Thesis for the Degree of Master of Science in Physics

Automorphic Forms in String Theory

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Abstract

In this thesis we give an introduction to automorphic forms in string theory by examining a well-known case in ten-dimensional Type IIB superstring theory. An automorphic form, constructed as a non-holomorphic Eisenstein series $\mathcal{E}_{3/2}^{\mathrm{SL}(2,\mathbb{Z})}$, is known to encode all perturbative and non-perturbative quantum corrections in the genus expansion for the R^4-term included in the asymptotic string expansion for the effective action. Explicit calculations are shown, and group- and algebra theoretical aspects are thoroughly explained.

Furthermore, we study Type IIA superstring theory compactified on a rigid Calabi-Yau threefold, which is the topic of the recent paper [9]. Here, our focus is more on the explicit calculations and less on physical interpretations. A discrete group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$, called the *Picard modular group*, is believed to be a preserved symmetry of the quantum theory, and an automorphic form, constructed as a non-holomorphic Eisenstein series $\mathcal{E}_{3/2}^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}$, is conjectured to encode the quantum corrections to the metric of the hypermultiplet moduli space, which classically is a coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. To read off the loop corrections arising from the string coupling $g_s = \mathrm{e}^{\phi}$, as well as the non-perturbative instanton corrections, we want to rewrite the Eisenstein series as a Fourier series. The general Fourier series is decomposed into a constant, abelian and non-abelian part, referring to the action of the maximal nilpotent subgroup $\mathrm{H}_3(\mathbb{Z}) \subset \mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$. The main complication arises when trying to identify the coefficients in the non-abelian part of the Fourier expansion. We try to bring some clarity to this issue.

Keywords

Automorphic form, Eisenstein series, Fourier series, Type IIA and Type IIB superstring theory, coset space, double coset, modular group, Picard modular group.

Acknowledgements

"I feel a very unusual sensation – if it is not indigestion, I think it must be gratitude." – Benjamin Disraeli.

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“There is a theory which states that if ever anyone discovers
exactly what the Universe is for and why it is here,
it will instantly disappear and be replaced
by something even more bizarre and inexplicable.
There is another theory which states that this has already happened.”

–Douglas Adams (The Restaurant at the End of the Universe)

Contents

1

Introduction

1.1 Background

The cosmos is a remarkable place in every sense. At the risk of being too poetic, we shall immediately put on our “physical glasses” and switch to “scientific mode”. We see a world consisting of space(time) and interacting matter. The interaction is governed by four observed physical forces: electromagnetism, the strong force, the weak force and gravity. The world seems rather deterministic and classical until one zooms in on the microcosm where we see a world ruled by quantum mechanics.

Guided by principles of unification and reductionism, theoretical physicists have advanced in the quest to find a fundamental theory that would summarize the physical world in a simple way[1]. This line of action has been proven to be very fruitful in the past. For example, Newton unified the seemingly two different phenomena of the falling apple and the moon orbiting the earth, Maxwell unified electricity and magnetism, Einstein unified space and time and Feynman amongst others unified quantum mechanics and electromagnetism. The list can be made long. Naturally, one might think that the unification will go on and reduce the number of loose ends down to one unified theory of physics from which one would be able to obtain all other experimentally verified effective theories, that is, theories only valid as approximations in certain settings. The language used is that of mathematics and the most important tool is the identification and use of symmetries that somehow are a very fundamental principle of nature. In mathematics, symmetries are explained by groups and algebras.

Note that finding a unified fundamental theory, or a *theory of everything*, will not make us know “everything”, although very much. We would know

[1]Then naturally, one asks what a *simple* explanation is, but this philosophical question will not be answered here. Practically this question is often not a problem for theoretical physicists, or is it?!

only the “rules of the game”. To know the rules of chess does not imply that you are a chess master.

Today, we have on one hand by studying the microcosmos arrived at the *standard model*, based on the Lie group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, describing quite satisfactory the interaction of all observed particles neglecting gravity. On the other hand, by studying the macrocosm we have arrived at the *general theory of relativity* giving an explanation of how mass geometrically affects spacetime and causes gravity. The standard model is a quantum mechanical theory whereas the general relativity is not. Sadly, the two otherwise so successful theories cannot be combined in a consistent way. Another way of saying this is that we have not found a good way of quantizing gravity. Physicists have instead began trying to find a more fundamental theory, incorporating both the standard model and the general relativity.

The most promising candidate for a fundamental theory of everything today is string theory. Firstly, a reason for this is that string theory is a quantum mechanical theory that naturally incorporates gravity. In some sense one could actually say that string theory predicts general relativity when assuming Lorentz invariance! Secondly, the string theory works quite well as a perturbative field theory in line with the standard model. And it generically includes supersymmetry, the symmetry between bosons and fermions, that is of some physicists believed to exist in nature. As this is being written, scientists at LHC[2] are making measurements that will hopefully soon give an answer to the physical relevance of supersymmetry. Thirdly, string theory has a great potential of being a fundamental theory drawing from the fact that the only free dimensionful parameter in string theory is the string length l_s, whereas for instance in the standard model there are 19 adjustable parameters such as the lepton masses and the quark masses.

The fundamental constituents in string theory are one-dimensional strings that vibrate in different modes, much as a cello string can vibrate in different harmonics. The different vibrations are supposed to give rise to the different particles such as the quarks and the electron. The strings can be open and closed. If we hypothetically had a particle accelerator with infinite energy we would be able to probe into the inner “stringy” structure of the particles. However, the energies required are way beyond practical reach in the foreseeable future. One has instead to seek methods giving evidence for the strings more indirectly. Besides the strings there are higher dimensional objects in string theory, the most prominent examples being the NS5-brane and the D-branes.

Since we have encountered both bosonic and fermionic particles in the world, for a theory of everything we need also a string theory that incorporates both bosonic and fermionic strings. When constructing such a theory, a so-called *superstring* theory, one comes to the conclusion that it naturally includes supersymmetry and that it must exist in ten-dimensional spacetime to be consistent. It turns out that there are five superstring theories, namely:

[2]The *Large Hadron Collider*.

- Type I
- Type IIA
- Type IIB
- $E_8 \times E_8$ heterotic
- SO(32) heterotic.

The physicists E. Witten, C. M. Hull and P. K. Townsend, amongst others, discovered that all five theories are related to one another via certain duality transformations and that there is a unifying theory of the superstring theories in eleven dimensions called *M-theory* [28, 41].

An apparent problem is to get from the ten-dimensional superstring theory to an effective four-dimensional theory, capable of explaining our observed world. A reasonable scenario is that the six extra spatial dimensions are curled up and very small, so that they are not directly seen unless in extreme circumstances. Depending on what these extra dimensions look like, one gets different effective four-dimensional theories as candidates for the observed universe. This process is called *compactification* and can be seen as a generalization of Kaluza-Klein theory, which tries to unify gravity with electromagnetism by making one spatial dimension compact in a five-dimensional spacetime.

Despite the successes of string theory there are many subtleties and some problems with using the theory as a theory of everything. The most obvious problem is perhaps the difficulty of making experimental verifications. Another problem is the anthrophic landscape, which refers to the immensely large number of possible configurations ($\sim 10^{500}$(!)) of string theories that could be a theory explaining the observed universe [5, 17]. This derives from the fact that one has a large freedom in choosing the space to compactify the theory on. String theoreticians have come up with possible solutions to this problem using multiverse ideas combined with the anthropic principle [14]. This is a hot field of ongoing research and it depends strongly on the most recent experimental data acquired from outer space, by measuring the background radiation, down to the LHC measuring probably never before seen particles.

Many people believe that the significance of string theory is still very high although one may have to let go of the expectation of it to give a unique theory of everything. Instead, one can view string theory as a *framework*, much as quantum mechanics, in which other important theories can be formulated. Also notable is the ongoing progress with the AdS/CFT correspondence, first proposed by J. Maldacena in [32], and its recent applications to for example quantum chromodynamics.

For a first introduction to string theory the reader is referred to [42] and [30].

1.2 Prerequisites

Mathematics

The reader should have taken an introductory course in Lie groups and Lie algebras. However, many of the algebraic parts of the thesis are more carefully explained in the appendices. The reader should also have taken courses in Fourier analysis and complex analysis.

Physics

The reader is recommended to have basic knowledge of special and general relativity, quantum field theory, supersymmetry and string theory.

1.3 Constants and Definitions

Mathematics

The real and imaginary part of a complex number τ are denoted as

$$\Re(\tau) \qquad \text{and} \qquad \Im(\tau). \tag{1.1}$$

Summations will for simplicity be denoted as:

$$\begin{aligned} \sum_{n} &= \sum_{n\in\mathbb{Z}}, \\ \sum_{n\neq 0} &= \sum_{n\in\mathbb{Z}\setminus\{0\}}, \\ \sum_{(m,n)\neq(0,0)} &= \sum_{m}\sum_{n} \qquad \text{except } m=n=0. \end{aligned} \tag{1.2}$$

The latter sum can also be denoted by a prime. This holds in general for a sum over components of a vector $\vec{m} = (a, b, \ldots)$:

$$\sum_{\vec{m}}{}' = \sum_{a}\sum_{b}\ldots \qquad \text{except } a=b=\ldots=0. \tag{1.3}$$

The *sign function* is denoted as

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases} \tag{1.4}$$

There will appear a number of different kinds of special functions. They will all be properly introduced in the text; an exception is the *Gamma function*, which is defined for positive integers as

$$\Gamma(n) = (n+1)!, \qquad \forall n \in \mathbb{N}. \tag{1.5}$$

For complex numbers with a positive real part there is the standard integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t, \qquad \forall z \in \mathbb{C} \text{ and } \Re(z) > 0. \tag{1.6}$$

As convention we will write groups in roman type, e.g.: G, N, A, K and their corresponding Lie algebras as $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{k}$. Group representatives are written in italics e.g.: V, N, A. $\oplus$ stands for *direct sum* of vector spaces, and will be used when decomposing Lie algebras into vector subsets. The group K will in this thesis always be the *maximal compact subgroup* of G. For simplicity we will adopt the following notation

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{k}, \tag{1.7}$$

which simply means

$$\{\forall x \in \mathfrak{k}, y \in \mathfrak{p} \mid [x, y] = z \in \mathfrak{k}\}. \tag{1.8}$$

Physics

Throughout the thesis we will use the natural units $c = \hbar = 1$ for elegance and simplicity. As mentioned before, the only dimensionful parameter in string theory is the *string length* l_s. It is related to the *slope parameter* (or *worldsheet coupling*) α', and also to the *string tension* T_0 through

$$\alpha' = l_s^2, \qquad T_0 = \frac{1}{2\pi l_s^2}. \tag{1.9}$$

Any of the three parameters is as good to use as any other, but it is common to use α' which we will do henceforth.

All string theories contain a scalar field ϕ called the dilaton, which governs the strength of string interactions. Indeed, the string coupling constant g_s is by definition determined by the vacuum expectation value of the dilaton $g_s := \mathrm{e}^{\langle\phi\rangle}$.

1.4 Automorphic Forms

This thesis have two purposes:

- to give an introduction to the automorphic forms in string theory that is intended to be understood by students with a master's degree in fundamental physics. The introduction will be made by studying an already well-known case in Type IIB superstring theory. We will introduce general mathematical techniques that can be used in other cases as well, including the one below.

- to investigate further into a problem in the recent paper [9], where one studies Type IIA superstring theory compactified on a rigid Calabi-Yau threefold. More specifically, we will try to identify the coefficients in a non-abelian Fourier expansion of a specific automorphic form.

For some generality, we state the definition of an automorphic form: given a Lie group G one can construct the coset space

$$\mathrm{G}/\mathrm{K}, \tag{1.10}$$

where K is the *maximal compact subgroup* of G. A function f living on the coset G/K is an automorphic form if it:

i. transforms under a discrete group $\mathrm{G}(\mathbb{Z}) \subset \mathrm{G}$ as

$$f(\gamma \cdot x) = j_\gamma(x) f(x), \qquad \gamma \in \mathrm{G}(\mathbb{Z}) \quad \text{and} \quad x \in \mathrm{G}/\mathrm{K} \tag{1.11}$$

where $j_\gamma(x)$ is called the *factor of automorphy*

ii. is an eigenfunction of all the Casimir operators of G

iii. is satisfying some conditions at infinity.

The domain of the automorphic forms is then the double coset

$$\mathrm{G}(\mathbb{Z})\backslash \mathrm{G}/\mathrm{K}, \tag{1.12}$$

which is parameterized by a number of parameters (these will be related to the scalar fields in the string theory in question). This space is called the *moduli space*. The automorphic forms can be scalar-valued or vector-valued.

In this thesis we will deal with a special case of automorphic forms constructed as non-holomorphic[3] Eisenstein series $\mathcal{E}_s$, which form a subset of the so-called *Maass wave forms* (the other subset being *cusp forms*, which are deemed non-physical for supersymmetric reasons). They are scalar-valued and the important properties (for our purposes) of these functions are:

i. invariance under a discrete group $\mathrm{G}(\mathbb{Z}) \subset \mathrm{G}$, that is

$$\mathcal{E}_s(\gamma \cdot x) = \mathcal{E}_s(x), \qquad \gamma \in \mathrm{G}(\mathbb{Z}) \quad \text{and} \quad x \in \mathrm{G}/\mathrm{K} \tag{1.13}$$

ii. the functions are eigenfunctions of the Laplace-Beltrami operator[4] on the coset G/K, that is

$$\Delta_{\mathrm{G}/\mathrm{K}} \mathcal{E}_s = \lambda \mathcal{E}_s, \qquad \lambda \in \mathbb{C} \tag{1.14}$$

[3] A holomorphic function is a complex-valued function that is *complex-differentiable* in a neighborhood of every point in its domain; a non-holomorphic function is not.

[4] The Laplace-Beltrami operator on G/K is defined as $\Delta_{\mathrm{G}/\mathrm{K}} = |g|^{-1/2}\partial_i(\sqrt{|g|}g^{ij}\partial_j)$ where g_{ij} is the metric on G/K.

iii. the functions are well-behaved at infinity (cusp). In practice this means that our Eisenstein series should not diverge in the low-coupling limit $g_s = e^{\phi} \longrightarrow 0$. The dilatonic field will be the only non-compact argument of our functions.

I.e., the factor of automorphy is identically one, and one of the Casimir operators is the Laplace-Beltrami operator on G/K, which is quadratic in the derivatives and will have a physical importance.[5] The subscript s is called the *order* of the series and we will later see the significance of its value. The double cosets (1.12) that we will treat and construct Eisenstein series on are for the Type IIB case $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and for the Type IIA case $\mathrm{SU}(2,1;\mathbb{Z}[i])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$.

For an introduction to automorphic forms and Eisenstein series in string theory see for instance [37], [34] and [8]. N. A. Obers and B. Pioline were the first to investigate the relevance of the automorphic forms in string theory more rigorously in [33]. There are also the books by Terras [39] and [40] that explains the automorphic forms from a more mathematical point of view in a quite elementary way.

1.5 Cosets and Double Cosets

The superstring theories, as almost all physical theories, rely on symmetries of various kinds. The language used to describe these are that of continuous and discrete Lie groups, and their corresponding Lie algebras. In this thesis it is important to fully understand the coset spaces (1.10) and the double coset spaces (1.12), specifically for the cases mentioned above: $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $\mathrm{SU}(2,1;\mathbb{Z}[i])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. A large part of the thesis is devoted to this understanding. The group- and algebra theoretic details are treated quite extensively in the appendices.

To get a feeling for the process in constructing a double coset, we now provide a cursory explanation of a way to see this in the case of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. First of all, one can show, by constructing a Riemannian metric on the Lie group $\mathrm{SL}(2,\mathbb{R})$, that the group is isomorphic to a three-dimensional hyperbolic space with a compact direction. When dividing out the maximal compact subgroup SO(2), one effectively takes away the compact dimension. The resulting coset space is isomorphic to the upper complex half-plane $\mathbb{H}$, parameterized by the complex variable $\tau = \tau_1 + i\tau_2$ with $\tau_2 > 0$.[6]

[5] The functions are still eigenfunctions of *all* possible Casimir operators.

[6] Actually, to be exact, the isomorphism is

$$\mathbb{H} \simeq \mathrm{PSL}(2,\mathbb{R})/\mathrm{SO}(2), \tag{1.15}$$

where the P stands for *projective* and $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\mathbb{1},-\mathbb{1}\}$; this is also discussed in Section 2.1. The coset $\mathrm{SL}(2,\mathrm{R})/\mathrm{SO}(2)$ is isomorphic to *two* disjoint two-dimensional hyperbolic surfaces. We will henceforth always refer to the projective special linear group when writing $\mathrm{SL}(2,\mathbb{R})$, which is a common convention in the literature.

The result of dividing out the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ is then that one enforces periodicity conditions on this half-plane. The whole group $\mathrm{SL}(2,\mathbb{Z})$ is generated by two elements[7]: a translation and an involution, which act on the complex number as $\tau \longmapsto \tau + 1$ and $\tau \longmapsto -1/\tau$, respectively. Points in the upper half-plane that are related to each other via the two transformations are identified. This yields a *fundamental domain*, which we can pick as

$$D = \{\forall \tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } |\Re(\tau)| \leq 1/2\}, \tag{1.16}$$

that essentially represents the double coset. See figure 1.1.

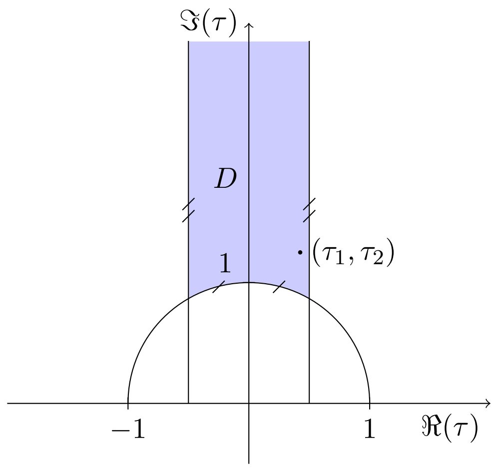

Figure 1.1: The fundamental domain in the upper complex plane representing $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

As a side remark, the involution transformation above is precisely the S-duality symmetry of Type IIB superstring theory. The dilaton field (coupling constant) is related to the parameter $\tau_2 = \mathrm{e}^{-\phi}$, so one sees the fantastic fact that S-duality relates a theory with large coupling constant to one with small coupling constant and vice versa.

1.6 String Interactions

Obviously a very important thing in string theory is to know how strings interact, how they scatter. The question one likes to answer is: if two strings collide what are the possible final states and what is the likelihood of these states? The physical quantity of interest is the scattering cross section. The cross section is proportional to the absolute square of the *scattering amplitude* A, which includes all the dynamics of the interaction. A is the non-trivial

[7]For more information see Appendix D.

factor that we want to calculate for different interactions in the string theory, whereas the rest of the cross section is given by known field theory methods. Having an explicit expression (in our case we will have a perturbative and non-perturbative series expansion) for the action of the theory in question makes it possible to calculate A.

To calculate the scattering amplitude in superstring theories turns out to be a hard task. As in ordinary quantum field theory one uses perturbative methods, but in string theory there are also important non-perturbative contributions originating from so-called *instantons*, which are objects localized in spacetime.

At first sight, it seems insurmountable to calculate scattering amplitudes more than up to a small order, but it turns out that one can use the string dualities (S, T and U) as a tool for doing this. The dualities are realized as discrete groups $G(\mathbb{Z})$. The fact that the superstring theories are invariant under $G(\mathbb{Z})$ restricts the possible form, and also value, of the terms in the perturbative and non-perturbative expansion. Here the automorphic forms become important. They are by construction functions of the physical fields in the theory, as well as invariant under $G(\mathbb{Z})$. They can in some circumstances encode the terms in the string action expansion.

1.7 The Type IIB Case

As mentioned before we will treat a case of Type IIB superstring theory, which will serve as a good introductory example of the significance of automorphic forms. The method and calculations follow [23] and [34]. We will study the *effective* theory, that is, the theory valid as an approximation in the low-energy limit, for small α'. The effective theory can be used to describe how the massless states interact, e.g., the bosonic graviton interactions. The effective quantum theory action consists of a classical part plus additional terms originating from quantum corrections. Moreover, there are non-perturbative corrections due to the presence of instantons. For Type IIB superstring theory it occurs that the classical part of the effective action is Type IIB *supergravity*. This is one of the very attractive features of superstring theory.

In the effective theory for small α' one can write an asymptotic series expansion of a closed string scattering amplitude as follows[8]

$$A = \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} (\alpha')^{n-4} g_s^{2(g-1)} A_{(n,g)}, \tag{1.17}$$

where g denotes the genus of the string worldsheet and $A_{(n,g)}$ denotes the amplitude of order $n-4$ in α' for the string worldsheet with genus g. The expansion (1.17) represents an effective string action containing terms to all orders in α' and g_s. The leading order $(n, g) = (0, 0)$ gives the ten-dimensional

[8] See [25] and the references there in.

Type IIB supergravity. If we for simplicity only consider the terms made of the Riemann tensor $R_{\mu\nu\rho\sigma}$, we have the following ten-dimensional Einstein-Hilbert action in the string frame[9]

$$S_{(0,0)} = (\alpha')^{-4} \int \mathrm{d}^{10}x \sqrt{|g|}\, \mathrm{e}^{-2\phi} R. \tag{1.18}$$

$g^{\mu\nu}$ is the metric on the Minkowski space. Let us step a bit further in the n power series. By supersymmetric reasons the terms $S_{(1,0)}$ and $S_{(2,0)}$ can be shown to be zero [25]. The next non-zero contribution is the term $S_{(3,0)}$, which can be explicitly calculated, coming from the tree-level four-graviton scattering, to be

$$S_{(3,0)} = (\alpha')^{-4} \int \mathrm{d}^{10}x \sqrt{|g|}\, \mathrm{e}^{-2\phi} (\alpha')^3 R^4, \tag{1.19}$$

where R^4 is a certain combination of the curvature scalar R, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$. As can be seen from dimensional analysis, the term has eight derivatives on the metric $g^{\mu\nu}$ since α' has the dimension of length squared[10]. The particular form of R^4 is not relevant for the discussion. One can step further in the n-expansion but it quickly gets very difficult to find explicitly the correct terms in the action, although there are several methods for simplifying these calculations.

When going to higher order in g_s, that is adding loop-corrections, one sees that there are no corrections to the first term $n = 0$. However, there are corrections for $n = 3$ to the R^4-term in (1.19)

$$S_{(3,g)} = (\alpha')^{-4} \int \mathrm{d}^{10}x \sqrt{|g|}\, \mathrm{e}^{-2\phi} (\alpha')^3 \sum_{g=0}^{\infty} c_g \mathrm{e}^{2(g-1)\phi} R^4, \tag{1.20}$$

with unknown coefficients c_g. It turns out that one can determine those coefficients and as a bonus also determine non-perturbative contributions, which are not seen in the asymptotic expansion. Namely, the quantum corrections are encoded in an automorphic form. This relies on the fact that the Type IIB superstring theory is invariant under the S-duality group $\mathrm{SL}(2,\mathbb{Z})$ [41]. The invariance shall hold for the effective theory as well, and it must hold individually for each order n in the asymptotic expansion since the discrete group transformations does not change orders of α'.

One can show that the additional non-perturbative corrections are coming from D-brane instantons. The instanton corrections are needed to make the

[9] We can make a Weyl rescaling of the metric tensor $g_{\mu\nu} \longrightarrow \mathrm{e}^{\phi/2} g_{\mu\nu}$ in the string action. This does not change the physics, i.e., scattering cross sections, and it means that we can write the action in different equivalent ways. The case when we have a dilaton factor $\mathrm{e}^{-2\phi}$ in front of the Riemann curvature scalar is referred to as the *string frame*, the other case without this factor is referred to as the *Einstein frame*.

[10] There is always a factor α'^{-4} making the measure dimensionless. This can be seen from the fact that there should be a factor $1/G_{\mathrm{N}}^{(10)}$ together with the R-term, and we have that $16\pi G_{\mathrm{N}}^{(10)} = (2\pi)^7 (\alpha')^4 g_s^2$.

asymptotic expansion valid for large g_s, and it is very crucial that they actually pop up in the automorphic forms. The non-perturbative nature of the factors is seen from the fact that they have a dependence of g_s of the form e^{-1/g_s}, which lack a Taylor expansion around $g_s = 0$, and therefore the factors are not seen in (1.17).

Green and Gutperle in [23] made it reasonable to think that the exact expression of the g_s-terms for order $n = 3$ should be of the form[11]

$$S_{(3,<\infty)} = (\alpha')^{-1} \int \mathrm{d}^{10}x \sqrt{|g|}\, f(\tau_1, \tau_2) R^4, \tag{1.21}$$

where $f(\tau_1, \tau_2)$ is a function parameterized on the coset upper half plane

$$\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{H} \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2), \tag{1.22}$$

invariant under the discrete duality group $\mathrm{SL}(2,\mathbb{Z})$. There are two scalar fields in the Type IIB superstring theory, the dilaton e^{ϕ} and an axion χ, and they are related to the real and complex part of τ as $\tau_1 = \chi$ and $\tau_2 = \mathrm{e}^{-\phi}$. The function $f(\tau_1, \tau_2)$ should be an eigenfunction of the Laplace-Beltrami operator on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ due to supersymmetric reasons, and it should be well-behaved in the low-coupling limit $g_s = \tau_2^{-1} \longrightarrow 0$. I.e., $f(\tau_1, \tau_2)$ is an automorphic form on the double coset $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

The first two coefficients, to the tree-level and one-loop term with $g = 0, 1$ have been calculated by explicit methods [24, 26] to

$$c_0 = 2\zeta(3) \quad \text{and} \quad c_1 = 4\zeta(2), \tag{1.23}$$

where $\zeta(z)$ denotes the Riemann zeta function. This implies that the function $f(\tau_1, \tau_2)$ must include these leading terms in g_s, that is

$$f(\tau_1, \tau_2) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \dots. \tag{1.24}$$

Note that the orders of $g_s = \mathrm{e}^{\phi} = \tau_2^{-1}$ does not match the asymptotic expansion. This is because we are in the Einstein frame. However, the factors differ with an order $g_s^2 = \tau_2^{-2}$, which must be unchanged after a Weyl-rescaling. We will see that there is indeed a unique way of determining the function (1.24) by constructing it as a non-holomorphic Eisenstein series $\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}$. The correct leading terms are recovered by a specific choice of the parameter s.

1.8 The Type IIA Case

It is tempting to study the automorphic forms in other similar cases. With the knowledge from the Type IIB case we will be able to treat the more intricate case where we compactify ten-dimensional Type IIA superstring theory on a

[11] Here we have with a Weyl-rescaling transformed to the *Einstein frame*. For the classical action in this frame, there is a global SL(2,$\mathbb{R}$) invariance as we will show in Chapter 2.

so-called rigid Calabi-Yau threefold. The resulting four-dimensional theory possesses an $\mathcal{N} = 2$ supersymmetry, which includes, amongst others, a universal hypermultiplet with four scalar fields. We will confine our studies to this universal hypermultiplet. In the classical theory, the moduli space of the scalar fields is a coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2) \times \mathrm{U}(1))$. The question is then how this classical symmetry group will be broken when taking into account quantum corrections.

We follow [9] where one conjectures that the quantum corrections leave an invariance under the discrete group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ called the *Picard modular group*. There are reasons then to think that, like in the Type IIB case above, there should exist an automorphic form, in this case on the double coset $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2) \times \mathrm{U}(1))$, that somehow sits in an asymptotic expansion of the string action. However, unlike the Type IIB case where we studied corrections to the R^4-term, we here investigate how the moduli space metric in the sigma model for the scalar fields is deformed by the quantum corrections. It will not be possible to directly see how this automorphic form should sit in the effective action as in the Type IIB case. For this, more advanced theory is needed; in [9] certain twistor techniques are used. Unfortunately, this is beyond the scope of this thesis. In Chapter 6 we will construct the automorphic form sought after, as a non-holomorphic Eisenstein series $\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}$.

1.9 Outline of the Thesis

We begin in Chapter 2 to show how the symmetric coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ for the scalar fields in Type IIB supergravity manifests itself. We will then shortly discuss how symmetric coset spaces arise when one compactifies Type IIA supergravity. In particular the space $\mathrm{SU}(2,1)$, when compactifying on a certain rigid three-dimensional complex Calabi-Yau manifold.

To get some more understanding from the group-theoretical point of view, we will in Chapter 3 describe how one can construct an action on a coset. We will then produce the nonlinear sigma model actions appearing in the Type IIB and Type IIA cases by choosing explicitly the cosets $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $\mathrm{SU}(2,1)/(\mathrm{SU}(2) \times \mathrm{U}(1))$. However, the method is general and can be used with any coset space.

In Chapter 4 we will show how the quantum corrections and instanton corrections to a classical part of a string theory are connected with the appearance of invariance of the theory under discrete symmetry groups. We will give the background on how to construct the automorphic forms as Eisenstein series that encode the quantum corrections with help of the double coset associated with the theory in question. To read off the physics we need to find an equivalent expression for the Eisenstein series in the form of a Fourier series. The Fourier series is found using a nilpotent subgroup described in Section 4.1.

In Chapters 5 and 6 we will explicitly derive the general Fourier series and the Eisenstein series for the two cases Type IIB and Type IIA and extract the precious Fourier coefficients needed for the effective string action expansions. However, in the Type IIA case we have not found a complete solution for this yet, the complication being when identifying the Fourier coefficients in the non-abelian term. Section 6.4 is devoted to this problem.

In Section 6.5 we elaborate on the possibility of writing the series of products of Hermite polynomials and Whittaker functions, appearing in the non-abelian term in the general Fourier series (6.79). The motive is to make it easier to connect to the non-abelian part of the Eisenstein series (6.118).

In Appendix A we review some important Lie algebra theory that is used throughout the thesis. In Appendices B and C the important Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$, as well as their corresponding Lie algebras are treated carefully. In particular the Iwasawa decompositions are given and their maximal compact subgroups are derived. In Appendices D and E we give a detailed review of the discrete groups $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$. In Appendix F we derive the important Laplace-Beltrami operators. In Appendix G we derive a Poisson resummation formula that will be used when rewriting the Eisenstein series to get to the Fourier series expansion. Finally, in Appendix H we examine the Whittaker functions that are solutions to a differential equation showing up in the Type IIA case when demanding the general Fourier series to be an eigenfunction of the Laplace-Beltrami operator on the coset space.

2

Coset Symmetries of Actions

2.1 Type IIB Supergravity

We begin by studying symmetry properties of the scalar fields in the bosonic sector of Type IIB supergravity, which is a ten-dimensional theory that describes the low-energy limit of Type IIB superstring theory. The supermultiplet in Type IIB supergravity includes a graviton ($g_{\mu\nu}$), two scalar fields (ϕ,χ), two antisymmetric tensors (B_2,C_2), one "self-dual" four-form (C_4), one complex Weyl gravitino (ψ^μ) and one complex Weyl dilatino (λ). Remember that in Section 1.7 we wrote down only the part consisting of the curvature scalar. Now, we will instead restrict ourselves to the two scalar fields and we have the following action in the Einstein frame:

$$S_{(\phi,\chi)} \propto (\alpha')^{-4} \int \mathrm{d}^{10}x\sqrt{|g|}\left[-\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\phi}\partial_\mu\chi\partial^\mu\chi)\right]. \tag{2.1}$$

There is a $\mathrm{SL}(2,\mathbb{R})$-invariance of the action that can be seen by first combining the scalar fields into a complex field

$$\tau = \tau_1 + \mathrm{i}\tau_2 = \chi + \mathrm{i}\mathrm{e}^{-\phi}. \tag{2.2}$$

Using this substitution in (2.1) yields

$$\begin{aligned} \partial_\mu\tau &= \partial_\mu\chi - \mathrm{i}\mathrm{e}^{-\phi}\partial_\mu\phi \\ \tau_2^2 &= \mathrm{e}^{-2\phi} \implies \\ \implies S_{(\phi,\chi)} &\propto \int \mathrm{d}^{10}x\sqrt{|g|}\left(-\frac{1}{2}\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{\tau_2^2}\right), \end{aligned} \tag{2.3}$$

omitting the slope parameter factor. In this parameterization the $\mathrm{SL}(2,\mathbb{R})$-invariance is now realized as a *Möbius transformation* with parameters (a, b, c, $d \in \mathbb{R}$) on τ

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}. \tag{2.4}$$

It is a fractional linear transformation, but it transforms the scalar fields nonlinearly. In the next chapter this symmetry is made linear by introducing a gauge field.

The Möbius transformations are in a one-to-two correspondence to the group elements

$$\mathrm{SL}(2,\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \qquad a,b,c,d \in \mathbb{R}, \tag{2.5}$$

which can be seen by performing two group actions and comparing with two transformations. It is one-to-*two* since the group elements g and $(-\mathbb{1}) \cdot g$ correspond to the same Möbius transformation. So actually, we should rather consider the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathbb{1}\}$ that is in a one-to-one correspondence with the Möbius transformations. As mentioned in a footnote above, we will be a bit sloppy and just write $\mathrm{SL}(2,\mathbb{R})$ instead of $\mathrm{PSL}(2,\mathbb{R})$ everywhere. We verify that the action term indeed is invariant under the transformation

$$\begin{aligned}
\tau \longmapsto \frac{a\tau + b}{c\tau + d} \implies& \\
\partial_\mu \tau \longmapsto \frac{a\partial_\mu \tau}{c\tau + d} - \frac{c\partial_\mu \tau (a\tau + b)}{(c\tau + d)^2} &= \frac{\partial_\mu \tau \overbrace{(ad - bc)}^{=1}}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2} \\
\Im(\tau) = \frac{1}{2\mathrm{i}}(\tau - \bar{\tau}) \longmapsto \frac{1}{2\mathrm{i}}\left(\frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) &= \frac{\Im(\tau)}{|c\tau + d|^2} \\
\implies \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{\Im(\tau)^2} \longmapsto \frac{\partial_\mu \tau \partial^\mu \bar{\tau} \frac{1}{(c\tau + d)^2 (c\bar{\tau} + d)^2}}{\Im(\tau)^2 \frac{1}{|c\tau + d|^4}} &= \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{\Im(\tau)^2}.
\end{aligned} \tag{2.6}$$

To understand the symmetry more thoroughly we can use the fact that the action (2.3) is an example of a *nonlinear sigma model*. In general, the dynamical part of the classical action for scalar fields is a nonlinear sigma model.

In a nonlinear sigma model we consider mappings ξ from a Riemannian space X to another Riemannian space M, called the *target space*. Let us call the metric tensor on these spaces $g^{\mu\nu}$ and γ^{ab}, respectively, and call the coordinates on the spaces x^μ ($\mu = 1, ..., p = \dim X$) and ξ^a ($a = 1, ..., q = \dim M$). The general nonlinear sigma model action then reads

$$S = -\frac{1}{2} \int_X \mathrm{d}^p x \sqrt{|g|} g^{\mu\nu}(x) \partial_\mu \xi^a(x) \partial_\nu \xi^b(x) \gamma_{ab}(\xi(x)). \tag{2.7}$$

When varied we get equations of motion that give ξ^α as functions of x^μ.

Let us write our action (2.3) on the nonlinear sigma model form. The space X corresponds to the ten-dimensional Minkowski space. The target space coordinates are nothing but the real and complex parts of the complex field $(\xi^1, \xi^2) = (\tau_1, \tau_2) = (\chi, \mathrm{e}^{-\phi})$. We have

$$S_{(\phi,\chi)} = -\frac{1}{2}\int \mathrm{d}^{10}x\sqrt{|g|}\left[\frac{\partial_\mu\tau\partial^\mu\bar{\tau}}{\tau_2^2}\right] = -\frac{1}{2}\int \mathrm{d}^{10}x\sqrt{|g|}g^{\mu\nu}\left[\frac{\partial_\mu\tau_1\partial_\nu\tau_1 + \partial_\mu\tau_2\partial_\nu\tau_2}{\tau_2^2}\right] =$$
$$= -\frac{1}{2}\int \mathrm{d}^{10}x\sqrt{|g|}g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\gamma_{ab}, \tag{2.8}$$

where the metric on the target space is identified as

$$\gamma_{ab} = \frac{1}{\tau_2^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.9}$$

This we recognize as the Poincaré metric on the upper half complex plane [13]

$$\mathbb{H} = \{\forall\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{C} \mid \Im(\tau) > 0\}. \tag{2.10}$$

Indeed our τ lies in the upper half plane since $\Im(\tau) = \tau_2 = \mathrm{e}^{-\phi} > 0$. This shows that the target space M is isomorphic to $\mathbb{H}$, which is a two-dimensional hyperbolic space. Now, it is known that the metric preserving automorphisms, i.e. *isometries*, of the upper half plane is precisely the Möbius transformations (2.4). This is the reason for the invariance (2.6). The isometry group to $\mathbb{H}$ is $\mathrm{SL}(2,\mathbb{R})$ as can be seen from the correspondence between group elements and Möbius transformations (2.5).

