in this fashion will fail to yield a closed set of equations. As such we will finally have to close the equation system by manipulating the momentum equation to produce a vorticity equation, describing the flux-surface-average component of the electrostatic potential $\varphi$.

## 3.1 Ordering Scheme

In order to derive TKRMHD from the Fokker-Planck equation (2.24) we will perform a multiscale expansion as described in Section 2.2. To this end we will need to analyse relevant scales and carefully choose how to order them with respect to each other for our present purpose of describing strong, gradient-driven plasma edge ion turbulence.

In any tokamak there are three obvious length scales that are set *a priori* by its design. These are the typical tokamak scale $a$ of the magnetic field and the typical ion and electron gyroradii $\rho_i$ and $\rho_e$. We also introduce two additional scales $l_\perp$ and $l_\parallel$ to describe the fluctuating fields and distributions, one perpendicular and one parallel. This is because the intense mean magnetic field creates strong anisotropy by heavily restricting cross-field transport while leaving parallel transport relatively unhindered.

In choosing how to order these length scales with respect to each other it makes sense, because parallel motion is so unconstrained, to set

$$l_\parallel \sim a, \tag{3.1}$$

where here and henceforth $\sim$ is to be interpreted as "is of the same order as". In contrast $l_\perp$, which for us will roughly correspond to the width of the edge region, is considerably smaller. Specifically, after using the strong magnetisation to assign

$$\epsilon \equiv \frac{\rho_i}{l_\perp} \ll 1 \tag{3.2}$$

as our fundamental expansion parameter we will set the anisotropy to be

$$\frac{l_\perp}{l_\parallel} \sim \epsilon. \tag{3.3}$$

In terms of the conventional geometric tokamak quantities, the safety factor $q$ and the tokamak major radius $R$, we have $a \sim qR$. Therefore these two orderings correspond to the scaling relation

$$l_\perp \sim \sqrt{\rho_i q R} \tag{3.4}$$

for the width of the edge region described by TKRMHD.

Proceeding to the time scales, we wish to describe plasma bulk transport which near the edge occurs on the ion streaming time scale. Therefore we choose

$$\omega \sim \frac{v_{th,i}}{l_\parallel}, \tag{3.5}$$

where

$$v_{th,i} = \sqrt{\frac{2T_i}{m_i}} \tag{3.6}$$

is the ion thermal velocity so that, by using (3.3), we have

$$\omega \sim \epsilon^2 \Omega_i, \tag{3.7}$$

where

$$\Omega_i = \frac{Z_i e \tilde{B}}{m_i c} \tag{3.8}$$

is the typical ion gyrofrequency. With this timescale set we then order the collision timescale to be the same,

$$\frac{\nu_{ii}}{\omega} \sim 1. \tag{3.9}$$

By doing so we can capture the phenomena in the entire range from weakly-collisional to collisionless physics.

Of course, we are more interested in perpendicular drifts across the field than parallel drifts, since these give rise to transport out of the tokamak. It is known that anomalous transport is the dominant transport mechanism in tokamaks. This transport is induced by the $\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}$ drift and is associated with the nonlinear ion timescale

$$\mathbf{u}_E \cdot \nabla \sim \frac{|\mathbf{u}_E|}{l_\perp} = \frac{1}{l_\perp} \left| \frac{c}{B} \tilde{\mathbf{b}} \times \nabla \delta\varphi \right|, \tag{3.10}$$

which we must therefore require to be of the same order as (3.5). This requires us to consider the electric field as large by setting

$$\frac{e\varphi}{T_i} \sim 1. \tag{3.11}$$

This ordering, in turn, requires us to consider $\mathcal{O}(1)$ fluctuations in the distribution functions,

$$\frac{\delta f_s}{f_s} \sim 1, \tag{3.12}$$

because of the large Boltzmann response to the electric field [29]. This is only natural because of the steep gradients and low density in the edge. These conditions cause such strong turbulence that it becomes fruitless to attempt to discern any quiescent equilibrium on the timescale of interest.

The fact that the density and temperature is so low in the edge region makes it reasonable that we should treat the plasma $\beta$ as small, which indeed is the case in present tokamaks. Indeed, in order for our plasma not to be ballooning unstable it can at most be of order

$$\beta = 8\pi \sum_s \frac{n_s T_s}{\tilde{B}^2} \sim \frac{l_\perp}{l_\parallel} \sim \epsilon. \tag{3.13}$$

This is because this instability is driven by pressure and stabilised by magnetic curvature which are characterised by $l_\perp$ and $l_\parallel$ respectively [3, 30]. Naturally, we choose to make use of this maximal ordering so as to capture all relevant physics from when the plasma is stable below this limit to when instabilities arise.

As we have previously mentioned, we will be interested in ion-scale phenomena because these dominate transport. The electrons, however, are still a very important part of the system and we must choose how order their dynamics with respect to the ion dynamics. Though they may differ substantially in a tokamak, there is no inherent reason to order ion and electron temperatures differently in a scaling sense. Therefore we should order their ratio as

$$\frac{T_e}{T_i} \sim 1. \tag{3.14}$$

Then, with our $\beta$-ordering (3.13), if we do not wish to discard electron kinetic effects in Alfvén waves we must order the electron/ion mass ratio as

$$\frac{m_e}{m_i} \sim \beta \sim \epsilon. \tag{3.15}$$

With this ordering electron and ion parallel motion will explicitly separate in our ordering scheme. This will prove useful and allow us to solve the electron dynamics to lowest order by separating similar terms for electrons and ions by order $\epsilon^{1/2}$.

All that remains now is to order the fluctuating fields with respect to the mean fields. Of course, since we specify a non-rotating plasma the mean electric field vanishes and we need only consider the magnetic field. Naturally, by our $\beta$-ordering (3.13), $\tilde{\mathbf{B}}_1$ should be small compared to $\tilde{\mathbf{B}}_0$ because the low pressure of the plasma should not seriously affect the intense external confining field. Indeed, for this reason it must vary on the perpendicular scale $l_\perp$ and parallel scale $l_\parallel$. This implies that

$$\frac{|\tilde{\mathbf{B}}_1|}{|\mathbf{B}_0|} \sim \frac{l_\perp}{l_\parallel} \sim \epsilon, \tag{3.16}$$

because the magnetic field lines are displaced by at most $l_\perp$ over a distance of $l_\parallel$ and these expressions are then both expressions for the tangent of the field lines.

Summarising the orderings we have chosen in this Section we will in our multiscale expansion use

$$\frac{l_\perp}{l_\parallel} \sim \beta \sim \frac{m_e}{m_i} \sim \frac{|\tilde{\mathbf{B}}_1|}{|\mathbf{B}_0|} \sim \frac{\rho_i}{l_\perp} \equiv \epsilon, \quad \frac{e\varphi}{T_i} \sim \frac{T_e}{T_i} \sim \frac{\delta f_s}{f_s} \sim 1, \quad \frac{\omega}{\Omega_i} \sim \frac{\nu_{ii}}{\Omega_i} \sim \epsilon^2. \tag{3.17}$$

With these in hand, we can make use of them in the next Section and finally begin to derive the equations that constitute TKRMHD.

## 3.2 Derivation of Equation System

Having introduced our ordering scheme (3.17) we now return to the Fokker-Planck equation (2.24) in Catto-transformed variables (2.21)-(2.23). First we note that, to lowest order, the derivatives present in this equation are given by [10]

$$\dot{\mathbf{R}}_s = v_\parallel \tilde{\mathbf{b}} + \frac{c}{B}\mathbf{b} \times \nabla \left( \varphi - \frac{\mathbf{v} \cdot \tilde{\mathbf{A}}}{c} \right) + \mathcal{O}(\epsilon^2 v_{th,s}) \tag{3.18}$$

$$\dot{\varepsilon}_s = Z_s e \frac{\partial}{\partial t} \left( \varphi - \frac{\mathbf{v} \cdot \tilde{\mathbf{A}}}{c} \right) \tag{3.19}$$

$$\dot{\vartheta} = \Omega_s + \mathcal{O}(\epsilon \Omega_s) \tag{3.20}$$

where to lowest order

$$\tilde{\mathbf{b}} = \mathbf{b} + \frac{1}{B} \nabla \times \tilde{\mathbf{A}} = \mathbf{b} - \frac{1}{B}\mathbf{b} \times \nabla \tilde{A}_\parallel. \tag{3.21}$$

Second, because our orderings render us unable to discern any quiescent mean distribution within the large-amplitude turbulence we will expand $f_s$, $\varphi$, and $\tilde{\mathbf{A}}$ like

$$f_s = f_s^{(0)} + f_s^{(1/2)} + f_s^{(1)} + ... \tag{3.22}$$

where the superscript indicate the $\epsilon$-order. The reason we expand in powers of $\epsilon^{1/2}$ instead of $\epsilon$ is because $v_{th,e}$ is $\epsilon^{1/2}$ larger than $v_{th,i}$ in our ordering (3.17). Therefore terms proportional to $v_{th,s}$ will differ by this order for electrons and ions, requiring us to expand like this to not disregard interactions between electron and ions arising from these terms.

With these preliminary considerations we can now proceed by matching terms order by order in the Fokker-Planck equation. Thankfully we only need repeat this process until the lowest order dynamic equation where we will be able to close our equation system. For ions, the fourth term in (2.24) dominates to $\mathcal{O}(\epsilon v_{th,i} f/l_\parallel)$. Thus we have

$$\frac{\partial f_s^{(0)}}{\partial \vartheta} = 0, \quad \frac{\partial f_s^{(1/2)}}{\partial \vartheta} = 0 \tag{3.23}$$

so the ions are gyrophase independent up to $\mathcal{O}(\epsilon f)$.

Proceeding to $\mathcal{O}(\epsilon v_{th,i}/l_\parallel)$ many new terms enter, including the time derivative, and so we find the dynamical equation

$$\begin{aligned} &\frac{\partial f_i^{(0)}}{\partial t} + \left[v_\parallel \mathbf{b} + \frac{c}{B}\mathbf{b} \times \nabla\left(\varphi - \frac{\mathbf{v}\cdot\tilde{\mathbf{A}}}{c}\right)\right] \cdot \frac{\partial f_i^{(0)}}{\partial \mathbf{R}_i} \\ &+ Z_i e \frac{\partial}{\partial t}\left(\varphi - \frac{\mathbf{v}\cdot\tilde{\mathbf{A}}}{c}\right)\frac{\partial f_i^{(0)}}{\partial \varepsilon_i} + \Omega_i \frac{\partial f_i^{(1)}}{\partial \vartheta} = C[f_i^{(0)}]. \end{aligned} \tag{3.24}$$

However, the equation above contains $f_i^{(1)}$, of which we have no knowledge. Thankfully, this term is annihilated by the the gyroaverage (2.26) and so after applying it we are left with the first equation of TKRMHD, the ion kinetic equation:

$$\begin{aligned} &\frac{\partial f_i^{(0)}}{\partial t} + \left[v_\parallel \mathbf{b} + \frac{c}{B}\mathbf{b} \times \nabla\left(\varphi - \frac{v_\parallel \tilde{A}_\parallel}{c}\right)\right] \cdot \frac{\partial f_i^{(0)}}{\partial \mathbf{R}_i} \\ &+ Z_i e \frac{\partial}{\partial t}\left(\varphi - \frac{v_\parallel \tilde{A}_\parallel}{c}\right)\frac{\partial f_i^{(0)}}{\partial \varepsilon_i} = C[f_i^{(0)}]. \end{aligned} \tag{3.25}$$

Here one should remember that all quantities are specifically to be evaluated at $\mathbf{R}_i$ and that partial derivatives are to be taken keeping other Catto-transformed variables fixed. In particular, this means that the time-derivative is not the usual one, but also contains a piece owing to the rapid variation of $\varphi$.

Having determined the TKRMHD ion kinetic equation we now turn to the electrons. Similar to ions the electron Fokker-Planck equation (2.24) is dominated by the gyrophase dependent term and so to zeroth order $f_e^{(0)}$ is gyrophase independent. However, because $v_{th,e}$ and thus $\nu_{ee}$ is $\mathcal{O}(\epsilon^{-1/2})$ larger than their ion equivalents, for electrons the collision term and terms linear to $\mathbf{v}$ are moved from second to first order compared to the ion equation. Accordingly the $\mathcal{O}(\epsilon^{1/2} v_{th,i} f/l_\parallel)$ electron Fokker-Planck equation becomes

$$\left[v_\parallel \mathbf{b} - \frac{\mathbf{b}}{B} \times \nabla(\mathbf{v}\cdot\tilde{\mathbf{A}})\right] \cdot \frac{\partial f_e^{(0)}}{\partial \mathbf{R}_e} + \frac{e}{c}\frac{\partial}{\partial t}(\mathbf{v}\cdot\tilde{\mathbf{A}})\frac{\partial f_e^{(0)}}{\partial \varepsilon_e} + \Omega_e \frac{\partial f_e^{(1/2)}}{\partial \vartheta} = C[f_e^{(0)}]. \tag{3.26}$$

After now applying the gyroaverage (2.26) and using equation (3.21) this becomes

$$v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial f_e^{(0)}}{\partial \mathbf{R}_e} + \frac{e v_\parallel}{c}\frac{\partial \tilde{A}_\parallel}{\partial t}\frac{\partial f_e^{(0)}}{\partial \varepsilon_e} = C[f_e^{(0)}]. \tag{3.27}$$

Equation (3.27) is a further constraint on the lowest order electron distribution $f_e^{(0)}$ that we can use to examine its structure. The first step we will perform to this end is, for reasons that will shortly becomes transparent, multiplying it by $1+\ln f_e^{(0)}$ to transform it into

$$v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}_e} f_e^{(0)} \ln f_e^{(0)} + \frac{ev_\parallel}{c} \frac{\partial \tilde{A}_\parallel}{\partial t} \frac{\partial}{\partial \varepsilon_e} f_e^{(0)} \ln f_e^{(0)} = (1 + \ln f_e^{(0)}) C[f_e^{(0)}]. \tag{3.28}$$

Proceeding by integrating this over all velocities the second term vanishes as can be seen by using the explicit expression for this operation in Catto-transformed variables (2.25). In order to also remove the first term we will be forced to assume the well-behaved closed field geometry described in Section 2.6. Upon then also applying the flux surface average and using (2.25) the first term can be changed into

$$\begin{aligned}\left\langle \int \mathrm{d}^3 v v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}_e} f_e^{(0)} \ln f_e^{(0)} \right\rangle_{\tilde{\psi}} &= \left\langle \tilde{\mathbf{B}} \cdot \frac{\partial}{\partial \mathbf{R}_e} \int \mathrm{d}^3 v \frac{v_\parallel}{\tilde{B}} f_e^{(0)} \ln f_e^{(0)} \right\rangle_{\tilde{\psi}} \\ &= \left\langle \tilde{\mathbf{B}} \cdot \nabla \int \mathrm{d}^3 v \frac{v_\parallel}{\tilde{B}} f_e^{(0)} \ln f_e^{(0)} \right\rangle_{\tilde{\psi}}\end{aligned} \tag{3.29}$$

to lowest order. This is clearly of the form (2.43) and so vanishes as desired. Thus, because the velocity integral of $C[f_e^{(0)}]$ vanishes by particle conservation, the original constraint (3.27) has been reduced to

$$\left\langle \int \mathrm{d}^3 v \mathbf{v} \ln f_e^{(0)} C[f_e^{(0)}] \right\rangle_{\tilde{\psi}} = 0. \tag{3.30}$$

To lowest order we can evaluate $f_e^{(0)}$ at $\mathbf{r}$ in (3.30) at which point it becomes a well-known constraint which arises in the proof of the Boltzmann H-theorem. The only function whose velocity-dependence satisfies it is a general Maxwellian with an arbitrary flow [10, 14]. However, because we have restricted our attention to stationary plasmas and because the only possible Boltzmann response we can have is that arising from $\varphi$, we can specify that

$$f_e^{(0)} = \eta_e(\mathbf{r}) \left( \frac{m_e}{2\pi T_e(\mathbf{r})} \right)^{3/2} \exp\left( -\frac{\varepsilon_e}{T_e(\mathbf{r})} \right) \tag{3.31}$$

where, to lowest order, $T_e$ is the electron temperature and $\eta_e$ is related to the electron density $n_e$ through

$$\eta_e = n_e \exp\left( -\frac{e\varphi}{T_e} \right). \tag{3.32}$$

Having deduced the Maxwellian nature of $f_e^{(0)}$ we now insert (3.31) into (3.27) to determine the properties of $T_e$ and $\eta_e$. After dividing by $f_e^{(0)}$ we find, to lowest order,

$$v_\parallel \tilde{\mathbf{b}} \cdot \nabla \left( \ln \eta_e - \frac{3}{2} \ln T_e \right) + \frac{\varepsilon_e}{T_e} v_\parallel \tilde{\mathbf{b}} \cdot \nabla \ln T_e - \frac{ev_\parallel}{cT_e} \frac{\partial \tilde{A}_\parallel}{\partial t} = 0. \tag{3.33}$$

This equation must obviously hold for arbitrary $\mathbf{v}$ so we can from the highest $v$-power, only present in the second term, conclude that $T_e$ must be an exact flux function,

$$T_e = T_e(\tilde{\psi}). \tag{3.34}$$

The fact that no electron temperature variation can occur on the exact flux surfaces should come as no surprise. By our ordering (3.17) the rapid thermal velocity of electrons causes electron convection along field lines to occur on a timescale much shorter than the processes we are attempting to describe, which occur on the sound time.

