Thesis for the degree of Master of Science

Chiral Perturbation Theory, Weak Interactions and the Nuclear Two-body Axial Vector Current

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Cover The two Feynman diagrams that give the leading contribution to the two-body axial vector current. The lines going from top to bottom represent nucleons, the dashed lines pions and the wavy lines the external field, e. g., vector bosons.

Abstract

In this thesis I give a practical introduction to chiral perturbation theory. This is an effective field theory of pions and nucleons. It is governed by the chiral symmetry stemming from the lightness of the up- and down quarks in quantum chromodynamics. The validity region comprises energies up to the rho meson mass. The theory is expressed as an infinite series of chiral invariant interactions, whose strengths are expressed in an infinite number of low energy constants. The interactions can be ordered and I identify the most important ones.

Special care has to be taken when including nucleons in chiral perturbation theory because of the scale introduced by the nucleon mass. To facilitate straightforward calculations I work in heavy-baryon chiral perturbation theory. In this formalism the nucleons are considered to be very heavy and the nucleon mass only appears in next-to-leading order corrections.

The pions and nucleons are coupled to the charged vector bosons of the weak interactions. This interaction is determined entirely by the chiral symmetry. As an example, I compute the decay rate of charged pions. Experimental data for this observable can be used to fix one low energy constant.

Finally, I compute the two-body axial vector current of nucleons in heavy-baryon chiral perturbation theory with the long wavelength approximation. This current complements the leading order one-body current operator and gives the first correction to the Gamow-Teller operator from the nuclear environment. I provide both a detailed derivation and an explicit expressions for this two-body current operator.

Keywords: Chiral perturbation theory, effective field theory, meson exchange currents, Gamow-Teller operator, two-body axial vector current, nuclear currents, two-body currents, Heavy baryon chiral perturbation theory

Contents

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Conventions

I will work with a system of units such that

$$c = \hbar = 1.$$

This will make formulas simpler as it will not be necessary to write out these common factors. Also, energy, mass, momentum and frequency then have the same dimension while time and length have the inverse dimension.

I will frequently write terms like $a^i a^i$ or $p_\mu p^\mu$ with a repeated index. The meaning is that there is a sum $\sum_i a^i a^i$ or $\sum_{\mu=0}^{3} p_\mu p^\mu$. Greek letters such as μ, ν, ρ, σ are Lorentz indices and run from 0 to 3; roman letters such as a, b, $c\dots$ denote isospin indices; and i, j, $k\dots$ are space indices – unless something else is stated.

To raise and lower Lorentz indices the Minkowski metric $g^{\mu\nu} = g_{\mu\nu}$ will be used and for it I will use the mostly minus convention,

$$g_{\mu\nu} = \begin{cases} 1 \text{ for } \mu = \nu = 0 \\ -1 \text{ for } \mu = \nu = 1,2,3 \end{cases}.$$

This means that the contraction of two four-vectors p^μ and k^μ is written

$$p_\mu k^\mu = p^\mu k^\nu g_{\mu\nu} = p^0 k^0 - \boldsymbol{p}\cdot\boldsymbol{k}.$$

I will use σ^i to denote an operator in spin space while τ^a will be an operator in isospin space. These operators can be represented by the Pauli matrices, which are Hermitian and traceless complex 2×2 matrices,

$$\sigma^1 = \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, I will use the Weyl representation of the Dirac matrices. These are 4×4 matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Contractions between the Dirac matrices and other four-vectors are very common and to shorten expressions I will use the Feynman slash notation,

$$\not{p} = p^\mu \gamma_\mu.$$

Chapter 1

Introduction

An important scientific principle is that a theory should not be any more complicated than necessary. This idea is present in Newton's Principia [New46] where he writes that no more causes for things than are necessary should be invented, and that this in turn leads to the necessity of assigning to many things the same cause. Put in another way, a good scientific theory should be as simple as possible while describing as many things as possible. This idea has so far been very fruitful.

The explanations for almost all known phenomena are believed to have been collected into the description of just four different forces. Two of them are manifest in the macroscopic world, namely electromagnetism and gravity. The two other forces, the weak- and the strong force, can only be seen at very short length scales. This does not mean that they are unimportant. The strong force is responsible for keeping the nuclei of atoms together. It also keeps the quarks together in protons, neutrons and all other strongly interacting particles. The weak force is responsible for some decays of unstable particles, one example is beta decay.

The theories of electromagnetim, the weak force and gravity have all been very successful in describing their part of the world. These theories are quite simple and describe very many different things with excellent accuracy. The theory of the strong force, on the other hand, has not been as successful in terms of Newton's principle so far. In many cases it has been necessary to create problem specific models for each application. Many of these models are phenomenological, i. e., they are not derived from first principles but are instead based on observations of the particular subject that is studied. This leads to problems in understanding and makes it hard to make predictions.

One cause for this lack of success is that there are a very large number of particles that interact strongly. In 1930 when Pauli suggested the existence of the neutrino there were only three known particles; the proton, the photon and the electron [Arn01]. Out of these only the proton is affected by the strong force. In 1932 Chadwick found the neutron [Cha32], the second known strongly interacting particle. In the thirty years that followed several hundred more particles were found in cosmic radiation and accelerator experiments. The majority of these particles were strongly interacting. They are collected under the common name of hadrons.

In 1964, in order to explain the large number of hadrons, Gell-Mann [GM64] and Zweig [Zwe64] independently proposed that they were not fundamental particles at all. A large simplification could be made if they were composite particles made up of a new kind of particles, quarks. These quarks would need to have quite peculiar properties, for example they would need to have an electric charge that is a fraction of the electron charge. Initially there where three proposed quarks, this number has now been expanded to six.

Hadrons are defined by the quarks they contain. The ones containing two quarks are called mesons and those with three quarks baryons. The quarks are believed to be confined within the hadrons. Free quarks have never been observed, which gives strong support to this conjecture.

A theory called quantum chromodynamics(QCD) was created to account for the force between the quarks. In this theory the quarks interact by exchanging gluons. Both the quarks and the gluons carry a type of charge called color.

It is very hard to calculate properties of hadrons using QCD. At very short length scales, the quarks in a hadron are almost unaffected by the strong force – this phenomenon is called asymptotic freedom – and calculations can be made using ordinary methods. But at longer length scales the strength of the force increases enough to make it impossible to use the standard computational methods. This is necessary if the quarks are to be confined within the hadrons. Because of the strength of the force it is an unsolved problem how to derive the properties of hadrons and their interactions from QCD. Based on the success of the theory at high energy it is believed to be the correct theory also at low energy.

The neutrino, which Pauli suggested, was needed to explain the apparent loss of energy in beta decay. This reaction is an example of a weak interaction. Almost all particles mentioned so far, and actually all known particles, except for the gluons, interact with the weak force. This force also allows many more interactions to occur than the strong and electromagnetic interactions. On the other hand it is much weaker at low energies, which is the reason for its name. The large number of allowed interactions makes the weak interactions able to facilitate decays that the strong and electromagnetic force can not. On the other hand it makes these particles more long lived since the force is weaker. A weaker force gives a lower probablility of decay and thus a longer lifetime.

Between 1960 and 1970 electroweak theory got its current form through the work of Glashow, Weinberg and Salam [Gla61, SW64, Wei67]. This theory unifies the electromagnetic and the weak interactions into one theory. Just as the electromagnetic force is mediated by a vector boson called the photon the weak force is mediated by vector bosons. The difference is that unlike the photon these vector bosons are very heavy. The range of a force is limited by the mass of the exchange particle. This makes the weak force a very short ranged force while the electromagnetic force has unlimited range since the photon is massless.

Together these two theories, QCD and the unified electroweak theory, make up the Standard Model of particle physics. Everything from the binding of atoms and molecules to the highest energy accelerator experiments can be explained by these theories.

1.1 Effective field theory

The Standard Model is a fundamental field theory. This implies that it aims to explain phenomena on all length scales, from the macroscopic world down to the smallest possible lengths. It is also a very simple theory utilizing only 20 parameters [Lan09]. But, this does not mean that it is trivial to use this theory for everything.

In contrast to this type of theory we have effective field theories (EFTs). These are theories that do not aim to explain everything but instead focus on a specific set of problems at a certain length scale or, equivalently, at a certain energy scale. Length- and energy scales can be related. To examine a shorter length scale a shorter wavelength probe and thus a higher frequency is needed. A quantum of higher frequency will have a higher energy, and in this way energy scales and length scales can be related.

The first step in creating an EFT is to determine the relevant energy scale. It is also necessary to determine a highest possible energy, the cutoff energy. Nothing above the cutoff energy will be treated explicitly in the EFT; the theory will only be valid up to this energy. At higher energies, or shorter lengths, the EFT will loose all predictive power. This is similar to how a theory about macroscopic sound waves is unable to predict how the individual molecules in the air will move.

At a certain energy scale some particles will be more important than others. Those that have a mass lower than the cutoff energy of the theory will be able to be real particles, while those with a higher energy will be restricted to be virtual particles as they can not acquire enough energy to become real particles. As a consequence, the second step amounts to selecting the relevant degrees of freedom (fields or particles) that are active in the energy range of the theory. All other particles are excluded.

When writing down the interactions of the remaining particles there will be more interactions than before the removal of the higher mass particles.When the removed particles where included explicitly they could show up as virtual intermediate particles allowing interactions that would not be possible when only considering the remaining low energy particles. In fact, in order to be consistent with the higher energy theory, the removal of the high mass particles must generate all kinds of interactions among the remaining ones as long as the interactions are consistent with the symmetries of the underlying theory.

A conjecture by Weinberg [Wei79] states that writing down the most general Lagrangian containing the selected matter content and obeying the relevant symmetries yields the most general S-matrix consistent with the fundamentals of quantum field theory and the imposed symmetry. This means that among all the possible EFTs containing a set of selected particles with some symmetry there will be at least one that will give the same predictions for observable quantities as a higher energy theory for these particles.

In practice, it is impossible to consider the infinite number of interactions for any given process. To be able to make predictions there must be a way to determine what interactions give the largest contributions. It turns out that the interactions can be put into a series expansion in the energy scale of the process divided by the cutoff energy of the EFT [Pic98]. The relative strength of these interactions are often not given by symmetry. The so called low energy constants (LECs) that parametrize this must be determined from the higher energy or fundamental theory. In the cases where it is not possible or known how to solve the high-energy theory the LECs can be determined from experiment.

For any calculation all interactions up to the chosen order will need to be considered. Of course, going to a higher order will give a more precise result but will at the same time involve a larger number and more complex interactions.

An EFT may coexist with the Standard Model even though the details differ. This is possible because of one of the basic principles of quantum mechanics. It is impossible to measure how a process takes place, it is only possible to measure the outcome. This is because a measurement of the details of a process will disturb the process enough to change it into another process. This gives us a freedom to choose any theory to describe a process as long as it gives the same outcome.

The reason for using EFTs is that they provide ways to describe processes or phenomena in an easier way than in the fundamental field theory. EFTs also extend the applicability of quantum field theory. It is possible to use EFTs in cases where the fundamental theory has not been solved as in the case of QCD, or is not a quantum field theory like gravity [Don94].

An example of an EFT is Euler-Heisenberg theory, which is a low energy theory of quantum electrodynamics. The photon is massless so at low energy it can be chosen as the active degree of freedom while removing the electron. This will then generate all photon-photon interactions that are consistent with the symmetries of quantum electrodynamics. In this case new 4-photon and higher vertices are generated in an expansion in the energy or frequency of the photons divided by the electron mass [Har01]. This theory will then of course only be valid under the electron-positron pair production energy of approximately 1 MeV. In this case the LECs determining the strength of the interactions among the photons can be calculated from quantum electrodynamics. The advantage of this approach is that it is much easier to compute the self-interaction of the electromagnetic field from this EFT than it is from the full theory of quantum electrodynamics.

A similar process can be applied to QCD to describe the interactions between hadrons. We take all hadrons that have a mass below a certain cutoff energy and find whatever symmetries we can impose on them from QCD and create the most general Lagrangian invariant under that symmetry. This yields a tool for calculating properties and interactions of hadrons even though the underlying theory is unsolved. This last fact makes it impossible to calculate the LECs from QCD at the moment; they must be determined from experiments or by using numerical methods.

A valid question is why this would be expected to be more successful than low energy QCD. The reason is that in taking the hadrons as the basic particles of our theory we have absorbed the largest part of the color force. All hadrons are color neutral and because of this the force between hadrons should be much weaker than the force between quarks. An analogy is the force between electrically charged particles, which is much stronger than the residual forces between neutral bound states of charged particles such as the van der Waals force.

1.2 A self-consistent framework for nuclear physics

One use for the EFTs of QCD is to provide a self-consistent and general framework for nuclear physics. Historically, nuclear physics has been developed as different phenomenological models. Collectively this is called the standard nuclear physics approach (SNPA).

The shortcomings of the SNPA are characterized by Krane [Kra87]:

> Nuclear physics lacks a coherent theoretical formulation that would permit us to analyze and interpret all phenomena in a fundamental way [...] As a result, we must discuss nuclear physics in a phenomenological way, using a different formulation to describe each different type of phenomenon, such as α decay, β decay, direct reactions, or fission.

Another problem is that these phenomenological models do not provide a connection to the Standard Model of particle physics. This means that the Standard Model can not be tested using the SNPA and also that the SNPA is not derived from the Standard Model.

With the advent of more powerful computers an alternative method has appeared. Based on brute force calculations many properties of nuclei have been calculated using realistic nucleon-nucleon potentials as the only input. This has been possible for some nuclei with up to 20 nucleons at this point [FRN11]. Many of these calculations have until recently been based on phenomenological SNPA inter-nucleon potentials.

In these calculations the nuclear wave functions and the excitation energies can be found, for example, by diagonalizing the Hamiltonian in an appropriate basis. These wave functions can then be used to calculate other properties such as cross-sections, decay rates as well as static ones such as radii, electromagnetic moments and so forth. To perform the dynamical calculations the interacting parts of the nuclear wave function must be found. This is called a nuclear current.

Recently it has become possible to use potentials derived from EFT to perform these ab-initio calculations. This enables the use of the same EFT to derive both the nuclear wave function and the interaction nuclear currents so that calculations can be done in a self-consistent way. This is important because intermediate results are often model dependent.

The infinite number of LECs in EFTs limit their predictive power. In principle these constants should be derivable from the underlying theory on which the EFT is based. For the case of low energy theories of QCD it should in principle be possible to utilize lattice field theory to compute the low energy constants.

Lattice field theory is a non-perturbative framework for quantum field theory. The basic idea is that by discretizing space-time, that is, formulating the theory on a lattice, computations can be made by, for example, Monte Carlo integration [Thi07]. QCD calculations using lattice field theory are collected under the term lattice QCD.

Lattice QCD is only dependent on the parameters of QCD, which are the quark masses and the strong coupling constant. If the LECs could be computed using this approach the infinite number of parameters of the EFTs would be dependent only on the finite number of QCD parameters.

Currently lattice QCD has not reached the level where the LECs of strong force EFTs can be computed to sufficient precision [EHM09]. Instead, experimental results must be used to fix the LECs. This limits the predictive power somewhat since it is then necessary to use a number of results as input to determine the values of the LECs. Still, once all LECs to certain order has been fixed, all other calculations have no free parameters and are fully predictive.

1.3 Purpose

I will focus on the EFT part of this program for the self-consistent framework.

There are two goals that I wish to achieve. The first is to give the reader a basic understanding of low energy hadron weak interactions. To achieve this an EFT is described including pions and later adding nucleons. This work will be limited to EFTs for nucleons and pions; kaons and other strange hadrons will be excluded.

The second goal is to give a detailed derivation of the nuclear two-body axial vector current. This current operator is important to provide an accurate description of the weak interactions in nuclei, for example, beta decay and neutrino nucleus interactions. The derivation of this current operator also serves as an example of an application of the EFT. The two-body axial vector current is only part of the full weak current. I will not go into detail about these other currents but focus my work on the two-body current.

This work will be performed using an EFT called Chiral Perturbation Theory (ChPT). It is a low energy theory of QCD that describes the interactions of pions and nucleons. I will describe it in detail in the following chapters.

Chapter 2

Pion-only chiral perturbation theory

The goal of any EFT is to descibe a certain set of processes. To create the theory the energy scale of these processes must first be identified. I want to describe the weak interactions of the lightest hadrons and nuclei. The lightest hadrons are the pions with a mass of approximately 140 MeV. The nucleons are much heavier with a mass of about 1 GeV. As a consequence, in a system with a center-of-momentum energy much below the nucleon mass and only incoming pions there will not be any resulting nucleons. In this chapter I will consider only such systems. How to include nucleons will be covered in the next two chapters.

An important symmetry of pions and nucleons is the chiral symmetry. Chiral symmetry has its roots in QCD and this symmetry is central to the low energy interactions of hadrons. The symmetry will be described in detail in this chapter. The most general theory of pions with the chiral symmetry is called chiral perturbation theory (ChPT). It is governed by a chiral invariant Lagrangian with an infinite number of terms representing an infinte series of interactions.

I will show how this Lagrangian can be ordered in a series in p/Λ where p is the external pion four-momentum and Λ is the high energy cutoff of this theory. There is an additional expansion in the symmetry breaking parameters, the quark masses m_q. Because of the quark masses' role in creating the pion masses each quark mass factor counts as p^2.

The cutoff is given by the next massive particle that can be created by the strong interactions. The kaons can be ignored since they must be produced in pairs due to conservation of strangeness, which leaves the rho-meson as the next lowest mass state. The result is a series expansion in p/M_ρ. The low energy constants of the theory need to be determined from experiment or by computational methods, e. g., lattice QCD, since low energy QCD has not been solved.

Using this Lagrangian I will show how to couple the pions to the vector bosons of the weak interaction. As an example of how this is performed I will show a detailed calculation of the decay rate of charged pions. This will demonstrate how to get to the explicit interaction Lagrangian and also how an experimental value can be used to determine one of the low energy constants of an EFT.

2.1 Energy scales of QCD

When choosing an energy scale and a cutoff for an EFT the matter content of the underlying theory plays a major part. In general, it is not fruitful to arbitrarily choose an energy scale and cutoff. Instead, in order to create a successful EFT, the mass spectrum must be studied in detail.

In the case of QCD there is a very large number of hadrons. It would be helpful if we could study only a subset of all of these particles. Luckily, as we can see in table 2.1, the pions are by far the lightest mesons. We will choose them as the matter content of our EFT. This also fixes the energy scale of our theory to that of the pion mass, about 140 MeV.

Table 2.1: Selected meson masses [PDG10]

Meson	Mass
π^0	135 MeV
$\pi^\pm$	140 MeV
$K^\pm$	493 MeV
$K^0, \bar{K}^0$	498 MeV
η	547 MeV
ρ	775 MeV

There are three different pions, which differ by their electric charge. The two charged pions are each other's antiparticles and thus have the same properties except for their charge. The neutral pion is slightly lighter, and it is its own antiparticle. None of the pions are stable even though they are the lightest strongly interacting particles. The charged pions decay weakly, mostly to muons and muon neutrinos; this gives them a comparatively long half life. The neutral pion mostly decays electromagnetically to two photons, which makes its half life much shorter. All pions are pseudoscalar particles, i.e., they are represented by scalar fields that pick up an extra minus sign under parity transformations.

