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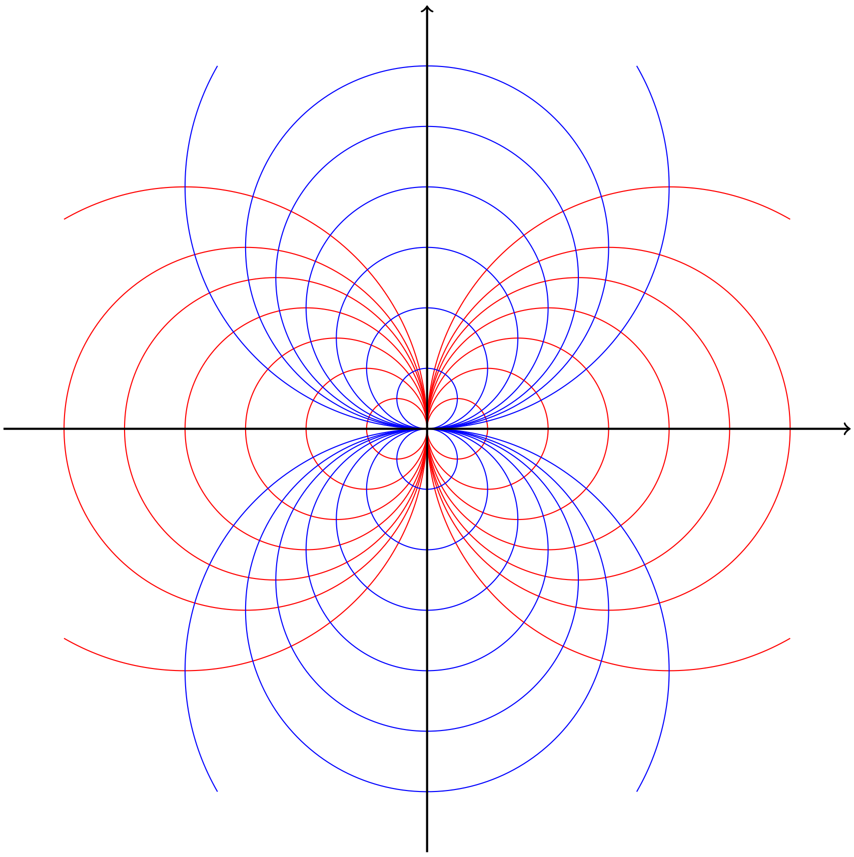

Aspects of Spin 3 Bianchi Identities in 2 + 1 Dimensional Conformal Higher-Spin Theories

Master's thesis in Physics and Astronomy

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Department of Fundamental Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
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Cover: A finite special conformal transformation of a Cartesian grid, for a description, see chapter 3.

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Abstract

The higher spin theories in $2+1$ dimensions considered in this thesis are of great physical interest since they are an important part in our understanding of both string theory, the AdS/CFT correspondance and M-theory. These conformal higher-spin theories are introduced by obtaining a conformal basis to the spin-2 algebra $so(2,3)$. This algebra neatly generalizes to the higher spin algebra, giving rise to a theory containing fields of all spins. Looking at the projection of the vacuum equations of motion, $F=0$, and the Bianchi identities, $DF=0$, onto the conformal basis, the content of these equations is explored. Using the conformal spin-2 basis, the curvature equations from $2+1$ dimensional (conformal) general relativity is obtained as a confirmation that the method used is correct. A similar projection onto the conformal spin-3 basis is found and it is shown that assuming only the parabolic part of $F=0$ is enough to satisfy the Bianchi identity. Hence, the Bianchi identity allows for coupling the system to matter.

Keywords: higher-spin theory, conformal gravity, Chern-Simons, gauge theory

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Vincent Ericsson, Gothenburg, March 2016

Contents

1

Introduction

1.1 Invitation

Nature is today known to have four fundamental forces: the electromagnetic, strong, and weak forces and gravity. On a classical level these are all described using field theories such as Yang-Mills or general relativity. The problem with classical physics is that it does not work on small scales, or at high energies where physics is governed by quantum mechanics. In the case of the electromagnetic, strong and weak forces, this poses no problem and theories such as quantum electrodynamics and the standard model have been shown to agree exceptionally well with experimental results. There are, however, trouble when trying to include gravity into these small scales, or in other words, when trying to quantize gravity. Using quantum field theories similar to those used for the other forces gives a perturbational theory which is non-renormalizable. Renormalizability is the property that makes it possible to remove any infinities of a theory. As of today, there is only one theory that allows gravity to be quantized and that is string theory.

String theory is a theory where the nature of the fundamental objects considered is changed. Instead of thinking of pointlike particles, string theory uses small vibrating strands of energy, or, strings, where different modes of oscillation (different excitations) of the strings correspond to different particles. This means that, instead of several different pointlike particles, there is one string whose vibrational modes create what on a larger scale is seen as different particles.

The thing is that string theory is a rather complicated theory and to get a better understanding of it, string theory is often studied in different limits, where it, in some way, is easier to understand. Some of these are the low-energy field-theory limit and a double limit leading to AdS/CFT. A more intricate one is the zero-tension limit believed to give a so-called higher-spin theory.

AdS/CFT is a limit where spacetime features two different theories, one gravitational (string) theory in the bulk of AdS (Anti-de Sitter space) and one conformal field theory (CFT) of one dimension lower than the AdS theory, where the CFT lives on the boundary of the AdS spacetime.

The higher-spin limit comes from the infinite tower of higher spin string states of which most are massive, but taking the limit where the tension, T, of the strings goes to zero, the mass of these higher spin states go to zero as well ($M \sim T \sim \frac{1}{\sqrt{\alpha'}}$). In this way, the higher-spin limit gives rise to an infinite tower of massless fields with all integer spins from 0 to ∞ (for the bosonic string, the superstring contains in addition all half integer spins). To ensure unitary, these fields have more symmetry

than the string theory it was obtained from (each higher-spin state will, in fact, have its own gauge symmetry). Such higher-spin theories have been constructed, independent of string theory, by Vasiliev [1]. These Vasiliev theories all live on AdS spacetimes.
In the field theory limit of M-theory one has also uncovered an eleven dimensional supergravity theory. M-theory is known to contain all known string theories in ten dimensions (due to various dualities). The last limit to be discussed here is this M-theory. It might be misleading to call it a limit of string theory since, in reality, it is the other way around where the string theories of today are limits to the more fundamental M-theory. However, instead of strings, this field theory has $2 + 1$ dimensional solutions, the so-called M_2-branes as its fundamental objects (which include the enigmatic M_5 branes of dimension $5 + 1$ not further discussed in this thesis). The ultimate goal when studying $2 + 1$ dimensional conformal field theories is to understand these M_2-branes better.

1.1.1 AdS/CFT

The AdS/CFT correspondance is a conjecture proclaimed by Maldacena in 1997 [2] stating that a gravitational (string) theory on AdS spacetime is dual to a conformal field theory in one less dimension. It may seem strange that these vastly different theories are dual; e.g. they have different dimensions and while one theory is strongly coupled, the other one is weakly coupled. However, the correspondence can be, to some extent, justified by a few arguments from string theory.
In type IIB string theory, there is a 5-form field strength giving a solution $AdS_5 \times S_5$ to the vacuum equations. This string theory allows for so-called D3 branes, which have two interpretations. The first is from the open string's point of view. Here, a stack of D3 branes results in a 4 dimensional conformal field theory, a result originating from the open strings' connection to the D3 branes. On the other hand, when regarding closed strings, they see D3 branes as charged objects that source the fields of the theory. In some way, a D3 brane sourcing gravity can be seen as a black hole. A very interesting aspect of the two theories now obtained is that the 5 dimensional AdS spacetime and the 4 dimensional conformal field theory have the same global symmetries $SO(2, 4)$, which is a crucial part of the duality.
The second, perhaps simpler version of this stringy AdS/CFT, is the HS/vector model duality. It is a similar duality exhibiting a higher spin theory in the bulk and an vector-sigma model on the boundary. It was first observed by Sundborg [3] that Vasiliev's theory [1] exhibit the correct set of higher-spin fields for this to work. This connection comes from considering a vector sigma model containing N scalar fields in the large N limit ($N \to \infty$), also called the semi-classical limit. This theory can be shown to contain $SO(N)$ invariant conserved higher-spin currents that couple to gauge fields now living in the bulk.

1.1.2 Higher-Spin Theory

Higher-spin theory is a subject of its own and concerns the study of massless gauge fields with all integer spins. However, three dimensions is an exception where theories without scalars exist, that is where spins $2, 3, 4, \ldots$ can make up a consistent theory by themselves. The theory discussed in this thesis is of this kind. As previously mentioned, these higher spin fields appears in the zero tension limit of string theory. There are a number of reasons why this theory is interesting. It may be a new way to understand geometry, e.g. the big bang gets a new interpretation when considering it in terms of higher-spin theory, without the spacetime singularity[4]. There is also the fact that the classical version of higher-spin theory lives somewhere right between string theory and general relativity. Because of this, a quantized version of it might be very interesting when trying to understand the AdS/CFT correspondance and it might in fact be the key to prove the duality.

1.1.3 M-theory

String theory emerged in the beginning of the 1970s by trying to answer questions in hadron physics. It describes how all particles, both matter and force carriers, are different vibrational modes of some fundamental one-dimensional object called a string. It did, however, run into a problem as more versions of the equations describing string theory were discovered, ending with a total of five different string theories. Each of these string theories exhibits different characteristics, but worst of all; each one of them appeared to be correct.

This was solved in 1994 by Witten [5], who found that each of these five string theories described different aspects of one and the same, more fundamental theory. This more fundamental theory is called M-theory and is an 11-dimensional theory featuring vibrating two-dimensional membranes (M_2-branes) instead of strings (as mentioned above, there are also fundamental M_5 branes in M-theory). The different string theories arise by reducing M-theory to 10 dimensions and using a web of dualities.

The surface of these two-dimensional membranes gives rise to a conformal field theory in $2 + 1$ dimensions and in order to understand M-theory, it is crucial to understand these $2 + 1$-dimensional conformal field theories.

1.1.4 Conformal Higher-Spin Theory in 2+1 Dimensions

The higher-spin theories that are concerned in this thesis are conformal higher-spin theories in three spacetime dimensions. A theory of this kind is related to much of what is said above and the interesting parts of it are due to two very important developements some years ago. First is BLG [6]–[8], an attempt to construct a conformal field theory on a stack of M_2-branes with $\mathcal{N} = 8$ supersymmetries, which was not entirely successful. Then there is ABJM [9], which is known to be the correct theory for such a stack, but with $\mathcal{N} = 6$ supersymmetries. Both these three-dimensional conformal field theories live on flat, three-dimensional Minkowski spacetime and are parity conserving. Although, as a crucial ingredient, they both contain parity breaking Chern-Simons terms.

In AdS/CFT, Chern-Simons terms arise from Neumann boundary conditions in AdS, as argued by Witten[10]. The question is: Can also the higher spin fields in a higher-spin theory be given such unusual boundary conditions and if so, what kind of boundary field theory would this lead to? Many recent results [11]–[15] indicate that the answer is yes and that the models appearing on the boundary are of the kind studied in this thesis.

1.2 Overview

The aim of this thesis is to present an introduction to higher-spin theory for the higher under-graduate level student, which is reflected in the order the various topics are presented. First is the basic theory needed to relate the theory presented here to other theories, where general relativity is the one most commonly refered to. This is needed to understand the arguments in later chapters. Following this are the actual calculations coupling Chern-Simons for spin 2 to general relativity. These are carried out in detail, since they are needed to follow the tougher problems which are the real aim of this thesis. In the appendices, extra calculations are found, some to clarify arguments, others because they are necessary for a coherent overall picture, but have no obvious place among the other chapters.

1.2.1 Cartan Formulation of General Relativity

General relativity is in this thesis presented in a slightly more mathematical version than usual. It describes the physics in the same way, but the theory is expressed differently and the calculations are performed in another way. Hence it is of importance to know how they are done and what the difference to the formulation generally taught to students is. This chapter mostly concern the formulation of general relativity, while the Chern-Simons description of spin 2 is left to later chapters.

1.2.2 Conformal Symmetry

It is important to know the symmetries of the system considered and the symmetry group exploited in this thesis is $SO(2,3)$. First it is approached in an easy to grasp way without much group theory, and using the generators that are defined to preserve the Minkowski metric (up to a multiplying factor), the conformal algebra is worked out.
This way of obtaining the conformal algebra is easy to understand, but not easily generalized to higher spins. Instead, a method relying on the fact that $so(2,3) \simeq sp(4,\mathbb{R})$ and functions on a phase space with this symmetry, is used. This method is a bit tricky, but relatively easy to generalize to higher spins than 2.

1.2.3 Spin 2

Spin 2 represents ordinary general relativity, here introduced in another way, using Chern-Simons theory, than commonly done and these are the techniques which will be used in higher-spin calculations. The equations are projected onto the conformal basis elements obtained in the conformal symmetry chapter and the content of the equations are worked out. The calculations are performed in detail to make it easy for the reader to follow all steps, since a good understanding of the spin-2 calculations are needed before moving on to higher spins.

1.2.4 Spin 3

Here, the spin-3 equations are presented and solved in their linearized versions. There are both more and tougher equations than in the spin-2 case. A good understanding of the techniques must hence be assumed in order to follow the calculations presented here.

1.3 The reader

The reader of this thesis is assumed to have a basic knowledge of general relativity in order to follow most calculations and some argumentation. For a complete understanding, a base in theoretical physics including general relativity, field theory and some string theory will be needed. For the uninitiated reader, it may be beneficial to read through the appendices before reading chapters 4 and forward.

2

Cartan Formulation of General Relativity

The Cartan formalism (sometimes called the tetrad formalism in 4 spacetime dimensions) is not a theory on its own, but rather a way to formulate existing theories of for instance gravity [16]–[18]. There is a possibility to use the Cartan formalism to include torsion into general relativity, but this will not be done here.
The main difference between the usual formalism of general relativity and the Cartan formalism is the object that defines the structure (curvature) of spacetime. In general relativity this is the metric tensor $g_{\mu\nu}$, where spacetime is observed as a bunch of coorinates related by the metric. The Cartan formalism has a slightly different way of observing spacetime. Here, the object defining it is called a vielbein $e_\mu{}^a$ (vierbein and dreibein in four and three dimensions respectively) and spacetime is seen from an observational point of view. Instead of seeing a bunch of coordinates, put an observer at every point in spacetime and let every observer see space from their local Lorentz frames. This creates a so-called tangent space at every point where spacetime is flat Minkowski space. The curvature of spacetime is found by relating two of these local frames by continuously moving one frame to the other and at every point observe how the frame changes to represent a new local frame. The connection between the tangent space and the curvature is described by the above mentioned vielben $e_\mu{}^a$. These two formulations of General Relativity may seem very different or very similar, but they lead to the same physics since there is a very direct way of connecting the metric with the vielbein

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} = e_\mu{}^a e_{\nu a}. \tag{2.1}$$

Note that $e_\mu{}^a$ has $\frac{D(D-1)}{2}$ extra degrees of freedom compared to $g_{\mu\nu}$, however, these can be removed by a local Lorentz transformation. The extra degrees of freedom can also be removed via a gauge choise in the Cartan formulation and is discussed further in section 4.2.1. One way to interpret this is that the metric is obtained by observing the curvature in the (μ, ν) directions from the same local frame.
These two formulations might lead to the same physics, but they do it in different ways. Einstein's formulation assumes the principle of general covariance, which says that the form of the laws governing the physics must be invariant under general coordinate transformations of some symmetry group. In Cartan formalism, the principle of general covariance is supplemented by local gauge invariance of the Lagrangian and hence, the manner in which the physics is obtained is different.
The lagrangian used in this thesis is the one from Chern-Simons theory [19]. It looks as follows

$$S_{\text{CS}} = k \int \left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A \right) = k \int A \wedge F, \tag{2.2}$$

where $F = dA + A \wedge A = DA$ is the field strength of the theory, D the exterior covariant derivative and A is the gauge field. The variation of this expression gives the equations of motion of free space

$$\frac{\delta S_{\text{CS}}}{\delta A} = F = 0. \tag{2.3}$$

If the theory instead is coupled to other (matter) fields, then $F \neq 0$.
For general relativity in four dimensions the action is generalized (the original theory is three-dimensional) to

$$S_{\text{CS}} = \frac{1}{\kappa^2} \int e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd}, \tag{2.4}$$

where R_{ab} is the 2-form Riemann tensor.
When a variation is introduced to e^a, the theory states the equations of motion as $R = 0$ (R is the Ricci scalar). In general relativity this means that in absence of matter (or other field), the spacetime is not curved. The Chern-Simons action will be regarded more closely in appendix E.

2.1 Properties of the vielbein $e_\mu{}^a$

In Einstein's formulation of gravity ([17], [20]), one very useful feature of the metric tensor is that it can be used to raise or lower indices. The vielbein serves a similar purpose in the Cartan formulation and depending on how it is used, it can raise and lower indices or switch between curved (μ, ν, σ, ...) and flat (a, b, c, ...) ones. To raise or lower indices, it is enough to relate the vielbein to the metric tensor (see eq. (2.1)), a continuation of this to raise or lower flat indices is simply

$$\eta_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu} = e^\mu{}_a e_{\mu b}. \tag{2.5}$$

The $e^\mu{}_a$ used here is just the inverse of $e_\mu{}^a$ and in the same way as for the metric tensor this can be ignored when doing index manipulations. If an index is changed from flat to curved it means that there is a vielbein hidden in the notation

$$e_\mu{}^a V_a = V_\mu, \qquad e^\mu{}_a U_\mu = U_a. \tag{2.6}$$

Many times, combinations of these index operations are used and some short-hand expressions are very useful. To raise or lower an index and on the same time switch between curved and flat index, the following notation is used

$$e_\nu{}^a g^{\mu\nu} = e^{\mu a}, \qquad e_\mu{}^b \eta_{ab} = e_{\mu a}, \qquad e_\mu{}^a e^\mu{}_b = \delta^a_b. \tag{2.7}$$

For any field featuring both curved and flat indices, the order is very important. The curved ones will always be written first to allow for the index manipulation in eq. (2.6), hence, it might be more correct to say that the curved content of a field will lie in its first indices, even if they are represented by flat indices.

2.2 The spin connection $\omega^a{}_b$

The affine connection plays an important role in general relativity when curvature is considered and is the connection used for covariant derivation of fields with curved indices. Now, when flat indices are introduced, there is another connection used in the covariant derivative. It is called the spin connection, denoted $\omega_\mu{}^a{}_b$, and works as

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b. \tag{2.8}$$

Even though many fields exibit both curved and flat indices, this thesis will almost exclusively rely on the spin connection. The affine connection (in torsion-free space) vanishes when equations are written using differential forms since this creates an anti-symmetry in the indices of the covariant derivative and the field. These are the lower two indices in the affine connection, which are known to be symmetric (at least in torsion-free space)

$$D_{[\mu} V^a_{\nu]} = \partial_{[\mu} V^a_{\nu]} + \omega_{[\mu}{}^{ab} V_{\nu]b} - \Gamma^\sigma_{[\mu\nu]} V^a_\sigma = \partial_{[\mu} V^a_{\nu]} + \omega_{[\mu}{}^{ab} V_{\nu]b}. \tag{2.9}$$

2.2.1 Uniqueness of the affine connection

Yet another argument for the equivalence of the Cartan and Einstein formulations of general relativity would be to show that the affine connection Γ obtained from the usual expression

$$\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\lambda} \left[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right] \tag{2.10}$$

in Einstein formulation is the same as the one found from the Cartan formalism. Just enter eq. (2.1) into eq. (2.10) and put this equal to the expression of the covariant derivative acting on a vielbein. Then solve for the affine connection. The covatiant derivative is known to look as

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a = 0, \tag{2.11}$$

and is here assumed to be zero in the same way that $D_\sigma g_{\mu\nu} = 0$. The spin connection ω can be expressed in terms of the vielbeins as (see appendix C for derivation)

$$\omega_\mu{}^{bc} = 2e^{\nu[b} \partial_{[\mu} e_{\nu]}{}^{c]} - e^{\sigma[b} e_\lambda{}^{c]} g^{\nu\lambda} e_{\mu a} \partial_\sigma e_\nu{}^a, \tag{2.12}$$

and the expression for the affine connection, taking the way via the spin connection, becomes

$$\begin{aligned}
\Gamma^\sigma_{\mu\nu}(\omega(e)) &= e^\sigma{}_a \partial_\mu e_\nu{}^a + 2e^\sigma{}_a e^{\lambda[a} \partial_{[\mu} e_{\lambda]}^{b]} e_{\nu b} + e^\sigma{}_a e^{\lambda[a} e_\rho{}^{b]} e_{\mu c} e_{\nu b} \partial^\rho e_\lambda{}^c = \\
&= e^\sigma_a \partial_\mu e^a_\nu + \tfrac{1}{2}\Big[e^\sigma{}_a e^{\lambda a} \partial_\mu e_\lambda{}^b e_{\nu b} - e^\sigma{}_a e^{\lambda b} \partial_\mu e_\lambda{}^a e_{\nu b} + e^\sigma{}_a e^{\lambda a} \partial_\lambda e_\mu{}^b e_{\nu b} - \\
&\qquad - e^\sigma{}_a e^{\lambda b} \partial_\lambda e_\mu{}^a e_{\nu b} + \Big] + \\
&\qquad + \tfrac{1}{2}\Big[e^\sigma{}_a e^{\rho b} e^{\lambda a} e_{\mu c} e_{\nu b} \partial_\rho e_\lambda{}^c - e^\sigma{}_a e^{\rho a} e^{\lambda b} e_{\mu c} e_{\nu b} \partial_\rho e_\lambda{}^c\Big] = \\
&= \tfrac{1}{2}\Big[e^\sigma{}_a \partial_\mu e_\nu{}^a + e^\sigma{}_a \partial_\nu e_\mu{}^a\Big] + \tfrac{1}{2} g^{\sigma\lambda}\Big[e_{\nu a} \partial_\mu e_\lambda{}^a + e_{\mu a} \partial_\nu e_\lambda{}^a\Big] - \\
&\quad - \tfrac{1}{2}\Big[e_{\nu a} \partial^\sigma e_\mu{}^a + e_{\mu a} \partial^\sigma e_\nu{}^a\Big].
\end{aligned} \tag{2.13}$$

Ideally, this equals the expression obtained by combining eqs. (2.1) and (2.10)

$$\begin{aligned}\Gamma^{\sigma}_{\mu\nu}(g(e)) &= \tfrac{1}{2}g^{\sigma\lambda}\left[\partial_\mu(e_\nu{}^a e_{\lambda a}) + \partial_\nu(e_\mu{}^a e_{\lambda a}) - \partial_\lambda(e_\mu{}^a e_{\nu a})\right] = \\ &= \tfrac{1}{2}g^{\sigma\lambda}\left[e_{\lambda a}\partial_\mu e_\nu{}^a + e_{\nu a}\partial_\mu e_\lambda{}^a + e_{\lambda a}\partial_\nu e_\mu{}^a + e_{\mu a}\partial_\nu e_\lambda{}^a - e_{\nu a}\partial_\lambda e_\mu{}^a - e_{\mu a}\partial_\lambda e_\nu{}^a\right] = \\ &= \tfrac{1}{2}\Big[e^\sigma{}_a\partial_\mu e_\nu{}^a + e^\sigma{}_a\partial_\nu e_\mu{}^a\Big] + \tfrac{1}{2}g^{\sigma\lambda}\Big[e_{\nu a}\partial_\mu e_\lambda{}^a + e_{\mu a}\partial_\nu e_\lambda{}^a\Big] - \\ &\quad - \tfrac{1}{2}\Big[e_{\nu a}\partial^\sigma e_\mu{}^a + e_{\mu a}\partial^\sigma e_\nu{}^a\Big].\end{aligned} \tag{2.14}$$

Fortunatelly, they are the same

$$\Gamma(\omega(e)) = \Gamma(g(e)). \tag{2.15}$$

Hence, either of the expressions (eqs. (2.10) and (2.11)) can be used to obtain an expression for the affine connection.

3
Conformal Symmetry

When using a Lagrangian to describe the physics of a system, it is of interest to know what symmetries this Lagrangian is invariant under, in other words, which transformations can be performed on the fields in a Lagrangian that results in a Lagrangian equal to the one the transformation was performed on. These symmetries of the Lagrangian depend on the theory considered. In special relativity, the symmetry group is the Poincaré group while the standard model has a symmetry group that also involves gauge symmetries related to $SU(3) \times SU(2) \times U(1)$. The symmetry considered in this thesis is $SO(2,3)$ viewed as a conformal symmetry in $2+1$ dimensions as explained below. For a discussion to why conformal symmetries are interesting, see [21].

Conformal symmetries are those generated by the transformations that leave the Minkowski metric invariant up to a (local) scale factor i.e. transformations that preserve angles

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu, \qquad \delta\eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = c(x)\eta_{\mu\nu}, \tag{3.1}$$

here expressed in terms of continuous transformations. There are four types of continuous transformations, and one discrete that have this property (see e.g. [22], [23]). The first two continuous transformations are simply the Poincaré transformations of translation and Lorentz rotations, while the other two are scalings and the special conformal transformations (here shown as infinitesimal transformations)

$$\begin{array}{ll} \epsilon^\mu(x) = \xi^\mu & \text{translations, } c(x) = 0, \\ \epsilon^\mu(x) = x_\nu \omega^{[\nu\mu]} & \text{rotations, } c(x) = 0, \\ \epsilon^\mu(x) = \lambda x^\mu & \text{scalings, } c(x) = 2\lambda, \\ \epsilon^\mu(x) = 2(a\cdot x)x^\mu - x^2 a^\mu & \text{special conformal, } c(x) = a\cdot x. \end{array} \tag{3.2}$$

The discrete transformation is called inversion and is the tensorial variant of $x \to \frac{1}{x}$

$$x^\mu \to \frac{x^\mu}{x^2}. \tag{3.3}$$

To get an idéa of how the conformal transformations work, an example of each transformation is shown in figs. 3.1 – 3.5.

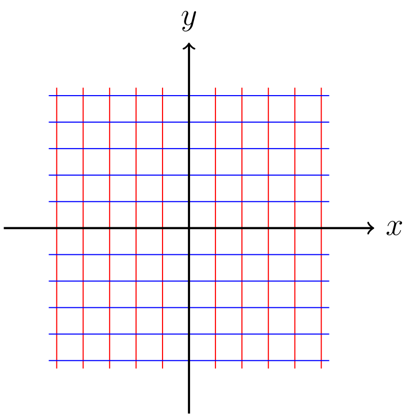

Figure 3.1: A Cartesian grid prior to any transformation.

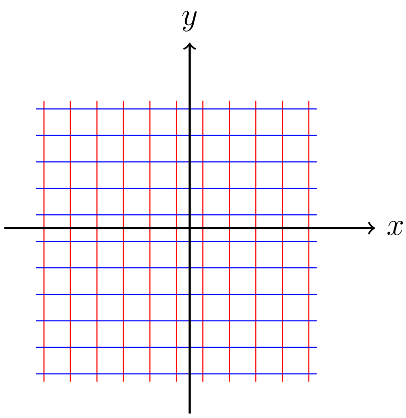

Figure 3.2: The grid from fig. (3.1) after a translation.

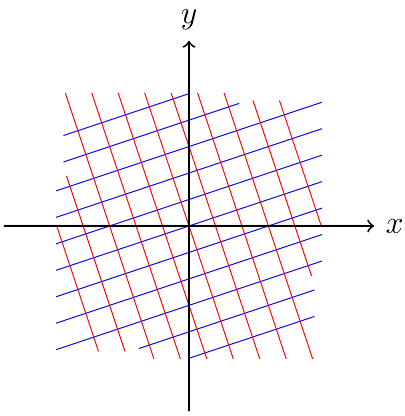

Figure 3.3: The grid from fig. (3.1) after a rotation.

