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Hypergeometric Functions and Their Generalizations to Higher Dimensions

A study of the classical hypergeometric function and its generalizations associated with root systems

Master's thesis in Engineering Mathematics and Computational Science

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Abstract

The hypergeometric differential equation is a classical ODE of second order, and it was already studied by Gauss. The hypergeometric function is classically defined as the solution to this equation that is analytic at $x = 0$. With this definition it is not obvious how to generalize the hypergeometric function to higher dimensions. With a shift in perspective we can arrive at the same differential equation by studying a certain eigenvalue problem of polynomials of so called Dunkl operators. These are easier to generalize and will lead us to hypergeometric functions associated with so called root systems in higher dimension.

Keywords: Hypergeometric differential equation, Hypergeometric function, Root system, Dunkl operators.

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Oskar Johansson, Gothenburg, June 2024

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1
Introduction

The hypergeometric differential equation is classically written as

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + (c-(1+a+b)x)\frac{\partial}{\partial x} - ab\right] f = 0, \tag{1.1}$$

where $x \in \mathbf{C}$. It is a second order differential equation so we know that the solution space is 2 dimensional. Only one of these solutions are analytic at $x = 0$, and this solution can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \quad \text{where } (\alpha)_n = \begin{cases} \alpha(\alpha+1)...(\alpha+n-1), & \text{if } n \neq 0 \\ 1, & \text{if } n = 0 \end{cases}. \tag{1.2}$$

This function is the main focus in this report. We will look at how we arive at this solution, and how it might be possible to generalize to higher dimensions.

1.1 Background

We will begin by motivating where the name hypergeometric comes from. A geometric series can be defied as a series on the form

$$\sum_{n=0}^{\infty} c_n \tag{1.3}$$

where the ratio

$$\frac{c_{n+1}}{c_n} = x \in \mathbf{C} \tag{1.4}$$

is constant. It is a well known fact that this series converges whenever $|x| < 1$, in that case (1.3) can be simplified to

$$f(x) = c_0 \sum_{n=0}^{\infty} x^n = \frac{c_0}{1-x}. \tag{1.5}$$

In this setting we may refer to f as the geometric function. A natural generalisation to this would be to consider the case when $\frac{c_{n+1}}{c_n} = P(n)$, where $P(n)$ is some polynomial in n. For example if we let

$$\frac{c_{n+1}}{c_n} = n, \tag{1.6}$$

we get the solution $c_n = kn!$, for some constant k. However the series

$$k \sum_{n=0}^{\infty} n! \tag{1.7}$$

diverges very fast. Introducing higher powers of n in $P(n)$ is going to make the sum diverge even faster, so this generalisation is not very interesting. The next meaningful generalisation is to let $\frac{c_{n+1}}{c_n} = R(n)$ where $R(n)$ is some rational function in n. Let us try and find a general form for c_n in this case. In the complex plane any polynomials can be factored into linear factors. Since a rational function is a ratio between two polynomials we can write $R(n)$ in the form

$$R(n) = d\frac{(a_1 + n)...(a_s + n)}{(b_1 + n)...(b_r + n)}. \tag{1.8}$$

Using the recursive relation $c_{n+1} = R(n)c_n$, one immediately gets the general formula

$$c_n = c\frac{(a_1)_n...(a_s)_n}{(b_1)_n...(b_r)_n}x^n. \tag{1.9}$$

This natural generalisation to the geometric series lead to the following definition.

Definition 1.1.1. A hypergeometric function is a function on the form

$$f(x) = c\sum_{n=0}^{\infty} \frac{(a_1)_n...(a_s)_n}{(b_1)_n...(b_r)_n}x^n. \tag{1.10}$$

This set of functions include a vast variety of functions.

Example 1.1.2. Both $f(x) = \ln(1 + x)$ and $f(x) = e^x$ are hypergeometric by definition 1.1.1. This can be seen by looking at their Taylor expansions. In the case of the exponential function $c_n = \frac{x^n}{n!}$, so that $\frac{c_{n+1}}{c_n} = \frac{x}{n}$. Which is definitely rational. By the same logic $\ln(1 + x)$, the trigonometric functions, and their inverses are all hypergeometric by definition 1.1.1. This can be easily seen by looking at their Taylor series. Also note that if we set a_i to be a negative integer for some i we can make the sum (1.10) terminate after any finite number of terms. Hence polynomials are in the class of hypergeometric functions as well.

Remark 1.1.3. Sometimes the notation ${}_pF_q(a_1, ..., a_p, b_1, ..., b_p; x)$ is used to specify which class of hypergeometric functions one is referring to. What is meant in this case is

$$\frac{c_{n+1}}{c_n} = \frac{(n + a_1)...(n + a_p)}{(n + b_1)...(n + b_q)(n + 1)}, \tag{1.11}$$

so that

$$ {}_pF_q(a_1, ..., a_p, b_1, ..., b_p; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{x^n}{n!}. \tag{1.12} $$

Note that ${}_pF_q$ contains an $n!$ in the denominator, and is missing the c in front of the sum. So this notation is referring to a specific class of hypergeometric functions.

Remark 1.1.4. What exactly is defined as the hypergeometric function varies between literature. The Gauss hypergeometric function is a special case of a hypergeometric function, and is typically defined as

$$ F(a, b, c; x) = {}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n. \tag{1.13} $$

However in some literature this is defined as the hypergeometric function. This function is what we are interested in generalizing, so from now in if we refer to the hypergeometric function, we refer to the Gauss hypergeometric function. If we refer to the general form defined in definition 1.1.1, that will be specifically stated. However we do mention here that there is a wider family of function we could look at.

Remark 1.1.5. Setting $(a, b, c) = (1, 1, 1)$ we notice that ${}_2F_1$ reduces to

$$ {}_2F_1(1, 1, 1; x) = \sum_{i=0}^{\infty} x^n = \frac{1}{1-x}, \tag{1.14} $$

so the regular geometric function is in fact included in ${}_2F_1$.

1.2 Motivation

The function ${}_2F_1$ is named after Gauss, that is however a misnomer because the first person to study the function was actually Euler. Both Gauss and Riemann also contributed to the study of ${}_2F_1$ however. The function is a solution to the Fuchsian differential equation

$$ \left[x(1-x)\frac{\partial^2}{\partial x^2} + (c - (1 + a + b)x)\frac{\partial}{\partial x} - ab \right] f = 0, \tag{1.15} $$

or more precisely it is a solution which is regular at 0. Equation (1.1) has three singularities, at 0, 1 and ∞. It is also characterized by its integral representation

$$ {}_2F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b} dt. \tag{1.16} $$

This function has many applications that we wont look into. It comes up many times in harmonic analysis and representation theory, and is also central for Kleins method for solving a quintic equation. It provides solutions to many differential equations, such as for example Bessel, Hermite, Airy and Painleve. More applications are listed in [11] and chapter 15 of [3].

Another application that we wont look further into is that a close relative to the hypergeometric function is an eigenfunction to the Laplace operator in hyperbolic space **H**. Hyperbolic space in this case is the upper half of the complex plane **H** with distance formula $ds = y^{-2}\sqrt{(dx)^2 + (dy)^2}$. For more details on this see [15].

All these applications have challenged mathematicians to generalize ${}_2F_1$ to higher dimensions, and this is still an active area of research. The Gelfand–Kapranov–Zelevinskii hypergeometric functions [4] is one example of a generalisation to higher dimensions, which is still an active research topic. Many generalizations to higher dimension has been done, see for example [12]. One of particular interest to me is hypergeometric functions associated with root systems, and their connection to Lie algebras. See for example [7] and [6].

1.3 Aim

This thesis will look further into hypergeometric functions associated with root systems. Although we will not look into the connection to Lie algebras, and we will not attempt to solve the most general problem for a general root system. We will however look in detail into the simplest non trivial example, which is the case with the root system A_1. Then we will see how this example might generalize for an arbitrary n, so that we will arrive at the hypergeometric function associated with A_n

The first part of the project focuses on understanding how we build the solutions to (1.1) and understand how ${}_2F_1$ is constructed. Equation (1.1) has three singular points, at 0, 1 and ∞. We will look at what solutions look like near these points and what their connections are. The general source for this part of the project is [11].

Then the main focus is to understand the generalization to higher dimensions, associated with root systems. This generalisation was originally studied by Heckman and Opdam [12]. The main goal of the project is to understand this generalization.

As technical background I will first study reflection groups and root systems, using mostly [10] and [9]. Root systems is a general construct, but my aim is to phrase everything in the simplest possible nontrivial example, which is A_n. I will use A_n as a running example throughout the project whenever root systems come up.

I will proceed to study the Heckman Opdam hypergeometric function. First I will have to study some of the technical tools needed, such as for example so called Dunkl operators. For this purpose I will mostly look at [1] and [13]. However the main focus will be on understanding Heckman and Opdams generalisation. How it is constructed, and why it is a reasonable generalisation of ${}_2F_1$.

2

Preliminaries

In this chapter we go through some of the preliminaries required to follow the project. Mainly theory regarding ODEs and symmetric polynomials.

2.1 General theory on ODEs

Most of this chapter will focus on theory regarding ordinary differential equations. With theory regarding the fundamental matrix of an ODE, and also theory on what happens if the differential equation has singularities. This theory will be essential for the hypergeometric differential equation (1.1) since it has precisely three singularities at $0, 1$ and ∞. The hypergeometric function ${}_2F_1$ is a solution which is regular at the singularity at 0. We will go through exactly what that means. Most of this chapter follows chapter 1 and 2 of [16] and chapter 1 of [11].

A system of N first order ordinary differential equations is a system on the form

$$y_j' = \sum_{j=1}^{N} a_{ij}(x)y_i + f_j(x), \quad j = 1, ..., N, \tag{2.1}$$

where $a_{ij}(x)$ is holomorphic functions in some region R in the complex plane. Equation (2.1) can be rewritten in matrix form as

$$y' = A(x)y + f(x). \tag{2.2}$$

This is the type of differential equations we will study in this project. Or more precisely where the matrix A has some singularities. Note that there is no need to study differential equations of higher order since they can all be reduced to (2.2) by introducing more variables. If we have a differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_0(x)y = f(x) \tag{2.3}$$

of order n, we can introduce the new variables $y_j = y^{(j)}$ and we then get the first order system

$$\begin{gathered} y_j' = y_{j+1}, \quad j = 0, ..., n-1 \\ y_n = -a_{n-1}(x)y_{n-1} - ... - a_0(x)y_0 + f(x), \end{gathered}$$

which is on the form (2.2). If we want to study higher order differential equations, it is therefore sufficient to study equation (2.2). This is useful because equation (2.2) has the following uniqueness theorem.

Theorem 2.1.1. If $A(x)$ and $f(x)$ are both holomorphic in a region R, and if $a \in R$ and $\alpha \in \mathbf{C}^n$. Then equation (2.2) has exactly one solution such that $y(a) = \alpha$.

Proof. We do not give a proof here since it is a standard textbook result, a proof can be found in [16]. □

If we now instead look at the homogeneous equation

$$y' = A(x)y, \quad y \in \mathbf{C}^n, A(x) \in \mathbf{C}^{n\times n} \tag{2.4}$$

then theorem 2.1.1 can be used to show that the dimension of the space of solutions to (2.4) is n.

Theorem 2.1.2. There are n linearly independent solution to (2.4).

Proof. First let $a \in R$ and $v_j(x)$ be the solution to (2.4) such that $v_j(a) = e_j$, $j = 1, ..., n$. Here e_j is the jth standard unit vector in $\mathbf{C}^n$. First note that such solutions exists and are unique by theorem 2.1.1. Also note that these solutions are linearly independent. Since if they where not there would exist complex numbers b_j, $j = 1, ..., n$, not all zero, such that

$$\sum_{j=1}^{n} b_j v_j(x) \equiv 0. \tag{2.5}$$

However that would imply that

$$\sum_{j=1}^{n} b_j v_j(a) = \sum_{j=1}^{n} b_j e_j \equiv 0, \tag{2.6}$$

which is clearly impossible since the basis vectors are definitely linearly independent.

On the other hand assume now that there are $n + 1$ linearly independent soulution $\{v_j\}_{j=1}^{n}$ to (2.4). In that case we assume that

$$\sum_{j=1}^{n+1} b_j v_j(x) \equiv 0 \implies b_j = 0, j = 1, ..., n + 1. \tag{2.7}$$

Fix some $a \in R$, the equation

$$\sum_{j=1}^{n+1} b_j v_j(a) \equiv 0 \tag{2.8}$$

clearly has a solution $\{\tilde{b}_j\}_{j=1}^{n}$ where not all $\tilde{b}_j$ are 0. This is because $v_j(a)$ are n dimensional vectors, and there are $n + 1$ of them. Now study the function

$$\tilde{y} := \sum_{j=1}^{n+1} \tilde{b}_j v_j(x). \tag{2.9}$$

It clearly solves (2.4) since it is a linear combination of solutions. It also satisfies $\tilde{y}(a) = 0$. However since $y \equiv 0$ also solves (2.4) and coincides with $\tilde{y}$ at $x = a$, theorem 2.1.1 implies $\tilde{y} \equiv 0$. However this contradicts our assumption in (2.7). Hence we have shown that it is always possible to find n linearly independent solutions, and that any set of $n + 1$ solutions is always linearly dependent. □

With this result in mind we can define a fundamental matrix $V(x)$ corresponding to (2.4). This is an $n \times n$ matrix where each column is a solution to (2.4) and where all the columns are linearly independent. We now point out that the matrix $V(a)$ has to be inevitable for every $a \in R$, since if it where not, $v_j(a)$ would be linearly dependent. So there would exist $\{\tilde{b}_j\}_{j=1}^n$ not all zero such that

$$\sum_{j=1}^{n+1} \tilde{b}_j v_j(a) = 0, \tag{2.10}$$

however this would imply that

$$\sum_{j=1}^{n+1} \tilde{b}_j v_j(x) \equiv 0. \tag{2.11}$$

So in this case V is not a fundamental matrix. Also note that any linear combination of the rows of $V(x)$ is also a solution to (2.4). Algebraically this means that for all $C \in \mathbf{C}^n$ the function $V(x)C$ also solves (2.4). In fact all solutions can be written in this form. Let y be the solution to (2.4) such that $y(a) = \alpha$. Then $y(x)$ and $V(x)V^{-1}(a)\alpha$ both solve (2.4) and coincide at $x = a$. So they are in fact the same solution.

A similar procedure can be done in the general case (2.2). However the calculations are a bit more tricky. We will not go through the calculations here since it is not needed for our purposes. We will just skip to the final result. The general solution to (2.2) with initial condition $y(a) = \alpha$ becomes

$$y(x) = V(x)V^{-1}(a)\alpha + V(x) \int_a^x V^{-1}(t)f(t)dt. \tag{2.12}$$

For a proof and all the details, see for example [16].

2.1.1 Singular points in the coefficients

So far everything has been holomorphic in the region R. However one key characteristic of the hypergeometric differential equation is that is has three singularities, at 0, 1 and ∞. It is therefore of interest to us to know how to handle these singularities.

Definition 2.1.3. A matrix $A : \mathbf{C} \to \mathbf{C}^{n\times n}$ is said to be holomorphic at $x_0 \in \mathbf{C}$ if

$$\lim_{h\to 0} \frac{A(x_0 + h) - A(x_0)}{h} \tag{2.13}$$

exists. The matrix $A(x)$ is said to have a pole of order m at $x = x_0$ if $\tilde{A}(x) = (x-x_0)^m A(x)$ can be extended to a holomorphic function at x_0 such that $\tilde{A}(x_0) \neq \mathbf{0}$.

Definition 2.1.4. The differential equation

$$y' = A(x)y, \tag{2.14}$$

where $y \in \mathbf{C}^n$ and $A(x) \in \mathbf{C}^{n\times n}$ is said to have a singularity of order m at the point x_0 if $A(x)$ has a pole of order m at x_0. Singularities of order 1 are often called regular singular points.

Example 2.1.5. The differential equation

$$xy'' - y = 0 \tag{2.15}$$

has a regular singular point at $x = 0$. To see this we write (2.15) in matrix form as

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{1}{x} & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}. \tag{2.16}$$

We see that the matrix $A(x)$ has a pole of order 1 at $x = 0$.

Let us begin by looking at an example where the theory from the previous chapter fails. Consider the differential equation $y' = \frac{y}{x}$. This equations is solved by for example $y = x$, but also $y \equiv 0$. These 2 solutions coincide at $x = 0$, however they are not identical. This is not a contradiction to theorem 2.1.1 however, because the theorem requires $A(x)$ to be holomorphic. In this example $A(x) = \frac{1}{x}$, so $x = 0$ is a singularity and theorem 2.1.1 can not be applied there. The theorem still works perfectly fine in the region $\mathbf{C} \setminus \{0\}$. In that case the unique solution satisfying $y(a) = \alpha$ is $y = \frac{\alpha}{a}x$. Which is precisely on the form $V(x)V^{-1}(a)\alpha$, since the fundamental matrix in this case is $V(x) = x$. We can also see from this example that even if the coefficients have a singularity at some point, the solution itself might not, but in many cases it will. If we tweak this example a little bit and instead study the equation $y' = \frac{y}{2x}$, this equation is solved by for example $y = x^{\frac{1}{2}}$. This function is not single valued in the complex plane, so we also might get branch points around a singular point.

Let us assume that the matrix A has an isolated singularity at $x = 0$, which can be achieved by a simple translation. Look at some set of fundamental solutions $y(x) = V(x)$ at some point a near 0. If we analytically continue this solution along some curve that enclose 0, we might not get back to $V(a)$ after one lap. We then get a new solution $V_1(x)$. However since these are both solutions there must be a constant matrix C such that $V_1(x) = V(x)C$, the matrix C is called the circuit matrix. In the special case that C is diagonizable it is possible to do even more. Then there exist a diagonal matrix D such that all possible fundamental solutions take the form $Y(x) = S(x)x^D$, where S is single valued and D is diagonal. Each individual solution then has the form $y_j(x) = s_j(x)x^{\lambda_j}$ for some λ_j. And the possible multivaluedness comes from the possible branches of x^{λ_j} around zero. The λ_js are called circuit exponents. For derivations of these results see [16]. It is in general not easy to calculate $S(x)$ and D. However in the special case that the pole is of first order there exists methods to do so. In that case the differential equation can naturally be rewritten as

$$xY' = A(x)Y. \tag{2.17}$$

If we assume that $A(0)$ has no two eigenvalues that differ from each other by a positive integer, then the solution is going to take the form $Y = S(x)x^{A(0)}$, and were $S(x)$ can be calculated explicitly from $A(x)$. We wont go into the details here. Again for derivations and all the details see [16]. In the hypergeometric differential equation the poles are of first order. It is even a Fuchsian equation, which allows us to draw some extra conclusions. We will now go into what this means. We will also give some theorems and results about Fuchsian equations that will be needed later on. However we leave the proofs out. For proof see for example [11].

Definition 2.1.6. A differential equation is said to be Fuchsian if all singularities are of first order, and if the differential equation one obtains after the change of variables $t = \frac{1}{x}$ has a singularity of first order at $t = 0$.

Theorem 2.1.7. For every Fuchsian equation of degree r with a regular singularity at $x = a$, there exists $s_1, ..., s_r \in \mathbf{C}$ and $g_1(x), ..., g_r(x)$ holomorphic and non zero in a neighborhood of a such that $(x-a)^{s_1}g_1(x)$,...,$(x-a)^{s_r}g_r(x)$ spans the set of solutions to the Fuchsian equation in a neighborhood of $x = a$.

Definition 2.1.8. The numbers $s_1, ..., s_r$ from definition 2.1.7 are called the characteristic exponents at the singularity a.

For a giver Fuschain equation it is often useful to store all the singularities and the corresponding characteristic exponents in to one matrix. This matrix is called the Riemann scheme for the equation. That is, for a Fuch equation with singularities $a_1, ..., a_m = \infty$, and characteristic exponents corresponding to a_n being $\lambda_{1,n}, ..., \lambda_{r,n}$, then the Riemann scheme is the following matrix

$$\begin{pmatrix} a_1 & ... & a_m \\ \lambda_{1,1} & ... & \lambda_{m,1} \\ ... & ... & ... \\ \lambda_{1,r} & ... & \lambda_{m,r} \end{pmatrix}. \tag{2.18}$$

Theorem 2.1.1 (Fuch relation)**.** For any Fuchsian equation, the sum of all characteristic exponents satisfy

$$\sum_{i=1}^{m}\sum_{j=1}^{r}\lambda_{i,j} = \frac{(m-2)r(r-1)}{2}, \tag{2.19}$$

for the Riemann scheme defined in equation (2.18).

From now on let us focus our attention to second order Fuchsian equations with three singularities on the Riremannsphere. In that case $m = 3$ and $r = 2$ so that Fuch relation reads

$$\sum_{i=1}^{3}\sum_{j=1}^{2}\lambda_{i,j} = 1, \tag{2.20}$$

and the corresponding Riemannscheme is

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} \\ \lambda_{1,2} & \lambda_{2,2} & \lambda_{3,2} \end{pmatrix}. \tag{2.21}$$

It is trivial that every Fuchsian equation has precisely one Riemannscheme, however the reverse is not true in general.