Using (3.34) in (3.33) we find the constraint

$$\tilde{\mathbf{b}} \cdot \nabla \ln \eta_e = \frac{e}{cT_e} \frac{\partial \tilde{A}_\parallel}{\partial t}. \tag{3.35}$$

This is an important result because it implies that the parallel electric field can be expressed as

$$\tilde{\mathbf{b}} \cdot \tilde{\mathbf{E}} = -\tilde{\mathbf{b}} \cdot \nabla \left( \delta\varphi + \frac{T_e}{e} \ln \eta_e \right) = -\tilde{\mathbf{b}} \cdot \nabla \lambda \tag{3.36}$$

so that Faraday's law in turn can be expressed as

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times (\mathbf{u}_{eff} \times \tilde{\mathbf{B}}) \tag{3.37}$$

after defining the effective velocity field $\mathbf{u}_{eff}$ through

$$\mathbf{u}_{eff} = \frac{c}{\tilde{B}} \tilde{\mathbf{b}} \times (\nabla\lambda - \tilde{\mathbf{E}}) = -\frac{c}{B} \tilde{\mathbf{b}} \times \nabla \left( \frac{T_e}{e} \ln \eta_e \right). \tag{3.38}$$

This effective velocity can, in a close analogy to the flux freezing of ideal MHD, be shown to transport closed curves such that they keep enclosing the same amount of magnetic flux [18]. Thus we find that the exact flux surfaces $\tilde{\psi}$ are convected through

$$\left( \frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla \right) \tilde{\psi} = \left. \frac{\partial \tilde{\psi}}{\partial t} \right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} = 0 \tag{3.39}$$

and similarly for $\tilde{\alpha}$ and $\tilde{l}$. Furthermore this implies that the nice toroidal flux surface topology, which we assumed were present to derive (3.31), are conserved [24, 31]. Importantly then, though the magnetic field may change substantially the flux-surface average will continue to be a permissible, consistent operation from moment to moment. The system, as described by the TKRMHD equations, will not evolve over time to a state that is incompatible with the TKRMHD assumptions.

Having solved for the lowest order electron distribution $f_e^{(0)}$ we now proceed to the second order in the electron Fokker-Planck equation (2.24) which, after being gyroaveraged like (3.25) and (3.27), is given by

$$\begin{aligned} &\frac{\partial f_e^{(0)}}{\partial t} + \frac{c}{B}\mathbf{b} \times \nabla\varphi \cdot \frac{\partial f_e^{(0)}}{\partial \mathbf{R}_e} - e\frac{\partial \varphi}{\partial t}\frac{\partial f_e^{(0)}}{\partial \varepsilon_e} + v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial f_e^{(1/2)}}{\partial \mathbf{R}_e} + \frac{ev_\parallel}{c}\frac{\partial A_\parallel^{(0)}}{\partial t}\frac{\partial f_e^{(1/2)}}{\partial \varepsilon_e} \\ &- \frac{v_\parallel}{B}\mathbf{b} \times \nabla A_\parallel^{(1/2)} \cdot \frac{\partial f_e^{(0)}}{\partial \mathbf{R}_e} + \frac{ev_\parallel}{c}\frac{\partial A_\parallel^{(1/2)}}{\partial t}\frac{\partial f_e^{(0)}}{\partial \varepsilon_e} = C[f_e^{(0)} + f_e^{(1/2)}]. \end{aligned} \tag{3.40}$$

There are two obvious problems with this equation in its current form. Firstly, it contains $A_\parallel^{(1/2)}$ and $f_e^{(1/2)}$ of which we have no knowledge. Secondly, it is a dynamic equation for the full lowest order electron distribution $f_e^{(0)}$, even though we already know by its explicit form (3.31) that all information about it is contained in the two functions $\eta_e$ and $T_e$. Both troubles can be remedied by neglecting the unnecessary information contained in (3.40) through taking two of its velocity moments.

To obtain two dynamic equations for $n_e$ and $T_e$ from (3.40) we will multiply it by 1 and $\varepsilon_e/T_e - 3/2$ respectively before integrating over all velocities. In both cases we will find that the sixth and seventh term vanish because there is no zeroth order electron flow and $f_e^{(0)}$ is even in $v_\parallel$. Similarly, the collisional term vanish, by electron number and energy conservation. That leaves the fourth and fifth terms containing higher order variables. Introducing the lowest order parallel electron flow

$$u_{\parallel e} = \int \mathrm{d}^3 v v_\parallel f_e^{(1/2)} \tag{3.41}$$

and dropping the superfluous superscript on $\tilde{A}_\parallel$ we discover that

$$\int \mathrm{d}^3 v v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial f_e^{(1/2)}}{\partial \mathbf{R}_e} + \int \mathrm{d}^3 v \frac{e v_\parallel}{c} \frac{\partial \tilde{A}_\parallel}{\partial t} \frac{\partial f_e^{(1/2)}}{\partial \varepsilon_e} = \tilde{\mathbf{B}} \cdot \frac{\partial}{\partial \mathbf{R}_e} \left( \frac{u_{\parallel e}}{\tilde{B}} \right) \tag{3.42}$$

because, from (2.25), the second term vanishes by virtue of its integrand being a complete $\varepsilon_e$-derivative. Similarly, we find, after introducing the shifted, normalised, lowest order parallel electron energy flow

$$q_{\parallel e} = \int \mathrm{d}^3 v \left( \frac{\varepsilon_e}{T_e} - \frac{3}{2} \right) v_\parallel f_e^{(1/2)}, \tag{3.43}$$

using integration of $\varepsilon_e$ by parts on the second term, and accounting for the extra term arising from pulling $\varepsilon_e$ through the $\mathbf{R}_e-$derivative that

$$\begin{aligned} &\int \mathrm{d}^3 v \left( \frac{\varepsilon_e}{T_e} - \frac{3}{2} \right) v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial f_e^{(1/2)}}{\partial \mathbf{R}_e} + \int \mathrm{d}^3 v \left( \frac{\varepsilon_e}{T_e} - \frac{3}{2} \right) \frac{e v_\parallel}{c} \frac{\partial \tilde{A}_\parallel}{\partial t} \frac{\partial f_e^{(1/2)}}{\partial \varepsilon_e} \\ &= \tilde{\mathbf{B}} \cdot \frac{\partial}{\partial \mathbf{R}_e} \left( \frac{q_{\parallel e}}{\tilde{B}} \right) + \frac{e u_{\parallel e}}{T_e} \left( \tilde{\mathbf{b}} \cdot \frac{\partial \varphi}{\partial \mathbf{R}_e} - \frac{1}{c} \frac{\partial \tilde{A}_\parallel}{\partial t} \right). \end{aligned} \tag{3.44}$$

Returning to the integral of the first three terms in equation (3.40), we now find it convenient to switch back to the usual $(\mathbf{r}, \mathbf{v})$-coordinates. With our orderings the different derivatives are related through

$$\frac{\partial}{\partial \mathbf{R}_e} = \nabla + e \nabla \varphi \frac{\partial}{\partial \varepsilon_e} + \mathcal{O}\left( \frac{\epsilon}{L_\parallel} \right), \tag{3.45}$$

and

$$\left. \frac{\partial}{\partial t} \right|_{\mathbf{R}_e, \varepsilon_e, \mu_e, \vartheta} = \left. \frac{\partial}{\partial t} \right|_{\mathbf{r}, \mathbf{v}} + e \frac{\partial \varphi}{\partial t} \frac{\partial}{\partial \varepsilon_e} + \mathcal{O}\left( \epsilon \omega \right). \tag{3.46}$$

Therefore we find to lowest order that these three terms in $(\mathbf{r}, \mathbf{v})$-coordinates, because $\nabla \varphi \cdot \mathbf{u}_E = 0$, can be combined into the simple expression

$$\left( \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) f_e^{(0)}. \tag{3.47}$$

Using the explicit Maxwellian form of (3.31) to evaluate the integrals it becomes a simple matter to deduce that

$$\int \mathrm{d}^3 v \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) f_e^{(0)} = \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) n_e \tag{3.48}$$

and

$$\begin{aligned} &\int \mathrm{d}^3 v \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2}\right) \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) f_e^{(0)} \\ &= -\frac{e\varphi}{T_e} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) n_e + \frac{3}{2} n_e \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \ln T_e. \end{aligned} \tag{3.49}$$

Having reached this point we can now combine equations (3.42), (3.44), (3.48), and (3.49) to reduce (3.40) into the two dynamic equations

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \ln n_e + \frac{1}{n_e} \tilde{\mathbf{B}} \cdot \nabla \left(\frac{u_{\parallel e}}{B}\right) = 0 \tag{3.50}$$

and

$$\begin{aligned} &- \frac{e\varphi}{T_e} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) n_e + \frac{3}{2} n_e \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \ln T_e \\ &+ \tilde{\mathbf{B}} \cdot \nabla \left(\frac{q_{\parallel e}}{B}\right) = \frac{e u_{\parallel e}}{T_e} \left(\tilde{\mathbf{b}} \cdot \nabla\varphi + \frac{1}{c} \frac{\partial \tilde{A}_\parallel}{\partial t}\right) \end{aligned} \tag{3.51}$$

where we have divided the first equation by the electron density $n_e$.

Now equations (3.50) and (3.51) still contain $u_{\parallel e}$ and $q_{\parallel e}$ of which we currently have no knowledge. We can remedy this by returning to Ampère's law (2.10) which, in its lowest order form, allows us to determine that

$$u_{\parallel e} = \frac{c}{e 4\pi} \mathbf{b} \cdot \nabla \times \tilde{\mathbf{B}}. \tag{3.52}$$

This result naturally expresses the fact that, because the mass ratio and $\beta$ is small, the current is almost entirely carried by the electrons and, importantly, allows us to eliminate $u_{\parallel e}$. Even so, we will keep it in its current form in our equation system for notational convenience. There is no similar relation we can use to find an expression for $q_{\parallel e}$, but this is no problem. Because $T_e$ is a flux function we can flux surface average equation (3.51) to produce

$$\begin{aligned} &- \left\langle \frac{e\varphi}{T_e} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) n_e \right\rangle_{\tilde{\psi}} + \left\langle \frac{3}{2} n_e \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \ln T_e \right\rangle_{\tilde{\psi}} \\ &= \left\langle \frac{e u_{\parallel e}}{T_e} \left(\tilde{\mathbf{b}} \cdot \nabla\varphi + \frac{1}{c} \frac{\partial \tilde{A}_\parallel}{\partial t}\right) \right\rangle_{\tilde{\psi}} \end{aligned} \tag{3.53}$$

where $q_{\parallel e}$ has been eliminated. One might be worried that in so doing we would discard necessary information about on-surface variation. However we have determined that $T_e$ is an exact flux function. Therefore we do not lose any information in switching to this equation to determine the evolution of $T_e$ because the in-surface variation of $n_e$ is still described by equation (3.50).

At this point, we have in our hand the four equations, (3.25), (3.35), (3.50), and (3.53), in which there appear five variables, $f_i^{(0)}$, $n_e$, $T_e$, $\tilde{A}_\parallel$, and $\varphi$. Of these, we are currently only missing an equation to specify the evolution of $\varphi$. To this end we can naturally return to the quasineutrality condition (2.11). This condition, together with the expression (3.32) for the electron density and the corresponding constraint (3.35) for $\eta_e$, clearly makes it possible to determine the on-surface variation of $\varphi$. However, there remains a freedom in choosing how to divide the flux-surface-dependence of $n_e$ between $\varphi$ and $\eta_e$, which prevents this approach from fully determining $\varphi$. Therefore, we need to employ some other method to determine the flux-surface variation of $\varphi$, and this will turn out to be a vorticity equation.

## 3.3 The Vorticity Equation

In order to describe the perpendicular variation of $\varphi$ in TKRMHD we will have to turn away from the Fokker-Planck equation which we up to this point have been focusing on. Instead our starting point will be the momentum equation

$$\frac{\partial}{\partial t}\left(\sum_s m_s \int \mathrm{d}^3 v \mathbf{v} f_s\right) + \nabla\cdot\left(\sum_s \int \mathrm{d}^3 v m_s \mathbf{v}\mathbf{v} f_s\right) = \frac{1}{c}\mathbf{j}\times\tilde{\mathbf{B}}, \tag{3.54}$$

from which we will derive a vorticity equation that will turn out to contain precisely the information about $\varphi$ we are missing.

To commence our investigation of the vorticity equation, we find it convenient to rewrite the momentum equation by separating out the $\mathbf{u}_E$-flow through

$$\mathbf{w} = \mathbf{v} - \mathbf{u}_E. \tag{3.55}$$

Upon then defining

$$n_s \mathbf{u}_s = \int \mathrm{d}^3 v \mathbf{v} f_s \tag{3.56}$$

and

$$n_s \mathbf{U}_s = \int \mathrm{d}^3 w \mathbf{w}_s f_s \tag{3.57}$$

we can transform (3.54) into the form

$$\sum_s n_s m_s \left(\frac{\partial}{\partial t} + \mathbf{u}_E\cdot\nabla\right)\mathbf{u}_s - \mathbf{X} = -\nabla\cdot\mathbf{p} + \frac{1}{c}\mathbf{j}\times\tilde{\mathbf{B}} \tag{3.58}$$

where

$$\mathbf{X} = \sum_s \left(\nabla\cdot\boldsymbol{\pi}_s - n_s m_s \mathbf{U}_s \nabla\cdot\mathbf{U}_s + n_s m_s \mathbf{U}_s\cdot\nabla\mathbf{u}_E\right) \tag{3.59}$$

and, because of gyrophase-independence, the total stress tensor

$$\int \mathrm{d}^3 w m_s \mathbf{w}\mathbf{w} f_s \tag{3.60}$$

has been divided into the gyrotropic pressure tensor

$$\mathbf{p} = p_\perp \mathbf{I} + (p_\parallel - p_\perp)\tilde{\mathbf{b}}\tilde{\mathbf{b}} \tag{3.61}$$

and the non-diagonal part $\boldsymbol{\pi}$ [25].

The momentum equation in the form of (3.58) contains the full information of all quantities to all orders. Of course this is unnecessary for our purposes so we restrict our attention by applying the orderings of (3.17). The first thing we then find is that by estimating $\mathbf{u}_s$ as $\mathbf{u}_E$ the left hand side is much smaller than the right hand side, $\mathcal{O}(\epsilon^2 p/l_\perp)$ as compared to $\mathcal{O}(p/l_\perp)$. This allows us to deduce an expression for the perpendicular current which will shortly prove useful,

$$\mathbf{j}_\perp = \frac{c}{B}\tilde{\mathbf{b}} \times \nabla p_\perp - \frac{c(p_\parallel - p_\perp)}{B}\tilde{\mathbf{b}} \times (\tilde{\mathbf{b}} \times (\nabla \times \tilde{\mathbf{b}})) + \mathcal{O}(\epsilon^2 cB/l_\parallel) \tag{3.62}$$

where we have used $\tilde{\mathbf{b}} \cdot \nabla\tilde{\mathbf{b}} = -\tilde{\mathbf{b}} \times (\nabla \times \tilde{\mathbf{b}})$ and $\tilde{B} = B + \mathcal{O}(\epsilon^2 B)$.

