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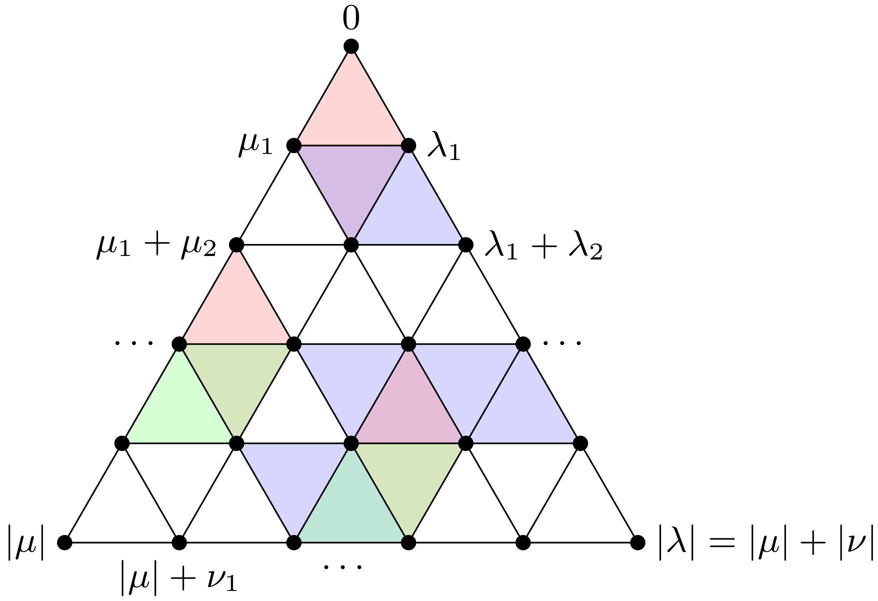

Hives and Hermitian Matrices

Examining a Hive Construction Proposal

Master's thesis in Engineering Mathematics and Computational Science

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CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2023
www.chalmers.se

Master's thesis 2023

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Cover: Knutson and Tao's hive model with a few rhombus inequalities highlighted.

Typeset in LaTeX
Printed by Chalmers Reproservice
Gothenburg, Sweden 2023

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Abstract

Given two Hermitian matrices, M and N, what can be said about the eigenvalues of their sum $L = M + N$? In 1962, A. Horn conjectured that a set of recursive inequalities would fully characterize the eigenspectrum of the sum in terms of the eigenspectra of the summands. Just before the turn of the millennium the saturation conjecture was proven, and as a consequence, Horn's conjecture was established. In their proof, A. Knutson and T. Tao introduced a combinatorial object, known as a *hive*, that rephrases Horn's inequalities into more tractable expressions. The path from Hermitian matrices to hives, however, remains largely unexplored. A proposal for a hive construction, that is, a mapping from matrices to hives, was put forth by G. Appleby and T.Whitehead in 2014, but the proof of the construction's validity has since come under question. In this thesis, the hive construction proposal by Appleby and Whitehead serves as a basis for a reformulated hive construction, adapted to the restricted setting of pairs of simultaneously diagonalizable Hermitian matrices. Equipped with this modified formulation, we proceed to prove algebraically that the construction indeed generates hives for simultaneously diagonalizable Hermitian matrices generally, thereby providing a mapping from matrix triples to hives under these special circumstances.

Keywords: Hives, Hive constructions, Horn's problem, the Saturation conjecture, Hermitian matrices, Littlewood-Richardson coe□cients.

Acknowledgements

Working on this thesis has been the true highlight of my time at Chalmers, and for that I am deeply grateful to my supervisor, Jakob Björnberg, whose idea this thesis was and who has throughout the project provided so much insight, encouragement and been most generous with his time. Thank you for giving me a glimpse into the truly fascinating world of mathematical research and percolation theory, it has been a privilege.

I would also like to thank all the amazing people I have met during my studies, for all the great collaborations, the entertaining lunch table discussions and the sharing of knowledge that has greatly supplemented lectures.

Lastly, I would like to thank my family and friends for being there, and of course, Ben, for his unparalleled support.

Karin Furufors, Gothenburg, May 2023

Contents

Nomenclature

Below is the nomenclature of indices, sets, parameters, and variables that have been used throughout this thesis.

Indices

p	The row of a hive node, $p \quad \{0, \cdots, n\}$
q	The column of a hive node, $q \quad \{0, \cdots, n\}$

Sets

$\mathcal{H}_n$	The set of Hermitian $n \times n$ matrices
$\mathcal{H}_n^{SD}$	The set of pairs of simultaneously diagonalizable Hermitian $n \times n$ matrices
$\mathcal{U}_n$	The set of unitary $n \times n$ matrices
$\mathcal{D}_n$	The set of diagonal $n \times n$ matrices
(I, J, K)	The sets of indices for Horn's inequalities
T_r^n	The set of admissible triples of index sets (I, J, K) with cardinality $r = \lvert I\rvert, \lvert J\rvert, \lvert K\rvert$ for matrices of size n
S_n	The set of permutations for a vector with n-entries
V_i	The set of differences between the sum of labels of obtuse angles and the sum of labels of acute angles in the vertical rhombus inequalities, indexed from top to bottom and left to right
L_i	The set of differences between the sum of labels of obtuse angles and the sum of labels of acute angles in the left rhombus inequalities, indexed from top to bottom and left to right
R_i	The set of differences between the sum of labels of obtuse angles and the sum of labels of acute angles in the right rhombus inequalities, indexed from top to bottom and left to right
$\mathcal{F}_{qp}(W)$	The set of all pairs of subspaces (U, V), $U, V \quad W$ such that $U \quad V$

$P_{(p,q)}$	The set of pairs of matrices $\{(P, Q) : P^2 = P = P\ , Q^2 = Q = Q\ , PQ = QP = Q, \text{rank}(P) = p, \text{rank}(Q) = q\}$
$S_{(p,q)}$	The set of pairs of sequences $\{((s_i)_{i=1}^n, (t_i)_{i=1}^n) : 0 \leq t_i \leq s_i \leq 1, \sum_{i=1}^n s_i = p, \sum_{i=1}^n t_i = q\}$

Parameters

n	The dimension of matrices in the triple $(M, N, L = M + N)$
μ	The vector of eigenvalues of matrix M in weakly decreasing order
ν	The vector of eigenvalues of matrix N in weakly decreasing order
λ	The vector of eigenvalues of matrix L in weakly decreasing order
σ	A permutation of n elements
ν_σ	The vector ν under the permutation σ
β	The unsorted vector $\beta = \mu + \nu_\sigma$
$c_{\mu,\nu}^{\lambda}$	The Littlewood-Richardson coe□cient for partitions (μ, ν, λ) with shape λ/μ and weight ν

Variables

$h_{i,j}$	The hive node at row i, column j
$h_{i,j,k}$	The hive node at row $n - i$ and on the j-th diagonal below the main diagonal and the k-th diagonal below the anti-diagonal in the triangular hive description

Acronyms

AW	Refers to the original hive construction proposed by G. Appleby and T. Whitehead
AWD	Refers to the reformulation of the *AW*-construction for diagonal matrices used in this thesis

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1
Introduction

The subject of this thesis is rooted in a deceptively simple question from linear algebra: Given two Hermitian matrices, M and N, what can be said about the eigenvalues of their sum $L = M + N$? This question goes back to the 19th century and is sometimes referred to as *Horn's problem*, after the mathematician who would eventually formulate a conjecture for its answer. Part of the allure of Horn's problem is, in the words of the authors of [4], its 'appearance under various guises'. The problem, however simple in formulation, notably transcends the subject of linear algebra and appears in places for reasons not yet fully understood. It has connections to fields such as representation theory, symplectic geometry and Schubert calculus [6].

The conjecture from 1962 by Alfred Horn was preceded by a number of different results in the form of inequalities of eigenvalues, and is itself constituted by a rather complicated recursive set of eigenvalue inequalities. In 1998, Allen Knutson and Terence Tao showed that Horn's conjecture was true, which is to say, gives a full description of the constraints on the eigenvalues of the sum [10]. Knutson and Tao also introduced a combinatorial model known as a *hive*, which along with a later equivalent called a *honeycomb*, rephrase Horn's inequalities into more tractable expressions. Their proof of the so called *saturation conjecture* also showed that the existence of three Hermitian matrices M, N and $L = M + N$ with integral spectra μ, ν and λ, hinges on the existence of a an integral hive and vice versa.

The object of interest here is the hive model, an example of which can be seen on the cover page. The model is essentially a triangular tessellation of an equilateral triangle, where the vertices are nodes to be assigned labels. The outer nodes have values prescribed from the eigenvalues of the Hermitian matrix triple $(M, N, L = M + N)$. The question is whether there exists a way to label the inner nodes, without violating any of the accompanying conditions in the form of inequalities. More specifically, we are interested in the idea of a *hive construction*, that is, an explicit map from the eigenvalues (μ, ν, λ) of a Hermitian matrix triple to the inner nodes of the hive.

Although the saturation conjecture was established over two decades ago, the subject of hive constructions is still relatively unexplored. In an article from 2014 [1], G. Appleby and T. Whitehead put forward a proposal for a hive construction as well as a proof that does not seem to hold up, as pointed out in an article by J. Lombard [11]. While the proof was lacking, the construction has not formally been disproven either and for this reason we have chosen to explore this construction algebraically, for the special case of simultaneously diagonalizable matrices, in the hopes of either proving or disproving it. The merit of a functional hive construction would lie in a sort of concretization of the saturation conjecture, which is an existence theorem that through an indirect path between Hermitian matrices and hives states that such a relationship exists.

1.1 Aim

The aim of this thesis is to examine the hive construction proposal in [1] in the simpler, special case of diagonal matrices. This will be done in steps by first translating it into a more explicit form and then proving it for a number of special cases of matrices such as diagonal matrices with specific structure or of smaller dimensions. Finally we prove the validity of the construction in the general diagonal case.

1.2 Outline

In Chapter 2, *Theory*, some useful theorems and definitions, chiefly related to Hermitian matrices, are presented for the convenience of the reader. The chapter also includes short introductions of the most central topics for this thesis: Horn's problem, the saturation conjecture, the hive model and hive constructions, with special attention given to the hive construction referred to as the Appleby-Whitehead construction. Chapter 3 serves as an introductory exercise for the reader to become better acquainted with hives by showing, for the two- and three-dimensional cases, how Horn's inequalities can be derived from the hive model. The results of the thesis are presented in Chapter 4. It begins with a reformulation of the *AW*-construction into the diagonal case and a re-establishing of the validity of the hive's borders. In the subsequent sections, we present proofs of the construction's validity in a few special cases, building towards the main proof of the thesis, in Section 4.3, that the construction actually *does* generate valid hives for diagonal matrices. The thesis ends with Chapter 5, *Conclusion*, in which the algebraical findings are discussed and put into context, as well as some suggestions made for future research. A detailed and rather lengthy proof of one of the special cases mentioned in Chapter 4 is included in Appendix A. Finally, a short note on the search for counter-examples to the construction appears in Appendix B.

2

Theory

In this chapter we lay a theoretic foundation for the following chapters, starting with recalling some useful results, mainly on the subject of Hermitian matrices. We then give some background to Horn's problem and the saturation conjecture in order to contextualize the subject of this thesis. Finally we describe hives, hive constructions and discuss the specific construction that will be at the center of our investigations in Chapter 4.

2.1 Useful Results

From beginning to end, we are exclusively concerned with Hermitian matrices, as they are the matrices for which Horn's problem and the hive model are defined. It is therefore suitable to recall their definition.

DEFINITION 2.1 (Hermitian matrix). *A complex square matrix H is said to be Hermitian if H is equal to its conjugate transpose, $H \ = H$.*

We also recall the definition of the conjugate transpose.

DEFINITION 2.2 (Conjugate Transpose). *The conjugate transpose, or Hermitian transpose, of a $m \times n$ complex matrix A is an $n \times m$ matrix obtained by transposing A and applying the complex conjugate on each entry. It is denoted A .*

We note that Hermitian matrices possess the following properties.

REMARK. *The sum of two Hermitian matrices is Hermitian.*

REMARK. *Hermitian matrices have real eigenvalues.*

The minimal and maximal eigenvalues of Hermitian matrices can be expressed through the following theorem which can be found in *Matrix Theory: Basic Results and Techniques* by Fuzhen Zhang [15, p.267].

THEOREM 2.3 (Rayleigh-Ritz Theorem). *Let H be a Hermitian matrix. Then the smallest eigenvalue λ_{min} is given by:*

$$\lambda_{min}(H) = \min_{x \ x=1} x \ Hx$$

and the largest eigenvalue λ_{max} is given by:

$$\lambda_{max}(H) = \max_{x \;\; x=1} x \;\; Hx.$$

Another theorem on the same theme is the following due to K. Fan, which can be found in *Inequalities: Theory of Majorization and Its Applications* by Marshall, Olkin and Arnold [12, p.785].

THEOREM 2.4 (Fan). *Let H be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1(H) \geq \cdots \lambda_n(H)$. Then:*

$$\sum_{i=1}^{k} \lambda_i(H) = \max_{UU \;\; = I_k} \operatorname{tr}(UHU \;\;)$$

and

$$\sum_{i=1}^{k} \lambda_{n-i+1}(H) = \min_{UU \;\; = I_k} \operatorname{tr}(UHU \;\;)$$

where $k = 1, \cdots, n$ where the extrema are over $k \times n$ complex matrices satisfying $UU \;\; = I_k$.

Two other sets of matrices that will play minor roles in this thesis are unitary and normal matrices.

DEFINITION 2.5 (Unitary matrix). *An invertible complex square matrix U is said to be unitary if its conjugate transpose U is also its inverse:*

$$U \;\; U = UU \;\; = UU^{-1} = I,$$

where I is the identity matrix.

DEFINITION 2.6 (Normal matrix). *A complex square matrix N is said to be normal if and only if it commutes with its conjugate transpose:*

$$N \;\; N = NN \;\; .$$

REMARK. *All Hermitian matrices are normal.*

In Chapter 4 we will be occupied with diagonalizable matrices and for that the following theorem, which can for example be found in [15, p.81], is needed.

THEOREM 2.7 (Spectral Decomposition Theorem). *Let A be a $n \times n$ complex matrix with eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then A is normal if and only if, A is unitarily diagonalizable; that is, there exists a unitary matrix U such that:*

$$U \;\; AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

In particular, A is Hermitian if and only if the λ_i are all real and is positive semi-definite if and only if the λ_i are all non-negative.

We also state the definition of simultaneously diagonalizable normal matrices.

Definition 2.8. *(Simultaneously Diagonalizable Normal Matrices) Two normal matrices A and B, among those the Hermitian matrices, are said to be simultaneously diagonalizable if and only if, there exists a unitary matrix U where:*

$$U\ AU,$$
$$U\ BU,$$

are both diagonal.

We now recall the definitions of permutation matrices.

Definition 2.9. *(Permutation matrix) A square matrix is called a permutation matrix if it has exactly one 1 in each row and column and 0s elsewhere.*

As well as orthogonal projections.

Definition 2.10. *(Orthogonal Projection) A square complex matrix A is called an orthogonal projection if:*

$$A^2 = A = A\ .$$

The following theorem concerning properties of orthogonal projections can found in [15, p.128].

Theorem 2.11. *For any $A \quad \mathrm{M}_n$ the following are equivalent:*

1. *A is an orthogonal projection matrix; that is: $A^2 = A = A$.*
2. *$A = U\ \mathrm{diag}(1, \cdots, 1, 0 \cdots, 0)U$ for some unitary matrix U.*
3. *$\ x - Ax\ \leq\ x - Ay\ $ for every pair $(x, y)\quad \mathrm{C}^n$.*
4. *$A^2 = A$ and $\ Ax\ \leq\ x\ $ for every $x\quad \mathrm{C}^n$.*
5. *$A = A\ A$.*

The trace is used frequently throughout the thesis and so we give its definition.

Definition 2.12. *(Trace) The trace of a square $n \times n$ matrix A is defined as:*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} = A_{11} + A_{22} + \cdots + A_{nn}.$$

What follows are some of the properties of the trace that we will have use for.

Remark. *For any real or complex matrix A with eigenvalues λ, the trace is the sum of eigenvalues:*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$

Remark. *The trace is a linear mapping:*

$$\begin{aligned} \mathrm{tr}(A + B) &= \mathrm{tr}(A) + \mathrm{tr}(B), \\ \mathrm{tr}(cA) &= c\,\mathrm{tr}(A), \end{aligned}$$

for a scalar c.

Remark. *The trace is invariant under circular shifts:*

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB).$$

For the later proofs, set theory will play a small but crucial role and so we recall some fundamentals.

Theorem 2.13 (Distributive Law of Set Theory).

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ (B \cap C) \cup A &= (B \cup A) \cap (C \cup A), \\ (B \cup C) \cap A &= (B \cap A) \cup (C \cap A). \end{aligned}$$

2.2 Horn's Problem

The fundamental question present throughout the thesis comes from linear algebra and dates all the way back to the eighteen hundreds:

> *Given two Hermitian matrices M and N, what can be said about the eigenvalues of their sum $L = M + N$?*

The first half of the 1900's saw the evolution of this problem in the form of necessary inequalities expressed through the eigenvalues of the matrices, culminating in a conjecture by mathematician Alfred Horn in 1962 [8]. In this section we attempt to give a brief history of the problem, heavily inspired by W. Fulton's account in [6].

If M and N are Hermitian matrices of dimension n then their sum $L = M + N$ is also a Hermitian matrix. Let μ and ν be the vectors containing the eigenvalues of M and N, sorted in weakly decreasing order such that: $\mu_1 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \cdots \geq \nu_n$, and let λ be the defined analogously for L. That the eigenspectra $\mu, \nu, \lambda \in \mathbb{R}$ follows on account of them being Hermitian. The first constraint on the eigenvalues λ follows trivially from the linearity of trace, $\mathrm{tr}(A) + \mathrm{tr}(B) = \mathrm{tr}(A + B)$, implying that the sum of eigenvalues in μ and ν must equal the sum of eigenvalues in λ.

Theorem 2.14 (Linearity of trace). *Let M, N, L be three matrices with respective eigenvalues μ, ν, λ such that $M + N = L$, then the sum of eigenvalues λ must equal the sums of eigenvalues μ and ν:*

$$|\lambda| = |\mu| + |\nu|.$$

The notation $|\mu| = \sum_i^n \mu_i$ will be used frequently throughout the rest of the thesis.

Apart from this basic condition, it was not until 1912 that the first progress was made, in the form of an inequality formulated by H. Weyl [6]:

Theorem 2.15 (Weyl's inequality). *Let $M, N, L = M + N$ be three Hermitian $n \times n$ matrices with respective eigenvalues μ, ν, λ ordered: $\mu_1 \geq \cdots \geq \mu_n$, $\nu_1 \geq \cdots \geq \nu_n$, $\lambda_1 \geq \cdots \geq \lambda_n$. Then the following inequality holds:*

$$\lambda_{i+j-1} \leq \mu_i + \nu_j,$$

whenever $i + j - 1 \leq n$.

Weyl's inequality gives an upper bound for λ, expressed as a sum of elements from μ and ν. This will be a common pattern for later inequalities as well. In Section 3.1 we will show that Weyl's inequality, together with the linearity of trace, are both necessary and sufficient constraints for Horn's problem in the special case of Hermitian matrices of size 2. In general, however, it is not enough to characterize λ.

In 1949, K. Fan established another useful inequality [6]:

Theorem 2.16 (Fan's inequality). *Let $M, N, L = M + N$ be three Hermitian $n \times n$ matrices with respective eigenvalues μ, ν, λ ordered: $\mu_1 \geq \cdots \geq \mu_n$, $\nu_1 \geq \cdots \geq \nu_n$, $\lambda_1 \geq \cdots \geq \lambda_n$. Then the following inequality holds:*

$$\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} \mu_i + \sum_{i=1}^{r} \nu_i \quad \text{for any } r < n.$$

A year later in 1950 V. B. Lidskii presented a geometric condition on the eigenvalues. This was shown in 1955 by H. Wielandt to be equivalent to the following inequality [6]:

Theorem 2.17 (Lidskii-Wielandt's inequality). *Let $M, N, L = M + N$ be three Hermitian $n \times n$ matrices with respective eigenvalues μ, ν, λ ordered: $\mu_1 \geq \cdots \geq \mu_n$, $\nu_1 \geq \cdots \geq \nu_n$, $\lambda_1 \geq \cdots \geq \lambda_n$. Then the following inequality holds:*

$$\sum_{i \in I} \lambda_i \leq \sum_{i \in I} \mu_i + \sum_{i=1}^{r} \nu_i,$$

for every subset I of $\{1, \cdots, n\}$, with $|I| = r, r < n$.

L. Freede and R. C. Thompson would later generalize Weyl's, Fan's and Lidskii-Wielandts' inequalities in 1971 with the following [6]:

Theorem 2.18 (Freede and Thompson's inequality). *Let $M, N, L = M + N$ be three Hermitian $n \times n$ matrices with respective eigenvalues μ, ν, λ ordered: $\mu_1 \geq \cdots \geq \mu_n$, $\nu_1 \geq \cdots \geq \nu_n$, $\lambda_1 \geq \cdots \geq \lambda_n$. Then the following inequality holds:*

$$\sum_{j=1}^{k} \lambda_{i_j + p_j - j} \leq \sum_{j=1}^{k} \mu_{i_j} + \sum_{j=1}^{k} \nu_{p_j}$$

for any choice of indices $1 \leq i_1 < \cdots < i_k \leq n$ *and* $1 \leq p_1 < \cdots < p_k \leq n$ *such that* $i_k + p_k - k \leq n$*.*

From the above expression we obtain Weyl's inequality by setting $k = 1$, Fan's inequality by setting $i_j = p_j = j$. The Lidskii-Wielandt inequality can be expressed by letting $p_j = j$ and $I = \{i_1, \cdots, i_k\}$.

How many more such necessary inequalities would need to be found? In what is frequently described as a 'remarkable' article from 1962, American mathematician Alfred Horn sought to answer this question through a systematic study. It resulted in Horn's conjecture, which is a recursive set of inequalities that Horn believed would fully characterize λ. Below we have used the formulation of Horn's conjecture presented in [2]. First let us define what we mean by *admissible triples* of index sets.

Definition 2.19 (Admissible Triples). *Let* (M, N, L) *be three Hermitian matrices of size* n *such that* $L = M + N$*. Let the eigenvalues of* (M, N, L) *be called* (μ, ν, λ) *and be sorted in weakly decreasing order. For eigenvalue inequalities of the form:*

$$\sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j \geq \sum_{k \in K} \lambda_k$$

a triple (I, J, K) *of subsets of* $\{1, 2, \cdots, n\}$ *for which the inequality holds is called admissible. The set* T_r^n *is the set of all admissible triples of cardinality* $|I| = |J| = |K| = r$ *for matrices of size* n*.*

Now we move on to Horn's conjecture.

Theorem 2.20 (Horn's conjecture). *In order to characterize the eigenvalues* λ *of the sum of two Hermitian matrices* M *and* N *of size* n*, with eigenspectra:* $\sigma(M) = \mu$ *and* $\sigma(N) = \nu$*, all sorted in weakly decreasing, Theorem 2.14 together with inequalities of the form:*

$$\sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j \geq \sum_{k \in K} \lambda_k, \text{ for } (I, J, K) \in T_r^n$$

are sufficient, where the set of admissible triples T_r^n *can be described by the following induction on* r*. For* $r = 1$ *let:*

$$(I, J, K) \in T_r^n \quad \text{if} \quad i + j - 1 = k, \tag{2.1}$$

for $i \in I, j \in J, k \in K$*. For* $r > 1$ *let:*

$$(I, J, K) \in T_r^n \quad \text{if} \quad \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \binom{r+1}{2}, \tag{2.2}$$

and for all $1 \leq p \leq r - 1$*:*

$$(U, V, W) \in T_p^r \quad \text{if} \quad \sum_{u \in U} i_u + \sum_{v \in V} j_v \leq \sum_{w \in W} k_w + \binom{p+1}{2}. \tag{2.3}$$

These inequalities are quite intricate and are not generally minimal for larger n [6]. The number of inequalities grow quickly with the dimension n. When $n = 7$ there are 2062 of Horn's inequalities with some dependency [2]. The conjecture was proven by Horn himself in the cases when $n = 3, 4$, and a proof for the general case was announced in 1982 by B. V. Lidskii, son of V. B. Lidskii, but the detailed proof never surfaced and so the conjecture would remain open until shortly before the new millennium [14], when it was proven by A. Knutson and T. Tao in 1998 [3]. A timeline for these events is given in Figure 2.1.

