Finite Hypergroups and Their Representation Theory

Kandidatarbete inom civilingenjörsutbildningen vid Chalmers

Adam Breitholtz
Björn Söderstedt Franzén
Joel Sjögren
John Moberg

Institutionen för matematiska vetenskaper
Chalmers tekniska högskola
Göteborgs universitet
Göteborg 2017

Finite Hypergroups and Their Representation Theory

Kandidatarbete i matematik inom civilingenjörsprogrammet Teknisk fysik vid Chalmers

Adam Breitholtz   Björn Söderstedt Franzén

Kandidatarbete i matematik inom civilingenjörsprogrammet Teknisk matematik vid Chalmers

Joel Sjögren   John Moberg

Handledare: Mahmood Alaghmandan
Examinator: Maria Roginskaya   Marina Axelson-Fisk

Institutionen för matematiska vetenskaper
Chalmers tekniska högskola
Göteborgs universitet
Göteborg 2017

Populärvetenskaplig presentation

Inom matematiken försöker man ofta bryta ned komplexa koncept till dess kärna för att lättare kunna förstå dem och för att se vad som faktiskt spelar roll. På samma sätt försöker man ofta hitta mönster för att kunna förena saker som ser annorlunda ut men som i grunden beter sig på samma sätt.

Ett exempel på ett koncept som uppkom efter att man noterade att det fanns många objekt som hade samma fundamentala struktur är det matematiska begreppet *grupp*. I matematiken är en grupp en samling saker och något sätt att kombinera dem. Detta är ett mycket abstrakt begrepp – vilken typ av saker, och vad menas med att kombinera dem? Det kan faktiskt vara nästan vilka saker som helst så länge det finns ett “vettigt” sätt att kombinera dem. Till exempel kräver man att samma operation alltid ska ge samma resultat.

Ett exempel på en grupp som alla har stött på (även om du troligtvis inte har tänkt på det som en grupp) är de positiva och negativa heltalen

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

tillsammans med addition som sättet att kombinera tal. Till exempel har vi att $-2+1=-1$ vilket alltid är sant oavsett vilken tid på dygnet man utför additionen. Ytterligare ett krav på sättet att kombinera saker är att om man kombinerar tre saker ska det inte spela någon roll i vilken ordning man gör det. Till exempel är $(2+3)+5=5+5=10$ samma sak som $2+(3+5)=2+8=10$, d.v.s. parenteserna spelar ingen roll. Utan detta krav kan det bli väldigt märkligt och svårt att räkna på, vilket motiverar kravet.

Grupper dyker upp på väldigt många ställen, särskilt inom matematik men också inom fysik och kemi. Till exempel förklarar teorin kring grupper många symmetrier i fysiken och hur vissa elementarpartiklar beter sig. 1963 fick den ungersk-amerikanska fysikern Eugene Wigner nobelpriset i fysik för sina upptäcker kring symmetrier som kom tack vare tillämpningen av matematiska grupper. Att studera grupper grundar sig alltså inte bara i matematisk nyfikenhet utan har också praktiska tillämpningar även om de också är abstrakta.

Ett mer fysikaliskt exempel på en grupp skulle kunna vara en samling partiklar som tillåts interagera genom att kollidera med varann. Sättet att kombinera dem är då att de krockar med varandra och bildar en ny partikel. Det viktiga är att när två partiklar krockar så blir det alltid samma partikel på samma sätt som att $2+3$ alltid är lika med 5.

En *hypergrupp* utvidgar gruppbegreppet genom att ta bort det kravet. Det behöver alltså i någon mening inte vara bestämt exakt vad som händer när två saker kombineras, utan vi nöjer oss med att tilldela sannolikheter för olika utfall. Om vi försökte tänka oss heltalen som en hypergrupp skulle det kunna innebära att $2+3=5$ med 50% sannolikhet, men med 50% sannolikhet blir det $2+3=-5$ istället!

Låt oss återgå till det fysikaliska (men orealistiska) exemplet om partiklar. Hypergruppen innehåller en samling partiklar som kan kombineras. Den beskriver då vilken typ av partikel kollisionen bildar på så sätt att den anger sannolikheter för vad som händer. Det handlar alltså inte om ett system där vi vet precis vad som händer, utan det är snarare slumpen som styr.

Som tidigare nämnts försöker vi inom matematiken ofta tänka på saker på olika sätt för att därigenom försöka få en djupare förståelse. Ett verktyg som är mycket välanvänt inom gruppteori är *representationer* vilket, precis som det låter, är ett annat sätt att representera grupper. En grupp har flera olika representationer som alla har olika för- och nackdelar. Denna möjlighet att representera grupper är mycket användbar då det kan ge upphov till enklare metoder att behandla grupper med.

Med denna bakgrund kan vi tala om vad detta arbete handlar om. Vi vill ta den representationsteori som redan är välkänd och välstuderad för grupper och se om vi kan få den att fungera även för hypergrupper. Vår frågeställning skulle alltså kunna formuleras som: Kan vi representera hypergrupper på samma sätt som grupper, och är den representationen givande? Om det är möjligt så skulle det kunna utgöra ett kraftfullt verktyg för att förstå denna abstrakta struktur bättre, men också andra områden i matematiken. Kanske kan teorin till och med göra att vi får en bättre förståelse för den huvudsakliga gruppteorin.

Osäkerheten i utfall för hypergrupper leder till en rad svårigheter som inte uppkommer i gruppfallet. Detta då delar av representationsteorin för grupper har stor användning av att vi vet vad som kommer hända när två saker kombineras. Detta gäller ju som bekant inte längre för hypergrupper vilket gör att vi måste försöka hitta på nya sätt att angripa problemet.

Dessa svårigheter till trots har vi lyckats med det vi gav oss på och konstruerat en representationsteori för hypergrupper vilket detaljeras i vår rapport. Det vi funnit är att samma saker som gäller för grupper även gäller för hypergrupper i stor utsträckning, och i andra fall kan man utnyttja särskilda strukturer i hypergruppen för att komma runt det. Resultatet är en representationsteori för hypergrupper som är bekant för alla som har sysslat med representationsteorin för ändliga grupper.

Sammanfattning

En hypergrupp är en algebraisk struktur som generaliserar gruppbegreppet genom att, i någon mening, ge operationen en probabilistisk tolkning där multiplikationen inte alltid ger samma resultat. Istället ger multiplikationen en linjärkombination av element från hypergruppen där koefficienterna summerar till 1. För två hypergruppelement c_i och c_j definieras multiplikationen alltså

$$c_i c_j = \sum_k n_{ij}^k c_k,$$

där vi summerar över alla element c_k i hypergruppen. De icke-negativa talen n_{ij}^k kallas strukturkonstanter och för hypergrupper måste det gälla att $\sum_k n_{ij}^k = 1$, vilket ger upphov till den probabilistiska tolkningen.

I motsats till gruppfallet där en unik invers alltid finns så leder detta till en svagare form av invers. Det gäller fortfarande att varje element har en unik motsvarighet till invers – involution – men kravet är bara att identiteten ska ingå med nollskild koefficient i linjärkombinationen. Specialfall av hypergrupper är välstuderade, men den nuvarande generella teorin bygger på abstrakt harmonisk analys, och det ändliga fallet har i stort sett förbisetts.

I denna rapport ämnas det ändliga fallet utredas närmare med speciellt fokus på utvecklingen av representationsteorin. Vi inleder med grundläggande definitioner och användbara begrepp såsom viktfunktionen då den är nödvändig för att kringgå de svårigheter som uppstår som följd av det försvagade kravet på inverterbarhet. Viktfunktionen definieras som

$$w(c_z) = \frac{1}{n_{z^*z}^0},$$

alltså ungefär ett mått på "hur mycket identitet fås" vid multiplikationen av c_z med dess involution. Den talar alltså om hur pass inverterbar c_z är. Viktfunktionen visar sig vara användbar därför att den är en så kallad Haarfunktion som flitigt används i den mer generella teorin. Vi visar här att viktfunktionen är lika med sin involution. Värt att notera är att detta inte gäller i det generella fallet och är ett av de resultat som gör de ändliga hypergrupperna mer lätthanterliga.

Genom att sedan utgå från den redan välkända representationsteorin för ändliga grupper utvecklar vi representationsteorin för ändliga hypergrupper. Flera av bevisteknikerna som traditionellt sett används i gruppfallet är inte längre tillämpbara vilket tvingar oss att utnyttja hypergruppens speciella struktur. I representationsteorin representerar vi element i hypergruppen som linjära avbildningar på ett ändligdimensionellt vektorrum. Detta gör att vi kan utnyttja metoder från linjär algebra för undersöka hypergruppens egenskaper.

Vi kommer fram till att alla representationer är nedbrytbara och kan skrivas som en direkt summa av irreducibla representationer. Irreducibla representationer kan liknas vid primtal i den mening att de bygger upp alla andra representationer och i sig inte kan brytas ned ytterligare. Kapitlet om representationsteori kulminerar i Schurs ortogonalitetsrelationer vilka talar om skalärprodukten mellan irreducibla representationer vilket vi formulerar i termer av hyperdimensionen av representationen. Relationerna säger att för två inekvivalenta irreducibla representationer φ och ρ av samma hypergrupp gäller

$$\text{(i)}\ \langle \varphi_{ij}, \rho_{kl} \rangle = 0,$$

$$\text{(ii)}\ \langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} \frac{1}{k(\varphi)} & \text{om } (i,j) = (k,l) \\ 0 & \text{annars,} \end{cases}$$

där $k(\varphi)$ är hyperdimensionen av representationen φ. Hyperdimensionen definieras som inversen av en skalärprodukt mellan en representation och sig själv. I gruppfallet är hyperdimensionen lika med det associerade vektorrummets dimension, vilket rättfärdigar namnet.

Vi introducerar sedan konceptet karaktär av en representation som spåret av den motsvarande matrisen. Det visar sig att matrisens spår kodar mycket information om den underliggande representationen. Detta leder till ett av rapportens huvudresultat vilket är ortogonaliteten mellan karaktärer vilken lyder

$$\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} \frac{\deg \varphi}{k(\varphi)} & \text{om } \varphi \sim \rho \\ 0 & \text{annars.} \end{cases}$$

Vi definierar även den reguljära representationen, som visar sig vara mycket användbar för att finna information om hypergruppen, och undersöker dess nedbrytning.

Därefter begränsar vi oss till att betrakta kommutativa hypergrupper och bevisar bland annat att alla hypergrupper av ordning strikt mindre än fem är kommutativa. Det är värt att notera att detta är ett så starkt resultat som möjligt eftersom att det finns exempel på icke-kommutativa hypergroupper av ordning fem. Vi utvecklar även Fourieranalys på kommutativa hypergrupper vilket resulterar i att en kommutativ hypergrupp helt bestäms av sin karaktärstabell upp till isomorfi.

Teoretiska tillämpningar presenteras i ett appendix. Ett återkommande tema är minskning av antal element via identifiering av symmetrier. Först härleds klasshypergruppen av en ändlig grupp, vilken är den största kommutativa delstrukturen av den associerade gruppalgebran. Några exempel på aritmetik ges. Karaktärshypergruppen härleds därefter, vilket är hypergruppen som kan genereras utifrån karaktärerna hos en finit grupp. Det visar sig att även karaktärerna från en kommutativ hypergrupp under vissa förutsättningar kan generera en kommutativ hypergrupp, som då kallas den duala hypergruppen.

Slutligen härleds hypergruppen av en distanstransitiv graf. Distanstransitivitet är ett starkt krav på symmetri och det visar sig att man med hänsyn till denna symmetri kan konstruera en hypergrupp som har färre element än antalet noder i grafen. En naturlig fråga är huruvida en distanstransitiv graf bestäms entydigt av sin hypergrupp, och svaret är negativt.

Abstract

A hypergroup is an algebraic structure generalizing the concept of a group. This is done by adding a sense in which the multiplication can be interpreted as probabilistic by letting the operation range over an algebra. This leads to weakening the requirement of invertibility that we have for ordinary groups. While hypergroups have been studied in full generality, finite hypergroups are an interesting special case which can be dealt with by more elementary methods.

In this report we restrict ourselves to finite hypergroups and develop the representation theory of hypergroups by trying to generalize the well-known representation theory of finite groups. While many proofs transfer immediately, some proofs that depend on the invertibility of group elements must be modified. We prove the Schur orthogonality relations for hypergroup representations, and establish character orthogonality. Finally, we restrict ourselves to commutative finite hypergroups and prove some interesting results about such objects. This naturally leads to the development of Fourier analysis on finite hypergroups, using similar techniques as in the finite group case.

In the appendices we consider several examples of hypergroups coming from finite groups and distance-transitive graphs. All of these hypergroups are commutative, and therefore the entire body of results apply to them.

Contents

Preamble

A logbook has been written continuously over the course of the project detailing the main activities of the group and also recording the individual time spent.

Contribution

All group members have contributed to most of the central parts of the report. A slightly more specific list of the chapters that the authors feel they contributed most to follows:

Adam: 1, 2, 3, C, D
Björn: 3, 4, A.2, A.3, B.3, B.4 C.1, D
Joel: 2, 3, A.1, A.3, B, D
John: 1, 2, 3, C, D

Some special mentions: Björn worked out the decomposition of the regular representation into irreducibles. Joel wrote a collection of scripts for doing arithmetic.
We make no claims of originality.

Special thanks

Thank you to our supervisor Mahmood for your great help and encouragement. Without your guidance and enthusiasm we would never have gotten this far.

1 Introduction

A group is an algebraic structure that has been studied for several centuries. The history of group theory is riddled with household names such as Lagrange, Cauchy, and Galois. Today, finite groups are well-understood and ubiquitous in both mathematics and applications. However, the classification of finite simple groups was finished only recently (in the 21th century) and is considered one of the greatest mathematical accomplishments of the last century.

The classification of finite simple groups was made possible by the development of representation theory, in which group elements are represented by matrices or linear transformations. This effectively allows us to transform hard problems in group theory to well-known linear algebra. While Gauss studied concepts later encompassed in representation theory, it was Frobenius who first formulated the representation theory of finite groups at the end of the 19th century. Practically as a result of our deep familiarity with linear algebra, representation theory is of great importance with countless applications in mathematics and the natural sciences.

A hypergroup is an algebraic structure that has been studied by many mathematicians under different names and definitions. Hypergroups generalize the concept of groups by weakening the requirement of invertibility in addition to giving the structure a probabilistic taste. While the operation on a group always results in another element in the group, the operation on a hypergroup leads to a linear combination of elements of the hypergroup. As hypergroups generalize groups, all groups can be viewed as hypergroups, but the converse obviously does not hold, resulting in "less" structure and a more general object.

To try to make sense of this somewhat abstract structure, we can make a physical (unrealistic but useful) analogy. Imagine a collection of particles that interact by colliding with each other. In the group case, each collision would result in another group element. However, in the case of hypergroups this is not the case – each collision returns in a combination of particles. This could instead be probabilistically interpreted – we get a probability distribution over the set of particles. Assume that we have some particle, call it "photon", which is always absorbed in collisions. Now consider that there might be "anti-particles" in the set; these anti-particles would in collision with their respective particle produce a photon with some probability. If this were a group, we would view the photon as the identity and the anti-particle as the inverse, and the collision would guarantee the identity. This is not the case for hypergroups and we only produce the identity with some probability.

The purpose of the present report is to develop and present a theory of finite hypergroups and their representation theory, accessible to interested undergraduate students with a solid background in linear algebra. Our method has been to study the representation theory of finite groups and to attempt a generalization. Aside from the abstract theory, concrete examples of hypergroups will be given, but only so far as to demonstrate the theory. Theoretical applications of hypergroups to the study of groups and graphs, which might otherwise overwhelm the reader, are presented in an appendix.

In Section 2 we present the basic definitions pertaining to the theory of hypergroups and prove some useful results about their structure. In Section 3 we translate the representation theory of finite groups to hypergroups leading up to the Schur orthogonality relations and the decomposition of the regular representation. A short detour into the character theory of hypergroups is then conducted, leading to the orthogonality of characters. In Section 4 we constrain ourselves to the study of commutative hypergroups, proving some interesting theorems that this extra constraint leads to. Furthermore, we develop Fourier analysis on commutative hypergroups which ultimately leads to a method of characterizing any commutative hypergroup by its character table up to isomorphism.

2 Hypergroups

This section will supply us with the basic definitions and terminology of finite hypergroups. We will also develop some results about the structure of hypergroups which are of crucial importance for the rest of the text. Many of the basic definitions are largely inspired by [1].

The reader is advised to look over the definition of a $*$-algebra in Appendix C or, alternatively, to overlook the details and treat hypergroup elements by the laws of matrix addition, multiplication, and transposition.[1]

Definition 2.1. A *finite hypergroup* is a finite set $K = \{c_0, c_1, \ldots, c_{m-1}\}$ of basis vectors spanning a $*$-algebra A_K over $\mathbb{C}$ with neutral element c_0 satisfying

- K is a basis of A_K.
- $K^* = K$.
- The *structure constants* $n_{ij}^k \in \mathbb{C}$ defined by

  $$c_i c_j = \sum_k n_{ij}^k c_k$$

  have unique involutions

  $$\begin{aligned} c_i^* = c_j &\iff n_{ij}^0 > 0, \\ c_i^* \neq c_j &\iff n_{ij}^0 = 0, \end{aligned}$$

  and are *positive* and *normalized*

  $$n_{ij}^k \geq 0, \quad \sum_k n_{ij}^k = 1$$

  for all indices i, j, and k.

The definition tells us that for each element c_i in a hypergroup, there must exist an *involution* c_i^* which, when multiplied by c_i, may "take us" to the neutral element. Thus c_i^* can be interpreted as some sort of inverse of c_i. In general, multiplying an element by its involution will only result in a linear combination containing c_0. If K is a group, the structure constants satisfy that for all i, j, k, n_{ij}^k =1 if $c_i c_j = c_k$, otherwise 0. In this case all elements will have an inverse, namely their involution.

We use $|K| := m$ to denote the number of basis vectors, and when writing a basis vector $c_i \in K$ it will be implicit that the index is in range, $0 \leq i < |K|$.

Remark. We will drop the finite in finite hypergroup for brevity and refer to it only as hypergroup. Note that this is at odds with most of the literature that uses hypergroup to denote a more general case.

Definition 2.2 (Commutative)**.** Let K be a hypergroup. We say that K is *commutative* if $c_i c_j = c_j c_i$ for all indices i and j.

Note that in the case where K is a group the term *abelian* is usually used instead of commutative.

Definition 2.3 (Cayley table)**.** Let K be a hypergroup. By the *Cayley table* of K we mean the table describing the multiplication acting on all pairs of elements in K, thus fully describing the hypergroup. Note that the order of the operation will matter unless K is commutative, thus the convention is to take the vertical elements first.

For example, if $K = \{c_0, \ldots, c_{m-1}\}$, the Cayley table will look like Table 1.

[1] For example, the involution works like conjugate transposition, i.e. it holds that $(c_i c_j)^* = c_j^* c_i^*$, $(c^*)^* = c$, $(a + b)^* = a^* + b^*$.

Table 1: The Cayley table of a hypergroup K.

$\cdot$	c_0	$\cdots$	c_j	$\cdots$
c_0	c_0	$\cdots$	c_j	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
c_i	c_i	$\cdots$	$\sum_k n_{ij}^k c_k$	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

Table 2: Cayley table of a hypergroup $K = \{c_0, c_1, c_2\}$.

	c_0	c_1	c_2
c_0	c_0	c_1	c_2
c_1	c_1	$\frac{1}{3}c_0 + \frac{2}{3}c_2$	c_1
c_2	c_2	c_1	$\frac{1}{2}c_0 + \frac{1}{2}c_2$

Example 2.4. Table 2 displays the Cayley table of a commutative hypergroup $K = \{c_0, c_1, c_2\}$. Note that the coefficients of all products sum to one as required for it to be a hypergroup.

Actually, K comes from a family of hypergroups called the *class hypergroups*, which are described in detail in Appendix A.1. This specific hypergroup is the so called class hypergroup of S_3. We will return to this example later in the text.

Definition 2.5. Let K be a hypergroup and $c_i \in K$. By the *involution* i^* of the index i, we mean the unique index solving $c_{i^*} = {c_i}^*$.

The somewhat abstract definition of a hypergroup is illuminated by the following theorem which constitutes an alternative, perhaps more concrete, definition [2]. In the main part of the present text, we only use the first part of the theorem; the converse is used in Appendix B to construct examples of hypergroups.

Theorem 2.6. *Let K be a hypergroup with structure constants n_{ij}^k and index involution $i \mapsto i^*$. Then the following holds*

$$\sum_k n_{ij}^k = 1 \quad \forall i, j \tag{1}$$

$$n_{ij}^k \geq 0 \quad \forall i, j \tag{2}$$

$$\sum_t n_{ij}^t n_{tk}^l = \sum_t n_{it}^l n_{jk}^t \quad \forall i, j, k, l \tag{3}$$

$$n_{i0}^k = n_{0i}^k = \begin{cases} 1, & \text{if } i = k \quad \forall i, k \\ 0 & \text{otherwise} \quad \forall i, k \end{cases} \tag{4}$$

$$n_{ij}^0 > 0 \iff j = i^* \quad \forall i, j \tag{5}$$

$$n_{ij}^k = n_{j^* i^*}^{k^*} \quad \forall i, j, k. \tag{6}$$

Conversely, for any $(n_{ij}^k)_{0 \leq i,j,k < m}$ and $*$ satisfying these equations, define $A_K := \mathbb{C}^m$, $K := \{e_i \mid 0 \leq i < m\}$ with operations multiplication and involution given for $a, b \in A_K$ and $0 \leq k < m$ by

$$(ab)_k := \sum_{i,j} a_i b_j n_{ij}^k \tag{7}$$

$$(a^*)_i := \overline{a_{i^*}}. \tag{8}$$

Then K is a hypergroup.

Proof.
(1) and (2) follow from the hypergroup being normalized ($\sum_k n_{ij}^k = 1$) and positive ($n_{ij}^k \geq 0$) respectively.
(3): $\sum_{t,l} n_{ij}^t n_{tk}^l c_l = (\sum_t n_{ij}^t c_t)c_k = (c_i c_j)c_k = c_i(c_j c_k) = c_i(\sum_t n_{jk}^t c_t) = \sum_{t,l} n_{it}^l n_{jk}^t c_l$
(4): $n_{i0}^k = n_{0i}^k$ as it is the structure constant that describes the "chance" that the multiplication $c_0 c_i = c_i c_0 = c_i$ is equal to c_k which only occurs when $c_i = c_k$ i.e when $i = k$.
(5) is one of the axioms reformulated in terms of Definition 2.5.
(6): $(c_i c_j)^* = \begin{cases} \sum_k \overline{n_{ij}^k} c_k^* = \sum_k n_{ij}^k c_k^* \\ c_j^* c_i^* = \sum_k n_{j^* i^*}^k c_k = \sum_k n_{j^* i^*}^{k^*} c_k^* \end{cases}$

It now remains to prove that if (1)-(6) are satisfied and K and A_K are defined as above, then K is a hypergroup. We require that K satisfy the conditions of a hypergroup.
Being positive and normalized follows from (2) and (1) respectively.
K is obviously a basis of A_K. $c_i^* = c_j \iff n_{ij}^0 > 0$ and $c_i^* \neq c_j \iff n_{ij}^0 = 0$ follow from (5).
$K = K^*$ follow from the definition of K. □

Remark. When we write $\mathbb{C}^k$ where k is a non-negative integer we mean the vector space of k-tuples of complex numbers.

Lemma 2.7. *The structure constants of any hypergroup satisfy*

$$n_{ij}^k n_{kk^*}^0 = n_{ii^*}^0 n_{jk^*}^{i^*} \quad \text{and} \quad \frac{n_{ji}^k}{n_{i^*i}^0} = \frac{n_{j^*k}^i}{n_{k^*k}^0},$$

for any indices i, j, and k.

Proof. To prove this, we use associativity, uniqueness of the involution and the fact that K is a basis. Look at the coefficient of c_0 in $(c_i c_j)c_k^* = c_i(c_j c_k^*)$ and $(c_i^* c_j^*)c_k = c_i^*(c_j^* c_k)$. □

Definition 2.8 (Weight, stationary element)**.** Let K be a hypergroup with structure constants (n_{ij}^k). By the *weight function* $w : K \to \mathbb{C}$ we mean $w(c_i) := 1/n_{i^*i}^0$. By the *weight* of the hypergroup we mean $w(K) := \sum_i w(c_i)$. By the *stationary element* we mean $\omega := \sum_i w(c_i)c_i$.

Observe that the weight function gives some measure of "how invertible" an element is. If K is a group then all weights are unity since it implies that all elements have an inverse.

As might be expected, the lack of invertibility prevents us from using the same proof techniques as in the group case. The next theorem however often works as a substitute when rearranging terms. It is based on a discussion of *Haar functions* by Lasser [3].