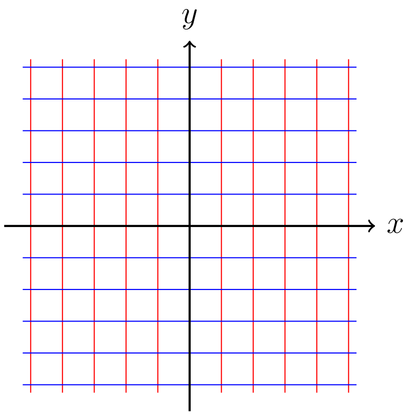

Figure 3.4: The grid from fig. (3.1) after a scale transformation.

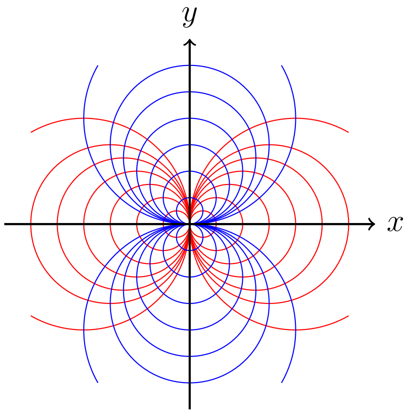

Figure 3.5: The grid from fig. (3.1) after a special conformal transformation.

3.1 Generators of conformal transformations

The easiest way to find the generators of the conformal transformations is to use the continuous conformal transformations in eq. (3.2) written on partial notation. The inversion cannot be generated since it is a discrete transformation and because of this it will only be considered when looking at finite transformations. The four generators of conformal symmetry is denoted P_μ, $M_{\mu\nu}$, D, and K_μ, representing translations, rotations, scalings, and special conformal transformations respectively

$$P_\mu = \partial_\mu, \tag{3.4}$$
$$M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \tag{3.5}$$
$$D = x^\mu \partial_\mu, \tag{3.6}$$
$$K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu. \tag{3.7}$$

The commutators of these generators are what is called the algebra of the symmetry group. Figuring out this algebra is important for later computations and is not that tricky. There are two things to remember, the first one being that partial derivatives commute

$$[P_\mu, P_\nu] = [\partial_\mu, \partial_\nu] = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0 \tag{3.8}$$

and the second one is how an x^μ-derivative acting on x^ν works inside a commutator, as e.g. in

$$\begin{aligned} [P_\mu, D] &= \partial_\mu \left(x^\nu \partial_\nu\right) - x^\nu \partial_\nu \partial_\mu = \partial_\mu x^\nu \partial_\nu + x^\nu \partial_\mu \partial_\nu - x^\nu \partial_\nu \partial_\mu = \\ &= \tfrac{\partial x^\nu}{\partial x^\mu} \partial_\nu = \delta^\nu_\mu \partial_\nu = \partial_\mu = P_\mu. \end{aligned} \tag{3.9}$$

Knowing this, the rest is down to shufling terms and result in

$$[P_\mu, P_\nu] = 0, \tag{3.10}$$
$$[M_{\mu\nu}, P_\lambda,] = -2\eta_{\lambda[\mu} P_{\nu]}, \tag{3.11}$$
$$[D, P_\mu] = -P_\mu, \tag{3.12}$$
$$[P_\mu, K_\nu] = 2\eta_{\mu\nu} D - 2M_{\mu\nu}, \tag{3.13}$$
$$[M_{\mu\nu}, M^{\sigma\lambda}] = 4\delta^{[\sigma}_{[\nu} M_{\mu]}{}^{\lambda]}, \tag{3.14}$$
$$[D, M_{\mu\nu},] = 0, \tag{3.15}$$
$$[M_{\mu\nu}, K_\sigma] = -2\eta_{\sigma[\mu} K_{\nu]}, \tag{3.16}$$
$$[D, D] = 0, \tag{3.17}$$
$$[D, K_\mu] = K_\mu, \tag{3.18}$$
$$[K_\mu, K_\nu] = 0. \tag{3.19}$$

This is one way to define the $so(2,3)$ conformal algebra, which can, but not easily, be generalized to higher spins. In the next section the algebra will be re-introduced in a way more suited for higher spin extensions.

3.2 The $so(2,3)$ conformal algebra in the qp-basis

The $so(2,3)$ higher spin conformal algebra is now introduced in the way it will be used throughout this thesis. It uses a product of two $SL(2,R)$ spacetime spinors (denoted q^α and p_α) to construct the spin-2 algebra presented in the previous section. The pq-realization [24], [25] of spin 2 uses the group theoretical observation that $so(2,3) \simeq sp(4,\mathbb{R})$. This means that the 4-component real objects (q^α, p_α) can be used to create the $SO(2,3)$ symmetry group, and writing out all bilinear forms and working out their Poisson brackets results in the $so(2,3)$ spin-2 Lie algebra. The commutators, rather than the Poisson bracket, would be used in a quantized theory.
First of all, the $so(1,2)$ subgroup of the algebra is obtained via the bilinear form

$$M^a = -\tfrac{1}{2}(\gamma^a)_\alpha{}^\beta(q^\alpha p_\beta) \tag{3.20}$$

and is extended to the full $so(2,3)$ algebra by including the rest of the possible bilinear forms of q^α and p_α

$$\begin{aligned} P^a &= -\tfrac{1}{2}(\gamma^a)_{\alpha\beta}(q^\alpha q^\beta), && (3.21)\\ K^a &= -\tfrac{1}{2}(\gamma^a)^{\alpha\beta}(p_\alpha p_\beta), && (3.22)\\ D &= -\tfrac{1}{2}q\cdot p, && (3.23) \end{aligned}$$

where the gamma matrices (γ^a) are given in appendix A.2. These are the basis elements of the full $so(2,3)$ algebra, which is obtained by finding the Poisson brackets between the possible combinations of the basis elements. The Poisson bracket is defined as

$$\{f(q,p), g(q,p)\}_{\mathrm{PB}} = \frac{\partial f}{\partial q^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q^\alpha}\frac{\partial f}{\partial p_\alpha}, \tag{3.24}$$

which directly implies the canonical relations $\{q^\alpha, p_\beta\}_{\mathrm{PB}} = \delta^\alpha_\beta$, $\{p_\alpha, q^\beta\}_{\mathrm{PB}} = -\delta^\beta_\alpha$ and $\{p_\alpha, p_\beta\}_{\mathrm{PB}} = \{q^\alpha, q^\beta\}_{\mathrm{PB}} = 0$.
An example of a Poisson bracket that needs to be figured out is $\{M^a, P^b\}_{\mathrm{PB}}$. It is done in the following way

$$\begin{aligned} \{M^a, P^b\}_{\mathrm{PB}} &= \{-\tfrac{1}{2}(\gamma^a)_\alpha{}^\beta(q^\alpha p_\beta), -\tfrac{1}{2}(\gamma^b)_{\gamma\delta}(q^\gamma q^\delta)\}_{\mathrm{PB}} = \\ &= \tfrac{1}{4}(\gamma^a)_\alpha{}^\beta(\gamma^b)_{\gamma\delta}\{(q^\alpha p_\beta), (q^\gamma q^\delta)\}_{\mathrm{PB}}, \end{aligned} \tag{3.25}$$

where the remaining Poisson bracket follows from

$$\begin{aligned} \{q^\alpha p_\beta, q^\gamma q^\delta\}_{\mathrm{PB}} &= q^\alpha\{p_\beta, q^\gamma q^\delta\}_{\mathrm{PB}} + \{q^\alpha, q^\gamma q^\delta\}_{\mathrm{PB}} p_\beta = \\ &= q^\alpha q^\gamma\{p_\beta, q^\delta\}_{\mathrm{PB}} + q^\alpha\{p_\beta, q^\gamma\}_{\mathrm{PB}} q^\delta + q^\gamma\{q^\alpha, q^\delta\}_{\mathrm{PB}} p_\beta + \{q^\alpha, q^\gamma\}_{\mathrm{PB}} q^\delta p_\beta = \\ &= -\delta^\delta_\beta q^\alpha q^\gamma - \delta^\gamma_\beta q^\alpha q^\delta, \end{aligned} \tag{3.26}$$

using the trick $[AB, C] = A[B,C] + [A,C]B$ known from commutator calculations. When considering the Poisson bracket, it follows from the product rule of differentiation. Hence eq. (3.25) becomes

$$\begin{aligned}\{M^a, P^b\}_{\text{PB}} &= -\tfrac{1}{4}(\gamma^a)_\alpha{}^\beta(\gamma^b)_{\gamma\delta}\left(\delta^\delta_\beta q^\alpha q^\gamma + \delta^\gamma_\beta q^\alpha q^\delta\right) = \\ &= -\tfrac{1}{4}\left((\gamma^a)_\alpha{}^\beta(\gamma^b)_{\gamma\beta} q^\alpha q^\gamma + (\gamma^a)_\alpha{}^\beta(\gamma^b)_{\beta\delta} q^\alpha q^\delta\right) = \\ &= -\tfrac{1}{2}\left(\gamma^a\gamma^b\right)_{\alpha\beta} q^\alpha q^\beta = -\tfrac{1}{4}\left([\gamma^a,\gamma^b] + \{\gamma^a,\gamma^b\}\right)_{\alpha\beta} q^\alpha q^\beta = \\ &= -\tfrac{1}{2}\left(\epsilon^{ab}{}_c(\gamma^c)_{\alpha\beta} + \eta^{ab}\epsilon_{\alpha\beta}\right) = -\tfrac{1}{2}\epsilon^{ab}{}_c(\gamma^c)_{\alpha\beta} q^\alpha q^\beta = \epsilon^{ab}{}_c P^c.\end{aligned} \tag{3.27}$$

The other brackets are worked out using the same techniques and result in (note that the Poisson bracket from now on will be denoted with square brackets, the same way as commutators)

$$[P^a, P^b] = 0, \tag{3.28}$$
$$[M^a, P^b] = \epsilon^{ab}{}_c P^c, \tag{3.29}$$
$$[D, P^a] = P^a, \tag{3.30}$$
$$[P^a, K^b] = -2\epsilon^{ab}{}_c M^c - 2\eta^{ab} D, \tag{3.31}$$
$$[M^a, M^b] = \epsilon^{ab}{}_c M^c, \tag{3.32}$$
$$[D, M^a] = 0, \tag{3.33}$$
$$[M^a, K^b] = \epsilon^{ab}{}_c K^c, \tag{3.34}$$
$$[D, D] = 0, \tag{3.35}$$
$$[D, K^a] = -K^a, \tag{3.36}$$
$$[K^a, K^b] = 0. \tag{3.37}$$

Notice that these are similar, but not equal to the commutators in eqs. (3.10) to (3.19). The change from $M_{\mu\nu}$ to M^a is due to the fact that in three dimensions, two anti-symmetric indices ab can be rewritten as one index c using a Levi-Civita symbol. Here, this is used to obtain an M with only one index as

$$M^a = -\tfrac{1}{2}\epsilon^a{}_{bc} M^{bc} \qquad \Leftrightarrow \qquad M^{ab} = \epsilon^{ab}{}_c M^c. \tag{3.38}$$

The differences in sign could be cared for by changing signs in eq. (3.23), but the sign is here kept common for all basis elements.

Note that on several places in this thesis, there will appear a parentheses after some fields, e.g. $F^a{}_{(1,3)}$, this parentheses $_{(q,p)}$ states how many q's and p's, respectively, the basis element the field is projected on has. For example, the part of the field strength F projected onto the spin-2 basis element P^a is denoted by $F^a{}_{(2,0)}$, since the basis element contains two q's and no p. The F will have as many flat indices as the basis element for the notation to be unambiguous, e.g. the projection onto M^a and D will be $F^a{}_{(1,1)}$ and $F_{(1,1)}$ respectively.

Also note that the D basis element provides a grading of the number of q'q and p's. Denote the number of q's with m and p's with n in a generator T, then $[D, T] = \frac{m-n}{2}T$, note that T can feature indices. This will be consistent even in higher spins.

3.3 The higher-spin algebra

The conformal higher spin algebra is obtained as an extension of the spin-2 conformal algebra obtained in the previous section. This is done by including polynomial terms of p_α and q^α of degree higher than 2 (all polynomials of degree 2 gave the spin-2 algebra). The spin-3 extension will be obtained by including polynomials of degree 4, the spin-4 extension by polynomials of degree 6 and so on. Why a polynomial of degree 4 corresponds to spin 3 is discussed in section 5.2. Using odd degrees of the polynomials would correspond to fermions, which are possible, but much harder to examine and will not be done here. Worth noting is that the bracket between two elements of the spin-3 algebra results in an element of the spin-4 algebra. This means that the spin-3 algebra is not closed, as in the case with the spin-2 algebra. To get a closed higher-spin algebra, all higher-spin terms must be included, this is called the universal enveloping algebra of $so(2,3)$ and is here denoted $hs(so(2,3))$.

If the case of pure AdS_3 had been considered instead of a 2+1 dimensional conformal spacetime, the algebra would have been obtained by $sl(N,\mathbb{R}) \times sl(N,\mathbb{R})$ and would have included the closed algebra for all spin states from spin 2 to spin N.

The qp polynomials for the higher-spin algebra basis elements will be on the form

$$\begin{aligned} T^{a\ldots b\ldots c}{}_{(r,s)} = \Big(-\tfrac{1}{2}(\gamma^a)_{\alpha_1\alpha_2}(q^{\alpha_1}q^{\alpha_2})\Big)\ldots\Big(-\tfrac{1}{2}(\gamma^b)_{\alpha_r}{}^{\beta_1}(q^{\alpha_r}p_{\beta_1})\Big)\ldots \\ \ldots\Big(-\tfrac{1}{2}(\gamma^c)^{\beta_{s-1}\beta_s}(p_{\beta_{s-1}}p_{\beta_s})\Big) \end{aligned} \tag{3.39}$$

Note that the expression is symmetric in all upper and lower indices respectively and that there is no middle term if both r and s are even.

An important aspect when calculating the algebra for these bases is how many of the spinors that appear in the different expressions. This is a kind of grading of the fields which is denoted by $G(N)$, where N is the number of spinors. As an example, $G(2)$ gives the spin-2 algebra, while $G(4)$ gives spin 3. Now remember eq. (3.24), it is clear that the bracket removes two of the spinors, one for each derivative, this can be expressed as

$$[G(N), G(M)] = G(M+N-2). \tag{3.40}$$

This means that finding the bracket between two elements of the spin-2 basis results in another element in the spin-2 basis, but when considering higher spins, this is not the case. In, for instance, spin 3, two elements of the spin-3 basis results in a spin-4 element. Hence, for the algebra to be closed, the theory either has to be restricted to spin-2, or all higher spins has to be included. Worth noting is that a bracket between any $G(N)$ and a spin-2 element results in an element of the grade N.

3.3.1 The spin-3 basis

Using eq. (3.39), the spin-3 basis elements are obtained as

$$P^{ab} = \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}(q^\alpha q^\beta)(q^\gamma q^\delta), \tag{3.41}$$

$$\tilde{P}^{ab} = \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)_\gamma{}^\delta(q^\alpha q^\beta)(q^\gamma p_\delta), \tag{3.42}$$

$$\tilde{M}^{ab} = \tfrac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)^{\gamma\delta}(q^\alpha q^\beta)(p_\gamma p_\delta), \tag{3.43}$$

$$\tilde{K}^{ab} = \tfrac{1}{4}(\gamma^a)_\alpha{}^\beta(\gamma^b)^{\gamma\delta}(q^\alpha p_\beta)(p_\gamma p_\delta), \tag{3.44}$$

$$K^{ab} = \tfrac{1}{4}(\gamma^a)^{\alpha\beta}(\gamma^b)^{\gamma\delta}(p_\alpha p_\beta)(p_\gamma p_\delta). \tag{3.45}$$

Using the Fierz identities from eqs. (A.8) and (A.10), it is clear that P^{ab} and K^{ab} are both symmetric and traceless. Similarly, $\tilde{P}^{ab}$ and $\tilde{K}^{ab}$ are traceless (see eq. (A.8)), while $\tilde{M}^{ab}$ are neither traceless nor symmetric. However, it would be more convenient if all of the basis elements in eqs. (3.41) to (3.45) would be symmetric and traceless. The way this is solved is to introduce a new basis where the elements in eqs. (3.41) to (3.45) are kept, but symmetrized and made traceless, where the anti-symmetric parts of $\tilde{P}^{ab}$, $\tilde{M}^{ab}$, and $\tilde{K}^{ab}$ are found in this new basis as the elements $\tilde{P}^a$, $\tilde{M}^a$ and $\tilde{K}^a$ respectively, while the trace of $\tilde{M}^{ab}$ is found in $\tilde{D}$. These are introduced as

$$\tilde{M}^{[ab]} = \epsilon^{ab}{}_c\tilde{M}^c, \qquad \tilde{M}^a = -\tfrac{1}{2}\epsilon^a{}_{bc}M^{ab}, \tag{3.46}$$

$$\tilde{P}^{[ab]} = -\tfrac{1}{2}\epsilon^{ab}{}_c\tilde{P}^c, \qquad \tilde{P}^a = \epsilon^a{}_{bc}P^{ab}, \tag{3.47}$$

$$\tilde{K}^{[ab]} = -\tfrac{1}{2}\epsilon^{ab}{}_c\tilde{K}^c, \qquad \tilde{K}^a = \epsilon^a{}_{bc}K^{ab}, \tag{3.48}$$

where the signs follow from the Fierz identities in eq. (A.9) for eq. (3.46) and from eq. (A.10) for eqs. (3.47) and (3.48). Similarly, the trace of $\tilde{M}^{ab}$ follows from eq. (A.11)

$$M_a{}^a = 2\tilde{D}. \tag{3.49}$$

Hence, the *qp*-form of these new basis elements are

$$\tilde{P}^a = \tfrac{1}{4}(\gamma^a)_{\alpha\beta}q^\alpha q^\beta(q\cdot p), \tag{3.50}$$

$$\tilde{M}^a = \tfrac{1}{4}(\gamma^a)_\alpha{}^\beta q^\alpha p_\beta(q\cdot p), \tag{3.51}$$

$$\tilde{D} = \tfrac{1}{4}(q\cdot p)^2, \tag{3.52}$$

$$\tilde{K}^a = \tfrac{1}{4}(\gamma^a)^{\alpha\beta}p_\alpha p_\beta(q\cdot p). \tag{3.53}$$

Together the nine equations in eqs. (3.41) to (3.45) and eqs. (3.50) to (3.53) make up the spin-3 basis, where all basis elements with two indices are symmetric and traceless.

3.3.2 The spin-3 algebra

In section 3.2, the spin-2 algebra was obtained using the *qp*-basis. Here the same is done for the spin-3 algebra. The calculations are very similar, with just a few exceptions. Still, one calculation will be shown here as well to make the procedure clear.
Note that all brackets considered here are between one spin-2 and one spin-3 element, since brackets between two spin-3 elements results in a spin-4 element (remember eq. (3.40)). The one shown is

$$\begin{aligned}\left[P^a, \tilde{M}^{bc}\right] &= \left[-\tfrac{1}{2}(\gamma^a)_{\alpha\beta}q^\alpha q^\beta, \tfrac{1}{4}(\gamma^b)_{\gamma\delta}(\gamma^c)^{\sigma\lambda}q^\gamma q^\delta p_\sigma p_\lambda\right] = \\ &= -\tfrac{1}{8}(\gamma^a)_{\alpha\beta}(\gamma^b)_{\gamma\delta}(\gamma^c)^{\sigma\lambda}\left[q^\alpha q^\beta, q^\gamma q^\delta p_\sigma p_\lambda\right].\end{aligned} \tag{3.54}$$

Where the bracket is given by

$$\left[q^\alpha q^\beta, q^\gamma q^\delta p_\sigma p_\lambda\right] = q^\gamma q^\delta q^\alpha p_\sigma \delta^\beta_\lambda + q^\gamma q^\delta q^\alpha p_\lambda \delta^\beta_\sigma + q^\gamma q^\delta q^\beta p_\sigma \delta^\alpha_\lambda + q^\gamma q^\delta q^\beta p_\lambda \delta^\alpha_\sigma. \tag{3.55}$$

Hence, eq. (3.54) is

$$\begin{aligned}&-\tfrac{1}{8}(\gamma^a)_{\alpha\lambda}(\gamma^b)_{\gamma\delta}(\gamma^c)^{\sigma\lambda}q^\gamma q^\delta q^\alpha p_\sigma - \tfrac{1}{8}(\gamma^a)_{\alpha\sigma}(\gamma^b)_{\gamma\delta}(\gamma^c)^{\sigma\lambda}q^\gamma q^\delta q^\alpha p_\lambda - \\ &-\tfrac{1}{8}(\gamma^a)_{\lambda\beta}(\gamma^b)_{\gamma\delta}(\gamma^c)^{\sigma\lambda}q^\gamma q^\delta q^\beta p_\sigma - \tfrac{1}{8}(\gamma^a)_{\sigma\beta}(\gamma^b)_{\gamma\delta}(\gamma^c)^{\sigma\lambda}q^\gamma q^\delta q^\beta p_\lambda = \\ &= -\tfrac{1}{4}(\gamma^b)_{\gamma\delta}\left((\gamma^a\gamma^c)_\alpha{}^\sigma q^\gamma q^\delta q^\alpha p_\sigma + (\gamma^a\gamma^c)_\alpha{}^\lambda q^\gamma q^\delta q^\alpha p_\lambda\right) = \\ &= -\tfrac{1}{2}(\gamma^b)_{\gamma\delta}(\gamma^a\gamma^c)_\alpha{}^\sigma q^\gamma q^\delta q^\alpha p_\sigma = -\tfrac{1}{2}(\gamma^b)_{\gamma\delta}\left(\epsilon^{ac}{}_d(\gamma^d)_\alpha{}^\sigma + \eta^{ac}\delta^\sigma_\alpha\right)q^\gamma q^\delta q^\alpha p_\sigma = \\ &= -\tfrac{1}{2}\epsilon^{ac}{}_d(\gamma^b)_{\gamma\delta}(\gamma^d)_\alpha{}^\sigma q^\gamma q^\delta q^\alpha p_\sigma - \tfrac{1}{2}\eta^{ac}(\gamma^b)_{\gamma\delta}q^\gamma q^\delta(q\cdot p) = \\ &= -2\epsilon^{ac}{}_d\tilde{P}^{bd} - 2\eta^{ac}\tilde{P}^b.\end{aligned} \tag{3.56}$$

This might seem correct, but look more closely at the expressions used. All places where $\tilde{P}^{ab}$ appears in terms of gamma matrices, the version used is the one that is not yet symmetrized and traceless, and this must be compensated for. The symmetry of $\tilde{M}^{bc}$ is added by simply symmetrizing the *bc* indices in the final result. Removing the anti-symmetry in the term $\tilde{P}^{bd}$ is harder, but accomplished by separating the symmetric and anti-symmetric parts of $\tilde{P}^{bd}$ and then use eq. (3.47) for the anti-symmetric part

$$\begin{aligned}-2\epsilon^{ac}{}_d\tilde{P}^{bd} &= -2\epsilon^{ac}{}_d\tilde{P}^{(bd)} + -2\epsilon^{ac}{}_d\tilde{P}^{[bd]} = -2\epsilon^{ac}{}_d\tilde{P}^{(bd)} + \epsilon^{ac}{}_d\epsilon^{bd}{}_e\tilde{P}^e = \\ &= -2\epsilon^{ac}{}_d\tilde{P}^{(bd)} + \eta^{ab}\tilde{P}^c - \eta^{bc}\tilde{P}^a.\end{aligned} \tag{3.57}$$

The symmetric part is here denoted $\tilde{P}^{(bd)}$. Similarly, the trace is removed by taking the *bc*-trace of eq. (3.56)

$$-2\epsilon^a{}_{cd}\tilde{P}^{cd} - 2\tilde{P}^a = -4\tilde{P}^a \tag{3.58}$$

and removing this times $\frac{1}{3}\eta^{bc}$ from eq. (3.56). All in all, the final expression for the bracket is

$$\left[P^a, \tilde{M}^{bc}\right] = -2\epsilon^{a(b}{}_d\tilde{P}^{c)d} - \eta^{a(b}\tilde{P}^{c)} + \tfrac{1}{3}\eta^{bc}\tilde{P}^a. \tag{3.59}$$

This shows the most important techniques for calculating brackets between spin-3 basis elements. Working out all combinations of one spin-2 and one spin-3 basis elements gives the following non-zero brackets

$$\big[P^a, \tilde{P}^{bc}\big] = \epsilon^{a(b}{}_d P^{c)d}, \tag{3.60}$$
$$\big[P^a, \tilde{P}^{b}\big] = -P^{ab}, \tag{3.61}$$
$$\big[P^a, \tilde{M}^{bc}\big] = -2\epsilon^{a(b}{}_d \tilde{P}^{c)d} - \eta^{a(b}\tilde{P}^{c)} + \tfrac{1}{3}\eta^{bc}\tilde{P}^a, \tag{3.62}$$
$$\big[P^a, \tilde{M}^{b}\big] = -\tilde{P}^{ab} + \tfrac{3}{2}\epsilon^{ab}{}_c \tilde{P}^c, \tag{3.63}$$
$$\big[P^a, \tilde{D}\big] = -2\tilde{P}^a, \tag{3.64}$$
$$\big[P^a, \tilde{K}^{bc}\big] = 3\epsilon^{a(b}{}_d \tilde{M}^{c)d} - 3\eta^{a(b}\tilde{M}^{c)} + \eta^{bc}\tilde{M}^a, \tag{3.65}$$
$$\big[P^a, \tilde{K}^{b}\big] = -\tilde{M}^{ab} - 3\epsilon^{ab}{}_c \tilde{M}^c - \tfrac{8}{3}\eta^{ab}\tilde{D}, \tag{3.66}$$
$$\big[P^a, K^{bc}\big] = -4\epsilon^{a(b}{}_d \tilde{K}^{c)d} - 6\eta^{a(b}\tilde{K}^{c)} + 2\eta^{bc}\tilde{K}^a, \tag{3.67}$$
$$\big[M^a, P^{bc}\big] = 2\epsilon^{a(b}{}_d P^{c)d}, \tag{3.68}$$
$$\big[M^a, \tilde{P}^{bc}\big] = 2\epsilon^{a(b}{}_d \tilde{P}^{c)d}, \tag{3.69}$$
$$\big[M^a, \tilde{P}^{b}\big] = \epsilon^{ab}{}_c \tilde{P}^c, \tag{3.70}$$
$$\big[M^a, \tilde{M}^{bc}\big] = 2\epsilon^{a(b}{}_d \tilde{M}^{c)d}, \tag{3.71}$$
$$\big[M^a, \tilde{M}^{b}\big] = \epsilon^{ab}{}_c \tilde{M}^c, \tag{3.72}$$
$$\big[M^a, \tilde{K}^{bc}\big] = 2\epsilon^{a(b}{}_d \tilde{K}^{c)d}, \tag{3.73}$$
$$\big[M^a, \tilde{K}^{b}\big] = \epsilon^{ab}{}_c \tilde{K}^c, \tag{3.74}$$
$$\big[M^a, K^{bc}\big] = 2\epsilon^{a(b}{}_d K^{c)d}, \tag{3.75}$$
$$\big[D, P^{ab}\big] = 2P^{ab}, \tag{3.76}$$
$$\big[D, \tilde{P}^{ab}\big] = \tilde{P}^{ab}, \tag{3.77}$$
$$\big[D, \tilde{P}^{a}\big] = \tilde{P}^{a}, \tag{3.78}$$
$$\big[D, \tilde{K}^{ab}\big] = -\tilde{K}^{ab}, \tag{3.79}$$
$$\big[D, \tilde{K}^{a}\big] = -\tilde{K}^{a}, \tag{3.80}$$
$$\big[D, K^{ab}\big] = -2K^{ab}, \tag{3.81}$$
$$\big[K^a, P^{bc}\big] = -4\epsilon^{a(b}{}_d \tilde{P}^{c)d} + 6\eta^{a(b}\tilde{P}^{c)} - 2\eta^{bc}\tilde{P}^a, \tag{3.82}$$
$$\big[K^a, \tilde{P}^{bc}\big] = 3\epsilon^{a(b}{}_d \tilde{M}^{c)d} + 3\eta^{a(b}\tilde{M}^{c)} - \eta^{bc}\tilde{M}^a, \tag{3.83}$$
$$\big[K^a, \tilde{P}^{b}\big] = \tilde{M}^{ab} - 3\epsilon^{ab}{}_c \tilde{M}^c + \tfrac{8}{3}\eta^{ab}\tilde{D}, \tag{3.84}$$
$$\big[K^a, \tilde{M}^{bc}\big] = -2\epsilon^{a(b}{}_d \tilde{K}^{c)d} + \eta^{a(b}\tilde{K}^{c)} - \tfrac{1}{3}\eta^{bc}\tilde{K}^a, \tag{3.85}$$
$$\big[K^a, \tilde{M}^{b}\big] = \tilde{K}^{ab} + \tfrac{3}{2}\epsilon^{ab}{}_c \tilde{K}^c, \tag{3.86}$$
$$\big[K^a, \tilde{D}\big] = 2\tilde{K}^a, \tag{3.87}$$
$$\big[K^a, \tilde{K}^{bc}\big] = \epsilon^{a(b}{}_d K^{c)d}, \tag{3.88}$$
$$\big[K^a, \tilde{K}^{b}\big] = K^{ab}. \tag{3.89}$$
$$\tag{3.90}$$

Note that the grading discussed last in section 3.2 holds true for the spin-3 basis as well.