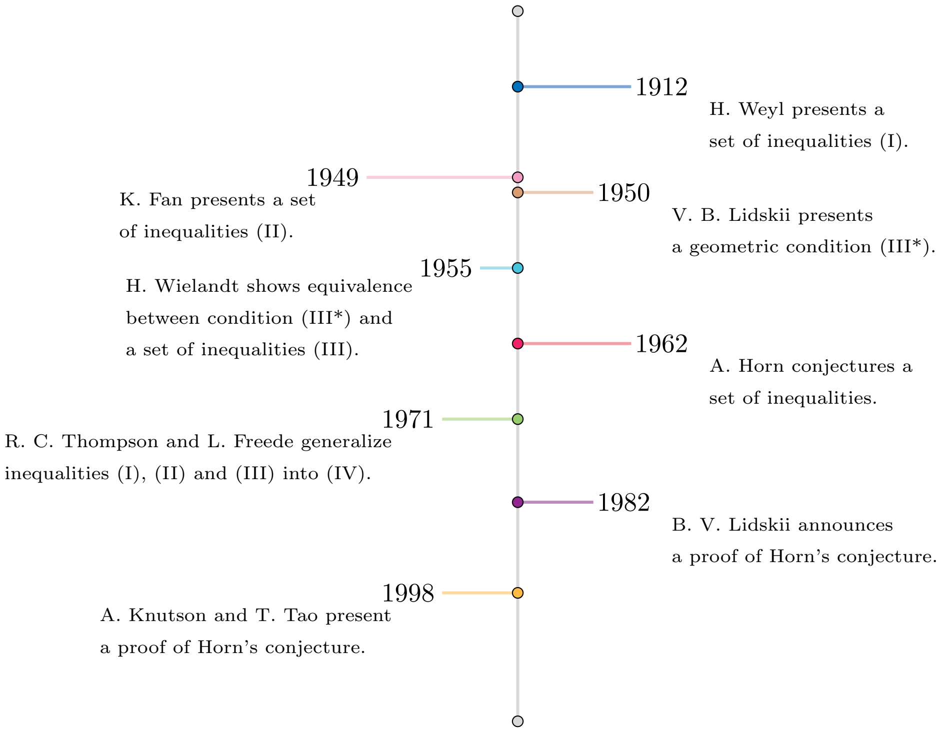


Figure 2.1: A timeline of significant results related to Horn's problem.

2.3 Littlewood-Richardson Coefficients and the Saturation Conjecture

Related to Horn's conjecture is the notion of Littlewood-Richardson coefficients. These numbers are non-negative integers that play a role in quite a few phenomena across various fields, among those in the topology of Grassmann varieties, in combinatorics connected to Young tableaux, and they are central to representation theory. Although originally stated in 1934 by D. E. Littlewood and A. R. Richardson, the rule has since been formulated in many different ways over the years and was not fully proven until the 1970's [9]. A definition of the Littlewood-Richardson coefficients which can be found in [13, p.175], is given below.

Definition 2.21 (Littlewood-Richardson coefficients). *For Schur functions: s_μ, s_ν, s_λ, described by partitions (λ, μ, ν), the Littlewood-Richardson coefficient* $c^{\lambda}_{\mu,\nu}$

is the natural number such that:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda.$$

The Littlewood-Richardson rule offers a combinatoric interpretation of these coefficients. A possible formulation of the Littlewood-Richardson rule would be through the use of diagrams, where the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$, for the three *partitions* (μ, ν, λ) is given by the number of ways in which a Littlewood-Richardson tableaux of skew shape λ/μ and weight ν can be filled out [7]. In essence these tableaux are made up from boxes arranged in certain ways, where the entries must satisfy certain rules.

A bijection between the Littlewood-Richardson rule and Knutson and Tao's hives is given by A. S. Buch in [3].

The saturation problem asks if the fact that $c^{N\lambda}_{N\mu,N\nu} > 0$ for some $N \geq 1, N \in \mathbb{N}$, implies that $c^\lambda_{\mu,\nu} > 0$ and vice verse, although the reverse implication is considered trivial. The relationship was proven by A. Knutson and T.Tao in 1998 [3].

Theorem 2.22 (The Saturation Conjecture). *Let T_n be the set of triples of partitions (μ, ν, λ) of length n for which the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu} > 0$. Then the following is true:*

$$(\mu, \nu, \lambda) \in T_n \iff (N\mu, N\nu, N\lambda) \in T_n,$$

for $N \in \mathbb{N}$, i.e. T_n is saturated in $\mathbb{Z}^{3n}$.

An important consequence of Knutson and Tao's proof of the saturation conjecture is the affirmation of Horn's conjecture. For hives, this also means that if all eigenvalues in a triple (μ, ν, λ) are integers, in order for the corresponding Hermitian matrices $(M, N, L = M + N)$ to exist, there must exist at least one integral hive and the other way around. Furthermore, the multiplicity of integral hives equals the Littlewood-Richardson coefficient [3].

Theorem 2.23 (Multiplicity of Integral Hives). *The number of possible integral hives with borders made from partitions (μ, ν, λ) is equal to the corresponding Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$.*

2.4 Hives

We now introduce the reader to the objects playing a key role in this thesis, *hives*. The hive model is, along with another model called the *Honeycomb* model, a combinatorial object introduced by Knutson and Tao in 1998 as part of their proof of the saturation conjecture [10]. They offer an alternative way of expressing Horn's inequalities, which are otherwise difficult to work with.

The hive model consists of a large equilateral triangle that is tessellated with smaller equilateral triangles. For one dimension this would just be the outer triangle, but for every dimension another layer is added so that the number of small triangles

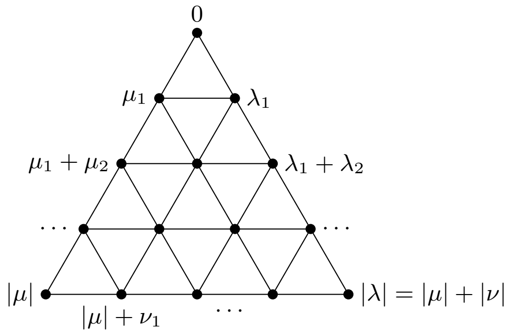


Figure 2.2: A hive of arbitrary size. The number of nodes on each side is equal to the dimension of the matrices plus one.

becomes n^2. We think of the vertices of the triangles as nodes that can each be assigned a value. The Horn inequalities are echoed in part by the conditions of the left, right and bottom border of the large triangle, and in part by the so called *rhombus inequalities*.

Take three Hermitian matrices M, N and $L = M + N$ of size n, and let them have eigenvalues spec(M) = μ, spec(N) = ν and spec(L) = λ, where (μ, ν, λ) are sorted in weakly decreasing order such that: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The hive model takes these eigenvalues, (μ, ν, λ), as parameters when forming the borders. The very top node is labelled zero. The nodes along the left border are labelled with the cumulative sum of entries in μ as shown in Figure 2.2. This makes the bottom left node $\sum_{i=1}^{n} \mu_i = |\mu|$. Along the bottom border the nodes are filled in with the cumulative sums of ν added to $|\mu|$. Finally, along the right border the cumulative sums of the entries of λ are assigned to the nodes from top to bottom, so that the bottom right node is $\sum_{i=1}^{n} \lambda_i = |\lambda|$, which due to the linearity of trace we know must be equal to $|\mu| + |\nu|$.

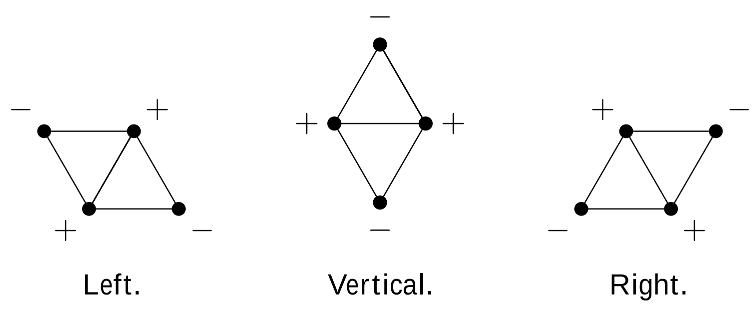


Figure 2.3: The three types of rhombus inequalities. The sum of the labels on the nodes marked with + must be greater than or equal to the sum of the labels on the nodes marked with −.

Now the second part of the hive specification are the rhombus inequalities. A rhombus in this case refers to the union of any two adjacent small triangles. These occur in three orientations, left, vertical and right, as seen in Figure 2.3. For every rhombus in the large triangle, regardless of how it is slanted, the sum of the labels on the two nodes on the obtuse angles of the rhombus must be greater than or equal to the sum of the labels on the nodes on the acute angles. If all vertices can be labelled without violating any inequality then that labelling, or *hive filling*, is a valid hive of type (μ, ν, λ). If the eigenvalues as well as all the interior nodes are integers, the hive is integral, which as we know from Section 2.3 counts towards the triple's Littlewood-Richardson coefficient.

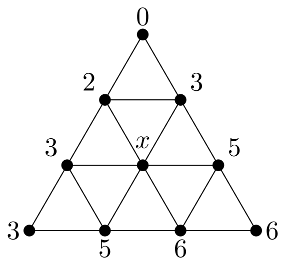


Figure 2.4: A hive of dimension 3, with $\mu = (2, 1, 0)$, $\nu = (2, 1, 0)$ and $\lambda = (3, 2, 1)$. The only two possible values for the interior node x are 4 and 5.

For illustrative purposes we include a very small example of a hive, which we borrow from A. S. Buch in [3]. For three Hermitian matrices with eigenvalues: $\mu = (2, 1, 0)$, $\nu = (2, 1, 0)$ and $\lambda = (3, 2, 1)$, the hive borders would be filled out as shown in Figure 2.4. If there exists such a triple of Hermitian matrices $(M, N, L = M + N)$ with integral spectra there must be, according to the saturation conjecture, a way to fill in the interior nodes using integer labels. For three dimensions there is only a single interior node. The interior node x in the hive shown in Figure 2.4 is constrained by the following rhombus inequalities.

Vertical rhombus inequalities:

$$
\begin{aligned}
2 + 3 \geq 0 + x &= 5 \geq x, \\
3 + x \geq 2 + 5 &= x \geq 4, \\
x + 5 \geq 3 + 6 &= x \geq 4.
\end{aligned}
$$

Left rhombus inequalities:

$$
\begin{aligned}
2 + x \geq 3 + 3 &= x \geq 4, \\
3 + 5 \geq x + 3 &= 5 \geq x, \\
x + 6 \geq 5 + 5 &= x \geq 4.
\end{aligned}
$$

Right rhombus inequalities:

$$
\begin{aligned}
3 + x \geq 2 + 5 &= x \geq 4, \\
x + 5 \geq 3 + 6 &= x \geq 4, \\
5 + 6 \geq x + 6 &= 5 \geq x.
\end{aligned}
$$

These rhombus inequalities can be reduced to two simple conditions: $x \geq 4$ and $x \leq 5$. Both $x = 4, 5$ satisfy the rhombus inequalities and so for these eigenvalues there are exactly two possible integral hives, one with interior node $x = 4$ and the other with $x = 5$, meaning that the Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu} = 2$.

2.5 Hive Constructions

For a Hermitian matrix triple $(M, N, L = M + N)$ with eigenspectra (μ, ν, λ) there can be zero, one or several ways to fill out a hive, depending on the eigenvalues. While the hive object offers a convenient way of checking Horn's inequalities, it does not provide an explicit mapping from eigenvalues to a hive filling. This type of generative algorithm could be useful for computing the Littlewood-Richardson coefficient for integral hives, something that in general is a $\#P$-complete problem [11]. A *hive construction* would be a set of instructions, or explicit expressions, for finding a hive filling, provided of course that the matrix triple has at least one such hive filling.

According to [11], V. I. Danilov and G. A. Koshevoy were the first to suggest an explicit constructive conjecture for hives in their 2003 article *Discrete Convexity and Hermitian Matrices* [5], and were successful in showing that their conjecture holds in the special cases of 2-dimensional (note that there is no inner node in this case) and 3-dimensional Hermitian matrices, as well as commuting Hermitian matrices generally. As the article by Danilov and Koshevoy is written in Russian, it is considered only indirectly in this thesis through the comments made by J. Lombard in [11]. Here we will focus on another hive construction, namely the *Appleby-Whitehead* construction.

2.5.1 The Appleby-Whitehead construction

In 2014 American mathematicians Glenn Appleby and Tamsen Whitehead published an article entitled: *Honeycombs from Hermitian Matrix Pairs* [1], in which they presented a construction that they claimed could generate a hive from a matrix pair, thereby addressing this gap in the literature.

In their article each hive node, $h^{AW}_{i,j,k}$, is indexed by three indices (i, j, k) which indicate where in the tessellation the node is located and here the superscript *AW* (for *Appleby-Whitehead*) is added for clarity. The first index indicates on what row the node sits, with $i = n$ being the top row and $i = 0$ being the bottom row. The other two indices, j and k, can be thought of as the number of diagonals below the main diagonal and the anti-diagonal respectively, the node is located. Together the indices sum up to $i + j + k = n$ for every node. The indexation can be seen in Figure 2.5.

Moving on to the actual construction. Let H_n denote the set of all Hermitian matrices of dimension n. The usual premise of two Hermitian matrices, $M, N \in H_n$ and their sum $M + N = L \in H_n$ is the starting point for the construction. The eigenvalues are denoted spec$(M) = \mu$, spec$(N) = \nu$ and spec$(L) = \lambda$ and are sorted in weakly descending order. Let W be a n-dimensional complex vector space, and let U be a p-dimensional subspace and V be a q-dimensional subspace, such that

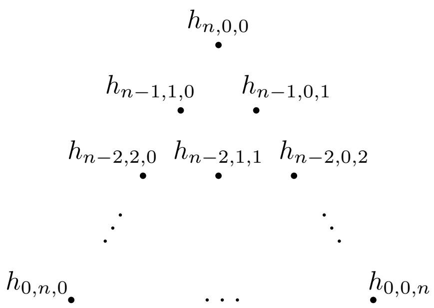


Figure 2.5: Indices in the *AW*-construction. The first index gives the row with n being the top and 0 being the bottom. The second and third indices designate which diagonals below the main diagonal and below the anti-diagonal respectively, the node sits at, with the top diagonals indexed as 0 and the bottom as n.

$U \quad V$. Let $\pi_U : W \to U$ denote the orthogonal projection, and $M|_U$ be the Hermitian operator on M such that $M|_U = \pi_U \circ M \circ \pi_U$. Then the construction is defined as follows.

Construction 2.24 (*AW*). *For any pair of $n \times n$-dimensional Hermitian matrices M and N, set the node at position (i, j, k) to be given by:*

$$h^{AW}_{i,j,k} = \max_{(U,V) \quad F_{k,(j+k)}} [\text{tr}(M|_V]) + \text{tr}(N|_U)],$$

where $i = n - j - k$ and $F_{pq}(W)$ is the set of pairs of subspaces (U, V) of the n-dimensional complex vector space W such that $U \quad V$, $\dim(U) = p$, $\dim(V) = q$ *and $p, q \quad \mathbb{Z} : p \leq q \leq n$.*

Appleby and Whitehead give a proof that the construction gives valid left, right and bottom borders for the hive model.

Proposition 2.25. *The borders in Construction 2.24 satisfy the boundary conditions of a hive.*

Appleby and Whitehead also sketch out a proof of the entire construction, broken down into proving the three types of rhombus inequalities generally, but the sketch is much less explicit than the proof of the borders and ultimately falls short of a proof. The critique the construction has received by the author of [11] will be covered in the following subsection.

2.5.2 Critique of the Appleby-Whitehead construction

The Appleby-Whitehead construction has received critique in an article from 2017 entitled *Honey from the Hives: A Theoretical and Computational Exploration of Combinatorial Hives* by John Lombard. The article claims that the optimization involved in finding the interior nodes in the *AW*-construction, i.e. the maximization of traces over all possible subspaces, will not result in hives for all cases. The rearrangement of subspaces used in the proof that the construction satisfies the right rhombus inequality given by G. Appleby and T. Whitehead in [1] is not generally true [11]. Without imposing a certain structure of the matrices involved, the dimension counting arguments in the proof do not hold, according to Lombard. The proof of the *AW*-construction does however work for reduced cases such as when matrices M and N are both sorted diagonal matrices, or when $M = N$ [11]. While the proof of the *AW*-construction itself is flawed, as pointed out by Lombard, there is no counter-proof.

3

Deriving Horn's Inequalities from the Hive

Before addressing the Appleby-Whitehead construction, which is the central focus of Chapter 4, we would like to first get better acquainted with the notion of hives as well as Horn's inequalities from Theorem 2.20. For that reason this chapter is devoted to the derivation of the latter from the former in the 2- and 3-dimensional cases.

3.1 2 × 2 - Hermitian Matrices

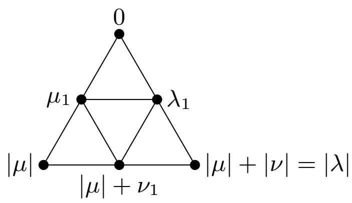


Figure 3.1: A hive of size 2. In the two-dimensional case there are no inner nodes.

Let us begin with the simplest case imaginable, namely a triple (M, N, L) of matrices of two dimensions: M, N, L H_2. Using the inductive procedure described in Section 2.2, the set of 'admissible triples' of cardinality $r = 1$, T_1^2, is given by:

$$T_1^2 : i + j - 1 = k.$$

From this we get the triples and the corresponding eigenvalue inequalities:

$$T_1^2 : (I, J, K) = \begin{Bmatrix} (\{1\}, \{1\}, \{1\}), \\ (\{1\}, \{2\}, \{2\}), \\ (\{2\}, \{1\}, \{2\}). \end{Bmatrix} = \begin{cases} \mu_1 + \nu_1 & \geq \lambda_1, \\ \mu_1 + \nu_2 & \geq \lambda_2, \\ \mu_2 + \nu_1 & \geq \lambda_2. \end{cases}$$

These inequalities can all be equivalently found using Weyl's inequality, Theorem 2.15. As per Horn's conjecture, the linearity of trace equality should be added to the above inequalities in order to complete them:

$$\mu_1 + \mu_2 + \nu_1 + \nu_2 = \lambda_1 + \lambda_2. \qquad \text{(Linearity of trace)}$$

In the case when $n = 2$, Weyl's inequalities and the linearity of trace are su□cient to characterize λ.

Does the hive model describe equivalent inequalities? The first condition the hive model imposes on the eigenvalues is that on the boundary the eigenvalues must satisfy:

$$|\mu| + |\nu| = |\lambda|,$$

which we of course recognize as the linearity of trace condition. Moreover, in the case when $n = 2$ there is one rhombus inequality of each kind. The vertical rhombus inequality is given by:

$$\mu_1 + \lambda_1 - (\mu_1 + \mu_2 + \nu_1) \geq 0$$

which simplifies to:

$$\lambda_1 \geq \mu_2 + \nu_1$$

and by substituting λ_1 with $(\mu_1 + \mu_2 + \nu_1 + \nu_2 - \lambda_2)$ we get an inequality of the form:

$$(\mu_1 + \mu_2 + \nu_1 + \nu_2 - \lambda_2) \geq \mu_2 + \nu_1$$

which we recognize as one of Weyl's inequalities:

$$\mu_1 + \nu_2 \geq \lambda_2.$$

From the left rhombus inequality we get:

$$\mu_1 + (\mu_1 + \mu_2 + \nu_1) - (\mu_1 + \mu_2) - \lambda_1 \geq 0$$

which simplifies to another of Weyl's inequalities:

$$\mu_1 + \nu_1 \geq \lambda_1.$$

Finally the right rhombus inequality contributes with the following:

$$\lambda_1 + (\mu_1 + \mu_2 + \nu_1) - \mu_1 - (\lambda_1 + \lambda_2) \geq 0$$

resulting in the last of Weyl's inequalities:

$$\mu_2 + \nu_1 \geq \lambda_2.$$

In conclusion, the hive model in two dimensions gives rise to the same inequalities as Horn's conjecture.

3.2 3 × 3 - Hermitian Matrices

When $n = 2$ there is no inner node and so it would be interesting to also attempt to derive the inequalities included in Horn's conjecture for when $n = 3$, seen in Figure

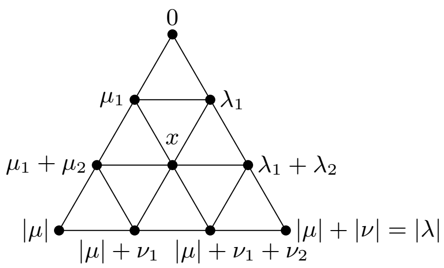


Figure 3.2: A hive of size 3. When $n = 3$ there is only one interior node x.

3.2. First we present them using Theorem 2.20 where the sets of admissible triples given by T_1^3 are of the form:

$$T_1^3 : i + j - 1 = k.$$

This gives the following triples of indices which imply the inequalities on the right:

$$T_1^3 : (I,J,K) = \left\{\begin{array}{l} (\{1\},\{1\},\{1\}), \\ (\{2\},\{1\},\{2\}), \\ (\{1\},\{2\},\{2\}), \\ (\{3\},\{1\},\{3\}), \\ (\{1\},\{3\},\{3\}), \\ (\{2\},\{2\},\{3\}). \end{array}\right\} = \left\{\begin{array}{ll} \mu_1 + \nu_1 \geq \lambda_1, & \text{(W.1)} \\ \mu_2 + \nu_1 \geq \lambda_2, & \text{(W.2)} \\ \mu_1 + \nu_2 \geq \lambda_2, & \text{(W.3)} \\ \mu_3 + \nu_1 \geq \lambda_3, & \text{(W.4)} \\ \mu_1 + \nu_3 \geq \lambda_3, & \text{(W.5)} \\ \mu_2 + \nu_2 \geq \lambda_3. & \text{(W.6)} \end{array}\right.$$

All of these inequalities can be found using Weyl's inequalities. Now the sets for the inequalities in T_2^3, where the sets all have cardinality 2, are given by:

$$T_2^3 : \sum_{u \in U} i_u + \sum_{v \in V} j_v \leq \sum_{w \in W} k_w + \binom{2+1}{2} \qquad \sum_{\substack{i \in I \\ |I|=2}} i + \sum_{\substack{j \in J \\ |J|=2}} j \leq \sum_{\substack{k \in K \\ |K|=2}} k + 3$$

From this we get the following triples of indices which give rise to the inequalities on the right:

$$T_2^3 : (I,J,K) = \left\{\begin{array}{l} (\{1,2\},\{1,2\},\{1,2\}), \\ (\{1,2\},\{1,2\},\{1,3\}), \\ (\{1,2\},\{1,2\},\{2,3\}), \\ (\{1,3\},\{1,2\},\{1,3\}), \\ (\{1,3\},\{1,2\},\{2,3\}), \\ (\{1,2\},\{1,3\},\{1,3\}), \\ (\{1,2\},\{1,3\},\{2,3\}), \\ (\{1,3\},\{1,3\},\{2,3\}), \\ (\{2,3\},\{1,2\},\{2,3\}), \\ (\{1,2\},\{2,3\},\{2,3\}). \end{array}\right\} = \left\{\begin{array}{ll} \mu_1 + \mu_2 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_2, & (3.1) \\ \mu_1 + \mu_2 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_3, & (3.2) \\ \mu_1 + \mu_2 + \nu_1 + \nu_2 \geq \lambda_2 + \lambda_3, & (3.3) \\ \mu_1 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_3, & (3.4) \\ \mu_1 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_2 + \lambda_3, & (3.5) \\ \mu_1 + \mu_2 + \nu_1 + \nu_3 \geq \lambda_1 + \lambda_3, & (3.6) \\ \mu_1 + \mu_2 + \nu_1 + \nu_3 \geq \lambda_2 + \lambda_3, & (3.7) \\ \mu_1 + \mu_3 + \nu_1 + \nu_3 \geq \lambda_2 + \lambda_3, & (3.8) \\ \mu_2 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_2 + \lambda_3, & (3.9) \\ \mu_1 + \mu_2 + \nu_2 + \nu_3 \geq \lambda_2 + \lambda_3. & (3.10) \end{array}\right.$$

Out of these we note that the second and third inequalities are implied by the first, the fifth is implied by the fourth and the seventh is implied by the sixth. Therefore these ten inequalities can be reduced to only (3.1), (3.4), (3.6), (3.8), (3.9) and (3.10). We repeat them below:

$$\mu_1 + \mu_2 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_2, \quad \text{(Fan)}$$
$$\mu_1 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_3, \quad \text{(Lidskii-Wielandt)}$$
$$\mu_1 + \mu_2 + \nu_1 + \nu_3 \geq \lambda_1 + \lambda_3, \quad \text{(Lidskii-Wielandt)}$$
$$\mu_1 + \mu_3 + \nu_1 + \nu_3 \geq \lambda_2 + \lambda_3, \quad \text{(Horn)}$$
$$\mu_2 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_2 + \lambda_3, \quad \text{(Lidskii-Wielandt)}$$
$$\mu_1 + \mu_2 + \nu_2 + \nu_3 \geq \lambda_2 + \lambda_3, \quad \text{(Lidskii-Wielandt)}$$

with the name in the parentheses referencing which other inequality expresses the same condition. One of these, labelled *Horn*, cannot be found from the other inequalities but is part of Horn's inequalities [2]. Finally, as with $n = 2$, we add to this the linearity of trace:

$$\mu_1 + \mu_2 + \mu_3 + \nu_1 + \nu_2 + \nu_3 = \lambda_1 + \lambda_2 + \lambda_3. \quad \text{(Linearity of trace)}$$