Theorem 2.9. *Let K be a hypergroup spanning an algebra A_K and let $c_j \in K$. For any linear map $f : A_K \to V$*

$$\sum_i f(c_j c_i)w(c_i) = \sum_i f(c_i)w(c_i),$$
$$\sum_i f(c_i c_j)w(c_i^*) = \sum_i f(c_i)w(c_i^*).$$

For any bilinear or sesquilinear map $g : A_K \times A_K \to V$

$$\sum_i g(c_j c_i, c_i)w(c_i) = \sum_i g(c_i, c_j^* c_i)w(c_i),$$
$$\sum_i g(c_i c_j, c_i)w(c_i^*) = \sum_i g(c_i, c_i c_j^*)w(c_i^*).$$

Proof.

$$\sum_i f(c_j c_i)w(c_i) = \sum_{i,k} \frac{n_{ji}^k}{n_{i^*i}^0} f(c_k) = \sum_{i,k} \frac{n_{j^*k}^i}{n_{k^*k}^0} f(c_k) = \sum_k \frac{1}{n_{k^*k}^0} f(c_k) = \sum_k f(c_k)w(c_k)$$

$$\sum_i g(c_j c_i, c_i) w(c_i) = \sum_{i,k} \frac{n_{ji}^k}{n_{i^*i}^0} g(c_k, c_i) = \sum_{i,k} \frac{n_{j^*k}^i}{n_{k^*k}^0} g(c_k, c_i) = \sum_k g(c_k, c_j^* c_k) w(c_k).$$

We used the hypergroup multiplication rule and linearity of the maps three times, and the second identity of Lemma 2.7, to prove the first and third identities. The third identity also relies on all n_{ij}^k being real in the case where g is sesquilinear. The second and fourth identities are similar. □

The following proposition tells us that the weight of an element is equal to the weight of its involution, and we will be using it implicitly and frequently in the coming chapters. It also justifies the term "stationary element".

Proposition 2.10. *Let K be a hypergroup. If $c_i \in K$ then $w(c_i) = w(c_i^*)$ and $c_i\omega = \omega = \omega c_i$.*

Proof. Define $\omega_* := \sum_j c_j w(c_j^*)$. Use the identity mapping $f(x) := x$ in Theorem 2.9 to get $c_i\omega = \omega$ and $\omega_* c_i = \omega_*$. By distributivity and using that $c_j \mapsto c_j^*$ is an involution it then follows that

$$w(K)\omega = \omega_*\omega = \omega_* w(K),$$

and since $0 < w(K) < \infty$ it follows by cancellation that $\omega = \omega_*$. Compare the coefficients in front of c_i on both sides of $\omega = \omega_*$ to get $w(c_i) = w(c_i^*)$. Finally $c_i\omega = \omega = \omega_* = \omega_* c_i = \omega c_i$. □

This concludes the definitions and results we need from the theory of hypergroups to proceed to develop its representation theory.

3 Representation Theory

In this section we develop a representation theory of hypergroups. It is largely a generalization of the representation theory of finite groups, based on Steinberg's textbook [4]. Much of the theory survives this generalization with only minor changes. Proofs that required little or no change are presented in the appendix for brevity.

We begin by developing some basic concepts and results of representation theory. Among the most basic and important concepts are the so called irreducible representations, which we will show are orthogonal with respect to a certain inner product. This is called the Shur orthogonality relations, and is among our most important results. Using this we show that the representations of a hypergroups are in a sense made of, or decomposed as, the irreducible representations. Then we proceed to introduce the character of a representation, and prove an orthogonality of characters. At the end of this section we prove that the decomposition of representations is unique, and show how the the degrees of the irreducible representations relate to the order of the hypergroup.

We begin by defining representations as a concept, and other related concepts and terminology.

Definition 3.1 (Representation)**.** Let K be a hypergroup with structure constants (n_{ij}^k), and let V be a finite-dimensional vector space. We say that $\varphi : K \to \mathrm{End}(V)$ is a *representation* if

- $\varphi(c_0) = I$ (the identity mapping)
- $\varphi(c_i)\varphi(c_j) = \sum_k n_{ij}^k \varphi(c_k)$ (a simple multiplication rule).

To avoid trivialities we also require that V is non-trivial.

Definition 3.2 (Degree)**.** Let K be a hypergroup and $\varphi : K \to \mathrm{End}(V)$ a representation. By the *degree* of φ we mean the dimension of the vector space V, i.e. $\deg\varphi = \dim V$.

Proposition 3.3. *Let K be a hypergroup spanning an algebra A_K. If $\varphi : K \to \mathrm{End}(V)$ is a representation of K then its linear extension $\tilde{\varphi} : A_K \to \mathrm{End}(V)$ satisfies $\tilde{\varphi}(ab) = \tilde{\varphi}(a)\tilde{\varphi}(b)$.*

Proof. Linear extension is possible because K is a basis for A_K. If $a, b \in A_K$ decompose $a = \sum a_i c_i$ and $b = \sum b_i c_i$.

$$\begin{aligned} ab &= \sum_i a_i c_i \sum_j b_j c_j = \sum_{i,j} a_i c_i b_j c_j = \sum_{i,j} a_i b_j c_i c_j = \sum_{i,j,k} a_i b_j n_{ij}^k c_k \\ \tilde{\varphi}(ab) &= \sum_{i,j,k} a_i b_j n_{ij}^k \varphi(c_k) \stackrel{*}{=} \sum_{i,j} a_i b_j \varphi(c_i)\varphi(c_j) = \sum_{i,j} a_i \varphi(c_i) b_j \varphi(c_j) \\ &= \sum_i a_i \varphi(c_i) \sum_j b_j \varphi(c_j) = \tilde{\varphi}(a)\tilde{\varphi}(b). \end{aligned}$$

We used the simple multiplication rule of the representation in step $\stackrel{*}{=}$. □

Henceforth we make no distinction between a representation φ and its linear extension $\tilde{\varphi}$. We now move on to the notion of equivalent representations, which in some sense can be thought of as being the same. We also need the notion of a morphism between representations, which may seem daunting, but can be thought of as a generalized "change-of-basis" map – generalized since it is not required to be invertible.

Definition 3.4 (Morphism)**.** Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be representations. We say that $T : V \to W$ is a *morphism* if it is linear and $T\varphi(c_i) = \rho(c_i)T$ for all $c_i \in K$. We denote this by $T \in \mathrm{Hom}(\varphi, \rho)$.

Definition 3.5 (Equivalence)**.** Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be representations. We say that φ and ρ are *equivalent* if there exists an invertible $T \in \mathrm{Hom}(\varphi, \rho)$, i.e. $\varphi = T^{-1}\rho T$. We denote this by $\varphi \stackrel{T}{\sim} \rho$ or simply $\varphi \sim \rho$.

Proposition 3.6. *Let K be a hypergroup. Then $\sim$ is an equivalence relation of representations of K.*

Proof. See appendix D. □

Theorem 3.7. *Let $\varphi : K \to \mathrm{End}(V)$ be a representation. Then there is an inner product $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that if $c_i \in K$ and $u, v \in V$ then $(\varphi(c_i)u, v) = (u, \varphi(c_i^*)v)$.*

The theorem is proved for groups by what is called an "averaging trick" construction in [4]. For hypergroups, this average needs to be weighted – by the weight function.

Proof. Because V is finite-dimensional, there is at least one inner product $[\cdot, \cdot]$ on V. Define

$$(u, v) := \sum_i [\varphi(c_i)u, \varphi(c_i)v] w(c_i).$$

$(\cdot, \cdot)$ is obviously symmetric and bilinear, and it is indeed positive definite; dropping most terms gives the estimate $(u, u) \geq [\varphi(c_0)u, \varphi(c_0)u]w(c_0) = [u, u]$. The identity follows as stated, from Theorem 2.9 by choosing $g(x, y) := [\varphi(x)u, \varphi(y)v]$:

$$(\varphi(c_j)u, v) = \sum_i g(c_i c_j, c_i) w(c_i) = \sum_i g(c_i, c_i c_j^*) w(c_i) = (u, \varphi(c_j^*)v).$$

□

Corollary 3.8. *Let $\varphi : K \to \mathrm{End}(V)$ be a representation, and let $(\cdot, \cdot) : V \times V \to \mathbb{C}$ be as in Theorem 3.7. Then there exists an orthonormal basis of V with respect to $(\cdot, \cdot)$. For $c_z \in K$, the matrix of $\varphi(c_z^*)$ with respect to this basis is given by $\varphi(c_z^*)_{ij} = \overline{\varphi(c_z)_{ji}}$.*

Proof. To find an orthonormal basis one needs a vector space with an inner product. Then there exists a basis on that space to which we can apply Gram-Schmidt to make it orthonormal. Call the orthonormal basis (e_i). For the second identity we we use Theorem 3.7 to show that

$$\varphi(c_z^*)_{ij} = \langle e_i, \varphi(c_z^*)e_j \rangle = \langle \varphi(c_z)e_i, e_j \rangle = \overline{\langle e_j, \varphi(c_z)e_i \rangle} = \overline{\varphi(c_z)_{ji}}.$$

□

Definition 3.9 (Direct sum)**.** By the (*external*) *direct sum* $\varphi \oplus \rho : K \to \mathrm{End}(V \oplus W)$ of two representations $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ we mean

$$(\varphi \oplus \rho)(c_i) := \varphi(c_i) \oplus \rho(c_i)$$
$$((A \oplus B)(v, w) := (Av, Bw) \text{ as in linear algebra.})$$

Note that in this definition $(\varphi(c_i)v, \rho(c_i)w)$ is just a pair and has nothing to do with the inner product mentioned in Theorem 3.7 and Corollary 3.8.

Example 3.10. In the matrix sense the direct sum of two matrices A and B works as follows.

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Proposition 3.11. *If φ and ρ are representations then $\varphi \oplus \rho$ is a representation.*

Proof. See appendix D. □

Now we wish to introduce irreducibility, which is a central concept of representation theory. An irreducible representation can be likened to the prime numbers, as we will later see that all representations are equivalent to a direct sum of irreducible representations. Before we can define irreducibility, we need to introduce the notion of invariance.

Definition 3.12 (Invariant subspace)**.** Let $\varphi : K \to \mathrm{End}(V)$ be a representation, and let U be a subspace of V. We say that U is an *invariant subspace* of V if for all $c_i \in K$

$$u \in U \implies \varphi(c_i)u \in U.$$

Definition 3.13 (Irreducibility, Reducibility)**.** Let $\varphi : K \to \mathrm{End}(V)$ be a representation. We say that φ is *irreducible* if the only invariant subspaces of V are $\{0\}$ and V. If φ is not irreducible, we call it *reducible*.

Now that we have defined irreducibility, we will provide an example, and some simple results about irreducible representations.

Proposition 3.14. *Let $\varphi : K \to \mathrm{End}(V)$ be a representation. If $\deg(\varphi) = 1$, then φ is irreducible.*

Proof. If $\deg(\varphi) = 1$, then $\dim(V) = 1$. Thus the only subspaces of V are $\{0\}$ and V itself, thus φ is irreducible. □

Example 3.15. Recall $K = \{c_0, c_1, c_2\}$ from Example 2.4. We will now determine all irreducible representations of K.

By the definition of a representation, the representations $\varphi : K \to \mathbb{C}$ must satisfy $\varphi(c_i)\varphi(c_j) = \sum_k n_{ij}^k \varphi(c_k)$ where n_{ij}^k are the structure constants. Using this together with the structure constants from Table 3a, and letting $\varphi_i := \varphi(c_i)$ temporarily, we see that $\varphi_0 = 1$ always, and we end up with the system of equations

$$\begin{aligned} \varphi_1^2 &= \frac{1}{3} + \frac{2}{3}\varphi_2 \\ \varphi_2\varphi_1 &= \varphi_1 \\ \varphi_2^2 &= \frac{1}{2} + \frac{1}{2}\varphi_2. \end{aligned}$$

Assuming $\varphi_1 \neq 0$, the second equation forces $\varphi_2 = 1$, and then we have $\varphi_1^2 = 1$, thus $\varphi_1 = -1$ or $\varphi_1 = 1$. On the other hand, if $\varphi_1 = 0$, we get $\varphi_2 = -\frac{1}{2}$.

We end up with three solutions, and since these are all the solutions, they are all the irreducible representations of this hypergroup.

Proposition 3.16. *Let $\varphi : K \to \mathrm{End}(V)$ be a mapping such that $\varphi(c_i) = I$ for all $c_i \in K$. Then φ is a representation. It is irreducible if and only if $\deg(\varphi) = 1$.*

Proof. The mapping trivially satisfies the requirement that $\varphi(c_0) = I$. For the other requirement, note that $\varphi(c_i)\varphi(c_j) = 1 = \sum_k n_{ij}^k = \sum_k n_{ij}^k \cdot 1 = \sum_k n_{ij}^k \varphi(c_k)$, and so φ is a representation. By Proposition 3.14 it is irreducible if $\deg(\varphi) = 1$.

Now assume that $\deg(\varphi) > 1$. Then $\dim(V) > 1$, and so there exists a subspace U of V such that $\dim(\{0\}) < \dim(U) < \dim(V)$. Now note that for any $u \in U$ and $c_i \in K$, $\varphi(c_i)u = u \in U$. Thus U is an invariant subspace, and so φ isn't an irreducible representation if $\deg(\varphi) > 1$. □

Definition 3.17 (Trivial representation)**.** Let K be a hypergroup with elements $(c_i)_{0\leq i<|K|}$. If φ is a representation of K such that $\varphi(c_i) = 1$ for all i, then φ is called the *trivial representation* of K.

Remark. In cases where we have a list of numbered irreducible representations, then a representation numbered with a zero is always assumed to be the trivial representation. For example, given a list of irreducible representations $(\varphi^{(i)})$, then $\varphi^{(i)}$ is the trivial representation if and only if $i = 0$.

We now introduce the notion of decomposabillity, which can be seen as way in which a representation is made of other representations. This can be compared to the factorization of integers by primes. And just as the integers have their primes, a representation can be decomposed into irreducible representations.

Definition 3.18 (Decomposability)**.** Let $\varphi : K \to \mathrm{End}(V)$ be a representation. We say that φ is *decomposable* if there are non-trivial invariant subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$.

We now formulate a theorem which shows that equivalent representations share many important features, which justifies why one might think of them as being the same.

Theorem 3.19 (Theorem of Equivalent Representations)**.** *Let $\varphi : K \to End(V)$ and $\rho : K \to End(W)$ be equivalent representations, i.e. $\varphi \overset{T}{\sim} \rho$. Then:*

(i) If U is an invariant subspace of V, then TU is an invariant subspace of W.

(ii) If ρ is irreducible, then φ is irreducible.

(iii) If ρ is decomposable, then φ is decomposable.

(iv) If ρ is completely reducible, then φ is completely reducible.

Proof. See appendix D. □

Proposition 3.20. $\mathrm{Hom}(\varphi, \rho)$ *is a linear subspace of* $\mathrm{Hom}(V, W)$.

Proof. See appendix D. □

Note that I, the identity mapping, is a morphism from any φ to itself, i.e $\varphi \in \mathrm{Hom}(\varphi, \varphi)$. The following lemma is used to prove Schur's lemma, and although neither of them is proved in the main text, they are both interesting results. Schur's lemma will eventually be used to prove Schur's orthogonality relations.

Lemma 3.21. *If $T \in \mathrm{Hom}(\varphi, \rho)$, then* $\ker T$ *and* $\operatorname{im} T$ *are invariant.*

Proof. See appendix D. □

Theorem 3.22 (Schur's lemma)**.** *Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be irreducible representations.*

(i) If $T \in \mathrm{Hom}(\varphi, \rho)$ is nonzero, then T is invertible.

(ii) If $T \in \mathrm{Hom}(\varphi, \varphi)$, then $T = \lambda I$ for some scalar $\lambda \in \mathbb{C}$.

Proof. See appendix D. □

We now aim to show that every representation decomposes into a direct sum of irreducible representations. A representation which is decomposed in this way is called completely decomposable, which is an important concept in representation theory.

Definition 3.23 (Completely reducible/decomposable)**.** Let $\varphi : K \to \mathrm{End}(V)$ be a representation. We say that φ is *completely reducible* if $V = \bigoplus_{0 \le i < n} V_i$ where V_i are invariant subspaces and each restriction $\varphi : K \to \mathrm{End}(V_i)$ is irreducible. Similarly, we say that φ is *completely decomposable* if $\varphi \sim \bigoplus_{0 \le i < n} \varphi^{(i)}$ where $\varphi^{(i)}$ are irreducible representations.

Lemma 3.24. *Every representation is either irreducible or decomposable.*

Proof. It should suffice[2] to prove that every reducible representation is decomposable. Therefore, suppose $\varphi : K \to \mathrm{End}(V)$ is reducible: U is an invariant subspace and $0 < U < V$. Pick an inner product $(\cdot,\cdot) : V \times V \to \mathbb{C}$ as in Theorem 3.7. The orthogonal complement

$$U^{\perp} := \{v \in V \mid \forall u \in U : (v,u) = 0\}$$

is an invariant subspace: if $v \in U^{\perp},\ c_i \in K,\ u \in U$ then $(\varphi(c_i)v, u) = (v, \varphi(c_i^*)u) = 0$ according to Theorem 3.7 and since U is invariant; so $\varphi(c_i)v \in U^{\perp}$. Finally $V = U \oplus U^{\perp}$ in the internal sense, by elementary linear algebra. $\square$

Armed with the above, the proof of complete reducibility and decomposability falls out naturally as in the group case [4]. In the group case, the following theorem is called *Maschke's theorem.*

Theorem 3.25. *Every representation is completely reducible and completely decomposable.*

Proof. See appendix D. $\square$

Definition 3.26. Let K be a hypergroup. Define $\mathbb{C}^K$ to be the vector space

$$\mathbb{C}^K := \{f : K \to \mathbb{C}\}$$

with point-wise addition and scalar multiplication.

We now proceed to define another inner product $\langle \cdot,\cdot \rangle$ and a linear map P which will be used to acquire the final pieces to prove the Schur orthogonality relations. P is defined using the "averaging trick" of Theorem 3.7. The name P comes from it being a projection in the group case.

Proposition 3.27. *Let K be a hypergroup. Define $\langle \cdot,\cdot \rangle$ on the linear space $\mathbb{C}^K$ by*

$$\langle a, b \rangle := \frac{1}{w(K)} \sum_z b(c_z)\overline{a(c_z)} w(c_z).$$

Then $\langle \cdot,\cdot \rangle$ is an inner product.

Proof. See appendix D. $\square$

Definition 3.28. Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be representations. Define $P : \mathrm{Hom}(V,W) \to \mathrm{Hom}(\varphi,\rho)$ by

$$P(T) := \frac{1}{w(K)} \sum_i \rho(c_i) T \varphi(c_i^*) w(c_i).$$

It is not immediately obvious from the definition that P maps all elements in $\mathrm{Hom}(V,W)$ to elements in $\mathrm{Hom}(\varphi,\rho)$, but the following proposition verifies that this indeed is the case. This property of P is crucial to the way it is used in the proof of the Schur orthogonality relations.

Proposition 3.29. *Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be representations. Then $P : \mathrm{Hom}(V,W) \to \mathrm{Hom}(\varphi,\rho)$ is a linear map.*

[2] We use the principle of excluded middle, according to which "any statement is true or false." In particular any given representation should be irreducible or reducible. Thus, if every reducible representation is decomposable, the theorem follows. However, when it comes to computations, this principle is flawed, and this is sometimes discussed among mathematicians. For more information, read about constructive mathematics.

Proof. If $T \in \mathrm{Hom}(V, W)$ and $c_j \in K$ define $g(x, y) := \rho(x)T\varphi(y^*)$. Theorem 2.9 gives

$$P(T)\varphi(c_j) = \frac{1}{w(K)} \sum_i g(c_i, c_j^* c_i)w(c_i) = \frac{1}{w(K)} \sum_i g(c_j c_i, c_i)w(c_i) = \rho(c_j)P(T).$$

The proof of linearity is obvious. □

The property of P stated above is not its only interesting property. It is also connected to the irreducible representations through the inner product. To be prove this, we first need the following Lemma.

Lemma 3.30. *Let $A \in M_{rm}(\mathbb{C})$, $B \in M_{ns}(\mathbb{C})$, and $E_{lj} \in M_{mn}(\mathbb{C})$, where E_{lj} is a matrix with a 1 in position (l, j), and 0 elsewhere. Then*

$$(AE_{lj}B)_{ki} = a_{kl}b_{ji},$$

where $A = (a_{ij})$ and $B = (b_{ij})$.

Proof. See appendix D. □

We can now move on to the following proposition, which shows that the inner product of the representations is encoded in the "projection matrix".

Proposition 3.31. *Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be representations, where $\deg V = n$, and $\deg W = m$. Let $E_{lj} \in M_{mn}(\mathbb{C})$ be a matrix with a 1 in position lj and 0 elsewhere. Then $\langle \varphi_{ij}, \rho_{kl} \rangle = P(E_{lj})_{ki}$.*

Proof. By Corollary 3.8, we have $\overline{\varphi_{ij}(c_z)} = \varphi_{ji}(c_z^*)$. Using this and Lemma 3.30, we compute

$$\langle \varphi_{ij}, \rho_{kl} \rangle = \frac{1}{w(K)} \sum_z \rho_{kl}(c_z)\overline{\varphi_{ij}(c_z)}w(c_z) = \frac{1}{w(K)} \sum_z (\rho(c_z)E_{lj}\varphi(c_z^*))_{ki}w(c_z) = P(E_{lj})_{ki}.$$

□

In order to formulate the Schur orthogonality relations, we need a concept which is not needed in the group case; namely that of a *hyperdimension*. In the group case the hyperdimension of a representation is just the dimension of its associated vector space, which justifies the name.

Definition 3.32 (Hyperdimension)**.** Let $\varphi : K \to \mathrm{End}(V)$ be an irreducible representation, and choose a basis for V as in Corollary 3.8. By the *hyperdimension* of φ we mean

$$k(\varphi) := \frac{1}{\langle \varphi_{ts}, \varphi_{ts} \rangle},$$

for any indices t and s.

The Schur orthogonality relations show in particular that the hyperdimension is well-defined, i.e. the value of $k(\varphi)$ is independent of the choice of t and s.

Theorem 3.33 (Schur Orthogonality Relations)**.** *Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be inequivalent irreducible representations. Then:*

$$(i)\ \ \langle \varphi_{ij}, \rho_{kl} \rangle = 0,$$

$$(ii)\ \ \langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} \frac{1}{k(\varphi)} & \text{if } (i, j) = (k, l) \\ 0 & \text{otherwise,} \end{cases}$$

and $k(\varphi)$ is well-defined.

Proof. (i) By Proposition 3.31, $\langle \varphi_{ij}, \rho_{kl} \rangle = P(E_{lj})_{ki}$, and by Proposition 3.29, $P(E_{lj}) \in \mathrm{Hom}(\varphi, \rho)$. As Theorem 3.22 states that all $T \in \mathrm{Hom}(\varphi, \rho)$ are either zero or invertible, and as the existence of such an invertible T would imply $\varphi \sim \rho$, we conclude that $P(E_{lj}) = 0$, so that $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$.

(ii) Using the same theorems again, we see that

$$\langle \varphi_{ij}, \varphi_{kl} \rangle = P(E_{lj})_{ki} \quad \text{and} \quad P(E_{lj}) = \lambda I.$$

From this, we can easily see that $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$ when $i \neq k$, and also that as long as $i = k$, the value of $\langle \varphi_{ij}, \rho_{kl} \rangle$ is independent of the value of i and k.

We may now show the following

$$\begin{aligned}\langle \varphi_{ij}, \varphi_{kl} \rangle &= \frac{1}{w(K)} \sum_z \overline{\varphi_{ij}(c_z)} \varphi_{kl}(c_z) w(c_z^*) = \frac{1}{w(K)} \sum_z \varphi_{ji}(c_z^*) \varphi_{kl}(c_z) w(c_z^*) \\ &= \frac{1}{w(K)} \sum_z \varphi_{ji}(c_z) \varphi_{kl}(c_z^*) w(c_z) = \frac{1}{w(K)} \sum_z \varphi_{ji}(c_z) \overline{\varphi_{lk}(c_z)} w(c_z^*) = \langle \varphi_{lk}, \varphi_{ji} \rangle\end{aligned}$$

which proves that $\langle \varphi_{ij}, \varphi_{kl} \rangle = 0$ when $j \neq l$ as well, and also that as long as $j = l$, $\langle \varphi_{ij}, \varphi_{kl} \rangle$ is independent of the value of j and l. From this we may state

$$\langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} \frac{1}{k(\varphi)} & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise.} \end{cases}$$

□

Using this orthogonality we can now demonstrate that there are only a finite number of inequivalent irreducible representations.

Corollary 3.34. *Let K be a hypergroup. Then the number of inequivalent irreducible representations of K is finite, and if $(\varphi^{(i)})$ is a complete list of inequivalent irreducible representations, then $1 \leq \sum_i \deg(\varphi^{(i)})^2 \leq K$.*

Proof. The first inequality follows from that $\varphi^{(0)}(c_j) := 1$ for all $c_j \in K$ is an irreducible representation of K by Proposition 3.16. For the second, note that $\{\varphi_{ij}^{(k)},\ 0 \leq i,j,k < |K|\}$ is an orthogonal set by Theorem 3.33. And since this set has $\sum_i \deg(\varphi^{(i)})^2$ elements, and all elements are in $\mathbb{C}^K$, we know that $\sum_i \deg(\varphi^{(i)})^2 \leq \dim(\mathbb{C}^K) = |K|$. □

Later we will see that this final inequality is actually an equality.
As we now know that $k(\varphi)$ is well-defined we derive an upper limit to its value.