The problem of having a dominating right hand side in the momentum equation must be remedied when we now wish to proceed and arrive at a vorticity equation for $\varphi$ by taking $\nabla \cdot (\tilde{\mathbf{b}} \times (3.58)/\tilde{B})$. Thankfully, in this process we will be able to make the dominating terms on the right vanish to appropriate order. This is because Ampère's law (2.10) can be used twice to manipulate the right hand side, to $\mathcal{O}(\epsilon^2 B/L_\parallel)$, like

$$\begin{aligned}&-\frac{\tilde{\mathbf{b}}}{B} \times \nabla \cdot ((p_\parallel - p_\perp)\tilde{\mathbf{b}}\tilde{\mathbf{b}}) = (p_\parallel - p_\perp)\frac{\tilde{\mathbf{b}}}{B} \times \left(\tilde{\mathbf{b}} \times (\nabla \times \tilde{\mathbf{b}})\right)\\ &= (p_\parallel - p_\perp)\left(\frac{1}{B}\nabla \times \tilde{\mathbf{b}} - \nabla \times \frac{\tilde{\mathbf{b}}}{B} - \frac{4\pi}{cB^2}\mathbf{j}_\perp\right) = (p_\parallel - p_\perp)\left(\frac{4\pi}{cB^2}(\mathbf{j} - \mathbf{j}_\perp) - \frac{1}{B}\nabla \times \tilde{\mathbf{b}}\right).\end{aligned} \tag{3.63}$$

As such, using $\nabla \cdot \mathbf{j} = 0$ which follows from quasineutrality (2.11) and charge conservation, we find that

$$\begin{aligned}&\nabla \cdot \left[\frac{\tilde{\mathbf{b}}}{\tilde{B}} \times \left(-\nabla \cdot \mathbf{p} + \frac{1}{c}\mathbf{j} \times \tilde{\mathbf{B}}\right)\right]\\ &= \nabla \cdot \left[-p_\perp \nabla \times \left(\frac{\tilde{\mathbf{b}}}{B}\right) - \left(\frac{1}{c} - 4\pi\frac{p_\parallel - p_\perp}{cB^2}\right)\mathbf{j}_\parallel - \frac{p_\parallel - p_\perp}{B}\nabla \times \tilde{\mathbf{b}}\right].\end{aligned} \tag{3.64}$$

so that our vorticity equation becomes

$$\begin{aligned}&\nabla \cdot \left[\sum_s \frac{n_s m_s}{B}\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right)\tilde{\mathbf{b}} \times \mathbf{u}_s\right] - \nabla \cdot \left(\frac{\tilde{\mathbf{b}}}{B} \times \mathbf{X}\right)\\ &= \nabla \cdot \left[-p_\perp \nabla \times \left(\frac{\tilde{\mathbf{b}}}{B}\right) - \left(\frac{1}{c} - 4\pi\frac{p_\parallel - p_\perp}{cB^2}\right)\mathbf{j}_\parallel - \frac{p_\parallel - p_\perp}{B}\nabla \times \tilde{\mathbf{b}}\right].\end{aligned} \tag{3.65}$$

Even after having used (3.64), the right hand side of (3.65) is still one order of $\epsilon$ larger than the left hand side. Thankfully, we can lower the relative order of these terms one power of $\epsilon$ further by applying the flux-surface average. Upon so doing, it is clear that the third term and part of the first term come to be of the form (2.46). The second term, similarly, comes to be of the form (2.43), as can be seen by using the explicit form of the divergence in Catto-transformed variables (2.47). Thus these terms vanish under the flux surface average and we are left with the remaining part of the first term which takes the form

$$\left\langle \nabla \cdot \left( \frac{1}{B^2} p_\perp \tilde{\mathbf{b}} \times \nabla B \right) \right\rangle_{\tilde{\psi}} . \tag{3.66}$$

Using the identity [32]

$$\nabla B \times \tilde{\mathbf{b}} = \frac{4\pi}{c} \mathbf{j} - B \nabla \times \tilde{\mathbf{b}}, \tag{3.67}$$

we can rewrite (3.66) into

$$-\left\langle \mathbf{j} \cdot \nabla \left( \frac{4\pi p_\perp}{cB^2} \right) \right\rangle_{\tilde{\psi}} , \tag{3.68}$$

after which the dominating right hand side, as promised, has been reduced one power of $\epsilon$ further.

In order to arrive at the final form of our flux-surface averaged vorticity equation from (3.65), we now use that the lowest order perpendicular particle flow is given by

$$n_s \mathbf{u}_{\perp s} = \frac{c}{B} \mathbf{b} \times \left[ \nabla \left( \frac{p_{\perp s}}{Z_s e} \right) + n_s \nabla \varphi \right] \tag{3.69}$$

to find that the first term on the left hand side becomes

$$\left\langle \nabla \cdot \left[ \sum_s \frac{n_s m_s}{B} \left( \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) \left( \frac{\nabla p_{\perp_s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right] \right\rangle_{\tilde{\psi}} . \tag{3.70}$$

Upon then employing manipulations similar to those of appendix B in [25], we find that the second term to appropriate order becomes

$$\left\langle \frac{1}{2B\Omega_s} \left[ (\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla \nabla \right] \int \mathrm{d}^3 w \frac{m_s w_\perp^2}{2} w_\parallel f_s \right\rangle_{\tilde{\psi}} . \tag{3.71}$$

Finally, using (3.62) to separate perpendicular and parallel currents in (3.68), which will prove useful in the next Chapter, we arrive at the final vorticity equation

$$\begin{aligned} &\left\langle \nabla \cdot \left( \sum_s \frac{n_s m_s}{B} \left( \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) \left( \frac{\nabla p_{\perp_s}}{n_s m_s \Omega_s} + \frac{c \nabla \varphi}{B} \right) \right) \right\rangle_{\tilde{\psi}} \\ &= \left\langle \frac{1}{2B\Omega_s} \left[ (\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla \nabla \right] \int \mathrm{d}^3 w \frac{m_s w_\perp^2}{2} w_\parallel f_s \right\rangle_{\tilde{\psi}} \\ &- \left\langle j_\parallel \tilde{\mathbf{b}} \cdot \nabla \left( \frac{4\pi p_\perp}{cB^2} \right) - \frac{8\pi p_\perp}{B^4} \tilde{\mathbf{b}} \times \nabla p_\perp \cdot \nabla B - \frac{4\pi (p_\parallel - p_\perp)}{B^3} \nabla \times \tilde{\mathbf{b}} \cdot \nabla p_\perp \right\rangle_{\tilde{\psi}} , \end{aligned} \tag{3.72}$$

where, like previously mentioned, $j_\parallel$ to lowest order is given by $eu_{\parallel e}$. This equation determines the flux-surface variation of the electrostatic potential $\varphi$, which, together with the quasineutrality condition (2.11) for its on-surface variation, completely determines $\varphi$.

At this point we have completed the TKRMHD equations. (3.25), (3.35), (3.50), (3.53), and (3.72), together with the quasineutrality condition, fully describe the dynamics of $f_i^{(0)}$, $n_e$, $T_e$, $\tilde{A}_\parallel$, and $\varphi$. They self-consistently describe ITG-like turbulence of large amplitude occurring in the near-edge region inside the LCFS. Therefore they should prove suitable for describing tokamaks operating in L-mode. They possess several important features which fluid or gyrofluid equations, the common edge plasma descriptions [33, 34], generally lack. These are their fully kinetic nature, their ability to capture the collisional to weakly collisional transition, and the fact that they are capable of being applied to general tokamak geometries.

# 4

# Shallow Gradient Toroidal Kinetic Reduced Magnetohydrodynamics

It is known that the tokamak core is well-described by gyrokinetics, but that this description becomes increasingly inapplicable closer to the LCFS. In particular, the steeper gradients drive the turbulence of this region to become stronger than in the core, with longer wavelengths and larger amplitudes. This failure and the need to accurately describe this region for safe tokamak operation was the imperative reason for developing the TKRMHD equations of the previous Chapter. Of course, they are inadequate to describe the extreme ELM-physics of H-mode operation as described in their sister equations of Abel and Hallenbert [25]. Nevertheless, by virtue of the orderings (3.17), which are tailored to describe the less extreme end of applicable physics in the edge region, they should, in most configurations, be able to describe some part of the edge.

Because physics happens on a continuous spectrum of scales and phenomena we should be able to smoothly match the gyrokinetic core onto the TKRMHD edge. This matching is what we will be concerned with in the next two Chapters. In this Chapter we will explore what becomes of the TKRMHD equations as the driving gradients are made shallower and the turbulence, in response, weakens. To this end, we will employ a secondary multiscale expansion, subsidiary to the one we used in the previous Chapter which will yield a nontrivial shallow gradient limit of TKRMHD. This expansion will, significantly, make it possible to distinguish a quiescent equilibrium from the turbulent motion. Because of this, we will have to make heavy use of the mathematical tools of Chapter 2 to split the TKRMHD equations into two different sets of equation describing both the rapid evolution of fluctuating quantities and the slow evolution of mean quantities. In this Chapter we will only deal with the former, leaving the latter for Chapter 6.

## 4.1 Shallow Gradient Subsidiary Expansion

With the aim of seeing how our TKRMHD equations (3.25), (3.35), (3.50), (3.53), and (3.72) behave closer to the gyrokinetic core as the gradients become shallower we now look to find a suitable subsidiary expansion to apply. Initially, we must decide how this subsidiary expansion will change the fundamental ordering (3.17). As we have already hinted, we expect the turbulence, being largely driven by the gradients, to reduce in amplitude until a quiescent equilibrium emerges. Therefore we introduce the separation (2.28) of mean and fluctuating quantities in this subsidiary expansion. In so doing, we also separate the typical turbulent length scales $l_\perp$ and $l_\parallel$ from the equilibrium length scales $L_\perp$ and $L_\parallel$ because they will come to differ as we leave the edge region.

As (3.2) is the defining relation for the expansion parameter $\epsilon$ in the TKRMHD multiscale expansion it is natural that we return and modify it in our subsidiary expansion. The lengthening of $L_\perp$ is naturally considered to be as the fundamental difference from the original expansion. Thus, we choose to introduce the small subsidiary expansion parameter $\xi$ through

$$\frac{\rho_i}{L_\perp} = \epsilon\xi. \tag{4.1}$$

Of course, in order for it to be valid for us to consider this a subsidiary expansion we must require that $\epsilon \ll \xi$ and that as $\xi \to 1$ we recover the initial ordering (3.17). The former condition will prove beneficial as it will allow us to instantly disregard any terms arising which are of higher order in $\epsilon$, even when they are of lower order in $\xi$.

Equipped with the fundamental ordering (4.1), and having introduced these different length scales, we now must order them with respect to each other. Additionally, we must also choose how to order the size of the fluctuations with respect to mean quantities. However, these are really all the changes we can make from the original ordering (3.17) because we are still interested in similar phenomena occurring on the same ion parallel streaming time scale. Therefore we will keep $\beta$, $m_e$, $m_i$, $T_e$, $T_i$, $\omega$, $\Omega_i$, and $\nu_{ii}$ ordered the same.

To guide us in choosing the remaining orderings we note that we wish to describe the strongest turbulence which gives rise to most of the transport. As explained in Barnes et al [29], this is the so called outer scale turbulence which is characterised by two conditions: that the turbulent parallel length scale is the same as the system scale,

$$l_\parallel \sim L_\parallel \sim a, \tag{4.2}$$

and that the linear drive is comparable to the nonlinear decorrelation, which for our stationary plasma where $\varphi = \delta\varphi$ can be expressed as

$$\frac{c}{B}\tilde{\mathbf{b}} \times \nabla\delta\varphi \cdot \nabla F_s \sim \frac{c}{B}\tilde{\mathbf{b}} \times \nabla\delta\varphi \cdot \nabla\delta f_s. \tag{4.3}$$

Having employed the outer scale conditions, we are almost at a point where we can uniquely determine how to order all unknown quantities with respect to each other. We only need to additionally use quasineutrality in the form of [29]

$$\frac{e\delta\varphi}{T_s} \sim \frac{\delta f_s}{F_s}, \tag{4.4}$$

and fix the nonlinear timescale to be the ion parallel streaming timescale (3.5). Then we find that

$$\frac{e\delta\varphi}{T_s} \sim \frac{\delta f_s}{F_s} \sim \frac{l_\perp}{L_\perp} \sim \xi^2. \tag{4.5}$$

In conclusion, to summarise what we have determined in this Section the subsidiary expansion ordering is given by

$$\beta \sim \frac{m_e}{m_i} \sim \epsilon, \quad \frac{e\delta\varphi}{T_i} \sim \frac{T_e}{T_i} \sim \frac{l_\parallel}{L_\parallel} \sim \frac{a}{l_\parallel} \sim 1, \quad \frac{\omega}{\Omega_i} \sim \frac{\nu_{ii}}{\Omega_i} \sim \epsilon^2,$$
$$\frac{e\delta\varphi}{T_s} \sim \frac{\delta f_s}{F_s} \sim \frac{l_\perp}{L_\perp} \sim \xi^2, \quad \frac{L_\perp}{l_\parallel} \sim \frac{|\tilde{\mathbf{B}}_1|}{|\mathbf{B}_0|} \sim \frac{\rho_i}{l_\perp} \equiv \frac{\epsilon}{\xi}. \tag{4.6}$$

## 4.2 Fluctuation Equations

Equipped with the subsidiary ordering (4.6), we can now return to full TKRMHD equations and see how they change. Because we have separated the distribution functions into mean and fluctuating quantities it is clear that we must extract equations governing the evolution of both from the original equations. We can achieve this by using the turbulence average (2.37) to separate the two from each other.

Initially however, by turning to the ion kinetic equation (3.25), we will not need the turbulence average. This is because, after expanding into fluctuating and non-fluctuating parts, to lowest order in $\epsilon$ and $\xi$ (3.25) reduces to

$$v_\parallel \tilde{\mathbf{b}} \cdot \frac{\partial F_i}{\partial \mathbf{R}_i} = C[F_i], \tag{4.7}$$

after dropping superfluous superscripts and using (3.21) for the lowest order expression of $\tilde{\mathbf{b}}$. Equation (4.7) obviously contains only $F_i$ and so we can use it to deduce information about the equilibrium, knowing nothing more about the fluctuations. Indeed, it is clearly of a similar but simpler form than the electron constraint (3.27). By exactly the same argument as the one we used for the electrons we immediately deduce that the ion equilibrium in the subsidiary expansion is Maxwellian,

$$F_i = N_i(\mathbf{r}) \left( \frac{m_e}{2\pi T_i(\mathbf{r})} \right)^{3/2} \exp\left( -\frac{\varepsilon_i}{T_i(\mathbf{r})} \right). \tag{4.8}$$

Inserting this back into equation (4.7) then yields, similarly to the electrons, the result that $N_i(\mathbf{r}) = N_i(\tilde{\psi})$ and $T_i(\mathbf{r}) = T_i(\tilde{\psi})$ are exact flux functions of $\tilde{\mathbf{b}}$. As the Boltzmann response is purely fluctuating, we see that the mean distribution depends spatially only through $\tilde{\psi}$. In particular, this allows us to write

$$\frac{\partial F_i}{\partial \mathbf{R}_i} = \frac{\partial F_i}{\partial \tilde{\psi}} \nabla \tilde{\psi}, \tag{4.9}$$

to lowest order. This will be convenient for comparisons to gyrokinetics as it is in the same form as [10].

Having deduced the Maxwellian nature of the ion equilibrium we now proceed to determine the equations governing the turbulence. We do this by subtracting the turbulence-averaged part of to the TKRMHD equations, which will eliminate any terms not containing fluctuating quantities. Starting with the ion kinetic equation (3.25) it is a simple matter to determine that this procedure leaves us with

$$\begin{aligned}\frac{\partial\delta f_i}{\partial t}&+\left[v_\parallel\tilde{\mathbf{b}}+\frac{c}{B}\mathbf{b}\times\nabla\left(\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right]\cdot\frac{\partial\delta f_i}{\partial\mathbf{R}_i}\\&+\frac{\partial F_i}{\partial\tilde{\psi}}\frac{c}{B}\mathbf{b}\times\nabla\left(\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\cdot\nabla\tilde{\psi}-\frac{Z_ieF_i}{T_i}\frac{\partial}{\partial t}\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)=C[f_i],\end{aligned}\tag{4.10}$$

after using (4.9).

Having determined how the ions behave in the subsidiary expansion, we now wonder if we can deduce any information about how the electron dynamics change. To this end, we turn first to equation (3.35) for the evolution of $\tilde{A}_\parallel$. Because $A_\parallel$, and indeed all mean quantities, by our ordering is assumed to be quiescent so its time derivative is slow, we find that this equation, to lowest order, reduces to a constraint on $\eta_e$,

$$\tilde{\mathbf{b}}\cdot\nabla\eta_e=0.\tag{4.11}$$

Unsurprisingly we find that $\eta_e$ is a flux function and so the electron Maxwellian (3.31), like the ions, is a flux function to lowest order.