3.4 The $SO(2,d)$ symmetry group

The generators of any group of type $SO(2,d)$ (using metric $\eta^{MN} = \text{trace}(-1,1,\ldots,1,-1)$) can be expressed as

$$(J^{MN})_{PQ} = \delta_P^{[M}\delta_Q^{N]}, \tag{3.91}$$

where P, Q are the indices of the generating matrices and M, N the indices of the algebra. With correct normalization this gives the commutation relations

$$[J^{MN}, J_{PQ}] = -4\delta_{[P}^{[M} J^{N]}{}_{Q]}, \tag{3.92}$$

where P, Q, M, N now all are indices of the algebra. These are not the commutation relations (bracket relations, remember that the theory is studied at a classical level) obtained in eqs. (3.28) to (3.37) for the conformal $so(2,3)$ algebra, only the generator M_{ab} commutated with itself follows this pattern (remember from eq. (3.38) how M_{ab} relates to M_a). However, the entire algebra can be cast on this form. First, let the indices μ and ν run over the first d dimensions ($\mu,\nu = 0,1,\ldots,d-1$) and define the generators J_{MN} as follows (the comma is for clarity, it does not denote derivation)

$$J_{\mu,\nu} = M_{\mu\nu}, \tag{3.93}$$
$$J_{\mu,d} = \tfrac{1}{2}\left(K_\mu - P_\mu\right), \tag{3.94}$$
$$J_{\mu,d+1} = \tfrac{1}{2}\left(K_\mu + P_\mu\right), \tag{3.95}$$
$$J_{d+1,d} = D. \tag{3.96}$$

Note that the J's are anti-symmetric. This combination of the algebra in eqs. (3.28) to (3.37) commutes as eq. (3.92) and is a confirmation that the algebra obtained indeed is the $so(2,3)$ Lie algebra.

3.5 Finite conformal transformations

The infinitesimal continuous conformal transformations have now been studied in detail and even tough this thesis does not use the finite ones, they deserve to be at least mentioned. The problem with the finite transformations is that they are much harder to express than the infinitesimal ones. The exceptions being the translations and the inversion

$$x^\mu \to x^\mu + a^\mu, \qquad x^\mu \to \frac{x^\mu}{x^2} \tag{3.97}$$

where a^μ is a constant finite vector describing the translation. To obtain the special conformal transformations, perform an inversion followed by a translation and another inversion. After some algebra the special conformal transformation becomes

$$x^\mu \to \frac{x^\mu - x^2 a^\mu}{1 - 2(a\cdot x) + a^2x^2}. \tag{3.98}$$

To be sure that this is the special conformal transformations used before, rewrite it as

$$\frac{x^\mu - x^2 a^\mu}{1 - 2(a\cdot x) + a^2x^2} = x^\mu + \frac{2(a\cdot x)x^\mu - a^2x^2x^\mu - x^2a^\mu}{1 - 2(a\cdot x) + a^2x^2} \tag{3.99}$$

and keep only the lowest powers of a^μ, leaving $2(a\cdot x)x^\mu - x^2 a^\mu$ in the enumerator and 1 in the denominator, which is the expression used for generating special conformal transformations in eq. (3.7). The finite versions of rotations and scalings are also obtained through inversions and translations. Doing this is hard, but the idéa is that the commutator between the generators of translation and special conformal transformations consists of an anti-symmetric part and a trace. These parts will correspond to rotations and scale transformations respectively. Working this out begins with taking the two group elements $g = P = e^{P_\mu a^\mu}$ and $h = K = e^{K_\mu b^\mu}$ and commuting these (here, it is done to first order)

$$ghg^{-1}h^{-1} - 1 = a^\mu b^\nu [P_\mu, K_\nu]. \tag{3.100}$$

But the special conformal transformation h can be written in terms of a translation and two inversions. h can hence be written as $h = IgI$, changing the commutator to

$$g_a I g_b I g_a^{-1} I g_b^{-1} I = a^\mu b^\nu [P_\mu, K_\nu]. \tag{3.101}$$

Taking the trace ($\eta^{\mu\nu}[P_\mu, K_\nu]$) and the anti-symmetric part ($\epsilon^{\sigma\mu\nu}[P_\mu, K_\nu]$) gives scale transformations and rotations respectively, to first order in a^μ and b^ν. To get the exact expressions, the entire spectrum of commutators from $[e^{P_\mu a^\mu}, e^{K_\nu b^\nu}]$ must be considered.

4

Spin-2

Before higher spins are introduced it is important to understand the spin-2 case. Even though both the Einstein and Cartan formulations [17] may be familiar to the reader, this chapter will introduce the most important techniques which is later used to obtain results in the higher-spin sector. It also gives a simpler interpretation for the results obtained since spin 2 can be thought of in terms of general relativity. Similar calculations can be found in [24]–[26].

4.1 Projection of $F = 0$

The vacuum equations of motion (obtained in eq. (2.3)) for spin-2 are $F = 0$. This will not be solved in its current form, but instead projected onto the different basis elements T_A of a spin-2 system. The basis elements are the ones found in section 3.2

$$T_A = (P_a, M_a, D, K_a)\,, \tag{4.1}$$

which is a complete basis for the Lie algebra of $so(2,3)$. First, the gauge field A is projected onto this basis

$$A^A T_A = A^a P_a + A^a M_a + AD + A^a K_a = e^a P_a + \omega^a M_a + bD + f^a K_a, \tag{4.2}$$

where e^a, ω^a, f^a and b are the different basis components of the gauge field and will from now on be the objects used instead of the gauge field itself. The notation used here is not coincidental, the P_a projected field is indeed the dreibein e^a and ω^a is the spin-connection from the Cartan formulation of gravity seen in section 2.2. f^a will similarly get a physical mening as the Schouten tensor (see appendix F), while b soon will be set to zero via a gauge choice.
While the projection of the gauge field is very simple, the projection of the field strength F is a little bit trickier. The field strength is connected to the gauge field in the known way

$$F = dA + A \wedge A = \left(dA^A + A^B A^C f_{BC}{}^{A}\right) T_A, \tag{4.3}$$

where $f_{BC}{}^{A}$ is the structure coefficient describing the bracket between T^B and T^C, defined as $[T_B, T_C] = f_{BC}{}^{A} T_A$. Selecting all terms corresponding to a specific basis element T_A is hence the projection of F onto T_A. Written in terms of brackets this look as follows

$$P_a : \quad F\Big|_{P_a} = de^a P_a + (\omega^a \wedge e^b)[M_a, P_b] + b \wedge e^a [D, P_a], \tag{4.4}$$

$$M_a : \quad F\Big|_{M_a} = d\omega^a M_a + (\omega^a \wedge \omega^b)[M_a, M_b] + (e^a \wedge f^b [P_a, K_b])\Big|_{M_a}, \tag{4.5}$$

$$D : \quad F\Big|_{D} = dbD + (e^a \wedge f^b [P_a, K_b])\Big|_{D}, \tag{4.6}$$

$$K_a : \quad F\Big|_{K_a} = df^a K_a + (\omega^a \wedge f^b)[M_a, K_b] + (b \wedge f^a)[D, K_a]. \tag{4.7}$$

Note that only the non-zero brackets are written out. The brackets are known from eqs. (3.28) to (3.37) and entered into the equations to find

$$P_a : \quad (de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c + b \wedge e^a)P_a = 0, \tag{4.8}$$

$$M_a : \quad (d\omega^a + \epsilon^a{}_{bc}\omega^b \wedge \omega^c - 2\epsilon^a{}_{bc} e^b \wedge f^c)M_a = 0, \tag{4.9}$$

$$D : \quad (db - 2\eta_{ab} e^a \wedge f^b)D = 0, \tag{4.10}$$

$$K_a : \quad (df^a + \epsilon^a{}_{bc}\omega^b \wedge f^c - b \wedge f^a)K_a = 0. \tag{4.11}$$

These can be simplified further by remembering the definition of the Cartan covariant derivative (eq. (2.8)) and the Riemann tensor

$$P_a : \quad De^a + b \wedge e^a = 0, \tag{4.12}$$

$$M_a : \quad R^a - 2\epsilon^a{}_{bc} e^b \wedge f^c = 0, \tag{4.13}$$

$$D : \quad db - 2e^a \wedge f_a = 0, \tag{4.14}$$

$$K_a : \quad Df^a - b \wedge f^a = 0, \tag{4.15}$$

where the basis elements have been removed for clarity and the fact that they play no role for the content of the equations. The exterior derivative d in eq. (4.14) could be exchanged for a covariant derivative, but since b has no flat index, it is omitted to avoid confusion.

4.2 Projection of δA

It is interesting to see how F transforms under a gauge transformation δA, since knowing this may give useful information about the equations. Specifically, it might give the opportunity to set some part of A to zero by a clever gauge choice.
δA is a non-abelian gauge transformation and hence looks as follows

$$\delta A = D\Lambda = d\Lambda + [A, \Lambda] \,. \tag{4.16}$$

Here Λ is the gauge parameter defining the transformation. It can be projected onto the basis elements in the same way as A was in eq. (4.2)

$$\Lambda = P_a \Lambda^a{}_{(2,0)} + M_a \Lambda^a{}_{(1,1)} + D\Lambda_{(1,1)} + K_a \Lambda^a{}_{(0,2)}. \tag{4.17}$$

The small parentheses after the Λ's denote the field content as $_{(q,p)}$, where q denote the number of q spinors in the basis that Λ is projected onto and similarly for p.

Hence $\Lambda|_D = \Lambda_{(1,1)}$, since the bilinear form used for the basis element D (eq. (3.23)) contained one p spinor and one q spinor, and the number of indices on the Λ's are the same as on the basis element it is projected onto. This allows for the projection of the entire expression in eq. (4.16), which is done in a similar, but not identical, way to how the equation $F = 0$ was projected in eqs. (4.4) to (4.7). Since the commutator in eq. (4.16) is between two different fields, care must be taken to include all its brackets

$$\begin{aligned} P_a: \quad \delta e^a &= d\Lambda^a{}_{(2,0)} + \omega^a[M_a, P_b]\Big|_{P_a}\Lambda^b{}_{(2,0)} + e^a[P_a, M_b]\Big|_{P_a}\Lambda^b{}_{(1,1)} + \\ &\quad + b[D, P_a]\Big|_{P_a}\Lambda^a{}_{(2,0)} + e^a[P_a, D]\Big|_{M_a}\Lambda_{(1,1)}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} M_a: \quad \delta\omega^a &= d\Lambda^a{}_{(1,1)} + \omega^a[M_a, M_b]\Big|_{M_a}\Lambda^b{}_{(1,1)} + e^a[P_a, K_b]\Big|_{M_a}\Lambda^b{}_{(0,2)} + \\ &\quad + f^a[K_a, P_b]\Big|_{M_a}\Lambda^b{}_{(2,0)}, \end{aligned} \tag{4.19}$$

$$D: \quad \delta b = d\Lambda_{(1,1)} + e^a[P_a, K_b]\Big|_{M_a}\Lambda^b{}_{(0,2)} + f^a[K_a, P_b]\Big|_{M_a}\Lambda^b{}_{(2,0)}, \tag{4.20}$$

$$\begin{aligned} K_a: \quad \delta f^a &= d\Lambda^a{}_{(0,2)} + \omega^a[M_a, K_b]\Big|_{M_a}\Lambda^b{}_{(0,2)} + f^a[K_a, M_b]\Big|_{M_a}\Lambda^b{}_{(1,1)} + \\ &\quad + b[D, K_a]\Big|_{M_a}\Lambda^a{}_{(0,2)} + f^a[K_a, D]\Big|_{M_a}\Lambda_{(1,1)}. \end{aligned} \tag{4.21}$$

Note that the Λ's are 0-forms and hence, no wedges appear between them and the other fields. If Λ instead had been a 1-form, it had been necessary to use the wedge product and also to change some signs in eqs. (4.18) to (4.21) (this stems from the nature of the wedge product and will have to be considered in section 4.4).
After entering the brackets into eqs. (4.18) to (4.21), noting the covariant derivatives and some simplifications

$$P_a: \quad \delta e^a = D\Lambda^a{}_{(2,0)} - \epsilon^a{}_{bc}e^b\Lambda^c{}_{(1,1)} + b\Lambda^a{}_{(2,0)} - e^a\Lambda_{(1,1)}, \tag{4.22}$$

$$M_a: \quad \delta\omega^a = D\Lambda^a{}_{(1,1)} - 2\epsilon^a{}_{bc}e^b\Lambda^c{}_{(0,2)} + 2\epsilon^a{}_{bc}f^b\Lambda^c{}_{(2,0)}, \tag{4.23}$$

$$D: \quad \delta b = d\Lambda_{(1,1)} - 2e^a\Lambda_{a(0,2)} + 2f^a\Lambda_{a(2,0)}, \tag{4.24}$$

$$K_a: \quad \delta f^a = D\Lambda^a{}_{(0,2)} - \epsilon^a{}_{bc}f^b\Lambda^c{}_{(1,1)} - b\Lambda^a{}_{(0,2)} + f^a\Lambda_{(1,1)}. \tag{4.25}$$

4.2.1 Choosing the spin-2 gauge condition

Looking at eqs. (4.22) to (4.25) there is a choice to use one of the Λ's to put some of the fields equal to zero. The eqs. (4.12) to (4.15) simplifies the most by using $\Lambda^a{}_{(0,2)}$ to gauge the b field to zero. Fields that can be set to zero in this way are called Stückelberg fields.
However, note that the remaining parameters $\Lambda^a{}_{(2,0)}$, $\Lambda^a{}_{(1,1)}$, and $\Lambda_{(1,1)}$ can be identified as translations, rotations, and scale transformations, respectively, which are some of the conformal symmetries. The $\Lambda^a{}_{(0,2)}$ parameter can in the same way be identified as the special conformal transformations, but it is used up to set $b = 0$. In some sense, it still has this role as a special transformation, but not at the linear level considered here and an extended discussion of how this works is outside the scope of this thesis.
To make the gauge choice, let $\Lambda^a{}_{(0,2)}$ take on such a value that $b = 0$. This is however not enough. Since δb also depends on $\Lambda^a{}_{(2,0)}$ and $\Lambda_{(1,1)}$, these transformations will

change the value of b. The solution to this is to make a so-called compensating gauge transformation, where the $\Lambda^a{}_{(0,2)}$ is tuned to always cancel δb. After b is set to zero, any new transformation on the fields leads to a change in $\Lambda^a{}_{(0,2)}$ according to

$$\begin{aligned} \delta b_\mu = \partial_\mu \Lambda_{(1,1)} - 2\Lambda_{\mu(0,2)} + 2f_{\mu\nu}\Lambda^\nu{}_{(2,0)} = 0 \quad \Rightarrow \\ \Rightarrow \Lambda_{\mu(0,2)} = \tfrac{1}{2}\partial_\mu \Lambda_{(1,1)} + f_{\mu\nu}\Lambda^\nu{}_{(2,0)}. \end{aligned} \tag{4.26}$$

If $\Lambda^a{}_{(0,2)}$ is, everywhere it appears, replaced in the way eq. (4.26) describes, b will be put to zero independent on which transformations are made. This changes the projection of the gauge transformations in eqs. (4.22) to (4.25) into

$$\begin{aligned} \delta e^a_\mu &= D_\mu \Lambda^a{}_{(2,0)} - \epsilon^a{}_{\mu c}\Lambda^c{}_{(1,1)} - e_\mu{}^a \Lambda_{(1,1)}, \\ \delta \omega_\mu{}^a &= D_\mu \Lambda^a - \epsilon^{\sigma a}{}_\mu \partial_\sigma \Lambda_{(1,1)} - 2\epsilon^{\sigma a}{}_\mu f_{\sigma\nu}\Lambda^\nu{}_{(2,0)} + 2\epsilon^a{}_{bc} f_\mu{}^b \Lambda^c{}_{(2,0)}, \\ \delta f_\mu{}^a &= \tfrac{1}{2} D_\mu \Big(e^{\sigma a}\partial_\sigma \Lambda_{(1,1)}\Big) + D_\mu\Big(e^{\sigma a} f_{\sigma\nu}\Lambda^\nu{}_{(2,0)}\Big) - \epsilon^a{}_{bc} f_\mu{}^b \Lambda^c + f_\mu{}^a \Lambda_{(1,1)}. \end{aligned} \tag{4.27}$$

Note that there also is the possibility to use $\Lambda^a{}_{(1,1)}$ to set the anti-symmetric part of $e_\mu{}^a$ to zero, removing the extra degrees of freedom it has compared to $g_{\mu\nu}$ and letting it take the role of the metric instead of the dreibein.

4.3 Solving $F = 0$

As stated before, the equations of motion are written as $F = 0$ and to solve this equation gives information of the vacuum state. The projections in eqs. (4.12) and (4.13) allow to solve for $\omega_\mu{}^a$ and $f_\mu{}^a$ in terms of $e_\mu{}^a$ and $\omega_\mu{}^a$, respectively. The projection in eq. (4.14) is an identity and may be used to perform calculations, while the last projection eq. (4.15) is an equation describing the dynamics of the system. These facts are now shown explicitly.

4.3.1 The P_a projection

The expression for the projection of the equation $F = 0$ was found in section 4.1 and is restated here

$$de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0. \tag{4.28}$$

To solve it, it is easier to implement the anti-symmetry using a Levi-Civita symbol rather than differential froms

$$\epsilon^{\mu\nu\lambda}\left(\partial_\mu e_\nu{}^a + \epsilon^a{}_{bc}\omega_\mu{}^b e_\nu{}^c\right) = 0. \tag{4.29}$$

multiplying the Levi-Civita symbols simplifies the expression

$$\epsilon^{\mu\nu\lambda}\partial_\mu e_\nu{}^a + \omega^{a\lambda} - e^{\lambda a}\omega^b{}_b = 0. \tag{4.30}$$

The trace of ω is obtained by multiplying the entire expression by $e_{\lambda a}$ and then solve for $\omega_\nu{}^\nu$

$$2\omega^c{}_c = e_{\lambda a}\epsilon^{\mu\nu\lambda}\partial_\mu e_\nu{}^a = \epsilon^{\mu\nu}{}_a \partial_\mu e_\nu{}^a. \tag{4.31}$$

This results in an expression that is correct, up to flatness of the indices

$$\omega^{a\lambda} = \tfrac{1}{2} e^{\lambda a} e_{\sigma c}\epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^c - \epsilon^{\mu\nu\lambda}\partial_\mu e_\nu{}^a. \tag{4.32}$$

Giving the final expression

$$\begin{aligned}\omega_\lambda{}^a &= \tfrac{1}{2} e_\lambda{}^a e_{\sigma c} \epsilon^{\mu\nu\sigma} \partial_\mu e_\nu{}^a - \epsilon^{\mu\nu\sigma} e_\sigma{}^a e_{\lambda c} \partial_\mu e_\nu{}^a = \\ &= \tfrac{1}{2} \epsilon^{\mu\nu\sigma} \left(e_\lambda{}^a e_{\sigma c} - 2 e_\sigma{}^a e_{\lambda c} \right) \partial_\mu e_\nu{}^c.\end{aligned} \tag{4.33}$$

4.3.2 The M_a projection

The M_a projected equation was found in eq. (4.13) and can be used to solve $f_\mu{}^a$ in terms of $\omega_\mu{}^a$, here there will not be any explicit $\omega_\mu{}^a$ terms, but they constitute the Riemann tensor R^a. The expression can be written as

$$\begin{gathered} R^a - 2\epsilon^a{}_{bc} e^b \wedge f^c = 0, \\ R^a = 2\epsilon^a{}_{bc} e^b \wedge f^c, \qquad R_{\mu\nu}{}^a = 4\epsilon^a{}_{bc} e_{[\mu}{}^b f_{\nu]}{}^c, \end{gathered} \tag{4.34}$$

which when multiplied with $\epsilon_a{}^{\nu\sigma}$ becomes

$$R_\mu{}^\sigma = -8\delta^{\nu\sigma}_{bc} e_{[\mu}{}^b f_{\nu]}{}^c = -8\delta^{[\nu}_{[\mu} f_{\nu]}{}^{\sigma]} = 2f_\mu{}^\sigma + 2\delta^\sigma_\mu \mathrm{Tr}\left[f\right]. \tag{4.35}$$

The trace is found by contracting μ and σ

$$R = 2\mathrm{Tr}\left[f\right] + 6\mathrm{Tr}\left[f\right] = 8\mathrm{Tr}\left[f\right] \quad \Rightarrow \quad \mathrm{Tr}\left[f\right] = \tfrac{1}{8} R \tag{4.36}$$

and the expression becomes

$$f_{\mu\sigma} = \tfrac{1}{2} R_{\mu\sigma} - \tfrac{1}{8} g_{\mu\sigma} R = \tfrac{1}{2} S_{\mu\sigma}. \tag{4.37}$$

The term $S_{\mu\sigma}$ is the so-called Schouten tensor recognized from three-dimensional general relativity.

4.3.3 The D projection

$$2e^a \wedge f_a = 0 \tag{4.38}$$

is obtained from eq. (4.14) and is quickly seen to be

$$e_{[\mu}{}^a f_{\nu]a} = -f_{[\mu\nu]} = 0. \tag{4.39}$$

This states that the Shouten tensor is symmetric as expected since it consists of the Ricci tensor $R_{\mu\nu}$ and the metric $g_{\mu\nu}$, which both are symmetric.

4.3.4 The K_a projection

Lastly, the projection in eq. (4.15) is considered

$$Df^a = 0. \tag{4.40}$$

This is just the anti-symmetrized covariant derivative acting on f^a, also known as the Cotton tensor $C_{\mu\nu}$

$$\epsilon_\mu{}^{\sigma\rho} \left(D_\sigma f_{\rho\nu} \right) = \epsilon_\mu{}^{\sigma\rho} D_\sigma \left(R_{\rho\nu} - \tfrac{1}{4} g_{\rho\nu} R \right) = C_{\mu\nu} = 0, \tag{4.41}$$

known as the Cotton equation. A theory that has a Cotton tensor equal to zero is conformally flat, which is not the case for a theory coupled to matter. Instead, a theory coupled to matter looks as $C_{\mu\nu} = T_{\mu\nu}(\phi, \psi)$.

4.4 Solving the spin-2 Bianchi identity $DF = 0$

The Bianchi identity must always be fulfilled and may seem trivial with $F = 0$. It could, however, reveal some interesting physics by solving it without assuming that $F = 0$ and if a subset of the projected equations of $F = 0$ is used while all projections of $DF = 0$ still are identities, it becomes interesting. Especially if the Cotton equation $C_{\mu\nu} = 0$ (eq. (4.41)) does not have to be assumed for DF to equal zero. If this is the case, the Cotton equation can pick up a right hand side which would allow it to couple to e.g. matter.
The projection of the Bianchi identity is done similarly to the equation $F = 0$ by the expression

$$DF = dF + A \wedge F - F \wedge A = 0 \tag{4.42}$$

and is again projected onto the conformal basis elements P_a, M_a, D and K_a

$$P_a: \quad DF\Big|_{P_a} = DF^a{}_{(2,0)} - \epsilon^a{}_{bc} F^b{}_{(1,1)} \wedge e^c - F_{(1,1)} \wedge e^a = 0, \tag{4.43}$$

$$M_a: \quad DF\Big|_{M_a} = DF^a{}_{(1,1)} - 2\epsilon^a{}_{bc} e^b \wedge F^c{}_{(0,2)} + 2\epsilon^a{}_{bc} F^b{}_{(2,0)} \wedge f^c = 0, \tag{4.44}$$

$$D: \quad DF\Big|_{D} = DF_{(1,1)} - 2e_a \wedge F^a{}_{(0,2)} + 2F^a{}_{(2,0)} \wedge f_a = 0, \tag{4.45}$$

$$K_a: \quad DF\Big|_{K_a} = DF^a{}_{(0,2)} - \epsilon^a{}_{bc} F^b{}_{(1,1)} \wedge f^c + F_{(1,1)} \wedge f^a = 0, \tag{4.46}$$

where the gauge $b = 0$ is implemented and the different components of F are obtained from section 4.1

$$\begin{aligned} F^a{}_{(2,0)} &= De^a, \\ F^a{}_{(1,1)} &= R^a - 2\epsilon^a{}_{bc} e^b \wedge f^c, \\ F_{(1,1)} &= -2e_a \wedge f^a, \\ F^a{}_{(0,2)} &= Df^a. \end{aligned} \tag{4.47}$$

As of now, the equations eqs. (4.43) to (4.46) are all still identities but that might not be the case after subjecting them to constraints. Note that it is important to remember all brackets and remember to use the correct sign when calculating the projections of eq. (4.42).

4.4.1 Zero torsion condition

Start with the minimal assumption, which is the zero torsion condition $De^a = 0$. The minimal assumption is the realisation where the least amount of projections of $F = 0$ is assumed. It means to only assume the highest graded equation, the P_a-component of F, to be zero ($F^a{}_{(2,0)} = 0$). Hence changing the equations to

$$\begin{aligned} P_a: \quad & DF\Big|_{P_a} = -\epsilon^a{}_{bc} F^b{}_{(1,1)} \wedge e^c - F_{(1,1)} \wedge e^a = 0, \\ M_a: \quad & DF\Big|_{M_a} = DF^a{}_{(1,1)} - 2\epsilon^a{}_{bc} e^b \wedge F^c{}_{(0,2)} = 0, \\ D: \quad & DF\Big|_{D} = DF_{(1,1)} - 2e_a \wedge F^a{}_{(0,2)} = 0, \\ K_a: \quad & DF\Big|_{K_a} = DF^a{}_{(0,2)} - \epsilon^a{}_{bc} F^b{}_{(1,1)} \wedge f^c + F_{(1,1)} \wedge f^a = 0. \end{aligned} \tag{4.48}$$

These are hopefully all identities, but that has to be made sure before any conclusions can be drawn.