After choosing the energy scale and matter content we need to determine the cutoff energy. This simply corresponds to the mass of the lightest excluded particle. The kaons must be produced in pairs so the lightest single particle that can be produced is the rho meson. Thus, the theory will be valid up to the rho meson mass but to assure reasonable convergence we should remain well below that energy; I will go into more detail about the series expansion in section 2.5.

2.2 Accidental symmetries of QCD

The next step is to find the relevant symmetries for the matter content of the theory. Since the underlying theory is QCD we look and see if there are any symmetries that our theory can inherit. Specifically, since we are interested in the dynamics of pions, the focus will be on the quark sector of QCD since the defining property of a hadron is its quark content.

Quarks are the matter particles of QCD. They have never been observed by themselves and are thought to be confined within the hadrons. There are 6 flavors of quarks which can be seen in table 2.2. Furthermore, each quark also has a color, which can be red, green or blue. This is of course not a real color but is just a name for the charge of the strong force known as the color charge.

Table 2.2: Quark flavors [PDG10]

Flavor	Charge	Mass
up	$+2/3$	2.5 MeV
down	$-1/3$	5 MeV
strange	$+2/3$	101 MeV
charm	$-1/3$	1.27 GeV
top	$+2/3$	172 GeV
bottom	$-1/3$	4.2 GeV

The color charge arises from an SU(3) gauge symmetry. This local symmetry gives rise to the force particles of QCD, that is, the gluons. These are massless vector bosons, in a way similar to the photon, but with the fundamental difference that the gluons themselves carry a color charge. There are 8 different color combinations of gluons corresponding to the 8 generators of SU(3).

The dynamics are governed by the QCD Lagrangian [Lan09]

$$\mathscr{L}_{\text{QCD}} = -\frac{1}{4}\sum_i F^i_{\mu\nu}F^{i\mu\nu} + \sum_r \bar{q}_{r\alpha} i\gamma^\mu i D_{\mu\ \beta}^{\ \alpha} q_r^\beta - \sum_r \bar{q}_r^{\ \alpha} q_{r\alpha} m_r. \tag{2.1}$$

In the first term, $F^i_{\mu\nu}$ is the field strength of the gluon fields. The index i runs over the 8 gluon fields. This term contains both the kinetic energy terms for the gluons and the gluon-gluon interaction. It does not contain any quark fields so we will ignore it.

In the second term, q_r^β are the quark fields. The index r runs over the quark flavors and the index β runs over the colors. $D_{\mu\ \beta}^{\ \alpha}$ is the color covariant derivative. It contains the kinetic term for the quarks and also the coupling to the gluon fields. The gluon field coupling is there to ensure SU(3) gauge invariance of the term. We note that the covariant derivative $D_{\mu\ \beta}^{\ \alpha}$ does not contain a flavor index r so its action is the same on all 6 quark flavors.

The last term, which gives rise to the quark masses is not really part of the QCD Lagrangian but comes from the coupling to the Higgs field and the spontaneous symmetry breaking in the weak sector of the Standard Model. In our case we can write these terms like this since we are not concerned with the details of gauge symmetry or spontaneous symmetry breaking.

As can be seen in table 2.2, the three lighter quarks; the up-, down- and strange quark; are much lighter than most hadrons and than the three heavier quarks. Of these three quarks the up- and the down quark are especially light.

Compared to the cutoff energy scale at the rho meson mass of ~ 770 MeV these two lightest quarks are almost massless. So for the next part we will ignore the heavier quarks and consider these two lightest quarks to be massless. Then, ignoring the gluon part, we have the Lagrangian

$$\begin{aligned}\mathscr{L}'_{\mathrm{QCD}} &= \sum_{r\in\{\mathrm{u,d}\}} \bar{q}_{r\alpha} i\gamma^\mu iD_{\mu\ \beta}^{\ \alpha} q_r^\beta \\ &= \bar{\boldsymbol{q}}_\alpha i\gamma^\mu iD_{\mu\ \beta}^{\ \alpha} \boldsymbol{q}^\beta.\end{aligned} \tag{2.2}$$

We have put the quark fields in a vector $\boldsymbol{q}^\alpha = (u^\alpha, d^\alpha)$. If we make a unitary transformation between the two quark flavors we get

$$\boldsymbol{q}^\alpha \to U\boldsymbol{q}^\alpha \tag{2.3}$$

$$\mathscr{L}'_{\mathrm{QCD}} \to \bar{\boldsymbol{q}}_\alpha U^\dagger i\gamma^\mu iD_{\mu\ \beta}^{\ \alpha} U\boldsymbol{q}^\beta. \tag{2.4}$$

Since the covariant derivative acts the same on all the different flavors we can just commute the unitary transformation through it;

$$\begin{aligned}\bar{\boldsymbol{q}}_\alpha U^\dagger i\gamma^\mu iD_{\mu\ \beta}^{\ \alpha} U\boldsymbol{q}^\beta &= \bar{\boldsymbol{q}}_\alpha U^\dagger U i\gamma^\mu iD_{\mu\ \beta}^{\ \alpha} \boldsymbol{q}^\beta \\ &= \bar{\boldsymbol{q}}_\alpha i\gamma^\mu iD_{\mu\ \beta}^{\ \alpha} \boldsymbol{q}^\beta \\ &= \mathscr{L}'_{\mathrm{QCD}}.\end{aligned} \tag{2.5}$$

This transformation leaves the Lagrangian unchanged which means that there is an approximate U(2) symmetry of the light quarks in QCD. But it is possible to find a larger symmetry than this. If we take a closer look at how the Dirac matrices connect the spinor components we see that the Lagrangian can be divided into two independent parts. We write each Dirac spinor as $q = (q_L, q_R)$ where q_L, q_R are two-component Weyl spinors. This decomposition can be seen explicitly by inserting the Weyl spinors into the Lagrangian,

$$\begin{aligned}\bar{\boldsymbol{q}}\gamma^\mu D_\mu \boldsymbol{q} &= \boldsymbol{q}^\dagger\gamma^0\gamma^\mu D_\mu \boldsymbol{q} \\ &= \begin{pmatrix}\boldsymbol{q}_L \\ \boldsymbol{q}_R\end{pmatrix}^\dagger \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \begin{pmatrix}0 & \sigma^\mu \\ -\bar{\sigma}^\mu & 0\end{pmatrix} D_\mu \begin{pmatrix}\boldsymbol{q}_L \\ \boldsymbol{q}_R\end{pmatrix} \\ &= \boldsymbol{q}_L^\dagger \sigma^\mu D_\mu \boldsymbol{q}_L - \boldsymbol{q}_R^\dagger \bar{\sigma}^\mu D_\mu \boldsymbol{q}_R.\end{aligned} \tag{2.6}$$

Since we have ignored the quark mass terms there is no part of the Lagrangian connecting q_L to q_R. Now, by the same argument as above each of these two terms can be independently flavor rotated. So we have two independent U(2) symmetries, i.e., a $U(2) \times U(2)$ symmetry. We will call this group $U(2)_L \times U(2)_R$ where the group $U(2)_L$ acts on the left handed spinors and $U(2)_R$ on the right handed spinors. This decomposition is manifest because of the choice of the Weyl representation of the Dirac algebra. Naturally, it is possible to achieve this decomposition in any representation.

However, this symmetry only holds on the classical level. During quantization parts of the symmetry are destroyed by an anomaly and only a $SU(2)_L \times SU(2)_R \times U(1)_V$ symmetry is left [PS95, p. 672]. The subscript V on $U(1)_V$ denotes that it acts on the whole Dirac spinor at the same time.

$SU(2)$ is the group of all unitary 2×2 matrices with determinant 1. The generators are three anti-Hermitian traceless matrices, $i\tau_i$ with τ_i being the Pauli matrices. This means that any $SU(2)$ matrix r can be written as $r = \exp i\Theta_i \tau_i$. The direct product group $SU(2) \times SU(2)$ is defined as the set of all pairs of elements in $SU(2)$ with the product defined as

$$(a, b)(c, d) \equiv (ac, bd). \tag{2.7}$$

However, we will use a different parametrization of the group than just independently rotating the left handed and the right handed fields. We will write it as $SU(2)_V \times SU(2)_A$. The vector subgroup, $SU(2)_V$, corresponds to both fields being rotated the same way with one element written as (v, v), $v \in SU(2)$. The axial vector subgroup in turn corresponds to the fields being rotated the opposite way (a, a^{-1}), $a \in SU(2)$. An element in $SU(2)_V \times SU(2)_A$ is written $\left(va, va^{-1}\right)$ and any element (l, r) in $SU(2)_L \times SU(2)_R$ can be written this way.

We can see this for a given element $(l, r) \in SU(2)_L \times SU(2)_R$ by choosing $a \in SU(2)_A$ such that

$$l = ra^2 \Leftrightarrow la^{-1} = ra. \tag{2.8}$$

If we then choose $v \in SU(2)_V$ such that $l = va$ we have

$$v^{-1} l a^{-1} = 1, \tag{2.9}$$

but by equation 2.8 we have that

$$v^{-1} ra = v^{-1} l a^{-1} = 1. \tag{2.10}$$

Extracting r from this equation gives,

$$r = va^{-1}, \tag{2.11}$$

just like we wanted to show.

If we go back to table 2.2 we see that the strange quark mass is also comparatively low, both compared to the other quarks and to the typical hadron mass of about 1 GeV. Including the strange quark would yield an approximate $SU(3) \times SU(3) \times U(1)$ symmetry. This is also a usable theory since the strange quark mass is low compared to the hadron mass scale. Most of the results presented here can be carried over to that theory with some minor modifications.

2.3 Chiral symmetry of pions

In the previous section we saw that QCD has an $SU(2) \times SU(2) \times U(1)$ symmetry. Our theory of pions should have this symmetry as well. In the following, we will see how to implement the transformation of this group on the pions and use that to create a chiral-invariant Lagrangian.

There is, however, one more thing we need to take into account. Nature does not seem to fulfill the full chiral symmetry. Only the $SU(2)_V \times U(1)_V$ part of it seems to be present when looking at particles found in nature. Thus, if we still want to believe in QCD, the symmetries must be hidden in some way. This can be realized in the framework of spontaneous symmetry breaking where the ground state, i.e., the vacuum, breaks the symmetry. This is believed to be the case in low energy QCD [GL84]. As we will see, this can also give an explanation for the very low masses of the light mesons which gives further support to this idea.

Spontaneous symmetry breaking is a mechanism where the dynamics of a theory destroys the invariance of the vacuum under one or more of its symmetries. For example, this may occur by some field taking a non-vanishing vacuum expectation value. When applying a symmetry transformation this expectation value is changed. Therefore, the symmetry has been broken.

According to Goldstone's theorem there will be a massless, scalar particle for each broken continuous symmetry [GSW62]. These particles are called Goldstone bosons. If we break $SU(2)_V \times SU(2)_A$ to $SU(2)_V$ we have broken the $SU(2)_A$ subgroup. This subgroup contains three independent symmetries so there should be three massless bosons. The three pions are unnaturally light and have the correct quantum numbers; so they are good candidates to be the Goldstone bosons.

The quark masses are only small, they do not vanish completely. The consequence of this is that the chiral symmetry is only approximate. This gives an explanation for the non-vanishing mass of the pions. Since the broken symmetry is only approximate the pions are only approximate Goldstone bosons.

The question is now how these bosons will transform under $SU(2)_V \times SU(2)_A$. We will use the conventions of Scherer [Sch03] regarding the transformations of the pions. The ground state must be invariant only under the unbroken $SU(2)_V$ part of the chiral group.

We introduce three real scalar fields ϕ_a which are linear combinations of the pion fields π^-, π^+ and π^0. To each field ϕ_a we pair a generator of SU(2), i. e., a Pauli matrix τ_a,

$$\phi = \tau_a \phi_a = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}. \tag{2.12}$$

We exponentiate this matrix to get an $SU(2)$ matrix with the fields ϕ_a as transformation parameters,

$$U = \exp\left(i\frac{\phi}{f_\pi}\right) = \exp\left(i\frac{\phi_a \tau_a}{f_\pi}\right). \tag{2.13}$$

The new constant f_π is the pion decay constant; it is one of the LECs of this theory of pions. It is possible to identify this group element with a representative of a left coset in $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R / \mathrm{SU}(2)_V$. We write the group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ as $\{(l, r) : l \in \mathrm{SU}(2), r \in \mathrm{SU}(2)\}$ and the $\mathrm{SU}(2)_V$-subgroup as $\{(v, v) : v \in \mathrm{SU}(2)\}$. We can then write an arbitrary coset as

$$(\tilde{l}, \tilde{r})\mathrm{SU}(2)_V = (1, \tilde{r}\tilde{l}^{-1})(\tilde{l}, \tilde{l})\mathrm{SU}(2)_V = (1, \tilde{r}\tilde{l}^{-1})\mathrm{SU}(2)_V. \tag{2.14}$$

Thus the coset can be represented by the element $\tilde{r}\tilde{l}^{-1}$. We will write a matrix representation with the same capital letter, for example, $L = \mathscr{D}(l)$. The representative of the coset then has the matrix representation $\tilde{R}\tilde{L}^\dagger$. Since $\tilde{L}, \tilde{R}$ represent arbitrary elements of SU(2) also the product $\tilde{R}\tilde{L}^\dagger$ represents an arbitrary element and because of this we can identify it with the matrix U.

To get the transformation behaviour of U we look at the left multiplication of a group element, $(l, r) \in \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, and a left coset represented by u,

$$\begin{aligned}
(l, r)(1, u)\mathrm{SU}(2)_V &= (l, ru)\mathrm{SU}(2)_V \\
&= (1, rul^{-1})(l, l)\mathrm{SU}(2)_V \\
&= (1, rul^{-1})\mathrm{SU}(2)_V.
\end{aligned} \tag{2.15}$$

Thus we can let U transform as

$$U \to RUL^\dagger. \tag{2.16}$$

If the transformation is part of the unbroken group $\mathrm{SU}(2)_V$ the ground state should be invariant. The ground state is the same as the vacuum state, i.e. all fields $\phi_i = 0$ and $U_0 = 1$. We let this transform under $\mathrm{SU}(2)_V$,

$$U_0 \to VU_0V^\dagger = V1V^\dagger = VV^\dagger = 1 = U_0. \tag{2.17}$$

We see that our construction of U_0 is invariant is this case. On the other hand for a transformation in the broken subgroup $\mathrm{SU}(2)_A$ we get

$$U_0 \to A^\dagger U_0 A^\dagger = A^\dagger A^\dagger. \tag{2.18}$$

The vacuum is not invariant which is consistent with spontaneous symmetry breaking.

It can also be shown [Sch03] that the pion fields ϕ_a transform linearly under the unbroken subgroup $\mathrm{SU}(2)_V$. This is not true for the broken subgroup $\mathrm{SU}(2)_A$.

2.4 Building blocks for interactions

The next point to address is how to use the pion matrix U to create chiral invariant Lagrangian terms. The Lagrangian is a Lorentz scalar, and as such all of its terms must be Lorentz scalars. Since the QCD Lagrangian is invariant under parity transformations, the ChPT Lagrangian should also have this symmetry.

We need some way to turn the pion matrix U into a scalar while also ensuring that it is invariant under the required symmetry transformations. One way to turn a matrix into a scalar is to take its trace. The trace of only one pion matrix, $\mathrm{Tr}\, U$ is a scalar but it is not chiral invariant. But, if we take a pair of matrices A, B that transform as $A, B \to RAL^\dagger, RBL^\dagger$ we can form a chiral invariant,

$$\mathrm{Tr}\left(AB^\dagger\right) \to \mathrm{Tr}\left(RAL^\dagger LB^\dagger R^\dagger\right) = \mathrm{Tr}\left(RAB^\dagger R^\dagger\right) = \mathrm{Tr}\left(R^\dagger RAB^\dagger\right) = \mathrm{Tr}\left(AB^\dagger\right). \quad (2.19)$$

Here we have exploited the cyclic property of the trace. The trace of the product $UU^\dagger$ is trivial since U is unitary.

One allowed term is then $\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$. But to create an accurate low energy theory of QCD the global $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry must be upgraded to a local (or gauge) symmetry [Leu94].

This means that the transformations L, R are made to be dependent on spacetime. The transformation of U is the same as in the global case,

$$U(x) \to R(x)U(x)L(x)^\dagger. \quad (2.20)$$

But for terms containing derivatives of U we also get derivatives of the transformation matrices;

$$\begin{aligned} \partial_\mu U(x) \to& \partial_\mu\left(R(x)U(x)L^\dagger(x)\right) \\ &= \partial_\mu R(x)U(x)L^\dagger(x) \\ &+ R(x)\partial_\mu U(x)L^\dagger(x) \\ &+ R(x)U(x)\partial_\mu L^\dagger(x). \end{aligned} \quad (2.21)$$

This destroys the invariance of $\mathrm{Tr}\left(\partial_\mu U(x)\partial^\mu U^\dagger(x)\right)$. In the following, I will not write the spacetime dependence explicitly.

To repair the invariance we introduce the covariant derivative D_μ, which transforms in a way as to cancel the terms that destroy the invariance. Following Scherer [Sch03] we introduce the gauge fields R_μ and L_μ. Their transformation properties are given by

$$R_\mu \to RR_\mu R^\dagger + iR\partial_\mu R^\dagger, \quad (2.22)$$

$$L_\mu \to LL_\mu L^\dagger + iL\partial_\mu L^\dagger. \quad (2.23)$$

The covariant derivative acting on U is defined as

$$D_\mu U \equiv \partial_\mu U - iR_\mu U + iUL_\mu. \quad (2.24)$$

We look at how this transforms,

$$\begin{aligned} D_\mu U \to& \partial_\mu(RUL^\dagger) - i(RR_\mu R^\dagger + iR\partial_\mu R^\dagger)RUL^\dagger \\ &+ iRUL^\dagger(LL_\mu L^\dagger + iL\partial_\mu L^\dagger) \\ &= \partial_\mu RUL^\dagger + R\partial_\mu UL^\dagger + RU\partial_\mu L^\dagger \\ &- iRR_\mu UL^\dagger + R\partial_\mu R^\dagger RUL^\dagger \\ &+ iRUL_\mu L^\dagger - RU\partial_\mu L^\dagger. \end{aligned} \quad (2.25)$$

The unwanted terms containing the derivative on $L^\dagger$ are immediately cancelled. To see the cancellation of the terms with the derivative on R we need to use that R is unitary,

$$R^\dagger R = 1, \tag{2.26}$$

$$\begin{aligned} 0 = \partial_\mu \left(R^\dagger R\right) = \partial_\mu R^\dagger R + R^\dagger \partial_\mu R; \\ \Rightarrow \partial_\mu R^\dagger R = -R^\dagger \partial_\mu R. \end{aligned} \tag{2.27}$$

Then we can rewrite the term $R\partial_\mu R^\dagger RUL^\dagger = -\partial_\mu RUL^\dagger$, which cancels the remaining unwanted terms. So we are left with

$$R\partial_\mu UL^\dagger - iRR_\mu UL^\dagger + iRUL_\mu L^\dagger = RD_\mu UL^\dagger. \tag{2.28}$$

This is the same transformation property as for the pion matrix U. As a consequence $D_\mu U$ is a building block that can be used in the same way to create a chiral-invariant Lagrangian.