Example 2.1.9. Let us consider first order differential equations ($r = 1$) with two singularities ($m = 2$). Fuch relation then tells us that the sum of all exponents is zero. Assume that the two singularities are 0 and ∞. Consider the Riemannscheme

$$\begin{pmatrix} 0 & \infty \\ -1 & 1 \end{pmatrix}. \tag{2.22}$$

Let us see if we can find Fuchsian equations with (2.22) as its corresponding Riemannscheme. Since the equation has to be of first order it has to take the form

$$a_1(x)f'(x) + a_2(x)f(x) = 0. \tag{2.23}$$

Since the singularities are 0 and ∞ we must have $a_1(0) = 0$ and $a_2(\infty) = \infty$. Set $a_1(x) = x$ and $a_2(x) = ax + b$, where $a \neq 0$. The differential equation

$$xf'(x) + (ax + b)f(x) = 0 \tag{2.24}$$

can be solved by integrating factor and has the solution

$$f(x) = cx^{-b}e^{-ax}. \tag{2.25}$$

We have already said that the characteristic exponent at 0 is -1 (meaning that y has to contain a factor x^{-1}) and that the exponent at ∞ is 1 (meaning that y has to contain a factor $\frac{1}{x}$). This forces $b = 1$. However a can still vary freely as long as $a \neq 0$. Hence we have found a full family of Fuchsian equations

$$xf'(x) + (1 + ax)f(x) = 0 \quad a \neq 0 \tag{2.26}$$

that all correspond to the same Riemannscheme (2.22). Hence we have shown that the correspondence between Riemannschemes and Fuchian equations is not bijective in general.

For general Fuchsian equations with an arbitrary number of singularities the equations is not fully determined by its Riemannscheme, however there is one exception. For second order Fuchsian equations with three singularities there is precisely a one to one correspondence between Fuchsian equations and corresponding Riemannscheme.

Theorem 2.1.10. For every 3×3 matrix on the form (2.21) which satisfy Fuch relation, there is precisely one Fuchsian equation that has (2.21) as its corresponding Riemannscheme.

Proof. We only do the sketch of the proof here, the full proof can be found in [11], page 28.

In the first step of the proof we prove that the differential equation

$$\left[\frac{d^r}{dx^r} + a_1(x)\frac{d^{r-1}}{dx^{r-1}} + ... + a_r(x)\right] f(x) = 0 \tag{2.27}$$

is Fuchsian with regular singular points at $x_1, ..., x_m, x_{m+1} = \infty$ if and only if

$$a_i(x) = \frac{p_i(x)}{\prod_{k=1}^{m}(x - x_k)^i} \quad i = 1, ..., r \tag{2.28}$$

where $p_i(x)$ is a polynomial of degree at most $i(m-1)$. We can then write p_i in the form

$$p_i(x) = A_{0,i} + A_{1,i}x + ... + A_{i,i}x^i, \quad i = 1, ..., r. \tag{2.29}$$

If we write down a relationship between the characteristic exponents $s_{k,i}$, $k = 1, ..., m$, $i = 1, ..., r$, the singularities x_k, $k = 1, ..., m$ and the coefficients $A_{j,i}$, $j = 1, ..., i$, $i = 1, ..., r$ we are going to see that this relationship is invertible precisely if $k = r = 2$, which proves the theorem. □

Remark 2.1.11. Theorem 2.1.10 states that there is a unique Fuchsian equation corresponding to the Riemannscheme 2.21. This means that we should be able to write down that Fuchsian equation explicitly. This equation is called the Papperitz Riemann equation and it takes the form [11]

$$f'' + \left(\frac{1-\lambda_{1,1}-\lambda_{1,2}}{x-a_1} + \frac{1-\lambda_{2,1}-\lambda_{2,2}}{x-a_2} + \frac{1-\lambda_{3,1}-\lambda_{3,2}}{x-a_3}\right) f'$$

$$+ \frac{\left(\frac{\lambda_{1,1}\lambda_{1,2}(a_1-a_2)(a_1-a_3)}{x-a_1} + \frac{\lambda_{2,1}\lambda_{2,2}(a_2-a_1)(a_2-a_3)}{x-a_2} + \frac{\lambda_{3,1}\lambda_{3,2}(a_3-a_1)(a_3-a_2)}{x-a_3}\right) f}{(x-a_1)(x-a_2)(x-a_3)} = 0.$$

Theorem 2.1.12. Let $E(f(x), x) = 0$ be a Fuchsian equation with a regular singular point at $x = a \in \mathbf{C}$ and characteristic exponents at a being $s_1, ..., s_r$. Also let $b \in \mathbf{C}$ and T be a conformal map from a neighborhood of a to a neighborhood of b such that $T(a) = b$. Then $E(f(T(x)), T(x)) = 0$ is a Fuchsian equation with a singularity at $x = b$ and the corresponding characteristic exponents at b are still $s_1, ..., s_r$.

Proof. For a proof see [11]. □

2.2 Symmetric polynomials

Later on we are going to look at a certain set of operators called Dunkl operators. Or more precisely symmetric polynomials in Dunkl operators. First we therefore need to clarify some notation that will be used.

Definition 2.2.1. Let $\mathbf{T}$ be a commutative ring and let $T = (T_1, ..., T_n) \in \mathbf{T}^n$. The set of polynomials in T with coefficients in the field F is denoted by $F[T]$. That is, elements $f \in F[T]$ will have the form

$$\sum c_\alpha T_1^{\alpha_1} ... T_n^{\alpha_n} \quad \text{where } c_\alpha \in F. \tag{2.30}$$

Often when we talk about a specific polynomial p we write $p(T)$.

Example 2.2.2. The set $\mathbf{C}[x]$ is the set of all polynomials with coefficients in $\mathbf{C}$. The set $\mathbf{C}[x][\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}]$ is the set of polynomial differential operators with polynomial coefficients, that is the elements of $\mathbf{C}[x][\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}]$ will be on the form

$$\sum c_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \quad \text{where } c_\alpha \in \mathbf{C}[x_1, ..., x_n] \tag{2.31}$$

Later on in this report we will need to use so called symmetric polynomials. What is meant by symmetric is that the polynomial is S_n invariant, that is, the polynomial does not change if we permute the variables. Let us formalize this idea

Definition 2.2.3. Let G be a group that acts on $\mathbf{T}^n$. The set $F[T]^G$ is the set of G invariant polynomials in T with coefficients in F. That is, it is the largest subset of $F[T]$ with the property that

$$f \in F[T]^G \implies f(T) = f(g(T)) \quad \forall g \in G. \tag{2.32}$$

Example 2.2.4. If we let $G = S_n$ we get an algebra of polynomials $F[T]^{S_n}$ called *symmetric polynomials*. These are polynomials that remain the same even if we permute the variables, or formally $p \in F[T]^{S_n} \iff p(T) = p(s(T))$ for all $s \in S_n$.

Example 2.2.5. Examples of polynomials in $\mathbf{R}[x_1, x_2]^{S_2}$ are $x_1 + x_2$, $x_1^2 x_2 + x_2^2 x_1$ and $x_1 x_2$. Examples of polynomials that are in $\mathbf{R}[x_1, x_2]$ but not in $\mathbf{R}[x_1, x_2]^{S_2}$ are x_1, $x_1^2 + x_2$, $2x_1 + x_2$ and so on. These polynomials change if we swap x_1 and x_2.

Later on we are going to need at set of generators for the set of symmetric polynomials $\mathbf{C}[x_1, ..., x_n]^{S_n}$. What this means is that we want a finite set $H \subset \mathbf{C}[x_1, ..., x_n]^{S_n}$ such that every polynomial in $\mathbf{C}[x_1, ..., x_n]^{S_n}$ can be written as a finite combination of sums and products of elements in H. That is taken care of by the following theorem.

Theorem 2.2.6. If T_i and T_j commute for all i and j then the set $F[T_1, ..., T_n]^{S_n}$ is generated by $\{p_i\}_{i=1}^n$ where $p_i(\mathbf{T}) = \sum_{j=1}^n T_j^i$, $i = 0, ..., n$.

Proof. See [14] □

Remark 2.2.7. More precisely what theorem 2.2.6 is saying is that every polynomial $p \in F[T_1, ..., T_n]^{S_n}$ can be written as a finite combination of sums and products of p_is and elements in F. Or more compactly

$$p \in F[T_1, ..., T_n]^{S_n} \implies p \in F[p_1, ..., p_n] \quad (2.33)$$

This means that there is a one to one correspondence between symmetric polynomials in n variables, and all polynomials in n variables.

Corollary 2.2.8. Let $\mathbf{x} = [x_1, ..., x_n]$ and $\mathbf{p} = [p_1, ..., p_n]$ where $p_i(\mathbf{x}) = \sum_{j=1}^n x_j^i$, $i = 0, ..., n$. For each $p \in \mathbf{C}[\mathbf{x}]^{S_n}$ there is precisely one $\tilde{p} \in \mathbf{C}[x]$ such that $\tilde{p}(\mathbf{p}) = p(\mathbf{x})$.

Definition 2.2.9. For each $p \in \mathbf{C}[\mathbf{x}]^{S_n}$ we call the polynomial $\tilde{p}$ from corollary 2.2.8 the generating polynomial associated with p

Example 2.2.10. Let $p(x) = x_1^2 x_2 + x_2^2 x_1 \in \mathbf{C}[x]^{S_2}$, then the generating polynomial associated with $p(x)$ is $\tilde{p}(\mathbf{p}) = \frac{1}{2}\left(p_1 p_2 - p_1^3\right)$. This is because

$$x_1^2 x_2 + x_2^2 x_1 = \frac{1}{2}\left((x_1 + x_2)(x_1^2 + x_2^2) - (x_1 + x_2)^3\right). \tag{2.34}$$

If we instead consider $q(x) = x_1 x_2 \in \mathbf{C}[x]^{S_2}$ then the corresponding generating polynomial is $\tilde{q}(\mathbf{p}) = \frac{1}{2}\left(p_1^2 - p_2\right)$ since

$$x_1 x_2 = \frac{1}{2}\left((x_1 + x_2)^2 - (x_1^2 + x_2^2)\right). \tag{2.35}$$

3

The hypergeometric function and differential equation in 1 dimension

This chapter generally follows [11]. We have already seen the hypergeometric differential equation

$$x(1-x)\frac{d^2 f}{dx^2} + (c - (a+b+1)x)\frac{df}{dx} - abf = 0. \tag{3.1}$$

Let us study the singularities in the coefficients. Following the procedure in the previous chapter we can introduce a new variable $f_2 = \frac{df}{dx}$. Then if we let $\mathbf{f} = (f\ f_2)^T$ the equation can be rewritten as

$$\mathbf{f}'(x) = A(x)\mathbf{f} \tag{3.2}$$

where

$$A(x) = \frac{1}{x(1-x)} \begin{pmatrix} 0 & x(1-x) \\ ab & -(c-(a+b+1)x)) \end{pmatrix}. \tag{3.3}$$

We see that this matrix has three regular singular points, at 0, 1 and ∞. It is hence a Fuchsian equation. The hypergeometric differential equation has the associated Rieman scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{pmatrix}. \tag{3.4}$$

We will now look at this Riemann scheme and see why this example in some sense is the leading example for all Fuchsian equations of second order with three singularities.

3.1 Motivating the hypergeometric differential equation

We have already seen in theorem 2.1.10 that second order Fuchsian equations with three singularities on the Riemannsphere are special since there is a one to one correspondence between Fuchsian equations and Riemannschemes in that case. Let

$$E(f(x), x) = 0 \tag{3.5}$$

be a Fuchsian equation with singularities a_1, a_2 and ∞ and characteristic exponents $\sigma_{a_1,1}$, $\sigma_{a_1,2}$, $\sigma_{a_2,1}$, $\sigma_{a_2,2}$, $\sigma_{\infty,1}$ and $\sigma_{\infty,2}$. Then let T be a conformal map on the Riemann sphere such that $T(a_1) = 0$, $T(a_2) = 1$ and $T(\infty) = \infty$. By theorem 2.1.12 the Fuchsian equation

$$E(f(T(x)), T(x)) = 0 \tag{3.6}$$

has singularities at 0, 1 and ∞. However the characteristic exponents will be the same as in equation (3.5). To study all Fuch equations of degree 2 with 3 singular points it is therefore sufficient to look at equation with a Riemann scheme on the form

$$\begin{pmatrix} 0 & 1 & \infty \\ \sigma_{0,1} & \sigma_{1,1} & \sigma_{\infty,1} \\ \sigma_{0,2} & \sigma_{1,2} & \sigma_{\infty,2} \end{pmatrix}. \tag{3.7}$$

Now let

$$P\left(\begin{pmatrix} a_1 & a_2 & a_3 \\ \sigma_{a_1,1} & \sigma_{a_2,1} & \sigma_{a_3,1} \\ \sigma_{a_1,2} & \sigma_{a_2,2} & \sigma_{a_3,2} \end{pmatrix}, x\right) \tag{3.8}$$

denote the set of solutions to the Fuchsian equation with Riemann scheme

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \sigma_{a_1,1} & \sigma_{a_2,1} & \sigma_{a_3,1} \\ \sigma_{a_1,2} & \sigma_{a_2,2} & \sigma_{a_3,2} \end{pmatrix}. \tag{3.9}$$

From theorem 2.1.7 we know that locally around 0 the set of solutions is of the form

$$\{x^{\sigma_{0,1}} h_{0,1}(x), x^{\sigma_{0,2}} h_{0,2}(x)\}, \tag{3.10}$$

where $h_{0,1}$ and $h_{0,2}$ are holomorphic and non zero in a neighborhood of 0. Similarly around 1 the set of solutions look like

$$\{(x-1)^{\sigma_{1,1}} h_{1,1}(x), (x-1)^{\sigma_{1,2}} h_{1,2}(x)\}, \tag{3.11}$$

and around ∞ they look like

$$\left\{\left(\frac{1}{x}\right)^{\sigma_{\infty,1}} h_{\infty,1}(x), \left(\frac{1}{x}\right)^{\sigma_{\infty,2}} h_{\infty,2}(x)\right\}. \tag{3.12}$$

Here $h_{1,1}$ and $h_{1,2}$ are holomorphic and non zero near 1, and $h_{\infty,1}$ and $h_{\infty,2}$ are holomorphic and non zero near ∞. We can take out a factor of $x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}}$ and get

$$P\left(\begin{pmatrix} 0 & 1 & \infty \\ \sigma_{0,1} & \sigma_{1,1} & \sigma_{\infty,1} \\ \sigma_{0,2} & \sigma_{1,2} & \sigma_{\infty,2} \end{pmatrix}, x\right) = x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}} G(E, x),$$

where $G(E, x)$ is a set of functions on the form

$$G(E,x) = \begin{cases} \left\{\frac{h_{0,1}(x)}{(x-1)^{\sigma_{1,1}}}, x^{\sigma_{0,2}-\sigma_{0,1}} \frac{h_{0,2}(x)}{(x-1)^{\sigma_{1,1}}}\right\} & \text{near } x = 0 \\ \left\{\frac{h_{1,1}(x)}{x^{\sigma_{0,1}}}, (x-1)^{\sigma_{1,2}-\sigma_{1,1}} \frac{h_{1,2}(x)}{x^{\sigma_{0,1}}}\right\} & \text{near } x = 1 \\ \left\{\frac{\left(\frac{1}{x}\right)^{\sigma_{\infty,1}} h_{\infty,1}(x)}{x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}}}, \frac{\left(\frac{1}{x}\right)^{\sigma_{\infty,2}} h_{\infty,2}(x)}{x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}}}\right\} & \text{near } x = \infty \end{cases}. \tag{3.13}$$

Notice that $G(E,x)$ correspond to a set of functions associated with the Riemann scheme

$$G = \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \sigma_{\infty,1} + \sigma_{0,1} + \sigma_{1,1} \\ \sigma_{0,2} - \sigma_{0,1} & \sigma_{1,2} - \sigma_{1,1} & \sigma_{\infty,2} + \sigma_{0,1} + \sigma_{1,1} \end{pmatrix}. \tag{3.14}$$

From theorem 2.1.10 we know that there is precisely one Fuchsian equation with Riemann scheme G. Hence we get the relation

$$P\left(\begin{pmatrix} 0 & 1 & \infty \\ \sigma_{0,1} & \sigma_{1,1} & \sigma_{\infty,1} \\ \sigma_{0,2} & \sigma_{1,2} & \sigma_{\infty,2} \end{pmatrix}, x\right) = x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}} P\left(G, x\right). \tag{3.15}$$

The exact expression for G is not so important. The main result is that we were able to reduce the system to a Riemann scheme of the form

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \tag{3.16}$$

By conversion one calls the two exponents at ∞ for a and b and the second exponent at 0 for $1-c$. Then Fuch relation forces the last exponent at 1 to be $c-a-b$. Hence we get the Riemann scheme (3.4).

Remark 3.1.1. In the step where we factored out $x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}}$ we forced two exponents to be zero. It is important to think through why we precisely can force two exponents to be zero, and no more in general. What it means for the system to have an exponent s at the singularity a is that the solution contains a factor $(x-a)^s$. It might hence be tempting to say that we can factor out $(x-a_i)^{s_{j,i}}$ for every single pair of singularity and characteristic exponent. Hence forcing all remaining exponents to be zero.

This is not the case. It is easy to see why we cannot factor out any exponents at ∞. The factor $(x-\infty)^s$ does not make any sense. The meaning of a singularity at infinity is that the equation we get after the change of variable $t = \frac{1}{x}$ has a singularity at $t = 0$. So we need to do a change of variable before factorizing. Which might affect the other singularities.

And to see why we cannot factor out two exponents for the same singularity. Note that P stands for a full set of functions. Let us look at a simple example. Say that P is on the form $\{x^2, x^5\}$. Then we can factor out x^2 and get $x^2\{1, x^3\}$. But no matter how we factor we can never force both exponents to be 0 at the same time. By this logic, two is precisely the number of exponents that we can force to be zero by this kind of factorization.

We have now shown that every second order Fuchsian equation with three regular singularities on the Riremmsphere can be studied by studying Riemannschemes on the form

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{pmatrix}. \tag{3.17}$$

However by remark 2.1.11 we know what the corresponding Papperitz Riemann equation is. Plugging all values in we get

$$f'' + \left(\frac{1-(1-c)}{x} + \frac{1-(c-a-b)}{x-1} + \frac{1-a-b}{x-a_3} \right) f'$$

$$+ \frac{\left(\frac{ab(a_3)(a_3-1)}{x-a_3}\right) f}{x(x-1)(x-a_3)} = 0.$$

If we multiply by $x(1-x)$ and let $a_3 \to \infty$ we get the equation

$$x(1-x)f''(x) + (c-(a+b+1)x))f'(x) - abf(x) = 0, \tag{3.18}$$

which we recognize as the hypergeometric differential equation. These calculations shows why the hypergeometric differential equation is such a leading example. Every second order Fuchsian equation with three regular singularities can be transformed into the hypergeometric differential equation under the right change of variable and renaming of parameters. The same is not true for any other type of Fuchsian equation.

3.2 Frobenius method

Now that we have seen why the hypergeometric equation is of interest, let us try and actually solving it. We know that one of the singularities is $x = 0$ and that one of the corresponding characteristic exponents is zero. Hence there must be a solution holomorphic around zero. Or in other words, there exists a solution with a Maclaurin expansion. Assume that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{3.19}$$

solves (1.1). We are going to ignore questions regarding convergence so far and just assume that we can swap differentiation and summation. We will come back to convergence later on and see that the solution converges absolutely in a neighborhood of zero, so that all steps are justified. We see that

$$f'(x) = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n \tag{3.20}$$

and

$$f''(x) = \sum_{n=0}^{\infty} c_{n+2}(n+1)(n+2)x^n. \tag{3.21}$$

Plugging this into equation (1.1) we get

$$x(1-x)\sum_{n=0}^{\infty} c_{n+2}(n+1)(n+2)x^n + (c-(a+b+1)x)\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n - ab\sum_{n=0}^{\infty} c_n x^n = 0. \tag{3.22}$$

Multiplying all brackets together and combining like powers together we eventually see that this can be simplified to

$$\sum_{n=0}^{\infty} \left(c_{n+1}(n+1)(n+c) - c_n(n+a)(n+b)\right) x^n = 0. \tag{3.23}$$

This now leads to the system of equations

$$c_{n+1}(n+1)(n+c) - c_n(n+a)(n+b) = 0 \quad \forall n \geq 0. \tag{3.24}$$

Rearranging we get

$$\frac{c_{n+1}}{c_n} = \frac{(n+a)(n+b)}{(n+c)(n+1)} \implies c_n = \frac{(a)_n(b)_n}{(c)_n n!}. \tag{3.25}$$

Hence the solution is

$$f(x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \tag{3.26}$$

which we recognise as ${}_2F_1(a, b, c; x)$.