Proposition 3.35. *Let K be a hypergroup and $\varphi : K \to \mathrm{End}(V)$ an irreducible representation. Then $w(K) \geq k(\varphi)$.*

Proof. Using the definition of hyperdimension, we compute

$$\begin{aligned}\frac{1}{k(\varphi)} = \langle \varphi_{ii}, \varphi_{ii} \rangle &= \frac{1}{w(K)} \sum_z \varphi_{ii}(c_z) \overline{\varphi_{ii}(c_z)} w(c_z) \\ &= \frac{1}{w(K)} \sum_z |\varphi_{ii}(c_z)|^2 w(c_z) \geq \frac{1}{w(K)} |\varphi_{ii}(c_0)|^2 w(c_0) = \frac{1}{w(K)}\end{aligned}$$

and thus $w(K) \geq k(\varphi)$. □

We have now introduced and derived the most fundamental parts of the representation theory of finite hypergroups. Among the chief results we have the Schur orthogonality, and the fact that all representations can be decomposed into a direct sum of irreducible representations. To show that this decomposition is unique we will need to first introduce the concept of the character of a representation, which is also interesting in and of itself.

3.1 Character Theory

In this section, we will be looking at the characters of representations. We will see that a lot of information about the hypergroup is encoded in the characters of the irreducible representations. The main results in this section will be the orthogonality of characters, the regular representation, and the uniqueness of decomposition.

We begin by defining the character of a representation.

Definition 3.36 (Character)**.** Let $\varphi : K \to \mathrm{End}(V)$ be a representation. The *character* $\chi_\varphi : K \to \mathbb{C}$ is defined as

$$\chi_\varphi(c_k) = \mathrm{tr}(\varphi(c_k)).$$

By an *irreducible character*, we mean the character of an irreducible representation.

We will now proceed to prove some basic properties of characters.

Proposition 3.37. *Let $\varphi : K \to \mathrm{End}(V)$ be a representation. Then $\chi_\varphi(c_0) = \deg\varphi$.*

Proof. By the definition of character, $\chi_\varphi(c_0) = \mathrm{tr}(\varphi(c_0)) = \mathrm{tr}(I) = \deg\varphi$. □

Proposition 3.38. *Let K be a hypergroup. If φ and ρ are representations of K, then $\chi_{(\varphi\oplus\rho)} = \chi_\varphi + \chi_\rho$.*

Proof. This follows from the definition of character and direct sum. □

Although the following theorem is not very hard to show, it is important as it shows that equivalent representations have the same character which is a coveted propery.

Theorem 3.39. *Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be representations. If $\varphi \overset{T}{\sim} \rho$, then $\chi_\varphi(c_k) = \chi_\rho(c_k)$ for all $c_k \in K$.*

Proof. By equivalence, $\varphi(c_k) = T^{-1}\rho(c_k)T$ holds for all $c_k \in K$. Recall that always $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Therefore,

$$\chi_\varphi(c_k) = \mathrm{tr}(\varphi(c_k)) = \mathrm{tr}(T^{-1}\rho(c_k)T) = \mathrm{tr}(TT^{-1}\rho(c_k)) = \mathrm{tr}(\rho(c_k)) = \chi_\rho(c_k).$$

□

Now that we have established some of the basic properties of characters we move on to their orthogonality relations, which is an important result.

Theorem 3.40 (Orthogonality of characters)**.** *Let $\varphi : K \to \mathrm{End}(V)$ and $\rho : K \to \mathrm{End}(W)$ be irreducible representations. Then*

$$\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} \frac{\deg\varphi}{k(\varphi)} & \text{if } \varphi \sim \rho \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\varphi \sim \rho$ then $k(\varphi) = k(\rho)$.

Proof. Observe that $\chi_\varphi = \sum_i \varphi_{ii}$ and $\chi_\rho = \sum_i \rho_{ii}$. The terms internal to these sums are pairwise orthogonal by Theorem 3.33, and the sums are orthogonal to each other unless $\varphi \sim \rho$ by Theorem 3.33, in which case by Theorem 3.39 and the Pythagorean theorem for inner product spaces we get

$$\langle \chi_\varphi, \chi_\rho \rangle = \langle \chi_\varphi, \chi_\varphi \rangle = \langle \sum_i \varphi_{ii}, \sum_i \varphi_{ii} \rangle = \sum_i \langle \varphi_{ii}, \varphi_{ii} \rangle = \sum_{0 \le i < \deg(\varphi)} \frac{1}{k(\varphi)} = \frac{\deg(\varphi)}{k(\varphi)}.$$

□

Following this theorem we now define a useful construction called the regular representation, which we will see later, "contains" all irreducible representations in some sense, and will aid us in several ways in other theorems.

Definition 3.41 (Regular representation)**.** Let K be a hypergroup. By its *regular representation* $L : K \to \mathrm{End}(\mathbb{C}^{|K|})$ we mean the k-indexed family of transformations given in matrix form by

$$L(c_k)_{ij} := n^i_{kj}.$$

Despite its name, it is not obvious that L is a representation at all. The next proposition proves that it is a representation.

Proposition 3.42. *Let K be a hypergroup. Then the regular representation $L : K \to \mathrm{End}(\mathbb{C}^{|K|})$ is a representation.*

Proof. $L(c_0) = I$ since $L(c_0)_{ij} = n^i_{0j} = \begin{cases} 1 \text{ if } & i = j, \\ 0 \text{ otherwise,} \end{cases}$ by Theorem 2.6 (4).

For the second requirement, use Theorem 2.6 (3):

$$(L(c_g)L(c_k))_{ij} = \sum_x L(c_g)_{ix} L(c_k)_{xj} = \sum_x n^i_{gx} n^x_{kj} = \sum_y n^y_{gk} n^i_{yj} = \sum_y n^y_{gk} L(c_y)_{ij}.$$

□

Example 3.43. We now want to continue to use Example 2.4 to contextualise the regular representation. We therefore calculate the regular representation of all the elements of K which gives us the following matrices.

$$L(c_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad L(c_1) = \begin{pmatrix} 0 & 1/3 & 0 \\ 1 & 0 & 1 \\ 0 & 2/3 & 0 \end{pmatrix} \quad L(c_2) = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1/2 \end{pmatrix}$$

We now define the notion of multiplicity in order to show in what sense the decomposition of a representation is unique. As a special case of this the decomposition of L will be studied, which will be used to derive an interesting result. When talking about multiplicities, $m\varphi$ is to be understood as the direct sum $m\varphi := \bigoplus_{0 \le k < m} \varphi = \varphi \oplus \cdots \oplus \varphi$.

Definition 3.44 (Multiplicity)**.** If $\varphi \sim \bigoplus_{0 \le i < s} m_i \varphi^{(i)} = m_0 \varphi^{(0)} \oplus \cdots \oplus m_{s-1} \varphi^{(s-1)}$, where $\varphi^{(i)}$ are inequivalent irreducible representations, then m_i is called the *multiplicity* of $\varphi^{(i)}$ for φ.

Using this notion of multiplicity, the next proposition shows what is meant by the decomposition being unique, and proves that it is unique.

Proposition 3.45. *If $\varphi \sim \bigoplus m_i \varphi^{(i)}$ and $\rho \sim \bigoplus r_j \rho^{(j)}$, where $(\varphi^{(i)})$ is a list of inequivalent irreducible representations, and $(\rho^{(j)})$ is another list of inequivalent irreducible representations, and $\varphi \sim \rho$, then $m_i = r_j$ whenever $\varphi^{(i)} \sim \rho^{(j)}$. Thus the multiplicity of a representation is well-defined and respects equivalence.*

Proof. Using the orthogonality of characters we get

$$\langle \chi_{\varphi^{(i)}}, \chi_\varphi \rangle = \sum_k m_k \langle \chi_{\varphi^{(i)}}, \chi_{\varphi^{(k)}} \rangle = m_i \langle \chi_{\varphi^{(i)}}, \chi_{\varphi^{(i)}} \rangle$$
$$\langle \chi_{\rho^{(j)}}, \chi_\rho \rangle = \sum_k r_k \langle \chi_{\rho^{(j)}}, \chi_{\rho^{(k)}} \rangle = r_j \langle \chi_{\rho^{(j)}}, \chi_{\rho^{(j)}} \rangle .$$

Since the characters in both equations are equal by Theorem 3.39, we get $m_i = r_j$. □

Now that we have demonstrated the uniqueness of the decomposition, we return to the regular representation.

Theorem 3.46. *If K is a hypergroup, then the regular representation is decomposed as*

$$L \sim \bigoplus_{0 \le i < s} \deg(\varphi^{(i)}) \varphi^{(i)} = \deg(\varphi^{(0)}) \varphi^{(0)} \oplus \cdots \oplus \deg(\varphi^{(s-1)}) \varphi^{(s-1)}$$

where $(\varphi^{(i)})_{0 \le i < s}$ is a complete list of inequivalent irreducible representations of K.

Proof. Let $d_i := \deg(\varphi^{(i)})$ and $k_i := k(\varphi^{(i)})$. By Theorem 3.25 we can decompose $L \sim \bigoplus_i m_i \varphi^{(i)}$. Taking the trace of this we get $\chi_L = \sum_i m_i \chi_{\varphi^{(i)}}$ so that by Theorem 3.40 $\langle \chi_{\varphi^{(i)}}, \chi_L \rangle = m_i d_i / k_i$. On the other hand, by definition,

$$\langle \chi_{\varphi^{(i)}}, \chi_L \rangle = \frac{1}{w(K)} \sum_z \chi_L(c_z) \overline{\chi_{\varphi^{(i)}}(c_z)} w(c_z).$$

Expanding $\chi_L(c_z) = \sum_u n_{zu}^u$ by definition, rewriting $n_{zu}^u w(c_z) = n_{uu^*}^{z^*} w(c_u)$ by Lemma 2.7 and rewriting $\overline{\chi_{\varphi^{(i)}}(c_z)} = \chi_{\varphi^{(i)}}(c_z^*)$ by algebraic manipulation, we continue

$$\langle \chi_{\varphi^{(i)}}, \chi_L \rangle = \frac{1}{w(K)} \sum_{z,u} n_{uu^*}^{z^*} \chi_{\varphi^{(i)}}(c_z^*) w(c_u),$$

where $\sum_z n_{uu^*}^{z^*} \chi_{\varphi^{(i)}}(c_z^*) = \chi_{\varphi^{(i)}}(c_u c_u^*) = \operatorname{tr}(\varphi^{(i)}(c_u)\varphi^{(i)}(c_u)^*) = \sum_{x,y} \varphi_{xy}^{(i)}(c_u) \overline{\varphi_{xy}^{(i)}(c_u)}$ so that

$$\langle \chi_{\varphi^{(i)}}, \chi_L \rangle = \sum_{x,y} \langle \varphi_{xy}^{(i)}, \varphi_{xy}^{(i)} \rangle = \frac{d_i^2}{k_i}.$$

Equating these formulas for $\langle \chi_{\varphi^{(i)}}, \chi_L \rangle$ we get $m_i d_i / k_i = d_i^2 / k_i$ and thus $m_i = d_i$. □

Using the decomposition of L, we are able to prove an equality which we previously stated without proof.

Corollary 3.47. *Let K be a hypergroup, and let $(\varphi^{(i)})$ be a complete list of irreducible representations of K. Then $|K| = \sum_i \deg(\varphi^{(i)})^2$.*

Proof. By Theorem 3.46, $L \sim \bigoplus_{0 \le i < s} \deg(\varphi^{(i)}) \varphi^{(i)}$. Using this together with the basic properties of characters, we have

$$|K| = \deg L = \chi_L(c_0) = \sum_i \deg(\varphi^{(i)}) \chi_{\varphi^{(i)}}(c_0) = \sum_i \deg(\varphi^{(i)})^2.$$

□

Armed with this equality, we can prove that the irreducible representations can be used as a basis for $\mathbb{C}^K$, which is shown in the following theorem.

Theorem 3.48. *Let K be a hypergroup. If $(\varphi^{(i)})$ is a complete list of inequivalent irreducible representations, then the set $\{\varphi_{ij}^{(k)},\ 0 \le i, j, k < |K|\}$ is a basis of $\mathbb{C}^K$.*

Proof. By Corollary 3.34 there are finitely many irreducible representations, and thus there is such a thing as a complete list. The set $\{\varphi_{ij}^{(k)},\ 0 \le i, j, k < |K|\}$ has $\sum \deg(\varphi^{(i)})$ elements, and by the Schur orthogonality (Theorem 3.33) the set is orthogonal. Note that all the elements in this set are functions in $\mathbb{C}^K$. And as $|K| = \sum \deg(\varphi^{(i)})^2$ by Corollary 3.47, and $\dim(\mathbb{C}^K) = |K|$, the theorem follows. □

We have now constructed the framework for a representation theory of hypergroups complete with a consideration of the character theory as well. We will now proceed to consider the more restricted category of commutative hypergroups which will allow us to define a notion of Fourier analysis on hypergroups.

4 Commutative Hypergroups

In this section we first consider theorems which implies commutativity of a hypergroup. We then restrict ourselves to the study of commutative hypergroups, and of their representation theory. This restriction is imposed as it leads to further tools and theorems that one might find useful. Furthermore, a portion of the hypergroups that may be of particular interest are commutative. After this we proceed to develop a notion of Fourier analysis on hypergroups, which is then used to develop a method for computing the structure constants of a commutative hypergroup from its characters.

Proposition 4.1. *Let K be a hypergroup. If all its irreducible representations have degree one, the hypergroup is commutative.*

Proof. Let $(\varphi^{(i)})$ be a complete list of irreducible representations. Let $d_i := \deg(\varphi^{(i)})$ and $k_i := k(\varphi^{(i)})$. Having degree one, those representations commute. By Theorem 3.46, $L \sim \bigoplus_i d_i \varphi^{(i)}$. Then

$$\varphi^{(i)}(c_s c_t) = \varphi^{(i)}(c_s)\varphi^{(i)}(c_t) = \varphi^{(i)}(c_t)\varphi^{(i)}(c_s) = \varphi^{(i)}(c_t c_s)$$
$$L(c_s c_t) = \bigoplus_i d_i \varphi^{(i)}(c_s c_t) = \bigoplus_i d_i \varphi^{(i)}(c_t c_s) = L(c_t c_s).$$

It is readily verified that $L(c_s c_t)_{u0} = \sum_v n_{st}^v L(c_v)_{u0} = \sum_v n_{st}^v n_{v0}^u = n_{st}^u$. Thus we have that

$$c_s c_t = \sum_u n_{st}^u c_u = \sum_u L(c_s c_t)_{u0} c_u = \sum_u L(c_t c_s)_{u0} c_u = \sum_u n_{ts}^u c_u = c_t c_s.$$

□

Using this, we may prove the following proposition, which shows that all hypergroups of order strictly less than five must be commutative. This proposition is as general as possible, in the sense that a non-commutative hypergroup of order five is given in [5].

Proposition 4.2. *All hypergroups of order strictly less than five are commutative.*

Proof. Let K be a hypergroup of order strictly less than 5, with a complete list of irreducible representations $(\varphi^{(i)})$. Let $d_i := \deg(\varphi^{(i)})$ and $k_i := k(\varphi^{(i)})$. We have shown in Corollary 3.47 that

$$|K| = \sum_i d_i^2.$$

Since the mapping $\varphi(c_k) = 1$ for all c_k is an irreducible representation for any hypergroup, it must be one of the $\varphi^{(i)}$. We may assume that we have numbered the irreducible representations so that $\varphi^{(0)}(c_k) = 1$ for all c_k. We now have

$$|K| = 1 + \sum_{i \neq 0} d_i^2.$$

Now we see that all d_i must be 1, because if we assume that at least one of them is strictly greater than one we get

$$1 + \sum_{i \neq 0} d_i^2 \geq 1 + 2^2 = 5 > |K|$$

which is a contradiction. And so $d_i = 1$ for all i, which by Proposition 4.1 gives us that K is commutative. □

The previous results show conditions that imply commutativity. From this point on we will instead restrict ourselves to the study of commutative hypergroups. We start with an important result for commutative hypergroups, which will be used frequently thoughout the rest of the text.

Proposition 4.3. *Let K be a hypergroup. If K is commutative and φ is an irreducible representation of K, then φ has degree 1.*

Proof. Since all $\varphi(c_i)\varphi(c_j) = \varphi(c_j)\varphi(c_i)$, every $\varphi(c_i)$ is an endomorphism of φ, and so is equal to some $\lambda_i I$ by Theorem 3.22. Choose a nonzero vector $v \in V$. From $\varphi(c_i)v = \lambda_i v$ and irreducibility it follows that $\mathbb{C}v = V$. □

Corollary 4.4. *If K is a commutative hypergroup, then there are exactly $|K|$ irreducible representations of K.*

Proof. This follows from Proposition 4.3 and Corollary 3.47. □

Since $\deg(\varphi) = 1$ for all irreducible representations of commutative hypergroups, the irreducible representations are equal to the irreducible characters. We will therefore not make any distinction between these in the text about commutative hypergroups. Using this, we may now show that the irreducible characters form a basis for $\mathbb{C}^K$.

Theorem 4.5. *Let K be a commutative hypergroup. Then the irreducible characters form a basis for $\mathbb{C}^K$.*

Proof. As stated above, the irreducible characters are equal to the irreducible representations. Theorem 3.48 now implies that the irreducible characters are a basis for $\mathbb{C}^K$. $\square$

We have now exemplified conditions which imply commutativity, and have then conversely derived some facts that are implied by commutativity. We will now proceed to study Fourier analysis on commutative hypergroups.

4.1 Fourier Analysis on Hypergroups

In this section we explore the theory of Fourier analysis on hypergroups which will culminate in a method for computing the structure constants of a commutative hypergroup from its character table. The section is largely based on Steinberg [4]. Note that the only differences from the group case is that $k(\varphi)$ is not necessarily equal to 1, the exchange of the order of the group with the weight of the hypergroup, and the addition of the weight function to the Fourier transform.

We begin by defining the dual space of a hypergroup.

Definition 4.6 (Dual space)**.** Let K be a hypergroup. By the *dual space* of K, denoted by $\hat{K}$, we mean the set of all irreducible characters $\varphi : K \to \mathbb{C}$.

Note that these are equal to the representations themselves as long as K is commutative, and so one can speak of the hyperdimension of a character. We will now introduce the Fourier transform, which is a function on the dual space.

Definition 4.7 (Fourier transform)**.** Let K be a commutative hypergroup, and let $f : K \to \mathbb{C}$. We define the *Fourier transform* $\hat{f} : \hat{K} \to \mathbb{C}$ for $\varphi \in \hat{K}$ as

$$\hat{f}(\varphi) := w(K) \langle \varphi, f \rangle = \sum_{c_z \in K} f(c_z)\overline{\varphi(c_z)}w(c_z).$$

Example 4.8. If $\varphi^{(i)}, \varphi^{(j)} \in \hat{K}$, then

$$\widehat{\varphi^{(j)}}(\varphi^{(i)}) = w(K) \langle \varphi^{(i)}, \varphi^{(j)} \rangle = \begin{cases} \frac{w(K)}{k(\varphi^{(i)})} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.9. The regular representation satisfies

$$\begin{aligned} \widehat{L_{ij}}(\varphi) &= \sum_{c_z} L_{ij}(c_z)\overline{\varphi(c_z)}w(c_z) = \sum_{c_z} n_{zj}^{i}\overline{\varphi(c_z)}w(c_z) = \sum_{c_z} n_{ji^*}^{z^*}\overline{\varphi(c_z)}w(c_i) \\ &= w(c_i) \sum_{c_z} n_{ji^*}^{z}\varphi(c_z) = w(c_i)\overline{\varphi(c_i)}\varphi(c_j). \end{aligned}$$

Just as one might expect from Fourier analysis on $\mathbb{R}$, there is not only a Fourier transform, but also an inverse Fourier transform.

Theorem 4.10 (Fourier inversion)**.** *Let K be a commutative hypergroup and $f : K \to \mathbb{C}$, then*

$$f = \frac{1}{w(K)} \sum_{\varphi \in \hat{K}} k(\varphi)\hat{f}(\varphi)\varphi.$$

Proof. Recall, from linear algebra, that the orthogonal projection of a vector v into the span of a list of pairwise orthogonal vectors $(u)_{0 \le i < n}$ is given by $\sum_{0 \le i < n} \frac{\langle u,v \rangle}{\langle u,u \rangle} u$. Also, the projection of a vector u already spanned by (v) is u. Thus, since the characters are an orthogonal basis for $\mathbb{C}^K$ by Theorem 3.40 and Theorem 4.5, so we have

$$\begin{aligned} f &= \sum_{\varphi \in \hat{K}} \frac{\langle \varphi, f \rangle}{\langle \varphi, \varphi \rangle}\varphi = \sum_{\varphi \in \hat{K}} k(\varphi) \langle \varphi, f \rangle \varphi = \frac{1}{w(K)} \sum_{\varphi \in \hat{K}} w(K)k(\varphi) \langle \varphi, f \rangle \varphi \\ &= \frac{1}{w(K)} \sum_{\varphi \in \hat{K}} k(\varphi)\hat{f}(\varphi)\varphi \end{aligned}$$

$\square$

Proposition 4.11. *The map from f to $\hat{f}$ is an invertible and linear transformation.*

Proof. The invertibility follows from Theorem 4.10. The linearity follows from computing

$$\begin{aligned}(\widehat{nf_1 + mf_2})(\varphi) &= w(K)\langle \varphi, nf_1 + mf_2\rangle = nw(K)\langle \varphi, f_1\rangle + mw(K)\langle \varphi, f_2\rangle = \\ &= n\hat{f}_1 + m\hat{f}_2,\end{aligned}$$

and so the map is linear. $\square$

Now that we have a Fourier transform, we define a product on the space of functions from K to $\mathbb{C}$, i.e. $\mathbb{C}^K$, which we will later see corresponds to pointwise multiplication under the Fourier transform.

Definition 4.12 (Convolution product)**.** Let K be a commutative hypergroup. Then we define

$$(a * b)(c_z) := \sum_{c_x \in K} a(c_z c_x^*) b(c_x) w(c_x),$$

for functions $a, b \in \mathbb{C}^K$.

Proposition 4.13. *Let K be a commutative hypergroup. The convolution product is commutative, i.e.*

$$(a * b)(c_z) = (b * a)(c_z) \quad (c_z \in K)$$

for all functions $a, b \in \mathbb{C}^K$.

Proof. We compute

$$\begin{aligned}(a * b)(c_z) &= \sum_{c_x \in K} a(c_z c_x^*) b(c_x) w(c_x) = \sum_{c_s, c_x \in K} a(c_s) b(c_x) n^s_{zx^*} w(c_x) = \sum_{c_s, c_x \in K} a(c_s) b(c_x) n^x_{s^* z} w(c_s) \\ &= \sum_{c_s \in K} a(c_s) b(c_s^* c_z) w(c_s) = \sum_{c_s \in K} a(c_s) b(c_z c_s^*) w(c_s) = (b * a)(c_z),\end{aligned}$$

where we have used Theorem 2.9 and the fact that K is commutative. $\square$

Now we are ready to demonstrate the earlier claim that the convolution product corresponds to pointwise multiplication under the Fourier transform.

Theorem 4.14. *Let K be a commutative hypergroup. Then the Fourier transform and convolution product satisfy*

$$\widehat{(a * b)}(\varphi) = \hat{a}(\varphi) \cdot \hat{b}(\varphi) \quad (\varphi \in \hat{K})$$

Proof. Use Theorem 2.9 with $g(x, y) := \overline{\varphi(x)} a(y)$, and note that $\varphi(c_z c_x) = \varphi(c_z)\varphi(c_x)$.

$$\begin{aligned}\widehat{a * b}(\varphi) &= \sum_{z,x} \overline{\varphi(c_z)} a(c_z c_x^*) b(c_x) w(c_x) w(c_z) = \sum_{z,x} \overline{\varphi(c_z c_x)} a(c_z) b(c_x) w(c_x) w(c_z) \\ &= \sum_{z} \overline{\varphi(c_z)} a(c_z) w(c_z) \sum_{x} \overline{\varphi(c_x)} b(c_x) w(c_x) = \hat{a}(\varphi) \cdot \hat{b}(\varphi).\end{aligned}$$

$\square$

We have now developed a basic theory of Fourier analysis on commutative hypergroups. In the rest of this section we will use this theory to derive a method for obtaining the structure constants of a hypergroup from its characters.

To achieve this we first need to derive a number of results, starting with the following.