We now want to derive the six inequalities above as well as the six inequalities by Weyl and the linearity of trace from the hive model with $n = 3$. From the hive we can get the boundary condition:

$$\mu_1 + \mu_2 + \mu_3 + \nu_1 + \nu_2 + \nu_3 = \lambda_1 + \lambda_2 + \lambda_3, \quad \text{(BC)}$$

which of course corresponds to the linearity of trace. There are three vertical rhombus inequalities when $n = 3$, each one involving the interior node x. We state them below:

$$\mu_1 + \lambda_1 \geq x, \quad \text{(V.1)}$$
$$\mu_1 + \mu_2 + x \geq \mu_1 + |\mu| + \nu_1 \qquad x \geq \mu_1 + \mu_3 + \nu_1, \quad \text{(V.2)}$$
$$x + \lambda_1 + \lambda_2 \geq \lambda_1 + |\mu| + \nu_1 + \nu_2 \qquad x \geq \lambda_1 + \lambda_3 - \nu_3, \quad \text{(V.3)}$$

and in each one we have moved x to be alone on one side. We do the same thing with the three left-slanted rhombus inequalities:

$$\lambda_1 + x \geq \mu_1 + \lambda_1 + \lambda_2 \qquad x \geq \mu_1 + \lambda_2, \quad \text{(L.1)}$$
$$|\mu| + \nu_1 + x \geq \mu_1 + \mu_2 + |\mu| + \nu_1 + \nu_2 \qquad x \geq \mu_1 + \mu_2 + \nu_2, \quad \text{(L.2)}$$
$$|\mu| + \nu_1 + \nu_2 + \lambda_1 + \lambda_2 \geq x + |\mu| + |\nu| \qquad \lambda_1 + \lambda_2 - \nu_3 \geq x, \quad \text{(L.3)}$$

as well as the three right-slanted rhombus inequalities:

$$\mu_1 + x \geq \mu_1 + \mu_2 + \lambda_1 \qquad x \geq \mu_2 + \lambda_1, \quad \text{(R.1)}$$
$$\mu_1 + \mu_2 + |\mu| + \nu_1 \geq |\mu| + x \qquad \mu_1 + \mu_2 + \nu_1 \geq x, \quad \text{(R.2)}$$
$$x + |\mu| + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_2 + |\mu| + \nu_1 \qquad x \geq \lambda_1 + \lambda_2 - \nu_2. \quad \text{(R.3)}$$

We can now derive Weyl's six inequalities: (W.1)-(W.6) by combining pairs of the rhombus inequalities. We have:

$$(R.2) \geq x \geq (R.1) \;=\; (R.2) \geq (R.1),$$

and so we get:

$$\mu_1 + \mu_2 + \nu_1 \geq \mu_2 + \lambda_1,$$

which simplifies to:

$$\mu_1 + \nu_1 \geq \lambda_1. \tag{W.1}$$

We can derive (W.2)-(W.6) similarly:

$$\begin{aligned}
(R.2) \geq (L.1) &= \mu_1 + \mu_2 + \nu_1 \geq \mu_1 + \lambda_2 = \mu_2 + \nu_1 \geq \lambda_2, && \text{(W.2)}\\
(V.1) \geq (R.3) &= \mu_1 + \lambda_1 \geq \lambda_1 + \lambda_2 - \nu_2 = \mu_1 + \nu_2 \geq \lambda_2, && \text{(W.3)}\\
(L.3) \geq (L.2) &= \lambda_1 + \lambda_2 - \nu_3 \geq \mu_1 + \mu_2 + \nu_2 = \\
&\quad \lambda_1 + \lambda_2 \geq \lambda_1 + \lambda_2 + \lambda_3 - (\mu_3 + \nu_1) = \mu_3 + \nu_1 \geq \lambda_3, && \text{(W.4)}\\
(V.1) \geq (V.3) &= \mu_1 + \lambda_1 \geq \lambda_1 + \lambda_3 - \nu_3 = \mu_1 + \nu_3 \geq \lambda_3, && \text{(W.5)}\\
(L.3) \geq (V.2) &= \lambda_1 + \lambda_2 - \nu_3 \geq \mu_1 + \mu_3 + \nu_1 = \\
&\quad \lambda_1 + \lambda_2 \geq \lambda_1 + \lambda_2 + \lambda_3 - (\mu_2 + \nu_2) = \mu_2 + \nu_1 \geq \lambda_2. && \text{(W.6)}
\end{aligned}$$

We can derive the inequality by Fan from (R.2) and (R.3):

$$\begin{aligned}
(R.2) \geq x \geq (R.3) &= \mu_1 + \mu_2 + \nu_1 \geq \lambda_1 + \lambda_2 - \nu_2 \\
&= \mu_1 + \mu_2 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_2. && \text{(Fan)}
\end{aligned}$$

Horn's inequality can be derived from the condition on the boundaries (BC) combined with (L.2) and (V.1). From (BC) we have:

$$(\mu_1 + \mu_2 + \nu_2) + \mu_3 + \nu_1 + \nu_3 = \lambda_1 + \lambda_2 + \lambda_3,$$

where if we replace the parenthesis with x, by (L.2), we will now have an upper bound for the left hand side:

$$x + \mu_3 + \nu_1 + \nu_3 \geq \lambda_1 + \lambda_2 + \lambda_3,$$

by (V.1) we can replace x with $\mu_1 + \lambda_1$ and still have an upper bound for the left hand side:

$$(\mu_1 + \lambda_1) + \mu_3 + \nu_1 + \nu_3 \geq \lambda_1 + \lambda_2 + \lambda_3,$$

and from this upper bound we get inequality (3.8):

$$\mu_1 + \mu_3 + \nu_1 + \nu_3 \geq \lambda_2 + \lambda_3. \tag{Horn}$$

We can get (3.4) by rearranging the terms in (BC):

$$\mu_1 + \mu_2 + \mu_3 + \nu_1 + \nu_2 - \lambda_3 = \lambda_1 + \lambda_2 - \nu_3,$$

and replacing the right hand side with a lower bound x by (L.3):

$$\mu_1 + \mu_2 + \mu_3 + \nu_1 + \nu_2 - \lambda_3 \geq x,$$

and by (R.1) we replace x with $\mu_2 + \lambda_1$ to obtain an even lower bound:

$$\mu_1 + \mu_2 + \mu_3 + \nu_1 + \nu_2 - \lambda_3 \geq \mu_2 + \lambda_1,$$

which is the same as (3.4):

$$\mu_1 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_1 + \lambda_3.$$

If we instead want to derive (3.6) we can do the following:

$$\begin{aligned}(R.2) \geq x \geq (V.3) \quad &= \quad \mu_1 + \mu_2 + \nu_1 \geq \lambda_1 + \lambda_3 - \nu_3 \\ &= \quad \mu_1 + \mu_2 + \nu_1 + \nu_3 \geq \lambda_1 + \lambda_3.\end{aligned}$$

Inequality (3.9) can be derived by first rearranging the terms in (BC):

$$(\mu_1 + \lambda_2) - \lambda_2 + \mu_2 + \mu_3 + \nu_1 + \nu_2 + \nu_3 = \lambda_1 + \lambda_2 + \lambda_3,$$

then letting x be an upper bound for the expression in the parenthesis as given in (L.1):

$$x - \lambda_2 + \mu_2 + \mu_3 + \nu_1 + \nu_2 + \nu_3 \geq \lambda_1 + \lambda_2 + \lambda_3,$$

by (L.3) we put $\lambda_1 + \lambda_2 - \nu_3$ instead of x to achieve a greater bound:

$$(\lambda_1 + \lambda_2 - \nu_3) - \lambda_2 + \mu_2 + \mu_3 + \nu_1 + \nu_2 + \nu_3 \geq \lambda_1 + \lambda_2 + \lambda_3,$$

we now have:

$$\mu_2 + \mu_3 + \nu_1 + \nu_2 \geq \lambda_2 + \lambda_3,$$

which is the same as (3.9).

The final Lidskii-Wielandt inequality (3.10) can be found from rearranging the terms in (BC) so that:

$$(\mu_1 + \mu_3 + \nu_1) + \mu_2 + \nu_2 + \nu_3 = \lambda_1 + \lambda_2 + \lambda_3,$$

and replacing the parenthesis with an upper bound x by (V.2):

$$x + \mu_2 + \nu_2 + \nu_3 \geq \lambda_1 + \lambda_2 + \lambda_3,$$

and by (V.1) we can replace x with $\mu_1 + \lambda_1$ for a greater bound:

$$\mu_1 + \lambda_2 + \mu_2 + \nu_2 + \nu_3 \geq \lambda_1 + \lambda_2 + \lambda_3,$$

which is the same as (3.10):

$$\mu_1 + \mu_2 + \nu_2 + \nu_3 \geq \lambda_2 + \lambda_3.$$

With this we have demonstrated a way to derive Horn's inequalities from the three-dimensional hive model.

4

The Appleby-Whitehead Construction

In this chapter we introduce a modified version of the hive construction by G. Appleby and T. Whitehead, discussed in Subsection 2.5.1. This modified construction is restricted to the specialized setting of pairs of simultaneously diagonalizable Hermitian matrices, (M, N) H_n^{SD}. In Section 4.1 we prove that the reformulation is a valid translation of the original for diagonal matrices. This is followed by the investigation of the construction in a few special cases of diagonalizable matrices in Section 4.2. The special cases covered are pairs of matrices, (M, N) H_n^{SD}, that can be diagonalized such that both matrices are sorted in weakly decreasing order, or that can be diagonalized such that one is in weakly decreasing order and the other weakly increasing, and also generally for (M, N) $H_{n=3}^{SD}$ and (M, N) $H_{n=4}^{SD}$. In Section 4.3 we state and prove the principal result of this thesis, Theorem 4.10, which says that the adapted construction generates valid hives for pairs of simultaneously diagonalizable Hermitian matrices generally.

4.1 The Diagonal Case

We would like to turn the attention to a special case of Hermitian matrices, namely the simultaneously diagonalizable Hermitian matrices H_n^{SD}. Let the pair (M, N) H_n^{SD} be given, with $\mu = \text{spec}(M)$ and $\nu = \text{spec}(N)$. Let their sum be called $L = M + N$ with $\text{spec}(L) = \lambda$.

Then since M, N are Hermitian by the Spectral Decomposition Theorem there exists a unitary matrix U U_n that diagonalizes both M and N in terms of their eigenvalues such that:

$$U \; M U = \text{diag}(\bar{\sigma}(\mu)),$$
$$U \; N U = \text{diag}(\hat{\sigma}(\nu)),$$

where denotes the conjugate transpose and $\bar{\sigma}, \hat{\sigma}$ S_n are two permutations of the eigenvalues. Hives depend only on the eigenvalue triple (μ, ν, λ) of the matrices (M, N, L). By diagonalizing the matrices on the left hand we do not change the spectrum of $M + N$:

$$\text{spec}(M + N) = \text{spec}(U \; (M + N)U),$$

the spectrum is identical to that of the same matrices but multiplied by a permutation matrix and its conjugate,

$$
\begin{aligned}
&= \text{spec}(\Pi_\mu U\ (M\ + N\ )U\,\Pi_\mu),\\
&= \text{spec}(\Pi_\mu U\ M\,U\,\Pi_\mu + \Pi_\mu U\ N\,U\,\Pi_\mu),\\
&= \text{spec}(\Pi_\mu\ \text{diag}(\bar{\sigma}(\mu))\,\Pi_\mu + \Pi_\mu\ \text{diag}(\hat{\sigma}(\nu))\,\Pi_\mu),
\end{aligned}
$$

and to simplify, we choose Π_μ such that it arranges the elements in diag($\bar{\sigma}(\mu)$) in weakly decreasing order,

$$= \text{spec}(\text{diag}(\mu) + \Pi_\mu\ \text{diag}(\hat{\sigma}(\nu))\,\Pi_\mu).$$

The elements in $\hat{\sigma}(\nu)$ are now also reordered but we cannot ensure any particular order in general. Let the new reordering of the vector ν under some permutation $\tilde{\sigma}$ be called $\tilde{\sigma}(\nu)$:

$$= \text{spec}(\text{diag}(\mu) + \text{diag}(\tilde{\sigma}(\nu))).$$

This means that instead of considering $M\,$, $N\,$ and $L\,$, we can, for the sake of hives which only depend on the eigenvalues themselves, consider only the *diagonal* matrices M = diag(μ), N = diag($\tilde{\sigma}(\nu)$) and L = diag(μ) + diag($\tilde{\sigma}(\nu)$). For convenience, $\tilde{\sigma}(\nu)$ will also be denoted $\nu_{\tilde{\sigma}}$.

The eigenvalues of $L = M + N$ are the elements in the vector:

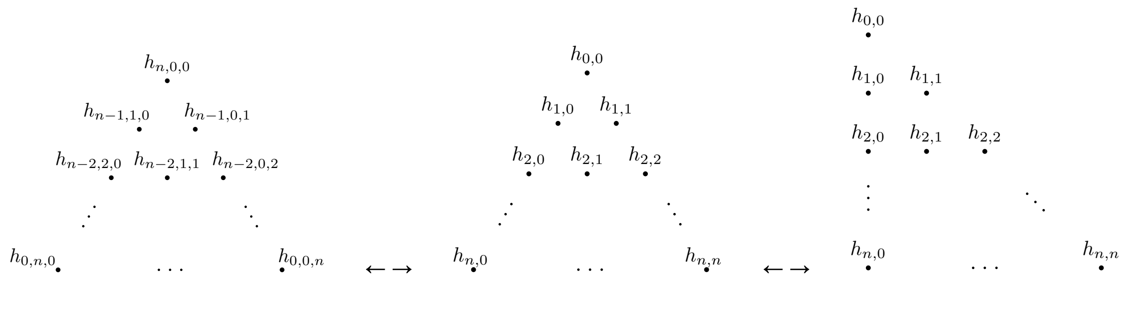
$$\beta = \mu + \nu_{\tilde{\sigma}}$$

where μ is sorted in weakly decreasing order and $\nu_{\tilde{\sigma}}$ is unsorted in general. Consequently, β will be an unsorted version of λ, since the order will depend on the permutation $\tilde{\sigma}$. Instead of the matrices $(M\,, N\,, L\,)$ we now consider diagonal matrices (M, N, L) of the form:

$$M = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix},\ N = \begin{pmatrix} \nu_{\tilde{\sigma}(1)} & 0 & \dots & 0 \\ 0 & \nu_{\tilde{\sigma}(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_{\tilde{\sigma}(n)} \end{pmatrix},\ L = \begin{pmatrix} \mu_1 + \nu_{\tilde{\sigma}(1)} & 0 & \dots & 0 \\ 0 & \mu_2 + \nu_{\tilde{\sigma}(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n + \nu_{\tilde{\sigma}(n)} \end{pmatrix}.$$

The next thing we would like to do is to derive a version of the *AW*-construction for diagonal matrices. But first, notation must be discussed. The use of three indices per node, $h_{i,j,k}$ is superfluous. We can equivalently describe each node by row and column as seen in Figure 4.1, with $h_{p,q}$ where the first index instead gives the row from 0 to n and the second the column from 0 to n. Transitioning from $h^{AW}_{i,j,k}$, where we recall that i gives the row, starting from the bottom, and j and k indicate where in respect to the main diagonal and the anti-diagonal the node sits. If we go from $h^{AW}_{i,j,k}$ to $h^{AW}_{p,q}$, we get our new indices $p = j + k = n - i$ and $q = k$, meaning that:

$$h^{AW}_{i,j,k} = h^{AW}_{(j+k),k}.$$



Original notation. *Row-column notation.* *As a lower triangular matrix.*

Figure 4.1: On the far left is the labelling of nodes used by the authors of the original *AW*-construction and the two figures on the right show the equivalent labelling used in this thesis, first shaped like a hive and then as a lower triangular matrix.

Moreover, no alterations are necessary to the formulation of the sets involved in the construction when making this transition.

In Construction 2.24, each node is defined as a simultaneous maximization over the traces of the two matrices M and N projected onto subspaces V and U. If we restrict ourselves to allowing only pairs of diagonal matrices $(M, N) \in D_n$, which as we now know in this context extends to pairs of simultaneously diagonalizable Hermitian matrices (M', N'), we propose the following construction.

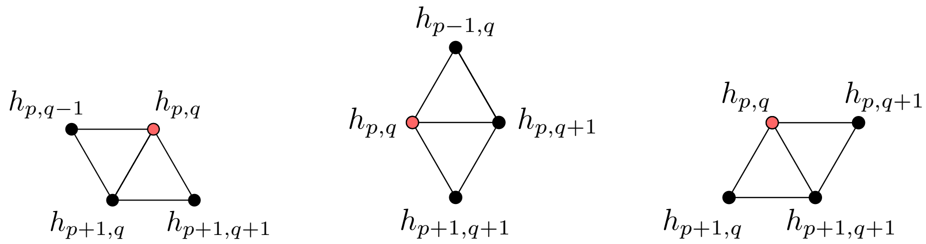
Construction 4.1. *(AWD) For any pair of $n \times n$ diagonal matrices M and N, set the node at position (p, q) to be given by:*

$$h^{AWD}_{p,q} = \max_{J,\ I,|J|=q,|I|=p} \left(\sum_{i \in I} \mu_i + \sum_{j \in J} \nu_{\sigma(j)} \right),$$

where μ is the vector of the eigenvalues of M in weakly decreasing order, and $v_\sigma = \sigma(\nu)$ is some order of the eigenvalues of N that depends on the diagonalization.

The superscript *AWD* is added so that we can distinguish between the regular Appleby-Whitehead construction and the modified Appleby-Whitehead construction for diagonal matrices.

We conclude this section by noting that each of the three types of rhombus inequalities can be expressed generally, using the row-column notation. Each rhombus can be referred to by one of its nodes, with the ones highlighted in red in Figure 4.2 being used for this purpose. In the case of the vertical inequality, all such inequalities are indexed by their left middle node, whereas in the left and right rhombus inequalities the top right and top left nodes, respectively, are used. Let $V_{p,q}$, $L_{p,q}$ and $R_{p,q}$ refer to the difference between the sum of the obtuse and acute nodes in the rhombus of corresponding type with reference index at position p, q. We can now formulate the rhombus inequalities generally. It is for these general expression we will compose our proofs.



Left. Vertical. Right.

Figure 4.2: The three types of rhombus inequalities expressed generally with the red node as reference node for the labelling of quantities: $V_{p,q}$, $L_{p,q}$ and $R_{p,q}$.

The *vertical rhombus inequality* is given by:

$$\underbrace{h^{AWD}_{p,q} + h^{AWD}_{p,q+1} - h^{AWD}_{p-1,q} - h^{AWD}_{p+1,q+1}}_{V_{p,q}} \geq 0 \quad q \quad \{0, \cdots, p-1\}. \tag{4.1}$$

The *left rhombus inequality* is given by:

$$\underbrace{h^{AWD}_{p,q} + h^{AWD}_{p+1,q} - h^{AWD}_{p,q-1} - h^{AWD}_{p+1,q+1}}_{L_{p,q}} \geq 0 \quad q \quad \{1, \cdots, p\}. \tag{4.2}$$

The *right rhombus inequality* is given by:

$$\underbrace{h^{AWD}_{p,q} + h^{AWD}_{p+1,q+1} - h^{AWD}_{p,q+1} - h^{AWD}_{p+1,q}}_{R_{p,q}} \geq 0 \quad q \quad \{0, \cdots, p-1\}. \tag{4.3}$$

4.1.1 Translating the *AW*-construction for diagonal matrices

Now that we have introduced a new version of Appleby and Whitehead's construction, the *AWD*-construction, we would like to show that it is a valid reformulation of the original *AW*-construction in the diagonal case. We claim that this is the case.

PROPOSITION 4.2. *For diagonal matrices the construction defined in 2.24 (h^{AW}) is equivalent to the construction defined in 4.1 (h^{AWD}).*

PROOF OF PROPOSITION 4.2. First, let us restate the preliminaries for h^{AW}. Let W be a n-dimensional complex vector space, with subspaces U and V, such that $\dim(U) = a$ and $\dim(V) = b$, with $U \quad V$ and naturally $a \leq b \leq n$. Let F_{ab} be the set of all such pairs (U, V). From Construction 2.24 nodes are set to:

$$h^{AW}_{i,j,k} = \max_{\substack{(U,V) \quad F_{k,(j+k)} \\ U \quad V \\ \dim(U)=k \\ \dim(V)=j+k}} \left(\mathrm{tr}[M|_V] + \mathrm{tr}[N|_U]\right), \quad k \leq (j+k) \leq n.$$

The first thing we would like to do is rewrite this in the row-column-notation, introduced in Section 4.1,

$$= h^{AW}_{(j+k),k} = \max_{\substack{(U,V) \;\; \mathsf{F}_{k,(j+k)} \\ U \quad V \\ \mathrm{dim}(U)=k \\ \mathrm{dim}(V)=j+k \\ k\leq(j+k)\leq n}} \left(\mathrm{tr}[M|_V] + \mathrm{tr}[N|_U]\right),$$

which as we know is an equivalent notation and does not affect the structure. Set $p = (j + k)$ and $q = k$,

$$h^{AW}_{p,q} = \max_{\substack{(U,V) \;\; \mathsf{F}_{q,p} \\ U \quad V \\ \mathrm{dim}(U)=q \\ \mathrm{dim}(V)=p \\ q\leq p\leq n}} \left(\mathrm{tr}[M|_V] + \mathrm{tr}[N|_U]\right)$$

We recall that $M|_V$ was the notation used for an the composition of the matrix with orthogonal projections: $\pi_V : W \to V$:

$$= \max_{\substack{(U,V) \;\; \mathsf{F}_{q,p} \\ U \quad V \\ \mathrm{dim}(U)=q \\ \mathrm{dim}(V)=p \\ q\leq p\leq n}} \left(\mathrm{tr}[\pi_V \circ M \circ \pi_V] + \mathrm{tr}[\pi_U \circ N \circ \pi_U]\right)$$

Note that π_U and π_V are orthogonal projections, and as such, are linear mappings. Every linear mapping can be represented by a matrix and we now let P be the matrix that describes π_U and Q be the matrix that describes π_V:

$$= \max_{\substack{(P,Q) \;\; \mathsf{P}_{(p,q)} \\ q\leq p\leq n}} \left(\mathrm{tr}[PMP] + \mathrm{tr}[QNQ]\right)$$

where,

$$\mathsf{P}_{(p,q)} = \{(P,Q) : P^2 = P = P \;, Q^2 = Q = Q \;, \\ PQ = QP = Q, \mathrm{rank}(P) = p, \mathrm{rank}(Q) = q\}.$$

Now, we use the cyclic property of the trace, $\mathrm{tr}[ABC] = \mathrm{tr}[BCA]$, to write:

$$= \max_{\substack{(P,Q) \;\; \mathsf{P}_{(p,q)} \\ q\leq p\leq n}} \left(\mathrm{tr}\left[MP^2\right] + \mathrm{tr}\left[NQ^2\right]\right),$$

and since we have projection matrices, $P^2 = P, Q^2 = Q$:

$$= \max_{\substack{(P,Q) \;\; \mathsf{P}_{(p,q)} \\ q\leq p\leq n}} \left(\mathrm{tr}[MP] + \mathrm{tr}[NQ]\right),$$

Because M and N are diagonal matrices, the diagonal entries in MP and NQ become the product of diagonal entries in the respective matrices, and so the traces are simply the sums of products of diagonal entries,

$$= \max_{\substack{(P,Q)\ \ P_{(p,q)} \\ q \le p \le n}} \left(\mathrm{tr}[\mathrm{diag}(\mu)P] + \mathrm{tr}[\mathrm{diag}(\nu_\sigma)Q]\right),$$

$$= \max_{\substack{(P,Q)\ \ P_{(p,q)} \\ q \le p \le n}} \left(\sum_{i=1}^{n} \mu_i P_{i,i} + \sum_{i=1}^{n} \nu_{\sigma(i)} Q_{i,i}\right).$$

We have arrived at a new expression for the *AW*-construction, which will be useful in showing $h^{AW} = h^{AWD}$. The proof of Proposition 4.2 will be resumed once two supporting lemmas have been established.

Lemma 4.3. *The sequences of diagonal elements,* $((P_{i,i})_{i=1}^n, (Q_{i,i})_{i=1}^n)$, *in matrices* $(P,Q)\ \ P_{(p,q)}$, *belong to the set* $S_{(p,q)}$ *of sequences of the form:*

$$S_{(p,q)} = \{((s_i)_{i=1}^n, (t_i)_{i=1}^n) : 0 \le t_i \le s_i \le 1, \sum_{i=1}^{n} s_i = p, \sum_{i=1}^{n} t_i = q\}.$$

Proof of Lemma 4.3. P and Q are projection matrices and as such are idempotent: $P^2 = P$, $Q^2 = Q$, as well as Hermitian: $P = P\ $, $Q = Q\ $. If v is an eigenvector of P with eigenvalue γ then:

$$\gamma v = Pv = P^2 v = P(Pv) = P(\gamma v) = \gamma(Pv) = \gamma^2 v \ = \ \ \gamma^2 = \gamma \underset{\substack{\gamma\ \ \text{R since P} \\ \text{is Hermitian}}}{=} \gamma \quad \{0,1\}.$$

Matrices P and Q are Hermitian with non-negative eigenvalues, which means that they are positive semi-definite $P \ge 0$, $Q \ge 0$. We take:

$$0 \le (P-Q)^2 = P^2 + Q^2 - 2PQ = P + Q - 2Q = P - Q$$

and since:

$$P - Q \ge 0 \ = \ \ P \ge Q,$$

and so $P_{i,i} \ge Q_{i,i},\ \ i$. It now remains to show $0 \le P_{i,i} \le 1$ and that the diagonal entries have the right sums.