Remark 3.2.1. Equation (3.24) gives the right solution for every triplet for parameters (a, b, c), however in the simplification (3.25) we divided by c_n, $n+c$ and $n+1$. This assumes that none of these factors are zero. Since n runs over the non negative integers $n+1$ can never be 0. Problem occurs however if c is a non positive integer. Equation (3.24) still works for this case. What will happen however is that at some point there will be some $c_n = 0$ followed by $c_{n+1} \neq 0$. Hence the quotient $\frac{c_{n+1}}{c_n}$ cannot be formed. This case can still be solved, but we have to treat it a bit different. For our purposes this case is not so relevant, so we will not look further into it. We just mention that in order to cover the full picture one would have to treat this case separately.

We now need to justify that the series (3.26) converges in a neighborhood of zero. This can easily be verified by the quotient test. We see that

$$\frac{c_{n+1}x^{n+1}}{c_n x^n} = \frac{(n+a)(n+b)}{(n+c)(n+1)} x \to x, n \to \infty \tag{3.27}$$

Hence the series is absolutely convergent in the open disk $\{x \in \mathbf{C} : |x| < 1\}$. It should not come as a surprise that the radius of convergence is one. Remember that the three singularities are 0, 1 and ∞. Hence 1 is simply the distance from 0 to the nearest singularity. It should also be noted that the series does not always diverge outside this open disk. If $a = -1$ for example ${}_2F_1(-1, b, c; x)$ reduces to $1 + \frac{b}{c}x$, which is holomorphic in all of $\mathbf{C}$. In general however a radius of convergence of one is the most we can hope for. The only exceptions will in fact be if either a or b is a non positive integer [11].

3.3 Kummers 24 solutions

The hypergeometric differential equation is of order 2 and therefore has 2 linearly independent solutions everywhere. We know that ${}_2F_1$ is one of these solutions locally around zero, and we also know what the characteristic exponents are near each singularity. Let us try and explore what the other solution looks like. We begin by looking at a particular example of fixed parameter values a, b and c

Example 3.3.1. Let $c = a = 2$ and $b = 1$. The hypergeometric differential equation then reads

$$x(1-x)f'' + (2-4x)f' - 2f = 0. \tag{3.28}$$

The Gauss hypergeometric function reduces to a geometric series ${}_2F_1(2,1,2;x) = \sum_{i=0}^{\infty} x^n = \frac{1}{1-x}$. Note that ${}_2F_1$ has an exponent of 0 at $x = 0$, an exponent of $-1(= c-a-b)$ at $x = 1$, and an exponent of $1(= b)$ at $x = \infty$. The other solution should therefore have an exponent of $-1(= 1-c)$ at $x = 0$, an exponent of 0 at $x = 1$, and an exponent of $2(= a)$ at $x = \infty$. Let us first try to find both solutions locally around $x = 0$. In that case the other solution should be on the form

$$g(x) = x^{-1}\sum_{n=0}^{\infty} c_n x^n. \tag{3.29}$$

Plugging $g(x)$ into equation (3.28) and simplifying we get

$$x^{-1}\sum_{n=0}^{\infty}(n^2+3n+2)(c_{n+1}-c_n)x^n = 0, \tag{3.30}$$

which is solved by $c_n = c_0$ for all n. Therefore $g(x)$ takes the form

$$g(x) = x^{-1}\sum_{n=0}^{\infty} x^n = x^{-1}\frac{1}{1-x}. \tag{3.31}$$

Note that $g(x)$ in this case has an exponent of -1 at $x = 0$, an exponent of -1 at $x = 1$, and an exponent of 2 at $x = \infty$, which is almost what we are looking for. All that remains to find is a solution which is analytic at $x = 1$. However we notice that

$$x^{-1}\frac{1}{1-x} - \frac{1}{1-x} = \frac{1}{x}, \tag{3.32}$$

which is analytic at $x = 1$. Hence the full set of solutions to (3.28) is

$$f(x) = \frac{C_1}{x-x^2} + \frac{C_2}{1-x} \tag{3.33}$$

Example 3.3.1 was neat because it allowed us to find all solutions by just solving the problem locally around zero. And then it just so happened that the solutions that we found also had the right exponents at the other singularities. However we still do not have a general method for finding all solutions near all singularities.

According to theorem 2.1.12 the characteristic exponents do not change if a conformal map is applied to the dependent variable x. The hypergeometric differential equation is defined in all of $\mathbf{C} \setminus \{0,1\}$. The idea of Kummers 24 solutions is to consider conformal mappings from $\mathbf{C} \setminus \{0,1\}$ to itself, that permute the singularities. In that way we can use information about the solution around zero to gain information about solutions around other singularities. Note that the six transformations

$$\{T_n\}_{n=1}^{6} := x \mapsto \{x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x}{x-1}, \frac{x-1}{x}\} \tag{3.34}$$

do all belong to $\mathrm{Aut}(\mathbf{C} \setminus \{0,1\})$. Furthermore when $\{T_n\}_{n=1}^{6}$ acts on $\{0,1,\infty\}$ their action is isomorphic to S_3 (meaning that they permute the three elements in all six ways possible). Hence if we know what the set of solutions look like near one singularity, we can preform one of these automprohism to then get information about the solutions around a different singularity. However, we still need to be able to solve the resulting Fuchsian equation after the automorphism is applied, otherwise the trick is of no use. If we begin with a hypergeometric differential equation

$$E(a,b,c,f(x),x) = 0, \tag{3.35}$$

and then preform an automorphism T to get the new equation

$$E(a,b,c,f(T(x)),T(x)) = 0, \tag{3.36}$$

theorem 2.1.12 does not guarantee that equation (3.36) is a hypergeometric differential equation, only that it is Fuchsian with 3 regular singular points. However in section 3.1 we saw that it is possible to transform any Fuchsian equation with 3 regular singular points to a hypergeometric one. The trick was to take out a factor of $x^{\sigma_{0,1}}(x-1)^{\sigma_{1,1}}$ where $\sigma_{0,1}$ and $\sigma_{1,1}$ are exponents at 0 and 1 respectively. It is important to note here that we are factoring out exponents in the equation after the automorphism T is applied. Therefore in terms of the original equation the exponents are not necessarily at 0 and 1, but rather at $T^{-1}(0)$ and $T^{-1}(1)$.

Example 3.3.1. Say we are interested in solving the hypergeometric differential equation

$$x(1-x)f''(x) + (c-(1+a+b)x)f'(x) - abf(x) = 0 \tag{3.37}$$

locally around $x = 1$. We then need a conformal map T such that $T(1) = 0$, we can for example choose $T = 1-x$. We then do the change of variable $t = 1-x$ to the hypergeometric differential equation. The new equation reads

$$t(1-t)f''(t) + (a+b-c-(a+b+1)x)f'(t) - abf(t) = 0. \tag{3.38}$$

We recognise this as a hypergeometric differential equation with parameter values $(a, b, a+b-c)$, hence a solution to the hypergeometric equation locally around $x = 1$ is

$$f(x) =_2 F_1(a,b,a+b-c;1-x). \tag{3.39}$$

Example 3.3.2. Assume now that we want to find a solution to the hypergeometric differential equation locally around $x = \infty$. We then need a conformal map T such that $T(\infty) = 0$. We can pick $T(x) = \frac{1}{x}$. We also need to pick an exponent at $T^{-1}(0) = \infty$ to factor out. We can choose a for example. Now do the change of variable $t = \frac{1}{x}$ and make the anzts $f(t) = t^a g(t)$, if we plug this into the hypergeomtric differential equation we get the equation

$$t(1-t)g''(t) + (a+1-b-(2a-c+2)x)g'(t) - (a^2 - ac + a)g(t) = 0, \qquad (3.40)$$

which we recognise as a hypergeomteric differential equation with parameter values $(a, a-c+1, a+1-b)$. Hence the solution is

$$g(t) =_2 F_1(a, a-c+1, a+1-b; t). \qquad (3.41)$$

Since $t = \frac{1}{x}$ and $f(t) = t^a g(t)$ we get that a solution the hypergeometric equation locally around ∞ is

$$f(x) = x^{-a} F(a, a-c+1, a+1-b; \frac{1}{x}). \qquad (3.42)$$

We could keep in going finding more solutions. Every choice of T and which exponents to factor out is going to give a unique expression. This will in fact cover the full set of solutions by the following theorem.

Theorem 3.3.3 (Kummes 24 solution)**.** Let

$$E(a, b, c, f(x), x) = 0 \qquad (3.43)$$

be the hypergeometric differential equation with parameter values a, b, c. For every triplet $(T, \sigma_{T^{-1}(0)}, \sigma_{T^{-1}(1)})$ where $T \in \mathrm{Aut}(\mathbf{C} \setminus \{0, 1\})$ there exists $\tilde{a}, \tilde{b}, \tilde{c} \in \mathbf{C}$ such that

$$f(x) = T(x)^{\sigma_{T^{-1}(0)}} (T(x) - 1)^{\sigma_{T^{-1}(1)}} F(\tilde{a}, \tilde{b}, \tilde{c}; T(x)) \qquad (3.44)$$

is a solution to (3.43) locally around $T^{-1}(0)$. The parameter values $\tilde{a}, \tilde{b}, \tilde{c}$ can be found by doing the change of variable $t = T(x)$ and $f(t) = t^{\sigma_{T^{-1}(0)}} (t-1)^{\sigma_{T^{-1}(1)}} g(t)$ and rewriting equation (3.43) in the form of a hypergeometric equation

$$E(\tilde{a}, \tilde{b}, \tilde{c}, g(t), t) = 0. \qquad (3.45)$$

Furthermore the set of solutions that can be reached by this manner span the full set of solutions to (3.43).

Proof. Consider a particular triplet $(T, \sigma_{T^{-1}(0)}, \sigma_{T^{-1}(1)})$. Do the change of variable $t = T(x)$ and $f(t) = t^{\sigma_{T^{-1}(0)}} (t-1)^{\sigma_{T^{-1}(1)}} g(t)$ is equation (3.43) and consider the problem

$$E(T(x)^{\sigma_{T^{-1}(0)}} (T(x) - 1)^{\sigma_{T^{-1}(1)}} g(T(x)), T(x)) = 0. \qquad (3.46)$$

Equation (3.46) is a Fuchsian equation with 3 regular singular points at $0, 1, \infty$ and it has one characteristic exponent both at 0 and 1 necessarily being equal to 0. Hence equation (3.46) is a hypergeometric equation for $g(t)$, therefor there exists $\tilde{a}, \tilde{b}, \tilde{c}$ such that (3.46) is on the form

$$E(\tilde{a}, \tilde{b}, \tilde{c}, g(t), t) = 0. \tag{3.47}$$

Equation (3.47) is solved by

$$g(t) = F(\tilde{a}, \tilde{b}, \tilde{c}; t). \tag{3.48}$$

If we undo the change of variable $t = T(x)$ and $f(t) = t^{\sigma_{T^{-1}(0)}}(t-1)^{\sigma_{T^{-1}(1)}} g(t)$ we find that

$$f(x) = T(x)^{\sigma_{T^{-1}(0)}}(T(x)-1)^{\sigma_{T^{-1}(1)}} F(\tilde{a}, \tilde{b}, \tilde{c}; T(x)) \tag{3.49}$$

is a solution to (3.43). Also note that the two different choices for $\sigma_{T^{-1}(0)}$ will give 2 linearly independent functions locally around $x = T^{-1}(0)$. Therefore the span of all functions that can be reached by this manner is at least two. However we already know that the solution space to the hypergeometric differential equation is 2 dimensional. Hence these functions span the full set of solutions. □

Remark 3.3.4. Note that there are 24 possible triplets $(T, \sigma_{T^{-1}(0)}, \sigma_{T^{-1}(1)})$, hence giving rise to 24 different expressions. This can be seen since there are 6 automorphisms, times 2 choices for which exponent at zero to factor out, times 2 choices for which exponent at 1 to factor out. Hence there will always be 24 different expressions one can get in this way. Note that this does not mean that the hypergeometric differential equation has 24 linearly independent solutions. It always has two. Hence these 24 functions always has a span of dimension 2.

Remark 3.3.5. It is worth noting that problem can still arise for certain parameter values. Remember from earlier that ${}_2F_1(a, b, c; x)$ is not defined if c is a non positive integer. We can have similar thing happen here, although this time it is a bit more subtle because we do not know beforehand what the parameters of equation (3.46) will be. In example 3.3.1 for example we arrived at a hypergeometric function with parameter values $(a, b, a+b-c)$. This is not defined if $a+b-c$ is a non positive integer. However these special cases will only be countably many and for our purposes they will not play an important role. We just mention here that problem can arise for certain parameter values. For more details see [11].

Example 3.3.6. Since there are only a finite number of expressions we can get from Kummers 24 solutions (24 of them to be exact), it is not so hard to write out a list of what those 24 functions are. We will not go through the calculations, for more details see [11]. Locally around 0 the 8 solutions are

$$f(x) = \begin{cases} F(a,b,c;x) \\ (1-x)^{c-a-b}F(c-a,c-b,c;x) \\ x^{1-c}F(a-c+1,b-c+1,2-c;x) \\ x^{1-c}(1-x)^{c-a-b}F(1-a,1-b,2-c;x) \\ (1-x)^{-a}F(c-b,a,c;\frac{x}{x-1}) \\ (1-x)^{-b}F(c-a,b,c;\frac{x}{x-1}) \\ x^{1-c}(1-x)^{c-a-1}F(1-b,a+1-c,2-c;\frac{x}{x-1}) \\ x^{1-c}(1-x)^{c-b-1}F(1-a,b+1-c,2-c;\frac{x}{x-1}) \end{cases}, \tag{3.50}$$

locally around 1 the 8 solutions are

$$f(x) = \begin{cases} F(a,b,a+b-c+1;1-x) \\ x^{1-c}F(b+1-c,a+1-c,a+b+1;1-x) \\ x^{-a}F(a+1-c,a,a+b+1-c;\frac{x-1}{x}) \\ x^{-b}F(b+1-c,b,a+b+1-c;\frac{x-1}{x}) \\ (1-x)^{c-a-b}F(c-a,c-b,c+1-a-b;1-x) \\ x^{1-x}(1-x)^{c-a-b}F(1-a,1-b,c+1-a-b;1-x) \\ x^{b-c}(1-x)^{c-a-b}F(1-b,c-b,c+1-a-b;\frac{x-1}{x}) \\ x^{a-c}(1-x)^{c-a-b}F(1-a,c-a,c+1-a-b;\frac{x-1}{x}), \end{cases}, \tag{3.51}$$

and locally around ∞ they are

$$f(x) = \begin{cases} x^{-a}F(a,a-c+1,a+1-b;\frac{1}{x}) \\ (-x)^{b-c}(1-x)^{c-a-b}F(1-b,c-b,a+1-b;\frac{1}{x}) \\ (1-x)^{-a}F(a,c-b,a+1-b;\frac{1}{1-x}) \\ (-x)^{1-c}(1-x)^{c-1-a}F(1+a-c,1-b,a+1-b;\frac{1}{1-x}) \\ x^{-b}F(b,b-c+1,b+1-a;\frac{1}{x}) \\ (-x)^{a-c}(1-x)^{c-a-b}F(1-a,c-a,b+1-a;\frac{1}{x}) \\ (1-x)^{-b}F(b,c-a,b+1-a;\frac{1}{1-x}) \\ (-x)^{a-c}(1-x)^{c-b-1}F(b+1-c,1-a,b+1-a;\frac{1}{1-x}) \end{cases}. \tag{3.52}$$

An alternative formulation of theorem 3.3.3 wold be to say that these functions span the set of solutions. However that is not as practical of a result, because in practice when one is interested in finding other solutions one does so by doing the change of variable from theorem 3.3.3.

Let us revisit example 3.3.1 again, however this time we will try and find the other solution from Kummers 24 solutions.

Example 3.3.7. We again consider equation (3.28). We know that one solution is

$$ {}_2F_1(2,1,2;x) = \frac{1}{1-x}. \tag{3.53} $$

Similar to before, we want to find the other solution near $x = 0$. Only this time we are going to do so using theorem 3.3.3. The first thing we need to do is to pick $T \in \mathrm{Aut}(\mathbf{C} \setminus \{0,1\})$ such that $T(0) = 0$. We can for example pick $T(x) = x$. Then we need to choose an exponent at $T^{-1}(1) = 1$. The two exponents are 0 and $c-a-b = -1$, we can for simplicity choose 0. We also need to choose an exponent at $T^{-1}(0) = 0$. The two exponents at 0 are 0 and $1-c(=-1)$. We know that ${}_2F_1$ corresponds to an exponent of 0, therefore in order to find the other solution we need to pick the exponent 1.

Lemma 3.3.3 then tells us to consider the equation

$$ E\left(x^{-1}g(x), x\right) = 0, \tag{3.54} $$

where $E(f(x), x) = 0$ is an abbreviation for equation (3.28). If we let $f(x) = x^{-1}g(x)$ equation (3.54) reads

$$ x(1-x)g'' + (2-4x)g' - 2g = 0, \tag{3.55} $$

which we recognize is solved by

$$ g(x) ={}_2 F_1(2,1,2;x) = \frac{1}{1-x}. \tag{3.56} $$

Since $f(x) = x^{-1}g(x)$ we know that the other solution is

$$ f(x) = x^{-1}\frac{1}{1-x}. \tag{3.57} $$

Note that these calculations are very similar to the calculations in example 3.3.1, only this time we used a more systematic approach. These calculations also tell us where the factor n^2+3n+2 in equation (3.30) comes from. We note that $n^2+3n+2 = (n+3)(n+1)$, which means $(n^2+3n+2)(c_{n+1}-c_n) = (n+c)(n+1)c_{n+1}-(n+a)(n+b)c_n$ for $(a,b,c) = (2,1,2)$. Now we know that g solves a hypergeometric differential equation with precisely these parameter values, which explains why this polynomial shows up in the series (3.30).

4

Dunkl operators and root systems

One area where the hypergeometric function turns up naturally is in a certain eigenvalue problem associated with so called Dunkl operators. These Dunkl operators are defined from a root system. We will look at the connections to ${}_2F_1$ in coming chapters. In this chapter we will define the objects involved and lay down the technical tools needed to proceed.

4.1 Root systems

Root systems are a general construct that are the foundations of the generalizations of ${}_2F_1$. In this section we go through the basics of roots systems and reflection groups. Most of this section follows chapter 1 of [10] and chapter 3 of [9]. We begin by defining what a root system is.

Definition 4.1.1 (Root systems)**.** Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. A root system R is a non empty subset $R \subset V$ such that:

1. If $\alpha \in R$ then $-\alpha \in R$ and $\lambda\alpha \notin R$ for any $\lambda \neq -1, 1$.
2. Span(R)=V
3. If $\alpha \in R$ then R is closed under reflection through the hyperplane orthogonal to α. Or in other words; $\alpha, \beta \in R \implies \beta - 2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}\alpha \in R$.
4. If $\alpha, \beta \in R$, then the orthogonal projection of β on the line through α is an integer multiple of $\frac{\alpha}{2}$. Or in other words, $\alpha, \beta \in R \implies 2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbf{Z}$.

The simplest possible root system lives in one dimension and consists of 2 elements, α and $-\alpha$. This root system is called A_1. In two dimensions there are some more examples. One example is to fix some α, and then let R be set set of vectors with the same length as α, and whose angle with α is a multiple of $60°$. This root system is conveniently called A_2. A visualisation of A_1 and A_2 is shown in figure 4.1.

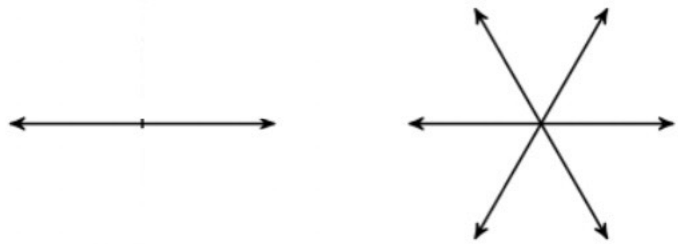

Figure 4.1: A representation of the root systems A_1 (to the left) and A_2 (to the right).

From this geometric description of A_1 and A_2 it is clear that they fulfill the definition of a root system. Now let us look at how to describe them algebraically. In the case of A_1, we can let V be the subspace of $\mathbf{C}^2$ where $e_1 + e_2 = 0$. The elements α and $-\alpha$ now becomes $e_1 - e_2$ and $e_2 - e_1$. Here $e_1, ..., e_n$ is a standard ON basis for $\mathbf{C}^n$. Similarly in the A_2 case we can let V be the subspace of $\mathbf{C}^3$ where $e_1 + e_2 + e_3 = 0$. In this case the corresponding 6 elements of A_2 are the 6 possible permutations of $e_i - e_j$, $i, j = 1, 2, 3$. We can easily verify that this is equivalent to the geometric description by computing the dot product between two such elements and see that every angle between two elements is a multiple of $60°$. However this algebraic description is possible to generalise to higher dimensions and lead to the following definition of A_n.

Definition 4.1.2. Let $V_n \subset \mathbf{C}^{n+1}$ be the hyperplane $\{x \in \mathbf{C}^{n+1} : x_1 + ... + x_{n+1} = 0\}$. We then define $A_n := \{e_i - e_j : i, j \in \{1, ..., n+1\},\ i \neq j\} \subset V_n$.