Before proceeding to the next order in (3.35), we momentarily change focus to equation (3.53) for the evolution of $T_e$. By using the explicit expression (3.52) to order $u_{\parallel e}$ we find that one term dominates the rest so that to lowest order

$$\left\langle\left(\frac{\partial}{\partial t}+\mathbf{u}_E\cdot\nabla\right)\ln T_e\right\rangle_{\tilde{\psi}}=0.\tag{4.12}$$

Here, because $T_e$ is a flux function, we find upon using the explicit Poisson bracket expression (2.45) that

$$\langle\mathbf{u}_E\cdot\nabla\ln T_e\rangle_{\tilde{\psi}}=\frac{\partial T_e}{\partial\tilde{\psi}}\left\langle\frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\tilde{\psi}\right\rangle_{\tilde{\psi}}=-c\frac{\partial T_e}{\partial\tilde{\psi}}\left\langle B\frac{\partial}{\partial\tilde{\alpha}}\left(\frac{\delta\varphi}{B}\right)\right\rangle_{\tilde{\psi}}\tag{4.13}$$

where we have used the fact that to lowest order the $\tilde{\alpha}$-derivative does not act on B. Because of axisymmetry, this is clearly of the form (2.43) and so vanishes. Therefore, we conclude that, in addition to being a flux function, the electron temperature is constant on the ion parallel streaming time scale,

$$\frac{\partial T_e}{\partial t}=0.\tag{4.14}$$

Now that we have explored the ion and electron equilibria, we return to equation (3.35). Using (3.32), with the new information that $\eta_e$ is a flux function, we find after proceeding to the next order past (4.11) that the evolution of $\delta A_\parallel$ is given by

$$\frac{e}{cT_e}\frac{\partial\delta A_\parallel}{\partial t}=\tilde{\mathbf{b}}\cdot\nabla\left(\frac{\delta\eta_e}{\eta_e}\right)=\tilde{\mathbf{b}}\cdot\nabla\left(\frac{\delta n_e}{n_e}-\frac{e\delta\varphi}{T_e}\right).\tag{4.15}$$

Proceeding to the final two equations (3.50) and (3.72) it is a straightforward procedure to extract the lowest order fluctuating piece from each. Starting with

(3.50) we find that it reduces to

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right)\delta n_e + \mathbf{u}_E \cdot \nabla n_e + \tilde{\mathbf{B}} \cdot \nabla \left(\frac{u_{\| e}}{B}\right) = 0. \tag{4.16}$$

Similarly, we find upon meticulously checking the order of every term in equation (3.72) and using (3.52) that two terms dominate so that it reduces to

$$\left\langle \sum_s \frac{n_s m_s}{B} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \left(\frac{\nabla_\perp^2 \delta p_{\perp_s}}{n_s m_s \Omega_s} + \frac{c \nabla_\perp^2 \delta \varphi}{B}\right) \right\rangle_{\tilde{\psi}} = 0. \tag{4.17}$$

At this point we have completely determined the equations governing the turbulence at the outer scale, in the shallow gradient TKRMHD limit our subsidiary expansion constitutes. The electron temperature has ceased to be an independent variable and our original five TKRMHD equations have been reduced to the four equations (4.10), (4.15), (4.16), and (4.17). However, by separating quantities into mean and fluctuating quantities we have introduced more variables whose variation, in this Section, has been treated simply as slow. Obviously we must also determine how they evolve, especially because they describe the bulk transport which is of primary importance for tokamak plasma confinement. However, this calculation is postponed to chapter 6 and in the next Chapter we instead examine a subsidiary expansion of gyrokinetics on the fluctuation time scale.

# 5

# Steep Gradient, Long Wavelength Gyrokinetics

In Chapter 3 we developed TKRMHD in order to explore some of the tokamak edge physics that conventional gyrokinetics fails to describe. Nevertheless, because conventional gyrokinetics has been so successful in describing the core we were guided to attempt to find an intermediary theory so that we can smoothly transition from the gyrokinetic core to the TKRMHD physics closer to the edge. In the previous Chapter we approached this from the TKRMHD side by performing a shallow gradient subsidiary expansion to arrive at a new set of equations. Now we will start from conventional gyrokinetics and attempt to reach that same set of equations.

We begin by presenting conventional gyrokinetics, in the absence of sonic rotation, and compare it to the TKRMHD limit of the previous Chapter. This comparison will then guide us in how we should approach the problem of reducing gyrokinetics to a matching set of equations. First, we will find that gyrokinetics' is too general in that it describes both ion and electron scale motion, treating them on equal footing. Therefore we will immediately turn to the reduced gyrokinetic model of Abel and Cowley [24] which separates out the relevant ion scale dynamics by taking the low mass ratio limit. We will then proceed by finding the low $\beta$-limit of these equations before employing a non-trivial long wavelength limit to arrive at the desired result with matching equations describing the turbulence. In this process we will specifically have to devote considerable efforts in manipulating the gyrokinetic equations to produce a vorticity equation. Having then matched the two set of turbulent equations onto each other we will in the next and final Chapter turn to the transport equations, matching these between the two sets of intermediary equations as well.

## 5.1 Conventional Gyrokinetics

Gyrokinetics is a fully kinetic theory describing plasma turbulence on scales comparable to the gyroradius [6, 35]. It can be derived in an entirely analogous manner to the way we derived TKRMHD in Chapter 3 by using a multiscale expansion. The standard gyrokinetic ordering by which this multiscale expansion is performed is, using the same notation as in the rest of this thesis, given by [6]

$$\frac{\omega}{\Omega_s} \sim \frac{\nu_s}{\Omega_s} \sim \frac{l_\perp}{l_\parallel} \sim \frac{|\delta \mathbf{B}|}{|\tilde{\mathbf{B}}|} \sim \frac{\delta f_s}{F_s} \sim \frac{1}{\sqrt{\omega t_{heat}}} \sim \frac{\rho_s}{a} \equiv \epsilon \ll 1 \tag{5.1}$$

while other dimensionless quantities, e.g. $\beta$ and $m_e/m_i$, are treated as order unity. Now we will not be using this ordering in order to derive gyrokinetics from scratch. Instead we will merely summarise the resulting equations in the absence of sonic flows, for a thorough derivation see [10].

With the gyrokinetic ordering (5.1), like in the subsidiary TKRMHD ordering (4.6), the rapid gyromotion is averaged away and the lowest order mean and fluctuating distributions become gyrophase independent. Furthermore, in the absence of rotation the mean distributions reduce to Maxwellians with density $n_s(\psi)$ and temperature $T_s(\psi)$ being mean flux functions. Their slow evolution in the gyrokinetic transport equations we will leave until Chapter 6 after we have completely matched the turbulent gyrokinetic equations to the TKRMHD limit of Chapter 3.

Turning to the fluctuating distribution $\delta f_s$, the non-adiabatic response

$$h_s = \delta f_s + \frac{Z_s e \delta\varphi}{T_s} F_s. \tag{5.2}$$

is usually separated out. Then the lowest order fluctuating quasineutrality condition becomes

$$\sum_s \frac{Z_s^2 e^2 n_s \delta\varphi}{T_s} = \sum_s Z_s e \int \mathrm{d}^3 v \, \langle h_s \rangle_{\mathbf{r}} \tag{5.3}$$

in accordance with (4.4). Here we note that this equation immediately determines the electrostatic potential fully, in contrast to the elaborate vorticity equation of TKRMHD. This is a large difference which will prove cumbersome to remedy when matching the two together.

The next equation of gyrokinetics is the evolution equation for $h_s$, known as the gyrokinetic equation:

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \left(v_\parallel \mathbf{b} + \mathbf{V}_{Ds} + \langle \mathbf{V}_\chi \rangle_{\mathbf{R}_s}\right) \cdot \frac{\partial}{\partial \mathbf{R}_s}\right] h_s - \langle C[h_s] \rangle_{\mathbf{R}_s} \\ &= \frac{Z_s e F_s}{T_s} \frac{\partial}{\partial t} \left\langle \delta\varphi - \frac{\mathbf{v}\cdot\delta\mathbf{A}}{c} \right\rangle_{\mathbf{R}_s} - \frac{\partial F_s}{\partial \psi} \langle \mathbf{V}_\chi \rangle_{\mathbf{R}_s} \cdot \nabla\psi\end{aligned} \tag{5.4}$$

where

$$\langle \mathbf{V}_\chi \rangle_{\mathbf{R}_s} = \frac{c}{B} \mathbf{b} \times \frac{\partial}{\partial \mathbf{R}_s} \left\langle \delta\varphi - \frac{\mathbf{v}\cdot\delta\mathbf{A}}{c} \right\rangle_{\mathbf{R}_s}, \tag{5.5}$$

is the fluctuating guiding centre drift velocity. This equation describe how the turbulent particles, or more particularly their guiding centres, drift and diffuse in the large-scale mean magnetic field and small scale self-generated fields while being driven by the gradients of the mean distributions.

The final equations needed to close gyrokinetics are obtained from Ampère's law and describe the evolution of $\delta\mathbf{B}$ which, within this ordering, is conveniently described by $\delta A_\parallel$ and $\delta B_\parallel$. The parallel and perpendicular fluctuating components of this law then, respectively, yields

$$\nabla_\perp^2 \delta A_\parallel + \frac{4\pi}{c} \sum_s Z_s e \int \mathrm{d}^3 v v_\parallel \, \langle h_s \rangle_{\mathbf{r}} = 0, \tag{5.6}$$

and
$$\nabla_\perp^2 \frac{\delta B_\parallel B}{4\pi} + \nabla_\perp \nabla_\perp : \sum_s \int \mathrm{d}^3 v \, \langle m_s \mathbf{v}_\perp \mathbf{v}_\perp h_s \rangle_\mathbf{r} = 0. \tag{5.7}$$

Equations (5.3), (5.4), (5.6), and (5.7) constitute the full gyrokinetic equations for $h_s$, $\delta\varphi$, $\delta A_\parallel$, and $\delta B_\parallel$ which we will now proceed to, by taking various limits, incrementally transform into (4.10), (4.15), (4.16), and (4.17).

## 5.2 Low-Mass-Ratio and Low-$\beta$ Limits

Proceeding from the gyrokinetic equations we commence by separating the ion and electron scale dynamics by taking the low mass ratio limit
$$\frac{m_e}{m_i} \to 0. \tag{5.8}$$

In addition to this limit we will fix the fluctuation timescales, including the nonlinear timescale, to be that of the ion parallel streaming time $\omega = v_{th,i}/a$. This leaves the ion kinetic equation unchanged but allows the electron kinetic equation to be solved and thus the fast electron dynamics to be eliminated like in TKRMHD, although the procedure is considerably more involved. Thankfully, this limit has been thoroughly explored by Abel and Cowley [24] and so we may simply quote the resulting equations, in their simpler non-rotating form.

In going to the low mass ratio limit it becomes convenient to introduce the exact flux surfaces already familiar to us from TKRMHD, with that same magnetic field (3.21). This is because then the solution for the ion scale fluctuating electron distribution which one obtains may be expressed as
$$\delta f_e = \frac{e\zeta(\mathbf{r})}{T_e} F_e + (\tilde{\psi} - \psi)\frac{\partial F_e}{\partial \psi} + \left(\frac{\varepsilon_e}{T_e} - \frac{3}{2}\right)\frac{\delta T_e(\tilde{\psi})}{T_e} F_e, \tag{5.9}$$

where $T_e(\psi)$ and $F_e(\psi, \mathbf{v})$ are mean flux functions. However, for this result to be valid one moderate constraint has to hold true. The electrons must trap and detrap rapidly enough so that there is no distinction to be made between trapped and passing electrons, so that they all sample the entire flux surface within a typical fluctuation period [24]. Because we are considering ITG-like turbulence in the edge region where electron and ion temperatures are of similar, moderate magnitude, unlike in the high-temperature core, the electron collision frequency $\nu_{ee}$ should greatly exceed the typical fluctuation frequency $\omega$. Therefore this constraint is not unreasonable as it indeed should hold true in the majority of cases.

The decomposition (5.9) is useful because then the fluctuating electron temperature, which is an exact flux function $\delta T_e(\tilde{\psi})$, and the field $\zeta$ fully describe the electron behaviour. This second quantity which describes part of the density fluctuation has physical importance in that it constitute the non-electric drift part of the field-line velocity
$$\mathbf{u}_{eff} = \frac{c}{B}\tilde{\mathbf{b}} \times \nabla(\delta\varphi - \zeta) \tag{5.10}$$

which we note coincides with the analogous TKRMHD effective velocity (3.38) in its subsidiary expansion. This velocity field, like (3.38) for TKRMHD, can be shown to conserve magnetic flux and so preserve the magnetic topology. Thus it follows for the exact Clebsch coordinates that they evolve through

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla\right) \tilde{\psi} = \left.\frac{\partial \tilde{\psi}}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} = 0 \tag{5.11}$$

and similarly for $\tilde{\alpha}$ and $\tilde{l}$.

Having found the solution (5.9) the electron kinetic equation reduces to two equations for $\zeta$ and $\delta T_e$. The first of these, after using the parallel Ampère's law (5.6) for the parallel electron flow, is given by

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla\right)\left(\frac{e\zeta}{T_e} - \frac{\delta B_\parallel}{B}\right) n_e + B\frac{\partial}{\partial \tilde{l}}\left(\frac{c}{4\pi e B}\nabla^2 \delta A_\parallel + \sum_{s=i}\frac{Z_s}{B}\int \mathrm{d}^3 v v_\parallel \left\langle h_s\right\rangle_{\mathbf{r}}\right) \\ &-\frac{cT_e}{eB}\left\{\frac{\delta B_\parallel}{B}, \frac{\delta T_e}{T_e}\right\} n_e = \hat{\mathbf{V}}_D \cdot \nabla(\tilde{\psi} - \psi)\left(\frac{\partial \ln N_e}{\partial \psi} + \frac{\partial \ln T_e}{\partial \psi}\right) n_e - \hat{\mathbf{V}}_D \cdot \nabla \frac{\delta T_e}{T_e} n_e \\ &+ \hat{\mathbf{V}}_D \cdot (\delta\varphi - \zeta)\frac{e n_e}{T_e} + c\frac{\partial}{\partial \tilde{\alpha}}\left(\zeta - \delta B_\parallel \frac{T_e}{eB}\right)\frac{\partial \ln N_e}{\partial \psi} n_e - c\frac{\partial \delta B_\parallel}{\partial \tilde{\alpha}}\frac{n_e T_e}{eB}\frac{\partial \ln T_e}{\partial \psi},\end{aligned} \tag{5.12}$$

where

$$\hat{\mathbf{V}}_D = -\frac{cT_e}{eB}\mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b} + \nabla \ln B), \tag{5.13}$$

is the Maxwellian-averaged electron magnetic drift velocity. The second equation, which determines $\delta T_e$, is given by

$$\begin{aligned}&\left\langle \left(\frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla\right)\left(\frac{3}{2}\frac{\delta T_e}{T_e} - \frac{\delta B_\parallel}{B}\right) n_e \right\rangle_{\tilde{\psi}} \\ &= -\left\langle \frac{7}{2}\hat{\mathbf{V}}_D \cdot \nabla(\tilde{\psi} - \psi)\frac{\partial}{\partial \tilde{\psi}}\left(\frac{\delta T_e}{T_e} + \ln T_e\right) n_e \right\rangle_{\tilde{\psi}} \\ &- \left\langle \hat{\mathbf{V}}_D \cdot \nabla(\tilde{\psi} - \psi)\frac{\partial \ln N_e}{\partial \tilde{\psi}} n_e \right\rangle_{\tilde{\psi}} + \left\langle \hat{\mathbf{V}}_D \cdot \nabla(\delta\varphi - \zeta)\frac{e n_e}{T_e} \right\rangle_{\tilde{\psi}}.\end{aligned} \tag{5.14}$$

Though there are many details we glossed over in the derivation of (5.12) and (5.14) by simply quoting the final equations there is one intermediary result of importance to us. It is found that

$$\frac{\partial \delta A_\parallel}{\partial t} = c\tilde{\mathbf{b}} \cdot \nabla(\zeta - \delta\varphi) \tag{5.15}$$

which can replace the parallel Ampère's law (5.6) in determining $\delta A_\parallel$.

