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A Tensor Formalism for Exceptional Geometry

Thesis for the degree Master of Science in Fundamental Physics

JOAKIM EDLUND

Department of Fundamental Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2014

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Abstract

Toroidal compactification of M-theory and its low energy limit, eleven-dimensional supergravity, possess hidden symmetries giving fields additional degrees of freedom. By extending the space-time to accomodate these extra degrees of freedom and constructing a generalised geometry on this space, the U-duality symmetry can be made manifest. The local diffeomorphism invariance is replaced by the larger exceptional groups which also happens to include gauge transformations. In this thesis, a tensor calculus for the exceptional generalised geometry is constructed. The geometrical concepts of diffeomorphisms, torsion, curvature, reducibility, tensors and tensor fields are given a generalised, covariant construction in the toroidally compactified, enhanced space-directions.

Acknowledgements

I want to thank my supervisor Martin Cederwall for all help and discussions while introducing me to the subject of this thesis. I also give my thanks to the other faculty members and students at the department for providing a friendly environment. I'm especially grateful for the collaboration with Anna Karlsson and for the discussions with Oskar Till. Thank you Erik Widén for proof-reading and giving feedback before my presentation.

Joakim Edlund, Göteborg 2014-03-25

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Chapter 1

Introduction

In the beginning of the 1990's, Hull, Townsend and Witten showed that the five known anomaly-free, perturbative superstring theories were connected and related to each other by dualities [1–3]. They could be thought of as different languages describing the same physics. At the low energy limit of the string theories are their respective supergravity field theories, which are also related in the same way by the dualities.

The superstring theories are only anomaly-free in 10 space-time dimensions, but it was known that a dimensional reduction of the 11-dimensional supergravity [4] theory on a circle gives the type IIA supergravity. Eleven is also the highest number of dimensions in which there exists a supergravity theory with Minkowskian metric containing no particles with spin higher than 2 [5–7]. The supergravity theory found in eleven dimensions also happens to be unique. These evidences led to the thoughts that the eleven-dimensional supergravity theory was in fact the low energy limit of a larger theory which was named M-theory. In this view, the genus expansion of each of the string theories corresponds to a different perturbative series in a particular limit of the string coupling constant, $g_s \to 0$, in the parameter space of M-theory.

Defining M-theory by the expansions of the superstring on each patch, using the dualities to move between patches, makes it possible to uncover some of the features of M-theory. The string theories are, however, only defined as asymptotic series in the low coupling limit, so all of the parameter space is not reachable by this approach.

The low energy limit of M-theory was proven to be the eleven-dimensional supergravity, since it is unique. By studying the low energy limits, dimensional reductions thereof and the duality symmetries, a lot of the structure of M-theory can be understood. The circle dimensional reduction of M-theory down to type IIA superstring theory and the branching rules of the field contents, showed that M-theory is not a string theory, the fundamental objects are higher-dimensional branes.

The dualities between different string theories relates different languages for the same physics while the symmetries of the theories can be used as a guide into M-theory. For example, a T-duality (the inversion of a radius on a d-dimensional torus) maps to each other the type IIA and IIB string theories, while a T-symmetry is an even number of such T-dualities (taking us back to the original theory). The type II theories and their M-theory extensions therefore has to be invariant under this T-symmetry of an even number of T-duality transformations.

The T-duality symmetry group of the type II string theories on a torus T^d is $O(d, d, \mathbb{Z})$, and the continuous version of this, $O(d, d, \mathbb{R})$ [8], is the symmetry of the low energy effective actions. T-duality is known to be the perturbative part of a larger duality, U-duality, which is a unification of the T- and S-dualities [9–13].

Cremmer and Julia, [14], discovered that the action of eleven-dimensional or type IIA supergravity compactified on torii, T^d, and the equation of motions for uncompactified type IIB supergravity exhibits continuous, global, non-compact symmetries described by the exceptional groups, $E_{d(d)}(\mathbb{R})$ and $Sl(2, \mathbb{R})$ [15–18]. These symmetries are out of reach for T-duality since they transform the scalar fields and the weak coupling regime. A discrete version of the exceptional groups, $E_{d(d)}(\mathbb{Z})$ [19], are identified as the U-duality group.

The duality symmetries can be made manifest by enlargening the dimension of the compactified space, accounting for the extra degrees of freedom introduced by the symmetries. This concept was first examined for T-duality and later for U-duality. The T-duality generalised space-time is doubled and is because of that called doubled geometry [20–35]. The U-duality groups are those of the exceptional groups and the generalised geometry of this space is called exceptional geometry [36–52].

The enlargening of space has however a drawback. Different directions in the enlarged space means different things. A translation in this space can turn a momentum into a winding number, or mix a field with gravity itself. To be able to study the structure and dynamics in this space we have to generalise the definitions of coordinate transformations by defining a generalised diffeomorphism [52].

In ordinary geometry, the diffeomorphism group is generated by infinitesimal diffeomorphism transformations by the Lie derivative. The active diffeomorphisms defines how tensors transforms on the manifold. Superstring theory compactified on torii mixes string momentum with its winding number, so ordinary geometry is not sufficient in describing their transformations. This can be solved by doubling the geometry, splitting the fields in a right- and lefthanded part and adding a dual torus to the manifold. To go down to the physical dimensions a slice of the enlarged space is chosen by a section condition. This is the concept used in the theory giving a manifest invariant construction of T-duality in doubled geometry.

Compactifying M-theory on torii, on the other hand, mixes the fields with gravity itself making the theory non-geometrical. Using the same scheme as in doubled geometry, the definition of a diffeomorphism transformation can be generalised to compensate for this extra complexity. Compactification on an 8-torus giving rise to the E_8 symmetry group on the torus involves the dual gravity which breaks the concept of geometry even further.

The purpose of this thesis is to construct the generalised geometry needed for studying the non-geometrical structures of compactified eleven-dimensional supergravity and the manifestation of the U-duality symmetry. We will focus on the compactification dimensions $4 \leq n \leq 8$. The diffeomorphism groups for $n = 3$ is not simply connected making the calculations a bit messy. For $n = 8$ the algebra fails to close under the generalised Lie bracket, more about this in chapter 7. Higher dimensional compactifications give rise to affine, extended exceptional groups and requires a different approach than the one presented.

The mathematical language of this thesis involves both Lie algebra, representation theory, group theory and differential geometry. In chapter 2 a short introduction to Lie algebra and representation theory is given. Since the purpose of this thesis is to reconstruct (a generalised) geometry, a brief recap on the construction of ordinary geometry and diffeomorphisms is given in chapter 3.

Since T-duality is a part of U-duality, the part visible in perturbation theory, all results from the construction of the U-duality exceptional geometry is also valid for the case of doubled geometry. The geometrical construction of double geometry can however serve as a more pedagogical way of introducing the concept of generalised geometries. This $O(d,d)$ doubled geometry for compactified type II string theories with T-duality is therefore shortly introduced chapter 4.

U-duality and toroidal compactification of M-theory is described in chapter 5 where the hidden symmetries and enlarged spacetime is motivated, setting the scene for the construction of the generalised geometry. Chapter 6 is devoted to the construction of the U-duality exceptional generalised geometry. We construct generalised diffeomorphisms, tensors, connections, curvature and tensor fields. The section condition, taking us back to n-dimensions, is also derived and discussed. This is the main work of the thesis and is based on **Paper 1**, published in Journal of High Energy Physics **07(2013)028**, which can found in the appendix.

The thesis ends with chapter 7 in a discussion with conclusions, reflections and what the future in this area can hold.

Chapter 2

Lie algebra and representations

Modern theoretical physics is about understanding and exploring symmetries of physical systems. Group theory and Lie algebra is the mathematical language for describing these symmetries. For a physicist, representations of Lie groups is one of the most important tools.

In this chapter, an introduction is given to the mathematical tools needed for an understanding of the work done in the thesis. It is to be viewed only as a short introduction to the areas, there are no proofs given. The concepts of groups, algebras, Lie algebras and their representations are introduced with their formal definitions and some of their important properties. For a more in-depth understanding the reader is encuraged to study 'Symmetries, Lie algebras and Representations' written by Jürgen Fuchs and Christoph Schweigert [53]. This chapter closely follows that book, from which the relevant concepts used in the thesis has been extracted.

2.1 Groups and Algebras

A **group**, G, is a set of elements together with a map (a product) $\circ : G \times G \to G$ with the following properties. The product has to be associative,

$$x \circ (y \circ z) = x \circ y \circ z = (x \circ y) \circ z, \text{ for all } x,y,z \in G.$$

There has to be an identity element, $e \in G$, satisfying

$$e \circ x = x = x \circ e, \text{ for all } x \in G.$$

And, to any element, x, there has to exist an inverse element x^{-1} in G such that

$$x \circ x^{-1} = e = x^{-1} \circ x.$$

The group is called **abelian** if the product is commutative, $x \circ y = y \circ x$, for all $x,y \in G$. Simple groups cannot be divided into smaller groups and can be completely classified.

A **Lie group** has a continuous set of elements, parameterized by a set of continuous parameters. The dimension of the group is the number of parameters, $d = dim(G)$. The group can be either finite- or infinite-dimensional. The finite groups can be divided into sets of groups with similar properties by the Cartan classification. In this classification scheme there are the classical groups: A_n, B_n, C_n, D_n with $n \in \mathbb{Z}^+$ and the exceptional cases: G_2, F_4, E_6, E_7, E_8. The indices denotes the rank of the groups and is the dimension of its Cartan subgroup, more about this in section 2.3. Examples of the infinite-dimensional groups are the Virasoro, Kac-Moody and Borcherds groups which will not be discussed in this thesis.

An **algebra** $\mathfrak{A}$ is a vector space over a field, F, together with a binary, bilinear operation $\circ : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. Because of the bilinearity constraint, the operation has to fulfill the following requirements

$$(x + y) \circ z = x \circ z + y \circ z$$
$$x \circ (y + z) = x \circ y + x \circ z$$
$$(\xi x) \circ (\eta y) = (\xi \eta) x \circ y$$

for all $x, y, z \in \mathfrak{A}$ and all elements ξ, η of the underlying field F. This definition is extremely general and in order to be of any interest, the operation is required to have more properties. A special case of an algebra which is of big importance for physicists is a Lie algebra.

A linearisation of a Lie group, G, gives its Lie algebra. A **Lie algebra**, $\mathfrak{g}$, is an algebra where the bilinear operator is called the **Lie bracket**, denoted $[\cdot, \cdot]$, which possesses the two additional properties:

$$[x, y] = -[y, x] \quad \text{(Antisymmetry)}$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{(Jacobi identity)}$$

for all $x, y \in \mathfrak{g}$. A Lie algebra, $\mathfrak{g}$, can always be directly constructed from an associative algebra, $\mathfrak{A}$, by defining the Lie bracket as the commutator with respect to the original operator

$$[x, y] := x \circ y - y \circ x. \tag{2.1}$$

The dimension of a Lie algebra is the dimension of $\mathfrak{g}$ considered as a vector space, $\mathrm{d} = \dim \mathfrak{g}$. For a finite dimensional Lie algebra, $\mathfrak{g}$, any basis, $\mathcal{B}$ of $\mathfrak{g}$ is spanned by a set of generators, T^a, written as

$$\mathcal{B} = \{T^a \,|\, a = 1, 2, \ldots, d\}. \tag{2.2}$$

The Lie bracket is uniquely determined if it is known on a basis, $\mathcal{B}$, because of the bilinearity. The Lie bracket, and therefore the Lie algebra, can then be defined abstractly through the expansion

$$[T^a, T^b] = \sum_{c=1}^{d} f^{ab}{}_c T^c$$

where the expansion coefficients are called the **structure constants** of the Lie algebra and depend on the basis. The anti-symmetry property of the Lie algebra can be expressed in terms of the structure constants as $f^{ab}{}_c = -f^{ba}{}_c$ and the Jacobi identity as $\sum_{c=1}^{d}(f^{ab}{}_c f^{cg}{}_e + f^{ga}{}_c f^{cb}{}_e + f^{bg}{}_c f^{ca}{}_e) = 0$.

The relation between Lie algebras and (symmetry) Lie groups is at this point perhaps a bit unclear. An example of the relation is that of the rotation group of three-dimensional space, the Lie group $SO(3)$, where infinitesimal rotations are described by the Lie algebra $\mathfrak{su}(2)$. Lie groups describes symmetries, while Lie algebras describes the structure of local one-parameter groups of symmetries.

A finite-dimensional Lie group has the algebraic structure of a group and is at the same time a differentiable manifold. The fact that it is a manifold makes it possible to relate them to linear spaces and to the Lie algebras. The mathematical language used on differentiable manifolds, differential geometry, involves invariant vector fields and describes their coordinate transformations on the manifold. A vector field can be thought of as differential operators acting on functions on the manifold. By defining a Lie bracket (called Lie derivative in this context) on the (Lie group) manifold between two vector fields in a way that makes the bracket fulfill the Lie bracket properties (bilinearity, closedness, antisymmetric, Jacobi identity), the vector space of all vector fields on the manifold becomes a Lie algebra.

Lie algebras gives an algebraic language to the analysis on Lie group manifolds and a lot of the information on the Lie group is carried in its Lie algebra. In fact, the only information about a Lie group not contained in its Lie algebra are topological properties. More about Lie algebra in the context of differential geometry on Lie groups in chapter 3.

A **homomorphism** from a Lie algebra, $\mathfrak{g}$, to a Lie algebra, $\mathfrak{h}$, is a linear map that carries the Lie brackets to Lie brackets and thereby preserves the algebraic structures

$$\begin{aligned}&\phi: \ \mathfrak{g} \to \mathfrak{h}\\ &[x,y] \mapsto \phi([x,y]) = [\phi(x), \phi(y)], \ \text{for all } x, y \in \mathfrak{g}.\end{aligned}$$

If a homomorphism is injective and surjective it is called an **isomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$, and the algebras are **isomorphic** to each other, $\mathfrak{g} \approx \mathfrak{h}$. The concept of isomorphism is important in classifying the Lie algebras.

A **subalgebra**, $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace, $\mathfrak{h} \subseteq \mathfrak{g}$ which itself is a Lie algebra (closed under the Lie bracket). Any Lie algebra has, at least, the two trivial subalgebras of the one-element subset, $\{0\}$, and the algebra itself. All other subalgebras of $\mathfrak{g}$ are called **proper** subalgebras.

If a set of Lie algebras, $\mathfrak{g}_i$, each is an ideal subalgebra of a Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{g}$ can be written as a **direct sum** of the subalgebras

$$\bigoplus_{i=1}^{n} \mathfrak{g}_{\mathfrak{i}} \equiv \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n.$$

The Lie algebras of most importance to a physicist are the abelian Lie algebras, the simple Lie algebras and their direct sums. An abelian Lie algebra is an algebra with $[x,y] = 0$ for all $x, y \in \mathfrak{g}$. A simple Lie algebra is a non-abelian algebra which contains no proper ideals. If each element in $\mathfrak{g}$ can be written as a commutator of two elements of $\mathfrak{g}$, the algebra is a semisimple Lie algebra and can be written as a direct sum of simple Lie algebras.

Any one-dimensional Lie algebra, $\mathfrak{g}$, is abelian because of the antisymmetry property of the Lie bracket of the single generator, T, $[T,T] = 0$. As a result of this there is, up to isomorphisms, only one unique one-dimensional Lie algebra, called $\mathfrak{u}(1)$. Any n-dimensional abelian Lie algebra is isomorphic to the direct sum of n one-dimensional Lie algebras

$$\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{u}(1).$$

2.2 Representations

A Lie algebra acts on a space, to a physicist it can be a space of physical states. Acting on a space, V, means that to any element $x \in \mathfrak{g}$ there is associated a map $R(x): V \to V$ such that the commutator of $R(x)$ and $R(y)$ reproduces the Lie bracket of x and y in $\mathfrak{g}$ as

$$R(x) \circ R(y) - R(y) \circ R(x) = R([x, y])$$

where the '$\circ$' is the composition of maps. V is now called a **representation space** (or a $\mathfrak{g}$-module) of $\mathfrak{g}$ and R is a **representation** of $\mathfrak{g}$. V can be an infinite-dimensional space even if the Lie algebra acting on it by some representation is finite-dimensional. In quantum mechanics, the Hilbert space is the space of physical states and $R(x)$ are linear mappings in this space.

The general linear Lie algebra $\mathfrak{gl}(V)$ is the vector space of all linear mappings $V \to V$ over F. For finite-dimensional spaces, V, with $dim(V) = n$ any element of $\mathfrak{gl}(V)$ can, after fixing a basis, be described by an $n \times n$-matrix and the composition of maps is simply the matrix multiplication. The set of all such matrices is denoted by $\mathfrak{gl}(n)$ and is an n^2-dimensional Lie algebra.

Any Lie algebra has at least one representation, the one that maps all elements of $\mathfrak{g}$ on the zero vector, this is the **singlet** representation of $\mathfrak{g}$. Another possibility is to represent $\mathfrak{g}$ on itself, this representation is called the **adjoint** representation and its dimension is equal to the dimension of the algebra. The adjoint representation is given by the structure constants of the algebra

$$[T^a, T^b] = f_c^{ab} T^c,$$

the entries of the adjoint representation seen as matrices is therefore

$$(R_{ad}(T^a))_c^b = f_c^{ab}.$$

Now, given a non-trivial representation of a Lie algebra, $\mathfrak{g}$, more representations can be constructed from it. Given a matrix representation, R, of a vector space, V, a new set of matrices can be defined by the transpose of the matrices, $R^+(x) := -(R(x))^t$ for all elements, x, in $\mathfrak{g}$. The new representation, R^+, is called the **conjugate** representation of R. Because of the transpose, the conjugate representation acts on the vector space dual to V, denoted V^* which is the space of linear forms.

Given two representation spaces, V and W of $\mathfrak{g}$, the **direct sum** of the vector spaces is also a representation space of $\mathfrak{g}$

$$R_V \oplus R_W \; : \quad ((R_V \oplus R_W)(x))(v \oplus w) := R_V(x)v + R_W(x)w.$$

The **tensor product**, $V \otimes W$, is also a representation of $\mathfrak{g}$

$$R_V \otimes R_W \; : \quad ((R_V \otimes R_W)(x))(v \otimes w) := (R_V(x)v) \otimes w + v \otimes (R_W(x)w).$$

More about the tensor product in section 2.7. A Lie algebra will typically have a lot of different representations, it is however possible to identify fundamental building blocks to which other representations can be decomposed. These building blocks can not be further decomposed and are therefore called **irreducible representations**, or irreps for short. An irreducible representation cannot be brought to a block-diagonal form by a change of basis.

Modules can be built out of irreducible ones using the direct sum, these modules are called fully reducible modules of a Lie algebra. This works the same for representations. A representation is said to be fully reducible if there is a basis of the underlying vector space in which all representation matrices are simultaneously of a block-diagonal form

$$R(x) = \begin{pmatrix} R_1(x) & 0 & \cdots & 0 \\ 0 & R_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n(x) \end{pmatrix}$$

2.3 The Cartan-Weyl and Dynkin bases

The structure constants of a Lie algebra depend on the chosen basis, and to be able to write them down explicitly, a choice of basis has to be made. Doing this in a systematic way will provide a tool to classify the finite-dimensional semisimple Lie algebras. For the semisimple finite-dimensional Lie algebras, there is a completely canonical basis, the **Cartan-Weyl basis**. In this section, $\mathfrak{g}$, is assumed to be semisimple and finite-dimensional and the underlying field, F, is algebraically closed if not otherwise stated.

A Lie algebra has a fixed number of generators, amongst these a certain number of them can be diagonalized simultaneously, this number is called the **rank** of the algebra. The diagonalizable generators are denoted H^i and has a zero Lie bracket among them,

$$[H^i, H^j] = 0, \text{ for } i, j = 1, 2 \ldots r. \tag{2.3}$$

These generators spans a Lie algebra called the **Cartan subalgebra**, $\mathfrak{g}_\circ$, of the algebra, $\mathfrak{g}_\circ = \text{span}_\mathbb{C}\{H^i | i = 1 \ldots r\}$. The Cartan subalgebra of a Lie algebra is the maximal abelian subalgebra consisting of only semisimple elements. A Lie algebra can have many different Cartan subalgebras but all are related by automorphisms. The dimension of the Cartan subalgebras, the rank r, is always the same for a given Lie algebra and is a property of it. For a physical system with $\mathfrak{g}$ as its symmetry algebra, the rank denotes the maximal number of quantum numbers used to label the states in the system.

The generators of the Lie algebra that are not used to span the chosen Cartan subalgebra can be written as linear combinations of each other to form the generators E^α with

$$[H^i, E^\alpha] = \alpha^i E^\alpha. \tag{2.4}$$

The set of generators $\{H^i, E^\alpha\}$ spans the Lie algebra and is a basis thereof, called the **Cartan-Weyl basis** for the Lie algebra $\mathfrak{g}$. The r-dimensional vector α^i (one for each generator E^α) of eigenvalues of E^α is called a **root vector** of $\mathfrak{g}$ relative to the basis of $\mathfrak{g}_\circ$. The root vectors lives in an r-dimensional **root space** spanned by the **simple roots** and the set of the root vectors is called the **root system**. The simple roots, denoted $\alpha^{(i)}$, are roots that cannot be written as linear combinations of the other roots and the number of them is exactly r. The generators E^α are also called **the step operators** to the roots α.