It is still not obvious that A_n really is a root system. We prove that by the following theorem.

Theorem 4.1.3. The set A_n as defined in definition 4.1.2 is a root system on V_n.

Proof. We prove this by checking all 4 criterion in definition 4.1.1:

1. If $\alpha \in A_n$ then $\alpha = e_i - e_j \implies -\alpha = e_j - e_i \in A_n$, and $\lambda\alpha = \lambda(e_i - e_j) \notin A_n$ for $\lambda \neq -1, 1$.
2. Span(A_n)=V_n since $A_n \subset V_n$ and dim(Span(A_n))=dim(V_n)=n.
3. $\alpha, \beta \in A_n \implies \alpha = e_i - e_j$, $\beta = e_k - e_l$. So

$$\beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha = e_k - e_l - 2\frac{\langle e_i - e_j, e_k - e_l \rangle}{\langle e_i - e_j, e_i - e_j \rangle}(e_i - e_j) =$$

$$= e_k - e_l - \langle e_i - e_j, e_k - e_l \rangle (e_i - e_j). \tag{4.1}$$

Here there are essentially three different cases; either all basis vectors e_i, e_j, e_k and e_l are different, there could be one overlap, or there could be 2 overlaps. These are the only cases since we must have $i \neq j$ and $k \neq l$ by definition. If there is no overlap we have $\langle e_i - e_j, e_k - e_l \rangle = 0$, so equation 4.1 reduces to $e_k - e_l \in A_n$. On the other hand, if for example $k = i$ equation 4.1 reduces to $e_j - e_l \in A_n$. Finally if $k = i$ and $l = j$ equation 4.1 reduces to $e_l - e_k \in A_n$. Note that we do not need to check any other cases, we do not need to check for example $k = j$. This is because swapping 2 of the indices can only have the effect of multiplying the whole expression by ± 1. So these 3 cases are sufficient to check.

4. $\alpha, \beta \in A_n \implies \alpha = e_i - e_j,\ \beta = e_k - e_l$. So

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = 2\frac{\langle e_i - e_j, e_k - e_l \rangle}{\langle e_i - e_j, e_i - e_j \rangle} = \langle e_i - e_j, e_k - e_l \rangle =$$

$$\langle e_i, e_k \rangle + \langle e_j, e_l \rangle - \langle e_i, e_l \rangle - \langle e_j, e_k \rangle \in \mathbf{Z}$$

$\square$

Example 4.1.4. There are many other examples of root systems other than A_n. Even in one dimension we could consider the set $B = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$, and this is a root system. In 2 dimensions there are many other examples other than A_2. For example $A_1 \times A_1$, $A_1 \times B$ or $B \times B$ to name a few. There are also more complicated classes of root systems called G_n and BC_n. Then it is possible to combine these to get new root systems. For example $BC_5 \times A_7$ is a 12 dimensional root system, and so on. The general study of all root systems is a bit outside the scope of this report and we will mostly stick to the A_n case. We will sometimes mention general root systems when necessary.

Remark 4.1.5. It is often useful to have the underlying vector space be all of $\mathbf{C}^{n+1}$. For that matter one often considers the extended version of $A_n^* := A_n \cup \frac{e_1 + \ldots + e_{n+1}}{n}$. Which has $V = \mathbf{C}^{n+1}$ as its underlying vector space. Note that the fact that A_n^* is a root system follows since A_n is a root system and that $\frac{e_1 + \ldots e_{n+1}}{n} \perp A_n$.

4.1.1 Some properties of root systems

There are a few terms and concepts we need to define before we can carry on. First of all is the notion of positive roots. Given a root system R it is always possible to decompose R into a set of positive roots R^+, and a set of negative roots R^- according to the following definition.

Definition 4.1.6. A subset R^+ of R is a set of positive roots if there exists $v \in V$ such that $\langle \alpha, v \rangle > 0$ for all $\alpha \in R^+$, and $\langle \alpha, v \rangle < 0$ for all $\alpha \in R \setminus R^+$. The set of negative roots is then defined as $R^- := R \setminus R^+$.

Note that this decomposition is not unique since it depends on the choice of the vector v. Geometrically we can take a hyperplane through the origin, not containing any roots. Then all the roots on one side are positive, and all on the other side are negative.

Lemma 4.1.7. The set $\{e_i - e_j\}_{i<j}$ is a set of positive roots for A_n

Proof. Take $v = \alpha_1 e_1 + ... + \alpha_n e_n 0$, where $\alpha_1 + ... + \alpha_n = 0$ and $\alpha_1 > ... > \alpha_n$, which obviously exists. □

It is sometime useful to have a basis consisting of roots, then the notion of simple roots can be useful.

Definition 4.1.8. An element of R^+ is called a simple root if it cannot be written as the sum of two elements in R^+. The set of simple roots is denoted R_s

Lemma 4.1.9. The set of simple roots form a basis of V. Furthermore every root can be written as a sum of simple roots with integer coefficients such that they are all either non positive or all non negative.

Proof. For a proof see [10]. □

Lemma 4.1.10. For the root system A_n and with R^+ as in lemma 4.1.7, the set of simple roots are $\alpha_i = \{e_i - e_{i+1}\}$, $i = 1, ..., n-1$.

Proof. It is trivial that $\alpha_i \in R^+$ for all i, and that α_i is not the sum of 2 elements in R^+. Finally if $\alpha = e_i - e_j$ with $i < j$ and $j \neq i+1$, then there exist an integer k between i and j. Hence $\alpha = (e_i - e_k) + (e_k - e_j)$ is the sum of 2 elements in R^+. □

Definition 4.1.11. [Fundamental Weyl chamber] The fundamental Weyl chamber associated with the root system R is the set $W = \{x \in V : \mathrm{Re}(\langle x, \alpha \rangle) < 0 \quad \forall \alpha \in R_s\}$

Example 4.1.12. The fundamental Weyl chamber for the root system A_{n-1} and with the positive roots as in lemma 4.1.10 is the set $W = \{x \in \mathbf{C}^n : i < j \implies \mathrm{Re}(x_i) < \mathrm{Re}(x_j)\}$

Proof. By definition $W = \{x \in \mathbf{C}^n : \text{Re}(\langle x, e_i - e_{i+1}\rangle) < 0 \quad i = 1, ..., n-1\}$. Expanding the dot product we see that $\text{Re}(\langle x, e_i\rangle) <\text{Re}(\langle x, e_{i+1}\rangle)$. Since this has to apply for every $i = 1, ..., n-1$ we must have $\text{Re}(x_i) <\text{Re}(x_j)$ whenever $i < j$. □

Definition 4.1.13. We now list some other terminology associated with root systems that will be relevant later on.

- The rank of a root system is the dimension of the vector space V
- The root lattice is the set of all points that can be written as a sum of roots with coefficients in $\mathbf{Z}$, commonly denoted Q.
- The positive root lattice is the set $Q^+ = \{\alpha \in Q : \langle \alpha, v\rangle > 0\}$

Example 4.1.14. We now list what the corresponding sets from definition 4.1.13 become in the A_n case. In the list below, let $\alpha = \pm(e_i - e_j)$.

- The rank of A_n is n
- The root lattice is the set $Q = \{n\alpha : n \in \mathbf{Z}\}$
- The positive root lattice is $Q^+ = \{n\alpha : n \in \mathbf{Z}_+\}$

Definition 4.1.15. Associated with each root system is a reflection group, called the Weyl group. It is the group generated by all reflections σ_α for $\alpha \in R$ where σ_α stands for the reflection through the hyperplane orthogonal to α. When we referee to the whole group often the notation σ_R is used.

Lemma 4.1.16. The group $\sigma_{A_{n-1}}$ is isomorphic to S_n.

Proof. The elements of $\sigma_{A_{n-1}}$ are on the form $\sigma_{e_i - e_j}$, which geometrically means reflection through the hyperplane $H_{ij} := e_i - e_j = 0$. Note that reflection through H_{ij} is equivalent to swapping the ith and the jth components in the vector. Hence the reflection group $\sigma_{A_{n-1}}$ is isomorphic to S_n. □

Remark 4.1.17. Often to save space, we do not write out $\sigma_{e_i - e_j}$, but rather just σ_{ij} instead.

4.2 Dunkl operators

We now look at a class of operators called Dunkl operators, which we will see later on relates to the hypergeometric function thorough a certain eigenvalue problem. The general sources for this section are [1] and [13]. We begin by defining what Dunkl operators are

Definition 4.2.1 (Dunkl operators)**.** A Dunkl operator T_ξ is an operator that depend on a root system R and a vector $\xi \in V$. It acts on functions in $\mathcal{C}^1(V)$ and is defined by

$$T_\xi := \partial_\xi + \sum_{\alpha \in R^+} k_\alpha \frac{1 + e^{\langle \alpha, x \rangle}}{1 - e^{\langle \alpha, x \rangle}} (s_{\sigma_\alpha} - 1)\alpha_\xi, \tag{4.2}$$

and

$$[s_{\sigma_\alpha} - 1] f(x) = f(\sigma_\alpha(x)) - f(x). \tag{4.3}$$

Here ∂_ξ is the directional derivative along ξ. Also k_α is a σ_R invariant function, in other words if there exists $\sigma \in \sigma_R$ such that $\sigma(\alpha_1) = \alpha_2$, then $k_{\alpha_1} = k_{\alpha_2}$. Also $\alpha_\xi = \frac{\langle \alpha, \xi \rangle}{\langle \xi, \xi \rangle} \xi$ stands for the orthogonal component of α along ξ with respect to the inner product of V.

Sometimes the notation T_R is used to talk about the whole set of Dunkl operators associated with the root system R.

Example 4.2.2. Let us see what $T_{A_{n-1}}$ looks like. The group $\sigma_{A_{n-1}} \approx S_n$ only has one orbit so $k_\alpha = k$ is just a constant. The set of positive roots is $\{e_{i'} - e_{j'} : i' < j'\}$. Finally from now on we only consider the extended root system A_{n-1} so that the underlying vector space is $V = C^n$. Then we can let $\xi = x_i$. The Dunkl operators then take the form

$$T_{x_i} = T_i = \frac{\partial}{\partial x_i} + k \sum_{i' < j'} \frac{1 + e^{x_{i'} - x_{j'}}}{1 - e^{x_{i'} - x_{j'}}} (s_{i'j'} - 1)(e_{i'} - e_{j'})_i. \tag{4.4}$$

What remains to calculate is the ith component of $e_{i'} - e_{j'}$. However this is easy, it is 1 if $i = i'$, -1 if $i = j'$ and zero otherwise. Hence T_i take the form

$$T_i = \frac{\partial}{\partial x_i} + k \sum_{j \neq i} (-1)^{\mathbf{1}_{j>i}} \frac{1 + e^{x_{\max(i,j)} - x_{\min(i,j)}}}{1 - e^{x_{\max(i,j)} - x_{\min(i,j)}}} (s_{ij} - 1). \tag{4.5}$$

However we note that this form is needlessly complicated because the fraction

$$\frac{1 + e^{x_i - x_j}}{1 - e^{x_i - x_j}} \tag{4.6}$$

is anti symmetric. Meaning that

$$\frac{1 + e^{x_i - x_j}}{1 - e^{x_i - x_j}} = -\frac{1 + e^{x_j - x_i}}{1 - e^{x_j - x_i}}. \tag{4.7}$$

Hence the min/max and the $(-1)^{\mathbf{1}_{j>i}}$ cancel each other out, and T_i can be written in the form

$$T_i = \frac{\partial}{\partial x_i} + k \sum_{j \neq i} \frac{1 + e^{x_i - x_j}}{1 - e^{x_i - x_j}} (s_{ij} - 1) \tag{4.8}$$

Dunkl operators are not local, meaning that in order to know the value of $Tf(x_0)$ it is in general not sufficient to only have local information about f in a neighborhood of x_0. There is however one exception. If the function f is σ_R invariant (meaning that $f(\sigma(x)) = f(x)$ for all $\sigma \in \sigma_R$ and all $x \in V$). In the particular case of A_{n-1} the Dunkl operators are local on S_n invariant functions (commonly called symmetric functions). On the set of symmetric functions the Dunkl operators are just equal to some differential operator. Sometimes it is of interest to just extract this differential operator, leading to the following definition.

Definition 4.2.3 (Res)**.** Let σ be a group and let T be an operator that is equal to some differential operator D on the set of σ invariant functions. Then the restriction of T to σ invariant functions is D (written $\text{Res}_\sigma(T)$=D). If $\sigma = S_n$ we often do not write out the group but rather just write Res(T)=D instead.

Example 4.2.4. The restriction of $\frac{\partial}{\partial x_i} s_{ij}$ to symmetric functions is $\text{Res}(\frac{\partial}{\partial x_i} s_{ij}) = \frac{\partial}{\partial x_i}$. Because to a symmetric function first swapping x_i and x_j, then differentiate with respect to x_i, is just the same as differentiating with respect to x_i. On the other hand $\text{Res}(s_{ij} \frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_j}$. This is probably easiest seen by looking at a concrete symmetric function. Let $f(x) = x_1^2 + x_2^2$, then

$$\frac{\partial}{\partial x_1} s_{1,2} f(x) = 2x_1 = \frac{\partial}{\partial x_1} f(x) \tag{4.9}$$

and

$$s_{1,2} \frac{\partial}{\partial x_1} f(x) = 2x_2 = \frac{\partial}{\partial x_2} f(x). \tag{4.10}$$

Theorem 4.2.5. The operators T_ξ and T_ν commute for all ξ and ν and for any root system.

Proof. For a proof see [13]. □

The general proof that Dunkl operators commute is a bit outside the scope of this report. However for some particular examples the result is not so difficult to justify.

Example 4.2.6. Consider the A_1 case, in that case there are 2 linearly independent Dunkl operators,

$$T_1 = \frac{\partial}{\partial x_1} + k\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}(s_{12}-1) \tag{4.11}$$

and

$$T_2 = \frac{\partial}{\partial x_2} + k\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}(s_{12}-1). \tag{4.12}$$

Let us try and compute the commutator $[T_1, T_2] = T_1T_2 - T_2T_1$. Let

$$b_{12} = \frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}(s_{1,2}-1), \tag{4.13}$$

we then get

$$[T_1, T_2] = \left(\frac{\partial}{\partial x_1} + kb_{1,2}\right)\left(\frac{\partial}{\partial x_2} + kb_{1,2}\right) - \left(\frac{\partial}{\partial x_2} + kb_{1,2}\right)\left(\frac{\partial}{\partial x_1} + kb_{1,2}\right). \tag{4.14}$$

Expanding the bracket we get

$$[T_1, T_2] = \frac{\partial^2}{\partial x_1 \partial x_2} + k\left(\frac{\partial}{\partial x_1}b_{1,2} + b_{1,2}\frac{\partial}{\partial x_2}\right) + k^2 b_{1,2}^2 \tag{4.15}$$

$$-\frac{\partial^2}{\partial x_1 \partial x_2} - k\left(\frac{\partial}{\partial x_2}b_{1,2} + b_{1,2}\frac{\partial}{\partial x_1}\right) - k^2 b_{1,2}^2. \tag{4.16}$$

A simple calculation shows that

$$\frac{\partial}{\partial x_2}\left(\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}(s_{1,2}-1)\right) = -\frac{\partial}{\partial x_1}\left(\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}(s_{1,2}-1)\right). \tag{4.17}$$

All terms therefore vanish and we get the result

$$[T_1, T_2] = 0. \tag{4.18}$$

The operators T_1 and T_2 commute, which is what theorem 4.2.5 predicts.

Since these operators commute we can use the theory on symmetric polynomials from chapter 2.2 on T, that is we can study $\mathbf{C}[T]$ and $\mathbf{C}[T]^{S_n}$. Let us compute some examples of these polynomials. From now on we only focus on the root system A_n.

Definition 4.2.7. The operator D_p^R is defined as $\mathrm{Res}_{\sigma_R}(p(T_R))$.

Example 4.2.8. Let $p(x) = x_1 + ... + x_n$. Let us try and compute $D_p^{A_{n-1}}$. By definition we get

$$D_p^{A_{n-1}} = \mathrm{Res}(p(T_{A_{n-1}})) = \mathrm{Res}\left(\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i} + k\sum_{j\neq i}\frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}(s_{ij}-1)\right)\right). \tag{4.19}$$

However $\text{Res}(s_{ij}-1)=0$, hence we get that

$$D_p^{A_{n-1}} = \frac{\partial}{\partial x_1} + ... + \frac{\partial}{\partial x_n}. \tag{4.20}$$

Lemma 4.2.9. Let $p_n(x) = x_1^2 + ... + x_n^2$, and

$$M_n = \Delta - 2k \sum_{i<j} \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right). \tag{4.21}$$

Then $D_{p_n}^{A_{n-1}} = M_n$.

Proof. For simplicity define

$$b_{ij} = \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}(s_{ij}-1), \tag{4.22}$$

so that

$$T_i = \frac{\partial}{\partial x_i} + 2k \sum_{i\neq j} b_{ij}. \tag{4.23}$$

We first notice that

$$\text{Res}(s_{ij}) = 1 \implies \text{Res}(b_{ij}) = 0. \tag{4.24}$$

Now we have

$$p_n(T_{A_{n-1}}) = \sum_{i=1}^{n} T_i^2 = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + 2k \sum_{i\neq j} b_{ij} \right)^2. \tag{4.25}$$

Expanding the square we get

$$p_n(T_{A_{n-1}}) = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + 2k \sum_{i\neq j} \left(\frac{\partial}{\partial x_i} b_{ij} + b_{ij} \frac{\partial}{\partial x_i} \right) + 4k^2 \sum_{i\neq j, i\neq l} b_{ij} b_{il} \right). \tag{4.26}$$

We now use that $\text{Res}(b_{ij})=0$ and conclude that

$$\text{Res}(p_n(T_{A_{n-1}})) = \text{Res} \left(\sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + 2k \sum_{i\neq j} \left(b_{ij} \frac{\partial}{\partial x_i} \right) \right) \right). \tag{4.27}$$

We now have that

$$b_{ij} \frac{\partial}{\partial x_i} = \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}(s_{ij}-1)\frac{\partial}{\partial x_i} = \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}} \left(\frac{\partial}{\partial x_j} s_{ij} - \frac{\partial}{\partial x_i} \right). \tag{4.28}$$

This means that

$$\text{Res}(p_n(T_{A_{n-1}})) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2k \sum_{i\neq j} b_{ij} = \Delta - 2k \sum_{i\neq j} \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right). \tag{4.29}$$

We finally notice that the term

$$\frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)=\frac{1+e^{x_j-x_i}}{1-e^{x_j-x_i}}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right) \tag{4.30}$$

is S_n invariant. Meaning that we can interchange i and j and still get the same operator. This means that instead of summing over the set $\{i \neq j\}$ we can just as well sum over $\{i < j\}$ and pull out a factor of 2. Hence

$$\mathrm{Res}(p_n(T_{A_{n-1}}))=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-2k\sum_{i\neq j}b_{ij}=\Delta-2k\sum_{i<j}\frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right). \tag{4.31}$$

Hence we get the dessiered result

$$D_{p_n}^{A_{n-1}}=M_n \tag{4.32}$$

□

Remark 4.2.10. The operators M_n turn up in physic when studying a system of n bodys that act on each other by gravity only. Although we wont look into applications here. For more details see [2].

5

The eigenfunctions of M_2 and their connection to the hypergeometric function

In this chapter we study the eigenfunctions to the operator M_2 as defined in the previous chapter. The calculations in this chapter are based on similar calculations in [8] and [2]. As a reminder we defined

$$M_n = \Delta - 2k \sum_{i<j} \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right). \tag{5.1}$$

In the case $n = 2$ we therefore get

$$M_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 2k \frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right). \tag{5.2}$$

We are now going to study the eigenfunctions to this operator.