We see that:

$$\sum_{i=1}^{n} P_{i,i} = tr(P) = (\text{number of eigenvalues} = 0),$$

which, since P is a projection matrix and thus diagonalizable, and so the dimensions spanned by the matrix equals the non-zero eigenvalues:

$$= \mathrm{rank}\,(P) = p.$$

Finally, if we now let γ be the vector of eigenvalues sorted in weakly decreasing order for P, and take a Hermitian matrix X with $\mathrm{rank}\,(X) = k$ and $X^2 = X$ we get using Theorem 2.4 that:

$$\sum_{i=1}^{k} \gamma_i = \max\left(\mathrm{tr}(PX)\right),$$

where if we choose X to be diagonal with entries given by a sequence of ones followed by a sequence of zeros, we derive an upper bound on the diagonal entries of P,

$$\sum_{i=1}^{k} \gamma_i = \max\left(\mathrm{tr}(PX)\right) \geq \mathrm{tr}(PX) = \sum_{i=1}^{k} P_{i,i}.$$

Because the identity matrix is also a projection matrix, and the difference between two projection matrices is a projection matrix, we get:

$$0 \leq (I - P)^2 = I^2 - 2P + P^2 = I - P = \quad 1 - P_{i,i} \geq 0,$$

and so we conclude that $0 \leq P_{i,i} \leq 1$. A similar argument can be made for Q. □

The next lemma is concerned with where the extreme points of the set of sequences $S_{(p,q)}$ are located.

Lemma 4.4. *For the set, $S_{(p,q)}$, the following two statements hold:*

(i) $S_{(p,q)}$ is convex, i.e., for every $(s,t) \in S_{(p,q)}$ and $(s',t') \in S_{(p,q)}$ and every $\theta \in [0,1]$, $\theta(s,t) + (1-\theta)(s',t') \in S_{(p,q)}$.

(ii) The extreme points of $S_{(p,q)}$, i.e the points that cannot be written as non-trivial convex combinations of other points, are of the form:

$$\begin{cases} s_i \in \{0,1\} & \text{with } \sum_{i=1}^{n} s_i = p, \text{ i.e. there are } p \text{ 1's}, \\ t_i \in \{0,1\} & \text{with } \sum_{i=1}^{n} t_i = q, \text{ i.e. there are } q \text{ 1's}. \end{cases}$$

Proof of Lemma 4.4. Let us first show (i) that $S_{(p,q)}$ is a convex set which is very straightforward. Let us take two pairs of sequences from this set $(s',t'), (s'',t'') \in S_{(p,q)}$ and take arbitrary $\theta \in [0,1]$. If we can show that the resulting pair of sequences:

$$(s,t) = \theta\,(s',t') + (1-\theta)\,(s'',t''),$$

belongs to $S_{(p,q)}$ then the set is convex. We have

$$s = (\theta s'_1 + (1-\theta)s''_1, \cdots, \theta s'_n + (1-\theta)s''_n)$$

and

$$t = (\theta t'_1 + (1-\theta)t''_1, \cdots, \theta t'_n + (1-\theta)t''_n).$$

First of all if $\sum_{i=1}^{n} s'_i = p$ and $\sum_{i=1}^{n} s''_i = p$ then:

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \theta s'_i + (1-\theta)s''_i = \theta \sum_{i=1}^{n} s'_i + (1-\theta)\sum_{i=1}^{n} s''_i = \theta p + (1-\theta)p = p,$$

and similarly,

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \theta t'_i + (1-\theta)t''_i = \theta \sum_{i=1}^{n} t'_i + (1-\theta)\sum_{i=1}^{n} t''_i = \theta q + (1-\theta)q = q.$$

Next, if $s_i', s_i'' \in [0,1]$ then $0 = \theta 0 + (1-\theta)0 = 0 \leq s_i = \theta s_i' + (1-\theta)s_i'' \leq \theta + (1-\theta) = 1$ and thus $s_i \in [0,1]$, $\forall i$. Same with t_i. Finally we need to show that $t_i \leq s_i$, which:

$$t_i = \theta t_i' + (1-\theta)t_i'' \leq \theta s_i' + (1-\theta)s_i'' = s_i,$$

follows from $t_i' \leq s_i'$ and $t_i'' \leq s_i''$.

We now want to show part (ii) of Lemma 4.4, that the extreme points of $S_{(p,q)}$ are sequences s_i and t_i made up from $\{0,1\}$. Let us start by showing that any $(s,t) \in S_{(p,q)} : s_i, t_i \in \{0,1\}, \forall i$ is extremal. Let (s,t) be such a pair of sequences and write:

$$(s,t) = (\theta s' + (1-\theta)s'', \theta t' + (1-\theta)t'')$$

for $(s',t'), (s'',t'') \in S_{(p,q)}, (s',t') \neq (s'',t'')$ and $\theta \in (0,1)$. We see directly that since $s', s'', t', t'' \geq 0$ and $\theta > 0$ that the sequences s' and s'' must both have zeros wherever s has zeros and the same goes for t. But if s' and s'' have $n-p$ zeros then since $s', s'' \leq 1$, they must have ones in all other positions in order for the elements to sum to p. Therefore $s' = s'' = s$, with the same for $t' = t'' = t$ with q. Since (s,t) cannot be written as convex combinations of two points in the set $S_{(p,q)}$ they are extreme points.

Let us now show that other points are not extremal. Take a point $(s,t) \in S_{(p,q)}$ which is not of the form in (ii). Then for at least one out of s and t, there must exist at least two indices, i and j, such that $s_i, s_j \notin \{0,1\}$ or $t_i, t_j \notin \{0,1\}$. First let us assume that $\exists i, j \in (1, \cdots n) : s_i, s_j \notin \{0,1\}$. Then let:

$$s' = (s_1, \cdots, s_i + \epsilon, \cdots, s_j - \epsilon, \cdots)$$

and

$$s'' = (s_1, \cdots, s_i - \epsilon, \cdots, s_j + \epsilon, \cdots)$$

this way we see that:

$$\sum_{k=1}^{n} s_k' = \sum_{k=1}^{n} s_k'' = \sum_{k=1}^{n} s_k,$$

and

$$s = \theta s' + (1-\theta)s'' \quad \text{for } \theta = \frac{1}{2} \in (0,1).$$

Let us now tentatively put:

$$t = t' = t''$$

and we realize that while $(s',t') \neq (s'',t'')$ since $s' \neq s''$, a potential problem arises if the changes at s_i or s_j disrupts the condition $t_i \leq s_i$ for either (s',t') or (s'',t'')). Since we add ϵ at an earlier index in s', the issue can only occur in s'', such that at some index in $l \in [1, \cdots, i]$ $s_l'' < t_l''$. If that is the case we can modify the point t'', so that $t_l'' = t_l - \epsilon$, but then we must add ϵ at a later index. Since we have subtracted an ϵ, we need to find an index where we can add ϵ. That index must occur at or after j and not be 1. We know that $s_j \leq 1$ since by assumption it is between zero and one, we must be able to add it to t_j. We get:

$$t \ \ = (t_1, \cdots, t_l - \epsilon, \cdots, t_j + \epsilon, \cdots), \quad l \leq i.$$

Now we must change t to balance the changes made to t :

$$t \ = (t_1, \cdots, t_l + \epsilon, \cdots, t_j - \epsilon, \cdots), \quad l \leq i.$$

We move on to showing that other points are not extremal. Let us first assume that we have a point (s, t) $\quad S_n$, where in s there exists an index i such that $s_i \quad (0, 1)$. If there exists one such element s_i, then there must exists at least one more s_j, since all the s-terms must sum to an integer, p. We now want to write (s, t) as the convex combination of $(s\ , t\ )$ and $(s\ \ , t\ \ )$, where $(s\ , t\ ) = (s\ \ , t\ \ )$. We get six di erent cases for how we need to modify the sequences. Let δ_i denote the vector $\delta_i = (0, \cdots, \underbrace{1}_{i}, \cdots, 0)$, with zeros everywhere except at the i-th position and $\epsilon > 0$, which can be made arbitrarily small.

Case 1: $s_i > t_i$ **and** $s_j > t_j$. Let:

$$s\ = s + \epsilon\delta_i - \epsilon\delta_j \text{ and } s\ \ = s - \epsilon\delta_i + \epsilon\delta_j,$$
$$t\ = t\ \ = t.$$

Case 2: $s_i > t_i > 0$ **and** $s_j = t_j$. Let:

$$s\ = s + \epsilon\delta_i - \epsilon\delta_j \text{ and } s\ \ = s - \epsilon\delta_i + \epsilon\delta_j,$$
$$t\ = t + \epsilon\delta_i - \epsilon\delta_j \text{ and } t\ \ = t - \epsilon\delta_i + \epsilon\delta_j.$$

Case 3: $s_i > t_i = 0$ **and** $s_j = t_j$. Let:

$$s\ = s + \epsilon\delta_i - \epsilon\delta_j \text{ and } s\ \ = s - \epsilon\delta_i + \epsilon\delta_j,$$
$$t\ = t + \epsilon\delta_k - \epsilon\delta_j \text{ and } t\ \ = t - \epsilon\delta_k + \epsilon\delta_j.$$

We cannot modify $t_i = 0$ by subtracting ϵ and since we need to make symmetric adjustments on the indices we choose, we instead find an index $k : t_k \quad (0, 1)$. How do we know that such an index k exists? Since $s_j = t_j \quad (0, 1)$ and $t_i = 0$, there must exist at least one more non-binary entry in t, since all entries should sum to an integral number q.

Case 4: $s_i = t_i$ **and** $s_j > t_j > 0$. Identical to case 2.

Case 5: $s_i = t_i$ **and** $s_j > t_j = 0$. Similarly to case 3 we get:

$$s\ = s + \epsilon\delta_i - \epsilon\delta_j \text{ and } s\ \ = s - \epsilon\delta_i + \epsilon\delta_j$$

$$t \ = t + \epsilon\delta_i - \epsilon\delta_k \text{ and } t \ = t - \epsilon\delta_i + \epsilon\delta_k,$$

where we again let t_k be an element $t_k \quad (0, 1)$.

Case 6: $s_i = t_i$ **and** $s_j = t_j$**.** Let:

$$s \ = s + \epsilon\delta_i - \epsilon\delta_j \text{ and } s \ = s - \epsilon\delta_i + \epsilon\delta_j$$
$$t \ = t + \epsilon\delta_i - \epsilon\delta_j \text{ and } t \ = t - \epsilon\delta_i + \epsilon\delta_j$$

We now let $\theta = \frac{1}{2} \quad (0, 1)$ and so for each of these cases we get that:

$$(s, t) = \frac{1}{2}(s \ , t \ ) + \frac{1}{2}(s \ , t \ ),$$

since the changes at indices i, j and k where applicable cancel each other out. In none of the cases $(s \ , t \ ) = (s \ , t \ )$. This was all based on an assumption that there existed at least one element in $s_i \ / \ \{0, 1\}$. If we instead assume that s is fully binary while there exist an i such that $t_i \quad (0, 1)$. These circumstances are quite simple to work with, since $s_i \geq t_i$ and if $t_i > 0$ it must be that $s_i = 1$, same with the entry $t_j \quad (0, 1)$ that must exist. For this case we can simply put:

$$s \ = s \ = s \text{ and}$$
$$t \ = t + \epsilon\delta_i - \epsilon\delta_j \text{ and } t \ = t - \epsilon\delta_i + \epsilon\delta_j,$$

this works since both $s_i = s_j = 1$ there is slack to both add and subtract an arbitrarily small ϵ from t_i and t_j. Once again, the ϵ's added and subtracted cancel each other so that $\frac{1}{2}(s \ , t \ ) + \frac{1}{2}(s \ , t \ ) = (s, t)$, while ensuring $(s \ , t \ ) = (s \ , t \ )$. □

Proof of Proposition 4.2 given Lemmas 4.3 and 4.4. Now we have shown that:

$$h^{AW}_{p,q} = \max_{\substack{(U,V) \ F_{q,p} \\ U \ V \\ \dim(U)=q \\ \dim(V)=p \\ q\leq p\leq n}} \left(\mathrm{tr}[M|_V] + \mathrm{tr}[N|_U]\right),$$

can be written,

$$= \max_{\substack{(P,Q) \ P_{(p,q)} \\ q\leq p\leq n}} \left(\sum_{i=1}^{n} \mu_i P_{i,i} + \sum_{i=1}^{n} \nu_{\sigma(i)} Q_{i,i}\right).$$

In the above expression we can always choose (P, Q) to be diagonal matrices with 0's and 1's as entries. But then we see that the position of the ones in (P, Q) correspond to the choice of indices in the pair of sets (I, J) used in the *AWD* expression. Therefore:

$$h^{AW}_{p,q} \geq h^{AWD}_{p,q}.$$

On the other hand, we know that the maximum of a linear function of variables, (s_i) and (t_i), over a convex set $S_{(p,q)}$, is attained at an extreme point and therefore:

$$h^{AW}_{p,q} \leq \max_{\left((s_i)^n_{i=1},(t_i)^n_{i=1}\right) \quad S(p,q)} \left(\sum_{i=1}^{n} \mu_i s_i + \sum_{i=1}^{n} \nu_{\sigma(i)} t_i \right) = h^{AWD}_{p,q}.$$

Taking these two bounds together means that in the diagonal case:

$$h^{AW}_{p,q} = h^{AWD}_{p,q}.$$

□

4.1.2 Verifying the borders

The borders in the AW-construction are shown explicitly to be correct in [1], and although strictly speaking redundant, in light of Proposition 2.25, we now verify that h^{AWD} also gives the correct borders.

We need to show that the nodes from Construction 4.1 are correct for the left, right and bottom border of the hive, regardless of the permutation σ of ν. The left border is made up of all nodes in the first column, i.e. when $q = 0$, and are thus given by:

$$h^{AWD}_{p,0} = \max_{\substack{J \quad I \\ |J|=0,|I|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right), \quad \text{for } p \quad \{0, \cdots, n\},$$

which since J is empty and μ is sorted from largest to smallest entry, are simply the cumulative sums of μ:

$$h^{AWD}_{p,0} = \max_{|I|=p} \left(\sum_{i \quad I} \mu_i \right) = \sum_{i=1}^{p} \mu_i,$$

which is the desired shape. The bottom border is made up from the nodes in the last row, i.e. when $p = n$, and so nodes are of the form:

$$h^{AWD}_{n,q} = \max_{\substack{J \quad I \\ |J|=q,|I|=n}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right), \quad \text{for } q \quad \{0, \cdots, n\}.$$

We note that since $p = n$ all nodes in μ are included in the maximization, reducing the simultaneous maximization into a maximization solely on the $\nu_{\sigma(j)}$ part. In addition to the sum of the elements in the μ-vector we once again we end up with a cumulative sum, this time of ν:

$$h^{AWD}_{p,q} = |\mu| + \max_{|J|=q} \left(\sum_{j \quad J} \nu_{\sigma(j)} \right) = |\mu| + \sum_{j=1}^{q} \nu_j,$$

thereby satisfying the requirements for the bottom border of the hive. Lastly, the right border is comprised of the nodes on the main diagonal, which is easily seen in the lower triangular matrix representation of the hive in Figure 4.1. Nodes along the right border are therefore of the form:

$$h^{AWD}_{p,p} = \max_{\substack{J \quad I \\ |I|=p, |J|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right), \quad \text{for } p \quad \{0, \cdots, n\},$$

where we note that the condition $J \quad I$ together with $p = q$ means that $I = J$,

$$= \max_{\substack{I=J \\ |I|=|J|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right), \quad \text{for } p \quad \{0, \cdots, n\},$$

and so, simplified,

$$= \max_{|I|=p} \left(\sum_{i \quad I} \mu_i + \sum_{i \quad I} \nu_{\sigma(i)} \right), \quad \text{for } p \quad \{0, \cdots, n\}.$$

Now, since we do not know the order of ν_σ in general, we will introduce some new notation to make computations go more smoothly. Let:

$$\beta = \mu + \nu_\sigma = (\mu_1 + \nu_{\sigma^{-1}(1)}, \cdots, \mu_n + \nu_{\sigma^{-1}(n)})$$

where in general we do not know the particular order of β. Only when the β's have been sorted in weakly descending order, we get λ:

$$\lambda = (\beta_{i_1}, \cdots, \beta_{i_n}),$$

with the order $i_1, \cdots, i_N$ s.t: $\beta_{i_1} \geq \cdots \geq \beta_{i_n}$.

Continuing with the right border, we can now use the fact that the two subsets are one and the same, to rewrite the maximization in terms of β:

$$h^{AWD}_{p,p} = \max_{|I|=p} \left(\sum_{i \quad I} \mu_i + \sum_{i \quad I} \nu_{\sigma(i)} \right) = \max_{|I|=p} \left(\sum_{i \quad I} \beta_i \right) = \sum_{i=1}^{p} \lambda_i.$$

The right border in h^{AWD} has now been shown to be made from the cumulative sums of λ, which is the correct form.

4.2 Special cases

Now that the correctness of the borders in h^{AWD} has been independently confirmed, we begin to take the first steps towards proving the construction in the general case. The process of proving a select few special cases of gradually increasing di□culty ultimately inspired the proof of the general case.

4.2.1 The identity permutation

Later on we want to examine h^{AWD} for an arbitrary permutation σ of ν, but for now we will content ourselves with studying pairs $(M, N)\quad H_n^{SD}$ that can be diagonalized such that $\sigma(\nu) = \nu$. Therefore, by the reasoning in Section 4.1, we can instead consider the matrices:

$$M = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad N = \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_n \end{pmatrix}, \quad L = \begin{pmatrix} \mu_1 + \nu_1 & 0 & \dots & 0 \\ 0 & \mu_2 + \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n + \nu_n \end{pmatrix}.$$

Importantly, because of the assumption of the identity permutation of ν we now know that in this case λ is of the form:

$$\lambda = \mu + \nu,$$

where the entries in λ are sorted in weakly decreasing order as a consequence of μ and ν being sorted that way.

PROPOSITION 4.5. *The construction $h_{p,q}^{AWD}$ gives valid hives for simultaneously diagonalizable Hermitian matrices that can be diagonalized such that both matrices are sorted in weakly decreasing order.*

With the identity permutation the maximization in Construction 4.1 can be simplified into summations over the first p and q entries in the respective vectors, since the vectors are both sorted in weakly decreasing order and these entries will maximize the sums:

$$h_{p,q}^{AWD} = \max_{J \quad I, |J|=q, |I|=p} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) = \sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q} \nu_j .$$

The borders of h^{AWD} have already been established and so what remains to prove is that the rhombus inequalities (4.1), (4.2) and (4.3) discussed in Section 2.4 are satisfied.

PROOF OF PROPOSITION 4.5. Any vertical rhombus inequality $V_{p,q}$ can be indexed by its left middle node, $h_{p,q}$. The vertical rhombus inequality can then be shown:

$$\begin{aligned} V_{p,q} &= h_{p,q}^{AWD} + h_{p,q+1}^{AWD} - h_{p-1,q}^{AWD} - h_{p+1,q+1}^{AWD} \\ &= \left(\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q} \nu_j \right) + \left(\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q+1} \nu_j \right) - \left(\sum_{i=1}^{p-1} \mu_i + \sum_{j=1}^{q} \nu_j \right) - \left(\sum_{i=1}^{p+1} \mu_i + \sum_{j=1}^{q+1} \nu_j \right), \end{aligned}$$

where the ν-terms cancel each other out,

$$= \sum_{i=1}^{p} \mu_i + \sum_{i=1}^{p} \mu_i + \sum_{i=1}^{p-1} \mu_i - \sum_{i=1}^{p+1} \mu_i = \mu_p - \mu_{p+1} \geq 0,$$

since $\mu_p \geq \mu_{p+1}$.

We can show for an arbitrary left rhombus inequality $L_{p,q}$, where $L_{p,q}$ is indexed by its right upper node $h_{p,q}$, that:

$$
\begin{aligned}
L_{p,q} &= h^{AWD}_{p,q} + h^{AWD}_{p+1,q} - h^{AWD}_{p,q-1} - h^{AWD}_{p+1,q+1} \\
&= \left(\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q} \nu_j\right) + \left(\sum_{i=1}^{p+1} \mu_i + \sum_{j=1}^{q} \nu_j\right) - \left(\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q-1} \nu_j\right) - \left(\sum_{i=1}^{p+1} \mu_i + \sum_{j=1}^{q+1} \nu_j\right),
\end{aligned}
$$

this time the μ-terms cancel each other out,

$$
= \sum_{j=1}^{q} \nu_j + \sum_{j=1}^{q} \nu_j - \sum_{j=1}^{q-1} \nu_j - \sum_{j=1}^{q+1} \nu_j = \nu_q - \nu_{q+1} \geq 0,
$$

since $\nu_q \geq \nu_{q+1}$.

For a right rhombus inequality $R_{p,q}$ indexed by its left upper node $h_{p,q}$ we get:

$$
\begin{aligned}
R_{p,q} &= h^{AWD}_{p,q} + h^{AWD}_{p+1,q+1} - h^{AWD}_{p+1,q} - h^{AWD}_{p,q+1} \\
&= \left(\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q} \nu_j\right) + \left(\sum_{i=1}^{p+1} \mu_i + \sum_{j=1}^{q+1} \nu_j\right) - \left(\sum_{i=1}^{p+1} \mu_i + \sum_{j=1}^{q} \nu_j\right) - \left(\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q+1} \nu_j\right) \\
&= 0,
\end{aligned}
$$

and so the right rhombus inequality is satisfied with equality. □

This completes the proof of the validity of Construction 4.1 for pairs of simultaneously diagonalizable Hermitian matrices where both matrices can be diagonalized with entries sorted in weakly decreasing order.

4.2.2 The reverse permutation

In this section we consider matrices M, N H_n^{SD} that can be diagonalized such that $\sigma(\nu) = (\nu_n, \nu_{n-1}, \cdots, \nu_1)$, which we will call the *reverse permutation*. It is therefore su□cient to consider diagonal matrices (M, N, L) of the form:

$$
M = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad N = \begin{pmatrix} \nu_n & 0 & \dots & 0 \\ 0 & \nu_{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_1 \end{pmatrix}, \quad L = \begin{pmatrix} \mu_1 + \nu_n & 0 & \dots & 0 \\ 0 & \mu_2 + \nu_{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n + \nu_1 \end{pmatrix}.
$$

The λ vector in this case will be comprised of entries of the form:

$$
\beta_i = \mu_i + \nu_{\sigma(i)} = \mu_i + \nu_{n-i+1},
$$

which unlike the previous case when σ was the identity permutation of ν cannot be assumed to be in weakly decreasing order. We can rewrite Construction 4.1 using β. Let the superscript β^k signify the k-last entries in β: $(\mu_{n-k+1} + \nu_k, ..., \mu_n + \nu_1)$ and the subscript β^k_l denote the l-th largest entry in the subvector β^k, even though its entries cannot be specified in general. An entry with only the subscript, β_k, will be used to refer to the k-th entry in β: $\beta_k = \mu_k + \nu_{n-k+1}$.

In the expression for entries given by $h^{AWD}_{p,q}$ it can be noted that since $J \quad I$, there are q indices that should be chosen from both μ and ν, which is the same as choosing q entries from the vector β. The remaining $p - q$ entries should be selected exclusively from the μ-vector and distinct from the μ's that will be part of the chosen β's. The simultaneous maximization of the sums of entries with indices $j \quad J$ and $i \quad I \setminus J$ can, due to the order of the entries in this special case, similarly to when σ was the identity permutation, be reduced to the maximization of their respective sums:

$$\begin{aligned} h^{AWD}_{p,q} &= \max_{J \quad I, |J|=q, |I|=p} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) \\ &= \max_{J \quad I, |J|=q, |I|=p} \left(\sum_{i \quad I \setminus J} \mu_i + \sum_{j \quad J} \beta_j \right) \\ &= \sum_{k=1}^{p-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p-q)}. \end{aligned} \tag{4.4}$$

This is because μ is sorted in weakly descending order and ν in weakly ascending order, which means that it will always be more beneficial to include the single μ's with as low index as possible, i.e from index 1 up to $p - q$. This does not risk overlooking a better β, since the ν's within $(1, ..., p - q)$ will always be smaller than the ones from the selected β within $(p - q, ..., n)$. This greatly simplifies the expression but there is still a subscript l on β which depends on the actual values of μ and ν. We make the following proposition.

