CHALMERS
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Holographic Duality and Strongly Interacting Quantum Matter

A Gravitational Approach to High T_c Superconductors

Master's thesis in Physics and Astronomy

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Department of Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
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Abstract

This thesis is devoted to the applications of holographic duality to condensed matter physics. It is centered around a 'bottom-up' approach where the starting point is the postulation of a reasonable gravitational bulk theory action, as opposed to the 'top-down' models where a specific duality is derived from a string theory setting. The main motivation for taking a holographic approach to condensed matter physics is the potential ability to perform reliable computations for strongly interacting quantum many-body systems, in the absence of a quasiparticle description. The duality maps a strongly coupled quantum field theory to a weakly interacting gravitational theory, which in principle can be solved perturabtively using ordinary general relativity. An introduction to some of the main topics of bottom-up holography is given. This includes a brief introduction to large N field theories, the AdS/CFT correspondance, the holographic dictionary, the holographic renormalization group, holographic thermodynamics, and the Hawking-Page transition and its interpretation in the light of AdS/CFT. Finally, a minimal bottom-up toy model for holographic superconductivity is studied. By imposing a mixed boundary condition at the boundary of AdS space, a dynamical photon is incorporated in the strongly coupled superconductor. This allows charged collective excitations, e.g. plasmons, to be studied. A linear response analysis of the minimal holographic superconductor is performed numerically, in an attempt to compute plasmon dispersion relations. It turns out that the mixed boundary condition, accounting for charged collective excitations, will likely have to be modified for this particular holographic superconductor model, since the computed plasmon dispersion relation indicates an instability at large momenta. The precise way in which the mixed boundary condition has to be modified remains unclear.

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Marcus Lassila, Gothenburg, April 2020

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1

Introduction

In the last decade, the recent applications of the holographic gauge/gravity duality to condensed matter physics has developed into a new promising research area [1]–[3]. In particular, the holographic method has the potential of dealing with certain strongly interacting quantum many-body systems, a regime inaccessible from a conventional field theory approach with a quasiparticle description. The field of holographic condensed matter physics or 'AdS/CMT'[1] is characterized by a cross-fertilization between many different areas of physics, including string theory, gravitational and black hole physics, quantum field theory, quantum information theory, and of course condensed matter physics. From a condensed matter perspective, it is the study of strongly interacting, long-range entangled, quantum many-body systems without quasiparticles. For the gravitational physicist it is the study of black hole geometries which are asymptotic to anti-de Sitter (AdS) space. In the context of quantum field theory (QFT) or string theory, it is the study of the statistical physics of *large N matrix field theories* and their dual gravitational description.

The holographic duality is rooted in the AdS/CFT correspondence or gauge/gravity duality which was originally proposed by Juan Maldacena in 1997 [4]. He argued that $\mathcal{N} = 4$ super-Yang-Mills theory in four spacetime dimensions, which is a conformal field theory (CFT) as well as a gauge theory, has a dual description in terms of supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$. He then proposed the AdS/CFT conjecture stating that a gravitational theory in AdS space has a dual description in terms of a CFT. The argument behind the AdS/CFT correspondence is based on considerations of an open/closed string duality in string theory. Although it technically has the status of a conjecture, plenty of empirical evidence supporting its claims has accumulated throughout the years and there is no doubt that the conjecture holds true.

Holographic duality is a generalized notion of an AdS/CFT-like correspondence. It identifies a 't Hooft large N quantum field theory[2] with a classical gravitational theory in one higher dimension. The best understood dualities are those where the gravitational spacetime is asymptotic to AdS space. In this case the QFT can be thought of as living on the 'conformal boundary' of the gravitating AdS spacetime. For this reason it is common to refer to the QFT as the boundary QFT or simply boundary theory. The gravitational theory is then properly referred to as the bulk theory. Note that the QFT does not have to be a CFT in a general holographic duality.

Many of the useful aspects of holography are generic and independent of the specific underlying duality. A specific duality may be motivated by a specific string theory con-

[1]A pun on 'AdS/CFT', the best understood and original version of gauge/gravity dualities. AdS stands for anti-de Sitter, the dynamical spacetime of the gravitational theory. CMT stands for condensed matter theory.

[2]The N here refers to the number of 'colour' degrees of freedom. A large N matrix field theory transforms in the adjoint of some gauge symmetry group, e.g. $U(N)$, in the limit of large N. We give an introduction to large N field theories in Chapter 2.

sideration, or it may simply be postulated by an educated guess. The former approach is commonly referred to as a top-down construction, whereas the latter approach is referred to as a bottom-up construction. Since the top-down construction is derived from a specific string theory model, it has the advantage of inheriting quantum consistency and UV completion. Furthermore, the gravitational theory and dual quantum field theory will be known in a top-down construction. The downside is that it may be difficult to find a string theory construction from which the desired condensed matter system can be obtained by a consistent truncation. The advantage of the bottom-up approach is that one may postulate a gravitational theory by considering which properties one want to impose on the dual QFT. One can then directly study the parameter space of the theory and compare with real life experiments. The downside is that such a theory is necessarily phenomenological and one cannot guarantee that it is UV complete, although UV completion is usually not a high priority for condensed matter theories since these generally deal with emergent low energy phenomena. In this thesis we will exclusively be considering bottom-up models, bypassing any explicit string theoretic construction and making the material accessible even for those readers unfamiliar to string theory.

The key feature of the large N limit is that it corresponds to a classical limit in the dual gravitational theory. Quantum gravity is still rather poorly understood and reliable computations cannot be performed in current theories of quantum gravity. The regime outside the large N limit is therefore also currently inaccessible. Although field theories with a large number of colour degrees of freedom seems rather artificial and irrelevant for real life condensed matter physics, many of the results from holography calculations seems to be rather generic and valid even for small and finite N. However, the large N limit is not only an unwanted artefact. It comes with a notion of a mean field and can for example allow one to work in a thermodynamical limit where it otherwise would not be possible. Ultimately though, experiments will have to confirm whether or not holographic condensed matter physics provide any useful models for real life physical systems such as the high-temperature superconductors.

Two important aspects of holography when it comes to its applications to condensed matter physics are the following:

- The extra dimension of the bulk spacetime geometrizes the renormalization group (RG) scale of the boundary QFT. The near boundary region captures the high energy, UV processes of the dual QFT, and the deep interior captures the emergent low energy, IR physics.

- Classical black holes are dissipative and have a thermodynamic interpretation. Adding a black hole to the interior of the bulk spacetime encodes for thermodynamic properties in the boundary QFT, e.g. a finite temperature and dissipative processes.

The identification of the extra radial dimension in the bulk spactime with the RG scale of the dual QFT allows us to extract the emergent low energy physics relevant for applications to condensed matter physics, while at the same time removing all of the dependence on the UV complete theory, which we are ignorant of in a bottom-up approach. In fact, all of the standard renormalization physics of QFT is captured in the dual gravitational description. This 'holograpic renormalization theory' is one of the main topics of Chapter 4. Knowing that the low energy physics is captured in the deep interior of the bulk spacetime, adding a black hole in this region should intuitively only affect the low energy processes. This encodes for dissipative processes and thermodynamic quantities, e.g. a temperature and entropy, in the effective low energy description of the boundary QFT.

1.1 T-linear resistivity of strange metals and Planckian dissipation

The primary benefit with the holographic approach to condensed matter physics is its potential ability to provide a description of strongly interacting quantum many-body systems in which reliable computations can be performed. This includes 'non-Fermi liquids' such as the strange metal normal state of cuprate high T_c superconductors. Ever since the discovery in the 1980s of the linear temperature dependence of the resistivity in this strange metal phase [3], [5], condensed matter theorists have sought for an explanation for this seemingly simple relation. No satisfactory such description have yet to be found from conventional condensed matter considerations, which are usually based on quasiparticle transport. Although these models can give rise to a linear temperature dependence they do not rule out other more complex dependencies and fail to explain why it is precisely the linear dependence that has been measured in the laboratory. Arguably, a linear temperature dependence reflects a very simple physical behavior and it is reasonable to assume that such a simple physical behaviour is rooted in a strong physical principle. Holography has provided a new take on this long lasting conundrum. It has been suggested that the electrons in the strange metal might be in a strongly interacting, maximally entangled state, where all of the electrons are entangled with one another [3]. Furthermore, it has been argued that the scattering rate of these electrons, as a function of temperature, reach the 'Planckian limit' where they dissipate energy and momentum at the fastest rate allowed by the uncertainty principle of quantum mechanics [6], [7]. This 'Planckian dissipation' is set by Planck's constant $\hbar$, suggesting that the principle behind the T-linear resistivity observed in the strange metals will involve new fundamental physics. Planckian dissipation also appears to be a generic property of the strongly interacting, maximally entangled, compressible quantum matter which are ideally described by holography. Moreover, these maximally entangled quantum states behaves as a perfect fluid characterized by a universal ratio of shear viscosity η by volume density of entropy s [8],

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}\,. \tag{1.1}$$

This minimal viscosity of certain strongly interacting quantum field theories was derived from a holographically dual gravitational AdS spacetime with a black hole. It is a consequence of basic properties of black holes, as is the Planckian dissipation in this context.

Understanding the strange metal phase of high T_c superconductors is essential for understanding the phase transition between the normal and superconducting state, as well as the mechanism behind the high T_c superconducting phase transition. The standard BCS theory works well for superconductors with low transition temperatures but fails in explaining the behaviour of high T_c superconductors. A better understanding of high T_c superconductors and their strange metal normal state is an important step towards being able to engineer materials which are superconducting at room temperature.

1.2 The objectives

The purpose of this thesis is twofold. First, it is intended as a fairly easy read introduction to the subject of holographic duality and its applications to condensed matter physics. In particular, it should be comprehensible for the undergraduate student who has taken courses in quantum field theory, general relativity, and condensed matter physics[3]. However, the reader which has not had any exposure to the path integral formalism or the renormalization group may need to complement some parts with additional reading. Although the holographic duality is derived from string theory, the bottom-up approach allows us to bypass the string theoretical framework more or less completely in this work. The only exceptions are our qualitative derivation of the AdS/CFT correspondance in 3.2 and the discussion of Wilson loops in 6.2 where some string theory terminology has been used. Both 3.2 and 6.2 may be skimmed over or even skipped completely without the loss of any vital information for chapters to come.

Second, the goal is to study dispersion relations in a holographic superconductor toy model. This toy model is a minimal bottom-up construction of holographic superconductivity originally introduced by Hartnoll, Herzog and Horowitz [9], [10]. The holographic superconductors are strongly coupled systems and they are candidates for modeling high T_c superconductors such as cuprates or pnictides. The high T_c cuprate superconductors are thought to consists of two dimensional superconducting layers stacked on top of each other, with the Coulomb interaction responsible for the dynamics between the layers [11], [12]. The Coulomb interaction gives rise to plasmon modes which has been studied previously in holographic models [12]–[16]. The new experimental technique of momentum-resolved electron energy-loss spectroscopy (M-EELS) has facilitated the measurment of plasmon properties in strange metals [17]. Thus, holographic plasmon physics is an area of holography where it should be possible to verify its predictions with experimental measurments. We study the linear response in our holographic superconductor toy model and compute dispersion relations of longitudinal quasi-normal modes. By incorporating the plasmon boundary condition derived in [13] we attempt to compute a plasmon dispersion relation for the superconductor. It turns out that the plasmon boundary condition will need to be modified for the superconductor model and the computed plasmon dispersion relation is not to be trusted.

1.3 Some notes on conventions

Here we summarize some of the conventions used in this thesis. When not stated otherwise, we work in natural units where $c = \hbar = 1$. For the signature of the metric we use positive signs for spatial coordinates and negative signs for time components. When it comes to dimensions in holographic models d is used as the number of spatial dimensions in the boundary QFT, i.e. $d + 1$ is the spacetime dimension of the boundary QFT and $d + 2$ is the spacetime dimension of the bulk gravity theory. If not explicitly stated otherwise, an integral measure $d^{d+2}x$ denotes a volume element of the bulk spacetime including the time and radial coordinates. Capital latin letters will be used for indices ranging over all of the $d + 2$ coordinates in the bulk spacetime whereas greek indices will range over the $d + 1$ 'flat' coordinates shared by the bulk and the boundary. Lower-case latin indices will range only over the d spatial boundary coordinates.

[3] Even just a basic solid state physics course will suffice as a condensed matter background.

2

Large N Field Theories

This chapter is devoted to a brief overview of large N field theories and their connection to the AdS/CFT correspondance and holography. Although the AdS/CFT correspondance is rooted in string theory, much of the physics it gives rise to is accessible without using any of the mathematical machinery of string theory by taking a bottom-up approach. In a bottom-up approach we only need to know about the classical gravity content of the holographic duality, and how to interpret gravitational quantities in terms of quantities in the dual QFT. The translation between the two sides of the duality is accomplished using the holographic dictionary which is the subject of Chapter 4. Here we give a brief discussion of the field-theoretical background of holography, which is essential for understanding what kind of quantum field theories and condensed matter systems one is dealing with in holography.

The quantum field theories described by holography are *matrix large N field theories*. These can for example be gauge field theories transforming in the adjoint representation of some symmetry group, e.g. $U(N)$, in the limit of large N . As fields transforming in the adjoint they are most conveniently represented as $N \times N$-matrices with gauge invariant interaction terms being functions of traces of the fields. One of the main advantages of the holographic approach is that a strongly coupled QFT correspond to a weakly coupled gravitational theory, as will be elaborated on in 3.2.

A familiar example for the high energy physicist of a matrix field theory is the non-Abelian Yang-Mills gauge theory, characterized by a $U(N)$ or $SU(N)$ gauge group[1]. The matrix fields transform in the adjoint of the gauge group. Part of the foundations of holography and the AdS/CFT correspondence was laid already in the 1970s by 't Hooft in [19]. There he considered a $U(N)$ gauge theory in the limit of large N with g^2N held fixed, where g is the $U(N)$ coupling constant. This limit is now commonly known as the *'t Hooft large N limit* and g^2N is referred to as the *'t Hooft coupling*. It allowed for a diagrammatic expansion in $1/N$ with the leading contributions coming from 'planar diagramms'. Furthermore, it was expected that the 't Hooft large N limit was somehow related to string theory. However, it remained until the discovery of the AdS/CFT correspondance [4] until the connection with string theory became clear.

There are other types of large N limits of quantum field theories, e.g. vector large N field theories where the fields transform in the vector representation of some symmetry group rather than the adjoint. We will consider such a vector large N limit in 2.2 as to demonstrate the existence of a saddle point description, a trait which the vector large N limit has in common with the 't Hooft matrix large N limit. The saddle point description in the matrix large N limit is, however, of a completely different kind.

[1] The $U(N)$ and $SU(N)$ gauge groups are related as $U(N) = (SU(N)/Z_N) \times U(1)$ [18]. The elements of Z_N are already included in $U(1)$ and are therefore excluded from $SU(N)$ by taking the quotient group $SU(N)/Z_N$. However, the Lie algebra of $SU(N)$ and $SU(N)/Z_N$ are identical. Thus, the $U(N)$ and $SU(N)$ gauge theories essentially differ only by a decoupled Maxwell field, i.e. a photon.

Since path (or functional) integrals will be used quite frequently in parts of this work we proceed with a short summary of the basic rules in this formalism. Then we derive a saddle point description of a vector large N theory in 2.2. In 2.3 we introduce some of the basic quantities of matrix large N field theories as well as the most essential properties for its application to bottom-up holography.

2.1 Correlation functions in the path integral representation

In general, physical observables in a QFT are constructed from the expectation values and multi-point correlation functions[2] of the field operators. These expectation values and multi-point functions are the basic set of observables that characterize the QFT. In the path integral formalism the partition function or vacuum amplitude is given by

$$Z_{\text{QFT}} = \int D\Phi \, e^{iI[\Phi]} \,, \tag{2.1}$$

where Φ denotes collectively the field degrees of freedom and $I[\Phi]$ is the microscopic action. The action is an integral over a Lagrangian density $\mathcal{L}(x)$. A generating functional for multi-point functions can be constructed from the partition function by adding source terms to the action,

$$Z_{\text{QFT}}[h_i] = \int D\Phi \, e^{iI[\Phi]+i\int d^{d+1}x \, h_i(x)\mathcal{O}_i(x)} \,, \tag{2.2}$$

where $h_i(x)$ is a set of external fields coupled to local field variables $\mathcal{O}_i(x)$ of the QFT, (the indices i are summed over). The QFT here is defined in $d+1$ spacetime dimensions. In the language of linear response theory, the external field $h_i(x)$ is referred to as the source and the field variable $\mathcal{O}_i(x)$ as the response. Any multi-point function of local operators can then be calculated by taking functional derivatives of the generating functional with respect to the sources, taking the limit of vanishing sources in the end,

$$\langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)...\mathcal{O}_{i_n}(x_n)\rangle = (-i)^n \frac{1}{Z_{\text{QFT}}} \frac{\delta}{\delta h_{i_1}(x_1)} \frac{\delta}{\delta h_{i_2}(x_2)} ... \frac{\delta Z_{\text{QFT}}[h_i]}{\delta h_{i_n}(x_n)}\bigg|_{h_i=0} . \tag{2.3}$$

Thus, the partition function contains all the relevant information about the spectrum of the QFT, a fact that will be important when translating a QFT observable to a dynamical field in the gravitating dual description, which is the subject of the next chapter. Note that only classical field variables goes in the path integral. The non-commuting nature of the field operators automatically comes out of the formalism when computing correlation functions using the generating functional. Also note that the generating functional (2.2) equals the expectation value of $\exp\Big(i\int d^{d+1}x \, h_i(x)\mathcal{O}_i(x)\Big)$, and some authors write

$$Z_{\text{QFT}}[h_i] \equiv \left\langle e^{i\int d^{d+1}x \, h_i(x)\mathcal{O}_i(x)} \right\rangle . \tag{2.4}$$

In the case of a free field theory the Lagrangian consists only of kinetic terms and mass terms which are both quadratic. Hence, the path integral then becomes Guassian and can be evaluated analytically. In the next section we demonstrate how to calculate such a Guassian functional integral.

[2]The expectation value is just the one-point correlation function.

2.2 The vector large N limit

In a conventional, non-holographic, approach to condensed matter physics, it is far more common to work with vector rather than matrix field theories. In contrast to the matrix large N field theories, the vector large N field theories are effectively free in the large N limit [1], [2]. Although the matrix field theories are the ones relevant for holography, we present here a brief demonstration of the saddle point description realized in the large N limit of vector field theories.

In particular, we consider a vector large N bosonic field theory with an $O(N)$ symmetry. The results for other types of vector field theories are similar since the vector large N mean field is rather generic [2]. Due to the imposed $O(N)$ symmetry, interactions are restricted to be functions of $\phi \cdot \phi = \phi_i \, \phi^i$, where ϕ_i is the bosonic vector field. For concreteness, we consider a theory with a ϕ^4 interaction term and an action given by

$$S = \int d^{d+1}x \left(-\frac{1}{2}\partial_\mu \phi_i \, \partial^\mu \phi^i - \frac{m^2}{2}\phi_i \, \phi^i - \frac{\lambda}{4!}\left(\phi_i \, \phi^i\right)^2 \right) , \tag{2.5}$$

with $i = 1, 2, ..., N$. For small values of the coupling constant λ the theory is weakly interacting and can be analyzed using a standard diagrammatic perturbation expansion in λ. However, for a strongly interacting theory characterized by a large λ, perturbation theory cannot be used. The usefulness of the vector large N field theories comes from the fact that such models are effectively free in the large N limit. The partition function can be evaluated in terms of saddle points of an effective action for a set of non-fluctuating operators obeying

$$\langle \mathcal{O}_{i_1} \, \mathcal{O}_{i_2} ... \mathcal{O}_{i_n} \rangle = \langle \mathcal{O}_{i_1} \rangle \, \langle \mathcal{O}_{i_2} \rangle \, ... \, \langle \mathcal{O}_{i_n} \rangle + \mathcal{O}\left(\frac{1}{N}\right) , \tag{2.6}$$

to leading order in large N. The theory at large but finite N can then be studied perturbatively, expanding in powers of $1/N$ around the classical large N limit solution.

The idea is to perform a Hubbard-Stratonovich transformation of the action (2.5). This transformation makes use of a standard Guassian integral to introduce an auxiliary field. The purpose is to linearize the quadratic term in the action. The partition function for our $O(N)$ bosonic field theory is

$$Z = \int \prod_k D\phi_k \, e^{iS} , \tag{2.7}$$

with the action S being given by (2.5). The Hubbard-Stratonovich transformation makes use of the functional generalization of the Guassian integral,

$$\frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} dy \, e^{-\frac{y^2}{2\alpha} - ixy} = e^{-\frac{\alpha}{2}x^2} , \tag{2.8}$$

valid for $\mathrm{Re}(\alpha) > 0$. Introducing an auxiliary scalar field $\sigma(x)$ to substitute for y and substituting $\phi_i \, \phi^i$ for x in (2.8), we have

$$\begin{aligned} Z &= \int \prod_k D\phi_k \, e^{i\int d^{d+1}x \left[-\frac{1}{2}\partial_\mu \phi_i \, \partial^\mu \phi^i - \frac{m^2}{2}\phi_i \, \phi^i - \frac{\lambda}{4!}\left(\phi_i \, \phi^i\right)^2 \right]} \\ &= \int \prod_k D\phi_k \, D\sigma \sqrt{\frac{6}{i\pi\lambda}} e^{i\int d^{d+1}x \left[-\frac{1}{2}\partial_\mu \phi_i \, \partial^\mu \phi^i - \frac{m^2}{2}\phi_i \, \phi^i + \frac{6}{\lambda}\sigma^2 - \phi_i \, \phi^i \, \sigma \right]} . \end{aligned} \tag{2.9}$$

The overall square root factor can be dropped since it will not affect any correlation functions generated by Z. Integrating the kinetic term by parts, assuming no boundary contributions, and then integrating out the ϕ_k fields, we get

$$\begin{aligned}
Z &= \int \prod_k D\phi_k \, D\sigma \, e^{i\int d^{d+1}x\left[\frac{1}{2}\phi_i\left(\partial^2 - m^2 - 2\sigma\right)\phi^i + \frac{6}{\lambda}\sigma^2\right]} \\
&= \int \prod_k D\phi_k \, D\sigma \, e^{\int d^{d+1}x \, d^{d+1}y \left[-\frac{i}{2}\phi_i(x)\delta^{(d+1)}(x-y)\left(-\partial^2_{(y)} + m^2 + 2\sigma(y)\right)\phi^i(y)\right]} \, e^{i\int d^{d+1}x \, \frac{6}{\lambda}\sigma^2} \\
&= \int D\sigma \left[\det\Big(2\pi i \, \delta^{(d+1)}(x-y)(-\partial^2_{(y)} + m^2 + 2\sigma(y))\Big)\right]^{-\frac{N}{2}} e^{i\int d^{d+1}x \, \frac{6}{\lambda}\sigma^2} \\
&= \int D\sigma \, \left[\det(2\pi i)\right]^{-\frac{N}{2}} e^{-\frac{N}{2}\operatorname{tr}\log\left[\delta^{(d+1)}(x-y)(-\partial^2_{(y)} + m^2 + 2\sigma(y))\right]} e^{i\int d^{d+1}x \, \frac{6}{\lambda}\sigma^2} \\
&= \int D\sigma \, e^{iN\int d^{d+1}x\left(\frac{6}{\hat\lambda}\sigma^2 + \frac{i}{2}\log\left(-\partial^2 + m^2 + 2\sigma\right)\right)} ,
\end{aligned} \tag{2.10}$$

where we have once again thrown away an overall constant factor $[\det(2\pi i))]^{-N/2}$ in the last step. The standard identity $\det A = e^{\operatorname{tr}\log A}$ was used to absorb the effect of the N functional determinants into a redefinition of the action. In the last step we have also redefined the coupling constant as $\hat\lambda = \lambda N$, in order to extract a factor of N from the action. Thus, after a Hubbard Stratonovich transformation the action reads

$$S = N \int d^{d+1}x \left(\frac{6}{\hat\lambda}\sigma^2 + \frac{i}{2}\log\left(-\partial^2 + m^2 + 2\sigma\right)\right) . \tag{2.11}$$

This is an effective action for the auxiliary field $\sigma(x)$, which must be a function of $\phi_i \, \phi^i$ in order to preserve the $O(N)$ symmetry. Since $\phi_i \, \phi^i$ is a sum of the squares of N fluctuating fields, we should expect that fluctuations of σ are subleading in N, (the argument here is essentially the central limit theorem).

In the large N limit, with $\hat\lambda$ held fixed, the dominating contributions to the partition function comes from the saddle points of the effective action (2.11). Furthermore, since the vector large N mean field is effectively free, we can study the regime of large but finite N by a perturbative diagrammatic expansion in $1/N$ around the mean field saddle point solution. The diagrammatic expansion has the structure of a conventional weak-coupling perturbation expansion. However, in the large N limit only a subset of the diagrams are summed over. In terms of our original model (2.5), this corresponds to each correlation function being approximated by its 'maximal loop decomposition' to leading order in large N [2]. Consequently, in the large N limit higher order correlation functions factorize into a product of expectation values of classical operators as in (2.6).

The vector large N limit has clearly some similarities with the classical limit $\hbar \to 0$. Restoring $\hbar$, the partition function is given by

$$Z = \int \prod_k D\phi_k \, e^{\frac{i}{\hbar}S} , \tag{2.12}$$

and in the limit $\hbar \to 0$ the theory can be semi-classically expanded around the saddle points of the action in powers of $\hbar$. However, in contrast to the classical limit, the saddle point equations in the large N limit depends on the coupling constant and can therefore capture non-trivial quantum physics which the semi-classical or weakly interacting theories are unable to capture. This is one of the motivations for considering such large N models.

2.3 Matrix large N field theories

We have now seen how the vector large N field theories allows for a mean field saddle point description of the theory. There is a novel kind of a saddle point description for the matrix large N field theories. In contrast to the vector large N theories, the mean field saddle point description of the matrix large N theories is encoded in a gravitating theory in one higher dimension.

In the strongly coupled gauge theories which are realized in several top-down constructions, the field operators are large $N \times N$ matrices Φ_I transforming in the adjoint of the gauge group. One then considers normalized gauge invariant operators of the form [20]

$$\mathcal{O}_i = \frac{1}{N}\,\mathrm{Tr}\,F_i\left(\Phi_I, \partial\Phi_I\right)\,. \tag{2.13}$$

Here $F_i(\Phi_I, \partial\Phi_I)$ are arbitrary functions of the matrix fields and their derivatives. Furthermore, F_i is defined without any traces or explicit dependence on N. The simplest example of such a function is a product of n matrix fields,

$$\mathcal{O}_i = \frac{1}{N}\,\mathrm{Tr}\left(\Phi_{I_1}\Phi_{I_2}...\Phi_{I_n}\right)\,. \tag{2.14}$$

Operators of the form (2.13) are called *single-trace operators* and in the large N limit, the expectation value of a product of single-trace operators factorize into the product of the expectation values of the single-trace operators [1], [21],

$$\left\langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}...\mathcal{O}_{i_n}\right\rangle = \left\langle \mathcal{O}_{i_1}\right\rangle\left\langle \mathcal{O}_{i_2}\right\rangle...\left\langle \mathcal{O}_{i_n}\right\rangle + \mathcal{O}\left(\frac{1}{N}\right)\,. \tag{2.15}$$

Thus, single-trace operators behave as classical variables in the large N limit. Note that in this product is not a multi-point correlation function, as all the single trace-operators should be evaluated at the same point in spacetime. This factorization further implies that the variance of these single-trace operators vanish in the large N limit, and consequently, the statistical ensemble of field configurations summed over in the partition function reduces to a single point [2], as first discovered by Witten [21]. He postulated that there should be a 'master field' formulation of matrix large N field theories where this localization to a single configuration is manifest. As it turns out this master field formulation seems to be a string theory, and it is deeply connected to the AdS/CFT correspondence.

One can also construct more general 'multi-trace' operators by forming products of single-trace operators. Then, to leading order in large N, the expectation values of any multi-trace operator factorizes into a product of expectation values of single-trace operators. Hence, multi-trace operators disappear from the spectrum in the large N limit. However, multi-trace operators can be used to deform a theory at large but finite N, effectively resulting in a change of boundary conditions for the bulk fields. We will discuss the topic of multi-trace interactions in 4.2.3.

Having constructed single-trace operators from the matrix fields, we consider an action

$$I[\mathcal{O}_i] = N^2 W[\mathcal{O}_i]\,. \tag{2.16}$$

Here the functional $W[\mathcal{O}_i]$ is defined with no dependence on N. The powers of N in the definitions (2.13) and (2.16) are chosen to guarantee the existence of a well-defined large N limit [20]. However, we will often omit extracting the explicit dependence of N in this way, and then we just have to keep in mind that single-trace operators goes as $1/N$ and the action goes as N^2.

In 3.2 we present a rather non-technical summary of the argument behind the AdS/CFT conjecture. The role of the large N limit and the limit of strong 't Hooft coupling in holography will then be made clear. However, it is worth to point out here why these limits are so essential for the applications of holographic duality to condensed matter physics. In general, the bulk theories in holography are full string theories of quantum gravity in higher dimensional spaces. These theories of quantum gravity are not yet well understood and one cannot use them to perform reliable holographic computations. However, there exists two key simplifying limits in which the holographic approach of describing quantum field theories and condensed matter systems by their dual gravitational theory becomes practical [1]:

- In the large N limit the AdS radius L is much larger than the Planck length scale l_P. As a consequence, effects due to quantum gravity which are suppressed by powers of l_P/L may be neglected, justifying us to work in the classical limit.

- In the limit of strong 't Hooft coupling $g^2N \to \infty$ the AdS radius is much larger than the string length scale l_s. Considering a derivative expansion of the bulk action one may keep only the terms that are at most quadratic in derivatives. Higher order derivatives are suppressed by powers of l_s/L. Moreover, an excited string state typically has a mass $m \sim 1/l_s$ and therefore most of the excitations acquire large masses in the strong coupling limit. Only a few low energy string states are left in the spectrum. This means that it is reasonable to consider only a small number of bulk fields having small masses.

Taking the large N limit and limit of strong 't Hooft coupling in the quantum field theory, the dual gravitational theory effectively reduces to ordinary general relativity on a bulk spacetime with one extra dimension. In addition there will be only a small content of light bulk fields which needs to be considered when describing low energy phenomena.

There are some caveats regarding the second point above. For one thing, bulk theories stemming from top-down constructions are generally supergravity defined on some ten- or eleven-dimensional spacetime. These higher dimensional spacetime manifolds generally have the form of an anti-de Sitter space times a compact space. In Maldacenas original holographic model [4] the bulk spacetime geomerty is $\mathrm{AdS}_5 \times \mathrm{S}^5$. In this case the compact space is a five dimensional sphere. This compact space can be considered an internal space which introduces an infinite tower of Kaluza-Klein modes to the bulk theory. With the inclusion of these modes the bulk theory may obtain a larger number of light fields [1]. In bottom-up holography one assumes that the internal space and the infinite tower of modes living there have been removed by a consistent truncation of the underlying string theory. Finding consistent truncations of string theory, reducing the dimensionality of the bulk space and removing undesired Kaluza-Klein modes, are problems left for the top-down holographist.

Another caveat to the second point above regarding the limit of strong 't Hooft coupling is that there may in principle exist a finite number of higher order derivative terms with an unsuppressed coupling. Such higher derivative terms effectively introduces additional bulk fields which often turns out to be ghosts. Fine tuning the bulk theory is required to turn these ghosts into well-behaved physical bulk fields. An infinite number of unsuppressed higher derivative terms would, however, correspond to a non-local action [1].

We end our discussion about matrix large N field theories and the 't Hooft large N limit here. For further information on the topic and an introduction to the planar diagrammatic structure and double line notation we refer to the original paper [19], but see also [2].

3

Black Holes, the Holographic Principle and the AdS/CFT Correspondance

In this chapter we give a brief review of some of the historical developments of black hole physics in the 20th centuary which eventually led to the proposition of 'the holographic principle' [22], [23]. We follow up with a qualitative derivation of the AdS/CFT correspondance [4], the first realization of the holographic principle in a model of quantum gravity and the best understood version of a holographic duality. The AdS/CFT correspondance is deduced from a string theory setting and thus require some concepts thereof. The readers who which may safely skip 3.2 where we discuss the argument behind the AdS/CFT correspondance at a qualitative level without going in to too much technicalities. In 3.3 we provide a short review of anti-de Sitter space since this spacetime is a fundamental part of the best understood versions of holographic duality.

3.1 A breif review of black hole physics and the holograpic principle

Much of the foundations of holography was laid already in the late 1960s and 1970s by the groundbreaking work on black holes by Bekenstein, Hawking, Penrose, and collaborators. This includes the 'singularity theorems', the 'no-hair theorems', the thermodynamic interpretation of black holes, Hawking's discovery that black holes quantum mechanically radiate, and Hawking's information paradox. The singularity theorems proved that ordinary matter in general relativity collapses and produces singular spacetimes, making the black hole solutions physically relevant contrary to what was previously assumed. The no-hair theorems stated that the black hole solutions in general relativity[1] are uniquely determined by the mass M, the charge Q, and the angular momentum J of the black hole. However, the area A_H of the horizon and the surface gravity[2] κ are other useful characterizing properties of a black hole to consider. A precise resemblance of the four laws of thermodynamics was found for the mechanics of black holes in [25]. More precisely, they found the following four laws:

1. The surface gravity κ of a stationary black hole is constant over the horizon.

[1] In 3+1 spacetime dimensions and when asymptotic to flat spacetime.

[2] For stationary spacetimes with a Killing horizon, the surface gravity is defined as the force required at infinity to hold a unit mass in place an infinitesimal distance above the horizon. For the case of non-Killing horizons we refer the interested reader to [24].

2. For adiabatic changes,
$$\delta M = \frac{\kappa}{8\pi G_{\mathrm{N}}}\delta A_H + \Omega\delta J + \Phi\delta Q\,, \tag{3.1}$$
where G_{N} is Newtons constant.

3. The area of the horizon never decreases with time, i.e. $\delta A_H \geq 0$.

4. It is impossible by any idealized procedure to reduce the surface gravity κ to zero by a finite sequence of operations.

These laws are the analogues of the zeroth through third laws of thermodynamics. This interpretation was further enhanced by the Bekenstein-Hawking formula for the entropy of a black hole [26], [27],
$$S_{\mathrm{BH}} = \frac{k_{\mathrm{B}} A_H}{4 l_{\mathrm{Planck}}} \equiv \frac{k_B A_H}{4\hbar G_{\mathrm{N}}}\,, \tag{3.2}$$
as well as Hawking's discovery [27] that black holes semi-classically radiate at a temperature,
$$T = \frac{\hbar\kappa}{2\pi}\,. \tag{3.3}$$
Here $l_{\mathrm{Planck}} = \sqrt{\hbar G_{\mathrm{N}}}$ is the Planck length in 3+1 dimensions, k_{B} is Boltzmann's constant, and $\hbar$ is of course Planck's constant. We have chosen units where the speed of light is one. With these formulas for the entropy and temperature of a black hole, and with the identification of M as the energy of a black hole by Einsteins $E = mc^2$ law, (3.1) is precisely the first law of thermodynamics, $\delta E = T\delta S$.

The thermodynamic interpretation of classical black holes led to an immediate contradiction. Equilibrium thermodynamics are macroscopic properties of a system emerging from the statistical mechanics of its microscopic constituents. The black hole entropy should measure the number of microstates giving rise to the same thermodynamic properties, in terms of the horizon area in units of Planck length squared. Hence, the black hole must have a microscopic description. This directly contradicts the no-hair theorems specifying that a black hole has no distinguishing features beyond its total mass, electric charge, and angular momentum. As a further consequence of these no-hair theorems, the Hawking radiation must be thermal, i.e. a mixed quantum state [2]. If a pure state then were to cross the horizon, and only a mixed state carrying less information could ever radiate from the black hole, information would be lost. This is Hawking's famous 'information paradox' [28]. A quantum theory without conservation of information would necessarily have to be non-unitary. Since unitarity is an essential feature of quantum theories as we know them, it would arguably be highly undesirable to discard the conservation of information when constructing a theory of quantum gravity. Although the information paradox to this day remains unsolved, it is generally believed that information somehow should not get lost when crossing the horizon of a black hole. Various speculative resolutions to the paradox have been proposed, e.g. a firewall[3] at the horizon [29] and the ER=EPR conjecture [30].