Lemma 5.0.1. For a given eigenvalue λ_{M_2}, the corresponding eigenspace to M_2 in a neighborhood[1] of $e^{x_1-x_2} = 0$ is spanned by functions on the form

$$f(x_1, x_2) = e^{\lambda_1 x_1 + \lambda_2 x_2} \sum_{i=0}^{\infty} c_n e^{n(x_1-x_2)}, \tag{5.3}$$

where $\lambda_1, \lambda_2 \in \mathbf{C}$ are constants such that

$$\lambda_{M_2} = \lambda_1^2 + \lambda_2^2 + 2k(\lambda_2 - \lambda_1). \tag{5.4}$$

Proof. We do a change of variables, let $r = x_1 + x_2$ and $s = x_1 - x_2$. Then

$$\frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x_1}\frac{\partial}{\partial s} = \frac{\partial}{\partial r} + \frac{\partial}{\partial s} \tag{5.5}$$

and

$$\frac{\partial}{\partial x_2} = \frac{\partial r}{\partial x_2}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x_2}\frac{\partial}{\partial s} = \frac{\partial}{\partial r} - \frac{\partial}{\partial s}. \tag{5.6}$$

[1] What is meant is a set that contains a set on the form $\{\mathbf{x} \in \mathbf{C}^2 : |e^{x_1-x_2}| < \epsilon\}$

Therefore

$$\frac{\partial^2}{\partial x_1^2} = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)^2 = \frac{\partial^2}{\partial r^2} + 2\frac{\partial^2}{\partial r \partial s} + \frac{\partial^2}{\partial s^2} \tag{5.7}$$

and

$$\frac{\partial^2}{\partial x_2^2} = \left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right)^2 = \frac{\partial^2}{\partial r^2} - 2\frac{\partial^2}{\partial r \partial s} + \frac{\partial^2}{\partial s^2}. \tag{5.8}$$

Hence

$$M_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 2k\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) = \tag{5.9}$$

$$= 2\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2} - 2k\frac{1+e^s}{1-e^s}\frac{\partial}{\partial s}\right). \tag{5.10}$$

We now let

$$L_1 = 2\frac{\partial^2}{\partial r^2} \tag{5.11}$$

and

$$L_2 = 2\left(\frac{\partial^2}{\partial s^2} - 2k\frac{1+e^s}{1-e^s}\frac{\partial}{\partial s}\right), \tag{5.12}$$

so that

$$M_2 = L_1 + L_2. \tag{5.13}$$

Since M_2 is a sum of operators that depends on r and s respectively we know that the solution is on the form

$$f(x_1, x_2) = \varphi(r)\phi(s) = \varphi(x_1 + x_2)\phi(x_1 - x_2). \tag{5.14}$$

with corresponding eigenvalue

$$\lambda_{M_2} = \lambda_{L_1} + \lambda_{L_2}. \tag{5.15}$$

For a fixed value of λ_{L_1}, the corresponding eigenspace is spanned by functions on the form

$$\varphi(r) = e^{\mu_1 r} \tag{5.16}$$

where $\mu_1 \in \mathbf{C}$ is such that $\lambda_{L_1} = 2\mu_1^2$.

In order to find the eigenfunction to L_2 we make another change of variables, let $z = e^s$. We then get

$$\frac{\partial}{\partial s} = \frac{\partial z}{\partial s}\frac{\partial}{\partial z} = z\frac{\partial}{\partial z} \tag{5.17}$$

and

$$\frac{\partial^2}{\partial s^2} = z\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\right) = z^2\frac{\partial^2}{\partial z^2} - z\frac{\partial}{\partial z}. \tag{5.18}$$

The problem of finding eigenfunctions to L_2 is then equivalent to solving the differential equation

$$2\left[z^2\frac{\partial^2}{\partial z^2} + x\frac{\partial}{\partial z} - 2k\frac{1+z}{1-z}z\frac{\partial}{\partial z}\right]\phi(z) = \lambda_{L_2}\phi(z). \tag{5.19}$$

We recognise this as a Fuchs equation with a singularity at $z = 0$, by theorem 2.1.7 in a neighborhood of $z = 0$ there is a basis of solutions on the form

$$\phi(z) = z^{\mu_2} \sum_{n=0}^{\infty} c_n z^n, \tag{5.20}$$

Hence we know that $\phi(s)$ has to be on the form

$$\phi(s) = e^{\mu_2 s} \sum_{n=0}^{\infty} c_n e^{ns}. \tag{5.21}$$

The problem of finding an eigenfunction to L_2 then reduces to

$$\left[2\left(\frac{\partial^2}{\partial s^2} - 2k\frac{1+e^s}{1-e^s}\frac{\partial}{\partial s}\right) - \lambda_{L_2}\right] e^{\mu_s} \sum_{n=0}^{\infty} c_n e^s = 0. \tag{5.22}$$

We now notice that if $e^s \to 0$ we must have $\lambda_{L_2} = 2(\mu_2^2 - 2k\mu_2)$ in order to have equality.

Hence the eigenspace is spanned by functions on the form

$$f(x_1, x_2) = \varphi(r)\phi(s) = e^{\mu_1(x_1+x_2)} e^{\mu_2(x_1-x_2)} \sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)}, \tag{5.23}$$

where μ_1 and μ_2 are such that

$$\lambda_{L_1} + \lambda_{L_2} = 2(\mu_1^2 + \mu_2^2 - 2k\mu_2) = \lambda_{M_2}. \tag{5.24}$$

If we let $\mu_1 = \frac{\lambda_1+\lambda_2}{2}$ and $\mu_2 = \frac{\lambda_1-\lambda_2}{2}$, then the corresponding basis of solutions is

$$f(x_1, x_2) = e^{\frac{\lambda_1+\lambda_2}{2}(x_1+x_2)} e^{\frac{\lambda_1-\lambda_2}{2}(x_1-x_2)} \sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)} \tag{5.25}$$

which if we simplify has the desired form

$$f(x_1, x_2) = e^{\lambda_1 x_1 + \lambda_2 x_2} \sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)}. \tag{5.26}$$

Where the corresponding eigenvalue is

$$\lambda_{M_2} = 2(\mu_1^2 + \mu_2^2 - 2k\mu_2) = \lambda_1^2 + \lambda_2^2 + 2k(\lambda_2 - \lambda_1). \tag{5.27}$$

□

Remark 5.0.2. In equation 5.19 we recovered a Fuchsian equation that looks almost like the hypergeometric. If we multiply by $\frac{1-z}{z}$ and rearrange the terms we get the differential equation

$$\left[z(1-z)\frac{\partial^2}{\partial z^2} + (1-2k-2kz)\frac{\partial}{\partial z} - \frac{1-z}{z}\lambda_{L_2}\right]\phi(z) = 0, \tag{5.28}$$

which almost looks like the hypergeometric differential equation, but the terms do not quiet match up. There is actually a change of variables to do that leads us to the hypergeometric differential equation. We are going to explore this connection more in section 5.1.

Now we know that there is an eigenspace spanned by functions on the form described in lemma 5.0.1, but we still do not know the values of the parameters c_n. We also do not know where the solutions are defined. In the proof we did the change of variables $z = e^{x_1-x_2}$ and solved the problem locally around $z = 0$. Hence all we know is that the basis defined in lemma 5.0.1 works in a neighborhood of $e^{x_1-x_2} = 0$, but where exactly do these functions converge? These questions are taken care of by the following two theorems.

Theorem 5.0.3. The function

$$f(x_1, x_2) = e^{\lambda_1 x_1 + \lambda_2 x_2} \sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)} \tag{5.29}$$

is an eigenfunction to M_2 if $\{c_n\}_{n=0}^{\infty}$ satisfies the recursive relation

$$((\lambda_1-\lambda_2)n + n^2 - 2kn)c_n = 2k\sum_{i=0}^{n-1} c_i((\lambda_1-\lambda_2)+2i) = 0 \quad \forall n > 0. \tag{5.30}$$

Proof. Let us now try and apply M_2 to f. We have

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 2k\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) - \lambda_{M_2}\right] f = 0. \tag{5.31}$$

If we assume that Re(x_1) <Re(x_2) then $|e^{x_1-x_2}| < 1$ and we have

$$\frac{1}{1-e^{x_1-x_2}} = \sum_{n=0}^{\infty} e^{n(x_i-x_j)}. \tag{5.32}$$

Note that this assumption is not a restriction since we solve the problem locally around $e^{x_1-x_2} = 0$. Now the equation reads

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - k(1+e^{x_1-x_2})\sum_{n=0}^{\infty} e^{n(x_i-x_j)}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) - \lambda_{M_2}\right] f = 0 \tag{5.33}$$

which we can simplify to

$$\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}-2k\left(1+2\sum_{n=1}^{\infty}e^{n(x_i-x_j)}\right)\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)-\lambda_{M_2}\right]f=0. \tag{5.34}$$

Then

$$\frac{\partial}{\partial x_1}f=\lambda_1 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}+e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n e^{n(x_1-x_2)}, \tag{5.35}$$

and

$$\frac{\partial}{\partial x_2}f=\lambda_2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}-e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n e^{n(x_1-x_2)}. \tag{5.36}$$

Furthermore

$$\frac{\partial^2}{\partial x_1^2}f=\lambda_1^2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}+2\lambda_1 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n e^{n(x_1-x_2)}+ \tag{5.37}$$

$$+e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n^2 e^{n(x_1-x_2)} \tag{5.38}$$

and

$$\frac{\partial^2}{\partial x_2^2}f=\lambda_2^2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}-2\lambda_2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n e^{n(x_1-x_2)}+ \tag{5.39}$$

$$+e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n^2 e^{n(x_1-x_2)}. \tag{5.40}$$

Plugging it all into one equation we get

$$\begin{gathered}
\lambda_1^2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}+2\lambda_1 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n e^{n(x_1-x_2)}+\\
+e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n^2 e^{n(x_1-x_2)}+\lambda_2^2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}\\
-2\lambda_2 e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n e^{n(x_1-x_2)}+e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n n^2 e^{n(x_1-x_2)}\\
-2k\left(1+2\sum_{n=1}^{\infty}e^{n(x_i-x_j)}\right)\left((2+\lambda_1-\lambda_2)e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}(1+n)c_n e^{n(x_1-x_2)}\right)\\
-(\lambda_1^2+\lambda_2^2+k(\lambda_2-\lambda_1))e^{\lambda_1 x_1+\lambda_2 x_2}\sum_{n=0}^{\infty}c_n e^{n(x_1-x_2)}=0.
\end{gathered}$$

We can multiply the bracket together, divide everything by $e^{\lambda_1 x_1+\lambda_2 x_2}$ and combine like terms to get

$$\sum_{n=0}^{\infty}(2(\lambda_1-\lambda_2)c_n n+2c_n n^2-4kc_n n \tag{5.41}$$

$$-2k(\lambda_1-\lambda_2)(c_0+...+c_{n-1})-8k(c_1+2c_2+...+(n-1)c_{n-1}))e^{n(x_1-x_2)}=0. \tag{5.42}$$

Giving a system of equations

$$2(\lambda_1 - \lambda_2)c_n n + 2c_n n^2 - 2kc_n n \tag{5.43}$$

$$-2k(\lambda_1 - \lambda_2)(c_0 + ... + c_{n-1}) - 4k(c_1 + 2c_2 + ... + (n-1)c_{n-1}) = 0 \quad \forall n > 0. \tag{5.44}$$

Grouping all terms containing c_n together and combining the rest into one sum gives the desired result

$$((\lambda_1 - \lambda_2)n + n^2 - 2kn)c_n = 2k\sum_{i=0}^{n-1} c_i((\lambda_1 - \lambda_2) + 2i) = 0 \quad \forall n > 0. \tag{5.45}$$

□

Remark 5.0.4. Note the similarity with how we solved the hypergeometric differential equation using Frobenius method. The difference there being that we could solve the recursive relation we got at the end explicitly.

Remark 5.0.5. In the proof we assumed that Re(x_1)<Re(x_2), if that inequality is reversed we can use the fact that

$$\frac{1+e^{x_1-x_2}}{1-e^{x_1-x_2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_j}\right) = \frac{1+e^{x_2-x_1}}{1-e^{x_2-x_1}}\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right) \tag{5.46}$$

and instead expand

$$\frac{1}{1-e^{x_2-x_1}} \tag{5.47}$$

as a geometric series. The analysis will be completely analogous, but with x_1 and x_2 interchanged.

We now have full control of what the eigenspace looks like in a neighborhood of $e^{x_1-x_2} = 0$. However we do not know how far the solutions extend. That is, we want to know where the series

$$\sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)} \tag{5.48}$$

converge. That is described in the following theorem.

Theorem 5.0.6. Let $\{c_n\}_{n=0}^{\infty}$ be defined as in 5.0.3, then the series

$$\sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)} \tag{5.49}$$

converge when Re(x_1)<Re(x_2)

Proof. Note that there exists constants C_1 and C_2 such that $(\lambda_1-\lambda_2)n+n^2-2kn \geq C_1 n^2$ and $(\lambda_1-\lambda_2)+2i \leq (\lambda_1-\lambda_2)+2n \leq C_2(1+n)$. From this we can deduce

$$C_1 n^2 |c_n| \leq C_2(1+n)k \sum_{i=0}^{n-1} |c_i|. \tag{5.50}$$

For $n \geq 1$ we have $1+n \leq 2n$ (when it comes to convergence of the series only asymptotic behavior matters, so the fact that this does not hold for $n=0$ is not a problem). We now set $C = \frac{2C_2}{C_1}$ and conclude that

$$|c_n| \leq \frac{C}{n} 2k \sum_{i=0}^{n-1} |c_i|. \tag{5.51}$$

Note that C depends on λ. Let $\epsilon > 0$, and let N_0 be an integer such that

$$2Ck \sum_{i=1}^{\infty} e^{-i\epsilon} \leq N_0. \tag{5.52}$$

Now choose $K = K_{\epsilon,\lambda}$ such that

$$|c_n| \leq K e^{n\epsilon} \quad \forall n < N_0. \tag{5.53}$$

We are now going to prove that $|c_n| \leq K e^{n\epsilon} \quad \forall n$. We will prove it by induction. Let $N > N_0$ and suppose $|c_n| \leq K e^{n\epsilon}$ for all $n < N$. Then

$$|c_N| \leq \frac{c}{N} 2k \sum_{i=0}^{N} |c_i| = \frac{C}{N} 2k \sum_{i=1}^{N} |c_{N-i}| \leq \frac{c}{N} 2k \sum_{i=1}^{N} K e^{N\epsilon - i\epsilon} = \tag{5.54}$$

$$= K e^{N\epsilon} \frac{c}{N} 2k \sum_{i=1}^{N} e^{-i\epsilon} \leq K e^{N\epsilon} \frac{C}{N} 2k \sum_{i=1}^{\infty} e^{-i\epsilon} \leq K e^{N\epsilon} \frac{N_0}{N} \leq K e^{N\epsilon}, \tag{5.55}$$

and the induction is done. From this estimate it trivially follows that the series converges whenever $\mathrm{Re}(x_2) > \mathrm{Re}(x_1)$. □

Remark 5.0.7. Note that convergence in $\mathrm{Re}(x_1) < \mathrm{Re}(x_2)$ is really the most we could hope for since at one point in the proof of theorem 5.0.3 we assumed that $\mathrm{Re}(x_1) < \mathrm{Re}(x_2)$.

5.1 Connection to the hypergeometric function

The calculations in this section is based on similar calculation in [5]. Let us specifically focus on the operator L_2 as defined in the proof of lemma 5.0.1, we are going to find the eigenfunction in a different way. We now make a new change of variable, let $x = \frac{1}{2} - \frac{1}{4}(e^s + e^{-s})$. Then

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s} \frac{\partial}{\partial x} = \frac{1}{4} \left(e^{-s} - e^{s} \right) \frac{\partial}{\partial x}. \tag{5.56}$$

and

$$\frac{\partial^2}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{1}{4}\left(e^{-s} - e^{s}\right)\frac{\partial}{\partial x}\right). \tag{5.57}$$

Using the product rule this evaluates to

$$\frac{\partial^2}{\partial s^2} = \frac{1}{4}\left(\left(e^{-s} - e^{s}\right)\frac{\partial^2}{\partial s \partial x} - (e^{-s} + e^{s})\frac{\partial}{\partial x}\right). \tag{5.58}$$

Replacing $\frac{\partial}{\partial s}$ with $\frac{1}{4}\left(e^{-s} - e^{s}\right)\frac{\partial}{\partial x}$ we see that

$$\frac{\partial^2}{\partial s^2} = \frac{1}{16}\left(e^{-s} - e^{s}\right)^2\frac{\partial^2}{\partial x^2} - \frac{1}{4}(e^{-s} + e^{s})\frac{\partial}{\partial x}. \tag{5.59}$$

By the definition of x we can conclude that

$$-\frac{e^{-s} + e^{s}}{4} = x - \frac{1}{2}, \tag{5.60}$$

and

$$\frac{(e^{-s} - e^{s})^2}{16} = -\frac{-(e^{-s} - e^{s})^2}{16} = -\frac{-e^{-4s} + 2 - e^{4s}}{16} = \tag{5.61}$$

$$-\left(\frac{1}{4} - \frac{1}{16}\left(e^{-4s} + 2 + e^{4s}\right)\right) = \tag{5.62}$$

$$= -\left(\frac{1}{2} - \frac{1}{4}\left(e^{-s} + e^{s}\right)\right)\left(\frac{1}{2} + \frac{1}{4}\left(e^{-s} + e^{s}\right)\right) = -x(1-x). \tag{5.63}$$

Therefore we can write $\frac{\partial^2}{\partial s^2}$ only in terms of x as

$$\frac{\partial^2}{\partial s^2} = -x(1-x)\frac{\partial^2}{\partial x^2} + \left(x - \frac{1}{2}\right)\frac{\partial}{\partial x}. \tag{5.64}$$

We are now ready to express L_2 in terms of x, we get

$$L_2 = 2\left(\frac{\partial^2}{\partial s^2} - 2k\frac{1 + e^{s}}{1 - e^{s}}\frac{\partial}{\partial s}\right) = \tag{5.65}$$

$$= 2\left(-x(1-x)\frac{\partial^2}{\partial x^2} + \left(x - \frac{1}{2}\right)\frac{\partial}{\partial x} - \frac{k}{2}\frac{1 + e^{s}}{1 - e^{s}}(e^{-s} - e^{s})\frac{\partial}{\partial x}\right). \tag{5.66}$$

We note that we can simplify

$$\frac{(1 + e^{s})(e^{-s} - e^{s})}{1 - e^{s}} = \frac{(1 + e^{s})(1 - e^{s})(1 + e^{s})e^{-s}}{1 - e^{s}} = \tag{5.67}$$

$$= 2 + e^{s} + e^{-s} = 4 - 4x. \tag{5.68}$$

Hence we can write L_2 as

$$L_2 = 2\left(-x(1-x)\frac{\partial^2}{\partial x^2} + \left(x - \frac{1}{2}\right)\frac{\partial}{\partial x} - 2k(1-x)\frac{\partial}{\partial x}\right). \tag{5.69}$$

Which we finally simplify to

$$L_2 = -2x(1-x)\frac{\partial^2}{\partial x^2} + (-1-4k+(2+4k)x)\frac{\partial}{\partial x}. \tag{5.70}$$

Lets try and find the eigenfunctions ϕ to this operator in terms of this new variable x. We get

$$\left[-2x(1-x)\frac{\partial^2}{\partial x^2} + (-1-4k+(2+4k)x)\frac{\partial}{\partial x}\right]\phi(x) = \lambda_{L_2}\phi(x) \tag{5.71}$$

which we may rewrite as

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}+2k-(1+2k)x\right)\frac{\partial}{\partial x} + \frac{\lambda_{L_2}}{2}\right]\phi(x) = 0. \tag{5.72}$$

Let $c = \frac{1}{2} + 2k$, $a = \frac{\lambda_1-\lambda_2}{2}$ and $b = \frac{\lambda_2-\lambda_1}{2} + 2k$, then

$$a+b = \frac{\lambda_1-\lambda_2}{2} + \frac{\lambda_2-\lambda_1}{2} + 2k = 2k \tag{5.73}$$

and

$$ab = -\frac{(\lambda_1-\lambda_2)^2}{4} - \frac{2k(\lambda_1-\lambda_2)}{2} = -\frac{\frac{(\lambda_1-\lambda_2)^2}{2} - 2k(\lambda_1-\lambda_2)}{2} = -\frac{\lambda_{L_2}}{2}, \tag{5.74}$$

so that the equation reads

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + (c-(1+a+b)x)\frac{\partial}{\partial x} - ab\right]\phi = 0, \tag{5.75}$$

which we recover as the hypergeometric differential equation. We therefore see that the problem of finding eigenfunctions to L_2, and solving the hypergeometric differential equation are equivalent. Meaning that the set of solutions must be the same, under this change of variables. We have now solved both of these problems, and received two very different looking solutions. Let us explore their connection.