Proposition 4.6. *The construction $h^{AWD}_{p,q}$ gives valid hives for simultaneously diagonalizable Hermitian matrices that can be diagonalized such that one matrix is sorted in weakly decreasing order and the other matrix in weakly increasing order.*

Proof of Proposition 4.6. We now want to verify the rhombus inequalities using this simplified expression (4.4). They can be proven in general as indexed relative to some fixed node, seen in Figure 4.2. Every vertical rhombus inequality, referred to by its left node, can be written:

$$\begin{aligned} V_{p,q} &= h^{AWD}_{p,q} + h^{AWD}_{p,q+1} - h^{AWD}_{p-1,q} - h^{AWD}_{p+1,q+1} \\ &= \left(\sum_{k=1}^{p-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p-q)} \right) + \left(\sum_{k=1}^{p-(q+1)} \mu_k + \sum_{l=1}^{q+1} \beta_l^{n-(p-(q+1))} \right) \\ &- \left(\sum_{k=1}^{(p-1)-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p-(q+1))} \right) - \left(\sum_{k=1}^{(p+1)-(q+1)} \mu_k + \sum_{l=1}^{q+1} \beta_l^{n-(p+1-(q+1))} \right) \\ &= \sum_{k=1}^{p-q} \mu_k + \sum_{k=1}^{p-q-1} \mu_k - \sum_{k=1}^{p-1-q} \mu_k - \sum_{k=1}^{p-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-p+q} + \sum_{l=1}^{q+1} \beta_l^{n-p+q+1} \\ &- \sum_{l=1}^{q} \beta_l^{n-p+q+1} - \sum_{l=1}^{q+1} \beta_l^{n-p+q} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{q} \beta_l^{n-p+q} + \sum_{l=1}^{q+1} \beta_l^{n-p+q+1} - \sum_{l=1}^{q} \beta_l^{n-p+q+1} - \sum_{l=1}^{q+1} \beta_l^{n-p+q} \\
&= \beta_{q+1}^{n-p+q+1} - \beta_{q+1}^{n-p+q} \geq 0.
\end{aligned}
$$

In the last row both entries are the $(q+1)$-th largest in their respective lists and since the former contains an additional entry to select among, its $(q+1)$-th entry must be at least as big as the latter's. The left rhombus inequality, as indexed by its upper right node, can be shown in the following way:

$$
\begin{aligned}
L_{p,q} &= h_{p,q}^{AWD} + h_{p+1,q}^{AWD} - h_{p,q-1}^{AWD} - h_{p+1,q+1}^{AWD} \\
&= \left(\sum_{k=1}^{p-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p-q)} \right) + \left(\sum_{k=1}^{p+1-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p+1-q)} \right) \\
&- \left(\sum_{k=1}^{p-(q-1)} \mu_k + \sum_{l=1}^{q-1} \beta_l^{n-(p-(q-1))} \right) - \left(\sum_{k=1}^{p+1-(q+1)} \mu_k + \sum_{l=1}^{q+1} \beta_l^{n-(p+1-(q+1))} \right) \\
&= \sum_{k=1}^{p-q} \mu_k + \sum_{k=1}^{p-q+1} \mu_k - \sum_{k=1}^{p-q+1} \mu_k - \sum_{k=1}^{p-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-p+q} + \sum_{l=1}^{q} \beta_l^{n-p+q-1} \\
&- \sum_{l=1}^{q-1} \beta_l^{n-p+q-1} - \sum_{l=1}^{q+1} \beta_l^{n-p+q} \\
&= \left(\sum_{l=1}^{q} \beta_l^{n-p+q-1} - \sum_{l=1}^{q-1} \beta_l^{n-p+q-1} \right) + \left(\sum_{l=1}^{q} \beta_l^{n-p+q} - \sum_{l=1}^{q+1} \beta_l^{n-p+q} \right) \\
&= \beta_q^{n-p+q-1} - \beta_{q+1}^{n-p+q} \geq 0.
\end{aligned}
$$

In the last row the two entries: $\beta_q^{n-p+q-1}$ and β_{q+1}^{n-p+q} are from lists that can diﬀer by at most one entry, depending on the inserted in the second list. If the additional entry in the second list is greater than the q-th largest entry in the first then the additional entry is inserted somewhere within $(1, q)$. That in turn pushes the previously smallest entry $\beta_q^{n-p+q-1}$ to become the the $q+1$-th largest entry β_{q+1}^{n-p+q}, which would make result in zero. If, on the other hand, the additional entry in $\beta^{n-p+q+1}$ is smaller or equal to β_q^{n-p+q} the order of entries is preserved and something smaller than or equal to is being subtracted from something greater than. Either way the inequality holds. Lastly the right rhombus inequality, indexed by its upper left node, is shown:

$$
\begin{aligned}
R_{p,q} &= h_{p,q}^{AWD} + h_{p+1,q+1}^{AWD} - h_{p+1,q}^{AWD} - h_{p,q+1}^{AWD} \\
&= \left(\sum_{k=1}^{p-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p-q)} \right) + \left(\sum_{k=1}^{(p+1)-(q+1)} \mu_k + \sum_{l=1}^{q+1} \beta_l^{n-(p+1-(q+1))} \right) \\
&- \left(\sum_{k=1}^{p+1-q} \mu_k + \sum_{l=1}^{q} \beta_l^{n-(p+1-q)} \right) - \left(\sum_{k=1}^{p-(q+1)} \mu_k + \sum_{l=1}^{q+1} \beta_l^{n-(p-(q+1))} \right) \\
&= \sum_{k=1}^{p-q} \mu_k + \sum_{k=1}^{p-q} \mu_k - \sum_{k=1}^{p-q+1} \mu_k - \sum_{k=1}^{p-q-1} \mu_k \\
&+ \sum_{l=1}^{q} \beta_l^{n-p+q} + \sum_{l=1}^{q+1} \beta_l^{n-p+q} - \sum_{l=1}^{q} \beta_l^{n-p+q-1} - \sum_{l=1}^{q+1} \beta_l^{n-p+q+1}
\end{aligned}
$$

$$= (\mu_{p-q} - \mu_{p-q+1}) + \left(\sum_{l=1}^{q} \beta_l^{n-p+q} - \sum_{l=1}^{q} \beta_l^{n-p+q-1}\right) + \left(\sum_{l=1}^{q+1} \beta_l^{n-p+q} - \sum_{l=1}^{q+1} \beta_l^{n-p+q+1}\right)$$
$$= (\mu_{p-q} - \mu_{p-q+1}) + \delta_1 + \delta_2$$

where:

$$\delta_1 = \sum_{l=1}^{q} \beta_l^{n-p+q} - \sum_{l=1}^{q} \beta_l^{n-p+q-1} \geq 0,$$
$$\delta_2 = \sum_{l=1}^{q+1} \beta_l^{n-p+q} - \sum_{l=1}^{q+1} \beta_l^{n-p+q+1} \leq 0.$$

In δ_1 the q-th largest entries from a list are subtracted from the q-th largest entry of the same list with an additional entry, therefore the di□erence must be greater than or equal to zero, whereas the opposite is true for δ_2. It is not directly evident whether $R_{p,q} \geq 0$ and the possible cases must be analysed separately. It holds that:

$$\delta_1 = \begin{cases} 0 & \text{when } \beta_{n-p+q} \leq \beta_q^{n-p+q-1} \\ \mu_{p-q+1} + \nu_{n-p+q} - \beta_q^{n-p+q-1} & \text{when } \beta_{n-p+q} > \beta_q^{n-p+q-1} \end{cases}$$

and:

$$\delta_2 = \begin{cases} 0 & \text{when } \beta_{n-p+q+1} \leq \beta_{q+1}^{n-p+q} \\ \beta_{q+1}^{n-p+q} - (\mu_{p-q} + \nu_{n-p+q+1}) & \text{when } \beta_{n-p+q+1} > \beta_{q+1}^{n-p+q}. \end{cases}$$

Depending on the relative sizes of the last three entries there are four cases.

Case 1: $\beta_q^{n-p+q} = \beta_q^{n-p+q-1}$ **and** $\beta_{q+1}^{n-p+q} = \beta_{q+1}^{n-p+q+1}$

The first case is when the list is expanded from $\beta^{n-p+q-1}$ with two entries: β_{n-p+q} and $\beta_{n-p+q+1}$. If these two entries are both equal to or smaller than $\beta_q^{n-p+q-1}$ then $\delta_1 = \delta_2 = 0$, resulting in:

$$\begin{aligned} R_{p,q} &= (\mu_{p-q} - \mu_{p-q+1}) + \delta_1 + \delta_2 \\ &= \mu_{p-q} - \mu_{p-q+1} + 0 + 0 \geq 0. \end{aligned}$$

Case 2: $\beta_q^{n-p+q} = \beta_q^{n-p+q-1}$ **and** $\beta_{q+1}^{n-p+q} = \beta_{q+1}^{n-p+q+1}$

The second case is when $\delta_1 \geq 0$ and $\delta_2 = 0$:

$$\begin{aligned} R_{p,q} &= (\mu_{p-q} - \mu_{p-q+1}) + \delta_1 + \delta_2 \\ &= \mu_{p-q} - \mu_{p-q+1} + \beta_q^{n-p+q} - \beta_q^{n-p+q-1} + 0 \geq 0. \end{aligned}$$

Case 3: $\beta_q^{n-p+q} = \beta_q^{n-p+q-1}$ **and** $\beta_q^{n-p+q} = \beta_q^{n-p+q+1}$

In the third case both inserted entries from expanding the list from $\beta^{n-p+q-1}$ to $\beta^{n-p+q+1}$ disrupted the order of the partial lists before them. This gives:

$$\begin{aligned}
R_{p,q} &= (\mu_{p-q} - \mu_{p-q+1}) + \delta_1 + \delta_2 \\
&= \mu_{p-q} - \mu_{p-q+1} + \beta_q^{n-p+q} - \beta_q^{n-p+q-1} + \beta_q^{n-p+q} - \beta_{q+1}^{n-p+q+1} \\
&= \mu_{p-q} - \mu_{p-q+1} + (\mu_{p-q+1} + \nu_{n-p+q}) - \beta_q^{n-p+q-1} + \beta_{q+1}^{n-p+q} - (\mu_{p-q} + \nu_{n-p+q+1}) \\
&= \nu_{n-p+q} - \nu_{n-p+q+1} + \beta_{q+1}^{n-p+q} - \beta_q^{n-p+q-1} \\
&= \nu_{n-p+q} - \nu_{n-p+q+1} \geq 0.
\end{aligned}$$

In the above equation $\delta_1 = 0$ was assumed, meaning that the order of the entries was changed when expanding the list from $\beta^{n-p+q-1}$ to β^{n-p+q} and consequently the entry at position q in the shorter list is moved down one place to $q + 1$ in the longer list, so that β_{q+1}^{n-p+q} and $\beta_q^{n-p+q-1}$ are the same entry.

Case 4: $\beta_q^{n-p+q} = \beta_q^{n-p+q-1}$ **and** $\beta_{q+1}^{n-p+q} = \beta_{q+1}^{n-p+q+1}$

$$\begin{aligned}
R_{p,q} &= (\mu_{p-q} - \mu_{p-q+1}) + \delta_1 + \delta_2 \\
&= \mu_{p-q} - \mu_{p-q+1} + 0 + \beta_q^{n-p+q} - \beta_{q+1}^{n-p+q+1} \\
&= \mu_{p-q} - \mu_{p-q+1} + \beta_q^{n-p+q} - \beta_{n-p+q+1} \\
&\geq \mu_{p-q} - \mu_{p-q+1} + (\mu_{p-q+1} + \nu_{n-p+q}) - (\mu_{p-q} + \nu_{n-p+q+1}) \\
&= \nu_{n-p+q} - \nu_{n-p+q+1} \geq 0.
\end{aligned}$$

Here the fact that $\delta_1 = 0$ means that $\beta_q^{n-p+q-1} = \beta_q^{n-p+q}$, which implies that $\beta_q^{n-p+q} \geq \beta_{n-p+q} = (\mu_{p-q+1} + \nu_{n-p+q})$. That $\delta_2 = 0$ means that $-\beta_{q+1}^{n-p+q+1}$ contributes with $-\beta_{n-p+q+1}$. In the end the result is non-negative. □

Now all three rhombus inequalities have been proven for the general case, proving that the expression h^{AWD} with the reverse permutation as ν_σ results in valid hives.

4.2.3 The general diagonal case for $n = 3$ and $n = 4$

In preparation for the general diagonal case we shall begin by studying the *AWD*-construction for Hermitian matrix pairs of arbitrary ordering but with a small and fixed number of dimensions. Trivially, we have the following.

PROPOSITION 4.7. *The Construction 4.1 defines valid hives for any triples of the form* $(M, N, L = M + N)$ *of diagonal Hermitian matrices of size* 2×2.

The proof is given in Subsection 4.1.2 as there is no inner node in this case, only borders. We therefore move on to the next smallest case, which is when $n = 3$.

Let $\mu = (\mu_1, \mu_2, \mu_3)$, where $\mu_1 \geq \mu_2 \geq \mu_3$, $\nu = (\nu_1, \nu_2, \nu_3)$ where $\nu_1 \geq \nu_2 \geq \nu_3$, and $\beta = \mu + \nu_\sigma$, where ν_σ is an arbitrary permutation of ν and:

$$\beta_1 = \mu_1 + \nu_{\sigma(1)},$$

$$\beta_2 = \mu_2 + \nu_{\sigma(2)},$$
$$\beta_3 = \mu_3 + \nu_{\sigma(3)},$$

with $\sigma(i)$ as a bijection between the indices for β and the indices for ν such that:

$$\beta_{\sigma^{-1}(i)} = \mu_{\sigma^{-1}(i)} + \nu_i.$$

Let the indices j_1, j_2, j_3 be the order of β's in weakly descending order and note that the vector of sorted β's are the λ's:

$$\lambda_1 = \beta_{j_1} \geq \lambda_2 = \beta_{j_2} \geq \lambda_3 = \beta_{j_3}.$$

A hive of size $n = 3$ with nodes given by:

$$h^{AWD}_{p,q} = \max_{\substack{J\ \ I\\ |J|=q\\ |I|=p}} \left(\sum_{i\ \ I} \mu_i + \sum_{j\ \ J} \nu_{\sigma(j)} \right), \qquad p \geq q,$$

is shown in Figure 4.3.

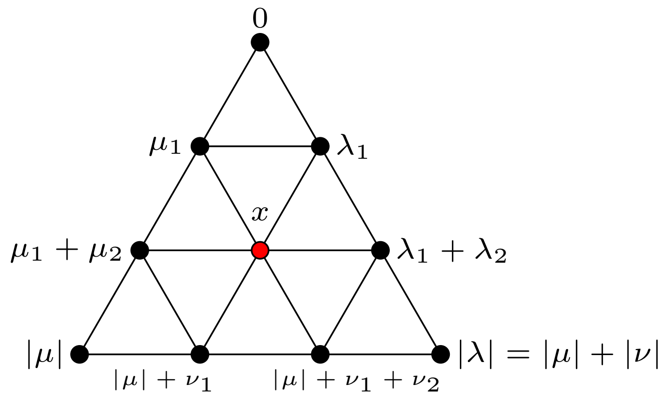


Figure 4.3: A hive of size $n = 3$, with λ's and the interior node x given by h^{AWD}.

The *AWD*-construction specifies the right border nodes as the cumulative sum of entries in λ. In the special case when $n = 3$ the only interior node is $h^{AWD}_{2,1}$ which we will from now on refer to as x. Node x should be the maximum sum of one μ_i and one β_j, for $i, j \quad (1, 2, 3)$ and importantly $i = j$:

$$x = h^{AWD}_{21} = \max_{\substack{J\ \ I\\ |J|=1\\ |I|=2}} \left(\sum_{i\ \ I} \mu_i + \sum_{j\ \ J} \nu_{\sigma(j)} \right) = \max_{|\{i,j\}|=2} \{\mu_i + \mu_j + \nu_{\sigma(j)}\} = \max_{|\{i,j\}|=2} \{\mu_i + \beta_j\}.$$

In the case when $n = 3$ there are three rhombus inequalities of each type: *vertical* V_i, *left* L_i and *right* R_i that the hive should satisfy. We can read them off the hive in Figure 4.4, where we let the indices i correspond to a left-to-right, top-to-bottom labelling of the inequalities. For clarity we enumerate all nine inequalities that need to be proven:

$$
\begin{aligned}
V_1 &= \mu_1 + \lambda_1 - x \geq 0, \\
V_2 &= (\mu_1 + \mu_2) + x - \mu_1 - (|\mu| + \nu_1) = x - \mu_1 - \mu_3 - \nu_1 \geq 0, \\
V_3 &= x + (\lambda_1 + \lambda_2) - \lambda_1 - (|\mu| + \nu_1 + \nu_2) = x + \lambda_2 - |\mu| - \nu_1 - \nu_2 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
L_1 &= x + \lambda_1 - \mu_1 - (\lambda_1 + \lambda_2) = x - \mu_1 - \lambda_2 \geq 0, \\
L_2 &= x + (|\mu| + \nu_1) - (\mu_1 + \mu_2) - (|\mu| + \nu_1 + \nu_2) = x - \mu_1 - \mu_2 - \nu_2 \geq 0, \\
L_3 &= (\lambda_1 + \lambda_2) + (|\mu| + \nu_1 + \nu_2) - x - |\lambda| = \lambda_1 + \lambda_2 - x - \nu_3 \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
R_1 &= \mu_1 + x - \lambda_1 - (\mu_1 + \mu_2) = x - \lambda_1 - \mu_2 \geq 0, \\
R_2 &= (\mu_1 + \mu_2) + (|\mu| + \nu_1) - x - |\mu| = \mu_1 + \mu_2 + \nu_1 - x \geq 0, \\
R_3 &= x + (|\mu| + \nu_1 + \nu_2) - (\lambda_1 + \lambda_2) - (|\mu| + \nu_1) = x + \nu_2 - \lambda_1 - \lambda_2 \geq 0.
\end{aligned}
$$

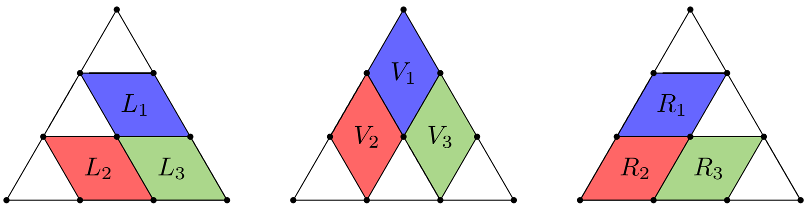


Figure 4.4: Reference images for the labelling of the three different kinds of rhombus inequalities for $n = 3$. Rhombus inequalities of the same type are separated in the figure by different colors.

We claim that:

Proposition 4.8. *The Construction 4.1 defines valid hives for any triples of the form* $(M, N, L = M + N)$ *of diagonal Hermitian matrices of size* 3×3.

Proof of Proposition 4.8. We begin by showing that $V_1 \geq 0$:

$$
\begin{aligned}
V_1 = &\mu_1 + \lambda_1 - x = \mu_1 + \lambda_1 - \max_{|\{i,j\}|=2}(\mu_i + \beta_j) \geq \mu_1 + \lambda_1 - \max(\mu_i + \beta_i) \\
\geq &\mu_1 + \lambda_1 - \mu_1 - \lambda_1 = 0.
\end{aligned}
$$

Continuing with V_2 we see that:

$$
\begin{aligned}
V_2 &= \mu_1 + \mu_2 + x - \mu_1 - |\mu| - \nu_1 = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) - \mu_1 - \mu_3 - \nu_1 \\
&= \max_{|\{i,j\}|=2}(\mu_i + \mu_j + \nu_{\sigma(j)}) - \mu_1 - \mu_3 - \nu_1 \\
&\geq \max_{i=\sigma^{-1}(1)}(\mu_i + \mu_{\sigma^{-1}(1)} + \nu_1) - \mu_1 - \mu_3 - \nu_1 \\
&= \max_{i=\sigma^{-1}(1)}(\mu_i) + \mu_{\sigma^{-1}(1)} - \mu_1 - \mu_3,
\end{aligned}
$$

where $\max_{i=\sigma^{-1}(1)}(\mu_i) = \mu_1$ unless $\mu_{\sigma^{-1}(1)} = \mu_1$. In either case:

$$\max_{i=\sigma^{-1}(1)}(\mu_i) + \mu_{\sigma^{-1}(1)} \geq \mu_1 + \mu_3,$$

making:

$$\max_{i=\sigma^{-1}(1)}(\mu_i) + \mu_{\sigma^{-1}(1)} - \mu_1 - \mu_3 \geq (\mu_1 + \mu_3) - \mu_1 - \mu_3 = 0.$$

The third and last vertical inequality can be proven:

$$\begin{aligned} V_3 &= x + (\lambda_1 + \lambda_2) - \lambda_1 - (|\mu| + \nu_1 + \nu_2) = x + \lambda_2 - (|\mu| + |\nu| - \nu_3) \\ &= x + \nu_3 - \lambda_1 - \lambda_3 = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \nu_3 - \lambda_1 - \lambda_3 \\ &\geq \max_{i=\sigma^{-1}(3)}(\mu_{\sigma^{-1}(3)} + \beta_i) + \nu_3 - \lambda_1 - \lambda_3 \\ &= \max_{i=\sigma^{-1}(3)}(\beta_i) + \mu_{\sigma^{-1}(3)} + \nu_3 - \lambda_1 - \lambda_3 \\ &= \max_{i=\sigma^{-1}(3)}(\beta_i) + \beta_{\sigma^{-1}(3)} - \lambda_1 - \lambda_3, \end{aligned}$$

we note that if $\beta_{\sigma^{-1}(3)} = \beta_{j_1}$ then $\max_{i=\sigma^{-1}(3)}(\beta_i) = \beta_{j_2}$ and if $\beta_{\sigma^{-1}(3)} = \beta_{j_1}$ then $\max_{i=\sigma^{-1}(3)}(\beta_i) = \beta_{j_1}$ and consequently:

$$\max_{i=\sigma^{-1}(3)}(\beta_i) + \beta_{\sigma^{-1}(3)} - \lambda_1 - \lambda_3 \geq (\beta_{j_1} + \beta_{j_3}) - \lambda_1 - \lambda_3 = 0.$$

Proceeding with the left rhombus inequalities, we have:

$$\begin{aligned} L_1 &= x + \lambda_1 - \mu_1 - (\lambda_1 + \lambda_2) = x - \mu_1 - \lambda_2 = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) - \mu_1 - \lambda_2 \\ &\geq \max_{i=1}(\mu_1 + \beta_i) - \mu_1 - \lambda_2 = \max_{i=1}(\beta_i) - \lambda_2 \geq \beta_{j_2} - \lambda_2 = 0. \end{aligned}$$

We then have:

$$\begin{aligned} L_2 &= x + (|\mu| + \nu_1) - (\mu_1 + \mu_2) - (|\mu| + \nu_1 + \nu_2) = x - \mu_1 - \mu_2 - \nu_2 \\ &= \max_{|\{i,j\}|=2}(\mu_i + \beta_j) - \mu_1 - \mu_2 - \nu_2 = \max_{|\{i,j\}|=2}(\mu_i + \mu_j + \nu_{\sigma(j)}) - \mu_1 - \mu_2 - \nu_2 \\ &\geq \max_{\substack{|\{i,j\}|=2 \\ i,j\leq 2 \\ \sigma(j)\leq 2}}(\mu_i + \mu_j + \nu_{\sigma(j)}) - \mu_1 - \mu_2 - \nu_2 = \max_{\substack{i\leq 2 \\ \sigma(i)\leq 2}}(\mu_1 + \mu_2 + \nu_{\sigma(i)}) - \mu_1 - \mu_2 - \nu_2 \\ &= \max_{\substack{i\leq 2 \\ \sigma(i)\leq 2}}(\nu_{\sigma(i)}) + \mu_1 + \mu_2 - \mu_1 - \mu_2 - \nu_2 \geq (\nu_2) - \nu_2 = 0, \end{aligned}$$

where we note that the restrictions on the maximization are possible. The first two say that one of i, j is 1 and the other 2. Since there are only three ν's in total it must be that $\sigma^{-1}(j_1) \leq 2$ or $\sigma^{-1}(j_2) \leq 2$ or both, making ν_2 a lower bound on $\max_{\substack{i\leq 2 \\ \sigma(i)\leq 2}}(\nu_{\sigma(i)})$.

The last left inequality can be shown as follows

$$\begin{aligned} L_3 =&(\lambda_1 + \lambda_2) + (|\mu| + \nu_1 + \nu_2) - x - |\lambda| = |\mu| + \nu_1 + \nu_2 - x - \lambda_3 \\ =&\lambda_1 + \lambda_2 - \nu_3 - \max_{|\{i,j\}|=2}(\mu_i + \beta_j) = \beta_{j_1} + \beta_{j_2} - (\mu_{i_0+\beta_{j_0}+\nu_3}), \end{aligned}$$

where $i_0 = j_0$,

$$\geq \beta_{j_1} + \beta_{j_2} - (\beta_{i_0} + \beta_{j_0}) \geq 0.$$

Now only the right rhombus inequalities remain. Beginning with R_1 we have:

$$\begin{aligned} R_1 =&\mu_1 + x - \lambda_1 - (\mu_1 + \mu_2) = x - \lambda_1 - \mu_2 = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) - \lambda_1 - \mu_2 \\ \geq&\max_{i=j_1}(\mu_i + \beta_{j_1}) - \lambda_1 - \mu_2 = \max_{i=j_1}(\mu_i) + \beta_{j_1} - \lambda_1 - \mu_2 \geq \mu_2 - \mu_2 = 0. \end{aligned}$$

R_2 is similarly straightforward:

$$\begin{aligned} R_2 =&(\mu_1 + \mu_2) + (|\mu| + \nu_1) - x - |\mu| = \mu_1 + \mu_2 + \nu_1 - x \\ =&\mu_1 + \mu_2 + \nu_1 - \max_{|\{i,j\}|=2}(\mu_i + \mu_j + \nu_{\sigma(j)}) \geq \mu_1 + \mu_2 + \nu_1 - (\mu_1 + \mu_2 + \nu_1) = 0. \end{aligned}$$

Finally we have:

$$\begin{aligned} R_3 =&x + (|\mu| + \nu_1 + \nu_2) - (\lambda_1 + \lambda_2) - (|\mu| + \nu_1) = x + \nu_2 - \lambda_1 - \lambda_2 \\ =&\max_{|\{i,j\}|=2}(\mu_i + \mu_j + \nu_{\sigma(j)}) + \nu_2 - \lambda_1 - \lambda_2 \geq \max_{\substack{|\{i,j\}|=2 \\ i,j\geq j_2 \\ \sigma(j)=2}}(\mu_i + \mu_j + \nu_{\sigma(j)}) + \nu_2 - \lambda_1 - \lambda_2 \\ \geq& \max_{\substack{i\geq j_2 \\ \sigma(i)=2}}(\mu_{j_1} + \mu_{j_2} + \nu_{\sigma(i)}) + \nu_2 - \lambda_1 - \lambda_2 = \max_{\substack{i\geq j_2 \\ \sigma(i)=2}}(\nu_{\sigma(i)}) + \mu_{j_1} + \mu_{j_2} + \nu_2 - \lambda_1 - \lambda_2 \\ \geq&(\lambda_1 + \lambda_2) - \lambda_1 - \lambda_2 = 0. \end{aligned}$$

□

After the *AWD*-construction had been proven for $n = 3$, we continued by examining it for $n = 4$. This meant proving 27 rhombus inequalities, a few of which: V_1, L_1 and R_1, were identical to the $n = 3$ case. An added di□culty was encountered when showing the rhombus inequalities that simultaneously included all three inner nodes: V_5, L_2 and R_3. These inequalities were proven not only for $n = 4$, but also for $n \geq 4$, using a somewhat more abstract approach, which ended up paving the way for proving the general diagonal case. We claim the following.