There is an added benefit of introducing the gauge fields L_μ, R_μ; they can be used to facilitate the coupling to external fields or particles that are not part of ChPT.

For the gauge fields we can write down the field strength tensors

$$f^R_{\mu\nu} \equiv \partial_\mu R_\nu - \partial_\nu R_\mu - i[R_\mu, R_\nu], \tag{2.29}$$

$$f^L_{\mu\nu} \equiv \partial_\mu L_\nu - \partial_\nu L_\mu - i[L_\mu, L_\nu]. \tag{2.30}$$

These transform as $f^R_{\mu\nu} \to Rf^R_{\mu\nu}R^\dagger$ and $f^R_{\mu\nu} \to Lf^L_{\mu\nu}L^\dagger$. Following Scherer [Sch03, p.102] who is following the convention of Gasser and Leutwyler we introduce the linear combination

$$\chi \equiv 2B_0(s + ip), \tag{2.31}$$

where s and p are scalar and pseudoscalar external fields respectively. These fields are matrices; we let them transform as $s, p \to RsL^\dagger, RpL^\dagger$. Concerning the quark masses we here employ a trick and let them be in s even though they will not have the correct chiral transformation. In fact the quark masses will contribute symmetry breaking terms.

We can now write down a few more building blocks transforming as $A \to RAL^\dagger$ [Sch03]:

$$\chi, \quad Uf^L_{\mu\nu}, \quad f^R_{\mu\nu}U. \tag{2.32}$$

Out of these only the combinations $\chi U^\dagger$ and $U\chi^\dagger$ have non-vanishing trace. In order to get a parity-invariant Lagrangian only the combination $\chi U^\dagger + U\chi^\dagger$ can be accepted [Sch03, pp.103-104].

2.5 Ordering

When creating the most general Lagrangian we will end up with an infinite number of terms containing products of U, the derivative of U and external fields including the quark masses. In front of each of these independently invariant terms there will be an unknown proportionality constant, an LEC. To be able to use this theory to make any kind of prediction we must have a way of ordering the terms by their relative contribution.

We start by dividing the Lagrangian into parts corresponding to the power of the quark masses and derivatives. Each derivative will generate a pion four-momentum p for the corresponding vertex in the Feynman rules and each quark mass is equivalent to two powers of the pion four-momentum [Sch03].

Because the dimension of each Lagrangian term must be fixed, each pion momentum or quark mass must be divided by some other mass or energy. The only other energy scale available is the cutoff $\Lambda = M_\rho$ and as such the expansion will be in p/M_ρ.

The contribution to the order of a diagram by a vertex of type i is [Wei79]

$$\nu_i = d_i + e_i - 2, \tag{2.33}$$

where d_i is the number of derivatives and e_i the number of external fields. We will denote the Lagrangian terms by the contribution to the order of the resulting vertices.

When determining the order of a diagram we need to look both at the order of the vertices included and the number of loops in the diagram. With V_i the number of vertices of type i and L the number of independent loops the order is given by [Wei79]

$$\nu = 2 + 2L + \sum_i V_i \nu_i \tag{2.34}$$

The order ν is the exponent in the expansion, that is, a diagram of order ν is proportional to $(p/\Lambda)^\nu$.

For example if we want all diagrams of order $\nu = 2$ we can have any number of vertices from $\mathscr{L}_0^\pi$ and no loops. At order $\nu = 4$ we can have any number of vertices from $\mathscr{L}_0^\pi$ and one vertex from $\mathscr{L}_2^\pi$ or one loop. When calculating any quantity to a given order all diagrams of lower orders need to be included.

The reason for counting the order of the Lagrangian terms the way that we do is to make it consistent with ChPT including multiple nucleons. In this case it becomes more clear how a certain Lagrangian term will contribute to a process no matter if it contains two, four or zero nucleons. Some authors(e.g. ref. [Sch03]) denote the order of Lagrangian terms by the number of derivatives d_i plus the number external fields e_i. In this case the order of each Lagrangian term is increased by 2.

2.6 Leading order Lagrangian

Of course, all terms in the Lagrangian must be Lorentz invariant as well as chiral invariant. The only available four-vector is the covariant derivative, D_μ. There are only two ways of turning four-vectors into scalars, either through contracting them with the metric $g_{\mu\nu}$ or the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$. This means that we can only have an even number of covariant derivatives in a Lagrangian term. If we also recall that the each quark mass counts as two powers of momentum, we can see that our expansion will only contain even order terms. Then the Lagrangian can be written as

$$\mathscr{L}^\pi = \mathscr{L}_0^\pi + \mathscr{L}_2^\pi + \cdots . \tag{2.35}$$

So to create $\mathscr{L}_0^\pi$ we want to find all terms of order p^2/M_ρ^2. This means that we can have either two derivatives or one quark mass.

There is one allowed term with two derivatives. As we saw in section 2.4, we need to take the trace of the resulting chiral matrix to get a chiral invariant term,

$$\mathrm{Tr}\left(D_\mu U \left(D^\mu U\right)^\dagger\right). \tag{2.36}$$

There is also a term including the quark masses. The quark masses are contained in χ, and we saw in section 2.4 that the only allowed expression is

$$\chi U^\dagger + U\chi^\dagger, \tag{2.37}$$

which, since χ contains one quark mass is of order p^2/M_ρ^2.

Combining these expressions we get the leading order Chiral perturbation theory Lagrangian,

$$\mathscr{L}_0^\pi = \frac{f_\pi^2}{4}\left(\mathrm{Tr}\left(D_\mu U \left(D^\mu U\right)^\dagger\right) + \mathrm{Tr}\left(\chi U^\dagger + U\chi^\dagger\right)\right). \tag{2.38}$$

This Lagrangian contains two LECs, the pion decay constant, f_π; and the parameter B_0, which is contained in χ. Both of these are free parameters that need to be fixed either by fitting them to experimental data or by deriving their values from QCD with, for example, lattice QCD. In the last section of this chapter we will see one possible way to determine the value of f_π.

To be able to make calculations we need the interaction Lagrangian. By this we mean the Lagrangian expressed in its constituent fields, for example, π, L_μ, and not the more abstract building blocks of the Lagrangian like U. When we have done this the Feynman rules can be easily read off.

2.7 Weak interactions of pions

In order to understand how to implement the interactions between the pions in ChPT with the vector bosons $W^\pm$, we will look at how the interaction is realized in the underlying theory, i. e, the Standard Model. By comparing the symmetries

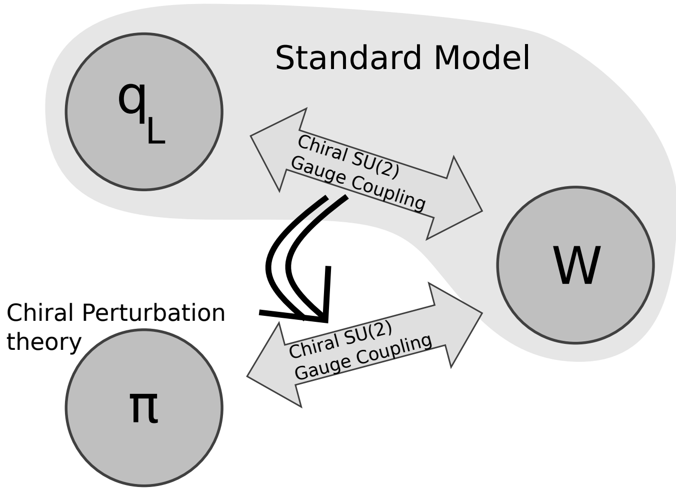


Figure 2.1: The coupling of pions and $W^{\pm}$ bosons follows from the coupling of quarks and $W^{\pm}$ bosons.

of the Standard Model with that of ChPT we can transfer the interactions of the quarks to the pions.

In the Standard model, the electroweak interaction is realized by postulating a local $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry of the lepton and quark fields, giving rise to four vector boson fields. Three of the vector boson fields are given masses by the Higgs-induced spontaneous symmetry breaking; one is left massless.

Another feature of the Standard Model is that it contains no regular fermion mass terms. Instead, all fermion masses are generated by Yukawa couplings to the Higgs field. With no mass terms, the natural basic building blocks for fermions are not the four-component Dirac spinors but the two-component Weyl spinors like we saw in section 2.2.

The $W^{\pm}$ bosons only interact with the left handed Weyl spinors of the quark fields. To get the interaction of the up- and down quarks we create an SU(2) doublet of the left handed fields

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \tag{2.39}$$

then for the physical up and down quarks the interaction term is [PS95, pp. 704, 714],

$$\mathscr{L}_0^{\mathrm{W}} = \bar{Q}_L \tilde{L}_\mu \gamma^\mu Q_L, \tag{2.40}$$

where,

$$\tilde{L}_\mu = \frac{V_{ud}\, g}{\sqrt{2}} \left(W_\mu^+ T^+ + W_\mu^- T^- \right). \tag{2.41}$$

Here $\tilde{L}_\mu$ is part of the gauge current of the SU(2) subgroup in the SU(2) × U(1) group. $W^\pm$ are the W-boson fields and the matrices T^+ and T^- give the coupling to the quarks fields,

$$T^{-\dagger} \equiv T^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.42}$$

while g is the weak interaction coupling strength. Even though the weak interaction has a universal strength due to gauge invariance there is an additional coupling constant V_{ud}. The reason is that the quark mass eigenstates are not the same as those that are created and destroyed by the interaction with the W-bosons; V_{ud} parametrizes this overlap.

This is nothing new, the SU(2) group acts on exactly the same quark doublet as the $\mathrm{SU}(2)_L$ subgroup of the chiral group. To implement the weak interactions of the pions we only need to identify the gauge fields in the Standard Model with the corresponding ones in ChPT as illustrated in figure 2.1. In ChPT we have already introduced the gauge field L_μ for the gauged $\mathrm{SU}(2)_L$ subgroup. To implement the weak interaction we simply identify

$$L_\mu \equiv \tilde{L}_\mu. \tag{2.43}$$

To get the vertices at leading order we set $R_\mu = 0$ in the effective Lagrangian (2.38) and expand the covariant derivative

$$\begin{aligned}\mathrm{Tr}\left(D_\mu U \left(D^\mu U\right)^\dagger\right) &= \mathrm{Tr}\left(\left(\partial_\mu U + iUL_\mu\right)\left(\partial^\mu U^\dagger - iL^\mu U^\dagger\right)\right) \\ &= \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger + iUL_\mu \partial^\mu U^\dagger - i\partial_\mu U L^\mu U^\dagger + UL_\mu L^\mu U^\dagger\right).\end{aligned} \tag{2.44}$$

The first term only contains the pion matrix U so it will not give rise to any interactions with the weak sector and the last term does not contribute at leading order since it is not linear in L_μ.

The pion matrix U is by definition unitary since it is an element of SU(2). Thus we can use the result of equation (2.27), $\partial_\mu U^\dagger U = -U^\dagger \partial_\mu U$. We use this to simplify equation (2.44). Together with the cyclic property of the trace we see that the second- and third term are equal. Thus, the relevant part of the Lagrangian can be written

$$\mathscr{L}_0^W = \frac{f_\pi^2}{2} \mathrm{Tr}\left(iL^\mu \partial_\mu U^\dagger U\right). \tag{2.45}$$

To get the interaction in terms of the pion fields we make an expansion of U in the matrix ϕ

$$U = \exp\left(\frac{i\phi}{f_\pi}\right) = 1 + \frac{i\phi}{f_\pi} + \mathscr{O}\left(\phi^2\right). \tag{2.46}$$

We insert this expression into the Lagrangian and keep terms linear in ϕ and L_μ. The matrix ϕ is Hermitian since it is a real linear combination of the Hermitian

Pauli matrices. We can then write the leading order interaction Lagrangian as

$$\begin{aligned}\mathscr{L}_0^{\pi W} &= \frac{f_\pi}{2}\,\mathrm{Tr}\left(L_\mu \partial_\mu \phi^\dagger\right) \\ &= \frac{g V_{ud} f_\pi}{2\sqrt{2}}\,\mathrm{Tr}\left(W_\mu^+ T^+ \partial^\mu \phi + W_\mu^- T^- \partial^\mu \phi\right).\end{aligned} \tag{2.47}$$

We insert the pion fields as in equation (2.12) and look at the first term of the trace,

$$\begin{aligned}\mathrm{Tr}\left(W_\mu^+ T^+ \partial^\mu \phi\right) &= \mathrm{Tr}\left(W_\mu^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \partial^\mu \begin{pmatrix} \pi_0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi_0 \end{pmatrix}\right) \\ &= \mathrm{Tr}\left(W_\mu^+ \partial^\mu \begin{pmatrix} \sqrt{2}\pi^- & -\pi^0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \sqrt{2} W_\mu^+ \partial^\mu \pi^-.\end{aligned} \tag{2.48}$$

The second term works out in the same way and we get the leading order interaction Lagrangian,

$$\mathscr{L}_0^{\pi W} = \frac{V_{ud} g f_\pi}{2}\left(W_\mu^+ \partial^\mu \pi^- + W_\mu^- \partial^\mu \pi^+\right). \tag{2.49}$$

This gives rise to two vertices in the Feynman rules coupling W bosons to pions. Each derivative of the pion fields gives a factor of $-ip^\mu$ for a field with momentum p^μ. We get the vertices

π^+, p^μ  $W^+ = \frac{V_{ud} g f_\pi}{2} p^\mu$ (2.50)

π^-, p^μ  $W^- = \frac{V_{ud} g f_\pi}{2} p^\mu.$ (2.51)

2.8 Decay of charged pions

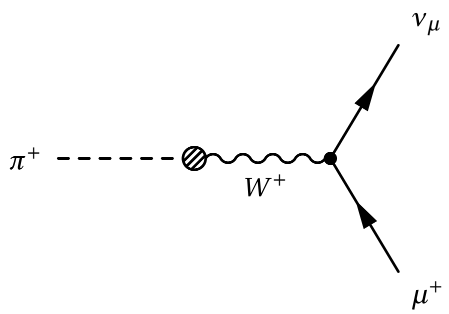


Figure 2.2: Tree level diagram giving the leading order contribution to the process $\pi^+ \to \mu^+ + \nu_\mu$

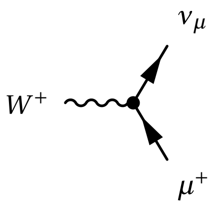
The overwhelming majority of positive pion decays go to a positive muon and a muon neutrino [PDG10]. By comparing the observed decay rate with one calculated from chiral perturbation theory we get a constraint on one of the LECs of ChPT.

At order p^2, i.e., at tree level with only vertices from $\mathscr{L}_0^\pi$, only the diagram in figure 2.2 contributes to the amplitude.

The W^+-boson propagator [PS95, p.743] can be approximated because of the very high mass of this particle compared to the mass of the pion. For a vector boson momentum $k^2 \approx m_\pi^2 << m_W^2$ we have

$$\frac{-i}{k^2 - m_W^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}\right) \approx \frac{i g_{\mu\nu}}{m_W^2}. \tag{2.52}$$

Further the vertex coupling the leptons to the W^+ is given by [PS95, p.705]



$$= -i\frac{g}{2\sqrt{2}}\gamma^\mu(1-\gamma^5). \tag{2.53}$$

Both the pion and the leptons couple to the W^+ with the strength g and with the approximated W-boson propagator it is convenient to use the Fermi constant

$$G_F = \frac{\sqrt{2}}{8}\frac{g^2}{m_W^2}. \tag{2.54}$$

Using these components we can write down the amplitude for the diagram in figure 2.2. With the incoming pion momentum p, muon momentum k' and neutrino momentum k we get

$$\begin{aligned} i\mathscr{M} &= \frac{V_{ud}\, g f_\pi}{2} p^\mu \frac{i g_{\mu\nu}}{m_W^2} \bar{u}^s(k)\left(-\frac{ig}{2\sqrt{2}}\right)\gamma^\nu\left(1-\gamma^5\right) v^r(k') \\ &= G_F V_{ud} f_\pi \bar{u}^s(k) \not{p}\left(1-\gamma^5\right) v^r(k'). \end{aligned} \tag{2.55}$$

This can be simplified by using standard methods, for further details look in appendix A. To compute the decay rate we need the square of the amplitude $|\mathscr{M}|^2$, which we will compute in the rest frame of the pion. With E_ν the neutrino energy and E_μ the muon energy we get

$$|\mathscr{M}|^2 = 8G_F^2 V_{ud}^2 f_\pi^2 m_\pi^2 \left(E_\nu E_\mu + \boldsymbol{k}\cdot\boldsymbol{k}'\right). \tag{2.56}$$

Combining the two-body phase space [PS95, p.107] with the square of the matrix element we get the decay rate,

$$\begin{aligned} \int d\Gamma &= \frac{1}{2m_\pi}\int d\Pi_2 |\mathscr{M}|^2 \\ &= \frac{1}{2m_\pi}\int d\Omega \frac{1}{16\pi^2}\frac{|\boldsymbol{k}|}{m_\pi} 8G_F^2 V_{ud}^2 f_\pi^2 m_\pi^2 \left(E_\nu E_\mu - \boldsymbol{k}^2\right). \end{aligned} \tag{2.57}$$

After performing the integration and simplifying the resulting expression we get the pion decay rate

$$\Gamma = \frac{G_F^2 V_{ud}^2 f_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 . \tag{2.58}$$

All parameters in this expression except for f_π are known from other experiments. By measuring the decay rate of charged pions we can now determine the value of the pion decay constant f_π. When we know the value we can reuse it in any other calculation dependent on f_π.

We could also turn this argument around. By determining f_π from another experiment, perhaps pion-pion scattering, we can predict the pion decay rate. If we find a way to determine B_0 from an experiment we have fixed all the LECs of leading order ChPT. This means that we could then predict the result of any experiment to leading order.

So, by using one set of observables to determine the LECs we can then predict the results of all other experiments. The result will always be approximative since there are an infinite number of higher order corrections which are ignored.

Chapter 3

Baryon chiral perturbation theory

Building on the previous chapter I will review how to add nucleons to chiral perturbation theory. Because of the different structure of the Lagrangian terms when including fermions it will be necessary to develop new transformation rules for the nucleons.