4.4.1.1 The (2,0) equation

Start with the (2,0)-equation by entering the expressions in eq. (4.47) for the different components of F into the P_a projection of DF

$$\begin{aligned} DF\Big|_{P_a} &= -\epsilon^a{}_{bc}F^b{}_{(1,1)} \wedge e^c - F_{(1,1)} \wedge e^a = \\ &= -\epsilon^a{}_{bc}\left(R^b - 2\epsilon^b{}_{de}e^d \wedge f^e\right) \wedge e^c + 2e_b \wedge f^b \wedge e^a. \end{aligned} \tag{4.49}$$

The Levi-Civita symbols can be simplified to $\epsilon^a{}_{bc}\epsilon^b{}_{de} = \delta^a_d\eta_{ce} - \delta^a_e\eta_{cd}$

$$\begin{aligned} DF\Big|_{P_a} &= -\epsilon^a{}_{bc}R^b \wedge e^c + 2e^a \wedge f^b \wedge e_b - 2e_b \wedge f^a \wedge e^b + 2e_b \wedge f^b \wedge e^a = \\ &= -\epsilon^a{}_{bc}R^b \wedge e^c = 0, \end{aligned} \tag{4.50}$$

where the last term vanish from symmetries of the Riemann tensor

$$\epsilon^a{}_{bc}R^b \wedge e^c = \tfrac{1}{2}R^a{}_b \wedge e^b = R_{[\mu\nu}{}^{ab}e_{\sigma]b} = R_{[\mu\nu}{}^a{}_{\sigma]} = R^a{}_{[\sigma\mu\nu]} = 0. \tag{4.51}$$

4.4.1.2 The (1,1) equations

The (1,1)-equations are quite easy to figure out

$$\begin{aligned} DF\Big|_{M_a} &= DF^a{}_{(1,1)} - 2\epsilon^a{}_{bc}e^b \wedge F^c{}_{(0,2)} = \\ &= DR^a - 2\epsilon^a{}_{bc}\left[De^b \wedge f^c - e^b \wedge Df^c\right] - 2\epsilon^a{}_{bc}e^b \wedge Df^c = \\ &= 2\epsilon^a{}_{bc}e^b \wedge Df^c - 2\epsilon^a{}_{bc}e^b \wedge Df^c = 0, \end{aligned} \tag{4.52}$$

$$\begin{aligned} DF\Big|_{D} &= DF_{(1,1)} - 2e_a \wedge F^a{}_{(0,2)} = \\ &= -2\left[De^a \wedge f_a - e_a \wedge Df^a\right] - 2e_a \wedge Df^a = \\ &= 2e_a \wedge Df^a - 2e_a \wedge Df^a = 0. \end{aligned} \tag{4.53}$$

4.4.1.3 The (0,2) equation

Now, the only one left is the (0,2)-equation, which is the one requiring most trickeries

$$\begin{aligned} DF\Big|_{K_a} &= DF^a{}_{(0,2)} - \epsilon^a{}_{bc}F^b{}_{(1,1)} \wedge f^c + F_{(1,1)} \wedge f^a = \\ &= DDf^a - \epsilon^a{}_{bc}\left(R^b - 2\epsilon^b{}_{de}e^d \wedge f^e\right) \wedge f^c - 2e_b \wedge f^b \wedge f^a. \end{aligned} \tag{4.54}$$

The Levi-Civita symbols are simplified as before

$$\begin{aligned} DF\Big|_{K_a} &= DDf^a - \epsilon^a{}_{bc}R^b \wedge f^c + 2e^a \wedge f_b \wedge f^b - 2e_b \wedge f^a \wedge f^b - 2e^b \wedge f_b \wedge f^a = \\ &= DDf^a - \epsilon^a{}_{bc}f^b \wedge R^c. \end{aligned} \tag{4.55}$$

Most terms vanished, but it remains to show that the two remaining are equal. Notice that the DDf-term contains two antisymmetrized covariant derivatives (they are covariant exterior derivatives), which means that they can be rewritten as their brackets. It can in turn be rewritten as a Riemann tensor through the Ricci identity [27]

$$\begin{aligned} DDf^a &= \left[D_{[\mu}, D_\nu\right] f_{\sigma]}{}^a = -R_{[\mu\nu\sigma]}{}^\lambda f_\lambda{}^a - R_{[\mu\nu}{}^{ab} f_{\sigma]b} = \\ &= -R_{[\mu\nu}{}^{ab} f_{\sigma]b} = -\epsilon^{ab}{}_c R_{[\mu\nu}{}^c f_{\sigma]b} = -\epsilon^{ab}{}_c R^c \wedge f_b \end{aligned} \tag{4.56}$$

which shows that also the fourth equation is an identity

$$DF\Big|_{K_a} = DDf^a - \epsilon^a{}_{bc} f^b \wedge R^c = -\epsilon^a{}_{bc} R^c \wedge f^b - \epsilon^a{}_{bc} f^b \wedge R^c = 0. \tag{4.57}$$

The conclusion that can be drawn form this is that if the zero torsion condition is assumed, the equation $DF = 0$ is still an identity and hence allows a non-homogenous Cotton equation.

4.4.2 The Schouten tensor

Assuming the zero torsion condition left the equation $DF = 0$ as an identity, but this minimal assumption will not allow other fields than ω^a to be solved in terms of the dreibeins. To make it possible to also solve f^a in terms of ω^a, another constraint has to be imposed. Since the equation used to solve f^a in terms of ω^a and e^a was $F^a{}_{(1,1)} = 0$, take this as the constraint. This also happens to be the field projection next in line according to the grading order. The equation $F^a{}_{(1,1)} = 0$ identified f^a as the Schouten tensor (eq. (4.37)) $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$. Before entering this into the equations, note that the constraint $F_{\mu\nu}{}_{(1,1)} = 4 f_{[\mu\nu]} = 0$ follows from $F^a{}_{(1,1)} = 0$. Just multiply it by $\epsilon^\nu{}_{\sigma a}$ to obtain

$$\begin{aligned} \epsilon^\nu{}_{\sigma a} F_{\mu\nu}{}^a{}_{(1,1)} &= \epsilon^\nu{}_{\sigma a} \left(R_{\mu\nu}{}^a - 4\epsilon^a{}_{bc} e_{[\mu}{}^b f_{\nu]}{}^c\right) = \tfrac{1}{2} R_{\mu\sigma} - 4\epsilon^\nu{}_{\sigma a} \epsilon^a{}_{bc} e_{[\mu}{}^b f_{\nu]}{}^c = \\ &= \tfrac{1}{2} R_{\mu\sigma} - 2\left(e^\nu{}_c e_{\sigma b} - e^\nu{}_b e_{\sigma c}\right)\left(e_\mu{}^b f_\nu{}^c - e_\nu{}^b f_\mu{}^c\right) = \\ &= \tfrac{1}{2} R_{\mu\sigma} - 2\left(g_{\mu\sigma} \mathrm{Tr}\,[f] - f_{\mu\sigma} - f_{\mu\sigma} + 3 f_{\mu\sigma}\right) = \\ &= \tfrac{1}{2} R_{\mu\sigma} - 2 g_{\mu\sigma} \mathrm{Tr}\,[f] - 2 f_{\mu\sigma} = 0, \end{aligned} \tag{4.58}$$

or after some rearranging of terms

$$f_{\mu\nu} = \tfrac{1}{4} R_{\mu\nu} - g_{\mu\nu} \mathrm{Tr}\,[f]\,. \tag{4.59}$$

The constraint $F_{\mu\nu}{}_{(1,1)} = 4 f_{[\mu\nu]} = 0$ follows since both $g_{\mu\nu}$ and $R_{\mu\nu}$ are symmetric. Now it is time to rewrite eqs. (4.43) to (4.46) using the new constraints $F^a{}_{(2,0)} = F^a{}_{(1,1)} = F_{(1,1)} = 0$

$$\begin{aligned} P_a:&\quad DF\Big|_{P_a} = 0 = 0, \\ M_a:&\quad DF\Big|_{M_a} = -2\epsilon^a{}_{bc} e^b \wedge F^c{}_{(0,2)} = 0, \\ D:&\quad DF\Big|_{D} = -2 e_a \wedge F^a{}_{(0,2)} = 0, \\ K_a:&\quad DF\Big|_{K_a} = DF^a{}_{(0,2)} = 0. \end{aligned}$$

As seen, the $(2,0)$-equation is trivial, but the other three require some work.

4.4.2.1 The M_a equation

Solving the M_a-equation is done by implementing the anti-symmetri of the curved-space indices with a Levi-Civita symbol and entering the expression for f^a

$$\begin{aligned}\epsilon^{\mu\nu\sigma}D_\mu F_{\nu\sigma}\Big|_{M_a} &= -2\epsilon^{\mu\nu\sigma}\epsilon^a{}_{bc}e_\mu{}^b F^c_{\nu\sigma(0,2)} = -2\epsilon^{\mu\nu\sigma}\epsilon^a{}_{bc}e_\mu{}^b D_\nu f_\sigma{}^c = \\ &= 2\left(e^{\sigma a}e^{\nu\, c} - e^{\nu a}e^{\sigma\, c}\right) D_\nu \left(R_\sigma{}^c - \tfrac{1}{4}e_\sigma{}^c R\right) = \\ &= D_\nu \left(R^{a\nu} - \tfrac{1}{4}e^{\nu a}R\right) - D^a\left(R - \tfrac{3}{4}R\right) = \\ &= \tfrac{1}{2}D^a R - \tfrac{1}{4}D^a R - D^a R + D^a \tfrac{3}{4}R = 0.\end{aligned} \tag{4.60}$$

4.4.2.2 The D equation

The D-equation is easy remembering that $f_{[\mu\nu]} = 0$

$$DF\Big|_D = -2e_a \wedge F^a{}_{(0,2)} = e_{[\mu}{}^a D_\nu f_{\sigma]a} = D_{[\sigma}f_{\mu\nu]} = 0. \tag{4.61}$$

4.4.2.3 The K_a equation

Lastly, the K_a-equation. It is simply two covariant exterior derivative acting on f^a. The expression for this was found in eq. (4.56). Again, use the Levi-Civita symbol for the anti-symmetry

$$\begin{aligned}\epsilon^{\mu\nu\sigma}D_\mu F_{\nu\sigma}\Big|_{K_a} &= \epsilon^{\mu\nu\sigma}D_\mu F^a_{\nu\sigma(0,2)} = \epsilon^{\mu\nu\sigma}D_\mu D_\nu f_\sigma{}^a = -\epsilon^{\mu\nu\sigma}\epsilon^a{}_{bc}R_{\mu\nu}{}^c f_\sigma{}^b = \\ &= -\epsilon^{\mu\nu\sigma}\epsilon^a{}_{bc}R_{\mu\nu}{}^c\left(R_\sigma{}^b - \tfrac{1}{4}e_\sigma{}^b R\right) = \\ &= -\epsilon^{\mu\nu\sigma}\epsilon^a{}_{bc}R_{\mu\nu}{}^c R_\sigma{}^b + \tfrac{1}{4}\epsilon^{\mu\nu\sigma}\epsilon^a{}_{bc}R_{\mu\nu}{}^c e_\sigma{}^b R = \\ &= -2\epsilon^a{}_{bc}R^{\sigma c}R_\sigma{}^b + \tfrac{1}{2}\epsilon^a{}_{bc}R^{bc}R = 0.\end{aligned} \tag{4.62}$$

The first term is zero, since $R^{\sigma c}R_\sigma{}^b$ is symmetric under the exchange $b \leftrightarrow c$ and is anti-symmetrized from the Levi-Civita symbol, and the second term is zero since the Ricci tensor is symmetric. This is good! The equation $DF = 0$ is still an identity with all of $F = 0$, except the projection that enforces the Cotton equation assumed. This means that there is no violation of the Bianchi Identity even if the theory is coupled to matter.

5

Spin 3

The first step towards higher spins is spin 3. Spin 3 is introduced in this chapter and the equations are solved in the same way as the spin 2 equations were solved in chapter 4. The main difference is the amount of work needed to solve the equations. Since there are 9 basis elements in the spin-3 algebra, there will be 9 projections of both the $F = 0$ equation and the Bianchi identity. These will be larger than the ones found in the spin-2 case. Hence, as a first approach, the entire equations will not be considered. Instead, the linearized versions of them are. The linearized equations are the original ones, with all terms not proportional to the dreibein e_μ^a removed, this also include reducing the covariant derivative to the partial one. This make them significantly easier to solve, while still preserving some physical properties. Similar calculations has been made in [24]–[26].

5.1 The projection of $F = 0$

The projection of the $F = 0$ equation onto the spin-3 generators are equivalent to what was done in section 4.1. First, the gauge field A is written out in its basis elements

$$A^A T_A = e^{ab} P_{ab} + \tilde{e}^{ab} \tilde{P}_{ab} + \tilde{e}^a \tilde{P}_a + \tilde{\omega}^{ab} \tilde{M}_{ab} + \tilde{\omega}^a \tilde{M}_a + \tilde{b} \tilde{D} + \tilde{f}^{ab} \tilde{K}_{ab} + \tilde{f}^a \tilde{K}_a + f^{ab} K_{ab}. \quad (5.1)$$

One important thing about these is that all fields with two flat indices (a and b) are symmetric and traceless. This also means that all equations with two free flat indices will be symmetric and traceless, since the basis elements they are projected onto are symmetric and traceless. The field strength F now follows from its definition

$$F = dA + A \wedge A = 0. \quad (5.2)$$

Which written out in therms of the different graded parts of F is five expressions, each containing the covariant derivative on the corresponding part of A with two brackets resulting in terms projected onto the respectively graded parts.

$$\begin{aligned}
F_{(4,0)} &= DA_{(4,0)} + [A_{(1,1)}, A_{(4,0)}] + [A_{(2,0)}, A_{(3,1)}] = 0, \\
F_{(3,1)} &= DA_{(3,1)} + [A_{(2,0)}, A_{(2,2)}] + [A_{(0,2)}, A_{(4,0)}] = 0, \\
F_{(2,2)} &= DA_{(2,2)} + [A_{(2,0)}, A_{(1,3)}] + [A_{(0,2)}, A_{(3,1)}] = 0, \\
F_{(1,3)} &= DA_{(1,3)} + [A_{(0,2)}, A_{(2,2)}] + [A_{(2,0)}, A_{(0,4)}] = 0, \\
F_{(0,4)} &= DA_{(0,4)} + [A_{(0,2)}, A_{(1,3)}] + [A_{(1,1)}, A_{(0,4)}] = 0.
\end{aligned} \quad (5.3)$$

Already now, this looks tougher than the spin-2 case, but this is only the conceptual equations, entering all the brackets into the respective equation reveals the full set of equations.

$$\begin{aligned} F^{ab}{}^{3\times5}_{(4,0)} &= De^{ab}{}_{(4,0)} + 2b_{(1,1)} \wedge e^{ab}{}_{(4,0)} + e^{c}{}_{(2,0)} \wedge \tilde{e}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - \\ &\quad - (e^{(a}{}_{(2,0)} \wedge \tilde{e}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]) = 0, \end{aligned} \tag{5.4}$$

$$\begin{aligned} F^{ab}{}^{3\times5}_{(3,1)} &= D\tilde{e}^{ab}{}_{(3,1)} - 2e^{c}{}_{(2,0)} \wedge \tilde{\omega}^{d(a}{}_{(2,2)}\epsilon^{b)}{}_{cd} - (e^{(a}{}_{(2,0)} \wedge \tilde{\omega}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) - \\ &\quad - 4f^{c}{}_{(0,2)} \wedge e^{d(a}{}_{(4,0)}\epsilon_{cd}{}^{b)} = 0, \end{aligned} \tag{5.5}$$

$$\begin{aligned} F^{a}{}^{3\times3}_{(3,1)} &= D\tilde{e}^{a}{}_{(3,1)} - e_{b(2,0)} \wedge \tilde{\omega}^{ba}{}_{(2,2)} + \tfrac{3}{2}e^{c}{}_{(2,0)} \wedge \tilde{\omega}^{d}{}_{(2,2)}\epsilon_{cd}{}^{a} - 2e^{a}{}_{(2,0)} \wedge \tilde{b}_{(2,2)} + \\ &\quad + 6f_{c(0,2)} \wedge e^{c(a}{}_{(4,0)} = 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} F^{ab}{}^{3\times5}_{(2,2)} &= D\tilde{\omega}^{ab}{}_{(2,2)} + 3e^{c}{}_{(2,0)} \wedge \tilde{f}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - (e^{(a}{}_{(2,0)} \wedge \tilde{f}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) + \\ &\quad + 3f^{c}{}_{(0,2)} \wedge \tilde{e}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} + (f^{(a}{}_{(0,2)} \wedge \tilde{e}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]) = 0, \end{aligned} \tag{5.7}$$

$$\begin{aligned} F^{a}{}^{3\times3}_{(2,2)} &= D\tilde{\omega}^{a}{}_{(2,2)} - 3e_{b(2,0)} \wedge \tilde{f}^{ba}{}_{(1,3)} - 3e^{b}{}_{(2,0)} \wedge \tilde{f}^{c}{}_{(1,3)}\epsilon_{bc}{}^{a} + \\ &\quad + 3f_{b(0,2)} \wedge \tilde{e}^{ba}{}_{(3,1)} - 3f^{b}{}_{(0,2)} \wedge \tilde{e}^{c}{}_{(3,1)}\epsilon_{bc}{}^{a} = 0 \end{aligned} \tag{5.8}$$

$$F^{3}_{(2,2)} = D\tilde{b}_{(2,2)} - \tfrac{8}{3}e^{a}{}_{(2,0)} \wedge \tilde{f}_{a(1,3)} + \tfrac{8}{3}f^{a}{}_{(0,2)} \wedge \tilde{e}_{a(3,1)} = 0, \tag{5.9}$$

$$\begin{aligned} F^{ab}{}^{3\times5}_{(1,3)} &= D\tilde{f}^{ab}{}_{(1,3)} - 4e^{c}{}_{(2,0)} \wedge f^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} - 2f^{c}{}_{(0,2)} \wedge \tilde{\omega}^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} + \\ &\quad + (f^{(a}{}_{(0,2)} \wedge \tilde{\omega}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) = 0, \end{aligned} \tag{5.10}$$

$$\begin{aligned} F^{a}{}^{3\times3}_{(1,3)} &= D\tilde{f}^{a}{}_{(1,3)} - 6e_{b(2,0)} \wedge f^{ba}{}_{(0,4)} + f_{b(0,2)} \wedge \tilde{\omega}^{ba}{}_{(2,2)} + \\ &\quad + \tfrac{3}{2}f^{b}{}_{(0,2)} \wedge \tilde{\omega}^{c}{}_{(2,2)}\epsilon_{bc}{}^{a} 2f^{a}{}_{(0,2)} \wedge \tilde{b}_{(2,2)} = 0, \end{aligned} \tag{5.11}$$

$$F^{ab}{}^{3\times5}_{(0,4)} = Df^{ab}{}_{(0,4)} + f^{c}{}_{(0,2)} \wedge \tilde{f}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} + (f^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) = 0. \tag{5.12}$$

The 5×3 in $F^{ab}{}^{3\times5}_{(q,p)}$ is the degrees of freedom in the equation. In the case of 5×3 the equation contains one curved index and two symmmetric and traceless flat indices, containing three and five degrees of freedom respectively. Note also that the $\mathrm{Tr}\,[]$ is shorthand for the trace of the term before it, e.g. $e^{(a}{}_{(2,0)} \wedge \tilde{e}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[] = e^{(a}{}_{(2,0)} \wedge \tilde{e}^{b)}{}_{(3,1)} - \frac{1}{3}\eta^{ab}e^{c}{}_{(2,0)} \wedge \tilde{e}_{c(3,1)}$. As said, the full set of equations is large and for the initial approach, the linearized ones are considered.

$$F^{ab}{}^{3\times5}_{(4,0)} = de^{ab}{}_{(4,0)} + e^{c}{}_{(2,0)} \wedge \tilde{e}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} = 0, \tag{5.13}$$

$$F^{ab}{}^{3\times5}_{(3,1)} = d\tilde{e}^{ab}{}_{(3,1)} - 2e^{c}{}_{(2,0)} \wedge \tilde{\omega}^{d(a}{}_{(2,2)}\epsilon^{b)}{}_{cd} - (e^{(a}{}_{(2,0)} \wedge \tilde{\omega}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) = 0, \tag{5.14}$$

$$\begin{aligned} F^{a}{}^{3\times3}_{(3,1)} &= d\tilde{e}^{a}{}_{(3,1)} - e_{b(2,0)} \wedge \tilde{\omega}^{ba}{}_{(2,2)} + \tfrac{3}{2}e^{c}{}_{(2,0)} \wedge \tilde{\omega}^{d}{}_{(2,2)}\epsilon_{cd}{}^{a} \\ &\quad - 2e^{a}{}_{(2,0)} \wedge \tilde{b}_{(2,2)} = 0, \end{aligned} \tag{5.15}$$

$$F^{ab}{}^{3\times5}_{(2,2)} = d\tilde{\omega}^{ab}{}_{(2,2)} + 3e^{c}{}_{(2,0)} \wedge \tilde{f}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - (e^{(a}{}_{(2,0)} \wedge \tilde{f}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) = 0, \tag{5.16}$$

$$F^{a}{}^{3\times3}_{(2,2)} = d\tilde{\omega}^{a}{}_{(2,2)} - 3e_{b(2,0)} \wedge \tilde{f}^{ba}{}_{(1,3)} - 3e^{b}{}_{(2,0)} \wedge \tilde{f}^{c}{}_{(1,3)}\epsilon_{bc}{}^{a} = 0, \tag{5.17}$$

$$
\begin{aligned}
F^{3}_{(2,2)} &= d\tilde{b}_{(2,2)} - \tfrac{8}{3} e^{a}{}_{(2,0)} \wedge \tilde{f}_{a}{}_{(1,3)} = 0, && (5.18)\\
F^{ab}{}^{3\times5}_{(1,3)} &= d\tilde{f}^{ab}{}_{(1,3)} - 4e^{c}{}_{(2,0)} \wedge f^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} + \left(f^{(a}{}_{(0,2)} \wedge \tilde{\omega}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]\right) = 0, && (5.19)\\
F^{a}{}^{3\times3}_{(1,3)} &= d\tilde{f}^{a}{}_{(1,3)} - 6e_{b}{}_{(2,0)} \wedge f^{ba}{}_{(0,4)} + \tfrac{3}{2} f^{b}{}_{(0,2)} \wedge \tilde{\omega}^{c}{}_{(2,2)}\epsilon_{bc}{}^{a} = 0, && (5.20)\\
F^{ab}{}^{3\times5}_{(0,4)} &= df^{ab}{}_{(0,4)} + \left(f^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]\right) = 0. && (5.21)
\end{aligned}
$$

5.2 The projection of δA

In section 4.2, the projections in eqs. (4.18) to (4.21) were found for a variation of the gauge field in spin 2. Here, the corresponding result in spin 3 for a variation of the gauge field is written out in eqs. (5.4) to (5.12). The terms appearing are all 1-forms, since the variations Λ are 0-forms. The structure of the terms are obtained through the brackets (section 3.3.2).

$$
\begin{aligned}
\delta e^{ab}{}^{3\times5}_{(4,0)} = {} & D\Lambda^{ab}{}_{(4,0)} + 2\Lambda^{c}{}_{(1,1)}e^{d(a}{}_{(4,0)}\epsilon_{cd}{}^{b)} + 2b_{(1,1)}\Lambda^{ab}{}_{(4,0)} + 2\Lambda_{(1,1)}e^{ab}{}_{(4,0)} + && (5.22)\\
& + e^{c}{}_{(2,0)}\tilde{\Lambda}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - \left(e^{(a}{}_{(2,0)}\tilde{\Lambda}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]\right) + \Lambda^{c}{}_{(2,0)}\tilde{e}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - \\
& - \left(\Lambda^{(a}{}_{(2,0)}\tilde{e}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]\right),
\end{aligned}
$$

$$
\begin{aligned}
\delta \tilde{e}^{ab}{}^{3\times5}_{(3,1)} = {} & D\tilde{\Lambda}_{(3,1)} - 2e^{c}{}_{(2,0)}\tilde{\Lambda}^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} - \left(e^{(a}{}_{(2,0)}\tilde{\Lambda}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]\right) - && (5.23)\\
& - 4f^{c}{}_{(0,2)}\Lambda^{d(a}{}_{(4,0)}\epsilon_{cd}{}^{b)} - \Lambda_{(1,1)}\tilde{e}^{ab}{}_{(3,1)} + 2\Lambda^{c}{}_{(2,0)}\tilde{\omega}^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} + \\
& + \left(\Lambda^{(a}{}_{(2,0)}\tilde{\omega}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]\right) + 4\Lambda^{c}{}_{(0,2)}e^{d(a}{}_{(4,0)}\epsilon_{cd}{}^{b)} + 2\Lambda^{c}{}_{(1,1)}\tilde{e}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)},
\end{aligned}
$$

$$
\begin{aligned}
\delta \tilde{e}^{a}{}^{3\times3}_{(3,1)} = {} & D\tilde{\Lambda}^{a}{}_{(3,1)} - e_{b}{}_{(2,0)}\tilde{\Lambda}^{ba}{}_{(2,2)} + \tfrac{3}{2}e^{b}{}_{(2,0)}\tilde{\Lambda}^{c}{}_{(2,2)}\epsilon^{a}{}_{bc} - 2e^{a}{}_{(2,0)}\tilde{\Lambda}_{(2,2)} + && (5.24)\\
& + 6f_{b}{}_{(0,2)}\Lambda^{ba}{}_{(4,0)} - \Lambda_{(1,1)}\tilde{e}^{a}{}_{(3,1)} - \Lambda^{b}{}_{(1,1)}\tilde{e}^{c}{}_{(3,1)}\epsilon^{a}{}_{,bc} + \Lambda_{b}{}_{(2,0)}\tilde{\omega}^{ba}{}_{(2,2)} - \\
& - \tfrac{3}{2}\Lambda^{b}{}_{(2,0)}\tilde{\omega}^{c}{}_{(2,2)}\epsilon^{a}{}_{bc} + 2\Lambda^{a}{}_{(2,0)}\tilde{\omega}_{(2,2)} - 6\Lambda_{b}{}_{(0,2)}e^{ba}{}_{(4,0)},
\end{aligned}
$$

$$
\begin{aligned}
\delta \tilde{\omega}^{ab}{}^{3\times5}_{(2,2)} = {} & D\tilde{\Lambda}^{ab}{}_{(2,2)} + 3e^{c}{}_{(2,0)}\tilde{\Lambda}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - 3\Lambda^{c}{}_{(2,0)}\tilde{f}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - && (5.25)\\
& - \left(e^{(a}{}_{(2,0)}\tilde{\Lambda}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]\right) + \left(\Lambda^{(a}{}_{(2,0)}\tilde{f}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]\right) + 3f^{c}{}_{(0,2)}\tilde{\Lambda}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - \\
& - 3\Lambda^{c}{}_{(0,2)}\tilde{e}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} + \left(f^{(a}{}_{(0,2)}\tilde{\Lambda}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]\right) - \left(\Lambda^{(a}{}_{(0,2)}\tilde{f}^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]\right) + \\
& + 2\Lambda^{c}{}_{(1,1)}\tilde{\omega}^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)},
\end{aligned}
$$