Calculations can sometimes be simplified by introducing another element for any root, α, as $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \in \mathfrak{g}_\circ^\star$ (in the dual algebra of the Cartan subalgebra). The vector $\alpha^\vee$ is now called the **coroot** or **dual root** of α. The dual space of the root space is called the **weight space** of $\mathfrak{g}$. The dual roots can be chosen as a basis of the root space

$$\mathcal{B} = \{\alpha^{(i)\vee} \,|\, i = 1, \ldots, r\}. \tag{2.5}$$

The basis of the weight space (dual to $\mathcal{B}$) now consists of the weights, $\Lambda_{(i)}$, that fulfills $\Lambda_{(i)}(\alpha^{(j)\vee}) = \delta_i^j$ and are called the **fundamental weights**. The basis of the weight space, $\mathcal{B}^\star = \{\Lambda_{(i)} \,|\, i = 1, \ldots r\}$ is called the **Dynkin basis** and the components of a weight in the Dynkin basis are called **Dynkin labels**.

2.4 The Cartan matrix

The whole structure of a finite, semisimple Lie algebra is encoded in the simple roots, spanning the root space. So, in order to classify and understand all possible semisimple Lie algebras up to isomorphisms, all we have to do is classify the possible Cartan subalgebras and their root systems.

The commutators among the generators spanning the Cartan subalgebra, H^i, and the step operators, E^α, can be encoded in the **Serre relations** expressed in the **Chevalley basis**. The Chevalley basis is a special case of the Cartan-Weyl basis and is constructed by the following identifications. Let h^i be the Cartan generators associated to the simple roots, e^i is identified with the step operators associated with the simple roots and f^i are identified with the step operators associated with the minus simple roots:

$$\begin{aligned} h^i &= H^{\alpha^{(i)}} \\ e^i &= E^{\alpha^{(i)}} \\ f^i &= E^{-\alpha^{(i)}}. \end{aligned}$$

The e^i and f^i should be normalised by $K(E^\alpha, E^{-\alpha}) = \frac{2}{(\alpha,\alpha)}$, where K is the **Killing form** which is a special form providing a metric on the root space and gives an isomorphism between $\mathfrak{g}_\circ$ and $\mathfrak{g}_\circ^\star$. The Killing form is defined as $K(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. These generators now obeys the commutation relations

$$\begin{aligned} [h^i, h^j] &= 0 \\ [h^i, e^j] &= A^{ji} e^j \\ [h^i, f^j] &= -A^{ji} f^j \\ [e^i, f^j] &= \delta^{ij} h^i \end{aligned}$$

and their multi-commutators satisfies the Serre relations

$$(\mathrm{ad}_{e^i})^{1-A^{ji}} e^j = 0 \tag{2.6}$$

$$(\mathrm{ad}_{f^i})^{1-A^{ji}} f^j = 0. \tag{2.7}$$

Where the multi-commutators are written in short-hand as $(\mathrm{ad}_x)^n = \mathrm{ad}_x \circ \mathrm{ad}_x \circ \cdots \circ \mathrm{ad}_x$, so that $(\mathrm{ad}_x)^2(y) = [x,[x,y]]$. These relations together with the commutation relations for the generators are known as the **Chevalley-Serre relations** and characterizes $\mathfrak{g}$ uniquely.

The only non-universal numbers in the presentation of the Lie algebra are given by the integer valued matrix A^{ij} in the Chevalley-Serre relations, called the **Cartan matrix**. The entire structure of the algebra can now be expressed by this matrix which is built up of the simple roots as

$$A^{ij} = 2\frac{(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})}, \tag{2.8}$$

where $(\cdot\,,\cdot)$ is the Euclidean inner product. Or, more simply, expressed in terms of the simple coroots as $A^{ij} = (\alpha^{(i)}, \alpha^{(j)\vee})$. The Cartan matrix is independent on the choice of the basis for the simple roots up to the numbering of its rows and columns. The classification of simple Lie algebras now amounts to the classification of their Cartan matrices. By analyzing the definition of the Cartan matrix, the following properties about Cartan matrices can be made:

- The diagonal elements of the Cartan matrix is always 2, since $A^{ii} = 2, \forall i = 1\ldots r$.
- If $A^{ij} = 0$ then $A^{ji} = 0$ because of the symmetry of the scalar product in root space.
- A^{ij} is always a positive integer.
- The difference of two simple roots is never a root.
- The matrix is always non-degenerate, $\det A \neq 0$. Or, in fact, $\det A > 0$.

The Cartan matrix of a simple Lie algebra also has to be irreducible; there should be no renumbering of the simple roots that puts the Cartan matrix in a block diagonal form. If this is the case, the Lie algebra is semisimple. All possible matrices fulfilling these properties (and that cannot be transformed into each other by a relabelling of rows or columns) corresponds to different (non-isomorphic) Lie algebras.

2.5 Dynkin diagrams

The categorisation of all simple Lie algebras is now reduced to finding all possible solutions to the requirements of the Cartan matrices, which is a purely combinatorial problem. The possible solutions can be described in a convenient way in the so called **Dynkin diagrams**, where each Cartan matrix is associated with a specific diagram consisting of vertices and connecting lines. Each node, or vertex, in the diagram represents a simple root, and two nodes, represented by $\alpha^{(i)}$ and $\alpha^{(j)}$, are connected by $\max\{|A^{ij}|, |A^{ji}|\}$ lines. The number of lines connecting two vertices denotes the angle between the corresponding simple roots. A single, double or triple line denotes an angle of $\frac{2\pi}{3}$, $\frac{3\pi}{4}$ and $\frac{5\pi}{6}$ respectively. Vertices that are not connected denotes two orthogonal simple roots. If $A^{ij} \neq 0$ and $|A^{ij}| > |A^{ji}|$ an arrow is added to the line connecting the ith and jth node.

As shortly discussed earlier, the Cartan classification of simple, finite-dimensional Lie algebras consists of the four infinite series (the classical Lie algebras) denoted by A_r $(r \geq 1)$, B_r $(r \geq 3)$, C_r $(r \geq 2)$ and D_r $(r \geq 4)$ together with the five exceptional cases G_2, F_4, E_6, E_7 and E_8. Where the subscripts denotes the rank of the groups. The classical algebras are isomorphic to the matrix algebras as $A_r \approx \mathfrak{sl}(r+1)$, $B_r \approx \mathfrak{so}(2r+1)$, $C_r \approx \mathfrak{sp}(r)$ and $D_r \approx \mathfrak{so}(2r)$.

All classified, simple, finite-dimensional Lie groups are listed, together with their dynkin diagrams and Cartan martices, in table 2.1 and 2.2.

Table 2.1: The classical Lie algebras with their respective Cartan matrices and Dynkin diagrams.

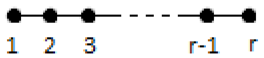
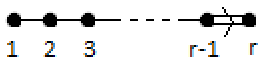
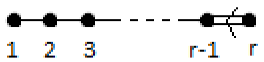
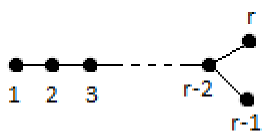
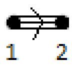
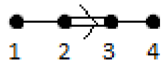
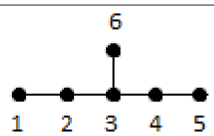
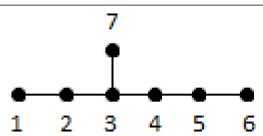
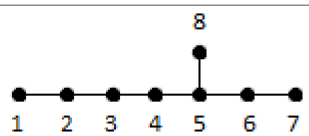
name	Dynkin diagram	Cartan matrix
A_r		$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$
B_r		$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$
C_r		$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -2 & 2 \end{pmatrix}$
D_r		$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}$

Table 2.2: The exceptional Lie algebras with their respective Cartan matrices and Dynkin diagrams.

name	Dynkin diagram	Cartan matrix
G_2		$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$
F_4		$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$
E_6		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$
E_7		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$
E_8		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$

2.6 Highest weight representations

As stated earlier, the representations of a Lie algebra are of most importance to physics. Any finite-dimensional representation of a simple Lie algebra is a heighest weight representation. Because of this, highest weight representations are found to be an interesting subclass of representations.

Each generator, H^i, of the Cartan subalgebra, $\mathfrak{g}_\circ$, of a semi-simple Lie algebra, $\mathfrak{g}$, spans the Cartan subalgebra of a specific subalgebra, $\mathfrak{sl}_\alpha(2)$, of $\mathfrak{g}$. This subalgebra is spanned by the generators H^i and $E^i_\pm$ of $\mathfrak{g}$ for a fixed i, corresponding to the simple root $\alpha = \alpha^{(i)}$. Because of this, it is possible to reduce the analysis of the heighest weight state for finite-dimensional representations of semi-simple Lie algebras, to the representation theory of $\mathfrak{sl}(2) \approx A_1$.

A_1 consists of the generators H and $E_\pm$, where H spans the Cartan subalgebra and the generators satisfies $[H, E_\pm] = \pm E_\pm$ and $[E_+, E_-] = H$. The representation space, V, for a representation, R, has a basis for which the generator $R(H)$ acts diagonally, this implies that each module V decomposes into **weight spaces**, $R_{(\lambda)}$ as

$$V = \bigoplus_\lambda V_{(\lambda)}, \quad V_{(\lambda)} = \{v \in V \mid R(H)v = \lambda \cdot v\}, \tag{2.9}$$

where the eigenvalues, λ, are the weights of V. Now, given a weight vector, $v \in V_{(\lambda)}$, then $R(E_\pm)v \in V_{\lambda\pm 2}$ and there is a weight Λ such that $V_{(\Lambda)} \neq 0$ but $V_{(\Lambda+2)} = 0$ for finite-dimensional representations. Such weights, Λ, are **maximal weights** and any weight vector, $v \in V_{(\Lambda)}$ are maximal weight vectors. If V is irreducible, Λ is a **highest weight**, v_Λ is a **highest weight vector** and V is a **highest weight module** denoted V_Λ.

A highest weight vector, v_Λ, is annihilated by the step-up operator, $E_+ v_\Lambda = 0$. All other weight vectors can be constructed from the highest weight vector using the step-down operator,

$$\begin{aligned} E_- v_\Lambda &= v_{\Lambda-2} \\ &\vdots \\ (E_-)^n v_\Lambda &= v_{\Lambda-2n} \\ &\vdots \\ (E_-)^N v_\Lambda &= v_{\Lambda-2N} \\ E_- v_{\Lambda-2N} &= 0 \end{aligned}$$

where the chain stops at a fixed N for finite-dimensional modules.

Continuing the analysis for simple Lie algebras, we can apply the representation theory of A_1 to each of the subalgebras, $\mathfrak{sl}_\alpha(2)$, for all $i = 1 \dots r$. In the same way as for A_1, there is a decomposition of any $\mathfrak{g}$-module, $V = \bigoplus_\lambda V_{(\lambda)}$, into weight spaces such that $R(H^i)v_\lambda = \lambda^i \cdot v_\lambda$. The eigenvalues are always integers and are grouped into the **weights** of V as an r-dimensional vector $\lambda = (\lambda^i)$. All the weights of a module V makes up its **weight system**. Λ is only a weight of a finite-dimensional module if it lies on the **weight lattice**, L_W, meaning that it can be written as an integral linear combination of the fundamental weights, $\Lambda = \sum_{i=1}^r \lambda^i \Lambda_{(i)}$.

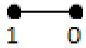
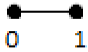
Module	Dynkin labels	Notation
3		[10]
$\bar{3}$		[01]
8		[11]

Table 2.3: Some $\mathfrak{sl}(3)$-modules and their Dynkin labels attached to the Dynkin diagram.

For any finite-dimensional module of $\mathfrak{g}$ maximal weights Λ can be found such that $R(E^\alpha)v_\Lambda = 0$ for all positive roots α and all $v_\Lambda \in V_{(\Lambda)}$. If there is only one such weight, it is the highest weight of the module. The Dynkin labels of a highest weight, Λ^i, are always positive integers.

The nodes of the Dynkin diagrams can not only denote the fundamental weights, $\Lambda_{(i)}$ or the simple roots, $\alpha^{(i)}$, they can also be used to visualise highest weight modules of a Lie algebra by attaching the Dynkin labels of the weight to the nodes of the diagram. An example of this is presented in table 2.3 for $\mathfrak{sl}(3)$.

2.7 Tensor products

Given two modules V and W of a Lie algebra $\mathfrak{g}$, we can also represent $\mathfrak{g}$ on the tensor product vector space $V \otimes W$ as seen earlier,

$$R_V \otimes R_W \; : \quad ((R_V \otimes R_W)(x))(v \otimes w) := (R_V(x)v) \otimes w + v \otimes (R_W(x)w). \tag{2.10}$$

A basis for the tensor product vector space, $V \otimes W$, can be constructed from two bases $\mathcal{B}_V = \{v_i\}$ of V and $\mathcal{B}_W = \{w_j\}$ of W as

$$\mathcal{B} = \{v_i \otimes w_j \,|\, v_i \in \mathcal{B}_V, w_j \in \mathcal{B}_W\}. \tag{2.11}$$

As seen from this construction of a basis for the tensor product, we can conclude that $V \otimes W$ and $W \otimes V$ are different vector spaces, and because of this, the modules are also different. The two modules are however isomorphic, and the tensor product is therefore said to be commutative (and associative) up to isomorphism.

If $W = V$, the vector space $V \otimes V$ splits into the two invariant subspaces

$$V_s = \{v \otimes v' + v' \otimes v \,|\, v, v' \in V\}, \tag{2.12}$$

$$V_a = \{v \otimes v' - v' \otimes v \,|\, v, v' \in V\} \tag{2.13}$$

where V_s is a symmetric and V_a antisymmetric element. The linearity property of the tensor product implies that these two subspaces are again $\mathfrak{g}$-modules, giving the direct sum decompositions

$$V \otimes V = V_s \oplus V_a, \tag{2.14}$$

$$R \otimes R = R_s \oplus R_a. \tag{2.15}$$

When forming the tensor product of two modules of a finite-dimensional, simple Lie algebra, the weights of the modules adds up. This implies that the weight system of $V \otimes V'$ consists of weights of the form $\lambda + \lambda'$. The tensor product $V_\Lambda \otimes V_{\Lambda'}$ for highest weight modules $V = V_\lambda$ and $V' = V_{\Lambda'}$ is again finite-dimensional, this implies that it is fully redicible into irreducible modules and can be decomposed as

$$V_\Lambda \otimes V_{\Lambda'} \approx \oplus_i \mathcal{L}^{\Lambda_i}_{\Lambda\Lambda'} V_{\Lambda_i} \tag{2.16}$$

where V_{Λ_i} are its irreducible submodules and $\mathcal{L}^{\Lambda_i}_{\Lambda\Lambda'}$ are non-negative integers called the tensor product coefficients which are related to the Clebsch-Gordan coefficients.

The decomposition of tensor products are tedious to do by hand and can (and will in the rest of the thesis) be calculated by computer algorithms.

2.8 Intertwiners

A physiscist is often interested in the singlet contributions from tensor products. The Lagrangian in a field theory, for example, has to be a singlet with respect to its manifest symmetry Lie algebra $\mathfrak{g}$, the fundamental, non-singlet fields of the theory has to be organized in multiplets that transforms correctly. This can be dealt with by using intertwiners (invariant tensors) between tensor product representations and singlets.

To start the analysis of intertwiners, we first fix a basis, $\mathcal{B} = \{v_{(i)}\}$ of the module V. Tensors are now described by its coordinates in this basis. The most simple tensor is the vector v^i with $i = 1, \ldots, \dim V$ in an irreducible $\mathfrak{g}$-module V. In the chosen basis, v is written as $v = \sum_i v^i v_{(i)}$. We can also form the conjugate module V^+ and describe the conjugate vector, $\tilde{v}$, in the dual basis, $\mathcal{B}^\star = \{\tilde{v}^{(i)}\}$, defined so that $\tilde{v}^{(i)} v_{(j)} = \delta^i_j$. Upper indices are called **covariant** and lower ones are called **contravariant**.

Representation matrices acts on the components of the vector and its dual companion as

$$v^i \mapsto (v')^i = \sum_j (R(x))^i{}_j v^j, \tag{2.17}$$

$$\tilde{v}_i \mapsto (\tilde{v}')_i = \sum_j (R^+(x))_i{}^j \tilde{v}_j. \tag{2.18}$$

In this context, an invariant tensor, t, for the tensor product $W = V^{\otimes m} \otimes (V^+)^{\otimes n}$ (a tensor with m covariant and n contravariant indices) is an intertwiner between the singlet of $\mathfrak{g}$ and the $\mathfrak{g}$-module W. It describes a way of forming a singlet out of irreducible modules. An example of an invariant tensor is the Killing form for the adjoint module. Another example is the Kronecker symbol which is an invariant tensor for the tensor product $V \otimes V^+$ for any finite-dimensinal module V.

For self-conjugate irreducible modules, where $V \approx V^+$, there has to exist an invariant tensor, e (an intertwiner between the singlet and $V \otimes V$). Because of this, there has to be a linear relation between the components of a vector v and the dual vector, $\tilde{v}$, given by the invariant tensor, $e = e_{ij}$, as $\tilde{v}_i = \sum_j e_{ij} v^j$ and an analogous tensor with upper indices $v^i = \sum_j e^{ij} \tilde{v}_j$ such that $\sum_j e_{ij} e^{jk} = \delta_i{}^k$. If e is symmetric, the representation corresponding to V is called **orthogonal** and if it's antisymmetric, the representation is said to be **symplectic**.

Table 2.4: Primitive invariants for the simple Lie algebras.

$\mathfrak{g}$	invariants
B_r, D_r	δ^{ij}
C_r	f^{ij}
G_2	δ^{ij}, f^{ijk}
F_4	δ^{ij}, d^{ijk}
E_6	d^{ijk}
E_7	f^{ij}, d^{ijkl}
E_8	$\delta^{ij}, f^{ijk}, t^{ijkl\ldots}$

Indices can be raised and lowered by contractions with e for self-conjugate modules, which is an often useful fact. This can however also be done for non-conjugate modules using the anti-symmetric Levi-Civita tensors, the drawback here is that the number of indices are changed.

Invariant tensors can be constructed from multiplication, summation and contractions of other invariant tensors. And as a fact, all invariant tensors can be constructed from a small number of algebraically independent invariant tensors, the so called **primitive invariants**. The primitive invariants for the simple Lie algebras are listed in table 2.8. The Kronecker symbol, $\delta^i{}_j$, and the upper and lower index Levi-Civita tensors, $\epsilon^{i_1 i_2 \ldots i_d}$ and $\epsilon_{i_1 i_2 \ldots i_d}$ are invariant tensors for all simple Lie algebras and are not listed in the table. All the invariants in the table are either symmetric or anti-symmetric, anti-symmetric are denoted by $f^{\cdots}$ and the symmetric ones are denoted by $d^{\cdots}$

A tensor product can always be split up to a summation of linearly independent invariant tensors. An example of this is the tensor product of a module V with its conjugate module for the Lie algebra $\mathfrak{sl}(n)$

$$\tilde{v}_i \otimes v^j = \delta^j{}_i S + \sum_a (R(T^a))^j{}_i A^a. \tag{2.19}$$

Here, both $\delta^j{}_i$ and $(R(T^a))^j{}_i$ are invariant tensors and it follows that S must transform as a singlet, while A^a transforms in the adjoint representation. This implies that the irreducible modules in the tensor product $V \otimes V^+$ for $\mathfrak{sl}(n)$ are the adjoint and singlet modules. This decomposition can also be understood on the level of matrices where it corresponds to the splitting of the matrix into a trace and a traceless part.

A **projection operator** is an invariant tensor corresponding to intermediate states in multiple tensor products. This can be illustrated in a Feynman diagram with two incoming states (Λ_1 and Λ_2) and two outgoing states (Λ_3 and Λ_4) with one intermediate state (Λ) between the vertices. The situation is described mathematically by the tensor product

$$(V_{\Lambda_1} \otimes V_{\Lambda_2}) \otimes (V_{\Lambda_3} \otimes V_{\Lambda_4}) = \sum_{\Lambda, \Lambda'} \mathcal{L}_{\Lambda_1\Lambda_2}{}^{\Lambda} \mathcal{L}_{\Lambda_3\Lambda_4}{}^{\Lambda'} V_\Lambda \otimes V_{\Lambda'}. \tag{2.20}$$

Each allowed state, Λ, in the tensor product corresponds to an intermediate state and for each irreducible module of these intermediate states there is associated an invariant tensor of the tensor product $V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_3} \otimes V_{\Lambda_4}$. The projection tensors are not always primitive invariants, but always linear combinations of such.

The invariant tensors can also be used to project between different representations of an algebra. For example, by using the symmetric primitive invariant d^{ijk} in E_6, two contravariant vectors, v_i and u_j in $\bar{\mathbf{27}}$ can be projected to a covariant vector, w^k in $\mathbf{27}$ as $w^k = v_i u_j d^{ijk}$.

As in the case of tensor products, the splitting of tensor products in irreducible modules or invariant tensors, or for the extraction of singlet contributions and so on is easily done with computer algorithms and is therefore the preferable and chosen way of doing it.

2.9 Subalgebras and branching

Symmetry breaking is a common phenomena in physics. When a system with a certain symmetry is perturbed in such a way that some part of the symmetry vanishes, the symmetry is said to be broken. If the full symmetry is described by the Lie algebra $\mathfrak{g}$ and the symmetry left after breaking by a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we are interested in how to decompose the modules of $\mathfrak{g}$ to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Another application of subalgebras, other than that of symmetry breaking, is for the analysis of physics where the symmetry is decribed by one of the exceptional Lie algebras. All classical Lie algebras are isomorphic to matrix algebras, this is not the case for the exceptional algebras and that makes them harder to do calculations in. Because of this, it can be useful (and sometimes the only way) to break the large exceptional symmetry down to one of its classical subalgebras and carry out the analysis in this more simple domain.

This section is more focused on the use of branching rules of modules into subalgebras, and not so much on the theory of how to find and construct various subalgebras of a given Lie algebra. Computer programs can be used to both find subalgebras and calculate the branching rules down to a known subalgebra of a given Lie algebra.