Bekenstein's realization that a black hole has an entropy proportional to the area of its event horizon, as given by (3.2), is remarkable. First, the Bekenstein-Hawking formula involves $\hbar$, k_{B}, G_{N}, and c (if c is not set to one), thus merging quantum mechanics,

[3] A firewall at the horizon is a paradox on its own known as the 'AMPS firewall paradox'.

statistical physics, gravity and special relativity in one formula. Second, it tells us that the entropy of a black hole is not extensive, i.e. it does not scale with the volume of the space it occupies. Instead the black hole entropy scales with the area of its confining surface. We refer the reader to [31] for a recent review of black hole entropy and some of the main developments that has occurred since Bekenstein's original proposal [26].

The observation that black hole entropy scales with the area led 't Hooft to propose that the observable degrees of freedom in a theory of quantum gravity in 3+1 spacetime dimensions could be stored in a 2 dimensional lattice evolving in time [22]. Susskind subsequently elaborated on this idea [23]. Supposedly, all the physical phenomena in our three dimensional world can be projected onto a two dimensional lattice at the spatial boundaries, with one discrete degree of freedom stored per Planck area. This idea became known as 'the holographic principle' after the analogy with an holographic image. The holographic principle poses some severe restrictions on possible theories of quantum gravity. Despite these restrictions a model of quantum gravity consistent with the holographic principle remained elusive, until the discovery of the AdS/CFT correspondance [4].

3.2 The AdS/CFT correspondance

The AdS/CFT correspondance is a conjecture stating that certain compactifications of M/string theory on AdS spacetimes are dual to various matrix large N conformal field theories. The prototype example is that of type IIB supergravity on $\mathrm{AdS}_5\times \mathrm{S}^5$ which is dual to $\mathcal{N}=4$ $U(N)$ super-Yang-Mills theory in 3+1 spacetime dimensions. $\mathcal{N}=4$ is an extended supersymmetry where the supersymmetry generators carries a usual spinor index but also an additional index $i=1,2,...,\mathcal{N}$ counting the number of independent spinor charges. Here we present an outline of the argument behind this particular duality of type IIB superstring theory and $\mathcal{N}=4$ super-Yang-Mills theory.

Any consistent string theory must posses both open and closed strings. For instance, a one loop diagram of an open string is equal to a tree level diagram of a closed string, after a reparametrization of the worldsheet coordinates [2]. In this way it is possible to view certain processes of a string theory from both an open and closed string perspective. This is known as the open/closed string duality.

In particular, we consider a large but fixed number N of D3 branes stacked parallel to each other in a 9+1 dimensional spacetime. Recall that a D-brane (Dirichlet brane) is a hyperplane spanned by the Dirichlet boundary conditions of an open string. A Dp-brane is an object with p spatial dimensions, e.g. a D0-brane is a point particle, a D1-brane is a string etc. Thus, we are considering three dimensional objects embedded in nine spatial dimensions, stacked upon each other. Furthermore, D-branes are non-perturbative *solitons* of string theory, i.e. they are localized finite energy states which can interact with each other.

D-branes are gravitating objects and how strongly they gravitate is determined by their tension as well as the strength of the gravitational force [1]. The tension of N D3-branes goes like N/g_s, where g_s is the string coupling constant. The strength of the gravitational force on the other hand goes as g_s^2. We define a coupling constant,

$$\lambda = 4\pi g_s N\,. \tag{3.4}$$

It can then be show that when $\lambda \ll 1$ gravitational effects can be neglected, whereas when $\lambda \gg 1$ the branes gravitate strongly [32]. The string coupling constant g_s determines the strength of the interactions between the strings. For small g_s we have free open strings on the branes and type IIB closed strings propagating over the whole spacetime.

The trick is to consider the low energy limit of this model of N parallel D3-branes on both sides of the open and closed string duality. More precisely, we consider energies smaller than the string energy scale $E \ll 1/l_s = 1/\sqrt{\alpha'}$, where l_s is the string length and α' is the slope parameter. Alternatively, we can think of the energies as being bounded and taking $\alpha' \to 0$. Moreover, one takes the limit of small λ on the open string side and the limit of large λ on the closed string side. By the open/closed string duality, the open and closed string sides describe the same physical system, albeit in different regimes of weak/strong coupling λ.

Consider first the low energy excitations on the open string side, in the limit of vanishing gravitational effects $\lambda \ll 1$. In this regime the D3-branes are fixed in a 9+1 dimensional flat spacetime. The low energy degrees of freedom are the excitations of open strings stretching between any of the N D3-branes and free closed strings propagating on the full spacetime[4]. The low energy open string excitations are massless gauge fields A_μ propagating on the branes. There are N^2 such open strings because each of the two endpoints may be fixed on any of the N D3-branes. In the low energy limit these N^2 open strings are described by $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 dimensions. This is a superconformal field theory with an action,

$$S_{\text{SYM}} = \frac{1}{4\pi g_s} \int d^4x \ \text{Tr}\Big(F^2 + D_\mu \Phi D^\mu \Phi + i\bar{\Psi}\gamma^\mu D_\mu \Psi + i\bar{\Psi}[\Phi, \Psi] - [\Phi, \Phi]^2\Big) \,. \tag{3.5}$$

Here F is the $U(N)$ field strength and $D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi]$ is the $U(N)$ covariant derivative. The super-Yang-Mills coupling constant g_{SYM} is related to the string coupling constant as $g^2_{\text{SYM}} = g_s$. There are six bosonic fields Φ and four fermionic fields Ψ, all transforming in the adjoint of $U(N)$. In $U(N)$ Yang-Mills theory one $U(1)$ Maxwell field decouples and the remaing theory is $SU(N)$ Yang-Mills theory [32]. The gauge group indices also labels which of the N D3-branes the corresponding string begin and end on. Using our definition of λ (3.4), the coupling in the action can be rewritten as N/λ. Thus, λ has the form of a 't Hooft coupling. In conclusion, the low energy excitations of N parallel D3-branes in the limit of small λ takes the form of a weakly interacting matrix large N theory, namely $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory in 3+1 dimensions, and free type IIB closed strings on a 9+1 dimensional flat space.

We now turn to the closed string side of the duality. In the low energy limit the excitations on the branes and the ones on the entire spacetime are still decoupled. However, in the limit of large $\lambda \gg 1$ the strong gravitational force causes the D3-branes to collapse upon themselves, forming a 'black brane'. Black branes are black hole solutions with translationally invariant planar horizons[5] [33], as opposed to the spherical horizons of the black holes that exist in our universe. Because the D-branes carry a Ramond-Ramond charge the black brane will be charged and thus described by an analogue to the Reissner-Nordström (RN) black hole solution. The horizon is defined as a surface of infinite gravitational redshift, and it is located an infinite distance away from any point on the spacetime at the end of an infinite throat [32]. The low energy excitations of the black brane system are partly the closed string excitations occuring in the near horizon

[4]The zero modes of the soliton are also part of the low energy excitations of the system.

[5]In a $d + 2$-dimensional spacetime the black brane horizon will be a d dimensional Euclidean space. However, in the following chapters we will simply refer to these objects as black holes and the spatial direction orthogonal to the horizon will be referred to as a radial direction.

region, due to the increasing redshift for an observer at infinity. This region is commonly referred to as the near horizon geometry. Low energy excitations of closed strings far away from the black brane, at an asymptotically flat spacetime, are the other part of the systems low energy excitations. In the case of a black brane formed by collapsing D3-branes, the near horizon geometry is that of AdS$_5\times$ S^5. The metric of this spacetime can be written as

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2\,, \tag{3.6}$$

where the horizon is located at $r = 0$, $\eta_{\mu\nu}$ is the Minkowski metric and $d\Omega_5^2$ is the metric of the five sphere. The AdS radius L is related to the string length l_s and the ten dimensional Planck length l_P as

$$L = \lambda^{1/4} l_s = (4\pi N)^{1/4} l_P\,. \tag{3.7}$$

In the limit of strong 't Hooft coupling $\lambda \gg 1$ and large N, the AdS radius is seen to be much larger than both the string length and the Planck length. As a consequence, in these limits one can neglect the effects of highly excited string states and quantum gravity, which are suppressed by powers of l_s/L and l_P/L, respectively. As mentioned, part of the low energy excitations will be described by the excitations of closed strings in the near horizon region. Since the closed string sector includes gravitons, these excitations includes ordinary classical gravitational perturbations of the near horizon geometry (3.6).

By the open/closed string duality, we have described the low energy excitations of N parallel D3-branes in two different limits. In the limit of small 't Hooft coupling λ they are described by weakly interacting $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory, a matrix large N superconformal field theory, plus low energy excitaions of free type IIB closed strings on a flat 9+1 dimensional spacetime. On the other hand, in the limit of large λ they are described by classical gravitational perturbations about the near horizon AdS$_5\times$ S^5 geometry, plus low energy excitations of free type IIB closed strings on a flat 9+1 dimensional spacetime similar to the ones on the open string side. In the low energy limit this decoupling occurs for all values of λ. Since the two sides should describe the same physical system in the two different regimes of $\lambda \ll 1$ and $\lambda \gg 1$, and since we have decoupled a free closed string sector in 9+1 flat spacetime on each side, it is reasonable to conjecture that type IIB superstring theory on AdS$_5\times$ S^5 should be a description of strongly interacting $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 dimensions. In other words, the classical gravitational dynamics of the near horizon geometry (3.6) should describe a strongly interacting matrix large N field theory. This is the effective classical 'master field' description of a matrix large N field theory postulated by Witten in [21].

The argument for the AdS/CFT conjecture is further enhanced by symmetry considerations. The $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 spacetime dimensions is a superconformal field theory, i.e. it possesses supersymmetry and is invariant under the conformal transformations. The symmetries of the dual type IIB superstring theory on AdS$_5\times$ S^5 must then have exactly these symmetries. The supersymmetry transformations of the super-Yang-Mills theory mixes the bosonic and fermionic fields among each other under rotations. This rotational symmetry is precisely the isometry group of the five sphere S^5. Furthermore, the conformal group in 3+1 spacetime dimensions is $SO(2, 4)$, which is precisely the isometry group of AdS$_5$ [1], [2], [32]. In fact, the boundary of AdS$_5$ at $r = \infty$ inherits the conformal symmetries of the isometry group. This motivates us to think of the $\mathcal{N} = 4$ super-Yang-Mills theory as living on this 'conformal boundary' of AdS$_5$ space.

The duality we have found here is between $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory in four spacetime dimensions and type IIB supergravity on $\mathrm{AdS}_5\times \mathrm{S}^5$, which is a ten dimensional spacetime. At first glance, this may seem to be inconsistent with the holographic principle which suggests that the gravitational theory should live in a space with one higher dimension than the quantum field theory. However, the five-sphere is a compact space and we can think of it as an internal space carrying excitations of massive modes living on AdS_5. For every field in the full ten dimensional spacetime we get an infinite tower of fields in AdS_5, one for each spherical harmonic on the five-sphere [1]. The mass of these *Kaluza-Klein* modes is proportional to the angular momentum of the spherical harmonics. In this way we interpret the four dimensional Yang-Mills theory as the dual of a gravitational theory on the five dimensional AdS_5. It is then rightfully referred to as a holographic duality, consistent with the holographic principle. There is an important caveat here, however. The Kaluza-Klein modes generically causes the emergent low energy field theory to grow extra dimensions, something which is highly undesired when considering applications to condensed matter physics. The most common way to deal with this issue is to find consistent truncations of string theory models, where it is possible to keep only a finite number of these Kaluza-Klein modes. Finding consistent truncations of string theory is an objective of the top-down approach to holography. We will not be considering these issues in this thesis, as we take the more pragmatic bottom-up approach to holography. The interested reader is referred to [1] and the subsequent references listed there.

We have now presented the argument for the specific duality between type IIB superstring theory on $\mathrm{AdS}_5\times \mathrm{S}^5$ and $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 spacetime dimensions. However, analogue arguments hold for many other types of supergravity theories defined on various spacetimes which are the product of an anti-de Sitter space and some compact space. These are found to be dual to other kinds of superconformal field theories. Moreover, holographic dualities are thought to exist which are even more general in the sense that the quantum field theory does not need to possess conformal symmetry. Much of the material in this thesis is based upon the existence of such generalized dualities.

3.3 Anti-de Sitter space

Anti-de Sitter space is a fundamental ingredient in AdS/CFT and plays a crucial role in the holographic dualities relevant for condensed matter applications. It is therefore appropriate to review some of its most important features in the context of holography. In the process some standard concepts of differential geometry will be introduced.

It is instructive to classify AdS space with regard to its symmetries. Recall that a diffeomorphism is a smooth, one-to-one map from a manifold to itself. It can be thought of as the active transformation of the corresponding passive coordinate transformation. A diffeomorphism under which the metric is invariant is called an isometry. The symmetries of a spacetime manifold are its isometries. The Lie derivative $\mathcal{L}_\xi$ gives the rate of change under a diffeomorphism along the integral curves of a vector ξ^μ. More precisely, under a diffeomorphism generated by ξ^μ the metric transforms as

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \,, \tag{3.8}$$

where ∇_μ is the covariant derivative. Thus, vectors ξ^μ that satisfy $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ generate isometries of the spacetime. These vectors are called Killing vectors and

$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ is called Killing's equation. For every symmetry of the spacetime there is a corresponding distinct Killing vector. Since the metric is a symmetric 2-tensor, it has $D(D+1)/2$ independent components in D dimensions. There can then be at most $D(D+1)/2$ independent Killing equations and consequently equally many Killing vectors. Spaces with this maximal number of Killing vectors are referred to as maximally symmetric spaces. There are exactly three maximally symmetric spacetimes, and these are Minkowski space, de Sitter space, and anti-de Sitter space. The Killing vectors form a group under rotations called the isometry group of the space. The isometry group of a D dimensional Minkowski space is the Poincaré group $ISO(1, D-1)$ consisting of the Lorentz transformations $SO(1, D-1)$ and spacetime translations. The isometry groups of de Sitter space and anti-de Sitter space are $SO(1, D)$ and $SO(2, D-1)$, respectively, the latter being isomorphic to the conformal group.

These Lorentzian maximally symmetric spacetimes have a Euclidean analogue. In particular, D dimensional Minkowski space is the Lorentzian verion of (flat) Euclidean space $\mathbb{R}^D$. Similarly, de Sitter space and anti-de sitter space in D dimensions are the Lorentzian versions of the sphere S^D and hyperboloid $\mathcal{H}^D$, respectively. The manifolds $\mathbb{R}^D$, S^D, and $\mathcal{H}^D$ are the three maximally symmetric spaces with Euclidean signature.

The three maximally symmetric spacetimes are solutions to Einstein's equation in vacuum with the addition of a cosmological constant Λ,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 0 \,. \tag{3.9}$$

Minkowski space, de-Sitter space, and anti-de Sitter space are the maximally symmetric solutions with $\Lambda = 0$, $\Lambda > 0$, and $\Lambda < 0$, respectively. In other words, anti-de Sitter space is the maximally symmetric spacetime with a negative cosmological constant.

3.3.1 Embedding and global coordinates

It is often useful to describe a certain topology, e.g. S^{D-1}, as a surface embedded in a higher dimensional Euclidean space $\mathbb{R}^D$. In this way the $D-1$ dimensional sphere is defined by the points that satisfy the constraint

$$x_1^2 + x_2^2 + ... + x_D^2 = R^2 \,, \tag{3.10}$$

where R is the radius of the sphere. The constraint equation makes the $SO(D)$ symmetry manifest, which is in fact the isometry group of S^{D-1}.

Similarly the $D-1$-dimensional hyperboloid $\mathcal{H}^{D-1}$ can be defined by the points in $\mathbb{R}^D$ satisfying

$$-x_1^2 + x_2^2 + x_3^2 + ... + x_D^2 = -R^2 \,. \tag{3.11}$$

Here it is possible to take an alternative perspective and view x_1 as a time coordinate. The $D-1$-dimensional hyperboloid is then viewed as a surface embedded in D-dimesional Minkowski space $\mathbb{R}^{1,D-1}$. The hyperboloid has a $SO(1, D-1)$ symmetry, as is directly seen from the defining constraint equation (3.11).

Now, D-dimensional anti de-Sitter space AdS_D can similarly be defined as a surface embedded in a higher dimensional space, albeit in an embedding space $\mathbb{R}^{2,D-1}$ with two time coordinates. The constraint equation for AdS_D is then given by

$$-x_{-1}^2 - x_0^2 + x_1^2 + x_2^2 + ... + x_{D-1}^2 = -L^2 \,. \tag{3.12}$$

The $SO(2,D-1)$ isometry group of AdS_D is again manifest in the constraint equation. Note, however, that even though the embedding space has two time directions the embedded AdS space has only one time direction. This is of course because the constraint equation (3.12) makes the two time coordinates dependent. Solving (3.12) amounts to finding a specific coordinate system for anti de-Sitter space. A particular solution to (3.12) is given by

$$\begin{aligned} X_{-1} &= L\cosh\mu\cos\tau\,, \\ X_0 &= L\cosh\mu\sin\tau\,, \\ X_1 &= L\sinh\mu\cos\theta_1\,, \\ X_2 &= L\sinh\mu\sin\theta_1\cos\theta_2\,, \\ X_3 &= L\sinh\mu\sin\theta_1\sin\theta_2\cos\theta_3\,, \\ &\vdots \\ X_{D-2} &= L\sinh\mu\sin\theta_1\sin\theta_2\ldots\cos\theta_{D-2}\,, \\ X_{D-1} &= L\sinh\mu\sin\theta_1\sin\theta_2\ldots\sin\theta_{D-2}\,, \end{aligned} \tag{3.13}$$

where $0\leq\mu<\infty$, $0\leq\tau<2\pi$, $0\leq\theta_i<\pi$ (for $i=1,\ldots,D-3$), and $0\leq\theta_{D-2}<2\pi$. The corresponding metric is

$$\begin{aligned} ds^2 &= -dX_{-1}^2 - dX_0^2 + dX_1^2 + \cdots + dX_{D-1}^2 \\ &= L^2\left(-\cosh^2\mu\, d\tau^2 + d\mu^2 + \sinh^2\mu\, d\Omega_{D-2}^2\right)\,, \end{aligned} \tag{3.14}$$

where $d\Omega_{D-2}^2$ is the metric on S^{D-2}. Note that the timelike coordinate τ is periodic, $\tau\sim\tau+2\pi$. For this reason it is common practise to extend the range of τ to the entire real line $\mathbb{R}$. The coordinates (3.13) with the extended range of τ is the universal cover of AdS_D [34].

It is often useful to map the hyperbolic coordinate μ to a finite range by a coordinate transformation $\sinh\mu=\tan\rho$, with $0\leq\rho\leq\pi/2$ for $D>2$ and $-\pi/2\leq\rho\leq\pi/2$ for $D=2$. The metric then takes the form

$$ds^2 = \frac{L^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_{D-2}^2\right)\,. \tag{3.15}$$

The overall factor $L^2/\cos^2\rho$ in the metric does not affect the topology of the space. Thus, a metric of the form

$$ds^2 \sim -d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_{D-2}^2 \tag{3.16}$$

describe a space which is topologically equivalent to AdS_D. Consider first the case of AdS_3. The topologically equivalent metric,

$$ds^2 \sim -d\tau^2 + d\rho^2 + \sin^2\rho\, d\theta^2\,, \tag{3.17}$$

describes a cylinder with radial direction ρ and longitudinal direction τ. The topology of AdS_3 is illustrated in Fig. 3.1. AdS_3 is the interior of the cylinder. For the single cover of AdS_3 where $0\leq\tau<2\pi$ the boundaries at the top and bottom of the cylinder should be identified. Thus, the topology of the single cover is that of a torus. On the other hand, the universal covering space of AdS_3 is obtained by stacking an infinite number of cylinders on top of each other. The shaded area in Fig. 3.1 is the Poincaré patch which is the topic of 3.3.2. Now, for the general case of AdS_D the S^1 coordinate θ is extended

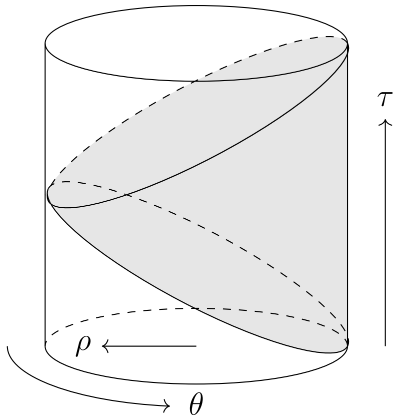

Figure 3.1: The topology of AdS_3. For the single cover of AdS_3 the top and bottom boundaries should be identified. For the universal cover an infinite number of copies should be stacked on top of each other, as the coordinate τ is unrolled on the real axis. The Poincaré patch covers the shaded area.

to hyperspherical coordinates on S^{D-2}. One may still picture the space as a cylinder, keeping in mind that there are $D-3$ additional spherical directions at every point on the cylinder.

The boundary at $\rho = \pi/2$ is strictly not part of the anti-de Sitter space. It does, however, inherit the full invariance under the conformal group $SO(2, D-1)$, the isometry group of AdS_D [35]. For this reason $\rho = \pi/2$ is referred to as the conformal boundary of AdS space. Note that in physical units this boundary is located an infinite distance away from any point in AdS_D.

3.3.2 Poincaré coordinates

In the context of the AdS/CFT correspondence there is a more convenient choice of coordinates than the global coordinates (3.13) considered above. These coordinates solves the AdS constraint equation (3.12) as follows:

$$\begin{aligned} X_{-1} &= \frac{1}{2z}\left(L^2 + z^2 - t^2 + \sum_{i=1}^{D-2} x_i^2\right) , \\ X_0 &= \frac{L}{z} t \, , \\ X_i &= \frac{L}{z} x_i \, , \qquad i = 1, 2, \ldots, D-2 \, , \\ X_{D-1} &= \frac{1}{2z}\left(L^2 - z^2 + t^2 - \sum_{i=1}^{D-2} x_i^2\right) , \end{aligned} \tag{3.18}$$

where $0 \leq z < \infty$, $-\infty < t < \infty$, and $-\infty < x_i < \infty$. In these coordinates the metric is

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + dx_1^2 + \cdots + dx_{D-2}^2\right) . \tag{3.19}$$

As opposed to the global coordinate system (3.13) the coordinates (3.18) only covers half of AdS space [34]. These coordinates are called *Poincaré coordinates* and the portion of

AdS space which they cover[6] is known as the *Poincaré patch*, see Fig. 3.1.

The conformal boundary is located at $z = 0$ in these coordinates. Furthermore, there is a 'horizon' of infinite redshift[7] at $z = \infty$. Up to an overall conformal factor the boundary is a Minkowski space $\mathbb{R}^{1,D-2}$. Since the quantum field theories in holography are defined on the boundary of the bulk space, the Poincaré coordinates are particularly useful for describing relativistic field theories in flat space.

In the global coordinates (3.13) the topology of the conformal boundary is $\mathbb{R}\times\mathrm{S}^{D-2}$, (assuming the temporal coordinate has been unrolled to the real line, otherwise the topology is $\mathrm{S}^1\times\mathrm{S}^{D-2}$). The spatial volume of the boundary is then finite and carries an associated length scale, namely the radius of the sphere (which is coincident with the AdS radius L itself). The conformal boundary of the Poincaré patch, however, has no such associated length scale since it is infinite. Thus, depending on if one uses global coordinates or Poincaré coordinates for AdS space, the conformal invariance of the dual field theory may or may not be broken. Note, however, that it takes two scales to break the scale invariance. In Chapter 5 we will introduce a temperature scale to the theory and then the global AdS coordinates will in fact break the conformal symmetry of the boundary field theory.

Sometimes we prefer to use a the coordinate $r \equiv L^2/z$ instead of the z-coordinate defined above. The metric is then

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2\,, \tag{3.20}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The 'horizon' is then located at $r = 0$ and the conformal boundary at $r = \infty$.

We end our discussion of anti-de Sitter space here. There is of course more to be said on this topic and the interested reader is referred to [34], [35].

[6]To be precise, the unshaded portion in Fig. 3.1 is also a Poincaré patch. It is covered by the same Poincaré coordinates (3.18) but with $-\infty < z \leq 0$.

[7]It can be thought of as a 'zero size' black hole horizon. In Chapter 5 we will add a black hole to the interior of the bulk spacetime, which is characterized by a finite size horizon of infinite redshift at some finite radius $z = z_+$.

4

The Holographic Dictionary

Having reviewed the essential features of large N field theories as well as some of the qualities of the AdS/CFT correspondance, we proceed with a somewhat more technical chapter devoted to the universal 'holographic dictionary'. This refers to the rules for translating the quantities of the gravitational bulk theory to the corresponding observables in the dual quantum field theory. The main ingredient in this dictionary is the GKPW formula, which will be introduced in 4.1. We use this GKPW formula to calculate the expectation value and two-point correlation function of a CFT operator in terms of the leading and subleading behavior of the dual AdS bulk field near the conformal boundary. We find that the subleading part is proportional to the expectation value of the CFT operator and that the ratio of the subleading and leading parts are proportional to the two-point function. These results turns out to be rather generic and independent of the specific holographic model.

One of the most essential properties of the holographic duality, at least when it comes to its application to condensed matter physics, is the notion that the renormalization group flow of the boundary QFT is geometrized by the gravitational bulk spacetime. More precisely, the extra radial dimension of the bulk is dual to the renormalization group scale of the boundary QFT. Processes close to the boundary correspond to the high energy, short distance, UV physics of the QFT whereas the deep interior encodes the low energy, large scale, IR physics of the QFT. This is the essence of *holographic renormalization theory*, a topic that we introduce briefly in 4.2. We conclude this chapter with a short discussion of the global/local symmetry correspondence in gauge/gravity dualities.

4.1 The GKPW formula

As mentioned in our discussion of large N field theories, the basic observables of a QFT are the multi-point correlation functions of field operators. In the case of matrix large N field theories, the basic observables are the expectation values and multi-point correlation functions of single-trace operators. Furthermore, the correlation functions can be generated from the path integral representation of the partition function by taking functional derivatives in the usual way.

For a theory with gravity it is in general difficult to define observables due to the fact that the spacetime itself is dynamical. However, when the spacetime has a boundary, observables can be defined on the boundary, since a boundary of a spacetime is not dynamical [1]. In AdS/CFT dualities and the applications of holography to condensed matter physics, the dynamical bulk spacetimes are asymptotically AdS, and as such they share the conformal boundary of AdS space. One can then consider various types of boundary value problems for the gravitational bulk theory where boundary values are specified for each bulk field. For instance, Dirichlet boundary values may be imposed on the conformal boundary, giving rise to *quasi-normal modes* (QNMs). Moreover, the partition function can be constructed as a function of the boundary values of the bulk fields. In particular, consider a set of bulk fields $\{\phi_i(x, r)\}$ with Dirichlet boundary values $\lim_{r\to\infty} \phi_i(x, r) = h_i(x)$ on the conformal boundary. Here r is the radial bulk coordinate and the argument x denotes collectively all remaining spacetime coordinates. The partition function of this bulk theory can then be written as

$$\mathrm{Z}_{\mathrm{Bulk}}[h_i(x)] = \int_{\phi_i \to h_i} \left(\prod_i D\phi_i \right) e^{iS[\phi_i]} \,, \tag{4.1}$$

where $S\left[\phi_i\right]$ is the bulk theory action.

A necessary requirement for a given matrix large N QFT and gravitational theory to be holographically dual is the existance of a one-to-one correspondance between observables in the two dual theories. Gubser, Klebanov and Polyakov, and independently Witten, discovered the right prescription for relating observables in the duality by equating the partition functions of the respective theories [36], [37],

$$\mathrm{Z}_{\mathrm{QFT}}[h_i(x)] = \int_{\phi_i \to h_i} \left(\prod_i D\phi_i \right) e^{iS[\phi_i]} \,. \tag{4.2}$$

This is commonly referred to as the GKPW formula. More precisely, the partition function of the gravitational bulk theory with bulk fields $\{\phi_i(x, r)\}$, taking boundary values $\{h_i(x)\}$, is equated to the generating functional of the QFT with sources $\{h_i(x)\}$ for the local single-trace operators $\{\mathcal{O}_i(x)\}$. In this way, the bulk fields $\{\phi_i\}$ are identified as the dual quantities corresponding to the QFT operators $\{\mathcal{O}_i\}$.

Since the large N limit in the QFT corresponds to the classical limit in the gravitational theory, the gravitational partition function can be evaluated semi-classically,

$$\mathrm{Z}_{\mathrm{Bulk}}[h_i(x)] = e^{iS[\phi_i^*]} \,, \tag{4.3}$$

where $\{\phi_i^*\}$ is a solution to the equations of motion subject to the boundary conditions $\lim_{r\to\infty} \phi_i^*(x, r) = h_i(x)$.

4.1.1 A CFT expectation value in AdS gravity

As a simple illustration of the GKPW rule in action we will calculate the expectation value of a local operator in a CFT from the dual AdS gravity description. In particular, we consider a real scalar field in an AdS background spacetime. The gravitational bulk theory action is taken to be

$$S = \int_{\mathrm{AdS}} d^{d+2}x \sqrt{-g} \left(-\frac{1}{2} g^{MN} \partial_M \phi \, \partial_N \phi - \frac{1}{2} m^2 \phi^2 \right) \,, \tag{4.4}$$

where g is the determinant of the AdS metric g_{MN}. Capital latin letters will be used for indices ranging over bulk coordinates whereas indices with greek letters range only over the 'flat' coordinates shared by the bulk spacetime and the boundary QFT. Integrating by parts, the action can be written as a sum of a bulk term and boundary term as follows:

$$S = \frac{1}{2} \int_{\mathrm{AdS}} d^{d+2}x \sqrt{-g} \, \phi \left(\nabla^2 - m^2 \right) \phi - \frac{1}{2} \oint_{\partial\mathrm{AdS}} d^{d+1}x \sqrt{-\gamma} \, \phi \, \partial_n \phi \,. \tag{4.5}$$

Here γ_{MN} is the induced metric on the boundary, defined as $\gamma_{MN} = g_{MN} - n_M n_N$ with n_M an outwards directed unit normal vector to the boundary. The normal derivative ∂_n is defined as $\partial_n = n_M g^{MN} \partial_N$. The following formula for the generalization of the Laplacian to curved spacetimes has also been used:

$$\nabla^2 \phi = \frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g} g^{MN} \partial_N \phi \right) . \tag{4.6}$$

We will use Poincaré coordinates where the AdS metric takes the form

$$ds^2 = g_{MN} dx^M dx^N = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 . \tag{4.7}$$

Here L is the AdS radius, $\eta_{\mu\nu}$ is the Minkowski metric, and r is the extra radial coordinate of the bulk, orthogonal to the coordinates x^μ shared by the boundary and the bulk. In these coordinates the 'horizon' of infinite redshift is located at $r = 0$ and the conformal boundary at $r = \infty$. It then follows that

$$\sqrt{-g} = \left(\frac{r}{L} \right)^d , \qquad n_M = \frac{L}{r} \delta_{Mr} , \qquad \partial_n = \frac{r}{L} \partial_r , \qquad \gamma_{\mu\nu} = \frac{r^2}{L^2} \eta_{\mu\nu} , \tag{4.8}$$

where δ_{Mr} is the Kronecker delta.

To employ the GKPW formula we want to evaluate the partition function of the gravitational theory semi-classically, as in (4.3). To this end, we need to solve the equation of motion for the real scalar field in the bulk. In fact, it suffices to find the asymptotic behavior of the solution as $r \to \infty$ since the leading near boundary part of the solution is what we will interpret as the source of the dual operator in the QFT.

The boundary term of the action (4.5) does not contribute to the equation of motion for the real scalar field, which is directly seen to be a Klein-Gordon equation,

$$\left(\nabla^2 - m^2 \right) \phi = 0 . \tag{4.9}$$

However, the boundary term will turn out to be useful later for dealing with a divergent term. To solve this wave equation we decompose the field into plane waves in the flat spacetime coordinates shared by the bulk and boundary,

$$\phi(r, t, x) = \varphi(r) e^{-i\omega t + ik\cdot x} . \tag{4.10}$$

Substituting this plane wave decomposition in (4.9), using (4.6) for the Laplacian as well as our choice of coordinates (4.7), we find

$$\begin{aligned}
(\nabla^2 - m^2)\phi &= \frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g} g^{MN} \partial_N \phi \right) - m^2 \phi \\
&= \left(\frac{L}{r} \right)^d \left[\partial_r \left(\left(\frac{r}{L} \right)^d \frac{r^2}{L^2} \partial_r \phi \right) - \partial_t \left(\left(\frac{r}{L} \right)^d \frac{L^2}{r^2} \partial_t \phi \right) + \partial_{x_i} \left(\left(\frac{r}{L} \right)^d \frac{L^2}{r^2} \partial_{x_i} \phi \right) \right] - m^2 \phi \\
&= \left(\frac{L}{r} \right)^d \left[\left(\frac{r}{L} \right)^{d+2} \partial_r^2 \phi + \frac{d+2}{L} \left(\frac{r}{L} \right)^{d+1} \partial_r \phi + \left(\frac{r}{L} \right)^{d-2} \left(-\partial_t^2 + \partial_{x_i}^2 \right) \phi \right] - m^2 \phi \\
&= \left(\frac{r}{L} \right)^2 \left[\varphi''(r) + \frac{d+2}{r} \varphi'(r) + \left(\frac{(\omega^2 - k^2) L^4}{r^4} - \frac{m^2 L^2}{r^2} \right) \varphi(r) \right] e^{-i\omega t + ik\cdot x} = 0 .
\end{aligned} \tag{4.11}$$

Thus, the wave equation has been reduced to an ordinary second order differential equation for the radial dependence of the scalar field,

$$\varphi''(r) + \frac{d+2}{r}\varphi'(r) + \left(\frac{(\omega^2 - k^2)\,L^4}{r^4} - \frac{m^2L^2}{r^2}\right)\varphi(r) = 0\,. \tag{4.12}$$

To find the asymptotic behavior of $\varphi(r)$ near the conformal boundary $r \to \infty$, we assume a power dependence: $\varphi(r) = \varphi_0 r^\alpha$, where φ_0 is a constant. Substituting this ansats into (4.12) results in

$$\alpha(\alpha-1)r^{\alpha-2} + \alpha\frac{d+2}{r}r^{\alpha-1} + \left(\frac{(\omega^2 - k^2)\,L^4}{r^4} - \frac{m^2L^2}{r^2}\right)r^\alpha = 0\,, \tag{4.13}$$

and in the limit $r \to \infty$ this reduces to a quadratic algebraic equation for α,

$$\alpha^2 + (d+1)\alpha - m^2L^2 = 0\,. \tag{4.14}$$

Solving this equation for α yields

$$\alpha_\pm = -\frac{d+1}{2} \pm \sqrt{\frac{(d+1)^2}{4} + m^2L^2}\,. \tag{4.15}$$

Since the scalar field is supposed to be real, the power of its r dependence better has to be real as well. This means that the expression in the square root is not allowed to be negative, resulting in a constraint on the mass squared of the scalar field known as the Breitenlohner-Freedman (BF) bound [38],

$$m^2L^2 \geq -\frac{(d+1)^2}{4}\,. \tag{4.16}$$

The AdS space can remain stable in the presence of scalar fields with negative mass squared as long as the BF bound is satisfied. If the BF bound is not satisfied the scalar field acquires perturbatively unstable tachyonic modes. Relativistic field theories with perturbatively unstable fluctuations are non-unitary [2], [39]. We will not consider the non-unitary theories associated with a BF bound breaking mass in the UV. However, violations of the BF bound in the IR will turn out to give rise to a dual gravitational description of spontaneous symmetry breaking in the boundary QFT. In the gravitational language, the symmetry breaking is captured in a Higgs mechanism, and this can give rise to superconducting states. We will elaborate on this point in our treatment of holographic superconductivity in Chapter 7.