$$
\begin{aligned}
\delta \tilde{\omega}^{a}{}^{3\times3}_{(2,2)} = {} & D\tilde{\Lambda}^{a}{}_{(2,2)} - 3e_{b}{}_{(2,0)}\tilde{\Lambda}^{ba}{}_{(1,3)} - 3e^{b}{}_{(2,0)}\tilde{\Lambda}^{c}{}_{(1,3)}\epsilon^{a}{}_{bc} + 3f_{b}{}_{(0,2)}\tilde{\Lambda}^{ba}{}_{(3,1)} - && (5.26)\\
& - 3f^{b}{}_{(0,2)}\tilde{\Lambda}^{c}{}_{(3,1)}\epsilon^{a}{}_{bc} + 3\Lambda_{b}{}_{(2,0)}\tilde{f}^{ba}{}_{(1,3)} + 3\Lambda^{b}{}_{(2,0)}f^{c}{}_{(1,3)}\epsilon^{a}{}_{bc} - 3\Lambda_{b}{}_{(0,2)}\tilde{e}^{ba}{}_{(3,1)} + \\
& + 3\Lambda^{b}{}_{(0,2)}\tilde{e}^{c}{}_{(3,1)}\epsilon^{a}{}_{bc} - \Lambda^{b}{}_{(1,1)}\tilde{\omega}^{c}{}_{(2,2)}\epsilon^{a}{}_{bc},
\end{aligned}
$$

$$
\delta \tilde{b}^{3}_{(2,2)} = D\tilde{\Lambda}_{(2,2)} - \tfrac{8}{3}e^{a}{}_{(2,0)}\tilde{\Lambda}_{a}{}_{(1,3)} + \tfrac{8}{3}\Lambda^{a}{}_{(2,0)}\tilde{f}_{a}{}_{(1,3)} + \tfrac{8}{3}f^{a}{}_{(0,2)}\tilde{\Lambda}_{a}{}_{(3,1)} - \tfrac{8}{3}\Lambda^{a}{}_{(0,2)}\tilde{e}_{a}{}_{(3,1)}, \quad (5.27)
$$

$$
\begin{aligned}
\delta \tilde{f}^{ab}{}^{3\times5}_{(1,3)} = {} & D\tilde{\Lambda}^{ab}{}_{(1,3)} - 4e^{c}{}_{(2,0)}\Lambda^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} - 2f^{c}{}_{(0,2)}\tilde{\Lambda}^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} + && (5.28)\\
& + \left(f^{(a}{}_{(0,2)}\tilde{\Lambda}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]\right) + 4\Lambda^{c}{}_{(2,0)}f^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} + 2\Lambda^{c}{}_{(0,2)}\tilde{\omega}^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} - \\
& - \left(\Lambda^{(a}{}_{(0,2)}\tilde{\omega}^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]\right) - 2\Lambda^{c}{}_{(1,1)}\tilde{f}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} + \Lambda_{(1,1)}\tilde{f}^{ab}{}_{(1,3)},
\end{aligned}
$$

$$\begin{aligned}
\delta \tilde{f}^{a}{}_{(1,3)}^{3\times3} &= D\tilde{\Lambda}^{a}{}_{(1,3)} - 6e_{b}{}_{(2,0)}\Lambda^{ba}{}_{(0,4)} + f_{b}{}_{(0,2)}\tilde{\Lambda}^{ba}{}_{(2,2)} + \tfrac{3}{2}f^{b}{}_{(0,2)}\tilde{\Lambda}^{c}{}_{(2,2)}\epsilon_{bc}{}^{a} + \\
&+ 2f^{a}{}_{(0,2)}\tilde{\Lambda}_{(2,2)} + 6\Lambda_{b}{}_{(2,0)}f^{ba}{}_{(0,4)} - \Lambda_{b}{}_{(0,2)}\tilde{\omega}^{ba}{}_{(2,2)} - \tfrac{3}{2}\Lambda^{b}{}_{(0,2)}\tilde{\omega}^{c}{}_{(2,2)}\epsilon_{bc}{}^{a} - \\
&- 2\Lambda^{a}{}_{(0,2)}\tilde{b}_{(2,2)} - \Lambda^{b}{}_{(1,1)}\tilde{f}^{c}{}_{(1,3)}\epsilon_{bc}{}^{a} + \Lambda_{(1,1)}\tilde{f}^{a}{}_{(1,3)},
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
\delta f^{ab}{}_{(0,4)}^{3\times5} &= D\Lambda^{ab}{}_{(0,4)} + f^{c}{}_{(0,2)}\tilde{\Lambda}^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} + (f^{(a}{}_{(0,2)}\tilde{\Lambda}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) - \\
&- \Lambda^{c}{}_{(0,2)}\tilde{f}^{d(a}{}_{(1,3)}\epsilon_{cd},{}^{b)} - (\Lambda^{(a}{}_{(0,2)}\tilde{f}^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) - 2\Lambda^{c}{}_{(1,1)}f^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} + \\
&+ 2\Lambda_{(1,1)}f^{ab}{}_{(0,4)}.
\end{aligned} \tag{5.30}$$

5.2.1 Choosing the spin-3 gauge condition

One choise, which is the one used throughout this chapter is to use the gauge parameters $\tilde{\Lambda}^{ab}{}_{(2,2)}$, $\tilde{\Lambda}^{a}{}_{(2,2)}$ and $\tilde{\Lambda}_{(2,2)}$ in eq. (5.24) to set the field $\tilde{e}_{\mu}{}^{a}$ to zero. The only other Stückelberg field is $\tilde{b}_{\mu}$, but it will not be set to zero. Instead the **5** and **3** part of $\tilde{\omega}_{\mu}{}^{a}$ in eq. (5.26) and the **5** part of $\tilde{f}_{\mu}\, a$ in eq. (5.29) are set to zero using $\tilde{\Lambda}^{ab}{}_{(1,3)}$, $\tilde{\Lambda}^{a}{}_{(1,3)}$ and $\tilde{\Lambda}^{ab}{}_{(0,4)}$ respectively.
The **5** and **3** parts refer to the degrees of freedom for the fields. For instance, the field $\tilde{\omega}_{\mu}{}^{a}$ has two indices, wich each has three degrees of freedom (one for each dimension). The total degrees of freedom of $\tilde{\omega}_{\mu}{}^{a}$ is hence $\mathbf{3}\times\mathbf{3} = \mathbf{5}+\mathbf{3}+\mathbf{1}$, where the **5** part represents the symmetric traceless part between the a and μ indices. Similarly, the **3** part corresponds to the anti-symmetric part and the **1** part to the trace. To summarize, the gauge choises used are

$$\tilde{e}_{\mu}{}^{a} = 0, \tag{5.31}$$

$$\tilde{\omega}_{\mu}{}^{a} = e^{a}_{\mu}\hat{\omega}, \tag{5.32}$$

$$\tilde{f}_{\mu}{}^{a} = \epsilon_{\mu}{}^{ab}\hat{f}_{b} + e^{a}_{\mu}\tilde{f}. \tag{5.33}$$

This has one consequence that is important to note. Since $\tilde{\omega}_{\mu}{}^{a}$ and $\tilde{f}_{\mu}{}^{a}$ now explicitly has a part depending on e^{a}_{μ} they must be included in the linearized equations eqs. (5.13) to (5.21).
Note that there is an option to use $\tilde{\Lambda}_{(3,1)}$ parameters to make $e_{\mu}{}^{ab}$ symmetric in the same way $\Lambda^{a}{}_{(1,1)}$ could be used to make $e_{\mu}{}^{a}$ symmetric in section 4.2.1. Hence, the $\tilde{\Lambda}_{(3,1)}$ parameters can be interpreted as the spin-3 Lorentz transformations and the symmetrized $e_{\mu}{}^{ab}$ can be seen as a spin-3 graviton $g_{\mu\nu\sigma}$. This shows that the four q and p spinors used result in spin 3.

5.3 Solving $F = 0$

As in the spin-2 case in section 4.3, some of the projected $F = 0$ equations not will be assumed when confirming the Bianchi Identity, but it is still interesting to find what they say. Especially, the assumed equations will provide convenient relations between the different fields to be used during calculations.
The projected equations were found in section 5.1, here the linearized versions of them will be used.
Note that all equations not solving one field in terms of another must be constraints following arguments in appendix H.

5.3.1 The P_{ab} projection

The first of the equations to be solved is the P_{ab}-projected one

$$F^{ab}{}^{3\times 5}_{(4,0)} = de^{ab} + e^c \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} = 0. \tag{5.34}$$

This equation can be used to express $\tilde{e}^{ab}$ in terms of e^{ab}. Start by writing out indices and implement the anti-symmetry using a Levi-Civita symbol

$$\epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \epsilon^{\mu\nu\sigma}e_\mu{}^c\tilde{e}_\nu{}^{d(a}\epsilon_{cd}{}^{b)} = \epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \epsilon^{c\nu\sigma}\tilde{e}_\nu{}^{d(a}\epsilon_{cd}{}^{b)} = 0, \tag{5.35}$$

simplifying the Levi-Civita symbols

$$\begin{aligned}\epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \tfrac{1}{2}\tilde{e}_\nu{}^{da}\left(e^\sigma{}_d e^{\nu b} - e^\nu{}_d e^{\sigma b}\right) + \tfrac{1}{2}\tilde{e}_\nu{}^{db}\left(e^\sigma{}_d e^{\nu a} - e^\nu{}_d e^{\sigma a}\right) =\\ = \epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \tfrac{1}{2}\tilde{e}^{b\sigma a} - \tfrac{1}{2}e^{\sigma b}\tilde{e}_\nu{}^{\nu a} + \tfrac{1}{2}\tilde{e}^{a\sigma b} - \tfrac{1}{2}e^{\sigma a}\tilde{e}_\nu{}^{\nu b} = 0.\end{aligned} \tag{5.36}$$

The trace of this is needed to continue solving the equation, this is obtained by multiplying the whole expression by $e_{\sigma b}$. Remember that $\tilde{e}^{ab}$ is symmetric and traceless in $a \leftrightarrow b$

$$\begin{aligned}e_{\sigma b}\left(\epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \tfrac{1}{2}\tilde{e}^{b\sigma a} - \tfrac{1}{2}e^{\sigma b}\tilde{e}_\nu{}^{\nu a} + \tfrac{1}{2}\tilde{e}^{a\sigma b} - \tfrac{1}{2}e^{\sigma a}\tilde{e}_\nu{}^{\nu b}\right) =\\ = \epsilon^{\mu\nu}{}_b\partial_\mu e_\nu{}^{ab} + \tfrac{1}{2}\tilde{e}_\sigma{}^{\sigma a} - \tfrac{3}{2}\tilde{e}_\nu{}^{\nu a} + \tfrac{1}{2}\tilde{e}^{a\sigma}{}_\sigma - \tfrac{1}{2}\tilde{e}_\nu{}^{\nu a} =\\ = \epsilon^{\mu\nu}{}_b e_\mu{}^{ab} - \tfrac{3}{2}\tilde{e}_\nu{}^{\nu a} = 0,\end{aligned} \tag{5.37}$$

which means that

$$\tilde{e}_\nu{}^{\nu a} = \tfrac{2}{3}\epsilon^{\mu\nu}{}_b\partial_\mu e_\nu{}^{ab}. \tag{5.38}$$

Now enter this into eq. (5.36).

$$\begin{aligned}\epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \tfrac{1}{2}\tilde{e}^{b\sigma a} - \tfrac{1}{2}e^{\sigma b}\tilde{e}_\nu{}^{\nu a} + \tfrac{1}{2}\tilde{e}^{a\sigma b} - \tfrac{1}{2}e^{\sigma a}\tilde{e}_\nu{}^{\nu b} =\\ = \epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \tfrac{1}{2}\tilde{e}^{b\sigma a} - \tfrac{1}{3}e^{\sigma b}\epsilon^{\mu\nu}{}_c\partial_\mu e_\nu{}^{ac} + \tfrac{1}{2}\tilde{e}^{a\sigma b} - \tfrac{1}{3}e^{\sigma a}\epsilon^{\mu\nu}{}_c\partial_\mu e_\nu{}^{bc} =\\ = \epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab} + \tilde{e}^{(ab)\sigma} - \tfrac{2}{3}\epsilon^{\mu\nu}{}_c e^{\sigma(a}\partial_\mu e_\nu{}^{b)c} = 0.\end{aligned} \tag{5.39}$$

Moving around the terms

$$\tilde{e}^{(ab)\sigma} = \tfrac{2}{3}\epsilon^{\mu\nu}{}_c e^{\sigma(a}\partial_\mu e_\nu{}^{b)c} - \epsilon^{\mu\nu\sigma}\partial_\mu e_\nu{}^{ab}, \tag{5.40}$$

where the following neat trick can be used

$$\tilde{e}^{cab} = \tilde{e}^{(cb)a} + \tilde{e}^{(ca)b} - \tilde{e}^{(ab)c}. \tag{5.41}$$

The expression becomes

$$\begin{aligned}\tilde{e}^{cab} &= \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{a(c}\partial_\mu e_\nu{}^{b)d} - \epsilon^{\mu\nu a}\partial_\mu e_\nu{}^{cb} + \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{b(c}\partial_\mu e_\nu{}^{a)d} - \epsilon^{\mu\nu b}\partial_\mu e_\nu{}^{ca} -\\ &\quad - \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{c(a}\partial_\mu e_\nu{}^{b)d} + \epsilon^{\mu\nu c}\partial_\mu e_\nu{}^{ab} =\\ &= \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{a(c}\partial_\mu e_\nu{}^{b)d} - 2\epsilon^{\mu\nu(a}\partial_\mu e_\nu{}^{b)c} + \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{b(c}\partial_\mu e_\nu{}^{a)d} -\\ &\quad - \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{c(a}\partial_\mu e_\nu{}^{b)d} + \epsilon^{\mu\nu c}\partial_\mu e_\nu{}^{ab} =\\ &= \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{ab}\partial_\mu e_\nu{}^{cd} - 2\epsilon^{\mu\nu(a}\partial_\mu e_\nu{}^{b)c} + \epsilon^{\mu\nu c}\partial_\mu e_\nu{}^{ab},\end{aligned} \tag{5.42}$$

where

$$e_\mu{}^{ab} = e_{\mu c}e^{cab}. \tag{5.43}$$

5.3.2 The $\tilde{P}_{ab}$ projection

$$F^{ab}{}^{3\times5}_{(3,1)} = d\tilde{e}^{ab} - 2e^c \wedge \tilde{\omega}^{d(a}\epsilon^{b)}{}_{cd} - (e^{(a} \wedge \tilde{\omega}^{b)} - \mathrm{Tr}\,[]) = 0. \tag{5.44}$$

This expression relates $\tilde{\omega}^{ab}$ to $\tilde{e}^{ab}$ in the same way $\tilde{e}^{ab}$ was related to e^{ab} in section 5.3.1 and if the same procedure is followed, the expression for $\tilde{\omega}^{ab}$ becomes

$$\tilde{\omega}^{cab} = \epsilon^{\mu\nu(a}\partial_\mu \tilde{e}_\nu{}^{b)c} - \tfrac{1}{2}\epsilon^{c\mu\nu}\partial_\mu \tilde{e}_\nu{}^{ab} - \tfrac{1}{3}\eta^{ab}\epsilon^{\mu\nu}{}_d \partial_\mu \tilde{e}_\nu{}^{cd}. \tag{5.45}$$

5.3.3 The $\tilde{P}_a$ projection

$$F^{a}{}^{3\times3}_{(3,1)} = -e_b \wedge \tilde{\omega}^{ba} + \tfrac{3}{2}e^c \wedge \tilde{\omega}^d \epsilon_{cd}{}^a - 2e^a \wedge \tilde{b} = 0. \tag{5.46}$$

The anti-symmetry of the curved indices can be represented with one index using a Levi-Civita symbol

$$\epsilon^{\mu\nu\sigma}\left(-e_{\mu b}\tilde{\omega}_\nu{}^{ba} + \tfrac{3}{2}e_\mu{}^c\tilde{\omega}_\nu{}^d\epsilon_{cd}{}^a - 2e_\mu{}^a\tilde{b}_\nu\right) = 0. \tag{5.47}$$

This equation has two free indices and hence $\mathbf{3}\times\mathbf{3} = \mathbf{5}+\mathbf{3}+\mathbf{1}$ degrees of freedom. These **5**, **3** and **1** parts can be examined separately by considering the symmetric, antisymmetric, and trace parts of the free indices to get the **5**, **3** and **1** parts, respectively. Start with the **1** part by multiplying eq. (5.47) with $e_{\sigma a}$

$$\begin{aligned}&\epsilon^{\mu\nu}{}_a\left(-e_{\mu b}\tilde{\omega}_\nu{}^{ba} + \tfrac{3}{2}e_\mu{}^c\tilde{\omega}_\nu{}^d\epsilon_{cd}{}^a - 2e_\mu{}^a\tilde{b}_\nu\right) = \\ &= -\epsilon^\nu{}_{ab}\tilde{\omega}_\nu{}^{ba} + \tfrac{3}{2}\epsilon^\nu{}_{ac}\tilde{\omega}_\nu{}^d\epsilon_d{}^{ac} - 2\epsilon^{\mu\nu}{}_\mu\tilde{b}_\nu = 0,\end{aligned} \tag{5.48}$$

which simplifyes to

$$\tilde{\omega}_\nu{}^\nu = e_\nu{}^\nu\hat{\omega} = 0 \qquad \Rightarrow \qquad \hat{\omega} = 0, \tag{5.49}$$

when the gauge $\tilde{\omega}_\mu{}^a = e_\mu{}^a\hat{\omega}$ is used, and hence, $\tilde{\omega}_\mu{}^a = 0$. Continue with the **3** part by multiplying eq. (5.47) by $\epsilon_{a\sigma\lambda}$

$$\begin{aligned}&\epsilon_{a\sigma\lambda}\epsilon^{\mu\nu\sigma}\left(-e_{\mu b}\tilde{\omega}_\nu{}^{ba} + \tfrac{3}{2}e_\mu{}^c\tilde{\omega}_\nu{}^d\epsilon_{cd}{}^a - 2e_\mu{}^a\tilde{b}_\nu\right) = \\ &\qquad = -\epsilon_{a\sigma\lambda}\epsilon_b{}^{\nu\sigma}\tilde{\omega}_\nu{}^{ba} - 2\epsilon_{a\sigma\lambda}\epsilon^{a\nu\sigma}\tilde{b}_\nu 0.\end{aligned} \tag{5.50}$$

Simplifying the Levi-Civita symbols

$$-\left(\delta^\nu_\lambda\eta_{ab} - e_{\lambda b}e^\nu{}_a\right)\tilde{\omega}_\nu{}^{ba} - 4\delta^\nu_\lambda\tilde{b}_\nu = -\tilde{\omega}_\lambda{}^a{}_a + \tilde{\omega}_{a\lambda}{}^a - 4\tilde{b}_\lambda = 0, \tag{5.51}$$

which gives the expression

$$\tilde{b}_\mu = \tfrac{1}{4}\tilde{\omega}_{\nu\mu}{}^\nu. \tag{5.52}$$

Now for the **5** part. What is done here is to symmetrize between the two free indices a and $\sigma = c$

$$\begin{aligned}\epsilon^{\mu\nu c}&e_{\mu b}\tilde{\omega}_\nu{}^{ba} + \epsilon^{\mu\nu a}e_{\mu b}\tilde{\omega}_\nu{}^{bc} + 2\epsilon^{\mu\nu c}e_\mu{}^a\tilde{b}_\nu + 2\epsilon^{\mu\nu a}e_\mu{}^c\tilde{b}_\nu = \\ &= \epsilon_b{}^{\nu c}\tilde{\omega}_\nu{}^{ba} + \epsilon_b{}^{\nu a}\tilde{\omega}_\nu{}^{bc} + 2\epsilon^{a\nu c}\tilde{b}_\nu + 2\epsilon^{c\nu a}\tilde{b}_\nu = \\ &= 2\epsilon_b{}^{\nu(a}\tilde{\omega}_\nu{}^{c)b} - 4\epsilon^{\nu(ac)}\tilde{b}_\nu = \\ &= 2\epsilon_b{}^{\nu(a}\tilde{\omega}_\nu{}^{c)b} = 0.\end{aligned} \tag{5.53}$$

5.3.4 The $\tilde{M}_{ab}$ projection

$$F^{ab}{}^{3\times5}_{(2,2)} = d\tilde{\omega}^{ab} + 3e^c \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) = 0. \tag{5.54}$$

Solving this using the procedure from section 5.3.1 gives

$$3\tilde{f}^{cab} = \epsilon^{\mu\nu c}\partial_\mu\tilde{\omega}_\nu{}^{ab} - 2\epsilon^{\mu\nu(a}\partial_\mu\tilde{\omega}_\nu{}^{b)c} + \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{ab}\partial_\mu\tilde{\omega}_\nu{}^{cd} + \tfrac{2}{3}\eta^{c(a}\hat{f}^{b)} - \tfrac{2}{9}\eta^{ab}\hat{f}^c \tag{5.55}$$

5.3.5 The $\tilde{M}_a$ projection

$$F^a{}^{3\times3}_{(2,2)} = d\tilde{\omega}^a - 3e_b \wedge \tilde{f}^{ba} - 3e^b \wedge \tilde{f}^c\epsilon_{bc}{}^a = 0. \tag{5.56}$$

Use a Levi-Civita symbol for anti-symmetry and examine the **1**, **3** and **5** parts of the equation separately. Start with the **1** part, the trace

$$\begin{aligned}
e_{\sigma a}\epsilon^{\mu\nu\sigma}&\left(-3e_{\mu b}\tilde{f}_\nu{}^{ba} - 3e_\mu{}^b\tilde{f}_\nu{}^c\epsilon_{bc}{}^a\right) = \epsilon^{\mu\nu}{}_a\left(3\tilde{f}_{\mu\nu}{}^a - 3\tilde{f}_\nu{}^c\epsilon_{\mu c}{}^a\right) = \\
&= 3\epsilon^{\mu\nu}{}_a\tilde{f}_{\mu\nu}{}^a + 6e^\nu{}_c\tilde{f}_\nu{}^c = 6e^\nu{}_c\tilde{f}_\nu{}^c = 6e^\nu{}_c\left(\epsilon_\nu{}^{cb}\hat{f}_b + e_\nu{}^c\hat{f}\right) = \\
&= 18\hat{f} = 0 \qquad \Rightarrow \qquad \hat{f} = 0.
\end{aligned} \tag{5.57}$$

using the gauge $\tilde{f}_\mu{}^a = \epsilon_\mu{}^{ab}\hat{f}_b + e_\mu{}^a\hat{f}$. Continue with the anti-symmetric **3** part

$$\begin{aligned}
\epsilon_{a\sigma\lambda}\epsilon^{\mu\nu\sigma}&\left(3\tilde{f}_{\mu\nu}{}^a - 3\tilde{f}_\nu{}^c\epsilon_{\mu c}{}^a\right) = \left(e^\mu{}_a\delta^\nu_\lambda - e^\nu{}_a\delta^\mu_\lambda\right)\left(3\tilde{f}_{\mu\nu}{}^a - 3\tilde{f}_\nu{}^c\epsilon_{\mu c}{}^a\right) = \\
&= 3\tilde{f}_{a\lambda}{}^a - 3\tilde{f}_{\lambda a}{}^a - 3\tilde{f}_\lambda{}^c\epsilon_{ac}{}^a + 3\tilde{f}_a{}^c\epsilon_{\lambda c}{}^a = \\
&= 3\tilde{f}_{a\lambda}{}^a + 3\tilde{f}_a{}^c\epsilon_{\lambda c}{}^a = 3\tilde{f}_{a\lambda}{}^a + 3\epsilon_a{}^{cb}\hat{f}_b\epsilon_{\lambda c}{}^a = \\
&= 3\tilde{f}_{a\lambda}{}^a + 6\hat{f}_\lambda = 0 \qquad \Rightarrow \qquad \hat{f}_a = -\tfrac{1}{2}\tilde{f}_{\mu a}{}^\mu.
\end{aligned} \tag{5.58}$$

Finish with the symmetric **5** part

$$\begin{aligned}
3\epsilon^{\mu\nu c}&\tilde{f}_{\mu\nu}{}^a + 3\epsilon^{\mu\nu a}\tilde{f}_{\mu\nu}{}^c - 3\epsilon^{\mu\nu c}\epsilon_\nu{}^{bd}\hat{f}_d\epsilon_{\mu b}{}^a - 3\epsilon^{\mu\nu a}\epsilon_\nu{}^{bd}\hat{f}_d\epsilon_{\mu b}{}^c = \\
&= 6\epsilon^{\mu\nu(a}\tilde{f}_{\mu\nu}{}^{c)} - 3\epsilon^{\mu\nu c}\epsilon_\nu{}^{bd}\hat{f}_d\epsilon_{\mu b}{}^a - 3\epsilon^{\mu\nu a}\epsilon_\nu{}^{bd}\hat{f}_d\epsilon_{\mu b}{}^c = \\
&= 6\epsilon^{\mu\nu(a}\tilde{f}_{\mu\nu}{}^{c)} - 3\left(e^{\mu b}\eta^{cd} - e^{\mu d}\eta^{bc}\right)\hat{f}_d\epsilon_{\mu b}{}^a - 3\left(e^{\mu b}\eta^{ad} - e^{\mu d}\eta^{ba}\right)\hat{f}_d\epsilon_{\mu b}{}^c = \\
&= 6\epsilon^{\mu\nu(a}\tilde{f}_{\mu\nu}{}^{c)} - 6\hat{f}^{(a}\epsilon_\mu{}^{c)\mu} + 6\hat{f}^\mu\epsilon_\mu{}^{(ac)} = \\
&= 6\epsilon^{\mu\nu(a}\tilde{f}_{\mu\nu}{}^{c)} = 0 \quad \Rightarrow \quad \epsilon^{\mu\nu(a}\tilde{f}_{\mu\nu}{}^{b)} = 0.
\end{aligned} \tag{5.59}$$

5.3.6 The $\tilde{D}$ projection

$$F^3_{(2,2)} = d\tilde{b} - \tfrac{8}{3}e^a \wedge \tilde{f}_a = 0, \tag{5.60}$$

which in tensors read

$$\begin{aligned}
\epsilon^{\mu\nu\sigma}\left(\partial_\mu\tilde{b}_\nu - \tfrac{8}{3}e_\mu{}^a\tilde{f}_{\nu a}\right) &= \epsilon^{\mu\nu\sigma}\partial_\mu\tilde{b}_\nu + \tfrac{8}{3}\epsilon^{\mu\nu\sigma}\tilde{f}_{\mu\nu} = \\
&= \epsilon^{\mu\nu\sigma}\partial_\mu\tilde{b}_\nu + \tfrac{8}{3}\epsilon^{\mu\nu\sigma}\epsilon_{\mu\nu a}\hat{f}^a = \\
&= \epsilon^{\mu\nu\sigma}\partial_\mu\tilde{b}_\nu - \tfrac{16}{3}\hat{f}^\sigma,
\end{aligned} \tag{5.61}$$

giving the relation between $\hat{f}_\mu$ and $\tilde{b}_\mu$

$$\hat{f}^a = \tfrac{3}{16}\epsilon^{\mu\nu a}\partial_\mu\tilde{b}_\nu \qquad \Rightarrow \qquad \tilde{f}_{\mu a}{}^\mu = -\tfrac{3}{8}\epsilon_a{}^{\mu\nu}\partial_\mu\tilde{b}_\nu. \tag{5.62}$$

5.3.7 The $\tilde{K}_{ab}$ projection

$$F^{ab}{}^{3\times 5}_{(1,3)} = d\tilde{f}^{ab} - 4e^c \wedge f^{d(a}\epsilon_{cd}{}^{b)} = 0. \tag{5.63}$$

Solving this using the procedure from section 5.3.1 gives

$$4f^{cab} = 2\epsilon^{\mu\nu(a}\partial_\mu \tilde{f}_\nu{}^{b)c} - \epsilon^{\mu\nu c}\partial_\mu \tilde{f}_\nu{}^{ab} - \tfrac{2}{3}\epsilon^{\mu\nu}{}_d \eta^{ab}\partial_\mu \tilde{f}_\nu{}^{cd}. \tag{5.64}$$

5.3.8 The $\tilde{K}_a$ projection

$$F^{a}{}^{3\times 3}_{(1,3)} = d\tilde{f}^{a} - 6e_b \wedge f^{ba} = 0. \tag{5.65}$$

Start with the **1** part

$$\begin{aligned}\epsilon^{\mu\nu a}\partial_\mu \tilde{f}_{\nu a} - 6\epsilon^{b\nu a} f_{\nu ab} &= \epsilon^{\mu\nu a}\partial_\mu \tilde{f}_{\nu a} = \\ &= \epsilon^{\mu\nu a}\partial_\mu \epsilon_{\nu a\lambda}\hat{f}^\lambda = \\ &= -2\partial_\mu \hat{f}^\mu = 0.\end{aligned} \tag{5.66}$$

Now the **3** part

$$\begin{aligned}\epsilon_c{}^{a\lambda}\epsilon^{\mu\nu c}\partial_\mu \tilde{f}_{\nu a} - 6\epsilon_c{}^{a\lambda}\epsilon^{b\nu c} f_{\nu ab} &= \left(e^{\nu a}g^{\lambda\mu} - e^{\mu a}g^{\lambda\nu}\right)\partial_\mu \tilde{f}_{\nu a} - 6\left(e^{\nu a}e^{\lambda b} - \eta^{ba}g^{\lambda\nu}\right) f_{\nu ab} = \\ &= \partial^\lambda \tilde{f}_\nu{}^\nu - \partial_\mu \tilde{f}^{\lambda\mu} - 6f_\nu{}^{\nu\lambda} + 6f_{\lambda a}{}^a = \\ &= -\partial_\mu \tilde{f}^{\lambda\mu} - 6f_\nu{}^{\nu\lambda} = 0.\end{aligned} \tag{5.67}$$

Lastly, the **5** part

$$\begin{aligned}\epsilon^{\mu\nu(a}\partial_\mu \tilde{f}_\nu{}^{b)} + 6\epsilon^{\mu\nu(a}f_{\mu\nu}{}^{c)} &= \partial^{(a}\tilde{f}^{b)} - 2\eta^{ab}\partial_\mu \hat{f}^\mu + 6\epsilon^{\mu\nu(a}f_{\mu\nu}{}^{c)} = \\ &= \partial^{(a}\tilde{f}^{b)} + 6\epsilon^{\mu\nu(a}f_{\mu\nu}{}^{c)} = 0.\end{aligned} \tag{5.68}$$

5.3.9 The K_{ab} projection

$$F^{ab}{}^{3\times 5}_{(0,4)} = df^{ab} = 0. \tag{5.69}$$

This is the last equation in graded order and is the one corresponding to section 4.3.4. It can hence be interpreted as the spin-3 Cotton equation, with $f_\mu{}^{ab}$ the spin-3 Schouten tensor.