A crucial fact that is essential for this thesis, is that the target space $M \simeq \mathbb{H}$ equipped with the metric (2.9) is also isomorphic to the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. To see this we begin with reminding ourselves of the fact that a group element $g \in \mathrm{SL}(2,\mathbb{R})$ acted on $\mathbb{H}$ through the Möbius transformation. Furthermore, we have that

$$\forall\tau, \upsilon \in \mathbb{H} \quad \exists\, g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mid \upsilon = \frac{a\tau + b}{c\tau + d}. \tag{2.11}$$

So, by the group action we can reach all points in the space from any other point. One then says that the group $\mathrm{SL}(2,\mathbb{R})$ acts *transitively* on $\mathbb{H}$. As was said before, the group $\mathrm{SL}(2,\mathbb{R})$ is also the *isometry group* of $\mathbb{H}$. However, there is also an *isotropy* subgroup $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$, i.e., a subgroup that leaves a point in the space fixed. Namely, $\tau = \mathrm{i}$ is a fixed point with respect to the action of

$$\mathrm{SO}(2) \ni k: \qquad \tau \longmapsto \frac{\tau\cos\theta + \sin\theta}{-\tau\sin\theta + \cos\theta}, \qquad \theta \in \mathbb{R} \tag{2.12}$$

Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively we can reach $\tau = \mathrm{i}$ with a group action from any other point in $\mathbb{H}$, i.e., no point is unique. This means that to get a one-to-one mapping between group elements and points in the space (needed for an isomorphism), we should identify all elements in $\mathrm{SL}(2,\mathbb{R})$ differing by the action of an element $k \in \mathrm{SO}(2)$, that is, construct the coset space

$\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. In general a *right coset* G/K is defined as the set of equivalence classes $[g] = \{gk \,|\, \forall k \in \mathrm{K}\}$. We could just as well work with left cosets $\mathrm{K}\backslash\mathrm{G}$, this is purely a matter of convention. The coset space G/K has the dimension $\dim \mathrm{G} - \dim \mathrm{K}$; in this case we have $\dim \mathrm{G/K} = 3 - 1 = 2$, which is indeed the dimension of $\mathbb{H}$.

A simpler example of the case above is the isomorphism

$$\mathrm{S}^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2).$$

S^2 means here the two-dimensional unit sphere embedded in $\mathbb{R}^3$. Choosing Euclidean coordinates (x, y, z), the manifold is given by the constraint

$$x^2 + y^2 + z^2 = 1.$$

The rotation group SO(3) acts transitively since by a three-dimensional rotation we can reach all points on the sphere from a given point. SO(3) is also the isometry group of S^2. The isotropy group is SO(2) as can be seen from the fact that the point $(1, 0, 0)$ is unchanged when performing rotations around the x-axis. This shows the isomorphism.

Importantly, for the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ we are interested in, SO(2) is the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. This holds in general for coset spaces that arise from toriodal compactification of eleven-dimensional supergravity as we will mention in the next section. The fact that we want to divide away the maximal compact subgroup will make it possible for us in the next chapter to construct the action corresponding to the coset by using the *Iwasawa decomposition*. We will choose a parameterization, with help of the so-called *Borel gauge*, of the scalar fields in the theory that will turn out to be the same parameterization as the first one stated in this section (2.1), i.e., using (ϕ, χ). Writing the nonlinear sigma model with the fields (ϕ, χ), we get the following moduli space metric

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\phi} \end{pmatrix}. \tag{2.13}$$

We will acquire this metric in the next chapter by starting with the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. A nonlinear sigma model where the target space is a manifold isomorphic to a coset G/K equipped with a G-invariant Riemannian metric, is also called a *quotient space* nonlinear sigma model.

For more on coset spaces, and their connection to supergravities, the reader is referred to [18].

2.2 Compactified Type IIA Supergravity

Type IIA supergravity is the low-energy limit theory of Type IIA superstring theory. The coset symmetries of the supergravity theory is therefore of high importance also for the superstring theory. When trying to connect to the real world, one needs to deal somehow with the six extra spatial dimensions of the superstring theories. The standard way is to follow T. Kaluza and O. Klein's idea and compactify some of the spatial dimensions. To get to four dimensions we need to compactify on a six-dimensional manifold of some kind. The topology of the manifold will give rise to different physical properties of the theory. Examples of possible manifolds to compactify on are the tori T^n and Calabi-Yau n-folds. The procedure of compactification, as well as the topological properties of the different manifolds, are beyond the scope of this thesis. We will merely state a few facts.

Type IIA supergravity and Type IIB supergravity are both sub-theories of *eleven-dimensional supergravity*, which is the low-energy limit theory of M-theory. E.g., Type IIA supergravity is derived by compactifying the eleven-dimensional theory on a circle S^1. When dimensionally reducing eleven-dimensional supergravity to $11-n$ dimensions, by compactifying on a T^n torus, one can show that the resulting scalar fields parameterize the symmetric cosets spaces G/K, where G is a non-compact group and K is the maximal compact subgroup of G, see Table 2.1. This was first done by E. Cremmer and B. Julia in their famous paper [15]. So, starting with Type IIA supergravity

Dim	G	K
11	1	1
10, Type IIA	$\mathrm{SO}(1,1;\mathbb{R})/\mathrm{Z}_2$	1
10, Type IIB	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$
9	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(1,1;\mathbb{R})$	$\mathrm{SO}(2)$
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{U}(2)$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{USp}(4)$
6	$\mathrm{O}(5,5;\mathbb{R})$	$\mathrm{USp}(4)\times\mathrm{USp}(4)$
5	E_6	$\mathrm{USp}(8)$
4	E_7	$\mathrm{SU}(8)$
3	E_8	$\mathrm{Spin}(16)$

Table 2.1: Coset symmetries arising from toroidal compactification of eleven-dimensional supergravity.

we could compactify on a T^6 torus to get to the four-dimensional spacetime. To compactify on a torus is the firstmost natural thing to do since the torus has a simple topology. However, the resulting theory has a large amount of supersymmetry, which needs to be broken in a physical reasonable theory. Promising candidates for which one can break supersymmetry in a controlled fashion are the Calabi-Yau threefolds CY_3 (they are complex and have six real dimensions).

The compactification of Type IIA supergravity on CY_3 leads to $\mathcal{N}=2$ supergravity in four dimensions. There are some different kinds of multiplets in this theory: a *gravity multiplet*, a *universal hypermultiplet*, a number of *vector multiplets* and a number of *hypermultiplets*. The number of vector-and hyper-multiplets depend on the topological properties of the Calabi-Yau manifold. The moduli space splits locally into a direct product

$$\mathcal{M}(\mathrm{CY}_3) = \mathcal{M}_{\mathrm{V}} \times \mathcal{M}_{\mathrm{H}}, \tag{2.14}$$

where the fields in the vector multiplets parameterize a complex special Kähler manifold $\mathcal{M}_{\mathrm{V}}$, and the fields in the hypermultiplets parameterize a real quaternionic-Kähler manifold $\mathcal{M}_{\mathrm{H}}$ [6]. We will throughout the whole thesis only treat the bosonic degrees of freedom in the theories (actually we treat only the scalar fields).

This thesis follows [9] where one studies the case where compactification of Type IIA is made on a so-called *rigid* Calabi-Yau threefold. We study this case to get more insight into the vast field of symmetries of superstring theories and the connection to, and application of, automorphic forms. Our main focus is not on connecting to the reality. In the special case of a rigid CY_3 it turns out that there is only one hypermultiplet, namely the universal hypermultiplet, and there are four real scalar fields in this multiplet: ϕ, χ, $\tilde{\chi}$ and ψ. The moduli space is again locally a direct product

$$\mathcal{M}_{\mathrm{V}} \times \mathcal{M}_{\mathrm{UH}}, \tag{2.15}$$

The rigid Calabi-Yau manifold is a special case in the sense that we get a *coset* moduli space, which appears in general only for torus compactifications. We will in the next chapter construct the nonlinear sigma model action on this coset.

In Type IIA superstring theory compactified on the rigid CY_3, the compactified supergravity theory above serves as the tree-level in the effective action.

3

Constructing an Action on a Coset Space

3.1 Through Gauging

One way of calculating the action $S_{\mathrm{G/K}}$ with fields parameterized on a coset space G/K is to construct an action for a group element $V(x) \in \mathrm{G}$, $x \in X$ (remember that X is the spacetime that our fields map from), so that it has a gauge symmetry that corresponds to the subgroup K. I.e.

$$S_{\mathrm{G/K}} \longrightarrow S_{\mathrm{G/K}} \quad \text{for} \quad V(x) \longmapsto V(x)k(x), \qquad \forall k(x) \in \mathrm{K}. \tag{3.1}$$

The physics will not depend on the transformation under the subgroup, i.e., it is a gauge transformation. Therefore the physical fields will parameterize the coset space. The action $S_{\mathrm{G/K}}$ will be invariant under $\mathrm{G}_{\mathrm{global}} \times \mathrm{K}_{\mathrm{local}}$-transformations, which are both linear transformations on V. We use the denotations *global* and *local* if the transformation is independent of X or not.

For a more careful review of the group- and algebra theoretical tools we will use later on, the reader is referred to Appendix A. We will work with the Lie algebra $\mathfrak{g}$ of G and write it in the Chevalley basis with a number of step operators e and f as well as Cartan subalgebra generators h. This is done explicitly in Appendix B for the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and in Appendix C for $\mathfrak{g} = \mathfrak{su}(2, 1)$. It is crucial that in our two coset space cases we have that K is the maximal compact subgroup of G. The corresponding algebra to K is denoted by $\mathfrak{k}$, and this maximal compact algebra is fixed by the Cartan involution τ on $\mathfrak{g}$, see Section A.4.

In the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ we have that $\mathfrak{k} = \mathfrak{so}(n)$. Therefore the elements $k \in \mathrm{K}$ obey $k^{\mathrm{T}}k = \mathbb{1}$ and all elements $T \in \mathfrak{k}$ are anti-symmetric, that is, $T = -T^{\mathrm{T}}$. In this case, the Cartan involution is also referred to as the

Chevalley involution[1], which we will denote as ω, and indeed it acts on the triple basis elements (e^i, f^i, h^i) as: $\omega(\cdot) = -(\cdot)^{\mathrm{T}}$, leaving the elements in $\mathfrak{so}(n)$ invariant. For simplicity we will first do the calculations in this section using the Chevalley involution, then we will explain how to modify our calculations to fit with the general case.

Beginning with an element $V \in G$, we can form the *bi-invariant* metric from the Maurer-Cartan form and Killing form (see for instance [18]) as

$$\mathrm{d}s^2 = \gamma_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b = \mathrm{Tr}[(V^{-1}\mathrm{d}V)(V^{-1}\mathrm{d}V)] \tag{3.2}$$

where $\mathrm{d} = \mathrm{d}x^\mu\partial_\mu$ is the *exterior* derivative. The metric tensor γ_{ab} is the same as in the nonlinear sigma model (2.8) where we had $\mathcal{L} = \partial_\mu\xi^a\partial^\mu\xi^b\gamma_{ab}$. This makes it possible to associate to $\mathrm{d}s^2$, a lagrangian $\mathcal{L}$ that has the same symmetry properties in the target space as $\mathrm{d}s^2$:

$$\mathrm{d}s^2 = -\frac{1}{2}\gamma_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b \longleftrightarrow \mathcal{L} = -\frac{1}{2}\gamma_{ab}\partial_\mu\xi^a\partial^\mu\xi^b. \tag{3.3}$$

Note that $V^{-1}\mathrm{d}V$ is invariant under global G-transformations acting from the left $V \longmapsto gV$, $g \in \mathrm{G}$. The metric (3.2) is invariant under both left and right global transformations (remember that the trace is cyclic); this is the reason it is called bi-invariant. We make the theory invariant under gauge transformations

$$V(x) \longmapsto V(x)k(x), \qquad k(x) \in K \tag{3.4}$$

by introducing a covariant derivative (henceforth the x-dependence is suppressed)

$$\mathrm{D}V = \mathrm{d}V - VA, \tag{3.5}$$

with a gauge potential A transforming as

$$A \longmapsto k^{\mathrm{T}}Ak + k^{\mathrm{T}}\mathrm{d}k \tag{3.6}$$

making the covariant derivative transform as it should

$$\mathrm{D}V \longmapsto (\mathrm{D}V)k.$$

Our gauged one-form $V^{-1}\mathrm{D}V$ then transform as

$$V^{-1}\mathrm{D}V \longmapsto k^{-1}V^{-1}\mathrm{D}Vk \quad \text{for} \quad V \longmapsto Vk,$$

and the gauged metric is then invariant under the local transformations. The relationship $A = -A^{\mathrm{T}}$ is unchanged by the transformation (3.6), as well as the trace of A (note that $\mathrm{Tr}(k^{\mathrm{T}}\mathrm{d}k) = 0$ since $k^{\mathrm{T}}\mathrm{d}k \in \mathfrak{k}$). This implies that A is in fact an element of the maximal compact subalgebra, $A \in \mathfrak{k}$. Using

[1] A Chevalley involution is the Cartan involution of a *split real form*, in this case $\mathfrak{sl}(n, \mathbb{R})$ is the split real form of $\mathfrak{sl}(n, \mathbb{C})$.

the covariant derivative (3.5) in (3.2) (with a factor of -1 that will give the correct factor in the end) yields

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{Tr}(V^{-1}\mathrm{d}VV^{-1}\mathrm{d}V - AV^{-1}\mathrm{d}V - V^{-1}\mathrm{d}VA + A^2) \\ &= -\mathrm{Tr}(V^{-1}\mathrm{d}VV^{-1}\mathrm{d}V - 2AV^{-1}\mathrm{d}V + A^2). \end{aligned} \tag{3.7}$$

The metric (lagrangian) must be stationary with respect to the gauge potential A due to Hamilton's principle. This will make it possible to solve for A

$$\begin{aligned} \delta(\mathrm{d}s^2) &= -\mathrm{Tr}((-2V^{-1}\mathrm{d}V + 2A)\delta A + (...)\delta G) = 0 \\ &\implies \mathrm{Tr}((-2V^{-1}\mathrm{d}V + 2A)\delta A) = 0. \end{aligned}$$

The anti-symmetry of A implies that the symmetric part of the first term $-2V^{-1}\mathrm{d}V$ will vanish when multiplied with δA. Moreover, since the variation δA is arbitrary (except for its anti-symmetric nature), the anti-symmetric part of the matrix inside the trace must be identically zero. We get the condition on A:

$$A - \frac{1}{2}(V^{-1}\mathrm{d}V - (V^{-1}\mathrm{d}V)^{\mathrm{T}}) = 0, \tag{3.8}$$

or expressed with the Chevalley involution ω (Cartan involution in the general case)

$$A = (V^{-1}\mathrm{d}V)_{\omega=1}. \tag{3.9}$$

Recall that the maximal compact subalgebra is defined by invariance under the Chevalley involution $\omega = 1$, i.e., $A = -A^{\mathrm{T}}$. So, we got the correct condition on A by using Hamilton's principle. Plugging (3.8) into the expression (3.7) gives

$$\mathrm{d}s^2 = -\frac{1}{2}\mathrm{Tr}\left[(V^{-1}\mathrm{d}V)^2 + V^{-1}\mathrm{d}V(V^{-1}\mathrm{d}V)^{\mathrm{T}}\right]. \tag{3.10}$$

This can be written more concisely if we introduce the *generalized metric*[2]

$$\mathcal{M} := VV^{\mathrm{T}}, \tag{3.11}$$

and we find

$$\mathcal{L}_{\mathrm{G/K}} \sim \mathrm{d}s^2 = \frac{1}{4}\mathrm{Tr}(\mathrm{d}\mathcal{M}^{-1}\mathrm{d}\mathcal{M}). \tag{3.12}$$

By construction this metric/action is invariant under $\mathrm{G}_{\mathrm{global}} \times \mathrm{K}_{\mathrm{local}}$- transformations.

Now, as was mentioned in the beginning of this chapter, the expression for the generalized metric (3.11) and (3.12) is only valid for the special case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(n)$. In general, the involution used to construct the

[2]The name derives from the fact that for the cases when $V \in \mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$ there is a direct correspondence with the metric in n dimensions written with veilbeins $g_{ij} = e_i^a e_j^b \delta_{ab}$, and $\mathcal{M} = VV^{\mathrm{T}}$. Both are elements in $\mathrm{GL}(n,\mathbb{R})$ and both are invariant under local $\mathrm{SO}(n)$-transformations of the veilbein and V respectively.

action is not the Chevalley involution ω, which here corresponded to ordinary matrix transpose, but a Cartan involution τ with another realization on the matrix representation. However, we can make all our expressions valid in the general case by changing the matrix transpose to a *generalized transpose* $\mathcal{T}$ defined as

$$(\cdot)^{\mathcal{T}} := -\tau(\cdot). \tag{3.13}$$

The proper definition of the generalized metric is thus

$$\mathcal{M} = VV^{\mathcal{T}}. \tag{3.14}$$

The action of global and local transformations ($V(x) \longmapsto gV(x)k(x)$) on $\mathcal{M}$ are

$$\begin{aligned} &\mathrm{G}: \ \mathcal{M} \longmapsto g\mathcal{M}g^{\mathcal{T}}, \qquad g \in \mathrm{G} \\ &\mathrm{K}: \ \mathcal{M} \longmapsto Vk(x)(Vk(x))^{\mathcal{T}} = Vk(x)k(x)^{\mathcal{T}}V^{\mathcal{T}} = \mathcal{M}, \qquad k(x) \in \mathrm{K}. \end{aligned} \tag{3.15}$$

3.2 Through Projection with the Involution

To explain the gauging procedure more generally and from a mathematical point of view, we follow [34] and make use of the Cartan involution. As a first thought, one can wonder why we cannot just pick a coset representative V and put in the non-gauged expression (3.2) and recover an acceptable metric. The reason is that although we choose $V \in \mathrm{G/K}$ the one-forms $V^{-1}\mathrm{d}V$, which lie in the algebra $\mathfrak{g}$, generally do not lie in the algebra of *interest*, namely $\mathfrak{t}$ (whose elements are fixed by the Cartan involution). So, what we essentially do when we gauge the theory and solve algebraically for the gauge potential and put it back into the expression, is that we write $V^{-1}\mathrm{d}V$ as an even and odd part under the Cartan involution τ (in the special case when $\mathfrak{g} = \mathfrak{so}(n)$ it is also an anti-symmetric and symmetric part as in (3.8))

$$\begin{aligned} V^{-1}\mathrm{d}V &= \frac{1}{2}(V^{-1}\mathrm{d}V + \tau(V^{-1}\mathrm{d}V)) + \quad (\in \mathfrak{k}) \\ &+ \frac{1}{2}(V^{-1}\mathrm{d}V - \tau(V^{-1}\mathrm{d}V)) = \quad (\in \mathfrak{p}) \\ &= \frac{1}{2}(V^{-1}\mathrm{d}V - (V^{-1}\mathrm{d}V)^{\mathcal{T}}) + \\ &+ \frac{1}{2}(V^{-1}\mathrm{d}V + (V^{-1}\mathrm{d}V)^{\mathcal{T}}), \end{aligned} \tag{3.16}$$

and throw away the even part $\in \mathfrak{k}$, which is exactly our gauge potential A. I.e., we *project* on the odd part under the involution τ. The decomposition of the algebra (3.16) is called the Cartan decomposition, see Appendix A.

One can directly verify that we get the same expression as in (3.10) by noting that

$$\begin{aligned} (V^{-1}\mathrm{d}V)^2 + V^{-1}\mathrm{d}V(V^{-1}\mathrm{d}V)^{\mathrm{T}} &= (V^{-1}\mathrm{d}V)((V^{-1}\mathrm{d}V) + (V^{-1}\mathrm{d}V)^{\mathrm{T}}) = \\ &= (V^{-1}\mathrm{d}V + (V^{-1}\mathrm{d}V)^{\mathrm{T}})(V^{-1}\mathrm{d}V + (V^{-1}\mathrm{d}V)^{\mathrm{T}}), \end{aligned}$$

since the anti-symmetric part $\in \mathfrak{k}$ of $V^{-1}\mathrm{d}V$ cancels when multiplied with the symmetric part $\in \mathfrak{p}$.

Writing the odd part as

$$P = \frac{1}{2}(V^{-1}\mathrm{d}V - \tau(V^{-1}\mathrm{d}V)), \tag{3.17}$$

we get a remarkably easy expression for the lagrangian ($\mathrm{d}s^2$)

$$\mathcal{L}_{\mathrm{G/K}} \sim \mathrm{Tr}(P \cdot P). \tag{3.18}$$

3.3 An Action on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$

We now use the method above to construct an action on the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, that indeed will turn out to be the same as (2.1). Since we are interested in the gauge invariant degrees of freedom we must somehow fix the gauge. A way of doing this is to use the Iwasawa decomposition G = NAK where N is a nilpotent subgroup, A is an abelian subgroup and K is the maximal compact subgroup. To fix the gauge we can put K = Id and hence consider $V = NA$. Due to the Iwasawa decomposition, for matrix groups, the coset representative V can be chosen to be an upper triangular matrix. This choice of parameterization is called the *Borel gauge*. A group action on V will in general destroy the upper triangular form (the gauge choice), so we need a compensating K-transformation to restore the form. Remember that in the coset G/K all elements differing by K-transformations are identified. A general transformation of $V \in$ G/K thus reads

$$V(x) \longmapsto gV(x)k(g,V(x)), \qquad g \in \mathrm{G},\ k(g,V(x)) \in \mathrm{K}. \tag{3.19}$$

This is a nonlinear realization of the group action since the compensating transformation $k(g,V(x))$ depends nonlinearly on $V(x)$.

For the particular case G = SL(2, ℝ) and K = SO(2) we have (see (B.12)):

$$\begin{aligned}
V = NA &= \begin{pmatrix} \mathrm{e}^{-\frac{\phi}{2}} & \chi\mathrm{e}^{\frac{\phi}{2}} \\ 0 & \mathrm{e}^{\frac{\phi}{2}} \end{pmatrix} \\
\Longrightarrow\ \mathcal{M} = VV^{\mathrm{T}} &= \begin{pmatrix} \mathrm{e}^{-\phi} + \chi^2\mathrm{e}^{\phi} & \chi\mathrm{e}^{\phi} \\ \chi\mathrm{e}^{\phi} & \mathrm{e}^{\phi} \end{pmatrix} \\
\Longrightarrow\ \mathrm{d}s^2 &= \frac{1}{4}\mathrm{Tr}(\mathrm{d}\mathcal{M}^{-1}\mathrm{d}\mathcal{M}) = \\
&= \frac{1}{4}\mathrm{Tr}\Bigg[\begin{pmatrix} \mathrm{e}^{\phi}\mathrm{d}\phi & -\mathrm{e}^{\phi}\mathrm{d}\chi - \chi\mathrm{e}^{\phi}\mathrm{d}\phi \\ -\mathrm{e}^{\phi}\mathrm{d}\chi - \chi\mathrm{e}^{\phi}\mathrm{d}\phi & (\chi^2\mathrm{e}^{\phi} - \mathrm{e}^{-\phi})\mathrm{d}\phi + 2\chi\mathrm{e}^{\phi}\mathrm{d}\chi \end{pmatrix} \\
&\qquad \begin{pmatrix} (\chi^2\mathrm{e}^{\phi} - \mathrm{e}^{-\phi})\mathrm{d}\phi + 2\chi\mathrm{e}^{\phi}\mathrm{d}\chi & \mathrm{e}^{\phi}\mathrm{d}\chi + \chi\mathrm{e}^{\phi}\mathrm{d}\phi \\ \mathrm{e}^{\phi}\mathrm{d}\chi + \chi\mathrm{e}^{\phi}\mathrm{d}\phi & \mathrm{e}^{\phi}\mathrm{d}\phi \end{pmatrix}\Bigg] = \\
&= -\frac{1}{2}(\mathrm{d}\phi^2 + \mathrm{e}^{2\phi}\mathrm{d}\chi^2).
\end{aligned} \tag{3.20}$$

Thus we recover the same lagrangian as we had for the scalar fields in Type IIB supergravity (2.8) when interpreting the lagrangian, with (3.3), as

$$\mathcal{L}_{\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)} = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\phi}\partial_\mu\chi\partial^\mu\chi). \tag{3.21}$$

3.4 An Action on $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$

When compactifying Type IIA supergravity on a rigid Calabi-Yau manifold it turns out, as was mentioned in Section 2.2, that the resulting symmetry group of the scalar fields is the coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. Therefore we want to make a similar explicit calculation of this action as in Section 3.3 above, to use later on when constructing the automorphic forms. For a review of $\mathrm{SU}(2,1)$ and its maximal compact subgroup, see Appendix C. $\mathrm{SU}(2,1)$ is eight-dimensional and $\mathrm{SU}(2)\times\mathrm{U}(1)$ is four-dimensional. Hence the coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$ is four-dimensional and we get four scalar fields in the theory: ϕ, χ, $\tilde{\chi}$ and ψ. ϕ is the dilaton and the other fields are referred to as axions. A convenient representation of the Lie algebra $\mathfrak{su}(2,1)$ is given in (C.32) and (C.33). It consists of two Cartan generators H_1 (compact) and H_2 (non-compact), three positive step-operators $X_{(1)}$, $\tilde{X}_{(1)}$ and $X_{(2)}$, and three negative step-operators $Y_{(-1)}$, $\tilde{Y}_{(-1)}$ and $Y_{(-2)}$.

First we need a coset representative V. It is calculated as before using the Iwasawa decomposition. This is explicitly done in Section C.5, and the result is

$$\begin{aligned} V = NA = \mathrm{e}^{\chi X_{(1)}+\tilde{\chi}\tilde{X}_{(1)}+2\psi X_{(2)}}\mathrm{e}^{-\phi H_1} = \\ = \begin{pmatrix} \mathrm{e}^{-\phi} & -\chi+\tilde{\chi}+\mathrm{i}(\chi+\tilde{\chi}) & \mathrm{e}^{\phi}(\mathrm{i}(\chi^2+\tilde{\chi}^2)+2\psi) \\ 0 & 1 & \mathrm{e}^{\phi}(\chi+\tilde{\chi}+\mathrm{i}(-\chi+\tilde{\chi})) \\ 0 & 0 & \mathrm{e}^{\phi} \end{pmatrix}. \end{aligned} \tag{3.22}$$

Now, we use the Cartan involution to project on the coset space. It is acting on the generators as

$$\tau(X) = -X^\dagger, \tag{3.23}$$

see (C.22). So, the odd part under the involution (belonging to the coset algebra) is

$$P = \frac{1}{2}(V^{-1}\mathrm{d}V - \tau(V^{-1}\mathrm{d}V)) = \frac{1}{2}(V^{-1}\mathrm{d}V + (V^{-1}\mathrm{d}V)^\dagger). \tag{3.24}$$

Our generalized metric reads

$$\begin{aligned} \mathcal{M} = VV^\dagger = \\ \begin{pmatrix} \mathrm{e}^{-2\phi}+2(\chi^2+\tilde{\chi}^2)+\mathrm{e}^{2\phi}((\chi^2+\tilde{\chi}^2)^2+4\psi^2) & (1+\mathrm{i})(\mathrm{i}\chi+\tilde{\chi})(1+\mathrm{e}^{2\phi}(\chi^2+\tilde{\chi}^2-2\mathrm{i}\psi)) & \mathrm{i}\mathrm{e}^{2\phi}(\chi^2+\tilde{\chi}^2-2\mathrm{i}\psi) \\ (-1-\mathrm{i})(\chi+\mathrm{i}\tilde{\chi})(1+\mathrm{e}^{2\phi}(\chi^2+\tilde{\chi}^2+2\mathrm{i}\psi)) & 1+2\mathrm{e}^{2\phi}(\chi^2+\tilde{\chi}^2) & (1+\mathrm{i})\mathrm{e}^{2\phi}(-\mathrm{i}\chi+\tilde{\chi}) \\ -\mathrm{i}\mathrm{e}^{2\phi}(\chi^2+\tilde{\chi}^2+2\mathrm{i}\psi) & (1+\mathrm{i})\mathrm{e}^{2\phi}(\chi-\mathrm{i}\tilde{\chi}) & \mathrm{e}^{2\phi} \end{pmatrix}. \end{aligned} \tag{3.25}$$

We can present it somewhat cleaner by defining two new complex variables

$$\xi := \chi + \tilde{\chi} + \mathrm{i}(\tilde{\chi} - \chi), \qquad \zeta := 2\psi + \frac{\mathrm{i}}{2}|\xi|^2, \tag{3.26}$$

this yields

$$\mathcal{M} = \begin{pmatrix} \mathrm{e}^{-2\phi} + |\xi|^2 + \mathrm{e}^{2\phi}|\zeta|^2 & \mathrm{i}\bar{\xi} + \mathrm{e}^{2\phi}\bar{\xi}\zeta & \mathrm{e}^{2\phi}\zeta \\ -\mathrm{i}\xi + \mathrm{e}^{2\phi}\xi\bar{\zeta} & 1 + \mathrm{e}^{2\phi}|\xi|^2 & \mathrm{e}^{2\phi}\xi \\ \mathrm{e}^{2\phi}\bar{\zeta} & \mathrm{e}^{2\phi}\bar{\xi} & \mathrm{e}^{2\phi} \end{pmatrix}. \tag{3.27}$$

The bi-invariant metric is preferably not calculated by hand. The result is (we choose a factor of $-\frac{1}{8}$ giving the correct factor in the end):

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{1}{8}(\mathrm{d}\mathcal{M}^{-1}\mathrm{d}\mathcal{M}) = \ldots = \\ &= \mathrm{d}\phi^2 + \mathrm{e}^{2\phi}(\mathrm{d}\chi^2 + \mathrm{d}\tilde{\chi}^2) + \mathrm{e}^{4\phi}(\chi\mathrm{d}\tilde{\chi} - \tilde{\chi}\mathrm{d}\chi + \mathrm{d}\psi)^2, \end{aligned} \tag{3.28}$$

from which we infer the lagrangian

$$\begin{aligned} &\mathcal{L}_{\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))} = \\ &= \partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\phi}(\partial_\mu\chi\partial^\mu\chi + \partial_\mu\tilde{\chi}\partial^\mu\tilde{\chi}) + \mathrm{e}^{4\phi}(\chi\partial_\mu\tilde{\chi} - \tilde{\chi}\partial_\mu\chi + \partial_\mu\psi)^2. \end{aligned} \tag{3.29}$$

4

Quantum Corrections, Instantons and Eisenstein Series

When adding quantum corrections to the classical supergravity theories to reach an effective superstring theory, the global symmetry $G(\mathbb{R})$ of the scalar fields in classical action is broken due to the appearance of D-branes (instantons) [28]. A discrete subgroup symmetry $G(\mathbb{Z}) \subset G(\mathbb{R})$ is conjectured to remain unbroken, and there is a possibility that the quantum corrections and the instanton effects are all encoded into automorphic forms. These forms can be constructed so that they fulfill all demands, namely by constructing them as Eisenstein series. The Eisenstein series are parameterized by a coset representative $V \in G/K$, which we can choose in the Borel gauge like in the preceding chapter, where G is a Lie group and K is its maximal compact subgroup. The series are manifestly invariant under the discrete subgroup $G(\mathbb{Z})$. I.e., they effectively live on the double coset space $G(\mathbb{Z})\backslash G/K$. Moreover, they are eigenfunctions to the Laplace-Beltrami operator on the coset space with a certain eigenvalue. This is required because of supersymmetric reasons. E.g., performing supersymmetry transformations on the R^4-term in Type IIB superstring theory takes one to another term in the lagrangian, but performing a transformation again gives partly back the first R^4 but with a Laplace-Beltrami operator acting on the coefficient (i.e. the automorphic function constructed as an Eisenstein series). This enforces the laplacian condition with a specific eigenvalue. Furthermore, the Eisenstein series are well-behaved in the weak coupling limit $g_s \longrightarrow 0$, which is physically demanded, as we will see proof of.

For the Type IIA case treated in Chapter 6, the story is a bit different. Unlike the Type IIB superstring theory, which has a double coset moduli space $G(\mathbb{Z})\backslash G/K$, the Type IIA compactified on a rigid CY_3 has a universal hypermultiplet moduli space, which classically is a coset space but will deform to some other kind of exact moduli space due to quantum corrections. This deformation should somehow be encoded in an automorphic form living on the double coset $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. The laplacian condition on this automorphic form is harder to justify, more on this in Chapter 6.

It is possible to write down the Eisenstein series explicitly with a so-called lattice method explained in Section 4.2. However, the series are then not on a form where we can read off the physics. This is because the fields mix in the series in one or several infinite sums in a non-trivial way. Depending on what kind of limit in the string theory we would like to study, we have to expand the Eisenstein series correspondingly. For instance, in the Type IIB case we would like to have an expression from which one can directly read off the perturbative contributions (tree-level and loop-terms) as well as the non-perturbative corrections. Specifically, we want to see the first two terms in (1.24), namely

$$2\zeta(3)\tau_2^{3/2}+4\zeta(2)\tau_2^{-1/2}. \tag{4.1}$$

This is accomplished by writing the Eisenstein series as a Fourier series with help of the maximal nilpotent subgroup $\mathrm{N}(\mathbb{Z})\subset\mathrm{G}(\mathbb{Z})$, introduced later in Section 4.1. The terms (4.1) are the zero mode of the Fourier series; they are called *constant* terms and they depend only on the dilaton $\tau_2=\mathrm{e}^{-\phi}$, which is unaffected by the actions of $\mathrm{N}(\mathbb{Z})$. In addition, we will in general have *abelian* and *non-abelian* terms referring to the action of $\mathrm{N}(\mathbb{Z})$. These two types of terms are then contributing non-perturbatively to the lagrangian, its contributions can be interpreted as coming from instantons of various kinds.

The general form of the Fourier series is derived from the requirements that the function must be an eigenfunction to the Laplace-Beltrami operator on the coset space G/K, and in the same time invariant under the nilpotent subgroup $\mathrm{N}(\mathbb{Z})\in\mathrm{G}(\mathbb{Z})$. This gives a number of differential equations of various types. In a dream scenario we would then be able to, by raw calculations, find the unknown coefficients making our general Fourier series invariant under the rest of the elements in the discrete subgroup $\mathrm{G}(\mathbb{Z})$. If that would succeed we would have the correct function sought after. This way proves to be very difficult much due to an involution transformation S in the two cases treated in this thesis. Our only working method by now is to use the exact function derived from the Eisenstein series to find the coefficients in the general Fourier series.

4.1 The Nilpotent Subgroup and the General Fourier Series

This section is based on Chapter 10 in [34]. It is a non-trivial task to choose the correct discrete subgroup $\mathrm{G}(\mathbb{Z})\subset\mathrm{G}(\mathbb{R})$ that is supposed to encode the quantum corrections and instanton corrections. The discrete group has to satisfy a number of properties drawn from the physics. A constraint on the discrete group that helps us to determine the structure of the general Fourier series of the automorphic forms, is the fact that there should be a nilpotent subgroup $\mathrm{N}(\mathbb{Z})\subset\mathrm{G}(\mathbb{Z})$, which is also a discrete subgroup of the nilpotent group N(ℝ) in the Iwasawa decomposition G=NAK of the real Lie group G(ℝ). This fact can be seen by studying the axion fields in the theory.

We are dealing with matrix algebras and in general we can write a nilpotent group element as an upper triangular matrix. E.g.

$$\mathrm{N}(\mathbb{R}) \ni N = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R}. \tag{4.2}$$

This particular three-by-three matrix happens to be a general element also in the three-dimensional *Heisenberg group* H_3; it will turn out that both in the Type IIB and Type IIA case our nilpotent groups will be of Heisenberg type. The Heisenberg groups H_n are special cases of nilpotent groups. For $\dim \mathrm{H} = 2m+1$, $m \in \mathbb{N}$, we can explicitly write a matrix representation as

$$\mathrm{H}_{2m+1} \ni h = \begin{pmatrix} 1 & \vec{a} & c \\ 0 & \mathbb{1}_m & \vec{b}^{\mathrm{T}} \\ 0 & 0 & 1 \end{pmatrix} \tag{4.3}$$

where $\vec{a}$ and $\vec{b}$ are some m-dimensional vectors, c is a real scalar and $\mathbb{1}_m$ is the $m \times m$ unit matrix. One acquires a discrete subgroup of a nilpotent group simply by restricting the field to the integer numbers ℤ. Our definition of N(ℤ) is then doing this procedure on the nilpotent group N(ℝ) in the Iwasawa decomposition of G(ℝ). Restricting the field to ℤ in (4.2) yields the *discrete* Heisenberg group $\mathrm{H}_3(\mathbb{Z})$.