We define Δ as the larger root of the equation $\Delta^2 - (d+1)\Delta - m^2L^2 = 0$, i.e.

$$\Delta = -\alpha_- = \frac{d+1}{2} + \sqrt{\frac{(d+1)^2}{4} + m^2L^2}\,. \tag{4.17}$$

For now we simply quote the fact that Δ is the scaling dimension of the CFT operator dual to the bulk scalar field[1] [2]. The near boundary behaviour of the on-shell bulk scalar field in momentum space can then be written as

$$\begin{aligned}\phi^*(\omega,k,r) &= \left[A(\omega,k) + \mathcal{O}\!\left(r^{-1}\right)\right]\left(\frac{r}{L}\right)^{-d-1+\Delta} + \left[B(\omega,k) + \mathcal{O}\!\left(r^{-1}\right)\right]\left(\frac{r}{L}\right)^{-\Delta} \\ &= A(\omega,k)\left(\frac{r}{L}\right)^{-d-1+\Delta} + B(\omega,k)\left(\frac{r}{L}\right)^{-\Delta} + \cdots\,,\end{aligned} \tag{4.18}$$

[1] We present an argument for this statement in 4.2.2.

where we have extracted factors of the AdS radius L from the implicit definitions of $A(\omega, k)$ and $B(\omega, k)$ in order to express the radial power dependence in a dimensionless quantity. Here $A(\omega, k)$ is the leading part of ϕ close to the conformal boundary and $B(\omega, k)$ is the subleading part. (The asterisk is used to denote that the field configuration is a solution to the equation of motion, i.e. evaluated on-shell.) Transforming back to position space, the asymptotic solution takes the following form:

$$\phi^*(x,r) = A(x)\left(\frac{r}{L}\right)^{-d-1+\Delta} + B(x)\left(\frac{r}{L}\right)^{-\Delta} + \cdots . \tag{4.19}$$

The GKPW formula now instruct us to identify $\lim_{r\to\infty}\phi$ as the source of the dual CFT single-trace operator $\mathcal{O}(x)$. There is an immediate problem here, however, since $-d-1+\Delta > 0$ and consequently ϕ diverges when approaching the conformal boundary. As mentioned before, the radial dimension in the bulk translates to the renormalization group scale of the boundary QFT; dynamics near the boundary correspond to high energy physics of the dual QFT while the deep interior of the bulk encodes for the low energy physics of the QFT. This divergence should therefore be interpreted as a short distance, UV divergence of the CFT. In the gravitational description, however, this divergence is simply a consequence of integrating over the infinite volume of the bulk spacetime. A direct way to handle such divergences is to regulate the theory, and this can be done quite nicely in the gravitational description.

A straightforward way to regulate the theory, as proposed by GKPW, is to introduce a cutoff boundary at a distance $r = \epsilon^{-1}$ and modify the theory such that the limit $\epsilon \to 0$ is well defined. Evaluating the action (4.5) on-shell, the bulk term vanishes. In our choice of AdS coordinates (4.7), using (4.8), the regulated on-shell action equals

$$\begin{aligned} S[\phi^*] &= -\frac{1}{2}\oint_{r=\epsilon^{-1}} d^{d+1}x\sqrt{-\gamma}\,\phi^*(x,r)\,\partial_n\phi^*(x,r) \\ &= -\frac{1}{2}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d+2}\phi^*(x,r)\,\partial_r\phi^*(x,r)\,. \end{aligned} \tag{4.20}$$

Inserting our near boundary solution (4.19), we get

$$\begin{aligned} S[\phi^*] = &-\frac{1}{2}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d+2}\left[A(x)\left(\frac{r}{L}\right)^{-d-1+\Delta} + B(x)\left(\frac{r}{L}\right)^{-\Delta} + \cdots\right] \\ &\times\left[\frac{\Delta-d-1}{L}A(x)\left(\frac{r}{L}\right)^{-d-2+\Delta} - \frac{\Delta}{L}B(x)\left(\frac{r}{L}\right)^{-1-\Delta} + \cdots\right] \\ = &-\frac{1}{2L}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d+2}\left[(\Delta-d-1)A(x)A(x)\left(\frac{r}{L}\right)^{2\Delta-2d-3}\right. \\ &\left. - (d+1)A(x)B(x)\left(\frac{r}{L}\right)^{-d-2} + \cdots\right] \\ = &\ \frac{1}{2L}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d}\left[(d+1-\Delta)A(x)A(x)\left(\frac{r}{L}\right)^{2\Delta-2d-1}\right. \\ &\left. + (d+1)A(x)B(x)\left(\frac{r}{L}\right)^{-d} + \cdots\right]. \end{aligned} \tag{4.21}$$

The problematic UV divergence is contained in the first term since all other terms are regular at $\epsilon \to 0$. To remove this divergence we make use of the fact that equations of motion derived from an action are independent of any boundary term. Furthermore, if a boundary term is completely defined in terms of boundary data, then it does not affect the boundary conditions of the bulk fields. Thus, we can add a boundary counterterm to the action in order to cancel the UV divergent term, (analogous to how one would add counterterm diagrams in renormalized perturbation theory in the context of QFT). In particular, we add the following boundary counterterm to our original action (4.5):

$$\begin{aligned} S_{\mathrm{ct}}[\phi^*] &= -\frac{d+1-\Delta}{2L}\oint_{r=\epsilon^{-1}} d^{d+1}x\sqrt{-\gamma}\phi^{*2}(x,r) \\ &= -\frac{d+1-\Delta}{2L}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d+1}\left[A(x)\left(\frac{r}{L}\right)^{-d-1+\Delta}+B(x)\left(\frac{r}{L}\right)^{-\Delta}+\cdots\right] \\ &\qquad\qquad \times\left[A(x)\left(\frac{r}{L}\right)^{-d-1+\Delta}+B(x)\left(\frac{r}{L}\right)^{-\Delta}+\cdots\right] \\ &= -\frac{d+1-\Delta}{2L}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d}\left[A(x)A(x)\left(\frac{r}{L}\right)^{2\Delta-2d-1}\right. \\ &\qquad\qquad \left. +2A(x)B(x)\left(\frac{r}{L}\right)^{-d}+\cdots\right]. \end{aligned} \tag{4.22}$$

The complete regulated on-shell action then equals

$$S[\phi^*] = \frac{2\Delta-d-1}{2L}\oint_{r=\epsilon^{-1}} d^{d+1}x\left(\frac{r}{L}\right)^{d+1}\left[A(x)B(x)\left(\frac{r}{L}\right)^{-d-1}+\cdots\right], \tag{4.23}$$

and we can now take the limit $\epsilon \to 0$, restoring the infinite volume of our AdS bulk space,

$$S[\phi^*] = \frac{2\Delta-d-1}{2L}\oint_{\partial\mathrm{AdS}} d^{d+1}x\, A(x)B(x)\,. \tag{4.24}$$

Now, the generating functional for the scalar CFT operator $\mathcal{O}(x)$ is given by

$$Z_{\mathrm{CFT}}[h(x)] = \int D\Phi\, e^{iS[\Phi]+i\int d^{d+1}x\, h(x)\mathcal{O}(x)}\,, \tag{4.25}$$

where $h(x)$ is the source of $\mathcal{O}(x)$ and Φ denotes collectively the field theory degrees of freedom, (of course, in the path integral everything is strictly speaking classical functions). The expectation value of the operator $\mathcal{O}(x)$ in the presence of the source $h(x)$ is given by

$$\langle\mathcal{O}(x)\rangle_h = -i\frac{1}{Z_{\mathrm{CFT}}[h]}\frac{\delta}{\delta h(x)}Z_{\mathrm{CFT}}[h]\,. \tag{4.26}$$

The GKPW rule now tells us that the gravitational partition function should equal the CFT generating funtional, and that the leading near boundary behavior $A(x)$ of the bulk scalar field should be interpreted as the source $h(x)$ of the dual CFT operator $\mathcal{O}(x)$. Using the semi-classically evaluated gravitational partition function (4.3), and the expression for the on-shell action (4.24), we find

$$\langle\mathcal{O}(x)\rangle_A = \frac{\delta S[\phi^*]}{\delta A(x)} = \frac{2\Delta-d-1}{2L}\left(B(x)+\frac{\delta B(x)}{\delta A(x)}A(x)\right). \tag{4.27}$$

Now, since ϕ^* is a solution to a linear equation, $B(x) = (\delta B(x)/\delta A(y))A(y)$ holds as a matrix equation[2] [1]. It follows that

$$\langle \mathcal{O}(x) \rangle_h = \frac{2\Delta - d - 1}{L} B(x) \,. \tag{4.28}$$

Hence, as a consequence of identifying the leading near boundary behavior $A(x)$ of the bulk field as the source of the CFT operator, the subleading behavior $B(x)$ is proportional to its excpectation value (or in the language of linear response theory, its response). This result is quite general; if one identifies the leading near boundary behavior of a bulk field as the source of the dual QFT operator, then the subleading behavior should be identified as its expectation value [1]. The source could for instance be the chemical potential and the expectation value the associated charge density.

4.1.2 A CFT two-point function in AdS gravity

Thus far, our analysis has been performed in position space. However, for the propose of e.g. studying bulk fluctuation modes in linear response theory, working with the momentum space representation is much more convenient. It is then useful to have a formula for the two-point correlation function in momentum space. Here we will continue the analysis of the previous section and derive an expression for the momentum space two-point function of the scalar CFT operator, in terms of the leading and subleading near boundary behaviours of the corresponding bulk field.

In the previous section, we showed that the regulated on-shell bulk action for our theory of a real scalar field in AdS space equals

$$S[\phi^*] = \frac{2\Delta - d - 1}{L} \oint_{\partial \mathrm{AdS}} d^{d+1}x \, \frac{1}{2} A(x) B(x) \,. \tag{4.29}$$

Writing A and B in their Fourier integral representations, we get an expression for the on-shell action in momentum space,

$$\begin{aligned} S[\phi^*] &= \frac{2\Delta - d - 1}{L} \oint_{\partial \mathrm{AdS}} d^{d+1}x \, \frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \, e^{ik\cdot x} A(k) \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \, e^{ip\cdot x} B(p) \\ &= \frac{2\Delta - d - 1}{2L} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \, A(k) \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \, B(p) \oint_{\partial \mathrm{AdS}} d^{d+1}x \, e^{i(k+p)\cdot x} \\ &= \frac{2\Delta - d - 1}{2L} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \, A(k) \int d^{d+1}p \, B(p) \delta(k+p) \\ &= \frac{2\Delta - d - 1}{L} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \, \frac{1}{2} A(k) B(-k) \,, \end{aligned} \tag{4.30}$$

where we have used the following standard identity of the Dirac delta function:

$$\delta(k) = \int \frac{d^d x}{(2\pi)^d} \, e^{ik\cdot x} \,. \tag{4.31}$$

Furthermore, in momentum space, the expression for the expectation value of the CFT operator $\mathcal{O}(k)$, in the presence of a source $h(k)$, is given by

$$\langle \mathcal{O}(k) \rangle_h = \frac{2\Delta - d - 1}{L} B(k) \,. \tag{4.32}$$

[2] In the sense that the variables x and y are viewed as indices, $B_x = (\delta B/\delta A)_{xy} A_y$.

Now, the semi-classical bulk theory partition function is given by $Z = \exp(iS[\phi^*])$, and the GKPW prescription equates it to the generating functional of the dual CFT. The leading near boundary behaviour $A(k)$ of the bulk scalar field should then be identified as the source $h(k)$ of the CFT operator. The two-point function of the operator $\mathcal{O}(k)$ can then be calculated in the bulk theory by taking two derivatives of the semi-classical partition function with respect to A, and setting $A = 0$ in the end. Using the expression (4.32) for the expectation value, and the fact that A and B are linearly proportional, we get

$$\begin{aligned} \langle \mathcal{O}(-k)\mathcal{O}(k)\rangle &= -i(2\pi)^{d+1}\frac{1}{Z}\frac{\delta^2 Z}{\delta A(k)\delta A(-k)}\Big|_{A=0} \\ &= \frac{1}{Z}\frac{\delta}{\delta A(k)}\left(\frac{2\Delta-d-1}{L}B(k)Z\right)\Big|_{A=0} \\ &= \frac{2\Delta-d-1}{L}\frac{\delta B(k)}{\delta A(k)} \\ &= \frac{2\Delta-d-1}{L}\frac{B(k)}{A(k)}\,. \end{aligned} \tag{4.33}$$

In other words, the two-point function of the scalar operator is given by the ratio of the subleading to the leading near boundary behavior of the corresponding bulk field[3]. This result turns out to be quite generic, valid also for other holographic models. For a careful derivation of this result directly from the GKPW formula, bypassing the linear response assumption that A and B are linearily proportional, we refer the reader to [2].

The retarded Green's function describes the causal linear response of the system to a small perturbation of the sources. In particular, consider small changes in a set of sources δh_i resulting in small changes in the corresponding responses $\delta\langle\mathcal{O}_i\rangle$. In momentum space, the retarded Green's function is then defined as

$$\delta\langle\mathcal{O}_i(k)\rangle = G^R_{\mathcal{O}_i\mathcal{O}_j}(k)\delta h_j(k)\,. \tag{4.34}$$

For the case at hand with a single scalar operator $\mathcal{O}(k)$ sourced by $h(k)$, the retarded Green's function is given by

$$G^R_{\mathcal{O}\mathcal{O}}(k) = \frac{\delta\langle\mathcal{O}(k)\rangle}{\delta h(k)}\,. \tag{4.35}$$

Now, the expectation value $\langle\mathcal{O}\rangle$ is given by (4.32), and with $h = A$ it follows that

$$G^R_{\mathcal{O}\mathcal{O}}(k) = \frac{2\Delta-d-1}{L}\frac{B(k)}{A(k)}\,. \tag{4.36}$$

We have thus found that the two-point function of the operator $\mathcal{O}$ equals its retarded Green's function, and it is proportional to the ratio of the subleading to leading near boundary behaviour of the dual bulk field.

Before moving on to the holographic renormalization group, some final remarks are in order. A solution to the bulk equations of motion is uniquely determined by a set of Dirichlet boundary conditions[4] at the conformal boundary and regularity conditions

[3] In position space, the two-point function formula reads $\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{2\Delta-d-1}{L}\frac{B(x_2)}{A(x_1)}$

[4] It is also possible to impose more general mixed boundary conditions corresponding to multi-trace interactions in the field theory, which are suppressed by large N.

at the zero (or finite) size horizon in the deep interior. More precisely, the regularity conditions are imposed on the Euclidean bulk fields obtained by performing a Wick rotation to imaginary time. When translated to real time, the retarded Green's function is obtained by analytically continuing the Euclidean Green's function from the upper half frequency plane. Under this analytic continuation the regularity conditions in the Euclidean theory translates into 'infalling' boundary conditions in real time, i.e. modes in the bulk must be falling into the horizon, as opposed to emerging out from it. If one instead want to consider the advanced Green's function, then one should impose outgoing boundary conditions on the bulk fields. However, since we only are interested in the retarded Green's function due to its causal structure, infalling boundary conditions should be imposed on the bulk fields at the horizon.

It is possible to evaluate the solution to the Klein-Gordon equation in the bulk explicitly using Green's functions or Bessel functions [2], [36], [37]. In particular, the leading and subleading near boundary behaviors of the real scalar field in the bulk have a fixed power law dependence in momentum space. The two-point function is then found to be proportional to $|k|^{2\Delta-d-1}$, which is exactly the scaling behavior expected for the two-point function of a scalar operator with scaling dimension Δ in a CFT. This is one of the many empirical evidences for the validity of the AdS/CFT correspondance.

Finally, three- or higher-point correlation functions are most conveniently dealt with using 'Witten diagrams'. Three-point functions in AdS/CFT was calculated in e.g. [40]. The formalism of Witten diagrams is beyond the scope of this thesis and will not be needed for our purposes. However, the interested reader is referred to [2], [37].

4.2 Holograpic renormalization theory

As we have stressed before, one of the most essential features of the holographic duality is the notion that the extra radial dimension in the bulk captures the renormalization physics of the boundary QFT. It allows us to integrate out the specific details of the short distance, UV physics and isolate the universal long distance, IR dynamics relevant for condensed matter systems. Beta functions and running couplings can be calculated directly from the gravitational theory. Finally, the structure of UV divergences can be determined and systematically removed. Here we only cover a few selected topics on holographic renormalization and refer the interested reader to [41] for additional material on the subject.

4.2.1 The holographic Wilsonian renormalization group

In holographic condensed matter physics or 'AdS/CMT', we are primarily interested in the emergent low energy phenomena of strongly interacting quantum systems with a huge number of microscopic constituents. The high energy physics of the underlying holographic theory is therefor rather unimportant. Indeed, we have mentioned that one of the advantages with the top-down approach to holography is that it guarantees quantum consistancy and UV completion. In the context of 'AdS/CMT', having a UV complete theory is of little practical importance and something which is only conceptually desired. This is good news for bottom-up holography since a bottom-up model only has to be the effective low energy description of a consistently truncated string theory, severely reducing the demands on the top-down implementation.

Returning to our example of a massive real scalar field in a pure AdS background, we show how one can take a Wilsonian perspective on renormalization and integrate out the high energy, short distance dynamics. In this way we can isolate the universal emergent low energy physics associated with an IR fixed point. This will allow us to relate the second order bulk equation of motion (4.9) to a first order renormalization group flow equation. We closely follow [1], which built on work from [42]–[44].

A QFT defined below some UV cutoff Λ is completely specified by its partition function,

$$Z^{\Lambda}_{\mathrm{QFT}} = \int_{\Lambda} D\Phi \, e^{iI_{\mathrm{eff.}}[\Phi]} \,. \tag{4.37}$$

Here Φ collectively denotes all the field theory degrees of freedom and the path integral is over all field configurations at energy scales below the cutoff Λ. The effective action $I_{\mathrm{eff.}}[\Phi] = I_0[\Phi] + I_{\mathrm{UV}}[\Phi]$ at the cutoff scale Λ is the sum of the original microscopic action $I_0[\Phi]$ and terms $I_{\mathrm{UV}}[\Phi]$ arising when integrating out the degrees of freedom at energy scales above Λ from the UV complete path integral[5],

$$Z_{\mathrm{QFT}} = \int D\Phi \, e^{iI_0[\phi]} \,. \tag{4.38}$$

Requiring the partition function to be independent of the cutoff scale gives rise to renormalization group equations for I_{UV}. In principle, by taking the cutoff scale to low energies we obtain the emergent low energy physics.

In the gravitational bulk, we consider again a cutoff to the volume of AdS. However, we switch from the previously used radial coordinate r to the 'inverse' radial coordinate $z = L^2/r$ defined in (3.18). The AdS metric is then given by (3.19) and the conformal boundary is located at $z = 0$ and the 'horizon' at $z = \infty$. The cutoff surface is placed at $z = \epsilon$. The bulk fields $\{\phi_i(x)\}$ now take on boundary values $\{\phi_i^\epsilon\}$ at this cutoff rather than at the conformal boundary of AdS. The resulting truncated gravitational partition function is then given by

$$Z^{\epsilon}[\phi_i^\epsilon] = \int_{\phi_i \to \phi_i^\epsilon} D\phi_i \, e^{iS[\phi_i]} \,. \tag{4.39}$$

The gravitational action $S[\phi_i]$ is of the form

$$S = \int_{z>\epsilon} d^{d+2}x \sqrt{-g}\mathcal{L}(\phi_i, \partial_M \phi_i) + S_{\mathrm{B}}[\phi_i, \epsilon] \,, \tag{4.40}$$

where $\mathcal{L}$ is the Lagrangian of the bulk theory and S_{B} denotes collectively all the boundary action terms defined at the cutoff surface $z = \epsilon$. The boundary action S_{B} can be interpreted as the terms arising from integrating out bulk field degrees of freedom in the region $z < \epsilon$ [43].

When regulating the bulk and boundary theories with a volume and UV cutoff, respectively, it is natural to generalize the GKPW formula (4.2) to

$$Z^{\epsilon}[\phi_i^\epsilon] = \int_{\Lambda} D\Phi \, e^{iI_{\mathrm{eff.}}[\Phi]} = \int_{\Lambda} D\Phi \, e^{iI_0[\Phi] + i\int d^{d+1}x \, \phi_i^\epsilon(x) \mathcal{O}_i(x)} \,. \tag{4.41}$$

In other words, the values of the bulk fields at the cutoff boundary $z = \epsilon$ acts as sources for local single-trace gauge invariant operators in an effective QFT valid at scales below the cutoff Λ. Furthermore, we will show that the boundary action S_{B} is related to the effective action I_{UV} by a Legandre transformation, (near a fixed point and up to some

[5] Although we will often simply refer to $I_{\mathrm{UV}}[\Phi]$ as the effective field theory action.

renormalization [43]). In fact, the gravitational boundary action will be interpreted as dual to the effective field theory action I_{UV}. However, there is a significant caveat to the generalized GKPW rule (4.41) as well as the identification of S_{B} with I_{UV} proposed above. The precise relation between the field theory cutoff Λ and the gravitational cutoff ϵ is in general not clear. Thus, the identification of integrating out high energy degrees of freedom above the cutoff Λ in the QFT with integrating out bulk degrees of freedom at $z < \epsilon$ cannot be made precise. In fact, determining the precise relation between Λ and ϵ would likely amount to a proof of holography [1].

In the following we restrict ourselves to the case of a single bulk scalar field for notational clarity. The analogous results for the general case of a set of bulk scalar fields are easily retrieved by adding a bunch of subscripts i on the fields, their boundary values and their dual operators. Now, the full bulk partition function (4.1) is related to the truncated partition function (4.39) by,

$$Z[h] = \int D\phi^\epsilon \, Z^\epsilon[\phi^\epsilon] Z^\epsilon_{\mathrm{UV}}[\phi^\epsilon, h] \,, \tag{4.42}$$

where

$$Z_{\mathrm{UV}}[\phi^\epsilon, h] = \int_{\phi\to\phi^\epsilon}^{\phi\to h} D\phi \, e^{iS[\phi]} \,, \tag{4.43}$$

is the bulk partition function integrated over the modes at $0 < z < \epsilon$ which are cut out of the truncated partition function, and with boundary conditions at each end. Invoking the generalized GKPW formula, we then have

$$Z[h] = \int D\phi^\epsilon \int_\Lambda D\Phi \, e^{iI_0[\Phi] + i\int d^{d+1}x\, \phi^\epsilon(x)\mathcal{O}(x)} Z_{\mathrm{UV}}[\phi^\epsilon, h] \,, \tag{4.44}$$

which is equated to the full quantum partition function (4.37), (note that (4.38) and (4.37) are equal since we require that the QFT partition function be independent of the cutoff scale.) Changing order of functional integrations, it follows that

$$\int_\Lambda D\Phi \, e^{iI_0[\Phi] + iI_{\mathrm{UV}}[\mathcal{O}]} = \int_\Lambda D\Phi \, e^{iI_0[\Phi]} \int D\phi^\epsilon \, e^{i\int d^{d+1}x\, \phi^\epsilon(x)\mathcal{O}(x)} Z_{\mathrm{UV}}[\phi^\epsilon, h] \,, \tag{4.45}$$

which gives us an expression for the effective field theory action $I_{\mathrm{UV}}[\mathcal{O}]$ for the single-trace operator $\mathcal{O}$ (built from the field theory degrees of freedom Φ) in terms of bulk quantities,

$$e^{iI_{\mathrm{UV}}[\mathcal{O}]} = \int D\phi^\epsilon \, e^{i\int d^{d+1}x\, \phi^\epsilon(x)\mathcal{O}(x)} Z_{\mathrm{UV}}[\phi^\epsilon, h] \,. \tag{4.46}$$

Taking the semi-classical limit yields the Legandre transform relation between the effective field theory action I_{UV} and the gravitational boundary action S_{B},

$$I_{\mathrm{UV}}[\mathcal{O}] = \int d^{d+1}x \, \phi^\epsilon_*(x)\mathcal{O}(x) + S_{\mathrm{B}}[\phi^\epsilon_*, \epsilon] \,, \tag{4.47}$$

where the vanishing of the first derivative at the saddle point implies that

$$\mathcal{O}(x) = -\frac{\delta S_{\mathrm{B}}[\phi^\epsilon_*, \epsilon]}{\delta \phi^\epsilon_*(x)} \,. \tag{4.48}$$

Furthermore, it directly follows from (4.47) that

$$\phi^\epsilon_*(x) = \frac{\delta I_{\mathrm{UV}}[\mathcal{O}]}{\delta \mathcal{O}(x)} \,. \tag{4.49}$$

Note that $\mathcal{O}$ as given by (4.48) is the canonical momentum along the radial direction conjugate to ϕ_*^ϵ, up to a sign.

Now, consider the previous bulk scalar field Lagrangian (4.4) generalized to an arbitrary potential,

$$\mathcal{L} = -\frac{1}{2} g^{MN} \partial_M \phi \, \partial_N \phi - V(\phi) \,, \tag{4.50}$$

in a static background spacetime with a metric of the form

$$ds^2 = g_{MN} dx^M dx^N = \gamma_{\mu\nu}(z) dx^\mu dx^\nu + g_{zz} dz^2 \,. \tag{4.51}$$

The action in the presence of a cutoff boundary has the form of a Lagrangian term plus a boundary term,

$$S = \int_{z>\epsilon} d^{d+2}x \sqrt{-g} \mathcal{L}(\phi, \partial_M \phi_i) + S_{\mathrm{B}}[\phi, \epsilon] \,. \tag{4.52}$$

Varying this action with respect to ϕ gives,

$$\begin{aligned} \delta S &= \int_{z>\epsilon} d^{d+2}x \sqrt{-g} \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_M \phi)} \delta(\partial_M \phi) \right) + \delta S_{\mathrm{B}}[\phi, \epsilon] \\ &= \int_{z>\epsilon} d^{d+2}x \sqrt{-g} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial(\partial_M \phi)} \right) \right) \delta\phi \\ &\quad + \int_{z=\epsilon} d^{d+1}x \sqrt{-\gamma} \, n_M \frac{\partial \mathcal{L}}{\partial(\partial_M \phi)} \delta\phi + \int_{z=\epsilon} d^{d+1}x \, \frac{\delta S_{\mathrm{B}}[\phi, \epsilon]}{\delta \phi} \delta\phi \,. \end{aligned} \tag{4.53}$$

Here we have integrated by parts in the last equality and n_M is an outwards directed unit normal vector to the cutoff boundary. Specifically, in our coordinates $n_M = -\sqrt{g_{zz}} \delta_{Mz}$. Requiring that the variation of the action vanish for any $\delta\phi$ which is zero at $z = \epsilon$, the Euler-Lagrange equations of motion follows from the 'bulk' term,

$$\frac{1}{\sqrt{-g}} \partial_M \left(\sqrt{-g} g^{MN} \partial_N \phi \right) - \frac{\partial V}{\partial \phi} = 0 \,. \tag{4.54}$$

Furthermore, we also get the following boundary condition at $z = \epsilon$:

$$\frac{\delta S_{\mathrm{B}}}{\delta \phi} = -\sqrt{-\gamma} n_z \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} = -\sqrt{-g} g^{zz} \partial_z \phi \equiv \Pi_z \,, \tag{4.55}$$

where we have used that $\gamma g_{zz} = g$ and in the last equality Π_z is precisely the conjugate momenta to ϕ under radial evolution, defined as

$$\Pi_z = \frac{\delta S}{\delta(\partial_z \phi)} \,. \tag{4.56}$$

In the large N limit the bulk partition function is evaluated at a saddle point as in (4.3). Thus, we are interested in evaluating the action (4.52) on a solution to (4.54) with boundary condition (4.55). Our choice of placing the cutoff boundary at $z = \epsilon$ is arbitrary, and hence the value of the on-shell action, as well as the solution itself, should be independent of ϵ. Requiring that the derivative of the on-shell action with respect to

ϵ vanish, we get the following flow equation:

$$\begin{aligned}
0 = \frac{d}{d\epsilon} S &= \frac{d}{d\epsilon} \int_{z>\epsilon} d^{d+2}x \sqrt{-g}\mathcal{L} + \partial_\epsilon S_{\rm B}[\phi, \epsilon] + \int_{z=\epsilon} d^{d+1}x \frac{\delta S_{\rm B}[\phi, \epsilon]}{\delta \phi(x, z)} \partial_z \phi(x, z) \\
&= \frac{d}{d\epsilon} \int_\epsilon^\infty dz \int d^{d+1}x \sqrt{-g}\mathcal{L} + \partial_\epsilon S_{\rm B}[\phi, \epsilon] + \int_{z=\epsilon} d^{d+1}x \, \Pi_z \, \partial_z \phi \\
&= -\int_{z=\epsilon} \int d^{d+1}x \sqrt{-g}\mathcal{L} + \partial_\epsilon S_{\rm B}[\phi, \epsilon] + \int_{z=\epsilon} d^{d+1}x \, \Pi_z \, \partial_z \phi \\
&= \partial_\epsilon S_{\rm B}[\phi, \epsilon] + \int_{z=\epsilon} d^{d+1}x \left(\Pi_z \, \partial_z \phi - \sqrt{-g}\mathcal{L} \right) ,
\end{aligned} \tag{4.57}$$

where we have used (4.55). The last integrand is exactly the Hamiltonian density $\mathcal{H}$ for evolution in the radial direction [43]. Thus, the Hamiltonian generates the flow of $S_{\rm B}$,

$$\partial_\epsilon S_{\rm B}[\phi, \epsilon] = -\int_{z=\epsilon} d^{d+1}x \, \mathcal{H} \, . \tag{4.58}$$

Note that this flow equation should be treated as a functional equation and one should not impose the equation of motion (4.54) when evaluating it. We can write this equation in a more explicit form using (4.50) and (4.55),

$$\begin{aligned}
\sqrt{g^{zz}} \partial_\epsilon S_{\rm B}[\phi, \epsilon] &= -\int_{z=\epsilon} d^{d+1}x \sqrt{g^{zz}} \left(\Pi_z \, \partial_z \phi - \sqrt{-g}\mathcal{L} \right) \\
&= -\int_{z=\epsilon} d^{d+1}x \sqrt{-g} \sqrt{g^{zz}} \left(-g^{zz} (\partial_z \phi)^2 + \frac{1}{2} g^{MN} \partial_M \phi \, \partial_N \phi + V(\phi) \right) \\
&= -\int_{z=\epsilon} d^{d+1}x \sqrt{-\gamma} \left(\frac{1}{2\gamma} \left(\frac{\delta S_{\rm B}}{\delta \phi} \right)^2 + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + V(\phi) \right) ,
\end{aligned} \tag{4.59}$$

where the greek indices run over the boundary coordinates only. The gravitational boundary action $S_{\rm B}$ can now be expanded in powers of ϕ as follows:

$$S_{\rm B}[\phi,\epsilon] = \sum_n \int d^{d+1}x \sqrt{-\gamma} \alpha_n(x, \epsilon) \phi^n(x, \epsilon) \, . \tag{4.60}$$

Inserting this expansion into the above flow equation specifies how the expansion coefficients α_n flow with the volume cutoff ϵ.

We can now derive the corresponding flow equation for the effective field theory action $I_{\rm UV}[\mathcal{O}]$, using the previously derived Legendre transform relation (4.47). Taking the derivative with respect to ϵ of the left hand side of (4.47), we get

$$\frac{d}{d\epsilon} I_{\rm UV}[\mathcal{O}] = \partial_\epsilon I_{\rm UV}[\mathcal{O}] + \int d^{d+1}x \frac{\delta I_{\rm UV}[\mathcal{O}]}{\delta \mathcal{O}(x)} \partial_\epsilon \mathcal{O}(x) \, . \tag{4.61}$$

The same derivative applied to the right hand side of (4.47) yields,

$$\int d^{d+1}x \left(\mathcal{O}(x) \partial_\epsilon \phi_*^\epsilon(x) + \phi_*^\epsilon(x) \partial_\epsilon \mathcal{O}(x) \right) + \partial_\epsilon S_{\rm B}[\phi_*^\epsilon, \epsilon] + \int d^{d+1}x \frac{\delta S_{\rm B}[\phi_*^\epsilon, \epsilon]}{\delta \phi_*^\epsilon(x)} \partial_\epsilon \phi_*^\epsilon \, . \tag{4.62}$$

Now, using (4.48) and (4.49) when equating (4.61) with (4.62), we get a quite simple result,

$$\partial_\epsilon I_{\rm UV}[\mathcal{O}] = \partial_\epsilon S_{\rm B}[\phi_*^\epsilon, \epsilon] \, . \tag{4.63}$$

Thus, the effective field theory action $I_{\rm UV}[\mathcal{O}]$ satisfy the same first order flow equation (4.59) as the gravitational on-shell boundary action $S_{\rm B}[\phi_*^\epsilon, \epsilon]$. Using (4.48), we can then write the flow equation for $I_{\rm UV}$ as follows:

$$\sqrt{g^{zz}}\partial_\epsilon I_{\rm UV}[\mathcal{O}] = \int d^{d+1}x\sqrt{-\gamma}\left(\frac{1}{2\gamma}\mathcal{O}^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_*^\epsilon\,\partial_\nu\phi_*^\epsilon + V(\phi_*^\epsilon)\right)\,, \tag{4.64}$$

where ϕ_*^ϵ is translated into field theoretical quantities by (4.49).

The effective field theory action $I_{\rm UV}[\mathcal{O}]$ can now be expanded in powers of the single-trace operator $\mathcal{O}$,

$$I_{\rm UV}[\mathcal{O}] = \sum_n \int d^{d+1}x\sqrt{-\gamma}\lambda_n(x,\Lambda)\mathcal{O}^n(x)\,. \tag{4.65}$$

The expansion coefficients λ_n are the coupling constants of the effective field theory. When inserted into the functional flow equation (4.64), we can find the beta functions for all couplings λ_n [1]. In conclusion, by varying the bulk volume cutoff ϵ, we have found first order renormalization group flow equations for running couplings in a QFT with some UV cutoff Λ. This shows that the interpretation of the bulk radial coordinate as being dual to the field theoretical RG parameter is certainly justified, although the exact relation between the cutoffs Λ and ϵ is in general not known.

4.2.2 Renormalization group flows and alternative quantization

An important aspect of the renormalization group in holography is that it allows us to depart from the conformal field theories in AdS/CFT dualities. There are many interesting statistical and condensed matter systems which are well described by conformal field theories [45], [46], in particular systems at their thermodynamic or quantum[6] critical points where a phase transition occurs. However, a lot of condensed matter systems of interest does not exhibit invariance under the conformal transformations. In particular, the superconductor toy model we will study in Chapter 7 exhibits both a nonzero temperature and chemical potential. This introduces two scales to the system which will break the conformal invariance. The renormalization group allows us to flow from the UV fixed point theories characterized by conformal invariance to IR fixed points describing novel emergent low energy physics. The interpretation of the radial direction as the RG scale enables us to follow the RG flow in the gravitational description; the change in geometry in the deep interior, departing from the asymptotic AdS near the conformal boundary, captures the emergent IR physics whilst reflecting the loss of conformal invariance.

The way in which we can induce such an RG flow is to turn on a relevant operator. Hence, we should identify which bulk fields are dual to relevant operators. Recall from 4.1.1 that the asymptotic behavior of a real scalar field in the bulk with action (4.4) is

$$\lim_{r\to\infty}\phi(x,r) = A(x)\left(\frac{r}{L}\right)^{-d-1+\Delta} + B(x)\left(\frac{r}{L}\right)^{-\Delta}\,, \tag{4.66}$$

where Δ was defined as the larger root of the equation $\Delta^2 - (d+1)\Delta - m^2L^2 = 0$,

$$\Delta = \frac{d+1}{2} + \sqrt{\frac{(d+1)^2}{4} + m^2L^2}\,. \tag{4.67}$$

[6] A quantum critical point signifies a zero temperature quantum phase transition.

