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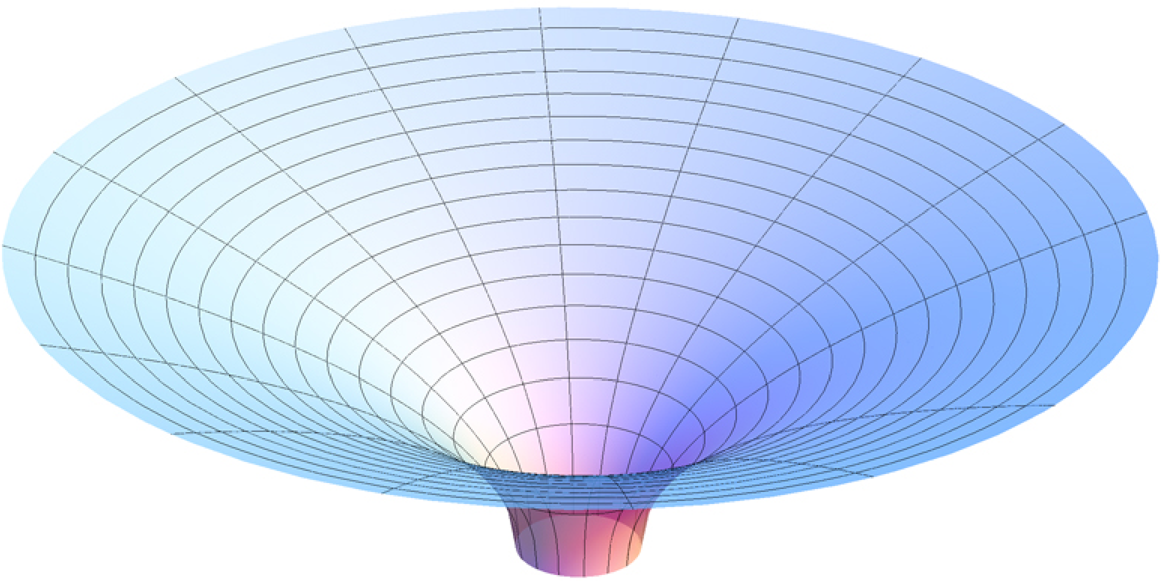

Resolution of the Big Bang Singularity in a Higher Spin Toy Model

From Chern-Simons Gauge Theory to Conformal Gravity and Higher Spin

Bachelor's Thesis in Engineering Physics

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Gothenburg, Sweden, June 5, 2015

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Abstract

This is a report describing introductory gauge theory, general relativity in Cartan formalism as well as their common features. It is shown that the Einstein-Hilbert action, which governs Einstein's theory of general relativity, in 2+1 dimensions can be written as a Chern-Simons gauge action. Using this, the gauge group is extended in two separate ways to produce two different generalisations of normal Einstein theory. First a theory of conformal gravity is produced and by selecting a particular gauge, the resulting algebraic equations are solved, leaving the Cotton equation. Secondly, a theory of gravity coupled to higher spin is constructed. This theory is then applied to a simple model of an empty universe, where it is shown that the Big Bang singularity is a gauge artifact that vanishes with a suitable gauge choice. Finally another extension to higher spin, with a canonical route to quantisation, using the conformal group is presented. It is shown how it is impossible for this theory to only include finite spins. Theories like this are central, for example in relation to the AdS/CFT correspondence, which is an area of active research today. As this text is a treatise meant for undergraduate students, it furthermore includes an introductory chapter as well as several supplementary appendices on the necessary mathematics used in the later chapters of this text.

Keywords: Gauge theory, 2+1 dimensional gravity, Cartan formalism, conformal gravity, singularity resolution, higher spin.

Sammanfattning

Detta är en rapport som behandlar inledande gaugeteori, allmän relativitetsteori i Cartanformalism samt deras gemensamma egenskaper. Det visas att Einstein-Hilbert verkan, vilken reglerar Einsteins generella relativitetetsteori, i 2+1 dimensioner kan skrivas som en Chern-Simons gaugeverkan. Med hjälp av detta utökas sedan gaugegruppen på två skilda sätt för att producera två olika generaliseringar av normal Einsteinteori. Först konstrueras en teori för konform gravitation och efter att ha valt en specifik gauge löses de uppkomna ekvationerna vilket resulterar i Cottonekvationen. Därnäst skapas en teori för gravitation kopplad till högre spinn. Denna teori appliceras sedan på en enkel modell av ett tomt universum där det visas att Big Bang-singulariten är en gaugeartefakt som försvinner med lämpligt gaugeval. Slutligen presenteras en annan utökning till högre spinn, försedd med en kanonisk kvantiseringsmetod, med hjälp av den konforma gruppen. Det visas att det är omöjligt för denna teori att enbart inkludera ändligt spinn. Teorier av detta slag är centrala, exempelvis i relation till AdS/CFT-korrespondensen, vilket är ett extremt aktivt forksningsområde idag. Då denna text är ämnad för tredjeårsstudenter på teknisk fysik på Chalmers eller motsvarande nivå inkluderas ytterligare ett inledande kapitel samt flera kompletterande appendix om den matematik som är nödvändig för de senare kapitlen.

Nyckelord: Gaugeteori, 2+1-dimensionell gravitation, Cartan-formalism, konform gravititation, singularitets upplösning, högre spinn.

Table of Contents

List of Symbols

A: Gauge connection

$A^{\mu}{}_{\nu}$: Gauge field

AdS_3: Anti-de Sitter space

$C^{\mu\nu}$: Cotton tensor

D: Scale generator

dx^{μ}: Differential one-form

dS_3: de Sitter space

$e^{\mu}{}_{a}$: Vielbein, frame field, dreibein, zweibein

$F^{\mu\nu}$: Field strength

$f^{ab}{}_{c}$: Structure factor

G_N: Newton's gravitational constant

$G_{\mu\nu}$: Einstein tensor

$g_{\mu\nu}$: Metric tensor

K_a: Special conformal generator

L: Lagrangian

$\mathcal{L}$: Lagrangian density

M_a: Rotation/boost generator

P_a: Translation generator

p: Generalised momenta

q: Generalised coordinate

$\dot{q}$: Generalised velocity

R: Ricci scalar, scalar curvature

$R^{a}{}_{b}$: Curvature form

$R_{\mu\nu}$: Ricci tensor

$R^{\mu}{}_{\nu\rho\sigma}$: Riemann tensor

S^d: D-sphere

$SL(n;\mathbb{R})$: Special linear group

$SO(n)$: Special orthogonal group

$SU(n)$: Special unitary group

$\mathcal{S}_{CS}$: Chern-Simons action

$\mathcal{S}_{EH}$: Einstein-Hilbert action

T^a: Torsion

$T_{\mu\nu}$: Stress energy tensor

$\Gamma^{\mu}{}_{\nu\rho}$: Christoffel symbol

γ^a: Gamma matrices

$\delta^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_p}$: Generalised Kronecker delta

ϵ: Grassmann number, $\epsilon^2 = 0$

$\varepsilon^{\mu\nu\rho}$: Levi-Civita tensor, $\varepsilon^{012} = 1$

ϵ^{abc}: Levi-Civita symbol, $\epsilon^{012} = 1$

$\eta_{\mu\nu}$: Minkowski metric tensor

Λ: Cosmological constant

$\Lambda^{\mu}{}_{\nu}$: Lorentz transform

$\tilde{\Lambda}^{\mu}{}_{\nu}$: Conformal transform

σ_n: Pauli matrices

ψ^i: Spinor field

$\bar{\psi}$: Spinor field conjugate

$\omega_{\mu}{}^{a}{}_{b}$: Spin connection

$\dagger$: Adjoint operator

$[\ ,\ ]$: Commutator

$\Box$: d'Alembert operator

∇: Gradient

*: Hodge Dual

$\cong$: Isomorphic

$\{\ ,\ \}_{PB}$: Poisson brackets

$\wedge$: Wedge product

$A_{(\mu\nu)}$: $\frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$

$A_{[\mu\nu]}$: $\frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$

D_{μ}: Covariant derivative

d: Exterior derivative

$Tr\,[\ ]$: Trace operator

Sammandrag

Inom den moderna fysiken kan alla fenomen beskrivas med hjälp av de fyra krafterna gravitation, elektromagnetism samt de starka- och svaga kärnkrafterna. De sista tre krafterna kan förenas i standardmodellen, men försök att inkludera även gravitation har ännu inte varit lika framgångsrika. För att åskådliggöra kopplingen mellan de båda teorierna, studerar vi det enklare fallet 2+1 dimensioner. Vi gör detta för att kunna uttrycka gravitation som en gaugeteori. Genom att därefter utöka denna teori till att även inkludera högre spinn vill vi slutligen kunna upplösa en enkel singularitet. Trots att detta är en relativt enkel modell finns det applikationer inom bland annat strängteori och i modeller för grafen och supraledning.

Målet med denna rapport är att introducera läsaren till detta område samt delge de matematiska verktyg som därvid behövs. För att förtydliga de centrala delarna i rapporten är den uppdelad i fyra huvudavsnitt. Den första delen är en kort, sammanfattande förklaring av den matematik som kommer användas senare. Därefter introduceras gaugeteori med centrala begrepp som gaugesymmetri och gaugefält. Den tredje delen behandlar gravitation enligt allmän relativitetsteori i $2 + 1$ dimensioner samt kopplar samman denna teori med en Chern-Simons gaugeteori. I det sista avsnittet utökas denna koppling till en ny teori för högre spinn. Denna appliceras sedan för att visa att det är möjligt att gauga bort singulariteter, någonting som i allmän relativitetsteori är omöjligt. Vidare inkluderas tre djupare matematiska appendix för den som är intresserad, samt ytterligare två appendix med intressanta beräkningar. Vi kommer nu ge en summering av de centrala koncepten som behandlas i denna rapport, frånsett de om den grundläggande matematiken bakom teorin.

Gaugeteori

För att kunna möjliggöra föreningen mellan standardmodellen och den allmänna relativitetsteorin måste vi studera gaugeteori då detta är centralt för standardmodellen. En gaugesymmetri är en global fältsymmetri som utökats till en lokal sådan. Dess symmetrier beskrivs av en kontinuerlig symmetrigrupp, en Liegrupp, som beskriver de transformationer under vilka fälten är invarianta. Standardmodellen är det främsta exemplet på en gaugeteori och då vi vill förena den med gravitation undersöker vi om den senare kan uttryckas som en gaugeteori.

Ursprungligen skapas en enkel, abelsk gaugeteori genom att låta en global transformation g, som verkar på en uppsamling fält enligt $g\psi$, bli lokal som $g(x)$. För att kunna mäta skillnaden i fälten själva och inte transformationselementen introduceras en kovariant derivata D_μ, vilken uppfyller $D_\mu\psi(x) \to (D_\mu\psi(x))' = e^{i\theta(x)}D_\mu\psi(x)$. Vid en enkel fastransformation $\psi \to \psi' = e^{i\theta(x)}\psi$ tar denna formen

$$D_\mu\psi(x) = (\partial_\mu + iA_\mu)\psi(x)$$

där A_μ är ett så kallat gaugefält vilket kan fixeras på lämpligt sätt genom att

specificera en gauge. Om man vill utöka denna metod till ickeabelska grupper där ordningen på transformationer spelar roll utnyttjar vi att man kan skriva denna transformation som $\psi(x)^i \to g\psi(x)^i = \exp(\theta^a T_a)\,\psi(x)^i$ där T_a är de infinitesimala generatorerna för gruppen. Den kovarianta derivatan tar då formen

$$D_\mu\psi(x)^i = (\partial_\mu + A^a_\mu T_a)\psi(x)^i$$

och vi får ett gaugefält för varje generator.

Efter att ha introducerat dessa fält kopplar vi nu dem samman i en ett-form kallad gaugekopplingen, $A = dx^\mu A^a_\mu T_a$. Med hjälp av denna konstruerar vi sedan den så kallade fältstyrkan

$$F = dA + A \wedge A$$

där d är den yttre derivatan och $\wedge$ är kilprodukten vilket kan ses som en generalisering av kryssprodukten. Från fältstyrkan kan vi sedan skapa en gaugeinvariant Lagrangefunktion

$$\mathcal{L} = Tr\left[F \wedge *F\right] = -Tr F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}{}^a F^{\mu\nu}{}_a$$

där spåret tas över en lämplig representation av Liegeneratorerna. Denna Lagrangefunktion kallas Yang-Mills Lagrangefunktion, men vi kan även skapa andra intressanta Lagrangefunktioner. För våra ändamål är den nästan gaugeinvarianta Chern-Simonsverkan, vilken i $2+1$ dimensioner ser ut som

$$\mathcal{S}_{CS} = \frac{k}{4\pi}\int_{M_3} Tr\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right],$$

speciellt intressant. Genom att variera denna verkan ger minsta verkans princip att rörelseekvationerna i Chern-Simonsteori ges av ”platthetsvillkoret”

$$F = dA + A \wedge A = 0.$$

Konform gravitation i $2+1$ dimensioner

Den centrala teorin för gravitation är Einsteins generella relativitetsteori, där rummet behandlas som en krökt Riemannmångfald. Här kan vi beskriva gravitationens interaktion med materia i Einstens fältekvationer

$$G_{\mu\nu} \equiv 8\pi G_N T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu},$$

där $R_{\mu\nu}$ är Riccitensorn, R är Ricciskalären, $g_{\mu\nu}$ är mångfaldens metrik och $T_{\mu\nu}$ är stressenergitensorn. Dessa ekvationer kan härledas från Einstein-Hilbertverkan vilken, med en kosmologisk konstant Λ, har utseendet

$$\mathcal{S}_{EH} = \frac{1}{16\pi G_N}\int (R - 2\Lambda)\sqrt{-g}\,\mathrm{d}^3x + \mathcal{S}_{\text{matter}}.$$

För att kunna koppla ihop vår gravitationsteori med gaugeteori använder vi oss av Cartanformalismen. Denna formalism bygger på att man förser den krökta mångfaldens tangentrum med en (Lorentz)ortonormal bas. Detta görs med hjälp av en samling transformationsmatriser $e^{\mu}{}_{a}$ vilka kallas vielbeins. Dessa matriser blir då en slags översättning mellan mångfaldens metrik $g_{\mu\nu}$ och Lorentzmetriken $\eta_{\mu\nu}$

$$g_{\mu\nu} = e_{\mu}{}^{a} e_{\nu}{}^{b} \eta_{ab}, \quad \eta_{ab} = e_{a}{}^{\mu} e_{b}{}^{\nu} g_{\mu\nu},$$

där vi använder grekiska index för tensorer i koordinatbasen och latinska för tensorer i den ortonormala basen. För att sedan koppla samman vektorer i olika tangentrum används en så kallad spinnkoppling $\omega_{\mu}{}^{a}{}_{b}$ vilken gör det möjligt att definiera en kovariant derivata D_{μ} i analogi med gaugeteori. I tre dimensioner kan man använda sig av Levi-Civita symbolen ε_{abc} för att skriva om spinnkopplingen som $\omega_{\mu}{}^{a}$ och man finner att Einstein-Hilbertverkan i 2+1 dimensioner tar följande form:

$$S_{EH}(e,\omega) = -\frac{1}{8\pi G_N} \int_{M_3} \left[e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \right) - \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right].$$

Denna verkan visar sig sedan vara ekvivalent med en Chern-Simonsverkan $\mathcal{S}_{CS}$. I fallet då $\Lambda = 0$ används exempelvis gaugekopplingen

$$A = dx^{\mu} e_{\mu}{}^{a} P_a + dx^{\mu} \omega_{\mu}{}^{a} M_a,$$

där P_a och M_a är generatorer för Poincarégruppen. Liknande konstruktioner används i de två andra fallen vilket resulterar i en tydlig koppling mellan gravitation i $2+1$ dimensioner och gaugeteori.

Efter att ha visat kopplingen mellan gravitation och gaugeteori är det naturligt att försöka generalisera den genom att utöka gaugegruppen. Vi väljer härvid att skapa en konform gravitationsteori genom att använda den konforma gruppen. För att göra detta introducerar vi två nya gaugefält b och f^a till de nya generatorerna som uppkommer för denna grupp. Efter att ha satt $b = 0$ som gauge finner vi att denna teori därmed producerar följande ekvationer:

$$\begin{aligned}
(P_a): &\quad de^a + \epsilon^a{}_{bc}\, e^b \wedge \omega^c = 0, \\
(M_a): &\quad d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\, \omega^b \wedge \omega^c - 2\epsilon^a{}_{bc}\, e^b \wedge f^c = 0, \\
(D): &\quad 2e^a \wedge f_a = 0, \\
(K_a): &\quad df^a + \epsilon^a{}_{bc}\, \omega^b \wedge f^c = 0.
\end{aligned}$$

Efter att ha använt att vielbeinmatriserna är inverterbara finner man att dessa ekvationer kan reduceras till Cottonekvationen

$$C^{\sigma\rho} = \varepsilon^{\mu\nu\sigma} D_{\mu}(R_{\nu}{}^{\rho} - \frac{1}{4}\delta_{\nu}{}^{\rho} R) = 0,$$

där $C^{\sigma\rho}$ är Cottontensorn. Precis då denna är noll är en mångfald konformt platt varför vi mycket riktigt har formulerat en konform gravitationsteori i 2+1 dimensioner.

Högre spinn

Genom att utöka gaugegrupperna på andra sätt kan vi skapa en teori där gravitation kopplas till högre spinnfält. Vi gör detta först med en naturlig utökning för fallet då $\Lambda < 0$ vilket beskrivs av en mångfald som kallas anti de Sitter, AdS_3. Då vi inkluderar spinn 3 introduceras nya gaugefält och generatorer vilket gör det möjligt att använda sig av nya gauges. Vi applicerar detta på en enkel 2+1-dimensionell Big-Bang-modell vilken kallas Milneorbifolden. Vi modifierar våra gaugepotentialer smått för att kunna applicera vår nya modell på detta fall och undersöker om vi kan välja en lämplig gauge så att singulariteten upplöses. Efter en del beräkningar finner vi att, med ett lämpligt gaugeval, krökningskalären kan skrivas som

$$R = \frac{24D^2(\phi)}{\left(\alpha^2 T^2 + 12D^2(\phi)\right)^2}.$$

Detta är en tydlig skillnad jämfört med tidigare, varvid den var noll överallt förutom i singulariteten där den var oändlig. Därmed är metoden en otvetydig framgång då vi finner att i denna teori är singulariteten ingenting annat än ett gaugeartefakt.

Då utökningen till högre spinn var så framgångsrik för AdS_3 vill vi nu utföra samma process för vår konforma gravitationsteori. Detta är dock inte lika naturligt och vi tvingas utnyttja en representation av den konforma Liealgebran med kanoniska variabler p_α och q^β. För att utöka denna teori används sedan nya generatorer bestående av högre polynom i p_α och q^β. Denna teori visar sig dock inte gå att trunkera vid ändliga spinn n och ger dessutom upphov till icke-integrerbara termer. Därför måste man kvantisera denna teori, vilket löser det senare problemet, men man måste fortfarande inkludera alla högre spinn till oändligheten. Teorier av detta slag studeras idag för att undersöka kopplingen mellan strängteori och M-teori samt i samband med AdS/CFT-dualiteten.

Slutsats

Vi har i denna rapport studerat likheterna mellan gauge- och gravitationsteorier i 2+1 dimensioner. Vidare skapade vi med det samband vi funnit en konform gravitationsteori vilken gav upphov till en samling rörelseekvationer. Dessa kunde lösas och slutresultatet blev Cottonekvationen. Trots att vi har begränsat oss till denna enkla 2+1-dimensionella modell existerar det fortfarande användningsområden, både teoretiska och praktiska. Konforma teorier av detta slag är intressanta för både sträng- och M-teori, men kan även appliceras vid modellerande av grafen, ett tvådimensionellt kollager. Då denna hängs upp deformeras den och för att beskriva dess svar till olika impulser kan det vara möjligt att en metrik samt en 2+1-dimensionell gravitationsteori måste införas för att beskriva den korrekt.

Vidare så har vi även funnit att utökandet till högre spinn är viktigt för att kunna lösa upp singulariteter, vilket sågs för Milneorbifolden. Detta är viktigt då det i vanlig allmän relativitetsteori är omöjligt att upphäva denna typ av singulariteter. Eftersom man kan tänka på högre spinn-teorier som den späninngslösa gränsen av strängteori så kan det finnas vidare användning av denna metod inom detta mer generella område. Användandet av högre spinn-fält på detta sätt har faktiskt undersökts vidare inom strängteori, med lovande resultat [1]. När vi å andra sdian försökte utöka den konforma teorin till högre spinn på liknande sätt, fann vi att det var omöjligt att trunkera denna vid ändliga spinn, varför alla högre spinn var tvungna att inkluderas. Detta är intressant då det idag undersöks om en liknande teori med högre spinn kan användas för att utreda kopplingen mellan sträng- och M-teori med hjälp av AdS/CFT-dualiteten.

1
Introduction

In our understanding of the universe, all phenomena can be explained by four fundamental forces: the strong and weak nuclear forces, the electromagnetic force, and the gravitational force. The two established theories that explain these are Einstein's theory of general relativity and the standard model of particle physics. As scientists are interested in a mathematical model that describes all areas of physics, an active area of research is the unification of these two theories into a theory of everything.

The standard model of particle physics describes the interaction of the strong, weak and electromagnetic forces. This theory is a gauge theory which means that it is a theory with a set of interal symmetries in its fundamental fields. These symmetries are described using the mathematical language of group theory, yielding symmetry gauge groups like the $U(1) \times SU(2) \times SU(3)$-group of the standard model. On top of this, the theory contains a set of scalar particles, the so called Higgs sector, that after symmetry breaking give rise to the the Higgs boson and mass in turn [2, 3, 4].

The other successful theory that is used extensively is Einstein's theory of general relativity. This is a geometric theory where gravity is described using a spacetime that becomes curved in the presence of matter. Furthermore, when used in cosmological models, even empty space can be curved. This represents the presence of vacuum energy and was initially introduced by Einstein in order to achieve a static universe. Even though Einstein's assumption of a static universe has been refuted, this concept is still in use today to describe dark energy and how it causes the Universe's expansion to accelerate [5]. Regardless of the success of general relativity, it is still a problematic theory. This can be seen by how it contains singularities like black holes and the Big Bang. These singularites are thought to be unphysical, but they are not easily remedied in the current theory.

When we want to unify these two theories, one approach is to attempt to formulate gravity as an ordinary gauge theory in order to make it more similar to the standard model. This is no easy task in general, but in the simpler $2+1$-dimensional case such a construction is possible. There are several reasons for this, including the fact that there are no propagating wave solutions, corresponding to gravitons, in 2+1 dimensions [6]. This simpler case can then be used as a guide in the developement of the more complicated theory in 3+1 dimensions[7]. In this thesis we will therefore attempt to do this by finding an equivalence between gravity and a Chern-Simons gauge theory. We will also attempt to extend this theory to a theory of conformal gravity. Even though this might seem like a limited field of study, there are actually a large number of applications.

One recent application where 2+1-dimensional gravity could be of importance is the new wonder material graphene. Suspended graphene can be seen as a curved 2 dimensional surface. Subjecting this surface to external impulses then necessitates the introduction of a metric and a stress energy tensor in analogy with gravity [8, 9]. Further applications can be found in solid state physics where it has been discovered that conformal symmetries arise at critical points in phase diagrams. Several interesting phenomena tied to this effect can be mentioned, including the fractional quantum Hall effect and superconductivity.

The final area we wish to study in this thesis is the introduction of higher spin to the setting of general relativity. This will be done using its gauge theory formulation, where we can extend the gauge group in a suitable way to describe gravity coupled to higher spin fields. This results in new degrees of freedom, and by using the gauge freedom this approach could possibly be used to gauge away singularities that normally causes unmanageable physics. It is clear that this attempt depends on our gauge formulation, as singularities normally are invariant in general relativity. We will attempt to use this method of resolving singularities on a specific 2+1-dimensional manifold. This is the Milne orbifold, which can be thought of as a simple model of the Big Bang. Finally we will explore a similar theory of higher spin in the conformal case, which has relevance to modern physics as such a theory is thought to be the tensionless limit of string theory.

One additional area where a 2+1-dimensional conformal theory of gravity is of relevance is string theory, or specifically its extension M-theory. In string theory, strings are lines in a high dimensional space and as they evolve in time they sweep out a surface in spacetime. On this 1+1 dimensional surface one can construct a field theory which becomes endowed with a rich mathematical structure [10]. M-theory on the other hand describes 2-branes on the same space. As these move through time they instead sweep out a $2+1$ dimensional surface, but the field theory that can be constructed on this surface is comparatively very poorly understood. Finally higher spin may be of use in the AdS/CFT duality, where the relationships between higher spin theories, conformal field theories, and string theory is still being investigated [6].

1.1 Reading guide

This thesis is divided into several chapters of different nature, but is most easily read in chronological order for clarity's sake. Initially in chapter 2, the mathematical tools needed to follow the rest of the chapters are briefly presented. Readers unfamiliar with the subjects presented can then visit the corresponding appendices A, B, and C where they are developed to a greater extent. In addition there are a few appendices of calculational nature that can be used in order to shed light on calculations in the main thesis.

Chapter 3 is devoted to gauge theory. Beginning with Maxwell's classical theory of electromagnetism, this chapter gradually introduces concepts like covariant derivatives and gauge transformations in order to describe more general gauge theories, culminating in Chern-Simons gauge theory.

In chapter 4 gravity is presented in both classical and the Cartan formalism. Using the latter it is shown how this theory in 2+1 dimensions can be expressed as a gauge theory. Finally this theory is extended to produce a theory of conformal gravity.

Continuing to chapter 5 the theory developed in the previous chapter is generalised to include fields of higher spin. This theory is then used to gauge away the singularity of a simple Big Bang-model, and various elements of the conformal higher spin theory is discussed. Following this chapter are some final conclusions with a brief discussion about applications and open problems related to this area.

2

Mathematical Preliminaries

Before we can proceed to the subject matter of this thesis, we need to take some time to ensure that the reader is familiar with the mathematics we will be employing. This is done in this chapter where we treat tensors and some selected topics from differential geometry, group theory, and analytical mechanics. Here we will only provide an introduction to these topics with educational examples; if the reader wants further information on these topics they are advised to consult appendices A, B and C, where their mathematical foundations are explored in greater detail. If the reader is already familiar with these concepts, they may proceed directly to the following chapter.

In this thesis we will mainly be exploring gauge theory, general relativity and subjects related to these fields. As general relativity is a geometrical theory of curved manifolds we need to understand their properties, which are explored using differential geometry. Furthermore, we will use the concepts of differential forms and exterior algebra from this subject extensively, so they too will be introduced here. The next concept we need to understand are symmetries, which are of profound importance in modern physics. As such we must have a firm understanding of group theory, which is the mathematical language used to describe symmetries. Of particular interest to us are continuous symmetries because of their fundamental importance in gauge theory. These are described by Lie groups and their infinitesimal Lie algebra, concepts which will be explored here. Finally we will discuss Lagrangians and the principle of least action from analytical mechanics. These concepts are used to determine the equations of motion and implement symmetries of a theory. As such, we will use them extensively when constructing new theories.

2.1 Tensors

Tensors are ubiquitous in modern physics as they allow theories to be expressed in a manifestly coordinate independent manner. This thesis is no different and they will be used extensively, necessitating their introduction. We will assume that the reader is comfortable with linear algebra and quickly go through how tensors are handled. Furthermore, we expect the reader to be at least somewhat familiar with tensors from special relativity. If this is not a case, [11] is recommended for a more thorough account.

2.1.1 Tensor notation and bases

In general, a tensor is a geometric object that is used to describe relations between vectors, covectors, matrices and/or other tensors. Consider a simple vector $\mathbf{v}$, which in a basis $\mathbf{e}_i, i = 1, ..., n$ is given by

$$\mathbf{v} = \sum_i v^i \mathbf{e}_i = v^i \mathbf{e}_i$$

where we have suppressed the summation over the indices according to the Einstein summation convention, a convention that will henceforth always be in use. This vector can intuitively be thought of as a direction, and as such it is a coordinate invariant object. However, the components v^i of $\mathbf{v}$ are often called a vector on their own and it is implicitly understood that they are attached to a basis. This basis can be chosen arbitrarily and as such it is important to know how these components transform when moving from one coordinate system to another. If the basis changes with a transformation matrix $T_i{}^j$ as

$$\mathbf{e}'_i = T_i{}^j \mathbf{e}_j$$

then clearly the components must change as

$$v'^i = T^{-1}{}^i{}_j v^j.$$

Because of this transformational property v^i is called a *contravariant* vector.

We can also form a dual basis $\mathbf{e}^i, i = 1, ..., n$ that transforms using the inverse matrix $T^{-1}{}^i{}_j$. A vector in this basis takes the form $\mathbf{u} = u_i \mathbf{e}^i$ and the components transform as

$$u'_i = T_i{}^j u_j$$

which is known as a *covariant* vector. These vectors are the simplest examples of a tensor. If we instead have a general tensor of type (n,m), we can understand it as a tensor product of vector spaces like

$$A = A^{i_1 \dots i_n}{}_{j_1 \dots j_m}\, \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n} \otimes \mathbf{e}^{j_1} \otimes \cdots \otimes \mathbf{e}^{j_m}.$$

This tensor transforms like the normal vector, once of each specific kind for each free index. Explicitly it will transform as

$$A'^{i_1 \dots i_n}{}_{j_1 \dots j_m} = T^{i_1}{}_{k_1} \dots T^{i_n}{}_{k_n} \left(T^{-1}\right)_{l_1}{}^{j_1} \dots \left(T^{-1}\right)_{l_m}{}^{j_m} A^{k_1 \dots k_n}{}_{l_1 \dots l_m}.$$

In this thesis we will encounter various kinds of tensors, including the field strength $F_{\mu\nu}$, the torsion tensor $T^\rho{}_{\mu\nu}$ and the Riemann curvature tensor $R^\rho{}_{\gamma\mu\nu}$. When working with mixed tensors like these there are several operations we can perform. Because tensors are a generalisation of vectors they maintain their linearity, but they can also be combined using the tensor product like

$$C^\mu{}_{\nu\rho} = A^\mu{}_\nu B_\rho.$$

Furthermore indices can be contracted in a generalisation of taking the trace of matrices,

$$Tr\left[A\right] = \sum_{\mu=1}^{n} A^\mu{}_\mu = A^1{}_1 + ... + A^n{}_n,$$

and in tensor notation this is expressed as ${A^\mu}_\mu$. This can be done for any tensor with both covariant and contravariant indices like ${C^\mu}_{\mu\rho}$. This process reduces the tensor from rank (n, m) to $(n-1, m-1)$ as the indices becomes dummy indices whose transformations cancel.

2.1.2 Metric

The metric is a tensor that makes it possible to introduce the scalar product between two vectors, which in turn introduces the concepts of length and angles. If a vector **v** and **u** has the scalar product $\langle \mathbf{v}|\mathbf{u}\rangle$, it can be written as

$$\langle \mathbf{v}|\mathbf{u}\rangle = v^\mu u^\nu \; \mathbf{e}_\mu \cdot \mathbf{e}_\nu = v^\mu u^\nu \, g_{\mu\nu}$$

where $g_{\mu\nu}$ is called the metric tensor. In normal Euclidean space the metric is defined as the elements of the identity matrix, for example in three dimensions

$$g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

whilst in Minkowski spacetime[1] the metric is denoted $\eta_{\mu\nu}$:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We will always be using $\eta_{\mu\nu}$ for the Lorentz metric while $g_{\mu\nu}$ will denote general metrics. Note how the fact that the Lorentz metric has a negative component is reflective of the fact that it has a timelike component. By diagonalising the general metric we can always reduce it to a diagonal form and we can see how many timelike components it has. This is true even though the metric may take many different shapes and even depend on the location in space.

Given some space with coordinates x^μ the metric is commonly determined by using the infinitesimal displacement $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. For example, using the spherical coordinates of normal Euclidean space

$$ds^2 = dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

we find that

$$g_{rr} = 1, \;\; g_{\theta\theta} = r^2, \;\; g_{\phi\phi} = r^2 \sin^2\theta,$$

while other components are zero. Additionally the metric is used as a method to identify a covariant with a contravariant tensor or vice versa, as

$$A_\mu = g_{\mu\nu} A^\nu,$$

an identification that will be used heavily throughout this thesis to raise or lower indices.

[1] The four dimensional spacetime of special relativity where the speed of light is $c = 1$

2.1.3 Special tensors

We will now introduce the reader to some special properties that are related to tensors and can be used in different calculations. A tensor $A_{\mu\nu}$ is called symmetric if the components doesn't change under a switch of indices, $A_{\mu\nu} = A_{\nu\mu}$. An obvious example of such a tensor is the metric $g_{\mu\nu}$ since the scalar product doesn't depend on the order of vectors. Similarly, a tensor $B_{\mu\nu}$ is antisymmetric if the switch produces a change in sign $B_{\mu\nu} = -B_{\nu\mu}$. It is important that a general tensor can be divided into its symmetric and antisymmetric parts. Using the definitions

$$A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$$

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$$

for the symmetric and antisymmetric parts respectively we have

$$A_{\mu\nu} = A_{(\mu\nu)} + A_{[\mu\nu]}$$

which is easily checked. We can extend this further using the generalised Kronecker delta

$$\delta^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_p} = \begin{cases} +\frac{1}{p!} & \text{if } \nu_1 \dots \nu_p \text{ are an even permutation of } \mu_1 \dots \mu_p \\ -\frac{1}{p!} & \text{if } \nu_1 \dots \nu_p \text{ are an odd permutation of } \mu_1 \dots \mu_p \\ 0 & \text{in all other cases.} \end{cases}$$

leading to

$$A_{[\mu_1 \dots \mu_n]\nu_1 \dots \nu_n} = \delta^{\rho_1 \dots \rho_n}_{\mu_1 \dots \mu_n} A_{\rho_1 \dots \rho_n \nu_1 \dots \nu_n}.$$

Another special object we will be using is the invariant Levi-Civita symbol ϵ which is totally antisymmetric. In normal three dimensions it is related to the cross product and the triple scalar product as we can write

$$(\mathbf{A} \times \mathbf{B})^\mu = \epsilon^{\mu\nu\rho} A_\nu B_\rho$$

and

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon^{\mu\nu\rho} A_\mu B_\nu C_\rho.$$

The the Levi-Civita symbol is important as we use it to define the volume element in general dimensions. It is defined as

$$\epsilon^{\mu\nu\rho} = \begin{cases} 1 & \text{if } \mu\nu\rho \text{ is an even permutation of } 123 \\ -1 & \text{if } \mu\nu\rho \text{ is an odd permutation of } 123 \\ 0 & \text{if two indices are equal.} \end{cases}$$

We can lower the indices of this object in normal tensor fashion using the metric, and in our case we will use the Lorentz metric $\eta_{\mu\nu}$ which means that $\epsilon_{\mu\nu\rho} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}\epsilon^{\alpha\beta\gamma}$ and thus

$$\epsilon_{\mu\nu\rho} = \begin{cases} -1 & \text{if } \mu\nu\rho \text{ is an even permutation of } 123 \\ 1 & \text{if } \mu\nu\rho \text{ is an odd permutation of } 123 \\ 0 & \text{if two indices are equal.} \end{cases}$$

We can also contract this object in the normal way and we will often be doing this using the Levi-Civita symbol again. The following identities arise as a result, and the will often be used in various calculations of this thesis:

$$\epsilon^{\mu\nu\rho}\epsilon_{\sigma\gamma\kappa} = -6\delta^{\mu\nu\rho}_{\sigma\gamma\kappa},$$

$$\epsilon^{\mu\nu\rho}\epsilon_{\sigma\gamma\rho} = -2\delta^{\mu\nu}_{\sigma\gamma},$$

$$\epsilon^{\mu\nu\rho}\epsilon_{\sigma\nu\rho} = -2\delta^{\mu}_{\sigma},$$

$$\epsilon^{\mu\nu\rho}\epsilon_{\mu\nu\rho} = -6.$$

2.2 Differential geometry

In special relativity the background space is the flat Minkowski space. When we switch over to general relativity, we need to extend our study to curved spaces. Because of this we need the mathematical language of differential geometry, the mathematical area focused on studying general geometric structures using analysis. The main objects of study are the differential manifolds, surfaces that locally resemble Euclidean (or Minkowski) space in each point in space. However, they do not need to look like Euclidean space globally as they can be curved in any fashion. On these manifolds we then introduce local coordinates x^μ which we in turn can use to define functions

$$f = f(x^1, \cdots, x^n)$$

upon the manifold. We will always be considering the manifolds to be smooth, meaning the functions will be infinitely differentiable. As an example of these concepts consider the surface of the sphere S^2 embedded into $\mathbb{R}^3$,

$$S^2 : \quad x^2 + y^2 + z^2 = L^2.$$

If we do not stray too far in any direction we can forget its global structure and think of it like $\mathbb{R}^2$, as one could think the earth is flat when living on only a part of its surface. For our coordinate system we can use the common spherical coordinates

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta,$$

and constraining them with the condition $r = L$. Thus $x^\mu = (\theta, \phi)$ are the coordinates of our sphere, whose metric follows directly from inserting (x,y,z) above into

$$ds^2 = dx^2 + dy^2 + dz^2 = L^2 d\theta^2 + L^2 \sin^2\theta d\phi^2.$$

There are two other manifolds that are of particular importance to this thesis. These are the de Sitter and Anti-de Sitter spaces dS_n and AdS_n, which are curved surfaces that locally resemble Minkowski space. Furthermore they are designed to act as large scale models of the Universem so they must possess all the properties we observe in nature. In particularm they both are maximally symmetric. That is, they are both homogeneous and isotropic, corresponding to the fact that the universe looks approximately the same throughout, without any preferred directions in space.