5.1.1 Connecting the solutions

One key property of the hypergeometric differential equation is that it has 2 linearly independent solutions, and since the problem of finding eigenfunctions to L_2 is equivalent to solving the hypergeometric differential equation, it two must have two linearly independent solutions. It is not so hard to see that this is the case. We saw in the proof of 5.0.1 that

$$\lambda_{L_2} = 2(\mu_2^2 - k\mu_2), \tag{5.76}$$

and that the corresponding eigenfunction is

$$\phi(x_1,x_2) = e^{\mu_2(x_1-x_2)}\sum_{n=0}^{\infty} c_n e^{n(x_1-x_2)}. \tag{5.77}$$

For a fixed value of λ_{L_2} there are two values of μ_2 that give the same eigenvalue, those being μ_2 and $\mu_2^{\text{new}} = k - \mu_2$. We saw from lemma 5.0.1 that for a given eigenvale λ_{M_2} the equation

$$M_2 f = \lambda_{M_2} f \tag{5.78}$$

has an infinite number of linearly independent solutions. The problem is that $\lambda_{M_2} = \lambda_{L_1} + \lambda_{L_2}$. However we do not know in advance how much weight is given to λ_{L_1} and λ_{L_2} respectively. Let us therefore assume that λ_{L_1} is fixed in advance. Then the solution space to the system of equations

$$\begin{cases} M_2 f = \lambda_{M_2} f \\ \mu_1 = \mu, \quad \mu \text{ fixed in advance} \end{cases} \tag{5.79}$$

should be equivalent to the solution space to the hypergeometric differential equation. However this is not a very practical system of equations because in order for it to make sense we have to explain what μ_1 is, and μ_1 was only a variable that we introduced while finding eigenfunctions to M_2. We would like an other equation that has the effect of fixing μ_1 without needing to define what it is. Remember that $\mu_1 = \frac{\lambda_1+\lambda_2}{2}$. If we define $P = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$, then it is not so hard to see that the equation

$$Pf = \lambda_P f \tag{5.80}$$

has the effect of fixing μ_1. In the sense that if λ_P has a fixed value. Then that also forces μ_1 to have a fixed value. Equation (5.80) is therefore equivalent to the equation $\mu_1 = \mu$. However equation (5.80) is much nicer to work with because it does not require us to define what μ_1 is. The system (5.79) can therefore be written as

$$\begin{cases} M_2 f = \lambda_{M_2} f \\ Pf = \lambda_P f. \end{cases} \tag{5.81}$$

Remember that

$$\mu_1 = \frac{\lambda_1 + \lambda_2}{2}. \tag{5.82}$$

So if we let

$$\lambda_1^{\text{new}} = \lambda_2 + 2k \tag{5.83}$$

and

$$\lambda_2^{\text{new}} = \lambda_1 - 2k, \tag{5.84}$$

we get the corresponding relations $\mu_1^{\text{new}} = \mu_1$ and $\mu_2^{\text{new}} = k - \mu_2$, which is precisely what we are looking for. Let ϕ be the eigenfunction to L_2 corresponding to λ_1 and λ_2 and let ϕ^{new} be the eigenfunction corresponding to λ_1^{new} and λ_2^{new}. Then it is clear that ϕ and ϕ^{new} span the eigenspace of L_2, which means that there is precisely one linear combination of ϕ and ϕ^{new} that correspond to the hypergeometric function.

We saw earlier in chapter 3 that what is so special about the hypergeometric function is that it is well behaved around $x = 0$. In this case, remember than $x = \frac{1}{2} - \frac{1}{4}\left(e^{x_1-x_2} + e^{x_2-x_1}\right)$, so that $x = 0$ is equivalent to Re(x_1)=Re(x_2). From theorem 5.0.6 convergence is only guaranteed if Re(x_1)<Re(x_2). And in fact both

ϕ and ϕ^{new} will blow up to ∞ as Re(x_1)-Re(x_2) $\to 0$. However the fact that they span the solution set to the hypergeometric differential equation tells us that the is precisely one linear combination

$$F = \sigma\phi + \sigma^{\text{new}}\phi^{\text{new}} \tag{5.85}$$

that does not have a singularity at Re(x_1)=Re(x_2), and that this function corresponds to the hypergeometric function. This particular linear combination is classified by the following theorem.

Theorem 5.1.1. Let $a(\lambda_1, \lambda_2) = \frac{\lambda_1-\lambda_2}{2}$, $b(\lambda_1, \lambda_2) = \frac{\lambda_2-\lambda_1}{2}+2k$ and $c(\lambda_1, \lambda_2) = \frac{1}{2}+2k$. Furthermore let

$$\sigma(\lambda_1, \lambda_2) = \frac{4^a\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}. \tag{5.86}$$

Then

$$\sigma(\lambda_1, \lambda_2)\phi(x_1, x_2) + \sigma(\lambda_1^{\text{new}}, \lambda_2^{\text{new}})\phi^{\text{new}}(x_1, x_2) \tag{5.87}$$

is the only linear combination of ϕ and ϕ^{new} that does not have a singularity at Re(x_1)=Re(x_2). Now let $x = \frac{1}{2} - \frac{1}{4}\left(e^{x_1-x_2} + e^{x_2-x_1}\right)$, then

$$F(a, b, c; x) = \sigma(\lambda_1, \lambda_2)\phi(x) + \sigma(\lambda_1^{\text{new}}, \lambda_2^{\text{new}})\phi^{\text{new}}(x), \tag{5.88}$$

where $F(a, b, c; x)$ is the hypergeometric function.

Proof. If we go back to where ϕ was originally defined, which is in equation (5.20). We see that ϕ is constructed as the solution to the Fuchsian equation

$$\left[z(1-z)\frac{\partial^2}{\partial z^2} + (1-2k-2kz)\frac{\partial}{\partial z} - \frac{1-z}{z}\lambda_{L_2}\right]\phi(z) = 0, \tag{5.89}$$

that has a characteristic exponent of μ_2 at $z = 0$. Using the fact that $\mu_2 = \frac{\lambda_1-\lambda_2}{2} = a$, this means that ϕ has to be on the form

$$\phi(z) = z^a g(z), \tag{5.90}$$

where g is holomorphic and non zero in a neighborhood of $z = 0$. If we substitute $\phi(z) = z^a g(z)$ in equation (5.89) we find that g has to satisfy

$$z(1-z)g''(z) + (1+a-b-(2+2a-c)z)g'(z) - a(1+a-c)g(z) = 0, \tag{5.91}$$

which we recognise as a hypergeometric equation, which is solved by

$$g(z) = F(a, 1+a-c, 1+a-b; z). \tag{5.92}$$

Hence ϕ has to take the form

$$\phi(z) = z^a F(a, 1+a-c, 1+a-b; z) \tag{5.93}$$

Remember that $z = e^s$ and $x = \frac{1}{2} - \frac{1}{4}(z + z^{-1})$. If we change to the variable x we find that ϕ takes the form

$$\phi(x) = 4^{-a}(-x)^{-a}F(a, 1+a-c, 1+a-b; x^{-1}). \tag{5.94}$$

Likewise we get the same expression for ϕ^{new} with a, b and c replaced with a^{new} b^{new} and c^{new}. We note that $a^{\text{new}} = b$, $b^{\text{new}} = a$ and $c^{\text{new}} = c$. Hence all that remains to prove is that

$$ {}_2F_1(a,b,c;x) = \sigma\Phi + \sigma^{\text{new}}\Phi^{\text{new}} = \tag{5.95}$$

$$ = \frac{4^a\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} 4^{-a}(-x)^{-a} F(a, 1+a-c, 1+a-b; x^{-1}) + \tag{5.96}$$

$$ + \frac{4^b\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} 4^{-b}(-x)^{-b} F(b, 1+b-c, 1+b-a; x^{-1}). \tag{5.97}$$

We already know from the theory of the hypergeometric function in one dimension that $F(a,b,c;x)$ is the only solution (up to a multiplicative constant) to the hypergeometric differential equation that does not have a singularity at $x = 0$. All that remains is then to verify that this linear combination is in fact equal to the hypergeometric function. There are several ways to do that, but one involves the integral representation of F. The hypergeometric function can be written like

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b} dt, \tag{5.98}$$

Which is derived in [11]. That then tells us that

$$ F(a, 1+a-c, 1+a-b; x^{-1}) = \frac{\Gamma(1+a-b)}{\Gamma(a)\Gamma(1-b)} \int_0^1 t^{a-1}(1-t)^{-b}(1-tx^{-1})^{c-a-1} dt \tag{5.99}$$

and

$$ F(b, 1+b-c, 1+b-a; x^{-1}) = \frac{\Gamma(1+b-a)}{\Gamma(b)\Gamma(1-a)} \int_0^1 t^{b-1}(1-t)^{-a}(1-tx^{-1})^{c-b-1} dt. \tag{5.100}$$

If we plug this into the linear combination $\sigma\Phi + \sigma^{\text{new}}\Phi^{\text{new}}$ we see that things nicely cancel out and we get the relation

$$ \sigma\Phi + \sigma^{\text{new}}\Phi^{\text{new}} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b} dt =_2 F_1(a,b,c;x), \tag{5.101}$$

which finishes the proof. □

Remark 5.1.2. Note that we have now developed 2 different ways of finding both solutions to the hypergeometric differential equation. One was given in the proof of lemma 3.3.3. There we said that if we choose to factor out two different exponents at $T^{-1}(0)$ we got 2 different solutions. The other method we just saw was to replace λ_1 and λ_2 by λ_1^{new} and λ_2^{new}. We arrived at these two methods in very different ways. We notice however that these two methods are actually equivalent. The two characteristic exponents at ∞ are a and b. Theorem 3.3.3 therefore tells us that we get both solutions by swapping a and b, and leaving c unchanged. On the other hand if we replace λ_1 and λ_2 by λ_1^{new} and λ_2^{new} we get the relation $a^{\text{new}} = b$, $b^{\text{new}} = a$ and $c^{\text{new}} = c$. This method therefore also corresponds to swapping a and b and leaving c unchanged. We therefore notice that the solution method introduced in this chapter is equivalent to the method of Kummers 24 solutions introduced in theorem 3.3.3.

6

Hypergeometric functions in higher dimensions

In the previous chapter we saw that the system of equations

$$\begin{cases} M_2 f = \lambda_{M_2} f \\ P f = \lambda_P f. \end{cases} \tag{6.1}$$

gives us a new way to arrive at the hypergeometric function. Let $p_1 = x_1 + x_2$ and $p_2 = x_1^2 + x_2^2$, From example 4.2.8 and lemma 4.2.9 we know that $D_{p_2}^{A_1} = M_2$ and $D_{p_1}^{A_1} = P$. Therefore an equivalent system of equations is

$$\begin{cases} D_{p_2}^{A_1} f = \lambda_{p_2} f \\ D_{p_1}^{A_1} f = \lambda_{p_1} f \end{cases} \tag{6.2}$$

where $D_p^R = \mathrm{Res}(p(T_R))$ as defined in definition 4.2.7. It is also obvious that if p_0 constant, then every function is an eigenfunction to $D_{p_0}^R$ with eigenvalue equal to that constant. In particular the function f also has to satisfy

$$D_{p_0}^{A_1} f = \lambda_{p_0} f. \tag{6.3}$$

Hence (6.1) may again be rewritten as

$$\begin{cases} D_{p_2}^{A_1} f = \lambda_{p_2} f \\ D_{p_1}^{A_1} f = \lambda_{p_1} f \\ D_{p_0}^{A_1} f = \lambda_{p_0} f. \end{cases} \tag{6.4}$$

We recognise p_0, p_1 and p_2 from theorem 2.2.6 as the three generators of $\mathbf{C}[x]^{S_2}$. Hence it is a natural step to study operators on the form D_p, where $p \in \mathbf{C}[x]^{S_n}$ is an arbitrary symmetric polynomial.

Lemma 6.0.1. Let q_1 and q_2 be polynomials assume that f satisfies

$$\begin{cases} D_{q_1} f = \lambda_{q_1} f \\ D_{q_2} f = \lambda_{q_2} f \end{cases} . \tag{6.5}$$

Then f also satisfies

$$\begin{cases} D_{q_1+q_2} f = (\lambda_{q_1} + \lambda_{q_2}) f \\ D_{q_1 q_2} f = \lambda_{q_1} \lambda_{q_2} f \end{cases} . \tag{6.6}$$

Proof. We have

$$D_{q_1+q_2} = \text{Res}((q_1 + q_2)(T)) = \text{Res}(q_1(T)) + \text{Res}(q_2(T)) = D_{q_1} + D_{q_2}, \tag{6.7}$$

and

$$D_{q_1 q_2} = \text{Res}(q_1 q_2(T))(f) = \text{Res}(q_1(T))\text{Res}(q_2(T))(f) = D_{q_1} D_{q_2}. \tag{6.8}$$

Applying $D_{q_1+q_2}$ to f we get

$$D_{q_1+q_2} f = D_{q_1} f + D_{q_2} f = (\lambda_{q_1} + \lambda_{q_2}) f. \tag{6.9}$$

Furthermore when we apply $D_{q_1 q_2}$ to f we get

$$D_{q_1 q_2} f = D_{q_1} D_{q_2} f = D_{q_1} \lambda_{q_2} f. \tag{6.10}$$

Using the fact that D_{q_1} is linear we get

$$D_{q_1 q_2} f = \lambda_{q_2} D_{q_1} f = \lambda_{q_1} \lambda_{q_2} f. \tag{6.11}$$

□

Theorem 6.0.2. Let $\tilde{\lambda} = [\lambda_{p_1}, \lambda_{p_2}]$ and $\lambda_p = \tilde{p}(\tilde{\lambda})$ for each $p \in \mathbf{C}[x]^{S_2}$, where $\tilde{p}$ is the generating polynomial from definition 2.2.9. Assume that f satisfies equation (6.4), then f also solves the infinite system of equations

$$D_p^{A_1} f = \lambda_p f \quad \forall p \in \mathbf{C}[x]^{S_2}. \tag{6.12}$$

Proof. Let $p \in \mathbf{C}[x]^{S_2}$ be arbitrary. From corollary 2.2.8 we know that $p(x) = \tilde{p}(p_1, p_2)$, hence

$$D_p^{A_1} = \text{Res}\left(p(T_{A_1})\right) = \text{Res}\left(\tilde{p}(p_1(T_{A_1}), p_2(T_{A_1}))\right). \tag{6.13}$$

However we know that $\text{Res}(p_1(T_{A_1}) = D_{p_1}^{A_1}$ and $\text{Res}(p_2(T_{A_1})) = D_{p_2}^{A_1}$. We therefore get

$$D_p^{A_1} = \tilde{p}(D_{p_1}^{A_1}, D_{p_2}^{A_1}). \tag{6.14}$$

We apply $D_p^{A_1}$ to f and use the result from lemma 6.0.1 to conclude that

$$D_p^{A_1} f = \tilde{p}(D_{p_1}^{A_1}, D_{p_2}^{A_1}) f = \tilde{p}(\lambda_{p_1}, \lambda_{p_2}) f. \tag{6.15}$$

By definition $\tilde{p}(\lambda_{p_1}, \lambda_{p_2}) = \lambda_p$ and the result follows. □

We now know that the function $f(x_1, x_2)$ from the previous chapter has to solve the following infinite system of equations.

$$D_p^{A_1} f = \lambda_p f \quad \forall p \in \mathbf{C}[x]^{S_2}. \tag{6.16}$$

Let us summarize the connection between ${}_2F_1$ and system (6.16) into one remark.

Remark 6.0.3 (Connection between ${}_2F_1$ and system (6.16))**.** Let $f(x_1, x_2)$ be the unique solution to the infinite system of PDEs

$$D_p^{A_1} f = \lambda_p f \quad \forall p \in \mathbf{C}[x]^{S_2} \tag{6.17}$$

that does not have a singularity at Re(x_1)=Re(x_2). Let $r = x_1 + x_2$ and $s = x_1 - x_2$. Now consider $f(x_1(r,s), x_2(r,s))$ and let

$$g(s) = f(x_1(0,s), x_2(0,s)). \tag{6.18}$$

Furthermore let $x = \frac{1}{2} - \frac{1}{4}\left(e^{x_1 - x_2} + e^{x_2 - x_1}\right)$ and for a given vale of x let $s(x)$ denote the set of possible s values corresponding to that x. Then the hypergeometric function satisfies

$$F(x) = g(s(x)). \tag{6.19}$$

The fact that $g(s(x))$ even is a well defined function is highly non trivial, and it is essentially what we dedicated all of the previous chapter to. Perhaps even less trivial is the fact that $g(s(x))$ takes the form

$$g(s(x)) = \sum_{n=0}^{\infty} \frac{(\frac{\lambda_1 - \lambda_2}{2})_n (\frac{\lambda_2 - \lambda_1}{2} + 2k)_n}{(\frac{1}{2} + k)_n n!} x^n \tag{6.20}$$

where λ_1 and λ_2 are such that $\lambda_1^2 + \lambda_2^2 + 2k(\lambda_2 - \lambda_1) = \lambda_{x_1^2 + x_2^2}$ and $\lambda_1 + \lambda_2 = \lambda_{x_1 + x_2}$.

There is no doubt that this connection is not of any help in the 1 dimensional case, since we already have a good understanding of the hypergeometric differential equation and its solutions. However the advantage of remark 6.0.3 is that it is possible to generalize to higher dimensions. Let us think about possible generalizations.

Remark 6.0.4. In remark 6.0.3 there were a few seemingly arbitrary choices. A few things that might consider generalizing are the following

1. Consider Dunkl operators associated with other root systems than A_1,
2. Consider other classes of polynomials other than $\mathbf{C}[x]^{S_n}$,
3. Consider other region than $\{x \in \mathbf{C}^2 : \text{Re}(x_1) < \text{Re}(x_2)\}$,
4. Make other changes of variables.

It may seem as if we have many different choices of how to generalize the hypergeometric function. However we can not change the points in remark 6.0.4 independent of each other. Say for example that we replace A_1 by an other root system R, then the Dunkl operators T_ξ are only defined for $\xi \in R$. Therefore the system

$$D_p^R f = \lambda_p \quad \forall p \in \mathbf{C}[x]^{S_n} \tag{6.21}$$

is not well defined for all root systems R. In order for the operator D_p^R to be well defined p must be in the set $\mathbf{C}[x]^{\sigma_R}$. Hence point 1 and 2 above can not be changed independently of each other. However we saw in lemma 4.1.16 that σ_{A_1} and S_2 are isomorphic. Therefore it is a natural generalization to always consider polynomials in $\mathbf{C}[x]^{\sigma_R}$, which gives the following definition of the hypergeometric differential equation associated with an arbitrary root system.

Definition 6.0.5 (Hypergeometric system of equation)**.** The system of equations

$$D_p^R \Phi = \lambda_p \Phi \quad \forall p \in \mathbf{C}[x]^{\sigma_R}, \tag{6.22}$$

is called the hypergeometric system of differential equations corresponding to the root system R.

This system of equations has been studied by G.J Heckman [13] and E.M Opdam [5]. They have found the following groundbreaking results.

Theorem 6.0.6. For each root system R there are $|\sigma_R|$ linearly independent solutions to the system of equations

$$D_p^R \Phi = \lambda_p \Phi \quad \forall p \in \mathbf{C}[x]^{\sigma_R}. \tag{6.23}$$

Each such solution is analytic in all of W. Furthermore, there is only one solution (up to a multiplicative constant) to (6.23) that can be extended to an analytic function in all of ∂W as well.

Proof. For proof see [13]. □

Example 6.0.7. Consider theorem 5.0.2 in the case $R = A_1$. In that case we have $\sigma_{A_1} \approx S_2$, and $|S_2| = 2! = 2$. We also know from example 4.1.12 that the fundamental Weyly chamber for A_1 is $W = \{x \in \mathbf{C}^2; \mathrm{Re}(x_1) < \mathrm{Re}(x_2)\}$, and hence $\partial W = \{x \in \mathbf{C}^2; \mathrm{Re}(x_1) = \mathrm{Re}(x_2)\}$. Theorem 6.0.6 therefore says that there are 2 linearly independent solutions to the system of equations

$$D_p^{A_1} f = \lambda_p f \quad \forall p \in \mathbf{C}[x]^{S_2}, \tag{6.24}$$

that both of these solutions converge in the region $\{x \in \mathbf{C}^2; \mathrm{Re}(x_1) < \mathrm{Re}(x_2)\}$ and that only one linear combination of these solutions is analytic at $\{x \in \mathbf{C}^2; \mathrm{Re}(x_1) = \mathrm{Re}(x_2)\}$. These are the same as the results from the precious chapter, only now we have them formulated for an arbitrary root system.

Theorem 6.0.6 tells us that point 3 in remark 6.0.4 can not be changed independent of point 1. Hence it is a natural choice to always consider the set W for an arbitrary root system. Now the only point in remark 6.0.4 that we have not considered is point number 4. In remark 6.0.3 there were many changes of variables before we arrived at the hypergeometric function. It is not obvious what would be the corresponding changes of variables for other root systems. However note that these changes of variables were only done to get the problem in the form of a Fuchsian equation. This allowed us to use the theory of Fuchsian equations to say which form the solution must have. However there is no corresponding theory in higher dimensions. The classical way to solve this problem is therefore to not do any changes of variables at all. Remember from definition 1.1.1 that the class of hypergeometric functions include a broad class of function. Often the functions $f(x_1, x_2)$, $f(x_1(r, s), x_2(r, s))$, $g(s)$ and $F(x)$ from remark 6.0.3 are all considered hypergeometric functions associated with A_1. The general problem is to understand the solution set to the hypergeomtric system of equations. In the A_1 case it was helpful to change the system into a single Fuchsian equation. However there is no theory of Fuschian equations in higher dimensions. Therefore one has to resort to other methods to solve the problem. Therefore point 4 in remark 6.0.4 will not lead to any interesting generalizations of the hypergeomeric functions. With theorem 6.0.6 in mind, the most natural generalization of the hypergeometric function is the following.

Definition 6.0.8. Let R be a root system on V with fundamental Weyl chamber W. Furthermore let $x \in V$ and let $\Phi(x)$ be the unique solution to the infinite system of equations

$$D_p^R \Phi = \lambda_p \Phi \quad \forall p \in \mathbf{C}[\mathbf{x}]^{\sigma_R} \tag{6.25}$$

in W that does not have a singularity at ∂W. Then Φ is called the hypergeometric function associated with the root system R.