Lemma 4.15. *Let K be a commutative hypergroup. Then the structure constants satisfy*

$$n^s_{ij} = \frac{w(c_s)}{w(K)} \sum_{\varphi \in \hat{K}} k(\varphi) \overline{\varphi(c_s)} \varphi(c_i) \varphi(c_j).$$

Proof. By the definition of the regular representation, we know that $n_{ij}^s = L_{sj}(c_i)$, and from Example 4.9 we know that $\widehat{L_{sj}}(\varphi) = w(c_s)\overline{\varphi(c_s)}\varphi(c_j)$. By using fourier inversion we now obtain the desired result:

$$n_{ij}^s = L_{sj}(c_i) = \frac{1}{w(K)} \sum_{\varphi \in \hat{K}} k(\varphi)\widehat{L_{sj}}(\varphi)\varphi(c_i) = \frac{w(c_s)}{w(K)} \sum_{\varphi \in \hat{K}} k(\varphi)\overline{\varphi(c_s)}\varphi(c_j)\varphi(c_i)$$

□

The preceding Lemma is already in itself a way to compute the structure constants, but requires that not only characters are known, but also the weight functions and hyperdimensions. We will soon show that these can be computed given the characters.

We now proceed to define the character table, and prove that it is invertible. This will be used in the method for acquiring the structure constants, but is also interesting in itself, and will also be used on several different occasions in the appendices.

Definition 4.16 (Character table)**.** Let K be a commutative hypergroup, and $(\varphi^{(i)})_{0 \le i < |K|}$ a complete list of irreducible characters of the hypergroup. By the *character table* of K, we mean a $|K| \times |K|$ matrix X with (i,j)-th coefficient

$$X_{ij} := \varphi^{(i)}(c_j).$$

Theorem 4.17. *The matrix X' defined by $X'_{ij} := X_{ij}\sqrt{\frac{w(c_j)k_i}{w(K)}}$, where $k_i := k(\varphi^{(i)})$, has orthonormal rows and columns. As a result of this, the character table is invertible.*

Proof. By Theorem 3.40 we see that the rows of X' are orthonormal. Thus the columns of X' are also orthonormal. □

Using this, we can derive a result which will come in handy in the appendix.

Corollary 4.18. *Let K be a commutative hypergroup, and $(\varphi^{(i)})_{0 \le i < |K|}$ a complete list of irreducible representations of the hypergroup. Then*

$$w(K) = \sum_{0 \le i < s} k(\varphi^{(i)}).$$

Proof. Let $k_i := k(\varphi^{(i)})$. By Theorem 4.17 we know that the columns of $X'_{ij} := X_{ij}\sqrt{\frac{w(c_j)k_i}{w(K)}}$ are orthonormal, and thus

$$1 = \sum_i \overline{\varphi^{(i)}(c_k)}\varphi^{(i)}(c_k)\frac{w(c_k)k_i}{w(K)}.$$

By setting $k = 0$ we get

$$w(K) = \sum_i \overline{\varphi^{(i)}(c_0)}\varphi^{(i)}(c_0)w(c_0)k_i = \sum_i k_i.$$

□

Now that we have introduced the character table, and proved that it is invertible, we return to the task of computing the structure constants. We have already shown how it can be computed given the characters, weight functions, and hyperdimensions. Now we wish to show that the weight functions and hyperdimensions can be computed from the character table, so that the structure constants can be computed given only the character table.

The following proposition will be used to define a system of equations which will be used to compute the weight functions and hyperdimensions.

Proposition 4.19. *Let K be a commutative hypergroup. Then*

$$\frac{1}{w(K)} \sum_{\varphi \in \hat{K}} \varphi(c_i)k(\varphi) = \begin{cases} 1 & \textit{if } \ i = 0, \\ 0 & \textit{otherwise}, \end{cases}$$

Proof. Since c_0 is the identity element, n_{i0}^0 is 1 if $i = 0$, and 0 otherwise. By Lemma 4.15 we therefore have

$$n_{i0}^0 = \frac{1}{w(K)} \sum_{\varphi \in \hat{K}} \varphi(c_i) k(\varphi) = \begin{cases} 1 \text{ if } \ i = 0, \\ 0 \text{ otherwise,} \end{cases}$$

□

Using this proposition, we are now able to compute the weight functions and hyperdimensions from the character table.

Proposition 4.20. *Let K be a hypergroup with irreducible representations $(\varphi^{(i)})$, where $\varphi^{(0)}$ is the trivial representation, and character table X. Then we can calculate $w(c_i)$, $k(\varphi^{(i)})$, and $w(K)$ for all i as*

$$w(K) = \frac{1}{(X^{-1})_{00}}, \quad w(c_i) = \frac{(X^{-1})_{i0}}{(X^{-1})_{00}}, \quad k(\varphi^{(i)}) = \frac{(X^{-1})_{0i}}{(X^{-1})_{00}}$$

Proof. By Theorem 3.33 we have that

$$\sum_{c_z \in K} \varphi^{(i)}(c_z) w(c_z) = \langle \varphi^{(0)}, \varphi^{(i)} \rangle = \begin{cases} w(K) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and by 4.19, we have

$$\sum_{\varphi^{(z)} \in \hat{K}} \varphi^{(z)}(c_i) k(\varphi^{(z)}) = \begin{cases} w(K) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

These expressions can be written as a linear system of equations on the form

$$XW = B \quad \text{and} \quad X^T H = B$$

where X is the character table, and W, H, and B are vectors such that $W_i := w(c_i)$, $H_i := k(\varphi^{(i)})$, $B_0 = 1$, and $B_i = 0$ when $i \neq 0$. Since the character table is invertible by Theorem 4.17, we have that

$$W = X^{-1} B \quad \text{and} \quad H = \left(X^{-1}\right)^T B$$

and thus

$$w(c_i) := W_i = \sum_v \left(X^{-1}\right)_{iv} B_v = \left(X^{-1}\right)_{i0} w(K)$$
$$k(\varphi^{(i)}) := H_i = \sum_v \left(\left(X^{-1}\right)^T\right)_{iv} B_v = \left(X^{-1}\right)_{0i} w(K)$$

Since we know that both $w(c_i)$ and $k(\varphi^{(i)})$ are nonzero for all i, both $\left(X^{-1}\right)_{i0}$ and $\left(X^{-1}\right)_{0i}$ must also be nonzero for all i. Using this, together with the fact that $w(c_0) = k(\varphi^{(0)}) = 1$, the result follows.

□

We now have the final result we need to compute the structure constants from the character table.

Theorem 4.21. *Let K be a commutative hypergroup with structure constants n_{ij}^s and character table X. Then the structure constants can be computed as follows:*

$$n_{ij}^s = \frac{\left(X^{-1}\right)_{s0}}{(X^{-1})_{00}} \sum_t \left(X^{-1}\right)_{0t} \overline{X_{ts}} X_{ti} X_{tj}$$

Proof. This follows from using Proposition 4.20 together with Lemma 4.15. □

We have now shown that all of the information of a commutative hypergroup is encoded in its character table, and how to compute the structure constants. In the following example, we return to the first and only example in the main text, wrapping it up by calculating its character table.

Example 4.22. Since the characters are the same as the irreducible representations we simply recall from Example 3.15 the irreducible representations of K. We input these into a table which results in Table 3b.

Table 3: Tables for $K = \{c_0, c_1, c_2\}$.

(a) Cayley table

	c_0	c_1	c_2
c_0	c_0	c_1	c_2
c_1	c_1	$\frac{1}{3}c_0 + \frac{2}{3}c_2$	c_1
c_2	c_2	c_1	$\frac{1}{2}c_0 + \frac{1}{2}c_2$

(b) Character table

	c_0	c_1	c_2
$\varphi^{(0)}$	1	1	1
$\varphi^{(1)}$	1	0	$-\frac{1}{2}$
$\varphi^{(2)}$	1	-1	1

For completeness, we calculate the inner product of the characters.

$$\begin{aligned} \langle \varphi^{(0)}, \varphi^{(0)} \rangle &= 1, \quad \langle \varphi^{(0)}, \varphi^{(1)} \rangle &= 0, \quad \langle \varphi^{(0)}, \varphi^{(2)} \rangle = 0, \\ \langle \varphi^{(1)}, \varphi^{(1)} \rangle &= \frac{1}{4}, \quad \langle \varphi^{(2)}, \varphi^{(1)} \rangle &= 0, \quad \langle \varphi^{(2)}, \varphi^{(2)} \rangle = 1. \end{aligned}$$

Thus by Theorem 3.33, the hyperdimensions are $k(\varphi^{(0)}) = 1$, $k(\varphi^{(1)}) = 4$, and $k(\varphi^{(2)}) = 1$.

This marks the end of the main text. We hope to have conveyed the general story of finite hypergroups and their representation theory. The interested reader can go on to read the appendix to see some more examples of hypergroups coming from groups and graphs, as well as how to construct them and the theory behind this.

References

[1] Wildberger NJ. Finite commutative hypergroups and applications from group theory to conformal field theory. Contemporary Mathematics. 1995;183:413–413.

[2] Wildberger NJ. Foundations of finite group theory for a (future) computer. The Mathematical Intelligencer. 2004;26(2):45–55.

[3] Lasser R. Discrete commutative hypergroups. Proceedings of the SIAM Summer School (on Orthogonal Polynomials, Harmonic Analysis, Approximation and Application). 2005;.

[4] Steinberg B. Representation theory of finite groups: an introductory approach. Springer Science & Business Media; 2011.

[5] Matsuzawa Y, Ohno H, Suzuki A, Tsurii T, Yamanaka S. Non-commutative hypergroup of order five. Journal of Algebra and Its Applications. 2015;p. 1750127.

[6] Wildberger NJ. MathFoundations150–172; 2015–2016. Available from: `https://www.youtube.com/watch?v=dyCRPT6iFBg&list=PLIljB45xT85AYIeGfDQwHM8i6PQEDnnTI`.

[7] Graham A. Kronecker Products and Matrix Calculus With Applications. Ellis Horwood Limited; 1981.

[8] Godsil C, Royle G. Algebraic Graph Theory. Springer Science & Business Media; 2001.

[9] Billington EJ, Oates-Williams S, Street AP. Combinatorial Mathematics IX: Proceedings of the Ninth Australian Conference on Combinatorial Mathematics Held at the University of Queensland, Brisbane, Australia, August 24-28, 1981. vol. 952. Springer; 2006.

[10] Durbin JR. Modern Algebra: An Introduction. 6th ed. John Wiley & Sons, Inc.; 2009.

[11] Lang S. Linear Algebra. Undergraduate texts in mathematics. Springer-Verlag, New York; 1987.

Appendix

A Hypergroups from Groups

In this appendix, we present two distinct ways to construct commutative hypergroups from an arbitrary finite group – the class hypergroup and the character hypergroup. The class hypergroup is constructed from the conjugacy classes of the group, and the character hypergroup is constructed from the irreducible characters of the group. We will also see that the irreducible characters of a commutative hypergroup generate a hypergroup under certain conditions. Finally, we show some examples.

A.1 Class Hypergroup

In this section we will demonstrate a way to generate a commutative hypergroup from the conjugacy classes of a finite group. But first we will need to introduce some notation, and derive some useful results. We shall find convenient the notion of a *multiset* from a finite group. This notion generalizes the notion of a subset, by allowing duplicates and even negative, fractional or complex multiplicities of elements. A multiset with fractional multiplicities summing to 1 is often interpreted as a probability distribution. For a recent and careful exposition of multiset theory, we refer to [6], but we also develop a minimal theory here, with [2] as a primary reference.

Definition A.1 (Multiset, Singleton, Product, Involution)**.** Let G be a finite group. By a *multiset from G* we mean a function $a : G \to \mathbb{C}$. We associate to each subset $S \subset G$ the multiset $S : G \to \mathbb{C}$ given by $S(g) := (1 \text{ if } g \in S \text{ else } 0)$. The *singletons* $[g] := \{g\} : G \to \mathbb{C}$, thus given by $[g](h) := (1 \text{ if } g = h \text{ else } 0)$ where $g, h \in G$. Obviously the set of singletons is a linear basis for multisets. The *product* of two singletons is defined as $[g][h] := [gh]$ and by linearity this multiplication is extended to arbitrary multisets from G. The *involution* of a singleton is defined as $[g]^* := [g^{-1}]$ and by linearity involution is also extended to arbitrary multisets from G.

Proposition A.2. *Let G be a finite group. Then the singletons from G form a hypergroup. This hypergroup is also a group, and as a group it is isomorphic to G.*

Proof. By trivial manipulations. □

What follows are the fundamental notions underlying the *class hypergroup* of a finite group. We aim to prove that the class hypergroup is indeed a hypergroup. We continue to follow [2] closely, but with more proofs.

Definition A.3 (Central, 1-class, Class hypergroup)**.** Let G be a finite group. A multiset $a : G \to \mathbb{C}$ is said to be *central* if $[x]a[x^{-1}] = a$ for all $x \in G$. By the *1-class* $c_g : G \to \mathbb{C}$ of an element $g \in G$ we mean the multiset $c_g := \frac{1}{|G|}\sum_{x\in G}[xgx^{-1}]$. By the *class hypergroup* $K(G)$ we mean the set of 1-classes from G.

Remark. It is possible for two different elements $g, h \in G$ to give the same 1-class $c_g = c_h : G \to \mathbb{C}$. In fact, the "more non-commutative" the group is, at least generally and relatively speaking, the fewer are the elements of $K(G)$. For instance the non-commutative *monster group M* has roughly 8×10^{53} elements, but its class hypergroup $K(M)$ has only 194 elements. Contrast this to the commutative case where the identity $c_g = [g]$ implies that $|K(G)| = |G|$.

Proposition A.4. *Let G be a finite group. The 1-classes of G are central.*

Proof. If $g, y \in G$ then

$$[y]c_g[y^{-1}] = [y]\frac{1}{|G|}\sum_{x\in G}[xgx^{-1}][y^{-1}] = \frac{1}{|G|}\sum_{x\in G}[yxgx^{-1}y^{-1}] = \frac{1}{|G|}\sum_{z\in G}[zgz^{-1}] = c_g$$

by definition and then the permutation $x \mapsto yx$. □

Definition A.5 (Conjugacy)**.** Let G be a finite group. Two elements $g, h \in G$ are said to be *conjugate*, denoted $g \sim h$, if there is an element $t \in G$ such that $g = tht^{-1}$. By the *conjugacy class* $Cl(g) \subset G$ of an element $g \in G$ we mean the set $Cl(g) := \{h \in G \;:\; g \sim h\}$. The set of conjugacy classes of G is denoted $G/\!\sim$ or $Cl(G)$.

Proposition A.6. *Let G be a finite group. Conjugacy is an equivalence relation. Furthermore, if $g, h \in G$ and $g \sim h$, then $g^{-1} \sim h^{-1}$.*

Proof. If $g \in G$ then $g = ege^{-1}$, where e is the neutral element of the group, and thus $g \sim g$. If $g \sim h$, say $g = xhx^{-1}$, then $h = x^{-1}g(x^{-1})^{-1}$ and thus $h \sim g$. If $g \sim h$, say $g = xhx^{-1}$, and $h \sim z$, say $h = yzy^{-1}$, then $g = xyzy^{-1}x^{-1} = (xy)z(xy)^{-1}$ and thus $g \sim z$. If $g \sim h$, say $g = xhx^{-1}$, then $g^{-1} = (xhx^{-1})^{-1} = xh^{-1}x^{-1}$ and thus $g^{-1} \sim h^{-1}$. □

Proposition A.7. *Let G be a finite group. Elements $g, h \in G$ are conjugate precisely when $c_g = c_h$.*

Proof. If $g \sim h$, say $g = tht^{-1}$, then

$$c_g = \frac{1}{|G|}\sum_{x \in G}[xtht^{-1}x^{-1}] = \frac{1}{|G|}\sum_{y \in G}[yhy^{-1}] = c_h$$

by the permutation $x \mapsto xt$. Conversely, if $c_g = c_h$ then in particular

$$\frac{1}{|G|}\sum_{x \in G}[xhx^{-1}](g) = c_h(g) = c_g(g) = \frac{1}{|G|}\sum_{x \in G}[xgx^{-1}](g) \geq \frac{1}{|G|}[ege^{-1}](g) = \frac{1}{|G|} > 0,$$

so $xhx^{-1} = g$ for at least one x, showing that $g \sim h$. □

Proposition A.8. *Let G be a finite group. If $g \in G$ then $c_g^* = c_{g^{-1}}$.*

Proof. Expand the definitions and use that $[xgx^{-1}]^* = [(xgx^{-1})^{-1}] = [xg^{-1}x^{-1}]$.

$$c_g^* = \left(\frac{1}{|G|}\sum_{x \in G}[xgx^{-1}]\right)^* = \frac{1}{|G|}\sum_{x \in G}[xgx^{-1}]^* = \frac{1}{|G|}\sum_{x \in G}[xg^{-1}x^{-1}] = c_{g^{-1}}.$$

□

Proposition A.9. *Let $a : G \to \mathbb{C}$ be a multiset. If $x, y \in G$ then $(a[x])(y) = a(yx^{-1})$ and $([x]a)(y) = a(x^{-1}y)$. If furthermore a is central and $x \sim y$ then $a(x) = a(y)$.*

Proof. Use that the singletons form a basis, and the definition of multiplication, to obtain

$$a[x] = \sum_z a(z)[z][x] = \sum_z a(z)[zx] \quad \text{and} \quad [x]a = [x]\sum_z a(z)[z] = \sum_z a(z)[xz].$$

Hence to get $(a[x])(y)$ you solve the equation $zx = y$ and get $(a[x])(y) = a(z) = a(yx^{-1})$. Similarly to get $([x]a)(y)$ you solve the equation $xz = y$ and get $([x]a)(y) = a(z) = a(x^{-1}y)$. If a is central and $x \sim y$, say $x = tyt^{-1}$, then by applying both of these results we derive

$$a(x) = a(tyt^{-1}) = (a[t])(ty) = ([t^{-1}]a[t])(y) = a(y).$$

□

Definition A.10 (Multiplicity, Total multiplicity)**.** Let $a : G \to \mathbb{C}$ be a multiset. By the *multiplicity* of an element $g \in G$ we mean $a(g)$. By the *total multiplicity* $\Sigma(a) \in \mathbb{C}$ of the multiset we mean $\Sigma(a) := \sum_{g \in G} a(g)$.

Proposition A.11. *Let G be a finite group. Then the total multiplicity function Σ is linear and satisfies $\Sigma(ab) = \Sigma(a)\Sigma(b)$. Furthermore, the total multiplicity of any singleton or 1-class is 1.*

Proof. If $a, b : G \to \mathbb{C}$ are multisets and $\alpha, \beta \in \mathbb{C}$ are complex numbers then

$$\Sigma(\alpha a + \beta b) = \sum_{g \in G} \alpha a(g) + \beta b(g) = \alpha \sum_{g \in G} a(g) + \sum_{g \in G} \beta b(g) = \alpha \Sigma(a) + \beta \Sigma(b).$$

This shows linearity. The total multiplicity of any singleton $[g] : G \to \mathbb{C}$ is

$$\Sigma([g]) = \sum_{h \in G} [g](h) = 1$$

since most terms are 0. Thus, for any $g, h \in G$,

$$\Sigma([g])\Sigma([h]) = 1 \times 1 = 1 = \Sigma([gh]) = \Sigma([g][h])$$

and by linearity the second statement follows. Finally, for any $g \in G$,

$$\Sigma(c_g) = \frac{1}{|G|} \sum_{x \in G} \Sigma([xgx^{-1}]) = \frac{1}{|G|} \sum_{x \in G} 1 = 1.$$

□

Proposition A.12. *Let G be a finite group. Then the 1-classes of G have nonnegative multiplicities, they have pairwise disjoint support, and they are constant on their support. Furthermore, all 1-classes $c_g : G \to \mathbb{C}$ of G satisfy*

$$c_g = \frac{1}{|Cl(g)|} \sum_{x \in Cl(g)} [x].$$

Proof. If $c_g : G \to \mathbb{C}$ is a 1-class then its multiplicities are nonnegative by linearity and the fact that the multiplicities of any singleton are 0 and 1. If $c_g(z) \neq 0$ and $c_h(z) \neq 0$ for some $g, h, z \in G$ then $g \sim z \sim h$ so that $c_g = c_h$ by Proposition A.7. The support of any $c_g : G \to \mathbb{C}$ is precisely $Cl(g)$ since if $c_g(h) \neq 0$ then $g = xhx^{-1}$ for some $x \in G$ so that $g \sim h$ and conversely if $h \in Cl(g)$, say $h = xgx^{-1}$, then the term $[xgx^{-1}](h)$ of $c_g(h)$ is nonzero. c_g is constant on $Cl(g)$ by Proposition A.4 and Proposition A.9. Finally, since both c_g and $\frac{1}{|Cl(g)|} \sum_{x \in Cl(g)} [x]$ have the same support, are constant on it, and have total multiplicity 1, they are equal: If $c_g(z) \neq 0$ then replacing $c_g(z)$ with the average nonzero value we get

$$c_g(z) = \frac{1}{|Cl(g)|} \sum_{x \in Cl(g)} c_g(x) = \frac{1}{|Cl(g)|} \Sigma(c_g) = \frac{1}{|Cl(g)|} = \frac{1}{|Cl(g)|} \sum_{x \in Cl(g)} [x](z).$$

□

Theorem A.13. *Let G be a finite group. Then the set $K(G)$ of 1-classes from G is a commutative hypergroup spanning precisely the central multisets from G.*

Proof. The space of all multisets from G, with multiplication as defined above, is a ring. For example associativity of multiplication follows by linearity from the derivation

$$([x][y])[z] = [xy][z] = [(xy)z] = [x(yz)] = [x][yz] = [x]([y][z]),$$

and the multiplicative neutral element is $c_0 := c_e = [e]$, where e is the neutral element of G.

To deal with involution, we expand the definitions:

$$c_g c_h = \frac{1}{|G|} \sum_{x \in G} [xgx^{-1}] \frac{1}{|G|} \sum_{y \in G} [yhy^{-1}] = \frac{1}{|G|^2} \sum_{x,y \in G} [xgx^{-1}yhy^{-1}]. \tag{9}$$

If c_g is a 1-class then, by (9) and Proposition A.8, $c_g c_g^*$ contains the term $\frac{1}{|G|^2}[ege^{-1}eg^{-1}e^{-1}] = \frac{1}{|G|^2} c_0$. Conversely, if $c_g c_h$ contains a positive multiple of c_0 then $xgx^{-1}yhy^{-1} = e$ for some

$x, y \in G$, so that $g^{-1} \sim h$ and thus $c_g^* = c_h$ by Proposition A.7. Furthermore by (9) and Proposition A.8

$$(c_g c_h)^* = \frac{1}{|G|^2} \sum_{x,y \in G} [xgx^{-1}yhy^{-1}]^* = \frac{1}{|G|^2} \sum_{x,y \in G} [yh^{-1}y^{-1}xg^{-1}x^{-1}] = c_h^* c_g^*.$$

Last of all, clearly $*$ is an involution, because applied to any singleton, $([x]^*)^* = [(x^{-1})^{-1}] = [x]$. Thus $K(G)$ generates a $*$-algebra.

To deal with the central multisets from G, first recall that every 1-class c_g is central by Proposition A.4. Next, conversely, show that every central multiset $a : G \to \mathbb{C}$ is a linear combination of 1-classes

$$a = \sum_{x \in G} a(x)[x] = \sum_{\substack{Cl(x) \in G/\sim \\ y \in Cl(x)}} a(y)[y] = \sum_{\substack{Cl(x) \in G/\sim \\ y \in Cl(x)]}} a(x)[y] = \sum_{Cl(x) \in G/\sim} a(x)|Cl(x)]|c_x$$

by dividing the group into conjugacy classes $Cl(x) \in G/\sim$, using Proposition A.9 and in the last step using the identity from Proposition A.12.

To show that $K(G)$ is a basis for its span, show that if a family $(c_g)_{g \in S}$ is linearly dependent, where S is some subset of G, then S contains two different but conjugate elements of G. If a family $(c_g)_{g \in S}$ is linearly dependent, where S is some subset of G, then there is an element $h \in S$ and scalars $(\alpha_g)_{g \in S, g \neq h}$ such that $c_h = \sum_{g \in S, g \neq h} \alpha_g c_g$, so since $c_h(h) \neq 0$, some term $\alpha_g c_g(h) \neq 0$ with $g \in S, g \neq h$. But then $c_g(h) \neq 0$ and $g \sim h$ while still $g \neq h$.

To evidence structure constants, it now suffices to show that the product of each couple of central multisets $a, b : G \to \mathbb{C}$ is central. If $a, b : G \to \mathbb{C}$ are central and $z \in G$, then

$$[z]ab[z^{-1}] = [z]a[z^{-1}][z]b[z^{-1}] = ab.$$

This proves closure in the subring of multisets spanned by $K(G)$. Since we have already shown that $K(G)$ is a basis for its span, structure constants exist and are uniquely determined.

Positivity of the structure constants follows from (9) since the different c_g all have nonnegative multiplicities and pairwise disjoint support by Proposition A.12. Normalization of the structure constants follows from Proposition A.11 since that proposition tells that 1-classes have total multiplicity 1 and $\Sigma(ab) = \Sigma(a)\Sigma(b) = 1 \times 1 = 1$. Thus $K(G)$ is a hypergroup spanning precisely the central multisets from G.