We also know that the metric of the universe is Lorentzian, so the metric of these manifolds are too. Finally we know how the Universe is expanding. This behaviour of space expanding (or contracting) is the reason these manifolds are used to model the Universe, as they realise this behaviour. Explicitly, AdS_n can be constructed using an embedding in $\mathbb{R}^{n-1,2}$. The metric of this space is given by

$$ds^2 = \sum_{k=1}^{n-1} dx_k^2 - dt_1^2 - dt_2^2,$$

and AdS_n is then parametrised in this space as

$$\sum_{k=1}^{n-1} x_k^2 - t_1^2 - t_2^2 = -l^2, \tag{2.1}$$

where l is a positive constant called the radius of curvature. This is a generalisation of a normal Euclidean hyperbolic surface embedded in a background space with two timelike directions. dS_n on the other hand is the generalization of a sphere, embedded into $\mathbb{R}^{n,1}$ like

$$\sum_{k=1}^{n} x_k^2 - t^2 = l^2, \tag{2.2}$$

where the background metric is given by

$$ds^2 = \sum_{k=1}^{n} dx_k^2 - dt^2.$$

Since we can never leave spacetime it is not possible to study it from the outside. As such when we study manifolds like AdS_n and dS_n, that are models of the Universe, we must forget how they can be embedded in a background space. Instead we are only interested in the internal properties of a manifold that can be determined entirely from within. This immediately introduces a problem which becomes apparent by studying S^2. We want to introduce vectors and tensors on this surface which is commonly done using the tangent planes to the embedded surface. However, such planes are obviously external and the vector space formed upon them diverges from the manifold itself, into the surrounding Euclidean space in which it is embedded. This naive version of a vector clearly fails and we need to find a generalisation. In order to remedy this we need to surrender our notion of vectors as pointing from one point of space to another and instead only think of them as quantities with direction and length. By doing this a solution arises when considering the functions f on the manifold, since we can clearly form the directional derivatives as

$$\frac{\partial}{\partial x^\mu} f(x) = \partial_\mu f,$$

which measures how much f varies along the coordinate curve of x^μ. However, we can also form other curves

$$x^\mu = x^\mu(\lambda),$$

parametrised by some parameter λ and form the directional derivative

$$\frac{d}{d\lambda} f(x) = \frac{dx^\mu}{d\lambda} \partial_\mu f(x) = V^\mu \partial_\mu f(x),$$

where we have employed some suggestive notation. Ignoring the arbitrary function we define the vector V as

$$V = V^{\mu}\partial_{\mu},$$

using the partial derivatives ∂_{μ} as the basis vectors. This is no arbitrary assignment as this is the best definition of a vector that can be chosen without leaving the internal strucutre off the manifold.

The partial derivatives we have just introduced clearly satisfy the linearity required of a vector space, and additionally they encode the direction and length we want our vectors to possess. Furthermore, under general coordinate transformations $x^{\mu} \to x'^{\mu}$ the partial derivatives are related to each other by the chain rule,

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \frac{\partial}{\partial x^{\nu}},$$

which is nothing but a transformation matrix acting upon the original partial derivatives. As the directional derivative along the curve λ clearly shouldn't depend on the coordinates the vector components must change in the inverse way,

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu},$$

showing how this is a contravariant vector like before. Because of their role in taking directional derivatives, the vector space spanned by the partial derivatives is called the tangent space. This is in analogy with the usual tangent plane of a surface embedded in Euclidean space as its vectors also are used for this purpose. Here we must note something very important. Using this approach the tangent vector space becomes intimately linked with the point at which the derivatives are taken. There is no obvious way to compare vectors of different spaces as they belong to separate tangent spaces. Later we will discuss how this necessitates the introduction of a covariant derivative which in turn will allow the creation of the Riemann curvature tensor, an intrinsic quantity that contains information about the curvature of the manifold.

Equipped with our tangent basis, naturally induced by the coordinate system, we also introduce its dual basis of one-forms. These are linear functionals that takes vectors from the tangent space as agrument and produce a scalar, and their vector space is called the cotangent space. The usual example is the gradient of a function. Recall how in Euclidean space we can form the gradient ∇f and the directional derivative in some direction specified by a vector $\mathbf{v}$ as

$$\nabla f \cdot \mathbf{v}.$$

In just the same way the gradient one-form df is defined as

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}. \tag{2.3}$$

Just like the for partial derivatives ∂_{μ} the coordinate system induces a natural basis of one-forms dx^{μ} defined by

$$dx^{\mu}(\partial_{\nu}) = \langle dx^{\mu}, \partial_{\nu}\rangle = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}.$$

Intuitively the partial derivatives can be thought of as arrows in the direction of the coordinate lines while the one-forms can be thought of as directed coordinate surfaces. The action upon some vector by these one-forms are then determined by how many coordinate surfaces it crosses. Writing df as

$$df = u_\mu dx^\mu,$$

and noting how the expression of equation (2.3) should be coordinate independent the transformation when changing coordinate system must be of the form

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

and

$$u'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} u_\nu$$

showing how the vector u really is a covariant vector. With these identifications the extension to tensors of higher rank is clear and the discussion from the previous section applies to this case. A general tensor T is thus written as

$$T = T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_n} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_m}$$

The question arises if we can determine if the manifold is curved like S^2 extended in $\mathbb{R}^2$, studying it only from within.

2.2.1 Differential forms and exterior algebra

In the previous section we discussed the generalisation of tensors to curved manifolds with the tangent and cotangent spaces. We denoted the basis elements of the latter as dx^μ and called them one-forms. This is no accident as the normal infinitesimal quantities we usually denote this way really should be thought of as linear functionals, like the one-forms. This furthermore implies that we really want to perform integration using these one-forms. This is exactly what we will do in later chapters where we will form Lagrangians to be integrated using these *differential forms*. Moreover, these come with a mathematical structure called exterior algebra that expresses the orientation of these elements. Recall how the orientation of a parallelogram in normal Euclidean space can be expressed using the antisymmetric cross product

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

of the vectors that span the parallelogram. Guided by this we define a differential n-form ω as

$$\omega = v_1 \wedge \dots \wedge v_n$$

where v_i are covariant vectors $v_\mu dx^\mu$ and $\wedge$ is the wedge product which changes sign under interchange of any of its adjacent arguments. This can just in analogy with the cross product be thought of as the oriented parallelepiped spanned by the covariant vectors.

We can form differential forms of higher rank by combining differential forms using the wedge product. If ω_1 is a n-form and ω_2 is a m-form we can form an $n+m$-form as

$$\omega_1 \wedge \omega_2 = (-1)^{nm}\, \omega_2 \wedge \omega_1$$

where the second expression is a consequence of the antisymmetry of the wedge product. This antisymmetry also ensures that whenever a differential form contains the same vector twice, it must be identically zero since by switching their order we find that

$$v \wedge v = -v \wedge v.$$

This is consistent with our view of these elements as spanning a parallelepiped since it can not be spanned by using the same vector twice. Because of this property differential forms should be thought of as the basis of the vector space of totally antisymmetric covariant tensors. Changing the usual basis $dx^\mu \otimes ... \otimes dx^\nu$ with $dx^\mu \wedge ... \wedge dx^\nu$ of some covariant tensor we can extract its totally antisymmetric part,

$$T = \frac{1}{n!} T_{\mu_1 ... \mu_n}\, dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} = \frac{1}{n!} T_{[\mu_1 ... \mu_n]}\, dx^{\mu_1} \wedge ... \wedge dx^{\mu_n}.$$

We will later be using this property to great extent in later chapters for curvature calculations. Because of the antisymmetry of the wedge product, if the coordinate basis consists of k elements there will be $k!/n!(k-n)!$ independent elements of the basis for n-forms. In particular there will be no differential form of higher rank than k as such forms must contain the same element at least twice.

There are two operations we will be using on differential forms in this thesis. The first one is the exterior derivative d which acts upon an n-form and produces an $(n+1)$-form. This can be thought of as the 1-form operator

$$d = dx^\mu \partial_\mu$$

which is understood to act from the left. Normally when taking the derivative of a tensor the product is not a tensor since the transformation of this derivative during the coordinate transformation $x^\mu \to x^{\mu'}$ is given by

$$\partial_\mu A_\nu \to \partial_{\mu'} A_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu A_\mu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}} A_\nu \tag{2.4}$$

which clearly is not the tensor transformation rule. However, the offending second term is symmetric and thus cancels because of the antisymmetry of the differential forms and the exterior derivative produces a proper antisymmetric tensor. This comes at a price as this operator can only be applied once with non-vanishing result. Indeed since partial derivatives commute we have

$$d^2 = dx^\mu \wedge dx^\nu\, \partial_\mu \partial_\nu = 0$$

since when performing the sum over the antisymmetric basis elements they cancel each other.

The second operation we will be performing on differential forms is called the Hodge dual. This operation is denoted by $*$ and is related to the volume element.

We said that we wanted to think of the basis element $dx^1 \wedge ... \wedge dx^n$ as the volume element $d^n x$, but these objects have completely different transformational properties. The former transforms like a tensor while the latter transform using the Jacobian,

$$d^n x' = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| d^n x.$$

This is resolved by creating an invariant volume element using the determinant of the metric, g. Forming the determinant on both sides of the transformation of $g_{\mu\nu}$,

$$g'_{\sigma\rho}(x') = g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial x'^{\sigma}} \frac{\partial x^{\nu}}{\partial x'^{\rho}},$$

we find that $g' = gJ^2$ and so

$$d^3 x \sqrt{g} = d^3 x' J \sqrt{\frac{g'}{J^2}} = d^3 x' \sqrt{g'}$$

is indeed invariant. We therefore identify this as the invariant volume form

$$\sqrt{|g|}\, d^n x = \varepsilon_{\mu_1 \dots \mu_n}\, dx^{\mu_1} \wedge ... \wedge dx^{\mu_n}$$

where

$$\varepsilon_{\mu_1 \dots \mu_n} = \sqrt{|g|}\, \epsilon_{\mu_1 \dots \mu_n}$$

is the Levi-Civita tensor. This is indeed a proper tensor unlike the invariant Levi-Civita symbol which actually is a tensor density. Now with this volume element in place we define the Hodge dual operator $*$ as the operator from a p-form to an $(n-p)$-form given by

$$*\left(\omega_{\mu_1 \dots \mu_n}\, dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}\right) = \frac{(-1)^t}{(n-p)!} \omega_{\mu_1 \dots \mu_n} \varepsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_n}\, dx^{\mu_{p+1}} \wedge ... \wedge dx^{\mu_n}.$$

where t is the number of negative terms in the signature of $g_{\mu\nu}$. The Hodge dual operator creates a dual differential form that can be combined with the first one to produce a differential form proportional to the volume form. In particular,

$$*1 = \sqrt{|g|}\, d^n x.$$

2.3 Group theory

Throughout the 20th century symmetries have played an important role in physics. Group theory is the mathematical language that can concisely describe symmetries which can be used to better understand the manifolds upon which physical theories are formulated. There exists two broad classes of groups, known as finite and continuous groups, the first having a limited number of elements whilst the second has an infinite number. The main focus of this thesis will be continuous groups and in particular Lie groups. These are continuous groups that also are differentiable manifolds and they are of fundamental importance for gauge theory.

Abstractly, a group $\mathcal{G}$ is a set of elements together with a, not necessarily commutative, group operation that combines two elements and produces a third, $ab = c$. This operation must be closed and associative, but furthermore there must exist a unique unit element e for which it is true that $ea = ae = a, \forall a \in \mathcal{G}$ and an inverse element $a^{-1}, a^{-1}a = aa^{-1} = e, \forall a \in \mathcal{G}$.

2.3.1 Matrix groups

The elements of a group are often represented using matrices, with the group operation in turn being represented by the matrix product. This representation is arbitrary as long as all the matrices are consistent with the group operation. We will be particularly interested in the following continuous matrix Lie groups: $SL(n, \mathbb{R})$, $SU(n)$ and $SO(n)$.

The first group is known as the special linear group $SL(n, \mathbb{R})$. This group contains all real $n \times n$ matrices with determinant 1. The second one is known as the special unitary group $SU(n)$ which contains all of the unitary $n \times n$ matrices, $A^\dagger A = AA^\dagger = I$, with determinant 1. The third group is known as the special orthogonal group $SO(n)$ and is of special interest as it describes rotations in n-dimensional space. This group leave an invariant scalar product in n-dimensional Euclidean space so distances are preserved during this rotation. This property is used to further extend this group to non-Euclidean space like Minkowski space where the metric is of different signature. Then it describes rotations and boosts that preserves the length of vectors and is denoted by $SO(n,m,\mathbb{R})$, where n is the number of positive eigenvalues of the metric and m is the number of negative eigenvalues.

2.3.2 Lie generators and Lie algebra

Because of their relationship with differentiable manifolds, the properties of Lie groups (G) can be described using their infinitesimal Lie algebra ($\mathfrak{g}$) from their *generators*. These are the basis elements of the tangent space of the group's manifold at the unit element. We can determine these generators by expressing a general element of the group using some set of parameters and then find the derivative of this parametrisation at the unit element. For example variation of $SO(3)$ can be expressed as three rotations around different axes,

$$g_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad g_y = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \quad g_z = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which upon differentiation at the unit element, $\theta = \phi = \psi = 0$, gives a representation of its three generators as

$$T_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad T_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad T_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

each corresponding to rotation around an axis.

In general a Lie group is characterised by a set of d generators T_a that describe the properties of the group by the infinitesimal Lie algebra. This describes the behaviour of the group when performing repeated, different infinitesimal transformations. It might be the case that the ordering of such transformations matter. This is then described using the commutators of the infinitesimal generators in the bilinear Lie bracket, written as

$$[T_a, T_b] = f_{ab}{}^c T_c\,,$$

where ${f_{ab}}^c$ is the structure constants which determines the Lie algebra. Because the tangent space is a vector space one is free to choose any basis resulting in a different structure constant. In particular, a constant of i is often used in quantum mechanics, but will rarely be done in this thesis. The structure constant for $SO(3)$ with the basis above is given by ${\epsilon_{ab}}^c$ where ϵ is of Euclidean signature so it has the same components with indices up or down, as can be checked by forming the commutator of the matrices.

It can happen that by identifying elements from one group with elements from another in a one-to-one correspondence, the group operation produces the same result for the two groups. If this is the case the groups are called isomorphic and are mathematically equivalent. The same can be true for groups' Lie algebra. If this is the case they do not need be totally isomorphic, but they must be locally isomorphic around their respective unit elements. For example, in appendix B we determine that the generators of $SU(2)$ can be expressed as the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By rescaling the basis by multiplying all elements by $-i/2$ and using the commutation relation for the Pauli matrices, we find that ${f_{ab}}^c = {\epsilon_{ab}}^c$. Thus the Lie algebras of $SU(2)$ and $SO(2{,}1)$ are the same and therefore isomorphic $\mathfrak{so}(2{,}1) \cong \mathfrak{su}(2)$. However their group manifolds are not isomorphic in turn, as $SU(2)$ is in fact a double covering of $SO(3)$. Some other isomorphic Lie algebras that will be of use for this thesis are $\mathfrak{so}(2{,}1) \cong \mathfrak{sl}(2)$ and $\mathfrak{so}(2{,}2) \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2{,}\mathbb{R})$ [1], where the notation for the last group means that it is made up of two non-interacting smaller parts.

2.3.3 The Lorentz, Poincaré and conformal groups

Recall how in the Minkowski space of special relativity, different observers are described using different inertial systems. These are related to each other via some Lorentz transform $({\Lambda^\mu}_\nu)$ that is a passive coordinate transformation giving different coordinates to the same events. The collection of these symmetry transformations is a Lie group itself called the Poincaré group. The allowed transformations of this group can be determined by finding the coordinate transformations that keep the metric invariant as

$$\eta_{\mu\nu}{\Lambda^\mu}_\rho{\Lambda^\nu}_\sigma = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} = \eta_{\rho\sigma}. \tag{2.5}$$

By expanding an infinitesimal new coordinate system in the old one as $x'^\mu = x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2)$ we can determine the generators of the Poincaré groups. Using this treatment it is discovered that only boosts/rotations and translations satisfy equation (2.5) and so

$$x'^\mu = {\Lambda^\mu}_\nu x^\nu + a^\mu.$$

In $2+1$ dimensions, the Lie algebra of the Poincaré group is given by

$$[P_a, P_b] = 0, \tag{2.6}$$

$$[M_a, P_b] = {\epsilon_{ab}}^c P_c, \tag{2.7}$$

$$[M_a, M_b] = \epsilon_{ab}{}^c M_c, \tag{2.8}$$

where M^a are the generators of boosts/rotations and P^a are the generators of translations. This Lie algebra will be very important later in this thesis when we investigate the similarities between gravity and gauge theory. We furthermore note that in absence of translations, the Lie algebra of the Lorentz transformations is nothing but $SO(n{,}1)$, the Lorentz group. This is because the Lorentz transformations are the isometry transformations of Minkowski space that leave the origin and metric fixed. Similarly, the de Sitter and Anti-de Sitter spacetimes which will be extensively used in this thesis have isometry groups themselves. Recall how we parametrised these manifolds in equations (2.1) and (2.2) as an embedded generalised sphere or hyperboloid respectively. Thus they possess symmetries of the space they are embedded in, so the isometry group of dS_n is given by $SO(n+1{,}1)$ and the isometry group of AdS_n is given by $SO(n{,}2)$.

In addition to the transformations of the Poincaré group determined by equation (2.5), there is one additional similar family of transformation that we will be interested in. These are the conformal transformations $(\tilde{\Lambda}^\mu{}_\rho)$that satisfy

$$\eta_{\mu\nu}\tilde{\Lambda}^\mu{}_\rho\tilde{\Lambda}^\nu{}_\sigma = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\mu}{\partial x'^\sigma} = A(x')\eta_{\rho\sigma} \tag{2.9}$$

for some arbitrary function $A(x)$. These are the transformations that no longer preserve length, but instead only preserve angles. In exactly the same way as the Poincaré case one can determine the generators of these transformations, calculations that are done in their entirety in appendix D. There are two new types of generators, D and K, in addition to the generators of the Poincaré group. The former is the infinitesimal generator of scaling transformations $x'^\mu = sx^\mu$ and the latter of the special conformal transformations

$$x'^\mu = \frac{x^\mu - c^\mu x^\sigma x_\sigma}{1 - 2b^\nu x_\nu + b^\rho b_\rho x^\gamma x_\gamma}$$

which correspond to an inversion followed by a translation followed by another inversion. In 2+1 dimensions the Lie algebra of this group is given by

$$\begin{aligned}
\left[M^a, P^b\right] &= \varepsilon^{ab}{}_c P^c \\
\left[M^a, K^b\right] &= \varepsilon^{ab}{}_c K^c \\
\left[M^a, M^b\right] &= \varepsilon^{ab}{}_c M^c \\
\left[P^a, K^b\right] &= -2\eta^{ab} D - 2\varepsilon^{ab}{}_c M^c \\
\left[D, P^a\right] &= P^a \\
\left[D, K^a\right] &= -K^a.
\end{aligned}$$

2.4 Analytical mechanics

Analytical mechanics is an improvement of the older Newtonian mechanics that instead describes the relevant physics using scalar quantities in integrals called actions.

This approach has been of great importance as it has been used extensively during the development of quantum physics and quantum field theories. In this thesis, all the equations of motion for the theories we discuss will be expressed using a Lagrangian, so we must obviously be familiar with them. To determine the equations of motion, a scalar quantity called the Lagrangian is introduced as $L = T - V$, where T and V are the kinetic and potential energies. Inserting the Lagrangian into the so called action integral $\mathcal{S} = \int L\, dt$, the actual development of the system is found by requiring that the fields of the Lagrangian be a stationary point of the action. That is, under infinitesimal variations $L \to L + \delta L$ the action must be invariant. Expressing the variation using generalised coordinates q^i and generalised velocities $\dot{q}^i$, the principle of stationary action yields that these quantities must behave according to the Euler-Lagrange equation

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \tag{2.10}$$

where this equation is applied to each generalised coordinate q^i. We now illustrate the use of the Lagrangian in a simple example that the reader should be familiar with: a mathematical pendulum.

A mathematical pendulum can be expressed as a particle of mass m, under influence of a gravitational acceleration, at a distant l from the origin. Even though we are in a 2D frame, the pendulum can be expressed with a single variable, the angular displacement θ. This means that we only have one degree of freedom and we only need one generalised coordinate q^1 to describe its motion. In Cartesian coordinates we can parametrise the pendulum's position using $l^2 = x^2 + y^2$ like

$$x = l \sin\theta$$

$$y = l \cos\theta,$$

which can be seen in figure 2.1.

Thus we want to find a Lagrangian that depends on only one coordinate θ from the definition above $L = T - V$. The kinetic and potential energies are given by $T = \frac{1}{2}mv^2$ and $V = mgh$ respectively, where v is the velocity of the pendulum and h is the distance above the zero energy potential in the direction of the gravitational acceleration. The square of the velocity can be expressed as the sum of the time derivatives on the Cartesian coordinates squared according to Pythagoras theorem,

$$v^2 = \dot{x}^2 + \dot{y}^2 \tag{2.11}$$

where

$$\dot{x} = \frac{d}{dt} l \sin\theta = \frac{d\theta}{dt}\frac{\partial}{\partial\theta} l \sin\theta = l\cos\theta\dot{\theta},$$

and similarly for y,

$$\dot{y} = -l\sin\theta\dot{\theta}.$$

Using this result in equation 2.11 gives us

$$v^2 = l^2\dot{\theta}^2.$$

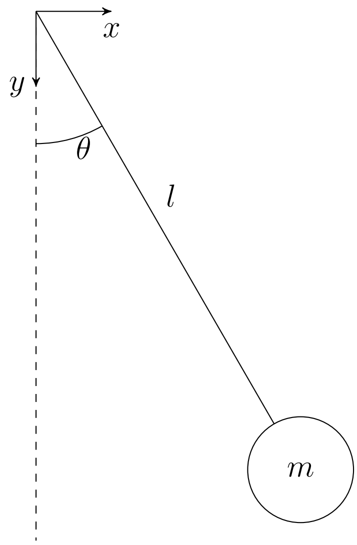


Figure 2.1: In this figure we can see how the pendulum of mass m hangs by a thread of length l by a angular displacement θ. We can also see how the position of the pendulum can be expressed in Cartesian and as well in the generalised angular displacement coordinate.

We furthermore define the ground potential level to be $y = 0$, resulting in

$$V = -mgl \cos\theta,$$

so our Lagrangian can be written as

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

We note that our generalised coordinate has dimension $[rad]$ instead of $[L]$. This is possible because generalised coordinates can have any dimension. By finally inserting the Lagrangian

$$L(q) = \frac{1}{2}ml^2\dot{q}^2 + mgl\cos q,$$

in the Euler-Lagrange equation 2.10, noting that

$$\frac{\partial L}{\partial q} = -mgl\sin\theta$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}}(\frac{1}{2}ml^2\dot{q}^2) = \frac{d}{dt}ml^2\dot{q} = ml^2\ddot{q}$$

we find after some cancellations that the principle of least action yields

$$\ddot{q} = \frac{g}{l}\sin q$$

as the equation of motion, just like expected.

Up to this point we have only considered Lagrangians for discrete particle systems. However this approach is not limited to this setting and we will in fact not be discussing any more discrete systems in this thesis. Instead we will consider Lagrangian describing fields. This cause the the Lagrangian, and its action, to be formed from a Lagrangian density $\mathcal{L}$ according to

$$\mathcal{S} = \int_t L dt = \int \mathcal{L} \, dt \, d^n x$$

where the integration of the final expression is performed over the entirety of space and time. We will not be picky and instead simply call the Lagrangian density the Lagrangian, understanding implicitly how it is to be integrated. With this Lagrangian comes a powerful theorem called Noether's theorem, which states that if the Lagrangian is invariant under a certain continuous transformation, ${q'}^i \to q + \delta q^i$, then there exists a conserved quantity related to this invariance. This is of great importance when formulating new theories and in particular we will use this when constructing gauge theories. If the Lagrangian (or more properly the action) is invariant under some transformation then the resulting equations of motions are too. This is a great guiding principle, when we experience some symmetry in nature we know that the Lagrangian must be invariant under said symmetry.

3

An Introduction to Gauge Theory

Throughout the last two centuries symmetries have played an increasingly important role in physics. These symmetries take many different forms, but can broadly be classified into two categories: internal and external. External symmetries are symmetries of the fundamental manifold the theory describes, such as rotations and translations. Internal symmetries on the other hand are symmetries of the fundamental fields of the theory. These internal symmetries describe redundancies in the theory where different configuration of the fields produce the same physics. One particular class of internal symmetries that have been found to be of fundamental importance to modern physics is gauge theories. These are theories which are invariant under some local transformations of the field, which means that the fields can be transformed in different ways at different points. These transformations are performed using elements of a particular choice of Lie group G, called the gauge group.

The first modern gauge theory was constructed by Weyl, who managed to recover classical electromagnetism by extending a global $U(1)$-symmetry of spinor fields to a local symmetry. Proceeding along similar lines physicists have managed to construct many new physical theories by extending to new gauge groups and settings. The standard model of physics describing the electro, weak, and strong forces is the foremost example of a successful gauge theory and its gauge group is given by $U(1) \times SU(2) \times SU(3)$. This theory and most modern theories with it are quantised since we now know the fundamental importance of quantum theories. However, in this introductory thesis we will only consider classical continuous gauge theories for simplicity. This is specially important for later chapters where we study gravity and its relation with gauge theories. Quantising gravity has proven extremely difficult and it is an active problem to this day.

3.1 Classical electromagnetism: the first gauge theory

The first example of a gauge theory to be discovered was Maxwell's classical theory of electromagnetism. In this theory, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are coupled to each other, to charges ρ and to currents $\mathbf{J}$. This coupling is governed by a set of differential equations called Maxwell's equations which in natural units, where $c = \epsilon_0 = \mu_0 = 1$, are given by

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho & \nabla \times \mathbf{E} + \tfrac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} - \tfrac{\partial \mathbf{E}}{\partial \mathbf{t}} &= \mathbf{J}. \end{aligned} \tag{3.1}$$

From the homogeneous equations above it is found that the electric and magnetic fields can be expressed from a scalar potential ϕ and a vector potential $\mathbf{A}$ as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{3.2}$$

However, these potentials are not unique as different potentials can give rise to the same fields. We can form new potentials ϕ' and $\mathbf{A}'$,

$$\phi' = \phi - \frac{\partial \psi}{\partial t}, \qquad \mathbf{A}' = \mathbf{A} + \nabla\psi \tag{3.3}$$

where ψ is an arbitrary differentiable function. Inserting these new potentials into equation (3.2) all new terms cancel and the resulting electric and magnetic fields remain the same. Because of this mathematical redundancy in this theory we can choose any particular combination of potentials using this *gauge transformation* in order to simplify calculations, a process known as *fixing a gauge*. This gauge freedom of Maxwell's theory was known long before its significance was understood, and it is easy to see why. At a cursory glance this seems like little more than a coincidence of this particular theory. It was not until the 20th century a deeper meaning was attributed to gauges as their mathematical framework was explored and we learned how to generalise from this simple example.

In order to proceed and discover how we can create new gauge theories we need to reformulate the classical theory into a more modern form. In particular, we want to reintroduce Maxwell's equations in a fashion where their symmetry under Lorentz transformations, a concept for which they were of fundamental importance in discovering, is immediately obvious. Furthermore we wish to express the dynamics of the theory in a Lagrangian, or more correctly Lagrangian density, since this standard formalism readily provides a way to quantise a field theory. In addition, the symmetries of the theory is contained in the invariance of the Lagrangian (or more properly the action of the Lagrangian) during said symmetry transformation. Thus the Lagrangian provides a method by which we can formulate theories with a particular symmetry, which in our case will be gauge symmetry.

Using our previous potentials ϕ and $\mathbf{A}$ we form a new four vector potential $A^\mu = (\phi, \mathbf{A})$. Using the relation from equation (3.2) we can then express the electric and magnetic field components as elements of an antisymmetric second order tensor $F^{\mu\nu}$, the field strength:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = 2\partial^{[\mu} A^{\nu]} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{3.4}$$

Explicitly, $E^i = F^{0i}$ and $B^i = \frac{1}{2}\epsilon^i{}_{jk}F^{jk}$ where ϵ^{ijk} is the Levi-Civita symbol and not the Lorentz-invariant Levi-Civita tensor. Using the antisymmetry of the field strength tensor the homogeneous Maxwell equations take the form

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \tag{3.5}$$

which is a so called Bianchi identity. Using equation (3.4) together with the fact that partial derivatives commute we can prove this identity like

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 2\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^{[\rho}A^{\sigma]} = 4\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^\rho A^\sigma = 4\epsilon_{\mu\nu\rho\sigma}\partial^{(\nu}\partial^{\rho)}A^\sigma = 0,$$

since the contraction of a symmetric and an antisymmetric tensor is zero.

In order to recover the other two Maxwell equations we need to combine the charge ρ and the current $\mathbf{J}$ into a four current $J^\mu = (\rho, \mathbf{J})$, entirely analogously to the four potential. Using this four current the inhomogeneous equations take the form

$$\partial_\nu F^{\mu\nu} = J^\mu. \tag{3.6}$$

However there is no direct way to derive this equation from what we have discussed so far. Instead we need to introduce a suitable Lagrangian for which this equation will be a stationary solution. Remembering how the symmetries of the theory are contained in the Lagrangian we require it to be Lorentz invariant. As such, we form the Lagrangian according to

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu = \\ &-\left(\partial^{[\mu}A^{\nu]}\right)\left(\partial_{[\mu}A_{\nu]}\right) + J^\mu A_\mu = -(\partial^\mu A^\nu)(\partial_{[\mu}A_{\nu]}) + J^\mu A_\mu\end{aligned} \tag{3.7}$$

where the last step is permitted since it's an contraction over two antisymmetric tensors.

In order to determine the equation of motion of A_μ we demand that the potential be a stationary point of the action

$$\mathcal{S} = \int \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right) \mathrm{d}^4x, \tag{3.8}$$

which means that the fields of the Lagrangian in equation (3.7) must satisfy the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0. \tag{3.9}$$

Inserting the last term into this equation we find that it contributes $-J^\nu$, while the first term requires some further calculations:

$$\begin{aligned}&\frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\rho A^\sigma)(\partial_{[\rho}A_{\sigma]}) = \eta^{\gamma\rho}\eta^{\xi\sigma}\frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\gamma A_\xi)(\partial_{[\rho}A_{\sigma]}) = \\ &\eta^{\gamma\rho}\eta^{\xi\sigma}\left(\delta^\mu_\gamma\delta^\nu_\xi\partial_{[\rho}A_{\sigma]} + \partial_\gamma A_\xi\delta^\mu_{[\rho}\delta^\nu_{\sigma]}\right) = 2\partial^{[\mu}A^{\nu]} = F^{\mu\nu}.\end{aligned}$$

Inserting this into equation (3.9) we see that it reduces to

$$\partial_\nu F^{\mu\nu} = J^\mu$$

which are the inhomogeneous Maxwell equations in tensor form.

Before we move on from this example we wish to study how the gauge transformation behaves. With our four vector potential $A^\mu = (\phi, \mathbf{A})$ we can write the gauge transformation as

$$A'^\mu = A^\mu + \partial^\mu \phi(x) \tag{3.10}$$

where ϕ, as before is an arbitrary differentiable scalar function of space and time. As mentioned earlier, the symmetries of the theory can be derived from the invariance of the theory's Lagrangian. So it is expected that this gauge transformation indeed should leave the Lagrangian in equation (3.7) invariant. We know from before that the field strength is expressed from the electric and magnetic fields which are invariant under this gauge transformation. So only the invariance of the second term remains to be checked. The change of this term during a gauge transformation is given by

$$J^\mu(A'_\mu - A_\mu) = J^\mu \partial_\mu \phi = \partial_\mu(\phi J^\mu) - \phi \partial_\mu J^\mu.$$

The first term is the divergence of the field and vanishes when integrated over space. Since we can express the four current from the antisymmetric field strength according to Maxwell's equation like above we furthermore have

$$\partial_\mu J^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0$$

and we see that the Lagrangian indeed, up to boundary terms, is invariant under this gauge transformation.

3.2 Creating new gauge theories

In the previous section we discussed the theory of electrodynamics and in particular how its gauge freedom can be seen from the invariance of its Lagrangian. We now wish to extend this concept and see what other Lagrangians possess gauge symmetries, or if we can extend them in such a way as to make them gauge invariant. In this process we expect to be able to develop the necessary tools we need in order to construct new gauge theories from scratch. In this endeavour we first turn to the free Dirac Lagrangian

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \tag{3.11}$$

where γ^μ are the Dirac gamma matrices[1], ψ is a spinor field and $\bar{\psi} = \psi^\dagger \gamma^0$. Spinors are a generalisation of vectors and tensors that are important in physics as they are used to describe fermions. They are supplied with a rich mathematical structure, but we will not be studying them in detail and only useing them incidentally.

[1]Their specific form is not of great importance to us here, but in three dimensions they are given by the matrices of equation (5.10)

The equation of motion for the free Dirac Lagrangian is given by the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0,$$

which is a relativistic generalisation of Schrödinger's equation and governs massive, relativistic spin $\frac{1}{2}$ particles. Importantly for us, this Lagrangian is invariant under the global phase transformation

$$\psi \to \psi' = e^{i\theta}\psi$$

which is a transformation with an element $g = e^{i\theta}$ of the Lie group $U(1)$. This invariance is unsurprising given how the probability density of the wave function is given by $\bar{\psi}\psi$, which is an absolute value. By specifying the initial phase of the state, the equations governing the theory will describe its development. A different initial configuration differing only by phase, which produces another solution, carries no new information as it describes the same physical state. As such, only phase differences carry any meaning which is governed by the equations of the theory.

Having noted the global symmetry of the free Dirac Lagrangian we ask ourselves if it is possible to extend it to a local symmetry of the field,

$$\psi \to \psi' = e^{i\theta(x)}\psi.$$

That is, we wonder if we can let the transformation change the phase of the wave function differently at different points in space. This transformation is called a gauge transformation of the first kind, and if our Lagrangian was invariant under this transformation too, even phase differences would lose their physical meaning. However, we see that this transformation produces a non-vanishing term, $-\bar{\psi}\gamma^\mu\psi\partial_\mu\theta(x)$, in the Lagrangian given by equation (3.11), because the derivatives act on the transformation elements. The fundamental cause of this problem can be seen by studying the directional derivative according to its definition

$$n^\mu \partial_\mu \psi(x) = \lim_{\epsilon \to 0} \frac{\psi(x + \epsilon n) - \psi(x)}{\epsilon}$$

where n^μ is a unit vector. This definition fundamentally depends on the field at two different points, which we now allow to transform differently. It is then clear that the derivative $\partial_\mu\psi(x)$ can not transform in the same way as $\psi(x)$, which is required for the Lagrangian to be invariant and instead an additional term proportional to the transformation parameter $\theta(x)$ arises.

In order to remedy the problem of normal partial derivatives we need to introduce a so called *covariant derivative* D_μ which transform in the same way as the field itself,

$$D_\mu\psi(x) \to (D_\mu\psi(x))' = e^{i\theta(x)} D_\mu\psi(x).$$

Replacing all the partial derivatives with this covariant derivative would then make the Lagrangian gauge invariant as required. In order to compensate for the new term the partial derivatives produced when acting on the transformation elements $e^{i\theta(x)}$, we need to introduce a another term A_μ, which is a gauge potential. By doing this we can let the transformation of the gauge potential negate that of the partial derivatives so that the covariant derivative as a whole transforms covariantly. We thus define this covariant derivative as[2]

$$D_\mu \psi(x) = (\partial_\mu + iA_\mu)\psi(x) \tag{3.12}$$

and find that upon performing our gauge transformation $\psi(x) \to e^{i\theta(x)}\psi(x)$ this object transforms as

$$D_\mu \psi(x) \to D_\mu e^{i\theta(x)}\psi(x) = e^{i\theta(x)}(\partial_\mu + i(\partial_\mu \theta(x)) + iA'_\mu)\psi(x).$$

In order for the covariant derivative to transform covariantly, we see from the expression above that the gauge potential has to transform as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \theta(x),$$

which is just how the gauge potential transformed according to equation (3.10) in the electromagnetic case, if we identify $\theta(x)$ with $-\phi(x)$. Recalling how we identified the transformation $e^{i\theta(x)}$ as an element g of $U(1)$ we have

$$ie^{i\theta(x)}\partial_\mu e^{-i\theta(x)} = \partial_\mu \theta(x),$$

so the transformation of A can be written as

$$A'_\mu = -igD_\mu g^{-1}.$$

This compensating transformation is called a gauge transformation of the second kind. Both this and the original transformation $\psi(x) \to e^{i\theta(x)}\psi(x)$ are gauge transformations, as they are continuous transformations of the fields in the theory. However, they transform entirely differently, their intimate link being given by the covariant derivative.