Now, to see why the knowledge of the discrete nilpotent group helps us when constructing the general Fourier series for the automorphic forms, we first recall that our physical fields are parameterized on cosets G/K for which we constructed coset representatives $V = NA$ in the Borel gauge. As an example we will study the Heisenberg group H_3 with a matrix representation as in (4.2). The axion fields are then α, β and γ. The abelian subgroup A is parameterized by one field; in our case this is the dilaton ϕ. Acting from the left on the coset representative V with a general element in the discrete nilpotent group yields

$$\begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & \alpha + m & \gamma + \beta m + p \\ 0 & 1 & \beta + n \\ 0 & 0 & 1 \end{pmatrix} A, \qquad m, n, p \in \mathbb{Z}, \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{4.4}$$

We see that the fields, the dilaton ϕ in our case, included in A are unaffected by the action of the discrete nilpotent group. When writing down a N(ℤ)-invariant function, the dependence on ϕ is arbitrary. There will be a constant term in the Fourier series only dependent on ϕ, and this is the zero-mode of the Fourier series. Furthermore, the action from the discrete nilpotent group on the axions is

$$\begin{aligned}\alpha &\longmapsto \alpha + m\\ \beta &\longmapsto \beta + n\\ \gamma &\longmapsto \gamma + p + \beta m.\end{aligned} \tag{4.5}$$

We see that the action is non-abelian since the transformation on γ includes β as well. For example, performing first the transformation $(m, n, p) = (1, 1, 1)$ and then $(m, n, p) = (1, 1, 0)$ is not the same as doing it the other way around. To simplify our calculations later on, it is convenient to decompose the group $\mathrm{N}(\mathbb{Z})$ into an abelian and a non-abelian part. Correspondingly we decompose the Fourier expansion into an abelian and a non-abelian part. In the example above, this amounts to:

$$f(\phi, \alpha, \beta, \gamma) = f^{(\mathrm{C})}(\phi) + f^{(\mathrm{A})}(\phi, \alpha, \beta) + f^{(\mathrm{NA})}(\phi, \alpha, \beta, \gamma). \tag{4.6}$$

This method will be used in the Type IIA case in Chapter 6 to help us divide our problem into smaller pieces and make it easier to identify the coefficients from the exact Eisenstein series later on. This type of decomposition of Fourier series is treated for the case SU(2,1) in [29]. D. Persson and P. Pioline made the first detailed treatment of a non-abelian Fourier series decomposition appearing in physics, in [36], where they treated automorphic forms on the double coset $\mathrm{SL}(3,\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.

Neglecting γ, we see that we have an abelian action on the fields α and β and we can directly write down a Fourier expansion of the part $f^{(\mathrm{A})}$, which is manifestly invariant under the discrete nilpotent subgroup, namely:

$$f^{(\mathrm{A})}(\alpha, \beta) = \sum_{l_1, l_2} C_{l_1, l_2} \mathrm{e}^{2\pi \mathrm{i}(l_1\alpha + l_2\beta)}. \tag{4.7}$$

So, for abelian nilpotent groups $\mathrm{N}(\mathbb{Z})$ it is easy to find the structure of the general Fourier series. See more below.

To find $f^{(\mathrm{NA})}(\phi, \alpha, \beta, \gamma)$ we need to take the last non-abelian transformation in (4.5) into account. We need a corresponding non-abelian Fourier series invariant under this transformation which is considerably harder to find; we have to treat case by case. For the Type IIA case see Section 6.1.

More generally, for a nilpotent group $\mathrm{N}(\mathbb{R})$ we can write an element as

$$n = \exp\left[\sum_{\alpha\in\Phi_+} \chi_\alpha e_\alpha\right], \tag{4.8}$$

where the positive step-operators e^α are enumerated by the positive roots $\alpha \in \Phi_+$, and χ_α denotes the different axion fields (using the physics terminology).

If it is the case that the group $\mathrm{N}(\mathbb{R})$ is *abelian*, the discrete group $\mathrm{N}(\mathbb{Z})$ will act on the axions as

$$\chi_\alpha \longmapsto \chi_\alpha + n_\alpha, \qquad n_\alpha \in \mathbb{Z}. \tag{4.9}$$

The Fourier series for an invariant function under these translations is

$$f(\chi_\alpha) = \sum_{l_\alpha \in \mathbb{Z}^D} C_{l_\alpha} \exp\left[2\pi \mathrm{i} \sum_{\alpha \in \Phi_+} l_\alpha \chi_\alpha\right], \tag{4.10}$$

where $D = \dim \mathrm{N}(\mathbb{R})$.

If the nilpotent group on the other hand is *non-abelian* we decompose the Fourier series into an abelian part and a non-abelian part. The abelian part will depend on all the fields that transform translationary as (4.9). Call those fields χ_i, $i = 1, \ldots, D$ (for some $D \in \mathbb{N}$), and we have

$$\chi_i \longmapsto \chi_i + m_i, \qquad m_i \in \mathbb{Z}. \tag{4.11}$$

The non-abelian part will depend on *all* the fields. Call the rest of the fields transforming in a mixed fashion (as γ in (4.5)) for ψ_j, $j = 1, \ldots, \tilde{D}$. We have then $\dim \mathrm{N}(\mathbb{R}) = D + \tilde{D}$. The fields ψ_j will transform in general as

$$\psi_j \longmapsto \psi_j + \tilde{m}_j + g_j(\chi_1, \ldots, \chi_D, \psi_1, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_{\tilde{D}}), \qquad \tilde{m}_j \in \mathbb{Z}, \tag{4.12}$$

where g_j is some linear combination of all the fields, except ψ_j, with integer coefficients (many coefficients are identically zero). The form of g_j depends on the type of nilpotent group. In our cases, g_j will not depend on the mixing fields ψ_j and we can then schematically write the non-abelian part of the Fourier expanded function as

$$f^{(\mathrm{NA})}(\chi_i, \psi_j) = \sum_{\vec{n}=(n_1,\ldots,n_{\tilde{D}}) \in \mathbb{Z}^{\tilde{D}}} C_{\vec{n}}(\chi_1, \chi_2, \ldots, \chi_D) \exp\left[2\pi \mathrm{i} \sum_{a=1}^{\tilde{D}} n_a \psi_a\right]. \tag{4.13}$$

However, a lot of constraints on the coefficients $C_{\vec{n}}$ to make the expression invariant under all discrete nilpotent transformations. Obviously, it must first be invariant under the translations of the fields χ_i, but then it must also cope with the transformations on ψ_j, which will yield exponential terms dependent on some of the fields χ_i. In the Type IIA case there is only one field ψ, and we make the function invariant under some of the types of transformations on this field by a shift in the summation.

4.2 Eisenstein Series

We want to be able to write down an automorphic form[1] on the moduli space $\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}(\mathbb{R})/\mathrm{K}$, which is a function invariant under the discrete subgroup $\mathrm{G}(\mathbb{Z})$ and whose arguments lie in the symmetric coset space $\mathrm{G}(\mathbb{R})/\mathrm{K}$. There are a number of different ways of constructing these automorphic forms; one can use a *lattice construction*, *Poincaré series* or *spherical vectors*.

[1]see Section 1.4.

In this thesis we will use the lattice construction method that will yield the automorphic form as an Eisenstein series $\mathcal{E}_s^{\mathrm{G}(\mathbb{Z})}$, whose properties were stated in Section 1.4. This method was first used by N. A. Obers and B. Pioline in [33] and is especially convenient for our purposes. For the definition of the series we invoke the Iwasawa decomposition G=NAK and construct the coset representative $V = NA \in \mathrm{G/K}$ in the Borel gauge, that we used when constructing the action on the coset space; see the end of Section 3.1 and Section 3.3. We form again the generalized metric (3.14)

$$\mathcal{M} := VV^{\mathcal{T}}, \tag{4.14}$$

where $\mathcal{T}$ was the generalized transpose defined in (3.13). The definition of the Eisenstein series of order s is then

$$\mathcal{E}_s^{\mathrm{G}(\mathbb{Z})}(\mathcal{M}) := \sum_{\vec{m}\in\Lambda_{\mathbb{Z}}}{}' \, \delta(\vec{m}\wedge\vec{m}) \left[\vec{m}^{\mathcal{T}}\cdot\mathcal{M}\cdot\vec{m}\right]^{-s}. \tag{4.15}$$

The summation is over a discrete lattice $\Lambda_{\mathbb{Z}}$ that is invariant under an action of the discrete group $\mathrm{G}(\mathbb{Z})$, i.e., elements in $\mathrm{G}(\mathbb{Z})$ acts as automorphisms of the lattice. The delta function provides the summation with a certain quadratic constraint

$$\vec{m}\wedge\vec{m} = 0, \tag{4.16}$$

which we will discuss further below. The parameter s that sits in the exponent is free for us to adjust and by this freedom we can get the series to be eigenfunctions to the Laplace-Beltrami operator on the coset with different eigenvalues. The series converges absolutely for $\Re(s) > n/2$ if we sum over an n-dimensional lattice $\lambda_{\mathbb{Z}}$ of integers [40].

By the definition (4.15) the function $\mathcal{E}_s^{\mathrm{G}(\mathbb{Z})}(\mathcal{M})$ fulfills the demands of living on the double coset space. First of all, it is a function on G/K since it is a function of the coset representative V. It is manifestly invariant under K since the generalized metric is invariant under these maximal compact subgroup transformations; see (3.15) where we also showed that $\mathcal{M}$ transforms covariantly under G

$$\mathcal{M} \longmapsto g\mathcal{M}g^{\mathcal{T}}.$$

The Eisenstein series is then also invariant under the discrete group because given an element $\gamma \in \mathrm{G}(\mathbb{Z})$ we get

$$\begin{aligned}\gamma: \quad \mathcal{E}_s^{\mathrm{G}(\mathbb{Z})}(\mathcal{M}) &\longmapsto \sum_{\vec{m}\in\Lambda_{\mathbb{Z}}}{}' \, \delta(\vec{m}\wedge\vec{m}) \left[\vec{m}^{\mathcal{T}}\cdot\gamma\mathcal{M}\gamma^{\mathcal{T}}\cdot\vec{m}\right]^{-s} = \\ &= \sum_{\vec{m}'\in\Lambda_{\mathbb{Z}}}{}' \, \delta(\vec{m}'\wedge\vec{m}') \left[\vec{m}'^{\mathcal{T}}\cdot\mathcal{M}\cdot\vec{m}'\right]^{-s} = \mathcal{E}_s^{\mathrm{G}(\mathbb{Z})}(\mathcal{M}),\end{aligned} \tag{4.17}$$

where we have just changed summation variable $\vec{m}' = \gamma^{\mathcal{T}}\vec{m}$, which runs through all of $\Lambda_{\mathbb{Z}}$ since $\gamma^{\mathcal{T}} \in \mathrm{G}(\mathbb{Z})$. Note that we must also demand the quadratic constraint to be invariant under the discrete transformation

$$\vec{m} \wedge \vec{m} = \gamma^{\mathcal{T}}\vec{m} \wedge \gamma^{\mathcal{T}}\vec{m}. \tag{4.18}$$

As was explained in Section 1.4 we seek automorphic forms that are eigenfunctions to the Laplace-Beltrami operator on the coset space G/K. This is precisely the reason why we need the quadratic constraint, which is a certain integer-valued product of the lattice vectors. Without digging any further we just state that for the Type IIB case, when we are dealing with the moduli space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, we do not need the quadratic constraint since the Eisenstein series are automatically eigenfunctions to the Laplace-Beltrami operator. Whereas in the Type IIA case we have the moduli space $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$ and on this space, when summing over gaussian integers $\vec{m} \in \mathbb{Z}[\mathrm{i}]^3$, the Eisenstein series are not eigenfunctions unless we impose the constraint

$$\vec{m} \wedge \vec{m} := \vec{m}^{\dagger} \cdot \eta \cdot \vec{m}. \tag{4.19}$$

Here

$$\eta = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 1 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}$$

is the defining metric for the group $\mathrm{SU}(2,1)$, i.e., with the property

$$g^{\dagger}\eta g = \eta, \qquad \forall g \in \mathrm{SU}(2,1), \tag{4.20}$$

showing also that the quadratic constraint is invariant under the discrete group (4.18) since in this case we have that the generalized transpose equals conjugate transpose, see Appendix C. For a more general treatment of the quadratic constraint see [33].

5

The Type IIB Case: $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$

With the Section 1.7 in mind, we will explicitly find the automorphic form

$$f(\tau_1,\tau_2) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \ldots, \tag{5.1}$$

which is supposed to sit in the $(n,g) = (3,{<}\infty)$ term in the expansion of the Type IIB action (1.21):

$$S_{(3,<\infty)} = (\alpha')^{-1}\int \mathrm{d}^{10}x\sqrt{|g|}\,\mathrm{e}^{-2\phi}f(\tau_1,\tau_2)R^4. \tag{5.2}$$

Remember that we are limiting ourselves to the scalar fields (ϕ,χ) in the Type IIB superstring theory. As was shown in Chapter 2, the Type IIB supergravity posesses a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset symmetry of the scalar fields. The parameterization of the coset space (upper half complex plane $\mathbb{H}$) was

$$\mathbb{H} \ni \tau = \tau_1 + \mathrm{i}\tau_2 = \chi + \mathrm{i}\mathrm{e}^{-\phi}, \qquad (\tau_2 > 0). \tag{5.3}$$

The classical coset symmetry is then broken when adding quantum corrections, as the R^4-term, yielding the effective string theory. The scalar fields are though still parameterizing the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and we have $f = f(\tau_1,\tau_2)$. There will be a surviving discrete symmetry coming from the fact that the Type IIB superstring theory is invariant under the S-duality group $\mathrm{G}(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$, as was mentioned in the introduction. The function $f(\tau_1,\tau_2)$ ought then to be a function on the double coset $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

As we will see, it is possible to construct $f(\tau_1,\tau_2)$ as an Eisenstein series to agree with all demands; this includes the demand of the correct constant part in (5.1) that had been calculated by other methods. For more mathematical details about the Lie groups the reader is referred to the Appendices B and D.

5.1 General Fourier Series

In Appendix D it is made apparent that the whole modular group $\mathrm{SL}(2,\mathbb{Z})$ is generated by combinations of two fundamental elements (transformations), a translation T and an involution S. Since we are working with a parameterization on the upper half plane $\mathbb{H}$, we are interested in the realization of the two transformations on the complex parameter τ. They act as following

$$T : \tau \longmapsto \tau + 1, \qquad S : \tau \longmapsto -\frac{1}{\tau}. \tag{5.4}$$

And when expressed in the real and complex component of τ

$$\begin{aligned} \Re(\tau) &= \tau_1 = \chi \\ \Im(\tau) &= \tau_2 = \mathrm{e}^{-\phi}, \end{aligned}$$

they act as

$$T : \tau_2 \longmapsto \tau_2 + 1, \qquad S : \begin{cases} \tau_1 \longmapsto -\frac{\tau_1}{\tau_1^2+\tau_2^2} \\ \tau_2 \longmapsto \frac{\tau_2}{\tau_1^2+\tau_2^2} \end{cases} . \tag{5.5}$$

If the function $f(\tau_1, \tau_2)$ is invariant under these transformations it is invariant under the whole group $\mathrm{SL}(2,\mathbb{Z})$.

The general Fourier series of $f(\tau_1, \tau_2)$ shall be diagonalized with respect to the discrete maximal nilpotent group $\mathrm{N}(\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{Z})$. As is explained in the appendix, the nilpotent group is generated solely by the T-transformation. The action on the axion χ (τ_1) is

$$\chi \longmapsto \chi + n, \qquad n \in \mathbb{Z}. \tag{5.6}$$

In this particularly easy case, with only one parameter, a manifest periodic Fourier expansion of a general function $f(\phi, \chi)$ with respect to the nilpotent discrete subgroup can directly be written down

$$f(\phi, \chi) = \sum_{n\in\mathbb{Z}} C_n(\phi)\mathrm{e}^{2\pi i n \chi}. \tag{5.7}$$

It is convenient for the subsequent calculations to use instead the components τ_1 and τ_2, and also to extract the constant term in the Fourier series:

$$f(\tau_1, \tau_2) = C_0(\tau_2) + \sum_{n\neq 0} C_n(\tau_2)\mathrm{e}^{2\pi i n \tau_1}. \tag{5.8}$$

Laplacian Eigenfunction Equation

The automorphic function is an eigenfunction to the Laplace-Beltrami operator on the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The Laplace-Beltrami operator is derived in F.1 to:

$$\Delta = \tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right). \tag{5.9}$$

The eigenfunction equation

$$\Delta f(\tau_1,\tau_2)=\lambda f(\tau_1,\tau_2) \tag{5.10}$$

will give us more information on the structure of the function. We will see later that the eigenvalue λ will depend on the parameter s in the Eisenstein series as $\lambda = s(s-1)$. Beginning with the constant term we get

$$\tau_2^2\frac{\partial^2}{\partial\tau_2^2}C_0(\tau_2)=\lambda C_0(\tau_2),\qquad (\tau_2>0). \tag{5.11}$$

This is a special case of Euler's differential equation and it is solved by making the ansatz

$$C_0(\tau_2)=\tau_2^m,\qquad m\in\mathbb{C}. \tag{5.12}$$

Plugging the ansatz into (5.11) we get

$$\begin{aligned}\tau_2^2\frac{\partial}{\partial\tau_2}(m\tau_2^{m-1}) =&\lambda\tau_2^m\\ m(m-1)\tau_2^m =&\lambda\tau_2^m \implies m(m-1)=\lambda \implies\\ \implies m =&\frac{1}{2}\pm\sqrt{\lambda+\frac{1}{4}}.\end{aligned} \tag{5.13}$$

The solution for the constant term is therefore

$$C_0(\tau_2)=A\tau_2^{\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}}+B\tau_2^{\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}}}, \tag{5.14}$$

where A and B are coefficients dependent on the discrete symmetries and are to be determined later with help of the Eisenstein series.

Applying (5.10) on the other terms yields

$$\left[\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)-\lambda\right]\sum_{n\neq0}C_n(\tau_2)\mathrm{e}^{2\pi\mathrm{i}n\tau_1}=0, \tag{5.15}$$

which gives the following differential equation for the coefficients

$$\left(\tau_2^2\frac{\partial^2}{\partial\tau_2^2}-4\pi n^2\tau_2^2-\lambda\right)C_n(\tau_2)=0. \tag{5.16}$$

This is a modified Bessel differential equation in disguise[1] and it is solved by first making a suitable change

$$\begin{aligned}C_n(\tau_2)=&\sqrt{\tau_2}\tilde{C}_n(\tau_2) \implies\\ \frac{\partial^2}{\partial\tau_2^2}C_n(\tau_2)=&\frac{\partial}{\partial\tau_2}\left(\frac{1}{2\sqrt{\tau_2}}\tilde{C}_n(\tau_2)+\sqrt{\tau_2}\frac{\partial}{\partial\tau_2}\tilde{C}_n(\tau_2)\right)=\\ =&\left(-\frac{1}{4}\tau_2^{-\frac{3}{2}}+\frac{1}{\sqrt{\tau_2}}\frac{\partial}{\partial\tau_2}+\sqrt{\tau_2}\frac{\partial^2}{\partial\tau_2^2}\right)\tilde{C}_n(\tau_2).\end{aligned} \tag{5.17}$$

[1] The modified Bessel differential equation has the general form:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-(x^2+\alpha^2)y=0.$$

Substituting this into (5.16) gives

$$\left(\tau^2\frac{\partial^2}{\partial\tau_2^2}+\tau_2\frac{\partial}{\partial\tau_2}-\left(4\pi^2n^2\tau_2^2+\lambda+\frac{1}{4}\right)\right)\tilde{C}_n(\tau_2)=0. \tag{5.18}$$

Now, we need to get rid of the factors sitting with the τ_2^2-term. This is done by a change of the argument

$$\begin{aligned}\tilde{\tau}_2 &= 2\pi|n|\tau_2 \implies \\ \implies \frac{\partial}{\partial\tau_2} &= 2\pi|n|\frac{\partial}{\partial\tilde{\tau}_2} \qquad \text{and} \\ \frac{\partial^2}{\partial\tau_2^2} &= 4\pi n^2\frac{\partial^2}{\partial\tilde{\tau}_2^2}.\end{aligned} \tag{5.19}$$

Finally, we get the differential equation

$$\left[\tilde{\tau}_2^2\frac{\partial^2}{\partial\tilde{\tau}_2^2}+\tilde{\tau}_2\frac{\partial}{\partial\tilde{\tau}_2}-\left(\tilde{\tau}_2^2+\lambda+\frac{1}{4}\right)\right]\tilde{C}_n(\tilde{\tau}_2)=0, \qquad n\in\mathbb{Z}\backslash\{0\}. \tag{5.20}$$

This is the standard form of the modified Bessel differential equation with the two linearly independent functions $I_\alpha(\tilde{\tau}_2)$ and $K_\alpha(\tilde{\tau}_2)$ as solutions, $\alpha^2 = \lambda + 1/4$. However, from the physics we require that the solution must not diverge in the weak-coupling limit, i.e., when $g_s = \mathrm{e}^{\phi} \longrightarrow 0$ or equivalently $\tau_2 = \mathrm{e}^{-\phi} \longrightarrow \infty$ and $\tilde{\tau}_2 \longrightarrow \infty$. This rules out the function I_α and we are left with the solution

$$\begin{aligned}\tilde{C}_n(\tilde{\tau}_2) &= D(n)K_\alpha(\tilde{\tau}_2) \implies \\ C_n(\tau_2) &= D(n)\sqrt{\tau_2}K_{\sqrt{\lambda+\frac{1}{4}}}(2\pi|n|\tau_2)\end{aligned} \tag{5.21}$$

with some unknown coefficients $D(n)$. Note that the possibility

$$\alpha = -\sqrt{\lambda+1/4}$$

is not taken into account due to the property: $K_\nu = K_{-\nu}$ of the modified Bessel function of the second kind.

Substituting (5.14) and (5.21) in the general Fourier series (5.8) yields

$$f(\tau_1,\tau_2) = A\tau_2^{\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}}+B\tau_2^{\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}}}+\sqrt{\tau_2}\sum_{N\neq 0}D(N)K_{\sqrt{\lambda+\frac{1}{4}}}(2\pi|N|\tau_2)\mathrm{e}^{2\pi\mathrm{i}|N|\tau_1}. \tag{5.22}$$

The capitalization of the summation variable will turn out to suit the subsequent calculations. The unknown coefficients A, B and $D(N)$ are to be determined from the Eisenstein series. They will have a structure making the function invariant under the involution S as well; that is, making the function invariant under the whole group $\mathrm{SL}(2,\mathbb{Z})$.

5.2 Eisenstein Series

The Eisenstein series is constructed with help of the defining formula (4.15):

$$\mathcal{E}_s^{G(\mathbb{Z})}(\mathcal{M}) = \sum_{m\in\Lambda_{\mathbb{Z}}}{}' \left[m^{\mathcal{T}} \cdot \mathcal{M} \cdot m\right]^{-s} . \tag{5.23}$$

For the case $\mathrm{G}(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$ we do not need the quadratic constraint $m \wedge m = 0$ since the function is automatically an eigenfunction to the Laplace-Beltrami operator, which we will verify below. The lattice $\Lambda_{\mathbb{Z}}$, left invariant by the action of the discrete group $\mathrm{SL}(2,\mathbb{Z})$, is nothing but the two-dimensional lattice of integers $\vec{m} = (m,n) \in \mathbb{Z}^2$. The generalized transpose $(\cdot)^{\mathcal{T}}$ is for the case $\mathrm{G} = \mathrm{SL}(n,\mathbb{R})$ ordinary matrix transpose, and we will henceforth denote it as usual $(\cdot)^{\mathrm{T}}$. The generalized metric $\mathcal{M} = VV^{\mathrm{T}}$ was already calculated in Appendix B for the construction of the metric in Section 3.3. We pick again the coset representative V in the Borel gauge

$$V = \begin{pmatrix} \mathrm{e}^{-\frac{\phi}{2}} & \chi\mathrm{e}^{\frac{\phi}{2}} \\ 0 & \mathrm{e}^{\frac{\phi}{2}} \end{pmatrix} \Longrightarrow$$
$$\mathcal{M} = VV^{\mathrm{T}} = \begin{pmatrix} \mathrm{e}^{-\phi} + \chi^2\mathrm{e}^{\phi} & \chi\mathrm{e}^{\phi} \\ \chi\mathrm{e}^{\phi} & \mathrm{e}^{\phi} \end{pmatrix} . \tag{5.24}$$

We get

$$\begin{aligned} \mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\chi,\phi) &= \sum_{\vec{m}\in\mathbb{Z}^2}{}' \left[\vec{m}^{\mathrm{T}} \cdot \mathcal{M} \cdot \vec{m}\right]^{-s} = \\ &= \sum_{(m,n)\neq(0,0)} [\mathrm{e}^{\phi}(m^2 + \mathrm{e}^{2\phi}(m\chi + n)^2)]^{-s} . \end{aligned} \tag{5.25}$$

Expressing the Eisenstein series in the real and complex part of τ instead, we get

$$\begin{aligned} \mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1,\tau_2) &= \sum_{(m,n)\neq(0,0)} [\tau_2^{-1}(m^2\tau_2^2 + (m\tau_1 + n)^2)]^{-s} = \\ &= \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{[m^2\tau_2^2 + (m\tau_1+n)^2)]^s} \end{aligned} \tag{5.26}$$

or

$$\boxed{\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1,\tau_2) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m\tau + n|^{2s}} .} \tag{5.27}$$

One can explicitly verify the invariance under the transformations of the discrete group T and S (5.4). Remember that an accompanying transformation of the lattice is needed to explicitly see the invariance. E.g., the T-transformation $\tau_1 \longmapsto \tau_1 + 1$ is cancelled by shifting the lattice parameter $n \longmapsto n - m$.

Applying the Laplace-Beltrami operator (5.9) on the Eisenstein series yields

$$\Delta \mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1, \tau_2) = \ldots = s(s-1) \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{[m^2\tau_2^2 + (m\tau_1 + n)^2)]^s}. \tag{5.28}$$

So, it is indeed an eigenfunction with the eigenvalue

$$\lambda = s(s-1). \tag{5.29}$$

Using the eigenvalue in the general Fourier series expression (5.22) gives

$$f(\tau_1, \tau_2) = A\tau_2^s + B\tau_2^{1-s} + \sqrt{\tau_2} \sum_{N\neq 0} D(N) K_{s-1/2}(2\pi|N|\tau_2) \mathrm{e}^{2\pi \mathrm{i}|N|\tau_1}. \tag{5.30}$$

This expression is valid for $s > 1/2$ and it will later turn out to be consistent with our purposes.

Now, how do we connect the two seemingly different expressions (5.30) and (5.26)? The Eisenstein series is exact, i.e., there are no undetermined coefficients as in the general Fourier series. Indeed, the Eisenstein series is the correct function sitting in the asymptotic expansion of the string action, but as was pointed out earlier there is no obvious way to read off the physics. We can not interpret the different terms in the series as terms in a perturbative expansion for the fields χ and ϕ, and in particular we do not see the first two terms (5.1). We need to rewrite the Eisenstein series (5.26) on the form (5.30). The unknown coefficients A, B and $D(N)$ in (5.30) will get specific values, precisely the values making the Fourier expansion invariant under the whole discrete group $\mathrm{SL}(2,\mathbb{Z})$. A trick to use for rewriting the Eisenstein series is needed, and luckily it exists; it is *Poisson resummation*.

5.3 Identification of the Fourier Coefficients

The following calculations are based on [34] and [33]. Without too much effort we can identify A already by extracting the term $m = 0$ from the sum (5.26):

$$\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1, \tau_2) = \sum_{n\neq 0} \frac{1}{n^{2s}} \tau_2^s + \sum_{m\neq 0} \sum_{n} \frac{\tau_2^s}{[m^2\tau_2^2 + (m\tau_1 + n)^2)]^s}, \tag{5.31}$$

and

$$A = \sum_{n\neq 0} \frac{1}{n^{2s}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 2\zeta(2s), \tag{5.32}$$

using the definition of the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}. \tag{5.33}$$

Comparing the coefficient (5.32) to (5.1) we see immediately that if we have $s = 3/2$ we will get the first correct term! However, we will still proceed and extract B and $D(N)$ keeping the parameter s unspecified.

The following steps are not trivial and comes from a sort of intelligent guessing. But the amount of trial and error needed for this calculation is fairly small compared to what is needed for the much more complicated Type IIA case treated in the next chapter. We know that the end result must include the modified Bessel function $K_{s-1/2}$. There are integral representations for this function that can be used. Furthermore, one of the sums over m or n must be separated and included in the coefficients so that the remaining one corresponds to the modes of the Fourier expansion.

We begin by utilizing an integral representation (see 3.381 no.4 in [22]) for the quotient

$$\frac{1}{(m^2\tau_2^2 + (m\tau_1 + n)^2)^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \mathrm{e}^{-(m^2\tau_2^2 + (m\tau_1+n)^2)x}\, \mathrm{d}x. \tag{5.34}$$

This is valid for $(m^2\tau_2^2 + (m\tau_1 + n)^2) > 0$ and $s > 0$, which is also the case. The aim is to use the following Poisson resummation formula

$$\sum_n \mathrm{e}^{-\frac{\pi}{x}(m\tau_1+n)^2} = \sqrt{x} \sum_{\tilde{n}} \mathrm{e}^{-\pi x \tilde{n}^2 - 2\pi \mathrm{i} \tilde{n} m \tau_1}. \tag{5.35}$$

This resummation helps us to move out τ_1 from the integral with the correct exponential form as we will see later. For a derivation of (5.35) see Appendix G. To use (5.35) we need to make a change of the integration variable in (5.34)

$$x \longrightarrow \frac{\pi}{x} \implies \mathrm{d}x \longrightarrow -\frac{\pi}{x^2}\mathrm{d}x \implies$$

$$\implies \frac{1}{(m^2\tau_2^2 + (m\tau_1 + n)^2)^s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{\mathrm{d}x}{x^{s+1}}\, \mathrm{e}^{-\frac{\pi}{x}(m^2\tau_2^2 + (m\tau_1+n)^2)}. \tag{5.36}$$

Applying (5.35) and inserting the resulting expression for the quotient into (5.31) gives

$$\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1, \tau_2) = 2\zeta(2s)\tau_2^s + \frac{\pi^s \tau_2^s}{\Gamma(s)} \sum_{m\neq 0} \sum_{\tilde{n}} \mathrm{e}^{-2\pi \mathrm{i}\tilde{n} m \tau_1} \int_0^\infty \frac{\mathrm{d}x}{x^{s+1/2}}\, \mathrm{e}^{-\pi x \tilde{n}^2 - \frac{\pi}{x} m^2 \tau_2^2}. \tag{5.37}$$

Extracting the term $\tilde{n} = 0$ in the sum yields

$$\begin{aligned}\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1, \tau_2) = 2\zeta(2s)\tau_2^s + \frac{\pi^s \tau_2^s}{\Gamma(s)} \sum_{m\neq 0} \int_0^\infty \frac{\mathrm{d}x}{x^{s+1/2}}\, \mathrm{e}^{-\frac{\pi}{x} m^2 \tau_2^2} + \\ + \frac{\pi^s \tau_2^s}{\Gamma(s)} \sum_{m\neq 0} \sum_{\tilde{n}\neq 0} \mathrm{e}^{-2\pi \mathrm{i}\tilde{n} m \tau_1} \int_0^\infty \frac{\mathrm{d}x}{x^{s+1/2}}\, \mathrm{e}^{-\pi x \tilde{n}^2 - \frac{\pi}{x} m^2 \tau_2^2}.\end{aligned} \tag{5.38}$$

The middle term can be further simplified using the integral representation (5.34) (this time backwards) and the definition of the Riemann zeta function (5.33). We obtain

$$\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1,\tau_2) = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s} + \\ + \frac{\pi^s\tau_2^s}{\Gamma(s)}\sum_{m\neq 0}\sum_{\tilde{n}\neq 0} \mathrm{e}^{-2\pi\mathrm{i}m\tilde{n}\tau_1}\int_0^\infty \frac{\mathrm{d}x}{x^{s+1/2}}\,\mathrm{e}^{-\pi x\tilde{n}^2-\frac{\pi}{x}m^2\tau_2^2}. \quad (5.39)$$

We can now identify the second coefficient B in (5.30) sitting with the factor τ_2^{1-s}:

$$B = 2\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1). \quad (5.40)$$

For the choice $s = 3/2$ we get the correct second term in (5.1)!

The remaining integral is conveniently of Bessel type (see 8.432 no. 6, [22]):

$$\int_0^\infty \frac{\mathrm{d}x}{x^{s+1/2}}\,\mathrm{e}^{-\pi x\tilde{n}^2-\frac{\pi}{x}m^2\tau_2^2} = 2\left|\frac{\tilde{n}}{m}\right|^{s-1/2}\tau_2^{1/2-s}K_{s-1/2}(2\pi|m\tilde{n}|\tau_2). \quad (5.41)$$

The last term in (5.39) becomes

$$\frac{2\pi^s}{\Gamma(s)}\sqrt{\tau_2}\sum_{m\neq 0}\sum_{\tilde{n}\neq 0}\left|\frac{\tilde{n}}{m}\right|^{s-1/2}K_{s-1/2}(2\pi|m\tilde{n}|\tau_2)\mathrm{e}^{-2\pi\mathrm{i}m\tilde{n}\tau_1}. \quad (5.42)$$

Looking at (5.30) we see that we want to sum over a variable $N = -m\tilde{n} \in \mathbb{Z}\backslash\{0\}$. It is actually possible to make this variable and summation change. Note first that the product $-m\tilde{n}$ indeed will take all values in the set $\mathbb{Z}\backslash\{0\}$. However, it will take values multiple times equal to the number of divisors to the product. For instance

$$\sum_{p\neq 0}\sum_{q\neq 0} f(|pq|) = \{P = pq\} = \sum_{P\neq 0}\sum_{q|P} f(|P|). \quad (5.43)$$

Having a quotient $p/q = pq/q^2$ as well we get

$$\sum_{p\neq 0}\sum_{q\neq 0}\left|\frac{p}{q}\right| f(|pq|) = \{P = pq\} = \sum_{P\neq 0}\sum_{q|P}\left|\frac{P}{q^2}\right| f(|P|). \quad (5.44)$$

Application of this on the last term (5.42) yields (using $N = -m\tilde{n}$):

$$\frac{2\pi^s}{\Gamma(s)}\sqrt{\tau_2}\sum_{N\neq 0}\sum_{m|N}\left|\frac{N}{m^2}\right|^{s-1/2}K_{s-1/2}(2\pi|N|\tau_2)\mathrm{e}^{2\pi\mathrm{i}N\tau_1} = \\ = \frac{2\pi^s}{\Gamma(s)}\sqrt{\tau_2}\sum_{N\neq 0}\mu_{1-2s}(N)|N|^{s-1/2}K_{s-1/2}(2\pi|N|\tau_2)\mathrm{e}^{2\pi\mathrm{i}N\tau_1}, \quad (5.45)$$

where we have used the so-called *instanton measure* defined as

$$\mu_t(N) := \sum_{m|N} |m|^t. \tag{5.46}$$

We can now identify the last coefficient(s) $D(N)$ in (5.30):

$$D(N) = \frac{2\pi^s}{\Gamma(s)} |N|^{s-1/2} \mu_{1-2s}(N). \tag{5.47}$$

Finally, collecting all terms we get:

$$\begin{aligned}\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1, \tau_2) = 2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s} + \\ + \frac{2\pi^s}{\Gamma(s)}\sqrt{\tau_2}\sum_{N\neq 0} \mu_{1-2s}(N)|N|^{s-1/2} K_{s-1/2}(2\pi|N|\tau_2)\mathrm{e}^{2\pi \mathrm{i} N\tau_1}.\end{aligned} \tag{5.48}$$

And as we have seen when comparing the terms with the coefficients A and B we need to put $s = 3/2$ yielding

$$\begin{aligned}\mathcal{E}_{3/2}^{\mathrm{SL}(2,\mathbb{Z})}(\tau_1, \tau_2) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \\ + \frac{2\pi^{3/2}}{\Gamma(3/2)}\sqrt{\tau_2}\sum_{N\neq 0} \mu_2(N)|N| K_1(2\pi|N|\tau_2)\mathrm{e}^{2\pi \mathrm{i} N\tau_1}.\end{aligned} \tag{5.49}$$

There are many indications that the Eisenstein series (5.49) is the correct function $f(\tau_1, \tau_2)$ sitting with the R^4-factor in the Type IIB action. The first two terms agreed with calculations by other methods, and the function has the correct symmetry properties as well as is an eigenfunction to the Laplace-Beltrami operator. It is fascinating that the discrete symmetry $\mathrm{SL}(2,\mathbb{Z})$ helped us to fully determine the function. That is, we have an expression for all loop-corrections to the order $n = 3$ ($1/\alpha'$ see (1.17)) as well as all instanton corrections. Cf. quantum field theory where we in general need to explicitly calculate loop diagrams to a specific order.