5.4 Solving the spin-3 Bianchi identity $DF = 0$

Using the method from section 4.4, the full projection of the spin-3 Bianchi identity $DF = 0$ is found. With gauge choices implemented as determined in section 5.2, the projections looks as

$$\begin{aligned} DF\big|_{P_{ab}} = {} & DF^{ab}{}_{(4,0)} - 2F^c{}_{(1,1)} \wedge e^{d(a}\epsilon_{cd}{}^{b)} + 2b \wedge F^{ab}{}_{(4,0)} - 2e^{ab} \wedge F_{(1,1)} + \\ & + e^c \wedge F^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - (F^{(a}{}_{(3,1)} \wedge e^{b)} - \mathrm{Tr}\,[]) - F^c{}_{(2,0)} \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} + \\ & + (F^{(a}{}_{(2,0)} \wedge \tilde{e}^{b)} - \mathrm{Tr}\,[]) = 0, \end{aligned} \tag{5.70}$$

$$\begin{aligned} DF\big|_{\tilde{P}_{ab}} = {} & DF^{ab}{}_{(3,1)} - 2e^c \wedge F^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge F^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) - \\ & - 4f^c \wedge F^{d(a}{}_{(4,0)}\epsilon_{cd}{}^{b)} - F_{(1,1)} \wedge \tilde{e}^{ab} + 2F^c{}_{(2,0)} \wedge \tilde{\omega}^{d(a}\epsilon_{cd}{}^{b)} + \\ & + (F^{(a}{}_{(2,0)} \wedge \tilde{\omega}^{b)} - \mathrm{Tr}\,[]) + 4F^c{}_{(0,2)} \wedge e^{d(a}\epsilon_{cd}{}^{b)} - \\ & - 2F^c{}_{(1,1)} \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} = 0, \end{aligned} \tag{5.71}$$

$$\begin{aligned} DF\big|_{\tilde{P}_{a}} = {} & DF^{a}{}_{(3,1)} - e_b \wedge F^{ba}{}_{(2,2)} + \tfrac{3}{2}e^b \wedge F^c{}_{(2,2)}\epsilon^a{}_{bc} - 2e^a \wedge F_{(2,2)} + \\ & + 6f_b \wedge F^{ba}{}_{(4,0)} - F_{(1,1)} \wedge \tilde{e}^a - F^b{}_{(1,1)} \wedge \tilde{e}^c\epsilon^a{}_{bc} + F_b{}_{(2,0)} \wedge \tilde{\omega}^{ba} - \\ & - \tfrac{3}{2}F^b{}_{(2,0)} \wedge \tilde{\omega}^c\epsilon^a{}_{bc} + 2F^a{}_{(2,0)} \wedge \tilde{b} - 6F_b{}_{(0,2)} \wedge e^{ba} = 0, \end{aligned} \tag{5.72}$$

$$\begin{aligned} DF\big|_{\tilde{M}_{ab}} = {} & DF^{ab}{}_{(2,2)} + 3e^c \wedge F^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - 3F^c{}_{(2,0)} \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} - \\ & - (e^{(a} \wedge F^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) + (F^{(a}{}_{(2,0)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) + \\ & + 3f^c \wedge F^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - 3F^c{}_{(0,2)} \wedge \tilde{e}^{d(a}\epsilon_{cd}{}^{b)} + (f^{(a} \wedge F^{b)}{}_{(3,1)} - \mathrm{Tr}\,[]) - \\ & - (F^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) - 2F^c{}_{(1,1)} \wedge \tilde{\omega}^{d(a}\epsilon_{cd}{}^{b)} = 0, \end{aligned} \tag{5.73}$$

$$\begin{aligned} DF\big|_{\tilde{M}_{a}} = {} & DF^{a}{}_{(2,2)} - 3e_b \wedge F^{ba}{}_{(1,3)} - 3e^b \wedge F^c{}_{(1,3)}\epsilon^a{}_{bc} + 3f_b \wedge F^{ba}{}_{(3,1)} - \\ & - 3f^b \wedge F^c{}_{(3,1)}\epsilon^a{}_{bc} + 3F_b{}_{(2,0)} \wedge \tilde{f}^{ba} + 3F^b{}_{(2,0)} \wedge f^c\epsilon^a{}_{bc} - \\ & - 3F_b{}_{(0,2)} \wedge \tilde{e}^{ba} + 3F^b{}_{(0,2)} \wedge \tilde{e}^c\epsilon^a{}_{bc} - F^b{}_{(1,1)} \wedge \tilde{\omega}^c\epsilon^a{}_{bc} = 0, \end{aligned} \tag{5.74}$$

$$\begin{aligned} DF\big|_{\tilde{D}} = {} & DF_{(2,2)} - \tfrac{8}{3}e^a \wedge F_a{}_{(1,3)} + \tfrac{8}{3}F^a{}_{(2,0)} \wedge \tilde{f}_a + \tfrac{8}{3}f^a \wedge F_a{}_{(3,1)} - \\ & - \tfrac{8}{3}F^a{}_{(0,2)} \wedge \tilde{e}_a = 0, \end{aligned} \tag{5.75}$$

$$\begin{aligned} DF\big|_{\tilde{K}_{ab}} = {} & DF^{ab}{}_{(1,3)} - 4e^c \wedge F^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} - 2f^c \wedge F^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} + \\ & + (f^{(a} \wedge F^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) + 4F^c{}_{(2,0)} \wedge f^{d(a}\epsilon_{cd}{}^{b)} + 2F^c{}_{(0,2)} \wedge \tilde{\omega}^{d(a}\epsilon_{cd}{}^{b)} - \\ & - (F^{(a}{}_{(0,2)} \wedge \tilde{\omega}^{b)} - \mathrm{Tr}\,[]) - 2F^c{}_{(1,1)} \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} + F_{(1,1)} \wedge \tilde{f}^{ab} = 0, \end{aligned} \tag{5.76}$$

$$\begin{aligned} DF\big|_{\tilde{K}_{a}} = {} & DF^{a}{}_{(1,3)} - 6e_b \wedge F^{ba}{}_{(0,4)} + f_b \wedge F^{ba}{}_{(2,2)} + \tfrac{3}{2}f^b \wedge F^c{}_{(2,2)}\epsilon_{bc}{}^a + \\ & + 2f^a \wedge F_{(2,2)} + 6F_b{}_{(2,0)} \wedge f^{ba} - F_b{}_{(0,2)} \wedge \tilde{\omega}^{ba} - \tfrac{3}{2}F^b{}_{(0,2)} \wedge \tilde{\omega}^c\epsilon_{bc}{}^a - \\ & - 2F^a{}_{(0,2)} \wedge \tilde{b} - F^b{}_{(1,1)} \wedge \tilde{f}^c\epsilon_{bc}{}^a + F_{(1,1)} \wedge \tilde{f}^a = 0, \end{aligned} \tag{5.77}$$

$$\begin{aligned} DF\big|_{K_{ab}} = {} & DF^{ab}{}_{(0,4)} + f^c \wedge F^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} + (f^{(a} \wedge F^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) - \\ & - F^c{}_{(0,2)} \wedge \tilde{f}^{d(a}\epsilon_{cd},^{b)} - (F^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) - 2F^c{}_{(1,1)} \wedge f^{d(a}\epsilon_{cd}{}^{b)} + \\ & + 2F_{(1,1)} \wedge f^{ab} = 0. \end{aligned} \tag{5.78}$$

However, these equations are very large and hard to solve, which is why the linearized versions of these is used here as well.

$$DF\Big|_{P_{ab}} = dF^{ab}{}_{(4,0)} + e^c \wedge F^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - (F^{(a}{}_{(3,1)} \wedge e^{b)} - \mathrm{Tr}\,[]) = 0, \tag{5.79}$$

$$DF\Big|_{\tilde{P}_{ab}} = dF^{ab}{}_{(3,1)} - 2e^c \wedge F^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge F^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) = 0, \tag{5.80}$$

$$\begin{aligned} DF\Big|_{\tilde{P}_{a}} = &\, dF^{a}{}_{(3,1)} - e_b \wedge F^{ba}{}_{(2,2)} + \tfrac{3}{2}e^b \wedge F^c{}_{(2,2)}\epsilon^a{}_{bc} - 2e^a \wedge F_{(2,2)} - \\ &- \tfrac{3}{2}F^b{}_{(2,0)} \wedge \tilde{\omega}^c\epsilon^a{}_{bc} = 0, \end{aligned} \tag{5.81}$$

$$\begin{aligned} DF\Big|_{\tilde{M}_{ab}} = &\, dF^{ab}{}_{(2,2)} + 3e^c \wedge F^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge F^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) + \\ &+ (F^{(a}{}_{(2,0)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) - (F^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) = 0, \end{aligned} \tag{5.82}$$

$$\begin{aligned} DF\Big|_{\tilde{M}_{a}} = &\, dF^{a}{}_{(2,2)} - 3e_b \wedge F^{ba}{}_{(1,3)} - 3e^b \wedge F^c{}_{(1,3)}\epsilon^a{}_{bc} + \\ &+ 3F^b{}_{(2,0)} \wedge \tilde{f}^c\epsilon^a{}_{bc} - F^b{}_{(1,1)} \wedge \tilde{\omega}^c\epsilon^a{}_{bc} = 0, \end{aligned} \tag{5.83}$$

$$DF\Big|_{\tilde{D}} = dF_{(2,2)} - \tfrac{8}{3}e^a \wedge F_{a(1,3)} + \tfrac{8}{3}F^a{}_{(2,0)} \wedge \tilde{f}_a = 0, \tag{5.84}$$

$$\begin{aligned} DF\Big|_{\tilde{K}_{ab}} = &\, dF^{ab}{}_{(1,3)} - 4e^c \wedge F^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} - (F^{(a}{}_{(0,2)} \wedge \tilde{\omega}^{b)} - \mathrm{Tr}\,[]) - \\ &- 2F^c{}_{(1,1)} \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} = 0, \end{aligned} \tag{5.85}$$

$$\begin{aligned} DF\Big|_{\tilde{K}_{a}} = &\, dF^{a}{}_{(1,3)} - 6e_b \wedge F^{ba}{}_{(0,4)} - \tfrac{3}{2}F^b{}_{(0,2)} \wedge \tilde{\omega}^c\epsilon_{bc}{}^a - \\ &- F^b{}_{(1,1)} \wedge \tilde{f}^c\epsilon_{bc}{}^a + F_{(1,1)} \wedge \tilde{f}^a = 0, \end{aligned} \tag{5.86}$$

$$DF\Big|_{K_{ab}} = dF^{ab}{}_{(0,4)} - (F^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) = 0. \tag{5.87}$$

Note that some terms containing $\tilde{\omega}^a$ or $\tilde{f}^a$ are included. After the gauge choises were made, these both obtained an explicit e^a term which must be included.

5.4.1 The minimal assumption

The minimal assumption in the spin-3 case is analoguous to the assumption in the spin-2 case (section 4.4.1), which means that the $_{(4,0)}$ part of F is set to zero, but the others are kept as they are. This changes the expression in eq. (5.79) to

$$DF\Big|_{P_{ab}} = e^c \wedge F^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - F^{(a}{}_{(3,1)} \wedge e^{b)} + \tfrac{1}{3}\eta^{ab}F^c{}_{(3,1)} \wedge e_c = 0, \tag{5.88}$$

while the others stay the same. The next thing to do is to check that these equations still are identities. This leads to quite a few long calculations that are similar to each other, hence only the first two equations will be detailed, since the others follow using the same techniques.

5.4.1.1 The first equation

Start by writing out eq. (5.88), implementing the anti-symmetry using a Levi-Civita symbol

$$\epsilon^{\mu\nu\sigma} e_\mu{}^c F_{\nu\sigma}{}^{d(a}{}_{(3,1)}\epsilon_{cd}{}^{b)} - \epsilon^{\mu\nu\sigma} F_{\mu\nu}{}^{(a}{}_{(3,1)} e_\sigma{}^{b)} + \tfrac{1}{3}\eta^{ab}\epsilon^{\mu\nu\sigma} F_{\mu\nu\sigma(3,1)}. \tag{5.89}$$

Use the dreibeins to change indices in the Levi-Civita symbols and contract the two Levi-Civita symbols in the first term

$$F^{(a}{}_\nu{}^{b)\nu}{}_{(3,1)} - F_\nu{}^{(ab)\nu}{}_{(3,1)} - \epsilon^{\mu\nu(a} F_{\mu\nu}{}^{b)}{}_{(3,1)} + \tfrac{1}{3}\eta^{ab}\epsilon^{\mu\nu\sigma} F_{\mu\nu\sigma(3,1)}. \tag{5.90}$$

The expressions for $F_\mu{}^{ab}{}_{(3,1)}$ and $F_\mu{}^a{}_{(3,1)}$ are found in eqs. (5.13) to (5.21)

$$F^{(a}{}_\nu{}^{b)\nu}{}_{(3,1)} = \partial^{(a}\tilde{e}_\nu{}^{b)\nu} - \epsilon_d{}^{\nu(a}\tilde{\omega}_\nu{}^{b)d} - \tfrac{1}{2}\eta^{ab}\tilde{\omega}_\nu{}^\nu - \tfrac{1}{6}\tilde{\omega}^{(ab)}, \tag{5.91}$$

$$F_\nu{}^{(ab)\nu}{}_{(3,1)} = \partial_\nu \tilde{e}^{(ab)\nu} - \tfrac{5}{3}\tilde{\omega}^{(ab)}, \tag{5.92}$$

$$F_{\mu\nu}{}^b{}_{(3,1)} = \tfrac{3}{2}\epsilon_{\mu d}{}^b\tilde{\omega}_\nu{}^d - \tilde{\omega}_{\nu\mu}{}^b - 2e_\mu{}^b\tilde{b}_\nu, \tag{5.93}$$

$$F_{\mu\nu\sigma(3,1)} = \tfrac{3}{2}\epsilon_{\mu d\sigma}\tilde{\omega}_\nu{}^d - \tilde{\omega}_{\nu\mu\sigma} - 2g_{\mu\sigma}\tilde{b}_\nu. \tag{5.94}$$

Collecting the terms from eq. (5.90) gives

$$\begin{aligned}\partial^{(a}\tilde{e}_\nu{}^{b)\nu} - \epsilon_d{}^{\nu(a}\tilde{\omega}_\nu{}^{b)d} - \tfrac{1}{2}\eta^{ab}\tilde{\omega}_\nu{}^\nu - \tfrac{1}{6}\tilde{\omega}^{(ab)} - \partial_\nu\tilde{e}^{(ab)\nu} + \tfrac{5}{3}\tilde{\omega}^{(ab)} - \\ - \tfrac{3}{2}\epsilon^{\mu\nu(a}\epsilon_{\mu d}{}^{b)}\tilde{\omega}_\nu{}^d + \epsilon^{\mu\nu(a}\tilde{\omega}_{\nu\mu}{}^{b)} + 2\epsilon^{\mu\nu(a}e_\mu{}^{b)}\tilde{b}_\nu + \tfrac{1}{2}\epsilon^{\mu\nu\sigma}\epsilon_{\mu d\sigma}\tilde{\omega}_\nu{}^d = \\ = \partial^{(a}\tilde{e}_\nu{}^{b)\nu} - \partial_\nu\tilde{e}^{(ab)\nu}.\end{aligned} \tag{5.95}$$

To tell that this is zero, the assumption $F^{ab}{}_{(4,0)} = 0$ must be used. It makes it possible to express $\tilde{e}_\mu{}^{ab}$ in terms of $e_\mu{}^{ab}$, which was found in section 5.3.1, and is

$$\tilde{e}^{cab} = \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\eta^{ab}\partial_\mu e_\nu{}^{cd} - 2\epsilon^{\mu\nu(a}\partial_\mu e_\nu{}^{b)c} + \epsilon^{\mu\nu c}\partial_\mu e_\nu{}^{ab}. \tag{5.96}$$

Hence, the expressions for $\tilde{e}_\nu{}^{a\nu}$ and $\tilde{e}^{(ab)\nu}$ are

$$\tilde{e}^{(ab)\nu} = \tfrac{2}{3}\epsilon^{\mu\sigma}{}_d e^{\nu(a}\partial_\mu e_\sigma{}^{b)d} - \epsilon^{\mu\sigma\nu}\partial_\mu e_\sigma{}^{ab}, \tag{5.97}$$

$$\tilde{e}_\nu{}^{a\nu} = \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\partial_\mu e_\nu{}^{ad}, \tag{5.98}$$

which when entered into the remaining terms of eq. (5.95) conveniently becomes

$$\partial_\nu\tilde{e}^{(ab)\nu} = \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\partial_\mu\partial^{(a}e_\nu{}^{b)d}, \tag{5.99}$$

$$\partial^{(a}\tilde{e}_\nu{}^{b)\nu} = \tfrac{2}{3}\epsilon^{\mu\nu}{}_d\partial_\mu\partial^{(a}e_\nu{}^{b)d}. \tag{5.100}$$

Hence, eq. (5.88) is an identity.

5.4.1.2 The second equation

The second equation of the spin-3 Bianchi identity is eq. (5.80)

$$\begin{aligned} DF\Big|_{\tilde{P}_{ab}} &= dF^{ab}{}_{(3,1)} - 2e^c \wedge F^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} - \\ &- \left(e^{(a} \wedge F^{b)}{}_{(2,2)} - \tfrac{1}{3}\eta^{ab} e^c \wedge F_{c(2,2)}\right) = 0. \end{aligned} \tag{5.101}$$

The main difference betwen this one and eq. (5.88) is that this one has the term with a derivative acting on some projection of F left. This derivative makes the calculations somewhat shorter. Written out it is

$$\begin{aligned} dF^{ab}{}_{(3,1)} &= dd\tilde{e}^{ab}{}_{(3,1)} - 2d\left(e^c \wedge \tilde{\omega}^{d(a}\epsilon^{b)}{}_{cd}\right) - d\big(e^{(a} \wedge \tilde{\omega}^{b)} - \tfrac{1}{3}\eta^{ab} e^c \wedge F_{c(2,2)}\big) = \\ &= -2de^c \wedge \tilde{\omega}^{d(a}\epsilon^{b)}{}_{cd} + 2e^c \wedge d\tilde{\omega}^{d(a}\epsilon^{b)}{}_{cd} - (de^{(a} \wedge \tilde{\omega}^{b)} - \tfrac{1}{3}\eta^{ab} de^c \wedge \tilde{\omega}_c) + \\ &\quad + (e^{(a} \wedge d\tilde{\omega}^{b)} - \tfrac{1}{3}\eta^{ab} e^c \wedge d\tilde{\omega}_c) = \\ &= 2e^c \wedge d\tilde{\omega}^{d(a}\epsilon^{b)}{}_{cd} + (e^{(a} \wedge d\tilde{\omega}^{b)} - \tfrac{1}{3}\eta^{ab} e^c \wedge d\tilde{\omega}_c). \end{aligned} \tag{5.102}$$

Where the assumption $F^a{}_{(2,0)} = de^a = 0$ has to be remembered. Now look at the other terms. They all contain a derivative in their F terms which match these derivatives up to the sign and hence, they all cancel. The terms not containing derivatives in $F^{ab}{}_{(3,1)}$ and $F^a{}_{(3,1)}$ are

$$F^{ab}{}_{(2,2)} \doteq 3e^c \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge \tilde{f}^{b)} - \tfrac{1}{3}\eta^{ab} e^c \wedge \tilde{f}_c), \tag{5.103}$$

$$F^a{}_{(2,2)} \doteq -3e_b \wedge \tilde{f}^{ab} - 3e^c \wedge \tilde{f}^d \epsilon_{cd}{}^a. \tag{5.104}$$

When these are entered into eq. (5.101), all terms consist of at least two dreibeins with wedges between them. Since the wedges anti-symmetrize and the dreibeins are interchangeable, eq. (5.101) equals zero.

5.4.2 Extending the minimal assumption

The most natural continuation of the minimal assumption is to continue according to the graded order and set the next order of the F projections to zero. The extension is hence given by $F^{ab}{}_{(3,1)} = F^{a}{}_{(3,1)} = 0$, this causes trouble in the third equation eq. (5.79), while the others are solved in using the same techiques that were used in the previous section.

$$DF\Big|_{P_{ab}} = 0 = 0, \tag{5.105}$$

$$DF\Big|_{\tilde{P}_{ab}} = -2e^c \wedge F^{d(a}{}_{(2,2)}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge F^{b)}{}_{(2,2)} - \mathrm{Tr}\,[]) = 0, \tag{5.106}$$

$$\begin{aligned} DF\Big|_{\tilde{P}_{a}} &= -e_b \wedge F^{ba}{}_{(2,2)} + \tfrac{3}{2}e^b \wedge F^{c}{}_{(2,2)}\epsilon^a{}_{bc} - 2e^a \wedge F_{(2,2)} - \\ &- \tfrac{3}{2}F^b{}_{(2,0)} \wedge \tilde{\omega}^c \epsilon^a{}_{bc} = 0, \end{aligned} \tag{5.107}$$

$$\begin{aligned} DF\Big|_{\tilde{M}_{ab}} &= DF^{ab}{}_{(2,2)} + 3e^c \wedge F^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - (e^{(a} \wedge F^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]) + \\ &+ (F^{(a}{}_{(2,0)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) - (F^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) = 0, \end{aligned} \tag{5.108}$$

$$\begin{aligned} DF\Big|_{\tilde{M}_{a}} &= DF^{a}{}_{(2,2)} - 3e_b \wedge F^{ba}{}_{(1,3)} - 3e^b \wedge F^{c}{}_{(1,3)}\epsilon^a{}_{bc} + \\ &+ 3F^b{}_{(2,0)} \wedge \tilde{f}^c \epsilon^a{}_{bc} - F^b{}_{(1,1)} \wedge \tilde{\omega}^c \epsilon^a{}_{bc} = 0, \end{aligned} \tag{5.109}$$

$$DF\Big|_{\tilde{D}} = DF_{(2,2)} - \tfrac{8}{3}e^a \wedge F_{a(1,3)} + \tfrac{8}{3}F^a{}_{(2,0)} \wedge \tilde{f}_a = 0, \tag{5.110}$$

$$\begin{aligned} DF\Big|_{\tilde{K}_{ab}} &= DF^{ab}{}_{(1,3)} - 4e^c \wedge F^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} - (F^{(a}{}_{(0,2)} \wedge \tilde{\omega}^{b)} - \mathrm{Tr}\,[]) - \\ &- 2F^c{}_{(1,1)} \wedge \tilde{f}^{d(a}\epsilon_{cd}{}^{b)} = 0, \end{aligned} \tag{5.111}$$

$$\begin{aligned} DF\Big|_{\tilde{K}_{a}} &= DF^{a}{}_{(1,3)} - 6e_b \wedge F^{ba}{}_{(0,4)} - \tfrac{3}{2}F^b{}_{(0,2)} \wedge \tilde{\omega}^c \epsilon_{bc}{}^a - \\ &- F^b{}_{(1,1)} \wedge \tilde{f}^c \epsilon_{bc}{}^a + F_{(1,1)} \wedge \tilde{f}^a = 0, \end{aligned} \tag{5.112}$$

$$DF\Big|_{K_{ab}} = DF^{ab}{}_{(0,4)} - (F^{(a}{}_{(0,2)} \wedge \tilde{f}^{b)} - \mathrm{Tr}\,[]) = 0. \tag{5.113}$$

5.4.2.1 The third equation

Before starting on the third equation, there is one important trick that will be used to solve it. It uses the fact that an anti-symmetrization over four indices, where the value of the expression takes on three values (up to a sign), is zero. Practically, this is used as

$$\epsilon^{abc}U^d - \epsilon^{bcd}U^a + \epsilon^{cda}U^b - \epsilon^{dab}U^c = 0, \tag{5.114}$$

which will be used to replace one term with three as

$$\epsilon^{abc}U^d = \epsilon^{bcd}U^a - \epsilon^{cda}U^b + \epsilon^{dab}U^c. \tag{5.115}$$

Now, when the tools are ready, start with the third equation, remove all terms containing two dreibeins and simplify the obtained expression

$$\epsilon_b{}^{\mu\nu}\partial_\mu\tilde{\omega}_\nu{}^{ab} + \tfrac{1}{2}\epsilon^{a\mu\nu}\partial_\mu\tilde{\omega}_{\sigma\nu}{}^{\sigma} \tag{5.116}$$

Here, use the trick just shown to cycle the indices in the first term to obtain

$$-\epsilon^{ab\nu}\partial_\mu\tilde{\omega}_{\nu b}{}^{\mu} - \tfrac{1}{2}\epsilon^{a\mu\nu}\partial_\mu\tilde{\omega}_{\sigma\nu}{}^{\sigma}. \tag{5.117}$$

As this expression stand, it does not seem to be zero, what must be done is to use eq. (5.45) to express the equation in terms of $\tilde{e}_\mu{}^{ab}$, which after simplification is

$$\partial_\mu\partial_\nu\tilde{e}^{(\mu a)\nu} - \partial_\mu\partial^{(\mu}\tilde{e}_\nu{}^{a)\nu} = -\partial_b\left(\partial^{(a}\tilde{e}_\nu{}^{b)\nu} - \partial_\nu\tilde{e}^{(ab)\nu}\right) \tag{5.118}$$

which is the same as eq. (5.95) and it was shown to be zero.