At first glance, the introduction of the gauge potential into the covariant derivative might seem like an ad hoc solution, but it is in fact of fundamental importance. When we allow the phase to transform differently at different points we allow ourselves to choose a specific gauge, the phase at each point in space, and the gauge potential carries information about this choice. This is a consequence of how the gauge potential mathematically serves the function of a so called connection which will be discussed in chapter 4. Nevertheless, the introduction of this new field is important as it can carry new physical meaning, previously absent in our theory. Replacing the partial derivatives in the free Dirac Lagrangian with the covariant derivative we see that the Lagrangian

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - A_\mu \bar{\psi}\gamma^\mu \psi \tag{3.13}$$

has fundamentally changed, as it now contains an additional term due the coupling between the fermion and our new gauge field. From this we can see how the approach

[2]One often introduces a dimensional coupling factor q into the covariant derivative as $D_\mu = \partial_\mu + iqA_\mu$. We will not do so here and instead such factors will always implicitly be part of the gauge potentials A_μ themselves.

of extending global symmetries to local ones results in modifications to the theory, as it can now include additional interacting fields.

However, the question arises if we can modify the Lagrangian further to produce even more interactions. As our entire aim is to produce a gauge theory, any additional terms we find must be gauge invariant. Having introduced an entirely new field A_μ, we now investigate what more gauge invariant terms we can form from this quantity. Thinking back to the electromagnetic case we remember that the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.14}$$

was a gauge invariant quantity involving only the gauge potential. This is a consequence of the fact that applying the covariant derivative to something that transforms covariantly produces an object that also transform covariantly. In particular we can form the commutator of the covariant derivative with itself,

$$[D_\mu, D_\nu]\phi(x) = 2\left(\partial_{[\mu}\partial_{\nu]} + i(\partial_{[\mu}A_{\nu]}) + iA_{[\nu}\partial_{\mu]} + iA_{[\mu}\partial_{\nu]} - A_{[\mu}A_{\nu]}\right)\phi(x) = iF_{\mu\nu}\phi(x),$$

as all terms but the second cancel. The left hand side transforms covariantly with an additional term $e^{i\theta(x)}$ which is precisely matched by that of the field $\psi(x)$ on the right hand side, $F_{\mu\nu}$ must be gauge invariant. As the field strength $F_{\mu\nu}$ measures the failure of covariant derivatives, in analogy with the Riemann curvature tensor of general relativity, it is also known as the curvature tensor. Adding this term to the Lagrangian in equation (3.13) we arrive at

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu\bar{\psi}\gamma^\mu\psi = \mathcal{L}_M + \mathcal{L}_{ED} + \mathcal{L}_C.$$

This is a three part Lagrangian: $\mathcal{L}_M$ describes the dynamics of fermion matter fields, $\mathcal{L}_{ED}$ describes the dynamics of the electrodynamic fields and $\mathcal{L}_C$ describes their coupling. Identifying $-\bar{\psi}\gamma^\mu\psi$ with J^μ we see that the coupling term of this Lagrangian corresponds to the source term in the original electrodynamical Lagrangian.

In principle we could add additional gauge invariant terms by forming higher dimensional polynomials of the fields strength $F_{\mu\nu}$ but such terms would require the insertion of additional dimensional factors so they still have dimension energy. Such terms are troublesome in quantum field theory as they produce terms that aren't renormalizable. Since ultimately a gauge theory should be quantised, we will not consider any such terms. Similarly, we could form terms by contracting the field strength with the Levi-Civita tensor, but such terms would not preserve parity, a feature we wish to maintain [12].

We can express that Lagrangian in the absence of matter another way by using differential forms. We do this by considering the gauge potential to be the components of the gauge potential one-form

$$A = dx^\mu A_\mu.$$

Similarly because of the antisymmetry of the field strength tensor, we can express it as a field strength two-form F,

$$F = \frac{1}{2}dx^\mu \wedge dx^\nu\, F_{\mu\nu}.$$

Looking back to the definition of the field strength in equation (3.14) we see that this is the same as

$$dA = dx^\mu \wedge dx^\nu \, \partial_{[\mu} A_{\nu]} = F$$

where d is the exterior derivative. Using the fact that two successive applications of the exterior derivative always yields 0 we find that

$$dF = 0.$$

Writing this expression explicitly in terms of the components of the three-form means that

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \, \partial_{[\mu} F_{\mu\rho]}$$

so, working in 4 dimensions, we recover the earlier Bianchi identity,

$$\epsilon^{\mu\nu\rho\gamma} \partial_\mu F_{\nu\rho} = 0.$$

Using the Hodge dual operator $*$ we can also recreate our original electrodynamics action of equation (3.8),

$$\mathcal{S} = \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right) dx^4,$$

in the language of differential forms. Denoting the source current one-form $dx^\mu J_\mu$ by J the action now takes the simplified form

$$\mathcal{S} = \int \frac{1}{2} F \wedge *F + *J \wedge A. \tag{3.15}$$

The equivalence of these two expressions is not immediately obvious but after using the definition of the Hodge dual we find that

$$\frac{1}{2} F \wedge *F + *J \wedge A = \frac{1}{8} F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge * \left(F_{\sigma\xi} dx^\sigma \wedge dx^\xi \right) + * \left(J_\sigma \; dx^\sigma \right) \wedge A_\gamma dx^\gamma =$$

$$\left(\frac{1}{16} F_{\mu\nu} F_{\sigma\xi} \epsilon^{\sigma\xi}{}_{\rho\gamma} + \frac{1}{6} J_\sigma A_\gamma \epsilon^\sigma{}_{\mu\nu\rho} \right) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\gamma.$$

Rewriting the volume element as $\epsilon^{\mu\nu\rho\gamma} dx^4$ and factoring it out we're left with

$$\epsilon^{\mu\nu\rho\gamma} \left(\frac{1}{16} F_{\mu\nu} F^{\sigma\xi} \epsilon_{\sigma\xi\rho\gamma} + \frac{1}{6} J^\sigma A_\gamma \epsilon_{\sigma\mu\nu\rho} \right) = -\frac{1}{4} F_{\mu\nu} F^{\sigma\xi} \delta^{\mu\nu}_{\sigma\xi} + J^\sigma A_\gamma \delta^\gamma_\sigma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\sigma J^\sigma$$

where in the last step we used the antisymmetry of the field strength tensor. Thus this form is indeed equivalent to the original and the equations of motion is, up to boundary terms, given by

$$d * F = *J,$$

which can be checked by calculating the variation of the action.

3.3 Extension to non-abelian gauge theories

In the last section we discussed the method of extending Lagrangians, invariant under global symmetries, to those that are locally gauge invariant through the use of the covariant derivative. This necessarily introduced a gauge potential where we saw that the Dirac Lagrangian reproduced electromagnetism. Even though this shed some further light on this theory we would now like to proceed and create new gauge theories. There is an seemingly obvious way to do this. Noting how the transformation symmetry that produced electromagnetism was $U(1)$ we can attempt to generalise by extending this method to other Lie groups $\mathcal{G}$. In this process we must be careful to take into consideration the fact that these groups might not be abelian like $U(1)$. Recall how an element $g \in \mathcal{G}$ can be expressed using the infinitesimal generators $T_a, a = 1, ..., n$ of the group as

$$g = \exp\left(\theta^a T_a\right).$$

From these generators the properties of the group can be found by using their Lie algebra, which is expressed using the Lie bracket

$$[T_a, T_b] = f_{ab}{}^c T_c$$

where the form factor $f_{ab}{}^c$ is antisymmetric in its lower indices. Generally the Lie bracket measures the failure of vectors to commute at the unit element of the group manifold. If this commutation fails to vanish we call the group non-abelian. For such groups it matters in what order two transformations are performed and this introduces additional complexity to our theory as we can't discard terms as easily as before. If we assume that our transformations are non-abelian we can form a another type of gauge theories.

To proceed with the creation of this theory we will ignore the explicit form of the Lagrangian and instead focus on the fundamental fields $\psi(x)^i$, arranged in a vector. Assuming that the Lagrangian is invariant under some global set of transformations $g \in \mathcal{G}$,

$$\psi(x)^i \to g\psi(x)^i = \exp\left(\theta^a T_a\right)\psi(x)^i.$$

we now let the transformation parameters become functions of the base space,

$$\theta^a \to \theta(x)^a,$$

so the global symmetry is extended to a local one, like we did for $U(1)$. These new degrees of freedom means that partial derivatives will produce an additional term per generator, when acting on the transformation parameter of the transformed fields,

$$\partial_\mu\psi(x)^i \to (\partial_\mu\psi(x)^i)' = \exp\left(\theta(x)^a T_a\right)\left(\partial_\mu + (\partial_\mu\theta(x)^a)T_a\right)\psi(x)^i. \tag{3.16}$$

As such we are once again forced to create a covariant derivative D_μ that only measure the difference in the fields themselves and not those caused by the transformation. The covariant transformation of this derivative means that the following relation must hold:

$$D_\mu\psi(x)^i \to (D_\mu\psi(x)^i)' = \exp\left(\theta(x)^a T_a\right)D_\mu\psi(x)^i. \tag{3.17}$$

When we study equation (3.16) we see that the unwanted terms breaking the invariance are Lie algebra valued, and as such the correction term of the covariant derivative must be too. Defining

$$D_\mu \psi(x)^i = (\partial_\mu + A_\mu^a T_a)\psi(x)^i$$

where A_μ^a is a collection of new gauge potential fields we find upon inserting eqation (3.16) into equation (3.17) that

$$\exp(\theta(x)^a T_a)\left(\partial_\mu + (\partial_\mu \theta(x)^a)T_a + A_\mu^{\prime a} T_a\right)\psi(x)^i = \exp(\theta(x)^a T_a)\left(\partial_\mu + A_\mu^a T_a\right)\psi(x).$$

Clearly A_μ^a then must transform according to

$$A_\mu^a \to A_\mu^{\prime a} = A_\mu^a - \partial_\mu \theta(x)^a,$$

or put more succinctly using $g^{-1} = \exp\left(-\theta(x)^a T_a\right)$,

$$A'_\mu = A_\mu^{\prime a} T_a = g D_\mu g^{-1}. \tag{3.18}$$

Once again we point out that this is a gauge transformation of a second kind while the the transformation using g on the original fields is a gauge transformation of the first kind.

Remembering how we for $U(1)$ could form a gauge invariant field strength tensor using the commutator of covariant derivatives, we attempt to do this again for the non-abelian case. We find that

$$[D_\mu, D_\nu]\phi(x)^i = 2\left(\partial_{[\mu}\partial_{\nu]} + (\partial_{[\mu}A_{\nu]}^a T_a) + A_{[\nu}^b \partial_{\mu]} T_b + A_{[\mu}^a \partial_{\nu]} T_a + \frac{1}{2}[A_\mu^a T_a, A_\nu^b T_b]\right)\phi(x)^i.$$

As earlier the first, third, and fourth terms vanish but the fifth term doesn't have to. Since the gauge fields are Lie algebra valued their commutator can be non-zero since the group is non-abelian. However, we still see that the expression on the right is not a derivative so we define the Lie algebra valued field strength tensor $F_{\mu\nu}$

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3.19}$$

where there is an implicit sum $F_{\mu\nu}^a T_a$ over the Lie generators. Writing this out explicitly using the structure factor f_{bc}^a we have

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c. \tag{3.20}$$

Because the field strength is the commutator of covariant derivatives it must yield a covariant quantity when applied to $\psi(x)^i$. This enables us to determine the transformational properties of the field strength:

$$[D_\mu, D_\nu]\psi(x)^i = F_{\mu\nu}\psi(x)^i \to \left(F_{\mu\nu}\psi(x)^i\right)' = F'_{\mu\nu} g\psi(x)^i = gF_{\mu\nu}\psi(x)^i \Rightarrow$$

$$F'_{\mu\nu} = gF_{\mu\nu}g^{-1}.$$

The field strength is no longer gauge invariant since it is Lie algebra valued. Only for abelian groups like $U(1)$ where the order of transformations doesn't matter will the field strength be invariant.

The non-invariance of the field strength might raise worries that we can not create gauge invariant Lagrangian involving the field strength describing the dynamics of the gauge fields themselves. However, we only need to perform a small adjustment to our previous expression $F^{\mu\nu}F_{\mu\nu}$ in order to ensure that it is gauge invariant. This expression transforms as

$$F^{\mu\nu}F_{\mu\nu} \to F'^{\mu\nu}F'_{\mu\nu} = gF'^{\mu\nu}g^{-1}gF'_{\mu\nu}g^{-1} = gF'^{\mu\nu}F'_{\mu\nu}g^{-1}.$$

Now recall how the Lie generators can be represented (in many different ways) as matrices. Choosing one particular representation we can form the trace of both sides of this transformation

$$Tr\left[F_{\mu\nu}\right] \to Tr\left[gF_{\mu\nu}g^{-1}\right] = Tr\left[F_{\mu\nu}\right]$$

where we have exploited the cyclicity of the trace, $Tr\left[ABC\right] = Tr\left[BCA\right]$. By simply applying the trace we have recovered an invariant expression which we can use as a Lagrangian.

We could in principle use the trace for any collection of covariant, representation matrix valued objects to produce gauge invariant objects. We will however mainly be studying compact groups, which in this instance means that the basis of the Lie algebra can be rescaled in such a way as to cause the trace to be of the form

$$Tr\left[T_a T_b\right] = \delta_{ab}.$$

In particular, all of the trace terms will have the same sign which means that all terms of the Lagrangian

$$-\frac{1}{4}Tr\left[F_{\mu\nu}F^{\mu\nu}\right]$$

can correspond to kinetic energy terms. We will in later chapters also consider non-compact groups where this is not the case. As such, the gauge fields in the Lagrangian lose this correspondence and must be thought of in a different way.

Let us now construct a gauge invariant Lagrangian for a collection of spinor fields ψ^i in analogue with the free Dirac Lagrangian of equation (3.11). In addition we will include the term derived above, resulting in the Lagrangian,

$$\mathcal{L} = \mathcal{L}_M + \mathcal{L}_F = i\bar{\psi}_i\gamma^\mu D_\mu\psi^i - m\bar{\psi}_i\psi^i - \frac{1}{4}Tr\left[F_{\mu\nu}F^{\mu\nu}\right],$$

The equations of motion from the spinor fields themselves is clearly an extended Dirac equation with the spinors coupled to the gauge fields:

$$i\gamma^\mu\partial_\mu\psi^i + i\gamma^\mu A_\mu\psi^i - m\psi^i.$$

Those obtained by varying the gauge potential are found by letting $A_\mu \to A_\mu + \delta A_\mu$, where we consider the variation to be gauge covariant. Recalling how the field strength is the commutator of covariant derivatives we see that $F_{\mu\nu} \to F_{\mu\nu} + D_\mu\delta A_\nu - D_\nu\delta A_\mu$ during this variation, so by using the antisymmetry of the field strength we find that

$$\delta\mathcal{S}_F = -\int Tr\left[F^{\mu\nu}D_\mu\delta A_\nu\right]dx^4.$$

However,

$$Tr\left[F^{\mu\nu}D_\mu\delta A_\nu\right] = D_\mu Tr\left[F^{\mu\nu}\delta A_\nu\right] - Tr\left[\delta A_\nu D_\mu F^{\mu\nu}\right],$$

and since δA_μ transforms covariantly, the first trace on the right hand is gauge invariant. But then we can replace the covariant derivative with a partial derivative, so this term results in a vanishing boundary term when inserted into the action, as it is nothing but a divergence. $\mathcal{L}_M$ clearly contributes

$$\delta A_\nu \bar{\psi}_i \gamma^\nu \psi^i = \delta A_\nu J^\nu$$

to the variation, so requiring that their sum be zero for any δA_ν yields

$$D_\mu F^{\mu\nu} = J^\nu, \tag{3.21}$$

The equation above is the analogy of the inhomogeneous Maxwell's equations (3.6) for the non-abelian case. Note that both of these expressions are Lie algebra valued so when explicitly written in their components and lowering indices this takes the form

$$D^\mu F^a_{\mu\nu} = J^a_\nu \quad \Leftrightarrow \quad \partial^\mu F^a_{\mu\nu} + f^a_{bc}\left(A^\mu\right)^b F^c_{\mu\nu} = J^a_\nu.$$

In the original, abelian Maxwell equation for $U(1)$ the second term on the left was zero since the structure factor was too. Now however, this additional term describes a new interaction between the gauge fields themselves. As such, the structure of the gauge group is shown to be of profound importance for the theory it describes.

In addition to equation (3.21) involving the source term above, we can also produce the analogy of the homogeneous equations which we identified as a so called Bianchi identity. This is done by adding three cyclic double commutators like

$$[D_\mu,[D_\nu,D_\rho]] + [D_\rho,[D_\mu,D_\nu]] + [D_\nu,[D_\rho,D_\mu]] = 0, \tag{3.22}$$

which is easily seen to be true by expanding all terms. Furthermore,

$$[D_\mu,[D_\nu,D_\rho]]\phi = D_\mu[D_\nu,D_\rho]\phi - [D_\nu,D_\rho]D_\mu\phi = (D_\mu F_{\nu\rho})\phi,$$

so

$$[D_\mu,[D_\nu,D_\rho]] = D_\mu F_{\nu\rho}.$$

Inserting this into equation (3.22) we find that

$$D_\mu F_{\nu\rho} + D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} = 0$$

or, expressed another way,

$$\varepsilon^{\mu\nu\rho} D_\mu F_{\nu\rho} = 0.$$

This is just the original Bianchi identity where partial derivatives have been replaced by covariant derivatives.

All of the discussion of this section have been done without specifying which gaugegroup we have been using. If we now specify that the group is $SU(2)$ and the fermion fields are those of the proton and neutron,

$$\psi^i = \begin{pmatrix} \psi^p \\ \psi^n \end{pmatrix},$$

we have the Yang-Mills theory of isospin. In general, if the gauge group is $SU(n)$ we are dealing with a Yang-Mills theory, but the isospin invariance of $SU(2)$ was the first non-abelian gauge theory constructed. This development was originally motivated by the fact that protons and neutrons in the absence of electromagnetic effect are nearly identical. The same process was later used for $SU(3)$ as a description of quantum chromodynamics [13].

As before we can express the Lagrangian and all its components in a more compact form using the language of differential forms. We form the Lie algebra valued gauge potential one-form A as

$$A = dx^\mu A^a_\mu T_a.$$

From this we also construct the Lie algebra valued field strength two-form F, which after studying equations, (3.19) and (3.20) is seen to be

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \left(\partial_{[\mu}A^a_{\nu]} + \frac{1}{2}f^a_{bc}A^b_\mu A^c_\nu\right)T_a dx^\mu \wedge dx^\nu = dA + A \wedge A.$$

The Yang-Mills Lagrangian furthermore takes the form

$$\mathcal{L}_{YM} = Tr\left[\frac{1}{2}F \wedge *F + *J \wedge A\right], \tag{3.23}$$

where J is the source current form, entirely in analogy with equation (3.15). The equations of motions from this Lagrangian take the form

$$D * F = *J$$

and we can write the Bianchi identity in a succinct form as

$$DF = 0,$$

where D is the covariant exterior derivative,

$$D = d + A \wedge [\ \ ].$$

3.4 Chern Simons gauge theory

Ignoring the source term, the compact Yang-Mills Lagrangian of equation (3.23) was proportional to $Tr\left[*F \wedge F\right]$ which ensures that it is easily integrated. Working in usual four-dimensional spacetime, we could just as easily construct a gauge invariant Lagrangian of the form

$$\mathcal{L}_C = Tr\left[F \wedge F\right],$$

since F is a two-form. This is known as the second Chern form and in general $Tr\left[F^n\right]$ is called the n-th Chern form. Unfortunately we find that a Lagrangian constructed from this theory yields no proper equations of motion, as all gauge potentials are stationary points of the Lagrangian. Under a variation δA we have

$$\delta F = d\delta A + \delta A \wedge A + A \wedge \delta A = D\delta A$$

so, because of the cyclicity of the trace,

$$\delta\mathcal{L}_C = Tr\left[D\delta A \wedge A + A \wedge D\delta A\right] = 2Tr\left[D\delta A \wedge A\right].$$

But

$$D\delta A \wedge A = D(\delta A \wedge A) - \delta A \wedge DA$$

and the first term on the right of this expression is a vanishing boundary term while the second term on the right is zero by the Bianchi identity $DF = 0$. The Lagrangian made using $F \wedge F$ is thus totally gauge invariant and the same result applies to general Chern forms.

The Chern forms are what is known as exact differential forms, which means that they can be expressed as $d\Omega$ where Ω is some differential form. Explicitly for the second Chern form Ω is given by

$$Tr\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right].$$

This form is known as Chern-Simons three-form and it can be thought of as a boundary term when performing the integral of $F \wedge F$ over $[0{,}1] \times S$ where S is some three dimensional spacetime manifold [14]. If we constrain ourselves from four to three dimensions we can use this to construct the three dimensional Chern-Simons action

$$\mathcal{S}_{CS} = \frac{k}{4\pi}\int_M Tr\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right], \tag{3.24}$$

integrated over some three dimensional manifold M. The reader should at this point be weary as this action clearly depends on the gauge potentials and not the gauge invariant field strength. However, the manifold over which the integration is performed is compact, the response produced by this action will be boundary terms and a term proportional to something called the winding number of the gauge transformation. This will be of the form $2\pi n, n \in \mathbb{N}$ so instead $\exp i\mathcal{S}_{CS}$ will be gauge invariant [1]. As such this action is most commonly used in quantum physics with its path integrals of this form. Having noted this we will henceforth neglect differing terms and consider the Chern-Simons action to be gauge invariant.

It remains for us to determine the equations of motion for the Chern-Simons action. Varying the potentials with δA the Lagrangian will change by

$$Tr\left[\delta A \wedge dA + A \wedge d\delta A + 2(\delta A \wedge A \wedge A)\right]$$

where we have used the cyclic invariance of the trace for the last term. Using the fact that

$$d(A \wedge \delta A) = dA \wedge \delta A - A \wedge d\delta A,$$

throwing away the vanishing boundary term and once again using the cyclic invariance of the trace we find that

$$\delta\mathcal{S}_{CS} = \frac{k}{4\pi}\int_M Tr(\delta A \wedge (dA + A \wedge A)).$$

Requiring the action be stationary we find that the equations of motion for the Chern-Simons action is given

$$F = dA + A \wedge A = 0. \tag{3.25}$$

We noted earlier that since the field strength measures the failure of covariant derivatives to commute it really is a measure of curvature. The vanishing of the field strength then means that A, in its role as a connection, is flat. This might be considered a restrictive condition that will render theories using this action uninteresting. However, when quantising such a theory this triviality disappears as several remarkable new features arise. We will in the next chapters also see that even in the classical case this theory can be used to interesting effect when we use it to investigate the similarities between gravity and gauge theory.

4

Conformal Gravity from Chern-Simons Gauge Theory in $2+1$ Dimensions

In the previous chapter we introduced the concept of gauge theories, both in their abelian and non-abelian forms. Now we proceed to a seemingly separate area of physics, Einstein's theory of general relativity. This theory has been of profound importance during the 20th century and has proven remarkably succesful when describing gravity. General relativity is a theory that is geometric at its core, describing how the key concept of curvature becomes intimately linked with the presence of matter. Its remarkable success describing gravity is only matched by the standard model describing the other forces. Unfortunately these theories have been found to be incompatible, even after decades of frantic attempts of unification by physicists. Amongst other things, the standard model is a gauge theory, while general relativity is not. However, there exists a connections between gravity and gauge theory, at least in 2+1 dimensions [1].

We will now be exploring the connection between gauge theory and gravity. In order to do so, we will be presenting the latter briefly, in the both standard and Cartan formalisms. Because of the geometrical nature of this theory, it can be formulated on general manifolds and so a description of these is important. The mathematical framework for this is complicated and large, but a short presentation of its foundations is given in appendix A. Having presented this theory, we will demonstrate its similarities with gauge theory. Specifically we will show how in 2+1 dimensions we can express the Einstein-Hilbert action, which governs the theory of general relativity, as a Chern-Simons action. Having noted this correspondence we then finally proceed to develop a generalisation to a conformal theory of 2+1-dimensional gravity.

4.1 Einstein's formulation of general relativity

The starting point of general relativity is a Riemannian manifold with (arbitrary) smooth coordinates x^μ and metric $g_{\mu\nu}$. The coordinates induce a natural covariant tangent vector basis ∂_μ and contravariant basis $\mathrm{d}x^\mu$. However, this basis is only local as there is no canonical way to relate different vectors at different places since they are part of different tangent vector spaces. Consider for example how the coordinate basis vector $\vec{r}$ in spherical coordinates is actually the $-\vec{r}$-vector when moved to its mirror point on the other side of the origin. This is illustrative, but the problem is in fact even deeper. The transportation process used is intrinsically euclidean and not available in general. In order to remedy this problem we need to introduce a connection between the different spaces, which is usually taken to be the special affine Levi-Civita connection. With this connection a covariant derivative D_μ of vector fields can be formed as

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho, \quad D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho{}_{\mu\nu} V_\rho,$$

where $\Gamma^\rho{}_{\mu\nu}$ is the so called Christoffel symbol. Acting on tensors of higher rank produces one additional term for each free index, each contracted using the Christoffel symbol as above.

As indicated by the name, the covariant derivative applied to a tensor results is a proper tensor of one higher rank, unlike the partial derivatives of a tensor which transformed according to equation (2.4). By using this requirement on $D_\mu V^\nu$, we can determine the transformational properties of the Christoffel symbol. It is found that in order to negate the contribution of the offending term the Christoffel symbol must transform as

$$\Gamma^\rho{}_{\mu\nu} \to \Gamma'^\rho{}_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\tau}{\partial x'^\nu} \Gamma^\sigma{}_{\gamma\tau} + \frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu},$$

so the Christoffel symbol is no proper tensor, only when it is combined with the partial derivative does it constitute a proper tensor [15]. As the connection introduces new degrees of freedom we restrict it to only include metric degrees of freedom. This is done by imposing the so called metric postulate

$$D_\mu \, g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} \, g_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} \, g_{\nu\sigma} = 0, \tag{4.1}$$

which ensures that the inner product of two vectors being parallel transported around the same path is conserved. That is, under this transportation vectors maintain their lengths and mutual angles.

The Levi-Civita connection is restricted in one more way as it is also specified to be free of torsion. This means that parallelograms formed by transporting infinitesimal vectors infinitesimally must be closed. Explicitly in terms of the Christoffel symbols this means that they must be symmetric in their lower indices since the torsion $T_{\mu\nu}{}^\rho$ is defined by

$$T_{\mu\nu}{}^\rho = 2\Gamma^\rho{}_{[\mu\nu]} = 0.$$

Using these two conditions the connection becomes uniquely fixed and we can express the Christoffel symbols in terms of the metric. Permutating indices in equation (4.1) and summing we find that

$$\begin{gathered} D_\rho \, g_{\mu\nu} - D_\mu \, g_{\nu\rho} - D_\nu \, g_{\rho\mu} = \\ \partial_\rho \, g_{\mu\nu} - \partial_\mu \, g_{\nu\rho} - \partial_\nu \, g_{\rho\mu} + 2 \left(\Gamma^\gamma{}_{(\mu\nu)} \, g_{\gamma\rho} + \Gamma^\gamma{}_{[\mu\rho]} \, g_{\gamma\nu} + \Gamma^\gamma{}_{[\nu\rho]} \, g_{\gamma\mu} \right) = 0, \end{gathered}$$

the antisymmetric terms cancel and upon applying $g^{\rho\sigma}$ on both sides we find that

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right).$$

The fundamental internal object containing the geometrical information about the manifold itself is the Riemann curvature tensor $R^{\rho}{}_{\gamma\mu\nu}$ which measures the failure of covariant derivatives to commute

$$[D_\mu, D_\nu] V^\rho = R^{\rho}{}_{\gamma\mu\nu} V^\gamma \Rightarrow R^{\rho}{}_{\gamma\mu\nu} = 2\partial_{[\mu} \Gamma^{\rho}{}_{\nu]\gamma} + 2\Gamma^{\rho}{}_{[\mu|\sigma|} \Gamma^{\sigma}{}_{\nu]\gamma}.$$

Using these conditions it is found that the Riemann curvature tensor satisfy the following symmetry relations

$$R_{\rho\gamma\mu\nu} = R_{\mu\nu\rho\gamma}, \quad R_{\rho\gamma\mu\nu} = -R_{\gamma\rho\mu\nu} = -R_{\rho\gamma\nu\mu}, \quad R_{\rho[\gamma\mu\nu]} = 0, \quad D_{[\sigma} R_{\rho\gamma]\mu\nu} = 0,$$

where the last two conditions are the so called first and second Bianchi identities. From the Riemann tensor we can produce two further objects by contracting over its indices, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, [14]

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{4.2}$$

These too carry information about the curvature of the manifold as they respectively measure how much the area or volume of a small circle or ball deviate from their Euclidean value.

The great triumph of Einstein's theory of general relativity was the way it expressed how the presence of matter and energy directly warps space itself by relating the stress energy tensor $T_{\mu\nu}$ to the curvature. This is done in the famous Einstein field equation

$$G_{\mu\nu} \equiv 8\pi G_N T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

where G_N is Newton's gravitational constant. This equation can be derived using the variational principle using the Einstein-Hilbert action which in 2+1 dimensions takes the form

$$\mathcal{S}_{EH} = \frac{1}{16\pi G_N} \int R\sqrt{-g}\, \mathrm{d}^3 x + \mathcal{S}_{\text{matter}}, \tag{4.3}$$

where g is the determinant of the metric, $g = \det(g_{\mu\nu})$.

4.2 The cosmological constant

In the previous section we introduced Einstein's equations which relates the presence of matter and energy to the curvature of space. These equations need to be modified on the cosmological scale with a so called cosmological constant Λ. It is a scalar curvature resulting from the presence of vacuum energy in space. It can be either positive or negative, resulting in either a repulsive or an attractive vacuum interaction. Originally the cosmological constant was introduced by Einstein so that his equations would model a static universe. However, this attempt was misguided as the Universe has been shown to be expanding. Nevertheless the cosmological constant is still of importance as it is used to model this expanding behaviour. After introducing this constant Einstein's field equation is modified to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + g_{\mu\nu}\Lambda,$$

and as such the Einstein-Hilbert action, equation (4.3), too is modified,

$$\mathcal{S}_{EH} = \frac{1}{16\pi G_N}\int (R - 2\Lambda)\sqrt{-g}\,\mathrm{d}^3x + \mathcal{S}_{\mathrm{matter}}. \tag{4.4}$$

We noted in chapter 2 how anti-de Sitter and de Sitter manifolds are used to model spacetime itself when vacuum energy causes space curve even in the absence of matter. Using this method the cosmological constant is found to directly depend upon the Ricci scalar of the respective spaces. In AdS_n this allows us to relate Λ with the radius of curvature l,

$$\Lambda = -\frac{(n-1)(n-2)}{2l^2}.$$

Because this is a negative curvature AdS_n describes an attractive vacuum interaction. dS_n on the other hand has constant positive curvature

$$\Lambda = \frac{(n-1)(n-2)}{2l^2}.$$

and thus describes a repulsive vacuum interaction. As the Universe is expanding we primarily use dS_n as a cosmological model. However, AdS_n is not unimportant, even though it produces the opposite effect. It has many applications and is often used in string theory and quantum field theories because of the recently conjectured AdS/CFT correspondence. This states that a string theory formulated on AdS space corresponds to a conformal field theory on its boundary[16]. This correspondence and its implications is a very active area of research today.

4.3 Cartan formalism

There is an alternate formalism of general relativity developed by Élie Cartan. This formalism takes advantage of the mathematical power of exterior algebra and the compact notation from differential geometry. Additionally, this compactness makes this approach coordinate independent. Recall how the mathematical basis of general relativity is a Euclidean manifold which when described by some coordinates x^μ induces a natural tangent basis ∂_μ and cotangent basis dx^μ. The information of the surface's geometry is contained in its metric $g_{\mu\nu}$, affine Levi-Civita connection and the Riemann curvature tensor $R^\mu{}_{\nu\rho\gamma}$. In Cartan or first order formalism we instead express the same information in the vielbein forms e_a, spin connection forms ω_{ab} and curvature form R_{ab}. The connection between the two methods is the fact that even though the manifold can be curved in any way its metric is still Lorentzian so we can always form local Lorentz frames of reference. This means that we can form a new orthonormal tangent basis

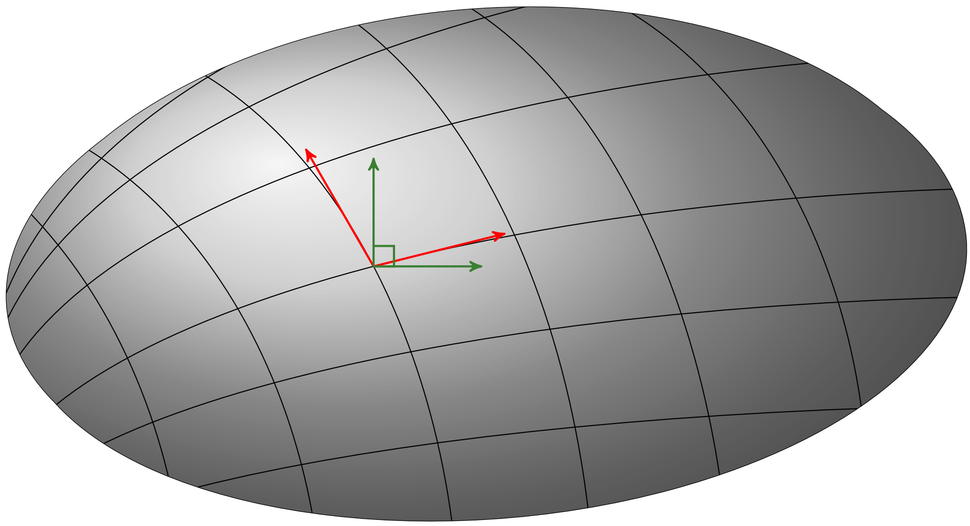


Figure 4.1: A curved Lorentzian manifold is described using a coordinate chart. This in turn induces a natural tangent basis (red) of partial derivatives ∂_μ that can be thought of as arrows pointing along the coordinate lines. Since the manifold is Lorentzian this basis can be transformed into an arbitrary orthonormal Lorentz basis (green) using transformational matrices called vielbeins $e_a^{\ \mu}$.

$$\vec{e}_a = e_a^{\ \mu}\partial_\mu, \quad a = 1, ..., n \ ,$$

using invertible matrices $e_a^{\ \mu}$. These matrices as well as their inverses $e_\mu^{\ a}$ are known as frame fields or vielbeins. The latter is German for many legs and one denotes them zweibeins, dreibeins and so forth depending on the dimension of the basis, which is illustrated in figure 4.1. Henceforth we will be using Greek indices when expressing something in the curved coordinate basis while Latin indices will be reserved for the Lorentz basis.

Using the vielbeins we can express any tensor in either the curved or the locally flat basis. In particular as the metric of the local frame is Lorentzian we have

$$g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}, \quad \eta_{ab} = e_a^{\ \mu} e_b^{\ \nu} g_{\mu\nu}.$$

These metrics can be used to raise or lower indices of their respective kind in the usual way. Furthermore because of this relation we can think about the the vielbeins as the square root of the metric, since taking the determinants on both sides of this expression, we find that $g = -e^2$.

It is important to note how the transformational properties differ for objects with curved indices from those with flat indices. If a tensor is expressed using flat indices it transforms like a scalar under curved coordinate transformations. Likewise we can choose a different local frame which manifests as a Lorentz transformation in the flat indices. During this process a tensor with curved indices transforms like a scalar. A tensor with mixed indices like the vielbeins transform accordingly for each index.

Just as for the original case with the curved tangent vectors, there is no canonical way of comparing vectors at different points since they belong to different tangent vector spaces. There we introduced the affine Levi-Civita connection and a covariant derivative acting on the curved indices. Now we need to perform the same thing for the flat ones as fundamentally the vector space is the same, only the basis differ. The covariant derivative on a vector with flat index takes the form

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b, \quad D_\mu V_a = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b$$

where the spin connection $\omega_\mu{}^a{}_b$ have the same function for flat indices as the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$ had for curved indices. As such it is not a proper tensor either and it transforms inhomogeneously under local Lorentz transforms of the orthonormal basis. When acting on a tensor of mixed type there will be one corresponding term, either Christoffel symbol or spin connection, for each index of either type. We can use this to apply the covariant derivative to the flat metric η_{ab} which, since the metric in the flat basis is Lorentzian throughout the manifold, must be 0:

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} = 0.$$

Since the first term is zero we find that the spin connection must be antisymmetric in its flat indices, since

$$\omega_{\mu ab} = -\omega_{\mu ba}.$$

It might seem like a small thing to change the basis in this fashion, but there is actually a very important reason. Without this approach it is impossible to introduce spinors to curved manifolds since the covariant derivative, in the original setting with Christoffel symbols, can not be used on spinors [17]. By working with the vielbeins this problem can be remedied, which explains the usage of the term spin connection.