We quoted the fact that Δ is the scaling dimension of the dual CFT operator. Here we define Δ_+ as the larger root and Δ_- as the smaller root to the aforementioned equation,

$$\Delta_\pm = -\alpha_\mp = \frac{d+1}{2} \pm \sqrt{\frac{(d+1)^2}{4} + m^2 L^2}\,. \tag{4.68}$$

For most bulk masses we need to take $\Delta = \Delta_+$ in (4.66) in order for $A(x)$ to be interpreted as the leading near boundary behaviour or boundary value of the bulk field. However, in this section we will see how there is a range of bulk masses which allows for an 'alternative quantization' scheme where we take Δ_- as the solution in (4.66). Finally, note how $\Delta_+ = d + 1 - \Delta_-$.

As mentioned earlier, one of the crucial justifications of the AdS/CFT correspondance is the fact that AdS space has an $SO(d+1, 2)$ isometry group, which is precisely the conformal group in $d+1$ spacetime dimensions. The $d+2$ dimensional pure AdS geometry (4.7) is perturbed by adding sources $\{h_i\}$ for CFT operators at the boundary. The resulting perturbations must preserve the isometries of the AdS background and therefore transforms under the $SO(d+1, 2)$ symmetry group, suggesting that these perturbations are dually described by excitations of a $d+1$ dimensional CFT. In particular, the AdS metric (4.7) has a scaling symmetry,

$$(t, x, r) \to \left(\lambda t, \lambda x, \frac{r}{\lambda}\right)\,. \tag{4.69}$$

A source $h(x)$ for a single-trace operator $\mathcal{O}(x)$ are added to the field theoretical generating functional in the following way:

$$Z_{\text{CFT}} = \int D\Phi\, e^{iI_0 + i\int d^{d+1}x\, h(x)\mathcal{O}(x)}\,, \tag{4.70}$$

where I_0 is the microscopic action of the CFT. In the case of our real scalar field in a pure AdS geometry, the source $h(x)$ is equated to the leading near boundary behaviour $A(x)$ in (4.66). The scalar field, just like the metric field, must be invariant under the isometries of AdS. In particular, it must be invariant under the scaling transformation (4.69), i.e. $\phi(x, r) \to \phi(\lambda x, r/\lambda)$. By (4.66), it then follows that $h(x)$ (which is identified as $A(x)$) transforms as $h(x) \to \lambda^{-d-1+\Delta} h(\lambda x)$ under this scaling transformation. Furthermore, the integral measure scales as $d^{d+1}x \to \lambda^{d+1} d^{d+1}x$. Thus, from the generating functional (4.70) we see that the CFT operator $\mathcal{O}(x)$ must transform as

$$\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)\,, \tag{4.71}$$

under the scaling transformation (4.69). This is precisely the scaling of a CFT operator with conformal dimension Δ [1], [2]. Note how the scaling dimension of the CFT operator is set by the mass of the bulk field by (4.67). This is a central result relating the scaling dimension of a scalar operator in the CFT to the mass of the dual scalar field in the bulk.

Recall that relevant and irrelevant operators are defined as operators which magnitude always increase and always decrease, respectively, under RG transformations which takes the system from a higher energy scale to a lower one. A marginal operator is neither relevant nor irrelevant. The scalar CFT operator dual to the real scalar field in the bulk is relevant if the scaling dimension $\Delta < d + 1$ [2]. Furthermore, it is irrelevant for $\Delta > d + 1$ and marginal for $\Delta = d + 1$. From the definition of Δ in terms of the bulk field mass squared (4.67), it follows that massive scalar fields with $m^2 > 0$ are dual to irrelevant operators. A massless scalar field is dual to a marginal operator, (although the operator dimension will in general acquire corrections [2]). A relevant operator, necessary for inducing RG flows to new IR fixed points, therefore has to be dual to a scalar field with a negative mass squared, $m^2 < 0$. In a flat spacetime such scalar fields would describe tachyons and signal an instability of the theory. However, as we have already argued in 4.1.1, AdS space can remain stable for scalar fields with negative mass squared as long as the BF bound (4.16) is satisfied. In fact, the BF bound precisely guaranties that the scaling dimension Δ is real. Thus, a relevant scalar operator in the CFT is dual to a scalar field in the bulk with a mass squared in the range $-\frac{(d+1)^2}{4} \leq m^2L^2 < 0$. Similar results for the relevancy of operators are found for fields with spin [2].

It has been established in [47] that a necessary condition for a scalar CFT operator to be unitary requires its scaling dimension to be bounded from below by $\Delta \geq (d-1)/2$, in $d+1$ spacetime dimensions. However, the lowest possible value of the scaling dimension according to (4.67) is $\Delta = (d+1)/2$ and it is realized precisely when the mass squared of the bulk scalar field equals the BF bound. Somehow, we still have to account for unitary operators with scaling dimension in the range $(d-1)/2 \leq \Delta \leq (d+1)/2$. For these values of the scaling dimension both asymptotic modes in (4.66) fall off sufficiently quickly to be normalizable [1], [2]. Thus, either mode can be taken to be the source, and the other the response. For scaling dimensions in the range between the unitary bound and the BF bound, we may identify the leading near boundary behavior of the bulk field with the response and the subleading behavior with the source. Quantizing the bulk theory having made these prescriptions leads to what is known as *alternative quantization*. In contrast, quantizing the bulk theory with our usual prescription of identifying the leading mode with the source and the subleading mode with the response, is commonly referred to as *standard quantization*. The alternatively quantized and standard quantized bulk theories are dual to two different boundary CFTs [48]. In fact, the alternatively quantized bulk scalar field is perhaps best interpreted as dual to a scalar CFT operator with scaling dimension Δ_-. As noted above, $\Delta_+ = d+1-\Delta_-$. Hence, taking Δ_- instead of Δ_+ as the scaling dimension precisely exchanges the role of the leading and subleading modes in (4.66).

Scalar operators having scaling dimensions in the range $(d-1)/2 \leq \Delta \leq (d+1)/2$ can occur only in the alternative quantization scheme, and when the mass of the dual bulk field is in the range $-\frac{(d+1)^2}{4} \leq m^2L^2 \leq -\frac{(d+1)^2}{4} + 1$. In standard quantization this range of the bulk field mass corresponds to a scaling dimension $\frac{d+1}{2} \leq \Delta_+ \leq \frac{d+1}{2} + 1$. The two different CFTs resulting from the two different quantization schemes are related by an RG flow induced by a certain 'double-trace deformation'. Double-trace deformations and more general multi-trace deformations is the subject of the next section. Here we just state that the composite operator[7] $\mathcal{O}^2_{\Delta_-}(x)$ is relevant and induce an RG flow to a new IR fixed point. The RG flows from double-trace deformations in matrix large N CFTs does not affect the bulk geometry [2]. In fact, this particular composite operator flow affects only the scaling dimension Δ_- of the single-trace operator itself, and it flows precisely to Δ_+. Hence, the emergent IR theory is 'standardly' quantized.

[7]The subscript Δ_- denotes that the operator has scaling dimension Δ_-, arising from an alternatively quantized bulk theory.

4.2.3 Multi-trace deformations and generalized boundary conditions

As we have emphasized in Chapter 2, one of the essential features of the t'Hooft matrix large N limit is the factorization property of products of gauge invariant single-trace operators, i.e. multi-trace operators. To leading order in large N , the expectation value of a multi-trace operator factorize into a product of the expectation values of the single-trace operators constituting the multi-trace operator. Hence, in the large N limit multi-trace operators vanish from the spectrum of the theory. However, at large but finite N we can deform the theory by adding a multi-trace interaction term to the action. Here we will show that this kind of *multi-trace deformation* of the QFT holographically corresponds to using a more general set of mixed boundary conditions for the bulk fields [2], [20], [49]. We also show how a double-trace deformation results in a Dyson re-summation of the Green's function, which results in the RPA (random phase approximation) formula for the Green's function. This RPA formula is actually obtained in the strict large N limit.

Consider again a matrix large N QFT with field degrees of freedom $\Phi(x)$, a single-trace scalar operator $\mathcal{O}(x)$ constructed from $\Phi(x)$, and an action $I_0[\Phi]$. To compute the generating functional of the QFT,

$$Z_{\mathrm{QFT}}[h] = \int D\Phi \, e^{iI_0[\Phi]+i\int d^{d+1}x\, h(x)\mathcal{O}(x)} \,, \tag{4.72}$$

the GKPW rule instructs us to compute the AdS bulk partition function with the scalar field $\phi(x,r)$ dual to $\mathcal{O}(x)$ taking boundary value $h(x)$ at the conformal boundary. With $A(x)$ and $B(x)$ being the leading and subleading near boundary behavior of $\phi(x,r)$, respectively, as in (4.66), the boundary condition is $A(x) = h(x)$. The generating functional (4.72) equals the partition function of the QFT in the presence of a source term $W[\mathcal{O}]$ added to the action, where $W[\mathcal{O}] = \int d^{d+1}x\, h(x)\mathcal{O}(x)$. As we showed in 4.1.1, $B(x)$ (up to an overall constant) equals the expectation value of $\mathcal{O}(x)$. If we write $W[B] = \int d^{d+1}x\, h(x)B(x)$, then the boundary condition is

$$A(x) = \frac{\delta W[B]}{\delta B(x)} \,. \tag{4.73}$$

A multi-trace deformation of the theory is obtained by adding a perturbation $W[x,\mathcal{O},\partial\mathcal{O}]$ to the action, where $W[x,\mathcal{O},\partial\mathcal{O}]$ is a local non-linear functional of $\mathcal{O}(x)$ and its derivatives. The partition function is then

$$Z_{\mathrm{QFT}} = \int D\Phi \, e^{iI_0[\Phi]+iW[x,\mathcal{O},\partial\mathcal{O}]} \,. \tag{4.74}$$

The prescription for incorporating multi-trace deformations in the gravitational description is to substitute $\mathcal{O}(x)$ for $B(x)$ in W, and then impose the boundary condition (4.73) on the bulk scalar field at the conformal boundary [20].

It is straightforward to generalize this result to a set of bulk scalar fields $\phi_i(x,r)$ with masses m_i, dual to a set of single-trace scalar operators $\mathcal{O}_i(x)$ with scaling dimensions Δ_i. The near boundary behavior of the bulk scalar fields are given by

$$\lim_{r\to\infty} \phi_i(x,r) = A_i(x)\left(\frac{r}{L}\right)^{-d-1+\Delta_i} + B_i(x)\left(\frac{r}{L}\right)^{-\Delta_i} \,. \tag{4.75}$$

The bulk masses m_i are related to the corresponding scaling dimensions Δ_i by[8]

$$\Delta_i = \frac{d+1}{2} + \sqrt{\frac{(d+1)^2}{4} + m_i^2 L^2}\,. \tag{4.76}$$

The leading and subleading behaviors $A_i(x)$ and $B_i(x)$ are identified with the sources and expectation values of the operators $\mathcal{O}_i(x)$, respectively. A multi-trace deformation $W[x, \mathcal{O}_i, \partial\mathcal{O}_i]$ of the QFT is incorporated in the bulk theory by imposing the boundary conditions

$$A_i(x) = \frac{\delta W[x, B_i, \partial B_i]}{\delta B_i}\,, \tag{4.77}$$

on the bulk fields at the conformal boundary of AdS.

Returning to the case of a single scalar operator $\mathcal{O}$, we consider a double-trace deformation,

$$W[\mathcal{O}] = \int d^{d+1}x \left(h(x)\mathcal{O}(x) + \frac{f}{2}\mathcal{O}^2(x) \right)\,, \tag{4.78}$$

where h is the source of $\mathcal{O}$ and f is a coupling constant. The boundary condition one should impose on the bulk scalar field is then

$$A(x) = \frac{\delta W[B]}{\delta B(x)} = h(x) + fB(x)\,. \tag{4.79}$$

Now, in 4.1.2, we derived the following formula for the retarded momentum space Green's function of the operator $\mathcal{O}$:

$$G_0(k) = \langle \mathcal{O}(-k)\mathcal{O}(k) \rangle = \frac{2\Delta - d - 1}{L}\frac{B(k)}{A(k)}\,. \tag{4.80}$$

The subscript 0 has been added to emphasis that this refers to the Green's function of the undeformed theory. We will now derive an expression for the retarded Green's function of the double-trace deformed theory, which we will denote by G. Thus, we should continue our analysis in momentum space.

As shown in 4.1.2, the regulated on-shell bulk action for our theory of a real scalar field in AdS space is given by

$$S[\phi^*] = \frac{2\Delta - d - 1}{L} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{2} A(k)B(-k)\,. \tag{4.81}$$

We begin by absorbing the factor of $(2\Delta - d - 1)/L$ into a redefinition of the dynamical field B, i.e. we take $B \to L/(2\Delta - d - 1)B$. Then the overall factor of $(2\Delta - d - 1)/L$ is removed from the on-shell action, and the undeformed Green's function is simply given by $G_0 = B/A$. Inverting this relation, we can write $A = B/G_0$. Now, substituting this expression for A in the regulated on-shell action gives

$$S[\phi^*] = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{2} B(k) \frac{1}{G_0(k)} B(-k)\,. \tag{4.82}$$

Transforming to momentum space, the double-trace potential (4.78) reads

$$W[B] = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left(h(k)B(-k) + \frac{f}{2}B(k)B(-k) \right)\,, \tag{4.83}$$

[8] Assuming masses satisfying the BF-bound. For masses between the unitary bound and the BF-bound the possibility of alternative quantization discussed in the previous section applies here too.

and we can add this potential directly to the action,

$$S[\phi^*] = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left(\frac{1}{2} B(k) \frac{1}{G_0(k)} B(-k) + h(k)B(-k) + \frac{f}{2} B(k)B(-k) \right) . \tag{4.84}$$

Now, varying the action with respect to the dynamical field B, we get the following equation of motion:

$$\frac{1}{G_0(k)} B(k) + h(k) + fB(k) = 0 \, , \tag{4.85}$$

where we have used that $G_0(k) = G_0(-k)$, in accordance with the well established result $G_0(k) \propto |k|^{2\Delta - d - 1}$, see e.g. [49]. Solving this equation for B, we find

$$B(k) = -\frac{G_0(k)}{1 + fG_0(k)} h(k) \, , \tag{4.86}$$

and substituting this expression for B in the action gives

$$\begin{aligned} S[\phi^*] &= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left(\frac{1}{2} \frac{G_0}{(1+fG_0)^2} h^2 - \frac{G_0}{1+fG_0} h^2 + \frac{f}{2} \frac{G_0^2}{(1+fG_0)^2} h^2 \right) \\ &= -\frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{G_0}{1+fG_0} h^2 \, , \end{aligned} \tag{4.87}$$

where $h^2 = h(k)h(-k)$ and $G_0 = G_0(k)$. The two-point correlation function of $\mathcal{O}$ for the deformed theory can now be calculated by differentiating the generating functional twice with respect to the source h, in the usual way. Because the respective partition functions of the bulk and boundary theories are set equal by the GKPW prescription, the two-point function can be calculated in the gravitational theory by differentiating the semi-classical bulk partition function $Z = \exp(iS[\phi^*])$ twice with respect to h, and setting h to zero in the end,

$$\begin{aligned} G(k) = \langle \mathcal{O}(-k)\mathcal{O}(k) \rangle &= -i(2\pi)^{d+1} \frac{1}{Z} \frac{\delta^2 Z}{\delta h(-k) \delta h(k)} \bigg|_{h=0} \\ &= \frac{G_0(k)}{1+fG_0(k)} \equiv G_0(k) \sum_{n=0}^{\infty} \left(-fG_0(k)\right)^n \, . \end{aligned} \tag{4.88}$$

This is precisely the RPA form of the Green's function that follows from a Dyson resummation [2]. Since we evaluated the gravitational partition function semi-classically, this result is exact in the large N limit.

4.3 The global/local symmetry correspondence

So far we have seen how expectation values, Green's functions and RG flows in the strongly interacting field theory can be computed in the dual gravitational theory. However, an obvious question arises here; which is the field theoretical operator corresponding to a given bulk field? This is in general a hard question to answer, even in a top-down construction where the bulk and boundary theories are known explicitly. One could even argue that it is not the right question to ask. Indeed, the bulk theory is in its own right a complete description of a strongly coupled quantum field theory. The bulk fields and their interactions define the QFT operators, their spectrum and correlation functions.

Even if the QFT operator algebra and Lagrangian where known, one would not be able to use this to perform reliable calculations, employing e.g. perturbation theory, due to the inherent strong coupling. However, it is still desirable to know how to interpret the quantities of the bulk theory in terms of familiar observables in quantum field theory, in order to determine which quantities are of interest to condensed matter applications. The most prominent approach to match up bulk fields with operators in the dual QFT is to appeal to symmetry arguments.

Up to this point we have only considered real scalar bulk fields dual to scalar single-trace operators. The GKPW rule is, however, valid also for fields with spin or internal symmetries. For instance, consider a $U(1)$ Maxwell gauge symmetry in the bulk spacetime, described by a Maxwell gauge field A_M. Under a local $U(1)$ transformation, the bulk gauge field transforms as $A_M \to A_M + \nabla_M \chi$, where χ is an arbitrary scalar function of the bulk spacetime coordinates. In particular, we consider a χ which is nonzero on the asymptotic boundary. This is called a 'large' gauge transformation. The gauge field in the bulk is coupled to a current J_μ on the boundary. Note that the bulk field index M run over one more value than the boundary current index μ, since the bulk spacetime has one more spatial dimension. In order to match the bulk field A_M with the boundary current J_μ we must make use of that A_M is a gauge field. It is possible to fix a gauge such that the component of the gauge field along the radial direction vanish, i.e. $A_r = 0$. This choice of gauge fixing is called radial gauge. We can then couple the gauge field to the current on the boundary by adding a standard 'JA-coupling' boundary term,

$$\int d^{d+1}x\sqrt{-\gamma}J^\mu A_\mu \,, \tag{4.89}$$

to the bulk action. Here γ is the induced metric on the conformal boundary. Under a 'large' gauge transformation, this coupling term transforms to

$$\int d^{d+1}x\sqrt{-\gamma}\left(A_\mu + \nabla_\mu\chi\right)J^\mu = \int d^{d+1}x\sqrt{-\gamma}\left(A_\mu J^\mu - \chi\nabla_\mu J^\mu\right)\,, \tag{4.90}$$

where we have integrated the second term by parts. For the bulk theory to be invariant under this gauge transformation, the action itself must be invariant. In particular, the coupling term (4.89) has to be gauge invariant. It follows that $\nabla_\mu J^\mu = 0$ must hold, i.e. the current operator J^μ must be conserved in order for the coupling (4.89) to be gauge invariant. The current is then naturally identified as the symmetry current of a global $U(1)$ symmetry, since Noether's theorem tells us that such a current is always conserved. Then according to the GKPW formula, the bulk gauge field A_M is dual to a conserved current J_μ of a global symmetry in the boundary field theory.

This is the standard approach of identifying a boundary QFT operator to a Maxwell field in the bulk. However, note that the condition for the coupling term (4.89) to be gauge invariant is only that the current is conserved. This guarantees at least a global symmetry in the boundary QFT, but it is only a minimal requirement. Indeed, in several top-down models, e.g. Maldacenas original type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ dual to $\mathcal{N}=4$ super-Yang-Mills theory in $d=4$ dimensions, the boundary QFT in fact possess local symmetry. A boundary QFT with only a global $U(1)$ symmetry does not contain a dynamical photon, i.e. it does not incorporate any electrodynamics into the theory. This would require a local $U(1)$ symmetry. Although one can only be sure to have a global symmetry in the boundary QFT, there are ways in which one can account for electrodynamics. The standard approach is to weakly gauge the theory, meaning that one explicitly weakly couple the boundary theory to a dynamical photon. This can be achieved by adding a $F^{\mu\nu}F_{\mu\nu}$ term to the QFT action, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. The effective action for the photon will then capture electrodynamical phenomena, see e.g. [10] where this is done for the superconductor toy model that we will study in Chapter 7.

Another way to incorporate a dynamical photon in the boundary QFT is to use the $U(1)$ field strength in the bulk to identify a $U(1)$ gauge theory in the boundary QFT, as proposed in [13]. This amounts to imposing a mixed boundary condition on the bulk Maxwell field at the conformal boundary. Moreover, this is equivalent to a (current-current) double-trace deformation of the QFT which results in an RPA form of the Green's function, as we saw in the previous section. Electrodynamics in the boundary QFT has been studied previously using the double-trace deformation approach [12], [14]. In our treatment of holographic superconductivity in Chapter 7, we will attempt to account for a dynamical photon in the boundary QFT by imposing the mixed boundary condition proposed in [13].

Now consider the metric field g_{MN} of the bulk spacetime. This is a symmetric 2-tensor field covariant under arbitrary diffeomorphisms of the spacetime. The gravitional theory itself is invariant under these diffeomorphisms. This diffeomorphism invariance reflects a redundancy in the description of the theory, namely, the freedom to choose a coordinate system. In this regard it is similar to gauge invariance, which also reflects a redundancy of the theory, manifesting itself in the freedom to choose a 'gauge'. In this way, it is natural to think of g_{MN} as a kind of gauge field. Just as the $U(1)$ gauge field is massless as long as the $U(1)$ symmetry is unbroken, so is the metric field massless unless the diffeomorphism invariance is broken.

We know that the metric field is sourced by the energy-momentum tensor and vice versa, as described by Einstein's equations. Because the energy-momentum tensor is a rank-2 tensor the metric field has spin-2. We are thus led to identify the bulk metric field as dual to the energy momentum tensor in the boundary QFT, in complete analogy with how the bulk Maxwell field A_M was identified as dual to the current J^μ in the boundary QFT. In particular, consider a pure gravitational theory in AdS space. The bulk metric field at the conformal boundary should be identified as the source for the dual CFT energy-momentum tensor. In a relativistic conformal field theory the spin-0 and spin-1 parts of the energy-momentum tensor are fixed by conformal invariance and Lorentz invariance, respectively [2]. The dynamical part of the energy-momentum tensor is thus a pure spin-2 field. Furthermore, the energy momentum tensor is a marginal operator with conformal dimension $\Delta = d+1$. There is a mass-scaling relation for spin-2 fields, analogous to the relation (4.67) for spin-0 fields, and the fact that a marginal CFT operator is dual to a massless bulk field holds also for spin-2 fields [2]. The bulk field dual to the energy-momentum tensor of the CFT should thus be a massless spin-2 field[9]. The only real candidate is the metric field, since general relativity is the only known consistent theory of a massless spin-2 field. The Lorentz invariance of the CFT energy-momentum tensor is captured by the conservation law $\nabla_\mu T^{\mu\nu} = 0$. Thus, similar to the bulk Maxwell field and dual conserved current, the bulk metric field is dual to a conserved current of a global symmetry in the boundary field theory, namely, the energy-momentum tensor which is the Noether current associated with global spacetime translations. This

[9]The spin here characterizes how the energy-momentum tensor of the CFT behaves under Lorentz transformations. Since the CFT can be thought of as defined on the boundary of the bulk spacetime, the dual bulk field should also be a spin-2 field. However, the bulk field spin characterizes how the field transforms under 'local' Lorentz transformations.

follows the standard identification scheme where a gauge field in the bulk correspond to a conserved current of the corresponding global symmetry in the boundary theory. In this way one identifies local symmetries in the bulk with global symmetries in the boundary. In this sense, a holographic duality can be thought of as a global/local symmetry duality.

5

Holographic Thermodynamics

Up to this point we have been exclusively dealing with a gravitational theory in pure AdS space, holographically dual to a conformal field theory 'livning' on the conformal boundary of AdS. Conformal field theories lack an intrinsic scale and are highly relevant for statistical and condensed matter physics in the context of describing the scale invariant physical systems realized at thermodynamic or quantum critical points, where a phase transition occurs [45], [46]. However, we would like to be able to model physical systems which does possess various scales, e.g. a temperature and chemical potential. In this chapter we introduce a temperature scale to our boundary field theory by introducing a black brane with a 'finite size' planar horizon in the bulk spacetime. The planar horizon is of course infinite but we use the linguistic of spherical black holes where a finite size horizon is located at a finite radius from the singularity. It is then appropriate to also refer to the black brane as a black hole. The thermodynamic properties of this black hole will translate into the same thermodynamic properties in the boundary theory. For instance, the boundary QFT will have a temperature coinciding with the Hawking temperature of the black hole, as well as an entropy coinciding with the Bekenstein-Hawking entropy.

Even with the inclusion of a black hole in the bulk spacetime, the boundary field theory will still be a conformal field theory, albeit at a finite temperature, since there is only this one temperature scale. To break the scale invariance a second scale has to be introduced. In the next chapter we break the scale invariance by considering a finite volume boundary space which comes with an associated length scale. This gives rise to a Hawking-Page phase transition in the bulk spacetime which is interpreted as a confinement-deconfinement transition in the dual QFT. Next, in our treatment of holographic superconductors in Chapter 7, the scale invariance is broken by considering a charged black hole, giving rise to a finite chemical potential in the boundary QFT.

This chapter starts of with introducing thermal field theories and the Schawarschild black hole in AdS space. The Hawking temperature of this black hole is calculated. We then proceed with calculating the free energy and entropy of this black hole solution, which is interpreted as the free energy and entropy of the boundary theory, respectively.

5.1 Black holes in AdS space and thermal field theories

A quantum field theory is related to a thermal field theory at equilibrium by a Wick rotation, $t \to -i\tau$. Here one considers the Euclidean time τ to be periodic with a periodicity $\beta = 1/(k_B T) = 1/T$, where T is the temperature of the thermal theory and k_B is Boltzmann's constant, which is one in our choice of units. The partition function of the thermal field theory is given by

$$Z_{\text{TFT}}(T) = \int_{\text{S}^1 \times \mathbb{R}^d} D\Phi e^{-I_E[\Phi]} \,, \tag{5.1}$$

where $I_E[\Phi]$ is the Euclidean field theory action and S^1 is the imaginary time circle. The holographic dictionary remains valid in the Euclidean signature. There is, however, a caveat here: the field theory is now defined on a nontrivial background space topology, $\text{S}^1 \times \mathbb{R}^d$, and the bulk space must therefore approach this geometry at spatial infinity. Using the GKPW formula to evaluate the thermal partition function semi-classically, we get

$$Z_{\text{TFT}}(T) = e^{-S_E[g^*_{MN}]} \,, \tag{5.2}$$

where S_E is the Euclidean Einstein-Hilbert action with a negative cosmological constant $\Lambda = -d(d+1)/(2L^2)$,

$$S_{\text{EH}} = -\frac{1}{2\kappa^2} \int d^{d+2}x \sqrt{g} \left(R + \frac{d(d+1)}{L^2} \right) \,, \tag{5.3}$$

and g^*_{MN} is a metric that solves Einstein's equation subject to appropriate boundary conditions respecting the thermal field theory background geometry. The solution is a black hole geometry with a metric

$$ds^2 = \frac{r^2}{L^2} f(r) d\tau^2 + \frac{L^2}{f(r) r^2} dr^2 + \frac{r^2}{L^2} dx_i^2 \,. \tag{5.4}$$

The function $f(r)$ is called the emblackening factor and the imaginary time coordinate τ is directly identified with the one in the dual thermal field theory. The periodicity of τ is then also the inverse temperature of the black hole Hawking radiation. The boundary condition at spatial infinity, $r \to \infty$, is to require that the black hole geometry asymptote to the $\text{S}^1 \times \mathbb{R}^d$ geometry of the thermal field theory. We must therefore require that $f(r) \to 1$ as $r \to \infty$. Rotating back to real time, we see that $g_{tt} = (r/L)^2 f(r)$ and the radius of the (outer) horizon is hence given by the largest solution to $f(r) = 0$. The emblackening factor can in general not be determined analytically except in some sufficiently simple theories including the present pure AdS gravity theory. In this case the emblackening factor is given by

$$f(r) = 1 - \frac{M}{r^{d+1}} \,, \tag{5.5}$$

where M is the mass of the black hole. The unique solution to $f(r) = 0$ is then $r_+ = M^{1/(d+1)}$. Thus, we can rewrite the emblackening factor as

$$f(r) = 1 - \left(\frac{r_+}{r} \right)^{d+1} \,. \tag{5.6}$$

The notion of a dynamical spacetime with a finite size black hole horizon as being a dual description of a QFT at a finite temperature is further justified by the interpretation of the bulk radial direction as the field theory RG energy scale. The high energy UV phenomena should be rather unaffected by a finite temperature given that the energy is 'high' compared to the temperature scale. On the other hand, the low energy IR dynamics is highly affected by a finite temperature. Sufficiently low energy, long wavelength processes will in fact be screened by the temperature, making it effectively act as an IR cutoff. This is perfectly encoded in the black hole horizon in the bulk since it only affects the geometry in the deep interior, corresponding to the low energy region of the dual QFT. The asymptotic near boundary geometry is not affected as was made sure by imposing $f(r) \to 1$ as $r \to \infty$. Furthermore, the horizon chops of the geometry in the deep interior at $r = r_+$, in perfect coherence with the low energy screening effect of the temperature.

We are now in a position to derive a relation between the horizon radius r_+ and the temperature T of the black hole. The metric (5.4) can be written as

$$ds^2 = h(r)d\tau^2 + \frac{1}{h(r)}dr^2 + \frac{r^2}{L^2}dx_i^2\,, \tag{5.7}$$

with

$$h(r) = g_{tt}(r) = g^{rr}(r) = \frac{r^2}{L^2}f(r)\,. \tag{5.8}$$

We expand the metric near the horizon while noting that $h(r_+) = 0$,

$$ds^2 = h'(r_+)(r - r_+)d\tau^2 + \frac{1}{h'(r_+)(r-r_+)}dr^2 + \frac{r_+^2}{L^2}dx_i^2 + \cdots \tag{5.9}$$

Next, we perform a change of variables,

$$\begin{aligned} \tau \to \varphi &= \frac{h'(r_+)}{2}\tau\,, \\ r \to \rho &= 2\sqrt{\frac{r-r_+}{h'(r_+)}}\,, \end{aligned} \tag{5.10}$$

in which the metric takes the following form:

$$ds^2 = \rho^2 d\varphi^2 + d\rho^2 + \frac{r_+^2}{L^2}dx_i^2 + \cdots \tag{5.11}$$

Note that $f'(r_+) = (d+1)/r_+ > 0$ and consequently $h'(r_+) > 0$, so we can omit the absolute value of $h'(r_+)$ when squaring ρ. However, in the general case one would have to replace every $h'(r_+)$ by its absolute value.

Since the imaginary time τ is defined on a circle, so is φ. Thus, the coordinates φ and ρ are identified as polar coordinates on a two-dimensional plane, with of course φ being the compact angular coordinate and ρ the radial coordinate. Note that the horizon is located at the origin $\rho = 0$ in these polar coordinates. In order to avoid a conical singularity at the horizon the coordinate φ must have a periodicity $\varphi \sim \varphi + 2\pi$. The definition of φ in terms of τ then directly implies that τ has a periodicity of $4\pi/h'(r_+)$. Now, recall that we have already made the identification $\tau \sim \tau + 1/T$. Thus, we arrive at an expression of the black hole temperature in terms of bulk quantities,

$$T = \frac{h'(r_+)}{4\pi} = \frac{r_+^2 f'(r_+)}{4\pi L^2} = \frac{(d+1)r_+}{4\pi L^2}\,. \tag{5.12}$$

It is worth mentioning that if we where to restore physical constants the formula for the temperature would read

$$T = \frac{\hbar c}{k_B}\frac{(d+1)r_+}{4\pi L^2}\,. \tag{5.13}$$

This result is in direct opposition to the analogous result for spherical Schwarzchild black holes asymptotic to flat space seen in our universe. The temperature of these black holes are inversely proportional to the horizon radius and, consequently, the larger black holes will be cooler than the smaller ones. Apparently this is not the case for the planar black holes in AdS space considered here, since the temperature here is linearly proportional to the horizon radius. A larger black hole will thus be hotter than a smaller one. However, this is again in perfect synchrony with the geometrization of the boundary theory RG scale. As the temperature increases, more and more low energy processes should be screened out by the temperature. This is precisely what happens in the bulk as the horizon radius grows; more and more of the IR region in the deep interior disappears behind the growing horizon. Moreover, with an increasing temperature, the dynamics at the temperature scale get increasingly closer to the UV dynamics. This is encoded in the growing horizon as well, since the near horizon region expands outwards toward the asymptotic boundary where the UV phenomena lives.

The temperature relation (5.13) was a consequence of demanding regularity of the Euclidean black hole geometry at the horizon. For bulk theories in a Euclidean signature with various field contents, one generally requires the different modes to be regular at the horizon. When Wick rotated back to real time these regularity conditions translates into intuitive 'infalling' boundary conditions at the horizon. The fields in the bulk must only be falling into the black hole and not emerging out from it. This encodes for dissipative processes in the dual field theory since any mode in the bulk will eventually disappear behind the horizon and thus acquire a finite lifetime.

5.2 Free energy and entropy

By the GKPW formula, the (Helmholtz) free energy in the semi-classical limit is given by

$$F = -T \ln Z_{\mathrm{QFT}}(T) = T S_E[g^*_{MN}]\,, \tag{5.14}$$

where S_E is the bulk theory Euclidean action and g^*_{MN} is the black hole geometry (5.4). Computing the free energy of the theory therefore amounts to evaluating the on-shell Euclidean action.

As is usually the case in holography, one has to take into consideration boundary contributions to the action. The $\mathrm{S}^1 \times \mathbb{R}^d$ geometry on the conformal boundary is fixed. We introduce a cutoff surface at $r = R$ and impose Dirichlet boundary conditions on the metric fluctuations δg_{MN} on this cutoff surface. In order to have a well defined variational problem in the limit $R \to \infty$, given the fixed conformal boundary, there should not be any boundary terms proportional to $\partial_r \delta g_{MN}$ after integrating the action by parts. This requires the addition of a 'Gibbons-Hawking-York' boundary term to the action [50], [51],

$$S_{\mathrm{GHY}} = \frac{1}{2\kappa^2} \int_{\mathrm{S}^1\times\mathbb{R}^d} d^{d+1}x \sqrt{h}(-2K)\bigg|_{r=R}\,. \tag{5.15}$$

Here h is the reduced determinant of the induced metric h_{MN} on the cutoff boundary and $K = h^{MN}K_{MN}$ is the trace of the extrinsic curvature K_{MN} of the induced metric, defined as

$$K_{MN} = h^K{}_M h^L{}_N \nabla_K n_L\,, \tag{5.16}$$

where ∇_M is the covariant derivative with respect to g_{MN}, and n^M is an outwards directed normal vector to the cutoff surface. The extrinsic curvature characterizes the way a submanifold is embedded in a manifold. Here it characterizes how the cutoff surface is embedded as a boundary of the bulk space.

Furthermore, even with the addition of a Gibbons-Hawking-York boundary term we will, as usual, encounter a divergence upon taking $R \to \infty$, resulting from integrating over an infinite volume. This infinite volume divergence correspond to a UV divergence in the boundary QFT and can be regulated directly in the bulk theory by the addition of a boundary counterterm to the action. This counterterm depends only on the intrinsic geometry of the boundary. Hence, it does not alter the bulk equations of motion. We generalize the convention in [50] to AdS geometries and choose the counterterm such that the on-shell action vanish for pure AdS space[1]. For our Schwarzchild black hole in AdS space (5.4), the counterterm is as follows:

$$S_{ct} = \frac{1}{2\kappa^2} \int_{\mathrm{S}^1 \times \mathbb{R}^d} d^{d+1}x \sqrt{h} \frac{2d}{L} \bigg|_{r=R} . \tag{5.17}$$

We will now show that this counterterm indeed precisely cancels the infinite volume divergance encountered in the unregulated on-shell action.