The last term in (5.49) encodes the instanton corrections. The non-perturbative nature of the term comes from the modified Bessel function. To see it we can use an asymptotic expression (see 11.127, [4]) valid in the low-coupling limit, i.e., for small g_s (large τ_2):

$$\begin{aligned}K_1(2\pi|N|\tau_2) \sim \sqrt{\frac{1}{4|N|\tau_2}}\,\mathrm{e}^{-\tau_2}(1 + \mathcal{O}(\tau_2^{-1})) = \\ = \sqrt{\frac{g_s}{4|N|}}\,\mathrm{e}^{-\frac{1}{g_s}}(1 + \mathcal{O}(g_s)).\end{aligned} \tag{5.50}$$

6

The Type IIA Case: $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$

The focus of this thesis is on the construction of the automorphic forms and the identification of the coefficients in the general Fourier series. Therefore, we will now give a purely schematic background to the physics. As was discussed in Section 2.2 the result of compactifying ten-dimensional Type IIA supergravity on a rigid Calabi-Yau threefold is that, amongst others, we get a universal hypermultiplet consisting of four scalar fields: ϕ, χ, $\tilde{\chi}$ and ψ. The fields parameterize the symmetric four-dimensional coset space $\mathcal{M}_{\mathrm{UH}} = \mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. We have already derived the lagrangian (3.29) for the scalar fields via the nonlinear sigma model with target space $\mathcal{M}_{\mathrm{UH}}$. The action reads

$$\begin{aligned} S_{(\phi,\chi,\tilde{\chi},\psi)} \propto \int \mathrm{d}^4x\sqrt{|g|}g^{\mu\nu}(x)\partial_\mu\xi^a(x)\partial_\nu\xi^b(x)\gamma_{ab}(\xi(x)) = \\ = \int \mathrm{d}^4x\sqrt{|g|}\Big(\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\phi}(\partial_\mu\chi\partial^\mu\chi + \partial_\mu\tilde{\chi}\partial^\mu\tilde{\chi}) + \\ + \mathrm{e}^{4\phi}(\chi\partial_\mu\tilde{\chi} - \tilde{\chi}\partial_\mu\chi + \partial_\mu\psi)^2\Big). \quad (6.1) \end{aligned}$$

This serves as the classical part, for the scalar field hypermultiplet, of the effective Type IIA superstring theory compactified on the rigid CY_3. Unlike in the Type IIB case in the last chapter, where we studied the quantum corrections sitting with the R^4-term in the effective action, we will now study how the quantum corrections deform the metric γ^{ab} of the moduli space for the scalar fields in the effective Type IIA superstring action. I.e., there will be corrections for the terms in (6.1), and we can no longer write down the action with help of the method described in Chapter 3 since the moduli space will no longer be a coset space. The moduli space after the quantum corrections is referred to as the *exact* moduli space $\mathcal{M}_{\mathrm{UH}}^{\mathrm{exact}}$.

In some way there ought to be an automorphic form f encoding the quantum corrections. This since due to dualities of the string theory the exact moduli space should have a discrete symmetry $\mathrm{G}(\mathbb{Z}) \subset \mathrm{SU}(2,1)$. By studying the physical properties of the scalar fields in the universal hypermultiplet, as well as topological properties of the rigid Calabi-Yau threefold, it is made reasonable in [9] that the discrete subgroup should be the *Picard modular group* defined as

$$\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}]) := \mathrm{SU}(2,1) \cap \mathrm{SL}(3,\mathbb{Z}[\mathrm{i}]). \tag{6.2}$$

This together with the fact that f should be a function of the fields in the universal hypermultiplet, which parameterize the coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$, points towards that f is an automorphic form on $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. We will construct a tentative proposal, given by a non-holomorphic Eisenstein series $\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}$, with help of the lattice sum method in Section 6.2. Before that, we want to know the general Fourier series structure, from which we later on can read off the different orders of loop-corrections in g_s (there will only be a tree-level and a one-loop term as in the Type IIB case) and instanton corrections coming in this case from D2- and NS5-branes; for more about the different kinds of instantons see [34] and [8]. The Fourier series structure is due to the fact that there is a maximal nilpotent subgroup $\mathrm{N}(\mathbb{Z}) = \mathrm{H}_3$ (the Heisenberg group) of the Picard modular group, as well as the constraint on $f(\phi,\chi,\tilde{\chi},\psi)$ that it must be an eigenfunction to the Laplace-Beltrami operator on $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. This laplacian constraint holds since the Eisenstein series we construct fulfill the Laplace eigenfunction equation. The eigenvalue will depend on the order of the Eisenstein series, and we will crave the specific order $s = 3/2$ that can be argued for by studying the fields in the series, see [34]. However, it is not yet made clear from supersymmetric reasons, as in the Type IIB superstring theory, that the Laplace-Beltrami operator is the correct operator to use. Instead, a corresponding constraint from supersymmetry is in this case that the moduli space $\mathcal{M}_{\mathrm{UH}}^{\mathrm{exact}}$ must be quaternionic-Kähler, see [34] and [7]. It may be that one should seek another type of automorphic form than the non-holomorphic Eisenstein series. In [9] one discusses the possibility of constructing an automorphic form attached to the quaternionic discrete series of $\mathrm{SU}(2,1)$.

The identification of the Fourier coefficients from the exact Eisenstein series $\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}$ prove to be difficult. As described in Section 4.1 the Fourier series is decomposed into three parts: a constant, an abelian and a non-abelian part. Most subtleties arise when trying to understand the non-abelian part; Section 6.4 is devoted to this problem.

We will in this thesis gloss over the apparent issue how to relate the automorphic form f with the metric on $\mathcal{M}_{\mathrm{UH}}^{\mathrm{exact}}$. In Type IIB superstring theory we had that the automorphic form (Eisenstein series) were the coefficient to the R^4-term in the effective action. In the compactified Type IIA case it is not as straightforward. In [9] one presents a way of doing this using so-called *twistor* techniques[1].

We will for convenience set $y = \mathrm{e}^{-2\phi}$ in most of the subsequent calculations.

6.1 General Fourier Series

The details of the Picard modular group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ are treated in Appendix E. The group is generated (non-minimally) by three translations, one reflection and one involution:

$$T_1 = \begin{pmatrix} 1 & -1+\mathrm{i} & \mathrm{i} \\ 0 & 1 & 1-\mathrm{i} \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 1 & 1+\mathrm{i} & \mathrm{i} \\ 0 & 1 & 1+\mathrm{i} \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$R = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & -1 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}. \tag{6.3}$$

As mentioned in Chapter 4, the general Fourier series is diagonalized with respect to the maximal nilpotent group $\mathrm{N}(\mathbb{Z}) \subset \mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ which, in this case, is isomorphic to the discrete Heisenberg group H_3 generated by the Heisenberg translations T_1, $\tilde{T}_1$ and T_2. A general action of the nilpotent group transforms the scalar fields as

$$\begin{aligned} T_{(a,b,c)}: \quad & \phi \longmapsto \phi \qquad (y \longmapsto y) \\ & \chi \longmapsto \chi + a \\ & \tilde{\chi} \longmapsto \tilde{\chi} + b \\ & \psi \longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi, \qquad a, b, c \in \mathbb{Z}. \end{aligned} \tag{6.4}$$

A function is then invariant under all translations if it is invariant under $T_{(1,0,0)}$, $T_{(0,1,0)}$, and $T_{(0,0,1)}$ separately. The nilpotent group is non-abelian as is seen from the transformation on ψ. Therefore, our Fourier expansion will be non-abelian and much more complicated than in the Type IIB case in the preceding chapter.

6.1.1 Using Invariance under the Nilpotent Subgroup

As described in 4.1, it is possible to divide the general Fourier series into a *constant*, *abelian* and *non-abelian* part with respect to the nilpotent group $\mathrm{N}(\mathbb{Z})$. This will be used to simplify our calculations considerably by chopping up the problem into smaller more manageable pieces. Using (6.4) we can directly read off the dependence on the four fields of the respective parts

$$f(y,\chi,\tilde{\chi},\psi) = f^{(\mathrm{C})}(y) + f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) + f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi). \tag{6.5}$$

[1]For more about twistors see for instance [1] and [2]

For the constant term, we cannot deduce anything from the discrete symmetries since it solely depends on the non-compact field $y=\mathrm{e}^{-2\phi}$. For the abelian term we can directly write down the general structure of the Fourier expansion using the requirement of invariance under the Heisenberg translations of χ and $\tilde{\chi}$:

$$f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) = \sum_{(l_1,l_2)\neq(0,0)} C^{(\mathrm{A})}_{l_1,l_2}(y)\mathrm{e}^{2\pi\mathrm{i}l_1\chi}\mathrm{e}^{2\pi\mathrm{i}l_2\tilde{\chi}}, \tag{6.6}$$

where $C^{(\mathrm{A})}_{l_1,l_2}(y)$ is some coefficient[2] whose structure will be explained in more detail later after we have applied the laplacian constraint. Note the important fact that the zero mode $l_1=l_2=0$ is not included because it is in fact $f^{(\mathrm{C})}(y)$.

Naturally, the general structure of the non-abelian term is not as simple to write down. The following calculation is based on [34]. We use first the $T_{(0,0,1)}$-transformation to see that we must have

$$f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sum_{k\neq0} C^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi})\mathrm{e}^{-4\pi\mathrm{i}k\psi}, \tag{6.7}$$

note that for $k=0$ we would recover the abelian term (6.6) again. The minus sign in the exponent is chosen just to fit with the calculations in [34] and [9]. Now, using $T_{(1,0,0)}$ on (6.7) yields

$$\sum_{k\neq0} C^{(\mathrm{NA})}_{k}(y,\chi+1,\tilde{\chi})\mathrm{e}^{-4\pi\mathrm{i}k\psi+4\pi\mathrm{i}k\tilde{\chi}}, \tag{6.8}$$

which implies the following constraint on the coefficient

$$C^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi}) = C^{(\mathrm{NA})}_{k}(y,\chi+1,\tilde{\chi})\mathrm{e}^{4\pi\mathrm{i}k\tilde{\chi}}. \tag{6.9}$$

We can, with help of (6.9), easily construct a function invariant under $T_{(1,0,0)}$ namely

$$C^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi})\mathrm{e}^{4\pi\mathrm{i}k\chi\tilde{\chi}}. \tag{6.10}$$

Indeed we have

$$\begin{aligned} T_{(1,0,0)}: C^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi})\mathrm{e}^{4\pi\mathrm{i}k\chi\tilde{\chi}} &\longmapsto C^{(\mathrm{NA})}_{k}(y,\chi+1,\tilde{\chi})\mathrm{e}^{4\pi\mathrm{i}k\chi\tilde{\chi}+4\pi\mathrm{i}k\tilde{\chi}} = \\ &= \{\text{use (6.9)}\} = C^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi})\mathrm{e}^{4\pi\mathrm{i}k\chi\tilde{\chi}}. \end{aligned} \tag{6.11}$$

This implies that there must be a Fourier expansion in χ. I.e.

$$C^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi})\mathrm{e}^{4\pi\mathrm{i}k\chi\tilde{\chi}} = \sum_{p\in\mathbb{Z}} C^{(\mathrm{NA})}_{k,p}(y,\tilde{\chi})\mathrm{e}^{2\pi\mathrm{i}p\chi}. \tag{6.12}$$

[2] We will use C throughout as a kind of coefficient in the Fourier series. Since there will be many coefficients we will use superscripts to denote if they correspond to the constant, abelian or non-abelian part; as well as subscripts denoting the dependence on the summation variables. The reader should also note that the arguments are important, e.g., $C^{(\mathrm{A})}(y,\chi,\tilde{\chi})$ is another type of function as $C^{(\mathrm{A})}(y,\chi)$,

Using this in (6.7) yields

$$f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sum_{k\neq 0}\sum_{p\in\mathbb{Z}} C^{(\mathrm{NA})}_{k,p}(y,\tilde{\chi})\mathrm{e}^{2\pi\mathrm{i}p\chi - 4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \tag{6.13}$$

The attentive reader notices that we could just as well have done these calculations starting with the shift $b=1$ in $\tilde{\chi}$ instead. This would in the end give

$$f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sum_{k\neq 0}\sum_{p\in\mathbb{Z}} \tilde{C}^{(\mathrm{NA})}_{k,p}(y,\chi)\mathrm{e}^{-2\pi\mathrm{i}p\tilde{\chi} - 4\pi\mathrm{i}k(\psi-\chi\tilde{\chi})}, \tag{6.14}$$

where we instead choose a minus sign in the Fourier expansion of the constructed function (in this case this function is $\tilde{C}^{(\mathrm{NA})}_{k}(y,\chi,\tilde{\chi})\mathrm{e}^{-4\pi\mathrm{i}k\chi\tilde{\chi}}$) to fit with later calculations. The two possibilities (6.13) and (6.14) are referred to as two types of *polarizations*. They are related through a Fourier transformation (see [34]), and as we will see later also through the rotation $R\in \mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$. We will henceforth mainly use the first choice of polarization (6.13).

Continuing from (6.13) we want to make it manifestly invariant under $T_{(0,1,0)}$ as well. An application of such a transformation gives

$$\begin{aligned} T_{(0,1,0)}:\quad f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) &\longmapsto \sum_{k\neq 0}\sum_{p\in\mathbb{Z}} C^{(\mathrm{NA})}_{k,p}(y,\tilde{\chi}+1)\mathrm{e}^{2\pi\mathrm{i}p\chi - 4\pi\mathrm{i}k(\psi+\chi\tilde{\chi}+2\chi)} = \\ &= \sum_{k\neq 0}\sum_{p\in\mathbb{Z}} C^{(\mathrm{NA})}_{k,p}(y,\tilde{\chi}+1)\mathrm{e}^{2\pi\mathrm{i}\chi(p-4k)+4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \end{aligned} \tag{6.15}$$

We see that if it was the case that the coefficients were related as

$$C^{(\mathrm{NA})}_{k,p} = C^{(\mathrm{NA})}_{k,p+4k}, \qquad \forall k\in\mathbb{Z}\backslash\{0\} \tag{6.16}$$

we could make a shift $p\longmapsto p+4k$ in the p summation to get rid of the extra term appearing in the exponent. Note that there is no way that the coefficient could soak up the exponent, since the former is independent of χ. A shift in the summation is therefore needed. Moreover, the coefficient must somehow be invariant when changing the argument with the accompanied shift $C^{(\mathrm{NA})}_{k,p}(y,\tilde{\chi})\longmapsto C^{(\mathrm{NA})}_{k,p+4k}(y,\tilde{\chi}+1)$. We will see later how this is solved.

The relation (6.16) tells us that there should for a fix k only be $4|k|$ independent constants $C^{(\mathrm{NA})}_{k,p}$. So guided by this we sum over p in the following way[3]:

$$\begin{aligned} \text{“}\sum_{p\in Z}\mathrm{e}^{2\pi\mathrm{i}p\chi} &= \sum_{n'=0}^{4|k|-1}\sum_{m\in\mathbb{Z}}\mathrm{e}^{\overbrace{2\pi\mathrm{i}(n'+4|k|m)}^{8\pi\mathrm{i}|k|\left(m+\frac{n'}{4|k|}\right)}\chi} = \\ &= \sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}}\mathrm{e}^{8\pi\mathrm{i}|k|n\chi}\text{”}, \end{aligned} \tag{6.17}$$

[3]We are dealing with absolute convergent series so we can sum in any order we like.

where our new parameter $n = m + n'/(4|k|)$ runs through all integers shifted with the fraction $n'/(4|k|)$. The function (6.13) modifies to

$$f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n,n'}(y,\tilde{\chi})\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \quad (6.18)$$

Acting with $T_{(0,1,0)}$ on (6.18) gives

$$\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n,n'}(y,\tilde{\chi}+1)\mathrm{e}^{\overbrace{8\pi\mathrm{i}|k|n\chi-8\pi\mathrm{i}k\chi}^{8\pi\mathrm{i}|k|(n-\mathrm{sgn}(k))\chi}-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \quad (6.19)$$

Here we see that the expression can be made invariant under $T_{(0,1,0)}$ if we shift the summation $n \longmapsto n + \mathrm{sgn}(k)$, and have it so that the coefficients do not depend on n, i.e., $C^{(\mathrm{NA})}_{k,n,n'} = C^{(\mathrm{NA})}_{k,n'}$. However, this is not the end of the story since we must also demand that the coefficient transforms to itself under $T_{(0,1,0)}$. Either it must be the case that

$$C^{(\mathrm{NA})}_{k,n'}(y,\tilde{\chi}) = C^{(\mathrm{NA})}_{k,n'}(y,\tilde{\chi}+1), \quad (6.20)$$

or that $C^{(\mathrm{NA})}_{k,n'}$ is not a function of $\tilde{\chi}$ but rather

$$C^{(\mathrm{NA})}_{k,n'} = C^{(\mathrm{NA})}_{k,n'}(y,\tilde{\chi}-n\cdot\mathrm{sgn}(k)). \quad (6.21)$$

The latter is a possibility since the translation $\tilde{\chi} \longmapsto \tilde{\chi}+1$ will be cancelled by the shift $n \longmapsto n+\mathrm{sgn}(k) \implies n\cdot\mathrm{sgn}(k) \longmapsto n\cdot\mathrm{sgn}(k)+1$. It turns out, as we will see later when solving the laplacian eigenfunction equation, that (6.21) is the case. Therefore, we will use this fact (unfortunately before we have come to the understanding) and rewrite the non-abelian term as (6.18) becomes

$$f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n'}(y,\tilde{\chi}-n\cdot\mathrm{sgn}(k))\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \quad (6.22)$$

Doing the same calculation as above for the other choice of polarization (6.14) gives

$$f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} \tilde{C}^{(\mathrm{NA})}_{k,n'}(y,\chi-n\cdot\mathrm{sgn}(k))\mathrm{e}^{-8\pi\mathrm{i}|k|n\tilde{\chi}-4\pi\mathrm{i}k(\psi-\chi\tilde{\chi})}, \quad (6.23)$$

And for this expression the $T_{(1,0,0)}$-transformation must be accompanied with the shift $n \longmapsto n+\mathrm{sgn}(k)$.

6.1.2 Using the Rotation R

The rotation R can be used to constrain the general Fourier series further. It acts as an *electric-magnetic* duality transformation on the fields χ and $\tilde{\chi}$:

$$R:\ (\chi,\tilde{\chi}) \longmapsto (-\tilde{\chi},\chi). \tag{6.24}$$

Applying R to the abelian term (6.6) yields

$$R:\ \sum_{(l_1,l_2)\neq(0,0)} C^{(\mathrm{A})}_{l_1,l_2}(y)\mathrm{e}^{2\pi\mathrm{i}l_1\chi}\mathrm{e}^{2\pi\mathrm{i}l_2\tilde{\chi}} \longmapsto \sum_{(l_1,l_2)\neq(0,0)} C^{(\mathrm{A})}_{l_1,l_2}(y)\mathrm{e}^{-2\pi\mathrm{i}l_1\tilde{\chi}}\mathrm{e}^{2\pi\mathrm{i}l_2\chi}. \tag{6.25}$$

Shifting the lattice in the transformed expression $(l_1,l_2)\longmapsto(-l_2,l_1)$ gives back the same exponents; this puts constraints on the coefficient $C^{(\mathrm{A})}_{l_1,l_2}$. Applying R two more times tells us, all in all

$$C^{(\mathrm{A})}_{l_1,l_2} = C^{(\mathrm{A})}_{-l_2,l_1} = C^{(\mathrm{A})}_{-l_1,-l_2} = C^{(\mathrm{A})}_{l_2,-l_1}, \tag{6.26}$$

which is seen as an invariance under $\pi/2$ rotations of the discrete lattice.

Acting with R on the non-abelian term (6.22) gives

$$\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n'}(y,\chi-n\cdot\mathrm{sgn}(k))\mathrm{e}^{-8\pi\mathrm{i}|k|n\tilde{\chi}-4\pi\mathrm{i}k(\psi-\chi\tilde{\chi})}. \tag{6.27}$$

I.e., we recover the other polarization (6.23) but with the coefficient corresponding to the first choice of polarization[4]. This implies an equality between the two coefficients, term by term since the terms in the expansion are linearly independent

$$C^{(\mathrm{NA})}_{k,n'} = \tilde{C}^{(\mathrm{NA})}_{k,n'}. \tag{6.28}$$

Acting with R a second time takes us back to the first polarization, but with different signs

$$\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n'}(y,-\tilde{\chi}-n\cdot\mathrm{sgn}(k))\mathrm{e}^{-8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \tag{6.29}$$

We will see what this implies on the coefficient later in the end of Section 6.1.3 when we know more precisely what kind of function of $\tilde{\chi}$ we are dealing with.

The involution $S\in\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ has a more involved effect on the fields, see (E.11) and (E.12), and as in the Type IIB case it is hard to apply it to the Fourier expansion and get any information on the coefficients.

[4] This was why we choose the minus sign in the exponent in (6.14), to make it easy to compare the coefficients.

To sum up, we have with help of the nilpotent subgroup $\mathrm{N}(\mathbb{Z})$, which is isomorphic to the Heisenberg group generated by the translations $T_{(a,b,c)}$, and the rotation R, which was also an element in the Picard modular group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$, arrived to the following structure of the general Fourier series expansion:

$$\begin{aligned} f(y,\chi,\tilde{\chi},\psi) &= f^{(\mathrm{C})}(y) + f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) + f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \\ &= f^{(\mathrm{C})}(y) + \sum_{(l_1,l_2)\neq(0,0)} C^{(\mathrm{A})}_{l_1,l_2}(y)\mathrm{e}^{2\pi\mathrm{i}l_1\chi}\mathrm{e}^{2\pi\mathrm{i}l_2\tilde{\chi}} + \\ &+ \sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n'}(y,\tilde{\chi}-n\cdot\mathrm{sgn}(k))\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \end{aligned} \tag{6.30}$$

In the next section we will see how the laplacian eigenfunction constraint further determines the unknown coefficients.

6.1.3 Laplacian Eigenfunction Equation

For reasons explained in the beginning of this chapter the automorphic form $f(y,\chi,\tilde{\chi},\psi)$ must be an eigenfunction to the Laplace-Beltrami operator on the coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. The Laplace-Beltrami operator is calculated in Appendix F to (scaled with $\frac{1}{4}$):

$$\Delta = y^2\partial_y^2 - y\partial_y + \frac{1}{4}y(\partial_\chi^2+\partial_{\tilde{\chi}}^2) + \frac{1}{2}y(\tilde{\chi}\partial_\chi - \chi\partial_{\tilde{\chi}})\partial_\psi + \frac{1}{4}y(y+\chi^2+\tilde{\chi}^2)\partial_\psi^2. \tag{6.31}$$

Applying the eigenfunction equation to the function (6.30) we get

$$\begin{aligned} \Delta f(y,\chi,\tilde{\chi},\psi) &= \Delta\left[f^{(\mathrm{C})}(y) + f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) + f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi)\right] = \\ &= \lambda\left[f^{(\mathrm{C})}(y) + f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) + f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi)\right], \end{aligned} \tag{6.32}$$

for some eigenvalue λ. As for the Type IIB case, λ will depend on the parameter s in the Eisenstein series. As we will see later in Section 6.2 the dependence is

$$\lambda = s(s-2). \tag{6.33}$$

To make our calculations more effective we will use this fact in this section as well. In particular, we require the order of the series to be $s = 3/2$, this since the tree-level and the one-loop term should differ with a factor of $g_s^2 = \mathrm{e}^{2\phi}$ coming from the orders in the asymptotic expansion (1.17). However, we will keep the variable s indefinite throughout the calculations to be as general as possible.

Equation (6.32) implies that the constant, abelian and non-abelian terms must satisfy the eigenfunction equations independently (with the same eigenvalue):

$$\begin{aligned} \Delta f^{(\mathrm{C})}(y) &= \lambda f^{(\mathrm{C})}(y) \\ \Delta f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) &= \lambda f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) \\ \Delta f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) &= \lambda f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi). \end{aligned} \tag{6.34}$$

We will now solve these differential equations one by one.

The Constant Term

The constant term depends solely on y so from (6.31) we get

$$(y^2\partial_y^2 - y\partial_y)f^{(\mathrm{C})}(y) = \lambda f^{(\mathrm{C})}(y), \qquad y > 0. \tag{6.35}$$

This is Euler's differential equation, which is solved by making an ansatz

$$f^{(\mathrm{C})}(y) = y^m. \tag{6.36}$$

We plug this into (6.35)

$$\begin{aligned} y^2 m(m-1)y^{m-2} - ymy^{m-1} - \lambda y^m &= 0 \\ (m(m-1) - m - \lambda)y^m &= 0 \\ m^2 - 2m &= \lambda. \end{aligned} \tag{6.37}$$

Using the knowledge of the eigenvalue $\lambda = s(s-2)$ we get

$$\begin{aligned} m^2 - 2m &= s(s-2) \\ (m-1)^2 = s(s-2) + 1 &= (s-1)^2 \\ m = 1 \pm (s-1) &= \begin{cases} s \\ 2-s. \end{cases} \end{aligned} \tag{6.38}$$

The general solution is

$$f^{(\mathrm{C})}(y) = C_1 y^s + C_2 y^{2-s} = C_1 \mathrm{e}^{-2s\phi} + C_2 \mathrm{e}^{-2(2-s)\phi}, \tag{6.39}$$

for some complex constants C_1 and C_2 that we need the Eisenstein series to determine. We see that for $s = 3/2$ the terms differ with a factor of $g_s^2 = \mathrm{e}^{2\phi}$, which is required from the asymptotic expansion as mentioned before.

The Abelian Term

The abelian term depends on y, χ and $\tilde{\chi}$ so from (6.31) we get

$$\left[y^2\partial_y^2 - y\partial_y + \frac{1}{4}y(\partial_\chi^2 + \partial_{\tilde{\chi}}^2)\right] f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) = \lambda f^{(\mathrm{A})}(y,\chi,\tilde{\chi}), \qquad y > 0. \quad (6.40)$$

Using the general Fourier expansion of the abelian term from (6.30) in (6.40) we get

$$\begin{aligned}\left[y^2\partial_y^2 - y\partial_y + \frac{1}{4}y(\partial_\chi^2 + \partial_{\tilde{\chi}}^2)\right] \sum_{(l_1,l_2)\neq(0,0)} C^{(\mathrm{A})}_{l_1,l_2}(y)\mathrm{e}^{2\pi\mathrm{i}l_1\chi}\,\mathrm{e}^{2\pi\mathrm{i}l_2\tilde{\chi}} = \\ = \sum_{(l_1,l_2)\neq(0,0)} \mathrm{e}^{2\pi\mathrm{i}l_1\chi}\,\mathrm{e}^{2\pi\mathrm{i}l_2\tilde{\chi}}\left[y^2\partial_y^2 - y\partial_y - y\pi^2(l_1^2+l_2^2)\right] C^{(\mathrm{A})}_{l_1,l_2}(y) = \\ = \lambda \sum_{(l_1,l_2)\neq(0,0)} C^{(\mathrm{A})}_{l_1,l_2}(y)\mathrm{e}^{2\pi\mathrm{i}l_1\chi}\,\mathrm{e}^{2\pi\mathrm{i}l_2\tilde{\chi}}. \quad (6.41)\end{aligned}$$

This implies a differential equation in the single non-compact field y, for the coefficient $C^{(\mathrm{A})}_{l_1,l_2}(y)$, that reads

$$\left[y^2\partial_y^2 - y\partial_y - (\pi^2(l_1^2+l_2^2)y + \lambda)\right] C^{(\mathrm{A})}_{l_1,l_2}(y) = 0. \quad (6.42)$$

To avoid a lot of cluttering we will simply define

$$Y(y) := C^{(\mathrm{A})}_{l_1,l_2}(y), \qquad c_1 := \pi^2(l_1^2+l_2^2), \quad (6.43)$$

and (6.42) becomes

$$\left[y^2\partial_y^2 - y\partial_y - (c_1 y + \lambda)\right] Y(y) = 0, \qquad c_1 > 0. \quad (6.44)$$

Although it may not be apparent, (6.44) can be transformed to a diffusion equation solved with modified Bessel functions. We first have to rewrite the equation in a new function

$$\begin{aligned}\tilde{Y}(y) = \frac{Y(y)}{y} \implies \\ \implies \partial_y Y = \partial_y(y\tilde{Y}) = \tilde{Y} + y\partial_y\tilde{Y} \qquad \text{and} \\ \partial_y^2 Y = \partial_y(\tilde{Y} + y\partial_y\tilde{Y}) = 2\partial_y\tilde{Y} + y\partial_y^2\tilde{Y}. \quad (6.45)\end{aligned}$$

Putting this into the differential equation (6.44) one finds

$$\begin{aligned}y^2(2\partial_y\tilde{Y} + y\partial_y\tilde{Y}) - y(\tilde{Y} + y\partial_y\tilde{Y}) - (c_1 y + \lambda)y\tilde{Y} = 0 \\ \implies \qquad y^2\partial_y^2\tilde{Y} + y\partial_y\tilde{Y} - (c_1 y + \lambda + 1)\tilde{Y} = 0. \quad (6.46)\end{aligned}$$

This is almost on the correct form, but the factor $c_1 y$ should be quadratic in the variable. This is fixed by the variable change

$$\xi = 2\sqrt{c_1 y}. \quad (6.47)$$

We get

$$\begin{aligned} y &= \frac{\xi^2}{4c_1}, \qquad \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} = \frac{2c_1}{\xi}\frac{\partial}{\partial \xi}, \\ \frac{\partial^2}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{2c_1}{\xi}\frac{\partial}{\partial \xi}\right) = \frac{4c_1^2}{\xi}\frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}\right) = \frac{4c_1^2}{\xi}\left(-\frac{1}{\xi^2}\frac{\partial}{\partial \xi} + \frac{1}{\xi}\frac{\partial^2}{\partial \xi^2}\right), \end{aligned} \tag{6.48}$$

and inserting this into (6.46) gives

$$\left[\xi^2\partial_\xi^2 + \xi\partial_\xi - (\xi^2 + 4(\lambda+1))\right]\tilde{Y}(\xi) = 0. \tag{6.49}$$

This equation is solved with the modified Bessel functions $I_\nu(\xi)$ and $K_\nu(\xi)$ with $\nu = \sqrt{4(\lambda+1)}$. However, only $K_\nu(\xi)$ fulfill the requirement that

$$K_\nu(\xi) \longrightarrow 0 \qquad \text{for} \qquad \xi \longrightarrow \infty \quad (y \longrightarrow \infty).$$

Since it also converges to zero exponentially fast in y, it takes care of the linear factor that we factored out in $\tilde{Y}(y) = Y(y)/y$. For $\lambda = s(s-2)$ we have

$$\nu = \sqrt{4(\lambda+1)} = 2\sqrt{(s-1)^2} = 2|s-1|, \qquad s \in \mathbb{R} \tag{6.50}$$

and the solution is

$$\begin{aligned} &\tilde{Y}(\xi) = K_{\nu=2|s-1|}\left(\xi = 2\sqrt{c_1}\sqrt{y} = 2\pi\sqrt{y}\sqrt{l_1^2+l_2^2}\right) \\ \Longrightarrow \quad &Y(y) = C^{(\mathrm{A})}_{l_1,l_2}(y) \propto y K_{2|s-1|}\left(2\pi\sqrt{y}\sqrt{l_1^2+l_2^2}\right). \end{aligned} \tag{6.51}$$

Putting all together we get the total solution to the abelian part

$$f^{(\mathrm{A})}(y,\chi,\tilde{\chi}) = \sum_{l_1,l_2\in\mathbb{Z}\setminus\{0\}} C^{(\mathrm{A})}_{l_1,l_2}\, y\, K_{2|s-1|}\left(2\pi\sqrt{y}\sqrt{l_1^2+l_2^2}\right) \mathrm{e}^{2\pi \mathrm{i} l_1 \chi}\, \mathrm{e}^{2\pi \mathrm{i} l_2 \tilde{\chi}}, \tag{6.52}$$

What is left is to determine the coefficients $C^{(\mathrm{A})}_{l_1,l_2}$, which we will be able to do from the Eisenstein series. As was shown in the end of Section 6.1 the invariance under rotations R: $(\chi,\tilde{\chi}) \longmapsto (-\tilde{\chi},\chi)$ implied invariance under $\pi/2$ rotations of the (l_1,l_2) lattice. As we see, this automatically holds for the modified Bessel function; for the coefficient $C^{(\mathrm{A})}_{l_1,l_2}$ we get that it is invariant under $(l_1,l_2) \longmapsto (-l_2,l_1)$.

The non-Abelian Term

Since the non-abelian term is dependent on all four fields, the whole Laplace-Beltrami operator has to be considered. The eigenfunction equation thus reads

$$\Big[y^2\partial_y^2 - y\partial_y + \frac{1}{4}y(\partial_\chi^2+\partial_{\tilde\chi}^2)+\frac{1}{2}y(\tilde\chi\partial_\chi-\chi\partial_{\tilde\chi})\partial_\psi+ \\ +\frac{1}{4}y(y+\chi^2+\tilde\chi^2)\partial_\psi^2\Big]\, f^{(\mathrm{NA})}(y,\chi,\tilde\chi,\psi)=\lambda f^{(\mathrm{NA})}(y,\chi,\tilde\chi,\psi). \tag{6.53}$$

We state again the general form (6.22) of the non-abelian term derived in the preceding section

$$f^{(\mathrm{NA})}(y,\chi,\tilde\chi,\psi)=\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k))\,\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}. \tag{6.54}$$

Using this in (6.53) yields

$$\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} \mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}\Big[y^2\partial_y^2-y\partial_y+\frac{1}{4}y\left(-16\pi^2(2|k|n-k\tilde\chi)^2+\right. \\ \left.+\partial_{\tilde\chi}^2-8\pi\mathrm{i}k\chi\partial_{\tilde\chi}+(4\pi\mathrm{i}k\chi)^2\right)+\frac{1}{2}y(\tilde\chi(2|k|n-k\tilde\chi)4\pi\mathrm{i}-\chi(\partial_{\tilde\chi}-4\pi\mathrm{i}k\chi))(-4\pi\mathrm{i}k)+ \\ +\frac{1}{4}y(y+\chi^2+\tilde\chi^2)(-4\pi\mathrm{i}k)^2\Big]\, C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k))= \\ =\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} \mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}\Big[y^2\partial_y^2-y\partial_y-4\pi^2k^2y^3+ \\ +y\left(-16\pi^2k^2(\tilde\chi-n\cdot\mathrm{sgn}(k))^2+\frac{1}{4}\partial_{\tilde\chi}^2\right)\Big]\, C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k))= \\ =\lambda\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k))\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}, \tag{6.55}$$

where we have for instance used

$$\left(\text{put}\quad(\ldots):=C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k))\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}\right)$$

$$\partial_{\tilde\chi}(\ldots)=\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}(\partial_{\tilde\chi}-4\pi\mathrm{i}k\chi)C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k))\qquad\text{and}$$

$$\partial_{\tilde\chi}^2(\ldots)=\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde\chi)}\left(\partial_{\tilde\chi}^2-8\pi\mathrm{i}k\chi\partial_{\tilde\chi}+(4\pi\mathrm{i}k\chi)^2\right)C^{(\mathrm{NA})}_{k,n'}(y,\tilde\chi-n\cdot\mathrm{sgn}(k)),$$

when moving the exponential functions through the Laplace-Beltrami operator. Immediately we notice two important facts about (6.55). First of all, we have eliminated the dependence on the fields χ and ψ in the Laplace-Beltrami operator, which means that we have only one differential equation in two variables left to solve! This separation of variables came from the structure of the non-abelian Fourier expansion, note especially the role of the mixed factor $e^{-4\pi i k(\psi+\chi\tilde{\chi})}$. Secondly, we automatically gained the factor $(\tilde{\chi} - n \cdot \mathrm{sgn}(k))$ that will significantly fit together with the requirement on the $\tilde{\chi}$-dependence on the coefficient from the accompanying shift of the lattice to the Heisenberg translation.