Now it is obvious that the covariant derivative must produce the same result when acting on a tensor, expressed in either flat or curved indices, since they are just different descriptions of the same object. For a vector $V = V^\mu \partial_\mu = V^a \vec{e}_a$ this implies that

$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\mu\nu} V^\nu,$$

and

$$e^\rho{}_a D_\mu V^a = e^\rho{}_a \partial_\mu V^a + e^\rho{}_a \omega_\mu{}^a{}_b V^b = e^\rho{}_a \partial_\mu (e^a{}_\nu V^\nu) + e^\rho{}_a e^b{}_\nu \omega_\mu{}^a{}_b V^\nu =$$
$$\partial_\mu V^\rho + V^\nu e^\rho{}_a \partial_\mu e^a{}_\nu + e^\rho{}_a e^b{}_\nu \omega_\mu{}^a{}_b V^\nu,$$

must be the same. Upon comparison this gives a relation between the Christoffel symbol and the spin connection,

$$\Gamma^\rho{}_{\mu\nu} = e^\rho{}_a \partial_\mu e^a{}_\nu + e^\rho{}_a e^b{}_\nu \omega_\mu{}^a{}_b. \tag{4.5}$$

This condition can be seen to be equivalent to

$$D_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\sigma{}_{\mu\nu} e^a{}_\sigma = 0,$$

by applying $e^{\rho}{}_{a}$ to both sides of this expression. The vanishing of the covariant derivative of the vielbeins is known as the tetrad postulate, even though it is no postulate at all. It is simply the statement that the spin connection is compatible with the Levi-Civita connection so that calculations in one basis will produce the same results as in the other.

Up to this point we have not introduced anything that differs drastically from the earlier formalism. Now however, guided by the antisymmetry of the Riemann curvature tensor and the torsion tensor, we want to apply the power of exterior algebra and differential forms to this setting. This is done by considering antisymmetric covariant (or mixed) tensors to be components of differential forms of suitable order. Thus the vielbeins $e_{\mu}{}^{a}$ can be thought of as a basis of vector valued one-forms

$$e^a = dx^\mu e_\mu{}^a,$$

and a general tensor $A_{\mu_1 \dots \mu_n}{}^{a_1 \dots a_n}$ antisymmetric in its curved indices as a tensor valued form

$$A^{a_1 \dots a_n} = \frac{1}{n!} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} A_{\mu_1 \dots \mu_n}{}^{a_1 \dots a_n},$$

where the $n!$-factor has been inserted so that the value for the $dx^1 \wedge .. \wedge dx^n$-term coincides with $A_{1..n}{}^{a_1..a_n}$ after performing the sum over contracted indices. In particular, we form the torsion form T^a and curvature form $R^a{}_b$ as

$$T^a = \frac{1}{2} dx^\mu \wedge dx^\nu T^a{}_{\mu\nu}, \qquad R^a{}_b = \frac{1}{2} dx^\mu \wedge dx^\nu R^a{}_{b\mu\nu}.$$

Since we formed the torsion and curvature tensors using the Christoffel symbols we should also be able to do the same for these objects using the spin connection, since it plays an entirely analogous role as a connection between the different tangent spaces. As such we construct the spin connection form $\omega^a{}_b$,

$$\omega^a{}_b = dx^\mu \omega_\mu{}^a{}_b,$$

noting that this is no tensor valued form since the spin connection does not transform tensorally. Now we should be able to express the torsion and curvature forms using this object. This is indeed the case and the construction is made explicit in the Cartan structure equations:

$$T^a = de^a + \omega^a{}_b \wedge e^b, \tag{4.6}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{4.7}$$

These equations are a general result of differential geometry and can be applied to more general manifolds replacing the vielbeins e^a with any basis of vector valued one-forms and the spin connection $\omega^a{}_b$ with any linear connection one-form [17].

Now we want to be able to use the exterior derivative d to general tensor valued differential forms. However, we know the importance of the connection for derivatives when working on a curved manifold so we generalise the concept to an exterior covariant derivative D. In this process the inherent antisymmetry of the exterior derivative means that all the Christoffel symbols of the covariant derivative cancel because they are symmetric. The spin connection on the other hand is antisymmetric and as such does not cancel. Hence we need to introduce an additional term which is precisely the connection form $\omega^a{}_b$. The covariant derivative of a vector valued form accordingly takes the form

$$DV^a = dV^a + \omega^a{}_b \wedge V^b, \quad DV_a = dV_a + \omega_a{}^b \wedge V_b.$$

As before, applying the covariant derivative to tensor valued form introduces one additional connection term for each additional free index.

Looking back at the definition of the connection and torsion forms we can see certain similarities. The curvature form is not the covariant derivative of the spin connection, since it is not tensor valued at all. The torsion form, on the other hand, is indeed the covariant derivative of the vielbein one-forms. Using this relation and the vanishing of the torsion form we can express the spin connection in terms of the vielbeins. Starting from

$$De^c = de^c + \omega^c{}_b \wedge e^b = dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^c + \omega_\mu{}^c{}_d e_\nu{}^d) = 0,$$

we find upon removal of the differential forms that

$$\partial_{[\mu} e_{\nu]}{}^c + \omega_{[\mu}{}^c{}_{|d|} e_{\nu]}{}^d = 0.$$

Acting upon this expression with $e_a{}^\mu e_b{}^\nu$ and moving the commutation brackets from μ, ν to a, b, we find that

$$e_{[a}{}^\mu e_{b]}{}^\nu (\partial_\mu e_\nu{}^c + \omega_\mu{}^c{}_d e_\nu{}^d) = 0.$$

Since the second term is

$$e_{[a}{}^\mu e_{b]}{}^\nu \omega_\mu{}^c{}_d e_\nu{}^d = \omega_{[a}{}^c{}_{b]},$$

we find upon lowering the c-index and using the antisymmetry of the spin connection in its last indices that

$$\omega_{[ab]c} = e_{[a}{}^\mu e_{b]}{}^\nu \partial_\mu e_{\nu c}$$

Once again using the antisymmetry of the spin connection we can recover ω_{abc} from this relation. Permuting indices and summing like

$$\omega_{[ab]c} - \omega_{[cb]a} + \omega_{[ca]b} = \omega_{a[bc]} + \omega_{b(ac)} + \omega_{c(ab)} = \omega_{abc}, \tag{4.8}$$

we find that

$$\omega_{abc} = e_{[a}{}^\mu e_{b]}{}^\nu \partial_\mu e_{\nu c} - e_{[c}{}^\mu e_{b]}{}^\nu \partial_\mu e_{\nu a} + e_{[c}{}^\mu e_{a]}{}^\nu \partial_\mu e_{\nu b}.$$

Often in calculations one instead keeps the differential forms and uses

$$\omega^a{}_b = dx^\mu \omega_\mu{}^a{}_b = dx^\mu e_\mu{}^c \omega_c{}^a{}_b = e^c \omega_c{}^a{}_b$$

so the torsion free condition becomes

$$de^a = -\omega_c{}^a{}_b \, e^c \wedge e^b = -\omega_{[c}{}^a{}_{b]} \, e^c \wedge e^b = \omega_{[cb]}{}^a \, e^c \wedge e^b. \tag{4.9}$$

One can then apply the exterior derivative to the vielbein forms, read off the components $\omega_{[ab]c}$ and insert them into equation (4.8). A few examples of such calculations are given in appendix E.

4.3.1 Expressing Einstein-Hilbert action in Cartan formalism

In 2+1 dimensions, we can use the totally antisymmetric Levi-Civita symbol to switch between two different dual tensors or tensor valued differential forms, effectively describing the same object using either one or two indices. In particular we define

$$\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}, \quad R^a = \frac{1}{2}\epsilon^{abc}R_{bc} = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c, \tag{4.10}$$

expressions that can both be inverted by multiplying with $-\epsilon_{ade}$. Recall how the Einstein-Hilbert action, equation (4.4), in a 2+1 dimensional matter free environment, is given by

$$\mathcal{S}_{EH}(g_{\mu\nu}) = \frac{1}{16\pi G_N}\int_M d^3xe(R-2\Lambda). \tag{4.11}$$

We now instead want to express it in the Cartan formalism. Using our new one-index differential forms the resulting action is given by

$$S_{EH}(e,\omega) = -\frac{1}{8\pi G_N}\int_M \left[e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c\right) - \frac{\Lambda}{6}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right]. \tag{4.12}$$

We will now show that the Lagrangian of equation (4.11) is equivalent to that of equation (4.12) by showing that the components of (4.12) are equal to those of (4.11). Noting how the expression in the parentheses is R_a, the first part of the Lagrangian (4.12) can be rewritten as

$$e^a \wedge \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c\right) = \frac{1}{2}\epsilon_{abc}e^a \wedge (d\omega^{bc} + \omega^b{}_d \wedge \omega^{dc}) =$$
$$\frac{1}{2}\epsilon_{abc}e^a \wedge R^{bc} = \frac{1}{4}\epsilon_{abc}e_\rho{}^a R_{\mu\nu}{}^{bc}\, dx^\rho \wedge dx^\mu \wedge dx^\nu =$$
$$\frac{1}{4}\epsilon_{abc}e_\rho{}^a \varepsilon^{\mu\nu\rho}R_{\mu\nu}{}^{bc}d^3x = \frac{1}{4}\epsilon_{abc}e_\rho{}^c \varepsilon^{\mu\nu\rho}R_{\mu\nu}{}^{ab}d^3x,$$

now using the fact $\epsilon_{abc}e_\rho{}^c\varepsilon^{\mu\nu\rho} = -e(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)$, where $e = det(e)$, we get

$$-\frac{1}{4}(e_a{}^\mu e_b{}^\nu - e_a{}^\nu e_b{}^\mu)eR_{\mu\nu}{}^{ab}d^3x = -\frac{1}{4}e(R_{\mu\nu}{}^{\mu\nu} - R_{\mu\nu}{}^{\nu\mu})d^3x = -\frac{1}{2}eR\, d^3x.$$

The second component of equation 4.12 can be rewritten as

$$\Lambda\epsilon_{abc}\, e^a \wedge e^b \wedge e^c = dx^\mu \wedge dx^\nu \wedge dx^\rho \Lambda e_\mu{}^a e_\nu{}^b e_\rho{}^c \epsilon_{abc} =$$

$$\Lambda dx^3 \varepsilon^{\mu\nu\rho}e_\mu{}^a e_\nu{}^b e_\rho{}^c \epsilon_{abc} = -6\Lambda d^3xe.$$

Thus the definitions are equivalent to

$$\mathcal{S}_{EH}(g_{\mu\nu}) = \mathcal{S}_{EH}(e,\omega),$$

where all terms are 1-form.

From the Einstein-Hilbert action, equation (4.12), it is possible to extract two equations of motion by either varying the spin connection or the frame field. The principle of stationary action then tells us that $\delta\mathcal{S}$ must be zero during this variation. Varying the spin connection we thus find that

$$\delta\mathcal{S} = \int_M e^a \wedge d\delta\omega_a + \frac{1}{2}\epsilon_{abc}\, e^a \wedge (\delta\omega^b \wedge \omega^c + \omega^b \wedge \delta\omega^c) = 0.$$

Using

$$d(e^a \wedge \delta\omega_a) = de^a \wedge \delta\omega_a - e^a \wedge d\delta\omega_a = 0,$$

we find that $de^a \wedge \delta\omega_a = e^a \wedge d\delta\omega_a$ since the left term is a vanishing boundary term. Inserting this into the variation above we find that

$$\begin{aligned}\delta\mathcal{S} = \int_M de^a \wedge \delta\omega_a + \frac{1}{2}\epsilon_{abc}\, e^a(\delta\omega^b \wedge \omega^c + \omega^b \wedge \delta\omega^c) = \\ = \int_M (de^c + \epsilon_{ab}{}^c\, e^a \wedge \omega^b) \wedge \delta\omega_c = 0.\end{aligned}$$

Noting that the expression inside the parenthesis is the same as the torsion form T_a according to equation (4.6), this equation of motion expresses nothing else than the fact, that the connection has vanishing torsion. If we instead vary the frame fields in equation (4.12) the principle of stationary action tells us that

$$\delta\mathcal{S} = \int_M \delta e^a \wedge \Big(d\omega_a + \frac{1}{2}\epsilon_{abc}\, \omega^b \wedge \omega^c\Big) - \frac{\Lambda}{6}\epsilon_{abc}\, \Big(\delta e^a \wedge e^b \wedge e^c + e^a \wedge \delta e^b \wedge e^c + e^a \wedge e^b \wedge \delta e^c\Big) =$$

$$\int_M \Big(d\omega_a + \frac{1}{2}\epsilon_{abc}\, \omega^b \wedge \omega^c\Big) \wedge \delta e^a - \frac{\Lambda}{2}\epsilon_{abc}\, e^b \wedge e^c \wedge \delta e^a = 0.$$

From this variation we obtain the equations of motion as

$$d\omega_a + \frac{1}{2}\epsilon_{abc}\, \omega^b \wedge \omega^c = \frac{\Lambda}{2}\epsilon_{abc}\, e^b \wedge e^c,$$

which upon comparison with equation (4.7) tells us that

$$R_a = \frac{\Lambda}{2}\epsilon_{abc}\, e^b \wedge e^c.$$

Note how these equations of motion are not dependent on the Cartan formalism since the Einstein-Hilbert action is the same, simply expressed in another fashion. These two equations of motion can be translated back to the original formulation and will still hold.

4.4 Gravity as a gauge theory

Having discussed general relativity in Cartan formalism, we can draw several parallels to the gauge theories we explored in the previous chapter. Recall how the the gauge transformation of the gauge potential A_μ according to equation (3.18) transformed as $A'_\mu = gD_\mu g^{-1}$. If we identify $\Gamma^\rho{}_{\mu\nu}$ as a matrix valued connection

$$\Gamma^\rho{}_{\mu\nu} = (A_\mu)^\rho{}_\nu$$

and the tensor transformation matrices

$$(U)^\rho{}_\sigma = \frac{\partial x'^\rho}{\partial x^\sigma}, \qquad (U^{-1})^\tau{}_\nu = \frac{\partial x^\tau}{\partial x'^\nu}.$$

as gauge transformation matrices, then the transformation of $\Gamma^\rho{}_{\mu\nu}$,

$$\Gamma^\rho{}_{\mu\nu} \to \Gamma'^\rho{}_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}\Gamma^\sigma{}_{\gamma\tau} + \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu},$$

is seen to be exactly a combined coordinate transformation on the space index μ and a gauge transformation:

$$(A_\mu(x))^\rho{}_\nu \to \left(A'_\mu(x')\right)^\rho{}_\nu = \frac{\partial x^\gamma}{\partial x'^\mu}\left(U\left(A_\gamma(x) + \partial_\gamma\right)U^{-1}\right)^\rho{}_\nu.$$

The identification of $\Gamma^\rho{}_{\mu\nu}$ with a connection also enables us to see a clear resemblance of the Riemann curvature tensor and the field strength of the gauge connection since

$$R^\rho{}_{\gamma\mu\nu} = 2\partial_{[\mu}\Gamma^\rho{}_{\nu]\gamma} + 2\Gamma^\rho{}_{[\mu|\sigma|}\Gamma^\sigma{}_{\nu]\gamma} = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^\rho{}_\gamma = (F_{\mu\nu})^\rho{}_\gamma.$$

This is entirely consistent as both of these objects measure the failure of covariant derivatives to commute, we even noted in chapter 3 that the field strength is also known as the curvature. Finally, and of most importance to us, this identification can be inserted into equation (4.5), the relationship between the Christoffel symbol and the spin connection, yielding

$$(A)^\rho{}_\nu = e^\rho{}_a \partial_\mu e^a{}_\nu + e^\rho{}_a \omega_\mu{}^a{}_b e^b{}_\nu.$$

But this is exactly a gauge transformation of $(\omega_\mu)^a{}_b$ using the gauge transformation $e^\rho{}_a$ [15].

Guided by this final property we will now attempt to further extend the similarities between gravity and gauge theories. We will do this by using the spin connection as some of the gauge fields of the gauge connection, but we will also be using the dreibeins for this task. This is significant as they come equipped with a condition not normally seen in gauge theories: they are invertible. Strictly speaking this property is significant enough that the resulting theory can not be considered a true gauge theory, but we will not be bothered. The explicit way we will go about showing the (almost) equivalence of gravity in 2+1 dimensions with a gauge theory is by writing the Einstein-Hilbert action of Cartan formalism as a Chern-Simons action. However, we note that the restriction to 2+1-dimensions is essential as this approach does not work in any other dimensions that 3. This is because we can not replace the Einstein-Hilbert action with a gauge-invariant action [7]. This in turn is partially related to the fact that the duality supplied by the Levi-Civita tensor, $A^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho}A_{\nu\rho}$, only exists in 3 dimensions [18].

4.5 Expressing Einstein-Hilbert action as a Chern-Simons Action

In the previous section we discussed how we want to express Einstein-Hilbert action as a Chern-Simons action. We will do this explicitly in two cases, when the cosmological constant is zero (Minkowski background space) and when it is negative (AdS_3 background space), but one could use the same method for positive cosmological constant too. For the ease of the reader we restate the Cherns-Simons action from equation (3.24):

$$\mathcal{S}_{CS} = \frac{k}{4\pi}\int_M Tr\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right], \tag{4.13}$$

where A is the gauge connection $A = dx^{\mu}A_{\mu}{}^{a}T_a$ and T_a are the generators of the gauge group. With some imagination, perhaps the reader can already see the similarity between this expression and equation (4.12).

We mentioned in the previous section how the roles of the gauge fields will be played by the dreibeins $e_{\mu}{}^{a}$, henceforth named frame fields, and the spin connection $\omega_{\mu}{}^{a}{}_{b}$. The gauge group whose generators we will let these gauge fields correspond to will be the isometry groups of the respective background manifolds. However, this is a problematic choice as these groups are non-compact. There was for a specific reason that we focused on Yang-Mills theory with the compact gauge groups $SU(n)$ in the previous chapter. It turns out that the trace in the gauge Lagrangians will contain terms of differing signs unless the the gauge group is compact. Intuitively this means that there will be energy terms with negative sign, a perplexing feature indeed. More rigorously in the context of quantum theory this non-compactness leads to non-unitarity because of how the negative norm states can propagate locally. However the theory we construct will be free of this problem since they lack any local degrees of freedom, so such states can in fact not propagate [1].

4.5.1 Connection for the Minkowski spacetime

We first demonstrate that the Einstein-Hilbert action in 2+1 dimensions can be written as a Chern-Simons gauge action in Minkowski space. This is because Minkowski space is the easiest case in which the connection between the two actions can be demonstrated, a consequence of how the cosmological constant in this case is zero. The isometry group of Minkowski space that we will be using as the gauge group of our Chern-Simons action is $SO(2,1)$. This group has two sets of generators: the translation generators (P_a) and the rotation/boost generators (M_a). Their Lie algebra is given by

$$\begin{aligned}\left[M_a, M_b\right] =& \epsilon_{ab}{}^{c} M_c,\\ \left[M_a, P_b\right] =& \epsilon_{ab}{}^{c} P_c,\\ \left[P_a, P_b\right] =& 0,\end{aligned}$$

as mentioned in section 2.3.3 and derived in appendix D. We construct the gauge connection by combining the spin connection and frame fields to the Lie generators in the following way

$$A = A^a T_a = e^a P_a + \omega^a M_a,$$

and let the trace relation for the generators be given by

$$Tr\left[P_a M_a\right] = \eta_{ab} \tag{4.14}$$

$$Tr\left[P_a P_b\right] = Tr\left[M_a M_b\right] = 0. \tag{4.15}$$

We can now rewrite the two components of the Chern-Simons action in equation (4.13), by using the Lie algebra and trace relations of the generators. The first part of the action is found to be

$$\begin{gathered} Tr\left[A \wedge dA\right] = Tr\left[\left(e^a P_a + \omega^a M_a\right) \wedge \left(de^b P_b + d\omega^b M_b\right)\right] = \\ e^a \wedge d\omega^b\, Tr\left[P_a M_a\right] + \omega^a \wedge de^b\, Tr\left[M_a P_b\right] = e^a \wedge d\omega_a + \omega^a \wedge de_a = 2e^a \wedge d\omega_a, \end{gathered} \tag{4.16}$$

where we in the last step have performed a partial integration and dropped the vanishing boundary term. Continuing with the second term, we find that is given by

$$Tr\left[A \wedge A \wedge A\right] = Tr\left[\left(e^a P_a + \omega^a M_a\right) \wedge \left(e^b P_b + \omega^b M_b\right) \wedge \left(e^c P_c + \omega^c M_c\right)\right].$$

When expanding this expression using equations (4.14) and (4.15) together with the relationships

$$e^b P_b \wedge e^c P_c = \frac{1}{2} e^b \wedge e^c \left[P_c, P_b\right] = 0,$$

and

$$\omega^a M_a \wedge \omega^b M_b = \frac{1}{2}\omega^a \wedge \omega^b \left[M_a, M_b\right] = \frac{1}{2}\epsilon_{ab}{}^{c}\, \omega^a \wedge \omega^b M_c,$$

we find that most terms can be eliminated. Keeping only terms that does not vanish we find that

$$\begin{gathered} Tr\left[A \wedge A \wedge A\right] = \\ Tr\left[e^a P_a \wedge \omega^b M_b \wedge \omega^c M_c + \omega^a M_a \wedge e^b P_b \wedge \omega^c M_c + \omega^a M_a \wedge \omega^b M_b \wedge e^c P_c\right]. \end{gathered} \tag{4.17}$$

Simplifying the expression above requires a rather lengthy calculation that we, in order to avoid cluttering, has moved to appendix F.1. We simply state the result here: each term is found to be $\frac{1}{2}\epsilon_{abc}\, e^a \wedge \omega^b \wedge \omega^c$. This means that we can write the second term as

$$\frac{2}{3}Tr\left[A \wedge A \wedge A\right] = \epsilon_{abc}\, e^a \wedge \omega^b \wedge \omega^c. \tag{4.18}$$

Inserting equation (4.16) and (4.18) into the Chern-Simons action of equation 4.13, we find that it takes the following form:

$$\mathcal{S}_{CS} = \frac{k}{2\pi}\int_M e^a \wedge d\omega_a + \frac{1}{2}\epsilon_{abc}\, e^a \wedge \omega^b \wedge \omega^c.$$

Upon comparison with equation (4.12) this is seen to be exactly the Einstein-Hilbert action of flat space, $\Lambda = 0$, if we let $k = -\frac{1}{4G_N}$. Therefore the Einstein-Hilbert action is equivalent to a Chern-Simons gauge action in 2+1 dimensions, as promised.

4.5.2 Connection for AdS$_3$

We now repeat the procedure of the previous section for AdS$_3$ instead. This differs from the previous case in two ways. First the cosmological constant is no longer zero and is instead given by $\Lambda = -\frac{1}{l^2} < 0$. Secondly the isometry group of AdS$_3$ is given by $SO(2{,}2)$ with Lie algebra $\mathfrak{so}(2{,}2) \cong \mathfrak{sl}(2;\mathbb{R}) \times \mathfrak{sl}(2;\mathbb{R})$ instead of $SO(2{,}1)$. This means that we can not directly relate the Einstein-Hilbert action with a Chern-Simons action and instead we must form a linear combination of two different Chern-Simon actions

$$\mathcal{S}_{(A,\tilde{A})} = \mathcal{S}_{CS}(A) - \mathcal{S}_{CS}(\tilde{A}). \tag{4.19}$$

The gauge connections of these separate actions will also be separate and non-interacting, in accordance with the isometry group. We form these connections using linear combinations of e and ω to construct two $\mathfrak{sl}(2;\mathbb{R})$ gauge connections [7],

$$A = \left(\omega + \frac{e}{l}\right) \qquad \tilde{A} = \left(\omega - \frac{e}{l}\right)$$

or after explicitly writing out the generators,

$$A = A^a T_a = \left(\omega^a + \frac{e^a}{l}\right) T_a, \qquad \tilde{A} = \tilde{A}^a T_a = \left(\omega^a - \frac{e^a}{l}\right) T_a.$$

The generators T_a are those of $\mathfrak{sl}(2;\mathbb{R})$ with Lie algebra

$$\left[T_a, T_b\right] = \epsilon_{ab}{}^c T_c$$

and we let the representation be such that their trace relation is given by

$$Tr\left[T_a T_b\right] = \frac{1}{2}\eta_{ab}.$$

Using the equations above, we can rewrite the four parts that form the components of $\mathcal{S}_{CS}(A)$ and $\mathcal{S}_{CS}(\tilde{A})$. Starting with the first term of $\mathcal{S}_{CS}(A)$, $A \wedge dA$, we find that

$$A \wedge dA = \left(e^a + \omega^a\right) T_a \wedge \left(de^b + d\omega^b\right) T_b =$$
$$\left(e^a \wedge de^b + e^a \wedge d\omega^b + \omega^a \wedge de^b + \omega^a \wedge d\omega^b\right) T_a T_b.$$

Applying the trace relations to this we find that

$$Tr\left[A \wedge dA\right] = \frac{1}{2}\left(e^a \wedge de_a + 2e^a \wedge d\omega_a + \omega^a \wedge d\omega_a\right). \tag{4.20}$$

The second term is likewise found to be

$$A \wedge A \wedge A = \left(\omega^a + \frac{e^a}{l}\right) T_a \wedge \left(\omega^b + \frac{e^b}{l}\right) T_b \wedge \left(\omega^c + \frac{e^c}{l}\right) T_c =$$
$$\left(\frac{1}{l^3} e^a \wedge e^b \wedge e^c + \frac{3}{l^2} e^a \wedge e^b \wedge \omega^c + \frac{3}{l} e^a \wedge \omega^b \wedge \omega^c + \omega^a \wedge \omega^b \wedge \omega^c\right) T_a T_b T_c,$$

so we determine that

$$Tr\left[A \wedge A \wedge A\right] =$$
$$\epsilon_{abc}\left(\frac{1}{4l^3} e^a \wedge e^b \wedge e^c + \frac{3}{4l^2} e^a \wedge e^b \wedge \omega^c + \frac{3}{4l} e^a \wedge \omega^b \wedge \omega^c + \frac{1}{4}\omega^a \wedge \omega^b \wedge \omega^c\right). \tag{4.21}$$

The calculations for $\mathcal{S}_{CS}(\tilde{A})$ are performed entirely analogous to those of $\mathcal{S}_{CS}(A)$ with the results

$$Tr\left[\tilde{A} \wedge d\tilde{A}\right] = \frac{1}{2}\left(e^a \wedge de_a - 2e^a \wedge d\omega_a + \omega^a \wedge d\omega_a\right), \tag{4.22}$$

and

$$Tr\left[\tilde{A} \wedge \tilde{A} \wedge \tilde{A}\right] =$$
$$\epsilon_{abc}\left(\frac{-1}{4l^3} e^a \wedge e^b \wedge e^c + \frac{3}{4l^2} e^a \wedge e^b \wedge \omega^c - \frac{3}{4l} e^a \wedge \omega^b \wedge \omega^c + \frac{1}{4}\omega^a \wedge \omega^b \wedge \omega^c\right). \tag{4.23}$$

Using the results of equations (4.20), (4.21), (4.22), and (4.23) in the combined action $S_{(A,\tilde{A})}$ of equation (4.19) we find that

$$S_{(A,\tilde{A})} = \frac{1}{2l\pi}\int_M e^a \wedge \left(d\omega^b + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c\right) + \frac{1}{6l^2}\epsilon_{abc} e^a \wedge e^b \wedge e^c =$$
$$\frac{1}{2l\pi}\int_M e^a \wedge \left(d\omega^b + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c\right) - \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c.$$

After comparison with the Einstein-Hilbert action $\mathcal{S}_{EH}$ of equation (4.12) this expression is seen to be the same if we set $k = \frac{-l}{4G_N}$. Thus we have shown that the Einstein-Hilbert action also is equivalent to a Chern-Simons gauge theory when the cosmological constant is negative. The same construction can also be used for dS_3 to show that the equivalence still holds when the cosmological constant is positive, but this requires inserting a cumbersome factor i and we will not do so here.

4.6 Extension to a theory of conformal gravity

We have previously shown that it is possible to connect Chern-Siomons action and Einstein-Hilbert action for anti-de Sitter space and Minkowski space. We will now extend the theory and create a field theory of conformal gravity. This is clearly a different theory than usual gravity, where distances play an important role. However, applications of theories similar to this one exist in many areas of physics, most notably string theory and condensed matter physics. We will create our conformal theory by extending the gauge group of our previous gauge theory of gravity, from the Poincaré group to the conformal group which is isomorphic to SO(3,2). We will then use the condition $F = dA + A \wedge A = 0$ of equation (3.25), the flatness condition of our Chern-Simons gauge action, and see what this entails for our theory.

By using the conformal Lie algebra, derived in appendix D and restated here for the ease of the reader,

$$\begin{aligned} \left[M_a, P_b\right] &= \epsilon_{ab}{}^c P_c, \\ \left[M_a, M_b\right] &= \epsilon_{ab}{}^c M_c, \\ \left[M_a, K_b\right] &= \epsilon_{ab}{}^c K_c, \\ \left[D, P_a\right] &= P_a, \\ \left[P_a, K_b\right] &= -2\epsilon_{ab}{}^c M_c - 2\eta_{ab} D, \end{aligned}$$

we create the gauge connection A with the generators T_a as $A = A^a T_a = e^a P_a + \omega^a M_a + bD + f^a K_a$ where we have used the spin connection and frame fields like before, but also introduced two new fields b and f^a. As before, we use the Chern-Simons action whose equations of motion according to equation (3.25) are the requirement that the connection is flat, $dA + A \wedge A = 0$. Explicitly writing this expression out in terms of the generators T^a we have

$$F^a T_a = dA^a T_a + \frac{1}{2} A^b \wedge A^c \left[T_b, T_c\right] = 0.$$

By inserting the commutator relations from above and separating the flatness condition into its four different parts, one for each generator, we find the following four equations:

$$\begin{aligned} (P_a): &\quad de^a + \epsilon^a{}_{bc}\, e^b \wedge \omega^c - e^a \wedge b = 0 \\ (M_a): &\quad d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\, \omega^b \wedge \omega^c - 2\epsilon^a{}_{bc}\, e^b \wedge f^c = 0 \\ (D): &\quad db - 2\eta_{ab}\, e^a \wedge f^b = 0 \\ (K_a): &\quad df^a + \epsilon^a{}_{bc}\, \omega^b \wedge f^c + f^a \wedge b = 0. \end{aligned}$$

Now it is tempting to choose a particularly simple gauge in order to solve these equations. The obvious choice is to set $b = 0$, but we know that we have an additional requirement that is not common to gauge theories: the frame fields must be invertible. Additionally there are subtle issues involving gauge transformations that are not simply connected to the identity with non-zero winding number, destroying the invariance of the action. However by forming infinitesimal gauge transformation and using the invertibility of the frame fields it is determined that there exists a family of gauge transformations that satisfy the invertibility condition and allow us to set $b = 0$ [19]. With this gauge the equations above take the form

$$(P_a): \quad de^a + \epsilon^a{}_{bc}\, e^b \wedge \omega^c = 0 \tag{4.24}$$

$$(M_a): \quad d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\, \omega^b \wedge \omega^c - 2\epsilon^a{}_{bc}\, e^b \wedge f^c = 0 \tag{4.25}$$

$$(D): \quad 2e^a \wedge f_a = 0 \tag{4.26}$$

$$(K_a): \quad df^a + \epsilon^a{}_{bc}\, \omega^b \wedge f^c = 0 \tag{4.27}$$

Recalling the expressions for the torsion form in equation (4.6), we see that equation (4.24) is nothing but the torsion free condition. Similarly upon comparison with (4.7) we see that the first part of equation (4.25) is the curvature form expressed in one-index form, $R^a = \frac{1}{2}\epsilon^a{}_{bc}R^{bc}$. This equations thus states that

$$R^a = 2\epsilon^a{}_{bc} e^b f^c.$$

Expanding this expression into its full form,

$$\epsilon^a{}_{bc} R_{\mu\nu}{}^{bc}\, dx^\mu \wedge dx^\nu = 2\epsilon^a{}_{bc} e_\mu{}^b f_\nu{}^c\, dx^\mu \wedge dx^\nu,$$

and then extracting the components from the differential forms and dropping $\epsilon^a{}_{bc}$ we arrive at

$$R_{\mu\nu}{}^{bc} = 8 e_{[\mu}{}^{[b} f_{\nu]}{}^{c]}.$$

We can actually use this equation to express $f_{\mu\nu}$ in terms of the metric $g_{\mu\nu}$. In order to this we first calculate the Ricci tensor, by contracting ν to c using $e_c{}^\nu$ as

$$R_\mu{}^b = 2(e_\mu{}^b f_c{}^c - e_c{}^b f_\mu{}^c - e_\mu{}^c f_c{}^b + e_c{}^c f_\mu{}^b)$$

which simplifies to

$$R_\mu{}^b = 2(e_\mu{}^b f_\rho{}^\rho + f_\mu{}^b)$$

since $e_c{}^b = \delta_c{}^b$ and $e_c{}^c = 3$ since we are in three dimensions. As the frame fields are invertible, we can multiply both sides with the inverse frame fields $e_b{}^\nu$ and then lower ν resulting in the following expression for the Ricci tensor,

$$R_{\mu\nu} = 2(f_{\mu\nu} + g_{\mu\nu} f_\rho{}^\rho). \tag{4.28}$$

Furthermore upon contraction of this expression it is found that that the Ricci scalar curvature is given by

$$R = 2(f_\rho{}^\rho + 3f_\rho{}^\rho) = 8f_\rho{}^\rho.$$

Reinserting this expression for $f_\rho{}^\rho$ into equation (4.28) and rearranging we find that

$$f_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - g_{\mu\nu}\frac{R}{4}\right). \tag{4.29}$$

The expression $R_{\mu\nu} - g_{\mu\nu}\frac{R}{4}$ is commonly know as the Schouten tensor, the traceless part of the Ricci tensor $R_{\mu\nu}$. As a side note we find that f is symmetric since dropping the forms from equation (4.26) means that

$$e_{[\mu}{}^{a} f_{\nu]a} = f_{[\mu\nu]} = 0.$$

Finally we see that (4.27) is the same as the exterior covariant derivative applied to f^c,

$$Df^c = 0.$$

As before we remove the differential forms and extract the components. After this procedure we find that this condition is equivalent to

$$D_{[\mu} f_{\nu]}{}^{a} = D_{[\mu}(R_{\nu]}{}^{\rho} - \frac{1}{4}\delta_{\nu]}{}^{\rho} R) = 0.$$

Multiplying with $\varepsilon^{\mu\nu\sigma}$ yields,

$$C^{\sigma\rho} = \varepsilon^{\mu\nu\sigma} D_{\mu}(R_{\nu}{}^{\rho} - \frac{1}{4}\delta_{\nu}{}^{\rho} R) = 0.$$

where $C^{\sigma\rho}$ is the so called Cotton tensor. Unlike Einsteins equations that consists of second order derivatives this is one of third order, because the metric consists of second order derivatives. An important aspect of the Cotton tensor is that it is the three dimensional analogue of an object called the Weyl tensor, an object that can be defined in dimensions higher than four. The vanishing of the respective object is a sufficient condition for the manifold of the theory to be conformally flat. Therefore, this result was expected as our theory lacks any coupled matter that could break the conformal flatness.

4.7 A word of caution

We have in this chapter studied the similarities between gauge theory and gravity, using an explicit construction with a Chern-Simons Lagrangian. In this construction we extended the Lorentz symmetry of the tangent space to a local symmetry as our gauge symmetry. However, general relativity is a much more symmetric theory than this as it is completely diffeomorphism invariant, a larger class of symmetries. Therefore we really should be using the diffeomorphism group as our gauge group, but unfortunately such a procedure is fraught with conceptual problems. First its Lie algebra consists of an infinite amount of generators and secondly it is a non-compact group. Both of these concerns are minor however, in comparison with the problems that arise during its quantisation. In the usual procedure of quantisation, spacetime itself becomes quantised. This results in fluctuating light-cones and thus leads to complications with causality, so we leave it be.

Even in the simple 2+1-dimensional case we have considered in this chapter there are subtle but important differences, which means that our theory in the strictest sense is not a gauge theory. We used the frame fields as gauge fields, but these come equipped with an extra condition of invertibility that is absent in usual gauge theory.

If were to attempt to quantise this theory like a normal gauge theory, this additional requirement is severely hindering. The canonical method for quantising a theory relies heavily on the important limit where the gauge fields approach zero. This is a limit that the invertibility condition clearly renders impossible. We will not be considering these issues further as we will be mostly be considering classical theories and only briefly discuss quantisation [6].