With the addition of the Gibbons-Hawking-York term and the counterterm, the total gravitational Euclidean action to be considered is

$$S_E = S_{\mathrm{EH}} + S_{\mathrm{GHY}} + S_{ct} , \tag{5.18}$$

where S_{EH} is the Euclidean Einstein-Hilbert action with a negative cosmological constant,

$$S_{\mathrm{EH}} = -\frac{1}{2\kappa^2} \int_{r_+}^{R} dr \int_{\mathrm{S}^1 \times \mathbb{R}^d} d^{d+1}x \sqrt{g} \left(R + \frac{d(d+1)}{L^2} \right) . \tag{5.19}$$

Here the R in the integrand of course denotes the Ricci scalar and not the cutoff radius. There should be no confusion about the meaning of R in the following calculation since we will evaluate the Ricci scalar and then R will always refer to the cutoff radius. In fact, the Ricci scalar and square root of the determinant of the metric (5.4) is

$$R = -\frac{(d+1)(d+2)}{L^2} , \qquad \sqrt{g} = \left(\frac{r}{L} \right)^d . \tag{5.20}$$

Thus, the Einstein-Hilbert action takes the following form,

$$\begin{aligned} S_{\mathrm{EH}} &= -\frac{1}{2\kappa^2} \int_0^{1/T} d\tau \int_{r_+}^{R} dr \int_{\mathbb{R}^d} d^d x \left(\frac{r}{L} \right)^d \left(-\frac{2(d+1)}{L^2} \right) \\ &= \frac{1}{2\kappa^2} \frac{2}{L^2 T} \int_{r_+}^{R} dr (d+1) \left(\frac{r}{L} \right)^d \int_{\mathbb{R}^d} d^d x \\ &= \frac{1}{2\kappa^2} \frac{2}{LT} \left[\left(\frac{R}{L} \right)^{d+1} - \left(\frac{r_+}{L} \right)^{d+1} \right] \int_{\mathbb{R}^d} d^d x . \end{aligned} \tag{5.21}$$

The normal vector n^M should be in the positive r-direction and satisfy $n^M n^N g_{MN} = 1$. It is straightforward to verify that it is indeed given by $n^M = r\sqrt{f}/L \delta^{Mr}$ where δ^{MN} is the Kronecker delta. Recall from 4.1.1 that the induced metric is defined as $h_{MN} = g_{MN} - n_M n_N$. The induced metric then exactly takes the form of the black hole metric

[1] In [50] they considered geometries asymptotic to Minkowski space and consequently imposed the condition that the on-shell action vanish for Minkowski space.

(5.4), but with a vanising rr-component. The reduced determinant h of the induced metric is computed by taking the determinant of $h_{\mu\nu}$, where the greek indices as usual run only over the coordinates orthogonal to the radial coordinate. Thus, $\sqrt{h} = (r/L)^{d+1}\sqrt{f}$. In the computation of the trace of the extrinsic curvature, we utilize the fact that the induced metric is orthogonal to the normal vector n^M and that $h_{\mu\nu} = g_{\mu\nu}$,

$$K = h_{MN}K^{MN} = h_{MN}h^{MK}h^{NL}\nabla_K n_L = h^{KL}\left(\partial_K n_L - \Gamma^I_{KL}n_I\right) = -h^{\mu\nu}\Gamma^r_{\mu\nu}n_r\,. \quad (5.22)$$

Here Γ^K_{MN} is the Cristoffel symbol with respect to g_{MN}. The two Cristoffel symbols we need are

$$\Gamma^r_{\tau\tau} = -\frac{r^3 f}{L^4}\left(f + \frac{rf'}{2}\right)\,, \qquad \Gamma^r_{x_i x_i} = -\frac{r^3 f}{L^4}\,. \quad (5.23)$$

Now, $n_r = g_{rr}n^r = L/(r\sqrt{f})$ and we can compute K,

$$\begin{aligned} K &= -h^{\tau\tau}\Gamma^r_{\tau\tau}n_r - \sum_{i=1}^{d} h^{x_i x_i}\Gamma^r_{x_i x_i}n_r \\ &= \frac{L^2}{r^2 f}\frac{r^3 f}{L^4}\left(f + \frac{rf'}{2}\right)\frac{L}{r\sqrt{f}} + d\frac{L^2}{r^2}\frac{r^3 f}{L^4}\frac{L}{r\sqrt{f}} \\ &= \frac{\sqrt{f}}{L}\left(d + 1 + \frac{rf'}{2f}\right)\,. \end{aligned} \quad (5.24)$$

The Gibbons-Hawking-York boundary term (5.15) is then given by

$$\begin{aligned} S_{\text{GHY}} &= \frac{1}{2\kappa^2}\int_{\text{S}^1\times\mathbb{R}^d} d^{d+1}x\left(\frac{r}{L}\right)^{d+1}\sqrt{f}\left[-2\frac{\sqrt{f}}{L}\left(d+1+\frac{rf'}{2f}\right)\right]\Big|_{r=R} \\ &= -\frac{1}{2\kappa^2}\left(\frac{R}{L}\right)^{d+1}\left[\frac{2f(d+1)}{L} + \frac{Rf'}{L}\right]\int_0^{1/T} d\tau \int_{\mathbb{R}^d} d^d x \\ &= -\frac{1}{2\kappa^2}\frac{1}{T}\left(\frac{r}{L}\right)^{d+1}\left[\frac{2(d+1)}{L}\left(1 - \left(\frac{r_+}{r}\right)^{d+1}\right) + \frac{d+1}{L}\left(\frac{r_+}{r}\right)^{d+1}\right]\int_{\mathbb{R}^d} d^d x\Big|_{r\to\infty} \\ &= -\frac{1}{2\kappa^2}\frac{d+1}{LT}\left[2\left(\frac{R}{L}\right)^{d+1} - \left(\frac{r_+}{L}\right)^{d+1}\right]\int_{\mathbb{R}^d} d^d x\,, \end{aligned} \quad (5.25)$$

where we have used that $f(R) = 1 - (r_+/R)^{d+1}$ in the third equality. Finally, the counterterm (5.17) is

$$\begin{aligned} S_{ct} &= \frac{1}{2\kappa^2}\int_{\text{S}^1\times\mathbb{R}^d} d^{d+1}x\left(\frac{r}{L}\right)^{d+1}\sqrt{f}\frac{2d}{L}\Big|_{r=R} \\ &= \frac{1}{2\kappa^2}\frac{2d}{LT}\left(\frac{R}{L}\right)^{d+1}\sqrt{1 - \left(\frac{r_+}{R}\right)^{d+1}}\int_{\mathbb{R}^d} d^d x \\ &= \frac{1}{2\kappa^2}\frac{2d}{LT}\left[\left(\frac{R}{L}\right)^{d+1} - \frac{1}{2}\left(\frac{r_+}{L}\right)^{d+1} + \mathcal{O}\left(\frac{1}{R^{d+1}}\right)\right]\int_{\mathbb{R}^d} d^d x\,, \end{aligned} \quad (5.26)$$

where we have expanded the square root in powers of $(r_+/R)^{d+1} \ll 1$. Now adding up (5.21), (5.25), and (5.26), we get an expression for the complete Euclidean bulk action, which we then substitute in (5.14) to calculate the free energy of the black hole solution.

The terms with a positive power of R cancel and we can thus take the limit $R \to \infty$. The result for the free energy is

$$F = -\frac{1}{2\kappa^2}\frac{1}{L}\left(\frac{r_+}{L}\right)^{d+1}\int_{\mathbb{R}^d} d^d x\,. \tag{5.27}$$

Note that the free energy indeed vanishes for pure AdS obtained by setting $r_+ = 0$. Furthermore, the free energy is proportional to the spatial volume of the conformal boundary on which the field theory is defined, as is the expected result for a quantum field theory. Using the relation (5.13) between the temperature and horizon radius, the free energy can be expressed as

$$F = -\frac{2\pi}{\kappa^2}\left(\frac{4\pi L}{d+1}\right)^d \frac{T^{d+1}}{d+1}\int_{\mathbb{R}^d} d^d x\,. \tag{5.28}$$

The entropy of the black hole and thermal field theory is then given by

$$S = -\frac{\partial F}{\partial T} = \frac{2\pi}{\kappa^2}\left(\frac{4\pi L}{d+1}\right)^d T^d \int_{\mathbb{R}^d} d^d x = \frac{1}{4G_N}\left(\frac{r_+}{L}\right)^d \int_{\mathbb{R}^d} d^d x\,, \tag{5.29}$$

where we have used that $\kappa = 8\pi G_\mathrm{N}$ with G_N being Newtons constant. This is in fact precisely the Bekenstein-Hawking entropy of the black hole (3.2), as is confirmed by calculating the 'area' of the horizon[2],

$$A = \int_{\mathbb{R}^d} d^d x \sqrt{g}\bigg|_{r=r_+} = \left(\frac{r_+}{L}\right)^d \int_{\mathbb{R}^d} d^d x\,. \tag{5.30}$$

We have thus shown that the entropy of our finite temperature boundary CFT is equal to the Bekenstein-Hawking entropy of the black hole in the bulk. The expressions we have derived for the free energy and entropy are written in terms of bulk quantities, (r_+,L, and G_N). In order to express the thermodynamic quantities in terms of field theory quantities one would need an explicit top-down construction of the duality. However, it is worth mentioning that in the canonical example of $3+1$ dimensional large N super-Yang-Mills theory dual to supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$, the black hole entropy is proportional to N^2. This suggest that the boundary gauge theory dual to an AdS black hole spacetime is in a deconfined state. This statement will be explored in the next chapter which is about the thermal 'Hawking-Page transition' and the related confinement-deconfinement transition in the dual field theory.

[2] It is more correct to use the word volume here as opposed to area, but we stick with the terminology of spherical black holes in 3+1 dimensional spacetime to guide our intuitions.

6

The Hawking-Page Transition and Confinement in Gauge Theories

In the previous chapter we introduced a temperature scale to our holographic duality by adding a black hole in the deep interior of the bulk spacetime. This induced a temperature in the boundary field theory coincident with the Hawking temperature of the black hole. Here we will introduce yet another scale to our system in a more rudimentary manner by considering a boundary space with a finite volume. In fact, this can be done in a natural and straightforward way by exchanging the Poincaré coordinates for AdS space, which we have been working with so far, for global coordinates. Recall from 3.3 that the Poincaré coordinates cover only the Poincaré patch of AdS space. The conformal boundary in Poincaré coordinates is a Minkowski space, making it a natural choice of coordinates when considering relativistic boundary QFTs. However, the global coordinates (3.13) cover the entire AdS space and the topology of the Euclidean boundary space is then $S^1 \times S^d$.

There are precisely two isotropic solutions to Einstein's equation in Euclidean signature which are asymptotic to 'global AdS', as proved already in the 1980s by Hawking and Page [52]. One is a Schwarzschild black hole solution with an imaginary time compactified on a circle. In the case of global AdS, however, the black holes are spherical as opposed to the planar black holes, or rather black branes, which are the corresponding solutions one obtains for the Poincaré patch of AdS space. The other solution is of course the global AdS solution (3.13), but with a periodic imaginary time. The global AdS solution with a periodic imaginary time is referred to as 'thermal AdS'. The corresponding boundary field theory will acquire a finite temperature despite the lack of a black hole horizon in the bulk spacetime.

We concluded the last chapter by stating that black hole entropy in the case of supergravity on $AdS_5 \times S^5$, dual to large N super-Yang-Mills theory in 3+1 dimensions, is proportional to N^2. Since N^2 counts the 'colour' degrees of freedom in the super-Yang-Mills gauge theory, this suggests that these degrees of freedom are not confined in a gauge invariant singlet, and that the black hole solution in fact describes the deconfined phases of the gauge theory. On the other hand, the thermal AdS solution has no horizon and consequently no entropy of order N^2. The confined states of the gauge theory should have an entropy of order unity, since the order N^2 degrees of freedom are then confined in a gauge invariant singlet state. In the large N limit, corresponding to the classical limit in the gravitational theory which we are working in, contributions of order one to the entropy will be suppressed by the leading large N contribution. The fact that the thermal AdS solution does not have an entropy of order N^2 suggests that it may describe the confined states of the boundary theory [2].

In order to determine which of the two competing thermal solutions is realized in a given system, one can compare their free energies as a function of temperature and AdS radius L. The solution with the lower free energy will correspond to the thermodynamically preferred state.

We will first analyse the thermodynamics of the two qualitatively different thermal solutions to Einstein's equation, and in particular, calculate their free energies. We will find that a thermal phase transition occurs at a particular critical temperature, which depends on the number of spatial dimensions and the AdS radius. This is in fact the 'Hawking-Page transition' [52]. We will then give a brief discussion of confinement in gauge theories, and its interpretation in holography. By computing a static quark potential, both in the zero and finite temperature case, we analyse the confined/deconfined nature of large N gauge theories in holographic duality. However, this computation is done in Poincaré coordinates, which correspond to an infinite volume conformal boundary, and therefore cannot be applied to the spacetimes involved in the Hawking-Page transition. We end the static quark potential analysis by briefly mention the result in [53] for the finite volume case, which suggests that the Hawking-Page transition precisely correspond to a confinement-deconfinement phase transition in the holographically dual QFT.

6.1 Thermodynamics of AdS space

The Euclidean metric of the two isotropic spacetimes asymptotic to 'global AdS' takes the following form [52], [53]:

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_d^2\,, \tag{6.1}$$

where $d\Omega_d^2$ is the metric of the d dimensional sphere S^d with unit radius. The emblackening factor is given by

$$f(r) = 1 + \frac{r^2}{L^2} - \omega_d M r^{-d+1}\,, \qquad \omega_d = \frac{2\kappa^2}{d\mathrm{Vol}(\mathrm{S}^d)}, \tag{6.2}$$

where M is the mass of the black hole and $\mathrm{Vol}(\mathrm{S}^d)$ is the volume of the d dimensional sphere. The thermal AdS solution is obtained by taking $M = 0$. Note that the black hole solution is asymptotic to pure AdS. The largest solution r_+ to $f(r) = 0$ with $M \neq 0$ determines the location of the horizon. Whereas the thermal AdS spacetime is defined for $0 \leq r < \infty$, the black hole spacetime is restricted to the region $r_+ \leq r < \infty$.

Expanding the metric (6.1) near the horizon and changing coordinates according to (5.10) (replacing h with f), we get

$$ds^2 = \rho^2 d\phi^2 + d\rho^2 + r_+^2 d\Omega_d^2 + \cdots \tag{6.3}$$

From (6.2) it directly follows that

$$f'(r_+) = 2\frac{r_+}{L^2} + (d-1)\omega_d M r_+^{-d}\,. \tag{6.4}$$

However, using that $f(r_+) = 0$ it also follows that

$$\omega_d M r_+^{-d} = \frac{1}{r_+} + \frac{r_+}{L^2} = \frac{L^2 + r_+^2}{r_+ L^2}\,, \tag{6.5}$$

and we can thus write

$$f'(r_+) = \frac{(d-1)L^2 + (d+1)r_+^2}{r_+ L^2} \,. \tag{6.6}$$

Since clearly $f'(r_+) > 0$, the absolute sign when squaring $\rho = 2\sqrt{(r - r_+)/f'(r_+)}$ can be omitted. In order to avoid a conical singularity at $\rho = 0$ we must require $\phi \sim \phi + 2\pi$. The Euclidean time $\tau = 2\phi/f'(r_+)$ then has a periodicity of $4\pi/f'(r_+)$ which is identified as the inverse temperature. Hence, the temperature of the black hole solution is given by

$$T = \frac{(d-1)L^2 + (d+1)r_+^2}{4\pi r_+ L^2} \,. \tag{6.7}$$

We now follow the prescription outlined in 5.2 to compute the free energies and determine the thermodynamically preferred solution.

Recall from 5.2 that the free energy is proportional to the on-shell Euclidean bulk action, $\beta F = S[g^*]$. For a pure gravitational theory in vacuum, the action consists of an Einstein-Hilbert term plus suitable boundary terms, in particular a Gibbons-Hawking-York term and a counterterm. We choose our counterterm such that the on-shell action vanish for the thermal AdS solution. Thus, we are essentially setting the free energy of the thermal AdS solution to zero. Consider the case of $d = 2$, i.e. 4 bulk spacetime dimensions. The counterterm that does the job is

$$S_{ct} = \frac{1}{2\kappa^2} \int_{\mathrm{S}^1 \times \mathrm{S}^2} d^3x \, \sqrt{h} \, \frac{4}{L} \sqrt{1 + \frac{L^2}{r^2}} \bigg|_{r=R} \,, \tag{6.8}$$

where h as usual is the determinant of the induced metric on the boundary. The Euclidean Einstein-Hilbert action (5.3) evaluated on the solution (6.1) is given by

$$S_{\mathrm{EH}} = \frac{3}{L^2\kappa^2} \int d^4x \sqrt{g} \,, \tag{6.9}$$

where we have used that $R = -(d+2)(d+1)/L^2$ is the Ricci scalar for the geometry (6.1). Thus, the on-shell Einstein-Hilbert action is simply given by a constant times the bulk volume. For the thermal AdS spacetime, we get

$$S_{\mathrm{EH}} = \frac{3}{L^2\kappa^2} \int_0^{\beta'} d\tau \int_0^R dr r^2 \int_{\mathrm{S}^2} d\Omega = \frac{4\pi\beta'}{L^2\kappa^2} R^3 \,, \tag{6.10}$$

and for the black hole spacetime,

$$S_{\mathrm{EH}} = \frac{3}{L^2\kappa^2} \int_0^{\beta} d\tau \int_{r_+}^R dr r^2 \int_{\mathrm{S}^2} d\Omega = \frac{4\pi\beta}{L^2\kappa^2} \left(R^3 - r_+^3\right) \,. \tag{6.11}$$

Now, the Gibbons-Hawking-York term is given by

$$S_{\mathrm{GHY}} = \frac{1}{2\kappa^2} \int_{\mathrm{S}^1 \times \mathrm{S}^2} d^3x \, \sqrt{h} \, (-2K) \bigg|_{r=R} \,. \tag{6.12}$$

The trace of the extrinsic curvature is

$$K = \frac{f'}{2\sqrt{f}} + \frac{2\sqrt{f}}{r} \,. \tag{6.13}$$

It is straightforward to verify that the total on-shell action $S_{\mathrm{EH}} + S_{\mathrm{GHY}} + S_{ct}$ vanish for the thermal AdS solution with $f(r) = 1 + r^2/L^2$ and arbitrary β'. For the black hole solution where $f(r) = 1 + r^2/L^2 - \omega_2 M r^{-1}$ and $\beta = 1/T$, with T given by (6.7), one finds

$$\begin{aligned} S_{\text{GHY}} &= \frac{1}{2\kappa^2}\int_0^\beta d\tau \int_{\text{S}^2} d\Omega R^2 \sqrt{f}\left(-\frac{f'}{\sqrt{f}} - \frac{4\sqrt{f}}{R}\right) \\ &= \frac{4\pi\beta}{L^2\kappa^2}\left(-3R^3 - 2RL^2 + \frac{3}{2}\omega_2 M L^2\right) , \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} S_{ct} &= \frac{1}{2\kappa^2}\int_0^\beta d\tau \int_{\text{S}^2} d\Omega \frac{4R^2}{L}\sqrt{f}\sqrt{1 + \frac{L^2}{R^2}} \\ &= \frac{4\pi\beta}{L^2\kappa^2}\left(2R^3 + 2RL^2 - \omega_2 M L^2\right) . \end{aligned} \tag{6.15}$$

The on-shell action for the black hole solution is then

$$S_{\text{BH}} = S_{\text{EH}} + S_{\text{GHY}} + S_{ct} = \frac{4\pi\beta}{L^2\kappa^2}\left(\frac{1}{2}\omega_2 M L^2 - r_+^3\right) . \tag{6.16}$$

Using (6.7) to evaluate β, as well as the fact that the thermal AdS on-shell action vanish with our choice of counterterm, we find

$$S_{\text{BH}} - S_{\text{AdS}} = \frac{16\pi^2 r_+^2 \left(L^2 - r_+^2\right)}{2\kappa^2\left(3r_+^2 + L^2\right)} , \tag{6.17}$$

where S_{AdS} denotes the thermal AdS on-shell action. In the general case of $d+2$ bulk spacetime dimensions, this expression would read [2], [53]

$$S_{\text{BH}} - S_{\text{AdS}} = \frac{4\pi \text{Vol}(\text{S}^d) r_+^d \left(L^2 - r_+^2\right)}{2\kappa^2\left((d+1)r_+^2 + (d-1)L^2\right)} . \tag{6.18}$$

The difference in on-shell actions $S_{\text{BH}} - S_{\text{AdS}}$ is positive[1] for $r_+ < L$ and negative for $r_+ > L$, indicating a phase transition at $r_+ = L$, corresponding to a critical temperature $T_c = d/(2\pi L)$ according to (6.7). Plotting the free energy F as a function of T indeed reveals a first order transition at T_c, characterized by a discontinuity in the derivative of F with respect to T, as shown in Fig.6.1. Here the constant values $d = 2$, $L = 1$, and $2\kappa^2 = 1$ have been used. For $T < T_c$ the thermal AdS space is the thermodynamically preferred solution and the free energy is zero. For $T > T_c$ the black hole geometry is the preferred solution with a negative free energy, monotonically decreasing in temperature. This phase transition between two different stationary spacetime configurations is known as the Hawking-Page transition, named after its discoverers [52]. Remarkably, it shows that a Schwarzschild black hole in AdS space is not stable at low temperatures; a cold black hole can 'uncollapse' into pure AdS space. The fact that black holes in AdS can become unstable is an essential feature when it comes to the applications of holography to condensed matter physics, as it enables a gravitational description of thermodynamical or quantum phase transitions. Indeed, in the minimal holographic superconductor toy model to be introduced in Chapter 7, the gravitational description of the superconducting phase transition utilizes the instability of a charged 'Reissner-Nordström' black hole to perturbations of a scalar field.

[1]Here we are assuming $d > 0$. For $d = 0$ it is negative for any r_+.

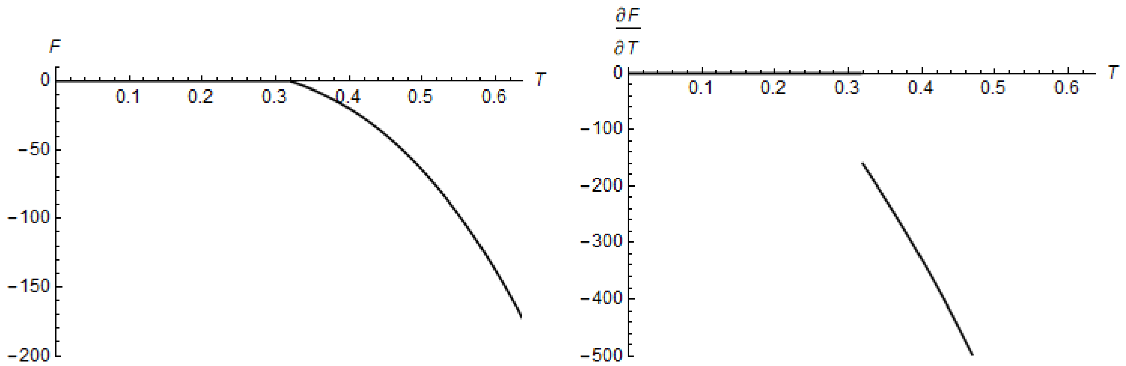

Figure 6.1: The free energy F of the preferred state as a function of temperature T (left) and the derivative of the free energy with respect to temperature (right). At $T_c = 1/\pi$ there is a first order phase transition from the thermal AdS solution to the black hole solution. The derivative is discontinuous at the critical temperature. The constant values $d = 2$, $L = 1$, and $2\kappa^2 = 1$ have been used.

One can now calculate the thermodynamic energy, $\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}$, as well as entropy of the black hole spacetime. The results are those which should be expected; the total energy of the black hole is equal to its mass, $\langle E \rangle = M$, and the entropy is in precise accordance with the Bekenstein-Hawking formula [52], [53].

It is possible to arrive at (6.18) by directly calculating the difference in on-shell actions for the black hole and thermal AdS spacetimes [52], [53]. The surface terms will not contribute because the black hole correction to the emblackening factor vanish too rapidly at infinity [53]. Furthermore, the infinite volume divergences cancels when computing the difference in actions. The difference in on-shell actions is then essentially the difference in bulk volumes, according to (6.9). However, there is one important caveat to keep in mind here. One has to make sure that the bulk geometries compared are dual to boundary theories at the *same* temperature. The black hole must have a fixed temperature given by (6.7), in order for the spacetime to be smooth. The thermal AdS spacetime, however, may be defined at any temperature, i.e. the periodicity of the imaginary time may be chosen arbitrarily. Thus, one introduces a cutoff surface at some radial distance $r = R$, and then define the imaginary time circle of the thermal AdS solution in terms of the black hole inverse temperature, in such a way that the two time circles have the same proper length at the cutoff surface. In the end of the calculation the limit $R \to \infty$ is taken, so the temperatures of the two solutions match at the conformal boundary.

Setting the proper lengths of the imaginary time circles at the cutoff radius equal for the thermal AdS and black hole solution, one finds

$$\beta' \sqrt{1 + \frac{R^2}{L^2}} = \beta \sqrt{1 + \frac{R^2}{L^2} - \omega_d M R^{-d+1}}\,, \tag{6.19}$$

where $\beta' = 1/T'$ and $\beta = 1/T$ are the inverse temperatures of the thermal AdS solution and black hole solution, respectively. When computing the difference in bulk volumes, this is the relation one should use to define β' in terms of β to make sure that the respective dual field theories are defined at the same temperature.

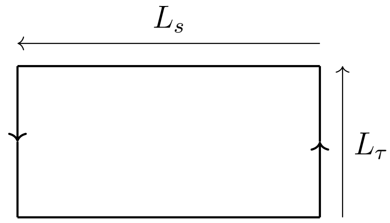

Figure 6.2: A standard rectangular Wilson loop contour for determining the static quark potential. The top and bottom sides are along a spatial direction, whereas the left and right sides are along the temporal direction.

6.2 Wilson loops and confinement

The Hawking-Page transition was discovered more than a decade before the AdS/CFT correspondence. However, it was not until the discovery of the latter that the meaning of the transition was fully understood. Witten showed in [53], by a Wilson loop calculation, that the Hawking-Page transition is the gravitational description of a confinement/deconfinement transition in the dual boundary field theory.

In gauge theory[2], the local gauge symmetry transformations are elements of a Lie group. A Lie group defines a differential manifold with non-trivial curvature and a notion of connections and covariant derivatives. The Wilson loop is a gauge invariant operator which depends on the holonomy[3] of the gauge connection. Here we are using terminology from differential geometry, but keep in mind that the gauge connection is really just the gauge field A_μ itself. Particles charged under the gauge symmetry group are commonly referred to as a quarks, in analogy with QCD. The quarks are in turn sources of the gauge fields.

A quark that is parallel transported around a closed contour γ on the group manifold will pick up a phase factor $\mathcal{P}\exp\left(i\oint_\gamma A\right)$, where $\mathcal{P}$ is the path ordering operator and $A = A_\mu dx^\mu$ is the gauge one-form. The path ordering is necessary for non-Abelian gauge theories, because of the non-commuting nature of the gauge fields. The Wilson loop is defined as

$$W(\gamma) = \mathrm{Tr}\,\mathcal{P}e^{i\oint_\gamma A}\,, \tag{6.20}$$

where the trace is taken over the fundamental representation of the gauge group. The Wilson loop can be thought of as the phase factor associated with the propagation of a quark anti-quark pair around a closed contour. Its expectation value allows us to determine whether the field theory vacuum is in a confining or deconfining state. In the zero temperature case, the textbook approach is to consider the contour to be a rectangle with two sides along the (imaginary) time direction, and two sides along a spatial direction, as indicated in Fig. 6.2. The length of the sides along the temporal and spatial direction are denoted as L_τ and L_s, respectively. This contour allows one to obtain the static potential $V(L_s)$ between an infinitely massive quark anti-quark pair[4], separated a distance L_s, from the expectation value of the Wilson loop. In the Euclidean

[2]For a more thorough review over the vast subject of gauge theory, we refer the reader to [54].

[3]The holonomy of a connection on a differential manifold essentially measures the amount a vector rotates under parallel transportation around a closed curve.

[4]This is in complete analogy with the Coulomb potential between two separated opposite electrical charges in the theory of electromagnetism. The infinite mass of the quarks is required in order to keep the separation distance between them fixed, in the presence of the force associated with the potential.

theory, when taking the limit $L_\tau \to \infty$, one finds

$$\lim_{L_\tau \to \infty} \langle W(\gamma) \rangle \sim e^{-L_\tau V(L_s)}\,, \tag{6.21}$$

Next, one also takes the limit $L_s \to \infty$; if the potential grows linearly (or faster) with L_s, the gauge theory is in a confining state, but if the potential goes to a constant value (or decays with L_s), the theory is in a deconfining state. We see that a confining state with a linearly growing potential gives rise to an area law,

$$\log \langle W(\gamma) \rangle \sim A(\gamma)\,, \tag{6.22}$$

where $A(\gamma)$ denote the minimal area enclosed by γ. On the other hand, a deconfining phase with $V(L_s) \to$ constant results in a perimeter law,

$$\log \langle W(\gamma) \rangle \sim L(\gamma)\,. \tag{6.23}$$

Moreover, the area and perimeter laws also hold for large but arbitrary spatial contours.

In the finite temperature case, the aforementioned area law still holds for spatial Wilson loops. However, a proper order parameter for confinement is the temporal Wilson-Polyakov loop [2], [53], [55], [56],

$$P(\mathbf{x}) = \mathrm{Tr}\left(\mathcal{P} e^{i\int_0^\beta A_\tau d\tau}\right)\,. \tag{6.24}$$

More precisely, its expectation value is an order parameter for deconfinement. A probe quark will induce a change in the free energy of the system. This change in free energy is measured by the expectation value of the Wilson-Polyakov loop; $\langle P(\mathbf{x}) \rangle \sim e^{-F(T)/T}$. An infinite amount of energy is required to separate quarks from a singlet state in the confining phase of a gauge theory, and as a consequence, the induced change in free energy by a probe quark should be infinite. Thus, $\langle P(\mathbf{x}) \rangle = 0$ for a gauge theory in the confining phase. In the deconfining phase, however, a probe quark will change the free energy by a finite amount. Hence, $\langle P(\mathbf{x}) \rangle \neq 0$ signals that the gauge theory is in a deconfining state.

A method for calculating the expectation value of Wilson loops in large N field theories was proposed in [57]. The string theoretical realisation of large N Wilson loops was also studied in [58]. Since the Wilson loop is a non-local operator, its dictionary entry cannot be obtained from the GKPW formula. Instead, one has to turn to the string theoretical foundation of the holographic duality. In string theory, a quark in the fundamental representation correspond to an endpoint of an open string, the endpoint being restricted to live on a D-brane. The string itself account for the forces acting on the quark. Consider an open string whose two end points corresponds to a quark and anti-quark, respectively. As the quark anti-quark pair propagates around the loop, the string will trace out a worldsheet. It is therefore reasonable to assume that the expectation value of the Wilson loop should go as

$$\langle W(\gamma) \rangle \sim e^{-S_{\mathrm{NG}}}\,, \tag{6.25}$$

where S_{NG} is the on-shell Euclidean Nambu-Goto action of the string, measuring the proper area of the string worldsheet. We choose coordinates σ^m, with $m = 0,\,1$, to parametrize the string worldsheet. The coordinates of the string in the bulk spacetime is denoted by $X^M = X^M(\sigma^0, \sigma^1)$, where $M = 0,\,1,\,\ldots,\,d+1$. The Nambu-Goto action is defined as

$$S_{\mathrm{NG}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{\det(G_{MN} \partial_m X^M \partial_n X^N)}\,, \tag{6.26}$$

where α' is the 'slope parameter' conventionally used in string theory, and G_{MN} is the bulk metric. Note that $h_{mn} = G_{MN}\partial_m X^M \partial_n X^N$ is the induced metric on the string worldsheet.

Although the endpoints of the string is restricted to the Wilson loop, defined on the boundary space where the gauge theory lives, the string itself may extend into the interior of the bulk. The contour γ on the boundary space is then extended into a surface, or rather worldsheet, in the bulk space. Determining this worldsheet amounts to finding the stationary points of the Nambu-Goto action (6.26).

6.2.1 Static quark potential at zero temperature

We use the dictionary entry (6.25) for the expectation value of the Wilson loop to compute the static quark anti-quark potential for a zero temperature gauge theory. This calculation was originally done in [57], and has been done in [2]. We consider the zero temperature field theory as living in an infinite flat space, and it is therefore convinient to choose AdS in Poincaré coordinates as the bulk space. Choosing a rectangle with one side in the temporal direction as contour for the Wilson loop, as illustrated in Fig. 6.2, and using (6.21), the potential is given by

$$V(L_s) = \lim_{L_\tau \to \infty} \frac{S_{\mathrm{NG}}}{L_\tau} \,. \tag{6.27}$$

The only subtlety in this calculation comes from the fact that the conformal boundary is located an infinite distance away. The area of the string worldsheet will therefore formally diverge. However, it is possible to regularize the on-shell Nambu-Goto action to obtain a finite result.

For simplicity, we consider the case of a five dimensional AdS bulk space. The pure AdS Euclidean metric in Poincaré coordinates is given by

$$ds^2 = r^2 \left(d\tau^2 + dx^2 + dy^2 + dz^2 \right) + \frac{dr^2}{r^2} \,, \tag{6.28}$$

where we have set the AdS radius to unity. In these coordinates, the Wilson loop rectangle can be taken as the boundary of the domain $-L_\tau/2 \leq \tau \leq L_\tau/2$, $-L_s/2 \leq x \leq L_s/2$, in the boundary space. The parametrization of the string worldsheet is conveniently chosen as $\sigma^0 = \tau$ and $\sigma^1 = x$. Furthermore, we choose to work in static gauge[5] where $X^\tau = \tau$, $X^x = x$. Since these are precisely the worldsheet parameters and X^M are the coordinates of the string worldsheet in the bulk space, it directly follows that X^y and X^z should be constants, assuming X^M are orthogonal coordinates. However, the string can move in the radial direction as well. We consider a static configuration and write $X^r \equiv R(x)$.

With this choice of coordinates and with the metric (6.28), the determinant of the intrinsic metric on the worldsheet can be evaluated as follows:

$$h = \det \left(G_{MN}(X) \frac{\partial X^M}{\partial \sigma^m} \frac{\partial X^N}{\partial \sigma^n} \right) = G_{\tau\tau}(R) \left(G_{xx}(R) + G_{rr}(R) R'^2 \right) = R^4 + R'^2 \,. \tag{6.29}$$

[5] The theory is invariant under reparametrizations of the worldsheet coordinates. The gauge freedom here is due to this reparametrization invariance.