The classification of subalgebras gives rules about which symmetries that can survive a breaking form a larger symmetry algebra. A subalgebra, $\mathfrak{h}$ of a Lie algebra, $\mathfrak{g}$ is said to be embedded in $\mathfrak{g}$, $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Some subalgebras can be found by studying the Dynkin diagram of a Lie algebra, an example is that of the E-series chain used in some grand unification theories

$$A_4 = E_4 \approx \mathfrak{sl}(5) \hookrightarrow D_5 = E_5 \approx \mathfrak{so}(10) \hookrightarrow E_6. \tag{2.21}$$

This chain of embeddings can be understood by simply covering nodes of the Dynkin diagram of E_6. By covering the rightmost node of E_6, the remaining nodes and lines are those of D_5, and analogous for $A_4 \hookrightarrow D_5$.

The branching rules controls how states in the larger algebra gets organized into modules of the subalgebra after the symmetry is broken,

$$V_\Lambda(\mathfrak{g}) \to \bigoplus_j V_{\lambda_j}(\mathfrak{h}). \tag{2.22}$$

In this decomposition, an irreducible highest weight module, V_Λ, of $\mathfrak{g}$ breaks to a direct sum of irreducible modules of the subalgebra $\mathfrak{h}$. After the decomposition, the total number of dimensions of the modules in the subalgebra has to be the same as that of the unbroken

$\mathfrak{g}$-module

$$\dim(V_\Lambda(\mathfrak{g})) = \sum_j \dim(V_{\lambda_j}(\mathfrak{h})). \tag{2.23}$$

The embedding of the standard model gauge symmetry in that of a larger Lie group used in the most simple case of a grand unification model, $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{u}(1) \hookrightarrow \mathfrak{sl}(5)$, can be used as an example of branchings. Here, the five- and ten-dimensional modules of $\mathfrak{sl}(5)$ is branched to the Standard model modules as

$$5 \to (3;\ 1;\ \frac{1}{3}) \oplus (1;\ 2;\ -\frac{1}{2}) \tag{2.24}$$

$$10 \to (\bar{3};\ 1;\ \frac{2}{3}) \oplus (3;\ 2;\ -\frac{1}{6}) \oplus (1;\ 1;\ -1). \tag{2.25}$$

As told before, calculating tensor products, finding subalgebras and branching rules can be made by hand using theory and some developed rules and techniques, this is however a tedious way of doing it and there are plenty of specialised computer programs developed for handling Lie algebras and their representation theory.

Chapter 3

Diffeomorphisms and geometry

Since the work of the thesis generalises the construction of ordinary gravity and its geometrical framework we give a short review of ordinary geometry and Einstein gravity. Geometry is the language of gravity, it considers how physics is changed when doing coordinate transformations. Coordinates do not exist in nature, it is merely a construction used in describing it. Because of this, a fundamental physical theory can not be coordinate dependant, it should be the same in all coordinate systems. This is called general covariance or diffeomorphism covariance and is the starting point for general relativity, formulating gravity in a coordinate independent way.

A **diffeomorphism** is an isomorphism on manifolds, and there are two different types of diffeomorphisms; active and passive diffeomorphisms. A passive diffeomorphism is a coordinate transformation, any theory can be made invariant under passive diffeomorphisms. Active diffeomorphisms, however, are the gauge symmetry of general relativity. The quantum mechanical theories QED and QCD are not invariant under active diffeomorphisms [54].

The recipe for constructing general relativity, a tensor formalism for gravity describing how tensor fields transform is as follows:

- Define a Lie derivative that gives the algebra of vector fields.
- Introduce an affine connection, Γ, a spin connection, Ω, and a covariant derivative, $D = \partial + \Gamma + \Omega$ such that $D_\mu V^\nu$ is a tensor.
- Define the torsion part of the affine connection and demand it to vanish.
- Introduce a covariant vielbein, $e_\mu{}^a$ which is a group element of $GL(n)$.
- Form a metric, $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$.
- Define curvature. The Riemann tensor, the Ricci tensor and the Ricci scalar.
- Form a stress tensor.
- Write down the Einstein equations for general relativity.

Some of these steps (the ones relevant for the rest of the thesis) are here shortly presented for ordinary geometry and will be generalised when constructing a U-duality invariant description of M-theory in chapter 6. We will try to discuss geometry in terms of group theory and Lie algebras, presenting the language used in chapter 6. The diffeomorphism group (gauge group) for general relativity in n dimensions is the general linear group $GL(n)$. There is also a local symmetry group, $SO(n)$, in geometry, that of local rotations (local Lorentz transformations).

3.1 The Lie derivative

The **Lie derivative** is an infinitesimal representation of the diffeomorphism group on tensor fields. It is a coordinate invariant construction that measures the change of a vector field, $v = v^\mu \partial_\mu$ when transported in the direction of flow of another vector field, $u = u^\mu \partial_\mu$,

$$L_u(v) = [u,v] \tag{3.1}$$

The Lie derivatives on a manifold, M, builds a Lie algebra representation of vector fields with the Lie bracket defined by the commutator of two Lie derivatives

$$L_{[u,v]} = [L_u, L_v]. \tag{3.2}$$

Diffeomorphisms are encoded in the Lie derivative, the variation of a tensor field $\Phi^{\cdots}{}_{\cdots}$ is given by $\delta_u \Phi^{\cdots}{}_{\cdots} = L_u \Phi^{\cdots}{}_{\cdots}$ where u^m is a gauge parameter vector. The Lie derivative of a vector, v^m, in the direction of a diffeomorphism parameter u^m is given by

$$L_u v^m = [u,v]^m = u^n \partial_n v^m - \partial_n u^m v^n. \tag{3.3}$$

The first term on the right hand side can be seen as a transport term and the second as a $\mathfrak{gl}(n)$ transformation term. The matrix element $\partial_n u^m \in \mathfrak{gl}(n)$ is valued in the fundamental representation of the Lie algebra.

3.2 Covariant derivative

We want the derivative of a tensor to transform as a tensor under $GL(n)$, this is not true for the ordinary derivative, ∂_μ, and we have to define a **covariant derivative**,

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \tag{3.4}$$

With a similar expression for a covariant vector index,

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\nu\mu} V_\rho. \tag{3.5}$$

Γ is a non-tensorial object called the **affine connection**. An affine connection connects nearby tangent spaces on smooth manifolds. There are infinitely many affine connections on any manifold and it can be defined in a way that it makes the covariant derivative behave the way we want. By defining the affine connection in the following way, we ensure that the covariant derivative of a tensor again is a tensor,

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}). \tag{3.6}$$

The tensor representations of $GL(n)$ behave like tensors under the subgroup of local rotations, the Lorentz group. There are however no representations of $GL(n)$ that behave like the spinors found in the Lorentz group. To be able to incorporate spinors in general relativity, we have to look at the problem in local, flat coordinates. This is done by employing a vielbein field, ${e_\mu}^a$ that describes a flat tangent space at every point of the manifold. The vielbein is a local inertial coordinate system. The coice of a vielbein breaks the symmetry from $GL(n) \to SO(n)$ (from $\mathfrak{g} \to \mathfrak{h}$ in a more generalised way, where $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$). We have now introduced a new, flat index that transforms under the local subgroup. In the same way as the affine connection, we have to insure that the derivative of a flat index transforms the way we want it to, the equivalent to the affine connection for flat indices are given by the **spin connection**. The spin connection is the gauge field generated by local Lorentz transformations and is defined as

$$\omega_\mu^{ab} \equiv e_\nu^a \partial_\mu e^{\nu b} + e_\nu^a e^{\sigma b} \Gamma^\nu_{\sigma\mu} \tag{3.7}$$

The spin connection transforms the flat indices while the affine connection transforms the curved ones.

3.3 Torsion and curvature

The affine connection is in itself not a tensor and its invariants are the torsion and curvature tensors. The affine connection generates a parallel transportation and the torsion part of the affine connection characterises how a tangent space is twisted when parallel transported along a geodesic. The curvature part is a measure, in the same way, of how the tangent space roll along a curve.

The **Riemann curvature tensor** is the only tensor that can be constructed from the metric tensor and its first and second derivatives, and is linear in the second order derivative. Its definition, in terms of the affine connection, is

$$R^\lambda{}_{\mu\nu\kappa} \equiv \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}. \tag{3.8}$$

Another tensor that can be formed from the curvature tensor is the **Ricci tensor**, which is a contracted Riemann tensor

$$R_{\mu\kappa} \equiv R^\lambda_{\mu\lambda\kappa}. \tag{3.9}$$

Contracting the Ricci tensor with the metric forms yet another invariant, the **Ricci scalar**,

$$R = g^{\mu\kappa} R_{\mu\kappa}. \tag{3.10}$$

The torsion part of the affine connection is defined as the antisymmetric part of it,

$$T^\lambda_{\mu\nu} \equiv 2\Gamma^\lambda_{[\mu\nu]}. \tag{3.11}$$

In general relativity, the torsion is set to zero. There are however versions of gravity that keeps the torsion part, but no evidence of this is seen in nature. If the affine connection was not torsion-free, the time derivative of the distance between two geodesic curves, separated by an infinitesimal parallel transport and parametrised in proper time, would not be zero.

When torsion is set to zero, the Riemann tensor measures the noncommutative part of the covariant derivative,

$$[D_\mu, D_\nu]V_\rho = R_{\mu\nu\rho}{}^\kappa V_\kappa. \tag{3.12}$$

3.4 Vielbein and metric

The metric is not invariant under $GL(n)$, the presence of a metric locally breaks $GL(n)$ to $SO(n)$. Both the vielbein and the metric is however covariantly constant,

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\rho\mu} = 0, \tag{3.13}$$

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a + \Omega_{\mu b}{}^a e_\nu{}^b = 0. \tag{3.14}$$

Through the vielbein, there is a relation between the $GL(n)$-metric and the flat Minkowski metric, η_{ab},

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}. \tag{3.15}$$

This equation makes it possible to use orthonormal bases locally in curved spacetime. The vielbein can be used to 'flatten' curved indices, and to go between flat and curved representations. A flat index is seen as a scalar from the $GL(n)$ point of view, it is transformed only by a Lorentz transformation and left untouched by the global symmetry group.

The infinitesimal change of the metric generated by a diffeomorphism vector field, $V^\mu(x)$, is given by its Lie derivative along, V^μ,

$$\delta g_{\mu\nu} = L_V g_{\mu\nu} = 2D_{(\mu} V_{\nu)}. \tag{3.16}$$

If $L_V g_{\mu\nu} = 0$, then V^μ is a **Killing vector field**, given by the Killing equation $D_{(\mu} V_{\nu)} = 0$. A maximally symmetric vector space has the largest number of Killing vectors. In a flat n-dimensional space, $\mathbb{R}^n$, there are $n(n+1)/2$ Killing vectors.

3.5 The Einstein-Hilbert action

The Einstein field equations are obtained by the principle of least action from the Einstein-Hilbert action defined by

$$S = \int R\sqrt{-g} d^n x + S_M, \tag{3.17}$$

where R is the Ricci scalar, $g = \det(g_{\mu\nu})$, S_M is the action from matter fields and the cosmological constant is omitted, other constants are set to 1. Varying with respect to the metric and setting it to zero minimises the action, the variation is given by

$$0 = \delta S = \int d^n x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)\partial g^{\mu\nu} + \frac{1}{2}\int d^n x\sqrt{-g} T_{\mu\nu} \partial g^{\mu\nu} + \text{b.t.} \tag{3.18}$$

Where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial S_M}{\partial g_{\mu\nu}}$ is defined as the **stress-energy-tensor** and b.t denotes boundary terms coming from the total derivative term in the variation of the Ricci scalar. The variations for the various curvature tensors used in the above calculation are as follows,

$$\delta R_{\mu\nu} = D_\rho(\delta\Gamma^\rho_{\nu\mu}) - D_\nu(\delta\Gamma^\rho_{\rho\mu}), \tag{3.19}$$

$$\delta R = R_{\mu\nu} g^{\mu\nu} + D_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^\rho_{\rho\mu}), \tag{3.20}$$

where the last term of the variation of the Ricci scalar is a divergence and generates a boundary term when integrated.

The solution to the variation of the Einstein-Hilbert action (3.18), using the expressions for the variation of curvature, is the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}, \tag{3.21}$$

describing the dynamics of the metric. G_N is the Newton gravitational constant.

Chapter 4

T-duality and doubled geometry

In this chapter, we give a short overview of the concept of formulating a string theory with manifest T-duality symmetry and diffeomorphism invariance in the non-geometrical background generated by the T-duality. The concept of a generalised geometry is more easily motivated for T-duality than that for the U-duality group since the 'hidden dimensions' are not as hidden and the group structure of the duality group is less complex.

Toroidally compactified string theories exhibits a special symmetry, T-duality, which relates different geometries for the compactified dimensions. The fields of a string theory compactified on a d-dimensional torus, T^d, will transform under the diffeomorphism group on the torus, $GL(d,\mathbb{Z}) = SL(d,\mathbb{Z})\times\mathbb{Z}_2$. The string theory on this torus is also symmetric under the T-dualities transforming under $O(d,d;\mathbb{Z})$. The T-duality exchanges string momentum and winding number of compactified string theories and creates a non-geometrical string background. These theories can be formulated geometrically by writing down the theory in an enlarged space where the diffeomorphisms and the gauge symmetries as well as the T-duality symmetries are made manifest. The geometrical description of a theory where the T-duality group, $O(d,d)$, is made manifest, is called doubled geometry.

In 10-dimensional string theory compactified on a torus, T^d, the momentum, p_i, and the winding number, w^i, of the string in the internal directions can be combined in the momenta $p_L^i = p^i + w^i$ and $p_R^i = p^i - w^i$. The conjugated coordinates, X_L^i and X_R^i, are given as coordinates on the compactification torus, X^i, and on a dual torus, $\tilde{X}^i$, by $X_L^i = X^i + \tilde{X}^i$ and $X_R^i = X^i - \tilde{X}^i$. So, for each coordinate, X^i with $i = 1\ldots d$, taking values on a circle with radius R_i, there is a coordinate $\tilde{X}^i$ on a circle with the T-dual radius $1/R_i$. The number of coordinates are effectively doubled, $X^M = \{\tilde{X}^i, X^i\}$, $M = 1\ldots 2d$ and parametrises a doubled space combining winding number and momenta.

The original theory can be formulated as a string theory on the doubled target space, T^{2d}, with coordinates X^i and $\tilde{X}^i$. By doing this, winding numbers are represented in a geometrical way on the dual torus and all transformations are diffeomorphisms on the double torus. The winding numbers and momentum charges of the string in the compactified, original theory are combined and transforms together in a $2d$-vector representation of the $O(d,d)$ symmetry in the doubled theory.

The massless fields of bosonic string theory are the metric g_{ij}, the two-form b_{ij} and the scalar dilaton ϕ. These fields are combined and encoded by purely geometrical objects in the doubled theory and transforms as tensors under $O(d,d)$. In the doubled theory, there is an $O(d,d)$-invariant generalised metric, G^{MN}, $ds_L^2 = 2dX^i\tilde{X}_i$ on the double torus that unifies g_{ij} and b_{ij}. An $O(d,d)$ singlet denotes the scalar dilaton ϕ through $e^{-2d} = \sqrt{-g}e^{-2\phi}$. When the two-form and the metric are combined in a generalised metric, transforming together under the duality group, the b and g fields are mixed with each other and a geometrical string state can translate to a non-geometrical configuration and momentum modes are mixed with winding numbers of the string.

The original string theory has been formulated in a doubled space in which the T-duality symmetry has been manifest and all of the original fields now transform as tensors under this symmetry group. In this framework of doubled space, it is possible to write down an $O(d,d)$ and gauge invariant spacetime action for doubled field theory without reference to the original fields. The doubled space now holds all possible T-duality configurations of the original theory and the physical space is a d-dimensional brane, a slice, of this larger theory. The T-duality is the choice of which subspace, $T^d \subset T^{2d}$, that is the physical space and is solved by a section condition. The concept of transformations, diffeomorphisms, in this doubled space has, however, to be generalised, since directions in the space no longer mean the same thing.

4.1 Generalised diffeomorphisms

The generalised diffeomorphisms in the doubled space are described by a generalised Lie derivative. The difference against the ordinary Lie derivative is that the projection on the adjoint representation of $GL(n)$ should now be replaced by that of $O(d,d)$. The generalised Lie derivative used in doubled geometry is called a **Dorfman bracket** and is given by

$$\mathcal{L}_U V^M = U^N\partial_N V^M - (\delta^M_Q\delta^P_N - \eta^{MP}\eta_{NQ})\partial_P U^Q V^N = L_U V^M + \eta^{MN}\eta_{PQ}\partial_N U^P V^Q, \quad (4.1)$$

where η is the $O(d,d)$-invariant metric and $(\delta^M_Q\delta^P_N - \eta^{MP}\eta_{NQ})$ is the projection on the adjoint representation of $O(d,d)$. From this generalised Lie derivative and a generalised version of the Lie bracket, the algebra of diffeomorphisms in the doubled spacetime can be derived. A full understanding of infinitesimal transformations in this doubled space is developed and the geometry in terms of curvature are given in a covariant description. The dynamics of fields in the doubled space are then encoded in a generalised version of the Einstein equations.

4.2 Section condition

The level matching constraint in closed string theory makes it necessary to introduce the following condition

$$\eta^{MN}\partial_M\partial_N = 0, \quad (4.2)$$

where the derivatives are acting on arbitrary gauge parameters and fields. This is called the **weak constraint**. In the concept of doubled geometry, a stronger version of this condition is needed since the product of two functions of (4.2) doesn't always satisfy the weak condition.

This stronger constraint, called the **strong section condition** is also derived from the closure of the algebra of diffeomorphisms and is written as

$$\eta^{MN}\partial_M \otimes \partial_N = 0, \tag{4.3}$$

where '$\otimes$' denotes that the derivatives are acting on different objects. The section condition is solved by simply picking out any pair of covectors in a d-dimensional subspace and can be interpreted as the choice of T-duality. It picks out d dimensions of the doubled space, the choice of physical space.

Chapter 5

M-theory and U-duality

Despite the fact that the superstring theories have to be formulated in 10 spacetime dimensions, type IIA superstring theory was argued to generate an extra compact dimension of radius $R_s \sim g_s^{2/3}$ at the strong coupling limit [55,56]. In this limit, there should therefore exist some eleven-dimensional quantum theory that, at low energies, reduces to an eleven-dimensional theory of supergravity. This is M-theory. Little is known about M-theory and a consistent quantum gravity theory in eleven dimensions is still missing, it is however known that a dimensional reduction on a circle of eleven-dimensional supergravity yields type IIA supergravity. Because of this, M-theory has to reduce to eleven-dimensional supergravity at small energies. Starting from this known eleven-dimensional supergravity theory, adding quantum concepts and studying the symmetries that we know M-theory should exhibit, it is possible to learn a lot about the mysterious theory.

The field content of the bosonic part of 11-dimensional supergravity includes the metric, g_{MN}, and a 3-form, $\mathcal{C}_{MNP}$ (denoted by $\mathcal{C}_3$). The action of the theory can be written as

$$S_{11} = \frac{1}{l_p^9} \int \mathrm{d}^{11}x\sqrt{-g}\left(R - \frac{l_p^6}{48}(d\mathcal{C})^2\right) + \frac{\sqrt{2}}{2^7 \cdot 3^2} \int \mathcal{C} \wedge d\mathcal{C} \wedge d\mathcal{C} \tag{5.1}$$

consisting of the Einstein-Hilbert term involving the Ricci scalar, a kinetic term for the 3-form gauge potential, $\mathcal{C}_3$, and the topological Wess-Zumino term which is required by supersymmetry.

Toroidally compactified string- or M-theory and their low energy supergravity field theories are all subject to T-duality, this duality is however only a small part of larger symmetry group, U-duality. This larger symmetry is a unification of the T- and S-dualities, hence its name. T-duality is the part of U-duality visible in perturbation theory, while the U-duality group relates different but equivalent M-theory backgrounds.

5.1 U-duality

The symmetries of M-theory compactified on a torus, T^n, can more easily be discussed in terms of its low energy effective action. Toroidal compactifications preserves the $\mathcal{N} = 1$,

D	n	$G_d = E_{n(n)}$	H_n
10	1	$\mathbb{R}^+$	1
9	2	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$U(1)$
8	3	$Sl(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$Sl(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}(\mathbb{R})$	$USp(8)$
4	7	$E_{7(7)}(\mathbb{R})$	$SU(8)$
3	8	$E_{8(8)}(\mathbb{R})$	$SO(16)$

Table 5.1: Cremmer-Julia symmetry groups and their maximal compact subgroups.

11-dimensional supersymmetry algebra of the action and the generators, Q_α decomposes to another superalgebra in $11 - n$ dimensions in representations of $SO(1, 10 - n) \times SO(n)$. The first factor in this group is the Lorentz group in the uncompactified dimensions and the second factor is called an R-symmetry which is part of the local supersymmetry.

5.1.1 Continuous symmetries

The low energy effective action of M-theory also has a continuous, global symmetry group, G_n, containing

$$SO(n-1,n-1,\mathbb{R}) \bowtie Sl(n,\mathbb{R}). \tag{5.2}$$

The bowtie denotes the group generated by the two non-commuting subgroups. The first factor is from T-duality and the second includes S-duality and is the modular group of T^n. The groups, G_n, generated in this way corresponds to the non-compact, normal real form of the exceptional groups, $G_n = E_{n(n)}(\mathbb{R})$, found by Cremmer and Julia [14]. For $n < 6$, the Dynkin diagrams of the exceptional groups can be extrapolated to find the groups in the lower dimensional compactifications. A list of the exceptional groups describing this global symmetry, together with their maximal compact subgroup, is found in table 5.1. The occurance of these groups can be motivated by a counting of the number of scalar fields in the theory, and a matching of this number with the dimension of the coset space G_n/H_d, this is done in section 5.2.