5

Higher Spin and Singularity Resolution

In modern physics the study of singularities and their resolution is of great importance. General relativity in its classical formulation is diffeomorphism invariant. This is problematic as diffeomorphisms preserve spacetime singularities. Therefore singularities are also invariants of the theory. This is a prime indicator that general relativity needs to be modified at short distances. However, this inability to resolve singularities is not shared with string theory where there exists several different ways to eliminate singularities. This comes at a cost however, since string theory, in its analytically manageable form, is intimately tied to the concept of supersymmetry. The only backgrounds that support supersymmetry are those that are time independent. Thus a resolution of cosmological singularities like the Big Bang, which inherently are time dependent, is difficult to achieve using this method [20].

Instead of attempting to use string theory directly to resolve singularities we will instead resume where we left off in the previous chapter. There we were able to formulate AdS-gravity in 2+1 dimensions as a Chern-Simons gauge theory with the gauge group $SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$. We now want to extend this theory to one of Chern-Simons gravity coupled to higher spins. After having developed this theory we want to use its gauge freedom in order to gauge away troublesome singularities, which then would be nothing more than gauge artifacts of our theory. At first glance this seems like an entirely unrelated approach,from string theory, but in fact there exists a link between the two methods. This link comes in the speculative belief that the tensionless limit $T \to 0$ of string theory is a theory of higher spin. Since general relativity is the other limit of string theory where the tension instead tends to infinity, these theories are all related which provides ample motivation for this approach [1, 21].

5.1 Higher spin for AdS$_3$

In the previous chapter, we discovered that the Einstein-Hilbert action in $2+1$ dimensions could be expressed as a Chern-Simons action. For the AdS$_3$ case, where $\Lambda = -1/l^2 < 0$, we explicitly constructed this action, in section 4.5.2, by forming two gauge potentials A and $\tilde{A}$ from the frame field and spin connections as

$$A = \omega + \frac{e}{l} = dx^\mu \left(\omega_\mu{}^a + \frac{1}{l} e_\mu{}^a \right) T_a,$$

$$\tilde{A} = \omega - \frac{e}{l} = dx^\mu \left({\omega_\mu}^a - \frac{1}{l} {e_\mu}^a \right) T_a,$$

where T_a are the generators of $SL(2;\mathbb{R})$ with Lie algebra

$$[T_a, T_b] = {\epsilon_{ab}}^c T_c.$$

Now we wish to extend this theory of spin 2 to one with higher spin. We can do this by extending the gauge group from $SL(2;\mathbb{R}) \times SL(2;\mathbb{R})$ to $SL(n;\mathbb{R}) \times SL(n;\mathbb{R})$ which couples spin 2 gravity to all spins from 3 up to n. In this particular case we will set $n = 3$, but the process is entirely analogous for higher n. By performing this extension we are forced to include five additional symmetric and traceless generators T_{ab}. Our new Lie algebra with these generators takes the form

$$\begin{aligned}
\Big[T_a, T_b\Big] &= {\epsilon_{ab}}^c T_c, \\
\Big[T_a, T_{bc}\Big] &= {\epsilon^d}_{a(b} T_{c)d}, \\
\Big[T_{ab}, T_{cd}\Big] &= -\left(\eta_{a(c} {\epsilon_{d)b}}^e + \eta_{b(c} {\epsilon_{d)a}}^e \right) T_e,
\end{aligned} \tag{5.1}$$

which indeed is the Lie algebra of $SL(3;\mathbb{R})$ [22]. Proceeding to $SL(3;\mathbb{R}) \times SL(3;\mathbb{R})$ we form new gauge connections A and $\tilde{A}$ as

$$\begin{aligned}
A &= dx^\mu \left({\omega_\mu}^a + \frac{{e_\mu}^a}{l} \right) T_a + dx^\mu \left({\tilde{\omega}_\mu}^{\ bc} + \frac{{e_\mu}^{bc}}{l} \right) T_{bc} \equiv \omega + \frac{e}{l}, \\
\tilde{A} &= dx^\mu \left({\omega_\mu}^a - \frac{{e_\mu}^a}{l} \right) T_a + dx^\mu \left({\tilde{\omega}_\mu}^{\ bc} - \frac{{e_\mu}^{bc}}{l} \right) T_{bc} \equiv \omega - \frac{e}{l},
\end{aligned} \tag{5.2}$$

where we have expressed the new degrees of freedom of this theory in the generalised frame field ${e_\mu}^{ab}$ and spin connection ${\tilde{\omega}_\mu}^{\ ab}$.

Using these new gauge potentials we will now investigate how the action has changed by increasing the degrees of freedom. As in spin 2, section 4.5.2, the Chern-Simons action for $SL(3;\mathbb{R}) \times SL(3;\mathbb{R})$ is

$$\mathcal{S}_{(A,\tilde{A})} = \mathcal{S}_{CS}[A] - \mathcal{S}_{CS}\left[\tilde{A}\right].$$

By expressing the gauge potentials as done in equation (5.2) it is possible to write the action as

$$\mathcal{S}_{(A,\tilde{A})} = \frac{k}{2\pi} Tr \int_M \left[e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e \right],$$

where $e = e^a T_a + e^{bc} T_{bc}$. To proceed from here we need to express the actions in terms of the frame fields, e^a and e^{ab}, as well as the spin connections, ω^a and $\tilde{\omega}^{ab}$. We do this by using a set of trace relations, which are given by [1]

$$\begin{aligned}
Tr\left[T_a T_b\right] &= 2\eta_{ab}, \\
Tr\left[T_a T_{bc}\right] &= 0, \\
Tr\left[T_{ab} T_{cd}\right] &= -\frac{4}{3}\eta_{ab}\eta_{cd} + 2\left(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}\right).
\end{aligned} \tag{5.3}$$

As the calculations are quite long, we calculate each of the term individually. Starting with $Tr\,[e \wedge R]$, we write the curvature form as[1]

$$
\begin{gathered}
R = d\omega + \omega \wedge \omega = \\
d\omega^a T_a + d\tilde{\omega}^{ab} T_{ab} + \frac{1}{2}\omega^a \wedge \omega^b \Big[T_a, T_b\Big] + \frac{1}{2}\tilde{\omega}^{ab} \wedge \tilde{\omega}^{cd} \Big[T_{ab}, T_{cd}\Big] + \omega^a \wedge \tilde{\omega}^{bc} \Big[T_a, T_{bc}\Big] = \\
\left(d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\, \omega^b \wedge \omega^c - \frac{1}{2}\left(\eta_{d(f}\epsilon_{g)e}{}^a + \eta_{e(f}\epsilon_{g)d}{}^a\right) \tilde{\omega}^{de} \wedge \tilde{\omega}^{fg}\right) T_a \\
+ \left(d\tilde{\omega}^{ab} + \epsilon^b{}_{hi}\, \omega^h \wedge \tilde{\omega}^{ia} + \epsilon^b{}_{jk}\, \omega^k \wedge \tilde{\omega}^{aj}\right) T_{ab} = \\
\left(R^a - \eta_{d(f}\epsilon_{g)e}{}^a\, \tilde{\omega}^{de} \wedge \tilde{\omega}^{fg}\right) T_a + D\tilde{\omega}^{ab} T_{ab},
\end{gathered}
$$

which has been simplified using the fact that the generalised frame field is symmetrical. Using this fact, the trace relationships, and equations (5.3), we can express the first term as

$$
Tr\,[e \wedge R] = 2e^a \wedge \left(R_a - \left(\eta_{d(f}\epsilon_{g)ea}\right) \tilde{\omega}^{de} \wedge \tilde{\omega}^{fg}\right) + 4e^{ab} \wedge D\tilde{\omega}_{ab}. \tag{5.4}
$$

Continuing with the second term we write out the combined frame field e as its components which results in that it can be expressed as

$$
Tr\,[e \wedge e \wedge e] = Tr\left[\left(e^a T_a + e^{de} T_{de}\right) \wedge \left(e^b T_b + e^{fg} T_{fg}\right) \wedge \left(e^c T_c + e^{hi} T_{hi}\right)\right].
$$

Proceeding with expanding this equation and removing the terms that are zero, using the Lie algebra given in equations (5.1) and the trace relations given in equations (5.3), we find that only the following four terms remain

$$
\begin{aligned}
Tr\,[e \wedge e \wedge e] = Tr\Big[&e^a T_a \wedge e^b T_b \wedge e^c T_c + e^a T_a \wedge e^{bc} T_{bc} \wedge e^{de} T_{de} \\
&+ e^{bc} T_{bc} \wedge e^a T_a \wedge e^{de} T_{de} + e^{bc} T_{bc} \wedge e^{de} T_{de} \wedge e^a T_a\Big].
\end{aligned} \tag{5.5}
$$

We will only perform the calculation for the first part of the expression $e^a T_a \wedge e^b T_b$ as most of these calculations are longer and more complex, instead these are presented in appendix F.2 for the curious reader to investigate. Using equation (5.1) we see that

$$
e^a T_a \wedge e^b T_b = \frac{1}{2}[T_a, T_b] e^a \wedge e^b = \frac{1}{2}\epsilon_{ab}{}^c\, e^a \wedge e^b\, T_c, \tag{5.6}
$$

which means that we can simplify the first term to

$$
Tr\left[e^a T_a \wedge e^b T_b \wedge e^c T_c\right] = \frac{1}{2}\epsilon_{ab}{}^d\, e^a \wedge e^b \wedge e^c\, Tr\,[T_c T_d] = \epsilon_{abc}\, e^a \wedge e^b \wedge e^c.
$$

Combining this result with that of those in appendix F.2 we find that the second term can be expressed as

$$
Tr\,[e \wedge e \wedge e] = \epsilon_{abc}\, e^a \wedge e^b \wedge e^c + \left(2\epsilon_{bcd}\eta_{ae} + \epsilon_{bad}\eta_{ce} - \epsilon_{eca}\eta_{bd}\right) e^a \wedge e^{bc} \wedge e^{de}. \tag{5.7}
$$

[1]In the following equation D is the spin 2 exterior covariant derivative $D = d + \omega\wedge$

Using these results, equations (5.7) and (5.5), we can formulate the action in terms of the frame fields and the spin connections like

$$\mathcal{S}_{(A,\tilde{A})} = \frac{k}{2\pi} Tr \int_M \left[e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e \right] =$$

$$\frac{k}{\pi} \int_M e^a \wedge R_a + \frac{1}{6l^2} \left(\epsilon_{abc}\, e^a \wedge e^b \wedge e^c \right) - \eta_{d(f}\epsilon_{g)ea}\, e^a \wedge \tilde{\omega}^{de} \wedge \tilde{\omega}^{fg}$$

$$+2e^{ab} \wedge D\tilde{\omega}_{ab} + \frac{1}{6l^2} \left(2\epsilon_{bcd}\eta_{ae} + \epsilon_{bad}\eta_{ce} - \epsilon_{eca}\eta_{bd} \right) e^a \wedge e^{bc} \wedge e^{de}.$$

If we set $k = \frac{-1}{8G_N}$ and $\frac{1}{l} = -\Lambda$, since we are working in AdS_3, we see that the two first terms are equal to the Einstein-Hilbert action, as seen in equation (4.12), whilst the other terms occur due to the extension to higher spin. We finally find

$$\mathcal{S}_{(A,\tilde{A})} = \frac{-1}{8\pi G_N} \int_M e^a \wedge R_a - \frac{\Lambda}{6} \epsilon_{abc}\, e^a \wedge e^b \wedge e^c - \eta_{d(f}\epsilon_{g)ea}\, e^a \wedge \tilde{\omega}^{de} \wedge \tilde{\omega}^{fg}$$

$$-\frac{\Lambda}{6} \left(2\epsilon_{bcd}\eta_{ae} + \epsilon_{bad}\eta_{ce} - \epsilon_{eca}\eta_{bd} \right) e^a \wedge e^{bc} \wedge e^{de} + 2e^{ab} \wedge D\tilde{\omega}_{ab}.$$

5.2 Resolution of the Milne singularity using higher spin

In the previous section we developed a gauge theory of gravity coupled to spin 3, with the purpose of being able to gauge away singularities. In this section we will do this for the Milne singularity, which can be seen as a simple model of the Big Bang in 2+1 dimensions. Therefore it is interesting if we could use the gauge freedom of our theory to resolve it. In $2+1$ dimensions the Milne orbifold has the metric

$$ds^2 = -dT^2 + r_c^2 dX^2 + \alpha^2 T^2 d\phi^2,$$

where ϕ and X takes values between 0 and 2π. As this model in addition can be related to a black hole, the parameters α and r_c can be related to its mass M and spin J like [20]

$$\alpha = \sqrt{M} \quad \text{and} \quad r_c = \sqrt{\frac{J^2}{4M}}.$$

The spacetime described by the Milne metric above has a singularity at $T = 0$ where the ϕ circle shrinks to a single point. In its entirety this spacetime appears like a double cone with the ends meeting at the singularity at $T = 0$. Spacetime thus gradually shrinks as time progresses until it collapses into a single point before it reverts and begin to expand again in a fashion similar to the Big Bang.

However, before applying our gauge theory to the manifold we wish to explore it a bit further. Using what we learned of Cartan formalism in section 4.3 we find that the dreibein one-forms are given by

$$e^0 = dT \quad e^1 = r_c dX \quad e^2 = \alpha T d\phi,$$

from which in turn we determine that

$$de^0 = 0, \qquad de^1 = 0, \qquad de^2 = \alpha\, dT \wedge d\phi = \frac{1}{T}\, e^0 \wedge e^2. \tag{5.8}$$

Inserting these expressions into the torsion free condition

$$T^a = de^a + {\omega^a}_b \wedge e^b = 0,$$

that according to equation (4.9) is the same as

$$de^a = e^b \wedge e^c {\omega_{[bc]}}^a,$$

we can determine the spin connections from the dreibein one-forms. Simply reading off the components of equation (5.8), remembering that the same component appears twice on the right hand side, we find that

$${\omega_{[bc]}}^0 = 0 \qquad {\omega_{[bc]}}^1 = 0 \qquad {\omega_{[02]}}^2 = \frac{1}{2T} = \omega_{[02]2}.$$

Using the antisymmetry of the spin connection in its final two indices allows us to now determine the spin connection from

$$\omega_{abc} = \omega_{[ab]c} - \omega_{[bc]a} + \omega_{[ca]b},$$

so we find that the only non-zero independent component is

$$\omega_{220} = \omega_{[22]0} - \omega_{[20]2} + \omega_{[02]2} = 2\omega_{[02]2} = \frac{1}{T}.$$

Using $\omega_{20} = e^2\omega_{220} = \alpha d\phi$ we now form the spin connection one forms with one index,

$$\omega^1 = \varepsilon^{120}\omega_{20} = \omega_{20} = \alpha d\phi, \qquad \omega^0 = \omega^2 = 0.$$

Inserting this into the expression for the Riemann curvature form of equation (4.10),

$$R^a = d\omega^a + \frac{1}{2}{\epsilon^a}_{bc}\omega^b \wedge \omega^c,$$

shows that this is identically zero. Thus the Riemann tensor must be too and the Milne orbifold, apart from its singularity at $T = 0$, must be flat.

The flatness of this space is a problem for us since we developed our gauge theory of gravity coupled to spin 3 on the AdS_3 setting where there is a constant negative curvature. As such we must modify our gauge theory in order to reapply it to the flat setting without curvature. We do this by introducing a Grassmann number ϵ and substituting this into our theory like

$$\frac{1}{l} \to \epsilon,$$

while demanding ϵ to satisfy $\epsilon^2 = 0$ [1]. This substitution changes the original Chern-Simons action by changing the gauge connections A and $\tilde{A}$. Calling these connections A^+ and A^- respectively, they now take the form

$$A^{\pm} = (\omega^a \pm \epsilon e^a)\, T_a = \pm(\epsilon dT)\, T_T + (\alpha d\phi \pm \epsilon\, r_c dX)\, T_X \pm (\epsilon\alpha T d\phi)\, T_\phi.$$

Now we extend these connections to our spin 3 theory using the generalised frame fields ${e_\mu}^{ab}$ and spin connections ${\tilde{\omega}_\mu}^{ab}$ in order to determine if it is possible to resolve the singularity with a suitable gauge. Actually we will use a different basis for the generators T_{ab}, W_n, $\quad n = -2, ..., 2$. These are related to each other through

$$\begin{aligned} T_{00} &= \tfrac{1}{4}(W_2 + W_{-2} + 2W_0), & T_{01} &= \tfrac{1}{4}(W_2 - W_{-2}), \\ T_{11} &= \tfrac{1}{4}(W_2 + W_{-2} - 2W_0), & T_{02} &= \tfrac{1}{2}(W_1 + W_{-1}), \\ T_{22} &= W_0, & T_{12} &= \tfrac{1}{2}(W_1 - W_{-1}). \end{aligned}$$

This change of basis is performed as the new basis parametrises the independent components of the traceless generators T_{ab} as the traceless condition

$$-T_{00} + T_{11} + T_{22} = 0$$

is obviously satisfied [22]. We denote the gauge fields, composed of the generalised frame fields and spin connections, for these generators by C^n and D^n and the spin 3 gauge connections of equation (5.2) now take the following form

$$A'^{\pm} = A^{+} + \sum_{n=2}^{2} (C^n \pm \epsilon D^n)\, W_n.$$

Proceeding from this point we simplify and assume that the higher spin fields are functions of T, independent of X and ϕ. The metric is given by

$$g_{\mu\nu} = Tr\left[e_\mu e_\nu\right],$$

which upon inclusion of the higher spin fields changes the original Milne metric to

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{4}{3} D^0_\mu D^0_\nu - 2D^1_\mu D^{-1}_\nu - 2D^{-1}_\mu D^1_\nu + 8D^2_\mu D^{-2}_\nu + 8D^{-2}_\mu 8D^2_\nu.$$

We now want to choose a simplifying gauge, but there are holonomic restrictions that must be considered. We will not study these in depth, and simply note that upon choosing $C^n_\mu = 0$ as an initial gauge, the holonomic constraints combined with the flatness condition of our action results in $D^0_\phi = 3(D^2_\phi + D^{-2}_\phi)$. We set $D^0_\phi = 3D^2_\phi$ and let all other fields be zero, resulting in the $\phi\phi$-component of the metric being only one that changes [20],

$$g'_{\phi\phi} = g_{\phi\phi} + 12(D^2_\phi),$$

$$ds'^2 = -dT^2 + r_c^2 dX^2 + \left(\alpha^2 T^2 + 12(D^2_\phi)\right) d\phi^2.$$

With this choice of gauge we have thus managed to change the metric and all that remains is to determine if this change has eliminated the singularity. Calculating the curvature as before using the frame fields we find that e^2 changes to e'^2,

$$e'^2 = \sqrt{\alpha^2 T^2 + 12(D^2_\phi)}\, d\phi.$$

and

$$de'^2 = \frac{\alpha^2 T}{\sqrt{\alpha^2 T^2 + 12(D^2_\phi)}}\, dT \wedge d\phi = \frac{\alpha^2 T}{\alpha^2 T^2 + 12(D^2_\phi)}\, e'^0 \wedge e'^2,$$

while remaining components are unchanged, $e'^0 = e^0$, $e'^1 = e^1$, $de'^0 = de^0 = 0$ and $de'^1 = de^1 = 0$. Just as before we included spin 3 to our theory, we find that there is only one independent component of the spin field,

$$\omega'_{220} = \frac{\alpha^2 T}{\alpha^2 T^2 + 12(D_\phi^2)}.$$

This in turn means that

$$\omega'^1 = \epsilon^{120}\omega'_{20} = e'^2\omega'_{220} = \frac{\alpha^2 T}{\alpha^2 T^2 + 12(D_\phi^2)}e'^2.$$

By applying the exterior derivative on ω'^1 we get

$$d\omega'^1 = d\left(\frac{\alpha^2 T}{\alpha^2 T^2 + 12(D_\phi^2)}e'^2\right) =$$

$$\frac{\partial}{\partial T}\left(\frac{\alpha^2 T}{\alpha^2 T^2 + 12(D_\phi^2)}\right)e'^0 \wedge e'^2 + \frac{\alpha^2 T}{\alpha^2 T^2 + 12(D_\phi^2)}e'^0 \wedge \frac{\partial}{\partial T}e'^2,$$

which means that

$$d\omega'^1 = \alpha^2\left(\frac{\alpha T^2 + 12(D_\phi^2) - 2\alpha T^2 + \alpha T^2}{\left(\alpha^2 T^2 + 12(D_\phi^2)\right)^2}\right)e'^0 \wedge e'^2.$$

From the expression above we can see that the curvature tensor for the Milne metric in spin-3 with our gauge is equal to

$$R'^0 = 0 \quad , \quad R'^1 = d\omega'^1 = \frac{12D^2(\phi)\alpha^2}{(\alpha^2T^2 + 12D^2(\phi))^2}e'^0 \wedge e'^2 \quad , \quad R'^2 = 0.$$

We can now calculate the Ricci scalar $(R)^2$, as defined in equation (4.2), using the fact that $R'^1 = \epsilon^{120}R'_{20} = R'_{20}$ and

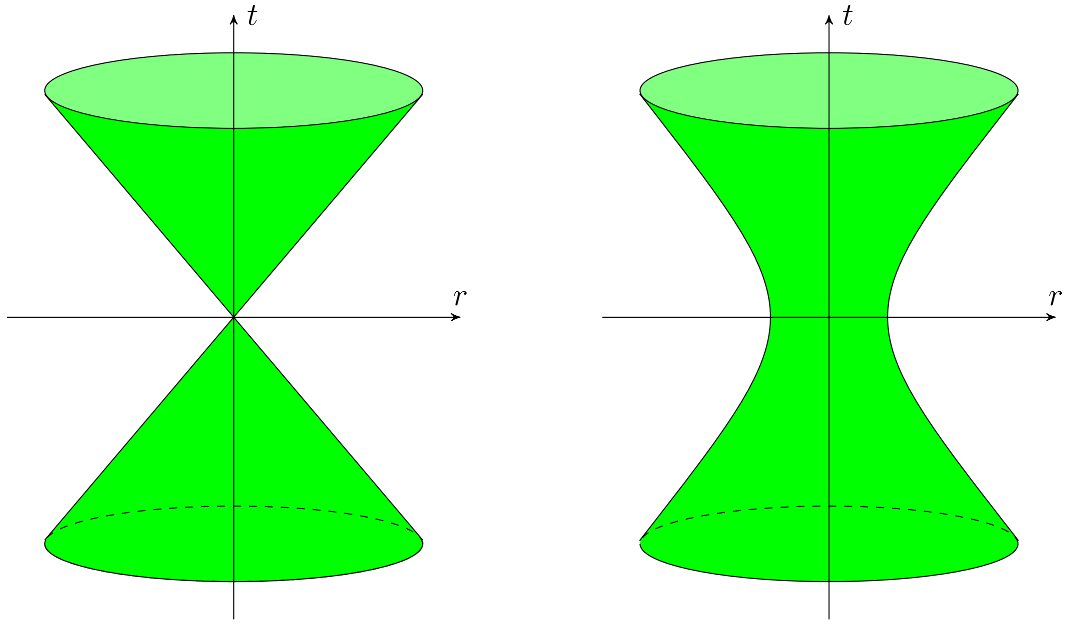
$$R'_{20} = R'_{2002}\, e^0 \wedge e^2 = -R'_{2020}e^0 \wedge e^2 = R'^{20}{}_{20}e^0 \wedge e^2.$$

As $R'^{20}{}_{20} = R'^{02}{}_{02}$ is the only independent non-zero component upon forming the Ricci scalar as $R^{ab}{}_{ab}$ this term appears twice whilst no others do, resulting in

$$R' = \frac{24D^2(\phi)\alpha^2}{(\alpha^2T^2 + 12D^2(\phi))^2}.$$

Thus the geometry has become smooth at $T = 0$ with a finite curvature scalar instead of crunching to a point like before. As all the necessary constraints were fulfilled this gauge has preserved all the symmetries of the Milne orbifold. This means that the singularity in this theory is nothing but a gauge artifact.

[2]The contractions can be performed using flat indices just as well as with curved ones.



(a) A representation of the Milne orbifold where we can clearly see how the metric has a singularity at $T = 0$
.

(b) After selecting a suitable gauge in a theory of gravity coupled to spin 3, the singularity is resolved and the manifold is seen to become smooth with finite curvature at $T = 0$.

Figure 5.1: The Milne orbifold is flat apart from at $T = 0$ where it crunches into a singularity. However, in a gauge theory of gravity coupled to spin 3 this is seen to be nothing but a gauge artifact and the singularity can be gauged away with a suitable gauge.

5.3 Conformal higher spin and its quantisation

In the previous section we saw how successful the Chern-Simons gauge theory version of gravity was by extending it to include higher spin. Guided by this we now want to repeat this procedure for our theory of conformal 2+1 dimensional gravity, from section 4.6. For AdS_3-gravity we included higher spin by replacing the AdS_3 isometry Lie group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ with $SL(n,\mathbb{R}) \times SL(n,\mathbb{R})$. However, we can not repeat this procedure for the conformal group as it is not possible to divide it into smaller, non-interacting parts which can be extended as naturally as $SL(2,\mathbb{R})$ could. Therefore we must find another way to include higher spin in this theory.

The way we choose to add higher spin fields into our conformal theory of gravity is guided by a specific representation of the conformal Lie algebra. This representation is given by [6],

$$
\begin{aligned}
M^a &= -\frac{1}{2}(\gamma^a)_\alpha^\beta\, q^\alpha p_\beta, \\
P^a &= -\frac{1}{2}(\gamma^a)_{\alpha\beta}\, q^\alpha q^\beta, \\
K^a &= -\frac{1}{2}(\gamma^a)^{\alpha\beta}\, p_\alpha p_\beta, \\
D &= -\frac{1}{2} q^\alpha p_\alpha,
\end{aligned} \tag{5.9}
$$

where the phase space variable q^α and p_α are the classical, canonical position and momentum variables and γ^μ are the gamma matrices that can be expressed in terms of the Pauli matrices

$$
\gamma^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.10}
$$

which satisfy $\gamma^0\gamma^1\gamma^2 = 1$. The matrix indices α, β of the gamma matrices are spinorial and hence they are raised and lowered in a peculiar fashion. This is done with the matrices $\epsilon^{\alpha\beta} = \gamma^0$ and its inverse $\epsilon_{\alpha\beta}$ defined by $\epsilon_{\alpha\beta}\epsilon\beta\gamma = \delta_\alpha^\gamma$. The spinorial indices are then raised or lowered from the left or right for the first and second index respectively[23]. With this representation we furthermore replace the commutator brackets of our previous Lie algebra representations with the Poisson bracket

$$
\{f(p,q), g(p,q)\}_{PB} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.
$$

This expression, described in greater detail in Appendix C, obeys the canonical Poisson bracket relations

$$
\{q^\alpha, p_\beta\}_{PB} = \delta^\alpha_\beta, \quad \{q^\alpha, q_\beta\}_{PB} = 0, \quad \{p^\alpha, p_\beta\}_{PB} = 0. \tag{5.11}
$$

We have henceforth dropped the notation PB on the brackets for simplicity.

In order to demonstrate that this indeed is a representation of the conformal Lie algebra we show the equivalence of the most complicated relation

$$
\left[P^a, K^b\right] = -2\eta^{ab}D - 2\varepsilon^{ab}{}_c M^c \;\Rightarrow\; \{P^a, K^b\} = -2\eta^{ab}D - 2\varepsilon^{ab}{}_c M^c. \tag{5.12}
$$

To start the calculations we simply insert the definitions of the generators P^a and K^b into $\{P^a, K^b\}$,

$$
\{P^a, K^b\} = \{-\frac{1}{2}(\gamma^a)_{\alpha\beta}\, q^\alpha q^\beta, -\frac{1}{2}(\gamma^b)^{\kappa\lambda}\, p_\kappa p_\lambda\},
$$

and then move the constant γ-matrices outside the brackets, resulting in

$$
\{P^a, K^b\} = \frac{1}{4}(\gamma^a)_{\alpha\beta}\, (\gamma^b)^{\kappa\lambda}\, \{q^\alpha q^\beta, p_\kappa p_\lambda\}. \tag{5.13}
$$

Using

$$
\{AB, CD\} = A\{B, C\}D + AC\{B, D\} + \{A, C\}DB + C\{A, D\}B,
$$

we now find that

$$\{q^\alpha q^\beta, p_\kappa p_\lambda\} = q^\alpha\{q^\beta, p_\kappa\}p_\lambda + q^\alpha p_\kappa\{q^\beta, p_\lambda\} + \{q^\alpha, p_\kappa\}p_\lambda q^\beta + p_\kappa\{q^\alpha, p_\lambda\}q^\beta, \quad (5.14)$$

which can be simplified by using the canonical Poisson bracket relations of equation (5.11),

$$\{q^\alpha q^\beta, p_\kappa p_\lambda\} = q^\alpha p_\lambda \delta^\beta_\kappa + q^\alpha p_\kappa \delta^\beta_\lambda + p_\kappa q^\beta \delta^\alpha_\kappa + p_\kappa q^\beta \delta^\alpha_\lambda.$$

Inserting this expression into equation (5.13) we find that

$$\{P^a,K^b\} = \frac{1}{4}(\gamma^a)_{\alpha\beta}(\gamma^b)^{\kappa\lambda}(q^\alpha p_\lambda \delta^\beta_\kappa + q^\alpha p_\kappa \delta^\beta_\lambda + p_\kappa q^\beta \delta^\alpha_\kappa + p_\kappa q^\beta \delta^\alpha_\lambda),$$

where the first term is easily seen to be

$$(\gamma^a)_{\alpha\beta}(\gamma^b)^{\beta\lambda}\, q^\alpha p_\lambda.$$

In order to simplify this further we perform the matrix multiplication $(\gamma^a)_{\alpha\beta}(\gamma^b)^{\beta\lambda}$. This multiplication produces a symmetric and an antisymmetric term like

$$(\gamma^a)_{\alpha\beta}(\gamma^b)^{\beta\lambda} = 2(\eta^{ab}(\delta)^\lambda_\alpha + \varepsilon^{ab}{}_c(\gamma^c)^\lambda_\alpha),$$

because

$$([\gamma^a,\gamma^b])^\beta_\alpha = 2\varepsilon^{ab}{}_c(\gamma^c)^\beta_\alpha$$
$$(\{\gamma^a,\gamma^b\})^\beta_\alpha = 2\eta^{ab}(\delta)^\beta_\alpha,$$

where in this case the bracket denote the anti-commutator[3] instead of the Poisson bracket.

The other three terms are found to be the same as the first one since the order of the canonical variables q and p does not matter. Summing these terms we thus find that

$$\{P^a, K^b\} = 2(\eta^{ab}(\delta)^\kappa_\alpha + \varepsilon^{ab}{}_c(\gamma^c))^\kappa_\alpha\, q^\alpha p_\kappa.$$

Identifying the generators D and M^c in this expression we finally recover

$$\{P^a, K^b\} = -2\eta^{ab}D - 2\varepsilon^{ab}{}_c\, M^c,$$

which is exactly the relationship of equation (5.12) as promised. Similarly, the assignments of equation (5.9) produces the correct results for all the other commutation relations, as can be verified by direct computation. Therefore this is indeed a representation of the conformal Lie algebra.

Having chosen this representation we can now see a possible way to include higher spin terms. This is done by adding new generators to those of equation (5.9) that are polynomials of the phase space variables q, p of higher order. Specifically we will add the polynomials of even order, which will correspond to spin $3, 4, ...$ for the polynomials of order $4, 6, ...$ and so on. We will not be considering the polynomials of odd order that describe half integer spins, since they are inherently spinorial. Only when we combine two phase space variables p_α, q^β can we transform their spinorial

[3] $\{A,B\} = AB + BA$.

indices to tensorial ones using the gamma matrices $(\gamma^\mu)_{\alpha\beta}$. If we would have wanted to include these terms properly, we would have had to describe their spinor nature properly. This requires a proper introduction of the concept of supersymmetry, so we abstain from this attempt [6].

In the previous section we did not need to introduce spins of higher order than 3 in order to achieve our goal of being able to gauge away singularities. Therefore we attempt to do the same in this case and extend our generators to include quartic terms, corresponding to spin 3. However, we immediately encounter a great problem. When we perform the commutator of two spin 3 generators using the Poisson bracket like before, the result is neither a cubic or quartic term. Instead the result is a polynomial of degree 6, which is one of the generators for spin 4. Indeed, a similar result is found to hold for spins of other orders as well. To see this we write the generators of spin $n+1$, $n \geq 1$ as

$$G(2n)^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s}|_{r+s=2n} = q^{\alpha_1} \ldots q^{\alpha_r} p_{\beta_1} \ldots p_{\beta_s},$$

where we have excluded the n gamma matrices that transform spinorial to tensorial indices. Calculating the commutator of two generators $G(2n)$ and $G(2m)$, using the Poisson bracket and commutation relations like those of equation (5.14), we find that only 2 phase-space variables cancel. Therefore we find that

$$\{G(2n), G(2m)\} = G(2(n+m)-2). \tag{5.15}$$

As can be seen from this formula, only the spin 2 case is a closed group representation, all generators of higher order result in terms of even higher order. If we attempt to include only spin 3 we find that we are forced to include spin 4 and so on, until eventually we have included spins of all integer orders. Unlike the case for AdS_3 with its $SL(n,\mathbb{R}) \times SL(n,\mathbb{R})$ gauge group, we can only extend the conformal algebra to include infinite spins.

Unfazed by the discovery that we must include all spins of higher order we now investigate the resulting theory. This will clearly result in a more complicated theory than the finite theory of the previous sections, but is otherwise allowed. We swiftly encounter problems however, when we attempt to form a field strength and a suitable action that would yield the equations of motion for this theory. This is exactly what we did for the restricted spin 2 case in section 4.6 and one could hope that the procedure would be entirely analogous now. Unfortunately there are terms that must be included in the Lagrangian which are found to be non-integrable. Clearly, this approach then must be flawed as the action of a theory must be well-defined, See [23] for more details.

Up to this point in this thesis we have only considered classical theories. Now however we must transition to a quantum theory in order to remedy the problems of the non-integrable terms of classical higher spin theory. We perform this transition by substituting the phase space variables q, p with quantum mechanical operators $\hat{p}, \hat{q}$ and the Poisson brackets

$$\{q^\alpha, p_\beta\}_{PB} = \delta^\alpha_\beta$$

by the commutator of the quantum operators. The operators are then specified to satisfy the canonical commutation relations

$$[\hat{q}^\alpha, \hat{p}_\beta] = i\hbar\delta^\alpha_\beta.$$

This quantisation method is canonical and was first used by Dirac to unify Heisenberg's matrix mechanics and Schrödinger's wave mechanics. Importantly for us, this method circumvents the problem we previously mentioned of quantisation when using the frame fields as gauge fields. However, it also necessitates some additional changes to our original theory. In the transition to the quantised theory, the generators need to be adjusted so that they are Hermitian operators and can correspond to observables. Therefore we redefine M^a as[4]

$$\hat{M}^a = -\frac{1}{4}(\gamma^a)^{\beta}_{\alpha}(\hat{q}^{\alpha}\hat{p}_{\beta} + \hat{p}_{\beta}\hat{q}^{\alpha}) = (\hat{M}^a)^{\dagger}.$$

Furthermore as the phase-space variables no longer are simple numbers but operators, they no longer commute. Because of this their order matters, and some of the calculational simplifications we used in the previous case are no longer valid. This will change the higher spin algebra fundamentally, as we will soon see.

Having quantised our theory we now introduce a position vacuum state defined by

$$\hat{q}^{\alpha}\left|0\right\rangle_q = 0, \quad \hat{p}_{\beta}\left|0\right\rangle_q \neq 0,$$

that is the state of lowest possible energy and corresponds to the unexcited fields, in analogy with the ground state of a harmonic oscillator. The introduction of this vacuum state enables us to find a connection to this theory with the seemingly unrelated Klein-Gordon equation. This equation is an extension of the Schrödinger equation that governs the behaviour of relativistic spin-less particles and for massless particles described by a scalar wave function ϕ it reads

$$\Box\phi = 0.$$

The vacuum state can now be used to an interesting effect in relation to this equation. By acting upon with both sides of the commutator

$$[\hat{P}_a,\hat{K}_a] = -2i\varepsilon_{ab}{}^{c}\hat{M}_c - 2i\eta_{ab}\hat{D}.$$

After inserting the definitions of equation (5.9) with p_{α} and q^{β} replaced by $\hat{p}_{\alpha}$ and $\hat{q}^{\beta}$ we find that $\hat{M}^a\left|0\right\rangle = 0$ and $\hat{D}\left|0\right\rangle = -\frac{i\hbar}{2}\left|0\right\rangle$. Inserting this into the expression above yields

$$[\hat{P}_a,\hat{K}_a]\left|0\right\rangle = -\eta_{ab}\left|0\right\rangle$$

in analogy with how the ground state in a harmonic oscillator has non-zero energy. Furthermore this term is found to be precisely needed when one want to extend the Klein-Gordan equation to a conformal version. We simply quote the resulting conformal Klein-Gordan that arise because of this:

$$\Box\phi - \frac{1}{8}R\phi = 0,$$

where $\frac{1}{8}$ is a dimension dependent constant and R is the Ricci curvature scalar[6].