To deal with commutativity, note that if $a : G \to \mathbb{C}$ is central and $g \in G$ then

$$a[g] = [g]a[g^{-1}][g] = [g]a.$$

From this it follows by linearity that a commutes with every multiset from G. In particular, any two 1-classes – central by Proposition A.4 – commute. Thus $K(G)$ is commutative.

□

Having derived the class hypergroup of a general finite group, we now turn to an example. This should give the reader more concrete insight and an ability to do arithmetic in the class hypergroup.

In order to work out examples, it is important to have an efficient notation. For this reason, we extend the singleton notation $[g] : G \to \mathbb{C}$ as follows. Group elements enclosed in square brackets, are used to denote the multiset which counts the number of occurrences of its argument. For example a multiset like $[\mathbf{0\ 0\ 0\ 1\ 6\ 7\ 7\ 0}] : G \to \mathbb{C}$, where $\mathbf{0}, \mathbf{1}, \mathbf{6}, \mathbf{7}$ are elements of some group G, satisfies by definition $[\mathbf{0\ 0\ 0\ 1\ 6\ 7\ 7\ 0}](\mathbf{0}) = 4$ and $[\mathbf{0\ 0\ 0\ 1\ 6\ 7\ 7\ 0}](\mathbf{6}) = 1$.

Example A.14. Let A_4 be the *alternating group on 4 elements.* Table 4 displays a multiplication table for this group, with rows and columns sorted by conjugacy class, in order to to also feature the hypergroup multiplication. Boxes containing group elements are multisets. To read it as a Cayley table for A_4, ignore the 1-classes c_i. The elements $\{\mathbf{0}, \ldots, \mathbf{11}\}$ of A_4 were named in lexicographic order, so that $\mathbf{0}$ is the identity. Group elements are written in a bold font to distinguish them from multiset multiplicities.

Table 4 encodes the 1-classes which constitute the class hypergroup $K(A_4)$:

$$c_0 = [\mathbf{0}], \quad c_1 = \frac{1}{4}[\mathbf{1\ 5\ 6\ 10}], \quad c_2 = \frac{1}{4}[\mathbf{2\ 4\ 7\ 9}], \quad c_3 = \frac{1}{3}[\mathbf{3\ 8\ 11}]$$

Table 4: Group multiplication table for the alternating group $A_4 = \{\mathbf{0}, \dots, \mathbf{11}\}$, with rows and columns sorted by conjugacy class, in order to to also feature the hypergroup multiplication.

$\times$		c_0	$4c_1$				$4c_2$				$3c_3$		
		0	**1**	**5**	**6**	**10**	**2**	**4**	**7**	**9**	**3**	**8**	**11**
c_0	**0**	**0**	**1**	**5**	**6**	**10**	**2**	**4**	**7**	**9**	**3**	**8**	**11**
$4c_1$	**1**	**1**	**2**	**7**	**9**	**4**	**0**	**8**	**11**	**3**	**6**	**10**	**5**
	5	**5**	**4**	**9**	**7**	**2**	**3**	**11**	**8**	**0**	**10**	**6**	**1**
	6	**6**	**7**	**2**	**4**	**9**	**8**	**0**	**3**	**11**	**1**	**5**	**10**
	10	**10**	**9**	**4**	**2**	**7**	**11**	**3**	**0**	**8**	**5**	**1**	**6**
$4c_2$	**2**	**2**	**0**	**11**	**3**	**8**	**1**	**10**	**5**	**6**	**9**	**4**	**7**
	4	**4**	**3**	**8**	**0**	**11**	**5**	**6**	**1**	**10**	**7**	**2**	**9**
	7	**7**	**8**	**3**	**11**	**0**	**6**	**5**	**10**	**1**	**4**	**9**	**2**
	9	**9**	**11**	**0**	**8**	**3**	**10**	**1**	**6**	**5**	**2**	**7**	**4**
$3c_3$	**3**	**3**	**5**	**1**	**10**	**6**	**4**	**2**	**9**	**7**	**0**	**11**	**8**
	8	**8**	**6**	**10**	**1**	**5**	**7**	**9**	**2**	**4**	**11**	**0**	**3**
	11	**11**	**10**	**6**	**5**	**1**	**9**	**7**	**4**	**2**	**8**	**3**	**0**

and lets us find structure constants, for example

$$
\begin{aligned}
3c_3 4c_1 &= \begin{bmatrix} \mathbf{5} & \mathbf{1} & \mathbf{10} & \mathbf{6} \\ \mathbf{6} & \mathbf{10} & \mathbf{1} & \mathbf{5} \\ \mathbf{10} & \mathbf{6} & \mathbf{5} & \mathbf{1} \end{bmatrix} \quad \text{(multiset copied from table)} \\
c_3 c_1 &= \frac{1}{3 \cdot 4}[\mathbf{5\ 1\ 10\ 6\ 6\ 10\ 1\ 5\ 10\ 6\ 5\ 1}] = \frac{1}{4}[\mathbf{1\ 5\ 6\ 10}] = c_1 \\
n_{31}^0 &= 0, \quad n_{31}^1 = 1, \quad n_{31}^2 = 0, \quad n_{31}^3 = 0 \\
c_3 c_3 &= \frac{1}{3 \cdot 3}[\mathbf{0\ 11\ 8\ 11\ 0\ 3\ 8\ 3\ 0}] = \frac{1}{3}[\mathbf{0}] + \frac{2}{9}[\mathbf{3\ 8\ 11}] = \frac{1}{3}c_0 + \frac{2}{3}c_3 \\
n_{33}^0 &= \frac{1}{3}, \quad n_{33}^1 = 0, \quad n_{33}^2 = 0, \quad n_{33}^3 = \frac{2}{3}.
\end{aligned}
$$

We can also find the structure constants n_{ij}^k by looking at the proportion of the elements of c_k in the multiset at the position for $c_i c_j$ in the table. For example, we see that one quarter of the elements of

$$
4c_1 4c_2 = \begin{bmatrix} \mathbf{0} & \mathbf{8} & \mathbf{11} & \mathbf{3} \\ \mathbf{3} & \mathbf{11} & \mathbf{8} & \mathbf{0} \\ \mathbf{8} & \mathbf{0} & \mathbf{3} & \mathbf{11} \\ \mathbf{11} & \mathbf{3} & \mathbf{0} & \mathbf{8} \end{bmatrix} \quad \text{(multiset copied from table)}
$$

belong to c_0, and the remaining three quarters belong to c_3. Thus simply,

$$
n_{12}^0 = \frac{1}{4}, \quad n_{12}^1 = 0, \quad n_{12}^2 = 0, \quad n_{12}^3 = \frac{3}{4}.
$$

Continuing in this way, we produce a much more succinct hypergroup multiplication table, namely Table 5.

We finish the section with a simple but pleasant result.

Proposition A.15. *Let G be a finite group. If $c_x \in K(G)$, then $w(c_x) = |Cl(x)|$.*

Proof. Let c_x and c_y be 1-classes of G, and let e be the identity of G. By Proposition A.12, we know that the 1-classes have disjoint support, and that any 1-class c_x of G can be written as $c_x = \frac{1}{|Cl(x)|}\sum_{y \in Cl(x)}[y]$. As $c_0 := [e]$ is the identity of $K(G)$, and the 1-classes have disjoint support, we have that $\frac{1}{w(c_x)} = (c_x c_x^*)(e)$. Now define $y := x^{-1}$. By Proposition A.6 we have

Table 5: Cayley table for the class hypergroup $K(A_4)$.

$\cdot$	c_0	c_1	c_2	c_3
c_0	c_0	c_1	c_2	c_3
c_1	c_1	c_2	$\frac{1}{4}c_0 + \frac{3}{4}c_3$	c_1
c_2	c_2	$\frac{1}{4}c_0 + \frac{3}{4}c_3$	c_1	c_2
c_3	c_3	c_1	c_2	$\frac{1}{3}c_0 + \frac{2}{3}c_3$

that each element in $Cl(x)$ has its inverse in $Cl(y)$, which also implies $|Cl(x)| = |Cl(y)|$. We now use this to compute

$$\frac{1}{w(c_x)} = (c_x c_x^*)(e) = (c_x c_{x^{-1}})(e) = \frac{1}{|Cl(x)|^2} \sum_{\substack{s \in Cl(x) \\ t \in Cl(y)}} [st](e) = \frac{1}{|Cl(x)|}$$

and thus $w(c_x) = |Cl(x)|$. □

Now that we have shown how to generate a commutative hypergroup from the conjugacy classes of finite groups we move on to hypergroups generated from characters of groups and commutative hypergroups.

A.2 Character Hypergroup

In this section we introduce ways to generate commutative hypergroups from characters of hypergroups. From a commutative hypergroup K we will generate its *dual* $\hat{K}$, which in certain cases will be a hypergroup, which we then call the *dual hypergroup* of K. Then we will, from a not necessarily abelian group G, generate a commutative hypergroup which we call the *character hypergroup* of G, and denote by $K(\hat{G})$. Finally, we define isomorphism between hypergroups, and study how certain hypergroups relate to each other. We will see that the dual is related by a transposition of the character table to K.

We begin by studying $\hat{K}$. Remember that if K is a hypergroup, $\hat{K}$ is the set of all irreducible characters of K. But first we introduce an operator on sets of functions $f : K \to \mathbb{C}$.

Definition A.16 (Pointwise multiplication)**.** Let K be a hypergroup, and let F be a set of functions $f : K \to \mathbb{C}$. Then we define an operator on F by *pointwise multiplication*, so that $\forall f_1, f_2 \in F$ the product is a function $(f_1 \cdot f_2) : K \to \mathbb{C}$ defined by

$$(f_1 \cdot f_2)(c_g) := f_1(c_g) \cdot f_2(c_g), \ \forall c_g \in K$$

Theorem A.17 (Dual hypergroup)**.** *Let K be a commutative hypergroup, and $\hat{K}$ the set of all its irreducible characters. Let $(c_i)_{i=0}^{|K|-1}$ and $(\varphi^{(i)})_{i=0}^{|\hat{K}|-1}$ be complete lists of elements in K and $\hat{K}$ respectively. Then pointwise multiplication of the elements in $\hat{K}$ satisfy*

$$\varphi^{(i)} \cdot \varphi^{(j)} = \sum_{t=0}^{s} m_{ij}^t \varphi^{(t)},$$

where $m_{ij}^t = \langle \varphi^{(t)}, \varphi^{(i)} \cdot \varphi^{(j)} \rangle \, k(\varphi^{(t)}) \in \mathbb{C}$. Furthermore, if and only if $0 \leq m_{ij}^t \ \forall i, j, t$, then $\hat{K}$ is a commutative hypergroup such that

$$w(\varphi_k) = k(\varphi_k), \quad w(\hat{K}) = w(K), \quad \text{and} \quad |K| = |\widehat{K}|.$$

The identity element is then $\varphi^{(0)}$, and involution is given by $\varphi^{(i)^*} = \overline{\varphi^{(i)}} \ \forall i$. When $\hat{K}$ is a hypergroup, we call it the dual hypergroup *of K.*

Proof. As K is commutative, all irreducible representations are of degree 1 by Proposition 4.3, and are equal to their characters. And since the characters span $\mathbb{C}^K$ by Theorem 4.5, we have

$$\varphi^{(i)}\varphi^{(j)} = \sum_k \frac{\langle \varphi^{(k)}, \varphi^{(i)}\varphi^{(j)} \rangle}{\langle \varphi^{(k)}, \varphi^{(k)} \rangle} \varphi^{(k)} = \sum_k \langle \varphi^{(k)}, \varphi^{(i)}\varphi^{(j)} \rangle \, k(\varphi^{(k)}) \varphi^{(k)} \tag{10}$$

Now define $m_{ij}^k := \langle \varphi^{(k)}, \varphi^{(i)} \cdot \varphi^{(j)} \rangle \, k(\varphi^{(k)})$. These are the structure constants of $\hat{K}$. We first show the normalisation of $\hat{K}$ by using that $\varphi^{(i)}(c_0) = 1 \ \forall \varphi^{(i)} \in \hat{K}$. By inserting this into equation (10) we get $1 = \sum_k n_{ij}^k$, thus confirming the normalisation. The associativity and commutativity follow from the definition of the operation, and that $\varphi^{(i)}(c_j)$ is a scalar for all i, j. We easily see that $\varphi^{(0)}$ satisfies the role of an identity.

Before we consider the existence of an involution, note that if ρ is an irreducible representation of K, then so is $\overline{\rho}$. Thus $\varphi^{(i)} \in \hat{K} \implies \overline{\varphi^{(i)}} \in \hat{K}$ With this in mind we study $m_{ij}^0 = \langle \varphi^{(0)}, \varphi^{(i)} \cdot \varphi^{(j)} \rangle = \langle \overline{\varphi^{(j)}}, \varphi^{(i)} \rangle$. By the orthogonality of characters (Theorem 3.40), we see that $\overline{\varphi^{(j)}} = \varphi^{(i)} \Leftrightarrow m_{ij}^0 \neq 0$, demonstrating the existence and uniqueness of the involution, and that it is given by $\varphi^{(i)^*} = \overline{\varphi^{(i)}}$.

We have now shown that $\hat{K}$ satisfies all the requirements of a hypergroup, except for $0 \leq m_{ij}^k$. Thus $0 \leq m_{ij}^k$ implies that $\hat{K}$ is a hypergroup.

Furthermore, assuming that $\hat{K}$ is a hypergroup, we have that $w(\varphi^{(k)}) = \frac{1}{n_{k^* k}^0} = \frac{1}{\langle \varphi^{(k)}, \varphi^{(k)} \rangle} = k(\varphi^{(k)})$. And as $w(\hat{K}) = \sum w(\varphi^{(i)})$ by definition, and $w(K) = \sum k(\varphi^{(i)})$ for commutative hypergroups by Corollary 4.18, we have that $w(K) = w(\hat{K})$. Finally, the fact that $|\hat{K}| = |K|$ follows from the fact that a commutative hypergroup has $|K|$ irreducible representations by Corollary 4.4. □

The things we have stated about $\hat{K}$ coming from a commutative hypergroup K especially hold for a finite abelian group G. However, we will later see that we will also be able to generate a commutative hypergroup from the characters of a finite group that isn't necessarily abelian. For a group G, this hypergroup group will be denoted by $K(\hat{G})$. But before we introduce $K(\hat{G})$, we first need to introduce a tensor product, as it will be needed to demonstrate that $K(\hat{G})$ is a hypergroup.

Definition A.18. A tensor (Kronecker) product between two matrices A and B is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

Proposition A.19. *The tensor product has the following properties*

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$$
$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

Proof. See Graham [7] for details. □

Proposition A.20. *Assume that φ and ρ are representations of the group. Then $\varphi \otimes \rho$, defined by $(\varphi \otimes \rho)(g) = \varphi(g) \otimes \rho(g)$, is also a representation of G.*

Proof. We need to show that $\varphi \otimes \rho$ maps the identity $e \in G$ onto the identity matrix, and that $\varphi \otimes \rho$ is a homomorphism from G.

$$(\varphi \otimes \rho)(e) = \varphi(e) \otimes \rho(e) = I \otimes I = I,$$

confirming the first property. We now look at

$$\begin{aligned}(\varphi \otimes \rho)(g)(\varphi \otimes \rho)(k) &= (\varphi(g) \otimes \rho(g))(\varphi(k) \otimes \rho(k)) = (\varphi(g)\varphi(k)) \otimes (\rho(g)\rho(k)) = \varphi(gk) \otimes \rho(gk) \\ &= (\varphi \otimes \rho)(gk),\end{aligned}$$

which verifies that $\varphi \otimes \rho$ is a representation of G. □

Now we are ready to define $K(\hat{G})$, which we will later see is equal to $\widehat{K(G)}$, in the sense that they are isomorphic. What this means will be explained later.

Definition A.21 (Character hypergroup)**.** Let G be a finite group, and $(\rho^{(i)})_{i=0}^{s-1}$ be a complete set of irreducible representations of G. For each $\rho^{(i)}$, define a function $\psi_i := \frac{\chi_{\rho^{(i)}}}{\deg \rho^{(i)}}$. The *character hypergroup* of G, denoted by $K(\hat{G})$, is the set of all such ψ_i. The operation on $K(\hat{G})$ is pointwise multiplication.

Note that if G is abelian, then $\hat{G}$ and $K(\hat{G})$ are the same sets. The following theorem verifies that $K(\hat{G})$ actually is a hypergroup.

Theorem A.22. *If G is a finite group, then $K(\hat{G})$ is a finite commutative hypergroup.*

Proof. Let $(\rho^{(i)})_{i=0}^{s-1}$ be a complete set of irreducible representations of G. By Proposition A.20 $\rho^{(i)} \otimes \rho^{(j)}$ is also a representation, and thus by Theorem 3.25 decomposes as

$$\rho^{(i)} \otimes \rho^{(j)} \sim \bigoplus_t M_{ij}^t \rho^{(t)}$$

where M_{ij}^t are non-negative integers. By Proposition A.19 we have

$$\chi_{\rho^{(i)}} \chi_{\rho^{(j)}} = \sum_t M_{ij}^t \chi_{\rho^{(t)}}$$

where $M_{ij}^t = \langle \chi_{\rho^{(t)}}, \chi_{\rho^{(i)}} \chi_{\rho^{(j)}} \rangle$ by the orthogonality of characters (Theorem 3.40).

By the definition of the elements $\psi_i \in K(\hat{G})$, we have

$$\deg(\rho^{(i)})\psi_i \deg(\rho^{(j)})\psi_j = \sum_t M_{ij}^t \deg(\rho^{(t)})\psi_t$$

Now define $m_{ij}^t := \frac{M_{ij}^t \deg(\rho^{(t)})}{\deg(\rho^{(i)})\deg(\rho^{(j)})}$. These are the structure constants of $K(\hat{G})$, so that

$$\psi_i \psi_j = \sum_t m_{ij}^t \psi_t$$

As ψ_i maps elements of G onto scalars, and the operation on $K(\hat{G})$ is given by pointwise multiplication, $K(\hat{G})$ is commutative and associative. The positivity of the strucutre constants follow from that M_{ij}^k is positive for all i, j, k. The identity is given by ψ_0. Let the identity of G be denoted e. Then the normalisation is verified by $1 = (\psi_i\psi_j)(e) = \sum_t m_{ij}^t \psi_t(e) = \sum_t m_{ij}^t$. Finally, since the conjugate of an irreducible character is an irreducible character, we have by the orthagonality of characters (Theorem 3.40) that $m_{ij}^0 \neq 0 \Leftrightarrow \psi_i = \overline{\psi_j}$. This verifies the existence and uniqueness of the involution, and that it is given by $\psi_i^* = \overline{\psi_i}$. We have now verified that $K(\hat{G})$ satisfies all the requirements of a hypergroup. $\square$

We will now move on to study how certain hypergroups relate to each other. For this we will need the concept of isomorphism between hypergroups.

Definition A.23 (Hypergroup isomorphism)**.** Let K_1 and K_2 be two hypergroups. Let $(c_i)_{0\leq i}^{s-1}$ and $(n_{ij}^k)_{0\leq i,j,k}^{s-1}$ be the elements and structure constants of K_1. We say that K_1 is *isomorphic* to K_2 if and only if there is a bijective mapping $\theta : K_1 \to K_2$ such that θ *preserves the operation*, by which we mean that

$$\theta(c_i)\theta(c_j) = \sum_k n_{ij}^k \theta(c_k) \quad \forall c_i, c_j.$$

If K_1 and K_2 are isomorphic, we denote this $K_1 \cong K_2$.

Note that in the group case hypergroup isomorphism is the same as regular group isomorphism. It is convenient to think of isomorphic hypergroups as being the same as each other. This is essentially true, since they in some sense share the same structure constants. What is meant by this is made explicit in the following Proposition.

Proposition A.24. *Let K_1 and K_2 be two hypergroups of order s with structure constants $(n_{ij}^k)_{0\leq i,j,k}^{s-1}$ and $(m_{ij}^k)_{0\leq i,j,k}^{s-1}$ respectively. If K_1 is isomorphic to K_2, then the elements in K_1 and K_2 can be renumbered so that $n_{ij}^k = m_{ij}^k$ for all i, j, k.*

Proof. Let $(c_i)_{0\leq i}^{s-1}$ and $(d_i)_{0\leq i}^{s-1}$ be the elements in K_1 and K_2 respectively. As K_1 and K_2 are isomorphic there is a bijective mapping $\theta : K_1 \to K_2$ such that $\theta(c_i)\theta(c_j) = \sum_k n_{ij}^k \theta(c_k)\ \forall c_i, c_j$. Since the mapping is bijective, we can without loss of generality renumber the elements in K_1 and K_2 so that $\theta(c_i) = d_i$ for all i. Now look at

$$\sum m_{ij}^k d_k = d_i d_j = \theta(c_i)\theta(c_j) = \sum n_{ij}^k \theta(c_k) = \sum n_{ij}^k d_k$$

which shows that $n_{ij}^k = m_{ij}^k$ for all i, j, k. $\square$

Just as in the group case, isomorphism is an equivalence relation, which is proved in the following Proposition.

Proposition A.25. *Hypergroup isomorphism is an equivalence relation.*

Proof. Define the hypergroups K_1, K_2 and K_3 with elements and structure constants (c_i, n_{ij}^k), (d_i, m_{ij}^k) and (e_i, l_{ij}^k) respectively. We need to show that isomorphism is reflexive($K_1 \cong K_1$), transitive($K_1 \cong K_2$ & $K_2 \cong K_3 \implies K_1 \cong K_3$) and symmmetric($K_1 \cong K_2 \implies K_2 \cong K_1$).

Isomorphism is symmetric as the map $\theta : K_1 \to K_2$ is bijective which implies that it is invertible which in turn implies that $K_2 \cong K_1$.

It is transitive as there exists θ such that $\theta(c_i)\theta(c_j) = \sum_k n_{ij}^k \theta(c_k)$ and there exists ψ such that $\psi(d_i)\psi(d_j) = \sum_k m_{ij}^k \psi(d_k)$. Since $\theta(c_m) \in K_2$ then we can construct the map $\psi \circ \theta : K_1 \to K_3$ which is bijective and preserves the operation.

Proof of reflexiveness is obvious. $\square$

That isomorphism is an equivalence relation will be useful later, together with the following Proposition.

Proposition A.26. *Let K_1 and K_2 be two commutative hypergroups such that $K_1 \cong K_2$. If $\hat{K_1}$ is a hypergroup, then $\hat{K_2}$ is a hypergroup, and $\hat{K_1} \cong \hat{K_2}$.*

Proof. Let $(c_i)_{0\leq i}^{s-1}$ and $(d_i)_{0\leq i}^{s-1}$ be the elements in K_1 and K_2 respectively. By Proposition A.24, we may assume without loss of generality that these elements can be numbered so that both hypergroups have the same structure constants $(n_{ij}^k)_{0\leq i,j,k}^{s-1}$. By Corollary 4.4 there are $|K| = |\hat{K}|$ for commutative hypergroups, and so we may denote $(\varphi^{(i)})_{0\leq i}^{s-1}$ to be a complete set of irreducible representations of K_1. Now for each $\varphi^{(i)}$, define $\rho^{(i)} : K_2 \to \mathbb{C}$ by $\rho^{(i)}(d_j) = \varphi^{(i)}(c_j)$ for all j. Each $\rho^{(i)}$ is an irreducible representation of K_2, since $\rho^{(i)}(d_k)\rho^{(i)}(d_g) = \varphi^{(i)}(c_k)\varphi^{(i)}(c_g) = \sum n_{kg}^t \varphi^{(i)}(c_t) = \sum n_{kg}^t \rho^{(i)}(d_t)$, and $\rho^{(i)}$ satisfies $\rho^{(i)}(d_0) = \varphi^{(i)}(c_0) = 1$. We know that $|\hat{K_2}| = |\hat{K_1}|$ as both K_1 and K_2 are commutative hypergroups of the same order. And since $\rho^{(i)} = \rho^{(j)} \implies \varphi^{(i)} = \varphi^{(j)} \implies i = j$, we have that $(\rho^{(i)})_{0\leq i}^{s-1}$ is a complete set of irreducible representations of K_2.

Let $(\mu_{ij}^k)_{0\leq i,j,k}^{s-1}$ and $(m_{ij}^k)_{0\leq i,j,k}^{s-1}$ be the structure constants of $\hat{K_1}$ and $\hat{K_2}$ respectively. We now define $\theta : \hat{K_1} \to \hat{K_2}$ by $\theta(\varphi^{(i)}) = \rho^{(i)}$ for all i. The mapping θ preserves the operation since, for all d_s, we have

$$\begin{aligned}(\theta(\varphi^{(i)})\theta(\varphi^{(j)}))(d_s) &= (\rho^{(i)}\rho^{(j)})(d_s) = (\varphi^{(i)}\varphi^{(j)})(c_s) = \sum \mu_{ij}^k \varphi^{(k)}(c_s) \\ &= \sum \mu_{ij}^k \rho^{(k)}(d_s) = \sum \mu_{ij}^k \theta(\varphi^{(k)})(d_s).\end{aligned}$$

And since this holds for all $d_s \in K_2$ and $0 \leq i, j \leq s-1$, $(\theta(\varphi^{(i)}))\theta(\varphi^{(j)}) = \sum \mu_{ij}^k \theta(\varphi^{(k)})$ for all i, j. Furthermore, $\sum m_{ij}^k \rho^{(k)} = \rho^{(i)}\rho^{(j)} = \theta(\varphi^{(i)})\theta(\varphi^{(j)}) = \sum \mu_{ij}^k \theta(\varphi^{(k)}) = \sum \mu_{ij}^k \rho^{(k)}$, so that $m_{ij}^k = \mu_{ij}^k$ for all i, j, k. Thus $0 \leq m_{ij}^k$ so that $\hat{K}_2$ is a hypergroup by Theorem A.17. And since both $\hat{K}_1$ and $\hat{K}_2$ are hypergroups, and there is a bijective mapping between them which preserves the operation, they are by definition isomorphic.