Remark 6.0.9. In general λ_p is defined in a very similar way as in the A_{n-1} case. That is, if $p_1, ..., p_r$ are generators for $\mathbf{C}[x]^{\sigma_R}$ then λ_p is defined as

$$\lambda_p = \tilde{p}(\lambda_{p_1}, ..., \lambda_{p_r}) \tag{6.26}$$

where $\tilde{p}$ is the generating polynomial for p with respect to the generators of $\mathbf{C}[x]^{\sigma_R}$.

We will not solve the general problem in definition 6.0.8. For more details on the hypergeometrc function associated with an arbitrary root system see for example [13], [8] and [5]. We will however limit our study to only consider the root system A_{n-1}, and try to explicitly write down the hypergeometric function in the following section.

Example 6.0.10. Let us try and consider a much simpler version of the hypergeometric system of differential equations, where the Dunkl operator T_i is replaced by $\frac{\partial}{\partial x_i}$. In that case, let $D = \left[\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right]$ and consider the infinite system of equations

$$p(D)\Phi(x) = \lambda_p \Phi(x) \quad \forall p \in \mathbf{C}[x]. \tag{6.27}$$

Note that there is no need to involve the operator Res here and that we can consider the whole class of polynomials $\mathbf{C}[x]$ in this case. What remains is to work out which values of λ_p give a consistent system. Let $p_1 \in \mathbf{C}[x]$ be arbitrary, we note that

$$p_1(D)\Phi(x) = p_1\left(\frac{\partial}{\partial x_1}\Phi(x), ..., \frac{\partial}{\partial x_n}\Phi(x)\right) = p(\lambda_1, ..., \lambda_n)\Phi(x), \tag{6.28}$$

where λ_i is such that

$$\frac{\partial}{\partial x_i}\Phi(x) = \lambda_i \Phi(x) \quad i = 1, ..., n. \tag{6.29}$$

The system of equations we want to solve is therefore

$$p(D) = p(\lambda)\Phi(x) \quad \forall p \in \mathbf{C}[x]. \tag{6.30}$$

However equation (6.30) is trivially solved by

$$\Phi(x) = e^{\langle \lambda, x \rangle} = e^{\lambda_1 x_x + ... \lambda_n x_n}. \tag{6.31}$$

If we instead considered a smaller set of polynomials, say $\mathbf{C}[x]^{S_n}$. Then the equation

$$p(D) = p(\lambda)\Phi(x) \quad \forall p \in \mathbf{C}[x]. \tag{6.32}$$

is still solved by

$$\Phi(x) = e^{\langle \lambda, x \rangle} = e^{\lambda_1 x_x + ... \lambda_n x_n}. \tag{6.33}$$

However this time there are $n!$ linearly independent solutions, because if we permute the λ_is we still get the same eigenvalues $p(\lambda)$. In general if we let σ be a group that acts on $\mathbf{C}^n$, then it is not so hard to see that the system of equations

$$p(D) = p(\lambda)\Phi(x) \quad \forall p \in \mathbf{C}[x]^\sigma \tag{6.34}$$

has $|\sigma|$ linearly independent solutions, those being

$$\Phi_s(x) = e^{\langle s(\lambda), x \rangle} \quad s \in \sigma. \tag{6.35}$$

This is because letting s act on λ for $s \in \sigma$ is by definition of $\mathbf{C}[x]^\sigma$ precisely all the ways we can act on λ and still leave $p(\lambda)$ invariant.

These results are precisely analogous for the hypergeometric functions and the corresponding Dunkl operators, and the reasoning is not so different. Although there are a lot of more details to fill in. However this example illustrates one example to think about the hypergeometric function. The hypergeometric function is to Dunkl operators what the exponential function is to partial derivatives, in the seance that they both span the eigenspace to polynomials of their respective operators.

6.1 The hypergeometric function associated with A_{n-1}

From theorem 4.1.16 we know that $\sigma_{A_{n-1}} \approx S_n$ and $W = \{x \in \mathbf{C}^n : i < j \implies \mathrm{Re}(x_i) < \mathrm{Re}(x_j)\}$. Theorem 6.0.6 therefore states that the system of equations

$$D_p^{A_{n-1}}\Phi = \lambda_p \Phi \quad \forall p \in \mathbf{C}[x]^{S_n}. \tag{6.36}$$

has $|S_n| = n!$ linearly independent solutions, that each such solution converge in the region $\{x \in \mathbf{C}^n : i < j \implies \mathrm{Re}(x_i) < \mathrm{Re}(x_j)\}$, and that there is only one linear combination of these solutions that is analytic at $\partial W = \{x \in \mathbf{C}^n : \mathrm{Re}(x_i) = \mathrm{Re}(x_j)$ for some i and $j\}$. In this section we will write out these functions explicitly, and hence construct the hypergeometric function associated with A_{n-1} for an arbitrary n.

We know that the generators of $\mathbf{C}[x]^{S_n}$ are $p_r(x) = x_1^r + ... + x_n^r$, for $r = 0, ..., n$ by theorem 2.2.6. Therefore equation 6.36 can be written as

$$D^{A_{n-1}}_{x_1^r+...+x_n^r}\Phi = \lambda_{x_1^r+...+x_n^r}\Phi \quad r = 1, ..., n. \tag{6.37}$$

Compared with the problem in 6.0.8 this is much simpler, because now we only have a finite list of equations, as opposed to an infinite system. However the equations in (6.37) still involves operators on the form

$$\mathrm{Res}\left(\sum_{i=1}^{n}(T_i)^r\right), \quad r = 0, ..., n. \tag{6.38}$$

We have this far only done this computation for $r = 0, 1, 2$. Unfortunately there is no neat formula for the rth power of a Dunkl operator.

Let us start by studying the case $r = 2$, in that case we have

$$D^{A_{n-1}}_{x_1^2+...+x_n^2} = M_n. \tag{6.39}$$

Let us start by finding the eigenfunctions to M_n, and see if these solutions also allow us to solve the problem for general r later on.

6.1.1 Calculating the eigenfunctions to M_n for general n

The calculations in this section are based on similar calculation in [8].

We want to find eigenfunctions to the operator

$$M_n = \Delta - 2k\sum_{i<j}\frac{1 + e^{x_i - x_j}}{1 - e^{x_i - x_j}}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right). \tag{6.40}$$

In the M_2 case we found the eigenfunctions by first reducing the problem to a Fuchsian equation. The we used the theory of Fuchsian equations developed in chapter

2 to prove which form the solutions must have. This was done in lemma 5.0.1. We then plugged that form into the equation and worked out the values of the parameters in the series. Unfortunately there is no theory of Fuchsian equations in higher dimensions. Hence the same approach will not work.

In higher dimensions we will solve the problem in reversed order. We will first assume that the solutions has a certain form, then we will work out the corresponding parameter values. Finally at the end will we show that the attained solution space has the right dimension, and that therefor every solution is accounted for. In the A_1 case we saw that the solution had the form

$$f(x_1, x_2) = e^{\lambda_1 x_1 + \lambda_2 x_2} \sum_{n=0}^{\infty} c_n e^{n(x_1 - x_2)}. \tag{6.41}$$

Using the slightly more compact notation of root systems we may as well write this function as

$$f(x) = e^{\langle x, \lambda \rangle} \sum_{\alpha \in Q_+} c_\alpha e^{\langle x, \alpha \rangle}, \tag{6.42}$$

where Q_+ is the positive root lattice. We will make the anzats (6.42) for other root systems as well. Let us write out (6.42) more explicitly for the root system A_{n-1}. Instead of summing over the positive root lattice we can sum over the set of simple roots (which is equivalent because every root is a sum of simple roots by lemma 4.1.9). Hence we are going to search for functions on the form

$$\Phi(x) = e^{\lambda_1 x_1 + ... + \lambda_n x_n} \sum_{k_1, ..., k_{n-1} \geq 0} c_{k_1, ..., k_{n-1}} e^{k_1(x_1 - x_2)} ... e^{k_{n-1}(x_{n-1} - x_n)}. \tag{6.43}$$

The eigenvalue λ_{M_n} can be found by considering the problem

$$[M_n - \lambda_{M_n}] \, \Phi = 0, \tag{6.44}$$

with M_n and Φ as above. Then if we let $e^{x_i - x_{i+1}} \to 0$ for all $i = 1, ..., n-1$ we see that in order to have equality we must have

$$\lambda_{M_n} = \lambda_1^2 + ... + \lambda_n^2 - k \sum_{i<j} (\lambda_i - \lambda_j) \tag{6.45}$$

What remains now is to calculate the coefficients $c_{k_1, ..., k_{n-1}}$.

Theorem 6.1.1. The function

$$\Phi(x) = e^{\lambda_1 x_1 + ... + \lambda_n x_n} \sum_{k_1, ..., k_{n-1} \geq 0} c_{k_1, ..., k_{n-1}} e^{k_1(x_1 - x_2)} ... e^{k_{n-1}(x_{n-1} - x_n)} \tag{6.46}$$

is an eigenfunction to M_n with corresponding eigenvalue

$$\lambda_{M_n} = \lambda_1^2 + ... + \lambda_n^2 - 2k \sum_{i<j} (\lambda_i - \lambda_j) \tag{6.47}$$

if and only if $c_{k_1,\ldots,k_{n-1}}$ satisfies the recursive relation

$$c_{k_1,\ldots,k_{n-1}}\left(k_1^2+2\lambda_1k_1+\sum_{i=2}^{n-1}\Big((k_i-k_{i-1})^2+2\lambda_i(k_i-k_{i-1})\Big)+k_{n-1}^2-2\lambda_nk_{n-1}\right)$$

$$-kc_{k_1,\ldots,k_{n-1}}\left((n-1)k_1+\sum_{i=2}^{n-2}((k_i-k_{i+1}-k_j+k_{j+1})+(n-1)k_{n-1}\right)$$

$$=2k\sum_{i<j}\sum_{m_1=0}^{k_i}\sum_{m_2=0}^{k_{i-1}}\sum_{m_3=0}^{k_j}\sum_{m_4=0}^{k_{j-1}}c_m(m_1-m_2-m_3+m_4+\lambda_i-\lambda_j), \tag{6.48}$$

where $c_m=c_{k_1,\ldots,k_{i-2},m_2,m_1,k_{i+1},\ldots,k_{j-2},m_4,m_3,k_{j+1},\ldots,k_{n-1}}$, and $k_0=k_n=0$. Furthermore, if $c_{0,\ldots,0}$ is fixed, then every $c_{k_1,\ldots,k_{n-1}}$ is uniquely determined by this relation.

Proof. We have

$$M_n=\Delta-2k\sum_{i<j}\frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right). \tag{6.49}$$

since $x\in W$ we know that $|e^{x_i-x_j}|<1$. Hence we can expand

$$\frac{1}{1-e^{x_i-x_j}} \tag{6.50}$$

as a geometric series. We get

$$M_n=\Delta-2k\sum_{i<j}\left((1+e^{x_i-x_j})\sum_{n'=0}^{\infty}e^{n'(x_i-x_j)}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\right)= \tag{6.51}$$

$$=\Delta-2k\sum_{i<j}\left(1+2\sum_{n'=1}^{\infty}e^{n'(x_i-x_j)}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\right). \tag{6.52}$$

Let us try and apply M_n to Φ. Applying the Laplace operator to Φ we get

$$\Delta\Phi(x)=\sum_{i'=1}^{n}\frac{\partial^2}{\partial x_{i'}^2}\Phi(x)= \tag{6.53}$$

$$=(\lambda_1^2+\ldots+\lambda_n^2)e^{\lambda_1x_1+\ldots+\lambda_nx_n}\sum_{k_1,\ldots,k_{n-1}\geq 0}c_{k_1,\ldots,k_{n-1}}e^{k_1(x_1-x_2)}\ldots e^{k_{n-1}(x_{n-1}-x_n)} \tag{6.54}$$

$$+\sum_{k_1,\ldots,k_n\geq 0}\Gamma(k_1,\ldots,k_n,\lambda_1,\ldots,\lambda_n)e^{k_1(x_2-x_1)}\ldots e^{k_{n-1}(x_n-x_{n-1})}=0 \tag{6.55}$$

where

$$\Gamma=c_{k_1,\ldots,k_n}\left(\sum_{i=1}^{n-1}(k_i^2-2\lambda_ik_i)+\sum_{i=2}^{n}(k_{i-1}^2+2\lambda_ik_{i-1})-2\sum_{i=2}^{n-1}(k_ik_{i-1})\right). \tag{6.56}$$

Now all that remains is to expand

$$-k\sum_{i<j}\left(1+2\sum_{n'=1}^{\infty}e^{n'(x_i-x_j)}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\right)\Phi(x). \tag{6.57}$$

If we assume that $k_0 = k_n = 0$ we get

$$\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Phi(x) = \tag{6.58}$$

$$e^{\lambda_1 x_1 + \ldots + \lambda_n x_n} \sum_{k_1,\ldots,k_{n-1} \geq 0} c_{k_1,\ldots,k_{n-1}}(k_i - k_{i-1} - k_j + k_{j-1}) e^{k_1(x_1-x_2)} \ldots e^{k_{n-1}(x_{n-1}-x_n)},$$

for all i and j. The term

$$-4k \sum_{i<j} \sum_{n'=1}^{\infty} e^{n'(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Phi(x), \tag{6.59}$$

expands to

$$4ke^{\lambda_1 x_1 + \ldots \lambda_n x_n} \sum_{i<j} \sum_{m_1=0}^{k_i} \sum_{m_2=0}^{k_{i-1}} \sum_{m_3=0}^{k_j} \sum_{m_4=0}^{k_{j-1}} c_m (m_1 - m_2 - m_3 + m_4 + \lambda_i - \lambda_j) e^{kx},$$

where $e^{kx} = e^{k_1(x_1-x_2)} \ldots e^{k_{n-1}(x_{n-1}-x_n)}$, $c_m = c_{k_1,\ldots,k_{i-2},m_2,m_1,k_{i+1},\ldots,k_{j-2},m_4,m_3,k_{j+1},\ldots,k_{n-1}}$ and it is understood that $k_0 = k_n = 0$. Finally we can plug all of these expanded forms into the equation

$$M_n \Phi = \lambda_{M_n} \Phi. \tag{6.60}$$

Combining like terms into one sum

$$\sum_{k_1,\ldots,k_{n-1} \geq 0} \alpha(k_1, \ldots, k_{n-1}) e^{k_1(x_1-x_2)} \ldots e^{k_{n-1}(x_{n-1}-x_n)} = 0 \tag{6.61}$$

gives a system of equations

$$\alpha(k_1, \ldots, k_{n-1}) = 0 \quad \forall k_1, \ldots, k_{n-1} \geq 0 \tag{6.62}$$

which is precisely equivalent to the desired recursive relation. The fact that every $c_{k_1,\ldots,k_{n-1}}$ is uniquely determined by $c_{0,\ldots,0}$ follows since the equation for $c_{k_1,\ldots,k_{n-1}}$ only involves parameters $c_{\tilde{k_1},\ldots,\tilde{k_{n-1}}}$ where $\tilde{k_i} < k_i$ for at least one index. Hence, by recursion $c_{0,\ldots,0}$ uniquely determines every $c_{k_1,\ldots,k_{n-1}}$. □

Remark 6.1.2. Note that in the case $n = 2$ the recursive relation in 6.1.1 reduces to

$$c_{k_1} \left(2\lambda_1 k_1 + k_1^2 - 2\lambda_2 k_1 \right) - kc_{k_1} (k_1 + k_1) = \sum_{m_1=0}^{k_1-1} c_{m_1}((\lambda_1 - \lambda_2) + 2m_1). \tag{6.63}$$

Note that is is the same as in theorem 5.0.3, only here we use k_1 instead of n because n is reserved for the dimension in the general case.

What remains now is to verify the convergence of Φ.

Theorem 6.1.3. The function Φ is convergent in W.

Proof. Call the right hand side of 6.48 RH and call the left hand side LH. Note that there exists constants C_1 and C_2 such that

$$\mathrm{RH} \geq C_1 \left(\sum |k_i - k_{i+1}|\right)^2 |c_{k_1,\ldots,k_{k-1}}| \tag{6.64}$$

and

$$\mathrm{LH} \leq C_2(1 + \sum |k_i - k_{i+1}|)|2k \sum_{i=1}^{n-2} \sum_{m_1=0}^{k_i-1} \sum_{m_2=0}^{k_{i+1}-1} |c_{k_1,\ldots,k_{i-1},m_1,m_2,k_{i+2}\ldots,k_{n-1}}| \tag{6.65}$$

This allows us to deduce

$$C_1 \left(\sum |k_i - k_{i+1}|\right)^2 |c_{k_1,\ldots,k_{n-1}}| \tag{6.66}$$

$$\leq C_2(1 + \sum |k_i - k_{i+1}|)|2k \sum_{i=1}^{n-2} \sum_{m_1=0}^{k_i-1} \sum_{m_2=0}^{k_{i+1}-1} |c_{k_1,\ldots,k_{i-1},m_1,m_2,k_{i+2}\ldots,k_{n-1}}|. \tag{6.67}$$

As long as $\sum |k_i - k_{i+1}| > 0$ we have $1 + \sum |k_i - k_{i+1}| \leq 2 \sum |k_i - k_{i+1}| > 0$. Hence with $C = \frac{2C_2}{C_1}$ we have

$$|c_{k_1,\ldots,k_{n-1}}| \leq \frac{C}{\sum |k_i - k_{i+1}|} 2k \sum_{i=1}^{n-2} \sum_{m_1=0}^{k_i-1} \sum_{m_2=0}^{k_{i+1}-1} |c_{k_1,\ldots,k_{i-1},m_1,m_2,k_{i+2}\ldots,k_{n-1}}|. \tag{6.68}$$

Let $\epsilon_1, \ldots, \epsilon_{n-1} > 0$ and let N_0 be an integer such that

$$2Ck \sum_{i_1,\ldots,i_{n-1}>0} e^{-i_1\epsilon_1 - \ldots - i_{n-1}\epsilon_{n-1}} < N_0. \tag{6.69}$$

Now choose K such that

$$|c_{k_1,\ldots,k_{n-1}}| < Ke^{k_1\epsilon_1 + \ldots + k_{n-1}\epsilon_{n-1}} \quad \forall k_1, \ldots, k_{n-1} : \sum |k_i - k_{i+1}| < N_0. \tag{6.70}$$

We want to prove that $|c_{k_1,\ldots,k_{n-1}}| < Ke^{k_1\epsilon_1 + \ldots + k_{n-1}\epsilon_{n-1}}$ for all $k_1, \ldots, k_{n-1}$. We do the proof by induction. Let $N > N_0$ and suppose that $|c_{k_1,\ldots,k_{n-1}}| < Ke^{k_1\epsilon_1 + \ldots + k_{n-1}\epsilon_{n-1}}$ for all $k_1, \ldots, k_{n-1}$ such that $\sum |k_i - k_{i+1}| < N$. Let $\bar{k_1}, \ldots, \bar{k_{n-1}}$ be such that $\sum |\bar{k_i} - \bar{k_{i+1}}| = N$. Then

$$|c_{\bar{k_1},\ldots,\bar{k_{n-1}}}| \leq \frac{C}{N} 2k \sum_{i=1}^{n-2} \sum_{m_1=0}^{\bar{k_i}-1} \sum_{m_2=0}^{\bar{k_{i+1}}-1} |c_{\bar{k_1},\ldots,\bar{k_{i-1}},m_1,m_2,\bar{k_{i+2}}\ldots,\bar{k_{n-1}}}|$$

$$= \frac{C}{N} 2k \sum_{i=1}^{n-2} \sum_{m_1=0}^{\bar{k_i}-1} \sum_{m_2=0}^{\bar{k_{i+1}}-1} |c_{\bar{k_1},\ldots,\bar{k_{i-1}},\bar{k_i}-m_1,\bar{k_{i+1}}-m_2,\bar{k_{i+2}}\ldots,\bar{k_{n-1}}}|$$

$$\leq \frac{C}{N}2k\sum_{i=1}^{n-2}\sum_{m_1=0}^{\bar{k}_i-1}\sum_{m_2=0}^{\bar{k}_{i+1}-1}Ke^{\bar{k}_1\epsilon_1+...+\bar{k}_{i-1}\epsilon_{i-1}+(\bar{k}_i-m_1)\epsilon_i+(\bar{k}_{i+1}-m_2)\epsilon_{i+1}+\bar{k}_{i+2}\epsilon_{i+2}+...+\bar{k}_{n-1}\epsilon_{n-1}}.$$

We notice that we can make the sum bigger by summing up to ∞ instead of $k_i - 1$ and $k_{i+1} - 1$. Pull everything that does not depend on m_1 and m_2 out of the sum and we get

$$|c_{\bar{k}_1,...,\bar{k}_{n-1}}| \leq Ke^{\bar{k}_1\epsilon_1+...+\bar{k}_{n-1}\epsilon_{n-1}}\frac{C}{N}2k\sum_{i_1,...,i_{n-1}>0}e^{-i_1\epsilon_1-...-i_{n-1}\epsilon_{n-1}} \tag{6.71}$$

$$\leq Ke^{\bar{k}_1\epsilon_1+...+\bar{k}_{n-1}\epsilon_{n-1}}\frac{N_0}{N} \leq Ke^{\bar{k}_1\epsilon_1+...+\bar{k}_{n-1}\epsilon_{n-1}}. \tag{6.72}$$

From this estimate it directly follows that Φ converges in all of W. □

Remark 6.1.4. We still have not shown that functions Φ on this form span the full set of eigenfunctions. Only that Φ is an eigenfunction in the region W. It is in fact the case that the entire eigenspace are spanned by functions on this form. We will come back to this in corollary 6.1.11.