5.1.2 Charge quantisation

The continuous symmetry groups, G_n, can not be a part of the quantum theory, since the gauge potentials transforms non-trivially under the group. States that are charged under the transforming potentials picks up quantised charges that breaks the symmetry to a quantised subgroup $E_{n(n)}(\mathbb{Z}) \subset E_{n(n)}(\mathbb{R})$. The U-duality group of toroidally compactified M-theory then proves to be generated by the T-duality $SO(n-1, n-1, \mathbb{Z})$ of the type IIA string theory and by the modular group of the torus, T^n; $Sl(n, \mathbb{Z})$ as

$$E_{n(n)}(\mathbb{Z}) = SO(n-1, n-1, \mathbb{Z}) \bowtie Sl(n, \mathbb{Z}). \tag{5.3}$$

The U-duality groups for M-theory are the discrete subgroups of the ones listed in table 5.1, with $\mathbb{R} \to \mathbb{Z}$.

5.2 Hidden symmetries and enlarged spacetime

The hidden symmetries of compactified 11D SUGRA was first discovered by Julia and Cremmer when they studied the enhanced R-symmetry, $SO(8)$, of 11 dimensional supergravity compactified on T^7. The occurance of the exceptional groups can, as already statad, be motivated by a counting of the degrees of freedom of the compactified theory and then by matching the number of scalar fields to that of the dimension of a moduli space G_n/H_n. H_n is the R-symmetry of the superalgebra, and in order to obtain a positive metric on the moduli space, it has to be the maximal compact subgroup of G_n. The dimension of the space together with this fact is enough to determine G_n.

The field content of the compactified theories has to be the same as that of the uncompactified theory. On compactification, the fields split in the compactified and uncompactified directions. After compactification on an n-torus, leaving $D = 11 - n$ dimensions extended, the field content of the uncompactified theory (g_{MN},$\mathcal{C}_{MNP}$) splits into the following parts when $M \to (\mu, m)$,

$$g_{MN}, \mathcal{C}_{MNP} \xrightarrow{M \to (\mu, m)} \begin{cases} g_{\mu\nu} & \text{gravity} \\ g_{\mu m} & \text{1-form} \\ g_{mn} & \text{scalars} \\ \mathcal{C}_{\mu\nu\lambda} & \text{3-form} \\ \mathcal{C}_{\mu\nu m} & \text{2-form} \\ \mathcal{C}_{\mu mn} & \text{1-form} \\ \mathcal{C}_{mnp} & \text{scalars} \end{cases} \tag{5.4}$$

where $M = 1 \ldots 11$, $\mu = 1 \ldots D$ and $m = 1 \ldots n$ and the time direction is left uncompactified in the greek indices. The description of the fields are given from the uncompactified directions point of view, so $g_{\mu\nu}$ is a metric for the uncompactified space and so on. Scalar fields thus comes from the internal directions of the metric, g_{mn}, and the three-form, $\mathcal{C}_{mnp}$. The number of scalars found in the symmetric metric, g_{mn} where m,$n = 1 \ldots n$ is $n(n+1)/2$. The number of scalars found in the antisymmetric three-form, $\mathcal{C}_{mnp}$, is a combinatorial problem on about chosing three indices out of n possibilities, the total number is therefore $\binom{n}{3}$. These are the only possibilities of finding scalars for $n \leq 5$.

In higher dimensional compactifications, we also have to take into account the dual forms. In 11D, the three-form, $\mathcal{C}_3$ is dual to a six-form $\mathcal{E}_6$ and upon compactification the vector (one-form) from the metric, $g_{\mu m}$ is dual to a nine-form with one index in the compactified directions, $\mathcal{K}_{1;8}$, by going through their field strengths. Because of this, 3-forms in $n = 6$ are dual to scalars, 2-forms are dual to 1-forms and 1-forms are dual to 2-forms for example.

The total counting of scalars for all dimensions of compactification are presented in table 5.2. Now, the dimension of the coset-spaces, G_n/H_n, should be the same of the total numbers of scalars. Knowing the dimensions of H_n, since they are the known R-symmetry groups, and using that $\dim(G_n) - \dim(H_n)$ should equal the number of scalars we see that it coincides with the dimensions of the exceptional Lie groups, $E_{n(n)}$. For $n = 7$, for example, $\dim(E_{7(7)}) - \dim(SU(8)) = 133 - 63 = 70$, matching the number of scalars.

The same type of counting can be done to find the dimensions of the vector and 2-form representations presented in table 5.3 and 5.4. Again, the number of vectors and 2-forms are seen to fit beautifully in representations of the exceptional Lie groups.

D	n	g	$\mathcal{C}_3$	$\mathcal{E}_6$	$\mathcal{K}_{1;8}$	total	scalar manifold
10	1	1				1	$\mathbb{R}^+$
9	2	3				3	$Sl(2,\mathbb{R})/U(1) \times \mathbb{R}^+$
8	3	6	1			7	$Sl(3,\mathbb{R})/SO(3) \times Sl(2,\mathbb{R})/U(1)$
7	4	10	4			14	$Sl(5,\mathbb{R})/SO(5)$
6	5	15	10			25	$SO(5,5,\mathbb{R})/SO(5) \times SO(5)$
5	6	21	20	1		42	$E_{6(6)}/USp(8)$
4	7	28	35	7		70	$E_{7(7)}/SU(8)$
3	8	36	56	28	8	128	$E_{8(8)}/SO(16)$

Table 5.2: The number of scalar fields and the corresponding scalar manifolds for compactified M-theory.

D	n	g	$\mathcal{C}_3$	$\mathcal{E}_6$	$\mathcal{K}_{1;8}$	total	vector representation
10	1	1				1	**1**
9	2	2	1			3	**3** of $Sl(2)$
8	3	3	3			6	$(\mathbf{3},\mathbf{2})$ of $Sl(3) \times Sl(2)$
7	4	4	6			10	**10** of $Sl(5)$
6	5	5	10	1		16	**16** of $SO(5,5)$
5	6	6	15	6		27	**27** of $E_{6(6)}$
4	7	7	21	21	7	56	**56** of $E_{7(7)}$
3	8	8	28	56	36	128	**248** of $E_{8(8)}$

Table 5.3: The number of vectors and charge representations for compactified M-theory.

D	n	g	$\mathcal{C}_3$	$\mathcal{E}_6$	$\mathcal{K}_{1;8}$	total	2-form representation
10	1		1			1	**1**
9	2		2			2	**2** of $Sl(2)$
8	3		3			3	$(\mathbf{3},\mathbf{1})$ of $Sl(3) \times Sl(2)$
7	4		4	1		5	**5** of $Sl(5)$
6	5		5	5		10	**10** of $SO(5,5)$
5	6		6	15	6	27	$\overline{\mathbf{27}}$ of $E_{6(6)}$
4	7		7	35	28	70	**133** of $E_{7(7)}$

Table 5.4: Counting of 2-forms.

5.3 Representations in $E_{n(n)}(\mathbb{Z})$

From the above counting of degrees of freedom, the representations of vectors, tensors and forms in G can be found. R_1 denotes the vector representation (an upper index) and are the parameters of generalised diffeomorphisms. Generalised momentum transforms in the $\bar{R}_1$-module (a lower index). The Dynkin labels for R_1 are taken to be $(10\ldots0)$, placing the representations at distinct nodes in the Dynkin diagrams as of table 2.2. Representations $R_k^{(n)}$ of $E_{n(n)}$ coincide with representations of form fields where R_k are possible representations for k-form fields in the uncompactified $11-n$ dimensions. The sequences, R_k, does not (as usual) stop at a finite k, it instead generates an infinite sequence of form fields responsible for the reducibility of transformations. A part of the sequences are listed in table 5.5.

The representation R_2 happens to be a part of the symmetric tensor product of two R_1's,

$$\otimes_s^2 R_1 = R_2 \oplus \ldots, \tag{5.5}$$

which will be of use when discussing the section condition of exceptional geometry in the next chapter in which bilinears in generalised momenta projected on $\bar{R}_2$ vanish. An example of this, for $n=6$,

$$\mathbf{27} \otimes_s \mathbf{27} = V^{(M}V^{N)} = C_{MNP}V^{(M}V^{N)} \oplus \cdots = R_2 \oplus \ldots \tag{5.6}$$

where the symmetric, invariant tensor C_{MNP} found in E_6 have been used to project out the R_2 part of the product.

The $GL(n,\mathbb{R})$ diffeomorphism group structure as well as the gauge transformations of the 3-form $\mathcal{C}_3$ should both be embedded in $E_{n(n)}$. In the structure of representations in the exceptional groups, the fields transforms together in the same representation of $E_{n(n)}$, mixing the space symmetries with those from the gauge, making it necessary to develop a generalised version of geometry.

n	R_1	R_2	R_3	R_4
3	$(\mathbf{3},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{5}$	$\overline{\mathbf{10}}$
5	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$
6	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\overline{\mathbf{351}'}$
7	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645} \oplus \mathbf{133}$
8	$\mathbf{248}$	$\mathbf{3875} \oplus \mathbf{1}$	$\mathbf{147250} \oplus \mathbf{3875} \oplus \mathbf{248}$	$\mathbf{6696000} \oplus \mathbf{779247} \oplus \mathbf{147250}$ $\oplus 2 \cdot \mathbf{30380} \oplus \mathbf{3875} \oplus 2 \cdot \mathbf{248}$

Table 5.5: Representations, $R_k^{(n)}$.

Chapter 6

Exceptional geometry

This chapter is the main topic and result of this thesis. Here, a tensor formalism for the exceptional geometry is constructed. The $GL(n)$ symmetry of ordinary geometry is here exchanged with the group $G = E_{n(n)} \times \mathbb{R}^+$ and the local group of rotations by the maximal compact subgroup H in G. A generalised, exceptional theory of gravity with manifest G-symmetry is constructed and a tensor formalism is constructed. First, a generalised Lie derivative and a Lie bracket is constructed from the required properties of such an object. We want it to form an algebra generating the diffeomorphisms on the enhanced space, this requirement determines the structure completely and the complete, generalised Lie derivative is given explicit exressions for various dimensions of compactification. The Jacobi identity is also checked and analysed. The section condition, making the connection to the physical space, is derived and examined. After the diffeomorphisms are under control, we turn our focus to forming invariant objects. A covariant derivative is introduced, defining an affine and a spin connection. Then follows the definition and analysis of curvature before we end up with a discussion on tensor fields.

The first six sections of this chapter closely follows "The gauge structure of generalised diffeomorphisms" [52] written by D. S. Berman, M. Cederwall, A. Kleinschmidt and D. C. Thompson and the rest of the sections of this chapter is based on **Paper I** found in the appendix.

6.1 Generalised diffeomorphisms

The generalised diffeomorphisms on the enhanced, exceptional space are gauge transformations that unifies tensor gauge transformations and diffeomorphisms. The U-duality mixes gravitational and tensorial degrees of freedom, and so does their gauge transformations. Diffeomorphisms in ordinary geometry are encoded in the Lie derivative. In the case of U-duality and its symmetry group, $E_{n(n)}$, the $\mathfrak{gl}(n)$-transformation part of the ordinary Lie derivative is assumed to be that of the Lie algebra $\mathfrak{e}_{\mathfrak{n}(\mathfrak{n})}$ plus a real scaling, $\mathbb{R}$. Tensors should transform as tensors under $E_{n(n)} \times \mathbb{R}$, and a generalised diffeomorphism has to be constructed on the form

$$\delta_U V^M = \mathcal{L}_U V^M = U^N \partial_N V^M - \alpha P_{(adj)}{}^M{}_N{}^P{}_Q \partial_P U^Q V^N + \beta \partial_N U^N V^M. \qquad (6.1)$$

Here $\mathcal{L}_U$ defines the generalised, exceptional, Lie derivative. This can be rewritten in a more general form as,

$$\mathcal{L}_U V^M = L_U V^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q = U^N \partial_N V^M + Z^{MN}{}_{PQ} \partial_N U^p V^Q \qquad (6.2)$$

where Y is an $E_{n(n)}$-invariant tensor, it takes two R_1-indices and picks out an R_2 (projects two R_1-indices on an R_2). The rewriting with Z will be a useful notation later. By equating the first expression of (6.2) with (6.1), an expression for Y is found to be

$$Y^{MN}{}_{PQ} = \delta^M_P \delta^N_Q - \alpha P_{(adj)}{}^M{}_{Q,}{}^N{}_P + \beta \delta^M_Q \delta^N_P. \qquad (6.3)$$

α and β are constants and the upper indices takes values in the coordinate representation, R_1 of $E_{n(n)}$. In the expression for Y, we can understand and motivate the structure of the generalised Lie derivative. The first part can be seen as an undoing of the $\mathfrak{gl}(n)$ part of the ordinary Lie derivative, the second part is the equivalence of this in the context of the exceptional Lie derivative: a projection on the adjoint representation of $E_{d(d)}$ together with a real scaling in the last part.

The generalised Lie derivative acting on tensors with an arbitrary number of upper (R_1) and lower ($\bar{R}_1$) indices is written as

$$\begin{aligned} \mathcal{L}_U W^{M_1 \dots M_p}{}_{N_1 \dots N_q} = {} & U^P \partial_P W^{M_1 \dots M_p}{}_{N_1 \dots N_q} \\ & + \sum_{i=1}^{p} Z^{M_i Q}{}_{RP} \partial_Q U^R W^{M_1 \dots M_{i-1} M_P M_{i+1} \dots M_p}{}_{N_1 \dots N_q} \\ & - \sum_{i=1}^{q} Z^{PQ}{}_{RN_i} \partial_Q U^R W^{M_1 \dots M_p}{}_{N_1 \dots N_{i-1} N_P N_{i+1} \dots N_q}. \end{aligned} \qquad (6.4)$$

The problem of constructing the generalised diffeomorphisms is now to find exclicit expressions for Y.

6.2 A generalised Lie bracket

As stated in chapter 3, the Lie derivative builds a Lie algebra representation of vector fields through the Lie bracket. The Lie derivative in ordinary geometry is already antisymmetric in U and V by its definition, so the algebra is generated by $L_U V = [U, V]$. This is unfortunately not true by definition in the case of our exceptional Lie derivative. The exceptional groups are not matrix Lie algebras, so there is no matrix commutation that can be used as a Lie bracket. In order for the Lie derivative to form an algebra, we have to make sure that the symmetric part of $\mathcal{L}_U$ vanishes. This is done in section 6.5 because we will need some more restrictions on Y in order to show this.

The generalised Lie bracket, denoted by $[\![U, V]\!]$, should be completely antisymmetric in U and V. So, we define it by adding an anti-symmetrisation of the U and V vectors in the Y part of the Lie derivative to the ordinary Lie bracket, resulting in the following construction

$$[\![U, V]\!] = [U, V]^M + \frac{1}{2} Y^{MN}{}_{PQ} (\partial_N U^P V^Q - \partial_N V^P U^Q). \qquad (6.5)$$

The defining tensor of the Lie derivative, Y, is completely defined by the construction of the generalised Lie bracket and that it should form an algebra,

$$[\mathcal{L}_U, \mathcal{L}_V] = \mathcal{L}_{[\![U,V]\!]}. \tag{6.6}$$

6.3 The section condition

The first equation obtained by commuting two of the generalised diffeomorphisms, eq (6.8), (as done in the next section) is $Y^{MN}{}_{PQ}\partial_M \ldots \partial_N \cdots = 0$, where the dots indicates that the derivatives are acting on different objects. A solution to this equation is called a **section condition**. It is called a weak section condition for the solution $P^{MN}_{(R_2)}{}_{PQ}\partial_M\partial_N\Phi = 0$, where the derivatives are acting on the same field Φ. The strong section condition is the solution where the derivatives are acting on different fields Φ and Φ', $P^{MN}_{(R_2)}{}_{PQ}\partial_M\Phi\partial_N\Phi' = 0$. $P_{(R_2)}$ denotes a projection on the R_2 representation (e.g we choose the part of the equation that is in R_2, R_2 defines the section condition). The strong section condition is the one important in our case and we write it in a shorthand notation as

$$(\partial \otimes \partial)|_{\bar{R}_2} = 0. \tag{6.7}$$

Any solution to the section condition picks out an n-dimensional subspace that is conserved by $GL(n)$, reducing the dimension of spacetime down to n. It provides a natural embedding of the n compact dimensions in D dimensional physical space and picks out a subsurface with the correct number of physical dimensions. U-duality is the choice of this subspace and can be seen of as a brane in the compactified, enhanced space directions. The section condition is a quadratic condition on momenta, a cõne in the momentum space and its solutions lies in the biggest possible linear subspace of the cône where $(p \times p')|_{\bar{R}_2} = 0$. The coordinate representation of E_n has to branch to the correct field content when the global symmetry is broken by the section condition.

These are the section conditions, explicitly expressed in the various dimensions with Φ and Φ' denoting some fields,

$$\begin{aligned}
n = 3 &: \varepsilon^{\alpha\beta}\partial_{a\alpha}\Phi\partial_{b\beta}\Phi' = 0,\;\; \alpha = 1{,}2, a = 1{,}2{,}3,\\
n = 4 &: \varepsilon^{abcde}\partial_{ab}\Phi\partial_{cd}\Phi' = 0,\;\; a = 1\ldots 5,\\
n = 5 &: \partial_\alpha\gamma^{a\alpha\beta}\partial_\beta = 0,\;\; a = 1\ldots 10, \alpha = 1\ldots 16,\\
n = 6 &: \varepsilon_{\alpha\beta}\partial^\alpha_a\partial^\beta_b + \varepsilon_{abcdef}\partial^{cd}\partial^{ef} = 0 \textbf{ and } \partial^{ab}\partial^\beta_b = 0,\;\; a = 1\ldots 6, \alpha = 1{,}2.
\end{aligned}$$

Solutions to the section condition can be studied in a linearised version with the use of pure spinors. This is, however, out of scope for this thesis.

$n = 4$ as an example

In $n = 4$, the U-duality group is $G_4 = SL(5) \times \mathbb{R}$ and the vector representation is $R_1 = \mathbf{10}$. So, vectors lies in R_1, $x^M \in R_1$, and derivatives in $\bar{R}_1$, $\partial_M \in \bar{R}_1$. R_2 is defined to be a part of the symmetric product of two R_1 (the section condition singles out this part in the symmetric product of two derivatives). Doing this product explicitly gives us that $\otimes^2_s R_1 = \bar{\mathbf{5}} \oplus \mathbf{10}$, it consists of one large and one small representation. R_2 is chosen to be $R_2 = \bar{\mathbf{5}}$ and this choice defines the section condition.

In the section condition, a quadratic momentum is projected on the $\bar{R}_2$ representation, $\partial^2|_{\bar{R}_2} = 0$. To make this projection, we write the $\bar{R}_1$-index of the derivative with two indices in $\bar{R}_2$ instead, $\partial_M = \partial_{(mn)}$ with $M = 1\dots10$ and $m,n = 1\dots5$. Momenta are now written as $\partial_{mn}f\partial_{pq}g = 0$ and all momenta $p, p'\dots$ span a linear momentum space, $p_{[mn}p'_{pq]} = 0$. Now, the section condition says that a part of this space of momenta is zero, the projection on $\bar{R}_2$. This effectively singles out a subspace of dynamics. The most general solution (modulo choice of 5) is $p_{m5} \neq 0$, so that $p_{mn} = 0$ if $m,n \neq 5$. This choice of one direction of five singles out 4 dimensions and is an embedding of $n = 4$ in the larger $D = 10$ space of momenta. The choice of subspace (choice of U-duality) fixes and breaks the U-duality and picks out the physical dimensions.

6.4 Closing the algebra

In order for the generalised Lie bracket to form an algebra, equation (6.6) has to hold. By letting both sides of the equation act on a vector, W, and solving for Y by moving indices around and matching terms with the same type of derivatives, we obtain the following set of equations for Y to fulfill.

$$Y^{MN}{}_{PQ}\partial_M \otimes \partial_N = 0, \tag{6.8}$$

$$(Y^{MN}{}_{TQ}Y^{TP}{}_{RS} - Y^{MN}{}_{RS}\delta^P_Q)\partial_{(N} \otimes \partial_{P)} = 0, \tag{6.9}$$

$$\begin{aligned}&(Y^{MN}{}_{TQ}Y^{TP}{}_{[SR]} + 2Y^{MN}{}_{[R|T|}Y^{TP}{}_{S]Q}\\ &- Y^{MN}{}_{[RS]}\delta^P_Q - 2Y^{MN}{}_{[S|Q|}\delta^P_{R]})\partial_{(N} \otimes \partial_{P)} = 0,\end{aligned} \tag{6.10}$$

$$\begin{aligned}&(Y^{MN}{}_{TQ}Y^{TP}{}_{(SR)} + 2Y^{MN}{}_{(R|T|}Y^{TP}{}_{S)Q}\\ &- Y^{MN}{}_{(RS)}\delta^P_Q - 2Y^{MN}{}_{(S|Q|}\delta^P_{R)})\partial_{[N} \otimes \partial_{P]} = 0.\end{aligned} \tag{6.11}$$

Here, '$\otimes$' denotes that the derivatives acts on different objects. The first equation (6.8) comes from the terms containing a derivative on W and resembles the section condition, discussed in the previous section. The second equation (6.9) are from terms multiplying $\partial^2 UV$ and $U\partial^2 V$ and can be rewritten in terms of Z as $(Z^{MN}{}_{TQ}Z^{TP}{}_{RS} + Z^{MP}{}_{RQ}\delta^N_S)\partial_{(N} \otimes \partial_{P)} = 0$. The third (6.10) and fourth (6.11) equations are from the mixed derivative terms with $\partial U \partial V$, one equation for each symmetrisation. At this stage, no symmetry properties has been assumed for Y, it is still a general expression without connection to a specific symmetry group. The equations can also be used in the case of doubled geometry for the T-duality group.