□

We now wish to show that if $\hat{K}$ is a hypergroup, then $\hat{\hat{K}} \cong K$. But before we do that, we will need the following proposition.

Proposition A.27. *Let K be a finite commutative hypergroup. Let $(c_i)_{i=0}^{s-1}$ and $(\varphi^{(i)})_{i=0}^{s-1}$ be the elements of K and $\hat{K}$ respectively. If $\hat{K}$ is a hypergroup, then the functions $\phi_{c_g} : \hat{K} \to \mathbb{C}$ defined by*

$$\phi_{c_g}(\varphi^{(i)}) := \varphi^{(i)}(c_g)$$

are irreducible representation of $\hat{K}$. Furthermore, the mapping $\theta : K \to \hat{\hat{K}}$ defined by $\theta(c_g) := \phi_{c_g}$ is bijective.

Proof. Denote the structure constants of $\hat{K}$ by m_{ij}^k so that $\varphi^{(i)}\varphi^{(j)} = \sum m_{ij}^k \varphi^{(k)}$. ϕ_{c_g} trivially satisfies $\phi_{c_g}(\varphi^{(0)}) = \varphi^{(0)}(c_g) = 1$. Now we compute

$$\phi_{c_g}(\varphi^{(i)})\phi_{c_g}(\varphi^{(j)}) = \varphi^{(i)}(c_g)\varphi^{(j)}(c_g) = (\varphi^{(i)}\varphi^{(j)})(c_g) = \sum m_{ij}^k \varphi^{(k)}(c_g) = \sum m_{ij}^k \phi_{c_g}(\varphi^{(k)})$$

and thus ϕ_{c_g} is a representation of $\hat{K}$, and as it is of degree 1, it is irreducible.

We now show that the mapping θ is bijective by contradiction. Assume that $\theta_{c_g} = \theta_{c_k}$ for some $c_g, c_k \in K$ such that $c_g \neq c_k$. This implies that $\varphi_i(c_g) = \varphi_i(c_k)\ \forall i$, which would mean that the character table had two identical columns. But as the character table is invertible by Theorem 4.17 this is a contradiction, and so the mapping is one-to-one. And as $|K| = |\hat{\hat{K}}|$ by Theorem A.17, the mapping is bijective. □

Corollary A.28. *Let K be a commutative hypergroup with character table X. If $\hat{K}$ is a hypergroup, then the representations of $\hat{K}$ can be numbered so that X^T is the character table of $\hat{K}$.*

Proof. Let $(c_i)_{0 \leq i < |K|}$ and $(\varphi)_{0 \leq i < |K|}$ be the elements in K and $\hat{K}$ respectively. For all i, define $\rho^{(i)} : \hat{K} \to \mathbb{C}$ by $\rho^{(i)}(\varphi^{(j)}) = \varphi^{(j)}(c_i)$ for all j. By Proposition A.27, the mapping defined by $\theta(c_i) := \rho^{(i)}$ for all i is a bijective mapping from K to $\hat{\hat{K}}$. Thus the character table of $\hat{K}$, let it be denoted by Y, can be defined for all i, j as $Y_{ij} := \rho^{(i)}(\varphi^{(j)}) = \varphi^{(j)}(c_i) = X_{ji}$. And thus $Y = X^T$. □

We are now ready to prove the following.

Proposition A.29. *Let K be a commutative hypergroup. If $\hat{K}$ is a hypergroup, then $\hat{\hat{K}}$ is also a hypergroup, and $\hat{\hat{K}} \cong K$.*

Proof. By Theorem A.17 we know that $|K| = |\hat{K}|$. Now assume that $(c_i)_{i=0}^{s-1}$ and $(\varphi_i)_{i=0}^{s-1}$ are the elements of K and $\hat{K}$ respectively. For each element in K, define a function $\phi_{c_g} : \hat{K} \to \mathbb{C}$ by $\phi_{c_g}(\varphi_i) := \varphi_i(c_g)$. By Proposition A.27, a mapping θ from K defined by $\theta(c_g) := \phi_{c_g}$ is a bijective mapping from K to $\hat{\hat{K}}$. Now we only need to show that θ preserves the operation. Let $(n_{ij}^k)_{0 \leq i,j,k}^{s-1}$ be the structure constants of K and compute

$$(\theta(c_i)\theta(c_j))(\varphi_t) = \varphi_t(c_i)\varphi_t(c_j) = \sum_k n_{ij}^k \varphi_t(c_k) = \sum_k n_{ij}^k \theta(c_k)(\varphi_t)$$

and since this is true for any $c_i, c_j \in K$ and $\varphi_t \in \hat{K}$, we have $\theta(c_i)\theta(c_j) = \sum_k n_{ij}^k \theta(c_k)$ for all c_i, c_j. Thus $\hat{\hat{K}}$ is isomorphic to K.

□

We previously stated that $K(\hat{G})$ in some sense was the same as $\widehat{K(G)}$. They are in fact isomorphic. To show this, we first need a classical result from the representation theory of groups.

Lemma A.30. *If G is a finite group, then the number of irreducible representations of G is equal to the number of conjugacy classes of G. As a consequence, the number of irreducible representations of G is equal to $|\widehat{K(G)}|$.*

Proof. That the number of irreducible representations is equal to the number of conjugacy classes is a classic result of the representation theory of finite groups. A proof can for example be found in [4].

The final statement follows from that $|K(G)|$ is equal to the number of conjugacy classes of G, and that since $K(G)$ is a commutative hypergroup, it has $|K(G)|$ irreducible representations by Corollary 4.4. $\square$

Theorem A.31. *If G is a finite group, then $\widehat{K(G)}$ is a hypergroup, and is isomorphic to $K(\widehat{G})$.*

Proof. Let the structure constants of $K(G)$ be (n_{ij}^k). By Lemma A.30, the number of irreducible representations of G is equal to $|\widehat{K(G)}|$. Thus we may denote the irreducible representations of G and $K(G)$ by$(\rho^{(i)})_{0\leq i<|\widehat{K(G)}|}$ and $(\varphi^{(i)})_{0\leq i<|\widehat{K(G)}|}$ respectively. Define for each $\rho^{(i)}$ the function $\tilde{\rho}^{(i)} : \{[g], g \in G\} \to \mathbb{C}$ by $\tilde{\rho}^{(i)}([g]) := \rho^{(i)}(g)\ \forall g \in G$. Now extend $\tilde{\rho}^{(i)}$ linearly so that $\tilde{\rho}^{(i)}\left(\sum[x]\right) = \sum \tilde{\rho}^{(i)}\left([x]\right)$. Now $\tilde{\rho}^{(i)}$ is a representation of $K(G)$. To see this, first note that $\tilde{\rho}^{(i)}(c_e) = \frac{1}{|G|}\sum_x \rho^{(i)}(xex^{-1}) = I$. For the second requirement of a representation we compute

$$\begin{aligned}\tilde{\rho}^{(i)}(c_g)\tilde{\rho}^{(i)}(c_k) &= \frac{1}{|G|^2}\sum_{x,y}\rho^{(i)}(xgx^{-1})\rho^{(i)}(yky^{-1}) = \frac{1}{|G|^2}\sum_{x,y}\rho^{(i)}(xgx^{-1}yky^{-1})\\ &= \tilde{\rho}^{(i)}\left(\frac{1}{|G|^2}\sum_{x,y}\{xgx^{-1}yky^{-1}\}\right) = \tilde{\rho}^{(i)}\left(c_g c_k\right)\\ &= \tilde{\rho}^{(i)}\left(\sum_s n_{gk}^s c_s\right) = \sum_s n_{gk}^s \tilde{\rho}^{(i)}\left(c_s\right)\end{aligned}$$

and thus $\tilde{\rho}^{(i)}$ is a representation of $K(G)$. The character of this representation satisfies $\chi_{\tilde{\rho}^{(i)}}(c_g) = \frac{1}{|G|}\sum_x \chi_{\rho^{(i)}}(xgx^{-1}) = \chi_{\rho^{(i)}}(g)$. Note that if $t \in G$ is in the same conjugacy class as g, then $\chi_{\rho^{(i)}}(t) = \chi_{\tilde{\rho}^{(i)}}(c_t) = \chi_{\tilde{\rho}^{(i)}}(c_g)$ since $c_t = c_g$. Let us now look at two inequivalent representations of G $\rho^{(i)}$, $\rho^{(j)}$. Since $\tilde{\rho}^{(i)}$ and $\tilde{\rho}^{(j)}$ are representations of $K(G)$, they decompose as $\tilde{\rho}^{(i)} \sim \bigoplus n_k\varphi^{(k)}$ and $\tilde{\rho}^{(j)} \sim \bigoplus m_k\varphi^{(k)}$ for some non-negative integers n_k, m_k by Theorem 3.25. Thus we have

$$\begin{aligned}\chi_{\rho^{(i)}}(g) &= \chi_{\tilde{\rho}^{(i)}}(c_g) = \sum_k n_k\varphi^{(k)}(c_g)\\ \chi_{\rho^{(j)}}(g) &= \chi_{\tilde{\rho}^{(j)}}(c_g) = \sum_k m_k\varphi^{(k)}(c_g).\end{aligned}$$

With this in mind, we use that by the orthogonality of characters on G (see Theorem 3.40)

$$\begin{aligned}0 = \langle\chi_{\rho^{(i)}}, \chi_{\rho^{(j)}}\rangle &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\rho^{(i)}}(g)}\chi_{\rho^{(j)}}(g) = \frac{1}{|G|}\sum_{s,t}n_s m_t\sum_{g\in G}\overline{\varphi^{(s)}(c_g)}\varphi^{(t)}(c_g)\\ &= \frac{1}{|G|}\sum_{s,t}n_s m_t\sum_{c_g\in K(G)}\overline{\varphi^{(s)}(c_g)}\varphi^{(t)}(c_g)|Cl(g)| = \sum_s n_s m_s\frac{1}{k(\varphi(s))}\end{aligned}$$

Where in the final step we used that for all $g \in G$, $|Cl(g)|$ is equal to the weight of $c_g \in K(G)$ by Proposition A.15, together with the Schur Orthogonality Relations (Theorem 3.33). Now we see that for this expression to be nonzero, n_i and m_i cannot both be nonzero at the same time. By comparing all represntations of G in this way, and using that the number of irreducible representations of G is equal to $|\widehat{K(G)}|$, we see that for each $\rho^{(i)}$, there is exactly one $\varphi^{(j)}$ such that $\tilde{\rho}^{(i)} \sim n_j\varphi^{(j)}$. And since the degrees of equivalent representations must be equal, $n_j = \deg \tilde{\rho}^{(i)} = \deg \rho^{(i)}$. And conversely, for each $\varphi^{(j)} \in \widehat{K(G)}$, there exists exactly one $\rho^{(i)}$ such that $\tilde{\rho}^{(i)} \sim \deg(\rho^{(i)})\varphi^{(j)}$. Since we have this bijection, we can assume without loss of generality that we have numbered the irreducible representations of G and $K(G)$ so that $\tilde{\rho}^{(i)} \sim \deg(\rho^{(i)})\varphi^{(i)}$ for all i. We now have

$$\varphi^{(i)}(c_g) = \frac{\chi_{\tilde{\rho}^{(i)}}(c_g)}{\deg(\rho^{(i)})} = \frac{\chi_{\rho^{(i)}}(g)}{\deg(\rho^{(i)})}$$

where $\psi_i := \frac{\chi_{\rho^{(i)}}}{\deg(\rho^{(i)})}$ are the elements in $K(\hat{G})$ by definition. Define a mapping $\theta : K(\hat{G}) \to \widehat{K(G)}$ by $\theta(\psi_i) = \varphi^{(i)}$. We know that this mapping is bijective. We now wish to show that it preserves the operation as well. Let (m_{ij}^k) and (μ_{ij}^k) be the structure constants of $K(\hat{G})$ and $\widehat{K(G)}$ respectively. We compute

$$\begin{aligned}(\theta(\psi_i)\theta(\psi_j))\,(c_g) &= \varphi^{(i)}(c_g)\varphi^{(j)}(c_g) = \psi_i(g)\psi_j(g) = (\psi_i\psi_j)\,(g) = \sum_k m_{ij}^k \psi_k(g) \\ &= \sum_k m_{ij}^k \varphi^{(k)}(c_g) = \sum_k m_{ij}^k \theta(\psi_k)(c_g)\end{aligned}$$

and since this is true for any $c_g \in K(G)$, we have $\theta(\psi_i)\theta(\psi_j) = \sum_k m_{ij}^k \theta(\psi_k)$, and so θ preserves the operation. We now need to show that $\widehat{K(G)}$ is a hypergroup. For this we use that $\sum m_{ij}^k \varphi^{(k)} = \sum m_{ij}^k \theta(\psi_k) = \theta(\psi_i)\theta(\psi_j) = \varphi_i\varphi_j = \sum \mu_{ij}^k \varphi_k$, which implies that $m_{ij}^k = \mu_{ij}^k$ for all i, j, k. This in turn implies that $0 \leq m_{ij}^k$ for all i, j, k, which by A.17 shows that $\widehat{K(G)}$ is a hypergroup . Thus $\widehat{K(G)}$ is isomorphic to $K(\hat{G})$.

□

We have now showed that $K(\hat{G})$ and $\widehat{K(G)}$ essentially are the same hypergroup. We will now proceed to show that the dual of the character hypergroup is isomorphic to the class hypergroup.

Theorem A.32. *If G is a finite group, then $\widehat{K(\hat{G})}$ is a hypergroup isomorphic to $K(G)$.*

Proof. By Theorem A.31 $K(\widehat{G}) \cong \widehat{K(G)}$. And since we know that $\widehat{K(G)}$ is a hypergroup, we know by Proposition A.29 that $\widehat{\widehat{K(G)}}$ is a hypergroup, and $\widehat{\widehat{K(G)}} \cong K(G)$. By using Proposition A.26, and that isomorphism is an equivalence relation, we now get that $\widehat{K(\hat{G})}$ is a hypergroup isomorphic to $K(G)$. □

We have now seen how pointwise multiplication on the dual of a hypergroup works, and when the dual is a hypergroup as well. We have also learned a way to generate hypergroups from the characters of any finite group. Then hypergroup isomorphism was introduced, and was used to show how certain hypergroups relate to each other. We now move on to examples of class and character hypergroups os S_3.

A.3 Class and Character Hypergroup of S_3

In this section, the class and character hypergroups of S_3 are presented as examples.

We denote the elements in $K(S_3)$ by $(c_i)_{0 \leq i < 3}$, and compute its Cayley table, see table Table 6, using the same method as the one used in Example A.14. The representations of $K(S_3)$ are denoted by $(\varphi)_{0 \leq i < 3}$, and the character table of $K(S_3)$ is shown in Table 7. The elements in $K(\hat{S_3})$ are denoted by $(\psi_i)_{0 \leq i < 3}$ and its Cayley table, see Table 8, is computed by using that its structure constants are given by $m_{ij}^t := \frac{\langle \chi_{\rho^{(t)}}, \chi_{\rho^{(i)}}\chi_{\rho^{(j)}} \rangle \deg(\rho^{(t)})}{\deg(\rho^{(i)})\deg(\rho^{(j)})}$, where $(\rho^{(i)})_{0 \leq i < 3}$ are the irreducible representations of G, see the proof of Theorem A.22.

Alternatively, since $\widehat{K(S_3)} \cong K(\hat{S_3})$ and therefore they have the same structure constants, the Cayley table of $K(\hat{S_3})$ can instead be computed as $m_{ij}^t = \langle \varphi^{(t)}, \varphi^{(i)} \cdot \varphi^{(j)} \rangle \, k(\varphi^{(t)})$ according to Theorem A.17. Note that the first of these inner products are inner products on G, whereas the second is an inner product on $K(S_3)$. The representations of $K(\hat{S}_3)$ are denoted by $(\rho^{(i)})_{0\leq i<3}$, and the character table of $K(\hat{S}_3)$ is shown in Table 9.

Finally, at the end of this section we present the regular representation, and its character, the weight functions, and the hyperdimensions, of $K(S_3)$.

Table 6: Cayley table for $K(S_3)$.

	c_0	c_1	c_2
c_0	c_0	c_1	c_2
c_1	c_1	$\frac{1}{3}c_0 + \frac{2}{3}c_2$	c_1
c_2	c_2	c_1	$\frac{1}{2}c_0 + \frac{1}{2}c_2$

Table 7: Character table for $K(S_3)$.

	c_0	c_1	c_2
$\varphi^{(0)}$	1	1	1
$\varphi^{(1)}$	1	0	$-\frac{1}{2}$
$\varphi^{(2)}$	1	-1	1

Table 8: Cayley table for $K(\hat{S}_3)$.

	ψ_0	ψ_1	ψ_2
ψ_0	ψ_0	ψ_1	ψ_2
ψ_1	ψ_1	$\frac{1}{4}\psi_0 + \frac{1}{2}\psi_1 + \frac{1}{4}\psi_2$	ψ_1
ψ_2	ψ_2	ψ_1	ψ_0

Table 9: Character table for $K(\hat{S}_3)$.

	ψ_0	ψ_1	ψ_2
$\rho^{(0)}$	1	1	1
$\rho^{(1)}$	1	0	-1
$\rho^{(2)}$	1	$-\frac{1}{2}$	1

$$L(c_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L(c_1) = \begin{pmatrix} 0 & 1/3 & 0 \\ 1 & 0 & 1 \\ 0 & 2/3 & 0 \end{pmatrix}, \quad L(c_2) = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1/2 \end{pmatrix}$$

$$\chi_L(c_0) = 3, \quad \chi_L(c_1) = 0, \quad \chi_L(c_2) = \frac{3}{2}$$

$$w(c_0) = 1, \quad w(c_1) = 3, \quad w(c_2) = 2, \quad w(K) = 6$$

$$k(\varphi_0) = 1, \quad k(\varphi_1) = 4, \quad k(\varphi_2) = 1$$

B Hypergroups from Graphs

B.1 Graph Hypergroup

The goal of this section is to construct and investigate hypergroups from *distance-transitive graphs*. We will begin by recalling some elementary definitions of graph theory. See [8] for a more thorough treatment.

Definition B.1 (Elementary definitions)**.**

- An *undirected graph* $\mathcal{G}$ is a finite set of vertices V and a set of edges E, where each edge is an unordered pair of distinct vertices. We denote the vertices of $\mathcal{G}$ by $V(\mathcal{G})$ and similarly its edges by $E(\mathcal{G})$.
- A pair of vertices x_0 and x_1 are *adjacent* if there is an edge $x_0x_1 \in E$ connecting them, and we denote this by $x_0 \sim x_1$.
- A *path* from u to v in a graph is a sequence of vertices $(w_i)_{0\leq i\leq n}$, where $w_0 = u$ and $w_n = v$, satisfying $w_i \sim w_{i+1}$ for $0 \leq i < n$. We say the path has *length* n.
- The *distance* $d(u, v)$ between two vertices u and v in a graph is the length of the shortest path between them.
- The *diameter* $\operatorname{diam}\mathcal{G}$ of the graph is the maximum distance between any two vertices, that is, $\operatorname{diam}\mathcal{G} := \max_{x_0,x_1\in V(\mathcal{G})} d(x_0, x_1)$.
- The *degree* of a vertex v in a graph $\mathcal{G}$ is $|v| := |\{e \in E(\mathcal{G}) \mid v \in e\}|$.
- A graph is *connected* if there is a path between every pair of vertices.

Definition B.2 (Isomorphism)**.** $\varphi : V(\mathcal{G}_0) \to V(\mathcal{G}_1)$ is a graph *isomorphism* if it is bijective and it holds that $x \sim y$ if and only if $\varphi(x) \sim \varphi(y)$.

Proposition B.3. *Let $\mathcal{G}$ be a connected, undirected graph. Then the distance function $d : V(\mathcal{G}) \times V(\mathcal{G}) \to \mathbb{N}$ is a metric; that is, for $x_0, x_1, x_2 \in V(\mathcal{G})$,*

(i) $d(x_0, x_1) = 0 \iff x_0 = x_1$

(ii) $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$

(iii) $d(x_0, x_1) = d(x_1, x_0)$.

Furthermore, the distance function respects isomorphism; that is, for $x_0, x_\infty \in V(\mathcal{G})$ and any graph isomorphism $\phi : V(\mathcal{G}) \to V(\mathcal{G}')$,

$$d(x_0, x_\infty) = d(\phi(x_0), \phi(x_\infty)).$$

Proof. (i) If $d(x_0, x_1) = 0$ then there is a path p_0 of length 0 between x_0 and x_1, so $(x_0) = p_0 = (x_1)$ and $x_0 = x_1$. If $x_0 = x_1$ then the path $p_0 := (x_0) = (x_1)$ goes between x_0 and x_1 and has length 0.
(ii) Since the graph is connected, there is a path p_0 of length $d(x_0, x_1)$ from x_0 to x_1, and similarly there is a path p_1 of length $d(x_1, x_2)$ from x_1 to x_2. Thus there is a path of length $d(x_0, x_1) + d(x_1, x_2)$, namely the composed path p_0p_1, from x_0 to x_2.
(iii) Path reversion is a bijection between paths from x_0 to x_1 and paths from x_1 to x_0, and preserves length.

As for isomorphisms, given a path $(x_i)_{0\leq i\leq n}$ in $\mathcal{G}$, the length of $(x_i)_{0\leq i\leq n}$ equals the length of $(\phi(x_i))_{0\leq i\leq n}$, so ϕ in an extended sense is a length-preserving bijection between paths from x_0 to x_∞ and paths from $\phi(x_0)$ to $\phi(x_\infty)$. Therefore, applying d and then min, we get $d(x_0, x_\infty) = d(\phi(x_0), \phi(x_\infty))$. □

Definition B.4 (Distance-transitive graph)**.** A connected undirected graph $\mathcal{G}$ is *distance-transitive* if, given any two ordered pairs of vertices (u, v) and (u', v') satisfying $d(u, v) = d(u', v')$, there is an isomorphism $g : V(\mathcal{G}) \to V(\mathcal{G})$ such that $g(u) = u'$ and $g(v) = v'$.

Using this basic group theory, we wish to demonstrate a way to generate a hypergroup from a distance-transitive graphs. With this goal in mind, we begin with the following definition.

Definition B.5 (Unnormalized structure constants)**.** Let $\mathcal{G}$ be a distance-transitive graph. Define the *unnormalized structure constants* $(N_{ij}^k)_{0\le i,j,k\le \operatorname{diam}\mathcal{G}}$ by choosing some $x_0, x_1 \in V(\mathcal{G})$ with $d(x_0, x_1) = i$ and setting

$$\begin{aligned} P_{ij}^k &:= \{x_2 \in V(\mathcal{G}) \ : \ d(x_1,x_2) = j,\ d(x_0,x_2) = k\} \\ N_{ij}^k &:= |P_{ij}^k|. \end{aligned}$$

As may be expected, these unnormalized structure constants will later be normalized, and will then be the structure constants of a hypergroup. But first we need to demonstrate that they are well defined.