6.1.2 Operators of higher degree

We have found a set of eigenfunctions to the operator $D^{A_{n-1}}_{x_1^2+...+x_n^2} = M_n$. We will now find eigenfunctions to $D^{A_{n-1}}_{x_1^r+...+x_n^r}$ for all other $r = 1, ...n$. However much of the work is already done. It turns out that we can reuse the solution Φ corresponding to M_n under some restriction of the parameter values $\lambda_1, ..., \lambda_n$. We begin by stating some useful results.

Lemma 6.1.5. If Φ is an eigenfunction to D^R_p, then $D^R_q\Phi$ is an eigenfunction to D^R_p for every polynomial q in the same number of variables as p

Proof. Note that D^R_p and D^R_q commute. This is because T_i and T_j commute for all i and j by theorem 4.2.5. Since all components commute, every polynomial in these components must commute. Hence we have

$$D^R_pD^R_q\Phi = D^R_qD^R_p\Phi = \lambda_pD^R_q\Phi. \tag{6.73}$$

□

Lemma 6.1.6. Let W be the fundamental Weyl chamber for A_{n-1} and let $q \in \mathbf{C}[x]^{S_n}$ be an arbitrary symmetric polynomial on $x \in W$. If $\Phi(x)$ is on the form

$$\Phi(x) = e^{\langle\lambda,x\rangle}\sum_{\alpha\in Q_+}c_\alpha e^{\langle\alpha,x\rangle}, \tag{6.74}$$

then $D_q^{A_{n-1}}\Phi(x)$ is also on the form

$$D_q^{A_{n-1}}\Phi = e^{\langle\tilde{\lambda},x\rangle} \sum_{\alpha\in Q_+} \tilde{c_\alpha} e^{\langle\alpha,x\rangle} \tag{6.75}$$

for possibly other parameter values $\tilde{\lambda}$ and $\tilde{c_\alpha}$.

Proof. By definition of $D_p^{A_{n-1}}$ an using theorem 2.2.6 we know that $D_p^{A_{n-1}}$, $p \in \mathbf{C}[x]^{S_n}$ are generated by operators on the form

$$D^{A_{n-1}}_{x_1^m+...+x_n^m} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + 2k \sum_{i<j} \frac{1+e^{x_i-x_j}}{1-e^{x_i-x_j}}(s_{ij}-1) \right)^m \quad m = 1, ..., n. \tag{6.76}$$

Since $x \in W$ we know that $|e^{x_i-x_j}| < 1$ whenever $i < j$, hence

$$i < j \implies \frac{1}{1-e^{x_1-x_2}} = \sum_{r=0}^{\infty} e^{r(x_i-x_j)}. \tag{6.77}$$

If we plug this into equation (6.76) we get

$$D^{A_{n-1}}_{x_1^m+...+x_n^m} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + 2k \sum_{i<j} \left(\left(1 + 2\sum_{r=1}^{\infty} e^{r(x_i-x_j)} \right) (s_{ij}-1) \right) \right)^m, \tag{6.78}$$

for $m = 1, ..., n$. We notice that the set of functions $\{ce^{k_1x_1+...+k_nx_n}\}$ is closed under action of the operators $\frac{\partial}{\partial x_i}$, s_{ij} and multiplication by $e^{m(x_i-x_j)}$. Hence if (6.78) is applied to

$$\Phi(x) = e^{\langle\lambda,x\rangle} \sum_{\alpha\in Q_+} c_\alpha e^{\langle\alpha,x\rangle} \tag{6.79}$$

the result is still going to be on the form

$$D_q^{A_{n-1}}\Phi = e^{\langle\tilde{\lambda},x\rangle} \sum_{\alpha\in Q_+} \tilde{c_\alpha} e^{\langle\alpha,x\rangle} \tag{6.80}$$

for possibly other parameter values $\tilde{\lambda}$ and $\tilde{c_\alpha}$. □

Theorem 6.1.7. The function Φ as defined in theorem 6.1.1 is an eigenfunction to D_q for every symmetric polynomial $q \in \mathbf{C}[x]^{S_n}$.

Proof. If we let $p = x_1^2 + ... + x_n^2$ and with Φ as in theorem 6.1.1 we see that lemma 6.1.5 implies that $D_q^{A_{n-1}}\Phi$ is an eigenfunction to M_n. However by lemma 6.1.6 we know that $D_q^{A_{n-1}}\Phi$ is on the form

$$D_q^{A_{n-1}}\Phi = e^{\langle\tilde{\lambda},x\rangle} \sum_{\alpha\in Q_+} \tilde{c_\alpha} e^{\langle\alpha,x\rangle}. \tag{6.81}$$

By theorem 6.1.1 we know that any function on the form 6.81 that is also an eigenfunction of M_2 must satisfy the recursive relation (6.48). Also from theorem 6.1.1

we know that every $\tilde{c_\alpha}$ is determined from $\tilde{c_0}$. Hence we get the relations $\tilde{\lambda} = \lambda$ and $\tilde{c_\alpha} = \tilde{c_0}\frac{c_\alpha}{c_0}$. This leads to the relation

$$D_q^{A_{n-1}}\Phi = \tilde{c_0}\Phi. \tag{6.82}$$

Hence Φ is an eigenfunction to $D_q^{A_{n-1}}$. □

Remark 6.1.8. It may seem pointless to consider the entire system of equations

$$D_p^{A_{n-1}}\Phi = \lambda_p\Phi \quad \forall p \in \mathbf{C}[x]^{S_n} \tag{6.83}$$

if the solution to one of these equations is automatically the solution to every equation. However that is not the case. Note that theorem 6.1.7 states that Φ is an eigenfunction to every operator $D_p^{A_{n-1}}$. However equation (6.83) does not only ask us to fins eigenfunctions, it asks us to find eigenfunctions corresponding to fixed eigenvalues. Hence each equation will still determine which set of parameter values $\lambda_1, ..., \lambda_n$ are possible. Compare this to the A_1 case, then we solved the system

$$\begin{cases} M_2 f = \lambda_{M_2} f \\ Pf = \lambda_P f \end{cases}. \tag{6.84}$$

The first equation $M_2 f = \lambda_{M_2} f$ was used to determine the form of the solution, however the second equation $Pf = \lambda_P f$ still played an important role in setting criterion on λ_1 and λ_2. This is a general phenomenon in higher dimensions as well.

All that remains to solve the hypergeometric system of equations is therefore to calculate which values of $\lambda_1, ..., \lambda_n$ make Φ an eigenfunction to D_p with eigenvalue λ_p for all symmetric polynomials p.

Theorem 6.1.9. Let Φ be as in theorem 6.1.1. Let $\lambda = \{\lambda_i + k(2i-n-1)\}_{i=1}^n$ and $\mathbf{k} = \{k(2i-n-1))\}$. Then Φ solves the system of equations

$$D_{x_1^r+...+x_n^r}^{A_{n-1}}\Phi = \lambda_{x_1^r+...+x_n^r}\Phi \quad r = 1, ..., n \tag{6.85}$$

if and only if $\lambda_1, ..., \lambda_n$ satisfy

$$p(\lambda) - p(\mathbf{k}) = \lambda_p \quad \forall p \in \mathbf{C}[x]^{S_n}. \tag{6.86}$$

Proof. Plugging Φ into the equation

$$D_p\Phi = \lambda_{D_p}\Phi \tag{6.87}$$

and letting $e^{\langle\alpha,x\rangle} \to 0$ all that remains are the constant terms

$$p(\lambda) - p(\mathbf{k}) = \lambda_{D_p} \tag{6.88}$$

□

A consequence of theorem 6.1.9 is that we can now work out the span of all such solutions Φ.

Corollary 6.1.10. For each fixed n tuple

$$\lambda_{x_1+...+x_n}, ..., \lambda_{x_1^n+...+x_n^n} \in \mathbf{C} \tag{6.89}$$

there are $n!$ linearly independent functions $\Phi(x)$ such that

$$D^{A_{n-1}}_{x_1^i+...+x_n^i}\Phi(x) = \lambda_{x_1^i+...+x_n^i}\Phi(x) \quad \forall i = 1, ..., n, \tag{6.90}$$

and where $\Phi(x)$ is in the form from theorem 6.1.1.

Proof. We know from theorem 6.1.7 that $\Phi(x)$ is an eigenfunction to each such operator. And by theorem 6.1.9 we know that $\Phi(x)$ is an eigenfunction with eigenvalue λ_p if the parameters $\lambda_1, ..., \lambda_n, k$ satisfy

$$p(\lambda) - p(\mathbf{k}) = \lambda_p. \tag{6.91}$$

It is clear that if $\Phi_1(x)$ and $\Phi_2(x)$ are both in the right form, but for different parameter values, then $\Phi_1(x)$ and $\Phi_2(x)$ are linearly independent. Therefore what remains is to prove that there are $n!$ unique solutions to

$$p(\lambda) - p(\mathbf{k}) = \lambda_p \quad p = x_1^i + ... + x_n^i \quad i = 1, .., n. \tag{6.92}$$

The polynomials p are by definition symmetric. Furthermore Φ need to be an eigenfunction to D_p for every symmetric polynomial. Hence the only way to leave $p(\lambda) - p(\mathbf{k})$ invariant is to permute the arguments of λ and $\mathbf{k}$. There are $n!$ such permutations, and hence $n!$ linearly independent solutions. □

This far we have not shown that the functions Φ span the full set of solutions. However with corollary 6.1.10 we are ready to do so.

Corollary 6.1.11. The functions $\Phi(x)$ where Φ is on the form from theorem 6.1.1 and where the parameters $\lambda_1, ..., \lambda_n, k$ satisfy

$$p(\lambda) - p(\mathbf{k}) = \lambda_p \quad \forall p \in \mathbf{C}[x]^{S_n} \tag{6.93}$$

span the full set of solutions to the hypergeometric system of equations for A_{n-1}.

Proof. We already know that the span of all such $\Phi(x)$ is of dimension $n!$. What remains to show is that the set of solutions to the hypergeometric system of equations is also of dimension $n!$. This can be done by rewriting the system as a matrix differential equation, where the relevant matrix is of size $n! \times n!$, and hence by theorem 2.1.2 has $n!$ linearly independent solutions. However we will not go through the details here, for more details see [8]. □

Remark 6.1.12. In the A_1 case we saw that there were 2 linearly independent solutions. We knew that λ_1 and λ_2 satisfied

$$\lambda_{M_2} = \lambda_1^2 + \lambda_2^2 + 2k(\lambda_2 - \lambda_1). \tag{6.94}$$

Rewriting a bit we get

$$\lambda_{M_2} = (\lambda_1 - k)^2 + (\lambda_2 + k)^2 - 2k^2, \tag{6.95}$$

which is precisely on the form $p(\lambda) - p(\mathbf{k})$. We saw earlier that the only way to get an other solutions that satisfy the whole hypergeometric system of equation was to let

$$\lambda_1^{\text{new}} = \lambda_2 + 2k \tag{6.96}$$

and

$$\lambda_2^{\text{new}} = \lambda_1 - 2k. \tag{6.97}$$

If we look back at equation (6.95) we see that this corresponds to swapping the values in the first and the second parenthesise, which according to theorem 6.1.9 is how we get all solutions.

Remark 6.1.13. In equation (6.45) we saw that $\lambda_1, ..., \lambda_n$ had to satisfy

$$\lambda_{M_n} = \lambda_1^2 + ... + \lambda_n^2 - 2k \sum_{i<j} (\lambda_i - \lambda_j). \tag{6.98}$$

If we complete the square we get the relation

$$\lambda_{M_n} = (k(1-n))^2 + (k(3-n))^2 + ... + (k(n-1))^2 - (k(1-n))^2 - ... - (k(1-n))^2$$

which is precisely in the form $p(\lambda) - p(\mathbf{k})$.

6.1.3 Constructing the right linear combination

We now know that there are $n!$ linearly independent solutions to the hypergeometric system of differential equations corresponding to A_{n-1}. What remains is to find which linear combination of these solutions is well behaved at ∂W.

Theorem 6.1.14. For $s \in S_n$ and $\alpha \in A_{n-1}$ we define

$$\delta_s(\alpha) = \begin{cases} 0, & \text{if } s(\alpha) \in R_+ \\ 1, & \text{if } s(\alpha) \in R_- \end{cases} . \tag{6.99}$$

Furthermore let

$$c_s(\lambda, k) = \prod_{\alpha \in R_+} \frac{2^{\langle \alpha, \lambda \rangle} \Gamma(\frac{1}{2} + 2k) \Gamma(2k - \langle \alpha, \lambda \rangle)}{\Gamma(2k - \frac{\langle \alpha, \lambda \rangle}{2}) \Gamma(2k + \frac{\delta_s(\alpha) - \langle \alpha, \lambda \rangle}{2})}. \tag{6.100}$$

Then

$$F(x) = \sum_{s \in S_n} c_s(s(\lambda), k) \Phi(s(\lambda), k; x) \tag{6.101}$$

is the only linear combination of

$$\Phi(s(\lambda), k; x) \quad s \in S_n \tag{6.102}$$

that does not have a singularity at ∂W.

Proof. For a proof see [13]. □

The proof of theorem 6.1.14 is very technical and unfortunately outside the scope of this project. We instead refer the reader to [13]. We will however consider a few special cases of theorem 6.1.14, to at least get a feeling why this is a plausible result.

Example 6.1.15. Let us consider theorem 6.1.14 in the A_1 case. Then group S_2 has 2 elements, the identity s_1 and the permutation $s_2 = (2\ 1)$. With $\lambda = (\lambda_1 - k, \lambda_2 + k)$ we see that $\Phi(s_1(\lambda), k; x)$ and $\Phi(s_2(\lambda), k; x)$ correspond to Φ and Φ^{new} from theorem 5.1.1 respectively. Furthermore the only positive root associated with A_1 is $\alpha = (1, -1)$. We get the relations

$$\langle \alpha, \lambda \rangle = \lambda_1 - \lambda_2 \tag{6.103}$$

and

$$\langle \alpha, \lambda^{\text{new}} \rangle = \langle (1, -1), (\lambda_2 + 2k, \lambda_1 - 2k) \rangle = \lambda_2 - \lambda_1 + 4k. \tag{6.104}$$

The unique linear combination in theorem 6.1.14 is therefore

$$F(x) = \frac{2^{\lambda_1 - \lambda_2} \Gamma(\frac{1}{2} + 2k) \Gamma(\lambda_2 - \lambda_1 + 2k)}{\Gamma(\frac{\lambda_2 - \lambda_1}{2} + 2k) \Gamma(2k + \frac{1 + \lambda_2 - \lambda_1}{2})} \Phi + \frac{2^{\lambda_2 - \lambda_1 + 4k} \Gamma(\frac{1}{2} + 2k) \Gamma(\lambda_1 - \lambda_2 - 2k)}{\Gamma(\frac{\lambda_1 - \lambda_2}{2}) \Gamma(\frac{\lambda_1 - \lambda_2}{2})} \Phi^{\text{new}}.$$

If we set $a = \frac{\lambda_1 - \lambda_2}{2}$, $b = \frac{\lambda_2 - \lambda_1}{2} + 2k$ and $c = \frac{1}{2} + 2k$ we see that

$$F(x) = \frac{4^a \Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} \Phi + \frac{4^b \Gamma(c) \Gamma(a - ab)}{\Gamma(ba) \Gamma(c - b)} \Phi^{\text{new}}, \tag{6.105}$$

which is precisely the same linear combination as in theorem 5.1.1.

The notation λ_j^{new}, $j = 1, 2$ was convenient in the previous chapter because there were only two linearly independent solutions to the hypergeometric equation. Those correspond to λ_j $j = 1, 2$, and λ_j^{new} $j = 1, 2$, respectively. However in the general case there are $n!$ solutions, one corresponding to each $s \in S_n$. Hence the notation λ_j^s, $j = 1, ..., n$ is more usefull in this case.

Definition 6.1.16. The value $\lambda_j^s \in \mathbf{C}$, $j = 1, ..., n$, $s \in S_n$ is the unique complex number such that the jth component of the vectors

$$((\lambda_i + k(2i - n - 1))_{i=1}^n)_j = (s((\lambda_i + k(2i - n - 1))_{i=1}^n))_j \tag{6.106}$$

coincide.

In other words, λ_j^s is the value such that if we replace λ_j by λ_j^s it has the same effect on the solution as applying s to $\lambda = \{\lambda_i + k(2i - n - 1)\}_{i=1}^n$. When we write down explicit solutions Φ to the hypergeometric system of equations it is simpler to work with λ_j^s $j = 1, ..., n$ because it is easier to replace λ_j by an other number, than to think of a permutation of the whole vector $(\lambda_i + k(2i - n - 1))_{i=1}^n$. Similarly to how we solved the problem in the A_1 case. Although theorem 6.1.14 becomes a lot more compact when formulated in terms of $s(\lambda)$ rather than λ_j^s, $j = 1, ..., n$.

Example 6.1.17. Consider the A_2 case. Then $\lambda = (\lambda_1 - 2k, \lambda_2, \lambda_3 + 2k)$. Let us consider the permutation $s_1 = (1\ 3\ 2)$, we want to work out expressions for $\lambda_1^{s_1}$, $\lambda_2^{s_1}$ and $\lambda_3^{s_1}$. By definition they need to satisfy

$$\begin{cases} \lambda_1^{s_1} - 2k = \lambda_1 - 2k \\ \lambda_2^{s_1} = \lambda_3 + 2k \\ \lambda_3^{s_1} + 2k = \lambda_2 \end{cases}. \tag{6.107}$$

Hence we get the result

$$\begin{cases} \lambda_1^{s_1} = \lambda_1 \\ \lambda_2^{s_1} = \lambda_3 + 2k \\ \lambda_3^{s_1} = \lambda_2 - 2k \end{cases}. \tag{6.108}$$

Example 6.1.18. Let us consider theorem 6.1.14 in the A_2 case. Then there are $3! = 6$ linearly independent solutions each corresponding to a permutation $s \in S_3$. The solutions take the form

$$\Phi(s(\lambda), k; x) = e^{\lambda_1^s x_1 + \lambda_2^s x_2 + \lambda_3^s x_3} \sum_{k_1, k_2 \geq 0} c_{k_1, k_2} e^{k_1(x_1 - x_2)} e^{k_2(x_2 - x_3)}. \tag{6.109}$$

To verify theorem 6.1.14 even in this case is very technical. However if we consider the extreme case where $\mathrm{Re}(x_1) << \mathrm{Re}(x_2) \approx \mathrm{Re}(x_3)$, and take a limit as $x_1 \to 0$ and $\mathrm{Re}(x_2) \to \infty$, then

$$e^{k_1(x_1-x_2)} = \begin{cases} 1 & \text{if } k_1 = 0 \\ 0 & \text{otherwise} \end{cases} . \tag{6.110}$$

In this case (6.109) therefore reduces to

$$\Phi(s(\lambda), k; x) = e^{\lambda_2^s x_2 + \lambda_3^s x_3} \sum_{k_2 \geq 0} c_{0,k_2} e^{k_2(x_2-x_3)}. \tag{6.111}$$

At this scale there are basically only going to be 2 linearly independent solutions, corresponding to the identity map s_0 and the permutation $s_1 = (1\ 3\ 2)$. From example 6.1.17 we know that

$$\begin{cases} \lambda_1^{s_1} = \lambda_1 \\ \lambda_2^{s_1} = \lambda_3 + 2k \\ \lambda_3^{s_1} = \lambda_2 - 2k \end{cases} . \tag{6.112}$$

Renaming $\tilde{x_1} = x_2$, $\tilde{x_2} = x_3$, $\tilde{\lambda_1} = \lambda_2$ and $\tilde{\lambda_2} = \lambda_3$ we see that the set of solutions reduces to that of A_1. Equation (6.112) is equivalent to

$$\begin{cases} \tilde{\lambda_1}^{\text{new}} = \tilde{\lambda_2} + 2k \\ \tilde{\lambda_2}^{\text{new}} = \tilde{\lambda_1} - 2k \end{cases} . \tag{6.113}$$

However we already know from example 6.1.15 that theorem 6.1.14 works in the A_1 case with this relation. We have therefore showed that theorem 6.1.14 holds for A_2 as well in a very particular limiting case.

Remark 6.1.19. The function F as defined in theorem 6.1.14 is called the Hypergeometric function associated with A_{n-1}.

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