5.4.3 The parabolic case

An approach that should work is to set the entire parabolic part of the algebra to zero of reasons explained in appendix G. Doing so reduces eqs. (5.79) to (5.87) considerably, but also makes them harder to sove

$$DF\Big|_{P_{ab}} = 0, \tag{5.119}$$

$$DF\Big|_{\tilde{P}_{ab}} = 0, \tag{5.120}$$

$$DF\Big|_{\tilde{P}_{a}} = 0, \tag{5.121}$$

$$DF\Big|_{\tilde{M}_{ab}} = 3e^c \wedge F^{d(a}{}_{(1,3)}\epsilon_{cd}{}^{b)} - \left(e^{(a} \wedge F^{b)}{}_{(1,3)} - \mathrm{Tr}\,[]\right) = 0, \tag{5.122}$$

$$DF\Big|_{\tilde{M}_{a}} = -3e_b \wedge F^{ba}{}_{(1,3)} - 3e^b \wedge F^c{}_{(1,3)}\epsilon^a{}_{bc} = 0, \tag{5.123}$$

$$DF\Big|_{\tilde{D}} = -\tfrac{8}{3}e^a \wedge F_{a(1,3)} = 0, \tag{5.124}$$

$$DF\Big|_{\tilde{K}_{ab}} = DF^{ab}{}_{(1,3)} - 4e^c \wedge F^{d(a}{}_{(0,4)}\epsilon_{cd}{}^{b)} = 0, \tag{5.125}$$

$$DF\Big|_{\tilde{K}_{a}} = DF^a{}_{(1,3)} - 6e_b \wedge F^{ba}{}_{(0,4)} = 0, \tag{5.126}$$

$$DF\Big|_{K_{ab}} = DF^{ab}{}_{(0,4)} = 0. \tag{5.127}$$

The first three of these equations does not need solving and the last three is the same as before. The three in the middle, however, is somewhat problematic. Especially the $\tilde{M}^a$ equation, which needs some extra trickery to solve.
Note that the $\tilde{f}$-terms are removed. They are gauged to have a part proportional to $e_\mu{}^a$, but the coefficient $\hat{f}$ in this proportionality were found to be zero in eq. (5.57).

5.4.3.1 The fourth equation

The fourth equation is equivalent to

$$\tfrac{3}{2}\epsilon^{c\mu\nu}\partial_\mu\tilde{f}_\nu{}^{da}\epsilon_{cd}{}^{b}+\tfrac{3}{2}\epsilon^{c\mu\nu}\partial_\mu\tilde{f}_\nu{}^{db}\epsilon_{cd}{}^{a}-\tfrac{1}{2}\epsilon^{a\mu\nu}\partial_\mu\tilde{f}_\nu{}^{b}-\tfrac{1}{2}\epsilon^{b\mu\nu}\partial_\mu\tilde{f}_\nu{}^{a}+\tfrac{1}{3}\eta^{ab}\epsilon^{c\mu\nu}\partial_\mu\tilde{f}_{\nu c}, \tag{5.128}$$

which after some Levi-Civita trickery and massageing becomes

$$3\partial^{(a}\tilde{f}_\nu{}^{b)\nu}-3\partial_\mu\tilde{f}^{(ab)\mu}-\partial^{(a}\hat{f}^{b)}+\tfrac{1}{3}\eta^{ab}\partial_\mu\hat{f}^\mu. \tag{5.129}$$

The last term is zero when using eq. (5.62) since

$$\partial_\nu\hat{f}^\nu=\tfrac{3}{16}\epsilon^{\mu\sigma\nu}\partial_\mu\partial_\nu\tilde{b}_\sigma=0 \tag{5.130}$$

and the first can be changed using eq. (5.58) giving

$$-7\partial^{(a}\hat{f}^{b)}-3\partial_\mu\tilde{f}^{(ab)\mu}, \tag{5.131}$$

but (using eq. (5.55))

$$\begin{aligned}3\partial_\mu\tilde{f}^{(ab)\mu}&=3\partial^{(a}\tilde{f}_\nu{}^{b)\nu}-\epsilon^{\mu\nu\sigma}\partial_\sigma\partial_\mu\tilde{\omega}_\nu{}^{ab}-\partial^{(a}\hat{f}^{b)}-\tfrac{1}{3}\eta^{ab}\partial_\sigma\hat{f}^\sigma=\\&=-7\partial^{(a}\hat{f}^{b)}\end{aligned} \tag{5.132}$$

and eq. (5.131) is zero.

5.4.3.2 The fifth equation

The fifth equations is solved in a similar way to the third equaiton in the previous case in section 5.4.2.1. Start with the equation

$$3\epsilon_b{}^{\mu\nu}\partial_\mu\tilde{f}_\nu{}^{ab}+3\epsilon^{b\mu\nu}\partial_\mu\tilde{f}_\nu{}^{c}\epsilon_{bc}{}^{a}=0. \tag{5.133}$$

Again, use the trick of index cycling on the first term, one of the obtained terms are zero, also simplify the last term and use eq. (5.58) to obtain

$$\tfrac{3}{2}\epsilon^{a\nu\mu}\partial_\mu\tilde{f}_{\sigma\nu}{}^{\sigma}-3\epsilon^{\nu a}{}_b\partial_\mu\tilde{f}_\nu{}^{\mu b}. \tag{5.134}$$

Now use eq. (5.55) to express the equation in terms of $\tilde{\omega}_\mu{}^{ab}$

$$\partial^2\tilde{\omega}_\nu{}^{a\nu}+\partial^a\partial_\mu\tilde{\omega}_\nu{}^{\mu\nu}-\partial^\mu\partial^\nu\tilde{\omega}^a{}_{\mu\nu}-\partial^\mu\partial^\nu\tilde{\omega}_{\mu\nu}{}^{a}, \tag{5.135}$$

which will be zero when one more step is taken using eq. (5.45) to express it in terms of $\tilde{e}_\mu{}^{ab}$.

5.4.3.3 The sixth equation

The sixth equation written out with a Levi-Civita symbol is

$$\begin{aligned}\epsilon^{a\nu\sigma}F_{\mu\nu a(1,3)}&=\epsilon^{a\nu\sigma}\left(\partial_\nu\tilde{f}_\sigma{}^{a}-6e_\nu{}^{b}f_{\sigma ab}\right)=\epsilon^{a\nu\sigma}\partial_\nu\tilde{f}_\sigma{}^{a}-6\epsilon^{ab\sigma}f_{\sigma ab}=\\&=\epsilon^{a\nu\sigma}\partial_\nu\tilde{f}_\sigma{}^{a}.\end{aligned} \tag{5.136}$$

Using the gauge choise for $\tilde{f}$

$$\epsilon^{a\nu\sigma}\partial_\nu\tilde{f}_\sigma{}^{a}=\epsilon^{a\nu\sigma}\epsilon_\sigma{}^{ab}\partial_\nu\hat{f}_b=-2e^{\nu b}\partial_\nu\hat{f}_b=\partial_\nu\hat{f}^\nu \tag{5.137}$$

and is zero using eq. (5.130). Hence the expression is an identity with the same conclusion in the spin-3 case as in the spin-2 case (section 4.4.2.3): There is no violation of the Bianchi Identity even if the theory is coupled to matter.

6

Outlook

Continuing on this work, there are some natural extensions. The two closest ones being to solve the non-linear versions of the equations and to look at even higher spins than three [25]. Another interesting aspect would be to explore the quantized versions [26], where the Poisson brackets are replaced with commutators, giving rise to ordering problems in the algebra.

Interesting is also to find out what these theories actually mean by adding higher-spin terms to the Lagrangian and se how/if this differs from the spin-2 case [24]. Here there would also be a goal to provide a closed expression for all the higher-spin terms included in the Lagrangian.

Perhaps quite a bit into the future, higher-spin theories can be placed more accurately in the landscape of theories of physics. It might be the key to understanding several advanced theories such as string theory, the AdS/CFT correspondance and M-theory, who knows? It will at least be very interesting to find out where this is heading!

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A

Conventions

A.1 The Levi-Civita symbol

The Levi-Civita symbol ϵ^{abc} is used extensively throughout this thesis and is a totally anti-symmetric tensor density with weight 0. When an even number of Levi-Civita symbols are found in a term, it is always possible to rewrite it as a number of Kroenecker-delta functions δ^a_b. How it looks depends on if there are any contractions between the different symbols.

$$\epsilon^{abc}\epsilon_{abd} = -2\delta^c_d, \tag{A.1}$$

$$\epsilon^{abc}\epsilon_{ade} = -2\delta^{[bc]}_{de} = \delta^c_d\delta^b_e - \delta^b_d\delta^c_e, \tag{A.2}$$

$$\epsilon^{abc}\epsilon_{def} = -6\delta^{[abc]}_{def} = \delta^a_d\delta^c_e\delta^b_f + \delta^c_d\delta^b_e\delta^a_f + \delta^b_d\delta^c_e\delta^c_f - \delta^a_d\delta^b_e\delta^c_f - \delta^b_d\delta^c_e\delta^a_f - \delta^c_d\delta^a_e\delta^b_f. \tag{A.3}$$

Note that in three dimensions, the Levi-Civita symbol can be used to rewrite two anti-symmetric indices as one index:

$$R_{\mu\nu}{}^a = -\tfrac{1}{2}R_{\mu\nu}{}^{bc}\epsilon_{bc}{}^a \qquad \Rightarrow \qquad R_{\mu\nu}{}^{ab} = \epsilon^{ab}{}_c R_{\mu\nu}{}^c. \tag{A.4}$$

For the Riemann tensor, there is also a possibility to use the Levi-Civita symbol to obtain the Ricci tensor

$$R_{\mu\nu} = -R_{\mu\sigma\lambda}\epsilon^{\sigma\lambda}{}_\nu. \tag{A.5}$$

A.2 Gamma matrices in 2+1 dimensions

The three gamma matrices γ^0, γ^1, and γ^2 constitutes a basis for the real traceless 2×2 matrices realized in terms of the Pauli-sigma matrices as

$$\begin{aligned}(\gamma^0)_\alpha{}^\beta &= (\epsilon)_\alpha{}^\beta = (i\sigma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\gamma^1)_\alpha{}^\beta = (\sigma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (\gamma^2)_\alpha{}^\beta &= (\sigma^3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned} \tag{A.6}$$

The full basis for real 2×2 matrices are obtained by including $\gamma^0\gamma^1\gamma^2 = \mathbb{1}_{2\times 2}$. The spinor indices $\alpha, \beta, \ldots$ are raised or lowered using ϵ. Note that with both indices down, all γ's are symmetric

$$(\gamma^0)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\gamma^1)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma^2)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.7}$$

To make caluculations with the gamma matrices, the Fierz identities are very useful

$$(\gamma^a)_{(\alpha\beta}(\gamma_a)_{\gamma)}{}^{\delta} = 0, \tag{A.8}$$

$$(\gamma^{[a})_{\alpha\beta}(\gamma^{b]})^{\gamma\delta} = \epsilon^{ab}{}_c(\gamma^c)_{(\alpha}{}^{(\gamma}\delta^{\delta)}_{\beta)}, \tag{A.9}$$

$$(\gamma^{[a})_{(\alpha\beta}(\gamma^{b]})_{\gamma)}{}^{\delta} = -\tfrac{1}{2}\epsilon^{ab}{}_c(\gamma^c)_{(\alpha\beta}\delta^{\delta}_{\gamma)}, \tag{A.10}$$

$$(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} = 2(\gamma^a)_{(\alpha}{}^{(\gamma}(\gamma_a)_{\beta)}{}^{\delta)}. \tag{A.11}$$

These are obtained by projection onto the basis elements. For instance, the first of these (eq. (A.8)) is found by projecting $\alpha\beta$ onto the symmetric basis $(\gamma^b)^{\alpha\beta}$, the anti-symmetric part of the basis can be ignored, since $\alpha\beta$ are symmetrized. This leads to

$$\begin{aligned}(\gamma^b)^{\alpha\beta}(\gamma^a)_{(\alpha\beta}(\gamma_a)_{\gamma)}{}^{\delta} &= (\gamma^b)^{\alpha\beta}(\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma}{}^{\delta} + 2(\gamma^b)^{\alpha\beta}(\gamma^a)_{\alpha\gamma}(\gamma_a)_{\beta}{}^{\delta} = \\ &= 2\eta^{ab}(\gamma_a)_{\gamma}{}^{\delta} + 2(\gamma^a\gamma^b\gamma_a)_{\gamma}{}^{\delta} = \\ &= 2(\gamma^b)_{\gamma}{}^{\delta} - 2(\gamma^b)_{\gamma}{}^{\delta} = 0.\end{aligned} \tag{A.12}$$

The second to last equality follows from writing out the expressions for $(\gamma^a\gamma^b\gamma_a)_{\gamma}{}^{\delta}$ and finding that these are $-(\gamma^b)_{\gamma}{}^{\delta}$. Using similar methods, the rest of the Fierz identities are found.

A.3 Groups vs. group algebra

There are many discussions in this thesis regarding different groups and their algebra. When a group is discussed in general, such as a gauge symmetry, the group is written with upper case letters, e.g. $SO(2,3)$. If the group is written with lower case letters, e.g. $so(2,3)$, it is the (Lie) algebra of the group that it refers to. In some arguments it is very important to separate the group from its algebra, since the arguments might be invalid if the group is considered instead of its algebra, or the other way around.

A.4 Equality

In some places in this thesis the symbol $\doteq$ is used instead of $=$. This is used when the equality only relates parts of the right hand side to the left hand side, or where the expressions contains the same information, but are not strictly equal. What the difference is in a specific case should be clear from the circumstances it is used.

B

Introduction to forms

B.1 Theory

The language of forms is sometimes used in physics, especially when dealing with differential equations. In short, (differential) forms are a convenient way of combining a quantity with its differential. As an example, consider some field A_μ and its differential dx^μ, the contraction of these can be expressed as

$$A = A_\mu dx^\mu, \tag{B.1}$$

where A is a so-called 1-form. It is a 1-form since it contains one differential. The integral over A_μ now takes the quite short appearance

$$\int A. \tag{B.2}$$

This is generalized to p-forms, which are forms containing p differentials. A higher dimensional p-form B consists of $B_{\mu_1 \dots \mu_p}$ and its differentials $dx^{\mu_1} \dots dx^{\mu_p}$ is in forms expressed as

$$B = \frac{1}{p!} B_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \tag{B.3}$$

where $\wedge$ denotes the wedge, or exterior, product defined below. Yet again, a very simple expression is obtained for the integral over B

$$\int B. \tag{B.4}$$

The wedge product introduced above is an anti-symmetrized product and hence exhibit the following

$$dx^{\mu_1} \wedge dx^{\mu_2} = -dx^{\mu_2} \wedge dx^{\mu_1} \tag{B.5}$$

for any two differentials in the product. It follows that if any p- and q-forms are interchanged, the sign of the expression changes as

$$A \wedge B = (-1)^{pq} B \wedge A, \tag{B.6}$$

for A a p-form and B a q-form. Another important object to introduce when considering forms is the exterior derivative, d, which is defined by

$$dA = \partial_\mu A \wedge dx^\mu \tag{B.7}$$

and turns any p-form into a $p+1$-form. It has the very useful property that $d^2 = 0$, since the partial derivatives acting on an expression commute and the wedge product is anti-symmetric, hence d^2 acting on any object must be zero. The calculation is quite short

$$d^2 A = \partial_\nu \partial_\mu A \wedge dx^\mu \wedge dx^\nu = -\partial_\nu \partial_\mu A \wedge dx^\nu \wedge dx^\mu = -\partial_\mu \partial_\nu A \wedge dx^\nu \wedge dx^\mu = 0. \quad \text{(B.8)}$$

The last equality follows from the fact that the first and last written-out expression is the same up to the sign (notice that the indices are relabeled) and must hence be zero.

Another important object in the language of forms is the *dual* of a form F, denoted $*F$. This will not be used extensively, but is nonetheless an imortant part of the form language. It takes a p-form in a d-dimensional space and creates a $(d-p)$-form and does this in a certain manner. In three dimensions the following is in some sense true

$$*(A \wedge B) = A \times B. \quad \text{(B.9)}$$

In general dimensions, it is not as easy, but the dual can be written in the following way for a k-form in n-dimensional space

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_{k+1}} \wedge \cdots \wedge e_{i_n}, \quad \text{(B.10)}$$

where $i_1, \ldots, i_n$ is an even permutation of $1, \ldots, n$. Uneven permutations follows from the definition of the wedge product.

B.2 Examples

B.2.1 The Bianchi identity of F

A good example on forms is the Bianchi identity of the electromagnetic tensor F, which states $DF = 0$. First, the form equivalence of $F^{\mu\nu}$ is obtained by acting, on some state Φ, twice with the covariant derivative $D = d + A$,

$$DD\Phi = (d+A)(d+A)\Phi = (d^2 + dA + Ad + A^2)\Phi = (dA + A^2)\Phi + Ad\Phi. \quad \text{(B.11)}$$

The terms that "stands on their own" in front of Φ are taken to be F, while the last term with a derivative acting on Φ is dropped. F is now expressed as

$$F = dA + A^2. \quad \text{(B.12)}$$

The Bianchi identity for this F is most easily found by first considering dF,

$$\begin{aligned} dF = d(dA + A^2) &= d^2 A + dA^2 = dAA - AdA = \\ &= dAA + A^3 - AdA - A^3 = (dA + A^2)A - A(dA + A^2) = \\ &= FA - AF = [F, A]. \end{aligned} \quad \text{(B.13)}$$

Continuing, care must be taken when writing out the expression for DF since

$$DF = (d + A)F \neq dF + AF. \quad \text{(B.14)}$$

What must be taken into account is that both A and F is matrix-valued and hence

$$DF = (d + A)F = dF + [A, F], \tag{B.15}$$

which immediately gives the Bianchi identity

$$DF = [F, A] + [A, F] = 0. \tag{B.16}$$

B.2.2 Maxwell's equations written in forms

Maxwell's equations in tensor notation is very neatly written as

$$\partial_{[\mu} F_{\nu\lambda]} = 0, \tag{B.17}$$

$$\partial_\mu F^{\nu\mu} = J^\nu, \tag{B.18}$$

but can be written even shorter using forms. Identifying the elements in F as two-forms, F can be written as

$$F = -E_x dt\wedge dx - E_y dt\wedge dy - E_z dt\wedge dz + B_x dy\wedge dz + B_y dz\wedge dx + B_z dx\wedge dy. \tag{B.19}$$

Now, since the first of Maxwell's equations is a derivative acting on F with anti-symmetric indices, the expression dF is a good place to start. It is

$$\begin{aligned} dF &= \partial_t F dt + \partial_x F dx + \partial_y F dy + \partial_z F dz = \\ &= -\partial_y E_x dt \wedge dx \wedge dy - \partial_z E_y dt \wedge dy \wedge dz + \partial_x E_z dt \wedge dx \wedge dz - \\ &\quad - \partial_z E_x dt \wedge dx \wedge dz + \partial_x E_y dt \wedge dx \wedge dy + \partial_y E_z dt \wedge dy \wedge dz + \\ &\quad + \partial_t B_x dt \wedge dy \wedge dz + \partial_t B_z dt \wedge dx \wedge dy + \partial_t B_y dt \wedge dz \wedge dx + \\ &\quad + \partial_x B_x dx \wedge dy \wedge dz + \partial_y B_y dy \wedge dz \wedge dx + \partial_z B_z dz \wedge dx \wedge dy = \\ &= \left(\partial_x E_y - \partial_y E_x + \partial_t B_z\right) dt \wedge dx \wedge dy + \\ &\quad + \left(\partial_x E_z - \partial_z E_x + \partial_t B_y\right) dt \wedge dx \wedge dz + \\ &\quad + \left(\partial_y E_z - \partial_z E_y + \partial_t B_x\right) dt \wedge dy \wedge dz + \\ &\quad + \left(\partial_x B_x + \partial_y B_y + \partial_z B_z\right) dx \wedge dy \wedge dz = \\ &= \left((\nabla \times \vec{E})_z + \tfrac{\partial B_z}{\partial t}\right) dt \wedge dx \wedge dy + \left((\nabla \times \vec{E})_y + \tfrac{\partial B_y}{\partial t}\right) dt \wedge dx \wedge dz + \\ &\quad + \left((\nabla \times \vec{E})_x + \tfrac{\partial B_x}{\partial t}\right) dt \wedge dy \wedge dz + \left(\nabla \cdot \vec{B}\right) dx \wedge dy \wedge dz, \end{aligned} \tag{B.20}$$

but it is known from Maxwell's equations that this equals zero, since putting $dF = 0$ and writing the terms out gives

$$\nabla \cdot \vec{B} = 0, \tag{B.21}$$

$$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t}. \tag{B.22}$$

These two equations are two of the ordinary Maxwell's equations and what is meant by the tensorial expression $\partial_{[\mu} F_{\nu\lambda]} = 0$.
For the rest of Maxwell's equations, first condider the expression for $*F$, which is

$$*F = E_x dy\wedge dz + E_y dz\wedge dx + E_z dx\wedge dy + B_x dt\wedge dx + B_y dt\wedge dy + B_z dt\wedge dz. \tag{B.23}$$

Now use this $*F$ instead of F and hit it with an exterior derivative

$$\begin{aligned} d{*}F &= \partial_t {*}F dt + \partial_x {*}F dx + \partial_y {*}F dy + \partial_z {*}F dz = \\ &= \left(-(\nabla \times \vec{B})_z + \tfrac{\partial E_z}{\partial t}\right) dt \wedge dx \wedge dy + \left(-(\nabla \times \vec{B})_y + \tfrac{\partial E_y}{\partial t}\right) dt \wedge dx \wedge dz + \\ &\quad + \left(-(\nabla \times \vec{B})_x + \tfrac{\partial E_x}{\partial t}\right) dt \wedge dy \wedge dz + \left(\nabla \cdot \vec{E}\right) dx \wedge dy \wedge dz, \end{aligned} \tag{B.24}$$

which if the following expression is introduced

$$J = \rho dx \wedge dy \wedge dz - J_x dt \wedge dy \wedge dz - J_y dt \wedge dz \wedge dx - J_z dt \wedge dy \wedge dz, \tag{B.25}$$

gives the last of Maxwell's equations through $d{*}F = J$. To put it all together, Maxwell's equations in form language can be written as

$$dF = 0, \tag{B.26}$$

$$d{*}F = J. \tag{B.27}$$

The first of these expression is recognized as the Bianchi Identity when generalized to curved space.

C

Zero-Torsion

The zero torsion condition $De^a = 0$ can be used to solve for ω in terms of $e_\mu{}^a$. Here it is made in general dimensions, where the zero torsion condition can be written as

$$\partial_{[\mu} e_{\nu]}{}^a + \omega_{[\mu}{}^a{}_{|b|} e_{\nu]}{}^b = 0. \tag{C.1}$$

Lower all indices and change $\mu, \nu \to b, c$

$$e^\mu{}_{[b} e^\nu{}_{c]} \partial_\mu e_{\nu a} + \omega_{[b|ad|} \delta^d_{c]} = e^\mu{}_{[b} e^\nu{}_{c]} \partial_\mu e_{\nu a} + \omega_{[b|a|c]} = e^\mu{}_{[b} e^\nu{}_{c]} \partial_\mu e_{\nu a} - \omega_{[bc]a} = 0, \tag{C.2}$$

hence, with some index relabeling

$$\omega_{[ab]c} = e^\mu{}_{[a} e^\nu{}_{b]} \partial_\mu e_{\nu c}. \tag{C.3}$$

Now, using another neat trick and rewrite

$$\omega_{[ab]c} + \omega_{[ca]b} - \omega_{[bc]a} = \omega_{abc}, \tag{C.4}$$

which is easily checked by remembering that ω_{abc} is anti-symmetric under $b \leftrightarrow c$ exchange. Using this expression for ω_{abc} results in the three terms

$$\omega_{abc} = e^\sigma{}_{[a} e^\nu{}_{b]} \partial_\sigma e_{\nu c} + e^\sigma{}_{[c} e^\nu{}_{a]} \partial_\sigma e_{\nu b} - e^\sigma{}_{[b} e^\nu{}_{c]} \partial_\sigma e_{\nu a}. \tag{C.5}$$

Raise b, c and convert a to μ to end up with the final expression

$$\begin{aligned} \omega_\mu{}^{bc} &= \delta^\sigma_\mu e^{\nu b} \partial_{[\sigma} e_{\nu]}{}^c + e^{\nu c} \delta^\sigma_\mu \partial_{[\nu} e_{\sigma]}{}^b - e^{\sigma[b} e^{c]\nu} e_{\mu a} \partial_\sigma e_\nu{}^a = \\ &= 2 e^{\nu[b} \partial_{[\mu} e_{\nu]}{}^{c]} - e^{\sigma[b} e^{c]\nu} e_{\mu a} \partial_\sigma e_\nu{}^a. \end{aligned} \tag{C.6}$$

D

Conformal invariance of Lagrangian terms

A conformal transformation is a transformation leaving the metric invariant up to a constant, i.e.

$$g_{\mu\nu} \to e^{\Omega(x)} g_{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} \to e^{-\Omega(x)} g^{\mu\nu}. \tag{D.1}$$

The term $\Omega(x)$ is some scalar-valued function of the position in spacetime. Below, some terms are shown and how they are seen to be invariant under such a transformation.

D.1 Invariance of anti-symmetric scalar

In four dimensions, the following scalar can be created

$$\epsilon^{\mu\nu\sigma\lambda} F_{\mu\nu} F_{\sigma\lambda}. \tag{D.2}$$

Its invariance is trivial since ϵ transforms as

$$\epsilon^{\mu\nu\sigma\lambda} \to \epsilon^{\mu\nu\sigma\lambda}. \tag{D.3}$$

Since there is no need for $F_{\mu\nu}$ to transform it can be left as it is, making invariance trivial.

D.2 Invariance of $F_{\mu\nu}F^{\mu\nu}$

To show conformal invariance of $F_{\mu\nu}F^{\mu\nu}$, start by writing out how the different parts of the expression transforms. First, notice how the scale factor $\sqrt{-g}$ transforms

$$\sqrt{-g} \to e^{\frac{D}{2}\Omega(x)} \sqrt{-g}. \tag{D.4}$$

The entire expression now transforms as

$$\sqrt{-g} F_{\mu\nu} F^{\mu\nu} \to e^{\frac{D-4}{2}\Omega(x)} \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \tag{D.5}$$

and is trivially invariant for $D = 4$ (but not for other dimensions).