There are issues that make it hard to create a consistent power-counting scheme in the nucleon-pion sector. The reason for this is that a new mass scale is introduced when including nucleons, namely the nucleon mass. I will explore an example of this problem in the end of the chapter with the calculation of the nucleon self-energy.

This chapter is minimal and I will not venture into nearly as much detail as the previous one. The reason being that I will not be making the major calculations in this formalism, instead it will only be used for further theoretical developments.

3.1 Building blocks

We will consider the nucleons to be point-like, spin-1/2 particles. As such we introduce Dirac spinor fields p and n for the proton and the neutron. Just as in the pion-only sector all the internal structure of the nucleons will be encoded in the LECs.

The nucleons are then inserted into a chiral vector, the nucleon doublet,

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}. \tag{3.1}$$

The nucleons are low energy QCD particles so they will transform under the chiral group. We will let them transform as

$$\Psi \to K(L, R, U)\Psi. \tag{3.2}$$

Here K is a function of the SU(2) transformations L, R and the pion matrix U,

$$K(L,R,U) = \sqrt{LU^\dagger R^\dagger} R\sqrt{U}. \tag{3.3}$$

This is also an SU(2) transformation [BKM95].

Since the pion matrix U is a local matrix the transformation of the nucleon field will be local. So, in order to get the correct transformation for the derivative of the nucleon field $\partial_\mu \Psi$ we need to use a covariant derivative [GSS88],

$$\begin{aligned} D_\mu \Psi &= \partial_\mu \Psi + \Gamma_\mu \Psi \\ \Gamma_\mu &= \frac{1}{2}\left[u^\dagger, \partial_\mu u\right] - \frac{i}{2} u^\dagger \left(V_\mu + A_\mu\right) u - \frac{i}{2} u \left(V_\mu - A_\mu\right) u^\dagger. \end{aligned} \tag{3.4}$$

The pion fields are contained in the matrix u, which is defined by

$$u^2 = U. \tag{3.5}$$

The gauge fields $V_\mu = R_\mu + L_\mu$ and $A_\mu = R_\mu - L_\mu$ are exactly the same gauge fields as the ones in chapter 2 and have the same transformation behavior. The transformation of the covariant derivative of the nucleon field is the expected

$$D_\mu \Psi \to K D_\mu \Psi. \tag{3.6}$$

We now have two covariant derivatives, which to use will be determined by the context. Both will be written as D_μ.

There is one more building block that we need for the nucleon ChPT Lagrangian. It is an axial vector object similar to the connection. It is called the chiral vielbein and is given by [GSS88]

$$\Delta_\mu = \frac{1}{2}\left\{u^\dagger, \partial_\mu u\right\} - \frac{i}{2} u^\dagger \left(V_\mu + A_\mu\right) u + \frac{i}{2} u \left(V_\mu - A_\mu\right) u^\dagger. \tag{3.7}$$

Under a chiral transformation the vielbein transforms covariantly

$$\Delta_\mu \to K \Delta_\mu K^\dagger. \tag{3.8}$$

3.2 Lagrangian

Any number of operators transforming as $B \to K B K^\dagger$ can be sandwiched between the nucleon fields, or covariant derivatives of the nucleons fields, to form a chiral invariant term. We want to have the minimal number of derivatives in order to get the lowest order Lagrangian.

There is one term with no derivatives,

$$\bar{\Psi}\Psi. \tag{3.9}$$

With one covariant derivative we get the term

$$\bar{\Psi}\gamma^\mu D_\mu \Psi. \tag{3.10}$$

The vielbein Δ_μ also contains one derivative. In order to create a parity invariant term we must contract it with another axial vector,

$$\bar{\Psi}\gamma^\mu\gamma_5\Delta_\mu\Psi. \tag{3.11}$$

Each of these three terms will have a preceding LEC and an arbitrary phase factor to follow conventions and ensure the reality of coupling constants. One of the LECs can be removed by redefining the nucleon field Ψ. With this in mind, the most general chiral-invariant Lagrangian with the smallest number of derivatives is [GSS88]

$$\mathscr{L}^{\mathrm{N}} = \bar{\Psi}\left(i\not{D} - m_N + ig_A\gamma^\mu\gamma_5\Delta_\mu\right)\Psi, \tag{3.12}$$

with m_N being the nucleon mass. A new LEC, the axial coupling constant g_A, has been introduced. It gives the coupling strength of the nucleon to a single pion as we will see in the next section.

This Lagrangian only describes the nucleons and their interactions with the pions. The dynamics and interactions of the pions are still described by the same Lagrangian of equation (2.38).

3.3 Nucleon self-energy

The purpose of this section is both to give an example and to show a problem with naive baryon ChPT. We will see that in the chiral limit, i. e., with zero pion mass, the nucleon mass is renormalized. This indicates that it will be hard to count the order of the contribution of a given diagram, more about this in the end of the section.

The Lagrangian (3.12) contains a free Dirac Lagrangian for each nucleon. With this as the free theory each nucleon will have the propagator,

$$iS_F(p) = \frac{i}{\not{p} - \mathring{m}_N + i\varepsilon}. \tag{3.13}$$

This expression is modified by the self-energy $\Sigma(p)$ in the usual way by including all irreducible diagrams in a geometric series (for more details see ref. [PS95, p. 220]),

$$iS_F(p) = \frac{i}{\not{p} - \mathring{m}_N - \Sigma(p) + i\varepsilon}. \tag{3.14}$$

For a real nucleon the four-momentum must satisfy $p^2 = m^2$. For a small perturbation the change in the nucleon mass should be small so that to zeroth order $m = \mathring{m}_N$. We will therefore evaluate $\Sigma(p)$ at $p^2 = \mathring{m}_N^2$.

Interaction Lagrangian and Feynman rules

To find the self-energy we will need an explicit expression of the interaction between nucleons and pions; we can derive it from the Lagrangian in (3.12). By deriving the interaction Lagrangian and the Feynman rules it will be possible to find and calculate the leading order self-energy diagrams.

There is an $NN\pi$ vertex in the leading order Lagrangian which comes from the term

$$\mathscr{L}_1^{\mathrm{N}\prime} = \bar{\Psi} g_A \gamma^\mu \gamma^5 i\Delta_\mu \Psi. \tag{3.15}$$

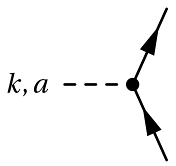
We set all external fields to zero and expand the pion matrix u in the vielbein

$$i\Delta_\mu = -\frac{1}{2f_\pi}\partial_\mu \phi^a \tau^a + \mathcal{O}(\pi^2). \tag{3.16}$$

Inserting this into the Lagrangian term gives the leading order $NN\pi$ Lagrangian

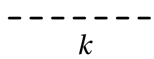
$$\mathscr{L}_1^{NN\pi} = -\bar{\Psi}\frac{g_A}{2f_\pi}\gamma^\mu\gamma^5\partial_\mu\phi^a\tau^a\Psi \tag{3.17}$$

This gives rise to the vertex



$$= \quad -\frac{g_A}{2f_\pi}\gamma^5 \not{k}\tau^a. \tag{3.18}$$

We will also need the pion propagator, which is the Klein Gordon propagator in momentum space [PS95],



$$= \quad \frac{i}{k^2 + i\varepsilon}. \tag{3.19}$$

Here the pion mass is set to zero since we do this calculation in the chiral limit.

Feynman diagrams and their evaluation

Using the $NN\pi$-vertex, the leading order contribution to the self-energy is given by the diagram in figure 3.1.

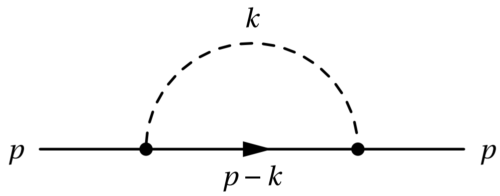


Figure 3.1: Nucleon self-energy diagram

When evaluating this diagram we get a factor 3 from summing over the Pauli matrices,

$$i\Sigma = \frac{3g_A^2}{4f_\pi^2}\int \frac{d^4k}{(2\pi)^4}\frac{\not{k}\gamma^5\left(\not{p} - \not{k} + \mathring{m}_N\right)\not{k}\gamma^5}{(k^2 + i\varepsilon)\left(\left(p-k\right)^2 - \mathring{m}_N^2 + i\varepsilon\right)}. \tag{3.20}$$

We will use dimensional regularization to control the divergent integrals. To avoid problems when including γ^5 we will use only the following identities to simplify the expression [Sch03],

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad g^\mu_\mu = n, \quad \{\gamma_\mu, \gamma_5\} = 0, \quad \gamma_5^2 = 1. \tag{3.21}$$

Starting with the numerator,

$$\not{k}\gamma^5 \left(\not{p} - \not{k} + \mathring{m}_N\right) \not{k}\gamma^5, \tag{3.22}$$

we can put it in a form that will cancel the denominator in a nice way. For the first term

$$\begin{aligned} k_\mu \gamma^\mu \gamma^5 p_\nu \gamma^\nu k_\rho \gamma^\rho \gamma^5 = k_\mu p_\nu k_\rho \gamma^\mu \gamma^\nu \gamma^\rho = \\ k_\mu p_\nu k_\rho \left(-\gamma^\mu \gamma^\rho \gamma^\nu + 2\gamma^\mu g^\rho \nu\right) = -k^2 \not{p} + 2\not{k} p_\nu k^\nu \end{aligned} \tag{3.23}$$

The second term,

$$\not{k}\gamma^5 \not{k}\not{k}\gamma^5 = \not{k}k^2. \tag{3.24}$$

And finally the third,

$$\not{k}\gamma^5 \mathring{m}_N \not{k}\gamma^5 = -k^2 \mathring{m}_N. \tag{3.25}$$

We assemble the whole numerator and put it in a more useful form.

$$-\left(\not{p} + \mathring{m}_N\right) k^2 + (p^2 - \mathring{m}_N^2)\not{k} - \left((p-k)^2 - \mathring{m}_N^2\right) \not{k} \tag{3.26}$$

The first and the last terms cancel parts of the denominator and the middle term is zero since we demand that $p^2 = \mathring{m}_N^2$. We can now assemble the expression for the self-energy,

$$i\Sigma = \frac{3g_A^2}{4f_\pi^2} \int \frac{d^4k}{(2\pi)^4} \left(\frac{-\not{p} - \mathring{m}_N}{(p-k)^2 - \mathring{m}_N^2 + i\varepsilon} - \frac{\not{k}}{k^2 + i\varepsilon} \right). \tag{3.27}$$

The last term is odd in $\not{k}$ and thus integrates to zero. We are now left with a single term with a quadratic polynomial in $\not{k}$ in the denominator. If we would just Wick-rotate and change to spherical coordinates we get a quadratic divergence of the integral. To control this divergence we use dimensional regularization. We Wick-rotate and change to Euclidean spherical coordinates. Then we take the number of dimensions to be integrated over as a complex parameter. For non-integer dimension the integrals will then converge, and we can take the limit when going to four dimensions in a controlled way.

Looking at just the integral we then have

$$\begin{aligned}
&\int \frac{d^n k}{(2\pi)^n} \frac{1}{(p-k)^2 - \mathring{m}_N^2 + i\varepsilon} \\
&= \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - \mathring{m}_N^2 + i\varepsilon} \\
&= -i \int \frac{d^n k_E}{(2\pi)^n} \frac{1}{k_E^2 + \mathring{m}_N^2 + i\varepsilon} \\
&= -i \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(1-n/2)}{\Gamma(1)} \left(\frac{1}{\mathring{m}_N^2}\right)^{1-n/2} .
\end{aligned} \tag{3.28}$$

The last integral comes from ref. [PS95, p.251]. Now by letting $n = 4 - \varepsilon$ we can find the behaviour near the $n = 4$ pole,

$$-i\left(\frac{1}{16\pi^2} + \frac{\varepsilon \ln 4\pi}{32\pi^2} + \mathcal{O}(\varepsilon^2)\right)\left(\frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)\right)\left(\mathring{m}_N^2 + \frac{\varepsilon \mathring{m}_N^2 \ln \mathring{m}_N^{-2}}{2} + \mathcal{O}(\varepsilon^2)\right). \tag{3.29}$$

We keep all terms up to order ε^0:

$$-i\frac{\mathring{m}_N^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \ln\frac{4\pi}{\mathring{m}_N^2}\right) + \mathcal{O}(\varepsilon). \tag{3.30}$$

The whole expression for the self-energy then becomes

$$\Sigma = \frac{3g_A^2}{4f_\pi^2}\left(\not{p} + \mathring{m}_N\right)\frac{\mathring{m}_N^2}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \ln\frac{4\pi}{\mathring{m}_N^2} + \mathcal{O}(\varepsilon)\right). \tag{3.31}$$

The self-energy clearly diverges when we let the number of dimensions approach 4 and ε approach 0. To absorb this infinity the nucleon mass must be renormalized [Sch03]. Also the coupling constant g_A is renormalized by pion loop diagrams [Sch03]. This means that loop diagrams contribute at order p which precludes the possibility to create a simple ordering scheme like in the pion-only sector. In that sector the loop would add two powers of momenta to the diagram which in this case would produce a diagram of order p^3.

The reason for this break-down is the nucleon mass which does not vanish in the limit of zero quark masses [BKM95]. Derivatives of the nucleon field will yield a nucleon four momentum that is of the same order as the nucleon mass, which in turn is of the same order as the cutoff energy of our theory. In essence, the nucleon momentum p is not a small momentum.

This does not mean that it is impossible to use baryon ChPT for calculations. As per usual it is necessary to include all terms up to the given order. For a one-loop calculation in baryon ChPT, there will be the three Lagrangian terms of order p, loop contributions of order p^2 and p^3, and finally counter terms of order p^2 and p^3 [BKM95].

In the next chapter we will see how the power counting can be restored and this picture simplified by considering the nucleons to be very heavy.

Chapter 4

Heavy baryon chiral perturbation theory

To make calculations in a practical manner we must have a way of ordering the contributions by powers of momenta as in pion-only chiral perturbation theory. As we saw in the previous chapter this is not possible with the direct inclusion of the nucleons.

This problem is solved by considering the nucleons to be very heavy, an approach pioneered by Jenkins and Manohar [JM91]. This removes the dependence on the nucleon mass from the leading order Lagrangian at the cost of removing manifest Lorentz covariance. Making this approximation the result is heavy baryon chiral perturbation theory. In this chapter I will describe how to construct this theory from the covariant formalism of the previous chapter.

Ultimiately I wish to describe properties of nuclei. These are bound states of nucleons and as such there are some complications that need to be considered. The presence of shallow bound states in itself indicates a breakdown of perturbation theory [Wei90]. I will go into some detail on how this can be controlled by only considering irreducible graphs in time-ordered perturbation theory.

I will also look at the ordering that arises in the heavy baryon formalism and present the leading order and next to leading order Lagrangian.

Together this will set the scene for computing the two-body axial vector current which is the topic of the next chapter.

4.1 Heavy baryons

The building blocks of the Lagrangian in the heavy baryon formalism are similar to those of the covariant baryon ChPT Lagrangian. The pion fields are treated in the same way as in the pion-only sector. The difference lies in the treatment of the nucleon fields.

By factoring out an on-mass-shell field already at the Lagrangian level the dependence on the nucleon mass can be eliminated in the leading order Lagrangian. By doing this the correspondence between the momentum expansion and the loop expansion is made as simple as in the pion-only case.

Since the nucleon is very heavy compared to the typical energy scale of m_π, it will leave an interaction with almost the same momentum that it entered with. Also, all external particles need to be on shell so we can write the nucleon momentum as

$$p^\mu = m_N v^\mu + k^\mu \tag{4.1}$$

where $v^\mu v_\mu = 1$. The two four-vectors are v^μ, the four-velocity of the nucleon, and k^μ, the small residual momentum of the nucleon, which is of the order of m_π.

The space dependent part of the positive energy solution to the Dirac equation with momentum $p = m_N v$ is just a plane wave $\exp(i m_N v^\mu x_\mu)$. By introducing the velocity projection operators $P_v^\pm$,

$$P_v^\pm \equiv \frac{1 \pm v^\mu \gamma_\mu}{2}, \tag{4.2}$$

we can decompose each nucleon field into two parts,

$$N = e^{i m_N v x} P_v^+ \Psi, \quad H = e^{i m_N v x} P_v^- \Psi. \tag{4.3}$$

For the special case of $v^\mu = (1, \mathbf{0})$ these parts correspond exactly to the light and heavy components of the spinor field [BKM95]. We have also factored out the dominant part of the solution, the plane wave with four-momentum $m_N v$.

Derivatives on the new field N yields $-ik^\mu$ [JM91] instead of the full nucleon momentum $-ip^\mu$ from the full nucleon field Ψ,

$$\begin{aligned} \partial_\mu N|p\rangle &= \partial_\mu \left(e^{i m_N v x} P_v^+ \Psi \right) |p\rangle = (i m_N v_\mu - i p_\mu) e^{i m_N v x} P_v^+ \Psi |p\rangle \\ &= -i k_\mu P_v^+ e^{i m_N v x} \Psi |p\rangle = -i k_\mu N |p\rangle. \end{aligned} \tag{4.4}$$

The heavy component field H can be completely eliminated from the Lagrangian at the cost of introducing an infinite series of corrections of increasing order in k/m_N. This process also completely removes the nucleon mass m_N from the leading order Lagrangian. For details see for example the lecture notes by Scherer [Sch03] or the review by Bernard et al. [BKM95]. For the fundamentals on heavy baryon fields see the short review by Georgi [Geo90].

4.2 Ordering

The motivation for the heavy baryon approach was to restore the power counting of the pion sector. This has been achieved but only for single nucleons. As we will see in the next section bound states introduce complications that also need to be accounted for.

The order of Lagrangian terms is characterised by three quantities. The small four-momentum p, which corresponds to pions, external fields or the residual four-momentum of the nucleons; the cutoff scale M_ρ; and the nucleon mass m_N. The latter two are of the same order of approximately $\Lambda = 1\,\text{GeV}$. There are two simultaneous expansions, one is the ordinary chiral expansion in small momenta over the cutoff p/M_ρ, and the other is a relativistic expansion in the small momenta divided by the nucleon mass p/m_N. Combined we get an expansion in p/Λ both for the terms of the Lagrangian and for the diagrams.

The point of using the heavy baryon approximation for the nucleon field is that it leads to a sensible ordering. Loops can be absorbed order for order as in the pion-only sector; for example, diagrams with vertices from the first order Lagrangian and one loop will only require the renormalization of terms in the third order Lagrangian.

The order of a vertex in equation (2.33) must be changed to account for the nucleon line [Wei92] resulting in

$$\nu_i = d_i + e_i + \frac{n_i}{2} - 2, \tag{4.5}$$

where d_i is the number of derivatives and pion masses, e_i the number of external fields and n_i the number of nucleon fields. We will continue to assign an index to the Lagrangian terms by the order of the resulting vertices.