The Nambu-Goto action is then given by

$$
\begin{aligned}
S_{\mathrm{NG}} &= \frac{1}{2\pi\alpha'} \int_{-L_\tau/2}^{L_\tau/2} d\tau \int_{-L_s/2}^{L_s/2} dx \sqrt{h} \\
&= \frac{L_\tau}{2\pi\alpha'} \int_{-L_s/2}^{L_s/2} dx\, R^2 \sqrt{1 + \frac{R'^2}{R^4}}\,.
\end{aligned}
\tag{6.30}
$$

Thus, the system is effectively one-dimensional and the action is the integral over a Lagrangian,

$$
\mathcal{L}(R(x), R'(x)) = \frac{L_\tau}{2\pi\alpha'} R^2 \sqrt{1 + \frac{R'^2}{R^4}}\,. \tag{6.31}
$$

The conjugate momentum to $R(x)$ is then

$$
P(x) = \frac{\partial \mathcal{L}}{\partial R'} = \frac{L_\tau}{2\pi\alpha'} \frac{R'}{R^2 \sqrt{1 + \frac{R'^2}{R^4}}}\,, \tag{6.32}
$$

and the Hamiltonian of the system is given by

$$
H(R, P) = P R' - \mathcal{L}(R, R') = -\frac{L_\tau}{2\pi\alpha'} \frac{R^2}{\sqrt{1 + \frac{R'^2}{R^4}}}\,. \tag{6.33}
$$

Since the Hamiltonian has no explicit x dependence, it is conserved with respect to 'evolution' in x,

$$
\frac{dH}{dx} = \frac{\partial H}{\partial R} \frac{dR}{dx} + \frac{\partial H}{\partial P} \frac{dP}{dx} = -P' R' + R' P' \equiv 0\,, \tag{6.34}
$$

where we have used Hamilton's equations,

$$
\frac{\partial H}{\partial R} = -P'\,, \qquad \frac{\partial H}{\partial P} = R'\,. \tag{6.35}
$$

It then follows directly from (6.33) that

$$
\frac{R^2}{\sqrt{1 + \frac{R'^2}{R^4}}} = \text{constant}\,. \tag{6.36}
$$

The endpoints of the string are attached to the Wilson loop on the boundary space where the radial coordinate is formally infinite, $R(-L_s/2) = R(L_s/2) = \infty$. A sketch of the string extending into the bulk is shown in Fig. 6.3. Now, starting from $x = -L_s/2$ and increasing x, the radial coordinate of the string decreases until it reaches a turning point, which by symmetry must be located at $x = 0$. At the turning point, $R'(x) = 0$ and $R(x)$ acquires its minimal value R_0. Increasing x further the radial coordinate will increase towards infinity at $x = L_s/2$. Evaluating (6.36) at $x = 0$, we can thus determine the constant,

$$
\frac{R^2}{\sqrt{1 + \frac{R'^2}{R^4}}} = R_0^2\,. \tag{6.37}
$$

Solving this equation for R', we get

$$
R' = \pm R^2 \sqrt{\frac{R^4}{R_0^4} - 1}\,, \tag{6.38}
$$

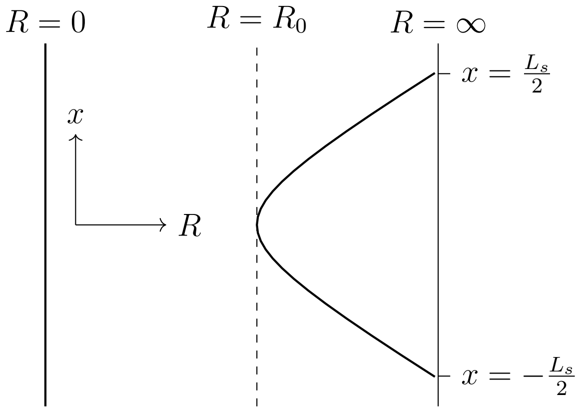

Figure 6.3: A string attached to the Wilson loop on the boundary space at $R=\infty$ and extending into the bulk space. The radial coordinate of the string takes its minimal value R_0 at the turning point of the string.

where the negative solution is valid for $-L_s \leq x \leq 0$, and the positive solution is valid for $0 \leq x \leq L_s/2$. Integrating the above differential equation from $x=0$ to $x=L_s/2$, we find a relation between L_s and R_0,

$$\begin{aligned} L_s &= 2\int_0^{L_s/2} dx = 2\int_{R_0}^{\infty} dR\,\frac{1}{R^2\sqrt{R^4/R_0^4-1}} = \frac{2}{R_0}\int_1^{\infty} dy\frac{1}{y^2\sqrt{y^4-1}} \\ &= \frac{2\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2 R_0}\,. \end{aligned} \tag{6.39}$$

Moreover, using (6.37) and (6.38) to change the integration variable from x to R, the Nambu-Goto action (6.30) can be rewritten as

$$\begin{aligned} S_{\mathrm{NG}} &= \frac{L_\tau}{2\pi\alpha'}\int_{-L_s/2}^{L_s/2} dx\,R^2\sqrt{1+\frac{R'^2}{R^4}} = \frac{L_\tau}{\pi\alpha'}\int_{R_0}^{\infty} dR\frac{R^2}{R_0^2\sqrt{R^4/R_0^4-1}} \\ &= \frac{L_\tau R_0}{\pi\alpha'}\int_1^{\infty} dy\frac{y^2}{\sqrt{y^4-1}}\,. \end{aligned} \tag{6.40}$$

This integral is divergent. As usual, the reason for this divergence has to do with the infinite volume of the bulk space. The properly regularized action is[6] [2], [57]

$$S_{\mathrm{NG}} = \frac{L_\tau R_0}{\pi\alpha'}\left[\int_1^{\infty} dy\left(\frac{y^2}{\sqrt{y^4-1}}-1\right)-1\right] = -\frac{2\sqrt{\pi}L_\tau R_0}{\Gamma(1/4)^2\alpha'} = -\frac{4\sqrt{2}\pi^2 L_\tau}{\Gamma(1/4)^4\alpha' L_s}\,. \tag{6.41}$$

where we have used (6.39) to write R_0 in terms of L_s in the last equality. Now, the heavy quark potential is given by (6.27),

$$V(L_s) = \lim_{L_\tau\to\infty}\frac{S_{\mathrm{NG}}}{L_\tau} = -\frac{4\sqrt{2}\pi^2}{\Gamma(1/4)^4\alpha' L_s}\,. \tag{6.42}$$

The potential is Coloumb-like with a $1/L_s$ behaviour. This is in fact a consequence of the conformal invariance [57]. Also note that $4\sqrt{2}\pi^2/(\Gamma(1/4)^4\alpha') > 0$ so the potential is of

[6] One way to evaluate the integral is to multiply the integrand by a factor of y^λ, integrate the two terms separately, and set $\lambda = 0$ at the end [57].

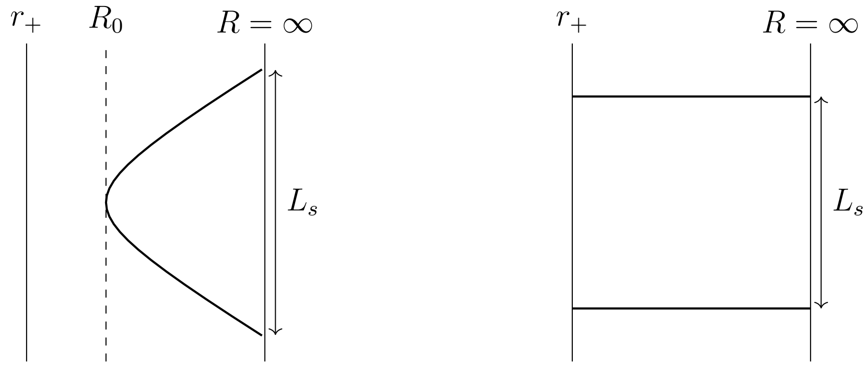

Figure 6.4: The two possible string configurations in the finite temperature case. On the left, the Wilson loop is not large enough for the string to reach the horizon before the turning point is reached. In this case the string does not feel the presence of the horizon and the situation is similar to the zero temperature case. On the right, the Wilson loop is large enough for the string to reach beyond the horizon and rupture into two disconnected strings, falling into the horizon.

the form $-c/L_s$ with a positive constant c, i.e. it is an attractive potential, as expected for a quark anti-quark pair. As mentioned earlier, the decay of the potential with L_s signals that the gauge theory is in a deconfining state. This is precisely what we expect for a boundary theory in infinite volume [53].

6.2.2 Static quark potential at finite temperature

The finite temperature case of the calculation in the previous section has been considered in [56], [59]. The finite temperature of the infinite volume boundary theory is accounted for by the addition of a black hole horizon in the bulk space. Here there are two qualitatively different string configurations possible, as depicted in Fig. 6.4. One possibility is that the Wilson loop on the boundary is small enough for the attached string not to reach the horizon in the bulk. In that case the string configuration is similar to that in the zero temperature case, with a turning point at some radial distance larger than the horizon radius. The other possibility is that the Wilson loop is large enough for the string to fall into the horizon and 'break' into two disconnected strings.

In the finite temperature case, the Euclidean bulk geometry is now given by

$$ds^2 = r^2 \left(f(r) d\tau^2 + dx^2 + dy^2 + dz^2 \right) + \frac{dr^2}{r^2 f(r)} , \tag{6.43}$$

with the emblackening factor

$$f(r) = 1 - \left(\frac{r_+}{r} \right)^4 . \tag{6.44}$$

As in the zero temperature case, we have set the AdS radius to unity. Consider first the configuration where the string does not reach the horizon and has a turning point at R_0, as shown on the left side of Fig. 6.4. Using the same coordinates as in the zero temperature case, the determinant of the intrinsic metric on the string worldsheet is now

$$h = \det\left(G_{MN}(X) \frac{\partial X^M}{\partial \sigma^m} \frac{\partial X^M}{\partial \sigma^m} \right) = R^2 f(R) \left(R^2 + \frac{R'^2}{R^2 f(R)} \right) = R^4 - r_+^4 + R'^2 . \tag{6.45}$$

It then follows that the Nambo-Goto action is given by

$$S_{\rm NG} = \frac{1}{2\pi\alpha'}\int_{-L_\tau/2}^{L_\tau/2} d\tau \int_{-L_s/2}^{L_s/2} dx\,\sqrt{h} = \frac{L_\tau}{2\pi\alpha'}\int_{-L_s/2}^{L_s/2} dx\,\sqrt{R^4 - r_+^4 + R'^2}\,. \tag{6.46}$$

Now, the conjugate momentum to $R(x)$ is

$$P(x) = \frac{\delta S_{\rm NG}}{\delta R'(x)} = \frac{L_\tau}{2\pi\alpha'}\frac{R'}{\sqrt{R^4 - r_+^4 + R'^2}}\,, \tag{6.47}$$

and the Hamiltonian is

$$H(R,P) = -\frac{L_\tau}{2\pi\alpha'}\frac{R^4 - r_+^4}{\sqrt{R^4 - r_+^4 + R'^2}}\,. \tag{6.48}$$

Since the Hamiltonian has no explicit x-dependence,

$$\frac{dH}{dx} = 0 \quad \Rightarrow \quad \frac{R^4 - r_+^4}{\sqrt{R^4 - r_+^4 + R'^2}} = \text{constant}\,. \tag{6.49}$$

Evaluating the above expression at the turning point $R = R_0$ where $R' = 0$, we find

$$\frac{R^4 - r_+^4}{\sqrt{R^4 - r_+^4 + R'^2}} = \sqrt{R_0^4 - r_+^4}\,. \tag{6.50}$$

Solving for R' yields

$$\frac{dR}{dx} = \pm\sqrt{\frac{(R^4 - r_+^4)(R^4 - R_0^4)}{R_0^4 - r_+^4}}\,, \tag{6.51}$$

and integrating this we obtain a formal expression for L_s,

$$\begin{aligned} L_s &= 2\int_0^{L_s/2} dx = 2\int_{R_0}^{\infty} dR\,\sqrt{\frac{R_0^4 - r_+^4}{(R^4 - r_+^4)(R^4 - R_0^4)}} \\ &= \frac{2}{R_0}\sqrt{1 - \frac{r_+^4}{R_0^4}}\int_1^\infty dy\,\frac{1}{\sqrt{\left(y^4 - \frac{r_+^4}{R_0^4}\right)(y^4 - 1)}}\,. \end{aligned} \tag{6.52}$$

Using (6.46) and (6.51), an expression for the static quark potential (6.27) follows,

$$\begin{aligned} V &= \frac{1}{2\pi\alpha'}\int_{-L_s/2}^{L_s/2} dx\,\sqrt{R^4 - r_+^4 + R'^2} = \frac{1}{\pi\alpha'}\int_{R_0}^{\infty} dR\,\sqrt{\frac{R^4 - r_+^4}{R^4 - R_0^4}} \\ &= \frac{R_0}{\pi\alpha'}\int_1^\infty dy\,\sqrt{\frac{y^4 - r_+^4/R_0^4}{y^4 - 1}}\,. \end{aligned} \tag{6.53}$$

Just as in the zero temperature case, this integral is divergent. The properly regularized static quark potential is given by [59]

$$V(R_0) = \frac{R_0}{\pi\alpha'}\left[\int_1^\infty \left(dy\,\sqrt{\frac{y^4 - r_+^4/R_0^4}{y^4 - 1}} - 1\right) - 1 + \frac{r_+}{R_0}\right]\,. \tag{6.54}$$

Now, in order to find the dependence of the static quark potential on the spatial side length L_s of the Wilson loop, one uses (6.52) to eliminate R_0 from (6.54) in favor of L_s. Here one has to resort to numerical methods, as opposed the zero temperature case where the analogue of this step could be done analytically. We will simply quote the results.

Note that the relations (6.52) and (6.54) are only valid for sufficiently small L_s, where the string has not reached the horizon and the string configuration is as illustrated on the left side of Fig. 6.4. For a given temperature, or equivalently, horizon radius r_+, there exists a length scale L_* for which the string does not reach the horizon if $L_s < L_*$. Furthermore, L_* should decrease with increasing temperature since r_+ increases with temperature. One finds from the numerical analysis that the static quark potential will have a Coulomb like behaviour, $V(L_s) \sim -1/L_s$ for $L_s < L*$. The area and perimeter laws for Wilson loops now tells us that the corresponding dual gauge theory is in a deconfined state.

For $L_s > L_*$, however, the string configuration will be that of two disconnected strings dropping through the horizon, as depicted on the right side of Fig. 6.4. In this case the static quark potential is simply given by [2]

$$V = -\frac{r_+}{\pi\alpha'}\,. \tag{6.55}$$

This is manifestly independent of L_s and the dual gauge theory should therefore again be in the deconfining phase. Thus, the infinite volume boundary theory is in a deconfined state at any temperature, zero or finite. This is the expected result [53]. In other words, no confinment-deconfinement phase transition can occur in a conformal gauge theory living on a boundary space with infinite spatial volume.

This is not the case for a finite volume boundary theory. Indeed, the 'global AdS' bulk spaces involved in the Hawking-Page transition have dual quantum field theories defined on finite volume spaces. If we consider the same finite temperature theory in global AdS coordinates, the conformal boundary has finite volume and the field theory living there is no longer a CFT. Here one indeed finds a phase transition between a confining and deconfining phase, as was argued in [53]. When $L_s < L_*$ the Wilson loop is sufficiently small to only probe the high energy, short distance processes. These are insensitive to the finite temperature, which affects only low energy phenomena. This is geometrically encoded in the bulk spacetime, where the string attached to the Wilson loop does not reach deep enough to feel the presence of the black hole horizon. In this case the boundary gauge theory is found to be in a confining state. On the other hand, when $L_s > L_*$ the Wilson loop will probe low energy, large distance physics which are highly sensitive to the finite temperature. The string in the bulk space now reach the horizon and breaks into two disconnected strings, signaling a transition from a confining to a deconfining state in the dual gauge theory. This confinement-deconfinement phase transition should be interpreted as the holographic dual to the Hawking-Page transition, which essentially is the transition between a thermal AdS space with a black hole horizon, and one without. The way in which this transition is geometrically described by a string stretching in the dynamical bulk spacetime once again supports the interpretation of the extra bulk dimension as the RG energy scale of the dual QFT.

In conclusion, the large N conformal field theories of AdS/CFT dualities are in a deconfining phase at any finite temperature when they live in infinite volume. Breaking the conformal invariance by introducing a length scale associated with a finite volume is enough to create a stable confining ground state. In the dual gravitational theory, this is described by the presence or absence of a black hole horizon. If a horizon exists, the dual QFT must be in a deconfining phase. Otherwise, a finite temperature QFT, which necessarily is dual to 'thermal AdS', must be in a confining phase. The infinite volume CFT at zero temperature is, however, deconfining. Since the Hawking-Page transition is the thermal phase transition between a 'thermal AdS' spacetime and an AdS black hole spacetime, we interpret it as the holographic dual of a confinment-deconfinement transition in gauge theory.

At first sight, the fact that the Hawking-Page transition, which is claimed to be a thermal phase transition, occurs as at finite volume seems to be in conflict with thermodynamic principle, stating that thermodynamic critical points can exist only in the thermodynamic limit[7]. The infinite volume associated with the thermodynamic limit suppresses thermal fluctuations. In the finite volume holographic duality, it is in fact the large N limit which is responsible for the existence of a thermal phase transition, as thermal fluctuations are suppressed at large N in a similar way as they are suppressed in the thermodynamic limit [2]. This is an instance where we explicitly see the mean field nature of the large N limit come to life.

[7]The thermodynamic limit is defined as the limit of a large number of microscopic degrees of freedom N, where the volume V of the space containing the microscopic degrees of freedom grows in proportion to N, i.e. $N \to \infty$ and $V \to \infty$ with $N/V =$ constant.

7

Holographic Superconductors

In this chapter we turn to the subject of holographic superconductivity, one of the earlier applications of the AdS/CFT correspondance to condensed matter physics. It was first discovered by Gubser in 2008 [60]. The same year a minimal bottom-up model for holographic superconductivity was constructed by Hartnoll, Herzog, and Horowitz [9], [10]. We will exclusively work with this minimal bottom-up toy model here.

The holographic duality maps a strongly coupled QFT with a superconducting ground state to a gravitational theory in one higher dimension. In the original model of a minimal holographic superconductor, the QFT is strictly speaking that of a charged 'superfluid' condansate, a state spontaneously breaking a global $U(1)$ symmetry. The charged superfluid does not have a dynamical photon which would require a local $U(1)$ symmetry, characteristic of Maxwell's theory of electrodynamics. In order to study properties emerging from a dynamical charge response such as charged collective excitations, e.g. plasmon modes, the theory must contain a dynamical photon. The standard procedure is to weakly gauge the superfluid theory, promoting it to a superconductor [2], [10]. However, we will take a different approach. By introducing a set of mixed boundary conditions on the conformal boundary of AdS, one can account for a dynamical photon in the dual QFT [13].

After introducing electrodynamical effects in this way, we will attempt to investigate the behaviour of charged collective excitations, or more specifically, plasmons, in holographic superconductors. In particular, we attempt to calculate dispersion relations for plasmon modes. It should be noted that introducing a set of mixed boundary conditions is equivalent to performing a multi-trace deformation of the theory, as explained in 4.2.3. The mixed boundary condition proposed in [13] is in fact equivalent to a double-trace deformation, resulting in a Dyson re-summation of the dynamical charge susceptibility. Plasmons in holographic theories have been studied previously using the double-trace deformation approach in e.g. [12], [14]. To the best of my knowledge, plasmons have not previously been studied in this holographic superconductor model. It turns out, however, that the mixed boundary condition most likely needs to be modified for this particular model.

The strongly coupled QFT describing a superconductor is charactarized by the existance of a critical temperature T_c, at which it undergoes a phase transition between a superconducting and normal state. Below the critical temperature the DC conductivity becomes infinite. This strongly interacting superconductor is a candidate model for the high T_c superconductors realized in e.g. cuprates and pnictides [3]. Since the discovery of high T_c superconductors in the 1980s, theoretical physicists have struggled with formulating a satisfying theory describing these systems. The well understood BCS (Bardeen-Cooper-Schrieffer) theory of superconductivity does not capture the behaviour of the high T_c superconductors accurately. In BCS theory, the superconductivity is explained by the formation Cooper pairs and their condensation into an Einstein-Bose condansate.

However, the mechanism responsible for the superconductiviy in high T_c superconductors remains unknown. The hope is that the holographic models of superconductors will shed some light on the issue of high T_c superconductivity.

The normal state of the holographic superconductor is a Reissner-Nordström (RN) metal, which is a type of 'strange metal'. We have not included the topic of RN black hole solutions and strange metals[1] in this thesis and will solely focus on the superconducting phase. For a complete understanding of the superconducting phase transition, however, an understanding of both the superconducting and normal phase is required. Understanding the competition between these two phases is a necessary requirement for understanding what determines the critical temperature T_c, and how it eventually could be raised to room temperature.

Again, the QFT describing the superconductor has a holographically dual gravitational description in terms of a dynamical bulk spacetime, having one additional spatial dimension. The gravitating bulk spacetime is taken to be asymptotic to AdS space, and the QFT is defined on the conformal AdS boundary of the bulk spacetime. The boundary QFT has a large N limit, in which quantum corrections in the bulk theory are suppressed. Since quantum effects in the gravitational theory are poorly understood, we are restricted to work in the large N limit. Any realistic theory of a strongly coupled superconducting system is not likely to have an infinite number of colour charges, which are associated with the large N limit. However, many of the physical properties of the system which can be studied in the large N limit are thought to be generic, insensitive to a departure from the large N limit to finite and small N.

In the minimal bottom-up model of a holographic superconductor to be studied, the field content in the bulk consists of a metric field, a Maxwell gauge field, and a charged scalar field. A charged scalar field is necessarily complex, as can easily be verified by calculating the associated Noether current. In the dual gravitational theory, the superconducting phase transition is described by an instability of a charged Reissner-Nordström black hole to perturbations by the charged scalar field. For Hawking temperatures below T_c, the black hole acquires scalar hair, i.e. an atmosphere of a charged scalar condensate forms around the horizon where the complex scalar field has non-vanishing amplitude. This does not violate the famous 'no-hair theorems' since these where derived for black holes in a four dimensional flat spacetime, and they are not valid for black holes in AdS space. In the boundary QFT description, the formation of scalar hair correspond to the dual charged scalar operator forming a 'superfluid' condensate. The formation of a charged scalar condensate around the black hole 'spontaneously' break the $U(1)$ gauge symmetry[2] in the bulk theory. The superfluid condensate in the quantum theory spontaneously break the global $U(1)$ symmetry associated with a conserved current (which is the dual quantity of the $U(1)$ gauge field in the bulk theory); in the condensed phase, the charged scalar operator acquires a nonzero expectation value in the absence of an external source.

Since the charged operator that condense is a scalar, the theory is that of an s-wave

[1]The interested reader is referred to e.g. [2] for an excellent introduction to the subject of strange metals and RN black holes in holography.

[2]It has been argued that a local continuous symmetry cannot be spontaneously broken [61], (see [62] for an introduction to spontaneous symmetry breaking.) What is happening around the black hole is that the scalar field acquires an negative effective mass squared, resulting in unstable tachyonic modes. This triggers a Higgs mechanism which break the $U(1)$ gauge symmetry. It is perhaps somewhat incorrect to attach 'spontaneous' to this symmetry breaking, even though it is not triggered by an external source. Many authors does, however, refer to this as spontaneous symmetry breaking.

superconductor, i.e. the superconducting condensate does not carry any angular momentum. s-wave superconductivity has been incorporated in a strongly coupled model of cuprate superconductors [63]. However, theories of unconventional p-wave and d-wave superconductors are also of importance for modeling experimentally observed high T_c superconductors. For instance, d-wave superconductivity plays an important role in cuprate high T_c superconductors [64]. Although many real life unconventional superconductors are of d-wave type, the effective low energy dynamics is often s-wave [1]. p-wave superconductivity has previously been incorporated in holographic models by promoting the $U(1)$ gauge field to a non-Abelian $SU(2)$ gauge field [65]–[67], but also by promoting the complex scalar field to a complex vector field, keeping the Maxwell gauge field [68], [69]. Holographic models of d-wave superconductors have been studied in [70]–[72]. For an introduction to various models of holographic superconductors with different angular momenta, see [73]. We will only study s-wave superconductivity here, though it would be interesting to incorporate a dynamical photon and study charged collective excitations in p-wave and d-wave holographic superconductors as well.

The outline of this chapter is as follows: First we introduce the minimal bottom-up model of a holographic superconductor, originally introduced in [9], [10]. We proceed by considering a stationary hairy black hole solution and associated boundary conditions. This solution will act as a background, which we will then perturb in order to analyse the plasmon modes in a linear response analysis. In particular, we will study the longitudinal modes of perturbations to the metric, gauge, and scalar fields, and implement the mixed boundary condition proposed in [13] to account for a Coulomb interaction in the dual QFT. The equations of motion cannot be solved analytically, and so we must resort to numerical methods. We will nevertheless outline how the numerical computations are made.

7.1 A minimal toy model of holographic superconductivity

When attempting to write down a physical theory, it is generally a good idea to start of by thinking about which symmetries are present. Superconducting quantum matter are found in crystal structures and therefore any realistic theory of such systems should be defined on a lattice. However, by defining the quantum theory on a lattice, the spatial translational invariance is broken. The symmetry correspondence then implies that the dual gravitational theory should not posses spatial translational invariance either. Solving Einstein's equations without translational symmetry is much more complicated due to the nonlinear structure of the equations. Holography with broken translational invariance has been an active research field under the last couple of years, and a lot of progress has been made since the discovery of holographic superconductivity. A minimal bottom-up model of a holographic superconductor has been studied under the presence of a periodic lattice potential in [74]. However, here we will stick to the Galilean continuum limit where translational symmetry and momentum conservation are present, which provides an accurate effective description of the low energy dynamics. Hence, our boundary theory will have a conserved energy-momentum tensor $T_{\mu\nu}$, which will be dual to the metric field g_{MN} in the bulk spacetime.

The boundary theories in AdS/CFT where originally relativistic. Recently, however, non-relativistic holographic dualities have been developed [75], [76]. We will consider a relativistic boundary QFT and hence impose Lorentz invariance, in order to avoid unnecessary complications. It would of course be interesting to generalize our analysis to the non-relativistic case.

The superconductivity is the result of a spontaneously broken global $U(1)$ symmetry. Thus, there should exist a corresponding conserved Noether current $\mathcal{J}^\mu$ in the boundary theory. The global current $\mathcal{J}^\mu$ should be dual to a $U(1)$ gauge field A_M in the bulk according to the global/local symmetry correspondence. The $U(1)$ symmetry, which is associated with a conserved number of particles, is spontaneously broken by the condensation of a charged scalar operator $\mathcal{O}$ at temperatures below a critical temperature T_c. This charged condensate has an indefinite amount of particles and forms a 'superfluid'. The expectation value of this superfluid is an order parameter of the broken symmetry and superconductivity. Now, the charged scalar operator is dual to a charged scalar field ϕ in the bulk, i.e. a complex scalar field. When the global $U(1)$ symmetry in the QFT is spontaneously broken by the charged superfluid, the $U(1)$ gauge symmetry in the gravitational theory should be broken as well. As a matter of fact, the only known way of constantly breaking a $U(1)$ gauge symmetry is through the Higgs mechanism, and this is indeed the mechanism behind the broken gauge symmetry in the gravitational theory. Thus, the superconducting phase transition in the D-dimensional quantum theory has a dual description in terms of a Higgs mechanism in a dynamical spacetime of dimension $D + 1$.

Having specified the field content in the bulk spacetime, the next step is to write down an action. The obvious ingredients for a minimal action is an Einstein-Hilbert term with a negative cosmological constant, kinetic terms for the gauge field and the complex scalar field, and a potential for the scalar field. The action then takes the form

$$S = \int d^{d+2}x\sqrt{-g}\bigg[\frac{1}{2\kappa^2}\left(R + \frac{d(d+1)}{L^2}\right) - \frac{1}{4g_F^2}F^{MN}F_{MN} - \left|\nabla_M\phi - iqA_M\phi\right|^2 - V(|\phi|)\bigg]. \tag{7.1}$$

Here ∇_M is the covariant derivative under spacetime diffeomorphisms. Note also that $D_M\phi = \nabla_M\phi - iqA_M\phi$ is a covariant derivative under $U(1)$ gauge transformations. For simplicity, we will set $2\kappa^2 = g_F^2 = 1$. Furthermore, we take the potential $V(|\phi|)$ to be only a mass term,

$$V(|\phi|) = m^2|\phi|^2\,, \tag{7.2}$$

and we will in particular consider the case of $m^2 = -2/L^2$. The mechanism behind the holographic superconductivity will be present despite neglecting any kind of self-interactions of the scalar field.

Recall from 4.2.2 that a bulk scalar field with a negative mass squared is dual to a relevant scalar operator. The value $m^2 = -2/L^2$ is above the BF bound in a boundary theory with two or more spatial dimensions, i.e. $d \geq 2$,

$$m^2L^2 \geq -\frac{(d+1)^2}{4} = -\frac{9}{4}\,. \tag{7.3}$$

We will restrict ourselves to the case of $d = 2$, i.e. a $3 + 1$ dimensional bulk spacetime dual to a QFT in $2 + 1$ dimensions. The theory should then be considered as a toy model for a superconducting plane or surface. Since the negative mass squared of the scalar field is above the BF bound, it does not introduce any tachyonic instability to the theory.

Moreover, it is below the upper limit on the mass squared corresponding to the unitary bound on the dual boundary theory operator,

$$m^2L^2 \leq -\frac{(d+1)^2}{4} + 1 = -\frac{5}{4}\,. \tag{7.4}$$

We are then free to choose either the standard or the alternative quantization scheme, i.e. either the leading or subleading behaviour of the bulk scalar field at the conformal boundary may be identified with the source of the dual QFT operator, while the other is identified with its expectation value. Note, however, that while our bottom-up approach allows us to choose a convenient potential for the complex scalar field, a string theoretic top-down model will specify the full potential completely.

In the case of $d = 2$ and with our choice of potential and dimensionful constants, the action (7.1) takes the form

$$S = \int d^4x\sqrt{-g}\left[R + \frac{6}{L^2} - \frac{1}{4}F^{MN}F_{MN} - |\nabla_M\phi - iqA_M\phi|^2 - m^2|\phi|^2)\right]\,. \tag{7.5}$$

Varying the action with respect to the different fields gives the equations of motion. One finds that the equation of motion for the scalar field is

$$(\nabla_M - iqA_M)\,(\nabla_N - iqA_N)\,g^{MN}\phi - m^2\phi = 0\,. \tag{7.6}$$

Maxwell's equations are

$$\nabla^M F_{MN} = iq\left[\phi^*\left(\nabla_N - iqA_N\right)\phi - \phi\left(\nabla_N + iqA_N\right)\phi^*\right]\,, \tag{7.7}$$

and Einstein's equations read

$$\begin{aligned} R_{MN} - \frac{1}{2}g_{MN}R - \frac{3}{L^2}g_{MN} &= \frac{1}{2}F_M{}^KF_{NK} - \frac{1}{8}g_{MN}F^2 - \frac{1}{2}g_{MN}|\nabla_K\phi - iqA_K\phi|^2 \\ &- \frac{1}{2}g_{MN}m^2|\phi|^2 + \frac{1}{2}\left[(\nabla_M\phi - iqA_M\phi)\left(\nabla_N\phi^* + iqA_N\phi^*\right) + M \longleftrightarrow N\right]\,. \end{aligned} \tag{7.8}$$

Here $M \longleftrightarrow N$ denotes the previous term but with indices M and N interchanged.

Having found the equations of motion for the minimal holographic superconductor, the next step is to find solutions to them. We will first look for a stationary black hole solution describing the equilibrium phases of the superconductor. This solution will then act as a background when we later proceed by analysing the linear response of the theory, in 7.4.

7.2 A stationary black hole solution

In order for a phase transition to be able to occur at a finite critical temperature T_c, the theory has to posses a scale. Thus, the conformal symmetry of the QFT in the AdS/CFT duality has to be broken. A stationary black hole solution naturally introduce a temperature scale to the theory; the Hawking temperature of the black hole is identified with the boundary field theory temperature. However, in the absence of another scale besides the temperature, all finite temperatures would be equivalent since there would be no other scale to compare the temperature with. Then, to properly break the scale invariance, an additional scale has to be introduced. By adding an electrical charge to the black hole we incorporate a finite charge density ρ, or equivalently, a finite chemical potential μ, to the boundary QFT. Charged black holes are described by a Reissner-Nordström solution.

We take the following ansatz for the metric:

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)e^{-\chi(z)}dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2\right). \tag{7.9}$$

Here z is the radial coordinate, (identified with the QFT renormalization scale,) with $z = z_+$ being the location of the planar horizon, and $z = 0$ being the location of the conformal boundary of AdS. The function $\chi(z)$ in the time component account for the backreaction of the scalar field on the geometry. One can consider the theory in the probe limit $q \to \infty$, where the $U(1)$ gauge field and scalar field decouples from the metric field. Then the matter fields does not backreact on the AdS geometry, which effectively becomes a static background space. This simplifies the theory considerably. However, we will consider the complete theory, including an interaction between the geometry and matter fields. Then, neither $\chi(z)$ nor the emblackening factor $f(z)$ can be determined analytically in the superconducting phase, and will have to be solved for numerically. We take the Maxwell gauge field to be a pure electrostatic potential, dependent only on the radial coordinate,

$$A = A_0(z)dt\,. \tag{7.10}$$

The complex scalar field is also assumed to only depend on z,

$$\phi = \phi(z)\,. \tag{7.11}$$

Moreover, the z-component of Maxwell's equation (7.7) implies that the phase of ϕ is constant. Hence, without loss of generality, we may take ϕ to be real valued.

7.2.1 An instability near the horizon

Looking at the action (7.5), we see that the coupling with the Maxwell field in the gauge covariant derivative effectively shifts the mass of the complex scalar field,

$$m_{\text{eff}}^2 = m^2 - \left|g^{tt}\right|q^2 A_0^2\,. \tag{7.12}$$

The term $|g^{tt}|q^2A_0^2$ will typically decay towards the conformal boundary of AdS, and consequently, $m_{\text{eff}} \approx m^2$ in the far UV region. However, in the deep IR the effective mass can fall below the BF bound for stability due to a negative contribution from the second term in (7.12). Indeed, $|g^{tt}| \to \infty$ when approaching the horizon and therefore the effective mass should be negative enough at some short distance from the horizon, assuming $A_0 \neq 0$ there. It turns out, however, that one must choose a gauge where $A_0 = 0$ on the horizon in order for the gauge one-form to be well defined there[3]. It is certainly plausible that a large enough q and A_0' slightly outside the horizon, and a small enough value of m^2, could be sufficient for shifting m_{eff}^2 below the stability bound. Although a rigorous proof of the existence of an instability in the IR region at small enough temperatures is still lacking, there are a lot of empirical evidence showing that

[3]The gauge one-form is defined as $A = A_M dx^M = A_0 dt$. The infinite redshift at the horizon means that the dt norm is infinite there, and therefore A_0 must be zero at the horizon as to keep A from blowing up there.

it indeed occurs in holographic models. Hence, the charged scalar field can induce an instability restricted to the IR region of the bulk spacetime. This kind of instability can then regulate itself as unstable modes condensate and backreact on the IR geometry, altering it in such a way that the theory is stabilized [1].

This is the mechanism behind the superconducting phase transition in the gravitational language[4]. Below a critical temperature, the complex scalar field acquires an effective mass squared below the BF bound in the deep interior, causing an instability of the black hole. The scalar field then forms a tachyonic condensate around the horizon, giving the black hole 'scalar hair'. This condensate backreacts on the near horizon geometry in such a way that the instability is removed. Above the critical temperature, the effective mass squared does not drop below the BF bound for stability in the deep interior. Consequently, the scalar field does not condansate near the black hole which remains 'bald' at these temperatures. In this high temperature phase, the amplitude of the scalar field is zero and the bulk spacetime is described by a Reissner-Nordström black hole solution in AdS space,

$$\begin{aligned} &\phi(z) = \chi(z) = 0\,, \\ &f(z) = 1 - \left(\frac{1}{z_+^3} + \frac{1}{2}Q^2 z_+\right) z^3 + \frac{1}{2}Q^2 z^4\,, \\ &A_0(z) = LQ\left(z_+ - z\right)\,, \end{aligned} \tag{7.13}$$

where Q is a free parameter of the solution, corresponding to the charge of the black hole [77].