D.3 Invariance of the kinetic term

Start with a massless kinetic term

$$\mathcal{L} = \sqrt{-g}\partial_\mu\phi\partial^\mu\phi \tag{D.6}$$

and continue with the conformal transformations

$$g_{\mu\nu} \to e^{2\Omega(x)}g_{\mu\nu}, \quad \sqrt{-g} \to e^{D\Omega(x)}\sqrt{-g}. \tag{D.7}$$

This makes $\mathcal{L}$ transform as

$$\mathcal{L} = \sqrt{-g}\partial_\mu\phi\partial^\mu\phi \to e^{(D-2)\Omega(x)}\sqrt{-g}\partial_\mu\phi'\partial^\mu\phi'. \tag{D.8}$$

To cancel the $e^{(d-2)\Omega}$-factor ϕ must transform as

$$\phi \to \phi' e^{(1-\frac{d}{2})\Omega}\phi = e^{\frac{2-d}{2}\Omega}\phi. \tag{D.9}$$

Performing the transformation, partial integration and dropping constant terms leaves

$$\sqrt{-g}\partial_\mu\phi\partial^\mu\phi \to \sqrt{-g}\left[\partial_\mu\phi\partial^\mu\phi + \left(\frac{d-2}{2}\right)^2\phi\phi\left(\partial_\mu\Omega\right)^2 + \left(\frac{d-2}{2}\right)\phi\phi\partial^2\Omega\right], \tag{D.10}$$

where d is the number of spacetime dimensions (not spatial dimensions). This is, as seen, not an invariant term. To fix it, another term must be added to the Lagrangian. The only scalar available in a massless theory related to coordinate transformation is the Ricci scalar. The Ricci scalar R transforms as (obtained by the transformation of the affine connection, followed by the Riemann tensor, Ricci tensor and finally the Ricci scalar)

$$R \to e^{-2\Omega}\left(R - 2(d-1)D^2\Omega + (d-1)(d-2)\left(\partial_\mu\Omega\right)^2\right), \tag{D.11}$$

making the entire Lagrangian invariant with the right pre-factors

$$\mathcal{L} = \sqrt{-g}\partial_\mu\phi\partial^\mu\phi + \frac{d-2}{2(d-1)}\sqrt{-g}\phi\phi R. \tag{D.12}$$

This transformation can also be made with vielbeins instead of with the metric. The vielbeins transform covariantly as

$$e^a_\mu \to e^\Omega e^a_\mu, \tag{D.13}$$

which is easily seen from its relation to the metric $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. This makes the spin connection transform as

$$\omega_\mu{}^a \to \omega_\mu{}^a + t_\mu{}^a \tag{D.14}$$

and following this through the Riemann tensor, the Ricci tensor and finally the Ricci scalar, the same transformation as above is obtained

$$R \to e^{-2\Omega}\left(R - 2(d-1)D^2\Omega + (d-1)(d-2)\left(\partial_\mu\Omega\right)^2\right). \tag{D.15}$$

Hence, both Einstein and Cartan gravity produces the same invariant Lagrangian terms.

E

Chern-Simons like variational general relativity

Here are some Chern-Simons and Chern-Simons like Lagrangians varied. First is two invariant general Chern-Simons Lagrangians from gauge theory, which is followed by the variation of some Chern-Simons like gravitational Lagrangians.

E.1 Chern-Simons Lagrangian

E.1.1 Three-dimensional abelian electromagnetic gauge invariance

Consider the integral over the field strength $F = A \wedge dA$ and introduce the gauge transformation $A \to A + d\Lambda$. This transforms the expression as

$$\begin{aligned} \int A \wedge dA \to \int (A + d\Lambda) \wedge (dA + dd\Lambda) = \int (AdA + d\Lambda \wedge dA) = \\ = \int [AdA + d(\Lambda \wedge dA)] = \int A \wedge dA, \end{aligned} \tag{E.1}$$

where the last equality follows from a total derivative. The dd-terms vanishes since two exterior derivatives on any field is identically zero (eq. (B.8)).

E.1.2 Three-dimensional non-abelian electromagnetic gauge invariance

The gauge-transformation is now written as

$$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g, \tag{E.2}$$

which, if the action from the abelian case is used, is not invariant and another term must be included. Adding this term gives the Chern-Simons action

$$S_{\text{CS}} = \int \mathrm{d}^3 x \epsilon^{\mu\nu\sigma} \mathrm{Tr} \left[\tfrac{1}{2} A_\mu \partial_\nu A_\sigma + \tfrac{1}{3} A_\mu A_\nu A_\sigma \right]. \tag{E.3}$$

Which now transforms as

$$S_{\text{CS}} \to S_{\text{CS}} - \frac{1}{3} \int \mathrm{d}^3 x \epsilon^{\mu\nu\sigma} \mathrm{Tr} \left[g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\sigma g \right], \tag{E.4}$$

where the extra term is called the *winding number* of the gauge transformation g and is most often imposed to be zero or a multiple of 2π. A way to understand the winding number is in terms of string theory. If two points are identified, e.g. to compactify the extra dimensions, and an open string has endings in both these points, it will behave as a closed string, but with winding number 2π. More technically, it tells the number of times a string is wound around space (a cylinder, a torus, etc) in such a way that it cannot be shrunk into a point.

E.2 Three-dimensional gravitational variation

A gravitational Chern-Simons like action in three dimensions is

$$S_{\text{CS}} = \frac{1}{\kappa^2}\int e^a \wedge R_a, \tag{E.5}$$

where

$$R_a = d\omega_a + \tfrac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \tag{E.6}$$

is the Ricci-tensor expressed in terms of the spin connection. The Ricci-tensor in general dimensions has two indices and the relation between these are given by eq. (A.4). The expression in eq. (E.5) can be varied with respect to both e^a and ω^a. Start vary with respect to ω^a, this produces a variation of R_a, which is

$$\delta R_a = d\delta\omega_a + \tfrac{1}{2}\epsilon_{abc}(\delta\omega^b \wedge \omega^c + \omega^b \wedge \delta\omega^c) = d\delta\omega_a + \epsilon_{abc}\omega^b \wedge \delta\omega^c. \tag{E.7}$$

The variation of the action is

$$\delta S = \frac{1}{\kappa^2}\int e^a \wedge \delta R_a = \frac{1}{\kappa^2}\int e^a \wedge (d\delta\omega_a + \epsilon_{abc}\omega^b \wedge \delta\omega^c), \tag{E.8}$$

which after partial integration (and dropping a boundary term) becomes

$$\delta S = \frac{1}{\kappa^2}\int (-de^a + \epsilon_{abc}e^b \wedge \omega^c) \wedge \delta\omega_a = \frac{1}{\kappa^2}\int (-de + e \wedge \omega) \wedge \delta\omega. \tag{E.9}$$

When the variation is put to zero, the zero-torsion condition is obtained

$$de + \omega \wedge e = 0. \tag{E.10}$$

The sign on $\omega \wedge e$ comes from the anti-symmetry of the wedge product.
If the variation instead is made on e^a and assuming that e^a and ω^a are independent

$$\delta S = \frac{1}{\kappa^2}\int \delta e^a \wedge R_a \quad \Rightarrow \quad R_a = 0. \tag{E.11}$$

For the case where ω^a is expressed in terms of e^a, see appendix F.2.4.

E.3 Four-dimensional gravitational variation

In four dimensions the Chern-Simons like action becomes

$$S_{\mathrm{CS}} = \frac{1}{\kappa^2} \int e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd}, \tag{E.12}$$

with

$$R^{cd} = d\omega^{cd} + \omega^a{}_c \wedge \omega^{cb} \tag{E.13}$$

or, written with tensors

$$R^{cd}_{\mu\nu} = \partial_\mu \omega^{cd}{}_\nu + 2\omega^a{}_{c[\mu}\omega^{cb}{}_{\nu]}. \tag{E.14}$$

A variation of this action with respect to ω results in

$$\delta S_{\mathrm{CS}} = \frac{1}{\kappa^2} \int e^a \wedge e^b \wedge \delta R^{cd} \epsilon_{abcd}, \tag{E.15}$$

where the variation of R^{ab} is

$$\delta R^{ab} = d\delta\omega^{ab} + \delta\omega^a{}_c \wedge \omega^{cb} + \omega^a{}_c \wedge \delta\omega^{cb} = D\delta\omega^{ab}. \tag{E.16}$$

The derivative on the $\delta\omega$-term is not wanted and is moved to one of the vielbeins through partial integration, leading to

$$\delta S_{\mathrm{CS}} = \tfrac{1}{\kappa^2} \int \epsilon_{abcd} e^a \wedge T^b \wedge \delta\omega^{cd}, \tag{E.17}$$

where T is the torsion coefficient and is given by

$$T^a = De^a = de^a + \omega^a{}_b \wedge e^b. \tag{E.18}$$

Since the variation must vanish for any $\delta\omega$, the following expression is obtained

$$\epsilon_{abcd} e^a \wedge T^b \doteq 6\epsilon_{abcd} e_{[\mu}{}^a T_{\nu\sigma]}{}^b = 0. \tag{E.19}$$

To get the same zero-torsion condition as in three dimensions, implement the anti-symmetry with the Levi-Civita symbol and make the contraction e^a_μ

$$\epsilon_{abcd}\epsilon^{\mu\nu\sigma\lambda} e_\mu{}^a T_{\nu\sigma}{}^b = 2\delta^{\nu\sigma\lambda}_{bcd} T_{\nu\sigma}{}^b = 12 T_{[bc}{}^b e^\lambda{}_{d]} = 0, \tag{E.20}$$

or

$$\begin{aligned} 6T_{[bc}e^\lambda{}_{d]}{}^b &= T_{bc}e^\lambda{}_d{}^b + T_{cd}{}^\lambda + T_{db}{}^b e^\lambda{}_c - T_{cb}{}^b e^\lambda{}_d - T_{dc}{}^\lambda - T_{bd}{}^b e^\lambda{}_c = \\ &= 2\left(T_{cd}{}^\lambda + T_{bc}{}^b e^\lambda{}_d + T_{db}{}^b e^\lambda{}_c\right) = 0. \end{aligned} \tag{E.21}$$

Closing λ with d makes the two last terms vanish (notice that D is the total dimension, not the covariant derivative)

$$\begin{aligned} e_\lambda{}^d\left(T_{cd}{}^\lambda + T_{bc}{}^b e^\lambda{}_d + T_{db}{}^b e^\lambda{}_c\right) &= T_{c\lambda}{}^\lambda + DT_{\lambda c}{}^\lambda + T_{\lambda c}{}^\lambda = \\ &= DT_{\lambda c}{}^\lambda = 0 \quad \Rightarrow \quad T_{\lambda c}{}^\lambda = 0, \end{aligned} \tag{E.22}$$

which makes the expression (E.21) state

$$T_{cd}{}^{\lambda} = 0. \tag{E.23}$$

The variation with respect to e^a is

$$\begin{aligned} \delta S &= \frac{1}{\kappa^2} \int \left(\delta e^a \wedge e^b + e^a \wedge \delta e^b \right) \wedge R^{cd} \epsilon_{abcd} = \\ &= \frac{2}{\kappa^2} \int \delta e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd} \quad \Rightarrow \quad \epsilon^{\mu\nu\sigma\lambda} \epsilon_{abcd} = \delta^{\lambda}_{d} R = 0, \end{aligned} \tag{E.24}$$

which means (with R the Ricci tensor 2-form)

$$R = 0. \tag{E.25}$$

F

The Schouten tensor

The Schouten tensor is mentioned in general relativity as a part of the Riemann tensor and appears in several places in this thesis, especially since it is a part of the Cotton tensor. Here, it is derived from an expression for the Riemann tensor and some features of the Cotton tensor is shown.

F.1 Derivation

The Riemann tensor $R_{\mu\nu}{}^{\sigma\lambda}$ in general relativity can be expressed as

$$R_{\mu\nu}{}^{\sigma\lambda} = W_{\mu\nu}{}^{\sigma\lambda} + 4\delta^{[\sigma}_{[\mu} S_{\nu]}{}^{\lambda]}. \tag{F.1}$$

Where $W_{\mu\nu}{}^{\sigma\lambda}$ is the Weyl tensor and $S_\mu{}^\nu$ is the Schouten tensor. However, in three dimensions, an anti-symmetry in two indices can be represented using one index by contraction using a Levi-Civita symbol. Since the Weyl tensor is identically zero for arbitrary curvature in three dimensions, all information of the Riemann tensor is contained in the Schouten tensor. The Riemann tensor is expressed in terms of the Schouten tensor as

$$R_{\mu\nu}{}^{\sigma\lambda} = 4\delta^{[\sigma}_{[\mu} S_{\nu]}{}^{\lambda]}, \tag{F.2}$$

which means that the Schouten tensor can be solved for in terms of the Riemann tensor. Multiply everything by $\epsilon_{\sigma\lambda\rho}$ (the factor of -2 comes from eq. (A.4))

$$-2R_{\mu\nu\rho} = 4\epsilon_{\sigma\lambda\rho}\delta^{\sigma}_{[\mu} S_{\nu]}{}^{\lambda} \tag{F.3}$$

and now by $\epsilon^{\tau\nu\rho}$

$$-2R_\sigma{}^\tau = -8\delta^{[\tau\nu]}_{\sigma\lambda}\delta^{\sigma}_{[\mu} S_{\nu]}{}^{\lambda} = -8\delta^{[\tau}_{[\mu} S_{\nu]}{}^{\nu]} = -2\left(S_\mu{}^\tau + \delta^\tau_\mu S\right) \tag{F.4}$$

where $S = S_\mu{}^\mu$. The trace of $S_\mu{}^\tau$ is obtain by closing μ and τ in the expression, giving

$$R = S + 3S = 4S \quad \Rightarrow \quad S = \tfrac{1}{4}R \tag{F.5}$$

and the final expression for the Schouten tensor is

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R. \tag{F.6}$$

F.2 The Cotton tensor

The Cotton Tensor

$$C_{\mu\nu} = \epsilon_\mu{}^{\sigma\rho} S_{\rho\nu} = \epsilon_\mu{}^{\sigma\rho} D_\sigma \left(R_{\rho\nu} - \tfrac{1}{4} g_{\rho\nu} R \right), \tag{F.7}$$

is a field taking on the role of the Weyl tensor in three spacetime dimensions in the sense that a conformally flat space has a vanishing Cotton tensor. Hence, it is of interest to not demand the Cotton tensor to be zero if the theory is to be coupled to other fields (matter, radiation, etc).

F.2.1 Symmetry of the Cotton tensor

The Cotton tensor is symmetric in its two indices, which can be shown by figuring out that their anti-symmetric part is zero. Contract it with the Levi-Civita symbol $\epsilon^{\mu\nu\lambda}$

$$\begin{aligned} \epsilon^{\mu\nu\lambda} C_{\mu\nu} &= \epsilon^{\mu\nu\lambda} \epsilon_\mu{}^{\sigma\rho} D_\sigma \left(R_{\rho\nu} - \tfrac{1}{4} g_{\rho\nu} R \right) = -2\delta^{\nu\lambda}_{\sigma\rho} D_\sigma R_{\rho\nu} - \tfrac{1}{2}\delta^\sigma_\lambda D_\sigma R = \\ &= -D^\sigma R_{\lambda\sigma} + D_\lambda R^\sigma{}_\sigma - \tfrac{1}{2}\delta^\sigma_\lambda D_\sigma R = -D^\sigma \left(R_{\sigma\lambda} - \tfrac{1}{2} g\sigma\lambda R \right) = 0, \end{aligned} \tag{F.8}$$

since the expression within the last set of parentheses is the contracted Bianchi Identity.

F.2.2 Trace of the Cotton tensor

Another thing that can be important to know about the Cotton tensor is the trace. It is obtained by contracting the two indices

$$g^{\mu\nu} C_{\mu\nu} = \epsilon^{\nu\sigma\rho} D_\sigma \left(R_{\rho\nu} - \tfrac{1}{4} g_{\rho\nu} R \right) = 0. \tag{F.9}$$

Both $R_{\rho\nu}$ and $g_{\rho\nu}$ is symmetric and hence, when they are contracted with the Levi-Civita symbol, the expression is zero.

F.2.3 Conformal invariance of the Cotton tensor

To make a conformal transformation of the Cotton tensor, note that it consists of R and $R_{\mu\nu}$, and the variation of these were found in appendix D. Hence, the variation of the Cotton tensor follows from those calculations and it is found to be conserved. This makes it a good candidate for a conserved current, especially since it can be found via a variation of the Lagrangian.

F.2.4 Variation of the Chern-Simons action

In appendix E.2 it was shown how to do a variation of a Chern-Simons like action where ω^a was independent of e^a. The case where ω^a is expressed in terms of e^a is shown here. Since the variation of ω^a must be covariant, there is only one real candidate for the variation

$$\begin{aligned} \omega_\mu{}^a &= \tfrac{1}{2}\epsilon^{\nu\sigma\lambda}\left(e_\mu{}^a e_{\lambda b} - 2e_\lambda{}^a e_{\mu b}\right)\partial_\nu e^b_\sigma \quad \Rightarrow \\ \Rightarrow \quad \delta\omega_\mu{}^a &= \tfrac{1}{2}\epsilon^{\nu\sigma\lambda}\left(e_\mu{}^a e_{\lambda b} - 2e_\lambda{}^a e_{\mu b}\right)D_\nu \delta e_\sigma{}^b. \end{aligned} \tag{F.10}$$

It is the expression for ω^a, but with a covariant derivative instead of a partial. Entering this into the variation of the Lagrangian

$$\delta\mathcal{L} = \delta\omega^a \wedge R_a = \tfrac{1}{2}\epsilon^{\nu\sigma\lambda}\left(e_\mu{}^a e_{\lambda b} - 2e_\lambda{}^a e_{\mu b}\right)D_\nu \delta e_\sigma{}^b R_{\kappa\tau a}\epsilon^{\mu\kappa\tau} \tag{F.11}$$

or

$$\delta\mathcal{L} = \tfrac{1}{2}\left(\epsilon^{\nu\sigma}{}_b\epsilon^{a\kappa\tau} - 2\epsilon^{\nu\sigma a}\epsilon_b{}^{\kappa\tau}\right)R_{\kappa\tau a}D_\nu\delta e_\sigma{}^b \tag{F.12}$$

Partial integration gives

$$\delta\mathcal{L} = -\tfrac{1}{2}\left(\epsilon^{\nu\sigma}{}_b\epsilon^{a\kappa\tau} - 2\epsilon^{\nu\sigma a}\epsilon_b{}^{\kappa\tau}\right)D_\nu R_{\kappa\tau a}\delta e_\sigma{}^b \tag{F.13}$$

which is zero for all variations δe, hence

$$\left(\epsilon^{\nu\sigma}{}_b\epsilon^{a\kappa\tau} - 2\epsilon^{\nu\sigma a}\epsilon_b{}^{\kappa\tau}\right)D_\nu R_{\kappa\tau a} = 0. \tag{F.14}$$

Now, recognizing that

$$\epsilon^{\mu\nu\sigma}R_{\mu\nu\sigma} = R, \qquad \epsilon^{\mu\nu}{}_a R_{\mu\nu b} = 2R_{ab} \tag{F.15}$$

gives

$$\epsilon^{\nu\sigma}{}_b D_\nu R - 4\epsilon^{\nu\sigma a}D_\nu R_{ba} = 0. \tag{F.16}$$

or

$$\epsilon^{\nu\sigma a}D_\nu\left(\eta_{ab}R - 4R_{ba}\right)0. \tag{F.17}$$

Shuffle terms and changing to curved indices now gives the Cotton equation

$$\epsilon_\mu{}^{\sigma\lambda}D_\sigma\left(R_{\lambda\nu} - \tfrac{1}{4}g_{\lambda\nu}R\right) = 0. \tag{F.18}$$

G
The parabolic subalgebra

The parabolic subalgebra discussed in this thesis is a subalgebra which emerge when keeping all brackets of the algebra that result in an element with $q \geq p$ (here q denotes the number of q^α spinors and similarly for p). When setting the parabolic part of F to zero (as done in section 5.4.3) all projected fields $F_{(q,p)}$ where $q \geq p$ are set to zero, while the fields $F_{(q,p)}$ with $q < p$ are kept. Note that if the bracket of one element consists of two q spinors (i.e. P^a) and another element has m q spinors and n p spinors, the result will be an element with $m+1$ q spinors and $n-1$ p spinors, moving a step up according to the grading order. This follows since the Poisson bracket removes one q and one p from the total number of q and p spinors of the elements that are included in the bracket yielding it non-zero. Similarly, if one element consists of only two p spinors, the result is one step down in the grading order. This means that if only the parabolic part of the equation $F = 0$ is assumed, i.e. all terms with $q \geq p$ are set to zero, there will not remain any elements whos' bracket is an element with a grade (transformation under D, or, equivalently $q-p$) higher or equal to zero. Hence, if the projection onto a closed subgroup of the algebra is removed, what is left of the Bianchi Identities should still be zero, which is exactly what is proved in this thesis.
A bit more technical, the full higher-spin algebra can be thought of in terms of the matrix G.2. It is a matrix containing all basis elements ordered according to their content of q and p spinors, where q spinors increase for each column and p spinors increase for each row.
In the matrix G.1, the elements along the diagonal forms a solvable algebra and it stays along the diagonal, while all off-diagonal elements are nilpotent, meaning that for each pair of nilpotent elemens J_0 and J, there exist an n such that $[\ldots[[J_0, J], J]\ldots, J] = 0$, where n is the number of brackets. In other terms, all off-diagonal terms will drift outwards, away from the diagonal and "escape" the matrix (looking at the spin-2 and spin-3 brackets, this is seen to be true). Before explaining how this relates to what is here called the parabolic part, the so-called Borel subgroup must be introduced. To understand this, start with the general linear group $GL(N)$, this group has the generators $(J^{mn})_{ij} = \delta_i^m \delta_j^n$ for all m and n.
The algebra of the general linear group viewed as in the matrix G.1 where the Cartan algebra (a solvable algebra), with elements called h, is placed on the diagonal ($m = n$) and elements above (or below) the diagonal, called e ($m > n$) and f ($m < n$) respectively, are nilpotent. The Borel algebra of this is all the h and e elements (B^+), or all h and f elements (B^-). In other words, taking all elements on the diagonal together with all elements above or below will result in the Borel algebra, which is the largest solveable subalgebra. This Borel algebra is also the minimal parabolic

algebra, where the parabolic algebra is any closed algebra equal to or larger than the Borel algebra, but smaller than the ambient (full) algebra. Hence, by picking the elements contained above the black line in the matrix G.1, a non-minimal parabolic algebra is obtained.

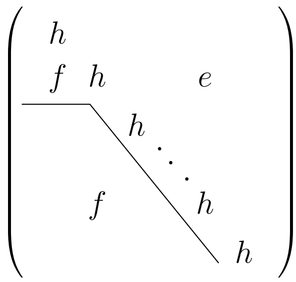

Figure G.1: The generator matrix for $GL(N)$.

What in the rest of this thesis is refered to as the parabolic part of the algebra is the maximal parabolic algebra of $SO3(3,2)$. The $SO(N)$ algebras are not as easily explained or visualized as the $GL(N)$ algebras. However, the maximal parabolic subgroup of $SO(3,2)$ is exactly the subgroup obtained by taking the terms with $q \geq p$. Including more terms would mean including terms drifting away to other terms and hence, the only algebra larger than the parabolic one is the ambient algebra.

	1	q	q^2	q^3	q^4	q^5	q^6	$\cdots$
1	$\mathbb{1}$	spin$(\frac{3}{2})$	P^a	spin$(\frac{5}{2})$	P^{ab}	spin$(\frac{7}{2})$	spin(4)	$\cdots$
p	spin$(\frac{3}{2})$	M^a, D	spin$(\frac{5}{2})$	$\tilde{P}^{ab}, \tilde{P}^a$	spin$(\frac{7}{2})$	spin(4)	$\cdots$	$\cdots$
p^2	K^a	spin$(\frac{5}{2})$	$\tilde{M}^{ab}, \tilde{M}^a, \tilde{D}$	spin$(\frac{7}{2})$	spin(4)	$\cdots$	$\cdots$	$\cdots$
p^3	spin$(\frac{5}{2})$	$\tilde{K}^{ab}, \tilde{K}^a$	spin$(\frac{7}{2})$	spin(4)	$\cdots$	$\cdots$	$\cdots$	$\cdots$
p^4	K^{ab}	spin$(\frac{7}{2})$	spin(4)	$\vdots$	$\ddots$	$\cdots$	$\cdots$	$\cdots$
p^5	spin$(\frac{7}{2})$	spin(4)	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\cdots$
p^6	spin(4)	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure G.2: The matrix of higher-spin basis elements.

The elements in the matrix where it says spin(Q) contains basis elements of spin $Q-1$, used to obtain fields of spin Q. For instance, the basis elements having spin 2 gives rise to spin-3 fields. The term $\mathbb{1}$ is the so-called central charge, an element commuting with all other. The central charge is obtained when promoting the q and p fields to operators ($q, p \to \hat{q}, \hat{p}$, quantizing the system), which gives the commutators such as

$$[\text{spin}(4), \text{spin}(4)] = \text{spin}(6) + \text{spin}(4) + \text{spin}(2), \tag{G.1}$$

where the central charge corresponds to spin 1 (see fig. G.2) and thus appears only when commutating an odd-spin field with an even-spin field, as in

$$[\mathrm{spin}(3), \mathrm{spin}(2)] = \mathrm{spin}(3) + \mathrm{spin}(1), \tag{G.2}$$

similarly to how the central charge (or conformal anomaly) appears through a double contraction in the Virasoro algebra in string theory.

H

Integrability of the system

Assume that the field strength $F = dA$ and that $A = d\Lambda$, making $F = 0$ (follows from eq. (B.8)). However, it is not clear if it is true the other way around as well, i.e. $F = 0 \Rightarrow A = d\Lambda$, but this is solved by the so-called Poincaré's lemma [28] stating that if $\partial_{[\alpha}\xi_{\beta]} = 0$, then (locally) $\xi_\alpha = \partial_\alpha \xi$.
To show this, start with the expressions

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda\xi_\beta(\lambda x)\right] = \xi_\beta(\lambda x) + \lambda x^\alpha \partial_\alpha \xi_\beta(\lambda x) \tag{H.1}$$

and

$$\partial_\beta\left[x^\alpha \xi_\alpha(\lambda x)\right] = \delta^\alpha_\beta \xi_\alpha(\lambda x) + x^\alpha \partial_\beta(\xi_\alpha(\lambda x)) = \xi_\beta(\lambda x) + \lambda x^\alpha \partial_\alpha \xi_\beta(\lambda x), \tag{H.2}$$

where the last equality comes from $\partial_{[\alpha}\xi_{\beta]} = 0$ and hence, these two expressions are equal.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda\xi_\beta(\lambda x)\right] = \partial_\beta\left[x^\alpha \xi_\alpha(\lambda x)\right]. \tag{H.3}$$

Now, ξ is given by

$$\xi(x) = \int_0^1 \mathrm{d}\lambda x^\alpha \xi_\alpha(x\lambda), \tag{H.4}$$

wich follows from differentiating the expression

$$\partial_\alpha \xi(x) = \int_0^1 \mathrm{d}\lambda \partial_\alpha(x^\beta \xi_\beta(x\lambda)) = \int_0^1 \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda\xi_\alpha(\lambda x)\right] = \xi_\alpha(x). \tag{H.5}$$

This means that locally $F = 0 \Rightarrow A = d\Lambda$, this is called the integrability condition, showing that the field strength F can be used to solve for the potential A. In three dimensions this can be done for all points in space, this follows from that there are as many equations in $F = 0$ as the number of degrees of freedom in A (remember that F is anti-symmetric). If there are more spacetime dimensions than three, the system becomes overdetermined and extra constraints have to be set on A.
The same argument as above also holds for the Bianchi identity, saying that if $dF = 0$, then locally $F = dA$.
Following from these arguments, the linear case studied in this thesis states that $F = dA + eA = 0$, which can be inverted when the theory considered is gravity since $e \neq 0$ (follows from $\det(g) \neq 0$), hence $A = e^{-1}dA$. Showing that when gravity is considered A can be expressed in terms of its derivative and the inverse dreibein. From this, since all equations relates the fields in A to some derivative of A, the parts of the equation $F = 0$ that does not solve one part of the field in terms of another must be identities. In other terms, the system is exactly soluble [29].