PROPOSITION 4.9. *The Construction 4.1 defines valid hives for any triples of the form* $(M, N, L = M + N)$ *of diagonal Hermitian matrices of size* 4×4.

For the interested reader, a detailed proof of Proposition 4.9 is given in Appendix A. For now we are satisfied with observing that the proof was on the whole largely similar to the proof of the three-dimensional case, and that the method used for the complicated inequalities strongly resembles the one used for proving the three general rhombus inequalities in Section 4.3.

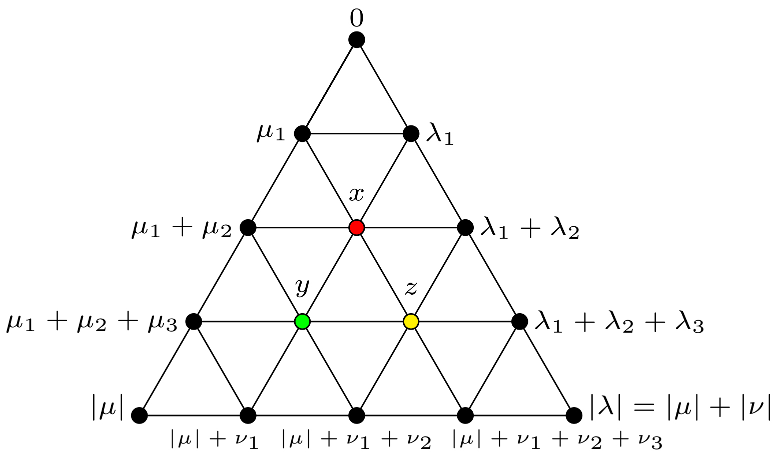


Figure 4.5: A hive of size $n = 4$ with λ's and interior nodes given by h^{AWD}.

4.3 The General Diagonal Case

We have thus far shown that the *AWD*-construction holds under special circumstances. It gives valid hives for pairs of diagonal matrices of any size provided that the entries are jointly sorted in ascending or descending order. On top of this we have shown that *AWD* is valid for diagonal matrices of size 3 and 4 independently of sorting constraints. In proving this last bit we also showed that a few of the rhombus inequalities held regardless of size: $V_1, V_5, L_1, L_2, R_1, R_3$, see Appendix A.

In our translation of the Appleby-Whitehead construction in diagonal case the nodes are given by:

$$h^{AWD}_{p,q} = \max_{\substack{J \quad I \\ |J|=q \\ |I|=p}} \left(\sum_{i \in I} \mu_i + \sum_{j \in J} \nu_{\sigma(j)} \right), \qquad p \geq q,$$

which after introducing the notation: $\beta_i = \mu_i + \nu_{\sigma(i)}$, for an arbitrary permutation ν_σ of ν, can be rewritten as:

$$h^{AWD}_{p,q} = \max_{\substack{B \cap M = \emptyset \\ |B|=q \\ |M|=p-q}} \left(\sum_{i \in M} \mu_i + \sum_{j \in B} \beta_j \right), \qquad p \geq q.$$

In Section 4.2.2 we derived expressions for the three rhombus inequalities generally:

$$V_{p,q} = h^{AWD}_{p,q} + h^{AWD}_{p,q+1} - h^{AWD}_{p-1,q} - h^{AWD}_{p+1,q+1} \quad q \in \{0, \cdots, p-1\},$$

$$L_{p,q} = h^{AWD}_{p,q} + h^{AWD}_{p+1,q} - h^{AWD}_{p,q-1} - h^{AWD}_{p+1,q+1} \quad q \in \{1, \cdots, p\},$$

$$R_{p,q} = h^{AWD}_{p,q} + h^{AWD}_{p+1,q+1} - h^{AWD}_{p,q+1} - h^{AWD}_{p+1,q} \quad q \in \{0, \cdots, p-1\},$$

where $p \quad \{1, \cdots, n-1\}$. The question remaining at this point is whether or not it is possible to know, without any insight into the actual values of μ and β, if the quantities $V_{p,q}, L_{p,q}, R_{p,q} \geq 0$. Can we beforehand know whether the two first maximizations in these expressions are greater than the two latter, being subtracted from them, based solely on the cardinalities of the sets being optimized over? We have now arrived at the main result of this thesis.

Theorem 4.10. *The Construction 4.1 defines a hive.*

We shall prove Theorem 4.10 in three parts, by showing that each type of rhombus inequality holds separately.

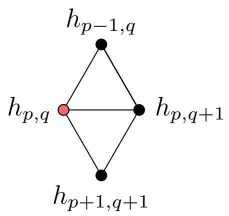


Figure 4.6: The vertical rhombus inequality from Figure 4.2. The expression $V_{p,q}$ is indexed after the left node, highlighted in red.

The vertical rhombus inequality in the general diagonal case:

$$V_{p,q} = h^{AWD}_{p,q} + h^{AWD}_{p,q+1} - h^{AWD}_{p-1,q} - h^{AWD}_{p+1,q+1},$$

with the corresponding rhombus pictured in Figure 4.6, can be rewritten from the maximizations of sums of μ's over a set I and sums of ν_σ's over a subset J of I, into maximizations of μ's and β's over two disjoint sets. The indices $j \quad J$ are also included in I since $J \quad I$, ensuring that both μ_j and $\nu_{\sigma(j)}$ exist in the maximization and allowing them to be grouped together as β_j.

We can therefore reformulate $V_{p,q}$ in the following steps:

$$V_{p,q} = \quad h^{AWD}_{p,q} + h^{AWD}_{p,q+1} - (h^{AWD}_{p-1,q} + h^{AWD}_{p+1,q+1}),$$

first by expanding the nodes,

$$= \left[\max_{\substack{J \quad I \\ |J|=q \\ |I|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) + \max_{\substack{J \quad I \\ |J|=q+1 \\ |I|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) \right] -$$

$$\left[\max_{\substack{J \quad I \\ |J|=q \\ |I|=p-1}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) + \max_{\substack{J \quad I \\ |J|=q+1 \\ |I|=p+1}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) \right],$$

then by incorporating μ_j and $\nu_{\sigma(j)}$ into β_j,

$$= \left[\max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q}} \left(\sum_{i \in A} \mu_i + \sum_{j \in B} \beta_j \right) + \max_{\substack{C\cap D= \\ |C|=p-q-1 \\ |D|=q+1}} \left(\sum_{i \in C} \mu_i + \sum_{j \in D} \beta_j \right) \right] -$$

$$\left[\max_{\substack{E\cap F= \\ |E|=p-q-1 \\ |F|=q}} \left(\sum_{i \in E} \mu_i + \sum_{j \in F} \beta_j \right) + \max_{\substack{G\cap H= \\ |G|=p-q \\ |H|=q+1}} \left(\sum_{i \in G} \mu_i + \sum_{j \in H} \beta_j \right) \right],$$

and finally by merging pairs of independent maximizations we arrive at our final expression,

$$= \max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q \\ C\cap D= \\ |C|=p-q-1 \\ |D|=q+1}} \underbrace{\left(\sum_{i \in A} \mu_i + \sum_{j \in B} \beta_j + \sum_{k \in C} \mu_k + \sum_{l \in D} \beta_l \right)}_{L(A,B,C,D)} -$$

$$\max_{\substack{E\cap F= \\ |E|=p-q-1 \\ |F|=q \\ G\cap H= \\ |G|=p-q \\ |H|=q+1}} \underbrace{\left(\sum_{i \in E} \mu_i + \sum_{j \in F} \beta_j + \sum_{k \in G} \mu_k + \sum_{l \in H} \beta_l \right)}_{R(E,F,G,H)}.$$

In order for the vertical rhombus inequality to be true we must show that:

Lemma 4.11. *(Vertical Rhombus Inequality in AWD)*

$$\max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q \\ C\cap D= \\ |C|=p-q-1 \\ |D|=q+1}} L(A,B,C,D) \geq \max_{\substack{E\cap F= \\ |E|=p-q-1 \\ |F|=q \\ G\cap H= \\ |G|=p-q \\ |H|=q+1}} R(E,F,G,H).$$

Proof of Lemma 4.11. We would like to find sets A_0, B_0, C_0, D_0 such that:

$$\begin{aligned} \text{LHS} = \max_{A,B,C,D} L(A,B,C,D) &\geq L(A_0,B_0,C_0,D_0) \\ &= R(E_0,F_0,G_0,H_0) = \max_{E,F,G,H} R(E,F,G,H) = \text{RHS}. \end{aligned}$$

The sets should have the following cardinalities:

$$|A| = p-q, \qquad |E| = p-q-1,$$
$$|B| = q, \qquad |F| = q,$$

$$|C| = p - q - 1, \qquad |G| = p - q,$$
$$|D| = q + 1, \qquad |H| = q + 1,$$

and satisfy the following conditions:

$$A \cap B = \ , \quad C \cap D = \ , \quad E \cap F = \ , \quad G \cap H = \ .$$

We begin by assuming that we have the sets that result in the optimal $R(E, F, G, H)$ and let the subscript zero denote optimality:

$$R(E_0, F_0, G_0, H_0) \text{ given by } \begin{cases} E_0 = \{e_1, ..., e_{p-q-1}\}, \\ F_0 = \{f_1, ..., f_q\}, \\ G_0 = \{g_1, ..., g_{p-q}\}, \\ H_0 = \{h_1, ..., h_{q+1}\}, \end{cases}$$

where the sets satisfy the conditions $E_0 \cap F_0 = \ $, $G_0 \cap H_0 = \ $ and have the correct cardinalities: $|E_0| = p - q - 1$, $|F_0| = q$, $|G_0| = p - q$, $|H_0| = q + 1$.

We would now like to redistribute the indices for the μ's in $R(E_0, F_0, G_0, H_0)$, i.e. the elements in the sets E_0 and G_0, into the sets A_0 and C_0 which hold the indices for μ's in the expression $L(A_0, B_0, C_0, D_0) = R(E_0, F_0, G_0, H_0)$ that we are trying to create, and similarly for the indices for β in B_0 and D_0. We will do this in two ways: A , B , C , D and $A \ , B \ , C \ , D$.

The first way is to assign E_0, F_0, G_0, H_0 to A , B , C , D and then move one element from C over to A :

$$L(A , B , C , D) = \begin{cases} A = E_0 \quad \{g_x\} = \{e_1, ..., e_{p-q-1}\} \quad \{g_x\}, \\ B = F_0 = \{f_1, ..., f_q\}, \\ C = G_0 \setminus \{g_x\} = \{g_1, ..., g_{p-q}\} \setminus \{g_x\}, \\ D = H_0 = \{h_1, ..., h_{q+1}\}. \end{cases}$$

We note that this redistribution is valid provided that $g_x \quad G_0 : \{g_x\} \cap E_0 = \ , \{g_x\} \cap F_0 = \ $. We check that the sets have the proper cardinalities:

$$|A | = |E_0 \quad \{g_x\}| \overset{g_x \cap E_0 =}{=} |E_0| + 1 = p - q,$$
$$|B | = |F_0| = q,$$
$$|C | = |G_0 \setminus \{g_x\}| = |G_0| - 1 = p - q - 1,$$
$$|D | = |H_0| = q + 1.$$

From the assumptions on the RHS we have that:

$$C \cap D = (G_0 \setminus \{g_x\}) \cap H_0 = \ ,$$

and using that : $\{g_x\} \cap E_0 = \{g_x\} \cap F_0 = \ $ we see that:

$$A \cap B = (E_0 \quad \{g_x\}) \cap F_0 = (E_0 \cap F_0) \quad (\{g_x\} \cap F_0) = \qquad = \ .$$

We now introduce the second redistribution of indices where the sets E_0, F_0, G_0, H_0 are assigned in a different way and an index is moved from B'' to D'':

$$L(A'', B'', C'', D'') = \begin{cases} A'' = G_0 = \{g_1, ..., g_{p-q}\}, \\ B'' = H_0 \setminus \{h_y\} = \{h_1, ..., h_{q+1}\} \setminus \{h_y\}, \\ C'' = E_0 = \{e_1, ..., e_{p-q-1}\}, \\ D'' = F_0 \cup \{h_y\} = \{f_1, ..., f_q\} \cup \{h_y\}. \end{cases}$$

This redistribution is valid provided that $\exists h_y \in H_0 : \{h_y\} \cap E_0 = \emptyset, \{h_y\} \cap F_0 = \emptyset$. We check once again that these sets have the right cardinalities:

$$\begin{aligned} |A''| &= |G_0| = p - q, \\ |B''| &= |H_0 \setminus \{h_y\}| = |H_0| - 1 = q, \\ |C''| &= |E_0| = p - q - 1, \\ |D''| &= |F_0 \cup \{h_y\}| \overset{h_y \cap F_0 = \emptyset}{=} |F_0| + 1 = q + 1. \end{aligned}$$

From the assumptions on the RHS we have that:

$$A'' \cap B'' = G_0 \cap (H_0 \setminus \{h_y\}) = \emptyset,$$

and using that : $\{h_y\} \cap E_0 = \{h_y\} \cap F_0 = \emptyset$ it holds:

$$C'' \cap D'' = E_0 \cap (F_0 \cup \{h_y\}) = (E_0 \cap F_0) \cup (E_0 \cap \{h_y\}) = \emptyset \cup \emptyset = \emptyset.$$

Both $L(A', B', C', D')$ and $L(A'', B'', C'', D'')$ feature the same indices for μ's and β's respectively as in $R(E_0, F_0, G_0, H_0)$ and so we can easily see that:

$$\begin{aligned} L(A', B', C', D') &= \sum_{i \in A'} \mu_i + \sum_{j \in B'} \beta_j + \sum_{k \in C'} \mu_k + \sum_{l \in D'} \beta_l \\ &= \sum_{i \in E_0 \cup \{g_x\}} \mu_i + \sum_{j \in F_0} \beta_j + \sum_{k \in G_0 \setminus \{g_x\}} \mu_k + \sum_{l \in H_0} \beta_l \\ &= \sum_{i \in E_0} \mu_i + \sum_{j \in F_0} \beta_j + \sum_{k \in G_0} \mu_k + \sum_{l \in H_0} \beta_l = R(E_0, F_0, G_0, H_0) \end{aligned}$$

and that:

$$\begin{aligned} L(A'', B'', C'', D'') &= \sum_{i \in A''} \mu_i + \sum_{j \in B''} \beta_j + \sum_{k \in C''} \mu_k + \sum_{l \in D''} \beta_l \\ &= \sum_{i \in G_0} \mu_i + \sum_{H_0 \setminus \{h_y\}} \beta_j + \sum_{k \in E_0} \mu_k + \sum_{l \in F_0 \cup \{h_y\}} \beta_l \\ &= \sum_{i \in E_0} \mu_i + \sum_{j \in F_0} \beta_j + \sum_{k \in G_0} \mu_k + \sum_{l \in H} \beta_l = R(E_0, F_0, G_0, H_0). \end{aligned}$$

Now if we can prove that the LHS can always be rewritten as either $L(A', B', C', D')$ or $L(A'', B'', C'', D'')$ with the corresponding restrictions on g_x and h_y, then we have shown that there is always a way to write LHS such that:

$$L(A, B, C, D) \geq L(A', B', C', D') = R(E_0, F_0, G_0, H_0)$$

or

$$L(A,B,C,D) \geq L(A\ \ ,B\ \ ,C\ \ ,D\ \ ) = R(E_0,F_0,G_0,H_0).$$

We are now going to argue why it is always the case for the vertical inequalities that at least one out of the following statements:

$$g_x \quad G : \{g_x\} \cap E = \quad , \{g_x\} \cap F =$$

or

$$h_y \quad H : \{h_y\} \cap E = \quad , \{h_y\} \cap F =$$

is true. If no such g_x or such h_y exists then G and H must be disjoint subsets of $E \quad F$ where E and F are disjoint. But if we look at the cardinality of $G \quad H$ versus the cardinality of $E \quad F$ we note that:

$$\begin{aligned} |G \quad H| &= |G| + |H| = p - q + q + 1 = p + 1 \\ &> |E \quad F| = |E| + |F| = p - q - 1 + q = p - 1, \end{aligned}$$

which gives a contradiction. We see that G and H cannot be contained within $E \quad F$ and that there exist at least two elements from $G \quad H$ that have no overlap with either E or F. Regardless of whether these elements belong to G or H, we know that there exists an element g_x or h_y required for creating either $L(A\ ,B\ ,C\ ,D\ )$ or $L(A\ \ ,B\ \ ,C\ \ ,D\ \ )$, and thereby a way to make the LHS equal to the RHS, ensuring that $V_{p,q} \geq 0$. □

The left rhombus inequality in the general diagonal case in given by:

$$L_{p,q} = h^{AWD}_{p,q} + h^{AWD}_{p+1,q} - h^{AWD}_{p,q-1} - h^{AWD}_{p+1,q+1},$$

with the corresponding rhombus pictured in Figure 4.7. We reformulate $L_{p,q}$ in the

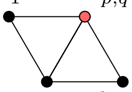


Figure 4.7: The left rhombus inequality from Figure 4.2. The expression $L_{p,q}$ is indexed after the top right node, highlighted in red.

same way as done with $V_{p,q}$:

$$\begin{aligned} L_{p,q} = &\quad h^{AWD}_{p,q} + h^{AWD}_{p+1,q} - (h^{AWD}_{p,q-1} + h^{AWD}_{p+1,q+1}) \\ = &\left[\max_{\substack{J \quad I \\ |J|=q \\ |I|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) + \max_{\substack{J \quad I \\ |J|=q \\ |I|=p+1}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) \right] - \\ &\left[\max_{\substack{J \quad I \\ |J|=q-1 \\ |I|=p}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) + \max_{\substack{J \quad I \\ |J|=q+1 \\ |I|=p+1}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) \right] \end{aligned}$$

$$= \left[\max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q}} \left(\sum_{i\ \ A} \mu_i + \sum_{j\ \ B} \beta_j \right) + \max_{\substack{C\cap D= \\ |C|=p-q+1 \\ |D|=q}} \left(\sum_{i\ \ C} \mu_i + \sum_{j\ \ D} \beta_j \right) \right] -$$

$$\left[\max_{\substack{E\cap F= \\ |E|=p-q+1 \\ |F|=q-1}} \left(\sum_{i\ \ E} \mu_i + \sum_{j\ \ F} \beta_j \right) + \max_{\substack{G\cap H= \\ |G|=p-q \\ |H|=q+1}} \left(\sum_{i\ \ G} \mu_i + \sum_{j\ \ H} \beta_j \right) \right]$$

$$= \max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q \\ C\cap D= \\ |C|=p-q+1 \\ |D|=q}} \underbrace{\left(\sum_{i\ \ A} \mu_i + \sum_{j\ \ B} \beta_j + \sum_{i\ \ C} \mu_i + \sum_{j\ \ D} \beta_j \right)}_{L(A,B,C,D)} -$$

$$\max_{\substack{E\cap F= \\ |E|=p-q+1 \\ |F|=q-1 \\ G\cap H= \\ |G|=p-q \\ |H|=q+1}} \underbrace{\left(\sum_{i\ \ E} \mu_i + \sum_{j\ \ F} \beta_j + \sum_{i\ \ G} \mu_i + \sum_{j\ \ H} \beta_j \right)}_{R(E,F,G,H)} .$$

In order for the left rhombus inequality to be true we must show that:

Lemma 4.12. *(Left Rhombus Inequality in AWD)*

$$\max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q \\ C\cap D= \\ |C|=p-q+1 \\ |D|=q}} L(A,B,C,D) \geq \max_{\substack{E\cap F= \\ |E|=p-q+1 \\ |F|=q-1 \\ G\cap H= \\ |G|=p-q \\ |H|=q+1}} R(E,F,G,H).$$

Proof of Lemma 4.12. We want to find sets A, B, C, D with cardinalities:

$$\begin{aligned} |A| &= p-q, & |E| &= p-q+1, \\ |B| &= q, & |F| &= q-1, \\ |C| &= p-q+1, & |G| &= p-q, \\ |D| &= q, & |H| &= q+1, \end{aligned}$$

and satisfying conditions:

$$A\cap B = \ \ , \quad C\cap D = \ \ , \quad E\cap F = \ \ , \quad G\cap H = \ \ .$$

We begin by assuming that we have the optimal $R(E, F, G, H)$ and let the subscript zero denote optimality:

$$R(E_0, F_0, G_0, H_0) \text{ with } \begin{cases} E_0 = \{e_1, ..., e_{p-q+1}\}, \\ F_0 = \{f_1, ..., f_{q-1}\}, \\ G_0 = \{g_1, ..., g_{p-q}\}, \\ H_0 = \{h_1, ..., h_{q+1}\}, \end{cases}$$

where the sets satisfy the conditions $E_0 \cap F_0 =$, $G_0 \cap H_0 =$ and have the appropriate cardinalities $|E_0| = p - q - 1$, $|F_0| = q$, $|G_0| = p - q$, $|H_0| = q + 1$.

We will find two ways of redistributing the indices of E_0, F_0, G_0, H_0, called $L(A, B, C, D)$ and $L(A, B, C\ s, D)$, so that they equal the optimal RHS and show that the prerequisites for at least one of these must be met.

We create the first sets:

$$L(A, B, C, D) = \begin{cases} A = G_0 = \{g_1, ..., g_{p-q}\}, \\ B = H_0 \setminus \{h_x\} = \{h_1, ..., h_{q+1}\} \setminus \{h_x\}, \\ C = E_0 = \{e_1, ..., e_{p-q+1}\}, \\ D = F_0 \quad \{h_x\} = \{f_1, ..., f_{q-1}\} \quad \{h_x\}. \end{cases}$$

This works provided $\{h_x\} \quad H_0 : \{h_x\} \cap F_0 =$, $\{h_x\} \cap E_0 =$, i.e. that there exists an element in H_0 that is neither part of E_0 nor F_0.

We therefore assume that $H_0 \quad E_0 \quad F_0$. Since $|H_0| = |F_0| + 2$ we can rule out that H_0 is a subset of F_0 only. This would mean that there exists an element $h_x \quad H_0 : \{h_x\} \cap E_0 = e_y$, or in other words that the element h_x belongs to both H_0 and E_0.

With this in mind we rearrange the sets:

$$L(A, B, C, D) = \begin{cases} A = E_0 \setminus \{e_y\} = \{e_1, ..., e_{p-q+1}\} \setminus \{e_y\}, \\ B = F_0 \quad \{h_x\} = \{f_1, ..., f_{q-1}\} \quad \{h_x\}, \\ C = G_0 \quad \{e_y\} = \{g_1, ..., g_{p-q}\} \quad \{e_y\}, \\ D = H_0 \setminus \{h_x\} = \{h_1, ..., h_{q+1}\} \setminus \{h_x\}. \end{cases}$$

In order for this to be valid we must have $h_x \cap F_0 =$ and $e_y \cap G_0 =$, which are true since $h_x = e_y$ belongs to both E_0 and H_0 and consequently $h_x \cap F_0 = e_y \cap G_0 =$. We also need $h_x \cap (E_0 \setminus \{e_y\}) =$ and $e_y \cap (H_0 \setminus \{h_x\}) =$ which we have since we know $h_x = e_y$ and there are no duplicate elements within a set. □

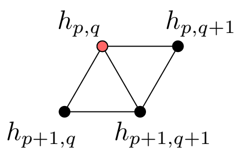


Figure 4.8: The right rhombus inequality from Figure 4.2. The expression $R_{p,q}$ is indexed after the top left node, highlighted in red.

The right rhombus inequality in the general diagonal case is given by:

$$R_{p,q} = h^{AWD}_{p,q} + h^{AWD}_{p+1,q+1} - h^{AWD}_{p,q+1} - h^{AWD}_{p+1,q},$$

with the corresponding rhombus pictured in Figure 4.8.