Proposition B.6. *Let $\mathcal{G}$ be a distance-transitive graph. Then the unnormalized structure constants are well-defined. That is, they are independent of the choice of x_0 and x_1.*

Proof. If $0 \le i,j,k \le \operatorname{diam}\mathcal{G}$ are fixed and $N_{ij}^k, P_{ij}^k, \widetilde{N}_{ij}^k, \widetilde{P}_{ij}^k$ are defined in terms of some $x_0, x_1, \widetilde{x}_0, \widetilde{x}_1 \in V(\mathcal{G})$ then some isomorphism $\phi : V(\mathcal{G}) \to V(\mathcal{G})$ takes $x_0 \mapsto \widetilde{x}_0$ and $x_1 \mapsto \widetilde{x}_1$ since $d(x_0,x_1) = i = d(\widetilde{x}_0, \widetilde{x}_1)$. ϕ can be restricted to a bijection $f : P_{ij}^k \to \widetilde{P}_{ij}^k$ since if $x_2 \in P_{ij}^k$ then

$$\begin{cases} d(x_0,x_1) = i \\ d(x_1,x_2) = j \\ d(x_0,x_2) = k \end{cases} \implies \begin{cases} d(\phi(x_0),\phi(x_1)) = i \\ d(\phi(x_1),\phi(x_2)) = j \\ d(\phi(x_0),\phi(x_2)) = k \end{cases} \implies \begin{cases} d(\widetilde{x}_0,\widetilde{x}_1) \quad\ = i \\ d(\widetilde{x}_1,\phi(x_2)) = j \\ d(\widetilde{x}_0,\phi(x_2)) = k \end{cases}$$

so that $f(x_2) = \phi(x_2) \in \widetilde{P}_{ij}^k$; similarly f is surjective; and finally f is injective since ϕ is injective. A bijection between P_{ij}^k and $\widetilde{P}_{ij}^k$ has been established, and thus $N_{ij}^k = \widetilde{N}_{ij}^k$. ☐

Corollary B.7. *Let $\mathcal{G}$ be a distance-transitive graph. Then all vertices of $\mathcal{G}$ have the same degree.*

Proof. The degree of any $x_0 \in V(\mathcal{G})$ is N_{01}^1. ☐

Definition B.8 (Graph hypergroup)**.** Let $\mathcal{G}$ be a finite distance-transitive graph. Define the structure constants $(n_{ij}^k)_{0\le i,j,k\le \operatorname{diam} G}$ by normalizing the unnormalized structure constants:

$$n_{ij}^k := \frac{N_{ij}^k}{N_{0j}^j}.$$

By the *graph hypergroup* $K(\mathcal{G})$, we mean the standard basis $\{e_i \mid 0 \le i \le \operatorname{diam} V(\mathcal{G})\}$ of the complex vector space $\mathbb{C}^{\operatorname{diam}\mathcal{G}+1}$ equipped with with operations multiplication and involution given according to the structure constants, for $a, b \in \mathbb{C}^{\operatorname{diam}\mathcal{G}+1}$ and $0 \le k \le \operatorname{diam}\mathcal{G}$ by

$$\begin{aligned} (ab)_k &:= \sum_{0\le i,j\le \operatorname{diam}\mathcal{G}} a_i b_j n_{ij}^k \\ (a^*)_k &:= \overline{a_i}. \end{aligned}$$

Remark. The denominator N_{0j}^j is always nonzero since we assume $0 \le j \le \operatorname{diam} G$.

Remark. Any $*$ was consciously omitted from $\overline{a_i}$.

One may interpret n_{ij}^k as the probability to end up at distance k from one's starting point x_0 by choosing at random a point x_1 on the circle with radius i centered at x_0 and then choosing at random a point x_2 on the circle with radius j centered at x_1.

It is our present task to show that the graph hypergroup is indeed a hypergroup, and that it is both commutative and *hermitian*.

Definition B.9. (Hermitian) A hypergroup with elements (c_i) is said to be *hermitian* if all $c_i^* = c_i$.

Theorem B.10. *Let $\mathcal{G}$ be a distance-transitive graph. Then the graph hypergroup $K(\mathcal{G})$ is a hypergroup, and it is both commutative and hermitian.*

Proof. To show that $K(\mathcal{G})$ is a hypergroup, it suffices to verify equations (1, 2, 3, 4, 5, 6) of Theorem 2.6.

(1) Normalization follows from the fact that if $0 \leq i, j \leq \operatorname{diam} \mathcal{G}$, then $(P_{ij}^k)_{0 \leq k \leq \operatorname{diam} \mathcal{G}}$ is a partition of P_{0j}^j, provided we use the same x_0, x_1 in the definition of all $(P_{ij}^k)_{0 \leq k \leq \operatorname{diam} \mathcal{G}}$ and use x_1, x_1 correspondingly in the definition of P_{0j}^j, as allowed by Proposition B.6.

(2) Nonnegativity follows from N_{ij}^k being natural numbers.

(4) Neutrality of e_0 follows from the fact that if $N_{0j}^k \neq 0$ then there are points $x_0, x_1, x_2 \in V(\mathcal{G})$ such that $d(x_0, x_1) = 0,\ d(x_1, x_2) = j,\ d(x_0, x_2) = k$, but then $x_0 = x_1$ and $j = k$.

(5) Uniqueness of the involution will be shown in the proof of hermitianness.

(6) The involution identity will follow from commutativity and hermitianness. Thus $K(\mathcal{G})$ is a hypergroup, if we can also verify associativity (3), which is the most difficult identity to prove.

The proof of commutativity is based on reversing paths. If $0 \leq i, j, k \leq \operatorname{diam} \mathcal{G}$ then there are in total $|V(\mathcal{G})| N_{0i}^i N_{ij}^k$ triples (x_0, x_1, x_2) with $d(x_0, x_1) = i,\ d(x_1, x_2) = j,\ d(x_0, x_2) = k$, since the choice of x_0, x_1 does not affect N_{ij}^k by Proposition B.6 and for each x_0 – of which there are $|V(\mathcal{G})|$ – there are N_{0i}^i permitted choices of x_1.

Similarly there are in total $|V(\mathcal{G})| N_{0j}^j N_{ji}^k$ triples (x_0, x_1, x_2) with $d(x_0, x_1) = j,\ d(x_1, x_2) = i,\ d(x_0, x_2) = k$. A bijection between these sets of triples is given by $(x_0, x_1, x_2) \mapsto (x_2, x_1, x_0)$. Thus

$$
\begin{aligned}
|V(\mathcal{G})| N_{0i}^i N_{ij}^k &= |V(\mathcal{G})| N_{0j}^j N_{ji}^k \\
n_{ij}^k = \frac{N_{ij}^k}{N_{0j}^j} &= \frac{N_{ji}^k}{N_{0i}^i} = n_{ji}^k.
\end{aligned}
$$

The proof of hermitianness is based on the graph being undirected. If $0 \leq i \leq \operatorname{diam} G$ then by definition of the diameter there are points $x_0, x_2 \in G$ such that $d(x_0, x_2) = \operatorname{diam} G$. Along a shortest path from x_0 to x_2 choose x_1 so that $d(x_0, x_1) = i$.

Since the structure constants are independent of x_0 and x_1 by Proposition B.6 we may use these ones. Thus $x_0 \in P_{ii}^0$ and $x_1 \in P_{0i}^i$, proving that $N_{ii}^0 > 0$, $N_{0i}^i > 0$ and $n_{ii}^0 > 0$. Conversely, if $n_{ij}^0 > 0$ then $N_{ij}^0 > 0$ and there is some point $x_2 \in P_{ij}^0$, meaning that $d(x_0, x_1) = i,\ d(x_1, x_2) = j,\ d(x_0, x_2) = 0$ so in particular $x_0 = x_2$ and

$$
i = d(x_0, x_1) = d(x_2, x_1) = d(x_1, x_2) = j.
$$

Finally, we turn to prove associativity (3). Use commutativity to reduce (3) to

$$
\sum_j N_{i_1 i_0}^j N_{j i_2}^k = \sum_j N_{i_1 i_2}^j N_{j i_0}^k.
$$

To this end, for fixed $i_0, i_1, i_2, k, x_0, x_1$ with $d(x_0, x_1) = i_1$, define sets whose cardinality is the left or right-hand side respectively:

$$
\begin{aligned}
S_L &:= \{(x_2, x_3)\ :\ d(x_1, x_2) = i_0,\ d(x_2, x_3) = i_2,\ d(x_0, x_3) = k\} \\
S_R &:= \{(x_1, x_3)\ :\ d(x_1, x_2) = i_2,\ d(x_2, x_3) = i_0,\ d(x_0, x_3) = k\}.
\end{aligned}
$$

To see that these sets have the same size, fix x_3 with $d(x_0, x_3) = k$. Choose some isomorphism $\phi : V(\mathcal{G}) \to V(\mathcal{G})$ that sends $x_1 \mapsto x_3$ and $x_3 \mapsto x_1$, according to distance-transitivity.

To each x_2 such that $(x_2, x_3) \in S_L$, ϕ arranges a point $\phi(x_2)$ such that $(\phi(x_2), x_3) \in S_R$, in a bijective manner. Thus $|S_L| = |S_R|$, and we have associativity. $\square$

Our task to derive the commutative hermitian graph hypergroup $K(\mathcal{G})$ has been completed, and we now turn to efficient computation of its structure constants n_{ij}^k. We will make use of the regular representation as given in Definition 3.41, along with matrix multiplication and the metric properties of the distance function. Some central concepts will now be defined.

Definition B.11 (Tridiagonal matrix, Diagonals)**.** An $(n+1)\times(n+1)$ matrix M is said to be *tridiagonal* if $M_{ij} \neq 0$ implies $|i-j| \leq 1$. The lists $(M_{i(i-1)})_{0<i\leq n}$, $(M_{ii})_{0\leq i\leq n}$, $(M_{i(i+1)})_{0\leq i<n}$ are called the *diagonals* of M.

Definition B.12 (Intersection array, Intersection matrix)**.** Let $\mathcal{G}$ be a distance-transitive graph. For fixed $x_0, x_1 \in V(\mathcal{G})$ define the *intersection array* ι by

$$\begin{aligned} \iota &:= ((a_i)_{0\leq i\leq \operatorname{diam}\mathcal{G}},\ (b_i)_{0\leq i<\operatorname{diam}\mathcal{G}}) \\ a_i &:= N_{i1}^i, \quad b_i := N_{i1}^{i+1}, \quad f_i := N_{i1}^{i-1}. \end{aligned}$$

Define the *intersection matrix* B to be the $(\operatorname{diam}\mathcal{G}+1)\times(\operatorname{diam}\mathcal{G}+1)$ tridiagonal matrix with diagonals $(f_i)_{0<i\leq \operatorname{diam} G}$, $(a_i)_{0\leq i\leq \operatorname{diam} G}$, $(b_i)_{0\leq i<\operatorname{diam} G}$. In pictures,

$$B := \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 & 0 \\ f_1 & a_1 & b_1 & \cdots & 0 & 0 \\ 0 & f_2 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{\operatorname{diam}\mathcal{G}-1} & b_{\operatorname{diam}\mathcal{G}-1} \\ 0 & 0 & 0 & \cdots & f_{\operatorname{diam}\mathcal{G}} & a_{\operatorname{diam}\mathcal{G}} \end{pmatrix}.$$

The intersection array ι actually gives sufficient information to characterize the graph hypergroup. That is, we can compute the structure constants from it. Furthermore, it turns out that the computation is pretty cheap. However, the intersection array does not characterize the entire distance-transitive graph, as for instance $\iota = ((6,4,4),(1,1,3))$ belongs to two different distance-transitive graphs [9], and therefore we briefly conclude that a graph hypergroup does not in general characterize its distance-transitive graph.

To support these claims, we need the following lemma.

Lemma B.13 (The reverse triangle inequality)**.** *Let $\mathcal{G}$ be a distance-transitive graph. If $x_0, x_1, x_2 \in V(\mathcal{G})$ then $|d(x_0,x_1) - d(x_1,x_2)| \leq d(x_0,x_2)$.*

Proof. Use the metric properties of Proposition B.3 a few times to get

$$\begin{aligned} d(x_0,x_1) &\leq d(x_1,x_2) + d(x_0,x_2) \\ d(x_1,x_2) &\leq d(x_0,x_1) + d(x_0,x_2). \end{aligned}$$

From here the result follows, as $a \leq b, -a \leq b$ implies $|a| \leq b$. $\square$

Theorem B.14. *Let $\mathcal{G}$ be a distance-transitive graph. Then $K(\mathcal{G})$ is characterized by the intersection array $\iota = (a,b)$; via the regular representation $L(c_z)_{ij} := n_{zj}^i$ and the following recurrence relations, where the intersection matrix B is computed via $f_i = b_0 - a_i - b_i$.*

$$L(c_0) = I \tag{11}$$

$$L(c_1) = B^T/b_0 \tag{12}$$

$$L(c_{i+2}) = (-f_{i+1}L(c_i) - a_{i+1}L(c_{i+1}) + L(c_1)L(c_{i+1})b_0)/b_{i+1}. \tag{13}$$

Proof. (11) is immediate since L is a representation by Proposition 3.42.

To show (12), start by showing that $L(c_1)$ is tridiagonal. If $L(c_1)_{ij} \neq 0$ then by definition $n_{1j}^i \neq 0$ and there is some $x_2 \in P_{1i}^j$, meaning we have $d(x_0,x_1) = 1, d(x_1,x_2) = i, d(x_0,x_2) = j$ for some $x_0, x_1 \in V(\mathcal{G})$. From the reverse triangle inequality we derive that $|i-j| = |d(x_1,x_2) - d(x_0,x_2)| \leq d(x_0,x_1) = 1$.

Now since both $L(c_1)$ and B are tridiagonal, it suffices to compare their entries along the three diagonals. By commutativity we have $L(c_1)_{ij} = n^i_{1j} = n^i_{j1} = N^1_{j1}/N^1_{01}$ and thus

$$\begin{aligned} L(c_1)_{ii} &= N^i_{i1}/N^1_{01} = a_i/N^1_{01} = B^T_{ii}/N^1_{01} \\ L(c_1)_{i(i+1)} &= N^{i+1}_{i1}/N^1_{01} = b_i/N^1_{01} = B^T_{i(i+1)}/N^1_{01} \\ L(c_1)_{i(i-1)} &= N^{i-1}_{i1}/N^1_{01} = f_i/N^1_{01} = B^T_{i(i-1)}/N^1_{01}. \end{aligned}$$

This shows (12), since by definition $b_0 = N^1_{01}$.
The equation $f_i = b_0 - a_i - b_i$ is another consequence of tridiagonality.
We finally turn to (13). By tridiagonality, for $0 \le i \le \operatorname{diam} \mathcal{G} - 2$, we have

$$\begin{aligned} c_1 c_{i+1} &= \sum_k n^k_{1(i+1)} c_k = (f_{i+1} c_i + a_{i+1} c_{i+1} + b_{i+1} c_{i+2})/b_0 \\ L(c_1)L(c_{i+1}) = L(c_1 c_{i+1}) &= (f_{i+1} L(c_i) + a_{i+1} L(c_{i+1}) + b_{i+1} L(c_{i+2}))/b_0. \end{aligned}$$

If we can show that $b_{i+1} \neq 0$, then rearrangement gives (13). There are points $x_0, x_2 \in V(\mathcal{G})$ with $d(x_0, x_2) = i + 2$ since $i + 2 \le \operatorname{diam} \mathcal{G}$. Define x_1 to be a point along a shortest path from x_0 to x_2 with $d(x_0, x_1) = i + 1$ and $d(x_1, x_2) = 1$. By Proposition B.6 we then have $x_2 \in P^{i+2}_{(i+1)1}$ and thus $b_{i+1} = N^{i+2}_{(i+1)1} > 0$. □

As suggested earlier, this characterization of $K(\mathcal{G})$ by ι is computationally efficient. This is because $L(c_1)$ is tridiagonal, as stated in the form (12), from which it follows that a single evaluation of (13) can be performed in $\mathcal{O}((\operatorname{diam} \mathcal{G})^2)$ time, provided that the intersection array ι has already been stored. The intersection array can be computed by first running the well-known *Dijkstra's algorithm*; or it can be looked up on the Internet for common graphs.

We end this theory section with a simple but pleasant result, which is analogous to the last proposition of the section about the class hypergroup, and then move on to some trivial and non-trivial examples of graph hypergroups.

Proposition B.15. *Let $\mathcal{G}$ be a distance-transitive graph. Then the weight of any element $c_i = e_i \in K(\mathcal{G})$ is $w(c_i) = N^i_{0i}$, the number of points at distance i from any fixed $x_0 \in V(\mathcal{G})$.*

Proof. If $0 \le i \le \operatorname{diam} \mathcal{G}$ then $N^0_{ii} = 1$, since at distance i from any given $x_0 \in V(\mathcal{G})$ there is a point $x_1 \in V(\mathcal{G})$, and there is only one point $x_2 \in V(\mathcal{G})$ with $d(x_0, x_2) = 0$. Thus we argue that

$$w(c_i) = 1/n^0_{ii^*} = 1/n^0_{ii} = 1/(N^0_{ii}/N^i_{0i}) = N^i_{0i}.$$

N^i_{0i} is almost by definition the number of points at distance i from any $x_0 \in V(\mathcal{G})$. □

We have now defined the graph hypergroup of a finite distance-transitive graph, and shown that it actually is a hypergroup. We have also demonstrated a method for computing the structure constants of this hypergroup. Most of the rest of this appendix will be devoted to presenting the graph hypergroups from certain graphs and families of graphs. But first we present a link to the source code used to generate some of the examples.

B.2 Source Code

Some procedures have been written to automate arithmetic with hypergroups. Available at

`https://github.com/Breitholtz/Slutrapport-kandidat/tree/master/procedures`

they allow computation of structure constants from distance-transitive graphs, and they also include a rudimentary ability to compute character tables of commutative hypergroups. Furthermore, Cayley tables of hypergroups are easily typeset in the LaTeX format.

This was done in Python 3 using the libraries `sympy` for equation solving and `networkx` for computing intersection arrays. Due to limitations of equation solving in `sympy`, the character tables can typically only be computed for small hypergroups.

B.3 Complete Graph

In this section we present the graph hypergroup of a *complete graph*. A complete graph is a graph such that there is an edge connecting every pair of vertices. A graph hypergroup from a complete graph with n vertices has 2 elements. The Cayley table is shown in Table 10, and the character table in Table 11.

Table 10: Cayley table of a hypergroup from a complete graph n vertices.

	c_0	c_1
c_0	c_0	c_1
c_1	c_1	$\frac{1}{n}c_0 + \frac{n-1}{n}c_1$

Table 11: Character table of a hypergroup from a complete graph with n vertices.

	c_0	c_1
$\varphi^{(0)}$	1	1
$\varphi^{(1)}$	1	$\frac{n-1+\sqrt{n^2-n+1}}{n}$

$$w(c_0) = 1, \quad w(c_1) = n, \quad w(K) = n+1, \quad k(\varphi^{(0)}) = 1, \quad k(\varphi^{(1)}) = \frac{n+1}{2\left(n^2-n+1+\sqrt{n^3-2n^2+2n-1}\right)}$$

B.4 (k,n)-graph

In this section, we consider a family of graphs which we call *(k,n)-graphs*.

Definition B.16 ((k,n)-graph)**.** By a *(k,n)-graph* we mean a graph in which we can colour the vertices using $k > 1$ colours, so that each colour is used on $n > 1$ vertices, and so that each pair of vertices has an edge connecting them if and only if they have different colours. The hypergroup generated from a (k,n)-graph is called a *(k,n)-hypergroup*.

Example B.17. The (3,4)-graph is shown in Figure 1. Note that 3 different vertex colours have been used, that there are 4 vertices of each colour, and that two vertices are connected if and only if they are of different colours.

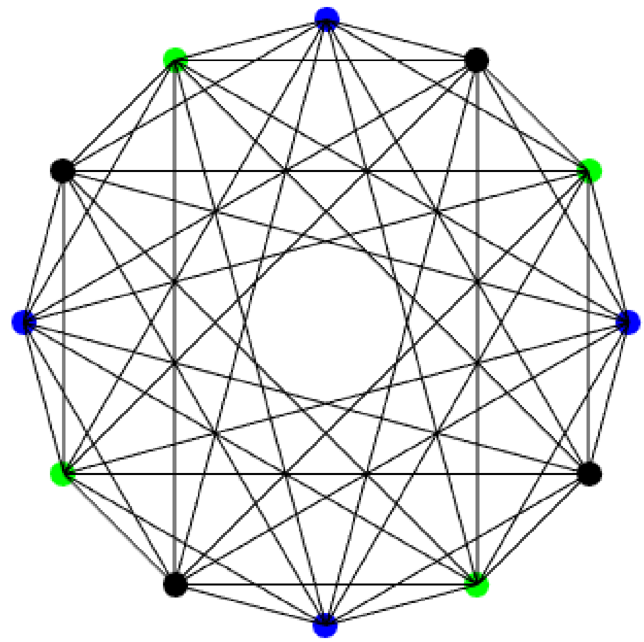

Figure 1: A (3,4)-graph where the vertices have been coloured black, blue, and green.

It can be shown that the (k,n)-graph is distance-transitive for all $k, n > 1$. Thus we can compute its graph hypergroup. The Cayley table of the hypergroup from a (k,n)-graph is shown in Table 12. The character table is shown in Table 13. The Cayley table of the dual hypergroup is shown in Table 14. Note that the dual hypergroup is generated by an (n,k)-graph. Thus the dual hypergroup of a (k,n)-hypergroup is an (n,k)-hypergroup.

Note also that if $k = 2$ and $n = 3$ the (k,n)-hypergroup is isomorphic to $K(S_3)$, which can easily be seen by comparing the Cayley tables in Table 6 and Table 12.

Table 12: Cayley table for the hypergroup generated from a (k, n)-graph

	c_0	c_1	c_2
c_0	c_0	c_1	c_2
c_1	c_1	$\frac{1}{(k-1)n}c_0 + \frac{k-2}{k-1}c_1 + \frac{n-1}{(k-1)k}c_2$	c_1
c_2	c_2	c_1	$\frac{1}{n-1}c_0 + \frac{n-2}{n-1}c_2$

Table 13: Character table for the hypergroup generated from a (k, n)-graph

	c_0	c_1	c_2
$\varphi^{(0)}$	1	1	1
$\varphi^{(1)}$	1	0	$-\frac{1}{n-1}$
$\varphi^{(2)}$	1	$-\frac{1}{k-1}$	1

$$w(c_0) = 1, \quad w(c_1) = (k-1)n, \quad w(c_2) = n-1, \quad w(K) = nk,$$
$$k(\varphi_0) = 1, \quad k(\varphi_1) = (n-1)k, \quad k(\varphi_2) = k-1$$

Table 14: Cayley table for the dual hypergroup of a (k,n)-hypergroup.

	$\varphi^{(0)}$	$\varphi^{(1)}$	$\varphi^{(2)}$
$\varphi^{(0)}$	$\varphi^{(0)}$	$\varphi^{(1)}$	$\varphi^{(2)}$
$\varphi^{(1)}$	$\varphi^{(1)}$	$\frac{1}{(n-1)k}\varphi^{(0)} + \frac{n-2}{n-1}\varphi^{(1)} + \frac{k-1}{(n-1)k}\varphi^{(2)}$	$\varphi^{(1)}$
$\varphi^{(2)}$	$\varphi^{(2)}$	$\varphi^{(1)}$	$\frac{1}{k-1}\varphi^{(0)} + \frac{k-2}{k-1}\varphi^{(2)}$

B.5 Platonic Solids

This section contains two examples of hypergroups constructed from the graphs of platonic solids, the octahedron and the dodecahedron. Attached are the Cayley tables, as well as some curious data about the hypergroup. ι contains the intersection numbers, w the weights of the hypergroup elements, and k the corresponding hyperdimensions. X is the character table.

The reader is invited to verify that the Cayley tables are symmetric; the rows of the character table are orthogonal, with respect to the inner product $\langle \cdot, \cdot \rangle$ given in the main text; and that in these commutative cases the sum of the weights equals the sum of the hyperdimensions.

We thank `User:Cyp` at Wikipedia for drawing platonic solids for anyone to use under the Creative Commons license CC BY-SA 3.0.

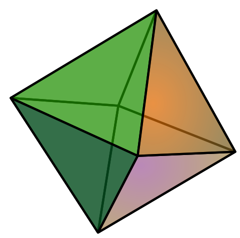

Table 15: Cayley table for the octahedral graph.

$\cdot$	c_0	c_1	c_2
c_0	c_0	c_1	c_2
c_1	c_1	$\frac{1}{4}c_0 + \frac{1}{2}c_1 + \frac{1}{4}c_2$	c_1
c_2	c_2	c_1	c_0

$$\iota = [4, 1; 1, 4], \quad w = [1, 4, 1], \quad k = [1, 2, 3]$$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

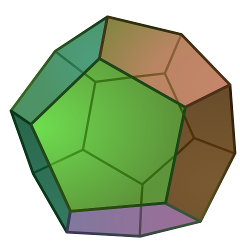

Table 16: Cayley table for the dodecahedral graph.