Equation (6.55) must hold term by term in the triple summation due to the linear independence of the variables in the exponent. We can also divide away the exponents and a factor of y. This yields

$$\Big[y\partial_y^2 - \partial_y - 4\pi^2k^2y - \frac{\lambda}{y} + \\ + \left(-16\pi^2k^2(\tilde{\chi} - n\cdot\mathrm{sgn}(k))^2 + \frac{1}{4}\partial_{\tilde{\chi}}^2\right)\Big] C^{(\mathrm{NA})}_{k,n'}(y, \tilde{\chi} - n\cdot\mathrm{sgn}(k)) = 0. \quad (6.56)$$

The differential equation is solvable by the method of separating variables. Assume

$$C^{(\mathrm{NA})}_{k,n'}(y, \tilde{\chi} - n\cdot\mathrm{sgn}(k)) = Y(y)\tilde{X}(\tilde{\chi} - n\cdot\mathrm{sgn}(k)), \quad (6.57)$$

we suppress the indices k and n', remembering that the functions Y and $\tilde{X}$ will depend on these summation variables. Inserting this in (6.56) as well as dividing by $Y\tilde{X}$ (assuming not equal to zero) gives

$$\underbrace{\frac{1}{4}\frac{\partial_{\tilde{\chi}}^2\tilde{X}}{\tilde{X}} - 16\pi^2k^2(\tilde{\chi} - n\cdot\mathrm{sgn}(k))^2}_{=-c_{\tilde{\chi}}^2} + \underbrace{Y^{-1}\left[y\partial_y^2Y - \partial_yY\right] - 4\pi^2k^2y - \frac{\lambda}{y}}_{=c_y^2} = 0, \quad (6.58)$$

where $c_{\tilde{\chi}}$ and c_y must be constants, since they depend on independent variables, satisfying

$$c_{\tilde{\chi}}^2 = c_y^2. \quad (6.59)$$

We have now to solve two differential equations depending on one variable each. We begin with the one for $\tilde{X}$ that reads

$$\left[\partial_{\tilde{\chi}}^2 + 4(-16\pi^2k^2(\tilde{\chi} - n\cdot\mathrm{sgn}(k))^2 + c_{\tilde{\chi}}^2)\right]\tilde{X}(\tilde{\chi} - n\cdot\mathrm{sgn}(k)) = 0. \quad (6.60)$$

This is a *parabolic cylinder differential equation*, but in a more standard form there is a pure "$-\tilde{\chi}^2$"-term in front of $\tilde{X}(\tilde{\chi})$. We can fix this by making a variable change

$$\hat{\chi} = \sqrt{8\pi|k|}(\tilde{\chi} - n\cdot\mathrm{sgn}(k)). \quad (6.61)$$

The derivative in the new variable is

$$\frac{\partial^2}{\partial\tilde{\chi}^2} = 8\pi|k|\frac{\partial^2}{\partial\hat{\chi}^2},$$

and our equation reads

$$\partial_{\hat{\chi}}^2 \tilde{X}(\hat{\chi}) + \left(-\hat{\chi}^2 + \frac{c_{\tilde{\chi}}^2}{2\pi|k|}\right)\tilde{X}(\hat{\chi}) = 0. \tag{6.62}$$

The equation is satisfied by *parabolic cylinder functions* [31]. However, we will make a standard substitution

$$\tilde{X}(\hat{\chi}) = \mathrm{e}^{-\frac{1}{2}\hat{\chi}^2}\hat{X}(\hat{\chi}), \tag{6.63}$$

which yields after simplification

$$\partial_{\hat{\chi}}^2 \hat{X}(\hat{\chi}) - 2\hat{\chi}\partial_{\hat{\chi}}\hat{X}(\hat{\chi}) + \underbrace{\left(-1 + \frac{c_{\tilde{\chi}}^2}{2\pi|k|}\right)}_{=2r}\hat{X}(\hat{\chi}) = 0. \tag{6.64}$$

In general, for an arbitrary parameter $r \in \mathbb{R}$, this equation is satisfied by *Hermite functions*, which can be expressed in terms of the confluent hypergeometric functions (for more see Section 6.5 and [11, 31]). However, to get *bounded* solutions we must restrict ourselves to a positive integer-valued parameter. The solutions are then the *Hermite polynomials* (appearing, for instance, in quantum mechanics when solving the Schrödinger equation for a linear harmonic oscillator):

$$H_r(\hat{\chi}) = H_r\left(\sqrt{8\pi|k|}\,(\tilde{\chi} - n\cdot\mathrm{sgn}(k))\right), \tag{6.65}$$

for $r \in \mathbb{N}\cup\{0\}$ with

$$r = -\frac{1}{2} + \frac{c_{\tilde{\chi}}^2}{4\pi|k|}, \tag{6.66}$$

and this is then a constraint on the constant $c_{\tilde{\chi}}$.

Our solution for $\tilde{X}(\tilde{\chi})$ reads

$$\tilde{X}(\tilde{\chi}) = \sum_{r=0}^{\infty} C_r \mathrm{e}^{-4\pi|k|(\tilde{\chi}-n\cdot\mathrm{sgn}(k))^2} H_r\left(\sqrt{8\pi|k|}(\tilde{\chi} - n\cdot\mathrm{sgn}(k))\right). \tag{6.67}$$

Where C_r are some r-dependent constants that will later be grouped together with constants from the second differential equation. As was desired, the function $\tilde{X}$ is really a function of $\tilde{\chi} - n\cdot\mathrm{sgn}(k)$; this may seem like a lucky coincidence but it is certainly not pure luck, rather, a sign that the Fourier expansion of the $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$-invariant function satisfying the Laplace condition is quite unique.

Finally, we solve the second differential equation for $Y(y)$ from (6.58) that reads

$$\left[y^2\partial_y^2 - y\partial_y - (4\pi^2k^2y^2 + c_y^2 y + \lambda)\right]Y(y) = 0. \tag{6.68}$$

This will turn out to be a *confluent hypergeometric equation*. Using the relationship between the constants (6.59), and $c_{\tilde{\chi}}^2 = 2\pi|k|(2r+1)$ from (6.66), we get

$$\left[y^2\partial_y^2 - y\partial_y - (4\pi^2k^2y^2 + 2\pi|k|(2r+1)y + \lambda)\right] Y(y) = 0. \tag{6.69}$$

One can put (6.69) on the form

$$M''_{p\mu}(x) + \left(-\frac{1}{4} + \frac{p}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) M_{p\mu}(x) = 0, \tag{6.70}$$

which is a self-adjoint equation satisfied by Whittaker functions (see for instance [4] p. 858). First we must factor out $\sqrt{y}$:

$$\begin{aligned} Y(y) = \sqrt{y}\,\hat{Y}(y) \implies \partial_y Y = \frac{1}{2\sqrt{y}}\hat{Y} + \sqrt{y}\partial_y\hat{Y} \implies \\ \partial_y^2 Y = -\frac{1}{4y^{\frac{3}{2}}}\hat{Y} + \frac{1}{\sqrt{y}}\partial_y\hat{Y} + \sqrt{y}\partial_y^2\hat{Y}. \end{aligned} \tag{6.71}$$

Inserting this into (6.69) yields

$$\begin{aligned} y^2\left(-\frac{1}{4y^{\frac{3}{4}}}\hat{Y} + \frac{1}{\sqrt{y}}\partial_y\hat{Y} + \sqrt{y}\partial_y^2\hat{Y}\right) - y\left(\frac{1}{2\sqrt{y}}\hat{Y} + \sqrt{y}\partial_y\hat{Y}\right) + \\ - \sqrt{y}\,\hat{Y}\left(4\pi^2k^2y^2 + 2\pi|k|(2r+1)y + \lambda\right) = 0, \end{aligned} \tag{6.72}$$

and after division by $y^2\sqrt{y}$ and some simplification we get

$$\partial_y^2\hat{Y} + \left(-\frac{1}{4}\cdot 16\pi^2k^2 - \frac{1}{y}2\pi|k|(2r+1) - \frac{1}{y^2}\left(\lambda + \frac{3}{4}\right)\right)\hat{Y} = 0. \tag{6.73}$$

To get rid of the factor $16\pi^2k^2$ (remember $k \neq 0$) in the first term in the parenthesis, we make the variable change

$$\begin{aligned} &\hat{y} = 4\pi|k|y \implies \partial_y^2 = 16\pi^2k^2\partial_{\hat{y}}^2 \implies \\ &\partial_{\hat{y}}^2\hat{Y}(\hat{y}) + \left(-\frac{1}{4} + \underbrace{\frac{1}{16\pi^2k^2}\frac{1}{\frac{\hat{y}}{4\pi|k|}}(-2\pi|k|)(2r+1)}_{=\frac{1}{\hat{y}}(-r-\frac{1}{2})} - \frac{1}{\hat{y}^2}\left(\lambda + \frac{3}{4}\right)\right)\hat{Y}(\hat{y}) = 0. \end{aligned} \tag{6.74}$$

This equation is indeed of the form (6.70) and it is satisfied by the two types of Whittaker functions

$$M_{p,\mu}(\hat{y}) = \mathrm{e}^{-\hat{y}/2}\hat{y}^{\mu+\frac{1}{2}}M\left(\mu - p + \frac{1}{2}, 2\mu+1; \hat{y}\right) \quad \text{and} \tag{6.75}$$

$$W_{p,\mu}(\hat{y}) = \mathrm{e}^{-\hat{y}/2}\hat{y}^{\mu+\frac{1}{2}}U\left(\mu - p + \frac{1}{2}, 2\mu+1; \hat{y}\right), \tag{6.76}$$

where M and U are the *confluent hypergeometric functions* of the first and second kind respectively. These functions are further discussed in Appendix H. The constants p and μ are identified as (see (6.70)):

$$\begin{aligned} p = -r - \frac{1}{2} \qquad &\text{and} \\ \frac{1}{4} - \mu^2 = -(\lambda + \frac{3}{4}) \implies \mu^2 = \lambda + 1&, \\ \lambda = s(s-2) \implies \mu = \pm(s-1)&. \end{aligned} \tag{6.77}$$

The solutions $M_{p\mu}(\hat{y})$ are not acceptable since they diverge in the low-coupling limit for $\hat{y} \longrightarrow \infty$ ($y \longrightarrow \infty$). On the other hand $W_{p,\mu}(\hat{y})$ converges in this limit. Actually, for $W_{p,\mu}$ we get the same solution independently of the sign on μ, so we can choose to use the plus sign. An important fact is that when using the normal definition of the Whittaker function $W_{p,\mu}$, the function is not well-defined for $c = 2\mu + 1 = 2(s-1) + 1 = 2s - 1 \in \mathbb{Z}$. Ironically, as mentioned earlier, we are interested precisely in the case $s = 3/2$. How to tackle this is treated in Appendix H.

The factor $Y(y)$ then has the form

$$Y(y) \propto \sqrt{y}\, W_{p,\mu}(4\pi|k|y) = \sqrt{y}\, W_{-r-\frac{1}{2},s-1}(4\pi|k|y). \tag{6.78}$$

Collecting everything, that is, using equations: (6.54), (6.67) and (6.78), we get the following structure for the non-abelian term:

$$\begin{aligned} f^{(\mathrm{NA})}(y,\chi,\tilde{\chi},\psi) = \sqrt{y} \sum_{k\neq 0} \sum_{n'=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} \sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',r}(s)\, \mathrm{e}^{-4\pi|k|(\tilde{\chi}-n\cdot\mathrm{sgn}(k))^2} \\ \times H_r\left(\sqrt{8\pi|k|}(\tilde{\chi} - n\cdot\mathrm{sgn}(k))\right) W_{-r-\frac{1}{2},s-1}(4\pi|k|y)\, \mathrm{e}^{8\pi\mathrm{i}|k|n\chi - 4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}, \end{aligned} \tag{6.79}$$

for some constants $C^{(\mathrm{NA})}_{k,n',r}(s)$ depending on s since different s yields different Whittaker functions. Note that we cannot factor out the whole r-dependence in $C^{(\mathrm{NA})}_{k,n',r}(s)$. Even though the coefficient C_r from (6.67) can be factored out, we have that the coefficients corresponding to the solutions to the differential equation in y also depends on r.

The expression (6.79) is invariant under the whole Heisenberg group of translations in the Picard modular group. To see the invariance under the $T_{(0,1,0)}$-transformation we must accompany it by a shift in the summation $n \longmapsto n + \mathrm{sgn}(k)$.

Knowing the exact structure of the Fourier expansion in all the fields we can now act with the rotation $R \in \mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ on our expression again and see what the requirement of invariance implies on the coefficients $C^{(\mathrm{NA})}_{k,n',r}(s)$. We state again the action of the rotation on the fields

$$R: \quad (\chi,\tilde{\chi}) \longmapsto (-\tilde{\chi},\chi). \tag{6.80}$$

Since the element is of order 4 ($R^4 = \mathrm{Id}$) in the Picard modular group, we will also get information from application of

$$R^2: \quad (\chi,\tilde{\chi}) \longmapsto (-\chi,-\tilde{\chi}) \qquad \text{and} \qquad R^3: \quad (\chi,\tilde{\chi}) \longmapsto (\tilde{\chi},-\chi). \tag{6.81}$$

As we saw in Section 6.1.2 invariance under a single R implied equality between the two choices of polarization (diagonalizations under the discrete Heisenberg group). As we have seen this does not constrain our coefficient $C^{(\mathrm{NA})}_{k,n',r}(s)$ further, but relates it to the coefficient $\tilde{C}^{(\mathrm{NA})}_{k,n',r}(s)$ corresponding to the other polarization. However, acting with R^2 on (6.79) yields

$$\begin{aligned} R^2: \quad f^{(\mathrm{NA})}(\phi,\chi,\tilde{\chi},\psi) \longmapsto \mathrm{e}^{-\phi} \sum_{k\neq 0} \sum_{n'=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} \sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',r}(s)\, \mathrm{e}^{-4\pi|k|(-\tilde{\chi}-n\cdot\mathrm{sgn}(k))^2} \\ \times H_r\left(\sqrt{8\pi|k|}(-\tilde{\chi}-n\cdot\mathrm{sgn}(k))\right) W_{-r-\frac{1}{2},s-1}(4\pi|k|\mathrm{e}^{-2\phi})\, \mathrm{e}^{-8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})} = \\ = \mathrm{e}^{-\phi} \sum_{k\neq 0} \sum_{n'=0}^{4|k|-1} \sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}} \sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',r}(s)\, \mathrm{e}^{-4\pi|k|(\tilde{\chi}+n\cdot\mathrm{sgn}(k))^2} \\ \times (-1)^r H_r\left(\sqrt{8\pi|k|}(\tilde{\chi}+n\cdot\mathrm{sgn}(k))\right) W_{-r-\frac{1}{2},s-1}(4\pi|k|\mathrm{e}^{-2\phi})\, \mathrm{e}^{-8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \end{aligned} \tag{6.82}$$

In the equality we have used the fact that the Hermite polynomials H_r are *even* for even r and *odd* for odd r. We get the same type of expression as in (6.79), except for the sign change in the shifted summation variable $n \longmapsto -n$ as well as the factor $(-1)^r$. One then wonders if it is possible making an equivalent summation with the change of sign of n. For $n' = 0$ we can let $n \longmapsto -n$ in the summation since in this case $n \in \mathbb{Z} = -\mathbb{Z}$. The invariance then implies

$$C^{(\mathrm{NA})}_{k,0,r} = (-1)^r C^{(\mathrm{NA})}_{k,0,r}, \tag{6.83}$$

which further implies that only even values of r are allowed for $n' = 0$. But letting $n \longmapsto -n$ for $n' > 0$ takes us to other values of n', this since $n \in \mathbb{Z} + \frac{n'}{4|k|} \neq \mathbb{Z} - \frac{n'}{4|k|}$. In fact, we have that

$$-n = -\mathbb{Z} - \frac{n'}{4|k|} = \mathbb{Z} + 1 - \frac{n'}{4|k|} = \mathbb{Z} + \frac{4|k| - n'}{4|k|} \tag{6.84}$$

giving the following constraint on the coefficient

$$C^{(\mathrm{NA})}_{k,n',r} = (-1)^r C^{(\mathrm{NA})}_{k,4|k|-n',r}, \qquad n' = 1,2,\ldots,4|k|-1. \tag{6.85}$$

6.2 Eisenstein Series

The Eisenstein series are constructed with help of the lattice method explained in Section 4.2:

$$\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}(\mathcal{M}) = \sum_{\substack{\vec{\omega}\in\mathbb{Z}[\mathrm{i}]^3\\ \vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0}}{}' \left[\vec{\omega}^\dagger\cdot\mathcal{M}\cdot\vec{\omega}\right]^{-s}. \tag{6.86}$$

In this case the generalized transpose equals conjugate transpose, and the generalized metric $\mathcal{M} = VV^\dagger$ was calculated explicitly in Section 3.4, see (3.25). The discrete group is the Picard modular group whose defining representation consists of matrices in $\mathrm{SL}(3,\mathbb{Z}[\mathrm{i}])$ with the additional constraint of preserving the metric η, see (E.1). The defining representation acts on the three dimensional lattice of gaussian integers $\vec{\omega} = (\omega_1,\omega_2,\omega_3) \in \mathbb{Z}[\mathrm{i}]^3\backslash\{0\}$. Unlike the Type IIB case we need this time a quadratic constraint

$$\vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega} = 0 \tag{6.87}$$

to make the Eisenstein series an eigenfunction to the Laplace-Beltrami operator on the symmetric coset space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. The reason is not so obvious but can be seen by rewriting the generalized metric, see [9]. As a side remark when constructing the Eisenstein series with help of the Poincaré series method, we get the same end result without having to take care of the quadratic constraint, which in this context may seem a bit ad hoc.

Using the two auxiliary complex variables ξ and ζ, defined in (3.26), we get

$$\begin{aligned}&\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}(\mathcal{M}) =\\ &= \sum_{\substack{\vec{\omega}\in\mathbb{Z}[\mathrm{i}]^3\\ \vec{\omega}^\dagger\cdot\eta\cdot\vec{\omega}=0}}{}' \mathrm{e}^{-2s\phi}\left[|\omega_1+\omega_2\xi+\omega_3\zeta|^2 + \mathrm{e}^{-2\phi}|\omega_2+\mathrm{i}\omega_3\bar{\xi}|^2 + \mathrm{e}^{-4\phi}|\omega_3|^2\right]^{-s},\end{aligned} \tag{6.88}$$

and one can show that the series with the quadratic constraint is an eigenfunction to the Laplace-Beltrami operator on $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$ with eigenvalue $\lambda = s(s-2)$:

$$\Delta\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}(\mathcal{M}) = s(s-2)\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}(\mathcal{M}). \tag{6.89}$$

6.3 Identification of the Fourier Coefficients

To connect to the constant, abelian and non-abelian part of the Fourier expansion calculated in the Section 6.1, we need to take care of the quadratic constraint to rewrite the summation. We will follow [9], but not do all the calculations, especially since these calculations are already done carefully in this paper.

The procedure is as follows. The Eisenstein series is first split into two pieces

$$\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}(\phi,\chi,\tilde{\chi},\psi) = \mathcal{E}_s^{(0)} + \mathcal{A}, \tag{6.90}$$

the part $\mathcal{E}_s^{(0)}$ corresponds to the part of the constant term with coefficient C_1, see (6.39). The summation of the remainder $\mathcal{A}$ is then rewritten using the Euclidean algorithm and Poisson resummation, thereby making it possible splitting it further

$$\mathcal{A} = \mathcal{D} + \mathcal{E}_s^{(\mathrm{NA})}. \tag{6.91}$$

The non-abelian part of the Eisenstein series $\mathcal{E}_s^{(\mathrm{NA})}$ will be treated lastly in Section 6.4, this part is the most tricky to connect with the Fourier expansion. Finally, the part $\mathcal{D}$ is split into

$$\mathcal{D} = \mathcal{E}_s^{(1)} + \mathcal{E}_s^{(\mathrm{A})}, \tag{6.92}$$

where $\mathcal{E}_s^{(1)}$ is identified with the second term in the constant part with coefficient C_2, and $\mathcal{E}_s^{(\mathrm{A})}$ is used to identify the coefficients $C_{l_1,l_2}^{(\mathrm{A})}$ in the abelian part (6.52).

The quadratic constraint has the following expression:

$$\vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = |\omega_2|^2 - 2\Im(\omega_1 \bar{\omega_3}) = 0. \tag{6.93}$$

If we put

$$\omega_1 = m_1 + \mathrm{i}m_2, \quad \omega_2 = n_1 + \mathrm{i}n_2 \quad \text{and} \quad \omega_3 = p_1 + \mathrm{i}p_2, \\ m_1, m_2, n_1, n_2, p_1, p_2 \in \mathbb{Z} \tag{6.94}$$

we get

$$\vec{\omega}^\dagger \cdot \eta \cdot \vec{\omega} = n_1^2 + n_2^2 + 2m_1p_2 - 2m_2p_1 = 0. \tag{6.95}$$

To get $\mathcal{E}_s^{(0)}$ in (6.90) is easy. We consider the terms in the summation with $\omega_3 = 0$. (6.95) then implies that $\omega_2 = 0$, which further implies that we are left with a summation over $\omega_1 = m_1 + \mathrm{i}m_2 \neq 0$. We get

$$\boxed{\mathcal{E}_s^{(0)} = \sum_{\omega_1 \in \mathbb{Z}[\mathrm{i}]}{}' \mathrm{e}^{-2s\phi}|\omega_1|^{-2s} = \mathrm{e}^{-2s\phi} \sum_{(m_1,m_2)\in\mathbb{Z}^2}{}' \frac{1}{(m_1^2+m_2^2)^s},} \tag{6.96}$$

and we can directly identify the first coefficient in the constant part

$$C_1 = \sum_{(m_1,m_2)\in\mathbb{Z}^2}{}' \frac{1}{(m_1^2+m_2^2)^s} = 4\zeta_{\mathbb{Q}(\mathrm{i})}(s), \tag{6.97}$$

where

$$\zeta_{\mathbb{Q}(\mathrm{i})}(s) := \frac{1}{4}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m_1^2+m_2^2)^s} \tag{6.98}$$

is the *Dedekind zeta function* over the gaussian integers.

Then, for the remainder $\mathcal{D}$, we have that $\omega_3 \neq 0$. Here we are faced with a choice of which summation variable to leave unconstrained. We follow [9] and keep the summation over $\omega_3 \neq 0$, thereby letting the quadratic constraint modify the summations over ω_1 and ω_2. Given an arbitrary $\omega_3 = p_1 + \mathrm{i}p_2$ we have from the constraint

$$n_1^2 + n_2^2 = 2(m_2 p_1 - m_1 p_2) \tag{6.99}$$

that since we are dealing with integers it must be the case that

$$2d | n_1^2 + n_2^2 \qquad \text{with} \qquad d = \gcd(p_1, p_2). \tag{6.100}$$

With this requirement fulfilled there exists integer solutions (m_1, m_2) to the Diophantine equation (6.99) acquired by first solving the *Bezout's identity*

$$q_2 p_1 - q_1 p_2 = d, \tag{6.101}$$

A solution (q_1, q_2) to this equation always exists and is found using the *Euclidean algorithm*. A particular solution $(\tilde{m}_1, \tilde{m}_2)$ to (6.99) is therefore

$$\begin{aligned} \tilde{m}_1 &= \frac{n_1^2+n_2^2}{2d} q_1 \\ \tilde{m}_2 &= \frac{n_1^2+n_2^2}{2d} q_2. \end{aligned} \tag{6.102}$$

A general solution is found by noting that given two pairs of solutions $(\tilde{m}_1, \tilde{m}_2)$ and $(\tilde{m}_1', \tilde{m}_2')$ we have that (using (6.99)):

$$\begin{aligned} \tilde{m}_2 p_1 - \tilde{m}_1 p_2 &= \tilde{m}_2' p_1 - \tilde{m}_1' p_2 \qquad \text{or} \\ p_1(\tilde{m}_2 - \tilde{m}_2') &= p_2(\tilde{m}_1 - \tilde{m}_1'), \end{aligned} \tag{6.103}$$

and dividing both sides by $d = \gcd(p_1, p_2)$, we see that it must be the case that $\tilde{m}_2 - \tilde{m}_2' = mp_2/d$ and $\tilde{m}_1 - \tilde{m}_1' = mp_1/d$ for $m \in \mathbb{Z}$. A general solution is therefore

$$\begin{aligned} m_1 &= \frac{n_1^2+n_2^2}{2d} q_1 + m\frac{p_1}{d} \\ m_2 &= \frac{n_1^2+n_2^2}{2d} q_2 + m\frac{p_2}{d}, \qquad m \in \mathbb{Z}, \end{aligned} \tag{6.104}$$

where (q_1, q_2) was any particular solution to (6.101). We have thereby made the summation over m_1 and m_2 into one over $m \in \mathbb{Z}$ only. We get that the summation in the term $\mathcal{A}$ is rewritten as

$$\mathcal{A} = \sum_{\omega_3 \neq 0} \sum_{\substack{\omega_2 \in \mathbb{Z}[\mathrm{i}] \\ 2d||\omega_2|^2}} \sum_{m\in\mathbb{Z}} (\ldots), \tag{6.105}$$

where we make the replacement (6.104) everywhere in the ellipsis. Furthermore, using an integral representation and performing a Poisson resummation on m, see [9], yields

$$\mathcal{A} = \frac{\pi^s}{\Gamma(s)} \mathrm{e}^{-2s\phi} \sum_{\tilde{m}\in\mathbb{Z}} \sum_{(p_1,p_2)\in\mathbb{Z}^2}{}' \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d}{|\omega_3|} \mathrm{e}^{-2\pi\mathrm{i}\tilde{m}\left(\frac{|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\tilde{l}_1\chi+\tilde{l}_2\tilde{\chi}+2d\psi\right)}$$
$$\times \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} \mathrm{e}^{-\pi t \frac{d^2}{|\omega_3|^2}\tilde{m}^2 - \frac{\pi}{t}\frac{|\omega_3|^2}{d^4}\mathrm{e}^{-4\phi}\left[d^2+\frac{e^{2\phi}}{4}\left((\tilde{l}_1+2d\tilde{\chi})^2+(\tilde{l}_2-2d\chi)^2\right)\right]^2}, \quad (6.106)$$

where we have defined the new variables

$$\begin{aligned} \tilde{l}_1 &:= \frac{d}{|\omega_3|^2}\left[(p_1-p_2)n_1+(p_1+p_2)n_2\right], \\ \tilde{l}_2 &:= \frac{d}{|\omega_3|^2}\left[(p_1+p_2)n_1-(p_1-p_2)n_2\right]. \end{aligned} \quad (6.107)$$

Here we can do the second split (6.91). The non-abelian part consists of the terms for which $\tilde{m} \neq 0$. We will treat this part in Section 6.4. Putting $\tilde{m} = 0$ we get $\mathcal{D}$, which is then

$$\mathcal{D} = \frac{\pi^s}{\Gamma(s)} \mathrm{e}^{-2s\phi} \sum_{(p_1,p_2)\in\mathbb{Z}^2}{}' \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d}{|\omega_3|}$$
$$\times \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} \mathrm{e}^{-\frac{\pi}{t}\frac{\mathrm{e}^{-4\phi}|\omega_3|^2}{d^4}\left[d^2+\frac{e^{2\phi}}{4}\left((\tilde{l}_1+2d\tilde{\chi})^2+(\tilde{l}_2-2d\chi)^2\right)\right]^2} \quad (6.108)$$

Making the split (6.92) is an involved calculation made in great detail in [9]. We will just state the results

$$\mathcal{E}_s^{(1)} = 4\frac{\pi^{3/2}\Gamma(s-1/2)\Gamma(2s-2)L(N,2s-1)}{\Gamma(s)\Gamma(2s-1)\zeta(2s-2)}\zeta_{\mathbb{Q}(\mathrm{i})}(s-1)\mathrm{e}^{-2(2-s)\phi}, \quad (6.109)$$

where $\zeta_{\mathbb{Q}(\mathrm{i})}$ is the Dedekind zeta function (6.98) and $L(N, 2s-1)$ is a Dirichlet series

$$L(N,s) := \sum_{d=1}^{\infty} N(d) d^{-s}, \quad (6.110)$$

including the combinatorial function $N(d) := \sharp\mathcal{F}(d)$, which is the cardinality of the set

$$\mathcal{F}(d) := \{n_1^0 + \mathrm{i}n_2^0 \ : \ n_1^2+n_2^2 = 0 \mod 2d, 0 \leq n_1^0 < d, 0 \leq n_2^0 < 2d\}. \quad (6.111)$$

We can directly read off the second coefficient of the constant part in the Fourier expansion

$$C_2 = 4\frac{\pi^{3/2}\Gamma(s-1/2)\Gamma(2s-2)L(N,2s-1)}{\Gamma(s)\Gamma(2s-1)\zeta(2s-2)}\zeta_{\mathbb{Q}(\mathrm{i})}(s-1). \tag{6.112}$$

Unfortunately, as is pointed out in [9], this value has the wrong sign, comparing with other calculations made in [3]. This points towards that we should seek another automorphic form to encode the quantum corrections to the metric. However, from a mathematical point of view it is interesting to continue with the calculations.

The abelian part reads

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{A})} &= 2\zeta_{\mathbb{Q}(\mathrm{i})}(s)\frac{\mathrm{e}^{-2\phi}}{\mathfrak{Z}(s)}\sum_{(l_1,l_2)\in\mathbb{Z}^2}{}' \mu_s(l_1,l_2)(l_1^2+l_2^2)^{s-1} \\ &\quad\times K_{2s-2}\left(2\pi\mathrm{e}^{-\phi}\sqrt{l_1^2+l_2^2}\right)\mathrm{e}^{2\pi\mathrm{i}(l_1\chi+l_2\tilde{\chi})},\end{aligned} \tag{6.113}$$

where we have defined an instanton measure

$$\mu_s(l_1,l_2) := \sum_{\omega_3'|l_2-\mathrm{i}l_1}|\omega_3'|^{2-2s}\sum_{z|\frac{l_2-\mathrm{i}l_1}{\omega_3'}}|z|^{4-4s}, \tag{6.114}$$

and a completed *Picard zeta function*

$$\mathfrak{Z}(s) := \zeta_{\mathbb{Q}(\mathrm{i})\star}(s)\beta_\star(2s-1), \tag{6.115}$$

introduced in [9]. It consists of the completed Dedekind zeta function (6.98) and the completed Dedekind beta function defined as

$$\begin{aligned}\zeta_{\mathbb{Q}(\mathrm{i})\star} &:= \pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}(\mathrm{i})}(s) \qquad \text{and} \\ \beta_\star(s) &:= \frac{\pi^{-\frac{s+1}{2}}}{4}\Gamma\left(\frac{s+1}{2}\right)\beta(s) \qquad \text{with} \\ \beta(s) &:= \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}, \qquad \text{for } \Re(s)>0.\end{aligned} \tag{6.116}$$

The abelian coefficients in (6.52) are then identified as

$$C_{l_1,l_2}^{(\mathrm{A})} = \frac{2\zeta_{\mathbb{Q}(\mathrm{i})}(s)}{\mathfrak{Z}(s)}\mu_s(l_1,l_2)(l_1^2+l_2^2)^{s-1}. \tag{6.117}$$

6.4 The non-Abelian Term

As was mentioned before we get the non-abelian term $\mathcal{E}_s^{(\mathrm{NA})}$ in the Eisenstein series, by considering the terms for which $\tilde{m} \neq 0$ in the expression (6.106) for $\mathcal{A}$, i.e.

$$\begin{aligned}\mathcal{E}_s^{(\mathrm{NA})} &= \frac{\pi^s}{\Gamma(s)} y^s \\ &\times \sum_{\tilde{m}\neq 0} \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2} \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d}{|\omega_3|} \mathrm{e}^{-2\pi\mathrm{i}\tilde{m}\left(\frac{|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\tilde{l}_1\chi+\tilde{l}_2\tilde{\chi}+2d\psi\right)} \\ &\times \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} \mathrm{e}^{-\pi t\frac{d^2}{|\omega_3|^2}\tilde{m}^2-\frac{\pi}{t}\frac{|\omega_3|^2}{d^4}y^2\left[d^2+\frac{1}{4y}\left((\tilde{l}_1+2d\tilde{\chi})^2+(\tilde{l}_2-2d\chi)^2\right)\right]^2}. \qquad (6.118)\end{aligned}$$

Remember here that (q_1, q_2) is any particular solution to the Bezout's identity

$$q_2p_1 - q_1p_2 = d,$$

and

$$\begin{aligned} y &= \mathrm{e}^{-2\phi}, \\ d &= \gcd(p_1, p_2), \\ \omega_2 &= n_1 + \mathrm{i}n_2, \\ \omega_3 &= p_1 + \mathrm{i}p_2. \end{aligned}$$

This term should equal the non-abelian part of the general Fourier series (6.79), which we here state again

$$\begin{aligned} f^{(\mathrm{NA})}(\phi,\chi,\tilde{\chi},\psi) &= \sqrt{y}\sum_{k\neq 0}\sum_{n'=0}^{4|k|-1}\sum_{n\in\mathbb{Z}+\frac{n'}{4|k|}}\sum_{r=0}^{\infty} C_{k,n',r}^{(\mathrm{NA})}(s)\mathrm{e}^{-4\pi|k|(\tilde{\chi}-n\cdot\mathrm{sgn}(k))^2} \\ &\times\ H_r\left(\sqrt{8\pi|k|}(\tilde{\chi}-n\cdot\mathrm{sgn}(k))\right)W_{-r-\frac{1}{2},s-1}(4\pi|k|y)\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}. \end{aligned} \qquad (6.119)$$

To identify $C_{k,n',r}^{(\mathrm{NA})}(s)$ we then need to do quite a change on the Eisenstein series, and supposedly on the Fourier series as well. The question is if we should begin massaging the Eisenstein series expression or the general Fourier series term.

As a first thing that could have some significance, we notice in (6.118) that the sum includes an integral representation of the modified Bessel function of the second kind; this fact was also used when modifying the abelian term above. The integral representation reads

$$2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}) = \int_0^\infty x^{\nu-1}\mathrm{e}^{-\frac{\beta}{x}-\gamma x}\,\mathrm{d}x, \qquad \text{for} \quad \Re(\beta),\Re(\gamma)>0. \quad (6.120)$$

Applying it to (6.118) yields

$$\mathcal{E}_s^{(\mathrm{NA})} = \frac{2\pi^s}{\Gamma(s)}\,\mathrm{e}^{-2s\phi}\sum_{\tilde{m}\neq 0}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\left(\frac{d}{|\omega_3|}\right)^{s+1/2}\left(\frac{|\tilde{m}|^2}{A(y,\chi,\tilde{\chi})}\right)^{s-1/2}$$
$$\times K_{s-1/2}\left(2\pi A(y,\chi,\tilde{\chi})\right)\mathrm{e}^{-2\pi\mathrm{i}(\hat{l}_1\chi+\hat{l}_2\tilde{\chi}+2\hat{m}\psi)}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{2\hat{m}d}(\hat{l}_1^2+\hat{l}_2^2)(q_1p_1+q_2p_2)}, \quad (6.121)$$

where we have introduced

$$\begin{aligned}\hat{m} &:= \tilde{m}d,\\ \hat{l}_1 &:= \tilde{m}\tilde{l}_1 = \frac{\hat{m}}{|\omega_3|^2}\left[(p_1-p_2)n_1+(p_1+p_2)n_2\right],\\ \hat{l}_2 &:= \tilde{m}\tilde{l}_2 = \frac{\hat{m}}{|\omega_3|^2}\left[(p_1+p_2)n_1-(p_1-p_2)n_2\right],\\ A(y,\chi,\tilde{\chi}) &:= |\hat{m}|\left[y+\left(\tilde{\chi}+\frac{\hat{l}_1}{2k}\right)^2+\left(\chi-\frac{\hat{l}_2}{2k}\right)^2\right].\end{aligned} \quad (6.122)$$

Note that unlike the summation variables l_1 and l_2 we had in the abelian part, the variables $\hat{l}_1$ and $\hat{l}_2$ are in general not integer-valued. The fact that we have a Bessel function in the summation could mean that we should aim for finding a similar function in (6.119) by rewriting the sum in r of the product of the Hermite polynomial and the Whittaker function. After all, in the abelian case we found a Bessel function in both the Fourier series term and the Eisenstein term, which made it possible to identify them both. A problem here is that the Eisenstein series (6.121) is not polarized as the general Fourier series expression (6.119), i.e., the fields χ and $\tilde{\chi}$ appear symmetrically in the Eisenstein series. We will discuss the possibility of doing the sum of the product in Section 6.5.