[4]Actually, rewriting M^a is not necessary for the spin 2 case. This is because $\hat{p}_{\beta}\hat{q}^{\alpha} = \hat{q}^{\alpha}\hat{p}_{\beta} + \left[\hat{P}_{\beta}, \hat{q}^{\alpha}\right]$ and the latter term is $i\hbar\delta^{\alpha}_{\beta}$ and when combined with the gamma matrix $(\gamma^a)^{\alpha}{}_{\beta}$ the result is $Tr\left[\gamma\right] = 0$, so the contribution of this term is zero. However, when one includes terms of higher spin this becomes important.

Up to this point we have only quantised our original conformal theory of spin 2. However, we performed the quantisation with the stated purpose of being able to apply it to the theory of higher spins. We now do this by extend our generators to all even polynomials of the phase space variables $\hat{q}^\alpha$ and $\hat{p}^\alpha$ just like we did for the classical case. In this process we need to remember that the order of the operators, unlike for the phase-space variables, matter. Accordingly, different elements from the generators of the same spin $G(2n)$ can now produce terms of different order and equation (5.15) must be modified to

$$[G(2n),G(2m)] = G(2(n+m)-2) + G(2(n+m)-6) + \ldots .$$

We can now express our one-form gauge connection $\hat{A}$ in our higher spin algebra as a sum of the generators of different spin as

$$\hat{A} = \sum_{n=1}^{\infty} (-i)^n dx^\mu \hat{A}_\mu(2n) G(2n),$$

where the term of $n = 1$ is simply our previous connection

$$\hat{A}(2) = e^a \hat{P}_a + \omega^a \hat{M}_a + b\hat{D} + f^a \hat{K}_a.$$

When we proceed to higher terms we have to include new fields for each new generators, just like how we used the generalised frame fields and spin connection of section 5.1. For the spin 3 case this looks like

$$\hat{A}(4) = e^{ab} \hat{\hat{P}}_{ab} + \bar{e}^{ab} \hat{\hat{P}}_{ab} + \bar{e}^a \hat{\hat{P}}_a + \omega^{ab} \hat{\hat{M}}_{ab} + \bar{\omega}^a \hat{\hat{M}}_a + \bar{b}\hat{D} + \bar{f}^a \hat{\hat{K}}_a + \bar{f}^{ab} \hat{\hat{K}}_{ab} + f^{ab} \hat{K}_{ab} \ ,$$

and accordingly for all terms of higher order where the number of indices continues to rise like this. Finally we can use this gauge connection to form the resulting field strength of this theory:

$$\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} = \sum_{n=1}^{\infty} \frac{1}{2} dx^\mu \wedge dx^\nu \hat{F}_{\mu\nu}(2n) G(2n).$$

This is as far as we will discuss this conformal theory of higher spin. It is an interesting theory that has applications in string theory, as it is thought that it can be derived from this field by taking its tensionless limit. There is currently active research being performed regarding the link between the AdS/CFT correspondence and theories of higher spin like this one. According to the AdS/CFT correspondence our original 3-dimensional conformal theory is equivalent to a string theory on AdS_4 and as such the extension to the higher spin should also have some relationship with this theory. Further applications include M-theory where higher spin could provide further links to string theory and possibly condensed matter physics. In this field theories describing the behaviour of matter become conformally invariant at the critical points of phase transitions, so extending these conformal theories using higher spin is a possibility [24].

6

Conclusions

We set out with the goal of using higher spin gauge theory to be able to resolve singularities. In this process we first explored the connection between gravity and gauge theory. In 4 dimensions there arises both conceptual and practical difficulties when attempting to unify gravity with the other fundamental gauge forces. In the 3-dimensional case however, this procedure is simplified considerably as the similarities between gravity and gauge theory become more apparent. In this thesis we have explored these similarities and demonstrated that the Einstein-Hilbert action, which governs general relativity, can be expressed as a Chern-Simons gauge action when restricted to the 2+1 dimensional case. The equivalence of the two actions holds for all signs of the cosmological constant, but requires different constructions in the different cases. One possible interesting application of this model is the practically two-dimensional material graphene. If suspended in space, graphene forms a curved surface and if subjected to external impulses this might necessitate the introduction of a metric and a 2+1-dimensional theory of gravity as a result[8, 9].

Guided by the equivalence of the Einstein-Hilbert action and the Chern-Simons gauge action we have also generalised its gauge theory formulation to one of conformal gravity in 2+1 dimensions. This was done by extending the gauge group from the isometry groups of AdS_3 and Lorentz space to the conformal group. Conformal symmetries and field theories of this kind are of great importance in string/M-theory, and are finding increasing relevance in condensed matter theory, where conformal symmetries arise at critical points like phase transitions [25]. Examples of note from this field where a theory of this could prove useful include superconductivity and the quantum Hall effect.

The similarity between 2+1-dimensional gravity and a Chern-Simons gauge theory allowed us to perform one additional generalisation. This was once again done by increasing the gauge group, this time in order to include higher spin fields. This approach was very successful and we used our extended theory to a simple three-dimensional model of Big Bang: the Milne orbifold. In normal general relativity, singularities of this kind are invariants and can not be resolved. However, we found that by selecting an appropriate gauge the singularity vanished, so the singularity is nothing but a gauge artifact in the higher spin model. Because of the close relationship between higher spin models and string theory[1] this approach could possibly also be used in the more general setting. Indeed, initial attempts along this lines have been successful [1].

The final theory we discussed in this thesis is an extension of conformal gravity

[1]Higher spin is thought to be the tensionless limit of string theory.

to include higher spins. We discovered how this theory could not be truncated at finite spins and briefly discussed how to quantise this theory. Conformal higher spin theories like this are now mainly studied in the context of the AdS/CFT duality. This duality was conjectured using a string theory on AdS-space, which resulted in indications that this theory was equivalent to a conformal theory of scalar fields upon the AdS boundary. This conjecture has revolutionised modern physics and Maldacena's thesis that introduced this idea has more than 10000 citations today [24]. An example of an area where the importance of this duality can be seen are strongly coupled conformal systems in 3 dimensions, $g \to \infty$. Systems of this sort are very difficult to model using standard methods of condensed matter physics, but using the duality, such theories can be transformed into a weakly coupled ($g \to g' = \frac{1}{g}$) theory of gravity with black holes in AdS_4. This type of theory is often much more manageable as standard perturbative approaches can be applied.

A

Differential Geometry

The mathematical language of general relativity and much of modern physics is based upon differential geometry, the study of calculus and linear algebra on differential manifolds. This allows theories to be expressed in a coordinate independent fashion which is desirable as coordinates are inherently arbitrary. In this appendix we will give a brief introduction to the subject with all the important definitions that we use in this thesis. This introduction is not mathematically rigorous and no proper proofs are presented. If the reader want to study the subject further, we recommend [17] and [14] where this field and its physical applications are presented in a more thorough way.

A.1 Topological spaces and manifolds

The foundation of differential geometry are differentiable manifolds, which are a special type of topological spaces. Topological spaces are of profound importance in mathematics as they are the most general mathematical spaces where the concepts of connectedness, convergence, and continuity can be defined [26].

Definition A.1.1 *A topological space is a point set X supplied with a collection of special subsets Θ called open sets. Furthermore X and its open sets are required to satisfy the following relations:*

i. *X and the empty set $\emptyset$ are both open.*

ii. *If $U, V \subseteq X$ are open, then so is their union $U \cup V$.*

iii. *If $U, V \subseteq X$ are open, then so is their intersection $U \cap V$.*

The collection of open sets Θ is called the topology of X, and by choosing a different collection Θ we can supply the same set X with a different topology. Proceeding from this rather abstract definition we say that if an element $x \in X$ also belongs to the open set U, then that set is a neighbourhood of x. This provides an intuitive way of "moving closer" to a point without any notion of length, which is just what we need for the concept of continuity. If (V_i) is a sequence of neighbourhoods of $x \in X$ that are also proper subsets of preceding V_i, then we can happily think of these sets as shrinking down, "closer" to x.

We now take a step back from this picture and instead turn to the global structure of our topological set where we define the concept of a topological cover.

Definition A.1.2 *A collection B of open sets $U_i \subseteq X$ such that all $x \in X$ belongs to the union of the open sets is called a cover of X. Furthermore, if all open sets in X can be expressed as a union of elements of B then B is said to constitute a basis for X.*

An example of an intuitive topological space where we can apply these definitions is the usual Euclidean space $\mathbb{R}^n$. There, the open sets $U_{\mathbf{x}}$ are the balls of points located within a distance R from the point $\mathbf{x_0}$, $U_{\mathbf{x}_0} = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}^0| < R\}$. This is called the metric topology and each ball $U_{\mathbf{x}_0}$ is, as expected, a neighbourhood of $\mathbf{x}$. The collection of all these balls constitutes both a basis and a cover of $\mathbb{R}^n$. This basis is clearly uncountable, however it is possible to form another basis that is countable. This basis is formed by restricting the original basis to only include balls of rational radius centred on points with rational coordinates.

We earlier discussed an intuitive way of thinking about closeness in a topological set which now can be used to define continuous maps. As topological spaces don't need to be equipped with any notion of length, continuity can in general not be defined similarly to the familiar epsilon-delta way. However, if we want to define continuity for general topological spaces our definition have to coincide with this notion when the space is endowed with a concept of length. Keeping this in mind we arrive at the following definition:

Definition A.1.3 *A map $f : X \to Y$ from one topological space to another is continuous if the inverse set $f^{-1}U \subseteq X$ of an open set $U \subseteq Y$ is open itself. If in addition f is bijective and f^{-1} is continuous then X and Y are homeomorphic and f is called an homeomorphism.*

Homeomorphisms are of central importance in topology, as it is a field dedicated to studying the properties that are invariant under general homeomorphisms. However, we will not delve much deeper into this fascinating subject.

With these concepts from topology we lack just one thing to define a manifold, the most common type of space upon which physical theories are formulated. This is the notion of a Hausdorff space, which is a topological space X for which there exist respective disjoint neighbourhoods U_x and U_y for all disjoint points $x, y \in X$. All spaces considered in this thesis will be Hausdorff as only the most bizarre and impractical spaces are non-Hausdorff [14].

Definition A.1.4 *A topological manifold of dimension n is a Hausdorff space X with a countable basis that is locally homeomorphic to $\mathbb{R}^n$. That is, for each point $x \in X$ there exists a neighbourhood U_α with an homeomorphism $\phi_x : U_\alpha \to U'_\alpha \subseteq \mathbb{R}^n$.*

Here we note the importance of our earlier statement that there exists a countable basis for $\mathbb{R}^n$. If this was not the case then $\mathbb{R}^n$, the most simple example of a manifold, would not be a manifold itself. Looking back at this definition we can further deduce that the neighbourhoods U_α, which are denoted patches, provide a cover of X which we call an atlas of X. The homeomorphisms ϕ_α, in turn, are called coordinate charts since the image of $y \in U_\alpha$ in the Euclidean space $\mathbb{R}^n$ naturally assign local coordinates $\phi_\alpha(y) = (x^1(y), ..., x^n(y))$ to all points in the neighbourhood U_α of x.

On the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, where two different coordinate charts ϕ_α and ϕ_β (and their inverses) are defined there exists a natural homeomorphism between open sets of $\mathbb{R}^n$. This is the transition map

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_{\alpha\beta}) \to \phi_\beta(U_{\alpha\beta}) \tag{A.1}$$

which we for obvious reasons call a coordinate transformation as it relates the coordinates y^i of a point $p \in U_{\alpha\beta}$ in the chart U_β to the coordinates x^i of the chart U_α as

$$y^i = y^i(x^1, ..., x^n),\ i = 1, ..., n. \tag{A.2}$$

It is these coordinate transformations that provide the connection between topology and differential geometry. On general topological spaces there does not seem to exist any natural notion of differentiability, but this concept is readily available in $\mathbb{R}^n$. By imposing additional constraints on both our coordinate charts and our coordinate transformations we can ensure that the calculus we wish to perform is valid. In order to describe these additional constraints we just need to develop a small bit of terminology.

If a coordinate transformation $\phi_{\alpha\beta}$ and its inverse $\phi_{\beta\alpha}$ as in equation (A.1) are C^k-differentiable, the coordinate charts ϕ_α and ϕ_β are said to be C^k-compatible. An atlas of C^k-compatible charts is then called an atlas of class C^k. When comparing two different atlases of class C^k one says that they are compatible with each other if each chart of the first is compatible with each chart of the second. Since the union of atlases is an atlas in turn, one can construct a maximal atlas of class C^k by forming the union of all atlases compatible with the first one. With this knowledge in hand we can now finally define a differentiable manifold.

Definition A.1.5 *A differentiable manifold of dimension n and class C^k is a n-dimensional topological manifold with a maximal atlas of class C^k. If $k = 0$ this is just a topological manifold while if $k = \infty$ the manifold is said to be smooth.*

This definition is of central importance as differential manifolds are the ubiquitous setting of physical theories in various forms. The first example that comes to mind is of course Euclidean (or Lorentzian) space itself as the space we exist in, but the applications reach much further. In analytical mechanics for example, very complex phase-spaces can be necessary as the complexities of the mechanical systems the theory describes increase. Finally we note that we tacitly assume that all manifolds of this thesis are smooth, simply calling them manifolds. Thereby we avoid any questions regarding differentiability, as this is not the focus of this thesis.

A.2 Tangent and cotangent spaces

We mentioned in the previous section how the goal of of differential geometry is to be able to perform calculus and linear algebra on general manifolds. From these fields there are some key concepts that we must introduce to this setting in order to achieve this. These concepts are functions, derivatives, and tangent vectors. Perhaps unsurprisingly there are some subtle issues involved with the generalisation of these concepts that need to be tackled. Luckily the concept of a (differentiable) function is easy enough to construct from our previous continuous maps between topological spaces. All that we need is a map $f : X \to Y$ between two manifolds X and Y with respective coordinate charts ϕ_X and ϕ_Y from their maximal atlases. We call this map differentiable of class C^k at a point $x \in X$ if the map $\hat{f} = \phi_Y \circ f \circ \phi_X : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k. In particular, a map $f : X \to \mathbb{R}$ from a general manifold X is called a real function or just a function over X. Adversely, a map $f : U \to X$ from an open interval $U = (a,b)$ of the real line to a general manifold X is called a parametrised curve in X. Finally there exist another very important class of maps between different manifolds called diffemorphisms:

Definition A.2.1 *A differentiable mapping ϕ whose inverse ϕ^{-1} also is differentiable and that is a bijective mapping between two manifolds is called a diffeomorphism. The manifolds are then called diffeomorphic.*

In this thesis all differentiable maps, and functions in particular, will be smooth so all derivative operations we wish to perform are allowed.

Functions were easily to define on manifolds, but the concept of an internal vector space is trickier. Howeever, there is a natural vector space we can associate with each point p of the manifold n-dimensional manifold X called the tangent space, T_p. In the intuitive sense, tangent vectors should allow us to take directional derivatives along their directions. We can create something along these lines by considering all the parametrised curves $\gamma(t) : \mathbb{R} \to X$ passing through p when $t = 0$. These curves should intuitively carry the information of directed tangent lines, and so we combine these curves with functions $f : X \to \mathbb{R}$ to produce a function $f \circ \gamma(t) : \mathbb{R} \to \mathbb{R}$. Now each of the curves allows us to form a derivative operator

$$\mathbf{v}(f) = \frac{d}{dt}\left(f \circ \gamma(t)\right)\bigg|_{t=0}$$

acting on the arbitrary functions f. The set of all these operators does indeed form a vectors because we can combine them linearly using the linearity of the derivative. Additionally these operators satisfy the Leibniz law of derivatives, a property so important we use it to define tangent vectors.

Definition A.2.2 *The tangent space T_p at $p \in X$ is the set of all linear maps from functions $f : X \to \mathbb{R}$ to $\mathbb{R}$ that satisfy*

i. $\mathbf{v}_p(\alpha f_1 + \beta f_2) = \alpha \mathbf{v}_p(f_1) + \beta \mathbf{v}_p(f_2), \quad \alpha, \beta \in \mathbb{R}.$

ii. $\mathbf{v}_p(f_1 f_2) = f_2 \mathbf{v}_p(f_1) + f_1 \mathbf{v}_p(f_2).$

A vector field, or simply vector, is an assignment of a vector in the tangent space T_p to each point $p \in X$.

With this definition it is clear that the tangent space T_p is inseparable from p. This means that one can not relate a vector in one tangent space to another, which differs from the geometrical vectors of Euclidean space that can be can freely translated. We can determine the dimensions of the tangent space by choosing a particular coordinate chart $\phi : X \to \mathbb{R}^n$. Denoting the coordinates by x^μ we find that

$$\frac{d}{dt}\left(f \circ \gamma(t)\right) = \frac{d}{dt}\Big((f \circ \phi^{-1})(\phi \circ \gamma(t))\Big) = \frac{d(\phi \circ \gamma)^\mu}{dt}\frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}$$

where the chain rule has been used for the last equality. Informally we can write this as

$$\frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu}$$

which shows that we can use the partial derivative ∂_μ as a basis for the tangent space. Clearly then the tangent space must be of the same dimension as the manifold X itself. We say that the coordinate chart induces a natural coordinate basis. Under a change of coordinate chart, a coordinate transformation, this natural basis clearly changes according to

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$$

according to the chain rule. This is exactly how a covariant basis transforms, so if the vector is to remain invariant under the coordinate transformation its components must transform in the inverse way. Therefore the vectors of the tangent space are contravariant vectors.

Having defined the tangent space there is one additional vector space that immediately follows. This is the dual space of T_p called the cotangent space T_p^*. The introduction of this field is motivated by the usual gradient of a function f, which enables us to form the directional derivative $\nabla f \cdot \mathbf{v}$ in the direction specified by the vector $\mathbf{v}$. We can think of this in another light as an operator that acts upon $\mathbf{v}$ and returns the the directional derivative of f in this direction. Guided by this we define a differential one-form as a member of the cotangent space.

Definition A.2.3 *The cotangent space T_p^* is the set of C^∞-linear maps $\omega : \mathbf{v} \to \mathbb{R}, \quad \mathbf{v} \in T_p$,*

$$\omega(f\mathbf{v} + g\mathbf{u}) = f\omega(\mathbf{v}) + g\omega(\mathbf{u}), \quad f, g \in C^\infty(\mathbb{R}^n), \qquad \mathbf{v}, \mathbf{u} \in T_p.$$

These maps are called differential forms or simply one-forms, as is the assignment of one one-form $\omega \in T_p^*$ to each $p \in X$.

One particular one form is the differential of a function f, df. This is the one-form defined by

$$df(\mathbf{v}) = \mathbf{v}(f) = \frac{d}{dt}\left(f \circ \gamma(t)\right)\bigg|_{t=0},$$

the analogous object of the gradient in differential geometry. Just like how the coordinate chart induced a natural coordinate basis ∂_μ in the tangent space, it induces a natural coordinate basis dx^μ in the cotangent space defined by

$$dx^\mu(\partial_\nu) = \langle dx^\mu, \partial_\nu \rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu.$$

Requiring this property to hold even after a change in coordinate chart tells us that this coordinate basis must transform according to

$$dx^\mu = \frac{\partial x^\mu}{\partial x^\nu} dx^\nu,$$

the transformation rule of a contravariant basis. This then means that the vectors of the cotangent space are covariant unlike the contravariant vectors of the tangent space.

Using vectors from both the tangent and cotangent spaces we can combine them using the tensor product to form tensors and tensor fields of any kind upon the manifold,

$$T = T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \, \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_m} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_m}, \quad \partial_{\mu_i} \in T_p, \quad dx^{\nu_i} \in T_p^*.$$

This is indeed a proper tensor space, and all the usual operations including the tensor product and contractions are present in their normal form. However one must be careful to note how these tensors have values in the tangent and cotangent spaces T_p and T_p^* that are different vector spaces for different points. One must therefore ensure that these operations can only be performed with tensors from the same point.

A.3 Exterior algebra and the exterior derivative

In 3 dimensions we can use the familiar cross product $\mathbf{v} \times \mathbf{u}$ to provide a sense of orientation to the parallelogram spanned by $\mathbf{v}$ and $\mathbf{u}$. The fundamental property of this orientation is contained in the identity

$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$$

and so we extend this concept to higher dimensional tensor spaces by introducing the concept of n-forms.

Definition A.3.1 *Let V be an m-dimensional vector space. An n-form then is a totally antisymmetric multilinear map $\omega : \underbrace{V \times \ldots \times V}_{ntimes} \to \mathbb{R}$. The set of n-forms is a vector space and is denoted by $\Lambda^n(V)$.*

Because an n-form is a totally antisymmetric map it is always zero if any of the vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ it acts upon is the same, just like how the cross product of something with itself is zero according to

$$\mathbf{v} \times \mathbf{v} = -\mathbf{v} \times \mathbf{v} \to \mathbf{v} = 0.$$

As there are m vectors in the basis of V the dimension of $\Lambda^n(V)$ is found to be $m!/(n!(m-n)!)$. The extension of the cross product is now made explicit by the introduction of the wedge product $\wedge$.

Definition A.3.2 *The wedge product is* $\wedge$ *is an operator* $\wedge : \Lambda^n(V) \times \Lambda^p(V) \to \Lambda^{n+p}(V)$ *that is required to be* C^∞*-bilinear, associative, and graded commutative:*

i. $(f_1\omega_1 + f_2\omega_2) \wedge (f_3\omega_3 + f_4\omega_4) =$
$f_1 f_3 \omega_1 \wedge \omega_3 + f_1 f_4 \omega_1 \wedge \omega_4 + f_2 f_3 \omega_2 \wedge \omega_3 + f_2 f_4 \omega_2 \wedge \omega_4, \quad f_i \in C^\infty$

ii. $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

iii. $\omega_1 \wedge \omega_2 = (-1)^{np} \omega_2 \wedge \omega_1, \quad \omega_1 \in \Lambda^n, \quad \omega_2 \in \Lambda^p.$

These algebraic properties of the wedge product are called the exterior algebra and it encapsulates the geometrical notion of orientation. As such, the n-forms really can be thought of as a n-dimensional parallelepipeds spanned by the one-forms.

In the setting of differential manifolds the fundamental maps from $V \in T_p$ to $\mathbb{R}$ is given by the differential forms $\omega \in T_p^*$. Therefore we call these objects differential n-forms. They are then nothing but a basis for the entirely antisymmetric covariant tensors

$$T = T_{\mu_1 \ldots \mu_n} dx^{\mu_1} \otimes \ldots \otimes \ldots dx^{\mu_n} = \frac{T_{\mu_1 \ldots \mu_n}}{n!} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}.$$

In order to proceed, recall how the differential of a function f was the one-form df. We now consider this to be the result of an operator d acting on the function f, which can be thought of as a zero-form. We call this the exterior derivative, an operator that we will also generalise to n-forms. In this process the exterior derivative also becomes the combined generalisation of all the common vector differential operations: the gradient, the curl and the divergence.

Definition A.3.3 *The exterior derivative is the unique map* $d : \Lambda^n(X) \to \Lambda^p n + 1(X)$ *that satisfy:*

i. $d : \Lambda^0 \to \Lambda^1$ *is the differential of* $f \in C^\infty$.

ii. $d(\alpha\omega_1 + \beta\omega_2) = \alpha d\omega_1 + \beta\omega_2, \quad \alpha, \beta \in \mathbb{R}, \quad \omega_1, \omega_2 \in \Lambda^n(X).$

iii. $d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \omega_2 + (-1)^{np} \omega_1 \wedge d(\omega_2), \quad \omega_1 \in \Lambda^n(X), \quad \omega_2 \in \Lambda^p(X).$

iv. $d^2\omega = 0.$

With this operation follows two terms of terminology. An n-form ω is called closed if $d\omega = 0$ and is called exact if $\omega = d\Omega$ for some $(n-1)$-form Ω.

A.4 The metric and the Hodge dual

After we have introduced the tangent space T_p to our differential manifold we want to be able to express the length and angles of these tangent vectors. This is done by defining an inner product or metric for the tangent space at which point it becomes a metric space.

Definition A.4.1 *The metric of T_p is a bilinear map $g : T_p \times T_p \to \mathbb{R}$ that also satisfy,*

i. $g(\mathbf{v},\mathbf{u}) = g(\mathbf{u},\mathbf{v}), \quad \mathbf{v},\mathbf{u} \in T_p$

ii. $g(\mathbf{v},\mathbf{u}) = 0 \forall \mathbf{u}$ *only when* $\mathbf{v} = 0$.

By choosing a suitable orthonormal basis $\mathbf{e_a}$ the metric map can be diagonalised to the following form:

$$g(\mathbf{e_a}, \mathbf{e_b}) = g_{ab} = \begin{cases} \pm 1, & a = b \\ 0, & a \neq b \end{cases}$$

The number of terms of respective sign determines the signature of the metric, (n,m). If all are of the same sign the manifold is called Riemannian, otherwise it is called pseudo-Riemannian, signifying that it has both space and timelike components. Furthermore the metric is defined in a similar fashion for covectors in the cotangent space.

In the previous section we mentioned how the n-forms could be thought of as a parallelepiped. As such, if the manifold is of dimensions m there is only one element in Λ^m and we identify it as a volume element η. We will only make on small modification and add a term $|\sqrt{|g|}|$ where g is the determinant of the metric. This is done in order to ensure that the volume element is invariant under coordinate transformations and so we have [17]

$$\eta = \sqrt{|g|} dx^1 \wedge .. \wedge dx^m.$$

With this volume element we can now define another dual operation from n-forms to $(m-n)$-forms.

Definition A.4.2 *The Hodge dual map is a unique linear map $* : \Lambda^n(X) \to \Lambda^p(X)$. Writing $\omega_1, \omega_2 \in \Lambda^n(X)$ as $v_1 \wedge ... \wedge v_n$ and $u_1 \wedge ... \wedge u_n$ respectively, the Hodge dual is specified by*

$$\omega_1 \wedge *\omega_2 = \langle \omega_1, \omega_2 \rangle \eta \quad \forall \omega_1, \omega_2 \in \Lambda^n(X)$$

where

$$\langle \omega_1,\omega_2 \rangle = det \left[g(v_i, u_j) \right]$$

.

Explicitly in terms of the basis dx^μ the Hodge dual of a differential p-form is given by the formula

$$* \left(dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \right) = \frac{(-1)^t}{(n-p)!} \varepsilon^{\mu_1 .. \mu_p}{}_{\mu_{p+1} .. \mu_n} dx^{\mu_p + 1} \wedge ... \wedge dx^{\mu_n}$$

where t is the number of negative signs in the metric's signature and ε is the Levi-Civita tensor related to the Levi-Civita symbol ϵ by $\sqrt{|g|}\epsilon$.

B

Group theory

This appendix serves as an introduction to group theory, which is a broad field of mathematics with many different applications. This is important in physics as we use it to describe symmetries in for example the standard model. Here we will introduce the basics of group theory before focusing on Lie groups, as their continuous transformations relate to symmetries. Finally we will present some groups that are important within the subject matter of this thesis.

We starty by presenting the definition of a group. This mathematical definition can concisely be stated as, [27]

Definition B.0.3 *A group is defined as a set ($\mathcal{G}$) which together with an operation ($*$) fulfils the following:*

i. **Closure** : *if a and $b \in \mathcal{G}$, so must the result $a * b \,\epsilon\, \mathcal{G}$.*

ii. **Associativity** : $\forall\, a,\ b, and c \in \mathcal{G}$*, the following relation* $(a*b)*c = a*(b*c)$ *must be met.*

iii. **Unit element** : *there exist an object $e \in \mathcal{G}$, so that $a * e = e * a = a$.*

iv. **Inverse element** : *for each element $a \in \mathcal{G}$, there is an element $b \in \mathcal{G}$ so that $a * b = b * a = e$.*

This definition provides with a wide amount of groups. An example of a group is all integers where addition is used as an operator. We can see that the property of closure is fullfilled because addition between two integers will result in a new integer. Furthermore it is associative since addition is. The unit element of this group is 0 since adding it to any integer will result in the same integer. Finally we understand the inverse element of an integer to be the corresponding negative integer as adding them together will result in 0.

An important property, that the previous example has, is the abelian property. It is defined as

Definition B.0.4 *An abelian group is a group for which every element in the set commute for the group operation as*

$$a * b = b * a.$$

This is important as we need to differentiate between groups where the order is important, which are known as non-abelian, and those where it is not. An example of a non-abelian group is matrix multiplication for (2×2) matrices.

Furthermore it is possible to have groups that are a part of a larger group. These are commonly known as subgroups and are defined as

Definition B.0.5 *If $\mathcal{H}$ and $\mathcal{G}$ are two groups with the same group operation $(*)$. Then $\mathcal{H}$ is subgroup of $\mathcal{G}$ if $\mathcal{H} \subseteq \mathcal{G}$.*

There are always two trivial subgroups, these are the group itself and the unit element.

As we want to be able to relate groups to each other we must introduce the concepts of homomorphism and isomorphism. These are defined as accordingly [28]

Definition B.0.6 *A map ϕ: $\mathbf{G} \rightarrow \mathbf{H}$ is a **homomorphism** if*

$$g_i * g_j = g_k$$

and

$$\phi(g_i) * \phi(g_j) = \phi(g_k),$$

*when g_i, g_j, $g_k \in \mathbf{G}$ and $\phi(g_i), \phi(g_j), \phi(g_k) \in \mathbf{H}$. If ϕ also is bijective it is an **isomorphism** denoted $\mathbf{G} \cong \mathbf{H}$*

B.1 Lie groups

In physics we are mostly interested in a class of groups called continuous groups. A property that all continuous groups have is that they contain a infinite uncountable set of elements. Just as the previous example of integers, this can be illustrated by the real numbers as they are uncountable. Of special interest are the continuous groups commonly known as Lie groups, as they are differential manifolds[1]. These are defined as follows[28]

Definition B.1.1 *A Lie group is a smooth manifold $(\mathcal{G})$ together with a smooth multiplication map*

$$(g_1, g_2) \ \epsilon \ \mathcal{G} \times \mathcal{G} \ \rightarrow \ g_1 g_2 \ \epsilon \ \mathcal{G}$$

and a smooth inverse map

$$g \ \epsilon \ \mathcal{G} \ \rightarrow g^{-1} \ \epsilon \ \mathcal{G}$$

that satisfy the group axioms in B.0.3.

The concept of Lie groups was introduced by the Norwegian mathematician Sophus Lie who had the idea that the transformation of an object can be understood by its infinitesimal transformation. Mathematically this is presented by introducing a generator, which is a infinitesimal transformation around the unit element for each group operation (U) controlled by a parameter ϕ [27].

We will now explain this further by deriving the generators and for this we need Lie's great idea that a group operation $U(\phi)$ either can be done by transforming ϕ at once or by splitting up the transformation as $\left(U\left(\frac{\phi}{n}\right)\right)^n$ [29]. An illustrative example is rotation where it is obvious that this operation either can be done by rotating ϕ at once or by rotating ϕ/n n times.

[1]See appendix A for more information.

An infinitesimal transformation around the unit element can be illustrated by

$$U(\delta\phi) = \mathbb{I} + i\delta\phi\mathcal{S}$$

and by using Lie's idea ($U(\phi) = \left(U\left(\frac{\phi}{n}\right)\right)^n$), we find the limit relation

$$U(\phi) = \lim_{n\to\infty}\left(\mathbb{I} + \frac{i\phi\mathcal{S}}{n}\right)^n. \tag{B.1}$$

This limit relation can be related to the limit of the natural logarithm base e's relation

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

As many of the generators can be expressed by matrix representation, we want to define the exponential in terms of the limit (B.1). This means that we can write the group element as

$$U(\phi) = \lim_{N\to\infty}\left(\mathbb{I} + \frac{i\phi\mathcal{S}}{N}\right)^N = \exp\left(i\phi\mathcal{S}\right). \tag{B.2}$$

By differentiating equation (B.2) and setting ϕ to zero, we find that a generator $\mathcal{S}$ equals

$$\mathcal{S} = -i\left[\frac{d\mathcal{U}(\phi)}{d\phi}\right]_{\phi=0}. \tag{B.3}$$

To illustrate what a generator is example, we will investigate rotations around an axis. One of these rotations can, in matrix form, be represented as

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

which means that the generator S for this rotation group are given by equation (B.3) as

$$\mathrm{S} = -i\left[\frac{d}{d\theta}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\right]_{\theta=0} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

As this rotation was only done around an axis we only find one generator. However, a rotation in three dimensional can rotate around three different axis and therefore has three generators.

By combining a groups different generators, it is possible to find relations between them. We will illustrate this by expanding a group operation U_a as

$$U_a = \exp(i\epsilon_a S_a) = 1 + i\delta\epsilon_a S_a - \frac{1}{2}\epsilon_a^2 S_a^2 + ...,$$

where ϵ_a is assumed small. It is now possible to combine two different group operators expansion and inverse expansion as

$$U_b^{-1}U_a^{-1}U_bU_a = 1 + \epsilon_a\epsilon_b\Big[S_a, S_b\Big] + ...,$$

where again ϵ_a and ϵ_b are assumed small. Since the group is closed (given by B.0.3), the right hand side of the equation must equal to a group element which give that $\Big[S_a, S_b\Big]$ either must equal a group generator or be zero. By this can we see that

$$\Big[S_j, S_k\Big] = i\, f_{jk}{}^l S_l$$

must be fulfilled, where $f_{jk}{}^l$ is called the structure factor. By knowing a groups generators and their commutation works we have extracted the groups Lie algebra $\mathfrak{g}$, which are given by B.1.2.

Definition B.1.2 *A Lie algebra* ($\mathfrak{g}$) *is a vector space which has a bilinear mapping* $[.,.]$:$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that:*

i. For all $X,\ Y\ \epsilon\ \mathfrak{g}$ *is*

$$[X,Y] = -[Y,X] \quad \textit{fulfilled}$$

ii. Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \textit{for all } X,\ Y,\ Z\ \epsilon\ \mathfrak{g}$$

The bilinear mapping that was mentioned in definition B.1.2 is commonly called the Lie brackets and we see that for matrix groups this is fulfilled by the commutation relation, so a Lie algebra for a matrix represented group is given by the generators and their commutation relations.

B.2 Special Lie groups

We will in this section introduce the reader to some special Lie groups that are used in this thesis. We will discuss groups such as the special linear group ($SL(n;R)$), the special orthogonal group ($SO(n)$) and the special unitary group ($SU(n)$). The *special* in the group name means that the matrix representation of a $n \times n$-matrix has a determinant which equals 1, because of this must their generators be traceless since

$$det(U) = exp\left(Tr\left[\ln U\right]\right) = exp\left(i\phi Tr\left[S\right]\right) = 1.$$

In table B.1 is the notation for their Lie algebra given and the dimension they represent [28].

Lie Group	Lie algebra	Dimension
$SL(n,\mathbb{R})$	$\mathfrak{sl}(\mathfrak{n},\mathbb{R})$	n^2-1
$SU(n)$	$\mathfrak{su}(\mathfrak{n})$	n^2-1
$SO(n)$	$\mathfrak{so}(\mathfrak{n})$	$\frac{1}{2}n\,(n-1)$

Table B.1: A table over different Lie Groups with their Lie algebra and dimension.

B.2.1 Special linear group, SL(n, $\mathbb{R}$)

The special linear group contains all the $n \times n$-matrices where the determinant equals 1 and can for $n = 2$ be expressed as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $ad - bc = 1$. It is possible to split this matrix as following

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-c}{\sqrt{a^2+b^2}} \\ \frac{c}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix} \begin{pmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \frac{1}{\sqrt{a^2+b^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{ab+cd}{a^2+c^2} \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

By writing dividing this matrix into parts we can extract the different generators

$$S_\theta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad S_x = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad S_r = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

Through the use of linear combinations of the generators we can express the Lie algebra as [1]

$$\left[T_a, T_b\right] = i\varepsilon_{ab}{}^{c}T_c,$$

which will be used in the thesis.

B.2.2 Special unitary group, SU(n)

The special unitary group contains all unitary matrices A which fullfill $A^\dagger A = AA^\dagger = \mathbb{I}$. But since we can write the different generators as

$$U^{-1} = \exp\left(-i\phi S^\dagger\right) = U^\dagger = \exp\left(-i\phi S\right),$$

must $S = S^\dagger$ and from this fact and that all *special* group are traceless we find that the generators for $n = 2$ must be

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices are commonly known as the Pauli matrices

B.2.3 Special orthogonal group

Another interesting group is the special orthogonal group, which represent by matrices that are orthogonal ($A^\top = A^{-1}$). This group is usually also called the rotation group since it represents rotation operations, we have actually already seen its $n = 2$ matrix representation in section B.1 as

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

and found its generator to be

$$S = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

By applying this to $SO(3)$, we see that a rotation around the x-axis is given by U as

$$U_x = \begin{pmatrix} 1 & & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & -\sin(\varphi) & \cos(\varphi) \end{pmatrix},$$

and the generator is

$$\mathcal{S}_1 = -i\left[\frac{dU_x(\varphi)}{d\varphi}\right]_{\varphi=0} = \begin{pmatrix} 0 & & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

and analogy, we find that

$$\mathcal{S}_2 = \begin{pmatrix} 0 & & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{S}_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The commutation relations for $SO(3)$ are given by

$$\Big[T_a, T_b\Big] = i\epsilon_{ab}{}^{c}T_c,$$

and by commuting the $SU(2)$ generators we see that $\mathfrak{su}(2) \cong \mathfrak{su}(2)$ since their Lie algebra has the same structure factor. However their group manifolds are not isomorphic in turn, as $SU(2)$ is in fact a double covering of $SO(3)$.
If the metric signature is not Euclidean space, for example Lorentzian metric, the group is denoted $SO(n,m\,;\mathbb{R})$ leaving invariant $\sum_i (x^i)^2 - \sum_j (x^j)^2$, where $i = 1,...,n$ and $j = n+1,...,n+m$.