Unlike the last few equations above, the final two equations of the low mass ratio limit of gyrokinetics are straightforwardly obtained. Simply inserting (5.9) into the quasineutrality condition (5.3) and the perpendicular force balance (5.7) yields

$$\frac{1}{n_e}\sum_{s=i}\frac{Z_s^2 e n_s \delta\varphi}{T_s} = -\frac{e\zeta}{T_e} - (\tilde{\psi} - \psi)\frac{\partial \ln N_e}{\partial \psi} + \sum_{s=i}\frac{Z_s}{n_e}\int \mathrm{d}^3 v \left\langle h_s\right\rangle_{\mathbf{r}}, \tag{5.16}$$

and

$$\nabla_\perp^2 \frac{\delta B_\parallel B}{4\pi} = \sum_{s=i} \nabla_\perp \nabla_\perp : \int \mathrm{d}^3 v m_s \left\langle \mathbf{v}_\perp \mathbf{v}_\perp h_s \right\rangle_{\mathbf{r}} + e n_e \nabla_\perp^2 (\zeta - \delta\varphi)$$
$$+ n_e \nabla_\perp^2 (\tilde{\psi} - \psi) \left( T_e \frac{\partial \ln N_e}{\partial \psi} + T_e \frac{\partial \ln T_e}{\partial \psi} \right) + n_e \nabla_\perp^2 \delta T_e, \tag{5.17}$$

respectively.

Here we will immediately reduce these equations further. We know that in TKRMHD the fluctuating parallel magnetic field does not enter while here it does. This is easily remedied by noting that in equation (5.17) all terms on the right hand side are of order $\mathcal{O}(\epsilon p_s / l_\perp^2)$ while the left hand side is of order $\mathcal{O}(\epsilon B^2 / l_\perp^2)$ by virtue of the gyrokinetic ordering (5.1). Of course $\beta$ is, in gyrokinetics unlike in TKRMHD, assumed to be of order unity so that they are of a similar size. We now wish to discard this general assumption by letting

$$\beta \to 0, \tag{5.18}$$

which is something we naturally wish to do when approaching the edge with its lower density and temperature. The reason we only now take this limit is because the reduced mass ratio subsidiary expansion was a necessary precursor, requiring the Alfvén speed to be smaller than the electron thermal speed, or equivalently $m_e/m_i \ll \beta$. Strictly speaking then we are then not freely taking the low $\beta$-limit but using a subsidiary expansion in small $\beta$ satisfying this condition. Nevertheless, the result is the same in that equation (5.17) becomes unbalanced so that to lowest order it reduces to

$$\nabla_\perp^2 \delta B_\parallel = 0, \tag{5.19}$$

whose only permissible solution is

$$\delta B_\parallel = 0. \tag{5.20}$$

Having taken the low $\beta$-limit and extracted from equation (5.17) that $\delta B_\parallel$ vanishes we can then discard it. As for the remaining equations there are no tricky pitfalls to be avoided as we simply have to remove all terms involving $\delta B_\parallel$. Doing this, the quasineutrality condition (5.16) and the ion kinetic equation (5.4) remain completely unchanged while (5.14) is only slightly modified to

$$\left\langle \left( \frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla \right) \frac{3}{2} \frac{\delta T_e}{T_e} n_e \right\rangle_{\tilde{\psi}} = - \left\langle \frac{7}{2} \hat{\mathbf{V}}_D \cdot \nabla (\tilde{\psi} - \psi) \frac{\partial}{\partial \tilde{\psi}} \left( \frac{\delta T_e}{T_e} + \ln T_e \right) n_e \right\rangle_{\tilde{\psi}}$$
$$- \left\langle \hat{\mathbf{V}}_D \cdot \nabla (\tilde{\psi} - \psi) \frac{\partial \ln N_e}{\partial \tilde{\psi}} n_e \right\rangle_{\tilde{\psi}} + \left\langle \hat{\mathbf{V}}_D \cdot \nabla (\delta\varphi - \zeta) \frac{e n_e}{T_e} \right\rangle_{\tilde{\psi}}. \tag{5.21}$$

Equation (5.12) however loses several terms and simplifies to

$$\left( \frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla \right) \frac{e\zeta}{T_e} n_e + B \frac{\partial}{\partial \tilde{l}} \left( \frac{c}{4\pi e B} \nabla^2 \delta A_\parallel + \sum_{s=i} \frac{Z_s}{B} \int \mathrm{d}^3 v v_\parallel \left\langle h_s \right\rangle_{\mathbf{r}} \right) - c \frac{\partial \zeta}{\partial \tilde{\alpha}} \frac{\partial \ln N_e}{\partial \psi} n_e$$
$$= \hat{\mathbf{V}}_D \cdot \nabla (\tilde{\psi} - \psi) \left( \frac{\partial \ln N_e}{\partial \psi} + \frac{\partial \ln T_e}{\partial \psi} \right) n_e - \hat{\mathbf{V}}_D \cdot \nabla \frac{\delta T_e}{T_e} n_e + \hat{\mathbf{V}}_D \cdot \nabla (\delta\varphi - \zeta) \frac{e n_e}{T_e}. \tag{5.22}$$

## 5.3 Outer Scale Long Wavelength Limit

At this point our modified gyrokinetic equation system consists of (5.15), (5.16), (5.21), (5.22), and finally the unmodified ion kinetic equation (5.4). Obviously we must further manipulate these five equations to reduce them into the four TKRMHD equations (4.10), (4.15), (4.16), and (4.17) we wish to match them onto. To this end, we will now perform a final subsidiary expansion of exactly the opposite kind to that which we used in Chapter 4 for TKRMHD, leaving a final similar ordering. Thus, we will steepen the perpendicular gradients, driving the small gyroscale turbulence to increase and its typical wavelength to lengthen into the typical outer scale turbulence of [29].

Because the gyrokinetic ordering (5.1) makes no distinction between the parallel and the perpendicular scales $L_\parallel$ and $L_\perp$ of the mean distributions it is logical to choose as an expansion parameter

$$\xi \equiv \frac{L_\perp}{L_\parallel}. \tag{5.23}$$

In so doing we separate the two scales as in the TKRMHD ordering (4.6). Hence we must clarify that the original expansion parameter in the gyrokinetic ordering (5.1) now refers to

$$\epsilon \equiv \frac{\rho_i}{L_\perp}. \tag{5.24}$$

This is the natural choice because the strength of the turbulence (in terms of transport), and consequently the size of the fluctuations, are controlled by this quantity.

We now turn to the outer scale conditions of (4.2), (4.3), and (4.4) we used in the TKRMHD subsidiary expansion. Obviously we wish to reuse them now in order to arrive at the same equations. Yet again we find that they will suffice in order to determine how the remaining length scales should be order with respect to each other and how large the fluctuations should be in comparison to the quiescent equilibrium. First we use the first condition to set the parallel length scales to be the system scale

$$L_\parallel \sim l_\parallel \sim a. \tag{5.25}$$

Then in order to fulfil the remaining two conditions we are obliged to use the ordering

$$\frac{e\delta\varphi}{T_s} \sim \frac{\delta f_s}{F_s} \sim \frac{l_\perp}{L_\perp} \sim \frac{\epsilon}{\xi}. \tag{5.26}$$

Obviously then as $\xi \to 0$ when the mean gradients steepen the turbulence increases in amplitude as a response to this increasing driving mechanism. Additionally, while we have already fixed the parallel turbulence wavelength to the long $a$-scale, this furthermore implies that

$$\frac{\rho_i}{l_\perp} \sim \xi \tag{5.27}$$

so that as $\xi \to 0$ the perpendicular turbulence wavelength also increases and we obtain a (particular) long wavelength limit.

The choices above constitute all the ordering changes which arises in our long-wavelength limit. The primary result then is the separation of the magnetic drift timescale

$$\frac{V_{Di}}{l_\perp} \tag{5.28}$$

from the other timescales present in the problem: the ion parallel streaming timescale, the fluctuation drift timescale, and the collisional timescale. All of these remain ordered as $v_{th,i}/l_\parallel$ while the magnetic drift slows to $\mathcal{O}(\xi v_{th,i}/l_\parallel)$. This is an intuitive result because the strong spatial anisotropy we have introduced in this subsidiary expansion does not change the mean magnetic field. As such, only the magnetic drift is not amplified and the magnetic drift timescale separates away.

The first major consequence of the diminishing magnetic drift can be seen by turning to equation (5.21). There all terms on the right-hand side are magnetic drift terms and thus the left hand side dominates. By using (5.11) the derivative can obviously be interchanged with the flux-surface average and so because all of $\delta T_e$, $T_e$, and $n_e$ are flux functions to lowest order we find that this equation reduces to the simple result

$$\left.\frac{\partial \delta T_e}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} = 0. \tag{5.29}$$

Thus, in accordance with the TKRMHD equations, $\delta T_e$ vanishes and we can drop this equation and focus on the remaining four.

The first equation in question is the ion kinetic equation (5.4) which becomes

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \left(v_\parallel \mathbf{b} + \langle \mathbf{V}_\chi\rangle_{\mathbf{R}_s}\right)\cdot\frac{\partial}{\partial \mathbf{R}_s}\right] h_s - \langle C[h_s]\rangle_{\mathbf{R}_s} \\ &= \frac{Z_s e F_s}{T_s}\frac{\partial}{\partial t}\left\langle \delta\varphi - \frac{\mathbf{v}\cdot\delta\mathbf{A}}{c}\right\rangle_{\mathbf{R}_s} - \frac{\partial F_s}{\partial \psi}\langle \mathbf{V}_\chi\rangle_{\mathbf{R}}\cdot\nabla\psi.\end{aligned} \tag{5.30}$$

Evaluating the gyroaverages in this equation it becomes exactly the TKRMHD ion kinetic equation (4.10) because they both describe the lowest order gyrophase independent ion distribution, $h_s$ and $\delta f_s$ respectively. This is because all $\mathbf{b}$ in that equation can to lowest order be switched to $\tilde{\mathbf{b}}$ (but not vice versa) and $\mathbf{b}$, as referred to here, is labelled as $\tilde{\mathbf{b}}$ there through (3.21).

Proceeding to the equation describing the parallel vector potential (5.15) it remains unchanged,

$$\frac{\partial \delta A_\parallel}{\partial t} = c\tilde{\mathbf{b}}\cdot\nabla(\zeta - \delta\varphi), \tag{5.31}$$

in a very similar form to the comparable TKRMHD equation (4.15) which it now is a trivial matter to find that it is equivalent to. Using (5.9) to find that

$$\delta n_e = \frac{e\zeta n_e}{T_e} + (\tilde{\psi} - \psi)\frac{\partial n_e}{\partial \psi}, \tag{5.32}$$

we find upon inserting this that the right hand side of (5.31) becomes

$$\tilde{\mathbf{b}}\cdot\nabla\left(\frac{\delta n_e}{n_e} - \frac{e\delta\varphi}{T_e}\right) - \tilde{\mathbf{b}}\cdot\nabla\left[(\tilde{\psi} - \psi)\frac{\partial \ln n_e}{\partial \psi}\right] = \mathbf{b}\cdot\nabla\left(\frac{\delta n_e}{n_e} - \frac{e\delta\varphi}{T_e}\right), \tag{5.33}$$

because, by using that $\tilde{\psi}$ and $n_e$ are exact flux functions, the final term is seen to be one order $\xi$ lower than the others. Inserting this back into (5.31) it becomes precisely the outer scale TKRMHD equation (5.9).

Turning now to equation (5.22) for $\zeta$ which becomes

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{u}_{eff} \cdot \nabla\right) \frac{e\zeta}{T_e} n_e + B\frac{\partial}{\partial \tilde{l}} \left(\frac{c}{4\pi e B}\nabla^2 \delta A_\parallel + \sum_{s=i} \frac{Z_s}{B} \int \mathrm{d}^3 v v_\parallel \left\langle h_s \right\rangle_\mathbf{r}\right) \\ &= c\frac{\partial \zeta}{\partial \tilde{\alpha}} \frac{\partial \ln N_e}{\partial \psi} n_e.\end{aligned} \tag{5.34}$$

we find that it does not take much work to prove that it is equivalent to the TKRMHD equation (4.16) for the density variation $\delta n_e$. Indeed the expression the $\tilde{l}$-derivative acts on is nothing but the lowest order electron parallel velocity $u_{e\parallel}$ as determined from the parallel Ampère's law (5.6) [24]. This term is then already the exact same term as the last in (4.16). Now the first term we can manipulate into

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \left[\delta n_e - (\tilde{\psi} - \psi) \frac{\partial n_e}{\partial \psi}\right] \\ &= \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \delta n_e + \mathbf{u}_E \cdot \nabla \psi \frac{\partial n_e}{\partial \psi} + \frac{\partial n_e}{\partial \psi}\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \tilde{\psi} \\ &= \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \delta n_e + \mathbf{u}_E \cdot \nabla n_e + \frac{\partial n_e}{\partial \psi} \frac{c}{B} \mathbf{b} \times \nabla \zeta \cdot \nabla \tilde{\psi}.\end{aligned} \tag{5.35}$$

where we, in order, have used first (5.10) and (5.32), then the slow variation of $n_e$ and $\psi$, and finally (5.11). Because the final term here is to lowest order the same as the right hand side of (5.34) through the Poisson bracket (2.45) we can thus rewrite (5.34) as

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \delta n_e + \mathbf{u}_E \cdot \nabla n_e + \tilde{\mathbf{B}} \cdot \nabla \left(\frac{u_{\parallel e}}{B}\right) = 0, \tag{5.36}$$

exactly in the form of (4.16).

## 5.4 The Vorticity Equation

In the previous Section we were able to match our outer scale gyrokinetics with three of the four outer scale TKRMHD equations in a straightforward fashion. We are now left with only one TKRMHD equation which we have yet to match onto: the vorticity equation (4.17) for the flux-surface averaged part of the electrostatic potential $\delta\varphi$. The procedure for this final step however will not be anywhere near as easy as the rest of the equations. It does not suffice to merely study the gyrokinetic quasineutrality condition, which still is given by

$$\frac{1}{n_e} \sum_{s=i} \frac{Z_s^2 e n_s \delta\varphi}{T_s} = -\frac{e\zeta}{T_e} - (\tilde{\psi} - \psi) \frac{\partial \ln N_e}{\partial \psi} + \sum_{s=i} \frac{Z_s}{n_e} \int \mathrm{d}^3 v \left\langle h_s \right\rangle_\mathbf{r}. \tag{5.37}$$

Instead we will have to return to before our final subsidiary expansion and use all four equations (5.22), (5.6), (5.16), and (5.4). This is because of a cancellation that will occur in our manipulations, forcing us to retain terms of higher order in $\xi$. In particular, we will have to include the magnetic drift terms, making it clear that the magnetic geometry enters our equations through the vorticity equation.