We reformulate $R_{p,q}$ similarly to $V_{p,q}$ and $L_{p,q}$:

$$\begin{aligned}
R_{p,q} = {} & h^{AWD}_{p,q} + h^{AWD}_{p+1,q+1} - (h^{AWD}_{p,q+1} + h^{AWD}_{p+1,q}) \\
= {} & \left[\max_{\substack{J\ I \\ |J|=q \\ |I|=p}} \left(\sum_{i\ I} \mu_i + \sum_{j\ J} \nu_{\sigma(j)} \right) + \max_{\substack{J\ I \\ |J|=q+1 \\ |I|=p+1}} \left(\sum_{i\ I} \mu_i + \sum_{j\ J} \nu_{\sigma(j)} \right) \right] - \\
& \left[\max_{\substack{J\ I \\ |J|=q+1 \\ |I|=p}} \left(\sum_{i\ I} \mu_i + \sum_{j\ J} \nu_{\sigma(j)} \right) + \max_{\substack{J\ I \\ |J|=q \\ |I|=p+1}} \left(\sum_{i\ I} \mu_i + \sum_{j\ J} \nu_{\sigma(j)} \right) \right] \\
= {} & \left[\max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q}} \left(\sum_{i\ A} \mu_i + \sum_{j\ B} \beta_j \right) + \max_{\substack{C\cap D= \\ |C|=p-q \\ |D|=q+1}} \left(\sum_{i\ C} \mu_i + \sum_{j\ D} \beta_j \right) \right] - \\
& \left[\max_{\substack{E\cap F= \\ |E|=p-q-1 \\ |F|=q+1}} \left(\sum_{i\ E} \mu_i + \sum_{j\ F} \beta_j \right) + \max_{\substack{G\cap H= \\ |G|=p-q+1 \\ |H|=q}} \left(\sum_{i\ G} \mu_i + \sum_{j\ H} \beta_j \right) \right] \\
= {} & \max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q \\ C\cap D= \\ |C|=p-q \\ |D|=q+1}} \underbrace{\left(\sum_{i\ A} \mu_i + \sum_{j\ B} \beta_j + \sum_{i\ C} \mu_i + \sum_{j\ D} \beta_j \right)}_{L(A,B,C,D)} - \\
& \max_{\substack{E\cap F= \\ |E|=p-q-1 \\ |F|=q+1 \\ G\cap H= \\ |G|=p-q+1 \\ |H|=q}} \underbrace{\left(\sum_{i\ E} \mu_i + \sum_{j\ F} \beta_j + \sum_{i\ G} \mu_i + \sum_{j\ H} \beta_j \right)}_{R(E,F,G,H)} .
\end{aligned}$$

In order for the right rhombus inequality to be true we must show that:

Lemma 4.13. *(Right Rhombus Inequality in AWD)*

$$\max_{\substack{A\cap B= \\ |A|=p-q \\ |B|=q \\ C\cap D= \\ |C|=p-q \\ |D|=q+1}} L(A,B,C,D) \geq \max_{\substack{E\cap F= \\ |E|=p-q-1 \\ |F|=q+1 \\ G\cap H= \\ |G|=p-q+1 \\ |H|=q}} R(E,F,G,H).$$

Proof of Lemma 4.13. We want to find sets A, B, C, D with cardinalities:

$$|A| = p - q, \qquad |E| = p - q - 1,$$
$$|B| = q, \qquad |F| = q + 1,$$
$$|C| = p - q, \qquad |G| = p - q + 1,$$
$$|D| = q + 1, \qquad |H| = q,$$

and satisfying conditions:

$$A \cap B = \quad , \qquad C \cap D = \quad , \qquad E \cap F = \quad , \qquad G \cap H = \quad .$$

We begin by assuming that we have the optimal $R(E, F, G, H)$ and let the subscript zero denote optimality:

$$R(E_0, F_0, G_0, H_0) = \begin{cases} E_0 = \{e_1, ..., e_{p-q-1}\}, \\ F_0 = \{f_1, ..., f_{q+1}\}, \\ G_0 = \{g_1, ..., g_{p-q+1}\}, \\ H_0 = \{h_1, ..., h_q\}, \end{cases}$$

where the sets satisfy the conditions $E_0 \cap F_0 = \quad$, $G_0 \cap H_0 = \quad$ and that $|E_0| = p - q - 1$, $|F_0| = q$, $|G_0| = p - q$, $|H_0| = q + 1$.

We will show this by creating two redistributions of indices $L(A\ , B\ , C\ , D\ )$ and $L(A\ \ , B\ \ , C\ \ s, D\ \ )$, and show that the prerequisites for at least one of these must be met.

We create the first sets:

$$L(A\ , B\ , C\ , D\ ) = \begin{cases} A\ = G_0 \setminus \{g_x\} = \{g_1, ..., g_{p-q-1}\} \setminus \{g_x\}, \\ B\ = H_0 = \{h_1, ..., h_q\}, \\ C\ = E_0 \quad \{g_x\} = \{e_1, ..., e_{p-q-1}\} \quad \{g_x\}, \\ D\ = F_0 = \{f_1, ..., f_{q+1}\}. \end{cases}$$

This works provided $\quad \{g_i\} \quad G_0 : \{g_i\} \cap E_0 = \quad , \{g_i\} \cap F_0 = \quad$, i.e. that there exists an element in G_0 that is neither part of E_0 nor F_0.

We therefore assume that $G_0 \quad E_0 \quad F_0$. Since $|G_0| = |E_0| + 2$ we can rule out that G_0 is a subset of E_0 alone. This would mean that there exists an element $\quad g_x \quad G_0 : \{g_x\} \cap F_0 = f_y$, or in other words that the element $g_x = f_y$ belongs to both G_0 and F_0.

With this in mind we rearrange the sets:

$$L(A\ \ , B\ \ , C\ \ , D\ \ ) = \begin{cases} A\ \ = E_0 \quad \{g_x\} = \{e_1, ..., e_{p-q-1}\} \quad \{g_x\}, \\ B\ \ = F_0 \setminus \{f_y\} = \{f_1, ..., f_q\} \setminus \{f_y\}, \\ C\ \ = G_0 \setminus \{g_x\} = \{g_1, ..., g_{p-q}\} \setminus \{g_x\}, \\ D\ \ = H_0 \quad \{f_y\} = \{h_1, ..., h_{q+1}\} \quad \{f_y\}. \end{cases}$$

In order for this to be valid we must have $g_x \cap E_0 = \quad$ and $f_y \cap H_0 = \quad$, which are true since $g_x = f_y$ belongs to both G_0 and F_0 and consequently does not belong to either H_0 or E_0, and so $g_x \cap E_0 = f_y \cap H_0 =$. We also need $g_x \cap (F_0 \setminus \{f_y\}) =$ and $f_y \cap (G_0 \setminus \{g_x\}) =$ which we have since we know $g_x = f_y$ and there are no duplicate elements within a set. □

This concludes the proof of Theorem 4.10.

5

Conclusion

The purpose of this project was to examine the Appleby-Whitehead hive construction in the restricted setting of simultaneously diagonalizable Hermitian matrices. We have presented a *reformulation* of the original construction in Subsection 4.1.1, where the limited scope enables a more explicit maximization expression for the hive nodes. We also gave a proof that the modified construction, the *AWD*-construction, is equivalent to the original *AW*-construction when considering simultaneously diagonalizable Hermitian matrices only. As with the original construction, the *AWD*-construction could be shown to fulfill the requirements on the borders of the hives.

Furthermore, we were able to prove algebraically that the *AWD*-construction generates valid hives for a few special cases. The construction held for matrices of arbitrary size provided that they could be diagonalized such that the entries of both matrices were in weakly descending order, or such that one matrix was in weakly descending order and the other in weakly ascending order. When proving these cases the order allowed for simpler expressions for the general rhombus inequalities, which could all be shown to hold. The construction was also proven for simultaneously diagonalizable Hermitian matrices of sizes $n = 3, 4$ generally. In these cases, the number of inequalities are finite, with there being 9 and 27 rhombus inequalities respectively, and so they could be proven one by one. Some of the more involved inequalities in $n = 4$ necessitated more abstract reasoning for which the reader is directed to Appendix A.

The series of proofs of special cases, in particular that of $n = 4$, gave way to the tools needed for the main result of the thesis, Theorem 4.10, which proves the validity of the *AWD*-construction for pairs of simultaneously diagonal Hermitian matrices generally. In view of this result we have accomplished what we set out to do, namely examining the Appleby-Whitehead hive proposal in a limited setting and providing a conclusive answer to its validity, with the proof affirming their proposal in these circumstances.

Ultimately, the broader question of whether or not the Appleby-Whitehead construction generates hives for Hermitian matrices generally remains open. The results in this project are only applicable for the special case of simultaneously diagonalizable Hermitian matrices and we cannot draw any conclusions outside of this.

While it is interesting in and of itself to find that the *AWD*-construction gives valid hives, the use of this appears limited. In the case where the eigenspectra of the three matrices are integral, the existence of a hive is equivalent to the Littlewood-Richardson coefficient being strictly positive. In the much simpler case with a pair of simultaneously diagonal Hermitian matrices with integral eigenspectra we know beforehand that a hive exists, since the sum of the matrix pair is a diagonal matrix and thus already know that the coefficient is $c^{\lambda}_{\mu,\nu} > 0$.

We note that there is no assumption in any of the proofs of the eigenspectra being integral and we can therefore conclude that the hive formulation holds for real eigenspectra as well.

The reformulated hive construction means that for simultaneously diagonalizable Hermitian matrices at least, there is an explicit way to go from matrices to a hive filling, which by itself might not offer any direct application; nevertheless it provides a previously missing link between Horn's problem and the saturation conjecture.

5.1 Outlook

The *AWD*-construction was only affirmed for the special setting of simultaneously diagonalizable Hermitian matrices, and we cannot draw conclusions about the general Hermitian case. This calls for future research in order to assess the correctness of the Appleby-Whitehead construction outside of this context. More broadly, the field of hive constructions has room for a lot innovation and it would be interesting to see new proposals for constructions.

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A

Detailed Proof of the General Diagonal Case for $n = 4$

Here we give the details of the proof of Proposition 4.9 stated in Subsection 4.2.3.

A hive of size $n = 4$ with borders and interior nodes (x, y, z) specified by the *AWD*-construction is shown in Figure 4.5. The interior nodes correspond to the maximizations:

$$x = h_{21}^{AWD} = \max_{\substack{J \;\; I \\ |J|=1 \\ |I|=2}} \left(\sum_{i \;\; I} \mu_i + \sum_{j \;\; J} \nu_{\sigma(j)} \right) = \max_{|\{i,j\}|=2} (\mu_i + \beta_j),$$

$$y = h_{31}^{AWD} = \max_{\substack{J \;\; I \\ |J|=1 \\ |I|=3}} \left(\sum_{i \;\; I} \mu_i + \sum_{j \;\; J} \nu_{\sigma(j)} \right) = \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k),$$

$$z = h_{32}^{AWD} = \max_{\substack{J \;\; I \\ |J|=2 \\ |I|=3}} \left(\sum_{i \;\; I} \mu_i + \sum_{j \;\; J} \nu_{\sigma(j)} \right) = \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k).$$

The rhombus inequalities are shown in Figure A.1 with labels assigned from left to right and top to bottom.

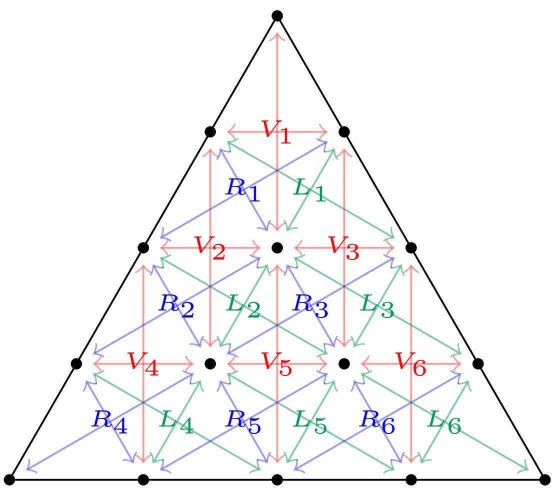


Figure A.1: The rhombus inequalities for $n = 4$ are labelled as illustrated in this figure. The colors red, green and blue are used to mark out vertical, left and right rhombus inequalities respectively.

In section 4.2.3 we claimed that the *AWD*-construction generates hives for matrix pairs of size $n = 4$. We will now prove this proposition.

PROOF OF PROPOSITION 4.9. We begin with the vertical rhombus inequalities. The first inequality is the same as when $n = 3$:

$$V_1 = \mu_1 + \lambda_1 - x = V_1^{(n=3)} \geq 0,$$

and holds therefore. In the second vertical rhombus inequality we have:

$$V_2 = x + (\mu_1 + \mu_2) - \mu_1 - y = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \mu_2 - \max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \beta_k)$$

where we note that the maximization on the right can be bounded from above by the maximization on the left:

$$\max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \beta_k) \leq \max_{|\{i,j\}|=2}(\mu_i + \beta_j + \mu_2) = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \mu_2.$$

This is because there are two stand-alone μ's included in the maximization and so at least one of them must be less than or equal to μ_2. We therefore have:

$$\begin{aligned} V_2 &= \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \mu_2 - \max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \beta_k) \\ &\geq \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \mu_2 - (\max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \mu_2) = 0. \end{aligned}$$

The third vertical rhombus inequality is given by

$$V_3 = x + (\lambda_1 + \lambda_2) - \lambda_1 - z = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_2 - \max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k),$$

where similarly to the previous inequality we note that the maximization on the right can be bounded from above by:

$$\max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) \leq \max_{|\{i,j\}|=2}(\mu_i + \beta_j + \lambda_2) = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_2,$$

since at least one of the β's in the maximum must be less than or equal to λ_2. From that it follows that the inequality holds:

$$\begin{aligned} V_3 &= \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_2 - \max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) \\ &\geq \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_2 - (\max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_2) = 0. \end{aligned}$$

For the fourth vertical rhombus inequality we have:

$$\begin{aligned} V_4 &= (\mu_1 + \mu_2 + \mu_3) + y - (\mu_1 + \mu_2) - (|\mu| + \nu_1) = y + \mu_3 - |\mu| - \nu_1 \\ &= \max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \beta_k) + \mu_3 - |\mu| - \nu_1, \end{aligned}$$

where we break β_k up into $\mu_k + \nu_{\sigma(k)}$:

$$= \max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \mu_k + \nu_{\sigma(k)}) + \mu_3 - |\mu| - \nu_1,$$

We note that:

$$\max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \mu_k + \nu_{\sigma(k)}) \geq \max_{\substack{|\{i,j\}|=2 \\ \sigma^{-1}(1)/\{i,j\}}}(\mu_i + \mu_j + \mu_{\sigma^{-1}(1)} + \nu_1) \geq \mu_1 + \mu_2 + \mu_4 + \nu_1,$$

since $\mu_{\sigma^{-1}(1)}$ has a the lower bound μ_4 and there is always a way to include μ_1, μ_2 in the maximization, either in the case where $\sigma^{-1}(1) \in \{3,4\}$ where we can choose the other μ's as μ_1, μ_2, or when $\sigma^{-1}(1) \in \{1,2\}$ we can choose the μ's to be the other index in $\{1,2\}$ and 3. Consequently:

$$V_4 \geq (\mu_1 + \mu_2 + \mu_4 + \nu_1) + \mu_3 - |\mu| - \nu_1 = 0.$$

The fifth rhombus inequality involves all three interior nodes: (x, y, z):

$$V_5 = y + z - x - (|\mu| + \nu_1 + \nu_2),$$

and is therefore a little more involved and will be held off until the end of this section.

Finally we have the sixth vertical rhombus inequality:

$$\begin{aligned} V_6 &= z + (\lambda_1 + \lambda_2 + \lambda_3) - (\lambda_1 + \lambda_2) - (|\mu| + \nu_1 + \nu_2 + \nu_3) \\ &= \max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) + \lambda_3 - |\mu| - |\nu| + \nu_4, \end{aligned}$$

where we exchange $|\mu| + |\nu|$ for $|\lambda|$ so that:

$$= \max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) + \nu_4 - \lambda_1 - \lambda_2 - \lambda_4,$$

and now by fixing i to be $\sigma^{-1}(4)$ we obtain a lower bound:

$$\begin{aligned} &\geq \max_{\substack{|\{j,k\}|=2 \\ \sigma^{-1}(4)/\{j,k\}}}(\mu_{\sigma^{-1}(4)} + \beta_j + \beta_k) + \nu_4 - \lambda_1 - \lambda_2 - \lambda_4 \\ &= (\mu_{\sigma^{-1}(4)} + \nu_4) + \max_{\substack{|\{j,k\}|=2 \\ \sigma^{-1}(4)/\{j,k\}}}(\beta_j + \beta_k) - \lambda_1 - \lambda_2 - \lambda_4, \end{aligned}$$

now depending on $\sigma^{-1}(4)$ the maximum will differ:

$$\sigma^{-1}(4) + \nu_4 + \max_{\substack{|\{j,k\}|=2 \\ \sigma^{-1}(4)/\{j,k\}}}(\beta_j + \beta_k) = \begin{cases} \lambda_1 + (\lambda_2 + \lambda_3), & \text{if } \sigma^{-1}(4) = j_1, \\ \lambda_2 + (\lambda_1 + \lambda_3), & \text{if } \sigma^{-1}(4) = j_2, \\ \lambda_3 + (\lambda_1 + \lambda_2), & \text{if } \sigma^{-1}(4) = j_3, \\ \lambda_4 + (\lambda_1 + \lambda_2), & \text{if } \sigma^{-1}(4) = j_4. \end{cases}$$

Regardless of the value of $\sigma^{-1}(4)$ we can therefore conclude that:

$$\sigma^{-1}(4) + \nu_4 + \max_{\substack{|\{j,k\}|=2 \\ \sigma^{-1}(4)/\{j,k\}}}(\beta_j + \beta_k) \geq \lambda_1 + \lambda_2 + \lambda_4,$$

ensuring that:

$$V_6 = \max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) + \nu_4 - \lambda_1 - \lambda_2 - \lambda_4 \geq \lambda_1 + \lambda_2 + \lambda_4 - \lambda_1 - \lambda_2 - \lambda_4 = 0.$$

We now continue with the left vertical inequalities. As with the first of the vertical inequalities, L_1 is simply the same inequality as when $n = 3$:

$$L_1 = x + \lambda_1 - \mu_1 - (\lambda_1 + \lambda_2) = L_1^{(n=3)} \geq 0.$$

The second left inequality however is more complicated as it contains all three of (x, y, z):

$$L_2 = x + y - (\mu_1 + \mu_2) - z,$$

and will like V_5 be saved for the end of in this section. We resume with L_3:

$$\begin{aligned} L_3 &= z + (\lambda_1 + \lambda_2) - x - (\lambda_1 + \lambda_2 + \lambda_3) \\ &= \max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) - \max_{|\{i,j\}|=2}(\mu_i + \beta_j) - \lambda_3, \end{aligned}$$

where

$$\max_{|\{i,j,k\}|=3}(\mu_i + \beta_j + \beta_k) \geq \max_{|\{i,j\}|=2}(\mu_i + \beta_j + \lambda_3) = \max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_3,$$

since in the maximization β_k can always be chosen to be at least (but not always precisely) λ_3, without limiting the maximization of the other two. After all, only one λ and one μ are to be chosen and the maximum will be the greatest out of $(\mu_1 + \lambda_1, \mu_2 + \lambda_1, \mu_1 + \lambda_2)$. This means that the only time λ_3 is relevant for the optimum is when $\beta_{j_1} = \lambda_3$, but then either λ_1, λ_2 can be separated instead. Therefore:

$$L_3 \geq (\max_{|\{i,j\}|=2}(\mu_i + \beta_j) + \lambda_3) - \max_{|\{i,j\}|=2}(\mu_i + \beta_j) - \lambda_3 = 0.$$

Moving on to the fourth left rhombus inequality we have:

$$\begin{aligned} L_4 &= y + (|\mu| + \nu_1) - (\mu_1 + \mu_2 + \mu_3) - (|\mu| + \nu_1 + \nu_2) \\ &= \max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \beta_k) - \mu_1 - \mu_2 - \mu_3 - \nu_2 \\ &= \max_{|\{i,j,k\}|=3}(\mu_i + \mu_j + \mu_k + \nu_{\sigma(k)}) - \mu_1 - \mu_2 - \mu_3 - \nu_2, \end{aligned}$$

where we put some restrictions on k and obtain a lower bound on the maximization:

$$\geq \max_{\substack{|\{i,j,k\}|=3 \\ k \leq 3 \\ \sigma(k) \leq 2}}(\mu_i + \mu_j + \mu_k + \nu_{\sigma(k)}) - \mu_1 - \mu_2 - \mu_3 - \nu_2.$$

This is possible since $\sigma(4)$ cannot be both 1 and 2 simultaneously and so there is always a way to choose k so that $\nu_{\sigma(k)} \geq \nu_2$ and $\mu_k \geq \mu_3$. The condition $k \geq 3$

ensures that the μ's can be chosen as $\mu_k \geq \mu_3$ as well as the two remaining out of (μ_1, μ_2, μ_3). We note that:

$$\max_{\substack{|\{i,j,k\}|=3 \\ k \leq 3 \\ \sigma(k) \leq 2}} (\mu_i + \mu_j + \mu_k + \nu_{\sigma(k)}) \geq \mu_1 + \mu_2 + \mu_3 + \nu_2,$$

and therefore

$$L_4 \geq (\mu_1 + \mu_2 + \mu_3 + \nu_2) - \mu_1 - \mu_2 - \mu_3 - \nu_2 = 0.$$

Let us now prove the fifth left rhombus inequality:

$$\begin{aligned} L_5 &= z + (|\mu| + \nu_1 + \nu_2) - y - (|\mu| + \nu_1 + \nu_2 + \nu_3) \\ &= \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) - \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) - \nu_3. \end{aligned}$$

We note that:

$$\begin{aligned} \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) &\geq \max_{\substack{|\{i,j,k\}|=3 \\ j \leq 3 \\ \sigma(j) \leq 3}} \left(\mu_i + \mu_j + \nu_{\sigma(j)} + \beta_k\right) \\ &\geq \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \nu_3, \end{aligned}$$

where in the first inequality we specify that we are choosing a β_j such that both elements in β_j have index three or lower. This is possible since there are four β's to choose from and μ_4 and ν_4 are part of at most two β's. The second inequality we get from moving the ν_3 out from the maximum as a lower bound on the $\nu_{\sigma(j)}$ chosen. The condition that $j \leq 3, j = k$ does not limit the maximization as there should be three μ's included in the sum, enforcing that the separate μ's should be greater than or equal to μ_3 regardless and with at least two β's to choose j from we can always choose the lesser β_j so that the maximum is attained on the right side. Thus we have:

$$L_5 \geq \left(\max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \nu_3\right) - \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) - \nu_3 = 0.$$

Next we prove that $L_6 \geq 0$. First we have:

$$L_6 = (|\mu| + \nu_1 + \nu_2 + \nu_3) + (\lambda_1 + \lambda_2 + \lambda_3) - z - |\lambda|,$$

which when inserting the expression for z is:

$$\begin{aligned} &= |\mu| + \nu_1 + \nu_2 + \nu_3 + \lambda_1 + \lambda_2 + \lambda_3 - \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) - |\lambda| \\ &= |\lambda| - \nu_4 + \lambda_1 + \lambda_2 + \lambda_3 - \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) - |\lambda| \\ &= \lambda_1 + \lambda_2 + \lambda_3 - \nu_4 - \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k). \end{aligned}$$

This can be expressed equivalently as:

$$L_6 = \max_{|\{i,j,k\}|=3} (\beta_i + \beta_j + \beta_k) - \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k + \nu_4) \geq 0,$$

which is clearly non-negative as whatever (i, j, k) are chosen in the maximum on the right can also be selected for the maximum on the left, with the accompanying ν naturally satisfying that $\nu \geq \nu_4$.

We now move on to the right rhombus inequalities. As with V_1 and L_1, the first right rhombus inequality can be solved the same way as when $n = 3$:

$$R_1 = \mu_1 + x - \lambda_1 - (\mu_1 + \mu_2) = R_1^{(n=3)} \geq 0.$$

In the second right rhombus inequality we have:

$$\begin{aligned} R_2 &= y + (\mu_1 + \mu_2) - x - (\mu_1 + \mu_2 + \mu_3) \\ &= \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) - \max_{|\{i,j\}|=2} (\mu_i + \beta_j) - \mu_3 \\ &\geq \max_{|\{i,j\}|=2} (\mu_i + \beta_j + \mu_3) - \max_{|\{i,j\}|=2} (\mu_i + \beta_j) - \mu_3. \end{aligned}$$

This inequality holds since it replaces a μ with a lower bound μ_3. The reason the μ's in the maximum have μ_3 as a lower bound is because there are three μ's included in the maximum in total and so for a stand-alone μ, i.e. a μ that is not part of a β, it cannot be smaller than μ_3. It follows that:

$$R_2 \geq \max_{|\{i,j\}|=2} (\mu_i + \beta_j) + \mu_3 - \max_{|\{i,j\}|=2} (\mu_i + \beta_j) - \mu_3 = 0.$$

The third right rhombus inequality:

$$R_3 = x + z - (\lambda_1 + \lambda_2) - y,$$

will be proven at the end of this section together with V_5 and L_2.