The conditions for Y may seem overdetermined and it may not be possible to find a Y solving these equations for any algebra. As soon will be shown, it is however possible to solve these equations, finding an expression for Y and closing the algebra for the exceptional cases where $3 \leq n \leq 7$. Now, finding explicit expressions for Y, satisfying these equations, depends on the dimension of the compactification space and its U-duality group.

The section condition (6.7) (the solution to the first equation) projects the derivatives on the R_2 representation. So, the easiest expression for Y satisfying the first equation (6.8) is that it is proportional to the projector on R_2 itself

$$Y^{MN}{}_{PQ} = kP^{MN}_{(R_2)}{}_{PQ}. \tag{6.12}$$

For $n \leq 6$, this proves to be correct and Y is then symmetric in pairs of indices. For $n = 7$, Y contain another term that also vanishes when contracted with the derivatives in the section condition.

Starting with $n \leq 6$ and entering equation (6.12) into equations (6.9), (6.10), (6.11) we see that they simplify when there is an extra symmetry property among the indices. Terms in equation (6.10) and (6.11) are combined in symmetrisations in three indices and the resulting expression also happens to solve equation (6.9). It turns out that for $n \leq 5$ using an expression satisfying (6.8), the other three equations are solved simultaneously if

$$Y^{(MN}{}_{TQ}Y^{P)T}{}_{RS} - Y^{(MN}{}_{RS}\partial_Q^{P)} = 0. \tag{6.13}$$

For $n = 6, 7$ this expression needs some extra terms because in those dimensions, there are some more possible projections present. Using the expression for Y in (6.12), we see that the index structure of equation (6.13) is $\bar{R}_1 \otimes \bar{R}_2(\otimes_s^3 R_1)$. As described in section 2.8, the number of singlets in the tensor product gives the number of possible invariant tensors, or the number of irreducible modules in $T = (\bar{R}_1 \otimes \bar{R}_2) \cap (\otimes_s^3 R_1)$.

For $n \leq 5$, there is only one invariant tensor in the product and the terms in (6.13) therefore has to be proportional to each other. The constant in (6.12) can then be found by taking a trace of the equation. When done, k is found to be $k = 2(n-1)$ and the full explicit expressions for Y when $n \leq 5$ are found and are presented in the end of this chapter.

When $n = 6, 7$, there are two invariant tensors in the equation, so the terms no longer has to be proportional and we have to add some extra terms. We know, from the table in section 2.8, that there is a symmetric, invariant tensor, c^{MNP} in E_6. Normalising this tensor so that $c^{MNP}c_{MNP} = 27$ (27 is the dimension of the vector representation), the projection on R_2 can be written as $P^{MN}_{(\mathbf{27})}{}_{PQ} = c^{MNP}c_{PQR}$. Adding this possibility to the equation with the correct normalisations and symmetrisations gives us

$$10P^{(MN}_{(\bar{\mathbf{27}})}{}_{TQ}P^{P)T}_{(\bar{\mathbf{27}})}{}_{RS} - P^{(MN}_{(\bar{\mathbf{27}})}{}_{RS}\delta_Q^{P)} - \frac{1}{3}c^{MNP}c_{QRS} = 0. \tag{6.14}$$

By cycling some indices around and using the section condition, the equation is seen to be solved by the same expression as for $n \leq 5$,

$$Y^{MN}{}_{PQ} = 2(n-1)P^{MN}_{(\bar{\mathbf{27}})}{}_{PQ}. \tag{6.15}$$

For $n = 7$, the U-duality group is E_7 and by looking at the table of invariant tensors we see that there is an antisymmetric invariant tensor, ε^{MN}, present. There is also a symmetric invariant tensor, c^{MNPQ}. The presence of the antisymmetric invariant tensor means that the projection on R_2 in equation (6.12) no longer is symmetric in indices. The full projector on $R_2 = \mathbf{133}$ has to be written as

$$P^{MN}_{(\mathbf{133})}{}_{PQ} = c^{MN}{}_{PQ} + \frac{1}{12}\delta_P^{(M}\delta_Q^{N)}. \tag{6.16}$$

Now, writing down the expression that simultaneously solves the four equations, the equivalent to equation (6.13), gives us the following equation,

$$12P^{(MN}_{(\mathbf{133})}{}_{TQ}P^{P)T}_{(\mathbf{133})}{}_{RS} - 4c^{MNPT}P_{(\mathbf{133})TQRS} - P^{(MN}_{(\mathbf{133})}{}_{RS}\delta_Q^{P)} = 0. \tag{6.17}$$

The expression for Y is found to be that of Y in the lower dimensions but with an antisymmetric term added,

$$Y^{MN}{}_{PQ} = 2(n-1)P^{MN}_{(R_2)}{}_{PQ} + \frac{1}{2}\varepsilon^{MN}\varepsilon_{PQ}. \tag{6.18}$$

The full expressions for Y are summarised in the following list,

$$\begin{aligned}
n=3 &: \quad Y^{i\alpha,j\beta}{}_{k\gamma,l\delta} = 4\delta^{ij}_{kl}\delta^{\alpha\beta}_{\gamma\delta},\\
n=4 &: \quad Y^{mn,pq}{}_{rs,tu} = 6\delta^{mnpq}_{rstu},\\
n=5 &: \quad Y^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\gamma^{\alpha\beta}_{a}\gamma^{a}_{\gamma\delta},\\
n=6 &: \quad Y^{MN}{}_{PQ} = 10c^{MNR}c_{PQR},\\
n=7 &: \quad Y^{MN}{}_{PQ} = 12c^{MN}c_{PQ} + \delta^{(M}_{P}\delta^{N)}_{Q} + \frac{1}{2}\varepsilon^{MN}\varepsilon_{PQ}.
\end{aligned} \tag{6.19}$$

Where the index notation are as follows: for $\underline{n=3}$; the U-duality group, $SL(3)\times SL(2)$, is not semisimple and $R_1 = (\mathbf{3},\mathbf{2})$ so we have one set of indices in each group $\alpha = 1,2$ and $i = 1,2,3$. For $\underline{n=4}$; $R_1 = \mathbf{10}$ and $R_2 = \bar{\mathbf{5}}$ and $m = 1,\ldots,5$. In $\underline{n=5}$; the vector representation is a spinorial representation, $\mathbf{16}$, and $R_2 = \mathbf{10}$, denoted by $a = 1,\ldots,10$ and $\alpha = 1,\ldots,16$. For $\underline{n=6}$; $R_1 = \mathbf{27}$ and $M = 1,\ldots,27$. Similarily for $\underline{n=7}$ where $R_1 = \mathbf{56}$ and $M = 1,\ldots,56$.

By a comparison, the constants α_n and β_n in the expression in (6.3) are found to be $\beta_n = \frac{1}{9-n}$ while α_n takes the values $3,4,6,12$ for $n = 4,5,6,7$.

Since T-duality is a part of U-duality, everything in this chapter holds also for doubled geometry. By comparing the exceptional Lie derivative with the one for doubled geometry 4.1 we see that an expression for Y in the case of doubled geometry is

$$Y^{MN}{}_{PQ} = \eta^{MN}\eta_{PQ}. \tag{6.20}$$

6.5 The symmetric part of $\mathcal{L}_U$

We define the symmetric part of the generalised Lie derivative as $((U,V)) = \frac{1}{2}(\mathcal{L}_U V + \mathcal{L}_V U)$ and want it to generate a zero transformation when acting on a vector W^N,

$$\mathcal{L}_{((U,V))}W^M = \cdots = -(Y^{M[N}{}_{PQ}Y^{|P|R]}{}_{[ST]} + Y^{M[N}{}_{[ST]}\delta^{R]}_{Q})\partial_N U^S \partial_R V^T W^Q. \tag{6.21}$$

In the $\ldots$-part of the equation, the definition of $((U,V))$ has been entered into the definition of the generalised Lie derivative and simplified. Then, equation (6.9) was used on the terms symmetric in indices sitting on derivatives, collecting terms and again simplified. In the excplicit expressions for Y in various dimensions (6.19) we see that Y is symmetric in pairs of indices for $n \leq 6$. Because of this, the expression vanishes trivially. For $n = 7$, however, Y is not symmetric. By entering the explicit expression for Y in (6.21) (the c- and δ-part of Y vanishes because they are symmetric in indices), the expression for $n = 7$ can be written as

$$\mathcal{L}_{((U,V))}W^M = -\frac{1}{4}\varepsilon^{NR}\varepsilon_{PQ}\partial_{[N}U^P\partial_{R]}V^Q W^M = 0, \tag{6.22}$$

generating a zero transformation on W^N by the section condition (6.8). The symmetric part of the generalised Lie derivative is thus not zero in itself, it does however generate a zero transformation when acting on an object, which is sufficient.

6.6 The Jacobi identity

The symmetric part of the generalised diffeomorphism does thus not vanish trivially, but its action gives a zero transformation. This is also happens to be true for the generalised Jacobi identity, which has to hold for a Lie bracket. We want the Jacobiator, $[\![U, V, W]\!] = [\![[\![U, V]\!], W]\!] + \{\text{cyclic}\}$, to generate a zero transformation, which can be seen as a generalised version of the Jacobi identity. By writing down one of the terms of the Jacobiator,

$$
\begin{aligned}
[\![[\![U, V]\!], W]\!] &= \frac{1}{2}(\mathcal{L}_{[\![U,V]\!]}W - \mathcal{L}_W[\![U, V]\!]) \\
&= \frac{1}{2}(\mathcal{L}_U\mathcal{L}_V W - \mathcal{L}_V\mathcal{L}_U W) - \frac{1}{4}(\mathcal{L}_W\mathcal{L}_U V - \mathcal{L}_W\mathcal{L}_V U).
\end{aligned}
$$

we see that the Jacobiator can be written in two ways. Because of the cyclicity of U, V and W in the expression, the Jacobiator can be written either as the first or second term of the first line,

$$
[\![U, V, W]\!] = \begin{cases} \frac{1}{4}\mathcal{L}_{[\![U,V]\!]}W + \text{cyclic} \\ \frac{1}{2}\mathcal{L}_W[\![U, V]\!] + \text{cyclic} \end{cases} \tag{6.23}
$$

By splitting the Jacobiator into two parts and expressing them in different ways according to the above possibilities, it can be written as

$$
[\![U, V, W]\!] = \frac{2}{3}[\![U, V, W]\!] + \frac{1}{3}[\![U, V, W]\!] = \frac{1}{6}(\mathcal{L}_{[\![U,V]\!]}W + \mathcal{L}_W[\![U, V]\!]) + \text{cyclic}. \tag{6.24}
$$

The last term is our definition of the symmetric part of the Lie derivative

$$
\frac{1}{6}(\mathcal{L}_{[\![U,V]\!]}W + \mathcal{L}_W[\![U, V]\!]) + \text{cyclic} = \frac{1}{3}(([\![U, V]\!], W)) + \text{cyclic}, \tag{6.25}
$$

which has been shown to generate a zero transformation, for $n \leq 7$. Our generalised Lie derivative now fulfills every requirement of being a Lie derivative and building an algebra. The generalised diffeomorphisms are taken care of and we now turn our attention to tensors and covariant derivatives.

6.7 Tensors and connections

Tensors should be tensors under $E_{n(n)} \times \mathbb{R}$ and in order to define the correct transformation of tensors, we need a covariant derivative, $D = \partial + \Gamma$. This is done in much the same way as in ordinary geometry, by the introduction of an affine connection $\Gamma_{MN}{}^P$ transforming upper and lower indices as,

$$
D_M V_N = \partial_M V_N + \Gamma_{MN}{}^P V_P, \tag{6.26}
$$

$$
D_M V^N = \partial_M V^N - \Gamma_{MP}{}^N V^P. \tag{6.27}
$$

The matrices Γ_M in $(\Gamma_M)_N{}^P$ are valued in the Lie algebra $\mathfrak{e}_{n(n)} \oplus \mathbb{R}$. The affine connection has to make sure that the covariant derivative of an arbitrary tensor again is a tensor. This requirement leads to the specific transformation rule of the affine connection,

$$\begin{aligned} \delta_\xi \Gamma_{MN}{}^P &= \mathcal{L}_\xi \Gamma_{MN}{}^P + Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R \\ &= \mathcal{L}_\xi \Gamma_{MN}{}^P - \partial_M \partial_N \xi^p + Y^{PQ}{}_{RN} \partial_M \partial_Q \xi^R. \end{aligned} \tag{6.28}$$

This can be rewritten by defining $\Delta_\xi = \delta_\xi - \mathcal{L}_\xi$ as a measure of the non-homogeneously transforming parts of objects, tensors should have $\Delta_\xi V^{\cdots}_{\cdots} = 0$. Equation (6.28) can then be written as

$$\Delta_\xi \Gamma_{MN}{}^P = Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R. \tag{6.29}$$

In the above definitions, the $\mathbb{R}$-weight was not taken in to account. An $E_{n(n)}$ invariant tensor also has to carry an $\mathbb{R}$-weight in order to be a tensor under the generalised diffeomorphisms. This charge is additive and behaves like a $U(1)$. Normalising so that a vector, R_1, has weight 1 and an $\overline{R}_1$-index carries -1, the E_6 invariant tensor c^{MNP} has to have weight 3 for example. This extra complexity makes it harder to lower and raising indices. It may also be convenient to use the duality $R_k \leftrightarrow \overline{R}_{9-n-k}$ to write representations with one lower instead of $8-n$ upper indices, this duality is indeed true but has the wrong weight and have to be compensated by adding an extra weight. This makes us consider generalised tensor densities by always specifying the $\mathbb{R}$-weight for an $E_{n(n)}$-module. This is the same as in ordinary geometry, where there is also both tensor and tensor densities. A one-form is the same as a $d-1$-form in the SL perspective, but not from a GL perspective because it scales differently. For the rest of the thesis, the term tensor is used for both, but one has to remember that objects always carries this extra weight. A covariant derivative takes tensors of weight w to ones with weight $w-1$, the definitions in (6.26) are extended to include the transformation of weights as

$$D_M W_N = \partial_M W_N + \Gamma_{MN}{}^P W_P - \frac{w+1}{|R_1|} \Gamma_{MP}{}^P W_N, \tag{6.30}$$

$$D_M W^N = \partial_M W^N - \Gamma_{MP}{}^N W^P - \frac{w-1}{|R_1|} \Gamma_{MP}{}^P W^N. \tag{6.31}$$

As written in chapter 3, torsion in ordinary geometry has a physical meaning, it is set to zero of physical reasons. In this context, we have the choice of defining the torsion part of the affine connection in a way that make our theory meaningful. We define torsion to be the irreducible modules in the affine connection that transforms homogeneously with the Lie derivative (the non-zero part of Z). Of the irreducible components of the affine connection, not all of them can appear in the inhomogeneous terms of equation (6.28). It is only the part occuring in $(\otimes_s^2 \overline{R}_1 \ominus \overline{R}_2) \otimes R_1$ that can include these terms. Now, by studying the overlap $[\overline{R}_1 \otimes \mathfrak{g}] \cap [(\otimes_s^2 \overline{R}_1 \ominus \overline{R}_2) \otimes R_1]$ we can find those modules appearing in both the affine connection and that are able to pick up the inhomogeneous terms. Doing this explicitly for each n, we conclude that the overlap consist of a small module, $\overline{R}_1$, and a larger module. The rest of the affine connection, the part defined as torsion, also consists of a small module, $\overline{R}_1$, and a larger module which coincides with $\overline{R}_{10-n}$. The torsion and non-torsion part of the affine connection, found in this way, is summarised in table 6.1.

Setting torsion to zero further determines our affine connection, but we need an expression for this constraint. We are looking for an object that transforms as a tensor and that only

n	Torsion	Non-torsion
3	$2(\overline{\mathbf{3}},\mathbf{2}) \oplus (\mathbf{6},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{2}) \oplus (\overline{\mathbf{3}},\mathbf{4}) \oplus (\mathbf{15},\mathbf{2})$
4	$\overline{\mathbf{10}} \oplus \overline{\mathbf{15}} \oplus \mathbf{40}$	$\overline{\mathbf{10}} \oplus \overline{\mathbf{175}}$
5	$\overline{\mathbf{16}} \oplus \mathbf{144}$	$\overline{\mathbf{16}} \oplus \overline{\mathbf{560}}$
6	$\overline{\mathbf{27}} \oplus \mathbf{351}'$	$\overline{\mathbf{27}} \oplus \overline{\mathbf{1728}}$
7	$\mathbf{56} \oplus \mathbf{912}$	$\mathbf{56} \oplus \mathbf{6480}$

Table 6.1: Parts of the affine connection that is torsion resp. non-torsion.

contains the torsion parts of the affine connection. An expression for torsion is found to be

$$T_{MN}{}^{P} = 2\Gamma_{[MN]}{}^{P} + Y^{PR}{}_{SN}\Gamma_{RM}{}^{S}. \tag{6.32}$$

It is more natural to write this as

$$T_{MN}{}^{P} = \Gamma_{MN}{}^{P} + Z^{PQ}{}_{RN}\Gamma_{QM}{}^{R}, \tag{6.33}$$

to make the adjoint property of the pair ${}_{N}{}^{P}$ manifest. Setting torsion to zero, a torsion-free connection thus satisfies

$$\Gamma_{MN}{}^{P} + Z^{PQ}{}_{RN}\Gamma_{QM}{}^{R} = 2\Gamma_{[MN]}{}^{P} + Y^{PQ}{}_{RN}\Gamma_{QM}{}^{R} = 0. \tag{6.34}$$

By contracting this equation with δ_P^N and using that $Z^{MP}{}_{PN} = \frac{|R_1|}{9-n}\delta_N^M$ another expression for torsion-freeness is found to be

$$\Gamma_{MN}{}^{N} + \frac{|R_1|}{9-n}\Gamma_{NM}{}^{N} = 0. \tag{6.35}$$

Or, by instead contracting with δ_P^M,

$$Y_{MN}{}^{QR}\Gamma_{QR}{}^{N} = -2\Gamma_{[NM]}{}^{N} = -(1 + \frac{|R_1|}{9-n})\Gamma_{NM}{}^{N}. \tag{6.36}$$

So if torsion is set to zero, the expression in (6.32) vanishes and the transformations kan be written covariantly as

$$\mathcal{L}_U V^M = U^N D_N V^M - \alpha P^{M}_{(adj)\,N,}{}^{P}{}_{Q} D_P U^Q V^N + \beta D_N U^N V^M. \tag{6.37}$$

Replacing the derivatives in our generalised Lie derivative with covariant derivatives, the generalised Lie derivative of a vector turns out not to contain any non-torsion part of the connection. This fact can also be used as an equivalent definition of torsion, as done in [46].

6.8 Vielbeins

We now turn our attention to the equivalent of the local group of rotations in ordinary geometry. For generalised exceptional geometry, the role of this group is played by the local subgroup H, which is the maximal compact subgroup $H_n = K(E_{n(n)})$, listed in table 5.1. Flat R_1 indices under H is denoted by $A, B, \ldots$ and the metric for the local, flat subgroup H is δ_{AB}.

In the same way as in ordinary geometry, a vielbein field, $E_M{}^A$, can be used to define a metric $G_{MN} = E_M{}^A E_N{}^B \delta_{AB}$. The vielbein is a group element of $E_{n(n)} \times \mathbb{R}^+$ and it has to be covariantly constant, meaning that a covariant derivative of a vielbein has to vanish. A spin connection, transforming flat indices, is introduced in the covariant derivative which is defined as

$$D_M E_N{}^A = \partial_M E_N{}^A + \Gamma_{MN}{}^P E_P{}^A - E_N{}^B \Omega_{MB}{}^A \equiv 0. \tag{6.38}$$

The covariantly constant property of the vielbein is called compability and equation (6.38) is called the compability equation. $(\Gamma_M)_N{}^P$ and $(E_N)^A$ (as matrices) are 1-forms in $\mathfrak{e}_{n(n)} \oplus \mathbb{R}$ while Ω is a 1-form in the maximally compact subgroup, $\mathfrak{h}$, of $\mathfrak{g}$. The choice of vielbein breaks $\mathfrak{g} \to \mathfrak{h}$, for example in $n = 6$ where $E_6 \to USp(8)$, the adjoint module of E_6 breaks as

$$\underbrace{78}_{\text{adj}} \to \underbrace{42}_{\text{g/h (MN)}} + \underbrace{36}_{\text{adj in USp(8)}} . \tag{6.39}$$

We now have two constraints on the connections; the vanishing of torsion and the compability equation. In ordinary geometry, both the affine and spin connection is completely determined by these conditions. In the case of exceptional geometry, this is not true, there will be parts of the connections that are undetermined making the covariant derivative not well-defined. To be able to use the connections when forming other objects and taking covariant derivatives, we have to make sure that we are using parts of the connections that are explicitly determined by the constraints. So, the question now is to find the undetermined parts and later make sure that those parts don't show up in calculations.