Now, in order to produce a vorticity equation we will turn to the identity

$$\frac{1}{n_i}\int \mathrm{d}^3 v F_i \left(\left\langle \langle\delta\varphi\rangle_{\mathbf{R}_i}\right\rangle_{\mathbf{r}} - \delta\varphi\right) = (\Gamma_0 - 1)\delta\varphi = \frac{T_i}{m_i\Omega_i^2}\nabla_\perp^2\delta\varphi \tag{5.38}$$

which follows by Taylor expanding $\delta\varphi$ from $\mathbf{r}$ to $\mathbf{R}_i$ and back while using

$$\langle \mathbf{v}_\perp\rangle_{\mathbf{r}} = 0, \quad \langle \mathbf{v}_\perp\mathbf{v}_\perp\rangle_{\mathbf{r}} = \frac{v_\perp^2}{2}(\mathbf{I} - \tilde{\mathbf{b}}\tilde{\mathbf{b}}). \tag{5.39}$$

With this identity it should be apparent how we should be able to produce a term of the same form as that in the vorticity equation (4.17). We will begin by introducing the substitution

$$h_i = (\tilde{\psi} - \psi)\frac{\partial F_i}{\partial\psi} + \frac{Z_i e}{T_i}F_i\,\langle\delta\varphi\rangle_{\mathbf{R}_i} + g_i \tag{5.40}$$

into the quasineutrality condition (5.16) to produce

$$n_e\frac{e\zeta}{T_e} = \sum_{s=i}\frac{Z_s^2 e T_i}{m_i T_s\Omega_i^2}n_s\nabla_\perp^2\delta\varphi + \sum_{s=i} Z_s\int \mathrm{d}^3 v\,\langle g_s\rangle_{\mathbf{r}}\,. \tag{5.41}$$

With equation (5.41) in hand we immediately proceed by inserting it into equation (5.22) after first changing $\mathbf{u}_{eff}$ to $\mathbf{u}_E$ in that equation, which is permissible to lowest order in $\epsilon$. In so doing we eliminate $\zeta$ and make the $\delta\varphi$-Laplacian appear more similar to the desired term in (4.17), as we are left with

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\right)\sum_{s=i}\frac{Z_s^2 e}{m_s\Omega_s^2}n_s\nabla_\perp^2\delta\varphi + \left(\frac{\partial}{\partial t} + \frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\right)\sum_{s=i} Z_s\int \mathrm{d}^3 v\,\langle g_s\rangle_{\mathbf{r}} \\
&+ B\frac{\partial}{\partial\tilde{l}}\left(\frac{1}{B}\left(\frac{c}{4\pi e}\nabla^2\delta A_\parallel + \sum_{s=i} Z_s\int \mathrm{d}^3 v v_\parallel\,\langle g_s\rangle_{\mathbf{r}}\right)\right) - c\frac{\partial\zeta}{\partial\tilde{\alpha}}\frac{\partial\ln N_e}{\partial\tilde{\psi}}n_e \\
&= \hat{\mathbf{V}}_D\cdot\nabla(\tilde{\psi} - \psi)\left(\frac{\partial\ln N_e}{\partial\psi} + \frac{\partial\ln T_e}{\partial\psi}\right)n_e + \hat{\mathbf{V}}_D\cdot\nabla(\delta\varphi - \zeta)\frac{e n_e}{T_e}.
\end{aligned} \tag{5.42}$$

At this point, we need only apply the flux surface average for the first term to completely match the term in the vorticity equation, up to a constant factor. In so doing the third term of (5.42) vanishes by being of the form (2.43). Similarly the sixth term also vanishes because, to lowest order in $\epsilon$, the $\tilde{\alpha}$-derivative does not act on any of $N_e$, $n_e$, or $B$ so it can be converted into a total-$\tilde{\alpha}$-derivative of the form (2.43). Thus we have

$$\begin{aligned}
&\left\langle\left(\frac{\partial}{\partial t} + \frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\right)\sum_{s=i}\frac{Z_s^2 e}{m_s\Omega_s^2}n_s\nabla_\perp^2\delta\varphi\right\rangle_{\tilde{\psi}} - \left\langle\hat{\mathbf{V}}_D\cdot\nabla(\delta\varphi - \zeta)\frac{e n_e}{T_e}\right\rangle_{\tilde{\psi}} \\
&+ \left\langle\left(\frac{\partial}{\partial t} + \frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\right)\sum_{s=i} Z_s\int \mathrm{d}^3 v\left(1 + \frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)g_s\right\rangle_{\tilde{\psi}} = 0
\end{aligned} \tag{5.43}$$

where we have only kept terms up to $\mathcal{O}(\xi\epsilon n_i v_{th,i}/l_\parallel)$. Thus we have expanded the integrand to second order and eliminated the fifth term because, to lowest order in $\epsilon$ $\left\langle \hat{\mathbf{V}}_D \cdot \nabla\tilde{\psi}\right\rangle_{\tilde{\psi}} = 0$ [24], so that term's order is lowered by two powers of $\xi$.

In order to proceed from this point we must now evaluate the $g_s$-integral up to second order in $\xi$, i.e. $v_{th,i}\epsilon\xi F_{0i}/L_\parallel$. Thus we return to the full ion gyrokinetic equation (5.4) into which we insert the substitution (5.40) to produce

$$\begin{aligned}
&\frac{\partial}{\partial t} g_i + (v_\parallel \mathbf{b} + \mathbf{V}_{Ds} + \langle \mathbf{V}_\chi \rangle_{\mathbf{R}_i} \cdot \nabla g_i - \langle C[h_i] \rangle_{\mathbf{R}_i} \\
&= -\frac{\partial F_i}{\partial \psi}\left[\left(v_\parallel \mathbf{b} + \mathbf{V}_{Di} + \langle \mathbf{V}_\chi \rangle_{\mathbf{R}_i}\right) \cdot \nabla(\tilde{\psi} - \psi) + \frac{\partial}{\partial t}(\tilde{\psi} - \psi) + \langle \mathbf{V}_\chi \rangle_{\mathbf{R}_i} \cdot \nabla\psi\right] \\
&- \frac{Z_i e}{T_i} F_i \left(v_\parallel \mathbf{b} + \mathbf{V}_{Di} + \langle \mathbf{V}_\chi \rangle_{\mathbf{R}_i} \cdot \nabla \langle \delta\varphi \rangle_{\mathbf{R}_i} + \frac{1}{c}\frac{\partial}{\partial t} \langle \mathbf{v} \cdot \delta\mathbf{A} \rangle_{\mathbf{R}_i}\right).
\end{aligned} \tag{5.44}$$

Evaluating the gyroaverages that appear in this equation and using equation (5.15) for $\delta A_\parallel$ it becomes a tedious if straightforward exercise to produce

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v_\parallel\left(\mathbf{b} - \frac{1}{B}\mathbf{b} \times \nabla\delta A_\parallel\right) \cdot \nabla + \frac{c}{B}\mathbf{b} \times \nabla\delta\varphi \cdot \nabla\right) g_i - \langle C[h_i] \rangle_{\mathbf{R}_i} \\
&+ \frac{\partial F_i}{\partial \psi}\left(\frac{\partial}{\partial t} + \frac{c}{B} \times \nabla\delta\varphi \cdot \nabla\right)\tilde{\psi} = -\mathbf{V}_{Di} \cdot \nabla\left((\tilde{\psi} - \psi)\frac{\partial F_i}{\partial \psi} + \frac{Z_i e}{T_i} F_i \delta\varphi + g_i\right) \\
&- \frac{c}{B} \times \nabla \frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\left(\delta\varphi - \frac{v_\parallel \delta A_\parallel}{c}\right) \cdot \nabla g_i \\
&- \frac{\partial F_i}{\partial \psi}\left[v_\parallel \mathbf{b} \cdot \nabla(\tilde{\psi} - \psi) + \frac{c}{B}\mathbf{b} \times \nabla\left(\frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)\left(\delta\varphi - \frac{v_\parallel \delta A_\parallel}{c}\right) \cdot \nabla\tilde{\psi}\right] \\
&- \frac{Z_i e}{T_i} F_i \left\{\left[v_\parallel \mathbf{b} + \frac{c}{B}\mathbf{b} \times \nabla\left(\left(1 + \frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)\left(\delta\varphi - \frac{v_\parallel \delta A_\parallel}{c}\right)\right)\right] \cdot \nabla \langle \delta\varphi \rangle_{\mathbf{R}_i}\right. \\
&\left. + v_\parallel\left(1 + \frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)\tilde{\mathbf{b}} \cdot \nabla(\zeta - \delta\varphi)\right\}.
\end{aligned} \tag{5.45}$$

Our next task is integrating equation (5.45) over all velocities. In doing so we must keep in mind that the quantities in this equation are to be evaluated at $\mathbf{R}_i$ and not the usual $\mathbf{r}$ which facilitates easy integration by allowing derivatives and integrals to be interchanged. To still make use of this we Taylor expand all quantities around $\mathbf{r}$, still only keeping terms to second order in $\xi$. In so doing we will find that all linear Taylor terms vanish by gyrophase independence if by nothing else. Furthermore, only the left hand side is of order $v_{th,i}\epsilon\xi^{-1}F_{0i}/L_\parallel$ and so needs to be expanded to second order which will result in two new terms arising. Taking this into consideration and recalling the definition of $\hat{\mathbf{V}}_D$ (5.13) the integration can be performed with the result

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\right)\int \mathrm{d}^3v\left(1+\frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)g_i+B\frac{\partial}{\partial\tilde{l}}\int \mathrm{d}^3v v_\parallel\frac{1}{B}\left(1+\frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)g_i\\
&+c\frac{\partial\zeta}{\partial\tilde{\alpha}}\frac{\partial n_i}{\partial\psi}=\frac{1}{Z_iT_e}\hat{\mathbf{V}}_D\cdot\nabla\left[(\tilde{\psi}-\psi)\frac{\partial p_i}{\partial\psi}+\frac{Z_ie\delta\varphi}{T_i}p_i\right]-\int \mathrm{d}^3v\mathbf{V}_{Di}\cdot\nabla g_i\\
&-\frac{c}{B}\mathbf{b}\times\nabla\int \mathrm{d}^3v\frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\cdot\nabla g_i-\frac{2}{3}\frac{\partial p_i}{\partial\psi}\frac{c}{B\Omega_i^2}\mathbf{b}\times\nabla\nabla_\perp^2\delta\varphi\cdot\nabla\tilde{\psi}\\
&-\frac{c}{B}\int \mathrm{d}^3v\mathbf{b}\times\nabla\left(\frac{1}{\Omega_i}\mathbf{b}\times\mathbf{v}\cdot\nabla\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right)\cdot\nabla\frac{1}{\Omega_i}\mathbf{b}\times\mathbf{v}\cdot\nabla g_i\\
&-\frac{c}{B}\mathbf{b}\times\nabla\int \mathrm{d}^3v\left(\frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right)\cdot\nabla g_i.
\end{aligned}\tag{5.46}$$

Of course we wish to quickly move past this ugly intermediate result because we are really only interested in its flux-surface averaged form which is given by

$$\begin{aligned}
&\left\langle\left(\frac{\partial}{\partial t}+\frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\right)\int \mathrm{d}^3v\left(1+\frac{v_\perp^2}{2\Omega_i^2}\nabla_\perp^2\right)g_i\right\rangle_{\tilde{\psi}}\\
&=\left\langle\hat{\mathbf{V}}_D\cdot\nabla\left(\frac{e\delta\varphi}{T_e}n_i\right)\right\rangle_{\tilde{\psi}}-\left\langle\int \mathrm{d}^3v\mathbf{V}_{Di}\cdot\nabla g_i\right\rangle_{\tilde{\psi}}\\
&-\left\langle\frac{c}{B}\mathbf{b}\times\nabla\int \mathrm{d}^3v\frac{v_\perp^2}{\Omega_i^2}\nabla_\perp^2\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\cdot\nabla g_i\right\rangle_{\tilde{\psi}}\\
&-\left\langle\frac{c}{B}\int \mathrm{d}^3v\mathbf{b}\times\nabla\left(\frac{1}{\Omega_i}\mathbf{b}\times\mathbf{v}\cdot\nabla\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right)\cdot\nabla\frac{1}{\Omega_i}\mathbf{b}\times\mathbf{v}\cdot\nabla g_i\right\rangle_{\tilde{\psi}}
\end{aligned}\tag{5.47}$$

where we used precisely the same arguments as in (5.42) to eliminate similar terms in the same fashion.

We can now insert equation (5.47) into our emerging vorticity equation (5.43). Before doing this however we note that the three terms

$$\hat{\mathbf{V}}_D\cdot\nabla(\delta\varphi-\zeta)\frac{en_e}{T_e}-\sum_{s=i}\hat{\mathbf{V}}_D\cdot\nabla\frac{Z_se\delta\varphi}{T_e}n_s+\sum_{s=i}Z_s\int \mathrm{d}^3v\mathbf{V}_{Ds}\cdot\nabla g_s,\tag{5.48}$$

by using the expressions (5.2), (5.32), and (5.40) to exchange $\zeta$ and $g_i$ for $\delta n_e$ and $\delta n_i$, can be combined into

$$\begin{aligned}
&\hat{\mathbf{V}}_D\cdot\nabla\left(\frac{e\delta\varphi}{T_e}n_e-\delta n_e+(\tilde{\psi}-\psi)\frac{\partial n_e}{\partial\psi}\right)-\sum_{s=i}\hat{\mathbf{V}}_D\cdot\nabla\frac{Z_se\delta\varphi}{T_e}n_s\\
&-\sum_{s=i}\frac{1}{T_e}\hat{\mathbf{V}}_D\cdot\nabla\left(Z_se(\delta\varphi-\langle\delta\varphi\rangle_{\mathbf{R}_i})n_s-(\tilde{\psi}-\psi)\frac{\partial p_s}{\partial\psi}\right)\\
&+\sum_{s=i}\frac{c}{eB}\Big(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\nabla\delta p_{\parallel s}+\mathbf{b}\times\nabla\ln B\cdot\nabla\delta p_{\perp s}\Big)\\
&=\frac{1}{T_e}\hat{\mathbf{V}}_D\cdot\nabla\left((\tilde{\psi}-\psi)\frac{\partial p}{\partial\psi}\right)+\frac{c}{eB}\Big(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\nabla\delta p_\parallel+\mathbf{b}\times\nabla\ln B\cdot\nabla\delta p_\perp\Big).
\end{aligned}\tag{5.49}$$

In the equation above the first term, by the same argument as the corresponding term in (5.43), is one order $\xi$ too small so we can drop it. As for the remaining terms they have a suggestive form, being very similar those we manipulated in Section 3.3 in order to derive the TKRMHD vorticity equation. Employing the same vector manipulations then we have that

$$\frac{\mathbf{b}}{B}\times(\mathbf{b}\cdot\nabla\mathbf{b})\cdot\nabla\delta p_\parallel+\frac{\mathbf{b}}{B}\times\nabla\ln B\cdot\nabla\delta p_\perp=\nabla\times\mathbf{b}\cdot\frac{\nabla\delta p_\parallel}{B}-\mathbf{b}\times\nabla\left(\frac{1}{B}\right)\cdot\nabla\delta p_\perp$$
$$=\nabla\times\mathbf{b}\cdot\frac{\nabla(\delta p_\parallel-\delta p_\perp)}{B}+\nabla\times\left(\frac{\mathbf{b}}{B}\right)\cdot\nabla\delta p_\perp=\nabla\cdot\left[\frac{(\delta p_\parallel-\delta p_\perp)}{B}\nabla\times\mathbf{b}+\delta p_\perp\nabla\times\left(\frac{\mathbf{b}}{B}\right)\right] \tag{5.50}$$

which are the fluctuating parts of two of the three terms under the divergence in (3.64), and importantly the term that does not vanish under the flux-surface average. This provides a clear indication that we are on the right track in re-deriving the vorticity equation, because in the subsidiary gyrokinetic ordering the fluctuating part dominates over the non-fluctuating part. Now the same argument we used to go from (3.64) to (3.72) still holds for the fluctuating part alone and we find that in the subsidiary gyrokinetic ordering (5.26) the right-hand side of (3.72) is still of higher order than the left hand side so the three terms of (5.48) can be neglected as small. Thus upon inserting equation (5.45) into (5.43) we currently have

$$\left\langle\left(\frac{\partial}{\partial t}+\frac{c}{B}\tilde{\mathbf{b}}\times\nabla\delta\varphi\cdot\nabla\right)\left(\sum_{s=i}\frac{Z_s^2e}{m_s\Omega_s^2}n_s\nabla_\perp^2\delta\varphi\right)\right\rangle_{\tilde{\psi}}$$
$$=\left\langle\sum_{s=i}\frac{cZ_s}{B}\int\mathrm{d}^3v\mathbf{b}\times\nabla\left(\frac{1}{\Omega_s^2}(\mathbf{b}\times\mathbf{v}\cdot\nabla)^2\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right)\cdot\nabla g_s\right\rangle_{\tilde{\psi}}$$
$$+\left\langle\sum_{s=i}\frac{cZ_s}{B}\int\mathrm{d}^3v\tilde{\mathbf{b}}\times\nabla\left(\frac{1}{\Omega_i}\mathbf{b}\times\mathbf{v}\cdot\nabla\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right)\cdot\nabla\frac{1}{\Omega_i}\mathbf{b}\times\mathbf{v}\cdot\nabla g_i\right\rangle_{\tilde{\psi}}. \tag{5.51}$$

Equipped with equation (5.51) it is clear the we must now find a means of converting the two terms on the right hand side into the perpendicular pressure term of (3.72). To this end we will employ another identity similar to (5.38) for the perpendicular pressure which is given by

$$\int\mathrm{d}^3vm_i\mathbf{b}\times\mathbf{v}g=-\frac{1}{\Omega_i}\nabla_\perp\delta p_\perp. \tag{5.52}$$

This identity is easy to derive from (5.2) and (5.32) by Taylor expanding the gyrophase independent $g$ from $\mathbf{R}_i$ to $\mathbf{r}$ before integrating to find this pressure moment. It is therefore natural that we use return to the ion gyrokinetic equation (5.44), multiply it $m_i\mathbf{b}\times\mathbf{v}$ and Taylor expand before integrating over all velocities. Keeping terms up to order $\mathcal{O}(\xi^{-1}\epsilon p_i/L_\parallel)$ we find that all terms on the right hand of (5.44) is of a higher order and so we are left with the left hand side which becomes