The fourth inequality can be shown as follows:

$$\begin{aligned} R_4 &= (|\mu| + \nu_1) + (\mu_1 + \mu_2 + \mu_3) - y - |\mu| \\ &= \nu_1 + \mu_1 + \mu_2 + \mu_3 - \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) \\ &= \nu_1 + \mu_1 + \mu_2 + \mu_3 - \max_{|\{i,j,k\}|=3} \left(\mu_i + \mu_j + \mu_k + \nu_{\sigma(k)}\right), \end{aligned}$$

we now replace the maximum by an upper bound:

$$\geq \nu_1 + \mu_1 + \mu_2 + \mu_3 - (\mu_1 + \mu_2 + \mu_3 + \nu_1) = 0,$$

proving the inequality.

In the fifth right rhombus inequality we have the following:

$$\begin{aligned} R_5 &= y + (|\mu| + \nu_1 + \nu_2) - z - (|\mu| + \nu_1) \\ &= \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \nu_2 - \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k), \end{aligned}$$

where we note that:

$$\max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) = \max_{|\{i,j,k\}|=3} \left(\mu_i + \beta_j + \mu_k + \nu_{\sigma(k)}\right)$$

$$\leq \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \mu_k + \nu_2) = \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \nu_2,$$

since in the maximum two distinct ν's must be chosen and therefore one of them must be smaller than or equal to ν_2. That is why fixing $\nu_{\sigma(k)}$ to be ν_2 constitutes a relaxation and consequently a non-strict increase of the maximum. By replacing the second maximum by this upper bound we can show that the inequality holds:

$$R_5 \geq \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \nu_2 - (\max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \nu_2) = 0.$$

Finally the sixth inequality can be shown:

$$\begin{aligned} R_6 &= z + (|\mu| + \nu_1 + \nu_2 + \nu_3) - (|\mu| + \nu_1 + \nu_2) - (\lambda_1 + \lambda_2 + \lambda_3) \\ &= \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) + \nu_3 - \lambda_1 - \lambda_2 - \lambda_3. \end{aligned}$$

Now if $\sigma^{-1}(j_4) = 3$ then:

$$\max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) + \nu_3 \geq \max_{\substack{|\{i,j,k\}|=3 \\ i,j,k\ \{j_1,j_2,j_3\} \\ \sigma(i)=3}} (\mu_i + \beta_j + \beta_k) + \nu_3 = \lambda_1 + \lambda_2 + \lambda_3.$$

If, however, $\sigma^{-1}(j_4) = 3$, then that means that $\nu_3 \geq \min\{\nu_{\sigma^{-1}(j_1)}, \nu_{\sigma^{-1}(j_2)}, \nu_{\sigma^{-1}(j_3)}\} = \nu_4$ and so:

$$\begin{aligned} \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) + \nu_3 &\geq \max_{\substack{|\{i,j,k\}|=3 \\ i,j,k\ \{j_1,j_2,j_3\} \\ \sigma(i)=4}} (\mu_i + \beta_j + \beta_k) + \nu_3 \\ &= \lambda_1 + \lambda_2 + \lambda_3 + \nu_3 - \nu_4. \end{aligned}$$

In either case we have:

$$\begin{aligned} R_6 &= \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) + \nu_3 - \lambda_1 - \lambda_2 - \lambda_3 \\ &\geq (\lambda_1 + \lambda_2 + \lambda_3) - \lambda_1 - \lambda_2 - \lambda_3 = 0. \end{aligned}$$

The more complicated inequalities, V_5, L_2 and R_3, highlighted in Figure A.2, involving all three inner nodes will now be discussed one by one. The reasoning will be a bit more involved but very similar between the three inequalities. We will begin by proving that $V_5 \geq 0$, not only for $n = 4$ but for all $n \geq 4$.

For $n = 4$, V_5 is of the form:

$$\begin{aligned} V_5 &= \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) + \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) - \max_{|\{i,j\}|=2} (\mu_i + \beta_j) - |\mu| - \nu_1 - \nu_2 \\ &= \max_{\substack{|\{i,j,k\}|=3 \\ |\{l,m,n\}|=3}} (\mu_i + \mu_j + \beta_k + \mu_l + \beta_m + \beta_m) \\ &\quad - \max_{\substack{|\{i,j\}|=2 \\ |\{k,l,m,n\}|=4}} (\mu_i + \beta_j + \mu_k + \mu_l + \beta_m + \beta_n), \end{aligned}$$

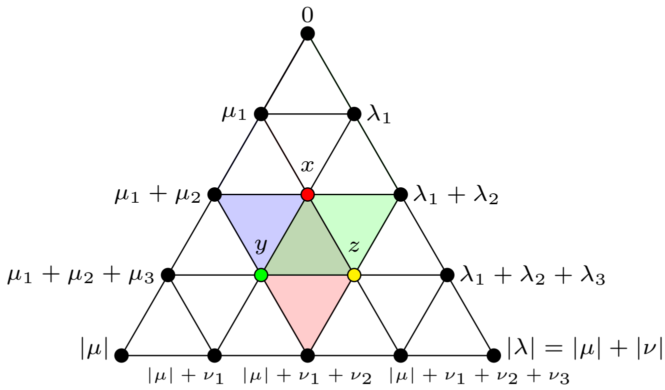


Figure A.2: A hive of size $n = 4$ with the inequalities that involved all three interior nodes highlighted in di erent colors.

where the indices $i, j, k, l, m, n \quad \{1, \ldots, n\}$. We can rewrite the maximization as:

$$V_5 = \max_{\substack{A,B,C,D \quad \{1,\ldots,n\} \\ A \cap B = \\ |A|=2, |B|=1 \\ C \cap D = \\ |C|=1, |D|=2}} \left(\sum_{i \quad A} \mu_i + \sum_{j \quad B} \beta_j + \sum_{k \quad C} \mu_k + \sum_{l \quad D} \beta_l \right) -$$

$$\max_{\substack{E,F,G,H \quad \{1,\ldots,n\} \\ E \cap F = \\ |E|=1, |F|=1 \\ G \cap H = \\ |G|=2, |H|=2}} \left(\sum_{i \quad E} \mu_i + \sum_{j \quad F} \beta_j + \sum_{k \quad G} \mu_k + \sum_{l \quad H} \beta_l \right).$$

Whenever $n > 4$, the bottom node in the rhombus inequality V_5 changes from $h_{42}^{n=4} = |\mu| + \nu_1 + \nu_2$ to:

$$h_{42}^{n>4} = \max_{\substack{J \quad I \\ |J|=2 \\ |I|=4}} \left(\sum_{i \quad I} \mu_i + \sum_{j \quad J} \nu_{\sigma(j)} \right) = \max_{\substack{G \cap H = \\ |G|=2 \\ |H|=2}} \left(\sum_{i \quad G} \mu_i + \sum_{j \quad H} \beta_j \right)$$
$$= \max_{|\{k,l,m,n\}|=4} \{\mu_k + \mu_l + \beta_m + \beta_n\},$$

which means that the maximizations in L_2 are the same for $n > 4$ as for $n = 4$. We now claim that:

$$\max_{\substack{A,B,C,D \quad \{1,\ldots,n\} \\ A \cap B = \\ |A|=2, |B|=1 \\ C \cap D = \\ |C|=1, |D|=2}} \underbrace{\left(\sum_{i \quad A} \mu_i + \sum_{j \quad B} \beta_j + \sum_{k \quad C} \mu_k + \sum_{l \quad D} \beta_l \right)}_{L(A,B,C,D)} \geq$$

$$\max_{\substack{E,F,G,H \quad \{1,\ldots,n\} \\ E \cap F = \\ |E|=1, |F|=1 \\ G \cap H = \\ |G|=2, |H|=2}} \underbrace{\left(\sum_{i \quad E} \mu_i + \sum_{j \quad F} \beta_j + \sum_{k \quad G} \mu_k + \sum_{l \quad H} \beta_l \right)}_{R(E,F,G,H)},$$

or in other words that $V_5 = L(A, B, C, D) - R(E, F, G, H) \geq 0$.

<u>Proof</u>: We argue that for *any* E, F, G, H satisfying the constraints on the RHS, we can construct *some* $A\ , B\ , C\ , D$ satisfying those on the LHS with:

$$L(A\ , B\ , C\ , D\ ) = R(E, F, G, H).$$

In particular:

$$\text{RHS} = \max_{E,F,G,H} R(E, F, G, H) = R(E_0, F_0, G_0, H_0) =$$
$$L(A_0, B_0, C_0, D_0) \leq \max_{A,B,C,D} L(A, B, C, D) = \text{LHS}.$$

To do this we let $E = \{a\}$, $F = \{c\}$, $G = \{b, d\}$, $H = \{e, f\}$, making:

$$R(E, F, G, H) = \mu_a + \beta_c + \mu_b + \mu_d + \beta_e + \beta_f,$$

with $|\{a, c\}| = 2$ and $|\{b, d, e, f\}| = 4$. This gives us the following assumptions to work with:

$\lvert G\rvert = 2$	$b = d,$	(I)
$\lvert H\rvert = 2$	$e = f,$	(II)
$E \cap F =$	$a = c,$	(III)
$G \cap H =$	$b = e, b = f,$	
	$d = e$ and $d = f.$	(IV)

We will prove the claim split into the two di erent cases:

Case 1: $G = E \quad F,$
Case 2: $G = E \quad F.$

Case 1: $G = E \quad F$
Since $G = E \quad F \qquad \{b, d\} = \{a, c\}$, we must have that either $b\ /\ \{a, c\}$ or $d\ /\ \{a, c\}$. Without loss of generality we let $b\ /\ \{a, c\}$, giving us the extra assumption on RHS:

$$b\ /\ \{a, c\} \qquad b = a \text{ and } b = c. \quad (V)$$

We then set $A\ = \{a, b\}$, $B\ = \{c\}$, $C\ = \{d\}$, $D\ = \{e, f\}$ which gives:

$$L(A\ , B\ , C\ , D\ ) = \mu_a + \mu_b + \beta_c + \mu_d + \beta_e + \beta_f$$
$$= \mu_a + \beta_c + \mu_b + \mu_d + \beta_e + \beta_f = R(E, F, G, H),$$

thereby proving the claim $V_5 \geq 0$ provided that the constraints on the LHS are fulfilled. We need to show the following:

$$|A\ | = 2 \qquad a = b,$$
$$|D\ | = 2 \qquad e = f,$$

$$A' \cap B' = \qquad a = c \text{ and } b = c,$$
$$C' \cap D' = \qquad d = e \text{ and } d = f.$$

By the extra assumption (V) for case 2 it follows that $|A'| = |\{a, b\}| = 2$ and from assumption (II) we see directly that $|D'| = |\{e, f\}| = |H| = 2$. We can also see that $A' \cap B' = \{a, b\} \cap \{c\} =$ from assumption (V) which says that $b = c$ and (III) which says that $a = c$. Finally, $C' \cap D' = \{d\} \cap \{e, f\} =$ is embedded in assumption (IV): $G \cap H = \{b, d\} \cap \{e, f\} =$.

Case 2: $G = E \quad F$
We start by setting $A' = \{b, d\}$, $B' = \{e\}$, $C' = \{a\}$ and $D' = \{c, f\}$. This would make:

$$\begin{aligned} L(A', B', C', D') &= \mu_b + \mu_d + \beta_e + \mu_a + \beta_c + \beta_f \\ &= \mu_a + \beta_c + \mu_b + \mu_d + \beta_e + \beta_f = R(E, F, G, H), \end{aligned}$$

provided that A', B', C' and D' satisfy the constraints on the LHS: $A' \cap B' =$ and $C' \cap D' =$ as well as $|A'| = |D'| = 2$. In this case we have the extra assumption on the RHS:

$$G = E \quad F \quad (V).$$

From assumption (IV) we get that $G \cap H = \{b, d\} \cap \{e, f\} =$ and so $A' \cap B' = \{b, d\} \cap \{e\} =$ follows from that. From (I) we get that $|A'| = |\{b, d\}| = |G| = 2$. It remains to show that C' and D' are disjoint and that $|D'| = 2$, which holds if $c = f$. From (V) we get that $G = E \quad F = \{a\} \quad \{c\}$, which we can put into (IV): $G \cap H = E \quad F \cap H = \{a\} \quad \{c\} \cap \{e, f\} =$, which gives us that $c = f \quad = \quad |D'| = 2$ and that $a = f$. Assumption (III) $E \cap F = \{a\} \cap \{c\}$ ensures that $a = c$ which combined with $a = f$ from assumption (IV) means that $C' \cap D' = \{a\} \cap \{c, f\} =$.
We have now shown that $V_5 \geq 0$ for both cases, and so $V_5 \geq 0$ for all $n \geq 4$.

We now move on to showing that $L_2 \geq 0$ for all $n \geq 4$. For $n \geq 4$, we have that L_2 is of the form:

$$\begin{aligned} L_2 &= \max_{|\{i,j\}|=2} (\mu_i + \beta_j) + \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) - \mu_1 - \mu_2 - \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) \\ &= \max_{\substack{|\{i,j\}|=2 \\ |\{k,l,m\}|=3}} (\mu_i + \beta_j + \mu_k + \mu_l + \beta_m) - \max_{\substack{|\{i,j\}|=2 \\ |\{k,l,m\}|=3}} (\mu_i + \mu_j + \mu_k + \beta_l + \beta_m). \end{aligned}$$

We claim that:

$$\max_{\substack{A,B,C,D \quad \{1,\ldots,n\} \\ A \cap B = \\ |A|=1, |B|=1 \\ C \cap D = \\ |C|=2, |D|=1}} \underbrace{\left(\sum_{i \quad A} \mu_i + \sum_{j \quad B} \beta_j + \sum_{k \quad C} \mu_k + \sum_{l \quad D} \beta_l \right)}_{L(A,B,C,D)} \geq$$

$$\max_{\substack{E,G,H \quad \{1,\ldots,n\} \\ |E|=2 \\ G \cap H = \\ |G|=1, |H|=2}} \underbrace{\left(\sum_{i \quad E} \mu_i + \sum_{k \quad G} \mu_k + \sum_{l \quad H} \beta_l \right)}_{R(E,G,H)},$$

or in other words, that $L_2 = L(A, B, C, D) - R(E, G, H) \geq 0$.

<u>Proof</u>: As with V_5 we argue that for *any* E, G, H satisfying the constraints on the RHS, we can construct *some* $A\ , B\ , C\ , D$ satisfying those on the LHS with:

$$\text{RHS} = \max_{E,G,H} R(E, G, H) = R(E_0, G_0, H_0) =$$
$$L(A_0, B_0, C_0, D_0) \leq \max_{A,B,C,D} L(A, B, C, D) = \text{LHS}.$$

Let $E = \{a, b\}$, $G = \{c\}$, $H = \{d, e\}$, so that:

$$R(E, G, H) = \mu_a + \mu_b + \mu_c + \beta_d + \beta_e,$$

with $|\{a, b\}| = 2$ and $|\{c, d, e\}| = 3$. From the constraints on RHS we get the following assumptions:

$$\begin{array}{rcl} |E| = 2 & a = b, & (I) \\ |H| = 2 & d = e, & (II) \\ G \cap H = & c = d, & (III) \\ \text{and} & c = e. & (IV) \end{array}$$

We shall prove the claim split into two cases:

Case 1: $H = E$,
Case 2: $H = E$.

Case 1: $H = E$
In this case $\{a, b\} = \{d, e\}$. Without loss of generality we can assume $a = d$ and $b = e$. Let us put $A\ = \{a\}$, $B\ = \{e\}$, $C\ = \{b, c\}$ and $D\ = \{d\}$, making:

$$L(A\ , B\ , C\ , D\ ) = \mu_a + \beta_e + \mu_b + \mu_c + \beta_d$$
$$= \mu_a + \mu_b + \mu_c + \beta_d + \beta_e = R(E, G, H),$$

provided that $|C\ | = |\{b, c\}| = 2$, $A\ \cap B\ =$ and $C\ \cap D\ =$, i.e. that:

$$b = c,$$
$$a = e,$$
$$b = d,$$
$$c = d.$$

We have that $b = e$ from the assumption specific to this case and so $b = e = c$ by (IV) and similarly $a = d = e$ by assumption (II). We also have that $b = e = d$ by assumption (II). Finally we have that $c = d$ follows from assumption (III).

Case 2: $H = E$
In this case we must assume that at least one out of $\{a, b\}$ is not equal to either d or e. Without loss of generality we assume that a / $\{d, e\}$ so that we have:

$$a = d, \quad (V)$$
$$a = e. \quad (VI)$$

We also make the observation that since by assumption (II) $d = e$, no index can be equal to both d and e. We can therefore make the additional assumption on b that:

$$b = d, \quad (VII)$$
or
$$b = e. \quad (VIII)$$

We first assume that $b = d$ and choose: $A = \{c\}$, $B = \{e\}$, $C = \{a, b\}$ and $D = \{d\}$, making:

$$L(A, B, C, D) = \mu_c + \beta_e + \mu_a + \mu_b + \beta_d = R(E, G, H) = \mu_a + \mu_b + \mu_c + \beta_d + \beta_e,$$

which ensures $L_2 \geq 0$ provided that $|C| = |\{a, b\}| = 2$, $A \cap B =$ and $C \cap D =$, or in other words that:

$$a = b,$$
$$c = e,$$
$$a = d,$$
$$b = d.$$

We then have that $a = b$ by assumption (I) and that $c = e$ by assumption (IV). Assumption (V) gives $a = d$ and (VII) gives $b = d$. If we instead assume $b = e$ we can choose our sets differently by switching places between d and e so that: $A = \{c\}$, $B = \{d\}$, $C = \{a, b\}$ and $D = \{e\}$, and:

$$L(A, B, C, D) = \mu_c + \beta_d + \mu_a + \mu_b + \beta_e = R(E, G, H) = \mu_a + \mu_b + \mu_c + \beta_d + \beta_e,$$

provided that $|C| = |\{a, b\}| = 2$, $A \cap B =$ and $C \cap D =$, i.e. that:

$$a = b,$$
$$c = d,$$
$$a = e,$$
$$b = e.$$

Then $a = b$ and $c = d$ still holds. We now have that $a = e$ which follows from (VI) and that $b = e$ by (VIII).

The last remaining inequality to be shown is that $R_3 \geq 0$ for $n \geq 4$. For $n \geq 4$, we have that R_3 is of the form:

$$R_3 = \max_{|\{i,j\}|=2} (\mu_i + \beta_j) + \max_{|\{i,j,k\}|=3} (\mu_i + \beta_j + \beta_k) - \max_{|\{i,j,k\}|=3} (\mu_i + \mu_j + \beta_k) - \lambda_1 - \lambda_2$$

$$= \max_{\substack{|\{i,j\}|=2 \\ |\{k,l,m\}|=3}} (\mu_i + \beta_j + \mu_k + \beta_l + \beta_m) - \max_{\substack{|\{i,j,k\}|=3 \\ |\{l,m\}|=2}} (\mu_i + \mu_j + \beta_k + \beta_l + \beta_m).$$

We claim that:

$$\max_{\substack{A,B,C,D \quad \{1,\dots,n\} \\ A\cap B= \\ |A|=1,|B|=1 \\ C\cap D= \\ |C|=1,|D|=2}} \underbrace{\left(\sum_{i \ A} \mu_i + \sum_{j \ B} \beta_j + \sum_{k \ C} \mu_k + \sum_{l \ D} \beta_l \right)}_{L(A,B,C,D)} \geq$$

$$\max_{\substack{E,F,H \quad \{1,\dots,n\} \\ |E|=2,|F|=1 \\ E\cap F= \\ |H|=2}} \underbrace{\left(\sum_{i \ E} \mu_i + \sum_{k \ F} \beta_k + \sum_{l \ H} \beta_l \right)}_{R(E,F,H)},$$

or in other words, that $R_3 = L(A, B, C, D) - R(E, F, H) \geq 0$.

<u>Proof</u>: We argue that for *any* E, F, H satisfying the constraints on the RHS, we can construct *some* $A\ , B\ , C\ , D$ satisfying those on the LHS with:

$$L(A\ , B\ , C\ , D\ ) = R(E, F, H).$$

In particular,

$$\text{RHS} = \max_{E,F,H} R(E, F, H) = R(E_0, F_0, H_0) =$$
$$L(A_0, B_0, C_0, D_0) \leq \max_{A,B,C,D} L(A, B, C, D) = \text{LHS}.$$

To do this we let $E = \{a, b\}$, $F = \{c\}$, $H = \{d, e\}$, so that:

$$R(E, F, H) = \mu_a + \mu_b + \beta_c + \beta_d + \beta_e,$$

with $|\{a, b, c\}| = 3$ and $|\{d, e\}| = 2$. This gives us the following assumptions to work with:

$$\begin{aligned} |E| = 2 \qquad & a = b, \quad (I) \\ |H| = 2 \qquad & d = e, \quad (II) \\ E \cap F = \qquad & a = c, \quad (III) \\ \text{and} \quad & b = c. \quad (IV) \end{aligned}$$

We shall prove the claim split into two cases:

Case 1: $E = H$,
Case 2: $E = H$.

Case 1: $E = H$

In this case $\{a, b\} = \{d, e\}$. Without loss of generality we can assume $a = d$ and $b = e$. We put $A = \{a\}$, $B = \{e\}$, $C = \{b\}$ and $D = \{c, d\}$, making:

$$\begin{aligned} L(A\,, B\,, C\,, D\,) &= \mu_a + \beta_e + \mu_b + \beta_c + \beta_d \\ &= \mu_a + \mu_b + \beta_c + \beta_d + \beta_e = R(E, F, H), \end{aligned}$$

provided that $|D\,| = |\{b, c\}| = 2$, $A \cap B =$ and $C \cap D =$. The constraints on the LHS can be expressed:

$$\begin{aligned} c &= d, \\ a &= e, \\ c &= b, \\ d &= b. \end{aligned}$$

From the assumption that $a = d$ and we can get $d = a = c$ by assumption (III) and $a = d = e$ by assumption (II). We also have $b = c$ from (IV). Finally, since $b = e = d$ by assumption (II) all conditions are met.

Case 2: $E = H$

In this case we must assume that at least one out of $\{a, b\}$ is not equal to either d or e. Without loss of generality we assume that $a\ /\ \{d, e\}$ so that we have:

$$a = d \text{ and } a = e \quad (V)\,.$$

We try $A = \{b\}$, $B = \{c\}$, $C = \{a\}$ and $D = \{d, e\}$, making:

$$\begin{aligned} L(A\,, B\,, C\,, D\,) &= \mu_b + \beta_c + \mu_a + \beta_d + \beta_e \\ &= R(E, F, H) = \mu_a + \mu_b + \beta_c + \beta_d + \beta_e, \end{aligned}$$

provided that $|D\,| = |\{d, e\}| = 2$, $A \cap B =$ and $C \cap D =$, i.e. that:

$$\begin{aligned} d &= e, \\ b &= c, \\ a &= d, \\ a &= e. \end{aligned}$$

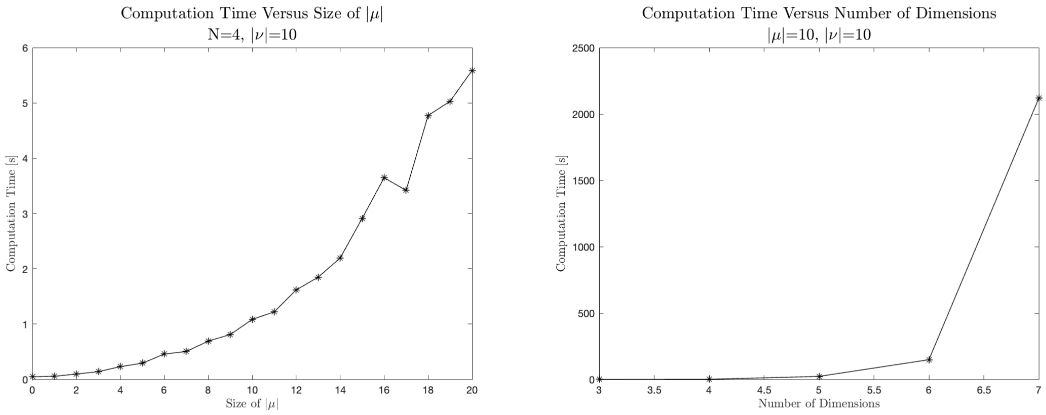
We then have that $d = e$ by assumption (II) and that $b = c$ by assumption (IV). The assumption on a that $a\ /\ \{d, e\}$ ensures that the last two constraints are fulfilled.

□

B

Search for Counter-Examples

Before Theorem 4.10 was shown, quite some time was devoted to exploring the *AWD*-construction through 'exhaustive searches', where we note the futility in performing exhaustive searches on the integers. In the end, no counter example was found, and when the general proof fell into place, the only thing worthwhile to note is perhaps the challenging time complexity of the problem. In the end, all unique permutations of integers with an eigenvalue sum of $|\mu| = 1, \cdots, 20$, $|\nu| = 1, \cdots, 20$ for dimensions in $n = 1, \cdots, 7$ were explored, and in each case the *AWD*-construction was successful in finding an integral hive filling. Figure B.1 shows how the computational time grew with the eigenvalue sum and the matrix dimension.



(a) **(b)**

Figure B.1: Graph (a) shows how time complexity grew with the sum of eigenvalues, $|\mu|$, when searching through all possible integral diagonal hives with $n = 4, |\nu| = 10$. Graph (b) shows how time complexity grew with the number of dimensions, n, when searching through all possible integral diagonal hives with $|\mu| = 10, |\nu| = 10$.

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