This analysis can be done by first eliminating the affine connection from the compability equation by using the vanishing of torsion and forming a specific combination of the compability equation containing Γ only through T. The resulting equation is

$$(D_M^{(\Omega)} E E^{-1})_N{}^P + Z^{PQ}{}_{RN} (D_Q^{(\Omega)} E E^{-1})_M{}^R = 0, \tag{6.40}$$

where $D^{(\Omega)}$ denotes a covariant derivative containing only the spin connection. We can also do the opposite, eliminating the spin connection by projecting the compability equation on the part in $\mathfrak{g}/\mathfrak{h}$. After lowering one index, the antisymmetric part is in $\mathfrak{h}$ while the symmetric part is in the coset $\mathfrak{g}/\mathfrak{h}$. The compability equation for the affine connection is thus,

$$(E^{-1} D_M^{(\Gamma)} E)_{(AB)} = 0. \tag{6.41}$$

This can also be written in terms of the generalised metric as

$$D_M G_{NP} = \partial_M G_{NP} + 2\Gamma_{M(NP)} = 0. \tag{6.42}$$

To be able to compare the parts of (6.40) with those of the spin connection, we have to decompose the equation into modules of H, since this is where Ω lives. By doing this, it is found that the content of equation (6.40) is smaller than that of the spin connection which has the structure of $R_1 \otimes \mathfrak{h}$. The missing module (the undetermined part) is called Σ. The same module is also found by doing a similar comparison for the affine connection and its compability equation. This undetermined part, Σ, is listed explicitly for the various dimensions in table 6.2. A well-defined covariant derivative of a module, U, now amounts to check if the undetermined part, Σ, don't show up in the tensor product, $R_1 \otimes U$.

n	H	Σ
4	$SO(5)$	$\mathbf{35} = (04)$
5	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$	$(\mathbf{4}, \mathbf{20}) \oplus (\mathbf{20}, \mathbf{4}) = (01)(03) \oplus (03)(01)$
6	$USp(8)/\mathbb{Z}_2$	$\mathbf{594} = (2100)$
7	$SU(8)/\mathbb{Z}_2$	$\mathbf{1280} \oplus \overline{\mathbf{1280}} = (1100001) \oplus (1000011)$

Table 6.2: The undetermined part of a torsion-free, compatible connection.

There are special cases of well-defined covariant derivatives where the connection is completely absent and $D_M = \partial_M$. As mentioned above, the $\mathbb{R}$-weight also has to be considered and an important property to analyse is to what the weight of a vector, W^M, has to be in order for the divergence $D_M W^M$ to be connection free. From the definition of the covariant derivative including the transformation of the weight in equation (6.30) it follows that

$$D_M W^M = \partial_M W^M - \Gamma_{MN}{}^M W^N + \frac{w-1}{9-n} \Gamma_{NM}{}^N W^M. \tag{6.43}$$

The two Γ-terms cancels for $w = 10 - n$ for a vector of weight 1. This can be expressed as

$$|G|^{-\frac{9-n}{2|R_1|}} D_M V^M = \partial_M (|G|^{-\frac{9-n}{2|R_1|}} V^M) \tag{6.44}$$

and will be of great importance when considering partial integration, measures and actions.

6.9 Curvature

Apart from torsion, the other tensorial part of an affine connection is the curvature. In this section, we seek to define and analyse curvature in a consistent way. Curvature is, schematically, expressed as $R \sim \partial\Gamma + \Gamma^2$. Since we want it to be a tensor, the transformation has to be $(\delta - \mathcal{L})R = \Delta R = \Delta(\partial\Gamma + \Gamma^2) = 0$. Now, starting from the Z-expression for the transformation of the affine connection in (6.29),

$$\Delta_\xi \Gamma_{MN}{}^P = Z^{PQ}{}_{RN} \partial_M \partial_Q \xi^R, \tag{6.45}$$

and taking a derivative, the following transformation of the derivative of the affine connection is found

$$\begin{aligned} \Delta_\xi \partial_M \Gamma_{NP}{}^Q = {} & Z^{QR}{}_{SP} \partial_M \partial_N \partial_R \xi^S \\ & + \Delta_\xi \Gamma_{MR}{}^Q \Gamma_{NP}{}^R - \Delta_\xi \Gamma_{MN}{}^R \Gamma_{RP}{}^Q - \Delta_\xi \Gamma_{MP}{}^R \Gamma_{NR}{}^Q. \end{aligned} \tag{6.46}$$

This expression can be rewritten to schematically resemble $\Delta(\partial\Gamma + \Gamma^2) = Z\partial^3$, the tensorial property of R now tells us to make the Z-term vanish. This can be done by an anti-symmetrisation of the equation in $[MN]$, cancelling the symmetric derivatives in the Z-term. Equation (6.46) is then reduced to

$$\Delta_\xi (\partial_{[M} \Gamma_{N]P}{}^Q + \Gamma_{[M|P|}{}^R \Gamma_{N]R}{}^Q) = -\Delta_\xi \Gamma_{[MN]}{}^R \Gamma_{RP}{}^Q = \frac{1}{2} Y^{RS}{}_{TN} \Delta_\xi \Gamma_{SM}{}^T \Gamma_{RP}{}^Q, \tag{6.47}$$

where the right hand side has been rewritten using the expression for torsion. By contracting the right hand side with δ_Q^N and symmetrising in (MP), it can be written as $\Delta_\xi(\frac{1}{4}Y^{RS}{}_{TQ}\Gamma_{SM}{}^T\Gamma_{RP}{}^Q)$. The whole expression in (6.47) can now be written as a total zero transformation, $\Delta(\dots) = 0$, where the dots are given by

$$\begin{aligned} R_{MN} = {} & \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P \\ & + \Gamma_{(MN)}{}^Q\Gamma_{PQ}{}^P - \Gamma_{P(M}{}^Q\Gamma_{N)Q}{}^P - \frac{1}{2}Y^{PQ}{}_{RS}\Gamma_{PM}{}^S\Gamma_{QN}{}^R, \end{aligned} \tag{6.48}$$

which consequencely transforms as a tensor. Using vanishing torsion, the last term can be rewritten and the curvature can be written as

$$\begin{aligned} R_{MN} = {} & \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P \\ & + \Gamma_{(MN)}{}^Q\Gamma_{PQ}{}^P - \frac{1}{2}\Gamma_{PM}{}^Q\Gamma_{QN}{}^P - \frac{1}{2}\Gamma_{P(M}{}^Q\Gamma_{N)Q}{}^P. \end{aligned} \tag{6.49}$$

Torsion defines the connection and the connection in turn defines the curvature. The curvature can also, to some extent, be defined in terms of vielbeins. The equation of motion for the geometry can be provided by the projection on $\mathfrak{g}/\mathfrak{h}$ of a curvature defined by vielbeins, this would be a an equivalent to a Ricci tensor. In order to sort these things out, we have to calculate to what extent curvature can be defined by vielbeins without containing the undetermined parts of the spin connection.

So, we have an expression for the Ricci-tensor, but it is to large to be expressed in terms of a metric, it has an extra Y-term. If we can show that the relevant part of the Ricci tensor (the projection on the coset) is well defined by connections determined by the equation of covariant vielbein, we have consistent equations of motion for the vielbein. Varying the curvature yields that

$$\delta R_{MN} = D_{(M}\delta\Gamma_{|P|N)}{}^P - D_p\delta\Gamma_{(MN)}{}^P. \tag{6.50}$$

Projecting this expression on $\mathfrak{g}/\mathfrak{h}$, examining the tensor product $\mathfrak{g}/\mathfrak{h} \otimes R_1$ and comparing with table 6.2 it is concluded that the undetermined parts of the connection does not show up in the tensor product. $R_{\{MN\}}$ is thus well defined and can be used as a Ricci tensor, where curly brackets around indices denotes a projection on $\mathfrak{g}/\mathfrak{h}$. The Ricci scalar, $R = G^{MN}R_{MN}$, obtained by contracting the Ricci tensor with the generalised metric is a part of $R_{\{MN\}}$ and is also well defined.

As seen in chapter 3, the curvature scalar shows up in the Lagrangian for geometry, and its variation gives the Einstein tensor. This does, however, involve partial integration which is not as straight forward in exceptional geometry as in ordinary geometry. The variation of the Ricci scalar is given by

$$\delta R = \delta(G^{MN}R_{MN}) = \delta G^{MN}R_{MN} + D_M(\delta\Gamma_N{}^{MN} - \delta\Gamma_N{}^{NM}). \tag{6.51}$$

The expression has to be multiplied with a scalar density from the measure to be able to discard the $D\partial\Gamma$-terms as boundary terms. From equation (6.44) it is concluded that this density has to carry weight $9-n$. So, the Lagrangian density has to be $\mathcal{L} = |G|^{-\frac{9-n}{2|R_1|}}R$ and the generalised Einstein equations, or the equation of motion for the metric, G_{MN}, takes the form

$$R_{\{MN\}} + \frac{9-n}{2|R_1|}G_{MN}R = 0. \tag{6.52}$$

This expression is the left hand side of the generalised Einstein equations with matter fields present which is the scope of the next section.

6.10 Generalised forms

In this section, a discussion regarding tensor fields will be carried out, the full analysis as done in the attached paper will not be done and the reader is therefore encouraged to read the more in-dept description there.

We want a way of describing field equations and gauge symmetries for the dynamics of fields in representations R_k on our locally realised generalised manifold and therefore need a description of some kind of generalised forms. For dimensionally reduces theories formulated in the uncompactified directions, the k-form gauge fields are known to come in the R_k-modules of the U-duality group. The theory of exceptional geometry is instead formulated on the internal directions of the compactification torus. The sequence of representations, R_k, comes from an infinite sequence of ghosts and are related to the generalised diffeomorphisms and their reducibilities as described in [52]. The nilpotenticity property of the derivative, ∂, taking an R_k-module to an R_{k-1}-module hints that the sequence of these modules are responsible for gauge transformations of tensor fields. This is analogous to the exterior derivative on ordinary forms that is nilpotent and maps a k-form to a $(k+1)$-form. As described in section 6.7, the infinite sequence of representations are symmetric under $R_k \leftrightarrow \overline{R}_{9-n-k}$ in just the same way as forms on an ordinary manifold. There are more evidence to further establish the analogy between ordinary forms and the sequence of modules $\{R_k\}_{k=1}^{8-n}$ in exceptional geometry that will be presented below.

To make a covariant description, the derivatives ought to be replaced by covariant versions, $D : R_k \to R_{k-1}$, and to make the connection with the ordinary exterior derivative, we have to examine the covariant derivative acting on our generalised forms. Ideally, the connections should be absent from the derivative and '$D \sim \partial$', in the same way as for the exterior derivative. In the special case of $R_2 \to R_1$ for $n \leq 6$, the covariant derivative can be expressed as

$$\begin{aligned} D_N W^{MN} &= \partial_N W^{MN} - \Gamma_{NP}{}^M W^{PN} - \Gamma_{NP}{}^N W^{MP} \\ &= \partial_N W^{MN} - \frac{1}{2(n-1)} \left(Y^{NP}{}_{RS} \Gamma_{NP}{}^M + Y^{MP}{}_{RS} \Gamma_{NP}{}^N \right) W^{RS} \\ &= \partial_N W^{MN}. \end{aligned}$$

For the proof of the general case of $R_k \to R_{k-1}$ see the attached paper, the derivative happens to be connection-free for $2 \leq k \leq 8-n$. For the cases $R_1 \to R_0$ and $R_{9-n} \to R_{8-n}$, the derivative is however not connection-free. There is a connection-free window amongst the modules and their behaviour is as if they live on an $(9-n)$-dimensional manifold in which the exterior derivative is acting in the wrong way! We are thus only allowed to describe gauge connections and field strengths within this connection-free window. Within this window, we can have a gauge field $A \in R_{k+1}$ with its field strength $F = DA = \partial A \in R_k$, a gauge symmetry parameter, $\Lambda \in R_{k+2}$, used in $\delta_\Lambda A = \partial \Lambda$ and a Bianchi identity $\partial F = \partial^2 A = 0$ in R_{k-1}, making it valid for $1 \leq k \leq 7-n$.

To be able to formulate the equation of motions for gauge fields, we need something resembling the duality operator on forms. The analogy to the Hodge duality operator on forms, which in our case is taking an $F \in R_k$ to $*F \in R_{9-n-k}$, can be defined naturally given a metric in two ways. The first of these is to simply lower the k indices with the metric and adjusting the $\mathbb{R}$-weight, resulting in

$$* F_{M_1 \ldots M_k} = |G|^{-\frac{9-n}{2|R_1|}} G_{M_1 N_1} \ldots G_{M_k N_k} F^{N_1 \ldots N_k}. \tag{6.53}$$

Another way is by using an invariant tensor, $\Sigma^{A_1 \ldots A_{9-n}}$, converting its indices with an inverse vielbein to 'curled' ones $(A_r \to M_r)$ and write

$$* F^{M_{k+1} \ldots M_{9-n}} = \Sigma^{M_1 \ldots M_{9-n}} G_{M_1 N_1 \ldots} G_{M_k N_k} F^{N_1 \ldots N_k}. \tag{6.54}$$

Using one of these definitions of the 'generalised Hodge dual' makes it possible to write the equation of motion for the gauge field A as

$$\partial * F = 0. \tag{6.55}$$

The field content of maximally supersymmetric generalised supergravity makes it necessary to place a gauge potential in R_1 which would have a field strength in R_0 which is outside the connection-free derivative window. Some arguments on how and why this would still work is given in the attached paper but needs further attention. With a covariant and well-defined description of generalised forms, describing matter fields and their equations of motion on the generalised locally realised manifold, a tensor calculus for exceptional geometry is complete.

Chapter 7

Conclusions and future work

In this thesis, the background and need for developing a generalised geometry based on the exceptional groups was presented. Following the steps of constructing ordinary geometry, the relevant objects and concepts has been generalised in the context of exceptional geometry. Transformation of tensors, definitions of a covariant derivative, vielbeins, affine and spin connection, metric, torsion, curvature, tensor fields and a generalisation of the Einstein equation was presented in a completely covariant way yielding a tensor calculus for exceptional generalised geometry.

The tensor calculus for generalised geometry in a local description is by this complete, but there is still a lot of work to be done in the area. Below is a list of (some) problems to be adressed.

- Global symmetries. The work done in the paper in the appendix of the thesis studies infinitesimal, local transformations. The concept of large, global diffeomorphisms and generalised mainfolds is still unsolved. Hohm and Zwiebach discussed this area in [57] in which they exponentiate the Lie algebra in double field theory to a large diffeomorphism. The topology is important for global questions and questions on how to patch together a manifold from many open areas has to be adressed.
- Exceptional supergeometry. A full geometric description of superspace involving fermions is still missing. Work towards this has been done by Coimbra et. al. in [46] where they examine minimal exceptional supergravity. M. Cederwall has published a paper [50], in which non-gravitational supermultiplets were constructed. Such multiplets has to be included in an extended supergravity theory.
- The section condition. The theory described by the exceptional geometry in the enhanced spacetime of the compaction-torus will at some point be included in the theory of the uncompactified directions. The section condition breaks the enhanced symmetry down to the physical degrees of freedom, but how do we interpret this? What is the section condition? Is the section condition a brane? Does it have dynamics? The section condition is a covariant constraint, but not its solutions.
- Integrability. In order to write down an action for the exceptional geometry, the question of integrability and a measure has to be adressed. What should the action be integrating over and how does this involve the section condition?

- <u>E_8 geometry</u>. In the thesis, the case of $n = 8$ is excluded, the algebra in this case fails to close. This is because dual gravity becomes present at this degree of compactification. Is an understanding of dual gravity needed to construct a geometrical description for $n = 8$, $E_{8(8)}$-symmetry?

- <u>Affine extensions</u>. The exceptional Lie groups only exists up to $n = 8$, by compactifying further to $n = 9, 10, 11$, the relevant symmetry groups may be described by the affine extensions of the exceptional Lie groups. The affine Lie group E_{11} has been discussed to be the symmetry group of uncompactified M-theory.

- <u>Other compactifications</u>. What about compactifications other than toroidal? Is it possible to find orbifolds large enough to not kill the dynamics? An example that don't work is $SL(n)$ where $R_1 = \Box$ and $\otimes_s^2 R_1 = \Box\Box$, there is only one representation in this product and there is no possibility of finding interesting solutions to the section condition. There are only two possibilities here, it either kills all dynamics or the dynamics stays in the original gravity. It doesn't single out a lower dimensional version. We want to find a physical situation where the U-duality is broken down to something smaller. In the symmetric product of two R_1, $\otimes_s^2 R_1 = R_2 \oplus$'something large', we want to have an R_2 that is small enough so that $\partial^2|_{\overline{R}_2} = 0$ has interesting solutions (picks out a subspace that is large enough).

M-theory still is a mysterious theory but there are a lot of different angles to attack and the search for the final theory of everything continues...

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CHALMERS

Gothenburg preprint
February, 2013

Exceptional geometry and tensor fields

Martin Cederwall, Joakim Edlund and Anna Karlsson

Fundamental Physics
Chalmers University of Technology
SE 412 96 Gothenburg, Sweden

Abstract: We present a tensor calculus for exceptional generalised geometry. Expressions for connections, torsion and curvature are given a unified formulation for different exceptional groups $E_{n(n)}$. We then consider "tensor gauge fields" coupled to the exceptional generalised gravity. Many of the properties of forms on manifolds are carried over to these fields.

email: martin.cederwall@chalmers.se, joaedl@student.chalmers.se, karann@chalmers.se

1. Introduction

The dualities of string theory or M-theory treat momenta and brane charges on an equal footing. By generalising space-time to include directions conjugate to brane charges, such symmetries can be made manifest, but obviously the concept of geometry has to be modified. There has been considerable progress in the understanding of such models recently, both in the context of U-duality [1-4], which is the main focus of the present paper, and T-duality. We refer to both types of theories as "generalised geometry"; doubled geometry [5-22] in the case of T-duality, and exceptional geometry [23-35] in the case of U-duality.

Turning to the state of the subject of exceptional geometry, it has been shown that it is possible to formulate the dynamics of a generalised metric, parametrising a coset G/H with $G = E_{n(n)} \times \mathbb{R}^+$ and H its maximal compact subgroup, in a manner which respects local symmetries, generalising and including diffeomorphisms [25-29,32,34]. There are also results on an underlying geometry and tensor formalism [32,35], but the covariant tensor calculus has so far been limited to $n = 4$ [35].

The purpose of the present paper is twofold. We give a universal (*i.e.*, valid for all $n \leq 7$) version of exceptional geometry, and a tensor formalism that agrees with the one given for $n = 4$ [35] and makes manifest the symmetry of ref. [32]. We also initiate an investigation of what may be thought of as differential geometry on a generalised manifold. A sequence of G modules, in many respect analogous to forms on ordinary manifolds, are given, and we describe how they may accommodate tensor (non-gravitational) gauge fields.

The paper is organised as follows. After some background on exceptional geometry in Section 2, we turn to the covariant construction of the generalised geometry in terms of vielbeins, connections and curvature in Sections 3-5. Section 6 deals with the dynamics of tensor fields coupled to generalised geometry. We summarise and point out some interesting questions in the concluding Section. Some conventions are given in an Appendix.

2. Preliminaries on exceptional geometry

As mentioned in the Introduction, we are concerned with a generalisation of geometry, where the traditional rôle of $GL(n)$ in ordinary geometry is subsumed by the group $G = E_{n(n)} \times \mathbb{R}^+$, and that of the locally realised rotation group by the maximal compact subgroup $H \subset G$.

Generalised momenta transform in a module $\overline{R}_1$ of G. A central identity in generalised geometry is the section condition. It states that bilinears in momenta projected on a certain module of G, $\overline{R}_2$, vanish. Although this condition is G-covariant, its solutions effectively single out n directions on which fields may depend.

It is well known how to form a generalised Lie derivative, governing the generalised diffeomorphisms, which effectively include tensor gauge transformation in addition to ordinary diffeomorphisms. The generalised diffeomorphisms, acting on a vector, take the form

$$\mathscr{L}_U V^M = L_U V^M + Y^{MN}{}_{PQ}\partial_N U^P V^Q \tag{2.1}$$

(L_U being the ordinary Lie derivative), which can be rewritten as

$$\begin{aligned}\mathscr{L}_U V^M &= U^N\partial_N V^M - \alpha P_{(\mathrm{adj})}{}^M{}_{N,}{}^P{}_Q\partial_P U^Q V^N + \beta\partial_N U^N V^M\\ &= U^N\partial_N V^M + Z^{MN}{}_{PQ}\partial_N U^P V^Q\ ,\end{aligned} \tag{2.2}$$

where $P_{(\mathrm{adj})}$ projects on the adjoint of $E_{n(n)}$ (we constrain the analysis to $n \geq 4$, where this group is simple). For $n \leq 6$, the tensor Y is proportional to the projection on R_2,

$$Y^{MN}{}_{PQ} = 2(n-1)P_{(R_2)}^{MN}{}_{PQ}\ , \tag{2.3}$$

and for $n = 7$ it contains an additional antisymmetric term $\frac{1}{2}\varepsilon^{MN}\varepsilon_{PQ}$. The constants α_n take the values $3, 4, 6, 12$ for $n = 4, 5, 6, 7$, respectively, while $\beta_n = \frac{1}{9-n}$.

The closure of the algebra of generalised diffeomorphisms relies on certain identities involving the invariant tensor Y. The simplest of these is the section condition itself,

$$Y^{MN}{}_{PQ}\partial_M \otimes \partial_N = 0\ , \tag{2.4}$$

where the $\otimes$ sign signifies that the two derivatives may act on any pair of fields. Another important identity is the nonlinear relation

$$\left(Y^{MN}{}_{TQ}Y^{TP}{}_{RS} - Y^{MN}{}_{RS}\delta^P_Q\right)\partial_{(N} \otimes \partial_{P)} = 0\ , \tag{2.5}$$

which can also be written

$$\left(Z^{MN}{}_{TQ}Z^{TP}{}_{RS} + Z^{MP}{}_{RQ}\delta^N_S\right)\partial_{(N} \otimes \partial_{P)} = 0\ . \tag{2.6}$$

Notice, that while eq. (2.5) manifests the R_2 and $\overline{R}_2$ projections of the index pairs ^{MN} and ${}_{RS}$, the form (2.6) manifests the $\mathfrak{g}$ projections in the pairs ${}^M{}_Q$ and ${}^P{}_R$.

The parameters of generalised diffeomorphisms come in R_1, and it was demonstrated in ref. [34] that the infinite sequences $\{R_k\}$ are responsible for the reducibility of the transformations. As we will see in Section 6, part of the sequence has many properties in common with forms in ordinary geometry, which is how we will be able to use them for constructing tensor fields. Before that is possible, we need to develop a tensor formalism.

n	R_1	R_2	R_3	R_4	R_5
3	$(\mathbf{3},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\overline{\mathbf{3}},\overline{\mathbf{2}})$
4	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{5}$	$\overline{\mathbf{10}}$	$\mathbf{24}$
5	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	
6	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$		
7	$\mathbf{56}$	$\mathbf{133}$			

Table 1: A partial list of modules $R_k^{(n)}$.