C

Analytical Mechanics

When studying the dynamics of a systems the equations of motion can be found by applying vectorial forces, which is known as Newtonian mechanics. A problem when using forces is that the theory is dependent on the coordinate system that describes the system. To overcome this problem, the use a theory, though completely equivalent, which can be described in general coordinates seems appropriate. This is done through Analytical mechanics, using the Lagrangian formalism, where two scalars are used instead of the vectorial forces that is the kinetic and the potential energy. Furthermore, there are several reasons why Lagrangian formalism is so convenient, the Lagrangian is a scalar function defined by the energy of the system and is completely invariant under all symmetries of the theory. It also makes the transition to quantum mechanics smooth since the operator of a quantum wave function can be described by path integrals which is a central part in Lagrangian mechanics. For readers interested in the finer details of this theory are referred to [30].

C.1 Generalised coordinates

Each physical system has a certain degree of freedom, which in for example a 2D mathematical pendulum is one, the angular displacement θ, since the length L is fixed. In Cartesian coordinates this would be described as

$$x = L\sin\theta$$
$$y = -L\cos\theta,$$

which can be reduced to one equation of motion, namely

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

where g is the acceleration that drives the motion. So, instead of describing the system in x and y, the system can be simplified by the coordinate θ. This shows that depending on the choice of coordinates the equations of motion can be simplified. The system can be reduced to as many independent general coordinates as the degree of freedom, denoted by q^i, where $i = 1, 2, ..., N$ where N is the degree of freedom. Therefore we can rewrite our initial system as

$$x^j = x^j(q^1, q^2, ..., q^N), \quad j = 1, 2, ..., k$$

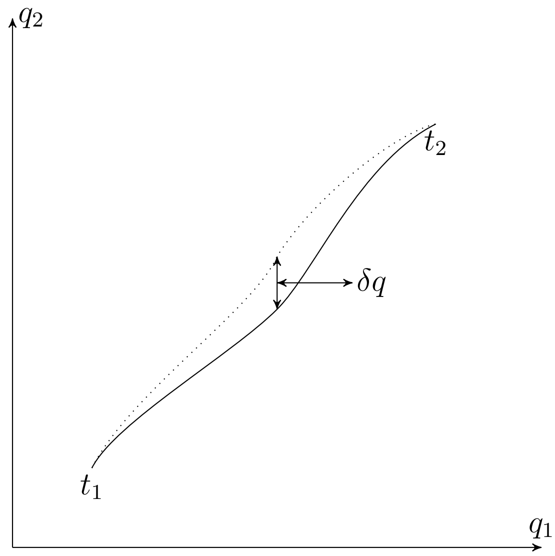


Figure C.1: This figure illustrates how variation principle works with fixed endpoints, the solid line is the path that minimises the action and the dotted line shows how a variation might look like from the extrema.

where k is the number of coordinates in the initial coordinate system. The system might also be dependent on time and we can therefore write the coordinates as

$$x^j = x^j(q^1, q^2, ..., q^N, t), j = 1,2,...,k.$$

We can also express the velocities v_i in terms of our new coordinates by applying the chain rule for differentiations,

$$v_i \equiv \frac{dx_i}{dt} = \sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}.$$

C.2 Lagrangian formalism

As we previously mentioned we will define a new function, the Lagrangian L as

$$L = T - V.$$

This function depends on our generalised coordinates and its velocities, and also time,

$$L = L(q, \dot{q}, t),$$

but as our coordinates may also depend on time $q = q(t)$, the Lagrangian depends on a function q, its time derivative and on time itself,

$$L = L[q(t), \dot{q}(t), t].$$

To get the motion of the system we need to integrate the Lagrangian over a time interval t_1 and t_2. This is expressed through a new object, called the action, and is defined by,

$$S = \int_{t_1}^{t_2} L\, dt.$$

As this integral depends on the path taken, defined through q, the action S is not a function but rather a functional, which takes a function as a parameter and yields a scalar. In order to determine how the system will behave over time, we can apply Hamilton's principle, which states that the true evolution of the system must be a stationary point of the action, where the endpoints are held fixed:

$$\frac{\delta S}{\delta q} = 0.$$

With the dependence of the Lagrangian more explicitly written,

$$S[q(t)] = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t]\, dt,$$

we can determine the variation of the action like

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0,$$

keeping the endpoints fixed, namely $\delta q(t_1) = \delta q(t_2) = 0$. This is illustrated in figure C.1 where we change the path but not the endpoints. So we have

$$\delta S = \delta \int_{t1}^{t2} L(q, \dot{q}, t)\, dt,$$

we want to rewrite the right-hand side of this expression to contain a variation δq. We can move the variation inside the integral and therefore,

$$\int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$

or

$$\int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d(\delta q)}{dt} \right) dt.$$

The first term contains δq whereas the second contains its time derivative, so we want to work with it to a better form. To remove the time derivative from δq we use partial integration,

$$\int \frac{\partial L}{\partial \dot{q}} \frac{d(\delta q)}{dt}\, dt = \int \frac{\partial L}{\partial \dot{q}} d(\delta q) = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t1}^{t2} - \int \left(\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}}\, dt \right) \delta q.$$

As previously mentioned we are not varying the path at the endpoints, so we cancel the first term and reinserting the result backwards to write the variation in terms of δq,

$$\delta I = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t1}^{t2} + \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\, dt = \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\, dt.$$

So we see for the condition

$$\frac{\delta S}{\delta q} = 0$$

we receive

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{C.1}$$

which is the Euler Lagrange equation.

To see that this is consistent with Newtonian mechanics we let $L = \frac{1}{2}m\vec{\dot{q}}^2 + V(\vec{q})$ where m is the mass of a particle, $\dot{q}$ its velocity and inserted into equation C.1 we receive

$$m\vec{\ddot{q}} - \frac{\partial V(\vec{q})}{\partial \vec{q}} = 0,$$

and if the potential $V(q)$ is conservative we can write $F = -\frac{d}{dq}V$ and therefore get

$$m\ddot{q} = F$$

which we identify as Newton's second law of motion.

C.3 Transformation

In the introductory text we wrote that the Lagrangian is invariant under some symmetries. To see this we make an infinitesimal transformation for our coordinates $q_a(t) \to q_a(t) + \delta q_a(t)$, the variation of the Lagrangian

$$0 = \delta L = \sum_a \left[\left(\frac{\delta L}{\delta \dot{q}_a}\right)\delta\left(\frac{dq_a}{dt}\right) + \frac{\delta L}{q_a}\delta q_a\right]$$

under the transformation mentioned above

$$\frac{dq_a}{dt} \to \frac{d}{dt}\left(q_a + \delta q_a\right) = \frac{dq_a}{dt} + \frac{d}{dt}\delta q_a, \qquad \delta\left(\frac{dq_a}{dt}\right) = \frac{d}{dt}\delta q_a.$$

If we insert these transformations in the variation of the Lagrangian,

$$0 = \frac{\delta L}{\delta \dot{q}}\frac{d}{dt}\delta q_a + \frac{\delta L}{\delta q_a}\delta q_a$$

and with the direct use of the Euler Lagrange equation

$$\frac{\delta L}{\delta q} = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right),$$

we get

$$0 = \left(\frac{\delta L}{\delta \dot{q}}\right)\frac{d}{dt}\delta q + \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right)\delta q.$$

This can be simplified using the revered product rule

$$0 = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\delta q\right),$$

so the motion for each coordinate q_a, expressed as Q_a, is

$$Q_a = \frac{\delta L}{\delta \dot{q}_a}\delta q_a,$$

which is invariant under the transformation $q_a \to q_a + \delta q_a$.

C.4 Lagrangian field theory

Lets consider an elastic rod with fixed length l containing n particles with a distance a from each other. Each particle can vibrate from its equilibrium with a force constant k and the displacement of particle i is described by ϕ_i and the kinetic energy can therefore be written as,

$$T = \frac{1}{2}\sum_{i=1}^{n} m\dot{\phi}_i,$$

where m is the mass of each particle. The potential terms is the result of $n+1$ springs being stretched or compressed from its equilibrium,

$$V = \frac{1}{2}\sum_{i=0}^{n} k\left(\phi_{i+1} - \phi_i\right)^2.$$

So the Lagrangian can be written as,

$$L = T - V = \frac{1}{2}\sum_{i=1}^{n} m\dot{\phi}_i - \frac{1}{2}\sum_{i=0}^{n} k\left(\phi_{i+1} - \phi_i\right)^2$$

If we increase the number of particles n to infinity $n \to \inf$ while keeping the length $l = (n+1)/a$ and also the mass per length $\mu = m/a$ fixed. The extension per unit length is $(\phi_{i+1} - \phi_i)/a$ and the force between two particles is $F = k(\phi_{i+1} - \phi_i) = ka(\phi_{i+1} - \phi_i)/a$ so ka is the Young's modulus Y which should be kept constant in the material. We can now rewrite the Lagrangian as,

$$L = \frac{1}{2}\sum_{i=1}^{n} a\left(\frac{m}{a}\dot{\phi}_i^2\right) - \frac{1}{2}\sum_{i=0}^{n} a(ka)\left(\frac{\phi_{i+1} - \phi_i}{a}\right)^2$$

If we describe the system with a continuous coordinate x we can replace $\phi_i \to \phi(x)$, as we take the limit $a \to 0$ and $n \to n$. Then the Lagrangian becomes an integral over the length of the rod,

$$L = \frac{1}{2}\int_0^l \mu\dot{\phi}^2 - Y(\partial_x\phi)^2\,dx,$$

where we have used,

$$\lim_{a\to 0}\frac{\phi_{i+1}-\phi_i}{a}=\lim_{a\to 0}\frac{\phi(x+a)-\phi(x)}{a}=\partial_x\phi.$$

So it seems that it is, in the continuous case, natural to describe the Lagrangian as a integral of a density, the Lagrangian density $\mathcal{L}$, which can in analogy to the example above be generalised to n dimensions. Our expression for the Lagrangian is then

$$L=\int_V \mathcal{L}\,dV$$

and our action S,

$$S=\int_{t_1}^{t_2} dt\int_V \mathcal{L}\,dV$$

C.5 Hamiltonian formalism

We start by defining the canonical momenta p as

$$p^i \equiv \frac{\partial L(q_j,\dot{q}_j,t)}{\partial \dot{q}^i}$$

which is used to reduce the equations of motion to first order. Substituting this into equation C.1 we get,

$$\dot{p}_i=\frac{\partial L}{\partial q_i}.$$

We need to change $(q,\dot{q},t)\to(q,p,t)$ which is done by Legendre transformation. To do this we write the differential of the Lagrangian L,

$$dL=\frac{\partial L}{\partial q_i}\,dq_i+\frac{\partial L}{\partial \dot{q}_i}\,d\dot{q}_i+\frac{\partial L}{\partial t}\,dt$$

and rewriting this with the canonical momenta we get

$$dL=\dot{p}_i\,dq_i+p_i\,d\dot{q}_i+\frac{\partial L}{\partial t}\,dt$$

The Hamiltonian H is formed from the Legendre transformation of the Lagrangian and therefore

$$H(q,p,t)=\dot{q}^i p^i-L(q,\dot{q},t)$$

and dH can be written as,

$$dH=\dot{q}_i\,dp_i-\dot{p}_i\,dq_i-\frac{\partial L}{\partial t}\,dt$$

and since dH also can be written as,

$$dH=\frac{\partial H}{\partial q_i}\,dq_i+\frac{\partial H}{\partial p_i}\,dp_i+\frac{\partial H}{\partial t}\,dt$$

which gives us the Hamilton's equation of motion

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}.$$

The Hamiltonian can also be extended to the continuous case, much like for the Lagrangian, and gives a Hamiltonian density $\mathcal{H}$ which can be written as

$$\mathcal{H} = q_i p_i - \mathcal{L}.$$

C.6 Poisson brackets

The time derivative of any function f dependent on the canonical variables (p,q) can we written as

$$\frac{d}{dt} f(q,p) = \dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial t}.$$

Through Hamilton's equation of motion we can rewrite this as

$$\frac{d}{dt} f(q,p) = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t}. \tag{C.2}$$

This can be simplified in terms of the so called Poisson brackets. Two functions a and b dependent on canonical variables (q,p) forms the Poisson bracket by

$$\frac{\partial a}{\partial q^i}\frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i}\frac{\partial b}{\partial q^i} = \{a,b\}_{PB}. \tag{C.3}$$

Using these bracket we can rewrite the time derivative of a function, as in equation (C.2), as

$$\frac{d}{dt} f = \{f,H\}_{PB} + \frac{\partial f}{\partial t}. \tag{C.4}$$

If we put the two functions a and b as either q or p and inserted in the definition for the Poisson brackets in equation C.3, we get the following canonical Poisson brackets relations:

$$\{q^i,p_j\}_{PB} = \delta^i_j$$
$$\{q^i,q_j\}_{PB} = 0$$
$$\{p^i,p_j\}_{PB} = 0$$

To simplify this we only write the non-vanishing term, that is when the Poisson brackets contain mixed variables, to $\{q^i,p_j\}_{PB} = \delta^i_j$.

C.7 Quantisation

From the work above, we can make the transition to quantum mechanics by changing the Poisson brackets with commutation of operators. The canonical variables q and p are changed to operators $\hat{q}$ and $\hat{p}$, so the the transition we want to make is,

$$(q, p) \rightarrow (\hat{q}, \hat{p})$$
$$\{q, p\}_{PB} \rightarrow [\hat{q}, \hat{p}]$$

In quantum mechanics we have the relation $[\hat{x}, \hat{p}_x] = i\hbar$, where $\hat{x}$ and $\hat{p}_x$ represents position and momenta, it seems appropriate to make the normalisation $i\hbar\{x,p\}_{PB} = [\hat{x}, \hat{p}]$. Using this this transition and considering equation (C.4), replacing f and H with quantum operators $\hat{f}$ and $\hat{H}$ yields a common equation,

$$\frac{d}{dt}\hat{f} = [\hat{f}, \hat{H}] + \frac{\partial}{\partial t}\hat{f}$$

which is the Heisenberg equation of quantum mechanics. This equation is central in the Heisenberg picture of quantum mechanics and in contrast to the Schrödinger picture where the quantum states evolve in time, the operators is the one that evolves in time.

D

Deriving the Lie Algebra for the Conformal Group

In this appendix we demonstrate how one can determine the Lie algebra of a symmetry transformation group. We will be doing this for the conformal group as it is of importance to this thesis. Conformal transformations are transformations that preserve the angles between vectors but can change their lengths. This condition can be conveniently stated, as seen in equation (2.9), using the transformational matrices

$$\tilde{\Lambda}^{\mu}{}_{\nu} \equiv \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}} \equiv \partial_{\nu} \tilde{x}^{\mu} \tag{D.1}$$

as

$$\tilde{\Lambda}^{\mu}{}_{\rho} \tilde{\Lambda}^{\nu}{}_{\sigma} \eta_{\mu\nu} = A(x') \eta_{\rho\sigma}, \tag{D.2}$$

where $A(x)$ is an arbitrary scalar function and η is the Lorentzian metric. This condition states that, upon the coordinate transformation $x^{\mu} \to \tilde{x}^{\mu}$, the metric is rescaled by an arbitrary function, but not otherwise changed. A special case of this condition is when $A = 1$ at which point the coordinate transformations are those that leave the metric unchanged. The transformations that satisfy these condition are exactly the Poincaré transformations and as such we will find that the Poincaré and Lorentz groups are subgroups of the conformal group.

D.1 Deriving the conformal group's generators

In order to derive the infinitesimal Lie algebra of the conformal group, we want to determine the transformations differing infinitesimally from the unit transformation. These will be parametrised in some way and the parametrisation will enable us to read off the generators. As such we expand the coordinate transformation like

$$\tilde{x}^{\mu} = \tilde{\Lambda} x^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) + \mathcal{O}(\varepsilon^2). \tag{D.3}$$

By inserting (D.1) and (D.3) into (D.2) while suppressing the dependence $\varepsilon(x)$ we find that

$$A(x)\eta_{\mu\nu} = \eta_{\rho\sigma}\Big(\delta^{\rho}_{\mu} + \partial_{\mu}\varepsilon^{\rho} + \mathcal{O}(\varepsilon^2)\Big)\Big(\delta^{\sigma}_{\nu} + \partial_{\nu}\varepsilon^{\sigma} + \mathcal{O}(\varepsilon^2)\Big) =$$

$$\eta_{\rho\sigma}\Big(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\rho}_{\mu}\partial_{\nu}\varepsilon^{\sigma} + \delta^{\sigma}_{\nu}\partial_{\mu}\varepsilon^{\rho} + \mathcal{O}(\varepsilon^2)\Big) = \eta_{\mu\nu} + \partial_{\nu}\varepsilon_{\mu} + \partial_{\mu}\varepsilon_{\nu} + \mathcal{O}(\varepsilon^2)$$

By neglecting terms of higher order and letting $A(x) - 1 = B(x)$ this equality can be rewritten as

$$B(x)\eta_{\mu\nu} = \partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu \tag{D.4}$$

In order to remove the arbitrary function $B(x)$ we note that $\eta^{\mu\nu}\eta_{\mu\nu} = d$, where d is the dimension, whence

$$B(x)\eta_{\mu\nu}\eta^{\mu\nu} = B(x)d = \eta^{\mu\nu}\partial_\nu \varepsilon_\mu + \eta^{\mu\nu}\partial_\mu \varepsilon_\nu = 2\partial^\sigma \varepsilon_\sigma \Rightarrow$$

$$B(x) = \frac{2}{d}\partial^\sigma \varepsilon_\sigma. \tag{D.5}$$

where we've written $\eta^{\rho\nu}\partial_\rho$ as ∂^ν and exchanged summation indices. We continue by inserting this expression into equation (D.4) with the result

$$\frac{2}{d}\partial^\sigma \varepsilon_\sigma \eta_{\mu\nu} = \partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu. \tag{D.6}$$

By acting on this expression with $\partial^\mu \partial^\nu$, noting that $\partial^\mu \partial^\nu \eta_{\mu\nu} = \partial_\mu \partial^\mu$ and using the commutativity of partial derivatives, we find that

$$\partial_\rho \partial^\rho \partial^\sigma \varepsilon_\sigma = 0. \tag{D.7}$$

If we instead apply $\partial_\rho \partial^\nu$ to equation (D.6) the result is

$$\frac{2}{d}\partial_\rho \partial_\mu (\partial^\sigma \varepsilon_\sigma) = \partial_\rho (\partial^\sigma \partial_\sigma \varepsilon_\mu) + \partial_\rho \partial_\mu (\partial^\sigma \varepsilon_\sigma).$$

Adding to this expression the same expression with switched free indices results in

$$\left(\frac{4}{d} - 2\right) \partial_\mu \partial_\rho (\partial^\sigma \varepsilon_\sigma) = \partial^\sigma \partial_\sigma (\partial_\rho \varepsilon_\mu + \partial_\mu \varepsilon_\rho).$$

By using equations (D.6) and (D.7) the right hand of this expression is found to be zero and we have

$$\partial_\nu \partial_\rho \partial^\sigma \varepsilon_\sigma = 0. \tag{D.8}$$

By finally acting on equation (D.6) with $\partial_\rho \partial_\sigma$ and using equation (D.8) we find that

$$\partial_\rho \partial_\sigma \partial_\mu \varepsilon_\nu = -\partial_\rho \partial_\sigma \partial_\nu \varepsilon_\mu.$$

But as partial derivatives commute we find after shuffling indices that

$$\partial_\rho \partial_\sigma \partial_\mu \varepsilon_\nu = \partial_\rho \partial_\mu \partial_\sigma \varepsilon_\nu = -\partial_\rho \partial_\mu \partial_\nu \varepsilon_\sigma = -\partial_\rho \partial_\nu \partial_\mu \varepsilon_\sigma = \partial_\rho \partial_\nu \partial_\sigma \varepsilon_\mu = \partial_\rho \partial_\sigma \partial_\nu \varepsilon_\mu.$$

This third derivative of ε is both symmetric and antisymmetric in its last indices and must therefore be identically zero. Consequently ε must be a polynomial of degree 2 or lower and we write

$$\varepsilon^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu + c^\mu{}_{\nu\rho} x^\nu x^\rho. \tag{D.9}$$

We now insert this expression into equation (D.6), our original constraint. First we find that

$$\partial_\nu \varepsilon_\mu = \partial_\nu b_{\mu\gamma} x^\gamma + \partial_\nu c_{\mu\gamma\sigma} x^\gamma x^\sigma = b_{\mu\nu} + c_{\mu\nu\sigma} x^\sigma + c_{\mu\gamma\nu} x^\gamma$$

and

$$\partial_\mu \varepsilon^\mu = \partial_\mu b^\mu{}_\nu x^\nu + \partial_\mu c^\mu{}_{\nu\gamma} x^\gamma x^\mu = b^\mu{}_\nu + c^\mu{}_{\mu\gamma} x^\gamma + c^\mu{}_{\nu\mu} x^\nu .$$

By demanding that our constraint be satisfied for each component independent of the others we first find that, by letting b and c be zero, there are no constrains on a. If instead a and c are zero we find that equation (D.6) take the form

$$\frac{2}{d} b^\sigma{}_\sigma \eta_{\mu\nu} = b_{\mu\nu} + b_{\nu\mu} .$$

Splitting $b_{\mu\nu}$ into its symmetric and antisymmetric part as $\omega_{\mu\nu}$ and $k_{\mu\nu}$, where $k_{\mu\nu} = k_{\nu\mu}$ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$, the expression above can be rewritten as

$$\frac{2}{d} k^\sigma{}_\sigma \eta_{\mu\nu} = 2k_{\mu\nu}$$

whence there are no constraints on $\omega_{\mu\nu}$. As $k^\sigma{}_\sigma$ is a constant scalar we furthermore find that $k_{\mu\nu} = \alpha\eta_{\mu\nu}$.
In order to find the constraint on $c_{\mu\nu\rho}$ we take the derivative of equation (D.6),

$$\frac{2}{d} \partial_\gamma \partial_\sigma \varepsilon^\sigma \eta_{\mu\nu} = \partial_\gamma \partial_\nu \varepsilon_\mu + \partial_\gamma \partial_\mu \varepsilon_\nu . \tag{D.10}$$

Permuting the indices into

$$\frac{2}{d} \partial_\mu \partial_\sigma \varepsilon^\sigma \eta_{\nu\gamma} = \partial_\mu \partial_\gamma \varepsilon_\nu + \partial_\mu \partial_\nu \varepsilon_\gamma \tag{D.11}$$

and

$$\frac{2}{d} \partial_\nu \partial_\sigma \varepsilon^\sigma \eta_{\gamma\mu} = \partial_\nu \partial_\mu \varepsilon_\gamma + \partial_\nu \partial_\gamma \varepsilon_\mu \tag{D.12}$$

and then by adding equations (D.10) and (D.11) and subtracting equation (D.12) we get

$$\frac{2}{d}\Big(\partial_\gamma \partial_\sigma \varepsilon^\sigma \eta_{\mu\nu} + \partial_\mu \partial_\sigma \varepsilon^\sigma \eta_{\nu\gamma} - \partial_\nu \partial_\sigma \varepsilon^\sigma \eta_{\gamma\mu}\Big) =$$
$$\cancel{\partial_\gamma \partial_\nu \varepsilon_\mu} + \partial_\gamma \partial_\mu \varepsilon_\nu + \partial_\mu \partial_\gamma \varepsilon_\nu + \cancel{\partial_\mu \partial_\nu \varepsilon_\gamma} - \cancel{\partial_\nu \partial_\mu \varepsilon_\gamma} - \cancel{\partial_\nu \partial_\gamma \varepsilon_\mu} .$$

Defining

$$\frac{1}{d} \partial_\gamma \partial_\sigma \varepsilon^\sigma \equiv b_\gamma$$

and noting that

$$\partial_\gamma \partial_\mu \varepsilon_\nu = c_{\nu\gamma\mu}$$

we find that

$$\Big(b_\gamma \eta_{\mu\nu} + b_\mu \eta_{\nu\gamma} - b_\nu \eta_{\gamma\mu}\Big) = c_{\nu\gamma\mu} .$$

So if we set a and b to zero, equation (D.9) equals

$$\varepsilon^\mu = \eta^{\nu\rho} c_{\nu\gamma\mu} x^\gamma x^\mu = c^\rho{}_{\gamma\mu} x^\gamma x^\mu = \eta^{\nu\rho}\Big(b_\gamma \eta_{\mu\nu} + b_\mu \eta_{\nu\gamma} - b_\nu \eta_{\gamma\mu}\Big) x^\gamma x^\mu =$$

$$\eta^{\nu\rho} \eta_{\mu\nu} b_\gamma x^\gamma x^\mu + \eta^{\nu\rho} \eta_{\nu\gamma} b_\mu x^\gamma x^\mu - \eta^{\nu\rho} \eta_{\gamma\mu} b_\nu x^\gamma x^\mu =$$

$$\delta^\rho_\mu b_\gamma x^\gamma x^\mu + \delta^\rho_\gamma b_\mu x^\gamma x^\mu - b^\rho x^\mu x_\mu =$$
$$b_\gamma x^\gamma x^\rho + b_\mu x^\rho x^\mu - b^\rho x^\mu x_\mu = 2b_\gamma x^\gamma x^\rho - b^\rho x^\mu x_\mu.$$

By adding up all the results, we get the following expression:

$$\varepsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \alpha x^\mu + 2b_\nu x^\nu x^\mu - b^\mu x^\nu x_\nu, \tag{D.13}$$

where α is an arbitrary scalar, a^μ and b^μ are arbitrary vectors and $\omega^\mu{}_\nu$ is an arbitrary antisymmetric tensor.

We can now find the generators for the conformal group acting on a scalar field $f(x)$. This field must be invariant under our infinitesimal conformal transformation $f'(x') = f'(\Lambda x) \Rightarrow f'(x) = f(\Lambda^{-1}x)$. But the inverse of a conformal transformation must be another conformal transformation whence, using a Taylor expansion in ε,

$$f(x) = f(x+\varepsilon) = f(x) + \varepsilon^\mu \partial_\mu f(x) + \mathcal{O}(\varepsilon^2) \approx$$
$$(1+\varepsilon^\mu\partial_\mu)f(x) = \left[1 + (a^\mu + \omega^\mu{}_\nu x^\nu + \alpha x^\mu + 2b_\nu x^\nu x^\mu - b^\mu x^\nu x_\nu)\partial_\mu\right] f(x) =$$
$$\left[1 + (a^\mu + \omega^{\mu\nu} x_\nu + \alpha x^\mu + 2b^\nu x_\nu x^\mu - b^\mu x^\nu x_\nu)\partial_\mu\right].$$

We now define the generators of the conformal group as

$$P_\mu = \partial_\mu \tag{D.14}$$
$$D = -x^\mu \partial_\mu \tag{D.15}$$
$$M_{\mu\nu} = -\frac{1}{2}\Big(x_\mu\partial_\nu - x_\nu\partial_\mu\Big) \tag{D.16}$$
$$K_\mu = \Big(2x_\mu x^\nu \partial_\nu - x^\nu x_\nu \partial_\mu\Big). \tag{D.17}$$

Since ω is antisymmetric we have

$$\omega^{\mu\nu} x_\nu \partial_\mu = \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu) = -\frac{1}{2}\omega^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

and $f(x)$ can be written as

$$f(x) = (1+\varepsilon^\mu\partial_\mu)f(x) = (1 + a^\mu P_\mu - \alpha D + \omega^{\mu\nu} M_{\mu\nu} + b^\mu K_\mu) f(x).$$

We note here that $M_{\mu\nu}$ are the generators of translations and are the only generators of the Lorentz algebra. P_μ on the other hand are the generator of translation, so by using them together with $M_{\mu\nu}$ we have the generators of the Poincaré algebra. D and K_μ are inherently conformal on the other hand as they are the generators of scale transformations and special conformal transformations respectively.

D.2 Forming the Lie algebra for the conformal group

In the previous section (D.1), we derived the conformal generators which we restate for convenience here

$$P^a = \partial^a, \tag{D.18}$$
$$M^{ab} = -\frac{1}{2}(x^a\partial^b - x^b\partial^a), \tag{D.19}$$
$$D = -x^a\partial_a, \tag{D.20}$$
$$K^a = 2x^a x^d\partial_d - x^d x_d\partial^a, \tag{D.21}$$

where they are expressed in the local indices because of standard convention.

When we now want to determine the Lie algebra of these generators by calculating their commutators, where there are many lengthy calculations. As such we perform some initial, simplifying calculations that will be used repeatedly. Specifically we will perform some calculations involving $\partial^c M^{ab}$ and $\partial^b K^c$, as they cause most of the lengthy calculations. We start with

$$\partial^c M^{ab} = \partial^c\Big(-\frac{1}{2}(x^a\partial^b - x^b\partial^a)\Big) =$$
$$-\frac{1}{2}\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \partial^c x^a\partial^b - \partial^c x^b\partial^a\Big) =$$
$$-\frac{1}{2}\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \eta^{c\alpha}\partial_\alpha x^a\partial^b - \eta^{c\beta}\partial_\beta x^b\partial^a\Big) =$$
$$-\frac{1}{2}\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \eta^{c\alpha}\delta^a_\alpha\partial^b - \eta^{c\beta}\delta^b_\beta\partial^a\Big) =$$
$$-\frac{1}{2}\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \eta^{ca}\partial^b - \eta^{cb}\partial^a\Big)$$

which means that

$$\partial^c M^{ab} = -\frac{1}{2}\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \eta^{ca}\partial^b - \eta^{cb}\partial^a\Big). \tag{D.22}$$

Proceeding with $\partial^b K^c$ we have

$$\partial^b K^c = \partial^b\Big(2x^c x^d\partial_d - x^d x_d\partial^c\Big) =$$
$$2\Big((\partial^b x^c)x^d\partial_d + x^c(\partial^b x_d)\partial^d + x^c x_d\partial^b\partial^d\Big) - 2x^b\partial^c - x^d x_d\partial^b\partial^c =$$
$$2\eta^{bc}x^d\partial_d + 2(x^c\partial^b - x^b\partial^c) + 2x^c x_d\partial^b\partial^d - x^d x_d\partial^b\partial^c.$$

which means that

$$\partial^b K^c = 2\eta^{bc}x^d\partial_d + 2(x^c\partial^b - x^b\partial^c) + 2x^c x_d\partial^b\partial^d - x^d x_d\partial^b\partial^c. \tag{D.23}$$

We now want to calculate the commutators of these generators which gives us the Lie algebra of the conformal group. First we note that $\Big[P^a, P^b\Big] = 0$ as partial derivatives commute and $\Big[D, D\Big] = 0$ because the commutator is antisymmetric and there is only one generator D. After these commutation relations we proceed with $\Big[M^{ab}, P^c\Big]$ which we can write as

$$\Big[M^{ab}, P^c\Big] = M^{ab}\partial^c - \partial^c M^{ab}$$

and by using (D.22) we find that

$$\Big[M^{ab}, P^c\Big] = -\frac{1}{2}\Big(\cancel{x^a\partial^b\partial^c} - \cancel{x^b\partial^a\partial^c}\Big) + \frac{1}{2}\Big(\cancel{x^a\partial^c\partial^b} - \cancel{x^b\partial^c\partial^a} + \eta^{ca}\partial^b - \eta^{cb}\partial^a\Big),$$

so

$$\Big[M^{ab}, P^c\Big] = \frac{1}{2}(\eta^{ca}P^b - \eta^{cb}P^a). \tag{D.24}$$

If we perform the same calculation for $\Big[M^{ab}, K^c\Big]$ we find that

$$\Big[M^{ab}, K^c\Big] = -\frac{1}{2}(x^a\partial^b K^c - x^b\partial^a K^c) - (2x^c x_d \partial^d M^{ab} - x^d x_d \partial^c M^{ab}) =$$

$$\begin{aligned}
&-\frac{1}{2}\Big(2x^a\eta^{bc}x^d\partial_d + 2x^a(x^c\partial^b - x^b\partial^c) + 2x^a x^c x_d\partial^b\partial^d - x^a x^d x_d \partial^b\partial^c \\
&\quad -2x^b\eta^{ac}x^d\partial_d - 2x^b(x^c\partial^a - x^a\partial^c) - 2x^b x^c x_d\partial^a\partial^d + x^b x^d x_d\partial^a\partial^c\Big) \\
&-\Big(2x^c x_d\Big(-\frac{1}{2}\Big)\Big(x^a\partial^d\partial^b - x^b\partial^d\partial^a + \eta^{da}\partial^b - \eta^{db}\partial^a\Big) \\
&\quad -x^d x_d\Big(-\frac{1}{2}\Big)\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \eta^{ca}\partial^b - \eta^{cb}\partial^a\Big)\Big).
\end{aligned}$$

Simplyfying by removing parentheses and cancelling terms, we have

$$\begin{aligned}
&-x^a\eta^{bc}x^d\partial_d - \cancel{x^a(x^c\partial^b - x^b\partial^c)} - \cancel{x^a x^c x_d\partial^b\partial^d} + \cancel{\frac{1}{2}x^a x^d x_d\partial^b\partial^c} \\
&+x^b\eta^{ac}x^d\partial_d + \cancel{x^b(x^c\partial^a - x^a\partial^c)} + \cancel{x^b x^c x_d\partial^a\partial^d} - \cancel{\frac{1}{2}x^b x^d x_d\partial^a\partial^c} \\
&+\cancel{x^c x_d x^a\partial^d\partial^b} - \cancel{x^c x_d x^b\partial^d\partial^a} + \cancel{x^c x_d\eta^{da}\partial^b} - \cancel{x^c x_d\eta^{db}\partial^a} \\
&-\cancel{\frac{1}{2}x^d x_d x^a\partial^c\partial^b} + \cancel{\frac{1}{2}x^d x_d x^b\partial^c\partial^a} - \frac{1}{2}x^d x_d\eta^{ca}\partial^b + \frac{1}{2}x^d x_d\eta^{cb}\partial^a
\end{aligned}$$

where we used the metric to raise indices. Thus we find that

$$\Big[M^{ab}, K^c\Big] = -x^a\eta^{bc}x^d\partial_d + x^b\eta^{ac}x^d\partial_d - \frac{1}{2}x^d x_d\eta^{ca}\partial^b + \frac{1}{2}x^d x_d\eta^{cb}\partial^a =$$

$$-\frac{1}{2}\eta^{bc}(2x^a x^d\partial_d - x^d x_d\partial^a) + \frac{1}{2}\eta^{ac}(2x^b x^d\partial_d - x^d x_d\partial^b) =$$

$$\frac{1}{2}(\eta^{ac}K^b - \eta^{bc}K^a).$$

Proceeding with $\Big[M^{ab}, D\Big]$ we have

$$\Big[M^{ab}, D\Big] = \frac{1}{2}\Big((x^b\partial^a - x^a\partial^b)x^e\partial_e - x^e\partial_e(x^b\partial^a - x^a\partial^b)\Big)\Big)$$

$$+\frac{1}{2}\Big((x^b\eta^{a\alpha}\partial_\alpha - x^a\eta^{b\beta}\partial_\beta)x^e\partial_e - x^e\delta^b_e\partial^a - x^e\delta^a_e\partial^b\Big) =$$

$$\frac{1}{2}\Big((x^b\eta^{a\alpha}\delta^e_\alpha\partial_e - x^a\eta^{b\beta}\delta^e_\beta\delta_e) - x^b\partial^a - x^a\partial^b\Big) =$$

$$\frac{1}{2}\Big(x^b\eta^{ae}\partial_e - x^a\eta^{be}\partial_e - x^b\partial^a - x^a\partial^b\Big) = \frac{1}{2}\Big(x^b\partial^a - x^a\partial^b - x^b\partial^a - x^a\partial^b\Big)$$

$$= 0.$$

Next we calculate $\Big[M^{ab}, M^{cd}\Big]$ using equation (D.22), which results in

$$\Big[M^{ab}, M^{cd}\Big] = -\frac{1}{2}\Big((x^a\partial^b - x^b\partial^a)M^{cd} - (x^c\partial^d - x^d\partial^c)M^{ab}\Big) =$$

$$\frac{1}{4}x^a\Big(x^c\partial^b\partial^d - x^d\partial^b\partial^c + \eta^{bc}\partial^d - \eta^{bd}\partial^c\Big) - \frac{1}{4}x^b\Big(x^c\partial^a\partial^d - x^d\partial^a\partial^c + \eta^{ac}\partial^d - \eta^{ad}\partial^c\Big)$$

$$-\frac{1}{4}x^c\Big(x^a\partial^d\partial^b - x^b\partial^d\partial^a + \eta^{da}\partial^b - \eta^{db}\partial^a\Big) + \frac{1}{4}x^d\Big(x^a\partial^c\partial^b - x^b\partial^c\partial^a + \eta^{ca}\partial^b - \eta^{cb}\partial^a\Big) =$$

$$\frac{1}{4}\Big(\eta^{ad}(x^b\partial^c - x^c\partial^b) + \eta^{bc}(x^a\partial^d - x^d\partial^a) - \eta^{ac}(x^b\partial^d - x^d\partial^b) - \eta^{bd}(x^a\partial^c - x^c\partial^a)\Big),$$

as all second derivatives cancel since they commute. Using the definition of M^{ab} given in equation (D.19) we therefore have

$$\Big[M^{ab}, M^{cd}\Big] = \frac{1}{2}\Big(\eta^{ad}M^{cb} + \eta^{bc}M^{da} - \eta^{ac}M^{db} - \eta^{bd}M^{ca}\Big).$$

Continuing with $\Big[P^a, K^b\Big]$ we use equation (D.23),

$$\Big[P^a, K^b\Big] = \partial^a K^b - (2x^b x^d\partial_d - x^d x_d\partial^b)\partial^a =$$

$$2\eta^{ab}x^d\partial_d + 2(x^b\partial^a - x^a\partial^b) + \cancel{2x^b x_d\partial^a\partial^d - x^d x_d\partial^a\partial^b} - \cancel{(2x^b x^d\partial_d - x^d x_d\partial^b)\partial^a} =$$

and using the definition of our generators we find that

$$\Big[P^a, K^b\Big] = -2\eta^{ba}D + 4M^{ab}.$$

When we now calculate $\Big[D, K^a\Big]$ we find that we can reuse much of our previous calculation:

$$\Big[D, K^a\Big] = -x_e\partial^e K^a + (2x^a x^d\partial_d - x^d x_d\partial^a)x_e\partial^e =$$

$$-x_e\Big(\partial^e K^a - (2x^a x^d\partial_d - x^d x_d\partial^a)\partial^e\Big) + \eta_{ce}(2x^a x^d\partial_d - \eta^{ab}x^d x_d\partial_b)x^c\partial^e =$$

$$-x_e\Big(\partial^e K^a - (2x^a x^d\partial_d - x^d x_d\partial^a)\partial^e\Big) + \eta_{ce}(2x^a x^d\delta^c_d - \eta^{ab}x^d x_d\delta^c_b)\partial^e =$$

$$-2x_e\Big(\eta^{ea}x^d\partial_d + (x^a\partial^e - x^e\partial^a)\Big) + 2x^a x^d\partial_d - x^d x_d\partial^a =$$

$$-2x^a x^e\partial_e + x^e_e\partial^a = -K^a.$$

The commutation $[K^a, K^b]$ vanishes after a few switches of indices while we after a short calculation determine what $\Big[D, P^a\Big]$ is,

$$\Big[D, P^a\Big] = -x^e\partial_e\partial^a + \partial^a x^e\partial_e = -x^e\partial_e\partial^a + \Big(\partial^a(x^e)\partial_e + x^e\partial_e\partial^a\Big) = \eta^{ac}\partial_c(x^e)\partial_e =$$

$$\partial^a = P.$$

Summarising these results we have

$$\Big[M^{ab}, P^c\Big] = \frac{1}{2}\Big(\eta^{ca}P^b - \eta^{bc}P^a\Big) \tag{D.25}$$

$$\Big[M^{ab}, M^{cd}\Big] = \frac{1}{2}\Big(\eta^{ad}M^{cb} + \eta^{bc}M^{da} - \eta^{ac}M^{db} - \eta^{bd}M^{ca}\Big) \tag{D.26}$$

$$\Big[M^{ab}, K^c\Big] = \frac{1}{2}\Big(\eta^{ca}K^b - \eta^{bc}K^a\Big) \tag{D.27}$$

$$\Big[P^a, K^b\Big] = 2\Big(-\eta^{ab}D + 2M^{ab}\Big) \tag{D.28}$$

$$\Big[D, P^a\Big] = P^a \tag{D.29}$$

$$\Big[D, K^a\Big] = -K^a \tag{D.30}$$

while other commutators are zero.