Each diagram is then characterised by its order ν which is defined as the power of the factor $(p/\Lambda)^\nu$ in that diagram. This order is given by [Wei92]

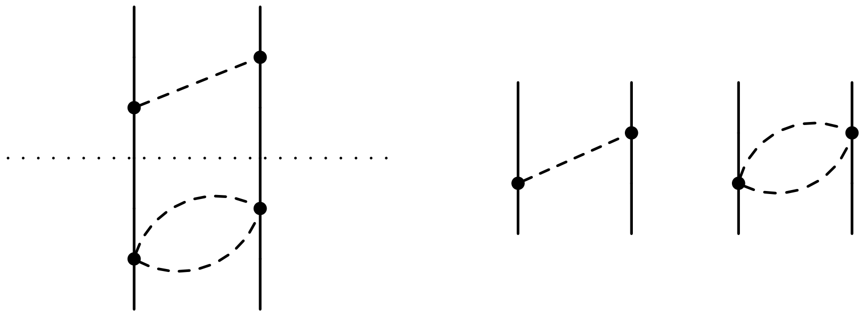
$$\nu = 4 - A - 2C + 2L + \sum_i V_i \nu_i \tag{4.6}$$

Here V_i is the number of vertices of type i, L the number of loops, A the number of nucleons. C is the number of separately connected parts in a diagram. In a process with A nucleons, a diagram can have up to A separately connected parts.

4.3 Considering several nucleons and nuclei

There are very loosely bound states of nucleons. An example is the deuteron that has a binding energy of only 2.2 MeV, which is very low compared to the typical momentum scale of m_π. These states will naturally play a very large role when considering more than one nucleon. Indeed, their existence implies a breakdown of perturbation theory in the form of infrared divergences [Wei90]. This means that the naive ordering of section 4.2 can not be correct for all Feynman diagrams.

It is possible to separate out the part that leads to the divergence. By abandoning the manifestly covariant formalism of Feynman diagrams, and instead considering time-ordered graphs, we can find the source of the divergence. In this framework it is meaningful to talk about intermediate states. The divergence of the perturbation series comes from intermediate states consisting only of nucleons [Wei90]. The solution is then simply to cut all graphs apart at each time where there is a nucleon-only intermediate state, an example can be seen in figure 4.1. We are then left with irreducible graphs, that is, graphs that have no purely nucleonic intermediate state. These graphs will be of the order indicated by equation (4.6).



(a) Reducible graph (b) Irreducible graphs

Figure 4.1: A reducible and corresponding irreducible graphs

The sum of irreducible graphs can then be used as an effective potential from which the S-matrix can be computed by the use of the Lippmann-Schwinger equation [Wei90].

A general graph consists of a number of irreducible graphs glued together by purely nucleonic intermediate states. Although we require that the full graph is connected, each component irreducible graph does not need to be fully connected [Wei92]. The result is that for a process involving A nucleons we must in principle consider one-, two- ... A-body graphs. From equation (4.6) we can see that a diagram is enhanced by $-2C$ powers of momenta where C is the number of separately connected parts. The leading contribution comes from graphs where all nucleons are disconnected from each other. In this case $C = A$. The next contribution in terms of connected graphs is when two nucleons are connected and the rest are spectators, which gives $C = A - 1$ separately connected parts. We call the leading contribution a one-body graph and the next order graph a two-body graph. For a higher N-body graph the number of disconnected parts is decreased as more nucleons are connected by pion lines. The number of disconnected parts in the general case is $C = A - N + 1$.

When considering interactions of nuclei we must take care to avoid double counting. In many cases we will have a nuclear wavefunction that is the result of solving the Schrödinger equation with an effective potential. This effective potential should then not be included in the interaction graphs since it has already been accounted for.

This is very similar to the ordinary amputation of the self energy from external lines in Feynman diagrams. This is also not part of the interaction but rather it gives the asymptotic particles their physical properties. For a bound state the binding energy is the self-energy of the bound state minus self-energy of the constituent particles if they were free. The requirement that external particles need to be on the mass shell is valid for the complete composite particle. For the constituent particles this condition is modified so that they should have the correct bound state wavefunction.

4.4 Lagrangian

At zeroth order we have the Lagrangian [BKM95],

$$\mathscr{L}_0^{\mathrm{HB}} = \bar{N}\left(i v^\mu D_\mu + 2 i g_A S^\mu \Delta_\mu\right) N + (\text{contact terms}), \tag{4.7}$$

where N is the two component heavy baryon field and S^μ is the spin matrix,

$$S^\mu = \frac{i}{2}\gamma_5 \sigma^{\mu\nu} v_\nu. \tag{4.8}$$

The other building blocks are carried over from the previous chapter. Note that the nucleon mass is not explicitly included in the leading order Lagrangian. The contact terms are not relevant for the calculations in this thesis and will be ignored.

We take the first order Lagrangian from ref. [BKM97] for the two-nucleon-field part and from ref. [PMS+03] for the four-nucleon-field part. The singlet gauge field has been excluded so we have the Lagrangian

$$\begin{aligned}\mathscr{L}_1^{\mathrm{HB}} = \bar{N}\Big\{ & \frac{v^\mu v^\nu - g^{\mu\nu}}{2m_N} D_\mu D_\nu + \frac{g_A}{m_N}\left\{S^\mu D_\mu, v^\nu \Delta_\nu\right\} + c_1 \operatorname{Tr}\left(\chi_+\right) \\ & + 4\left(c_2 - \frac{g_A^2}{8m_N}\right)\left(v^\mu i\Delta_\mu\right)^2 + 4c_3 i\Delta_\mu i\Delta^\mu + \left(4c_4 + \frac{1}{m_N}\right)\left[S^\mu, S^\nu\right] i\Delta_\mu i\Delta_\nu \\ & + c_5\left(\chi_+ - \frac{1}{2}\operatorname{Tr}\left(\chi_+\right)\right) - \frac{i\left(1+\kappa_v\right)}{4m_N}\left[S^\mu, S^\nu\right] f^+_{\mu\nu}\Big\} N \\ & - 4 i d_1 \bar{N} S^\mu \Delta_\mu N \bar{N} N + 2 i d_2 \varepsilon^{abc} \varepsilon_{\mu\nu\lambda\delta} v^\mu \Delta^{\nu a} \bar{N} S^\lambda t^b N \bar{N} S^\delta t^c N.\end{aligned} \tag{4.9}$$

Here we have, apart from the usual p/M_ρ terms, also terms that are part of the relativistic expansion, which are suppressed by powers of p/m_N. These terms are the result of the heavy baryon approximation and the elimination of the heavy component fields from the Lagrangian.

The singlet gauge field is needed to write the full electromagnetic current. Including it would give one more term and modify the covariant derivative. We are primarily interested in the charged weak currents so we avoid this unnecessary complication.

In this chapter we have seen how to overcome two problems involving divergences. First by identifying the nucleon as heavy we could remove the mass from the leading order Lagrangian; the result being a sensible power counting in the one nucleon sector. The extension to more than one nucleon introduced the problem of bound states. This was overcome by only considering the effective potential of irreducible time-ordered graphs. For the case of interactions we saw that only irreducible graphs will contribute to the amplitude.

In the next chapter we will see how to apply this Lagrangian to model weak interactions in nuclei. Then we will apply the tools developed in this chapter.

Chapter 5

Weak interactions in nuclei

In this chapter I will examine the weak interactions of systems of nucleons; of which the most familiar, and also the most important special case, is atomic nuclei. Just as for pions, the weak interactions of nucleons are determined by the chiral symmetry. This means that there is no need to introduce additional LECs to parametrize the coupling strength. The existing LECs together with the Standard Model weak interaction coupling constants are sufficient.

I will also compute the two-body axial vector current operator, which is the leading order two-body contribution to the weak currents. This current operator is important both for weak interaction phenomenology but also for determining LECs in HBChPT in order to create realistic three-nucleon potentials.

All computations will be performed in the HBChPT formalism, which was introduced in the previous chapter.

5.1 Weak interactions of nucleons

Perhaps the most important consequence of nuclear weak interactions is the beta decay of nuclei. This is a collective name for charge changing decays that do not change the mass number of the nuclei. There are three basic processes to be considered:

Beta decay

$$N(Z,A) \to N(Z+1,A) + e^- + \bar{\nu}_e$$

Electron capture

$$N(Z,A) + e^- \to N(Z-1,A) + \nu_e$$

Beta-plus decay

$$N(Z,A) \to N(Z-1,A) + e^+ + \nu_e$$

The neutrinos in these processes are there to conserve the electronic lepton number.

Another process is the scattering of nuclei and neutrinos. The neutrino only interacts weakly so to understand it we must understand nuclear weak interactions. Many neutrino experiments use the interactions between nuclei and neutrinos to detect neutrinos.

Just like in the case of pion-only weak interactions, which was covered in section 2.7, the nucleon weak interactions are completely determined by the chiral symmetry. This means that the HBChPT Lagrangian already contains the appropriate ingredients to facilitate the description of weak interactions. Once again it is the left-handed gauge field $L_\mu = V_\mu - A_\mu$ that can be identified with the current of charged vector bosons $W^\pm$.

5.2 Current operators

By four-current we mean a four-vector field satisfying some conservation principle. This means that we can associate a conserved charge to the field. A general four-current can be considered to consist of two components,

$$J^\mu(x) = (J^0(x), \boldsymbol{J}(x)), \tag{5.1}$$

where $J^0(x)$ is the charge density and $\boldsymbol{J}(x)$ is the three-vector current. The role of currents is to describe the source of force fields.

The most familiar and archetypal current is the electric four-current J^μ_{em}. The time component is the electric charge density, ρ, and the space components are the regular electric current j^i. This four-current is included in the electrodynamic Lagrangian through the interaction term: $J^\mu_{\text{em}} A_\mu$. In quantum electrodynamics the four-current is formed from the electron field, $J^\mu_{\text{em}} = ie\bar{\psi}\gamma^\mu\psi$; e is the electric charge and ψ the electron-positron spinor field.

The nuclear four-currents we will be looking at are very similar. By a weak four-current we mean a four-vector field that interacts with one of the weak fields. Just as the electromagnetic field is mediated by the photon, the weak field is mediated by the $W^\pm$- and Z bosons. Because of the very high masses of these bosons the field falls off in a very short distance from a weakly interacting particle.

The word *operator* in current operator refers to an operator in the quantum mechanical sense. This means that we must express the current as a linear operator on the Hilbert space of states. This operator must, in general, satisfy the same symmetries as the classical current, although in some cases quantization may destroy classical symmetries.

An important point to remember is that there is often no manifest classical current and corresponding force field to the quantum mechanical current operators. In the case of electromagnetism the force carrier is massless and thus the field has infinite range. This makes it easy to observe the field in the macroscopic world. In many other cases the force particles are massive and as a consequence the range of the field is often microscopic. This means that the classical description of the current and force field has very few applications. This also becomes apparent in how we derive the currents, not starting from classical physics, but instead from quantum mechanics.

We will look at the current operators that represent the weak currents of hadrons. To see how such an operator may come about we can examine a weak interaction of nucleons and leptons. Using the heavy vector meson approximation from chapter 2, the weak force can be described as a current-current interaction and the leading order matrix element can be written

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\langle N', l|J_H^{\mu}J_{l\mu}|N\rangle. \tag{5.2}$$

Here N describes the nucleon state and l the lepton state. If the interaction between the leptons and the nucleons can be considered to be very weak then the current J_H does not affect the leptons and vice versa. In this case the matrix element can be decomposed into a product of a nucleon matrix element and a lepton matrix element

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\langle N'|J_H^{\mu}|N\rangle\langle l|J_{l\mu}|0\rangle. \tag{5.3}$$

The lepton matrix element is easy to evaluate in the Standard Model. The nucleon matrix element is significantly harder. The reason is that the nucleon states hidden under the labels N, N' are much more complicated.

Let the nucleon states N and N' consist of A nucleons. The current operator J_H must operate on the whole space of A nucleons. But, recalling that the leading order contribution comes from minimally connected graphs, we can assume that the leading order contribution will come from the operators that affect as few nucleons as possible. Forgetting about the Lorentz index for a while we write the current operator as a sum

$$J_H = J_H^{1\mathrm{B}} + J_H^{2\mathrm{B}} + \cdots + J_H^{A\mathrm{B}}. \tag{5.4}$$

Each operator on the right hand side is a sum of all operators affecting one nucleon, a pair of nucleons, and so on.

The first two operators are written

$$J_H^{1\mathrm{B}} = \sum_i^A J_i, \quad J_H^{2\mathrm{B}} = \sum_{i>j}^{A,A} J_{ij}. \tag{5.5}$$

Here the first sum is over all nucleons and the second sum is over all pairs of nucleons. The operators J_i are one-body operators and J_{ij} are two-body operators.

The nucleon matrix element has been decomposed into different parts depending on the number of nucleons participating in each reaction. To simplify the calculations we insert a complete set of A-nucleon momentum states in the expression for the nucleon matrix element.

$$\begin{aligned}\langle N'|J_H^\mu|N\rangle = \int dp_1\cdots dp_A dp_1'\cdots dp_A' \times \\ \langle N'|p_1\cdots p_A\rangle\langle p_1\cdots p_A|J_H^\mu|p_1'\cdots p_A'\rangle\langle p_1'\cdots p_A'|N\rangle\end{aligned} \tag{5.6}$$

Given that we can write down the nucleon wavefunctions $\langle p_1\cdots p_A|N\rangle$ all that remains is to calculate the matrix element $\langle p_1\cdots p_A|J_H^\mu|p_1'\cdots p_A'\rangle$. These wavefunctions must of course be totally antisymmetric under interchange of the identical nucleons.

Remember that J_H is composed of a sum of N-body operators. The one-body operators will only affect one nucleon, it will thus be a function of only one nucleon momentum. For the rest of the momenta it will only give a momentum conserving delta function.

Let us take a closer look at the two-body matrix element,

$$\langle p_1\cdots\ p_A|J_H^{2\mathrm{B}}|p_1'\cdots p_A'\rangle = \sum_{i<j}^{A-1,A}\langle p_1\cdots\ p_A|J_{ij}|p_1'\cdots p_A'\rangle. \tag{5.7}$$

If we look at just one term of the sum on the right hand side we have

$$\langle p_1\cdots\ p_A|J_{12}|p_1'\cdots p_A'\rangle = \langle p_1p_2|J_{12}|p_1'p_2'\rangle\langle p_3\cdots p_A|p_3'\cdots p_A'\rangle. \tag{5.8}$$

All terms will be identical to this, only with different nucleon labels. The second factor will only contribute a product of momentum-conserving delta-functions. If we can compute the matrix element $\langle p_1p_2|J_{12}|p_1'p_2'\rangle$ as a function of the momenta p_1, p_2, p_1', p_2' it will only be a matter of summing the contributions from the different combinations of nucleons in order to get the matrix element from the full current operator J_H^{2B}.

Now that we have decomposed the general A-nucleon matrix element into one-body-, two-body- and many-body matrix elements we will have a closer look at the two-body matrix elements. First, to better understand the full picture, we will take a quick look at the leading order one-body operators and sketch how they can be realized in HBChPT.

If we go back to the specific goal of calculating weak currents we remember that it is only the left handed current that interacts weakly. In chapter 2 we saw that a left handed current can be written as $L^\mu = V^\mu - A^\mu$. When computing nuclear matrix elements it is more convenient to work in the latter basis.

In phenomenological models of the weak interactions the leading contributions come from the one-body Fermi and Gamov-Teller operators [Suh07]. The Fermi operator is also called the charge operator and it is the time component of the charge changing, four-vector nuclear current $V^{\mu\pm}$. The Gamov-Teller operator is the charge changing, axial vector current $\boldsymbol{A}^\pm$ of $A^{\mu\pm}$ and changes the spin state of the nucleus. These are nuclear operators and should not be confused with external vector- and axial vector fields.

These operators also give the leading order contribution in HBChPT. Let us take a closer look at the order of different diagrams to understand how this is. For a given process we will always have A nucleons. We take $A = 2$ since it is the minimal number to have two-body currents. We could just as well have taken any higher number, it does not have any impact on the following derivation. By inserting this into equation (4.6) we see that the order of the operators is given by

$$\nu = 2 - 2C + 2L + \sum_i V_i \nu_i. \tag{5.9}$$

It is evident that the only way to lower the order of a diagram is to increase the number of separately connected parts, C. For two nucleons the maximum number of separately connected parts is $C = 2$. Then, with no loops and only vertices with $\nu_i = 0$, we have the order

$$\nu = -2, \tag{5.10}$$

which we will also denote leading order (LO).

The one-body Fermi operator is derived from the first LO HBChPT Lagrangian term in equation (4.7),

$$\mathscr{L}_0^{\mathrm{HB}\prime} = \bar{N} i v^\mu D_\mu N,$$

while the Gamov-Teller operator comes from the second term,

$$\mathscr{L}_0^{\mathrm{HB}\prime\prime} = \bar{N} 2 i g_A S^\mu \Delta_\mu N.$$

Since loops only increase the order of a diagram the one-body Fermi and Gamov-Teller operators have the minimal order $\nu = -2$. There is a relativistic correction to the Gamov-Teller operator at order $\nu = 0$, or N^2LO [PMS+03].

5.3 Two-body axial vector current

The next correction to the Gamow-Teller operator comes from the two-body axial vector current $\hat{\boldsymbol{A}}_{12}^a$. The index a is a chiral vector, or isospin, index. The subscript numbers are the nucleon labels. The isospin index needs to be contracted with an isospin index on the external field. The charged vector bosons consist of the combinations $W^+ = W^1 + iW^2$ and $W^- = W^1 - iW^2$. The corresponding combinations of the axial vector current, $\hat{\boldsymbol{A}}_{12}^+ = \hat{\boldsymbol{A}}_{12}^1 + i\hat{\boldsymbol{A}}_{12}^2$ and $\hat{\boldsymbol{A}}_{12}^- = \hat{\boldsymbol{A}}_{12}^1 - i\hat{\boldsymbol{A}}_{12}^2$, give the two charge changing Gamow-Teller operators.

A two-body operator connects the two nucleons together, thus lowering the number of separately connected diagrams to $C = 1$. Then, with no loops, and only vertices with $\nu_i = 0$, we have the order $\nu = 0$ or N^2LO. Note that changing the number of nucleons A will change the number of separately connected diagrams; in the general case $C = A - 1$ for a two-body operator. This also changes the absolute order of the operator ν but the order relative to LO is unchanged. Thus, a two-body operator will always have its first possible contribution at $\mathrm{N^2LO}$ regardless of the total number of nucleons considered.

We will only consider the case where the momentum carried by the external field goes to zero. As we will see, then the first non-vanishing contribution to the two-body, axial vector current requires a vertex from the next to leading order Lagrangian $\mathscr{L}_1^{\mathrm{HB}}$. As a result the order is increased by one, giving $\nu = 1$ or $\mathrm{N^3LO}$.