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \frac{c}{B}\tilde{\mathbf{b}} \times \nabla\delta\varphi \cdot \nabla\right) \frac{1}{\Omega_i} \nabla_\perp \delta p_{\perp s} \\
&- \frac{m_i c}{\Omega_i B} \int \mathrm{d}^3 v \mathbf{b} \times \mathbf{v} \left(\mathbf{b} \times \nabla \left(\mathbf{b} \times \mathbf{v} \cdot \nabla \left(\delta\varphi - \frac{v_\parallel \delta A_\parallel}{c}\right)\right) \cdot \nabla g\right) \\
&+ m_i \int \mathrm{d}^3 v \mathbf{b} \times \mathbf{v} v_\parallel \tilde{\mathbf{b}} \cdot \nabla \left(\frac{1}{\Omega_i} \mathbf{b} \times \mathbf{v} \cdot \nabla g\right) - \frac{c}{\Omega_i} \nabla_\perp \left(\frac{\partial p_i}{\partial \psi} \frac{\partial \zeta}{\partial \tilde{\alpha}}\right) = 0.
\end{aligned} \tag{5.53}$$

Applying $B^{-1}\nabla\cdot$ and flux-surface averaging the equation above we find that the perpendicular pressure component of the vorticity equation is given by

$$\begin{aligned}
&\left\langle \frac{1}{B} \left(\frac{\partial}{\partial t} + \frac{c}{B}\tilde{\mathbf{b}} \times \nabla\delta\varphi \cdot \nabla\right) \nabla_\perp^2 \frac{\delta p_{\perp i}}{\Omega_i} \right\rangle_{\tilde{\psi}} \\
&+ \left\langle \frac{m_i c}{B^2 \Omega_i} \int \mathrm{d}^3 v \mathbf{b} \times \nabla \left((\mathbf{b} \times \mathbf{v} \cdot \nabla)^2 \left(\delta\varphi - \frac{v_\parallel \delta A_\parallel}{c}\right)\right) \cdot \nabla g \right\rangle_{\tilde{\psi}} \\
&+ \left\langle \frac{m_i c}{B^2 \Omega_i} \int \mathrm{d}^3 v \mathbf{b} \times \nabla \left((\mathbf{b} \times \mathbf{v} \cdot \nabla) \left(\delta\varphi - \frac{v_\parallel \delta A_\parallel}{c}\right)\right) \cdot \nabla (\mathbf{b} \times \mathbf{v} \cdot \nabla) g \right\rangle_{\tilde{\psi}} \\
&+ \left\langle \frac{m_i}{B} \nabla \cdot \int \mathrm{d}^3 v \mathbf{b} \times \mathbf{v} v_\parallel \tilde{\mathbf{b}} \cdot \nabla \left(\frac{1}{\Omega_i} \mathbf{b} \times \mathbf{v} \cdot \nabla g\right) \right\rangle_{\tilde{\psi}} = 0.
\end{aligned} \tag{5.54}$$

Summing this equation over all ion species and adding it to equation (5.51) multiplied by $(e/c)$ we then finally find that we have reproduced the outer scale TKRMHD vorticity equation (4.17) from gyrokinetics:

$$\begin{aligned}
&\left\langle \sum_s \frac{n_s m_s}{B} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \left(\frac{\nabla_\perp^2 \delta p_{\perp_s}}{n_s m_s \Omega_s} + \frac{c \nabla_\perp^2 \delta\varphi}{B}\right) \right\rangle_{\tilde{\psi}} \\
&+ \sum_{s=i} \left\langle \frac{m_s}{B} \nabla \cdot \int \mathrm{d}^3 v \mathbf{b} \times \mathbf{v} v_\parallel \tilde{\mathbf{b}} \cdot \nabla \left(\frac{1}{\Omega_s} \mathbf{b} \times \mathbf{v} \cdot \nabla g\right) \right\rangle_{\tilde{\psi}} \\
&= \left\langle \sum_s \frac{n_s m_s}{B} \left(\frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla\right) \left(\frac{\nabla_\perp^2 \delta p_{\perp_s}}{n_s m_s \Omega_s} + \frac{c \nabla_\perp^2 \delta\varphi}{B}\right) \right\rangle_{\tilde{\psi}} \\
&+ \left\langle \frac{1}{B \Omega_s} \left[(\nabla \cdot \mathbf{b}) \nabla^2 - (\nabla \mathbf{b}) : \nabla\nabla\right] \int \mathrm{d}^3 w \frac{m_s w_\perp^2}{4} w_\parallel f_s \right\rangle_{\tilde{\psi}} = 0.
\end{aligned} \tag{5.55}$$

To arrive at this result, in the final term we have moved the parallel derivative through the integral and carefully calculated gyroaverages with the unit vectors of equation (2.15) and further used the identities

$$\nabla \mathbf{e}_i \cdot \mathbf{b} = -\nabla \mathbf{b} \cdot \mathbf{e}_i, \tag{5.56}$$

which follow from the fact that these vectors are orthogonal, and

$$\mathbf{b} \cdot \nabla \ln B = -\nabla \cdot \mathbf{b}, \tag{5.57}$$

which follows from the vanishing divergence of $\mathbf{B}$, to move derivatives into the desired form.

Having reached this stage, we have found that the turbulent equations of TKRMHD and gyrokinetics can be smoothly matched onto each other in the intermediate outer scale limit. However, we are not quite done. In the final Chapter we will proceed to the transport equations to also match them up to their TKRMHD counterparts. Only then can we fully say that the intermediary limits fully describe the same physics.

# 6

# Matching on Transport Timescale

We now turn our attention to bulk plasma transport. Because both the electron and ion mean distributions are Maxwellian and thus characterised by the parameters $n_i, n_e, T_i$, and $T_e$ we will need to find four equations that determine the evolution of these quantities. We will find it convenient to begin with gyrokinetic transport because it is well understood with readily available transport equations. We will thus find it easy to simply order the various terms in these equations using our subsidiary expansions of gyrokinetics to eliminate small terms Furthermore, here our simplifying choice of no sonic rotation will drastically reduce the complexity of the full transport equations. With the resulting limit transport equations in hand, in a form whose physical interpretation is readily available, we can turn to our TKRMHD-limit and derive its transport equations from scratch. Guided by the gyrokinetic transport equations it will prove to be a short process to use the TKRMHD orderings to arrive at those same transport equations, fully completing the matching between TKRMHD and gyrokinetics.

## 6.1 Gyrokinetic Transport

In order to determine the gyrokinetic transport equations for our subsidiary expansion of the previous Chapter in a simple fashion we will immediately make use of two things: the lack of any sonic flows and the low $\beta$. These conditions allow us to treat the confining field geometry as fixed over the transport timescale because the low density, quiescent plasma then cannot give rise to sufficiently large self-generated field to seriously impact the mean field. As such, the ion density transport across flux surfaces in conventional gyrokinetics, as described in Abel et al [10], initially reduce to

$$\frac{\partial}{\partial t} \langle n_i \rangle_\psi + \frac{\partial}{\partial \psi} \langle \Gamma_i \rangle_\psi = 0, \tag{6.1}$$

where

$$\begin{aligned} \langle \Gamma_i \rangle_\psi = &\left\langle \int \mathrm{d}^3 v C[F_i] \left(\mathbf{R}_i - \mathbf{r}\right) \cdot \nabla \psi \right\rangle_\psi + \left\langle \int \mathrm{d}^3 v F_i^{(nc)} \mathbf{V}_{Ds} \cdot \nabla \psi \right\rangle_\psi \\ &- \langle n_i \rangle_\psi I(\psi) \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle_\psi}{\langle B^2 \rangle_\psi} + \left\langle \int \mathrm{d}^3 v h_i \left\langle \mathbf{V}_\chi \right\rangle_{\mathbf{r}} \cdot \nabla \psi \right\rangle_{turb,\psi} \end{aligned} \tag{6.2}$$

is the radial particle flux (because it crosses the flux surfaces $\psi$) and $F_i^{(nc)} \sim \epsilon F_i$ describes neoclassical corrections to the Maxwellian $F_i$. Here the first term represents classical collisional transport, while the second and third describe neoclassical transport arising from the non-uniformity of $\mathbf{B}$ and time-varying magnetic fields, and the final term comprises turbulent transport.

Now the radial flux in the form of (6.2) is obviously ordered with respect to the gyrokinetic orderings (5.1), so upon further applying our subsidiary orderings we can reduce it further. Of these terms we naturally expect the turbulent flux

$$\left\langle \int \mathrm{d}^3 v h_i \left\langle \mathbf{V}_\chi \right\rangle_{\mathbf{r}} \cdot \nabla\psi \right\rangle_{turb,\psi} \tag{6.3}$$

to dominate in our subsidiary expansion because the turbulent amplitude greatly increases. As such, we also expect that this term should set the subsidiary gyrokinetic transport order to be

$$\frac{1}{\tau_E} \sim \frac{\epsilon^2}{\xi^2}\frac{v_{th,i}}{l_\parallel}. \tag{6.4}$$

Going term by term to check our assumption (6.4), we first note that because of the important low mass ratio result that $\left\langle \mathbf{E}\cdot\mathbf{B}\right\rangle_\psi$ vanishes [24] the first neoclassical term vanishes. Using our final outer scale ordering (5.26) for the remaining two terms the first collisional term and the second neoclassical term are indeed found to be an order $\xi$ smaller than the final turbulent term. Thus, we find the unsurprising result that in the outer scale limit the strong turbulence dominates in determining transport through

$$\frac{\partial n_i}{\partial t} + \left\langle \nabla\cdot\left\langle \int \mathrm{d}^3 v h_i \left\langle \mathbf{V}_\chi \right\rangle_{\mathbf{r}} \right\rangle_{turb} \right\rangle_\psi = 0. \tag{6.5}$$

Proceeding to the heat transport equation the procedure is just the same. Starting from

$$\frac{3}{2}\frac{\partial}{\partial t}\left\langle n_i\right\rangle_\psi T_i + \frac{\partial}{\partial\psi}\left\langle q_i\right\rangle_\psi = P_i^{Ohm} + P_i^{comp} + P_i^{turb} + \left\langle C_i^{(E)}\right\rangle_\psi \tag{6.6}$$

where

$$\begin{aligned}\langle q_i\rangle_\psi = &\left\langle \int \mathrm{d}^3 v \frac{m_i v^2}{2} C[F_i](\mathbf{R}_s - \mathbf{r})\cdot\nabla\psi \right\rangle_\psi - \frac{5}{2} p_i I(\psi)\frac{\langle \mathbf{E}\cdot\mathbf{B}\rangle_\psi}{\langle B^2\rangle_\psi} \\ &+ \left\langle \int \mathrm{d}^3 v \frac{m_i v^2}{2} F_i^{(nc)} \mathbf{V}_{Di}\cdot\nabla\psi \right\rangle_\psi + \left\langle \int \mathrm{d}^3 v \frac{m_i v^2}{2} h_i \langle \mathbf{V}_\chi\rangle_{\mathbf{r}}\cdot\nabla\psi \right\rangle_{turb,\psi}\end{aligned} \tag{6.7}$$

is the radial heat flux and the terms on the right are different heat sources. Entirely analogous to $\Gamma_i$, it is clear that this flux to the same order (6.4) is dominated by its final turbulent term. As for the remaining source terms, the ohmic heating

$$P_i^{Ohm} = \langle \mathbf{E}\cdot\mathbf{B}\rangle_\psi \frac{Z_i e}{B}\int \mathrm{d}^3 v v_\parallel \hat{F}_{1i} \tag{6.8}$$

vanishes by the smallness of the mass ratio, the compressional heating

$$P_i^{comp} = \left\langle p_s \nabla\cdot\left(\frac{\partial\psi}{\partial t}\frac{\nabla\psi}{|\nabla\psi|^2}\right)\right\rangle_\psi \tag{6.9}$$

vanishes by the slowly varying magnetic field configuration, while the collisional energy transfer

$$C_i^{(E)} = \int \mathrm{d}^3 v \frac{m_s \mathbf{v}^2}{2} C[F_s] \tag{6.10}$$

between species again vanishes by the smallness of the mass ratio. The turbulent heating

$$P_i^{turb} = Z_i e \left\langle \int \mathrm{d}^3 v h_i \frac{\partial}{\partial t} \left( \delta\varphi - \frac{v_\parallel A_\parallel}{c} \right) \right\rangle_{turb,\psi}, \tag{6.11}$$

however is unsurprisingly of the same order as the turbulent heat flux so that

$$\frac{3}{2}\frac{\partial}{\partial t} n_i T_i + \left\langle \nabla \cdot \left\langle \int \mathrm{d}^3 v \frac{m_i v^2}{2} h_s \langle \mathbf{V}_\chi \rangle_{\mathbf{r}} \right\rangle_{turb} \right\rangle_{\psi} = Z_i e \left\langle \int \mathrm{d}^3 v h_i \frac{\partial}{\partial t} \left( \delta\varphi - \frac{v_\parallel \delta A_\parallel}{c} \right) \right\rangle_{turb,\psi} \tag{6.12}$$

is the ion heat transport equation for our subsidiary gyrokinetics expansion.

Because we have employed the low-mass ratio limit and focused on ion scale turbulence the gyrokinetic electron transport equations are most suitably treated in a different fashion, employing the exact instead of the lowest order flux surfaces. The electron particle transport equation, properly taking into account the small-mass ratio limit beyond conventional gyrokinetics, is then simply given by [24]

$$\frac{1}{\tilde{V}'} \frac{\partial}{\partial t}\bigg|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \tilde{V}' N_e + \frac{1}{\tilde{V}'} \frac{\partial}{\partial \tilde{\psi}} \tilde{V}' \langle \Gamma_e \rangle_{\tilde{\psi}} = 0. \tag{6.13}$$

Here, because the radial particle flux $\langle \Gamma_e \rangle_\psi$ contains an integral over the trapped particle distribution $g_{te}$, which vanishes in the small mass ratio and collisional limit under consideration here, it becomes a simple matter of using our ordering of slowly varying magnetic fields to move $\tilde{V}'$ through the time derivative to obtain

$$\frac{\partial}{\partial t}\bigg|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} N_e = 0. \tag{6.14}$$

The electron heat transport equation, similarly, is given by

$$\frac{3}{2\tilde{V}'} \frac{\partial}{\partial t}\bigg|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \tilde{V}' \langle N_e \rangle_{\tilde{\psi}} T_e + \frac{1}{\tilde{V}'} \frac{\partial}{\partial \tilde{\psi}} \tilde{V}' \langle q_e \rangle_{\tilde{\psi}} = P_e^{turb} + P_s^{comp} + C_s^{(E)}. \tag{6.15}$$

Here we can immediately deduce that to the transport order (6.4) the electron radial heat flux $\langle q_e \rangle_\psi$ vanishes for the same reason as $\langle \Gamma_e \rangle_\psi$, that the compressional heating $P_s^{comp}$ vanishes by the slow mean field evolution, and that the interspecies collisional heat transfer $C_s^{(E)}$ vanishes by the small mass ratio. That leaves us with the turbulent heating

$$P_s^{turb} = -e \left\langle \int \mathrm{d}^3 v \left( \frac{\varepsilon_e}{T_e} - \frac{3}{2} \right) \frac{\delta T_e}{T_e} F_e \frac{\partial}{\partial t}\bigg|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} \zeta \right\rangle_{turb,\tilde{\psi}} + e \langle N_e \zeta \nabla \cdot \mathbf{u}_{eff} \rangle_{turb,\tilde{\psi}}. \tag{6.16}$$

Clearly the first of these terms vanishes in the subsidiary expansion because it constrains $\delta T_e$ to be 0. The second term in turn is of order $\mathcal{O}(\xi)$ slower than the ion transport timescale (6.4) so that we, after using (6.14) and again disregarding derivatives acting on $V'$, find the electron heat transport equation

$$\left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} T_e = 0, \tag{6.17}$$

which concludes our subsidiary gyrokinetics transport equations. Before turning to match these up to the subsidiary TKRMHD transport equations in the next sectino, it is worth notiing that the electron transport equations in the forms of (6.14) and (6.17) principally expresses the fact that electrons equilibrate so quickly along the field line that they essentially become adiabatic.

## 6.2 Toroidal Kinetic Reduced Magnetohydrodynamics Transport

Equipped with the gyrokinetic transport equations we will now show that they coincide with the outer scale TKRMHD transport equations that we will now develop. Our starting point will be a return to the pre-subsidiary expansion TKRMHD equations, starting with the ion kinetic equation (3.25), in order to determine precisely how slow this transport is and what equations govern it. Though we return to the full equation before the subsidiary expansion, we will still keep the subsidiary expansion ordering, retaining terms of different orders so as to see what their influence is on the transport. However, we will neglect all higher order $\epsilon$-terms and will find this adequate because higher order $\xi$-terms will prove describe the dominant transport mechanism. Furthermore we will keep the confining magnetic field fixed over the transport timescale which is natural because our low $\beta$ should prevent the plasma from significantly altering the background field.