The substitutions (6.122) can be used in the first expression of the Eisenstein series (6.118) to clean it up. We also make a variable change

$$t \longrightarrow t\frac{|\omega_3|^2A}{k^2} \quad (>0). \quad (6.123)$$

in the integral to get rid of the fourth power in the exponential. This yields

$$\mathcal{E}_s^{(\mathrm{NA})} = \frac{\pi^s}{\Gamma(s)}y^s\sum_{\tilde{m}\neq 0}\sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}}\frac{d|\hat{m}|^{2s-1}}{|\omega_3|^{2s}}$$
$$\times\,\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{\tilde{m}|\omega_2|^2}{2|\omega_3|^2}(q_1p_1+q_2p_2)+\hat{l}_1\chi+\hat{l}_2\tilde{\chi}+2\hat{m}\psi\right)}\int_0^\infty\frac{\mathrm{d}t}{t^{s+1/2}}A^{1/2-s}\mathrm{e}^{-\pi\left(t+\frac{1}{t}\right)A}. \quad (6.124)$$

To connect to (6.119) we can try to transform (6.124) to the correct Fourier basis. I.e., we want to see a factor of the type

$$\mathrm{e}^{8\pi\mathrm{i}|k|n\chi-4\pi\mathrm{i}k(\psi+\chi\tilde{\chi})}, \quad (6.125)$$

and we want all χ- and ψ-dependence to lie in this factor (due to the choice of polarization (6.13)). Looking at the ψ-dependence in (6.124) there are reasons to think that somehow the summation variable $\hat{m} = \tilde{m}d$ is connected to the summation variable k. It is then possible to extract an exponential factor similar to (6.125) by first using an integral representation for the factor $A^{1/2-s}$ in the integrand (to lift up the dependence of χ to the exponent) and then performing a Fourier transform. We have that (3.381 no. 4 in [22]):

$$\frac{1}{\mu^\nu}\Gamma(\nu) = \int_0^\infty x^{\nu-1}\,\mathrm{e}^{-\mu x}\,\mathrm{d}x, \qquad \Re(\mu), \Re(\nu) > 0, \tag{6.126}$$

so in our case we get

$$A^{1/2-s} = \frac{1}{A^{s-1/2}} = \frac{1}{\Gamma(s-1/2)}\int_0^\infty u^{s-3/2}\,\mathrm{e}^{-Au}\,\mathrm{d}u, \quad \Re(A), \Re(s-1/2) > 0. \tag{6.127}$$

Making the rescaling $u \longrightarrow \pi u$ yields

$$A^{1/2-s} = \frac{\pi^{s-1/2}}{\Gamma(s-1/2)}\int_0^\infty u^{s-3/2}\,\mathrm{e}^{-\pi A u}\,\mathrm{d}u, \tag{6.128}$$

and putting this into (6.124) gives

$$\begin{aligned}
\mathcal{E}_s^{(\mathrm{NA})} &= \frac{\pi^{2s-1/2}}{\Gamma(s)\Gamma(s-1/2)}\, y^s \sum_{\tilde{m}\neq 0}\ \sideset{}{'}\sum_{(p_1,p_2)\in\mathbb{Z}^2}\ \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ 2d|n_1^2+n_2^2}} \frac{d|\hat{m}|^{2s-1}}{|\omega_3|^{2s}}\,\mathrm{e}^{-\frac{\pi\mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)} \\
&\quad\times \mathrm{e}^{-2\pi\mathrm{i}(\hat{l}_1\chi+\hat{l}_2\tilde{\chi})-4\pi\mathrm{i}\hat{m}\psi}\int_0^\infty\!\!\int_0^\infty \frac{\mathrm{d}t\,\mathrm{d}u}{t^{s+1/2}u^{3/2-s}}\mathrm{e}^{-\pi\left(u+t+\frac{1}{t}\right)A} = \\
&\quad= (\ldots) \\
&\quad\times\mathrm{e}^{-4\pi\mathrm{i}\hat{m}\psi-2\pi\mathrm{i}\hat{l}_2\tilde{\chi}}\int_0^\infty\!\!\int_0^\infty\frac{\mathrm{d}t\,\mathrm{d}u}{t^{s+1/2}u^{3/2-s}}\,\mathrm{e}^{-\pi|\hat{m}|\left(u+t+\frac{1}{t}\right)\left[y+\left(\tilde{\chi}+\frac{\hat{l}_1}{2\hat{m}}\right)^2\right]}f(y,\chi;t,u),
\end{aligned} \tag{6.129}$$

where we have collected all the χ-dependence in the function

$$f(y,\chi;t,u) = \mathrm{e}^{-\pi|\hat{m}|\left(u+t+\frac{1}{t}\right)\left(\chi-\frac{\hat{l}_2}{2\hat{m}}\right)^2-2\pi\mathrm{i}\hat{l}_1\chi}. \tag{6.130}$$

Now, we extract the mixed factor $\mathrm{e}^{-4\pi\mathrm{i}\hat{m}\chi\tilde{\chi}}$ that is supposed to be associated with $\mathrm{e}^{-4\pi\mathrm{i}k\chi\tilde{\chi}}$ in (6.125):

$$f(y,\chi;t,u) = \mathrm{e}^{-4\pi\mathrm{i}\hat{m}\chi\tilde{\chi}}g(y,\chi,\tilde{\chi};t,u) \tag{6.131}$$

with

$$g(y,\chi,\tilde{\chi};t,u) = \mathrm{e}^{-\pi|\hat{m}|\left(u+t+\frac{1}{t}\right)\left(\chi-\frac{\hat{l}_2}{2\hat{m}}\right)^2-2\pi\mathrm{i}\hat{l}_1\chi+4\pi\mathrm{i}\hat{m}\chi\tilde{\chi}}. \tag{6.132}$$

Looking again at (6.125), we see that we want to transform to a basis where the χ-dependence is of the form $\mathrm{e}^{8\pi\mathrm{i}|\hat{m}|n\chi}$. We will make a Fourier transform over χ to accomplish this; note that n will be a continuous integral variable that we somehow later need to connect to the discrete sum variable n in the general Fourier expansion.

$$\begin{aligned} g(y,\chi,\tilde{\chi};t,u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(y,n,\tilde{\chi};t,u)\mathrm{e}^{\mathrm{i}n\chi}\mathrm{d}n = \\ &= \{n\longrightarrow 8\pi|\hat{m}|n,\quad \mathrm{d}n\longrightarrow 8\pi|\hat{m}|\mathrm{d}n\} = \\ &= 4|\hat{m}|\int_{-\infty}^{\infty}\hat{g}(y,8\pi|\hat{m}|n,\tilde{\chi};t,u)\mathrm{e}^{8\pi|\hat{m}|n\mathrm{i}\chi}\mathrm{d}n, \end{aligned} \tag{6.133}$$

and

$$\begin{aligned} \hat{g}(y,8\pi|\hat{m}|n,\tilde{\chi};t,u) &= \int_{-\infty}^{\infty} g(y,\xi,\tilde{\chi};t,u)\mathrm{e}^{-8\pi|\hat{m}|n\mathrm{i}\xi}\mathrm{d}\xi = \\ &= \int_{-\infty}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}(4|\hat{m}|n-2\hat{m}\tilde{\chi}+\hat{l}_1)\xi-\pi X\left(\xi-\frac{\hat{l}_2}{2\hat{m}}\right)^2}\mathrm{d}\xi, \end{aligned} \tag{6.134}$$

where we have put

$$X = |\hat{m}|\left(u+t+\frac{1}{t}\right). \tag{6.135}$$

We solve this integral by first completing the square

$$\begin{aligned} &-2\pi\mathrm{i}(\overbrace{4|\hat{m}|n-2\hat{m}\tilde{\chi}+\hat{l}_1}^{=Y})\xi-\pi X\left(\xi^2-\frac{\hat{l}_2}{\hat{m}}\xi+\frac{\hat{l}_2^2}{4\hat{m}^2}\right) = \\ &= -\pi X\left(\xi^2+\left(\frac{2\mathrm{i}Y}{X}-\frac{\hat{l}_2}{\hat{m}}\right)\xi+\frac{\hat{l}_2^2}{4\hat{m}^2}\right) = -\pi X\left(\xi+\frac{\mathrm{i}Y}{X}-\frac{\hat{l}_2}{2\hat{m}}\right)^2-\pi\frac{Y^2}{X}-\pi\mathrm{i}\frac{Y\hat{l}_2}{\hat{m}} \\ &\Longrightarrow \hat{g}(y,8\pi|\hat{m}|n,\tilde{\chi};t,u) = \mathrm{e}^{-\pi\frac{Y^2}{X}-\pi\mathrm{i}\frac{Y\hat{l}_2}{\hat{m}}}\int_{-\infty}^{\infty}\mathrm{e}^{-\pi X\left(\xi+\frac{\mathrm{i}Y}{X}-\frac{\hat{l}_2}{2\hat{m}}\right)^2}\mathrm{d}\xi. \end{aligned} \tag{6.136}$$

We have an ordinary gaussian integral

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\pi X\left(\xi+\frac{\mathrm{i}Y}{X}-\frac{\hat{l}_2}{2\hat{m}}\right)^2}\mathrm{d}\xi = \frac{1}{\sqrt{X}},\qquad (X>0), \tag{6.137}$$

and we get

$$\begin{aligned} f(y,\chi;t,u) &= 4\hat{m}\mathrm{e}^{-4\pi\mathrm{i}|\hat{m}|\chi\tilde{\chi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{X}}\mathrm{e}^{-\pi\left[\frac{Y^2}{X}+\mathrm{i}\frac{Y\hat{l}_2}{\hat{m}}\right]} = \\ &= \frac{4|\hat{m}|}{\sqrt{|\hat{m}|\left(u+t+\frac{1}{t}\right)}}\int_{-\infty}^{\infty}\mathrm{e}^{8\pi|\hat{m}|\mathrm{i}n\chi-4\pi\mathrm{i}\hat{m}\chi\tilde{\chi}-\pi\mathrm{i}\frac{\hat{l}_2}{\hat{m}}(4|\hat{m}|n-2\hat{m}\tilde{\chi}+\hat{l}_1)} \\ &\qquad\times\mathrm{e}^{\frac{\pi(4|\hat{m}|n-2\hat{m}\tilde{\chi}+\hat{l}_1)^2}{|\hat{m}|\left(u+t+\frac{1}{t}\right)}}\mathrm{d}n. \end{aligned} \tag{6.138}$$

Finally, putting this back into (6.129) yields

$$\mathcal{E}_s^{(\mathrm{NA})} = \frac{4\pi^{2s-1/2}}{\Gamma(s)\Gamma(s-1/2)}\, y^s \sum_{\tilde{m}\neq 0} \sum_{(p_1,p_2)\in\mathbb{Z}^2}{}' \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d|\hat{m}|^{2s}}{|\omega_3|^{2s}}\, \mathrm{e}^{-\frac{\pi \mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)}$$
$$\times \int_0^\infty \int_0^\infty \frac{\mathrm{d}t\,\mathrm{d}u}{t^{s+1/2}u^{3/2-s}} \frac{1}{\sqrt{|\hat{m}|\left(u+t+\frac{1}{t}\right)}}\, \mathrm{e}^{-\pi|\hat{m}|\left(u+t+\frac{1}{t}\right)\left[y+\left(\tilde{\chi}+\frac{\hat{l}_1}{2\hat{m}}\right)^2\right]}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}n\, \mathrm{e}^{-\pi \mathrm{i}\frac{\hat{l}_2}{\hat{m}}(4|\hat{m}|n+\hat{l}_1)-\frac{4\pi|\hat{m}|\left[\tilde{\chi}-\frac{\hat{l}_1}{2\hat{m}}-2n\,\mathrm{sgn}(\hat{m})\right]^2}{u+t+\frac{1}{t}}}\, \mathrm{e}^{8\pi \mathrm{i}|\hat{m}|n\chi-4\pi \mathrm{i}\hat{m}(\psi+\chi\tilde{\chi})}. \tag{6.139}$$

At this point it could be fruitful to verify the invariance under the discrete group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$. More specifically, we will look into the invariance under the translation

$$T_{(0,1,0)}: \quad \begin{aligned} \tilde{\chi} &\longmapsto \tilde{\chi}+1 \\ \psi &\longmapsto \psi+\chi, \end{aligned} \tag{6.140}$$

which was one of the fundamental elements in the discrete nilpotent subgroup $\mathrm{N}(\mathbb{Z}) = \mathrm{H}_3$. We have

$$T_{(0,1,0)}: \mathcal{E}_s^{(\mathrm{NA})} \longmapsto$$
$$\frac{4\pi^{2s-1/2}}{\Gamma(s)\Gamma(s-1/2)}\, y^s \sum_{\tilde{m}\neq 0} \sum_{(p_1,p_2)\in\mathbb{Z}^2}{}' \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2 \\ 2d|n_1^2+n_2^2}} \frac{d|\hat{m}|^{2s}}{|\omega_3|^{2s}}\, \mathrm{e}^{-\frac{\pi \mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)}$$
$$\times \int_0^\infty \int_0^\infty \frac{\mathrm{d}t\,\mathrm{d}u}{t^{s+1/2}u^{3/2-s}} \frac{1}{\sqrt{|\hat{m}|\left(u+t+\frac{1}{t}\right)}}\, \mathrm{e}^{-\pi|\hat{m}|\left(u+t+\frac{1}{t}\right)\left[y+\left(\tilde{\chi}+1+\frac{\hat{l}_1}{2\hat{m}}\right)^2\right]}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}n\, \mathrm{e}^{-\pi \mathrm{i}\frac{\hat{l}_2}{\hat{m}}(4|\hat{m}|n+\hat{l}_1)-\frac{4\pi|\hat{m}|\left[\tilde{\chi}+1-\frac{\hat{l}_1}{2\hat{m}}-2n\,\mathrm{sgn}(\hat{m})\right]^2}{u+t+\frac{1}{t}}}\, \mathrm{e}^{8\pi \mathrm{i}|\hat{m}|n\chi-4\pi \mathrm{i}\hat{m}(\psi+\chi\tilde{\chi})-8\pi \mathrm{i}\hat{m}\chi}. \tag{6.141}$$

The way of getting back to the first expression and prove the invariance, is to do the shift

$$\begin{cases} \hat{l}_1 \longmapsto \hat{l}_1 - 2\tilde{m} \\ \hat{l}_2 \longmapsto \hat{l}_2, \end{cases} \qquad \text{by letting} \qquad \begin{cases} n_1 \longmapsto n_1 - p_1 + p_2 \\ n_2 \longmapsto n_2 - p_1 - p_2.^5 \end{cases} \tag{6.142}$$

We need to shift the integration variable n as well:

$$n \longmapsto n + \mathrm{sgn}(\hat{m}). \tag{6.143}$$

[5] Note that this shift is compatible with the constraint $2d|n_1^2+n_2^2$, we are "within" the same d:s since $2d|(n_1-p_1+p_2)^2+(n_2-p_1-p_2)^2$.

The factor $\mathrm{e}^{-2\pi\mathrm{i}\hat{l}_2}$ (remember that $\hat{l}_2$ is generally not an integer), coming from $\mathrm{e}^{-\pi\mathrm{i}\frac{\hat{l}_2}{\hat{m}}(4|\hat{m}|n+\hat{l}_1)}$ in the n-integral, cancels against a factor from the first exponential function since

$$\mathrm{e}^{-\frac{\pi\mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)}=\mathrm{e}^{-\pi\mathrm{i}\tilde{m}\frac{n_1^2+n_2^2}{p_1^2+p_2^2}(q_1p_1+q_2p_2)}, \tag{6.144}$$

and making the shift (6.142) on this term yields

$$\begin{aligned}&\mathrm{e}^{-\pi\mathrm{i}\tilde{m}\frac{n_1^2+n_2^2+2(p_1^2+p_2^2)+2(n_1(p_2-p_1)-n_2(p_1+p_2))}{p_1^2+p_2^2}(q_1p_1+q_2p_2)}=\mathrm{e}^{-\frac{\pi\mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)}\\ &\quad\times\overbrace{\mathrm{e}^{-2\pi\mathrm{i}\tilde{m}(q_1p_1+q_2p_2)}}^{=1}\mathrm{e}^{-\frac{2\pi\mathrm{i}\tilde{m}}{p_1^2+p_2^2}[n_1(p_2-p_1)(q_1p_1+q_2p_2)-n_2(p_1+p_2)(q_1p_1+q_2p_2)]}=\\ &=\{\text{use } q_2p_1-q_1p_2=d\}=\ldots=\mathrm{e}^{-\frac{\pi\mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)}\mathrm{e}^{2\pi\mathrm{i}\tilde{m}d\frac{n_1(p_1+p_2)+n_2(p_2-p_1)}{p_1^2+p_2^2}}=\\ &\qquad=\mathrm{e}^{-\frac{\pi\mathrm{i}\tilde{m}|\omega_2|^2}{|\omega_3|^2}(q_1p_1+q_2p_2)}\mathrm{e}^{2\pi\mathrm{i}\tilde{m}\hat{l}_2}.\end{aligned} \tag{6.145}$$

The other extra factors, appearing in (6.141), coming from the $T_{(0,1,0)}$-transformation cancels trivially by the shifts. It is interesting, bearing in mind the proposed comparisons $\hat{m}\leftrightarrow k$ as well as the integral n with the shifted summation n, that for the $T_{(0,1,0)}$-translation the needed shift (6.143) look the same as the one $n\longmapsto n+\mathrm{sgn}(k)$ needed in the general Fourier series.

6.5 Series of Products of Confluent Hypergeometric Functions

In this section we discuss the possibility of rewriting the infinite series (the sum in r) of products of the Hermite polynomials and Whittaker functions in the non-abelian part of the general Fourier series (6.79) or (6.119). Specifically, we seek an identity that gives the series of products as a Bessel function of the second kind K or a similar function. By rewriting the series of products we could make use of integral representations, of the special functions involved, that may make it possible to connect to the expression (6.139), where the Eisenstein series has the correct polarization.

We then dissect the function (6.119) and focus on the part involving dependence of r:

$$\sum_{r=0}^{\infty}C^{(\mathrm{NA})}_{k,n',r}(s)H_r\left(\sqrt{8\pi|k|}(\tilde{\chi}-n\cdot\mathrm{sgn}(k))\right)W_{-r-\frac{1}{2},s-1}(4\pi|k|y). \tag{6.146}$$

We simplify the expression by going back to the variables (6.61) and (6.74):

$$\begin{aligned}&x:=\hat{\chi}=\sqrt{8\pi|k|}(\tilde{\chi}-n\cdot\mathrm{sgn}(k))\in\mathbb{R}\qquad\text{and}\\ &\hat{y}:=4\pi|k|y>0,\end{aligned} \tag{6.147}$$

yielding

$$\sum_{r=0}^{\infty} C_{k,n',r}^{(\mathrm{NA})}(s) H_r(x)\, W_{-r-\frac{1}{2},s-1}(\hat{y}). \tag{6.148}$$

At this point we need to dive into the vast field in mathematics of hypergeometric functions. A standard mathematical reference is the *Bateman project* books [10,11]. A highly recommended book is also [38], which is self-contained. The Hermite polynomial and the Whittaker function are both special kinds of hypergeometric functions[6], which are a very general type of functions that can be written as a hypergeometric series. The hypergeometric series has the property that the quotient of two consecutive terms is a rational function of the index [11]. To find an identity involving, on one side, an expression of the type (6.148) we should start searching in the mathematics literature. There are several ways of writing the functions, and there are some different notations. We will now try to clarify this.

The generalized hypergeometric series ${}_pF_q$ is defined as

$${}_pF_q(\alpha_r;\rho_t;z) := \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n \; z^n}{(\rho_1)_n \ldots (\rho_p)_n \; n!}, \tag{6.149}$$

where we use the Pochhammer symbols defined in (H.9). Specifically, we have that the confluent hypergeometric function of the first kind M (H.8) is written

$$M(a,c;x) = {}_1F_1(a,c;x). \tag{6.150}$$

In some literature, as [11], the two confluent hypergeometric functions M and U are denoted Φ and Ψ, respectively.

There is a relation between the confluent hypergeometric function of the second kind U and the generalized hypergeometric function ${}_2F_0$ (see 6.6. (3) [11]):

$$x^{\alpha} U(\alpha, \alpha - \beta + 1; x) = {}_2F_0(\alpha, \beta; -1/x). \tag{6.151}$$

The Hermite Polynomial $H_r(x)$

The Hermite polynomials are a special kind of *parabolic cylinder functions* D_r with integer index r, these are further related to the confluent hypergeometric function of the second kind U (see 6.9.2 (32) in [11]):

$$H_r\left(\frac{x}{\sqrt{2}}\right) = 2^{r/2} \mathrm{e}^{\frac{x^2}{4}} D_r(x) = 2^{r-1/2} x\, U\left(\frac{1}{2} - \frac{r}{2}, \frac{3}{2}; \frac{x^2}{2}\right) \tag{6.152}$$

or

$$H_r(x) = 2^{r/2} \mathrm{e}^{\frac{\hat{x}^2}{2}} D_r(\sqrt{2}x) = 2^r x\, U\left(\frac{1}{2} - \frac{r}{2}, \frac{3}{2}; x^2\right). \tag{6.153}$$

[6]Remember that the Whittaker function W consists of a confluent hypergeometric function, as explained in Section 6.1.3.

Using (6.151) we have the identity

$$H_r(x) = (2x)^r\,{}_2F_0\left(\frac{1-r}{2}, -\frac{r}{2}; -\frac{1}{x^2}\right). \tag{6.154}$$

The Hermite polynomials are also related to the confluent hypergeometric function of the first kind $M = {}_1F_1$ (or Φ), but here we need to distinguish between even and odd r (see 10.13 (17) and (18) [10]):

$$\begin{aligned} H_{2r}(x) &= (-1)^r\frac{(2r)!}{r!}M(-r,1/2;x^2), \\ H_{2r+1}(x) &= (-1)^r\frac{(2r+1)!}{r!}2xM(-r,3/2;x^2), \qquad r = 0,1,2,\ldots. \end{aligned} \tag{6.155}$$

The Whittaker Function $W_{-r-\frac{1}{2},s-1}(\hat{y})$

We have first the relationship between W and U, given in (6.76)

$$W_{p,\mu}(\hat{y}) = \mathrm{e}^{-\hat{y}/2}\hat{y}^{\mu+\frac{1}{2}}U\left(\mu - p + \frac{1}{2}, 2\mu+1; \hat{y}\right). \tag{6.156}$$

With our parameters we get

$$W_{-r-\frac{1}{2},s-1}(\hat{y}) = \mathrm{e}^{-\hat{y}/2}\hat{y}^{s-1/2}U\left(r+s, 2s-1; \hat{y}\right). \tag{6.157}$$

Using (6.151) we can also relate the function to the generalized hypergeometric function ${}_2F_0$:

$$W_{-r-\frac{1}{2},s-1}(\hat{y}) = \mathrm{e}^{-\hat{y}/2}\hat{y}^{-r-1/2}\,{}_2F_0\left(r+s, r-s+2; -\frac{1}{\hat{y}}\right). \tag{6.158}$$

The Modified Bessel Function $K_\nu(z)$

The modified Bessel function of the second kind $K_\nu(z)$ is what potentially could be the right hand side of our identity sought after. The function is related to the confluent hypergeometric function of the second kind

$$K_\nu(z) = \sqrt{\pi}\,(2z)^\nu \mathrm{e}^{-z}U(\nu+1/2, 2\nu+1; 2z). \tag{6.159}$$

Ways of Writing the Series of Products

Using the different identities above, we can write the series of products (6.148) as

$$\begin{aligned} &\sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',r}(s)H_r(x)\,W_{-r-\frac{1}{2},s-1}(\hat{y}) = \\ x\mathrm{e}^{-\hat{y}/2}\hat{y}^{s-1/2}&\sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',r}(s)2^r\,U\left(\frac{1}{2}-\frac{r}{2},\frac{3}{2};x^2\right)U\left(r+s,2s-1;\hat{y}\right) = \\ \mathrm{e}^{-\hat{y}/2}&\sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',r}(s)(2x)^r\,{}_2F_0\left(\frac{1-r}{2},-\frac{r}{2};-\frac{1}{x^2}\right) \\ &\times \hat{y}^{-r-1/2}\,{}_2F_0\left(r+s, r-s+2; -\frac{1}{\hat{y}}\right). \end{aligned} \tag{6.160}$$

By dividing the sum into an even and an odd part we can use (6.155). Performing this, and writing the Whittaker function as a confluent hypergeometric function of the second kind, yields

$$\mathrm{e}^{-\hat{y}/2}\hat{y}^{s-1/2}\left[\sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',2r}(s)(-1)^r\frac{(2r)!}{r!}M(-r,1/2;x^2)\,U\,(2r+s,2s-1;\hat{y})+\right.$$
$$\left.+\sum_{r=0}^{\infty} C^{(\mathrm{NA})}_{k,n',2r+1}(s)(-1)^r\frac{(2r+1)!}{r!}2x\,M(-r,3/2;x^2)\,U\,(2r+1+s,2s-1;\hat{y})\right]. \tag{6.161}$$

Transformation of Series of Products

There are numerous published papers on transformations of different kinds of series involving confluent hypergeometric functions. What is usually the subject of these papers is that of *generating functions*. Unfortunately, most of them treat the first type of function $M = {}_1F_1$, whereas we would prefer an identity with at least one function of the second kind $U = \Psi$. E.g., in [19] one shows (Eq. 17)

$$\sum_{r=0}^{\infty}\frac{(c-a)_r(d-b)_r}{(c)_r(d)_r r!}M(a,c+r;z)M(b,d+r;z)z^r = \mathrm{e}^z\,{}_2F_2(a,b;c,d;z). \tag{6.162}$$

And in [16] one obtains (Eq. 7)

$$\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}(\sigma-\alpha)_r(\sigma-\beta)_r\Gamma(-\sigma-r+\alpha+\beta)$$
$$\times\, M(-r,\sigma-\beta;z)M(r+\sigma-\alpha,\beta;-x) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\sigma)}M(\alpha,\sigma;z-x). \tag{6.163}$$

The idea is that if we find a way of doing the r-sum expansion of the products of the Hermite polynomial and the Whittaker function, it would then determine the dependence on r in the coefficients $C^{(\mathrm{NA})}_{k,n',r}(s)$. This provided of course that we somehow reach the Eisenstein expression (6.121) or (6.139).

We have tried to make a similar calculation as in [16] including instead a function U on the left hand side, but without success. Note that we cannot use the expression (6.163) for two reasons

- Our Whittaker function, which is related to to U, cannot be written as a linear combination of M, as in (H.8). This since we are interested in the value $s = 3/2$ yielding the *logarithmic case*, see Section H.4, for which one has to use the integral representation of U (H.20).

- Assume that we did not have the logarithmic case and we then were able to use the expression for U (W) in terms of M. Still, we cannot divide the summation (6.163) into two parts (one for each M in U), since the two terms in the relation (H.8) have a different dependence on the summation variable r and the parameter s. Thereby, we will from (6.163) predict different r- and s-dependence of the coefficient $C^{(\mathrm{NA})}_{k,n',r}(s)$ when forcing the identity, which is not acceptable.

7

Conclusions

In this thesis we have provided a short route for understanding the significance of automorphic forms in string theory. There are two faces of the exposition: the mathematical aspect, i.e., how the calculations are made technically; and the physical aspect, i.e., in what physical settings do we need the mathematics. The focus of this thesis was slightly more on the explicit calculations and the needed mathematical machinery, including most notably: Lie algebra, discrete Lie groups, Fourier analysis and special functions solving differential equations.

We have examined two different cases in string theory where automorphic forms are essential. The first one was in ten-dimensional Type IIB superstring theory where an automorphic form is known to sum up all the quantum corrections to the R^4-term in the effective string action, including non-perturbative contributions from instantons. This case was well-suited as a first acquaintance with automorphic forms since, apart from being a well-known fact in the string theory community, it is not so hard[1] to treat from a mathematical point of view. In the second case, we followed the recent paper [9] where one studies how an automorphic form could encode the quantum corrections to the hypermultiplet moduli space metric in Type IIA superstring theory compactified on a rigid Calabi-Yau threefold. The goal was here to acquire more insight into an unsolved subtlety, namely how to determine the unknown non-abelian Fourier coefficients from the Eisenstein series $\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}$.

In the introduction, Section 1.4, we began to state the definition of an automorphic form. We confined ourselves to a certain kind of automorphic forms, constructed as non-holomorphic Eisenstein series $\mathcal{E}_s^{\mathrm{G}(\mathbb{Z})}$. The Eisenstein series were parameterized on double cosets $\mathrm{G}(\mathbb{Z})\backslash\mathrm{G}/\mathrm{K}$, so it was crucial to first study how coset symmetries G/K appear in the physics (more specifically in supergravity theories) in Chapter 2. A more mathematical treatment of actions of scalar fields with a coset moduli space G/K were given in Chapter 3,

[1] By this we mean "not so hard *compared* with other cases in string theory".

where we derived the actions on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$, which were the two coset spaces appearing in the Type IIB case and the compactified Type IIA case respectively.

In Chapter 4, we explained how the coset symmetries of the classical supergravity theories are broken by quantum corrections. As mentioned in Section 1.7, we study the effective string theories valid in the massless limit, i.e., for a small value of the slope parameter α'. In this case one can make an asymptotic expansion of the scattering amplitude (1.17); there is an expansion for the effective action with corresponding orders in the two parameters α' and the string coupling constant g_s. The tree-level in both Type IIB superstring theory, and Type IIA superstring theory compactified on a rigid Calabi-Yau threefold, corresponded to a supergravity theory for which the scalar fields had a coset moduli space G/K. Even though this moduli space is unaffected by quantum corrections in the Type IIB superstring theory, adding higher orders in g_s to Type IIB supergravity still spoils the global $\mathrm{SL}(2,\mathbb{R})$-invariance of the scalar fields in the theory. We get, for instance, the higher order R^4-term, which does not exhibit the continuous invariance under $\mathrm{SL}(2,\mathbb{R})$. However, due to dualities of the string theory, the effective theory is still invariant under a discrete subgroup $\mathrm{SL}(2,\mathbb{Z})\subset\mathrm{SL}(2,\mathbb{R})$. This was then the reason for the Eisenstein series $\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}$ on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ being the coefficient to R^4 in the effective action of Type IIB superstring theory.

For Type IIA superstring theory compactified on a rigid Calabi-Yau threefold, there was another story. Here the classical coset moduli space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$, for the scalar fields: ϕ, χ, $\tilde{\chi}$ and ψ in the hypermultiplet, was bound to be deformed when adding quantum corrections by stepping further in the g_s-expansion. In [9] they conjectured the quantum corrected theory to be invariant under the Picard modular group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$, and they argued that an automorphic form constructed as a non-holomorphic Eisenstein series on $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$ could hypothetically encode the quantum corrections to the metric on the moduli space.

To read off the perturbative and non-perturbative terms from the automorphic form in question we needed to rewrite it as a Fourier series. The general structure of the Fourier series was found using the maximal nilpotent subgroup $\mathrm{N}(\mathbb{Z})\subset\mathrm{G}(\mathrm{Z})$, as was explained in Section 4.1. To construct the Eisenstein series we used the lattice method, attributed to Obers and Pioline [33], explained in Section 4.2.

In Chapter 5 we found, following [23, 34, 37], the general Fourier series for the Type IIB case. We explicitly constructed the exact Eisenstein series $\mathcal{E}_s^{\mathrm{SL}(2,\mathbb{Z})}$ and identified the coefficients in the Fourier series by rewriting the Eisenstein series, using e.g., the Poisson resummation formula.

In Chapter 6 we followed [9, 34] and found the general Fourier series to the automorphic form on $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. We decomposed the Fourier series in a constant, abelian and non-abelian part and applied the laplacian condition to further determine the functional structure of the three parts. Specifically, the non-abelian part consisted of a series of products of Hermite polynomials and Whittaker functions. In Section 6.2 we presented a way of constructing the Eisenstein series $\mathcal{E}_s^{\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])}$, and decomposing the series accordingly to the Fourier series. Following [9] we extracted the coefficients in the constant and the abelian part of the Fourier series. However, the first loop term in the constant part appeared to have the wrong sign compared to other existing calculations. We moved quickly over this issue and continued the pursuit of the Fourier expansion for the Eisenstein series, even though the physics seemed to tell us that we need another automorphic form to correct the metric. For the non-abelian part of the Eisenstein series, we could not connect to the corresponding non-abelian part in the general Fourier series to find the unknown coefficients $C^{(\mathrm{NA})}_{k,n',r}(s)$. We performed a Fourier transform to get to the correct basis, although we had to assume that different summation variables were to be identified. In particular, we needed the integral over n in some way to equal a summation over shifted integers as in the non-abelian part of the Fourier series.

Finally, in Section 6.5 we tried to perform the summation of the product of Hermite polynomials and Whittaker functions by finding a similar identity in the mathematics literature. Such an identity could provide us with more clues of how to perform the non-trivial summations in the non-abelian part of the Eisenstein series. We had no success in finding this identity, however, we pointed out a few facts about the relations between the different special functions.

Appendices

A

A Touch of Lie Algebra

For a more thorough explanation of the following, the reader is referred to [21] and [27].

A.1 The Chevalley-Serre Presentation

For any finite-dimensional, semi-simple Lie algebra $\mathfrak{g}$ over the complex field there is a so-called *Chevalley-Serre presentation*. For a rank r algebra this presentation consists of $3r$ basic elements

$$\{e^i, f^i, h^i | i = 1, 2, \ldots, r\}. \tag{A.1}$$

The superscript i refers to the simple root $\alpha^{(i)} \in \Phi_s$, where Φ_s is the set of simple roots of $\mathfrak{g}$. Each triple building block $\{e^i, f^i, h^i\}$ makes up an own $\mathfrak{sl}(2, \mathbb{R})$-algebra with the commutation relations

$$[e^i, f^i] = h^i, \qquad [h^i, e^i] = 2e^i, \qquad [h^i, f^i] = -2f^i. \tag{A.2}$$

The *Chevalley relations* show how the $\mathfrak{sl}(2, \mathbb{R})$-algebras are intertwined with one another

$$\begin{aligned} [h^i, h^j] &= 0 \\ [h^i, e^j] &= A^{ji} e^j \\ [h^i, f^j] &= -A^{ji} f^j \\ [e^i, f^j] &= \delta_{ij} h^i, \qquad i, j = 1, 2, \ldots, r. \end{aligned} \tag{A.3}$$

The intertwining is encoded in the $r \times r$-matrix with integer entries A^{ij}, called the *Cartan matrix*. The Cartan matrix contains all important information of the algebra. The rows of the Cartan matrix coincide with the components of the simple roots in the Dynkin basis

$$A^{ij} = (\alpha^{(i)})^j. \tag{A.4}$$

It is the off-diagonal elements of A^{ij} that tells us how the different $\mathfrak{sl}(2,\mathbb{C})$ "talk" to one other. E.g., in the case

$$A^{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{A.5}$$

we have two separate $\mathfrak{sl}(2,\mathbb{C})$ with no connection. This algebra is simply

$$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}).$$

Whereas for example the algebra $\mathfrak{sl}(3,\mathbb{C})$, which will be treated in Appendix C, has a Cartan matrix

$$A^{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{A.6}$$

The complete Chevalley-Serre presentation finally consists of the *Serre relations*

$$\begin{aligned} \overbrace{[e^i, \dots, [e^i}^{1-A^{ji}\,\text{times}}, e^j] &= 0, \\ \overbrace{[f^i, \dots, [f^i}^{1-A^{ji}\,\text{times}}, f^j] &= 0, \end{aligned} \tag{A.7}$$

which also depends on the Cartan matrix. We can generate all elements in $\mathfrak{g}$ from multiple Lie brackets and linear combinations of the basic elements (A.1). The Serre relations (A.7) then puts constraints on the multiple Lie brackets, and for certain Cartan matrices this makes the algebra finite-dimensional. Note that, e.g., the infinite-dimensional *Kac-Moody* algebras are also compatible with the Chevalley-Serre presentation, but with other restrictions on the Cartan matrix. The Kac-Moody algebras can be defined and constructed with *finitely* many base-triples ($\mathfrak{sl}(2,\mathbb{C})$-algebras); all its additional elements are obtained by multiple commutations of these fundamental elements. It would not be possible to write down the infinite-dimensional algebras in the more standard way

$$[T_i, T_j] = {f_{ij}}^k T_k,$$

because we would need infinitely many structure constants ${f_{ij}}^k$. We will see an explicit example of the multiple commutators when treating $\mathfrak{sl}(3,\mathbb{C})$ in Appendix C.

As an example, the Cartan matrix for the matrix Lie algebra $\mathfrak{sl}(r+1,\mathbb{C})$, with $r \geq 1$, is

$$\mathrm{A}_r = \mathfrak{sl}(r+1,\mathbb{C}): \qquad A^{ij} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \tag{A.8}$$

From the Chevalley-Serre presentation one sees the important fact that the Lie algebra $\mathfrak{g}$, as a vector space, can be decomposed as

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_- \tag{A.9}$$

called the *triangular decomposition*[1] of $\mathfrak{g}$. Here the subspaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are generated solely by e^i- and f^i-elements respectively, and they are nilpotent subalgebras. $\mathfrak{h}$ is an abelian subalgebra called the *Cartan subalgebra* consisting of linear combinations of the h^i-elements. We can also write

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathbb{C}e^i \oplus \bigoplus_{i=1}^{r} \mathbb{C}h^i \oplus \bigoplus_{i=1}^{r} \mathbb{C}f^i. \tag{A.10}$$

For the matrix algebras that we will treat in this thesis, this decomposition simply states that a general element in the algebra can be written as a sum of an upper triangular, a diagonal and a lower triangular matrix.

By definition, a *positive* root that can be written as a linear combination of the simple roots $\alpha^{(i)}$ with positive integer coefficients. All elements in $\mathfrak{g}_+$ are then associated with a positive root, since, firstly we have that the Chevalley basis step-operators e^i are associated to a simple root each, and secondly all elements generated by multiple Lie brackets of the positive step-operators have a root that is a sum of the simple roots with positive integer coefficients. E.g., pick two Chevalley basis elements e^1 and e^2 in an algebra for which the commutator $[e^i, e^j]$ is non-zero. We then have for a Cartan subalgebra element h^k:

$$\begin{aligned}[h^k, [e^i, e^j]] &= -[e^i, [e^j, h^k]] - [e^j, [h^k, e^i]] = \\ &= (\alpha^{(j)})^k [e^i, e^j] - (\alpha^{(i)})^k [e^j, e^i] = (\alpha^{(i)} + \alpha^{(j)})^k [e^i, e^j],\end{aligned} \tag{A.11}$$

using the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \qquad \forall x, y, z \in \mathfrak{g}, \tag{A.12}$$

which is by definition satisfied by all Lie algebras $\mathfrak{g}$. The set of all positive roots is denoted as Φ_+. Correspondingly, the above definition for positive roots holds for the *negative* roots, interchanging positive to negative everywhere. The elements in the subalgebra $\mathfrak{g}_-$ are then associated to the negative roots Φ_-.