$\cdot$	c_0	c_1	c_2	c_3	c_4	c_5
c_0	c_0	c_1	c_2	c_3	c_4	c_5
c_1	c_1	$\frac{1}{3}c_0+\frac{2}{3}c_2$	$\frac{1}{3}c_1+\frac{1}{3}c_2+\frac{1}{3}c_3$	$\frac{1}{3}c_2+\frac{1}{3}c_3+\frac{1}{3}c_4$	$\frac{2}{3}c_3+\frac{1}{3}c_5$	c_4
c_2	c_2	$\frac{1}{3}c_1+\frac{1}{3}c_2+\frac{1}{3}c_3$	$\frac{1}{6}c_0+\frac{1}{6}c_1+\frac{1}{6}c_2+\frac{1}{3}c_3+\frac{1}{6}c_4$	$\frac{1}{6}c_1+\frac{1}{3}c_2+\frac{1}{6}c_3+\frac{1}{6}c_4+\frac{1}{6}c_5$	$\frac{1}{3}c_2+\frac{1}{3}c_3+\frac{1}{3}c_4$	c_3
c_3	c_3	$\frac{1}{3}c_2+\frac{1}{3}c_3+\frac{1}{3}c_4$	$\frac{1}{6}c_1+\frac{1}{3}c_2+\frac{1}{6}c_3+\frac{1}{6}c_4+\frac{1}{6}c_5$	$\frac{1}{6}c_0+\frac{1}{6}c_1+\frac{1}{6}c_2+\frac{1}{3}c_3+\frac{1}{6}c_4$	$\frac{1}{3}c_1+\frac{1}{3}c_2+\frac{1}{3}c_3$	c_2
c_4	c_4	$\frac{2}{3}c_3+\frac{1}{3}c_5$	$\frac{1}{3}c_2+\frac{1}{3}c_3+\frac{1}{3}c_4$	$\frac{1}{3}c_1+\frac{1}{3}c_2+\frac{1}{3}c_3$	$\frac{1}{3}c_0+\frac{2}{3}c_2$	c_1
c_5	c_5	c_4	c_3	c_2	c_1	c_0

$$\iota = [3,2,1,1,1;1,1,1,2,3], \quad w = [1,3,6,6,3,1], \quad k = [1,3,3,4,4,5]$$

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{\sqrt{5}}{3} & -1 \\ 1 & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{5}}{3} & -1 \\ 1 & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & 1 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \\ 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

C Algebra

Some basic knowledge of algebra group theory is practically necessary to understand and set our article in context. For example, the representation theory we build for hypergroups is originally a way to study and represent groups. This section will therefore contain some of the basics in group theory, as well as some basic definitions related to rings. The focus will primarily lie on what is necessary to understand our paper, and this section does in no way give a comprehensive view of group theory or algebra. Instead we suggest picking up a book in abstract algebra for further study, e.g. [10].

C.1 Group Theory

Definition C.1 (Group)**.** A *group* is a set G and a binary operation $\cdot$ satisfying:

(i) Closure: For all a, $b \in G$, the product $a \cdot b \in G$.

(ii) Associativity: For all a, b, $c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(iii) Identity: There exists an element $e \in G$ such that $a \cdot e = e \cdot a = e$ for all $a \in G$.

(iv) Inverse: For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

All readers will have encountered several groups in their mathematical careers, although not necessarily aware of it. Let us look at a few examples to solidify this somewhat abstract definition.

Example C.2. The set of integers $\mathbb{Z}$ with addition $(+)$ is a group. This can readily be verified by going through the requirements in the definition. Clearly the sum of two integers is an integer. Associativity is obvious. The identity is 0 and each element $a \in \mathbb{Z}$ has an inverse $-a$ for which $a + (-a) = 0$.

On the other hand, $\mathbb{Z}$ with multiplication $(\cdot)$ **is not** a group, since only 1 and -1 have inverses.

Example C.3. The set $S = \mathbb{Q} \setminus \{0\}$ (the rational numbers excluding zero) together with multiplication is a group. For all $\frac{p}{q}, \frac{s}{t} \in S$, $\frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt} \in S$, thus it is closed. Associativity follows from the associativity of regular multiplication, 1 is the identity element, and $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p} \in S$ so we have inverses.

Example C.4. The set of all rotations of a plane about a point p is a group. Closure holds, as two rotations by θ and ϕ degrees respectively corresponds to one rotation by $\theta + \phi$ degrees. Associativity holds as rotation is associative. The identity is simply rotation by zero degrees, and the inverse of a rotation by θ degrees is a rotation by $-\theta$ degrees.

Example C.5. The set $GL_n(\mathbb{C})$ consisting of all square matrices of order n with nonzero determinant is a group with matrix multiplication. Associativity follows from the associativity of matrix multiplication, the identity matrix is the identity of the group, all elements have an inverse since their determinant is nonzero, and finally closure is confirmed through $\det(AB) = \det(A)\det(B) \neq 0$ for all $A, B \in GL_n(\mathbb{C})$.

One important difference between the last example compared to the previous three is that matrix multiplication isn't commutative. The group operation is not commutative in general, but there are enough important special cases that we will make the following definition.

Definition C.6 (Abelian group)**.** A group G with operation $\cdot$ is called *abelian* if and only if

$$a \cdot b = b \cdot a \text{ for all } a, b \in G$$

Definition C.7 (Cayley table)**.** Let G be a finite group with operation $\cdot$. By the *Cayley table* of G we mean the table describing the binary operation $\cdot$ acting on all pairs of elements in G, thus fully describing the group. Note that the order of the operation will matter if G is not abelian, thus the convention is to take the vertical elements first.

For example, if $G = \{a, b, c\}$, the Cayley table will look like Table 17.

Table 17: The Cayley table of G.

$\cdot$	a	b	c
a	$a \cdot a$	$a \cdot b$	$a \cdot c$
b	$b \cdot a$	$b \cdot b$	$b \cdot c$
c	$c \cdot a$	$c \cdot b$	$c \cdot c$

Definition C.8 (Order)**.** Let G be a group. By the *order* of G we mean the cardinality $|G|$ of the underlying set, i.e. the number of elements in the group.

Definition C.9 (Symmetric group)**.** Assume X is a finite set. Denote by $S(X)$ the set of all bijections from X to itself, i.e. the permutations of its elements. It is then not hard to verify that $S(X)$ is a group under composition. We call this *the symmetric group on* X. When $X = \{1, 2, \ldots, n\}$, we write S_n and call it the symmetric group of degree n. It can be shown that $|S_n| = n!$.

In all groups we have seen so far, the identity element has been unique, and all elements have only had one unique inverse. It turns out that these are not just special cases, but it is true of groups in general.

Theorem C.10. *Let G be a group. Then*

(i) The identity element of G is unique, i.e. if $e, h \in G$ satifies

$$eg = ge = g \text{ for all } g \in G$$

and

$$hg = gh = g \text{ for all } g \in G$$

then

$$e = f.$$

(ii) The inverse of an element in G is unique, i.e if $a, b, c \in G$ satisfy

$$ab = ba = e$$

and

$$ac = ca = e$$

then

$$b = c.$$

Proof.

(i) Assume that $eg = g$ for all $g \in G$. Then, as special case of this, $eh = h$. Now assume that $gh = g$ for all $g \in G$. Then $eh = e$. From this it follows that $e = h$, which proves the uniqueness of the identity.

(ii) With a, b, c, e defined as above we have

$$b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c,$$

which proves the uniqueness of the inverse.

□

We can now speak about *the* identity of a group and *the* inverse of an element, which is important in our discussion of representations. Another concept we will need in our discussion of representations is that of the equivalence relation.

Definition C.11. A binary relation $\sim$ on a set S is an *equivalence relation* if it is

(i) Reflexive: if $a \in S$ then $a \sim a$

(ii) Symmetric: I $a, b \in S$ and $a \sim b$, then $b \sim a$.

(iii) Transitive: If $a, b, c \in S$, $a \sim b$, and $b \sim c$, then $a \sim c$.

Equivalence relations are an important concept with interesting properties, which we will discuss, but let us first look at a few examples.

Example C.12. Equality is an equivalence relation on $\mathbb{R}$. The reflexive property is easily verified, as every number is equal to itself. The symmetric propoerty also holds, as $a = b$ if and only if $b = a$. Finally, the transitive property holds, since $a = b$ and $b = c$ implies $a = c$.

Example C.13. Assume that G is a group, and define a relation $\sim$ on G by

$$a \sim b \text{ if and only if there is } g \in G \text{ such that } a = gbg^{-1}.$$

Then $\sim$ is an equivalence relation. Let us verify this by going through the different requirements of the definition.

Reflexive The identity e is in G, and $eae^{-1} = a$. Thus a$\sim$a.

Symmetric Assume $a \sim b$. Then there is a $g \in G$ such that $a = gbg^{-1}$. Multiplication by g from the right and g^{-1} from the left results in $b = g^{-1}ag = \left(g^{-1}\right) a \left(g^{-1}\right)^{-1}$ Thus $b \sim a$.

Transitive Assume $a, b, c \in G$, $a \sim b$, and $b \sim c$. Then there exist $g_1, g_2 \in G$ such that $a = g_1 b g_1^{-1}$ and $b = g_2 c g_2^{-1}$. This implies that $a = g_1 b g_1^{-1} = g_1 g_2 c g_2^{-1} g_1^{-1}$. As $g_1 g_2 \in G$ and $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ we conclude that $a \sim c$.

The equivalence relation in this last example is important, and will be used in the discussion of representations. If $\sim$ is as in this example, and $a \sim b$, we say that b is the *conjugate* of a, or that a and b are in the same *conjugacy class*.

An important property of equivalence relations is that partition sets into equivalence classes.

Definition C.14 (Partition)**.** Let S be a nonempty set. Then the collection of sets $S_1, \ldots, S_k$ forms a *partition* of S if

(i) $S_i \subseteq S$ for all $i = 1, \ldots, k$,

(ii) $S = \bigcup_{i=1}^{k} S_i$, and

(iii) if $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$.

Definition C.15. Let $\sim$ be an equivalence relation on a set S. Assume $a \in S$. By the *equivalence class* $[a]$ of a we mean

$$[a] = \{x \in S : a \sim x\}$$

Theorem C.16. *Let $\sim$ be an equivalence relation on a set S. Then the equivalence classes of $\sim$ form a partition of S.*

Proof. From the definition of equivalence classes, we know that they only contain elements from S, and thus are subsets of S. Assume that $a \in S$. We know that a belongs to at least one equivalence class, $[a]$. As this is true for every element in S, it follows that S is the union of the equivalence classes.

It remains to prove that two equivalence classes are either equal or disjoint. Assume that $[a] \cap [b] \neq \emptyset$. Then $\exists c$ such that $c \in [a]$ and $c \in [b]$. Let x denote any element in $[b]$, then we both have $b \sim c$ and $b \sim x$, which implies $c \sim x$, because $\sim$ is both symmetric and transitive. But we also have that $c \in [a] \implies a \sim c$. Using the transitive property again we get $a \sim x$, which implies $[b] \subseteq [a]$. In the same way it can be shown that $[a] \subseteq [b]$, which implies that $[a] = [b]$. $\square$

Knowing this, let us return to the equivalence relation from Example C.13.

Definition C.17. Let G be a group. The equivalence classes defined by

$$[a] = \{x \in G \mid \exists g \in G \ : \ a = gxg^{-1}\}$$

are called the conjugacy classes of G.

This concludes our treatment of group theory.

C.2 Rings and Fields

We will now proceed to define algebraic structures with even more structure than the group, starting with rings and ending with the $*$-algebra which is used in the definition of a hypergroup.

Definition C.18 (Ring)**.** A *ring* is a set R with two binary operations $+$ and $\cdot$, sometimes denoted $(R, +, \cdot)$, for which

(i) R is an abelian group under addition.

(ii) Multiplication is associative and there is a multiplicative identity, i.e. there is $1 \in R$ such that

$$a \cdot 1 = 1 \cdot a = a \text{ for all } a \in R.$$

(iii) Multiplication is distributive over addition, i.e.

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ for all } a, b, c \in R$$
$$(a + b) \cdot c = a \cdot c + b \cdot c \text{ for all } a, b, c \in R.$$

Definition C.19 ($*$-ring)**.** A *$*$-ring* or an *involutive ring* is a ring R with a map $* : R \to R$ such that $x \mapsto x^*$:

(i) is distributive over addition

$$(x + y)^* = x^* + y^*.$$

(ii) is an *anti-homomorphism*

$$(xy)^* = y^* x^*.$$

(iii) is an *involution*

$$(x^*)^* = x.$$

(iv) takes 1 to 1

$$1^* = 1.$$

Definition C.20 (Field)**.** A *field* is a ring $(F, +, \cdot)$ where we, in addition, demand that:

(i) The multiplication $\cdot$ is commutative.

(ii) All nonzero elements have *multiplicative inverses*, i.e. for all $a \in F \setminus \{0\}$ there is an $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

Example C.21. All readers will have encountered fields in the past, for example the rational numbers $\mathbb{Q}$ with ordinary addition and multiplication is a field with $a^{-1} = \frac{1}{a}$. In fact, $\mathbb{R}$ and $\mathbb{C}$ are also fields.

Definition C.22 (Module)**.** Let R be a ring, and 1_R its multiplicative identity. Then a *left R-module* consists of an abelian group $(M, +)$ and an operation $\left[\,\cdot \mid R \times M \to M\right]$ such that for all $r, s \in R$ and $u, v \in M$ the following holds:

(i) $1_R \cdot u = u$

(ii) $(r + s) \cdot u = r \cdot u + s \cdot u$

(iii) $(rs) \cdot u = r \cdot (s \cdot u)$

(iv) $r \cdot (u + v) = r \cdot u + r \cdot v$.

Definition C.23 (Associative algebra)**.** Let R be a commutative ring. By an *associative R-algebra* we mean an additive abelian group A which has the structure of both a ring and an R-module in such a way that the scalar multiplication satisfies the following:

- $r \cdot (u \cdot v) = (r \cdot u) \cdot v = u \cdot (r \cdot v)$ for all $r \in R$ and $u, v \in A$

In addition, A should have the following properties:

- Existence of identity: There exists $1 \in A$ such that $1 \cdot u = u = u \cdot 1$ for all $u \in A$.
- Associativity: $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ for all $u, v, w \in A$

Definition C.24 ($*$-algebra)**.** A *$*$-algebra* is a $*$-ring with involution $*$ that is an associative algebra over a $*$-ring R with involution $'$ such that

$$(rx)^* = r'x^* \quad \forall\, r \in R,\ x \in B$$

In this text, the $*$-ring R will be $\mathbb{C}$ with complex conjugation as the involution $'$.

C.3 Linear Algebra

While we assume that the reader is familiar with basic linear algebra, this section covers some of the more advanced notions used. It also repeats some concepts that are of higher importance for our text. For a more thorough treatment, we recommend turning to a book on linear algebra such as [11].

Definition C.25 (Vector space)**.** A *vector space* V over a field F is a set of objects called *vectors* which can be added together and multiplied by elements of F. In addition, the following properties must be satisfied:

(i) Closed under addition: If $u, v \in V$, then $u + v \in V$.

(ii) Closed under multiplication by scalar: If $c \in F$ and $u \in V$, then $c \cdot u \in V$.

(iii) Associativity: If $u, v, w \in V$, then $(u + v) + w = u + (v + w)$.

(iv) Zero vector: There is $\mathbf{0} \in V$ such that for any $u \in V$, $u + \mathbf{0} = \mathbf{0} + u = u$.

(v) Additive inverse: If $u \in V$, there is $-u \in V$ such that $u + (-u) = 0$.

(vi) Commutativity: If $u, v \in V$, then $u + v = v + u$.

(vii) Distributivity w.r.t. vector addition: If $c \in F$ and $u, v \in V$, then $c(u + v) = cu + cv$.

(viii) Distributivity w.r.t. field addition: If $a, b \in F$ and $u \in V$, then $(a + b)u = au + bu$.

(ix) Field identity: If $u \in V$ and $1 \in F$ is the multiplicative identity, then $1 \cdot u = u \cdot 1 = u$.

This definition looks overwhelming, but hopefully you have seen it before. We call the elements of F *scalars* and thus the multiplication is by a scalar. We will only be working with vector spaces over the familiar field $\mathbb{C}$.

Definition C.26 (Subspace)**.** Let V be a vector space over a field F, and W a subset of V. We say that W is a *linear subspace* of V if the following holds:

(i) Zero vector: $\mathbf{0} \in V$ is also in W, i.e. $\mathbf{0} \in V$.

(ii) Closed under addition: If $u, v \in W$, then $u + v \in W$.

(iii) Closed under multiplication by scalar: If $c \in F$ and $w \in W$, then $cv \in W$.

Definition C.27 (Homomorphism)**.** Let V and W be vector spaces over some field F. If $f : V \to W$ is a linear map, i.e. for any $u, v \in V$ and $c \in F$ it holds that

$$f(u+v) = f(u) + f(v) \quad \text{and} \quad f(cu) = af(u),$$

we say that f is a *homomorphism*. Let $\mathrm{Hom}(V, W)$ denote the set of all homomorphisms from V to W.

Definition C.28 (Endomorphism)**.** An *endomorphism* is a linear map from a vector space to itself, i.e. the endomorphisms of V are the homomorphisms from V to V. We denote the set of all these by $\mathrm{End}(V)$, so $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

Definition C.29 (General linear group)**.** The set of all invertible linear maps from a vector space V to itself forms a group, and we call it the *general linear group* of V, denoted $\mathrm{GL}(V)$.

Definition C.30 (Kernel and image)**.** Let V and W be vector spaces over a field F, and let $f \in \mathrm{Hom}(V, W)$. By the *kernel* of f we mean

$$\ker(f) = \{v \in V \mid f(v) = \mathbf{0}\},$$

and by the *image* of f we mean

$$\mathrm{im}(f) = \{f(v) \mid v \in V\}.$$

Note that $\ker(f) \subseteq V$ and $\mathrm{im}(f) \subseteq W$.

D Proofs Omitted from the Main Text

Due to space considerations, we have chosen to free the main text of proofs that are trivial but space-consuming, and proofs virtually identical to the proofs in [4]. The investigative reader can find these in this section.

Proof of Proposition 3.6. If φ is a representation then $\varphi \sim \varphi$ by I. If $\varphi \sim \rho$ by T then $\rho \sim \varphi$ by T^{-1}. If $\varphi_i \sim \varphi_j$ by T_i and $\varphi_j \sim \varphi_k$ by T_j then $\varphi_i \sim \varphi_k$ by the composition $T_j \circ T_i$. □

Proof of Proposition 3.11.

$$\begin{aligned}
(\varphi \oplus \rho)(c_0) &= \varphi(c_0) \oplus \rho(c_0) = I \oplus I = I \\
(\varphi \oplus \rho)(c_i)(\varphi \oplus \rho)(c_j) &= (\varphi(c_i) \oplus \rho(c_i))(\varphi(c_j) \oplus \rho(c_j)) \\
&= \varphi(c_i)\varphi(c_j) \oplus \rho(c_i)\rho(c_j) \\
&= \sum_k n_{ij}^k \varphi(c_k) \oplus \sum_k n_{ij}^k \rho(c_k) \\
&= \sum_k n_{ij}^k (\varphi(c_k) \oplus \rho(c_k)) = \sum_k n_{ij}^k (\varphi \oplus \rho)(c_k).
\end{aligned}$$

□

Proof of Theorem 3.19.

(i) For any $v \in U$, and $\forall c_i \in K$, we have $\varphi(c_i)v \in U$ because U is invariant. The equivalence implies that $\rho(c_i)Tv = T\varphi(c_i)v \in TU$, which proves that TU is invariant.

(ii) If $U \leq V$ is invariant then we wish to show that either $U = \{0\}$ or $U = V$. If $U \neq \{0\}$ then there is a nonzero $u \in U$, and then TU is nontrivial because T is invertible. But TU is invariant by (i), so $TU = W$ and therefore $U = V$.

(iii) If φ and ρ are two equivalent representations then

$$\exists\, T : V \to W \text{ s.t } \varphi = T^{-1}\rho T$$

Suppose that $W_1, W_2 < W$ invariant e.g $W = W_1 \oplus W_2$. We then have that $T\varphi = \rho T$. Now let $V_1 = T^{-1}(W_1)$, $V_2 = T^{-1}(W_2)$. We now claim that $V = V_1 \oplus V_2$. Take

$$v \in V_1 \cap V_2 \implies Tv \in W_1 \cap W_2 = \{0\} \implies Tv = 0 \implies v = 0$$

since T is injective. If we now take $v \in V \implies Tv = w_1 + w_2$ for some $w_1 \in W_1$, $w_2 \in W_2$ which in turn implies

$$v = T^{-1}w_1 + T^{-1}w_2 \in V_1 \oplus V_2 \implies v \in V_1 \oplus V_2 = V$$

It only remains to show V_1, V_2 invariant. Take

$$v \in V_i \implies \varphi v = T^{-1}\rho T v, \text{ but } Tv \in W_i \implies \rho T v \in W_i$$

as W_i invariant which in turn gives us $\varphi v \in T^{-1}(w_i) \in V_i$ for some $w_i \in W_i$

(iv) According to (i) we have that $\exists T_i$ s.t $T_i V_i \leq W\ \forall i$. Since φ decomposible this implies by (iii) that ρ is decomposible into the same amount of parts as φ. If we now assume that ρ_i is decomposible then it follows that φ_i is decomposible as well. However since φ_i is an irreducible representation this is impossible. Therefore ρ is completely reducible.

□

Proof of Proposition 3.20. 0 is a morphism because $0\varphi(c_i) = 0 = \rho(c_i)0$. If $\lambda_1, \lambda_2 \in \mathbb{C}$ and $T_1, T_2 \in \mathrm{Hom}(\varphi, \rho)$ and $c_i \in K$ then

$$\begin{aligned}(\lambda_1 T_1 + \lambda_2 T_2)\varphi(c_i) &= \lambda_1 T_1 \varphi(c_i) + \lambda_2 T_2 \varphi(c_i) \\ &= \lambda_1 \rho(c_i) T_1 + \lambda_2 \rho(c_i) T_2 = \rho(c_i)(\lambda_1 T_1 + \lambda_2 T_2).\end{aligned}$$

Linearity of $\rho(c_i)$ was used in the last step. □

Proof of Lemma 3.21. If $Tv = 0$ and $c_i \in K$ then $T\varphi(c_i)v = \rho(c_i)Tv = \rho(c_i)0 = 0$. If $Tv = w$ and $c_i \in K$ then $\rho(c_i)w = \rho(c_i)Tv = T\varphi(c_i)v$. □

Proof of Theorem 3.22. (i) Since T is nonzero, $Tv \neq 0$ for some v, which is then nonzero because T is linear. As $\ker T$ is invariant and not V, $\ker T = \{0\}$. As $\operatorname{im} T$ is invariant and not $\{0\}$, $\operatorname{im} T = W$. Being linear, this makes T injective and surjective, respectively, and thus invertible. (ii) By the fundamental theorem of algebra, $T - \lambda I$ has determinant zero for some $\lambda \in \mathbb{C}$. By linearity, $T - \lambda I$ is a morphism. By part (i), being non-invertible, $T - \lambda I$ must be zero. □

Proof of Theorem 3.25. For the first part, proceed recursively using Lemma 3.24: Given a representation $\varphi : K \to \mathrm{End}(V)$, reduce V as a sum $V_1 \oplus V_2$ of invariant subspaces if possible, and then treat $\varphi : K \to \mathrm{End}(V_1)$ and $\varphi : K \to \mathrm{End}(V_2)$ by the same procedure. This process will halt eventually because $\dim V_1 < \dim V$ and $\dim V_2 < \dim V$. Thus invariant subspaces V_i are constructed, such that and each restriction $\varphi : K \to \mathrm{End}(V_i)$ is irreducible and $V = \bigoplus V_i$.

For the second part: given $\varphi : K \to \mathrm{End}(V)$, decompose $V = \bigoplus V_i$ (internal sum) according to the first part. The bijection $T : V \to \bigoplus V_i$ (external sum) is given by $(T \sum_i v_i)_j = v_j$, where $v_i \in V_i$. Then $\varphi \sim \bigoplus \varphi_i$ (external sum) where by $\varphi_i : K \to \mathrm{End}(V_i)$ we mean the restriction of φ to V_i. Every φ_i is irreducible by the first part. □

Proof of Proposition 3.27. We want an inner product to follow the following axioms

- $\langle a, b\rangle = \overline{\langle b, a\rangle}$
- $\langle Aa, b\rangle = A\langle a, b\rangle$
- Suppose $c : K \to \mathbb{C}$ then $\langle a + c, b\rangle = \langle a, b\rangle + \langle c, b\rangle$
- $\langle a, a\rangle \geq 0$ with equality only when $a = 0$

We first check conjugate symmetry

$$\overline{\langle a, b\rangle} = \frac{1}{w(K)} \sum_z \overline{b(c_z)\overline{a(c_z)}w(c_z)} = \frac{1}{w(K)} \sum_z \overline{b(c_z)}a(c_z)\overline{w(c_z)} = \frac{1}{w(K)} \sum_z \overline{b(c_z)}a(c_z)w(c_z) = \langle b, a\rangle$$

We now show that it is linear in its first argument

$$\langle Aa + Bc, b\rangle = \frac{1}{w(K)} \sum_z b(c_z)\overline{(Aa(c_z) + Bc(c_z))}w(c_z) = \frac{1}{w(K)} \sum_z b(c_z)\overline{Aa(c_z)w(c_z)} +$$
$$\frac{1}{w(K)} \sum_z b(c_z)\overline{Bc(c_z)}w(c_z) = A\frac{1}{w(K)} \sum_z b(c_z)\overline{a(c_z)}w(c_z) + B\frac{1}{w(K)} \sum_z b(c_z)\overline{c(c_z)}w(c_z)$$

Finally we show that the scalar product is positive definite

$$\langle a, a\rangle = \frac{1}{w(K)} \sum_z \overline{a(c_z)}a(c_z)w(c_z) = \frac{1}{w(K)} \sum_z \underbrace{|a(c_z)|^2}_{\geq 0} \underbrace{w(c_z)}_{>0} \geq 0$$

We note that equality only occurs when $a(c_z) = 0$.

□

Proof of Lemma 3.30. By definition

$$(AE_{lj}B)_{ki} = \sum_{x,y} a_{kx}(E_{lj})_{xy}b_{yi},$$

but all terms are zero, except when $x = l$, $y = j$, which gives us the desired formula. $\square$