To get an expression for the two-body axial vector current we will evaluate the matrix element $\langle p_1 p_2 | \hat{\boldsymbol{A}}_{12}^a | p_1' p_2' \rangle$ in the formalism of HBChPT. In practice this is done by considering all irreducible graphs that couple to $\boldsymbol{A}$, the external axial vector field. This field is already included in the HBChPT Lagrangian as the space components of the axial gauge field A_μ. The calculation will be performed as the momentum q^μ that is carried by the external field goes to zero. The asymptotic states will be nuclei so there will be no need to consider reducible graphs since all purely hadronic subgraphs can be absorbed into the binding of the nuclei. To find all irreducible graphs we need to know which vertices exist. In the following we will see how to derive the vertices and the interaction Lagrangian.

5.3.1 Interaction Lagrangian

There is a large number of possible vertices that comes from the LO and NLO Lagrangians. But when the momentum of the external field goes to zero it turns out that all non-vanishing graphs can be built out of only three vertices.

Each vertex will be denoted by the external legs it has. For example, a vertex with two nucleon legs, one pion leg and one coupling to the external axial current will be called a $NN\pi\boldsymbol{A}$-vertex. With this notation in place we can list the candidate vertices:

$$\pi\boldsymbol{A}, \quad \pi\pi\boldsymbol{A}, \quad NN\pi, \quad NN\pi\pi, \quad NN\pi\boldsymbol{A}, \quad 4N\boldsymbol{A} \text{ and } 4N\pi.$$

These are the vertices that can be included in fully connected diagrams with four nucleon lines, one external field coupling and no loops. We will go through these one by one; deriving the interaction Lagrangian, structure and strength of the vertices.

In chapter 2 we saw how to get a πL_μ-vertex from $\mathscr{L}_0^\pi$ when computing the pion decay rate. The derivation of the $\pi\boldsymbol{A}$ vertex is very similar, and it is not possible to remove the momentum dependence of the vertex. This means that any diagram containing this vertex will be proportional to the momentum transfer $\boldsymbol{q}$ and will vanish when $\boldsymbol{q} \to 0$.

Because the $\pi\boldsymbol{A}$-vertex vanishes when $\boldsymbol{q} \to 0$ we can ignore the $NN\pi\pi$ and $4N\pi$-vertices because they would only contribute in conjunction with the $\pi\boldsymbol{A}$-vertex.

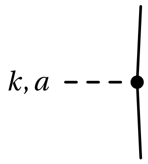
Turning to the next pion vertex $\pi\pi\boldsymbol{A}$ we look in the LO pion Lagrangian $\mathscr{L}_0^\pi$ of equation (2.38). The possible candidate term is

$$\mathscr{L}_0^{\pi\prime} = \frac{f_\pi^2}{4} \operatorname{Tr}\left(D_\mu U (D^\mu U)^\dagger \right). \tag{5.11}$$

The other term in the LO pion Lagrangian does not contain any coupling to the external axial vector field. This vertex vanishes, for more details see appendix B.1.

We have seen that there are no vertices coupling pions to the external field that gives a contribution at vanishing momentum transfer. Now we continue with the vertices coupling the external field to the nucleon fields instead. Too see these derivations in more detail see appendix B.1.

First is the $NN\pi$-vertex for which the derivation can be seen in appendix B.1.2. It comes from the second term of the LO Lagrangian $\mathscr{L}_0^{\mathrm{HB}\prime}$ and the resulting vertex is



$$= -\frac{g_A}{2 f_\pi} \boldsymbol{\sigma} \cdot \boldsymbol{k} \tau^a. \tag{5.12}$$

In all vertices and diagrams the baryon number and nucleon momenta flow from the bottom of the diagram(the in-state) to the top of the diagram(the out-state). In HBChPT we do not allow for the creation or annihilation of nucleons so this defines all fermion line directions completely.

There is no $NN\pi\boldsymbol{A}$-vertex in the LO Lagrangian (4.7). In HBChPT this vertex is kinematically suppressed, i. e., the Lorentz vector A_μ is contracted with the field velocity v^μ. Consequently there will be no contribution including $\boldsymbol{A}$; there will only be a vertex proportional to A^0.

We look instead in the two nucleon part of the NLO Lagrangian (4.9) for this vertex. Out of the eight terms in the Lagrangian only three contribute to this vertex. The two terms

$$c_1 \operatorname{Tr}(\chi_+), \quad c_5 \left(\chi_+ - \frac{1}{2} \operatorname{Tr}(\chi_+)\right), \tag{5.13}$$

can be ignored since they do not include the external field $\boldsymbol{A}$. The two terms proportional to $v^\mu \Delta_\mu$,

$$\frac{g_A}{m_N} \left\{S^\mu D_\mu, v^\nu \Delta_\nu\right\}, \quad 4\left(c_2 - \frac{g_A^2}{8 m_N}\right)\left(v^\mu i \Delta_\mu\right)^2, \tag{5.14}$$

can be ignored since, with $v^\mu = (1,0)$, they only contain A^0, not $\boldsymbol{A}$. Finally, the term

$$\frac{i\,(1+\kappa_v)}{4 m_N} \left[S^\mu, S^\nu\right] f^+_{\mu\nu} \tag{5.15}$$

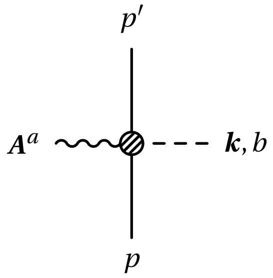
does not contribute when $q \to 0$ since the resulting vertex is proportional to q. What is left are the three terms that do contribute:

$$\begin{aligned} \mathscr{L}_1^{\mathrm{HB}\prime} = \bar{N}\Bigg\{ & \underbrace{\frac{v^\mu v^\nu - g^{\mu\nu}}{2 m_N} D_\mu D_\nu}_{\mathrm{A}} + \underbrace{4 c_3 i \Delta_\mu i \Delta^\mu}_{\mathrm{B}} \\ & + \underbrace{\left(4 c_4 + \frac{1}{m_N}\right)\left[S^\mu, S^\nu\right] i \Delta_\mu i \Delta_\nu}_{\mathrm{C}} \Bigg\} N \end{aligned} \tag{5.16}$$

When expanding this Lagrangian to find the explicit interaction Lagrangian we want to find any terms with one pion field, one external field and two nucleon fields. The details of the derivation of the interaction Lagrangian and the resulting vertex can be found in appendix B.1.3. The resulting interaction Lagrangian is

$$\begin{aligned}\mathscr{L}_1^{NN\pi\boldsymbol{A}} = \frac{i}{4f_\pi m_N}\left\{\boldsymbol{A}^a\cdot\nabla\phi^b\varepsilon^{abc}\bar{N}\tau^c N + 2\boldsymbol{A}_i^a\phi^b\varepsilon^{abc}\bar{N}\tau^c\partial_i N\right\} + \frac{2c_3}{f_\pi}\nabla\phi^a\cdot\boldsymbol{A}^a\bar{N}N \\ - \frac{1}{f_\pi}\left(c_4+\frac{1}{4m_n}\right)A_i^a\partial_k\phi^b\varepsilon^{ijk}\varepsilon^{abc}\bar{N}\sigma^j\tau^c N.\end{aligned} \tag{5.17}$$

Quantizing this Lagrangian yields the following vertex



$$\begin{aligned} = \; & \frac{i}{2f_\pi m_N}\frac{\boldsymbol{p}+\boldsymbol{p}'}{2}\varepsilon^{abc}\tau^c \\ & -\frac{2c_3}{f_\pi}\boldsymbol{k}\delta^{ab} \\ & +\frac{1}{f_\pi}\left(c_4+\frac{1}{4m_N}\right)(\boldsymbol{\sigma}\times\boldsymbol{k})\,\varepsilon^{abc}\tau^c.\end{aligned} \tag{5.18}$$

The LECs of this vertex are included in the nucleon-nucleon potential and one-body operators and are well-known.

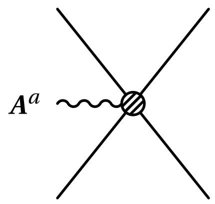
There is one type of vertex left that we have not discussed, the $4N\boldsymbol{A}$ vertex. It comes from the contact terms of the NLO Lagrangian (4.9),

$$\begin{aligned}\mathscr{L}_1^{\mathrm{HB}\prime\prime} = & -\underbrace{4id_1\bar{N}S^\mu\Delta_\mu N\bar{N}N}_{\mathrm{CT1}} \\ & +\underbrace{2id_2\varepsilon^{abc}\varepsilon_{\mu\nu\lambda\delta}v^\mu\Delta^{\nu a}\bar{N}S^\lambda t^b N\bar{N}S^\delta t^c N}_{\mathrm{CT2}}.\end{aligned} \tag{5.19}$$

From this Lagrangian we derive the interaction Lagrangian with exactly one external axial vector field and no pion fields. For details see appendix B.1.4.

$$\mathscr{L}_1^{4N\boldsymbol{A}} = d_1 A^{i,a}\bar{N}\sigma^i\tau^a N\bar{N}N + \frac{1}{2}d_2\varepsilon^{abc}\varepsilon^{ijk}A^{i,a}\bar{N}\sigma^j\tau^b N\bar{N}\sigma^k\tau^c N \tag{5.20}$$

This gives rise to the vertex



$$\begin{aligned} = \; & id_1\left(\boldsymbol{\sigma}_1\tau_1^a+\boldsymbol{\sigma}_2\tau_2^a\right) \\ & +id_2\left(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\right)\left(\tau_1\times\tau_2\right)^a.\end{aligned} \tag{5.21}$$

The coupling constant of this vertex is new for the two-body operators and also shows up in the three-nucleon potential.

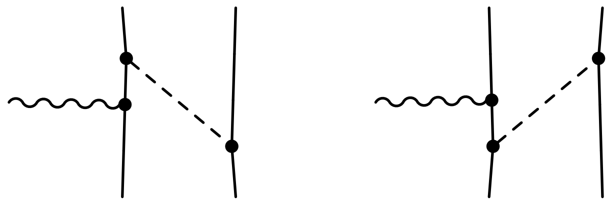


Figure 5.1: N^2LO terms that are canceled by recoil contributions

5.3.2 Irreducible graphs and Feynman diagrams

Now that we have all the contributing vertices we can compute the matrix elements of the current operator by writing down all diagrams and evaluating them. In doing this we will consider the two nucleons to be distinct; the crossed contributions must be handled in the anti-symmetrization of the initial- and final state wavefunction. The two nucleons will be labeled 1 and 2.

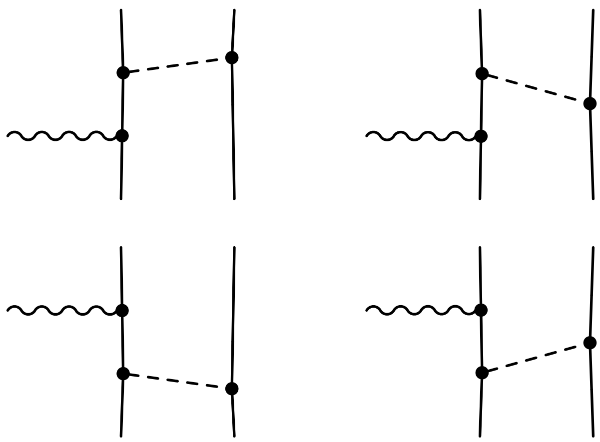


Figure 5.2: Recoil contributions

In figure 5.1 there are two graphs which show up at N^2LO. To evaluate these graphs we need to use time-ordered perturbation theory since they only form part of a Feynman diagram. However, these graphs are canceled by the recoil contributions in figure 5.2, which can be found in the set of reducible graphs at N^2LO. For details on how this cancelation takes place see Pastore et. al. [PSG08].

Instead, the first contribution comes at N^3LO. The reason that the order is increased by one is that there is no term generating a $NN\pi\boldsymbol{A}$ vertex in the LO Lagrangian; as we saw in the previous section this vertex is generated by three terms in the NLO Lagrangian. In total there will be two types of graphs that will give a contribution to the matrix element.

The first are the one pion exchange graphs, which can be seen in figure 5.3. Here the different time-orderings can be summed into one Feynman diagram.

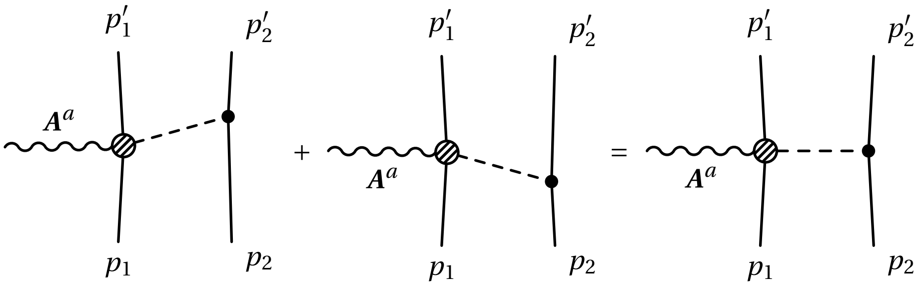


Figure 5.3: Meson exchange graphs and corresponding Feynman diagram

The blob represents the $NN\pi\boldsymbol{A}$ vertex in equation (5.18). Remember that in all diagrams fermion lines point from the bottom of the diagram to the top. We will also include the diagram where the external field is attached to the other nucleon. This amounts to exchanging the labels of the nucleons in the final expression.

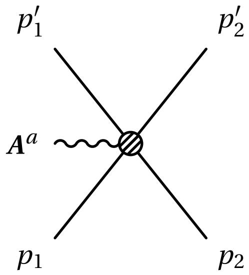


Figure 5.4: The 4N contact diagram contributing to the two-body axial vector current

The other diagram is the contact diagram in figure 5.4. This diagram represents the exchange of heavier, excluded particles between the nucleons. In this case only one time-ordering appears and the Feynman diagram is equal to the time-ordered perturbation theory graph. The $4N\boldsymbol{A}$-vertex is the vertex in equation (5.21). In this case the exchange of the nucleon labels is included already in the vertex and we need not take any further care to include all contributions.

We evaluate these diagrams. After having derived the vertices this is straightforward, we only need to take care to get the correct signs due to the directions of the momenta along the particle lines. We use general external nucleons and calculate the matrix element depending on the nucleon spin- and isospin states. The details of this calculation can be found in appendix C. The resulting two-body axial vector current of nucleons is,

$$\begin{aligned}\hat{\boldsymbol{A}}_{12}^{a} = &\frac{g_A}{2m_N f_\pi^2}\frac{\boldsymbol{\sigma}_2\cdot\boldsymbol{k}}{k^2-m_\pi^2}\left(\left(i\frac{\boldsymbol{p}_1+\boldsymbol{p}_1'}{2}(\tau_1\times\tau_2)^a\right)+\left(2\hat{c}_3\boldsymbol{k}\tau_2^a\right)\right.\\ &\left.+\left(\left(\hat{c}_4+\frac{1}{4}\right)(\tau_1\times\tau_2)^a\boldsymbol{\sigma}_1\times\boldsymbol{k}\right)\right)+(1\leftrightarrow 2)\\ &+\frac{g_A}{m_N f_\pi^2}\left(\hat{d}_1\left(\tau_1^a\boldsymbol{\sigma}_1+\tau_2^a\boldsymbol{\sigma}_2\right)+\hat{d}_2(\tau_1\times\tau_2)^a\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\right).\end{aligned} \tag{5.22}$$

This is an operator in the two nucleon momentum space. It is derived through quantum field theory methods but the matrix elements can be carried over into a quantum mechanical operator. Also, since we are working in the formalism of HBChPT the nucleons are already non-relativistic and there is no need to make any further non-relativistic reduction.

This exhausts the possibilities for the two-body axial vector current at zero momentum transfer. Together with one-body contributions up to N^3LO this gives the axial vector current and the Gamov-Teller operator up to N^3LO.

One way to interpret the two-body current is that the presence of other nucleons change the properties of a nucleon to some extent; when considering composite nuclei it is not enough to just sum the contributions from single nucleons. The two-body operators parametrize the first correction to the single nucleon vertex from the nuclear environment. Since, in EFT all allowed interactions exist, there will also be three-body, four-body, and contributions at higher orders.

The change of the nucleon interactions in the nuclear environment was historically attributed to the exchange of mesons between the nucleons. Pions, and also heavier mesons were considered to be exchanged. As a consequence the currents parametrizing these effects are called meson exchange currents(MEC). In ChPT we only consider the explicit exchange of pions; the exchange of heavier mesons is contained in the contact interactions.

Chapter 6

Conclusion

In the introduction I presented an idea for a self-consistent framework for nuclear physics. Based on an EFT of the strong force; it would form the foundation from which both nucleon potentials and nuclear currents could be derived. By using ab-initio methods nuclear wavefunctions can be calculated with only these nucleon potentials as input. To make these calculations the LECs, which are the parameters of the EFT, would have to be determined by some method. This can be done either by using lattice QCD or by fitting these parameters to experimental data.

As part of this framework, I have reviewed how to describe nucleons and pions in the formalism of EFT. In this process I have only assumed the chiral symmetry, the spontaneous breaking of chiral symmetry and the basic properties of pions and nucleons.

The resulting theory, ChPT, is not uniquely determined, instead it contains an infinite number of parameters. At increasing orders there is an ever expanding number of LECs. To be able to make predictions about a given process with a certain precision all interactions up to some order must be included. Consequently, all of the LECs that are relevant and occur in the Lagrangian up to that order must be known. Determining these parameters is one of the challenges of using ChPT.

It would be reasonable to think that by including the weak interactions the situation would only become more complicated. This is not the case. It is possible to realize the weak interaction of hadrons without introducing any LECs in addition to those that already exist in ChPT to begin with. This is achieved by observing that the left-handed chiral symmetry current of quarks in the Standard Model is the same current as the weak current of quarks. The interaction of this current is then transferred to the corresponding left-handed chiral symmetry current of pions and nucleons. The study of weak interactions in this context is interesting not only in its own right but is also useful for determining LECs. Adding weak interactions to ChPT reveals a new set of observables; these can be used to find the LECs by fitting the theoretical predictions to the experimental results. So, the inclusion of the weak interaction does in fact improve the situation with regard to determining the unknown parameters.

The most involved calculation I have performed is that of the two-body axial vector current. This current gives the leading two-body correction to the Gamow-Teller operator. The leading order contribution to the Gamow-Teller operator comes from one-body currents. However, in systems with multiple nucleons there is a correction from two-body currents. By comparing weak interaction processes in different nuclei the contribution from two-body currents can be determined. This can in turn be used to fix LECs. For example, neutron decay, which is a purely one-body process, can be compared with tritium beta decay, which is the lightest nucleus where two-body currents affect the process. By accounting also for the nuclear structure in the case of tritium it is possible to isolate the contribution coming from two-body currents.

Describing interactions with nuclear currents derived from ChPT is only one part of the self-consistent framework for nuclear physics. Another important part is deriving nucleon potentials which can be used as input to calculations of nuclear wavefunctions. By using HBChPT the nucleon potential can be derived from the same Lagrangian as the currents associated with interactions. The power counting scheme of HBChPT is valid also for potentials and give a natural ordering of the nuclear force. The leading contribution comes from the nucleon-nucleon force. At higher orders the three-nucleon force and forces involving larger numbers of nucleons appear. This is completely analogous to the ordering of nuclear current operators.