A.2 Real Forms

The above presentation is made to classify all *complex* semi-simple Lie algebras, that is, Lie algebras with a base field $\mathbb{C}$. The reason for this relies on the fact that the complex field is algebraically closed, i.e., any algebraic equation has a solution in the base field in contrast with the real numbers $\mathbb{R}$. When

[1]Also *Gauss decomposition* or *Cartan decomposition*.

choosing linearly independent elements of an algebra $\mathfrak{g}$ that make up a diagonal basis for the whole algebra, like the elements h^i above, we must require the eigenvalues to the adjoint action of h^i to be elements in the base field. Since we get the eigenvalues from a characteristic equation, which is an algebraic equation, we need an algebraically closed base field.

In physics one more often deals with the real Lie algebras; real Lie group symmetries show up since the physical fields are often real. Luckily, we can use the knowledge from the complex semi-simple Lie algebras to relate to the real Lie algebras in a special way. Taking a look at the relations (A.3) we see that all eigenvalues that are encoded in the Cartan matrix are real; actually, as was mentioned before, they are even integers in this basis. This implies that we can restrict ourselves to a real algebra from a complex one in the Chevalley-Serre presentation by just choosing the base field to be $\mathbb{R}$ and keeping the step-operators and Cartan subalgebra generators. The triangular decomposition (A.10) becomes

$$\mathfrak{g}_{\text{normal}} = \bigoplus_{i=1}^{r} \mathbb{R}e^i \oplus \bigoplus_{i=1}^{r} \mathbb{R}h^i \oplus \bigoplus_{i=1}^{r} \mathbb{R}f^i. \tag{A.13}$$

E.g., from $\mathfrak{sl}(n,\mathbb{C})$ we would get $\mathfrak{sl}(n,\mathbb{R})$. This real Lie algebra is called the *normal*[2] *real form* of the corresponding complex algebra.

By definition, a *real form* $\mathfrak{h}$ of a complex simple Lie algebra $\mathfrak{g}$ is a real algebra whose *complexification* $\mathfrak{h}_\mathbb{C}$ is isomorphic to $\mathfrak{g}$. The complexification is defined as

$$\mathfrak{h}_\mathbb{C} := \mathfrak{h} \oplus \mathrm{i}\mathfrak{h}, \tag{A.14}$$

where we now consider the algebra as a *real* vector field. The real dimension of the real form is then half the real dimension of the complex algebra. The isomorphism $\mathfrak{h}_\mathbb{C} \simeq \mathfrak{g}$ is equivalent to the statement that all elements $z \in \mathfrak{g}$ can uniquely be written as $z = x + \mathrm{i}y$ with $x, y \in \mathfrak{h}$.

A complex semi-simple Lie algebra has in general several different real forms. The normal real form above is always one of them, and this algebra corresponds to a Lie group that is as far as possible from being compact. Moreover, there is always a so-called *compact* real form, which has a negative definite Killing form, in an appropriate basis

$$B^{ab} = -\delta^{ab}. \tag{A.15}$$

As an example, $\mathfrak{sl}(2,\mathbb{C})$ has two real forms: the compact real form $\mathfrak{su}(2)$ and the non-compact normal real form $\mathfrak{sl}(2,\mathbb{R})$. In Appendix C we will treat the real forms of $\mathfrak{sl}(3,\mathbb{C})$: $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{su}(3)$ and $\mathfrak{su}(2,1)$.

Just as one classifies the complex semi-simple algebras with *Dynkin diagrams* there is a corresponding classification of the real forms with help of the so-called *Satake diagrams*. However, these diagrams will not be treated here.

[2] Sometimes also called the *split* real form.

One can show that the real forms are obtained with help of so-called *involutive automorphisms*, or *involutions*, on the algebra. These automorphisms will be explained more in the next section. The Satake diagrams gives information about these involutions.

A.3 Involutive Automorphisms

An automorphism ω of a Lie algebra $\mathfrak{g}$ is an isomorphic map from $\mathfrak{g}$ to itself. That is, it is one-to-one and onto (bijection). It further respects the Lie algebra structure, i.e.

$$\omega([x,y]) = [\omega(x), \omega(y)], \qquad \forall x, y \in \mathfrak{g}. \tag{A.16}$$

The set of all automorphisms of $\mathfrak{g}$ is denoted $\mathrm{Aut}(\mathfrak{g})$, and one can show that this is indeed a group with the composition of maps as composition rule. We denote multiple compositions of an automorphism by

$$\omega^n := \overbrace{\omega \circ \omega \circ \ldots \circ \omega}^{n \text{ times}}. \tag{A.17}$$

It can be the case that for an automorphism ω it holds that

$$\omega^N = \mathrm{Id}, \quad \text{for some } N \in \mathbb{N}. \tag{A.18}$$

The automorphism is then called *finite* and of order N (pick smallest possible natural number). If N does not exist, the automorphism is naturally called *infinite*.

We will interest ourselves with a special type of automorphisms called *involutions*. These are of order two, i.e.,

$$\omega^2 = \mathrm{Id}. \tag{A.19}$$

Now, any element x in $\mathfrak{g}$ can be decomposed into a part invariant $x_{(0)}$ and anti-invariant $x_{(1)}$ under the involution ω, as

$$x = x_{(0)} + x_{(1)} = \frac{1}{2}(x + \omega(x)) + \frac{1}{2}(x - \omega(x)). \tag{A.20}$$

This shows that we can decompose the whole algebra as a vector space into two subsets with eigenvalues ± 1 under the involution

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}. \tag{A.21}$$

In this decomposition only $\mathfrak{g}_{(0)}$ with eigenvalue 1 is a subalgebra, called the *fixed point algebra* of ω, as we will now show. Any element in $\mathfrak{g}_{(0)}$ can be written as $x + \omega(x)$ where x is some element in $\mathfrak{g}$, let us take the Lie bracket of two arbitrary elements in $\mathfrak{g}_{(0)}$:

$$[x + \omega(x), y + \omega(y)] = [x, y] + [x, \omega(y)] + [\omega(x), y] + [\omega(x), \omega(y)]. \tag{A.22}$$

The resulting element is also invariant under ω as can be seen by using the property (A.16). So, $\mathfrak{g}_{(0)}$ is closed under commutations and is therefore a subalgebra of $\mathfrak{g}$. However, doing the same calculation for two arbitrary elements in $\mathfrak{g}_{(1)}$ yields

$$[x - \omega(x), y - \omega(y)] = [x, y] + [y, \omega(x)] + [\omega(y), x] + [\omega(x), \omega(y)], \tag{A.23}$$

and we see that this resulting element is also invariant under ω. This tells us that the subset $\mathfrak{g}_{(1)}$ is not a subalgebra since it does not close. Furthermore, the commutation of one element in $\mathfrak{g}_{(0)}$ and one in $\mathfrak{g}_{(1)}$ yields an element in $\mathfrak{g}_{(1)}$, so we have the following bracket relations

$$[\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}] \subseteq \mathfrak{g}_{(0)}, \quad [\mathfrak{g}_{(0)}, \mathfrak{g}_{(1)}] \subseteq \mathfrak{g}_{(1)}, \quad [\mathfrak{g}_{(1)}, \mathfrak{g}_{(1)}] \subseteq \mathfrak{g}_{(0)}. \tag{A.24}$$

As a side remark, it generally holds for automorphisms of order N that they decompose the Lie algebra into N eigenspaces of the automorphism (this is a $\mathbb{Z}_N$-gradation of the algebra), and of these, only the invariant subset with eigenvalue 1 is a subalgebra.

If we have that $\mathfrak{g}$ is an algebra over $\mathbb{R}$, we can with the subsets $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(1)}$ form another real algebra by the vector space direct sum

$$\mathfrak{r} := \mathfrak{g}_{(0)} \oplus \mathrm{i}\mathfrak{g}_{(1)}. \tag{A.25}$$

This is an algebra since we have that it closes under the Lie bracket

$$\begin{aligned}&[\mathfrak{g}_{(0)} \oplus \mathrm{i}\mathfrak{g}_{(1)}, \mathfrak{g}_{(0)} \oplus \mathrm{i}\mathfrak{g}_{(1)}] = \\ &\quad = [\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}] + [\mathfrak{g}_{(0)}, \mathrm{i}\mathfrak{g}_{(1)}] + [\mathrm{i}\mathfrak{g}_{(1)}, \mathfrak{g}_{(0)}] + [\mathrm{i}\mathfrak{g}_{(1)}, \mathrm{i}\mathfrak{g}_{(1)}] = \mathfrak{g}_{(0)} \oplus \mathrm{i}\mathfrak{g}_{(1)},\end{aligned} \tag{A.26}$$

where in the last equality we have used the commutation relations (A.24). It is also a subalgebra of the complexification of $\mathfrak{g}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}. \tag{A.27}$$

One can then show that *all* real forms of $\mathfrak{g}_{\mathbb{C}}$ are obtained by considering all possible involutions of $\mathfrak{g}_{\mathbb{C}}$. There is a theorem that says that the finite-dimensional semi-simple Lie algebras are in a one-to-one correspondence with pairs $(\mathfrak{h}, \omega)$, where $\mathfrak{h}$ is a finite-dimensional semi-simple complex Lie algebra, and ω an involution of $\mathfrak{h}$ [21].

A.4 Cartan Decomposition

For all semi-simple Lie algebras $\mathfrak{g}_0$ over $\mathbb{R}$ there is a unique decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \tag{A.28}$$

called the *Cartan decomposition*, where $\mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$ and $\mathfrak{p}_0$ is a vector subspace. For a proof see [27]. The decomposition is accomplished by using an involution τ called the *Cartan involution*. Thus, τ decomposes the algebra as in the general case (A.21). Here $\mathfrak{k}_0$ is invariant under τ and $\mathfrak{p}_0$ has eigenvalue -1, i.e.,

$$\tau(k) = k, \qquad \tau(p) = -p, \qquad k \in \mathfrak{k}_0,\ p \in \mathfrak{p}_0. \tag{A.29}$$

The Killing form

$$B_\tau(x, y) = -B(x, \tau(y)) \tag{A.30}$$

has the property of being *strictly positive* definite showing that the subalgebra $\mathfrak{k}_0$ has a negative definite Killing form and is thus compact. In fact an involutive automorphism of a Lie algebra is by definition a Cartan involution if the bilinear form (A.30) is strictly positive definite.

The Cartan decomposition will be essential for us when constructing the symmetric coset spaces. We want to divide a specific Lie group by its maximal compact group and this group is obtained by exponentiation of the maximal compact algebra to the corresponding Lie algebra. The realization of the Cartan involution of the algebra elements is derived from the root system. In our cases we are interested in the two types of algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{C})$, in the latter case we will use the the Cartan involution again on the real form $\mathfrak{su}(2,1)$. For $\mathfrak{sl}(2,\mathbb{R})$, and in fact for all algebras $\mathfrak{sl}(n,\mathbb{R})$, the Cartan involution act as

$$\tau(x) = -x^{\mathrm{T}}, \qquad x \in \mathfrak{sl}(2,\mathbb{R}). \tag{A.31}$$

For $\mathfrak{sl}(3,\mathbb{C})$, and for all $\mathfrak{sl}(n,\mathbb{C})$, it acts on algebra elements as

$$\tau(x) = -x^{\dagger}, \qquad x \in \mathfrak{sl}(3,\mathbb{C}). \tag{A.32}$$

In the specific case when considering the Cartan involution acting on a split real form, it is also called the *Chevalley involution*. Loosely speaking, one can say that the Cartan involution picks out the compact part of an algebra.

A.5 Iwasawa Decomposition

It is possible to combine the Cartan decomposition of a real semi-simple Lie algebra with the triangular decomposition (A.9) of its complexification, thereby yielding the *Iwasawa decomposition*. Starting with the Lie algebra $\mathfrak{g}_0$ of a connected semi-simple real Lie group G, the decomposition reads

$$\mathfrak{g}_0 = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}_0, \tag{A.33}$$

where $\mathfrak{n}_0$ is a *nilpotent* subalgebra, $\mathfrak{a}_0$ is an abelian subalgebra (the maximal abelian subalgebra of $\mathfrak{p}_0$ in the Cartan decomposition (A.28)) and $\mathfrak{k}_0$ is the maximal compact subalgebra to $\mathfrak{g}_0$ given by the Cartan involution. The proof is lengthy, see for instance [27]. In practice, for the matrix groups we are dealing with, we choose $\mathfrak{a}_0$ to consist of diagonal matrices and $\mathfrak{n}_0$ as upper triangular matrices (positive step-operators with respect to the abelian subalgebra generators).

On the Lie group level the decomposition reads

$$\mathrm{G} = \mathrm{NAK}, \tag{A.34}$$

where then the subgroups N, A and K are derived from exponentiating the subalgebras $\mathfrak{n}_0$, $\mathfrak{a}_0$ and $\mathfrak{k}_0$ respectively. Note that the decomposition could equally well be made as $\mathrm{G} = \mathrm{KAN}$; we choose the decomposition (A.34) because it fits with our choice of right cosets G/K in this thesis.

For general linear matrix groups $\mathrm{G} = \mathrm{GL}_n(\mathbb{R})$ the nilpotent subgroup can be represented by an upper triangular matrix, the abelian subgroup by a diagonal matrix and the maximal compact group by an orthogonal matrix.

B

$\mathrm{SL}(2,\mathbb{R})$

We review some of the properties of the group $\mathrm{SL}(2,\mathbb{R})$, its algebra $\mathfrak{sl}(2,\mathbb{R})$ and the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The group $\mathrm{SL}(2,\mathbb{R})$ is the group of two-by-two matrices with real elements and unit determinant

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{R} \tag{B.1}$$

It is a three-dimensional, non-compact Lie group. To find the generators J^i of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ we use

$$A = \exp(\alpha_i J^i), \tag{B.2}$$

together with the unit determinant condition

$$1 = \det A = \det(\exp \alpha_i J^i). \tag{B.3}$$

Specifically, we can choose an element A that equals the exponentiation of just one generator: $A = \exp(\alpha J^{(a)})$. Then, we have for that particular element that

$$\begin{aligned} \det(1 + \alpha J + \frac{1}{2}\alpha^2 J^2 + ...) = \det U^{-1} \det(1 + \alpha D + \frac{1}{2}\alpha^2 D^2 + ...) \det U = \\ = \exp(\alpha(\lambda_1 + \lambda_2)) = 1, \end{aligned} \tag{B.4}$$

where we have diagonalized $J = U^{-1} D U$ and gained the eigenvalues λ_1 and λ_2. This implies $\lambda_1 + \lambda_2 = 0 \implies \mathrm{Tr}\, J = 0$. Thus, the generators are two-by-two, linearly independent, traceless matrices with real elements.

There is a particularly nice choice of basis

$$\begin{aligned} e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{B.5}$$

which is the Chevalley basis, see (A.3). It is the same basis as for $\mathfrak{sl}(2,\mathbb{C})$; recall from Appendix A that $\mathfrak{sl}(2,\mathbb{R})$ is nothing but the split real form of $\mathfrak{sl}(2,\mathbb{C})$. The generators e and f play the role of ladder operators and the generator h forms the abelian Cartan subalgebra, which in this case is one-dimensional. The commutation relations are

$$\begin{aligned} [e,f] &= h \\ [h,e] &= 2e \\ [h,f] &= -2f. \end{aligned} \tag{B.6}$$

B.1 The Maximal Compact Subgroup of $\mathrm{SL}(2,\mathbb{R})$

In Section A.4, we saw that we can use the Cartan involution τ to decompose a semi-simple real Lie algebra into a maximal compact subalgebra and an anti-invariant subset

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{k}_0 \oplus \mathfrak{p}_0 \qquad \text{with} \\ \mathfrak{k}_0 &= \{x \in \mathfrak{g}_0 | \tau(x) = x\} \qquad \text{and} \\ \mathfrak{p}_0 &= \{x \in \mathfrak{g}_0 | \tau(x) = -x\}. \end{aligned}$$

The Cartan involution of $\mathfrak{sl}(2,\mathbb{R})$ acts on the algebra elements as $\tau(x) = -x^{\mathrm{T}}$. Acting on the basis generators (B.5) we get

$$\tau(e) = -f, \qquad \tau(f) = -e, \qquad \tau(h) = -h. \tag{B.7}$$

Now, we want to find the maximal compact part $\mathfrak{k}_0$, which is invariant under the involution. By inspection we see that the element $e-f$ is fixed by τ and we have

$$e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}(2). \tag{B.8}$$

In fact, $e-f$ is a generator of $\mathfrak{so}(2)$ and so this algebra is the maximal compact subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. The Cartan decomposition of $\mathfrak{sl}(2,\mathbb{R})$ thus reads

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}(e-f) \oplus (\mathbb{R}h \oplus \mathbb{R}(e+f)). \tag{B.9}$$

We can explicitly verify that the anti-invariant part $\mathfrak{p} = \mathbb{R}h \oplus \mathbb{R}(e+f)$ is not a subalgebra since it does not close $[h, e+f] = 2(e-f) \in \mathfrak{so}(2)$.

Exponentiating the maximal compact subalgebra we get that the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SO}(2) \simeq \mathrm{U}(1)$.

B.2 Iwasawa Decomposition of $\mathrm{SL}(2,\mathbb{R})$

The Iwasawa decomposition of the algebra reads

$$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}e \oplus \mathbb{R}h \oplus \mathfrak{so}(2). \tag{B.10}$$

For the Lie group we have then

$$\mathrm{SL}(2,\mathbb{R}) = \mathrm{NAK} = \mathrm{e}^{\mathbb{R}e}\mathrm{e}^{\mathbb{R}h}\mathrm{e}^{\mathbb{R}(e-f)}. \tag{B.11}$$

Now, to form a coset representative V for the two-dimensional coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ we can simply put $\mathrm{K} = \mathrm{Id}$. We get

$$V = \mathrm{e}^{\chi e}\mathrm{e}^{-\frac{\phi}{2}h} = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\frac{\phi}{2}} & 0 \\ 0 & \mathrm{e}^{\frac{\phi}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\frac{\phi}{2}} & \chi\mathrm{e}^{\frac{\phi}{2}} \\ 0 & \mathrm{e}^{\frac{\phi}{2}} \end{pmatrix}. \tag{B.12}$$

This particular choice of an upper triangular matrix is called the *Borel gauge*. We could in fact just as well have chosen V in a lower triangular form instead. The two parameters χ and ϕ are the *axion* field and the *dilaton* field respectively when identifying this coset representative in the Type IIB superstring theory.

C

SU(2, 1)

In this appendix we will review the basics of the Lie group SU(2, 1), its Lie algebra $\mathfrak{su}(2,1)$ and the coset formed by dividing out the maximal compact subgroup. It will be similar to the previous section, the difference being that SU(2, 1) is a more complicated group and things will be less trivial. SU(2, 1) is the group of 3×3-matrices with complex entries preserving the metric

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{C.1}$$

i.e., for $A \in \mathrm{SU}(2,1)$ we have

$$A^\dagger G A = G. \tag{C.2}$$

This relation already implies $|\det A| = 1$. One picks $\det A = +1$ for the "special" group. The dimension for the group is $3^2 - 1 = 8$. We can make a change of basis with a similarity transformation acting on both sides of (C.2)

$$SA^\dagger GAS^{-1} = SA^\dagger S^{-1} SGS^{-1} SAS^{-1} = SGS^{-1}, \qquad \det S \neq 0. \tag{C.3}$$

When we let $A \longrightarrow SAS^{-1}$, the new metric that is preserved is exactly SGS^{-1}. Knowing that the determinant and the trace is preserved in the similarity transformation, we can schematically state what other different types of metrics we can transform to from the one above (C.1)

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 1 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}. \tag{C.4}$$

Elements in the Lie algebra $\mathfrak{su}(2,1)$ will be traceless due to the unit determinant condition as was shown in the previous section. We get additional information from the constraint (C.2). Consider an element in the group near the identity element to the first order

$$A = \mathrm{e}^{\alpha X} \approx \mathbb{1} + \alpha X, \qquad \alpha \in \mathbb{R},\ X \in \mathfrak{su}(2,1). \tag{C.5}$$

Plugging this into (C.2) gives

$$A^\dagger G A = (\mathbb{1} + \alpha X^\dagger) G (\mathbb{1} + \alpha X) = G + \alpha X^\dagger G + G\alpha X + \mathcal{O}(\alpha^2) = G \\ \Longrightarrow\ X^\dagger G + G X = 0. \tag{C.6}$$

At this point one could use (C.6) and the traceless condition to find eight linearly independent generators for the group. But we will instead use our knowledge of the Chevalley-Serre presentation in Appendix A, and the connection to real forms from complex algebras. This is more enlightening from an algebraic point of view. It turns out that $\mathfrak{su}(2,1)$ is actually one of the three real forms of $\mathfrak{sl}(3,\mathbb{C})$, the other two being $\mathfrak{su}(3)$ and $\mathfrak{sl}(3,\mathbb{R})$. Since the Chevalley basis for $\mathfrak{sl}(3,\mathbb{C})$ is straightforward to derive knowing the basis for $\mathfrak{sl}(2,\mathbb{C})$, as we will see below, it is natural to begin with $\mathfrak{sl}(3,\mathbb{C})$ and then find the generators to $\mathfrak{su}(2,1)$ through a certain involution. With help of the Cartan involution we will also be able to derive the maximal compact algebra to both $\mathfrak{sl}(3,\mathbb{C})$ and $\mathfrak{su}(2,1)$, as we saw in Appendix B. It turns out that the maximal compact subgroup of SU$(2,1)$ is SU$(2)\times$U(1). Furthermore, using the Iwasawa decomposition as in (B.10), we will be able to construct the coset space SU$(2,1)/($SU$(2)\times$U$(1))$. I am grateful to D. Persson for giving me important insights and lending me his precious notes on SU(2,1).

C.1 Chevalley Basis for $\mathfrak{sl}(3,\mathbb{C})$

As stated in Appendix A we have in general that a complex semi-simple Lie algebra $\mathfrak{g}$ in the Chevalley basis has generators satisfying the following commutation relations[1]:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, \\ [h_i, e_j] &= A_{ij} e_j, \\ [h_i, f_j] &= -A_{ij} f_j, \\ [h_i, h_j] &= 0. \end{aligned} \tag{C.7}$$

The indices i and j run from 1 to $r = \mathrm{Rank}(\mathfrak{g})$. The algebra $\mathfrak{sl}(3,\mathbb{C})$ is a natural extension of $\mathfrak{sl}(2,\mathbb{C})$; it is simply two $\mathfrak{sl}(2,\mathbb{C})$-algebras intertwined together. I.e., it consists of two base triples $\{e_i, f_i, h_i\}$ $(i = 1, 2)$ and has rank 2, but moreover we get two additional step-operators e_3 and f_3 from commuting the step-operators included in the base triples, e.g., $e_3 = [e_1, e_2]$. In total we have

[1] Unlike in Appendix A we will now use subscript indices.

eight generators[2]. The Cartan matrix for $\mathfrak{sl}(3,\mathbb{C})$ is

$$A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{C.8}$$

Note that the additional generators e_3 and f_3 are not included in the commutation relations (C.7). These relations only tell us how the base-triples commute, but they are included in the Serre relations (A.7).

The natural matrix realization of $\mathfrak{sl}(3,\mathbb{C})$ in the Chevalley basis is

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{C.9}$$

One can easily verify that these matrices indeed satisfy the algebra commutation conditions (C.7). We have thus decomposed the algebra as

$$\mathfrak{sl}(3,\mathbb{C}) = \sum_{i=1}^{3} \mathbb{C} f_i \oplus \sum_{i=1}^{2} \mathbb{C} h_i \oplus \sum_{i=1}^{3} \mathbb{C} e_i. \tag{C.10}$$

C.2 The Real Forms of $\mathfrak{sl}(3,\mathbb{C})$

For completeness, we explain how we get all the three real forms of $\mathfrak{sl}(3,\mathbb{C})$. But remember that this appendix is first and foremost devoted to the group SU(2,1) and its Lie algebra $\mathfrak{su}(2,1)$.

C.2.1 $\mathfrak{sl}(3,\mathbb{R})$

The easiest real form to derive is the normal form $\mathfrak{sl}(3,\mathbb{R})$, since, as we have seen before, it is practically given when we know the basis of $\mathfrak{sl}(3,\mathbb{C})$:

$$\mathfrak{sl}(3,\mathbb{R}) = \sum_{i=1}^{3} \mathbb{R} f_i \oplus \sum_{i=1}^{2} \mathbb{R} h_i \oplus \sum_{i=1}^{3} \mathbb{R} e_i, \tag{C.11}$$

with the same generators as above (C.9).

[2] We do not get any more generators because further commutation with the additional step-operators give zero.

C.2.2 $\mathfrak{su}(3)$

The algebra $\mathfrak{su}(3)$ is a compact real form of $\mathfrak{sl}(3,\mathbb{C})$. It is derived using the Cartan involution τ (A.32) on $\mathfrak{sl}(3,\mathbb{C})$ acting on the elements $x \in \mathfrak{sl}(3,\mathbb{C})$ as

$$\tau(x) = -x^\dagger. \tag{C.12}$$

The realization on the elements in the basis (C.9) is as follows:

$$\begin{aligned} &\tau(h_1) = -h_1, \qquad \tau(h_2) = -h_2, \\ \tau(e_1) = -f_1, \qquad &\tau(e_2) = -f_2, \qquad \tau(e_3) = -f_3, \\ \tau(f_1) = -e_1, \qquad &\tau(f_2) = -e_2, \qquad \tau(f_3) = -e_3, \end{aligned} \tag{C.13}$$

and importantly it acts on complex numbers as a complex conjugation, i.e.,

$$\tau(\mathrm{i}) = -\mathrm{i} \qquad \text{so e.g.} \qquad \tau(\mathrm{i}e_3) = \mathrm{i}f_3. \tag{C.14}$$

As required of an automorphism of the algebra, the involution preserves the commutation relations (C.7), e.g., $\tau(e_3) = -f_3$ is consistent because $\tau(e_3) = \sigma([e_1, e_2]) = (-1)^2[f_1, f_2] = -f_3$. There are eight linearly independent operators that are fixed under τ, namely

$$\mathrm{i}h_1,\ \mathrm{i}h_2,\ e_1 - f_1,\ e_2 - f_2,\ e_3 - f_3,\ \mathrm{i}(e_1 + f_1),\ \mathrm{i}(e_2 + f_2) \quad \text{and} \quad \mathrm{i}(e_3 + f_3).$$

The fact that they are fixed under τ, i.e. $X = -X^\dagger$, is exactly the condition on the generators of $\mathfrak{su}(3)$ that we can derive using the standard matrix definition of SU(3),[3] cf. (C.5) and (C.6). Thus

$$\mathfrak{su}(3) = \sum_{i=1}^{2} \mathbb{R}\mathrm{i}h_i \oplus \sum_{i=1}^{3} \mathbb{R}(e_i - f_i) \oplus \sum_{i=1}^{3} \mathbb{R}\mathrm{i}(e_i + f_i). \tag{C.15}$$

C.2.3 $\mathfrak{su}(2,1)$

We merely state the third possible involution[4] σ on $\mathfrak{sl}(3,\mathbb{C})$, and show that the elements fixed by this particular σ really is a basis for the algebra $\mathfrak{su}(2,1)$ (using real parameters). Our σ acts on the Chevalley basis generators as follows:

$$\begin{aligned} &\sigma(h_1) = h_2, \qquad \sigma(h_2) = h_1, \\ \sigma(e_1) = e_2, \qquad &\sigma(e_2) = e_1, \qquad \sigma(e_3) = -e_3, \\ \sigma(f_1) = f_2, \qquad &\sigma(f_2) = f_1, \qquad \sigma(f_3) = -f_3, \end{aligned} \tag{C.16}$$

[3] Note that we normally work with *Hermitian* generators (operators), i.e., $x = x^\dagger$, which is an important property in quantum mechanics. This comes from the fact that we write elements in the group as $\mathrm{e}^{\mathrm{i}\alpha X}$ instead of $\mathrm{e}^{\alpha X}$.

[4] It can be shown that there are only three possible involutions of $\mathfrak{sl}(3,\mathbb{C})$ giving its real forms.

and just like τ in the previous section, σ acts as a complex conjugation on numbers $\sigma(\mathrm{i}) = -\mathrm{i}$. We can form eight linearly independent generators H_1, H_2, X_1, X_2, X_3, Y_1, Y_2 and Y_3 that are fixed under σ

$$H_1 = h_1 + h_2, \qquad H_2 = \mathrm{i}(h_1 - h_2)$$

$$\begin{aligned} X_1 &= e_1 + e_2, & X_2 &= \mathrm{i}(e_1 - e_2), & X_3 &= \mathrm{i}e_3 \\ Y_1 &= f_1 + f_2, & Y_2 &= \mathrm{i}(f_1 - f_2), & Y_3 &= \mathrm{i}f_3. \end{aligned} \tag{C.17}$$

Using the realization (C.9) we write the generators explicitly

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -2\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix},$$
$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}. \tag{C.18}$$

We have eight linearly independent 3×3-matrices in agreement with what is demanded of a basis for $\mathfrak{su}(2,1)$. However, to see that the algebra actually is $\mathfrak{su}(2,1)$, we must also verify that the constraint (C.6) is satisfied when writing an element ξ in the basis (C.18) with parameters: $h_1, h_2, x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$, i.e.,

$$\begin{aligned} \xi &= h_1 H_1 + h_2 H_2 + x_1 X_1 + x_2 X_2 + x_3 X_3 + y_1 Y_1 + y_2 Y_2 + h Y_3 = \\ &= \begin{pmatrix} h_1 + \mathrm{i}h_2 & x_1 + \mathrm{i}x_2 & \mathrm{i}x_3 \\ y_1 + \mathrm{i}y_2 & -2\mathrm{i}h_2 & x_1 - \mathrm{i}x_2 \\ \mathrm{i}y_3 & y_1 - \mathrm{i}y_2 & -h_1 + \mathrm{i}h_2 \end{pmatrix}. \end{aligned} \tag{C.19}$$

Here an important question immediately pops up. What kind of metric G should we use in the equation (C.6), i.e., what metric G fits together with our newly found basis (C.18)? As was mentioned before, we could perform a similarity transformation and get different kinds of representations of the same group, preserving different kinds of metrics. This freedom manifests itself in this context as different ways of defining the involution σ, and different ways of realizing the algebra. There is nothing that says that G must be the natural metric (C.1). In fact, with our choice of realization (C.9) and involution (C.16), the basis (C.18) turns out to satisfy

$$\xi^\dagger G + G\xi = 0 \qquad \text{for} \qquad G = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{C.20}$$

This shows that the real form of $\mathfrak{sl}(3,\mathbb{C})$ formed by considering the fixed elements under the involution σ, as stated in (C.16), is indeed the algebra $\mathfrak{su}(2,1)$. A set of generators is given explicitly in (C.18). If we are interested in another basis, we just make a similarity transformation $\xi \to S\xi S^{-1}$ and $G \to SGS^{-1}$.

C.3 The Maximal Compact Subgroup of $\mathrm{SU}(2,1)$

We derive the subgroup from the algebra. The maximal compact subalgebra $\mathfrak{k}_0$ is defined as the subset of $\mathfrak{su}(2,1)$, fixed under the Cartan involution τ that we used to derive $\mathfrak{su}(3)$ above (remember that the Cartan involution selects the compact part of an algebra):

$$\mathfrak{k}_0 = \{x \in \mathfrak{su}(2,1) \mid \tau(x) = x\}. \tag{C.21}$$

Using the knowledge of how τ acts on the original basis generators, see (C.13) and (C.14), we can derive how it acts on the basis generators (C.17) for $\mathfrak{su}(2,1)$:

$$\begin{gathered} \tau(H_1) = -H_1, \qquad \tau(H_2) = H_2, \\ \tau(X_1) = -Y_1, \qquad \tau(X_2) = Y_2, \qquad \tau(X_3) = Y_3, \\ \tau(Y_1) = -X_1, \qquad \tau(Y_2) = X_2, \qquad \tau(Y_3) = X_3. \end{gathered} \tag{C.22}$$

It is straightforward to form all possible linearly independent invariant and anti-invariant combinations of the generators (the two Cartan generators are trivially already invariant and anti-invariant respectively). These are

$$\begin{aligned} &H_2, \\ &K_1 = X_1 - Y_1, \\ &K_2 = X_2 + Y_2, \\ &K_3 = X_3 + Y_3 \end{aligned} \tag{C.23}$$

and

$$\begin{aligned} &H_1, \\ &P_1 = X_1 + Y_1, \\ &P_2 = X_2 - Y_2, \\ &P_3 = X_3 - Y_3. \end{aligned} \tag{C.24}$$

The Cartan decomposition reads

$$\mathfrak{su}(2,1) = \mathfrak{k}_0 \oplus \mathfrak{p}_0 = \left(\mathbb{R}H_2 \oplus \sum_{i=1}^{3} \mathbb{R}K_i\right) \oplus \left(\mathbb{R}H_1 \oplus \sum_{i=1}^{3} \mathbb{R}P_i\right). \tag{C.25}$$

We are interested in the maximal compact subalgebra

$$\mathfrak{k}_0 = \mathbb{R}H_2 \oplus \sum_{i=1}^{3} \mathbb{R}K_i. \tag{C.26}$$

To reveal what this is, we calculate the commutation relations between the basis elements. This can be done by using either the commutation relations for the original basis elements for $\mathfrak{sl}(3,\mathbb{C})$, see (C.7), or by using the matrix realization (C.18) and calculate explicitly. It turns out that the commutation relations are

$$\begin{aligned} [H_2, K_1] &= 3K_2 & [K_3, K_1] &= -K_2 \\ [H_2, K_2] &= -3K_1 & [K_3, K_2] &= K_1 \\ [H_2, K_3] &= 0 & [K_1, K_2] &= 2(H_2 - K_3). \end{aligned}$$

No generator is diagonal in this basis and we cannot directly tell anything about the algebra from these relations. We have to make a suitable change of basis, and a particularly good one is as follows:

$$\begin{aligned} \tilde{u} &= \frac{1}{2}K_3 + \frac{1}{6}H_2 \\ \tilde{k}_1 &= \frac{1}{2}(H_2 - K_3) \\ \tilde{k}_2 &= \frac{1}{\sqrt{2}}K_1 \\ \tilde{k}_3 &= \frac{1}{\sqrt{2}}K_2. \end{aligned} \tag{C.27}$$

The commutation relations for the new basis elements are

$$\begin{aligned} [\tilde{u}, \tilde{k}_i] &= 0, \\ \left[\tilde{k}_i, \tilde{k}_j\right] &= 2\epsilon_{ijk}\tilde{k}_k, \qquad i,j,k = 1,2,3. \end{aligned}$$

Firstly, we see that since $\tilde{u}$ commutes with all other basis elements, i.e., it is an independent algebra. The algebra is $\mathfrak{u}(1)$ since $\tilde{u} = -\tilde{u}^\dagger$, which is the constraint on generators to a unitary group. Secondly, $\{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3\}$ is a basis to $\mathfrak{su}(2)$ since the commutation relations are fulfilled and also the constraints $\mathrm{Tr}(\tilde{k}_i) = 0$ and $\tilde{k}_i = -\tilde{k}_i^\dagger$. I.e.,

$$\mathfrak{k}_0 = \mathfrak{u}(1) \oplus \mathfrak{su}(2). \tag{C.28}$$

Finally, to get the maximal compact subgroup K to SU(2, 1) we just exponentiate the maximal compact subalgebra $\mathfrak{k}_0$

$$\mathrm{K} = \mathrm{e}^{\mathbb{R}\tilde{u}}\mathrm{e}^{\sum_{i=1}^{3}\mathbb{R}\tilde{k}_i} = \mathrm{U}(1) \times \mathrm{SU}(2). \tag{C.29}$$

C.4 Another Basis for $\mathfrak{su}(2,1)$

In the second case treated in this thesis, when we study Type IIA superstring theory compactified on a rigid Calabi-Yau threefold, it turns out that we should use another basis for $\mathfrak{su}(2,1)$ than the derived (C.18). Namely, one preserving instead the metric

$$G = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 1 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}. \tag{C.30}$$

The similarity transformation matrix S, taking us to this basis from the one derived above, is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}. \tag{C.31}$$

Acting on the generators (C.18) (as SXS^{-1}), we get the new generators (the Cartan generators are obviously unchanged)

$$\begin{aligned} H_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -2\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \\ X_{(1)} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{X}_{(1)} = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{(-2)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Y_{(-1)} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \quad \tilde{Y}_{(-1)} = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y_{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{C.32}$$

where we have now for convenience chosen to let the subscript be the eigenvalue of the generator in question when taking the Lie bracket with the non-compact Cartan generator H_1, e.g., $[H_1, X_{(-2)}] = -2X_{(-2)}$. Finally, to get to the right basis we need only to make a linear combination of the step-operators $X_{(1)}$ and $\tilde{X}_{(1)}$ as well as a sign change of $X_{(2)}$ (the same goes for the operators Y):

$$\begin{aligned} X_{(1)} &= \begin{pmatrix} 0 & -1+\mathrm{i} & 0 \\ 0 & 0 & 1-\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{X}_{(1)} = \begin{pmatrix} 0 & 1+\mathrm{i} & 0 \\ 0 & 0 & 1+\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Y_{(-1)} &= \begin{pmatrix} 0 & 0 & 0 \\ 1+\mathrm{i} & 0 & 0 \\ 0 & -1-\mathrm{i} & 0 \end{pmatrix}, \quad \tilde{Y}_{(-1)} = \begin{pmatrix} 0 & 0 & 0 \\ -1+\mathrm{i} & 0 & 0 \\ 0 & -1+\mathrm{i} & 0 \end{pmatrix}, \quad Y_{(-2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{C.33}$$

We do not bother renaming the generators; these are the ones which we will use henceforth.