3. Tensors and connections

The property (2.2) of the generalised Lie derivative on vectors ensures that it can be defined on a tensor carrying an arbitrary number of indices in R_1 and $\overline{R}_1$, with the transformation

$$\begin{aligned}\mathscr{L}_U W^{M_1\ldots M_p}{}_{N_1\ldots N_q} &= U^P\partial_P W^{M_1\ldots M_p}{}_{N_1\ldots N_q}\\ &\quad+\sum_{i=1}^{p} Z^{M_iQ}{}_{RP}\partial_Q U^R W^{M_1\ldots M_{i-1}PM_{i+1}\ldots M_p}{}_{N_1\ldots N_q}\\ &\quad-\sum_{i=1}^{q} Z^{PQ}{}_{RN_i}\partial_Q U^R W^{M_1\ldots M_p}{}_{N_1\ldots N_{i-1}PN_{i+1}\ldots N_q}\ ,\end{aligned} \tag{3.1}$$

so that tensor products and contractions respect the tensorial property.

Note that composition of tensors implies that the $\mathbb{R}$-weight is not freely assigned. Not any invariant $E_{n(n)}$ tensor is a tensor under generalised diffeomorphisms. For example, E_6 has an invariant tensor c^{MNP}. In order to be a tensor under generalised diffeomorphisms it would need to carry total $\mathbb{R}$ weight 3, if the weight of a vector is normalised to one. Otherwise it becomes a tensor density. On the other hand, $c^{MNP}c_{QRS}$ is a tensor.

We will introduce an affine connection, $\Gamma_{MN}{}^P$. As matrices $(\Gamma_M)_N{}^P$, Γ_M are valued in the Lie algebra $\mathfrak{g}=\mathfrak{e}_{n(n)}\oplus\mathbb{R}$. Note that this excludes any specific symmetry properties for the lower indices. Defining a covariant derivative $D = \partial + \Gamma$, the transformation rule of the connection should ensure that $D_M W^{\{N\}}{}_{\{P\}}$ is a tensor if $W^{\{N\}}{}_{\{P\}}$ is a tensor. We use the convention

$$\begin{aligned} D_M V_N &= \partial_M V_N + \Gamma_{MN}{}^P V_P \ , \\ D_M V^N &= \partial_M V^N - \Gamma_{MP}{}^N V^P \ , \end{aligned} \tag{3.2}$$

with the obvious generalisation to arbitrary number of indices.

The covariant derivatives of eq. (3.2) are valid for tensors, *i.e.*, for objects where each R_1 index is accompanied with a certain $\mathbb{R}$-weight w, which we may normalise to 1, and accordingly -1 for each $\overline{R}_1$ index. This is not always an ideal way of describing modules. One may for example want to use invariant tensors of $E_{n(n)}$ which do not have weight zero. One example is the duality $R_k \leftrightarrow \overline{R}_{9-n-k}$. It may sometimes be more convenient to represent, say, $R_{8-n} = \overline{R}_1$ with one lower index instead of $8-n$ upper ones. This amounts to considering "tensor densities", by specifying $E_{n(n)}$ module and $\mathbb{R}$-weight w. There is no acute need of distinguishing tensors and "tensor densities", and we will use the term "tensor" for both. The covariant derivatives (taking a tensor of weight w to one of weight $w-1$) on vectors and covectors, with natural generalisation to arbitrary index structures, are

$$\begin{aligned} D_M W_N &= \partial_M W_N + \Gamma_{MN}{}^P W_P - \tfrac{w+1}{|R_1|}\Gamma_{MP}{}^P W_N \ , \\ D_M V^N &= \partial_M V^N - \Gamma_{MP}{}^N V^P - \tfrac{w-1}{|R_1|}\Gamma_{MP}{}^P V^N \ . \end{aligned} \tag{3.3}$$

Demanding that the covariant derivative takes tensors to tensors immediately leads to the transformation rule for the connection,

$$\begin{aligned} \delta_\xi \Gamma_{MN}{}^P &= \mathscr{L}_\xi \Gamma_{MN}{}^P + Z^{PQ}{}_{RN}\partial_M\partial_Q\xi^R \\ &= \mathscr{L}_\xi \Gamma_{MN}{}^P - \partial_M\partial_N\xi^P + Y^{PQ}{}_{RN}\partial_M\partial_Q\xi^R \ . \end{aligned} \tag{3.4}$$

As mentioned, the generic $E_{n(n)}$ module for the affine connection is $\overline{R}_1 \otimes \mathfrak{g}$. Not all of the irreducible components of Γ can appear in the inhomogeneous terms of eq. (3.4). Only the part occurring in $(\vee^2\overline{R}_1 \ominus \overline{R}_2) \otimes R_1$ will pick up inhomogeneous transformation terms. We define:

Torsion is defined as the irreducible modules in the affine connection transforming homogeneously, i.e., with the generalised Lie derivative.

Defined in this covariant way, torsion can consistently be set to zero.

It is quite straightforward to verify that the overlap $[\overline{R}_1 \otimes \mathfrak{g}] \cap [(\vee^2\overline{R}_1 \ominus \overline{R}_2) \otimes R_1]$ generically consists of a small module, which is $\overline{R}_1$, and a big module, which is the largest module in the product of $\overline{R}_1$ and the adjoint. The torsion module, which is the rest of Γ, consists of a small module $\overline{R}_1$ and a bigger one (reducible for low n), which turns out to coincide with $\overline{R}_{10-n}$[1].

n	torsion	non-torsion
3	$2(\overline{\mathbf{3}},\mathbf{2})\oplus(\mathbf{6},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{2})\oplus(\overline{\mathbf{3}},\mathbf{4})\oplus(\mathbf{15},\mathbf{2})$
4	$\overline{\mathbf{10}}\oplus\overline{\mathbf{15}}\oplus\mathbf{40}$	$\overline{\mathbf{10}}\oplus\overline{\mathbf{175}}$
5	$\overline{\mathbf{16}}\oplus\mathbf{144}$	$\overline{\mathbf{16}}\oplus\overline{\mathbf{560}}$
6	$\overline{\mathbf{27}}\oplus\mathbf{351}'$	$\overline{\mathbf{27}}\oplus\overline{\mathbf{1728}}$
7	$\mathbf{56}\oplus\mathbf{912}$	$\mathbf{56}\oplus\mathbf{6480}$

Table 2: Torsion and non-torsion part of the affine connection.

We need explicit expressions for the torsion, or expressions that a torsion-free connection satisfies. It turns out that

$$T_{MN}{}^P = \Gamma_{MN}{}^P + Z^{PQ}{}_{RN}\Gamma_{QM}{}^R \tag{3.5}$$

transforms as a tensor. This is verified by direct insertion into the transformation rule (3.4) and use of the identity (2.6). A torsion-free connection obeys

$$\Gamma_{MN}{}^P + Z^{PQ}{}_{RN}\Gamma_{QM}{}^R = 0\ , \tag{3.6}$$

or, equivalently, $2\Gamma_{[MN]}{}^P + Y^{PQ}{}_{RN}\Gamma_{QM}{}^R = 0$. Note that the result from ordinary geometry is recovered for $Y = 0$.

It is straightforward to take a trace to determine which combination of the two $\overline{R}_1$'s is torsion and which is torsion-free. Contracting eq. (3.6) with δ_P^N and using $Z^{MP}{}_{PN} = \frac{|R_1|}{9-n}\delta_N^M$ shows that a torsion-free connection satisfies

$$\Gamma_{MN}{}^N + \tfrac{|R_1|}{9-n}\Gamma_{NM}{}^N = 0\ . \tag{3.7}$$

[1] It has been observed in ref. [32] that this torsion module can be identified with the embedding tensor of gauged supergravity. Work by Palmkvist [36] identifies a new class of algebras, symmetric under $R_p \to \overline{R}_{9-n-p}$ where torsion appears as R_{-1}.

On the other hand, contracting eq. (3.6) with δ_P^M gives

$$Y_{MN}{}^{QR}\Gamma_{QR}{}^N = -2\Gamma_{[NM]}{}^N = -\big(1+\tfrac{|R_1|}{9-n}\big)\Gamma_{NM}{}^N \ . \tag{3.8}$$

For $n<7$ this identity may be used to derive a "stronger" constraint. Since $Y_{MN}{}^{QR}\Gamma_{QR}{}^P$ can only contain the $\overline{R}_1$ part of a torsion-free connection[2], it must be proportional to $Y_{MN}{}^{PQ}\Gamma_{RQ}{}^R$, and the proportionality constant is determined from eq. (3.8). The resulting relation is

$$Y_{MN}{}^{QR}\Gamma_{QR}{}^P + Y_{MN}{}^{PQ}\Gamma_{RQ}{}^R = 0 \ . \tag{3.9}$$

This relation is useful for determining when covariant derivatives are connection-free; see below.

The generalised Lie derivative on a vector does not contain any non-homogeneously transforming connection, if one replaces the naked derivatives with covariant ones. This is verified by replacing the derivatives in $\mathscr{L}_U V$ of eq. (2.2) with covariant derivatives and checking that the connections come in the torsion combination of eq. (3.6). This property was used as a definition of torsion (equivalent to ours) in ref. [32].

Eq. (3.6) contains the torsion modules in the connection. The actual torsion-free connection cannot be obtained simply by adding a multiple of $T_{MN}{}^P$ to $\Gamma_{MN}{}^P$, since the different torsion modules take different eigenvalues under $\Gamma \to T$.

4. Vielbeins and compatible connections

The structure group $G = E_{n(n)} \times \mathbb{R}^+$ has a locally realised subgroup H, which in the signature we are using is the maximal compact subgroup $H = K(E_{n(n)})$. We denote R_1 indices under H by $A, B, \ldots$

Consider a vielbein (frame field) $E_M{}^A$, which is a group element of $E_{n(n)} \times \mathbb{R}^+$. Locally it represents an element of the coset G/H, so it should be considered modulo local H-transformations from the right. It can be used to form a metric $G_{MN} = E_M{}^A E_N{}^B \delta_{AB}$, where δ_{AB} is an H-invariant constant metric.

We want to impose that the vielbein is covariantly constant, when transported by a covariant derivative containing both affine and spin connections:

$$D_M E_N{}^A = \partial_M E_N{}^A + \Gamma_{MN}{}^P E_P{}^A - E_N{}^B \Omega_{MB}{}^A = 0 \ . \tag{4.1}$$

[2] Because $\overline{R}_2 \otimes R_1$ does not contain the big torsion-free connection module. This is not true for $n=7$, where $\overline{R}_2$ is the adjoint.

We now want to examine to what extent the connections are determined from the vanishing of torsion together with the compatibility equation (4.1). The affine connection can be eliminated from the equation by the use of the vanishing torsion condition — this simply amounts to forming a combination of eq. (4.1) that contains Γ through T of eq. (3.5). The result is

$$(D^{(\Omega)}_M EE^{-1})_N{}^P + Z^{PQ}{}_{RN}(D^{(\Omega)}_Q EE^{-1})_M{}^R = 0 \ . \qquad (4.2)$$

On the other hand, the spin connection can be eliminated by projecting the compatibility equation on its $\mathfrak{g}/\mathfrak{h}$ part. Note that when we talk about the local subgroup H we always mean the one defined by the vielbein. The projection is easy, since after lowering one index, the symmetric part of $\mathfrak{g}$ is $\mathfrak{g}/\mathfrak{h}$ and the antisymmetric part $\mathfrak{h}$. This leads to

$$(E^{-1}D^{(\Gamma)}_M E)_{(AB)} = 0 \ , \qquad (4.3)$$

or, equivalently,

$$D_M G_{NP} = \partial_M G_{NP} + 2\Gamma_{M(NP)} = 0 \ . \qquad (4.4)$$

To analyse the compatibility equations for the spin connection (4.2) and the affine connection (4.3), one must decompose into H-modules. One then finds that the content of eq. (4.2), which is identical to the torsion modules of Table 2, is smaller than the content of Ω, which is $R_1 \otimes \mathfrak{h}$. The missing module Σ is the "big" irreducible module in $R_1 \otimes \mathfrak{h}$, *i.e.*, the H-module whose highest weight is the sum of the highest weights of R_1 and $\mathfrak{h}$. Similarly, the same result is obtained from the compatibility for the affine connection, so there is always an undefined part (in the same module) of a torsion-free compatible affine connection. This is summarised in the table below, whose content agrees with ref. [32].

n	H	undetermined connection Σ
4	$SO(5)$	$\mathbf{35} = (04)$
5	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$	$(\mathbf{4},\mathbf{20}) \oplus (\mathbf{20},\mathbf{4}) = (01)(03) \oplus (03)(01)$
6	$USp(8)/\mathbb{Z}_2$	$\mathbf{594} = (2100)$
7	$SU(8)/\mathbb{Z}_2$	$\mathbf{1280} \oplus \overline{\mathbf{1280}} = (1100001) \oplus (1000011)$

Table 3: The undetermined part of a compatible torsion-free connection

This means, that if connection is not to represent independent degrees of freedom, one should only introduce covariant derivatives mapping between certain special pairs of modules. Consider two modules U and V under H (or its double cover), and let a covariant derivative map from one to the other. This means that $R_1 \otimes U \supset V$. We are then only allowed to do this for pairs where at the same time $\Sigma \otimes U \not\supset V$. Some such pairs ("spinor" and "gravitino" modules) were discussed in refs. [32,33], and we will encounter other ones later.

A special case of such well-defined covariant derivatives consists of situations where not only the Σ part of a connection is absent, but where connection is altogether absent, and a covariant derivative equals an ordinary derivative. Such connection-free actions of derivatives will be important for our description of tensor gauge fields in Section 6, but we will already at this point check what the weight of a vector W^M must be in order for the divergence $D_M W^M$ to be connection-free. From eq. (3.3) it follows that

$$D_M W^M = \partial_M W^M - \Gamma_{MN}{}^M W^N + \tfrac{w-1}{9-n}\Gamma_{NM}{}^N W^M \;. \tag{4.5}$$

The connection terms cancel for $w = 10 - n$, which can be expressed as

$$|G|^{-\frac{9-n}{2|R_1|}} D_M V^M = \partial_M \big(|G|^{-\frac{9-n}{2|R_1|}} V^M\big) \tag{4.6}$$

for a vector of weight 1. This result will have bearing on any discussion on measures and partial integration.

At this point, we would also like to comment on the relation between the present approach and the one used in a recent paper by Park and Suh [35]. There, the affine connection is subject to precisely the right number of constraints to make it uniquely determined from compatibility. In addition to the torsion condition, this procedure amounts to setting, by hand, the Σ module in Γ to zero. The resulting derivative with connection is then not fully covariant, but will behave as such acting between certain modules, the pairs described in the previous paragraph. We tend to prefer the present, geometric description, which allows for connections to transform as such (both with respect to generalised diffeomorphisms and local H transformations).

5. Curvature

We will now examine how curvature can be defined. We write the transformation rule (3.4) for the affine connection as

$$\Delta_\xi \Gamma_{MN}{}^P \equiv (\delta_\xi - \mathscr{L}_\xi)\Gamma_{MN}{}^P = Z^{PQ}{}_{RN}\partial_M\partial_Q\xi^R \ , \tag{5.1}$$

in order to manifest the inhomogeneous term. Tensors are characterised by $\Delta_\xi = 0$. This leads to the corresponding transformation of its derivative:

$$\begin{aligned}\Delta_\xi\partial_M\Gamma_{NP}{}^Q &= Z^{QR}{}_{SP}\partial_M\partial_N\partial_R\xi^S \\ &\quad + \Delta_\xi\Gamma_{MR}{}^Q\Gamma_{NP}{}^R - \Delta_\xi\Gamma_{MN}{}^R\Gamma_{RP}{}^Q - \Delta_\xi\Gamma_{MP}{}^R\Gamma_{NR}{}^Q \ .\end{aligned} \tag{5.2}$$

There are two possibilities to make the $\partial^3\xi$ terms vanish — antisymmetrisation $[MN]$ or symmetrisation and projection on $\overline{R}_2$. We have not found any way of directly using the $\overline{R}_2$ (although it will become clear below that it really is a specific combination of the two possibilities that leads to a tensor). Antisymmetrisation gives

$$\Delta_\xi\left(\partial_{[M}\Gamma_{N]P}{}^Q + \Gamma_{[M|P|}{}^R\Gamma_{N]R}{}^Q\right) = -\Delta_\xi\Gamma_{[MN]}{}^R\Gamma_{RP}{}^Q = \tfrac{1}{2}Y^{RS}{}_{TN}\Delta_\xi\Gamma_{SM}{}^T\Gamma_{RP}{}^Q \ , \tag{5.3}$$

where we have used the tensor property of the torsion of eq. (3.5) in the last step. This is a nice form that reduces to the covariant transformation of the Riemann tensor for ordinary geometry ($Y = 0$). The middle step clearly shows why an attempt to construct a “Riemann tensor” fails, when the torsion-free condition does not suffice to set $\Gamma_{[MN]}{}^P$ to zero. If however the expression on the right hand side of eq. (5.3) is contracted with δ_Q^N and symmetrised in (MP), it can be written as $\Delta_\xi(\frac{1}{4}Y^{RS}{}_{TQ}\Gamma_{SM}{}^T\Gamma_{RP}{}^Q)$. Therefore,

$$\begin{aligned}R_{MN} &= \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P \\ &\quad + \Gamma_{(MN)}{}^Q\Gamma_{PQ}{}^P - \Gamma_{P(M}{}^Q\Gamma_{N)Q}{}^P - \tfrac{1}{2}Y^{PQ}{}_{RS}\Gamma_{PM}{}^S\Gamma_{QN}{}^R\end{aligned} \tag{5.4}$$

transforms as a tensor. If we restrict to vanishing torsion, the last term may be rewritten using eq. (3.6), and the curvature takes the form

$$\begin{aligned}R_{MN} &= \partial_{(M}\Gamma_{|P|N)}{}^P - \partial_P\Gamma_{(MN)}{}^P \\ &\quad + \Gamma_{(MN)}{}^Q\Gamma_{PQ}{}^P - \tfrac{1}{2}\Gamma_{PM}{}^Q\Gamma_{QN}{}^P - \tfrac{1}{2}\Gamma_{P(M}{}^Q\Gamma_{N)Q}{}^P \ .\end{aligned} \tag{5.5}$$

An alternative way of deriving curvature is to start from the covariant constancy of the generalised vielbein, eq. (4.1). The procedure is to act with one more covariant derivative, and use only combinations where second derivatives on the vielbein are absent, due to either antisymmetry or the section condition. The result (which of course is zero) should be expressible as the difference of two tensors, of which the one expressed in terms of Ω should be manifestly a tensor, and the one expressed in Γ manifestly invariant under local transformations in H. Then the equality of the two expressions implies that each of them enjoys the property manifest in the other.

Acting with a second derivative on eq. (4.1) gives

$$\begin{aligned}0 &= \partial_M\partial_N E_P{}^A + \partial_M\Gamma_{NP}{}^Q E_Q{}^A - E_P{}^B\partial_M\Omega_{NB}{}^A \\ &\quad - (\Gamma_N\Gamma_M)_P{}^Q E_Q{}^A - E_P{}^B(\Omega_M\Omega_N)_B{}^A + 2(\Gamma_{(M}E\Omega_{N)})_P{}^A \ .\end{aligned} \tag{5.6}$$

Antisymmetrising in $[MN]$ gives

$$\begin{aligned}0 &= (\partial_{[M}\Gamma_{N]} + \Gamma_{[M}\Gamma_{N]})_P{}^Q E_Q{}^A \\ &\quad - E_P{}^B(\partial_{[M}\Omega_{N]} + \Omega_{[M}\Omega_{N]})_B{}^A \ ,\end{aligned} \tag{5.7}$$

exactly as in ordinary geometry. The expression $\partial_{[M}\Omega_{N]}$ on the second line is however not a tensor, since $\Gamma_{[MN]}{}^P$ is not torsion. One has to form some combination of terms so that the $\Gamma\Omega$ terms in eq. (5.6) combine with the $\partial\Omega$ terms into covariant derivatives $D^{(\Gamma)}$. They can then be converted into Ω using $D_M A_N = E_N{}^A D_M A_A$. This can be achieved with one contraction of indices and symmetrisation in the remaining two (as in the construction of the curvature above)[3]. The resulting curvature is identical to the one given in eq. (5.4), and its expression in terms of Ω is

$$\begin{aligned}R_{MN} &= E_{(M}{}^A\partial_{N)}\Omega_{BA}{}^B - E_{(M}{}^A E_{N)}{}^B E_C{}^P\partial_P\Omega_{AB}{}^C - \tfrac{1}{2}Y^{PA}{}_{B(M}E_{N)}{}^C\partial_P\Omega_{AC}{}^B \\ &\quad + \Omega_{(MN)}{}^A\Omega_{BA}{}^B - \Omega_{AM}{}^B\Omega_{BN}{}^A \\ &\quad - \tfrac{1}{2}Y^{AB}{}_{C(M}\left(\Omega_{|AB}{}^D\Omega_{D|N)}{}^C + \Omega_{|A|N)}{}^D\Omega_{BD}{}^C\right) \ .\end{aligned} \tag{5.8}$$

(Here, we have used vanishing torsion and restricted the calculation to $n \leq 6$. We have also converted indices with the vielbein.)

We do not have a direct proof that R_{MN} exhausts the possible curvature tensors, although we suspect that this is the case. It is however clear that it is large enough to

[3] Hohm and Zwiebach manage to form a 4-index tensor in the $O(d,d)$ situation, where one has access to an H-invariant metric [11]. We do not see how that construction generalises to the exceptional cases.

contain anything we need. For example, R_1 contains a 2-form in n dimensions, so there is enough room in R_{MN} for the modules of an ordinary Riemann tensor.