One of the advantages of using self-consistent interactions and potentials derived from the same EFT is that the LECs are common to both sectors. One example is that the two-body axial vector current plays an important part in determining LECs that parametrize the strength of the three-nucleon potential [GQN09]. Exactly the same combination of LECs that appears in the contact contribution to the current also shows up in one of the leading three-body graphs of the potential. Apart from beta decay calculations, determining the value of this combination of LECs is the main result stemming from the two-body axial vector current.

Currently it is not possible to describe nuclei with more than ~20 nucleons using ab-initio methods. This puts an upper limit on the applicability of the framework in terms of nuclear mass number. To describe heavier nuclei using ab-initio nuclear models either computational power needs to be increased or the methods of ab-initio calculations need to be improved. Some components and ideas in the framework can be used for heavier nuclei already today, but it would then be necessary to use phenomenological models for other parts. This would destroy the self-consistency of the approach.

The framework is only valid for nuclear physics and other low energy descriptions of the strong force. This is because ChPT can not describe the energy region above the rho mass. Perturbative QCD is unable to describe the energy region below the asymptotically free regime. In this intermediate region other methods and effective field theories must be used to give a complete description of the strong force.

There can be a question about the consistency of the approach since it has not been shown that ChPT is a low energy theory of QCD. This is related to the issue that the lowest mass excitations in QCD are not known, i. e., it has not been shown that the properties of the pions and nucleons are the result of solving QCD. One partial resolution to these problems lies in lattice QCD methods. This non-perturbative approach could serve as a bridge between low-energy QCD and ChPT not only by determining the LECs of ChPT, but also by providing a general test of the validity of ChPT [JLQCD07]. At the current date lattice QCD can not be used to replace experiments in determining the values of LECs, but progress is being made [EHM09].

In the last three decades great progress has been made toward a better description of the strong force. With ChPT or other EFTs many, if not all, low-energy strong interaction phenomena can potentially be described. This is done in a way that is compatible with the other two forces of the Standard Model and seemingly also with QCD.

The framework based on ChPT represents a step toward a simple, yet complete, description of the strong force at low energies. But there are remaining challenges. One lies in finding higher order potentials and interactions. To do this more LECs must be determined; here the weak interactions can play an important part. Another is extending the use of ab-initio methods to heavier nuclei. Lastly, the whole theory could be put on a firmer ground by showing that ChPT is a low-energy theory of QCD.

If all of these things can be achieved the result is a complete, self-consistent framework for nuclear physics that is fully compatible with the Standard Model. Considering the richness and complexities of hadron interaction phenomenology it can indeed count as a simple theory of the strong force at low energies.

Appendix A

Pion decay calculation

Here I will explain the details for evaluating the expression for the amplitude of the pion decay. For an incoming charged pion with momentum p that decays into a muon with momentum k' and a neutrino with momentum k we get the invariant amplitude

$$i\mathscr{M} = G_F V_{ud} f_\pi \bar{u}^s(k) \not{p} \left(1-\gamma^5\right) v^r(k'). \tag{A.1}$$

We square the amplitude and sum over the final state spins,

$$\begin{aligned}\sum_{r,s} |\mathscr{M}|^2 =& G_F^2 V_{ud}^2 f_\pi^2 \sum_{r,s} \bar{u}^s(k) p_\mu \gamma^\mu \left(1-\gamma^5\right) v^r(k') \\ & \times \bar{v}^r(k') p_\nu \gamma^\nu \left(1-\gamma^5\right) u^s(k).\end{aligned} \tag{A.2}$$

To be able to rearrange this product we write out the spinor indices,

$$\begin{aligned}\sum_{r,s} |\mathscr{M}|^2 =& G_F^2 V_{ud}^2 f_\pi^2 \sum_{r,s} \bar{u}_a^s(k) \gamma^\mu_{ab} \left(1-\gamma^5\right)_{bc} v_c^r(k') \\ & \times \bar{v}_d^r(k') \gamma^\nu_{de} \left(1-\gamma^5\right)_{ef} u_f^s(k).\end{aligned} \tag{A.3}$$

Next we use the completeness relations for the spinors [PS95, p.132];

$$\sum_s u^s(p) \bar{u}^s(p) = \not{p} + m \tag{A.4}$$

$$\sum_s v^s(p) \bar{v}^s(p) = \not{p} - m \tag{A.5}$$

and get everything in terms of Dirac matrices:

$$\begin{aligned}\sum_{r,s} |\mathscr{M}|^2 =& G_F^2 V_{ud}^2 f_\pi^2 p_\nu p_\mu \not{k}_{fa} \left(\not{k}' - m\right)_{cd} \gamma^\mu_{ab} \left(1-\gamma^5\right)_{bc} \gamma^\mu_{de} \left(1-\gamma^5\right)_{ef} \\ =& G_F^2 V_{ud}^2 f_\pi^2 p_\nu p_\mu k_\rho \operatorname{Tr}\left(\gamma^\rho \gamma^\mu \left(1-\gamma^5\right) \left(\not{k}' - m\right) \gamma^\nu \left(1-\gamma^5\right)\right)\end{aligned} \tag{A.6}$$

We use the linearity property of the trace to divide this expression into a sum of manageable terms,

$$\begin{aligned}
&\mathrm{Tr}\left(\gamma^\rho\gamma^\mu\left(1-\gamma^5\right)\left(\not{k}'-m\right)\gamma^\nu\left(1-\gamma^5\right)\right)=\\
&k'_\sigma\,\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu)-k'_\sigma\,\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu)\\
&-\mathrm{Tr}(\gamma^\rho\gamma^\mu m\gamma^\nu)-k'_\sigma\,\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^5)\\
&+\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^5 m\gamma^\nu)+\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^5 m\gamma^\nu\gamma^5)\\
&+\mathrm{Tr}(\gamma^\rho\gamma^\mu m\gamma^\nu\gamma^5)+k'_\sigma\,\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^5\gamma^\sigma\gamma^\nu\gamma^5).
\end{aligned} \tag{A.7}$$

All traces involving an odd number of gamma matrices or an odd number plus any number of γ^5 are zero. Then after anticommuting the γ^5 matrices to the right side we only have two kinds of non-zero traces,

$$\begin{aligned}
\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu)&=4\left(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}\right)\\
\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^5)&=-4i\varepsilon^{\sigma\nu\rho\mu}.
\end{aligned} \tag{A.8}$$

We put this back into (A.6) and get

$$\sum_{r,s}|\mathcal{M}|^2=G_F^2V_{ud}^2f_\pi^2p_\nu p_\mu k_\rho k'_\sigma 8\left(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\rho\nu}g^{\mu\sigma}+4i\varepsilon^{\sigma\nu\rho\mu}\right). \tag{A.9}$$

But,

$$\begin{aligned}
-p_\nu p_\mu k_\rho k'_\sigma 4i\varepsilon^{\sigma\nu\rho\mu}&=p_\nu p_\mu k_\rho k'_\sigma 4i\varepsilon^{\sigma\mu\rho\nu}\\
&=p_\mu p_\nu k_\rho k'_\sigma 4i\varepsilon^{\sigma\mu\rho\nu}\\
&=0,
\end{aligned} \tag{A.10}$$

because of the antisymmetry of $\varepsilon^{\mu\nu\rho\sigma}$. For the terms involving $g^{\mu\nu}$ we get different four-vector products,

$$|\mathcal{M}|^2=8G_F^2V_{ud}^2f_\pi^2\left(2p\cdot kp\cdot k'-k\cdot k'p\cdot p\right). \tag{A.11}$$

We specialize to the center of momentum frame so that $p=(m_\pi,0)$:

$$\begin{aligned}
|\mathcal{M}|^2&=8G_F^2V_{ud}^2f_\pi^2\left(2m_\pi^2E_\nu E_\mu-E_\nu E_\mu m_\pi^2+\boldsymbol{k}\cdot\boldsymbol{k}'m_\pi^2\right)\\
&=8G_F^2V_{ud}^2f_\pi^2m_\pi^2\left(E_\nu E_\mu+\boldsymbol{k}\cdot\boldsymbol{k}'\right)
\end{aligned} \tag{A.12}$$

Combining the two-body phase space with the square of the matrix element [PS95, p.107] we get the differential decay rate. Integrating this over all outgoing particle momenta we get the total decay rate,

$$\begin{aligned}
\Gamma=\int d\Gamma&=\frac{1}{2m_\pi}\int d\Pi_2\,|\mathcal{M}|^2\\
&=\frac{1}{2m_\pi}\int d\Omega\frac{1}{16\pi^2}\frac{|\boldsymbol{k}|}{m_\pi}8G_F^2V_{ud}^2f_\pi^2m_\pi^2\left(E_\nu E_\mu-\boldsymbol{k}^2\right).
\end{aligned} \tag{A.13}$$

The integrand has no angular dependance so we just get the surface area of the sphere from the integral. This results in

$$\Gamma=\frac{G_F^2V_{ud}^2f_\pi^2|\boldsymbol{k}|}{\pi}\left(E_\nu E_\mu-\boldsymbol{k}^2\right). \tag{A.14}$$

To get an expression for the momentum k of the outgoing particles we look at the total energy,

$$\begin{aligned} E_{tot} &= m_\pi \\ E_{tot} &= E_\nu + E_\mu = |\boldsymbol{k}| + \sqrt{\boldsymbol{k}^2 + m_\mu^2} \\ \Rightarrow |\boldsymbol{k}| &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi}. \end{aligned} \tag{A.15}$$

We insert this into the energy dependence of the decay rate,

$$\begin{aligned} |\boldsymbol{k}|\left(E_\nu E_\mu - \boldsymbol{k}^2\right) &= |\boldsymbol{k}|\left(|\boldsymbol{k}|\sqrt{\boldsymbol{k}^2 + m_\mu^2} - \boldsymbol{k}^2\right) \\ &= \boldsymbol{k}^2\left(\sqrt{\boldsymbol{k}^2 + m_\mu^2} - |\boldsymbol{k}|\right) \\ &= \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right)^2\left(\sqrt{\left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right)^2 + m_\mu^2} - \frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right). \end{aligned} \tag{A.16}$$

We take a closer look at the expression under the square root,

$$\begin{aligned} \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right)^2 + m_\mu^2 &= \frac{m_\pi^4 + m_\mu^4 - 2m_\pi^2 m_\mu^2 + 4m_\mu^2 m_\pi^2}{4m_\pi^2} \\ &= \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi}\right)^2. \end{aligned} \tag{A.17}$$

Going back to (A.16) we then get

$$\begin{aligned} |\boldsymbol{k}|\left(E_\nu E_\mu - \boldsymbol{k}^2\right) &= \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right)^2\left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} - \frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right) \\ &= \frac{1}{8}\left(\frac{m_\pi^2 - m_\mu^2}{m_\pi}\right)^2 \frac{2m_\mu^2}{m_\pi} \\ &= \frac{1}{4} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2. \end{aligned} \tag{A.18}$$

With this expression the total decay rate is

$$\Gamma = \frac{G_F^2 V_{ud}^2 f_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2. \tag{A.19}$$

Appendix B

Interaction Lagrangian

Here I collect some definitions and recurring calculations for later use.

Spin matrix

In all calculations we will use $v^\mu = (1, \mathbf{0})$, which implies,

$$S^\mu = \left(0, \frac{\boldsymbol{\sigma}}{2}\right). \tag{B.1}$$

Chiral vielbein

With $V_\mu = 0$,

$$i\Delta_\mu = \frac{i}{2}\left\{u^\dagger, \partial_\mu u\right\} + \frac{1}{2}u^\dagger A_\mu u + \frac{1}{2}u A_\mu u^\dagger. \tag{B.2}$$

If we keep only terms up to order $\partial_\mu \phi^a$ or A_μ,

$$i\Delta_\mu \approx \frac{1}{2}\left(-\frac{1}{f_\pi}\partial_\mu \phi^a + A^a_\mu\right)\tau^a. \tag{B.3}$$

Properties of the Pauli matrices

$$\left[\tau^a, \tau^b\right] = 2i\varepsilon^{abc}\tau^c \tag{B.4}$$

$$\left\{\tau^a, \tau^b\right\} = 2\delta^{ab} \tag{B.5}$$

B.1 Interaction Lagrangian and vertices

B.1.1 $\pi\pi A$ vertex

We start from the Lagrangian term in equation (2.38)

$$\mathscr{L}_0^{\pi\,\prime} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left(D_\mu U (D^\mu U)^\dagger\right). \tag{B.6}$$

For vanishing vector current $V_\mu = 0$ the covariant derivative is reduced to

$$D_\mu U = \partial_\mu U - i\left\{A_\mu, U\right\}. \tag{B.7}$$

We insert this into the Lagrangian,

$$\mathscr{L}_0^{\pi\prime} = \frac{f_\pi^2}{2}\,\mathrm{Tr}\left(\left(\partial_\mu U - i\left\{A_\mu, U\right\}\right)\left(\partial^\mu U^\dagger + i\left\{A^\mu, U^\dagger\right\}\right)\right), \tag{B.8}$$

and keep only terms with one power of A_μ,

$$i\frac{f_\pi^2}{2}\,\mathrm{Tr}\left(\partial_\mu U\left\{A^\mu, U^\dagger\right\} - \left\{A_\mu, U\right\}\partial^\mu U^\dagger\right). \tag{B.9}$$

By utilizing the cyclic property of the trace we can rewrite this as

$$i\frac{f_\pi^2}{2}\,\mathrm{Tr}\left(\partial_\mu U\left\{A^\mu, U^\dagger\right\} - \partial_\mu U^\dagger\left\{A^\mu, U\right\}\right). \tag{B.10}$$

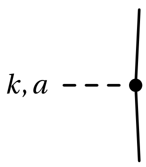
We expand the pion matrix,

$$U = 1 + i\frac{\phi^a\tau^a}{f_\pi} - \frac{(\phi^a\tau^a)^2}{2f_\pi} + \mathscr{O}(\phi^3), \tag{B.11}$$

and see that the first order term is anti-Hermitian while the second order term is Hermitian. This has the consequence that if we try to get one pion field from each first order term we will get one minus sign in each term from the Hermitian conjugate; these terms will cancel. The other way to get two pion fields is if both come from the derivative of the second order term, but this term is hermitian, and these terms will also cancel. This means that there is no $\pi\pi\boldsymbol{A}$-vertex at this order in ChPT.

B.1.2 $NN\pi$ vertex

We want to compute the two nucleon, one pion vertex:



(B.12)

The nucleon momentum flows from bottom to top and the momentum of the pion flows into the vertex.

This vertex comes from the leading order, two nucleon field Lagrangian:

$$\mathscr{L}_0^{\mathrm{HB}\prime} = \bar{N}2ig_A S^\mu\Delta_\mu N \tag{B.13}$$

We set all external fields to zero and use equation (B.3)

$$S^\mu i\Delta_\mu = \frac{\boldsymbol{\sigma}}{2}\cdot i\boldsymbol{\Delta} = -\frac{\boldsymbol{\sigma}}{2}\cdot\frac{1}{2f_\pi}\nabla\phi^a\tau^a. \tag{B.14}$$

This gives us the leading order interaction Lagrangian with two nucleon fields and one pion field,

$$\mathscr{L}_0^{\pi NN} = -\frac{g_A}{2f_\pi}\bar{N}\boldsymbol{\sigma}\cdot\nabla\phi^a\tau^a N. \tag{B.15}$$

Vertex

The derivative of the pion field yields a factor of $-i\boldsymbol{k}$ where $\boldsymbol{k}$ is the pion momentum. We get an overall phase factor of i from the perturbation series.

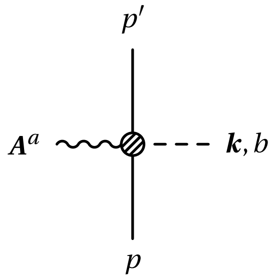
For an incoming pion with momentum $\boldsymbol{k}$ and isospin a we get the following vertex:

$$-\frac{g_A}{2f_\pi}\boldsymbol{\sigma}\cdot\boldsymbol{k}\tau^a \tag{B.16}$$

Here it is implied that σ acts on the spin part of the nucleon field and that τ acts on the isospin part.

B.1.3 $NN\pi\boldsymbol{A}$ vertex

Here we compute the two nucleon, one pion, one external axial vector field vertex in detail.



(B.17)

We stick with the convention of the nucleon momentum flowing from the bottom to the top but the pion momentum will be flowing out of the vertex.

Term A

The starting point is the Lagrangian term

$$\mathscr{L}_{1,A}^{\mathrm{HB}} = \bar{N}\frac{v^\mu v^\nu - g^{\mu\nu}}{2m_N}D_\mu D_\nu N. \tag{B.18}$$

With $v^\mu = (1,\mathbf{0})$ we get for the first factor

$$\frac{v^\mu v^\nu - g^{\mu\nu}}{2m_N} = \frac{1}{2m_N}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = \frac{1}{2m_N}\delta^{ij}. \tag{B.19}$$

Next we examine the expansion of the covariant derivative,

$$\begin{aligned} D_\mu(D_\nu N) &= (\partial_\mu + \Gamma_\mu)(D_\nu N) \\ &= (\partial_\mu + \Gamma_\mu)(\partial_\nu N + \Gamma_\nu N) \\ &= \partial_\mu\partial_\nu N + \partial_\mu\Gamma_\nu N + \Gamma_\nu\partial_\mu N + \Gamma_\mu\partial_\nu N + \Gamma_\mu\Gamma_\nu N. \end{aligned} \tag{B.20}$$

The term $\partial_\mu\partial_\nu N$ contains no pion-, or external fields so it will not contribute to the vertex. We set the external vector current $V_\mu = 0$ and simplify the connection

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger, \partial_\mu u\right] - \frac{1}{2} i u^\dagger A_\mu u + \frac{1}{2} i u A_\mu u^\dagger. \tag{B.21}$$

The first term, the commutator $\left[u^\dagger, \partial_\mu u\right]$ will not give rise to any of the vertices we are interested in since it is proportional to ϕ^2. To see this we expand the matrix u to first order in ϕ,

$$\begin{aligned} \left[u^\dagger, \partial_\mu u\right] &= \left[1 - i\frac{\tau^a\phi^a}{2f_\pi}, i\partial_\mu\phi^b\frac{\tau^b}{2f_\pi}\right] \\ &= \left[\frac{\tau^a\phi^a}{2f_\pi}, \partial_\mu\phi^b\frac{\tau^b}{2f_\pi}\right]. \end{aligned} \tag{B.22}$$

This means that the term $\Gamma_\mu\Gamma_\nu$ will not contribute either since it is at least of order ϕ^2 or A_μ^2.