C.5 Iwasawa Decomposition of $\mathrm{SU}(2,1)$

We now invoke the Iwasawa decomposition (B.10) of $\mathrm{SU}(2,1)$ to construct a coset representative for $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times \mathrm{U}(1))$. The algebra is decomposed as

$$\mathfrak{su}(2,1) = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}_0, \tag{C.34}$$

where $\mathfrak{n}_0$ is the nilpotent subalgebra, which we choose to consist of the positive step-operators $\{X_{(1)}, \tilde{X}_{(1)}, X_{(2)}\}$, $\mathfrak{a}_0$ is the non-compact abelian Cartan subalgebra spanned by H_1 and $\mathfrak{k}_0 = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is the maximal compact subalgebra derived in Section C.3. We can directly verify that this decomposition is legitimate by just counting the number of linearly independent generators, which is eight as it should be.

Now, a coset representative V for $\mathrm{SU}(2,1)/(\mathrm{SU}(2) \times \mathrm{U}(1))$ is formed, as in Section B.2, by throwing away the compact part and exponentiating only $\mathfrak{n}_0 \oplus \mathfrak{a}_0$:

$$V = NA = \mathrm{e}^{\chi X_{(1)} + \tilde{\chi}\tilde{X}_{(1)} + 2\psi X_{(2)}} \mathrm{e}^{-\phi H_1} \qquad \phi, \chi, \tilde{\chi}, \psi \in \mathbb{R}. \tag{C.35}$$

The parameters (fields), in particular the factor 2 with ψ, are chosen to fit with the physics. The axion fields are: $\chi, \tilde{\chi}$ and ψ, while ϕ is the dilaton as usual. We calculate V explicitly by using the representation (C.33) (with H_1 from (C.32)):

$$V = \exp(\Xi) \exp\left[\begin{pmatrix} -\phi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \phi \end{pmatrix}\right], \tag{C.36}$$

where we have put

$$\Xi = \chi \begin{pmatrix} 0 & -1+\mathrm{i} & 0 \\ 0 & 0 & 1-\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix} + \tilde{\chi} \begin{pmatrix} 0 & 1+\mathrm{i} & 0 \\ 0 & 0 & 1+\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix} + 2\psi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & \chi(-1+\mathrm{i}) + \tilde{\chi}(1+\mathrm{i}) & 2\psi \\ 0 & 0 & \chi(1-\mathrm{i}) + \tilde{\chi}(1+\mathrm{i}) \\ 0 & 0 & 0 \end{pmatrix}.$$

We have

$$\Xi^2 = \begin{pmatrix} 0 & 0 & 2\mathrm{i}(\chi^2 + \tilde{\chi}^2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{C.37}$$

and $\Xi^3 = 0$. Since $\mathrm{e}^{\Xi} = \mathbb{1} + \Xi + \frac{1}{2}\Xi^2 + \mathcal{O}(\Xi^3)$ we get

$$V = \begin{pmatrix} 1 & \chi(-1+\mathrm{i}) + \tilde{\chi}(1+\mathrm{i}) & 2\psi + \mathrm{i}(\chi^2 + \tilde{\chi}^2) \\ 0 & 1 & \chi(1-\mathrm{i}) + \tilde{\chi}(1+\mathrm{i}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\phi} \end{pmatrix} =$$
$$= \begin{pmatrix} \mathrm{e}^{-\phi} & -\chi + \tilde{\chi} + \mathrm{i}(\chi + \tilde{\chi}) & \mathrm{e}^{\phi}(\mathrm{i}(\chi^2 + \tilde{\chi}^2) + 2\psi) \\ 0 & 1 & \mathrm{e}^{\phi}(\chi + \tilde{\chi} + \mathrm{i}(-\chi + \tilde{\chi})) \\ 0 & 0 & \mathrm{e}^{\phi} \end{pmatrix}, \tag{C.38}$$

which is our coset representative in the Borel gauge.

C.6 The Coset Space $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$

Although we will not delve into this topic in this thesis, see instead [9], one can show that the coset $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$ is isomorphic to the complex hyperbolic upper half plane[5]

$$\mathbb{CH}^2 \cong \mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1)), \tag{C.39}$$

parameterized by $(z_1, z_2) \in \mathbb{C}^2$, and obeying the following constraint

$$\mathcal{F}(z_1, z_2) \equiv \Im(z_1) - \frac{1}{2}|z_2|^2 > 0, \tag{C.40}$$

where $\mathcal{F}$ is called the *height function*. The relation between the complex variables and the real variables used above is

$$\begin{aligned} z_1 &= 2\psi + \mathrm{i}\left(\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2\right) \\ z_2 &= \chi + \tilde{\chi} + \mathrm{i}(\tilde{\chi} - \chi). \end{aligned} \tag{C.41}$$

An important property of the coset is that it is a quaternionic-Kähler manifold, as was mentioned in Section 2.2, and it is Kähler with the Kähler potential

$$K(z_1, z_2) = -\log(\mathcal{F}(z_1, z_2)). \tag{C.42}$$

C.7 Heisenberg Translations

The nilpotent subgroup $\mathrm{N}(\mathbb{R})$ in C.35, which is formed by exponentiating the positive step operators: $X_{(1)}$, $\tilde{X}_{(1)}$ and $X_{(2)}$, is of great importance. This will become apparent when studying the discrete subgroup $\mathrm{N}(\mathbb{Z}) \subset \mathrm{N}(\mathbb{R})$, which is also a subgroup of the Picard modular group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ treated in Appendix E. As explained in Section 4.1, the discrete group $\mathrm{N}(\mathbb{Z})$ is essential when determining the general Fourier series for the automorphic forms.

The general nilpotent transformation is

$$\begin{aligned} n &= \mathrm{e}^{aX_{(1)} + b\tilde{X}_{(1)} + cX_{(2)}} = \\ &= \begin{pmatrix} 1 & a(-1+\mathrm{i}) + b(1+\mathrm{i}) & c + \mathrm{i}(a^2+b^2) \\ 0 & 1 & a(1-\mathrm{i}) + b(1+\mathrm{i}) \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R}. \end{aligned} \tag{C.43}$$

These transformations are also called *Heisenberg translations* since they form a Heisenberg subgroup. They apparently leave the coset representative $V = NA$ in the Borel gauge, and acting on V we see that they transform the fields in the following way

$$\begin{aligned} \phi &\longmapsto \phi \\ \chi &\longmapsto \chi + a \\ \tilde{\chi} &\longmapsto \tilde{\chi} + b \\ \psi &\longmapsto \psi + \frac{1}{2}c - a\tilde{\chi} + b\chi. \end{aligned} \tag{C.44}$$

[5] Cf. the isomorphism $\mathbb{CH} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

The dilaton field ϕ is unaffected since it sits safe in the Abelian part A in V, unreachable from an upper triangular operator acting from the left. The Heisenberg translations are non-Abelian as can be seen for the transformation of the field ψ that also includes the fields χ and $\tilde{\chi}$.

D

The Modular Group $\mathrm{SL}(2,\mathbb{Z})$

The modular group $\mathrm{SL}(2,\mathbb{Z})$ is naturally represented by two-by-two matrices with integer entries and unit determinant

$$\mathrm{SL}(2,\mathbb{Z}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det g = 1, \qquad a,b,c,d \in \mathbb{Z}. \tag{D.1}$$

It is a group with matrix multiplication as composition rule since the unit determinant property is unchanged under multiplication, and the inverse to the general element g is given by

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{D.2}$$

Similarly as for the Lie group $\mathrm{SL}(2,\mathbb{R})$, mentioned in Section 2.1, the discrete group acts on the complex upper half-plane as

$$g \cdot \tau = \frac{a\tau + b}{c\tau + d}. \tag{D.3}$$

And as in the case of $\mathrm{SL}(2,\mathbb{R})$, τ is invariant under the action of the element $-\mathbb{1} \in \mathrm{SL}(2,\mathbb{R})$, so to get a one-to-one correspondence between the group elements and the Möbius transformation, which we are interested in to connect to the physics, we need to consider $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm\mathbb{1}\}$ instead. By sloppiness we neglect the letter P.

The discrete group can be generated by various combinations of the two fundamental elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{D.4}$$

This can be proven by showing that all matrices in $\mathrm{SL}(2,\mathbb{Z})$ can be transformed to the identity matrix with help of combinations of S and T in a certain way

– it is a good exercise! For a more careful mathematical treatment of the generators of $\mathrm{SL}(2,\mathbb{Z})$ the reader is referred to [12].

The identity element $1 = \{\pm\mathbb{1}\}$ is given by

$$S^2 = 1 \qquad \text{and} \qquad (ST)^3 = 1. \tag{D.5}$$

Therefore, for example the inverse to T is given by

$$T^{-1} = (ST)^2 S = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{D.6}$$

The transformations act on τ as

$$T \cdot \tau = \tau + 1 \qquad \text{and} \qquad S \cdot \tau = -\frac{1}{\tau}. \tag{D.7}$$

I.e., T is a translation and S is an involution.

By only considering the translation T together with its inverse $T^{-1} = (ST)^2 S$ one can form a nilpotent subgroup $\mathrm{N}(\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{Z})$

$$\mathrm{N}(\mathbb{Z}) \ni g = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad n \in \mathbb{Z}, \tag{D.8}$$

which then also is a subgroup of the continuous nilpotent group $\mathrm{N}(\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{R})$, gained from the Iwasawa decomposition G=NAK of $\mathrm{SL}(2,\mathbb{R})$. The action of $\mathrm{N}(\mathbb{Z})$ on the axion χ is simply

$$\chi \longmapsto \chi + n, \qquad n \in \mathbb{Z}. \tag{D.9}$$

E

The Picard Modular Group $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$

In this appendix we explain more of the formalities of the discrete Picard modular group. We follow [9].

The defining representation of the Picard modular group is consisting of the matrices g with the following properties

$$g = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \qquad \det g = 1, \qquad g^\dagger \eta g = \eta \quad \text{and} \\ a,b,c,d,e,f,g,h,i \in \mathbb{Z}[\mathrm{i}]. \tag{E.1}$$

Here η is the metric, explicitly our choice is

$$\eta = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 1 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \tag{E.2}$$

i.e., the same metric as was argued for in the case of the Lie group $\mathrm{SU}(2,1)$, see Section C.4. The set $\mathbb{Z}[\mathrm{i}]$ is the *gaussian* integers

$$\mathbb{Z}[\mathrm{i}] = \{z \in \mathbb{C} \,|\, \Re(z), \Im(z) \in \mathbb{Z}\}. \tag{E.3}$$

To verify the group properties we first note that the unit determinant property is preserved by matrix multiplication. This also holds for the metric condition, since given two elements g and h in $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ we have

$$(gh)^\dagger \eta (gh) = h^\dagger g^\dagger \eta g h = h^\dagger \eta h = \eta. \tag{E.4}$$

An inverse to g in (E.1) is given by

$$g^{-1} = \begin{pmatrix} ei - hf & ch - bi & bf - ec \\ gf - di & ai - gc & dc - af \\ dh - ge & gb - ah & ae - db \end{pmatrix}. \tag{E.5}$$

E.1 T, R and S Transformations

We are interested in finding the nilpotent subgroup $\mathrm{N}(\mathbb{Z})$ for the Fourier expansion on the moduli space $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])\backslash\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$. Fortunately, we did most of the work already in Section C.7 where we found the nilpotent subgroup of $\mathrm{SU}(2,1)$. One can see directly by looking at the general transformation C.43, that the discrete nilpotent group is gained by only considering integer parameters a, b and c. I.e.,

$$\begin{aligned}\mathrm{N}(\mathbb{Z})\ni n =& \mathrm{e}^{aX_{(1)}+b\tilde{X}_{(1)}+cX_{(2)}} = \\ =& \begin{pmatrix}1 & a(-1+\mathrm{i})+b(1+\mathrm{i}) & c+\mathrm{i}(a^2+b^2)\\ 0 & 1 & a(1-\mathrm{i})+b(1+\mathrm{i})\\ 0 & 0 & 1\end{pmatrix},\\ & a,b,c\in\mathbb{Z}.\end{aligned} \tag{E.6}$$

And the resulting action on the fields is

$$\begin{aligned}\phi &\longmapsto \phi\\ \chi &\longmapsto \chi + a\\ \tilde{\chi} &\longmapsto \tilde{\chi}+b\\ \psi &\longmapsto \psi+\frac{1}{2}c - a\tilde{\chi}+b\chi, \qquad a,b,c\in\mathbb{Z}.\end{aligned} \tag{E.7}$$

In contrast with the relatively simple group $\mathrm{SL}(2,\mathbb{Z})$, the discrete nilpotent group is non-abelian. This complicates the derivation of the Fourier series. The nilpotent group $\mathrm{N}(\mathbb{Z})$ is isomorphic to the discrete Heisenberg group $\mathrm{H}_3(\mathbb{Z})$ in three dimensions. The elements are also called translations since they translate the fields, although the transformation on ψ is not really an ordinary translation for a and b not equal to zero. We will denote a general Heisenberg translation by $T_{(a,b,c)}$. The translations corresponding to putting $(a,b,c)=(1,0,0)$, $(a,b,c)=(0,1,0)$ and $(a,b,c)=(0,0,1)$ are called T_1, $\tilde{T}_1$ and T_2 respectively. These elements generate the whole of $\mathrm{H}_3(\mathbb{Z})$.

There are two other types of transformations in $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ of great interest. First there is a rotation, which for $\mathrm{SU}(2,1)$ is formed by exponentiating the compact Cartan generator H_2, and for $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$ is restricted to[1]

$$R = m\begin{pmatrix}\mathrm{i} & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & \mathrm{i}\end{pmatrix}, \qquad m=0,1,2,3. \tag{E.8}$$

On the scalar fields it acts as

$$(\chi,\tilde{\chi}) \longmapsto (-\tilde{\chi},\chi), \tag{E.9}$$

and it is apparent that the transformation is of order 4.

[1] In the case of $\mathrm{SU}(2,1)$ the parameter m can take all values in the interval $[0,4)$.

Secondly, there is an involution S (which is also an element in $\mathrm{SU}(2,1)$)

$$S = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & -1 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \tag{E.10}$$

that acts on the fields in a rather non-trivial way

$$\begin{aligned}
\phi &\longmapsto -\frac{1}{2}\ln\left[\frac{\mathrm{e}^{-2\phi}}{4\psi^2 + (\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2)^2}\right], \\
\chi &\longmapsto \frac{2\psi\tilde{\chi} - (\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2)\chi}{4\psi^2 + (\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2)^2}, \\
\tilde{\chi} &\longmapsto \frac{2\psi\chi + (\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2)\tilde{\chi}}{4\psi^2 + (\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2)^2}, \\
\psi &\longmapsto -\frac{\psi}{4\psi^2 + (\mathrm{e}^{-2\phi} + \chi^2 + \tilde{\chi}^2)^2}.
\end{aligned} \tag{E.11}$$

However, when considering the complex parameters C.41 instead, the effect of the transformation is more simple

$$z_1 \longmapsto -\frac{1}{z_1}, \qquad z_2 \longmapsto -\mathrm{i}\frac{z_2}{z_1}. \tag{E.12}$$

And one can verify that the transformation indeed is of order 2, worthy of the name involution.

E.2 Generating Group Elements

There is an important theorem by Francsics and Lax [20] that says that the Picard modular group is generated by the elements: T_1, T_2, R and S. One can understand why the translation $\tilde{T}_1$ is not needed by noting that $\tilde{T}_1 = RT_1R^{-1}$. However, from a physical point of view it is natural to construct the invariant functions with help of all three types of translations, and we will therefore use a non-minimal representation of $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$:

$$T_1 = \begin{pmatrix} 1 & -1+\mathrm{i} & \mathrm{i} \\ 0 & 1 & 1-\mathrm{i} \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 1 & 1+\mathrm{i} & \mathrm{i} \\ 0 & 1 & 1+\mathrm{i} \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & -1 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}. \tag{E.13}$$

A function is invariant under $\mathrm{SU}(2,1;\mathbb{Z}[\mathrm{i}])$-transformations, if and only if it is invariant under these five transformations. When constructing the Fourier series expansion of a general invariant function, we will first use the invariance under the discrete nilpotent subgroup (discrete Heisenberg group) generated by the first three elemensts. We will then see how R restricts the coefficients of the Fourier series even more. However, we will not be able to take into account the invariance under the involution S (that would determine the unknown function exactly); this particular transformation proves to be very difficult to use on the Fourier series[2].

[2]Further investigation could perhaps be fruitful.

F

Laplace-Beltrami Operators

F.1 On $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$

The metric on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\phi} \end{pmatrix}. \tag{F.1}$$

The Laplace-Beltrami operator acting on scalar functions is by definition

$$\Delta = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j). \tag{F.2}$$

In our case we have

$$g = \mathrm{e}^{2\phi} \implies \sqrt{|g|} = \mathrm{e}^{\phi}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-2\phi} \end{pmatrix}$$

and

$$\begin{aligned} \Delta = \mathrm{e}^{-\phi}\partial_i(\mathrm{e}^{\phi}g^{ij}\partial_j) &= \mathrm{e}^{-\phi}[\partial_\phi(\mathrm{e}^{\phi}\partial_\phi) + \partial_\chi(\mathrm{e}^{\phi}\mathrm{e}^{-2\phi}\partial_\chi)] = \\ &= \mathrm{e}^{-\phi}[\mathrm{e}^{\phi}\partial_\phi + \mathrm{e}^{\phi}\partial_\phi^2 + \mathrm{e}^{-\phi}\partial_\chi^2] = \partial_\phi + \partial_\phi^2 + \mathrm{e}^{-2\phi}\partial_\chi^2. \end{aligned} \tag{F.3}$$

Expressing instead the Laplace-Beltrami operator using the complex coordinate $\tau = \tau_1 + \mathrm{i}\tau_2 = \chi + \mathrm{i}\mathrm{e}^{-\phi}$, we get

$$\begin{aligned} \partial_\phi &= \frac{\partial \tau_2}{\partial \phi}\frac{\partial}{\partial \tau_2} = -\mathrm{e}^{-\phi}\frac{\partial}{\partial \tau_2} = -\tau_2\partial_{\tau_2} \\ \partial_\phi^2 &= \tau_2\partial_{\tau_2}(\tau_2\partial_{\tau_2}) = \tau_2(\partial_{\tau_2} + \tau_2\partial_{\tau_2}^2) \implies \end{aligned}$$

$$\Delta = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2). \tag{F.4}$$

F.2 On $\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))$

The metric on the coset space is

$$\mathrm{d}s^2 = \mathrm{d}\phi^2 + \mathrm{e}^{2\phi}(\mathrm{d}\chi^2 + \mathrm{d}\tilde{\chi}^2) + \mathrm{e}^{4\phi}(\mathrm{d}\psi + \chi\mathrm{d}\tilde{\chi} - \tilde{\chi}\mathrm{d}\chi)^2. \tag{F.5}$$

We make a variable change to simplify the calculation:

$$\begin{aligned} y = \mathrm{e}^{-2\phi} \implies \mathrm{d}\phi = -\frac{1}{2y}\mathrm{d}y \implies \mathrm{d}\phi^2 = \frac{1}{4y^2}\mathrm{d}y^2 \\ \implies \mathrm{d}s^2 = \frac{1}{4y^2}\mathrm{d}y^2 + y^{-1}(\mathrm{d}\chi^2 + \mathrm{d}\tilde{\chi}^2) + y^{-2}(\mathrm{d}\psi + \chi\mathrm{d}\tilde{\chi} - \tilde{\chi}\mathrm{d}\chi)^2. \end{aligned} \tag{F.6}$$

We have the metric tensor

$$g_{ij} = \begin{pmatrix} \frac{1}{4}y^{-2} & 0 & 0 & 0 \\ 0 & y^{-1} + y^{-2}\tilde{\chi}^2 & -y^{-2}\chi\tilde{\chi} & -y^{-2}\tilde{\chi} \\ 0 & -y^{-2}\chi\tilde{\chi} & y^{-1} + y^{-2}\chi^2 & y^{-2}\chi \\ 0 & -y^{-2}\tilde{\chi} & y^{-2}\chi & y^{-2} \end{pmatrix}, \quad i,j = (y,\chi,\tilde{\chi},\psi). \tag{F.7}$$

Determinant:

$$g = \frac{1}{4y^6} \qquad (> 0). \tag{F.8}$$

Inverse metric:

$$g^{ij} = \begin{pmatrix} 4y^2 & 0 & 0 & 0 \\ 0 & y & 0 & \tilde{\chi}y \\ 0 & 0 & y & -\chi y \\ 0 & \tilde{\chi}y & -\chi y & y(\chi^2 + \tilde{\chi}^2 + y) \end{pmatrix}, \quad i,j = (y,\chi,\tilde{\chi},\psi). \tag{F.9}$$

Straightforward calculation using the definition of the Laplace-Beltrami operator (F.2) gives:

$$\begin{aligned} &\Delta_{\mathrm{SU}(2,1)/(\mathrm{SU}(2)\times\mathrm{U}(1))} = \\ &= 4\left(y^2\partial_y^2 - y\partial_y + \frac{1}{4}y(\partial_\chi^2 + \partial_{\tilde{\chi}}^2) + \frac{1}{2}y(\tilde{\chi}\partial_\chi - \chi\partial_{\tilde{\chi}})\partial_\psi + \frac{1}{4}y(y + \chi^2 + \tilde{\chi}^2)\partial_\psi^2\right). \end{aligned} \tag{F.10}$$

G

Poisson Resummation Formula

In this appendix we will derive the Poisson resummation identity

$$\sum_n \mathrm{e}^{-\frac{\pi}{x}(m\tau_1+n)^2} = \sqrt{x}\sum_{\tilde{n}} \mathrm{e}^{-\pi x\tilde{n}^2-2\pi\mathrm{i}\tilde{n}m\tau_1}. \tag{G.1}$$

used in the Type IIB case for rewriting the expression (5.34).[1] We start with the general Poisson summation formula [35]

$$\sum_n f(t+nT) = \frac{1}{T}\sum_k \hat{f}\left(\frac{k}{T}\right)\mathrm{e}^{2\pi\mathrm{i}\frac{k}{T}t}, \tag{G.2}$$

where

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\mathrm{d}x, \tag{G.3}$$

is the ordinary Fourier transform of f. Using the formula (G.2) for the function $f(t) = \exp(-\alpha t^2)$ (for $\alpha > 0$), whose Fourier transform is $\hat{f}(\xi) = \sqrt{\frac{\pi}{\alpha}}\exp(-\frac{(\pi\xi)^2}{\alpha})$, yields

$$\sum_n \mathrm{e}^{-\alpha(t+nT)^2} = \sum_n \mathrm{e}^{-\alpha T^2\left(\frac{t}{T}+n\right)^2} = \frac{1}{T}\sum_k \sqrt{\frac{\pi}{\alpha}}\,\mathrm{e}^{-\frac{\pi^2}{\alpha}\left(\frac{k}{T}\right)^2+2\pi\mathrm{i}\frac{k}{T}t}. \tag{G.4}$$

To connect to (G.1) we identify first

$$\frac{\pi}{T^2\alpha} = x > 0, \tag{G.5}$$

giving

$$\sum_n \mathrm{e}^{-\frac{\pi}{x}\left(\frac{t}{T}+n\right)^2} = \sqrt{x}\sum_k \mathrm{e}^{-\pi x k^2+2\pi\mathrm{i}k\frac{t}{T}}. \tag{G.6}$$

[1] It is also used in the Type IIA case but we gloss over this part.

Finally, we see that we get (G.1) if we also put

$$\frac{t}{T} = m\tau_1. \tag{G.7}$$

H

The Whittaker Equation and Functions

In Chapter 4, we arrived to a Whittaker differential equation when separating the solution to the Laplace eigenfunction equation for the non-abelian term $f^{(\mathrm{NA})}(y, \chi, \tilde{\chi}, \psi)$. This appendix is devoted to deepen our understanding of the Whittaker equation and its solving functions.

H.1 Solutions to the Differential Equation

The starting differential equation, see (6.69), was

$$y^2 \partial_y^2 Y - y \partial_y Y - (4\pi^2 k^2 y^2 + 2\pi |k|(2r+1)y + \lambda) Y = 0. \tag{H.1}$$

After the extraction of a factor $\sqrt{y}$, i.e., setting $Y(y) = \sqrt{y}\,\hat{Y}(y)$, and a change of variables, $\hat{y} = 4\pi|k|y$, we arrived to the Whittaker equation, see (6.74):

$$\partial_{\hat{y}}^2 \hat{Y}(\hat{y}) + \left(-\frac{1}{4} + \frac{1}{\hat{y}} \left(-r - \frac{1}{2} \right) - \frac{1}{\hat{y}^2} \left(\lambda + \frac{3}{4} \right) \right) \hat{Y}(\hat{y}) = 0. \tag{H.2}$$

Using the known expression for the eigenvalue in terms of the order of the Eisenstein series $\lambda = s(s-2)$ yields

$$\partial_{\hat{y}}^2 \hat{Y}(\hat{y}) + \left(-\frac{1}{4} + \frac{1}{\hat{y}} \left(-r - \frac{1}{2} \right) - \frac{1}{\hat{y}^2} \left(\frac{1}{4} - (s-1)^2 \right) \right) \hat{Y}(\hat{y}) = 0. \tag{H.3}$$

Comparing to the general form of the Whittaker differential equation, see (6.70):

$$M''_{p\mu}(x) + \left(-\frac{1}{4} + \frac{p}{x} + \frac{\frac{1}{4} - \mu^2}{x^2} \right) M_{p\mu}(x) = 0, \tag{H.4}$$

we identify the coefficients

$$\begin{aligned} p &= -r - \frac{1}{2} \\ \mu &= \pm(s-1). \end{aligned} \tag{H.5}$$

The two linearly independent solutions to the Whittaker equation (H.4) are

$$M_{p,\mu}(x) = \mathrm{e}^{-x/2} x^{\mu+\frac{1}{2}} M\left(\mu - p + \frac{1}{2}, 2\mu+1; x\right) \qquad \text{and} \tag{H.6}$$

$$W_{p,\mu}(x) = \mathrm{e}^{-x/2} x^{\mu+\frac{1}{2}} U\left(\mu - p + \frac{1}{2}, 2\mu+1; x\right), \tag{H.7}$$

where the confluent hypergeometric functions M and U have the following form [4]:

$$\begin{aligned} M(a,c;x) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!} \qquad \text{and} \\ U(a,c;x) &= \frac{\pi}{\sin(\pi c)} \left[\frac{M(a,c;x)}{\Gamma(a-c+1)\Gamma(c)} - \frac{x^{1-c} M(a+1-c, 2-c; x)}{\Gamma(a)\Gamma(2-c)} \right]. \end{aligned} \tag{H.8}$$

Here we use the *Pochhammer symbols* to simplify the expressions

$$\begin{aligned} (a)_n = a(a+1)(a+2)...(a+n-1) = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} \\ (a)_0 = 1. \end{aligned} \tag{H.9}$$

The confluent hypergeometric functions are solutions to the *confluent hypergeometric equation*[1]:

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \tag{H.10}$$

As we see from the solutions (H.6), the confluent hypergeometric equation and the Whittaker equation are intimately related. Indeed, one obtains the Whittaker equation from the confluent hypergeometric equation when making the latter self-adjoint[2].

One immediately notices that not all values of the parameter c are allowed in the expressions for the solutions. This, since the confluent hypergeometric functions are undefined when $c = 0, -1, -2, ...$, for these values of c there is a zero in the denominator in the expansion of $M(a,c;x)$. Moreover, there is a possibility of a positive or negative infinity in the denominators in the expression for $U(a,c;x)$, depending on the Gamma functions, as well as a zero in the denominator due to the sinus function for *all* integer values of c. For these values of c that brings the solution to an indeterminate form, one has to consider a limit value, if it exists. As we will see below, we will get an indeterminate expression when we choose $s = 3/2$, which is the interesting case we want to study.

[1] Also known as *Kummer's equation*.

[2] The condition for a self-adjoint operator L is that $\langle u|Lu\rangle = \langle Lu|u\rangle$.

H.2 Asymptotic Behaviour of the Solutions

We are interested in solutions to the Whittaker equation that are well-behaved in the limit $y \longrightarrow \infty$, i.e., $x \longrightarrow \infty$. Beginning with the functions $M_{p\mu}(x)$ one can derive an asymptotic expansion valid for large x and $\Re(c) > \Re(a) > 0$, see [4]:

$$\frac{\Gamma(c)}{\Gamma(a)}\frac{\mathrm{e}^x}{x^{c-a}}\left(1+\frac{(1-a)(c-a)}{1!x}+\frac{(1-a)(2-a)(c-a)(c-a+1)}{2!x^2}+...\right), \tag{H.11}$$

where Γ is the usual *Gamma function*. The asymptotic behaviour for $M_{p\mu}(x)$ is then

$$M_{p\mu}(x) \propto \frac{\mathrm{e}^{x/2}}{x^{c-a-(\mu+1/2)}}(1+\mathcal{O}(x^{-1})). \tag{H.12}$$

Clearly, the function diverges for $x \longrightarrow \infty$ and is therefore not physically acceptable and can directly be ruled out. On the other hand the functions $U(a,c;x)$ have the asymptotic expansion, for $\Re(a) > 0$:

$$\frac{1}{x^a}\left(1+\frac{a(1+a-c)}{1!(-x)}+\frac{a(a+1)(1+a-c)(2+a-c)}{2!(-x)^2}+\dots\right). \tag{H.13}$$

So, for the second solution we have for large x:

$$W_{p,\mu}(x) = \mathrm{e}^{-x/2}x^{\mu+\frac{1}{2}-a}(1+\mathcal{O}(x^{-1})) \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \infty, \tag{H.14}$$

which is acceptable. Let us now therefore concentrate on the second solution.

H.3 The Kummer Transformation

We will now show that

$$W_{p,\mu} = W_{p,-\mu}, \tag{H.15}$$

as was mentioned in the end of Section 6.1.3. We will use the relationship between M and U in (H.8). For the so-called logarithmic case, described in the next section, the relation (H.15) still holds, but it must be shown using instead an integral representation of U (H.20).

Let $a_+ = \mu - p + 1/2$ and $c_+ = 2\mu + 1$ be the parameters for $U(a_+, c_+; x)$ included in $W_{p,\mu}$; and correspondingly $a_- = -\mu - p + 1/2$ and $c_- = -2\mu + 1$

the parameters for $U(a_-, c_-; x)$ included in $W_{p,-\mu}$. Indeed we have:

$$
\begin{aligned}
\sin(\pi c_-) = \sin(\pi - 2\pi\mu) = \sin\pi\cos(2\pi\mu) - \sin(2\pi\mu)\cos\pi = \sin(2\pi\mu) = -\sin(\pi c_+) \\
M(a_-, c_-; x) = M(-\mu - k + 1/2, -2\mu + 1; x) = M(a_+ + 1 - c_+, 2 - c_+; x) \\
(a_- - c_-)! = (-\mu - k + 1/2 - (2\mu + 1))! = (\mu - k + 1/2 - 1)! = (a_+ - 1)! \\
(c_- - 1)! = (-2\mu + 1 - 1) = (1 - c_+)! \\
x^{1-c_-} = x^{1-(-2\mu+1)} = x^{2\mu} \\
x^{-2\mu} = x^{1-c_+} \\
M(a_- + 1 - c_-, 2 - c_-; x) = M(a_+, c_+; x) \\
(a_- - 1)! = (-\mu - k + 1/2 - 1)! = (a_+ - c_+)! \\
(1 - c_-)! = (1 - (-2\mu + 1))! = (c_+ - 1)! \\
\Longrightarrow \\
U(a_-, c_-; x) = x^{1-c_-} U(a_+, c_+; x).
\end{aligned}
\tag{H.16}
$$

This is also known as a *Kummer transformation*. The relationship (H.16) implies:

$$
W_{p,-\mu} = \mathrm{e}^{-x/2} x^{-\mu+1/2} U(a_-, c_-; x) = \mathrm{e}^{-x/2} \overbrace{x^{-\mu+1/2} x^{2\mu}}^{=x^{\mu+1/2}} U(a_+, c_+; x) = W_{p,\mu}.
\tag{H.17}
$$

Hereby we set $\mu = s - 1$.

H.4 The Logarithmic Case $s = 3/2$

Now, let us discuss what happens with the confluent hypergeometric function U for $s = 3/2$, which we are particularly interested in coming from the required properties of the Eisenstein series. We have then

$$
\begin{aligned}
s = 3/2 &\Longrightarrow \mu = s - 1 = 1/2 \\
&\Longrightarrow c = 2\mu + 1 = 2
\end{aligned}
$$

and

$$
a = \mu - p + 1/2 = 1/2 - (-r - 1/2) + 1/2 = r + 3/2, \qquad r = 0, 1, 2, 3, \ldots
\tag{H.18}
$$

The case $c \in \mathbb{N}$ is referred to as the *logarithmic case* in [11]. Putting this into the definition of the second solution in (H.8) we get

$$
U\left(r + \frac{3}{2}, 2; x\right) = \frac{\pi}{\sin(\pi \cdot 2)} \left[\frac{M\left(r + \frac{3}{2}, 2; x\right)}{\Gamma\left(r + \frac{1}{2}\right)\Gamma(2)} - \frac{x^{-1} M\left(r + \frac{1}{2}, 0; x\right)}{\Gamma\left(r + \frac{3}{2}\right)\Gamma(0)} \right].
\tag{H.19}
$$

Clearly this is undefined, and we have to consider the limit $c \longrightarrow 2$. To show that the function actually converges in the limit, for all $x > 0$, one must use an integral representation of $U(a, c; x)$ instead (see [4] or 6.5. (2) [11]):

$$U(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-xt}\, t^{a-1}(1+t)^{c-a-1}\mathrm{d}t. \tag{H.20}$$

It is valid for $\Re(x) > 0$ and $\Re(a) > 0$. We have

$$\begin{aligned} 0 < \lim_{c \longrightarrow 2} \int_0^\infty \mathrm{e}^{-xt}\, t^{a-1}(1+t)^{c-a-1}\mathrm{d}t < \lim_{c \longrightarrow 2} \int_0^\infty \mathrm{e}^{-xt}\, \underbrace{(1+t)^{a-1}(1+t)^{c-a-1}}_{=(1+t)^{c-2}}\mathrm{d}t = \\ = \{t' = t+1\} = \mathrm{e}^x \lim_{c \longrightarrow 2} \int_1^\infty \mathrm{e}^{-xt'} t'^{c-2}\mathrm{d}t' < \mathrm{e}^x \lim_{c \to 2} \int_0^\infty \mathrm{e}^{-xt'} t'^{c-2}\mathrm{d}t' = \\ = \{\text{for } c > 1 \text{ and } x > 0\} = \mathrm{e}^x \lim_{c \to 2} \frac{\Gamma(c-1)}{x^{c-1}} < \infty. \end{aligned} \tag{H.21}$$

Since the limit is larger than zero and bounded from above, as well as *strictly increasing* with c (c can go from 1_+ to 2), the limit exists. I.e., $U(a, c, x)$ is well-defined for $a = r + 3/2$ and $c = 2$ if the limit is implicit in the definition.

Since $\Re(a) > 0$ the asymptotic expansion (H.13) is also valid[3]. The expression for large x is thus

$$W_{-r-\frac{1}{2};\frac{1}{2}}(x) = \mathrm{e}^{-x/2} x^{-r-\frac{1}{2}} \left(1 + \mathcal{O}\left(x^{-1}\right)\right). \tag{H.22}$$

[3] The asymptotic expansion is actually derived with help of the integral representation (H.20).

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