D.3 Simplifying the Lie algebra in 2+1 dimensions

In the previous section we derived the Lie algebra for the conformal group of dimension $D > 2$. We now specifically study the case $D = 3$ where we can treat our rotations in a special way. In 3 dimensions we know that we can express rotations in a plane by specifying a vector normal to said plane. We define our 3 rotations using

$$M^a = \frac{1}{2}\epsilon^a{}_{bc}M^{bc},$$

where $\epsilon^a{}_{bc}$ is the Levi-Civita symbol. Now, as M^{bc} is antisymmetric, we have

$$\epsilon^{de}{}_a M^a = \frac{1}{2}\epsilon^{de}{}_a\epsilon^a{}_{bc}M^{bc} = -\delta^{de}_{bc}M^{bc} =$$

$$-\frac{1}{2}\Big(\delta^d_b\delta^e_c - \delta^d_c\delta^e_b\Big)M^{bc} = -\frac{1}{2}(M^{de} - M^{ed}) = -M^{de}$$

which leads to

$$M^{bc} = -\epsilon^{bc}{}_a M^a.$$

By rewriting equation (D.25) we find that

$$\begin{aligned}
\Big[M^{ab}, P_c\Big] &= \frac{1}{2}\Big(\eta^{ca}P^b - \eta^{bc}P^a\Big)\\
-\epsilon^{ab}{}_d\Big[M^d, P^c\Big] &= \frac{1}{2}\Big(\eta^{ca}P^b - \eta^{bc}P^a\Big)\\
-\epsilon^f{}_{ab}\epsilon^{ab}{}_d\Big[M^d, P^c\Big] &= \frac{1}{2}\epsilon^f{}_{ab}\Big(\eta^{ca}P^b - \eta^{bc}P^a\Big)\\
2\delta^f_d\Big[M^d, P^c\Big] &= \frac{1}{2}\epsilon^f{}_{ab}\Big(\eta^{ca}P^b - \eta^{bc}P^a\Big)\\
\Big[M^d, P^c\Big] &= \frac{1}{4}\Big(\epsilon^{dc}{}_bP^b - \epsilon^d{}_a{}^cP^a\Big) = \frac{1}{2}\epsilon^{dc}{}_bP^b
\end{aligned}$$

We can by the similarity of equations (D.25) and (D.27) clearly see that this further implies

$$\Big[M^a, K^b\Big] = \frac{1}{2}\epsilon^{ab}{}_cK^c.$$

If we rewrite the left side of equation (D.26) we find that

$$\Big[M^{cd}, M^{ef}\Big] = (-\epsilon^{cd}{}_a)(-\epsilon^{ef}{}_b)\Big[M^a, M^b\Big] = \epsilon^{cd}{}_a\epsilon^{ef}{}_b\Big[M^a, M^b\Big].$$

If we then introduce $\epsilon^g{}_{cd}$ and $\epsilon^h{}_{ef}$ to both sides of (D.26), we find that the left hand side can be rewritten as

$$\epsilon^g{}_{cd}\epsilon^{cd}{}_a\epsilon^h{}_{ef}\epsilon^{ef}{}_b\Big[M^a, M^b\Big] = 4\delta^g_a\delta^h_b\Big[M^a, M^b\Big] = 4\Big[M^g, M^h\Big].$$

Therefore (D.26) can be written as

$$\Big[M^a, M^b\Big] = \frac{1}{8}\epsilon^a{}_{de}\epsilon^b{}_{fg}\Big(\eta^{dg}M^{fe} + \eta^{ef}M^{gd} - \eta^{df}M^{ge} - \eta^{eg}M^{fd}\Big). \qquad \text{(D.31)}$$

Starting with the first term on the right hand side we find that

$$\begin{gathered}
\epsilon^a{}_{de}\epsilon^b{}_{fg}\eta^{dg}M^{fe} = -\epsilon^a{}_{de}\epsilon^b{}_{fg}\eta^{dg}\epsilon^{fe}{}_cM^c =\\
-\epsilon^{age}\epsilon^{bf}{}_g\epsilon_{fec}M^c = 2\delta^{ag}_{cf}\epsilon^{bf}{}_gM^c = -\epsilon^{ba}{}_cM^c = \epsilon^{ab}{}_cM^c
\end{gathered}$$

By switching indices we find that

$$\epsilon^a{}_{de}\epsilon^b{}_{gf}\eta^{eg}M^{fd} = \epsilon^a{}_{ed}\epsilon^b{}_{fg}\eta^{eg}M^{fd} = \{d \leftrightarrow e, g \leftrightarrow f\} = \epsilon^a{}_{de}\epsilon^b{}_{gf}\eta^{df}M^{ge}$$

and similarly for the final two terms except with only one permutation producing a sign difference. Summing all these terms we finally arrive at

$$\Big[M^a, M^b\Big] = \frac{1}{2}\epsilon^{ab}{}_cM^c.$$

The last equation containing M^{ab}, and therefore the last we have to rewrite, is equation (D.28).

$$\begin{aligned}
\Big[P^a, K^b\Big] &= 2\Big(-\eta^{ab}D + 2M^{ab}\Big) = 2\Big(-\eta^{ab}D + 2(-\epsilon^{ab}{}_cM^c)\Big)\\
&= -2\eta^{ab}D - 4\epsilon^{ab}{}_cM^c
\end{aligned}$$

Summing up, we have derived the following Lie algebra for the conformal group in 2+1 dimensions

$$
\begin{aligned}
\left[M^a, P^b\right] &= \frac{1}{2}\epsilon^{ab}{}_c P^c, \\
\left[M^a, K^b\right] &= \frac{1}{2}\epsilon^{ab}{}_c K^c, \\
\left[M^a, M^b\right] &= \frac{1}{2}\epsilon^{ab}{}_c M^c, \\
\left[P^a, K^b\right] &= -2\eta^{ab} D - 4\epsilon^{ab}{}_c M^c, \\
\left[D, P^a\right] &= P^a, \\
\left[D, K^a\right] &= -K^a.
\end{aligned}
$$

After one final rescaling of M^a to $\frac{1}{2}M^a$ we finally have

$$
\begin{aligned}
\left[M^a, P^b\right] &= \epsilon^{ab}{}_c P^c, \\
\left[M^a, K^b\right] &= \epsilon^{ab}{}_c K^c, \\
\left[M^a, M^b\right] &= \epsilon^{ab}{}_c M^c, \\
\left[P^a, K^b\right] &= -2\eta^{ab} D - 2\epsilon^{ab}{}_c M^c, \\
\left[D, P^a\right] &= P^a, \\
\left[D, K^a\right] &= -K^a.
\end{aligned}
$$

E

Calculations of the Riemann Curvature Tensor using Cartan Formalism

In the standard coordinate dependent approach, the Riemann curvature tensor is calculated by first finding the Christoffel symbols from the metric using the metric postulate,

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right).$$

After this step is done the Riemann tensor is then determined from

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\mu\sigma}.$$

This is easily seen to be a cumbersome process that quickly become very lengthy with increasing dimension. Instead one can use the power of differential forms and their exterior algebra from the Cartan formalism to obtain the same result faster.

As documented in section 4.3, the objects that are used during the calculations of the curvature are the vielbein one-forms e^a and spin connection one-forms $\omega^a{}_b$. First one easily form the first of these objects as the transformational matrix between the coordinate and the orthonormal Lorentz basis. Then the latter is determined using the torsion free condition

$$de^a = \omega_{[cb]}{}^a \, e^c \wedge e^b \tag{E.1}$$

and the antisymmetry of the spin connection,

$$\omega_{abc} = \omega_{[ab]c} - \omega_{[cb]a} + \omega_{[ca]b}. \tag{E.2}$$

Finally one calculates the curvature form as

$$\frac{1}{2} R_{abcd} e^c \wedge e^d = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b \tag{E.3}$$

and read off the Riemann tensor R_{abcd}, expressed in the local Lorentz basis. We will perform these calculations for educational purposes for three different manifolds: the 2-sphere S_2, the 3-sphere S_3 and anti-de Sitter, ADS_3. Finally we will also determine the Ricci tensor R_{ab} and the Ricci scalar R for these spaces by simply contracting indices of the Riemann tensor.

E.1 S^2 in local coordinates

S^2 is the two dimensional surface of a three dimensional sphere with a radius of L. If we use the spherical coordinates

$$x = L\sin\theta\cos\varphi \tag{E.4}$$
$$y = L\sin\theta\sin\varphi \tag{E.5}$$
$$z = L\cos\theta, \tag{E.6}$$

and imposes the S^2-restriction

$$x^2 + y^2 + z^2 = L^2$$

,
we find that the metric is given by

$$g_{\mu\nu} = L^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin\theta^2 \end{pmatrix}.$$

This means that the zweibeins are given by

$$e_\theta{}^1 = L$$
$$e_\varphi{}^2 = L\sin\theta$$

while the rest are zero. Expressed like one-forms, $e^a = e_\mu{}^a dx^\mu$, this is equivalent to

$$e^1 = L\ d\theta$$
$$e^2 = L\sin\theta\ d\phi.$$

With this knowledge in hand, we proceed to find the spin connection. This is done in two steps, first solving equation E.1 for $\omega_{[cb]}{}^a$ and then using it to find ω_{abc} from equation E.2. To solve equation E.1 we first take the exterior derivative of the zweibei, which is

$$de^1 = 0$$
$$de^2 = L\cos\theta\ d\theta \wedge d\phi,$$

or when expressed using our zweibeins,

$$de^2 = L^{-1}\cot\theta\ e^1 \wedge e^2.$$

To find $\omega_{[cb]}{}^a$ we need to solve E.1 for all values of a. This gives the following two relationships

$$\omega_{[cb]}{}^1 = 0$$
$$\omega_{[cb]}{}^2\, e^c \wedge e^b = L^{-1}\cot\theta\ e^1 \wedge e^2,$$

which can be combined into

$$\omega_{[cb]2} = \delta^{12}_{cb} L^{-1} \cot\theta.$$

Using this we now find that the only non zero solutions of E.2 are

$$-\omega_{221} = \omega_{212} = -L^{-1}\cot\theta.$$

which means that the spin connection is given by

$$\omega_{abc} = -2L^{-1}\cot\theta\,\delta^2_a\delta^{12}_{bc},$$

or expressed as a one-form, $\omega_{ab} = e^c\omega_{cab}$,

$$\omega_{ab} = -2\cos\theta\delta^{12}_{ab}\,d\phi.$$

Using the one-form of the spin connection we can now derive the Riemann Tensor using equation E.3. It can easily be seen that the external derivative of the spin connection is

$$d\omega_{ab} = 2\sin\theta\delta^{12}_{ab}\,d\theta\wedge d\phi$$

or

$$d\omega_{ab} = 2L^{-2}\delta^{12}_{ab}\,e^1\wedge e^2.$$

Furthermore knowing that the exterior product is antisymmetric we find that

$$\omega_{ab}\wedge\omega^b{}_c = 0.$$

Therefore equation E.3 yields

$$\frac{1}{2}R_{abcd}\,e^c\wedge e^d = 2L^{-2}\delta^{12}_{ab}\,e^1\wedge e^2,$$

which means that the Riemann tensor for S_2 can be written as

$$R_{abcd} = 4\delta^{12}_{ab}\delta^{12}_{cd}L^{-2}. \tag{E.7}$$

Using this we find that the Ricci tensor is

$$R_{bd} = R^a{}_{bad} = L^{-2}(\delta^1_b\delta^1_d + \delta^2_b\delta^2_d) = L^{-2}\,\eta_{bd} \tag{E.8}$$

and the Ricci scalar is

$$R = 2L^{-2}. \tag{E.9}$$

E.2 $\mathbf{S}^3$

S^3 is the surface of a three-dimensional hypersphere with a radius of L. If we work in hyper-spherical coordinates

$$\begin{aligned}
x &= L\cos\theta \\
y &= L\sin\theta\cos\varphi \\
z &= r\sin\theta\sin\varphi\cos\psi \\
w &= r\sin\theta\sin\varphi\sin\psi,
\end{aligned}$$

we find the metric to be

$$g_{\mu\nu} = L^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta^2 & 0 \\ 0 & 0 & \sin\theta^2\sin\phi^2 \end{pmatrix},$$

This means that the dreibeins can be expressed in their one-form form as

$$\begin{aligned}
e^1 &= L\ d\theta \\
e^2 &= L\sin\theta\ d\varphi \\
e^3 &= L\sin\theta\sin\varphi\ d\psi.
\end{aligned}$$

We will now find the spin connection using equations E.1 and E.2 like we did for S^2. We find that the exterior derivative applied to the dreibeins is given by

$$\begin{aligned}
de^1 &= 0 \\
de^2 &= L\cos\theta\ d\theta\wedge d\varphi \\
de^3 &= L\cos\theta\sin\varphi\ d\theta\wedge d\psi + L\sin\theta\cos\varphi\ d\varphi\wedge d\psi,
\end{aligned}$$

or equivalently

$$\begin{aligned}
de^1 &= 0 \\
de^2 &= L^{-1}\cot\theta\ e^1\wedge e^2 \\
de^3 &= L^{-1}\cot\theta\ e^1\wedge e^3 + L^{-1}\cot\varphi\ e^2\wedge e^3.
\end{aligned}$$

Now we use equation E.1 with different values of a and find that

$$\omega_{[cb]}{}^1 e^c\wedge e^b = 0 \tag{E.10}$$

$$\omega_{[cb]}{}^2 e^c\wedge e^b = L^{-1}\cot\theta\ e^1\wedge e^2 \tag{E.11}$$

$$\omega_{[cb]}{}^3 e^c\wedge e^b = L^{-1}(\cot\theta\ e^1\wedge e^3 + \cot\varphi\ e^2\wedge e^3), \tag{E.12}$$

which in turn means that

$$\omega_{[cb]}{}^2 = \delta^{12}_{cb} L^{-1}\cot\theta \tag{E.13}$$

$$\omega_{[cb]}{}^3 = \delta^{13}_{cb}\ L^{-1}\cot\theta + \delta^{23}_{cb}\ L^{-1}\cot\varphi\csc\theta. \tag{E.14}$$

Using equation E.2 we now find that the spin connection is given by

$$\omega_{2bc} = -2L^{-1}\delta^{12}_{bc}\cot\theta$$
$$\omega_{3bc} = -2L^{-1}(\delta^{13}_{bc}\cot\theta + \delta^{23}_{bc}\cot\varphi\csc\theta)$$

which when combined with one-forms is the same as

$$\omega_{ab} = -2\delta^{12}_{ab}\ \cos\theta\ d\varphi - 2(\delta^{13}_{ab}\ \cos\theta\sin\varphi + \delta^{23}_{ab}\ \cos\varphi)\ d\psi.$$

We can now use the one-form of the spin connection to find the Riemann tensor using equation E.3. To do this we take the external derivative of the spin connection which results in

$$d\omega_{ab} = 2\delta^{12}_{ab}\ \sin\theta\ d\theta\wedge d\varphi + 2\delta^{13}_{ab}\ \sin\theta\sin\varphi\ d\theta\wedge d\psi - \tag{E.15}$$
$$2\delta^{13}_{ab}\ \cos\theta\cos\varphi\ d\varphi\wedge d\psi + 2\delta^{23}_{ab}\ \sin\varphi\ d\varphi\wedge d\psi \tag{E.16}$$

and similarly we find that

$$\omega_a{}^c\wedge\omega_{cb} = (2\delta^{13}_{ab}\cos\theta\cos\varphi\ - 2\delta^{23}_{ab}\cos^2\theta\sin\varphi)\ d\varphi\wedge d\psi.$$

This simplifies to

$$\omega_a{}^c\wedge\omega_{cb} = (2\delta^{13}_{ab}\cos\theta\cos\varphi\ - 2\delta^{23}_{ab}(1-\sin^2\theta)\sin\varphi)\ d\varphi\wedge d\psi.$$

Therefore we find that

$$\frac{1}{2}R_{abcd}\ e^c\wedge e^d = 2\delta^{12}_{ab}\ \sin\theta\ d\theta\wedge d\varphi + 2\delta^{13}_{ab}\ \sin\theta\sin\varphi\ d\theta\wedge d\psi + 2\delta^{23}_{ab}\ \sin^2\theta\sin\varphi\ d\varphi\wedge d\psi$$

which simplifies to

$$\frac{1}{2}R_{abcd}\ e^c\wedge e^d = 2L^{-2}\delta^{12}_{ab}\ e^1\wedge e^2 + 2L^{-2}\delta^{13}_{ab}\ e^1\wedge e^3 + 2L^{-2}\delta^{23}_{ab}\ e^2\wedge e^3,$$

whence we find that

$$R_{abcd} = 2\delta^{12}_{ab}\delta^{12}_{cd}L^{-2} + 2\delta^{13}_{ab}\delta^{13}_{cd}L^{-2} + 2\delta^{23}_{ab}\delta^{23}_{cd}L^{-2}.$$

This means that we find the Ricci tensor is given by

$$R_{ab} = R^c{}_{bcd} = 2L^{-2}(\delta^1_a\delta^1_b + \delta^2_a\delta^2_b + \delta^3_a\delta^3_b) = L^{-2}\,\eta_{ab},$$

and Ricci scalar by

$$R = \frac{6}{L^2}.$$

E.3 AdS_3

AdS_3 is the hyperboloid in $3+2$ dimensional space given by the restriction

$$x^2 + y^2 - w^2 - t^2 = -L^2.$$

If we use the parametrisation

$$\begin{aligned} w &= L\cosh\rho\cos\tau \\ t &= L\cosh\rho\sin\tau \\ x &= L\sinh\rho\sin\theta \\ y &= L\sinh\rho\cos\theta \end{aligned}$$

We find that the infinitesimal displacement in these coordinates is given by

$$ds^2 = L^2(-\cosh^2\rho\ d\tau^2 + \sinh^2\rho\ d\theta^2 + d\rho^2)$$

and so the metric is given by

$$g_{\mu\nu} = L^2 \begin{pmatrix} -\cosh^2\rho & 0 & 0 \\ 0 & \sinh^2\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we then choose the signature of the flat coordinates as $(-, +, +)$,

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we find that

$$\begin{aligned} e^1 &= L\cosh\rho\ d\tau \\ e^2 &= L\sinh\rho\ d\theta \\ e^3 &= L\ d\rho. \end{aligned}$$

We then proceed by taking the exterior derivative of these dreibeins like

$$\begin{aligned} de^1 &= -L\sinh\rho\ d\tau\wedge d\rho \\ de^2 &= -L\cosh\rho\ d\theta\wedge d\rho \\ de^3 &= 0. \end{aligned}$$

Using equation E.1 and choosing $a = 1$ we have

$${\omega_{[cb]}}^1\ e^c\wedge e^b = -L\sinh\rho\ d\tau\wedge d\rho$$

and hence

$$-{\omega_{[ab]}}^1 = 2\delta^{13}_{ab}\ L^{-1}\tanh\rho.$$

If we instead choose $a = 2$ we find that

$$\omega_{[cb]}{}^{2}\ e^c \wedge e^b = -L \cosh\rho\ d\theta \wedge d\rho$$

and thus

$$\omega_{[ab]}{}^{2} = -2\delta^{23}_{ab}\ L^{-1} \coth\rho.$$

Finally, by letting $a = 3$, we find that

$$\omega_{[ab]}{}^{3} = 0,$$

so upon combining these results we have,

$$\begin{aligned} \omega_{[ab]}{}^{1} &= -2\delta^{13}_{ab}\ L^{-1} \tanh\rho \\ \omega_{[ab]}{}^{2} &= -2\delta^{23}_{ab}\ L^{-1} \coth\rho \\ \omega_{[ab]}{}^{3} &= 0. \end{aligned}$$

and therefore

$$\begin{aligned} \omega_{[ab]1} &= 2\delta^{13}_{ab}\ L^{-1} \tanh\rho \\ \omega_{[ab]2} &= -2\delta^{23}_{ab}\ L^{-1} \coth\rho. \end{aligned}$$

Using equation E.2 we now find that

$$\begin{aligned} \omega_{1bc} &= -2\delta^{13}_{bc}\ L^{-1} \tanh\rho \\ \omega_{2bc} &= 2\delta^{23}_{bc}\ L^{-1} \coth\rho \end{aligned}$$

and so

$$\begin{aligned} \omega_{ab} &= -2\delta^{13}_{ab}\ \sinh\rho\ d\tau + 2\delta^{23}_{ab}\ \cosh\rho\ d\theta, \\ \omega_{a}{}^{b} &= -2\eta^{bc}\delta^{13}_{ac}\ \sinh\rho d\tau + 2\eta^{bc}\delta^{23}_{ac}\ \cosh\rho d\theta. \end{aligned}$$

If we solve for $\omega_{a}{}^{c} \wedge \omega_{cb}$ we get

$$\omega_{a}{}^{c} \wedge \omega_{cb} = 2\delta^{12}_{ab} \sinh\rho \cosh\rho\ d\tau \wedge d\theta,$$

or equivalently

$$\omega_{a}{}^{c} \wedge \omega_{cb} = 2L^{-2}\delta^{12}_{ab}\ e^1 \wedge e^2.$$

If we instead solve for $d\omega_{ab}$ we find that

$$d\omega_{ab} = 2\delta^{13}_{ab}\ \cosh\rho\ d\tau \wedge d\rho - 2\delta^{23}_{ab}\ \sinh\rho\ d\theta \wedge d\rho,$$

or

$$d\omega_{ab} = 2L^{-2}\delta^{13}_{ab}\ e^1 \wedge e^3 - 2L^{-2}\delta^{23}_{ab}\ e^2 \wedge e^3.$$

From this we determine that the curvature form is given by

$$R_{ab} = 2L^{-2}\delta^{12}_{ab}\ e^1 \wedge e^2 + 2L^{-2}\delta^{13}_{ab}\ e^1 \wedge e^3 - 2L^{-2}\delta^{23}_{ab}\ e^2 \wedge e^3,$$

which means that the Riemann tensor becomes

$$R_{abcd} = 4L^{-2}\delta^{12}_{ab}\delta^{12}_{cd} + 4L^{-2}\delta^{13}_{ab}\delta^{13}_{cd} - 4L^{-2}\delta^{23}_{ab}\delta^{23}_{cd}.$$

Contracting this we find that the Ricci tensor is given by

$$R_{ab} = R^{c}{}_{acb} = -2\eta_{ab}\,L^{-2}$$

and that the Ricci scalar is

$$R = -6L^{-2}.$$

F

Miscellaneous Calculations

F.1 Calculations for $Tr\,[A \wedge A \wedge A]$ for the Poincaré group

In the calculation of $Tr\,[A \wedge A \wedge A]$ in $2+1$-dimensional space for the Poincaré group, we find from (4.17) that

$$Tr\,[A \wedge A \wedge A] = \tag{F.1}$$
$$e^a \wedge \omega^b \wedge \omega^c\, Tr\,[P_a M_b M_c] + \omega^a \wedge e^b \wedge \omega^c\, Tr\,[M_a P_b M_c] + \omega^a \wedge \omega^b \wedge e^c\, Tr\,[M_a M_b P_c]\,.$$

The commutation relations for the Poincaré group in $2+1$-dimension are given by (2.6), (2.7) and (2.8) as

$$\begin{aligned}[P_a, P_b] &= 0\\ [M_a, P_b] &= \epsilon_{ab}{}^{c} P_c\\ [M_a, M_b] &= \epsilon_{ab}{}^{c} M_c\end{aligned} \tag{F.2}$$

and its trace relations are given in (4.14) and (4.15) as

$$\begin{aligned}Tr\,[P_a M_a] &= \eta_{ab}\\ Tr\,[P_a P_b] = Tr\,[M_a M_b] &= 0.\end{aligned}$$

The first part of equation (F.1), is

$$\begin{gathered}e^a \wedge \omega^b \wedge \omega^c\, Tr\,[P_a M_b M_c] =\\ \frac{1}{2} e^a \wedge \omega^b \wedge \omega^c\, Tr\left[P_a M_{(b} M_{c)}\right] = \frac{1}{2} e^a \wedge \omega^b \wedge \omega^c\, Tr\left[P_a\left[M_b, M_c\right]\right] =\\ \frac{1}{2}\epsilon_{ab}{}^{d} e^a \wedge \omega^b \wedge \omega^c\, Tr\,[P_a M_d] = \frac{1}{2}\epsilon_{bc}{}^{d}\eta_{da} e^a \wedge \omega^b \wedge \omega^c =\\ \frac{1}{2}\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c.\end{gathered}$$

The second part is

$$\omega^a \wedge e^b \wedge \omega^c\, Tr\,[M_a P_b M_c]\,,$$

and by rewrite the commutation relations (F.2) as $M_a P_b = P_b M_a + \epsilon_{ab}{}^{d} P_d$, this part can be written as

$$\omega^a \wedge e^b \wedge \omega^c \left(Tr\left[P_b M_a M_c\right] + Tr\left[\epsilon_{ab}{}^d P_d M_c\right]\right) =$$

$$\omega^a \wedge e^b \wedge \omega^c \left(\frac{1}{2} Tr\left[P_b M_{[a} M_{c]}\right] + Tr\left[\epsilon_{ab}{}^d P_d M_c\right]\right) =$$

$$\omega^a \wedge e^b \wedge \omega^c \left(\frac{1}{2} Tr\left[P_b \left[M_a, M_c\right]\right] + Tr\left[\epsilon_{ab}{}^d P_d M_c\right]\right) =$$

$$\omega^a \wedge e^b \wedge \omega^c \left(\frac{1}{2} Tr\left[P_b \epsilon_{ac}{}^d M_d\right] + Tr\left[\epsilon_{ab}{}^d P_d M_c\right]\right).$$

By performing the trace calculations, we find that the second part equals

$$\left(\frac{1}{2}\epsilon_{acb} + \epsilon_{abc}\right)\omega^a \wedge e^b \wedge \omega^c = \epsilon_{abc}\left(1 - \frac{1}{2}\right)\omega^a \wedge e^b \wedge \omega^c =$$

$$\frac{1}{2}\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c$$

The last part is

$$\omega^a \wedge \omega^b \wedge e^c\, Tr\left[M_a M_b P_c\right] = \frac{1}{2}\omega^a \wedge \omega^b \wedge e^c\, Tr\left[M_{(a} M_{b)} P_c\right]$$

$$\frac{1}{2}\omega^a \wedge \omega^b \wedge e^c\, Tr\left[\left[M_a, M_b\right] P_c\right] = \epsilon_{ab}{}^d \omega^a \wedge \omega^b \wedge e^c\, Tr\left[M_d P_c\right]$$

$$\epsilon_{ab}{}^d \eta_{dc} \omega^a \wedge \omega^b \wedge e^c = \epsilon_{abc} \omega^a \wedge \omega^b \wedge e^c =$$

$$\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c.$$

By summing up we find

$$Tr\left[A \wedge A \wedge A\right] = \frac{3}{2}\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c.$$

F.2 Calculations for specific parts of $e \wedge e \wedge e$ in Spin 3, AdS$_3$

In this section, we will evaluate the equation (5.5), here named (F.3), given in section 5.1 for spin 3

$$Tr\left[T_a e^a \wedge T_{bc} e^{bc} \wedge T_{de} e^{de}\right] + Tr\left[T_{bc} e^{bc} \wedge T_{de} e^{de} \wedge T_a e^a\right] + Tr\left[T_{bc} e^{bc} \wedge T_a e^a \wedge T_{de} e^{de}\right]. \tag{F.3}$$

and for this we the generators commutation and trace relations given by

$$\left[T_a, T_b\right] = \epsilon_{ab}{}^c T_c,$$
$$\left[T_a, T_{bc}\right] = \epsilon^d{}_{a(b} T_{c)d},$$
$$\left[T_{ab}, T_{cd}\right] = -\left(\eta_{a(c}\epsilon_{d)b}{}^e + \eta_{b(c}\epsilon_{d)a}{}^e\right) T_e,$$

and

$$Tr\left[T_aT_b\right] = 2\eta_{ab},$$
$$Tr\left[T_aT_{bc}\right] = 0,$$
$$Tr\left[T_{ab}T_{cd}\right] = -\frac{4}{3}\eta_{ab}\eta_{cd} + 2\left(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}\right).$$

By calculating

$$T_{bc}e^{bc} \wedge T_{de}e^{de} = \frac{1}{2}e^{bc} \wedge e^{de}\Big[T_{bc}, T_{de}\Big] =$$
$$-\frac{1}{2}\left(\eta_{b(d}\epsilon_{e)c}{}^{f} + \eta_{c(d}\epsilon_{e)b}{}^{f}\right)T_f.$$

and inserting this to equation (F.3) and perform the trace relation, we get the first term as

$$-e^a \wedge e^{bc} \wedge e^{de}\left(\eta_{b(d}\epsilon_{e)c}{}^{f} + \eta_{c(d}\epsilon_{e)b}{}^{f}\right)\eta_{af} =$$
$$-\frac{1}{2}e^a \wedge e^{bc} \wedge e^{de}\left(\eta_{bd}\epsilon_{eca} + \eta_{be}\epsilon_{dca} + \eta_{cd}\epsilon_{eba} + \eta_{ce}\epsilon_{dba}\right)$$

The second term is

$$Tr\left[T_{bc}e^{bc} \wedge T_{de}e^{de} \wedge T_ae^a\right],$$

by doing the same commutation and trace as before, we get

$$Tr\left[T_{bc}e^{bc} \wedge T_{de}e^{de} \wedge T_ae^a\right] = -e^a \wedge e^{bc} \wedge e^{de}\left(\eta_{b(d}\epsilon_{e)c}{}^{f} + \eta_{c(d}\epsilon_{e)b}{}^{f}\right)\eta_{fa},$$

which is the same as for the first term.
The third term in the expression is

$$Tr\left[T_{bc}e^{bc} \wedge T_ae^a \wedge T_{de}e^{de}\right],$$

to simplify we start of by using the commutation relation $T_{bc}T_a = T_aT_{bc} - \epsilon^{f}{}_{b(c}T_{a)f}T_{de}$, which give

$$Tr\left[e^{bc} \wedge e^a \wedge e^{de}\left(T_aT_{bc}T_{de} - \epsilon^{f}{}_{b(c}T_{a)f}T_{de}\right)\right],$$

where we see that the first part is

$$Tr\left[e^{bc} \wedge e^a \wedge e^{de}\left(T_aT_{bc}T_{de}\right)\right] = -Tr\left[T_ae^a \wedge T_{bc}e^{bc} \wedge T_{de}e^{de}\right].$$

We see that this is the almost the same as the first term, just differing with a negative sign.
Left to evaluate is

$$e^a \wedge e^{bc} \wedge e^{de}\left(\epsilon_f{}^{b(c}T_{a)f}T_{de}\right) = \frac{1}{2}e^a \wedge e^{bc} \wedge e^{de}\left(\epsilon_f{}^{bc}T_{af}T_{de} +_f{}^{ba}T_{cf}T_{de}\right) =$$
$$\frac{1}{2}e^a \wedge e^{bc} \wedge e^{de}\left(\epsilon_f{}^{bc}\left(-\frac{4}{3}\eta_{af}\eta_{de} + 2(\eta_{ad}\eta_{fe} + \eta_{ae}\eta_{fd})\right) + \epsilon_f{}^{ba}\left(-\frac{4}{3}\eta_{cf}\eta_{de} + 2(\eta_{ce}\eta_{fd} + \eta_{cd}\eta_{fe})\right)\right) =$$

$$e^a \wedge e^{bc} \wedge e^{de} \left(\epsilon_{ebc}\eta_{ad} + \epsilon_{dbc}\eta_{ae} + \epsilon_{dba}\eta_{ce} + \epsilon_{eba}\eta_{cd} \right).$$

So to sum up

$$Tr\left[T_a e^a \wedge T_{bc}e^{bc} \wedge T_{de}e^d e\right] + Tr\left[T_{bc}e^{bc} \wedge T_{de}e^{de} \wedge T_a e^a\right] + Tr\left[T_{bc}e^{bc} \wedge T_a e^a \wedge T_{de}e^{de}\right] =$$

$$-\frac{1}{2} e^a \wedge e^{bc} \wedge e^{de} \left(\epsilon_{eca}\eta_{bd} + \epsilon_{dca}\eta_{be} + \epsilon_{eba}\eta_{cd} + \epsilon_{dba}\eta_{ce} - 2\epsilon_{ebc}\eta_{ad} - 2\epsilon_{dbc}\eta_{ae} - 2\epsilon_{dba}\eta_{ce} - 2\epsilon_{eba}\eta_{cd} \right) =$$

$$e^a \wedge e^{bc} \wedge e^{de} \left(\epsilon_{ebc}\eta_{ad} + \epsilon_{dbc}\eta_{ae} + \frac{1}{2}\epsilon_{dba}\eta_{ce} + \frac{1}{2}\epsilon_{eba}\eta_{cd} - \frac{1}{2}\epsilon_{eca}\eta_{bd} - \frac{1}{2}\epsilon_{dca}\eta_{be} \right) =$$

$$e^a \wedge e^{bc} \wedge e^{de} \left(2\epsilon_{bcd}\eta_{ae} + \epsilon_{bad}\eta_{ce} - \epsilon_{eca}\eta_{bd} \right),$$

where we in the last step used that the frame fields are symmetric as $e^{ab} = e^{ba}$.

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