We are left with two contributing terms,

$$\Gamma_{(\mu}\partial_{\nu)}, \quad \partial_\mu\Gamma_\nu. \tag{B.23}$$

We can exchange the symmetrization of the indices μ, ν with a factor of 2 because this factor is contracted with a symmetric expression.

Looking at the explicit expression for Γ_μ, we already argued that $\left[u^\dagger, \partial_\mu u\right]$ does not contribute, so then all that could give a contribution is

$$\begin{aligned} -\frac{1}{2} i u^\dagger A_\mu u + \frac{1}{2} i u A_\mu u^\dagger = &\frac{i}{2}\left\{ -\left(1 - i\frac{\tau^a\phi^a}{2f_\pi}\right)\frac{\tau^b}{2}A_\mu^b\left(1 + i\frac{\tau^c\phi^c}{2f_\pi}\right)\right. \\ &\left. + \left(1 + i\frac{\tau^d\phi^d}{2f_\pi}\right)\frac{\tau^e}{2}A_\mu^e\left(1 - i\frac{\tau^f\phi^f}{2f_\pi}\right)\right\}. \end{aligned} \tag{B.24}$$

We are interested in the four terms with exactly one pion field:

$$\begin{aligned} -\frac{1}{4f_\pi}\left(\tau^a\phi^a\tau^b A_\mu^b - \tau^b A_\mu^b\tau^c\phi^c\right) &= -\frac{1}{4f_\pi}\phi^a A_\mu^b\left[\tau^a, \tau^b\right] \\ &= -\frac{i}{2f_\pi}\phi^a A_\mu^b\varepsilon^{abc}\tau^c \end{aligned} \tag{B.25}$$

Then the derivative of Γ_μ will contribute,

$$\partial_\nu\left(-\frac{i}{2f_\pi}\phi^a A_\mu^b\varepsilon^{abc}\tau^c\right) = -\frac{i}{2f_\pi}\left(\partial_\nu\phi^a A_\mu^b + \phi^a\partial_\nu A_\mu^b\right)\varepsilon^{abc}\tau^c. \tag{B.26}$$

For our applications we can drop the term proportional to $\partial_\nu A^b_\mu$ since we let the four-momentum of the external field go to zero.

Remembering the factor of 2 from the symmetrization we get the expression for the interaction Lagrangian by contracting with $\delta^{ij}/2m_N$

$$\mathcal{L}_{1,A}^{NN\pi\boldsymbol{A}} = \frac{i}{4f_\pi m_N}\left\{\boldsymbol{A}^a \cdot \nabla\phi^b \varepsilon^{abc}\bar{N}\tau^c N + 2\boldsymbol{A}^a_i \phi^b \varepsilon^{abc}\bar{N}\tau^c \partial_i N\right\}. \tag{B.27}$$

Vertex We get a factor of $-i\boldsymbol{p}$ for the derivative of incoming nucleon line and $i\boldsymbol{k}$ from the derivative of the pion since its momentum is pointing out of the vertex.

Since $q \to 0$ we can use momentum conservation to replace the pion momentum

$$\boldsymbol{k} = \boldsymbol{p} - \boldsymbol{p}'. \tag{B.28}$$

The expression for the vertex is

$$\begin{aligned}\frac{-i}{4f_\pi m_N}\left(\boldsymbol{k} - 2\boldsymbol{p}\right)\varepsilon^{abc}\tau^c &= \frac{i}{4f_\pi m_N}\left(2\boldsymbol{p} - \left(\boldsymbol{p} - \boldsymbol{p}'\right)\right)\varepsilon^{abc}\tau^c \\ &= \frac{i}{2f_\pi m_N}\frac{\boldsymbol{p} + \boldsymbol{p}'}{2}\varepsilon^{abc}\tau^c.\end{aligned} \tag{B.29}$$

Term B

We begin by setting $V_\mu = 0$ and expanding Δ_μ to first order in ϕ using (B.3),

$$\begin{aligned}\mathcal{L}_{1,B}^{\text{HB}} &= \bar{N}4c_3 i\Delta^\mu i\Delta_\mu N = \\ 4c_3\bar{N}&\left(\frac{1}{2}\left(-\frac{1}{f_\pi}\partial_\mu\phi^a + A^a_\mu\right)\tau^a\right)^2 N.\end{aligned} \tag{B.30}$$

We want the terms with a minimal number of pions that are linear in the external fields. We are also not interested in vertices containing only the pion fields. We keep only terms that contain exactly one pion field and one external field and get for the expression sandwiched by the nucleon fields:

$$-\frac{1}{4}\left\{\partial_\mu\phi^a\frac{\tau^a}{f_\pi}, A^{\mu,b}\tau_b\right\} = -\frac{1}{4f_\pi}\partial_\mu\phi^a A^{\mu,b}\left\{\tau^a,\tau^b\right\} = -\frac{1}{2f_\pi}\partial_\mu\phi^a A^{\mu,a} \tag{B.31}$$

We are only interested in the vertex involving the vector current $\boldsymbol{A}$, and we get an extra minus sign since the index on A^μ is raised. With this we get the interaction Lagrangian term

$$\mathcal{L}_{1,B}^{NN\pi\boldsymbol{A}} = \frac{2c_3}{f_\pi}\nabla\phi^a \cdot \boldsymbol{A}^a\bar{N}N. \tag{B.32}$$

Vertex Just as in the case of term A we we get a factor of $i\boldsymbol{k}$ for the derivative of the final state pion. The resulting contribution to the vertex is

$$-\frac{2c_3}{f_\pi}\boldsymbol{k}\delta^{ab}. \tag{B.33}$$

Term C

This is the final term to contribute to the $NN\pi\boldsymbol{A}$ interaction Lagrangian and vertex:

$$\mathscr{L}_{1,C}^{\text{HB}} = \bar{N}\left(4c_4 + \frac{1}{m_N}\right)\left[S^\mu, S^\nu\right] i\Delta_\mu i\Delta_\nu N \tag{B.34}$$

To simplify the following calculations we will put the factors $i\Delta_\mu$ into a commutator. We can see that this is allowed,

$$\left[S^\mu, S^\nu\right] i\Delta_\mu i\Delta_\nu = -\left[S^\nu, S^\mu\right] i\Delta_\mu i\Delta_\nu = -\left[S^\mu, S^\nu\right] i\Delta_\nu i\Delta_\mu, \tag{B.35}$$

and we can rewrite

$$\left[S^\mu, S^\nu\right] i\Delta_\mu i\Delta_\nu = \frac{1}{2}\left[S^\mu, S^\nu\right]\left[i\Delta_\mu, i\Delta_\nu\right]. \tag{B.36}$$

This is a general property of antisymmetric tensors. In any contraction between an arbitrary tensor $T^{\mu\nu}$ and an antisymmetric tensor $A^{\mu\nu}$ only the antisymmetrization $\frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$ will give a contribution.

We continue by looking at the commutator of the vielbein. We use (B.3) to expand to first order in ϕ or A,

$$\left[i\Delta_\mu, i\Delta_\nu\right] = \frac{1}{2}\left(-\frac{1}{f_\pi}\partial_\mu\phi^a + A_\nu^a\right)\frac{1}{2}\left(-\frac{1}{f_\pi}\partial_\nu\phi^b + A_\nu^b\right)\left[\tau^a, \tau^b\right]. \tag{B.37}$$

Some terms can be ignored since we are only interested in the terms which have one power of ϕ and one of A_μ. We are left with

$$\left[i\Delta_\mu, i\Delta_\nu\right] = -\frac{i}{2f_\pi}\left(\partial_\mu\phi^a A_\nu^b + A_\mu^a\partial_\nu\phi^b\right)\varepsilon^{abc}\tau^c + \mathscr{O}(\phi^2, A^2). \tag{B.38}$$

Next we turn our attention to the commutator of the spin matrix,

$$\left[S^\mu, S^\nu\right] = \frac{1}{4}\left[\sigma^i, \sigma^j\right] = \frac{1}{2} i\varepsilon^{ijk}\sigma^k. \tag{B.39}$$

Because both A_μ and ∂_μ are naturally covariant objects we get no sign change from the contractions when assembling the whole expression,

$$-\left(2c_4 + \frac{1}{2m_N}\right)\frac{i}{2f_\pi}\frac{1}{2} i\varepsilon^{ijk}\sigma^k\left(\partial_i\phi^a A_j^b + A_i^a\partial_j\phi^b\right)\varepsilon^{abc}\tau^c. \tag{B.40}$$

Exchanging both the space indices and the isospin indices on $\partial_i\phi^a A_j^b$ we pick up two minus signs, one from ε^{ijk} and one from ε^{abc}. Further we exchange the indices on ∂_j and σ^k, which gives a minus sign. Finally there is a factor i^2 that yields another minus sign. The interaction Lagrangian then becomes

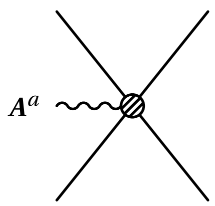
$$\mathscr{L}_{1,\text{C}}^{NN\pi\boldsymbol{A}} = -\frac{1}{f_\pi}\left(c_4 + \frac{1}{4m_n}\right)A_i^a\partial_k\phi^b\varepsilon^{ijk}\varepsilon^{abc}\bar{N}\sigma^j\tau^c N. \tag{B.41}$$

Vertex Also in this case the derivative of the pion fields give a factor $i\boldsymbol{k}$:

$$\frac{1}{f_\pi}\left(c_4 + \frac{1}{4m_N}\right)(\boldsymbol{\sigma}\times\boldsymbol{k})\,\varepsilon^{abc}\tau^c \tag{B.42}$$

B.1.4 $4N\boldsymbol{A}$ vertex

Here we compute the four-nucleon, one external axial vector field vertex in detail.



(B.43)

We label the nucleons 1 and 2 and consider them to be distinct. This means that operators acting on the internal nucleon state have a subscript index indicating which nucleon they act on, for example τ_1^a acts on the isospin space of nucleon 1.

Term CT1

We now turn to the four-nucleon terms in the first order Lagrangian. These terms will give rise to nucleon contact vertices. The ones we are interested in are four nucleon, one external field-vertices,

$$\mathscr{L}_{1,\mathrm{CT1}}^{\mathrm{HB}} = -4id_1\bar{N}S^\mu\Delta_\mu N\bar{N}N. \tag{B.44}$$

We do not want any pions in these vertices so we will use $u = 1 + \mathcal{O}(\phi)$. Together with setting $V^\mu = 0$ this simplifies the chiral vielbein immensely,

$$i\Delta_\mu = \frac{1}{2}A_\mu^a\tau^a + \mathcal{O}(\phi). \tag{B.45}$$

Using $S^\mu = (0, \sigma^i/2)$ we get to the interaction Lagrangian with one external field and no pion field,

$$\mathscr{L}_{1,\mathrm{CT1}}^{4N\boldsymbol{A}} = d_1 A^{i,a}\bar{N}\sigma^i\tau^a N\bar{N}N. \tag{B.46}$$

Vertex Depending on how we contract the baryon fields we get two different terms resulting in the vertex

$$id_1\left(\tau_1^a\boldsymbol{\sigma}_1 + \tau_2^a\boldsymbol{\sigma}_2\right). \tag{B.47}$$

Term CT2

The starting point is the second four-nucleon term of the first order Lagrangian

$$\mathscr{L}_{1,\mathrm{CT2}}^{\mathrm{HB}} = 2id_2\varepsilon^{abc}\varepsilon_{\mu\nu\lambda\delta}v^{\mu}\Delta^{\nu,a}\bar{N}S^{\lambda}\tau^{b}N\bar{N}S^{\delta}\tau^{c}N. \tag{B.48}$$

This term looks very complicated but since there are no pion fields it simplifies easily. We begin by observing,

$$\frac{\tau^a}{2}\Delta^{\nu,a} \equiv \Delta^{\nu} \Rightarrow \Delta^{\nu,a} = 2\tau^a\Delta^{\nu}. \tag{B.49}$$

As in the previous section $i\Delta_{\mu} = \frac{1}{2}A_{\mu}^{a}\tau^{a}$ with no pion fields. Also $v^{\mu} = (1,0)$ which means that the four dimensional Levi-Civita tensor is reduced to a three dimensional one,

$$\varepsilon_{\mu\nu\lambda\delta}v^{\mu} \to \varepsilon^{ijk}, \quad \nu \to i, \quad \lambda \to j, \quad \delta \to k. \tag{B.50}$$

We insert this into the Lagrangian term,

$$\mathscr{L}_{1,\mathrm{CT2}}^{4N\boldsymbol{A}} = 2d_2\varepsilon^{abc}\varepsilon^{ijk}A^{i,a}\bar{N}S^{j}\tau^{b}N\bar{N}S^{k}\tau^{c}N. \tag{B.51}$$

Using $\tau^a\tau^b = \delta^{ab}$ and plugging in $S^i = \sigma^i/2$ we get the final expression for the interaction Lagrangian,

$$\mathscr{L}_{1,\mathrm{CT2}}^{4N\boldsymbol{A}} = \frac{1}{2}d_2\varepsilon^{abc}\varepsilon^{ijk}A^{i,a}\bar{N}\sigma^{j}\tau^{b}N\bar{N}\sigma^{k}\tau^{c}N. \tag{B.52}$$

Vertex When writing down the vertex we must remember that we can exchange the pairs of baryon fields yielding a factor of two,

$$id_2\,(\tau_1\times\tau_2)^a\,\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2. \tag{B.53}$$

Appendix C

Current operators

The total current operator is given by

$$\hat{\boldsymbol{A}}_{12}^a = \hat{\boldsymbol{A}}_A^a + \hat{\boldsymbol{A}}_B^a + \hat{\boldsymbol{A}}_C^a + \hat{\boldsymbol{A}}_D^a + \hat{\boldsymbol{A}}_{CTA}^a + \hat{\boldsymbol{A}}_{CTB}^a. \tag{C.1}$$

In these calculations we will use

$$c_i = \frac{\hat{c}_i}{m_N},$$

and

$$d_i = \frac{\hat{d}_i g_A}{m_N f_\pi^2}.$$

C.1 One pion exchange currents

C.1.1 Term A

We insert the vertex (B.29) into the diagram in figure 5.3. We get a minus sign from exchanging the order of the cross product,

$$\begin{aligned} i\mathcal{M}_A &= \boldsymbol{A}^a \frac{i}{2f_\pi m_N} \frac{\boldsymbol{p}_1 + \boldsymbol{p}_1'}{2} \varepsilon^{abc} \tau_1^c \frac{i}{k^2 - m_\pi^2} \left(-\frac{g_A}{2f_\pi} \boldsymbol{\sigma}_2 \cdot \boldsymbol{k} \tau_2^b \right) \\ &= \boldsymbol{A}^a \frac{i g_A}{2m_N f_\pi^2} \frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{k^2 - m_\pi^2} i \frac{(\boldsymbol{p}_1 + \boldsymbol{p}_1')}{2} (\tau_1 \times \tau_2)^a. \end{aligned} \tag{C.2}$$

If we also take into account the exchange of nucleon 1 and 2 we get the nuclear current operator

$$\hat{\boldsymbol{A}}_A^a = \frac{g_A}{2m_N f_\pi^2} \frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{k^2 - m_\pi^2} \left(i \frac{\boldsymbol{p}_1 + \boldsymbol{p}_1'}{2} (\tau_1 \times \tau_2)^a \right) + (1 \leftrightarrow 2). \tag{C.3}$$

C.1.2 Term B

We insert the vertex (B.33) into the diagram in figure 5.3,

$$\begin{aligned} i\mathcal{M}_B &= \boldsymbol{A}^a \left(-\frac{2c_3}{f_\pi}\boldsymbol{k}\delta^{ab}\right)\frac{i}{k^2 - m_\pi^2}\left(-\frac{g_A}{2f_\pi}\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}\tau_2^b\right) \\ &= \boldsymbol{A}^a \frac{ig_A}{2m_N f_\pi^2}\frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{k^2 - m_\pi^2}\left(2c_3\boldsymbol{k}\tau_2^a\right). \end{aligned} \tag{C.4}$$

The corresponding nucleon momentum-space operator is then

$$\hat{\boldsymbol{A}}_B^a = \frac{g_A}{2m_N f_\pi^2}\frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{k^2 - m_\pi^2}\left(2\hat{c}_3\boldsymbol{k}\tau_2^a\right) + (1 \leftrightarrow 2). \tag{C.5}$$

C.1.3 Term C

This works in a very similar way to the previous term. We get a minus sign from the exchange of τ_1, τ_2 in the cross product.

$$\begin{aligned} i\mathcal{M}_C &= \boldsymbol{A}^a \left(\frac{1}{f_\pi}\left(c_4 + \frac{1}{4m_N}\right)(\boldsymbol{\sigma}_1 \times \boldsymbol{k})\,\varepsilon^{abc}\tau_1^c\right)\frac{i}{k^2 - m_\pi^2}\left(-\frac{g_A}{2f_\pi}\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}\tau_2^b\right) \\ &= \boldsymbol{A}^a \frac{ig_A}{2m_N f_\pi^2}\frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{k^2 - m_\pi^2}\left(\left(\hat{c}_4 + \frac{1}{4}\right)(\tau_1 \times \tau_2)^a \boldsymbol{\sigma_1} \times \boldsymbol{k}\right) \end{aligned} \tag{C.6}$$

We get the nuclear current operator

$$\hat{\boldsymbol{A}}_C^a = \frac{g_A}{2m_N f_\pi^2}\frac{\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}}{k^2 - m_\pi^2}\left(\left(\hat{c}_4 + \frac{1}{4}\right)(\tau_1 \times \tau_2)^a \boldsymbol{\sigma}_1 \times \boldsymbol{k}\right) + (1 \leftrightarrow 2). \tag{C.7}$$

C.2 Contact currents

C.2.1 Term CT1

The diagram follows from the vertex trivially

$$i\mathcal{M}_{CT1} = \boldsymbol{A}^a \frac{g_A}{m_N f_\pi^2} i\hat{d}_1\left(\tau_1^a \boldsymbol{\sigma}_1 + \tau_2^a \boldsymbol{\sigma}_2\right). \tag{C.8}$$

In this case the interchange of nucleons has already been accounted for and we get the operator

$$\hat{\boldsymbol{A}}_{CT1}^a = \frac{g_A}{m_N f_\pi^2}\left(\hat{d}_1\left(\tau_1^a \boldsymbol{\sigma}_1 + \tau_2^a \boldsymbol{\sigma}_2\right)\right). \tag{C.9}$$

In both this term and the next the interchange of the two nucleons is already accounted for by the vertex.

C.2.2 Term CT2

$$i\mathscr{M}_{CT2} = \boldsymbol{A}^a \frac{g_A}{m_N f_\pi^2} \left(i\hat{d}_2 \tau_1 \times \tau_2\right)^a \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 \tag{C.10}$$

$$\hat{\mathbf{A}}^a_{CT2} = \frac{g_A}{m_N f_\pi^2} \left(\hat{d}_2 \left(\tau_1 \times \tau_2\right)^a \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2\right) \tag{C.11}$$

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