7.2.2 Equations of motion and boundary conditions

With the metric field, Maxwell field, and complex scalar field being given by (7.9), (7.10), and (7.11), respectively, the equations of motion (7.6), (7.7), and (7.8) reduces to a set of coupled ordinary differential equations in the variable z. The equation of motion for the complex scalar field becomes

$$\phi'' + \left(\frac{f'}{f} - \frac{\chi'}{2} - \frac{2}{z}\right)\phi' + \left(\frac{q^2 A_0^2 e^{\chi}}{f^2} - \frac{m^2 L^2}{z^2 f}\right)\phi = 0\,, \tag{7.14}$$

where the prime denotes a derivative with respect to z. Maxwell's equations reduces to

$$A_0'' - \frac{\chi'}{2}A_0' + \frac{2L^2 q^2 \phi^2}{z^2 f} A_0 = 0\,. \tag{7.15}$$

From Einstein's equations we get two independent equations,

$$\begin{aligned} &f' - \frac{z}{4}\left(\frac{2q^2 A_0^2 \phi^2 e^{\chi}}{f} + \frac{12L^2(f-1) + 2m^2 L^4 \phi^2 + z^4 {A_0'}^2 e^{\chi} + 2L^2 z^2 f \phi'^2}{L^2 z^2}\right) = 0\,, \\ &\chi' + \frac{z}{4}\left(\frac{12}{z^2} - 2\phi'^2 + \frac{2m^2 L^4 \phi^2 + z^4 {A_0'}^2 e^{\chi} - 4L^2(3 + z f')}{L^2 z^2 f} - \frac{2q^2 A_0^2 \phi^2 e^{\chi}}{f^2}\right) = 0\,. \end{aligned} \tag{7.16}$$

[4] A near extremal charged black hole is also unstable to perturbations by a neutral scalar field. An explanation for this phenomenon accounts for the fact that the near horizon geometry of the extremal solution is $AdS_2 \times \mathbb{R}^2$, rather than AdS_4. The former geometry has a less negative BF bound than the latter, and can therefore cause a tachyonic instability for neutral scalar fields near the extremal black hole horizon, without causing any instability in the UV region. An instability in the UV would be pathological.

Together, (7.14)-(7.16) constitute a system of four coupled differential equations for the four unknown functions $f(z)$, $\chi(z)$, $A_0(z)$, and $\phi(z)$. The equations are first order in f and χ, and second order in A_0 and ϕ. Hence, six boundary conditions have to be supplied in order to create a well-posed boundary value problem; two boundary conditions for A_0, two for ϕ, and one for f and χ each.

One boundary condition comes directly from the infinite redshift at the black hole horizon, namely $f(z_+) = 0$. In fact, this defines the horizon radius z_+. We should then look for solutions where $f(z) \neq 0$ for any $z < z_+$. As mentioned earlier, we must also impose the boundary condition $A_0(z_+) = 0$ in order for the gauge one-form to be well-defined at the horizon.

The four coupled differential equations (7.14)-(7.16) are singular both at the horizon $z = z_+$ and on the conformal boundary at $z = 0$. We deal with the singular behaviour at these boundaries in the most simple and robust way possible, by introducing cutoff boundaries at $z = z_+ - \epsilon$ and $z = \epsilon$ for some small parameter ϵ. The equations are then solved numerically only in the region $\epsilon \leq z \leq z_+ - \epsilon$. Now, by expanding the equations of motion around the horizon we can impose consistent boundary conditions on the cutoff boundary at $z = z_+ - \epsilon$. To lowest order, one finds

$$\begin{aligned} f'(z_+) &= \frac{-12L^2 + 2m^2L^4\phi_+^2 + e^{\chi_+}E_+^2}{4L^2}\,, \\ \phi'(z_+) &= \frac{4m^2L^4\phi_+^2}{-12L^2 + 2m^2L^4\phi_+^2 + e^{\chi_+}E_+^2}\,, \end{aligned} \tag{7.17}$$

where $\phi_+ = \phi(z_+)$, $E_+ = A_0'(z_+)$, and $\chi_+ = \chi(z_+)$ are degrees of freedom on horizon. Together with the horizon radius r_+, these are the parameters on the horizon which are free for us to specify, and different choices yield different solutions. Solving the expanded equations around the horizon to higher orders gives successively higher order derivatives of the fields, evaluated at the horizon. The boundary conditions at the horizon is related to boundary conditions on the cutoff surface at $z = z_+ - \epsilon$ by

$$\begin{aligned} f(z_+ - \epsilon) &= f(z_+) - \epsilon f'(z_+) + \mathcal{O}(\epsilon^2) = \frac{-12L^2 + 2m^2L^4\phi_+^2 + e^{\chi_+}E_+^2}{4L^2}\epsilon + \mathcal{O}(\epsilon^2)\,, \\ \chi(z_+ - \epsilon) &= \chi(z_+) - \epsilon\chi'(z_+) + \mathcal{O}(\epsilon^2) = \chi_+ + \mathcal{O}(\epsilon)\,, \\ A_0(z_+ - \epsilon) &= A_0(z_+) - \epsilon A_0'(z_+) + \mathcal{O}(\epsilon^2) = -\epsilon E_+ + \mathcal{O}(\epsilon^2)\,, \\ A_0'(z_+ - \epsilon) &= A_0'(z_+) - A_0''(z_+ - \epsilon) + \mathcal{O}(\epsilon^2) = E_+ + \mathcal{O}(\epsilon)\,, \\ \phi(z_+ - \epsilon) &= \phi(z_+) - \epsilon\phi'(z_+) + \mathcal{O}(\epsilon^2) = \phi_+ - \frac{4m^2L^4\phi_+^2}{-12L^2 + 2m^2L^4\phi_+^2 + e^{\chi_+}E_+^2}\epsilon + \mathcal{O}(\epsilon^2)\,, \\ \phi'(z_+ - \epsilon) &= \phi'(z_+) + \mathcal{O}(\epsilon) = \frac{4m^2L^4\phi_+^2}{-12L^2 + 2m^2L^4\phi_+^2 + e^{\chi_+}E_+^2} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{7.18}$$

Here we have given explicit expressions for the boundary conditions only up to the zeroth or first order in ϵ, and omitted higher order terms since they are quite lengthy. When solving the equations numerically, however, we have included second order corrections as well.

The six boundary conditions (7.18) are sufficient for uniquely solving the coupled set of ODEs (7.14)-(7.16) numerically by integrating the equations from the near horizon boundary at $z = z_+ - \epsilon$, to the cutoff surface at $z = \epsilon$, near the conformal boundary.

However, some conditions also has to be imposed on the asymptotic near boundary behaviours of the fields.

The chemical potential μ and charge density ρ of the boundary QFT are identified as the leading and subleadning asymptotic behaviour, respectively, of the electrostatic potential A_0 near the conformal boundary. More precisely,

$$A_0(z) = \mu - \rho z + \cdots \quad \text{as} \quad z \to 0\,. \tag{7.19}$$

With our choice of mass, $m^2L^2 = -2$, the asymptotic behavior of the complex scalar field is [10]

$$\phi(z) = z\phi^{(1)} + z^2\phi^{(2)} + \cdots \quad \text{as} \quad z \to 0\,. \tag{7.20}$$

As pointed out earlier, the mass falls within the range where both the standard and alternative quantization schemes are admissible. In standard quantization, we identify the leading behaviour $\phi^{(1)}$ with the source of the dual charged scalar operator in the QFT, and the subleading part $\phi^{(2)}$ with its expectation value. In alternative quantization, the roles of $\phi^{(1)}$ and $\phi^{(2)}$ are interchanged, i.e. $\phi^{(1)}$ is identified with the expectation value and $\phi^{(2)}$ with the source. Here we will only consider the standard quantization theory.

In general, both the leading part, $\phi^{(1)}$, and the subleading part, $\phi^{(2)}$, are non-vanishing. Characteristic for the symmetry breaking charged scalar condensate is a nonzero expectation value in the absence of any external source. Indeed, an external source would explicitly break the symmetry. Thus, we are looking for solutions with

$$\phi^{(1)} = 0\,, \qquad \langle \mathcal{O}_2 \rangle = \sqrt{2}\phi^{(2)}\,, \tag{7.21}$$

where the subscript two on the QFT charged scalar operator $\mathcal{O}$ denotes its mass dimension, (in the alternative quantization the operator would have mass dimension one,) and the factor of $\sqrt{2}$ is a convenient normalization [9], [10]. Hence, we must impose the boundary condition $\lim_{z\to 0} \phi'(z) = 0$. Here one could remove one of the boundary conditions for ϕ or ϕ' on the horizon (7.18) and add the zeroth order boundary condition $\phi'(\epsilon) = 0$ on the cutoff surface near the conformal boundary. Then the system of ODEs could be solved using a shooting method. However, we will take a different approach to implementing this boundary condition.

By fixing χ_+ and E_+, and varying ϕ_+, we can solve the system of coupled differential equations (7.14)-(7.16), using the near horizon boundary conditions (7.18), to find out how the source $\phi^{(1)}$ depends on ϕ_+. Plotting $\phi^{(1)}$ versus ϕ_+ allows us to read off which values of ϕ_+, given a fixed χ_+ and E_+, results in a vanishing source $\phi^{(1)} = 0$. Such a plot is shown in Fig. 7.1. As seen in the figure, there will in general be several roots of $\phi^{(1)}(\phi_+) = 0$. For one thing, $\phi_+ = 0$ is always a solution but this just results in the Reissner-Nordström black hole solution (7.13) with a vanishing scalar field over the whole bulk space. However, there will also be multiple nonzero roots. The root closest to zero corresponds to the solution with a monotone scalar profile, while the larger roots correspond to scalar profiles with a higher number of nodes. The monotonic scalar profile is the configuration which minimizes the free energy, i.e. it is the thermodynamically preferred solution and represents the ground state.

We are interested in solutions where the bulk spacetime is asymptotic to AdS_4. This is necessary in order for the Hawking temperature of the black hole to coincide with the temperature of the boundary QFT. It is readily seen that the metric (7.9) is asymptotic to a pure AdS_4 metric, in Poincaré coordinates, if $f(z) \to 1$ and $\chi(z) \to 0$ as $z \to 0$. The correct asymptotic value of f is automatically embedded in the solutions to the equations

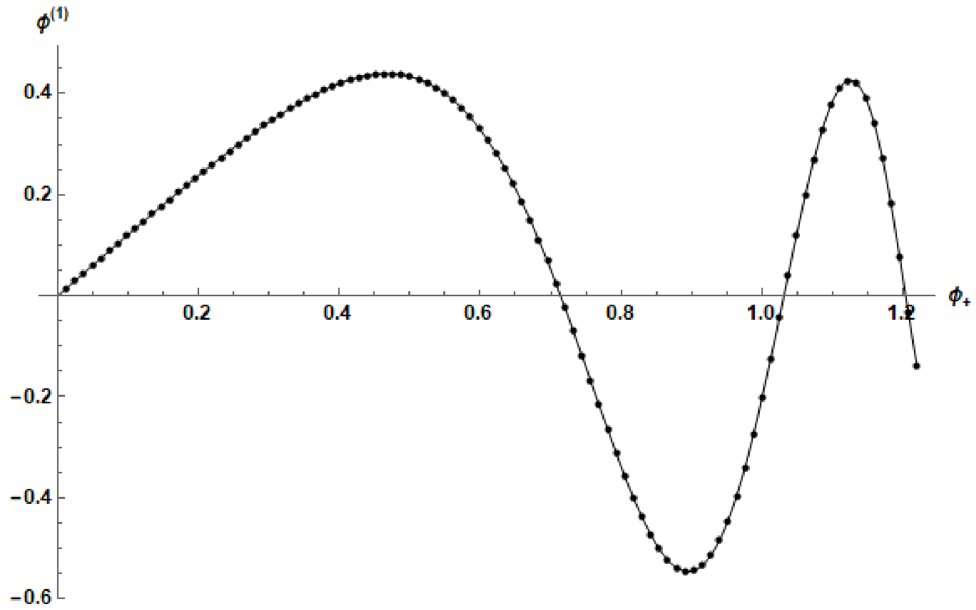

Figure 7.1: A plot of the source $\phi^{(1)}$ versus the value of the scalar field at the horizon ϕ_+, with $E_+ = 0.75$ and $\chi_+ = -1$ held fixed. The dotted points have been numerically computed by solving the equations of motion. The solid line is a polynomial fit to these points. Three nontrivial values of ϕ_+ for which $\phi^{(1)} = 0$ are captured in the plotted interval. Of these, the one at $\phi_+ \approx 0.714$ corresponds to a solution with a monotonic scalar field while the other two correspond to scalar fields with nodes. The monotonic profile represents the ground state.

of motion. However, the asymptotic value of χ is generally not zero. In order to set it to zero, we can utilize the following scaling symmetry of the metric, gauge field, and equations of motion:

$$t \to at\,, \qquad e^{\chi} \to a^2 e^{\chi}\,, \qquad A_0 \to A_0/a\,. \tag{7.22}$$

In practise, after solving the equations of motion (7.14)-(7.16) subject to the near horizon boundary conditions (7.18), we rescale the solution by

$$\chi(z) \to \chi(z) - \chi(\epsilon) + \epsilon\chi'(\epsilon)\,, \qquad A_0(z) \to A_0(z)\exp\left(\frac{\chi(\epsilon) - \epsilon\chi'(\epsilon)}{2}\right). \tag{7.23}$$

In addition to (7.22), there are two other scaling symmetries which allows us to set $L = z_+ = 1$ [2], [10].

7.2.3 Hawking temperature

The temperature in the boundary theory is identified as the Hawking temperature of the black hole in the bulk theory. The standard procedure for calculating the Hawking temperature is to perform a Wick rotation and compute the radius of the imaginary time circle. This was done in Chapter 5 for the case of a diagonal black hole metric with a radial component given by the inverse of the time component, $g_{rr}(r) = 1/g_{tt}(r)$. This is not the case for our stationary black hole geometry (7.9), and so the derivation has to be somewhat altered. Writing the Euclidean metric in the form

$$ds^2 = u(z)d\tau^2 + \frac{dz^2}{v(z)} + \frac{L^2}{z^2}d\mathbf{x}^2\,, \tag{7.24}$$

where $d\mathbf{x}^2 = dx_1 + \cdots + dx_d$, and expanding to the lowest order around the horizon $z = z_+$, defined by $u(z_+) = v(z_+) = 0$, we get

$$ds^2 = |u'(z_+)|(z_+ - z)d\tau^2 + \frac{dz^2}{|v'(z_+)|(z_+ - z)} + \frac{L^2}{z_+^2}d\mathbf{x}^2 + \cdots . \tag{7.25}$$

Here we have assumed a non-zero $u'(z_+)$ and $v'(z_+)$. Accounting for the possibility of negative $u'(z_+)$ and $v'(z_+)$, the absolute signs have to be included in order to guarantee the Euclidean signature of the metric. Performing the coordinate transformation

$$z \to \rho = 2\sqrt{\frac{z_+ - z}{|v'(z_+)|}}, \qquad \tau \to \varphi = \frac{\sqrt{|u'(z_+)v'(z_+)|}}{2}\tau , \tag{7.26}$$

introduces polar coordinates on the plane spanned by τ and z, i.e. the metric takes the form $ds^2 = \rho^2 d\varphi^2 + d\rho^2 + \cdots$. The origin in this coordinate system is at the horizon. Then, in order to avoid a conical singularity at the horizon, φ must be periodic with period 2π, implying that τ must have a period $4\pi/\sqrt{u'(z_+)v'(z_+)}$. The Hawking temperature of the black hole is given by the inverse period of τ,

$$T = \frac{\sqrt{|u'(z_+)v'(z_+)|}}{4\pi} . \tag{7.27}$$

Now, with the geometry (7.9) for the stationary black hole solution, we have

$$u'(z_+) = \frac{L^2}{z_+^2}f'(z_+)e^{-\chi_+} , \qquad v'(z_+) = \frac{z_+^2 f'(z_+)}{L^2} . \tag{7.28}$$

The temperature of the black hole is then

$$T = \frac{e^{-\chi_+/2}|f'(z_+)|}{4\pi} = \frac{\left|(-12L^2 + 2m^2L^4\phi_+^2)e^{-\chi_+/2} + e^{\chi_+/2}E_+^2\right|}{16\pi L^2} , \tag{7.29}$$

where we have used (7.17) to evaluate $f'(z_+)$ in the last equality.

7.3 Charged collective excitations and mixed boundary conditions

We will now derive a mixed boundary condition which introduces a dynamical photon in the boundary QFT, following [13] closely. This allows one to study charged collective excitations, e.g. plasmons, in a strongly interacting quantum theory from a holographically dual gravitational theory. Later, by implementing this boundary condition on longitudinal fluctuations of the Maxwell gauge field around the stationary black hole background, we will investigate the dispersion relation for plasmon modes in our holographic superconductor. It turns out, however, that the obtained plasmon dispersion relation contains an unphysical instability at large spatial momenta. The reason for this seems to be that there is a mixing of the Maxwell field with the scalar condensate at nonzero momenta, which alters the conductivity [10]. This is not accounted for in our derivation of the mixed boundary condition for plasmon modes, but it needs to be in order to get reliable results for nonzero momenta. How one should alter the derivation to incorporate the effects of a mixing with the condensate is still unclear. Nevertheless, we will present the analysis, culminating in a flawed plasmon dispersion relation in 7.4.3, anyway. Our analysis could perhaps be revisited in the future when the mixing with the condensate is better understood.

The holographic dictionary identifies a global $U(1)$ symmetry current $\mathcal{J}^\mu$ in the boundary QFT with a local $U(1)$ gauge field A_M in the gravitational bulk theory, (as usual, the greek indices range over the boundary coordinates $\mu = t, x_1, x_2, \ldots, x_d$, while capital latin indices range over the bulk coordinates $M = t, z, x_1, x_2, \ldots, x_d$. We are considering the general case of a $d+2$ dimensional bulk spacetime with a dual boundary QFT in $d+1$ dimensions.) Since the $U(1)$ symmetry is global on the boundary, $\mathcal{J}^\mu$ is a conserved Noether current. Our starting point is to use the holographic dictionary to identify a $U(1)$ gauge theory for the boundary QFT from the $U(1)$ gauge theory in the bulk. For this purpose it is convenient to work in radial gauge in the bulk, defined by $g_{Mz} = A_z = 0$.

An electromagnetic field strength two-form $\mathcal{F}$ and a conserved electric current one-form $\mathcal{J}$ for the boundary QFT is identified with corresponding bulk quantities according to

$$\mathcal{F} = \frac{1}{\sqrt{\lambda}} F\Big|_{\partial M}, \qquad \mathcal{J} = \sqrt{\lambda}\, l_n W\Big|_{\partial M}. \tag{7.30}$$

Here F and W are the bulk electromagnetic field strength and induction tensor[5], respectively, and l_n is a normal vector on the boundary ∂M. The parameter λ relates the coupling strengths of the boundary and bulk $U(1)$ gauge theories, which does not necessarily have to be equal. The bulk quantities in (7.30) are evaluated on the boundary ∂M of the bulk spacetime manifold M. For bulk spacetimes which are asymptotic to AdS, the boundary ∂M is the AdS conformal boundary. The electric current has the form

$$\mathcal{J} = \mathcal{J}_\mu dx^\mu = -\langle \rho \rangle\, dt + j_m dx^m\,, \tag{7.31}$$

where $\langle \rho \rangle$ is the expectation value of the internal charge density.

We now identify an induction tensor $\mathcal{W}$ for the boundary theory in the following way:

$$d \star (\mathcal{F} - \mathcal{W}) = \star \mathcal{J}\,, \tag{7.32}$$

where d is the exterior derivative and $\star$ is the 'Hodge star operator'. The reader unfamiliar with the formalism of differential forms is referred to e.g. [78] for a brief introduction. In tensor component notation, the above equation reads

$$\partial_\mu \left(\mathcal{F}^{\nu\mu} - \mathcal{W}^{\nu\mu}\right) = \mathcal{J}^\nu\,. \tag{7.33}$$

This identification states that the symmetry current $\mathcal{J}$ on the boundary is an internal one. Then (7.30) and (7.33) results in the standard macroscopic Maxwell's equations for the boundary theory, characteristic for a $U(1)$ gauge theory,

$$d\mathcal{F} = 0\,, \qquad d \star \mathcal{W} = \star \mathcal{J}_{ext}\,, \tag{7.34}$$

or in tensor component notation,

$$\partial_{[\mu} \mathcal{F}_{\nu\rho]} = 0\,, \qquad \partial_\mu \mathcal{W}^{\nu\mu} = \mathcal{J}^\nu_{\text{ext}}\,, \tag{7.35}$$

[5] The induction tensor is the analogue of the field strength using the displacement field D and the magnetic field strength H instead of the electric field E and magnetic field B.

where the square bracket on the indices denotes anti-symmetrization and $\mathcal{J}_{\text{ext}}$ is the external current, i.e. the current probed by external sources. The field strength and induction tensor for the boundary QFT can now be decomposed into electric and magnetic field components, and displacement field and magnetic field strength components, respectively,

$$\mathcal{F} = \boldsymbol{\mathcal{E}} \wedge dt + \star^{-1} (\boldsymbol{\mathcal{B}} \wedge dt) \ , \qquad \mathcal{W} = \boldsymbol{\mathcal{D}} \wedge dt + \star^{-1} (\boldsymbol{\mathcal{H}} \wedge dt) \ . \tag{7.36}$$

For instance, in the case of a three dimensional boundary QFT, which is the relevant case for our superconductor model,

$$\mathcal{F}^{ti} = \mathcal{E}_i \,, \qquad \mathcal{W}^{ti} = \mathcal{D}_i \,, \tag{7.37}$$

$$\frac{1}{2}\epsilon_{ij}\mathcal{F}^{ij} = \mathcal{B} \,, \qquad \frac{1}{2}\epsilon_{ij}\mathcal{W}^{ij} = \mathcal{H} \,, \tag{7.38}$$

where ϵ_{ij} is the Levi-Civita tensor and the indices $\{i, j\}$ range over the spatial boundary coordinates $\{x, y\}$. Note that the magnetic field is a (psudo) scalar in three spacetime dimensions. Furthermore, (7.37) is the definition of the electric and displacement fields in any dimension with $d > 0$.

The conductivity σ and dielectric function ε, which in general are tensorial quantities, are defined by the response to the electric field $\mathcal{E}_i$,

$$j_i = \sigma_{ij}\mathcal{E}_j \,, \qquad \mathcal{D}_i = \varepsilon_{ij}\mathcal{E}_j \,. \tag{7.39}$$

In general, both the conductivity and the dielectric function can have a complicated dependence on the electric and magnetic fields. For weak enough electromagnetic fields, the response of the medium to an applied electric field is linear. Then, in a linear response analysis, the conductivity and dielectric function can be taken as independent on the electromagnetic fields. Moreover, for an isotropic medium they are diagonal with all elements equal [79].

In a Minkowski space, Maxwell's equations (7.35) can be cast into the conventional form,

$$\begin{aligned} &\operatorname{div} \boldsymbol{\mathcal{B}} = 0 \,, &&\operatorname{div} \boldsymbol{\mathcal{D}} = \rho_{\text{ext}} \,, \\ &\operatorname{curl} \boldsymbol{\mathcal{E}} = -\dot{\boldsymbol{\mathcal{B}}} \,, &&\operatorname{curl} \boldsymbol{\mathcal{H}} = \boldsymbol{j}_{\text{ext}} + \dot{\boldsymbol{\mathcal{D}}} \,, \end{aligned} \tag{7.40}$$

By using the definitions of the conductivity and dielectric function (7.39) together with Maxwell's equations (7.40), and transforming to momentum space, one can derive the following relation between σ and ε [13]:

$$k_i \left(\varepsilon_{ij} - \delta_{ij} + \frac{\sigma_{ij}}{i\omega} \right) \mathcal{E}_j = 0 \,. \tag{7.41}$$

The conductivity can straightforwardly be obtained from the screened 'density-density' correlator, using the Kubo formula,

$$\sigma = \frac{i\omega}{k^2} \langle \rho\rho \rangle \ . \tag{7.42}$$

Charge density fluctuations are characterized by having a vanishing displacement field $\boldsymbol{\mathcal{D}} = 0$ despite having a nonzero electric field $\boldsymbol{\mathcal{E}} \neq 0$, in the absence of external sources, $\mathcal{J}_{\text{ext}} = 0$. By (7.39), they must have a dispersion relation $\omega(k)$ satisfying

$$\varepsilon(\omega, k) = 0 \,, \tag{7.43}$$

since otherwise a vanishing displacement field would imply a vanishing electric field as well. Furthermore, only the longitudinal modes give rise to charge density fluctuations, and therefore transverse modes do not need to be considered here. In particular, without loss of generality, we will look at the longitudinal mode in the x direction, for which the relation (7.41) between the conductivity and dielectric tensor reads

$$\varepsilon_{xx} = 1 - \frac{\sigma_{xx}}{i\omega} \,. \tag{7.44}$$

For simplicity, we will also assume vanishing magnetic fields, i.e. $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{H}} = 0$ and $\mathcal{F}^{ij} = \mathcal{W}^{ij} = 0$. From (7.31), (7.33), and (7.37), we then get

$$\dot{\mathcal{E}}_x + j_x = 0 \,, \tag{7.45}$$

for the charge density fluctuation with momentum in the x direction. In a homogeneous media, assuming the gauge choice $\mathcal{A}_t = 0$, we have

$$\dot{\mathcal{E}}_x = \partial_t \mathcal{F}^{tx} = -\partial_t \mathcal{F}_{tx} = -\partial_t^2 \mathcal{A}_x \,. \tag{7.46}$$

Transforming to momentum space, the condition (7.45) for the charge density fluctuation can then be written

$$\omega^2 \mathcal{A}_x + \mathcal{J}_x = 0 \,. \tag{7.47}$$

This condition can in fact be shown to be equivalent to (7.43) [13], [77].

A plasmon mode is a certain subset of the modes satisfying (7.43); they are the lowest energy self-sourcing propagating oscillations. In order to calculate dispersion relations for plasmon modes, we translate the conditions (7.43) and (7.47) for the boundary QFT into a boundary condition for the bulk fields. Plasmons are generated by the response to a perturbation in the electric field, and we therefore fix boundary conditions for bulk field fluctuations at the conformal boundary ∂M such that the fluctuation δA_x is nonzero. To avoid any overlap with the response from other fluctuations to the extent possible, we choose Dirichlet boundary conditions for the other components of metric and gauge field fluctuations at the conformal boundary. Having non-vanishing metric fluctuations at the conformal boundary could potentially even introduce an undesired dynamical graviton to the boundary QFT, in complete analogy with how the nonzero Maxwell fluctuation leads to a dynamical photon in the boundary theory. However, for the fluctuation of the complex scalar field, the amplitude will automatically vanish at spatial infinity. We should impose the same boundary condition for the scalar field fluctuation as for the background solution, i.e. that the normal derivative at the conformal boundary vanish, $\delta\phi' = 0$.

In general, the boundary condition for the plasmon modes would involve the bulk induction tensor and be dependent on the particular bulk model under consideration. In our minimal holographic superconductor model, the Maxwell kinetic term in the bulk action is such that

$$\delta \mathcal{J}_x \propto \delta A'_x \Big|_{\partial M} \,, \tag{7.48}$$

where the prime again denotes a normal derivative at the conformal boundary ∂M. With the identification (7.30) and the plasmon condition (7.47), it directly follows that the plasmon boundary condition is of the form

$$\Big(\omega^2 \, \delta A_x + p(\omega, k) \lambda \, \delta A'_x\Big) \Big|_{\partial M} = 0 \,, \tag{7.49}$$

where $p(\omega, k)$ is a function which depends on the specific form of the Maxwell kinetic term in the bulk theory. It is a mixed boundary condition with a linear relation between the source δA_x and the expectation value $\delta A'_x$. As such, it is equivalent to a double-trace deformation of the boundary QFT, obtained by adding a potential, quadratic in the corresponding QFT operator, to the QFT action, see 4.2.3.

The function $p(\omega, k)$ is often bounded or even completely independent of ω and k, at least in the more common and simpler holographic models. In the case of the minimal holographic superconductor model of relevance to us, we will assume it to be entirely independent of ω and k, and absorb the constant into a redefinition of λ. In other words, we effectively set $p(\omega, k) = 1$. In general, however, a bounded $p(\omega, k)$ will imply that for small ω, the second term in (7.49) will dominate and one has effectively a Neumann boundary condition. On the other hand, for large ω, the first term dominates and one has approximately a Dirichlet boundary condition. A large ω means that the fluctuation is rapidly oscillating, and the dynamical polarization of the mode can then safely be neglected. Thus, in this case a Dirichlet boundary condition is indeed the appropriate choice, perfectly captured by (7.49). Conversely, the dynamical polarization is essential for small ω, and it has been argued that a Neumann boundary condition in fact introduces a dynamical photon to the boundary theory [80].

7.4 Linear response analysis and dispersion relations

After having found a stationary black hole solution to our toy model of holographic superconductivity, we investigate its linear response. This describes how the system responds to small fluctuations in the fields. In particular, we will look at infinitesimal fluctuations in momentum space. The metric, gauge, and scalar fields is decomposed as follows:

$$\begin{aligned} g_{MN} &= \bar{g}_{MN}(z) + \int \frac{d\omega\, dk}{(2\pi)^2} h_{MN}(z) e^{-i\omega t + i k_x x}\,, \\ A_M &= \bar{A}_M(z) + \int \frac{d\omega\, dk}{(2\pi)^2} a_M(z) e^{-i\omega t + i k_x x}\,, \\ \phi &= \bar{\phi}(z) + \int \frac{d\omega\, dk}{(2\pi)^2} \delta\phi(z) e^{-i\omega t + i k_x x}\,, \end{aligned} \tag{7.50}$$

where $\bar{g}_{MN}(z)$, $\bar{A}_M(z)$, and $\bar{\phi}(z)$ are the stationary black hole background fields, and $h_{MN}(z)$, $a_M(z)$, and $\delta\phi(z)$ are infinitesimal fluctuations in momentum space. We have used the rotational symmetry to choose a wave vector in the x-direction, i.e. $k_x = k$ and $k_y = 0$. When performing the numerical computations, we consider a single fluctuation mode at a time. Furthermore, we factor out a z^{-2} divergence at the conformal boundary from the metric fluctuations, and a factor of $g_F L/\kappa = \sqrt{2}$ from the gauge field fluctuations, and absorb the factors of 2π into a redefinition of the fields. The decomposition for a single fluctuation mode is then written as

$$\begin{aligned} g_{MN} &= \bar{g}_{MN}(z) + \frac{1}{z^2} h_{MN}(z) e^{-i\omega t + i k_x x}\,, \\ A_M &= \bar{A}_M(z) + \sqrt{2}\, a_M(z) e^{-i\omega t + i k_x x}\,, \\ \phi &= \bar{\phi}(z) + \delta\phi(z) e^{-i\omega t + i k_x x}\,. \end{aligned} \tag{7.51}$$

We choose a radial gauge for the metric and gauge field fluctuations such that $g_{Mz} = a_z = 0$. To account for attenuation of the fluctuation waves in the media, ω is allowed to take on complex values.

Now, the fluctuations can be subdivided into two groups depending on their parity under the inversion symmetry $y \to -y$. The components $\{h_{tt}, h_{tx}, h_{xx}, h_{yy}, a_t, a_x\}$ have even parity and constitutes the longitudinal modes, describing the response parallel to the momentum flow. The odd parity fluctuations are $\{h_{ty}, h_{xy}, a_y\}$ and constitute the transverse modes, representing the response perpendicular to the momentum flow. Since plasmons are longitudinal self-sourced charged excitations, we restrict ourselves to study only longitudinal modes and set $h_{ty} = h_{xy} = a_y = 0$.

Substituting the decomposed fields (7.51) into the equations of motion (7.6)-(7.8) results in a system of seven linear coupled ordinary differential equations for the fluctuation fields. These equations are quite cumbersome and we omit writing them down here. Similar to the stationary black hole field equations (7.14)-(7.16), they are singular at the black hole horizon and at the conformal boundary at spatial infinity. The singular behaviour at the horizon is factored out from the fluctuations and parametrized as follows:

$$\begin{aligned} h_{\mu\nu} &\to (1-z)^{i\alpha} h_{\mu\nu} \,, \\ a_\mu &\to (1-z)^{i\alpha} a_\mu \,, \\ \delta\phi &\to (1-z)^{i\alpha} \delta\phi \,. \end{aligned} \tag{7.52}$$

Here α is a parameter which has to be solved for. Since the equations for the fluctuations are linear, a factor of $(1-z)^{i\alpha}$ can be factored out from the equations as well.

Expanding the equations for the fluctuations around the horizon, one can solve for α, the fluctuations and their derivatives, evaluated at the horizon, in terms of the stationary black hole background parameters as well as three independent degrees of freedom $h_{xx}(1)$, $a_x(1)$, and $\delta\phi(1)$. From the lowest order in the expansions, one finds

$$\alpha = \pm \frac{4\, e^{\chi_+/2}\, \omega}{-12 + 2m^2\phi_+^2 + e^{\chi_+} E_+^2} \,, \tag{7.53}$$

and $h_{tt}(1) = h'_{tt}(1) = h_{tx}(1) = a_t(1) = 0$ and $h_{yy}(1) = -h_{xx}(1)$. The two signs of α correspond to fluctuations which are either falling into the black hole or emerging out from it, respectively. Only the fluctuations which are falling into the horizon are physically relevant, and these modes correspond to an α given by (7.53) with the plus sign. The higher order terms in the expansion of the equations give successively higher order derivatives in the fluctuation fields. From these one can construct boundary conditions for the fluctuations at the near horizon cutoff surface $z = 1 - \epsilon$. The equations can then be solved numerically by integrating from $z = 1 - \epsilon$ to $z = \epsilon$. There are three linearly independent solutions which can be found by setting one of $h_{xx}(1)$, $a_x(1)$, and $\delta\phi(1)$ to unity and the remaining two to zero.

There are still four independent solutions for the fluctuation fields missing. It turns out that these amounts to pure diffeomorphisms and $U(1)$ transformations of the metric, the $U(1)$ gauge field, and the scalar field. We will refer to them as 'pure gauge solutions'.

7.4.1 Pure gauge solutions

The radial gauge does not completely eliminate all the gauge degrees of freedom. One also has to consider 'large' gauge transformations in the bulk, i.e. gauge transformations which are non-trivial on the boundary at spatial infinity. This will allow us to find pure gauge solutions to the fluctuation equations. Since the quantum field theory lives on this boundary, a large gauge transformation can indeed affect the physics of the field theory, even though it does not affect the physics in the bulk.

The large gauge transformations to be considered are spacetime diffeomorphisms and local $U(1)$ transformations of the Maxwell field and complex scalar field. Under a diffeomorphism generated by the vector ξ^M, the bulk fields transform as

$$\begin{aligned} g_{MN} &\to g_{MN} + \delta_\xi g_{MN} \,, \\ A_M &\to A_M + \delta_\xi A_M \,, \\ \phi &\to \phi + \delta_\xi \phi \,, \end{aligned} \tag{7.54}$$

where the change δ_ξ is given by the Lie derivative along the vector field ξ^M,

$$\begin{aligned} \delta_\xi g_{MN} &= \mathcal{L}_\xi g_{MN} = \xi^K \partial_K g_{MN} + g_{MK}\partial_N \xi^K + g_{KN}\partial_M \xi^K \,, \\ \delta_\xi A_M &= \mathcal{L}_\xi A_M = \xi^N \partial_N A_M + A_N \partial_M \xi^N \,, \\ \delta_\xi \phi &= \mathcal{L}_\xi \phi = \xi^M \partial_M \phi \,. \end{aligned} \tag{7.55}$$

The $U(1)$ gauge transformations are given by

$$\begin{aligned} A_M &\to A_M + \partial_M \Lambda \,, \\ \phi &\to e^{i\Lambda}\phi \,, \end{aligned} \tag{7.56}$$

for some arbitrary function Λ.

Since we are looking for pure gauge solutions to the fluctuation equations, we consider the gauge parameters ξ and Λ to be infinitesimal and oscillating quantities. More precisely, we write

$$\xi^M = \zeta^M(z) e^{-i\omega t + i k_x x} \,, \qquad \Lambda = \lambda(z) e^{-i\omega t + i k_x x} \,. \tag{7.57}$$

Then, to first order, only transformations of the background fields has to be considered. Moreover, the pure gauge solutions should be consistent with our choice of radial gauge, and therefore we must impose the constraint

$$\delta_\xi g_{Mz} = 0 \,, \qquad \delta_\xi A_z + \partial_z \Lambda = 0 \,. \tag{7.58}$$

Note that in order not to rule out the possibility of a nonzero $\delta_\xi A_z$ being canceled by $\partial_z \Lambda$, we only require that the z-component of a simultaneous diffeomorphism and $U(1)$ transformation vanish. The radial gauge constraint equations (7.58) specifies the functions $\zeta^M(z)$ and $\lambda(z)$ in (7.57) up to five integration constants. However, only four of these integration constants will enter the longitudinal pure gauge modes. Hence, there will be four independent pure gauge solutions to the equations for the fluctuations. A derivation of the pure gauge modes are given in Appendix A, and the analytical expressions for these are given in (A.25).