An important question is to what extent this curvature is defined in terms of a vielbein. This especially concerns its projection on $\mathfrak{g}/\mathfrak{h}$, since that part is a candidate for a "Ricci" or "Einstein" tensor, providing equations of motion for the geometry. A variation of the curvature gives at hand that

$$\delta R_{MN} = D_{(M}\delta\Gamma_{|P|N)}{}^{P} - D_P\delta\Gamma_{(MN)}{}^{P} \ . \tag{5.9}$$

There is nothing here that prevents the undefined module Σ from appearing in the second term. But if we consider the projection on $\mathfrak{g}/\mathfrak{h}$, we observe that $(\mathfrak{g}/\mathfrak{h})\otimes R_1 \not\supset \Sigma$, so the variation of R_{MN} does not contain the Σ part of $\delta\Gamma$. Thus, $R_{\{MN\}}$, the projection of R_{MN} on $\mathfrak{g}/\mathfrak{h}$, is well-defined, and can serve as a Ricci tensor[4].

From this it is also clear that the singlet, the curvature scalar $R = G^{MN}R_{MN}$ (which is part of $R_{\{MN\}}$), is well-defined in terms of the metric.

It is tempting to think of the curvature scalar as a Lagrangian for generalised gravity, whose variation should give an Einstein tensor. This of course has to rely on partial integration, since

$$\delta R = \delta(G^{MN}R_{MN}) = \delta G^{MN}R_{MN} + D_M(\delta\Gamma_N{}^{MN} - \delta\Gamma_N{}^{NM}) \ . \tag{5.10}$$

The $D\delta\Gamma$ terms cannot be discarded unless the expression is multiplied by a scalar density from the measure, and it follows from eq. (4.6) that this density must have weight $9-n$. So, if the Lagrangian density is

$$\mathscr{L} = |G|^{-\frac{9-n}{2|R_1|}}R \ , \tag{5.11}$$

the equations of motion for G_{MN}, the generalised Einstein's equations, become

$$R_{\{MN\}} + \tfrac{9-n}{2|R_1|}G_{MN}R = 0 \ . \tag{5.12}$$

[4] The independence of the Σ part of Γ cannot be observed by simply entering an expression for Γ in terms of its decomposition in H-modules into eq. (5.4). Then the $\Gamma\Gamma$ part of the second term would seems to contain Σ. One has to realise that the H subgroup defined by the vielbein/metric is special; only for this subgroup the covariant derivatives respect the decomposition into H-modules. We have checked in a couple of examples ($n=4,5$) that an explicit decomposition in H modules yields no Σ^2 in the $\mathfrak{g}/\mathfrak{h}$ part, but indeed terms linear in Σ.

For pure generalised gravity, this is of course equivalent to $R_{\{MN\}} = 0$, but in presence of matter fields, as in the following section, eq. (5.12) provides the left hand side of the generalised Einstein's equations.

We note that our density $|G|^{-\frac{9-n}{2|R_1|}}$ agrees with the one given in ref. [35] for $n = 4$. There, the density is written as "M^{-1}", where M is the determinant of a metric on the fundamental **5** of $SL(5)$. We have $-\frac{9-n}{2|R_1|} = -\frac{1}{4}$, but our G_{MN} is a metric on the module **10**. The double weight of G and the double size of the determinant together account for the factor 4 compared to ref. [35].

6. Tensor fields

It is well known that the k-form gauge fields in dimensionally reduced theories come in the modules R_k under the U-duality group. Here, we instead ask for the dynamics in the "internal" directions, *i.e.*, for the descriptions of fields in R_k on a generalised manifold (at least locally). We need to be able to describe gauge symmetry and field equations, as well as some counting of degrees of freedom. The resulting description provides the U-duality version of the spinor of Ramond–Ramond fields for T-duality and double field theory [37].

The sequences $\{R_k\}$ are symmetric under $R_k \leftrightarrow \overline{R}_{9-n-k}$ (and the proper reassignment of $\mathbb{R}$ weight), in analogy with forms. When we occasionally talk about modules R_k outside the window $1 \leq k \leq 8-n$, which *e.g.* are needed for the complete reducibility, we will take the ones for $k \geq 9-n$ to agree with the ones given in ref. [34], which agrees with the positive levels of a Borcherds algebra [38] (the precise reason for this will be the subject of a future publication [39]). For $k \leq 0$, we will assume that the symmetry around $k = \frac{9-n}{2}$ remains. Seen as objects with k upper indices, entities $F^{M_1 \ldots M_k}$ in R_k are in general neither totally antisymmetric nor symmetric, but have mixed symmetry. R_2 is always symmetric, but already R_3 is a module of mixed symmetry $\tiny\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$.

In ref. [34] it was shown how the R_k's arise as an infinite sequence of ghosts related to the generalised diffeomorphisms and its reducibility. An essential property is that a derivative, $\partial : R_k \to R_{k-1}$, is nilpotent, so the sequence forms a complex. With this knowledge, it seems natural that the same modules should be responsible for gauge transformations of tensor fields (and their reducibilities).

We will now proceed to show that the sequence of modules $\{R_k\}_{k=1}^{8-n}$ in many respects plays a rôle similar to that of forms on an ordinary manifold. An important piece of information is to what extent the affine connection takes part in the covariantised operation $D : R_k \to R_{k-1}$. Ideally, we would want connection to be absent, and "$D = \partial$", in analogy with the situation for the exterior derivative on forms.

It turns out that the derivative from R_k to R_{k-1} is connection-free for $2 \leq k \leq 8-n$. For some simple cases, like $R_2 \to R_1$ ($n \leq 6$), it is straightforward to show:

$$\begin{aligned} D_N W^{MN} &= \partial_N W^{MN} - \Gamma_{NP}{}^M W^{PN} - \Gamma_{NP}{}^N W^{MP} \\ &= \partial_N W^{MN} - \tfrac{1}{2(n-1)} \left(Y^{NP}{}_{RS} \Gamma_{NP}{}^M + Y^{MP}{}_{RS} \Gamma_{NP}{}^N \right) W^{RS} = 0 \ , \end{aligned} \tag{6.1}$$

with the use of eq. (3.9). For $R_3 \to R_2$, the proof is more involved, and relies on the hook (⊞) property of R_3. For higher k it is more convenient to use $R_{9-n-k} = \overline{R}_k$ and to treat them as tensor densities. For example, the covariant derivative from $\overline{R}_1$ with weight w to $\overline{R}_2$ is ($n \leq 6$)

$$Y_{MN}{}^{PQ} D_P W_Q = Y_{MN}{}^{PQ} \left(\partial_P W_Q - \tfrac{8-n-w}{9-n} \Gamma_{RP}{}^R W_Q \right) \ , \tag{6.2}$$

where eq. (3.9) has been used again, showing that the derivative $R_{8-n} \to R_{7-n}$ is connection-free ($n \leq 6$).

However, it is obvious from direct inspection that $R_1 \to R_0$ and $R_{9-n} \to R_{8-n}$ contain connection. Neither is it possible to make the complex finite by using singlets at $k = 0$ and $k = 9-n$; the corresponding derivatives also contain connection. These singlets actually both take the rôle one would have wanted from the other: the derivative $\mathbf{1} \to \overline{R}_1$ is connection-free for weight 0, and the divergence $R_1 \to \mathbf{1}$, as we have seen, is connection-free when the singlet has weight $9-n$. In some sense, it looks as though we had an $(9-n)$-dimensional manifold, but with an exterior derivative "acting the wrong way". To some extent, it becomes clearer from the diagrams in Appendix B what happens. They depict the action of an ordinary derivative on the modules R_k decomposed into $GL(n)$ modules. There are always two sequences containing forms. All sequences are finite, but the ones starting at R_1 (or lower) or ending at R_{8-n} (or higher) consist of the tensor product of a complex of forms with some non-trivial $GL(n)$ module.

The problematic situation at the limits of the connection-free window does not prevent us from describing gauge connections and their field strengths within the window. It makes it more complicated to describe a gauge field in R_1 (more about this below), and it seems to obstruct a complete covariant description of the full reducibility of the gauge transformations at any k.

Consider a gauge field A in R_{k+1}, $1 \leq k \leq 7-n$. It will have a field strength $F = \partial A$ in R_k. There is a gauge symmetry $\delta_\Lambda A = \partial\Lambda$ with parameter Λ in R_{k+2} and a Bianchi identity $\partial F = 0$ in R_{k-1}. (For $k = 7-n$ the above discussion shows a difficulty with the covariance of the gauge transformation, and similarly with the Bianchi identity for $k = 1$. We will for the moment ignore this issue.)

Given a metric, there is a natural duality operation, taking F in R_k to $*F$ in R_{9-n-k}. This can be written in two ways (analogous to lower or upper indices for ordinary forms).

One is obtained by simply lowering the k indices with the metric. This results in a tensor in $\overline{R}_k$ with weight $-k$. A tensor in R_{9-n-k} has weight $9-n-k$, so the weight has to be adjusted by an appropriate power of $|G|$. The correct dual field strength is

$$*F_{M_1\ldots M_k} = |G|^{-\frac{9-n}{2|R_1|}} G_{M_1N_1}\ldots G_{M_kN_k} F^{N_1\ldots N_k} \ . \tag{6.3}$$

The other way is to use an invariant tensor $\Sigma^{A_1\ldots A_{9-n}}$, which after conversion of indices with inverse vielbeins becomes a tensor $\Sigma^{M_1\ldots M_{9-n}}$ and write

$$*F^{M_{k+1}\ldots M_{9-n}} = \Sigma^{M_1\ldots M_{9-n}} G_{M_1N_1}\ldots G_{M_kN_k} F^{N_1\ldots N_k} \ . \tag{6.4}$$

The equation of motion for A can now be written

$$\partial * F = 0 \ . \tag{6.5}$$

Since only connection-free derivatives have been used for forming the field strengths and the equations of motion, it is clear that there are no problems with undefined connection. The metric enters only through the dualisation. There is a duality symmetry under $k \to 9-n-k$ exchanging equations of motion and Bianchi identities. Again, we find that a Lagrangian density $F * F$ with weight $9-n$ is necessary in order to make partial integration possible.

It may seem that it is problematic to use a gauge potential in R_1, since the field strength would belong to R_0, which is outside the connection-free window. For a number of reasons (one is the field content of maximally supersymmetric generalised supergravity, see below) one would still like to have potentials in R_1. Although we will leave the detailed formulation to future work, we would like to argue that it is meaningful to have such a potential. The argument is based on dimensional reduction of generalised gravity. We will consider linearised fields. The linearised degrees of freedom of generalised gravity lie in $\mathfrak{g}/\mathfrak{h}$. Consider the decomposition under "dimensional reduction", *i.e.*, when n is lowered by 1. We drop the singlet part, which is irrelevant for the argument, and do not consider the weights of resulting modules. Let us denote the module $\mathfrak{e}_{n(n)}/\mathfrak{k}(\mathfrak{e}_{n(n)})$ by ϕ_n. Under dimensional reduction, $\phi_n \to \phi_{n-1} \oplus R_1^{(n-1)} \oplus \mathbf{1}$. The R_1 in the lower-dimensional theory is a "generalised graviphoton", whose dynamics is dictated by generalised gravity in the higher dimension. We have not examined the details of this, but it clearly shows that one can have fields in R_1.

The following is also worth noticing about derivatives on R_1. Taking a derivative of a field A in R_1 gives $D_Q A^R = \partial_Q A^R - \Gamma_{QM}{}^R A^M$ We can use the Z-tensor to pick out the $\mathfrak{g}$ part:

$$Z^{PQ}{}_{RN} D_Q A^R = Z^{PQ}{}_{RN}(\partial_Q A^R - \Gamma_{QM}{}^R A^M) = Z^{PQ}{}_{RN}\partial_Q A^R + \Gamma_{MN}{}^P A^M \ , \tag{6.6}$$

where the the torsion-free property was used for the second term. If the free index pair ${}_N{}^P$ is projected on $\mathfrak{g}/\mathfrak{h}$, only well-defined connection enters. In addition, the $\mathfrak{g}/\mathfrak{h}$ part of the compatibility equation (4.4) tells us that the $\mathfrak{g}/\mathfrak{h}$-valued part of a compatible Γ_M contains a ∂_M and obeys the section condition. Therefore, even if the derivative $R_1 \to \mathfrak{g}/\mathfrak{h}$ contains connection, a field strength $F = (DA)|_{\mathfrak{g}/\mathfrak{h}}$ allows for a gauge invariance with parameter in R_2. Such an invariance is expected, since $R_1^{(n)} \to R_1^{(n-1)} \oplus R_2^{(n-1)} \oplus \mathbf{1}$ under dimensional reduction.

We would like to say some words about the counting of degrees of freedom, both off-shell and on-shell. The models we are dealing with are effectively euclidean field theories, so in a strict sense it is not meaningful to talk about local on-shell degrees of freedom. What we mean is the number of physical polarisations the on-shell fields would carry, had the model been formulated with another real form of G corresponding to Minkowski signature after solution of the section condition. This gives numbers that are of practical use, especially when it comes to supersymmetric models [33,40] and matching of bosonic and fermionic degrees of freedom.

The counting of off-shell degrees of freedom is straightforward. It is simply given by the number of field components subtracted with the number of gauge parameters. Here, the infinite reducibility has to be taken into account, and we thus know that the number of off-shell degrees of freedom of a gauge field in R_k is

$$N_k = \sum_{\ell=0}^{\infty} (-1)^\ell |R_{k+\ell}| \ . \tag{6.7}$$

Such sums are naïvely divergent (the terms are alternating but growing) but have a meaningful regularisation [41,34]. Of course, it is enough to perform the regularisation for N_1 and calculated the finite difference. The result for $1 \leq k \leq 8-n$ is

$$N_k^{(n)} = \begin{cases} |R_k^{(n-1)}| + 1 & k = 1 \ , \\ |R_k^{(n-1)}| & 2 \leq k \leq 8-n \ . \end{cases} \tag{6.8}$$

The numbers have the property $N_k^{(n)} = N_{10-n-k}^{(n)}$.

The on-shell number of degrees of freedom can safely be deduced from the observation that all the fields on the n-dimensional solution of the section condition are forms (ordinary massless tensor fields). Therefore, the number of on-shell degrees of freedom of a field in $R_k^{(n)}$ is obtained as $|R_k^{(n-2)}|$. The number of physical polarisations of a field is obtained by regarding the "same" field in "two dimensions less", just as the counting goes for massless fields in Minkowski space. Since $R_{k+1}^{(n-2)}=\overline{R}_{10-n-k}^{(n-2)}$, this counting agrees with the dualisation from a potential for F in $R_{k+1}^{(n)}$ to a potential for $*F$ in $R_{10-n-k}^{(n)}$.

The counting has been tested on a number of non-gravitational supermultiplets [40]. Here we will illustrate it by counting the bosonic degrees of freedom in the maximal generalised supergravity. Fields will transform under the $SL(11-n)$ or $SO(1,10-n)$ "R-symmetry" of the "reduced" directions, and behave as forms under these. If one associates R_k with a k-form for $k=1,\ldots[\frac{11-n}{2}]$, and asks for a selfduality for $R_{\frac{11-n}{2}}$ when n is odd, the resulting counting is as follows:

n	gen. gravity	scalar coset	R_k	total
4	2	28	$\binom{7}{1}\times 3+\binom{7}{2}\times 2+\binom{7}{3}\times 1=98$	128
5	6	21	$\binom{6}{1}\times 6+\binom{6}{2}\times 3+\frac{1}{2}\times\binom{6}{3}\times 2=101$	128
6	13	15	$\binom{5}{1}\times 10+\binom{5}{2}\times 5=100$	128
7	24	10	$\binom{4}{1}\times 16+\frac{1}{2}\times\binom{4}{2}\times 10=94$	128

Table 4: Counting of bosonic degrees of freedom for maximal supersymmetry.

Note that for $n=7$ also $R_2=R_{9-n}=\mathbf{133}$, which we have not discussed above, is needed. Maybe the dual of the well-defined derivative $R_1\rightarrow\mathfrak{g}/\mathfrak{h}$ can be of use. The appearance of fields as forms in R_k is well known. In the present context it can also be obtained from dimensional reduction. We have already seen that the generalised gravity on dimensional reduction gives rise to a generalised graviphoton in R_1. The generic rule for tensor fields is that $R_k^{(n)}$ gives rise to $R_k^{(n-1)}$ and $R_{k+1}^{(n-1)}$ (with an extra singlet for $k=1$ and $k=8-n$), in close analogy to form fields. This is how the binomial coefficients are sequentially built.

7. Conclusions

We have presented a tensor calculus for exceptional generalised geometry. This includes universal and covariant expressions for connections and curvatures. Our analysis agrees with ref. [32], but has manifest covariance, and with ref. [35] for $n=4$. We have also given details on tensor gauge fields and their coupling to exceptional geometry. Some technical issues remain concerning the "generalised graviphoton" field. Even if the local description in terms of a tensor calculus respecting infinitesimal transformation now seems complete, important questions concerning the concept of generalised manifolds remain open. Hohm and Zwiebach have discussed the issue of exponentiating the Lie derivative in double field theory to a large diffeomorphism, but there are many remaining questions. An important one is to find an integration measure.

In ref. [33], minimal exceptional supergravity was formulated. In an accompanying paper [40] non-gravitational supermultiplets based on the tensor fields we present here were constructed. Extended supergravity will demand inclusion of such multiplets. It would be very interesting to investigate the possibility of formulating such models as some generalised supergeometry. It is not clear which set of modules will accompany the R_k's in order to build the correspondence to "forms on superspace". Such a formulation, preferably in an off-shell version using pure spinor techniques generalising refs. [42,43], could perhaps provide a simultaneous manifestation of supersymmetry and U-duality.

Note added: The paper [44], which appeared on the completion of our work, specialises on $n = 7$ and has a substantial overlap with the present paper concerning the geometric analysis.

Acknowledgements: MC would like to thank Axel Kleinschmidt, Jakob Palmkvist and David Berman for discussions.

Appendix A: Notation

G and H denote throughout the paper the groups $G = E_{n(n)} \times \mathbb{R}^+$ and its compact subgroup $H = K(E_{n(n)})$, and their Lie algebras (and adjoint modules) are written $\mathfrak{g}$ and $\mathfrak{h}$. For the complement to $\mathfrak{h}$ in $\mathfrak{g}$ we use "$\mathfrak{g}/\mathfrak{h}$" (even if "$\mathfrak{g} \ominus \mathfrak{h}$" might have been more correct). A projection of a 2-index object on $\mathfrak{g}/\mathfrak{h}$ is denoted by curly brackets: $\{MN\}$.

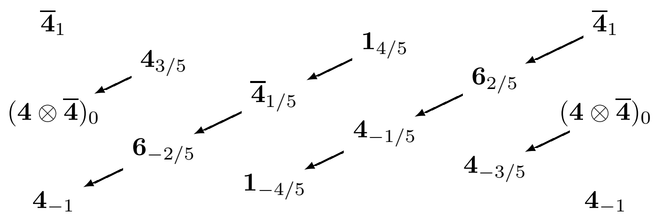
We use the notation $\vee$ for symmetrised tensor product. The dimension of a module R is denoted $|R|$. When a module in the sequence $\{R_k\}$ carries an upper index, $R_k^{(n)}$, it refers to n, the rank of the exceptional group.

Appendix B: The action of a derivative among the R_k

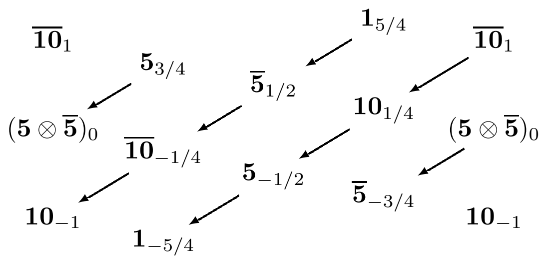
Below are diagrams showing the action of a derivative fulfilling the section condition on elements in R_k, $0 \leq k \leq 9-n$. The modules are split into modules of $SL(n) \times \mathbb{R}$. For $n = 6, 7$, there is an $SL(2)$ which is broken to $\mathbb{R}$ by the solution of the section condition.

Note that there are always two lines containing the ordinary n-dimensional forms. Other lines consist of the tensor product of the forms by some non-trivial module. Such lines begin at R_1 and end at R_{8-n}, and may be seen as responsible for the appearance of connection.

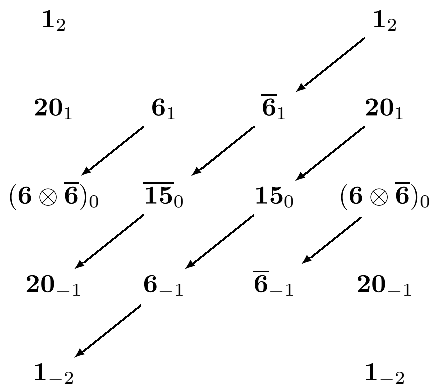
$n = 4$:



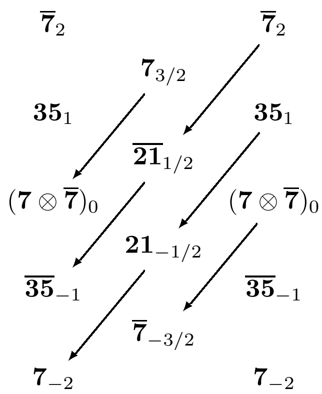
$n = 5$:



$\underline{n=6:}$



$\underline{n=7:}$



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A Tensor Formalism for Exceptional Geometry
JOAKIM EDLUND

Department of Fundamental Physics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
Telephone + 46 (0)31-772 1000