In studying the TKRMHD ion kinetic equation (3.25) we find it convenient to once again switch derivatives to the normal $\mathbf{r}, \mathbf{v}$-variables through (3.45) and (3.46), both because this will simplify integrals and because slow, large scale mean quantities are more intuitively understood through these. Using (3.45) and (3.46) to switch derivatives in the ion kinetic equation we find that it becomes

$$\begin{aligned}&\frac{\partial}{\partial t}(F_i+\delta f_i) - \frac{Z_i e v_\parallel}{c}\frac{\partial \delta A_\parallel}{\partial t}\frac{\partial}{\partial \varepsilon_i}(F_i+\delta f_i) - C[F_i+\delta f_i]\\&+\left[v_\parallel \tilde{\mathbf{b}} + \frac{c}{B}\mathbf{b}\times\nabla\left(\delta\varphi - \frac{1}{c}v_\parallel \delta A_\parallel\right)\right]\cdot\left(\nabla - Z_s e\nabla\delta\varphi\frac{\partial}{\partial\varepsilon_i}\right)(F_i+\delta f_i) = 0.\end{aligned} \tag{6.18}$$

Integrating this equation is a straightforward matter because upon using (2.25) it is immediately found that all terms involving $\varepsilon_i$-derivatives can be turned into exact derivatives which vanish under velocity integration. As for the remaining terms, the derivatives are now expressed in conventional $\mathbf{r}, \mathbf{v}$-variables which commute with the integral so that we arrive at

$$\begin{aligned}&\frac{\partial}{\partial t}(n_i+\delta n_i) + \frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla(n_i+\delta n_i) + \nabla\cdot(\tilde{\mathbf{b}}u_{\parallel i}n_i)\\&+\int \mathrm{d}^3v\left(v_\parallel\tilde{\mathbf{b}} - \frac{v_\parallel}{B}\mathbf{b}\times\nabla\delta A_\parallel\right)\cdot\nabla\delta f_i = 0,\end{aligned} \tag{6.19}$$

by defining

$$n_i u_{\|i} = \int \mathrm{d}^3 v v_\| f_i. \tag{6.20}$$

Equation (6.19) in its present form describes fully many different scales to the point of uselessness. We must now extract from it the relevant slow transport, which we will do by using the turbulence average of (2.37). Applying this operator the rapid, small scale fluctuation (in the perpendicular direction) is removed as any term linear in fluctuating quantities by definition vanish. Importantly, because this really is a fixed integration it commutes with derivatives so we can for example move a short-scale gradient $\mathcal{O}(l_\perp^{-1})$ through the turbulence average which then eliminates any such short-scale dependence in favour of the slower $\mathcal{O}(L_\perp^{-1})$-dependence. Of course this is not necessary as the turbulence average still eliminates such fast scales, but this manipulation makes it very apparent.

We now take advantage of this lowering of order when turbulence averaging (6.19) to produce

$$\begin{aligned} &\frac{\partial n_i}{\partial t} + \nabla \cdot \left\langle \delta n_i \frac{c}{B} \mathbf{b} \times \nabla \delta\varphi \right\rangle_{turb} - \nabla \cdot \left\langle \int \mathrm{d}^3 v \frac{v_\|}{B} \delta f_i \mathbf{b} \times \nabla \delta A_\| \right\rangle_{turb} + \left\langle \nabla \cdot (\tilde{\mathbf{b}} u_{\|i} n_i) \right\rangle_{turb} \\ &= \frac{\partial n_i}{\partial t} + \nabla \cdot \left\langle \int \mathrm{d}^3 v \frac{c}{B} \mathbf{b} \times \nabla \left( \delta\varphi - \frac{v_\| \delta A_\|}{c} \right) \delta f_i \right\rangle_{turb} + \left\langle \nabla \cdot (\tilde{\mathbf{b}} u_{\|i} n_i) \right\rangle_{turb} = 0, \end{aligned} \tag{6.21}$$

where the $u_{\|i}$-term have to be kept because we do not know that it vanishes to high enough order. Nevertheless, it can be removed by flux surface averaging so that we arrive at

$$\frac{\partial n_i}{\partial t} + \nabla \cdot \left\langle \int \mathrm{d}^3 v \frac{c}{B} \mathbf{b} \times \nabla \left( \delta\varphi - \frac{v_\| \delta A_\|}{c} \right) \delta f_i \right\rangle_{turb,\tilde{\psi}} = 0, \tag{6.22}$$

which is the TKRMHD ion particle transport equation. This equation matches the gyrokinetic ion particle transport equation (6.5) and defines the ion transport timescale to be given by

$$\frac{1}{\tau_E} \sim \xi^4 \frac{v_{th,i}}{l_\|}. \tag{6.23}$$

We now return to the ion kinetic equation (6.18) in order to find the next transport equation, the heat transport equation determining the evolution of $T_i$. To this end we proceed in a similar fashion by now taking the $m_i v^2/2$-moment of (6.18). Using Catto-transformed variables for partial integration in $\partial/\partial\varepsilon$-terms) yields

$$\begin{aligned} &\frac{3}{2}\left( \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right)(n_i + \delta n_i)(T_i + \delta T_i) + Z_i e \nabla \delta\varphi \cdot \int \mathrm{d}^3 v \left( v_\| \tilde{\mathbf{b}} - \frac{v_\|}{B} \mathbf{b} \times \nabla \delta A_\| \right) \delta f_i \\ &+ \int \mathrm{d}^3 v \frac{m_i v^2}{2} \left( v_\| \tilde{\mathbf{b}} + \frac{c}{B} \mathbf{b} \times \nabla \left( \delta\varphi - \frac{1}{c} v_\| \delta A_\| \right) \right) \cdot \nabla_{\mathbf{r}} \delta f_i \\ &+ \nabla \cdot (\tilde{\mathbf{b}} q_{\|i} n_i) + \int \mathrm{d}^3 v \frac{Z_i e v_\|}{c} \frac{\partial \delta A_\|}{\partial t} (F_i + \delta f_i) = 0, \end{aligned} \tag{6.24}$$

where

$$n_i q_{\|i} = \int \mathrm{d}^3 v v_\| v^2 f_i. \tag{6.25}$$

After then applying the turbulence average we are left with

$$\begin{aligned}&\frac{3}{2}\frac{\partial}{\partial t}n_iT_i+\int \mathrm{d}^3v\frac{m_iv^2}{2}\left\langle\frac{c}{B}\mathbf{b}\times\nabla\left(\delta\varphi-\frac{1}{c}v_\parallel\delta A_\parallel\right)\cdot\nabla_{\mathbf{r}}\delta f_i\right\rangle_{turb}+\left\langle\nabla\cdot(\tilde{\mathbf{b}}q_{\parallel i}n_i)\right\rangle_{turb}\\&+Z_ie\left\langle\nabla\delta\varphi\cdot\int\mathrm{d}^3v\left(v_\parallel\tilde{\mathbf{b}}-\frac{v_\parallel}{B}\mathbf{b}\times\nabla\delta A_\parallel\right)\delta f_i\right\rangle_{turb}+Z_ie\left\langle\int\mathrm{d}^3v\frac{v_\parallel}{c}\frac{\partial\delta A_\parallel}{\partial t}\delta f_i\right\rangle_{turb}=0,\end{aligned}\tag{6.26}$$

after dropping small terms.

Comparing (6.26) with the the gyrokinetic heat transport equation (6.12) we find that, except for the third term, only the fourth term does not immediately match. In order to remedy this we turn to the outer scale TKRMHD ion kinetic equation (4.10). In that equation we now change the derivatives, which is permissible to lowest order, before integrating over all velocities to produce

$$\begin{aligned}&\frac{\partial}{\partial t}\int\mathrm{d}^3v\delta f_i+\int\mathrm{d}^3v\left(v_\parallel\tilde{\mathbf{b}}+\frac{c}{B}\mathbf{b}\times\nabla\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\right)\cdot\nabla\delta f_i\\&+\int\mathrm{d}^3v\frac{c}{B}\mathbf{b}\times\nabla\delta\varphi\cdot\nabla\psi\frac{\partial F_i}{\partial\psi}-\int\mathrm{d}^3v\frac{Z_ie}{T_i}\frac{\partial}{\partial t}\delta\varphi F_i=0.\end{aligned}\tag{6.27}$$

From this point we proceed by multiplying this expression by $\delta\varphi$ before applying the turbulence average. Since it is an integral, we use it to integrate by parts in order to produce

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}\delta\varphi\int\mathrm{d}^3v\delta f_i\right\rangle_{turb}+\left\langle\nabla\delta\varphi\cdot\int\mathrm{d}^3v\left(v_\parallel\tilde{\mathbf{b}}-\frac{v_\parallel}{B}\mathbf{b}\times\nabla\delta A_\parallel\right)\delta f_i\right\rangle_{turb}\\&-\left\langle\int\mathrm{d}^3v\frac{c}{B}\mathbf{b}\times\nabla\frac{\delta\varphi^2}{2}\cdot\nabla\psi\frac{\partial F_i}{\partial\psi}\right\rangle_{turb}+\left\langle\int\mathrm{d}^3v\frac{Z_ie}{T_i}\frac{\partial}{\partial t}\frac{\delta\varphi^2}{2}F_i\right\rangle_{turb}=0.\end{aligned}\tag{6.28}$$

Here the third term vanishes by flux-surface averaging and the time-derivative in the fourth term can be commuted through the turbulence average to make it slow enough to be neglected compared to the first two terms. Putting this information together into (6.26) we find the outer scale TKRMHD ion heat transport equation becomes

$$\begin{aligned}&\frac{3}{2}\frac{\partial}{\partial t}n_iT_i+\int\mathrm{d}^3v\frac{m_iv^2}{2}\left\langle\frac{c}{B}\mathbf{b}\times\nabla\left(\delta\varphi-\frac{1}{c}v_\parallel\delta A_\parallel\right)\cdot\nabla_{\mathbf{r}}\delta f_i\right\rangle_{turb}+\left\langle\nabla\cdot(\tilde{\mathbf{b}}q_{\parallel i}n_i)\right\rangle_{turb}\\&=Z_ie\left\langle\int\mathrm{d}^3v\frac{\partial}{\partial t}\left(\delta\varphi-\frac{v_\parallel\delta A_\parallel}{c}\right)\delta f_i\right\rangle_{turb}\end{aligned}\tag{6.29}$$

which matches the GK ion heat transport equation (6.12) after flux-surface averaging to remove the $q_{\parallel i}$-term.

We now turn to the matter of determining the electron transport equations. We will obtain them in exactly the same fashion we did the ion transport equations (6.21) and (6.29), by returning to the TKRMHD equations before the subsidiary expansion, then applying said expansion and using the turbulence average. Starting with the electron particle transport equation we take as our starting point equation (3.50) in its flux-surface averaged form

$$\left\langle \left( \frac{\partial}{\partial t} + \mathbf{u_E} \cdot \nabla_{\mathbf{R}} \right) n_e \right\rangle_{\tilde{\psi}} = 0. \tag{6.30}$$

Using (3.39) with the fact that the effective velocity (3.38) in the subsidiary expansion becomes

$$\mathbf{u}_{eff} = \mathbf{u}_E - \frac{cT_e}{eBN_e} \mathbf{b} \times \nabla \delta n_e, \tag{6.31}$$

we can switch time derivatives to convert (6.30) into

$$\left\langle \left( \left.\frac{\partial}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} + \frac{cT_e}{eBN_e} \mathbf{b} \times \nabla \delta n_e \cdot \nabla \right) (N_e + \delta n_e) \right\rangle_{\tilde{\psi}} = 0. \tag{6.32}$$

Proceeding by applying the turbulence average at fixed $\tilde{\psi}$ this becomes

$$\left.\frac{\partial N_e}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} = 0, \tag{6.33}$$

to ion transport order, because the term quadratic in $\delta n_e$-term is of order $\xi^2$ too small.

The TKRMHD electron temperature equation is similarly obtained from equation (3.53) in the form

$$\begin{aligned} &- \left\langle e\varphi \left( \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) n_e \right\rangle_{\tilde{\psi}} + \left\langle \frac{3}{2} n_e \left( \frac{\partial}{\partial t} + \mathbf{u}_E \cdot \nabla \right) T_e \right\rangle_{\tilde{\psi}} \\ &= \left\langle e u_{\parallel e} \left( \tilde{\mathbf{b}} \cdot \nabla \varphi + \frac{1}{c} \frac{\partial \tilde{A}_{\parallel}}{\partial t} \right) \right\rangle_{\tilde{\psi}}. \end{aligned} \tag{6.34}$$

Once again using (3.52) to order the final term we find that it vanishes to ion transport order. Applying the turbulence average to the remaining two terms the first also vanishes to ion transport order upon using the order-lowering effect of moving derivatives through the turbulence average. That leaves the second term which, by using that $N_e$ is a flux function and precisely the same manipulations as for (6.30), reduces to

$$\left.\frac{\partial T_e}{\partial t}\right|_{\tilde{\psi},\tilde{\alpha},\tilde{l}} = 0. \tag{6.35}$$

At this point we have completed our task. The TKRMHD electron transport equation (6.33) and (6.35) clearly match the correspond gyrokinetic equations (6.14) and (6.17) at which point we have fully demonstrated that the two equations system coincide in their respective subsidiary limits. This is an important result for two reasons. First, on a conceptual level this demonstrates that while the gyrokinetic ordering (5.1) breaks down approaching the edge with its steepening gradients, gyrokinetics itself instead morphs into a different set of equations. On a practical level this intermediary matching may prove fruitful in future endeavours to reliably determine suitable boundary conditions for gyrokinetics transport simulations. Because these simulations sensitively depend on the boundary conditions, which in the majority of cases are currently set by *ad hoc* assumptions, the significance of this result can hardly be overstated.

# 7

# Conclusions

In this thesis we have developed the TKRMHD equation system, consisting of (3.25), (3.35), (3.50), (3.53), and (3.72), from a first-principles multiscale expansion. These equations constitute a self-consistent description of large amplitude ITG-like turbulence in the near-edge region inside the LCFS and so should be applicable for tokamak L-mode operation. Importantly they are fully kinetic, applicable in general geometry, and capable of capturing the weakly collisional to collisionless transition. These are advantages over the current main edge descriptions which are either in the form of fluid equations which inherently neglect kinetic effects, or gyrofluid equations whose validity becomes dubious at low collisionality.

One weakness of the TKRMHD equations is that they do not describe any obvious transport of buoyant filaments. As such they fail to desribe the origin of the observed plasma blobs being launched into the scrape-off layer [36]. Hence, either these must form even closer to the LCFS, or a sharp filamentary structure, governed by the H-mode sister equations of Abel and Hallenbert [25], must arise to produce them. Nevertheless, the TKRMHD equations still constitute an initial step towards a first principles description of an L-mode tokamak edge.

In addition to deriving the TKRMHD equations, we have also demonstrated that they can be made to smoothly transition onto the gyrokinetic core through an intermediary set of equations, both for fluctuations and for transport. This is important on a conceptual level because it proves that gyrokinetics does not abruptly break down at the edge but instead smoothly transitions into a different set of equations. Of course this is expected, but confirmation is still an encouraging result. Furthermore this is important on a practical level because current gyrokinetic transport simulations rely on *ad hoc* assumptions to produce suitable boundary conditions at the top of the pedestal. By instead employing TKRMHD to move closer to the LCFS and beyond, where the boundary conditions are much more readily determined, this conceptual weakness is remedied.

Though TKRMHD in the form of this thesis is a promising first step towards a description of an L-mode tokamak edge, it is by no means complete in its present form. In its current form the TKRMHD equations have two main deficiencies which arise from restricting assumptions we made in our derivation. First and foremost the severe restriction of no sonic rotation should be relaxed, since in the majority of realistic scenarios this condition should not hold. Secondly, but less vitally, by developing a barely collisional version of TKRMHD we could increase the range of applicabilty further. Apart from these immediate and natural extensions there is much else that could also be done in the pursuit of a fuller description of the L-mode edge. One natural thing would be to attempt to match TKRMHD onto another set of equations in the other direction, to a collisional theory in the open field line region beyond the LCFS. Then the entire tokamak plasma could be aptly described by smoothly matching these regions together with natural boundary conditions.

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