7.4.2 Boundary conditions and the determinant method

In total, we have found seven linearly independent solutions for the fluctuation fields, three numerical solutions and four analytical pure gauge solutions. Any linear combination of these will also solve the equations for the fluctuations. In order to get a well-posed boundary value problem, we also supply seven boundary conditions at the conformal boundary, one for each field component. The type of boundary conditions at the conformal boundary determines the type of mode described by the solution, e.g. Dirichlet boundary conditions give rise to quasi-normal modes. Then one has to find a particular linear combination of solutions which satisfy these boundary conditions at spatial infinity. The solutions depend on ω and k, and to find a dispersion relation for a given set of boundary conditions we iterate over ω and k, and look for values where there exists a non-trivial linear combination which satisfies the boundary conditions. To determine whether such a non-trivial linear combination exists, given a particular ω and k, we calculate a determinant which is zero only if a non-trivial solution exists.

The longitudinal modes of interest to us are primarily the plasmon modes, but we will also investigate the quasi-normal modes. To study the quasi-normal modes we must impose Dirichlet boundary condtitions at the conformal boundary for all fluctuations except $\delta\phi$. Indeed, $\delta\phi$ will automatically vanish on the conformal boundary since the fluctuation has the same asymptotic behavior (7.20) as the background scalar field. The correct boundary condition on $\delta\phi$ should be that its leading part (the source) vanish,

$$\delta\phi'(z) \to 0 \quad \text{as} \quad z \to 0\,. \tag{7.59}$$

To investigate the plasmon modes we change the Dirichlet boundary condition for a_x to the plasmon boundary condition

$$\omega^2 a_x + \lambda a_x' \to 0 \quad \text{as} \quad z \to 0\,, \tag{7.60}$$

which was derived in the previous section. Furthermore, we choose $\lambda = 1$ for simplicity.

Assume we want to impose the plasmon boundary condition, i.e. (7.59) and (7.60), and Dirichlet boundary conditions on the remaining field components. Given a complex ω and real k, a non-trivial linear combination of the seven linearly independent solutions for the fluctuations, satisfying the plasmon boundary condition, will exist if and only if

$$\begin{vmatrix} (h_{tt})_1 & (h_{tx})_1 & (h_{xx})_1 & (h_{yy})_1 & (a_t)_1 & (\omega^2 a_x + a_x')_1 & (\delta\phi')_1 \\ (h_{tt})_2 & (h_{tx})_2 & (h_{xx})_2 & (h_{yy})_2 & (a_t)_2 & (\omega^2 a_x + a_x')_2 & (\delta\phi')_2 \\ & & & \vdots & & & \\ (h_{tt})_7 & (h_{tx})_7 & (h_{xx})_7 & (h_{yy})_7 & (a_t)_7 & (\omega^2 a_x + a_x')_7 & (\delta\phi')_7 \end{vmatrix} = 0\,. \tag{7.61}$$

Iterating through a range of values for ω and k, we can determine a relation $\omega(k)$ for which the determinant is zero. The relation $\omega(k)$ is then the sought after dispersion relation for the plasmon mode. An analogous determinant with a Dirichlet boundary condition on a_x can also be computed, yielding a dispersion relation for a quasi-normal mode. We will now present dispersion relations for quasi-normal modes and a plasmon mode, numerically computed using this determinant method. However, it should be emphasised again that the plasmon boundary condition (7.60) most likely has to be modified for our holographic superconductor, due to a mixing of the gauge field with the scalar condensate at nonzero momenta. This is a plausible explanation as to why we find an instability in our plasmon dispersion relation for large k.

7.4.3 Dispersion relations for quasi-normal modes and plasmons

Using the determinant method explained in the previous section, we have computed dispersion relations for three quasi-normal modes. These dispersion relations are plotted

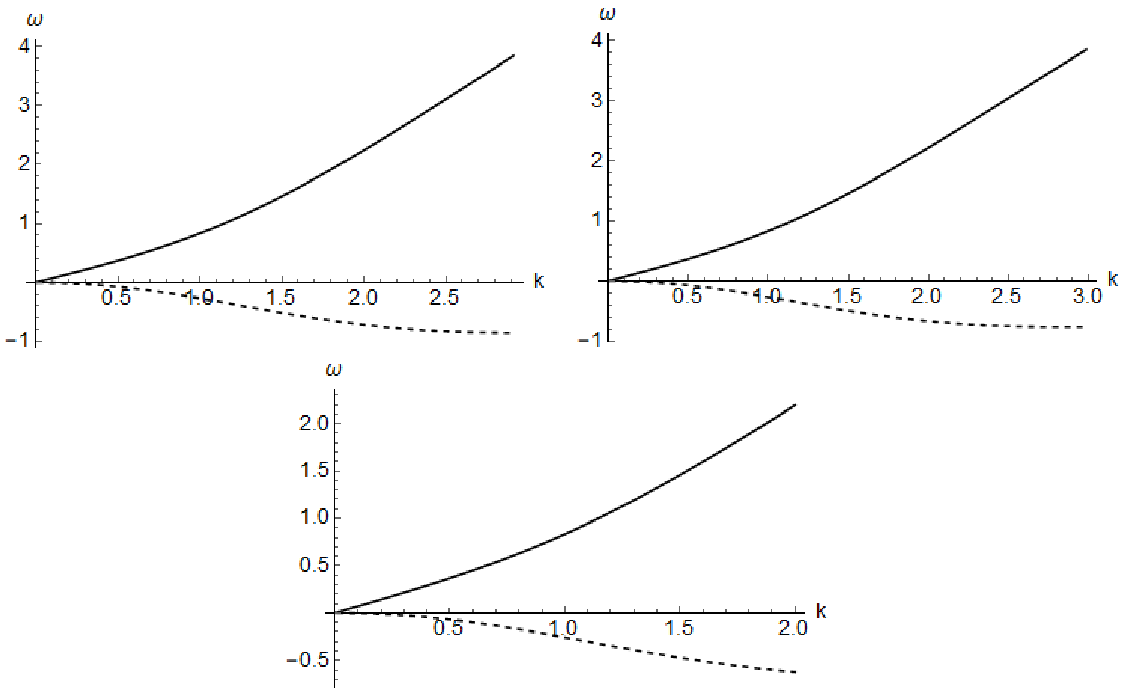

Figure 7.2: Dispersion relations for three quasi-normal modes. The solid lines shows the real part of ω and the dashed lines shows the imaginary part. (Left): $E_+ = 0.25$, $\phi_+ = 0.998\ldots$. (Right): $E_+ = 0.75$, $\phi_+ = 0.714\ldots$. (Bottom): $E_+ = 1.2$, $\phi_+ = 0.545\ldots$. In all figures $\chi_+ = -1$ and $q = 3$.

in Fig. 7.2. The three quasi-normal modes correspond to three different choices for the free parameters of our holographic superconductor model. Having made the choice $m^2L^2 = -2$ for the mass of the scalar field, and having set various other parameters to unity, we are left with four free parameters. One of these remaining parameters is the charge of the scalar field q, which couples the scalar field to the $U(1)$ gauge field via the gauge covariant derivative term in the action (7.1). We have chosen the value $q = 3$ in all our numerical computations. A natural extension of our analysis is therefore to study how the dispersion relations depend on q.

The other three remaining free parameters are values of background fields, evaluated at the black hole horizon. More precisely, they are ϕ_+, E_+, and χ_+, i.e. the value of the complex scalar field, the electric field $E_+ = A_0'(z_+)$, and the function χ in the time component of the metric, all evaluated at the horizon, respectively. However, to find a background solution having a vanishing source for the scalar operator in the dual QFT, using our method described above in 7.2.2, we specify only two of these parameters and solve for the third one such that the solution has a vanishing leading part of the scalar field near the conformal boundary. In particular, we have specified E_+ and χ_+, and then solved for ϕ_+ using plots similar to 7.1. Moreover, we have chosen $\chi_+ = -1$ in all cases. The three dispersion relations for quasi-normal modes shown in Fig. 7.2 correspond to the specified values of $E_+ = 0.25$, $E_+ = 0.75$, and $E_+ = 1.2$. This then fixes the value of the scalar field at the horizon to $\phi_+ = 0.998\ldots$, $\phi_+ = 0.714\ldots$, and $\phi_+ = 0.545\ldots$, respectively. The solid lines shows the real part of ω and the dashed lines shows the imaginary part. These dispersion relations for quasi-normal modes look physically reasonable. The negative imaginary part for nonzero momenta signals that the modes are dispersive, attenuating in the media. The reason for this is that the

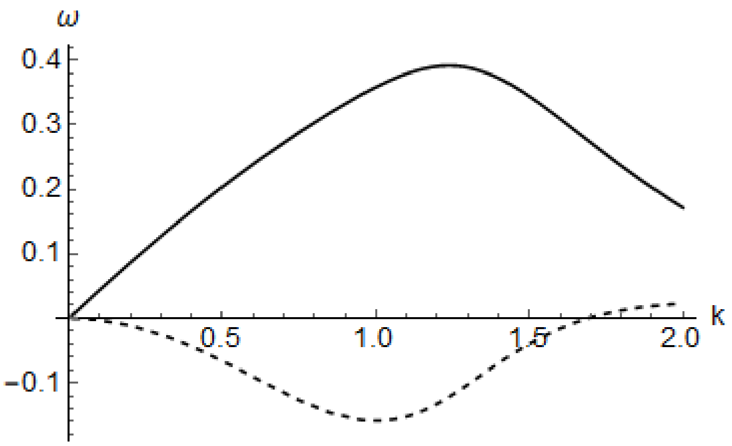

Figure 7.3: A dispersion relation for a plasmon mode. The solid line is the real part of ω and the dashed line is the imaginary part. For small k the dispersion relation is approximately linear. The positive $\mathrm{Im}(\omega)$ for large k indicates an instability. Such an instability is not expected to occur. The parameter values used are $q = 3$, $\chi_+ = -1$, $E_+ = 0.75$ and $\phi_+ = 0.714\ldots$.

oscillating factor $e^{-i\omega t+ik_x x}$ decreases in amplitude with increasing time when ω has a negative imaginary part.

Fig. 7.3 shows a dispersion relation for a mode using the plasmon boundary condition (7.60). Here we have used the parameter values $q = 3$, $\chi_+ = -1$, $E_+ = 0.75$, and $\phi_+ = 0.714\ldots$. Again, the solid line and dashed line shows the real part and imaginary part of ω, respectively. At $k \approx 1.7$ the imaginary part of ω obtains a positive value. This signals an instability due to the oscillating factor $e^{-i\omega t+ik_x x} = e^{\mathrm{Im}(\omega)t}e^{-i\,\mathrm{Re}(\omega)t+ik_x x}$, which blows up as $t \to \infty$ in the case of a positive imaginary part. The instability found at large k is not expected to occur, and is most likely a consequence of not accounting for a mixing between the gauge field and scalar condensate at nonzero momenta when deriving the plasmon boundary condition. We note also that the dispersion relation is approximately linear for small momenta. It would be interesting to see if this linearity remains after having resolved the problem of the instability at large k.

7.5 Conclusions and outlook

In this chapter we have presented a minimal bottom-up model of a holographic superconductor, a model originally proposed in [9], [10]. This model was constructed by postulating a bulk theory action (7.1), describing a classical gravitational theory in anti-de Sitter space, containing a $U(1)$ gauge field as well as a complex scalar field. The bulk theory is interpreted as the holographically dual description of a strongly interacting large N quantum field theory, living on the boundary of the bulk spacetime. The low energy processes in the QFT are described by the dynamics near the horizon of a charged 'Reissner-Nordström' black hole in the deep interior of the bulk spacetime. The scalar field can become tachyonic near the horizon, resulting in the formation of a charged scalar condansate around the horizon; the black hole aquires scalar 'hair'. The scalar condensate in the bulk is interpreted as a scalar 'superfluid' in the dual QFT, which is formed by the condensation of the charged scalar operator, in dual correspondence with the complex scalar field in the bulk, below a critical temperature T_c. The superfluid condensate is superconducting, and its formation below the critical temperature is then interpreted as a superconducting phase transition.

In this work we have not verified that a superconducting phase transition indeed occurs at some critical temperature T_c. However, this has been done in e.g. [10]. Indeed, by imposing a nonzero amplitude for the scalar field on the horizon, any ground state found by solving the equations of motion must be in the superconducting phase, since the ground state in the normal phase is described by a pure Reissner-Nordström solution with a vanishing scalar field profile. Another possibility is that the solution we found is not a ground state in the superconducting phase, but an excited state in the normal phase. However, a quick numerical calculation of the conductivity yields a similar result as in [10], strongly suggesting that the system indeed is in the superconducting phase, having an infinite DC conductivity. A plot of the conductivity with a somewhat low numerical accuracy is given in Appendix B.

Ruling out the possibility of having found a solution describing an excited state in the normal phase is also of importance for finding an explanation to the 'large k' instability of the plasmon dispersion relation. If the system where in an excited normal state, then it would be plausible that a high energetic fluctuation could knock the system back to a stable ground state. It would then be reasonable to expect the system to be unstable against fluctuations having large enough momenta. However, by confirming that the stationary black hole solution describes a ground state in the superconducting phase, this explanation for why the plasmon modes become unstable at large momenta can safely be discarded.

As we have commented on above, the reason for the unstable plasmon modes probably has to do with a mixing of the gauge field with the scalar condensate at nonzero momenta. This mixing introduces additional terms to the conductivity which plays an essential role in the derivation of the plasmon boundary condition. Determining the precise form of these additional terms, and how they alter the plasmon boundary condition, is a prerequisite for being able to compute reliable dispersion relations for plasmon modes in the holographic superconductor model. This would be an interesting research project for the future.

The bottom-up model of a holographic superconductor considered in this thesis is of s-wave type, i.e. the superfluid condensate does not carry any angular momentum. Although s-wave superconductivity are part of many experimentally studied superconductors, unconventional p- and d-wave superconductivity seems to be essential parts of real life high T_c superconductors as well. Extending the analysis of charge density fluctuations to holographic models of p- and d-wave superconductors, e.g. the models listed in [73], is a natural next step. Moreover, the experimentally studied high T_c cuprate superconductors consists of $2+1$ dimensional layers stacked on top of each other, and it has been argued in [11] that it is the Coulomb interaction (in $3+1$ dimensions) between the layers that give rise to the high critical temperature observed. The holographic superconductor toy model considered here is in fact a $2+1$ dimensional system, and it would therefore be interesting to stack up many layers of this model, coupled with a $3+1$ dimensional Coulomb interaction. Hopefully, this would shed some light on why the critical temperature is high in the cuprate superconductors.

Finally, holographic bottom-up models can provide a tractable description of the emergent low energy phenomena of condensed matter systems, at least in principle. It is, however, conceptually desired to also find top-down constructions for holographic dualities describing condensed matter systems. This would guarantee that the theory is

UV complete, although UV completion is of less relevance for applications to condensed matter physics, where one is primarily interested in the emergent low energy dynamics. A string theoretical top-down construction does, however, give some justification to the bottom-up model obtained by consistently truncating the top-down model. Furthermore, the bottom-up models are highly phenomenological, in the sense that they come with a larger number of free parameters, which at best can be determined by experimental data. Top-down models tend to be more constrained from the particular string theory they are derived from. For instance, the potential for the complex scalar field in our bottom-up superconductor model where free for us to specify, and we restricted our analysis to the case of a simple mass term. In a top-down construction, the precise form of the potential would be specified and not free for us to choose. For an introduction to the subject of superconductivity in top-down holography, we refer the reader to some of the earlier work done in [81]–[83], but see also [2].

A

Derivation of longitudinal pure gauge modes

Here we present the derivation of the longitudinal pure gauge modes of the holographic superconductor model. The diffeomorphism and $U(1)$ transformation parameters ξ^M and Λ are first determined from the radial gauge constraint equations for the metric and gauge fields. These are then used to calculate the longitudinal pure gauge modes for the metric, gauge, and scalar field.

Consider first the constraint equation $\delta_\xi g_{zz} = 0$. Using (7.9) and (7.55), we get

$$\begin{aligned} 0 = \delta_\xi g_{zz} &= \xi^K \partial_K g_{zz} + g_{zz}\partial_z \xi^z + g_{zz}\partial_z \xi^z \\ &= \xi^z \partial_z g_{zz} + 2 g_{zz}\partial_z \xi^z \\ &= \xi^z \partial_z \frac{L^2}{z^2 f} + \frac{2L^2}{z^2 f}\partial_z \xi^z \\ &= -L^2 \left[\left(\frac{2}{z^3 f} + \frac{f'}{z^2 f^2}\right)\zeta^z + L^2 \frac{2}{z^2 f}\zeta'^z\right] e^{-i\omega t + i k_x x}\,. \end{aligned} \tag{A.1}$$

This first order differential equation for ζ^z can be written as

$$\zeta'^z - \left(\frac{1}{z} + \frac{f'}{2f}\right)\zeta^z = 0\,. \tag{A.2}$$

It is straightforward to solve this equation using an integrating factor,

$$\frac{d}{dz}\left(e^{-\int^z dz'\left(\frac{1}{z'} + \frac{f'}{2f}\right)}\zeta^z\right) = \frac{d}{dz}\left(\frac{1}{z\sqrt{f}}\zeta^z\right) = 0\,. \tag{A.3}$$

Integrating this expression yields the following solution:

$$\zeta^z = c_1 z \sqrt{f}\,, \tag{A.4}$$

where c_1 is an integration constant.

Next we consider the radial gauge constraint equations $\delta_\xi g_{\mu z} = 0$ where $\mu = t, x, y$. Using the fact that the background metric (7.9) is diagonal, as well as the expression for the Lie derivative (7.55), we get

$$\begin{aligned} \delta_\xi g_{\mu z} &= \xi^K \partial_K g_{\mu z} + g_{\mu K}\partial_z \xi^K + g_{Kz}\partial_\mu \xi^K \\ &= g_{\mu\mu}\partial_z \xi^\mu + g_{zz}\partial_\mu \xi^z\,. \end{aligned} \tag{A.5}$$

For $\mu = t$, we have

$$\begin{aligned} 0 = \delta_\xi g_{tz} &= g_{tt}\partial_z\xi^t + g_{zz}\partial_t\xi^z \\ &= -\frac{L^2 f e^{-\chi}}{z^2}\partial_z\left(\zeta^t e^{-i\omega t + ik_x x}\right) + \frac{L^2}{z^2 f}\partial_t\left(\zeta^z e^{-i\omega t+ik_x x}\right) \\ &= -\frac{L^2}{z^2}\left(f e^{-\chi}\zeta'^t + \frac{i\omega c_1 z}{\sqrt{f}}\right) e^{-i\omega t+ik_x x}\,, \end{aligned} \tag{A.6}$$

where we have used (A.4) in the last step. Solving for ζ'^t, we find

$$\zeta'^t = -\frac{i\omega c_1 z e^{\chi}}{f^{3/2}}\,, \tag{A.7}$$

which can be directly integrated to yield a solution for ζ^t,

$$\zeta^t = -i\omega c_1 \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} + c_2\,. \tag{A.8}$$

Here c_2 is an another integration constant. Since the functions $f(z)$ and $\chi(z)$ can only be found numerically, the integral can only be evaluated numerically as well. The lower integration limit can be chosen arbitrarily in the interval $(0,1)$. However, since we are only interested in the behaviour of the pure gauge solutions near the boundary at spatial infinity, $z = 0$, it is convenient to choose the lower limit close to this boundary.

Next, for $\mu = x$, we have

$$\begin{aligned} 0 = \delta_\xi g_{xz} &= g_{xx}\partial_z\xi^x + g_{zz}\partial_x\xi^z \\ &= \frac{L^2}{z^2}\zeta'^x e^{-i\omega t+ik_x x} + \frac{ik_x L^2}{z^2 f}\zeta^z e^{-i\omega t+ik_x x} \\ &= \frac{L^2}{z^2}\left(\zeta'^x + \frac{ik_x c_1 z}{\sqrt{f}}\right) e^{-i\omega t+ik_x x}\,. \end{aligned} \tag{A.9}$$

It is straightforward to solve this equation for ζ'^x and integrate to get the following solution for ζ^x:

$$\zeta^x = -ik_x c_1 \int^z dz' \frac{z'}{\sqrt{f}} + c_3\,, \tag{A.10}$$

where c_3 is a third integration constant. Lastly, for $\mu = y$, we have

$$0 = \delta_\xi g_{yz} = g_{yy}\partial_z\xi^y + g_{zz}\partial_y\xi^z = \frac{L^2}{z^2}\zeta'^y e^{-i\omega t+ik_x x}\,. \tag{A.11}$$

We must then have $\zeta'^y = 0$, which implies that ζ^y is just an integration constant,

$$\zeta^y = c_4\,. \tag{A.12}$$

Consider now the radial gauge constraint equation for the gauge field, $\delta_\xi A_z + \partial_z\Lambda = 0$. Using (7.10), (7.55), and (7.57), we get

$$\begin{aligned} 0 = \delta_\xi A_z + \partial_z\Lambda &= \xi^N\partial_N A_z + A_N\partial_z\xi^N + \partial_z\Lambda \\ &= A_0\partial_z\xi^t + \partial_z\Lambda \\ &= \left(A_0\zeta'^t + \lambda'\right) e^{-i\omega t+ik_x x} \\ &= \left(-\frac{i\omega c_1 z e^{\chi} A_0}{f^{3/2}} + \lambda'\right) e^{-i\omega t+ik_x x}\,, \end{aligned} \tag{A.13}$$

where (A.7) was used in the last equality. Solving for λ' and integrating yields,

$$\lambda = i\omega c_1 \int^z dz' \frac{z' e^{\chi} A_0}{f^{3/2}} + c_5 \,, \tag{A.14}$$

where c_5 is an integration constant.

In conclusion, (A.4), (A.8), (A.10), (A.12), and (A.14) determine the gauge parameters (7.57) up to five integration constants:

$$\begin{aligned}
\xi^z &= c_1 z \sqrt{f} e^{-i\omega t + ik_x x} \\
\xi^t &= \left(-i\omega c_1 \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} + c_2 \right) e^{-i\omega t + ik_x x} \\
\xi^x &= \left(-ik_x c_1 \int^z dz' \frac{z'}{\sqrt{f}} + c_3 \right) e^{-i\omega t + ik_x x} \\
\xi^y &= c_4 e^{-i\omega t + ik_x x} \\
\Lambda &= \left(i\omega c_1 \int^z dz' \frac{z' e^{\chi} A_0}{f^{3/2}} + c_5 \right) e^{-i\omega t + ik_x x} \,.
\end{aligned} \tag{A.15}$$

However, we will find that only four of these constants will enter the longitudinal modes of the pure gauge solutions. Using (7.55) and (A.15), we now compute these longitudinal modes of the pure gauge solutions. First of, $\delta_\xi g_{\mu\mu}$ with $\mu = t,\, x,\, y$, is given by

$$\delta_\xi g_{\mu\mu} = \xi^K \partial_K g_{\mu\mu} + g_{\mu K} \partial_\mu \xi^K + g_{K\mu} \partial_\mu \xi^K = \xi^z \partial_z g_{\mu\mu} + 2 g_{\mu\mu} \partial_\mu \xi^\mu \,. \tag{A.16}$$

Using (7.9) for the background metric and (A.15) for ξ, the tt-component is given by

$$\begin{aligned}
\delta_\xi g_{tt} &= \xi^z \partial_z g_{tt} + 2 g_{tt} \partial_t \xi^t \\
&= \xi^z \partial_z \frac{-L^2 f e^{-\chi}}{z^2} + 2 \frac{-L^2 f e^{-\chi}}{z^2} (-i\omega) \xi^t \\
&= \xi^z L^2 \left(\frac{f\chi' - f'}{z^2} e^{-\chi} + \frac{2 f e^{-\chi}}{z^3} \right) + \frac{2i\omega L^2 f e^{-\chi}}{z^2} \xi^t \\
&= c_1 L^2 z \sqrt{f} \frac{z f \chi' - z f' + 2f}{z^3} e^{-\chi} e^{-i\omega t + ik_x x} \\
&\quad + \frac{2i\omega L^2 f e^{-\chi}}{z^2} \left(-i\omega c_1 \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} + c_2 \right) e^{-i\omega t + ik_x x} \\
&= L^2 \left[c_1 \left(\sqrt{f} \frac{z f \chi' - z f' + 2f}{z^2} e^{-\chi} + \frac{2\omega^2 f e^{-\chi}}{z^2} \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} \right) + c_2 \frac{2i\omega f e^{-\chi}}{z^2} \right] e^{-i\omega t + ik_x x} \,.
\end{aligned} \tag{A.17}$$

Similarly, for the xx- and yy-components, we get

$$\begin{aligned}
\delta_\xi g_{xx} &= \xi^z \partial_z g_{xx} + 2 g_{xx} \partial_x \xi^x \\
&= \xi^z \partial_z \frac{L^2}{z^2} + 2 \frac{L^2}{z^2} ik_x \xi^x \\
&= \frac{L^2}{z^2} \left[-2 c_1 \sqrt{f} + 2ik_x \left(-ik_x c_1 \int^z dz' \frac{z'}{\sqrt{f}} + c_3 \right) \right] e^{-i\omega t + ik_x x} \\
&= \frac{2L^2}{z^2} \left[c_1 \left(-\sqrt{f} + k_x^2 \int^z dz' \frac{z'}{\sqrt{f}} \right) + ik_x c_3 \right] e^{-i\omega t + ik_x x} \,,
\end{aligned} \tag{A.18}$$

and

$$\delta_\xi g_{yy} = \xi^z \partial_z g_{yy} + g_{yy}\partial_y \xi^y = \xi^z \partial_z \frac{L^2}{z^2} = -c_1 \frac{2L^2\sqrt{f}}{z^2} e^{-i\omega t + ik_x x} \,. \tag{A.19}$$

The remaining longitudinal metric mode is given by

$$\begin{aligned}
\delta_\xi g_{tx} &= \xi^K \partial_K g_{tx} + g_{tK}\partial_x \xi^K + g_{Kx}\partial_t \xi^K \\
&= g_{tt}\partial_x \xi^t + g_{xx}\partial_t \xi^x \\
&= -\frac{L^2 f e^{-\chi}}{z^2} ik_x \xi^t + \frac{L^2}{z^2}(-i\omega)\xi^x \\
&= -\frac{L^2}{z^2}\left[ik_x f e^{-\chi}\left(-i\omega c_1 \int^z dz' \frac{z' e^\chi}{f^{3/2}} + c_2\right) + i\omega\left(-ik_x c_1 \int^z dz' \frac{z'}{\sqrt{f}} + c_3\right)\right] e^{-i\omega t + ik_x x} \\
&= -\frac{L^2}{z^2}\left[c_1 \omega k_x \left(f e^{-\chi}\int^z dz' \frac{z' e^\chi}{f^{3/2}} + \int^z dz' \frac{z'}{\sqrt{f}}\right) + ik_x c_2 f e^{-\chi} + i\omega c_3\right] e^{-i\omega t + ik_x x} \,.
\end{aligned} \tag{A.20}$$

Now, the two longitudinal modes for the Maxwell gauge field is

$$\begin{aligned}
\delta A_t &= \delta_\xi A_t + \partial_t \Lambda = \xi^K \partial_K A_t + A_K \partial_t \xi^K - i\omega\Lambda \\
&= \xi^z A_0' - i\omega A_0 \xi^t - i\omega\Lambda \\
&= \left[c_1 z f A_0' - i\omega A_0\left(-i\omega c_1 \int^z dz' \frac{z' e^\chi}{f^{3/2}} + c_2\right) - i\omega\left(i\omega c_1 \int^z dz' \frac{z' e^\chi A_0}{f^{3/2}} + c_5\right)\right] e^{-i\omega t + ik_x x} \\
&= \left[c_1\left(z f A_0' - \omega^2 A_0 \int^z dz' \frac{z' e^\chi}{f^{3/2}} + \omega^2 \int^z dz' \frac{z' e^\chi A_0}{f^{3/2}}\right) - i\omega c_2 A_0 - i\omega c_5\right] e^{-i\omega t + ik_x x} \,,
\end{aligned} \tag{A.21}$$

and

$$\begin{aligned}
\delta A_x &= \delta_\xi A_x + \partial_x \Lambda = \xi^K \partial_K A_x + A_K \partial_x \xi^K + ik_x \Lambda \\
&= ik_x A_0 \xi^t + ik_x \Lambda \\
&= \left[ik_x A_0\left(-i\omega c_1 \int^z dz' \frac{z' e^\chi}{f^{3/2}} + c_2\right) + ik_x\left(i\omega c_1 \int^z dz' \frac{z' e^\chi A_0}{f^{3/2}} + c_5\right)\right] e^{-i\omega t + ik_x x} \\
&= \left[c_1 \omega k_x \left(A_0 \int^z dz' \frac{z' e^\chi}{f^{3/2}} - \int^z dz' \frac{z' e^\chi A_0}{f^{3/2}}\right) + ik_x c_2 A_0 + ik_x c_5\right] e^{-i\omega t + ik_x x} \,,
\end{aligned} \tag{A.22}$$

where we have used (7.10) for the background Maxwell field and (A.15) for ξ and Λ. Lastly, the scalar field transforms both under diffeomorphisms and $U(1)$ transformations. Since the gauge parameters are infinitesimal quantities, the $U(1)$ transformation of the scalar field reduces to

$$\phi \to e^{i\Lambda}\phi = (1 + i\Lambda)\phi = \phi + \delta_\Lambda \phi \,, \tag{A.23}$$

where $\Lambda = \lambda e^{-i\omega t + ik_x x}$ and λ is given by (A.14). The pure gauge mode for the scalar field is then

$$\begin{aligned}
\delta\phi &= \delta_\xi \phi + \delta_\Lambda \phi = \xi^M \partial_M \phi + i\Lambda\phi \\
&= \left[c_1 z \sqrt{f}\phi' + \left(-\omega c_1 \int^z dz' \frac{z' e^\chi A_0}{f^{3/2}} + ic_5\right)\phi\right] e^{-i\omega t + ik_x x} \,.
\end{aligned} \tag{A.24}$$

The modes (A.17)-(A.24) are determined only up to four integration constants c_1, c_2, c_3, and c_5. The integration constant c_4 only affect the transverse modes and is therefore unimportant in our analysis. Then, for notational clarity, we simply relabel c_5 by c_4.

Finally, since we defined the fluctuations of the metric and gauge field with additional factors of $1/z^2$ and $g_F L/\kappa = L\sqrt{2}$ in (7.51), the pure gauge solutions has to be redefined with the same factors. This can be accounted for by simply multiplying the pure gauge solutions (A.17)-(A.20) by z^2, and (A.21) and (A.22) by $1/(L\sqrt{2})$. The sought after longitudinal pure gauge modes are thus given by

$$
\begin{aligned}
\delta g_{tt} &= L^2 \left[c_1 \left(\sqrt{f}\,(zf\chi' - zf' + 2f)\, e^{-\chi} + 2\omega^2 f e^{-\chi} \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} \right) + 2ic_2 \omega f e^{-\chi} \right] e^{-i\omega t + ik_x x}\,, \\
\delta g_{tx} &= -L^2 \left[c_1 \omega k_x \left(f e^{-\chi} \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} + \int^z dz' \frac{z'}{\sqrt{f}} \right) + ik_x c_2 f e^{-\chi} + i\omega c_3 \right] e^{-i\omega t + ik_x x}\,, \\
\delta g_{xx} &= 2L^2 \left[c_1 \left(-\sqrt{f} + k_x^2 \int^z dz' \frac{z'}{\sqrt{f}} \right) + ik_x c_3 \right] e^{-i\omega t + ik_x x}\,, \\
\delta g_{yy} &= -2c_1 L^2 \sqrt{f} e^{-i\omega t + ik_x x}\,, \\
\delta A_t &= \frac{1}{L\sqrt{2}} \left[c_1 \left(zfA_0' - \omega^2 A_0 \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} + \omega^2 \int^z dz' \frac{z' e^{\chi} A_0}{f^{3/2}} \right) - i\omega c_2 A_0 - i\omega c_4 \right] e^{-i\omega t + ik_x x}\,, \\
\delta A_x &= \frac{1}{L\sqrt{2}} \left[c_1 \omega k_x \left(A_0 \int^z dz' \frac{z' e^{\chi}}{f^{3/2}} - \int^z dz' \frac{z' e^{\chi} A_0}{f^{3/2}} \right) + ik_x c_2 A_0 + ik_x c_4 \right] e^{-i\omega t + ik_x x}\,, \\
\delta\phi &= \left[c_1 z \sqrt{f} \phi' + \left(-\omega c_1 \int^z dz' \frac{z' e^{\chi} A_0}{f^{3/2}} + ic_4 \right) \phi \right] e^{-i\omega t + ik_x x}\,.
\end{aligned}
\tag{A.25}
$$

Setting either one of the four integration constants to unity and the others to zero results in four linearly independet solutions to the fluctuation equations.

B

Conductivity of the holographic superconductor

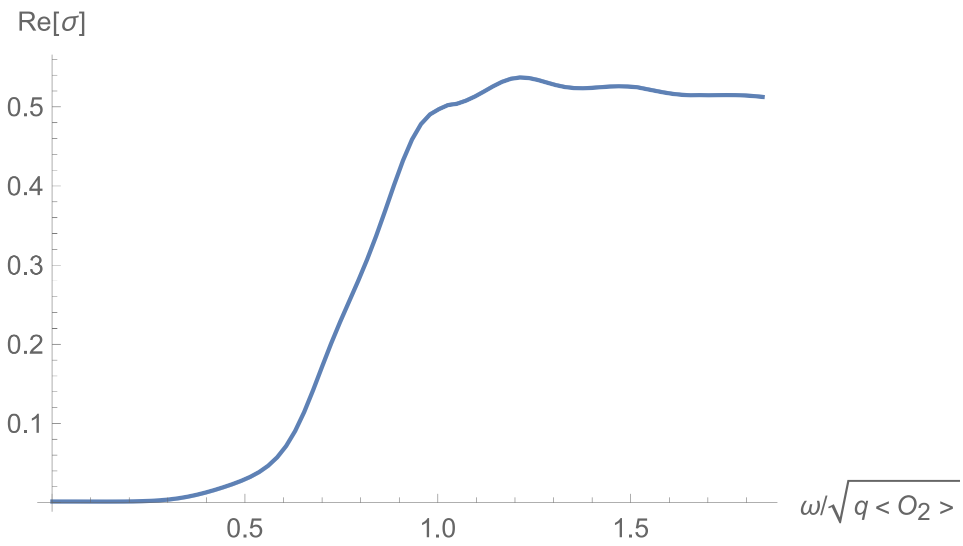

Figure B.1: A plot of the real part of the conductivity versus frequency, normalized by the scalar condensate. Here we have chosen $q = 3$ for the charge of the complex scalar field. There is a delta function at $\omega = 0$, invisible in the plot. The small oscillation in the curve indicates a numerical instability.

Fig. B.1 shows the result of a numerical computation of the conductivity. More precisely, it shows the real part of the conductivity σ as a function of the frequency ω normalized by the scalar condensate. We have chosen the particular value $q = 3$ for the charge of the complex scalar field in the bulk. The expectation value of the scalar operator is given by the subleading part of the complex scalar field near the conformal boundary, according to our identification (7.21). The plot qualitatively agrees with the result in [10], at least up to a factor of ~ 2, which could be a consequence of the factor $g_F L/\kappa = \sqrt{2}$ in our definition of the $U(1)$ gauge field fluctuation. There is a delta function at $\omega = 0$ which is invincible in the plot, meaning that the DC conductivity is formally infinite. This is evidence in support of the solution describing a superconducting ground state. The small oscillating behaviour of the curve is likely a sign of some numerical instability. The numerical accuracy could be improved upon, but the qualitative behaviour is enough to confirm that the solution found should be in the superconducting phase, which was the main purpose of this calculation.

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