CHALMERS
UNIVERSITY OF TECHNOLOGY

Flux Backgrounds and Generalised Geometry

Type II supergravity compactifications, exceptional generalised geometry and generalised G-structures in exceptional field theory

Master's thesis in Physics and Astronomy

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Department of Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
Gothenburg, Sweden 2019

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Master's Thesis 2019
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Typeset in LaTeX
Printed by Chalmers Reproservice
Gothenburg, Sweden 2019

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Abstract

In this thesis we study aspects of compactifications of mainly the type II supergravity theories. We begin with the study of classical approaches with a Kaluza-Klein compactification of the type II supergravity theories on a Calabi-Yau 3-fold, followed by a presentation of their orientifold variants, mirror symmetry, and the effects of allowing background fluxes on the moduli in the 4D effective field theory. The moduli fields can be stabilised by the presence of non-trivial background fluxes, perturbative corrections to the 10D theory and non-perturbative corrections to the 4D scalar potential. These corrections can be used to construct toy model de Sitter vacua as in the KKLT and large volume scenarios. We also introduce a compactification with so-called non-geometric fluxes, whose presence makes the metric of the internal manifold ill-defined. This is followed by a discussion of double field theory, which treats geometric and non-geometric fluxes on equal footing by extending spacetime in order to covariantise the T-duality group $O(d,d)$. We briefly discuss consistent truncations in the context of the generalised Scherk-Schwarz ansatz. This is followed by an introduction of exceptional field theory, which is also an extension of supergravity which covariantises the exceptional U-duality groups. This brings us to the formalism of exceptional generalised geometry where we formulate supersymmetric flux backgrounds as torsion-free generalised G-structures. The notion of generalised G-structures is then interpreted as generalised differential forms in exceptional field theory and used to describe vacua. The application to find consistent truncations to 4D is also discussed. This construction is believed to play an important role in the classification of supersymmetric backgrounds.

Acknowledgements

I would like to express my deep gratitude to my supervisor Martin Cederwall for his guidance and support. Thank you for the opportunity to explore this research area for two semesters. I also want to extend a special thanks to Torbjörn Nilsson for his help and never-ending support. Thanks to Jakob Palmqvist for taking time to answer questions during this period. Last but not least I want to thank my corridor mates Simon Ekhammar, Daniel Erkensten, Sebastian Bergström as well as my other classmates of MPPAS 2017 for discussions and coffee breaks.

Magdalena Eriksson, Gothenburg, June 2019

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1
Introduction

During the past half century a huge success for theoretical physics has been the construction of the Standard Model, which is a framework that unites the electromagnetic, strong and weak forces. The Standard Model describes how particles interact via these forces using the formalism of quantum field theory, which follows the laws of quantum mechanics and special relativity. The Standard Model has been extensively experimentally checked to surprising levels of accuracy, and is regarded as the most complete theory of fundamental interactions.

An open question in theoretical physics is how to include the fourth fundamental physical force – gravitation – into this theory. The failure of incorporating gravity is to a large extent due to the fact that it can not be renormalisable, i.e. the procedure of adding a finite number of counterterms to an action that cancel divergences does not work for gravity, but does for the Standard Model.

The most studied and arguably the most promising candidate for a uniting theory, a quantum theory of gravity, is string theory. This theory is based on vibrating one-dimensional objects called strings, and the discrete vibration modes of the quantised string viewed from large distances correspond to particles. The theory is regarded to incorporate gravitation in the sense that there is always a vibration mode of the closed quantised string that is massless and has spin two, which is interpreted as the graviton. The other fundamental forces are encoded as massless states corresponding to gauge bosons.

The spectrum of the quantised string also contains vibration modes that have negative mass, known as tachyons, and they imply an instability of the spacetime. These states can be removed using the notion of supersymmetry, which relates bosons and fermions, and result in what is called the superstring theories which by conformal invariance are forced to be ten-dimensional. There are five different superstring theories; Type I, Type IIA/B and two heterotic string theories with ten-dimensional gauge groups $SO(32)$ and $E_8 \times E_8$, and they are all related to one another via two types of duality, namely S- and T-duality. It was found that these superstring theories could be seen as different formulations of a more fundamental theory, called M-theory. The low-energy limit of the superstring theories are known as supergravity theories, and a common feature of all ten-dimensional supergravity theories is the presence of higher-rank gauge fields; the NSNS and RR fields, whose origin will be discussed later. The supergravity theories themselves are non-renormalisable but are viewed as effective field theories of the original string theory. As we will see, the supergravity theories can be extended to include T-duality by way of double field theory, and $T + S = U$-duality by way of exceptional field theory.

In order to connect the supergravity theories to the 3+1-dimensional world we experience every day, we turn to the concept known as compactification. This is based on assuming that spacetime can be viewed as having four large spacetime directions and the remaining six are wrapped together to form a very small compact space. This space is so small that we are unable to experience its presence. The interest is then in supergravity solutions which have topology

$$\mathcal{M} = \mathcal{M}_4 \times \mathcal{M}_6 \ , \tag{1.1}$$

where $\mathcal{M}_4$ is the four-dimensional external Lorentzian spacetime and $\mathcal{M}_6$ is a compact Riemannian internal manifold. As we will see in this thesis, the effect of the extra dimensions can still be seen in the four-dimensional effective field theory. For instance, when compactifying, the effective four-dimensional theory contains a large number of massless scalar fields that would give rise to long-range forces that are not experienced. These fields are called moduli and will be unrestricted unless stabilised, i.e. fixing their expectation value and giving them a mass, by mechanisms which will be presented in this thesis. In particular we will consider general string backgrounds in which the NSNS and RR fields take non-trivial expectation values, known as fluxes. As we will see, these can be used to stabilise the moduli fields.

Further, the spacetime solutions taking the form of eq. (1.1) should preserve some degree of the original supersymmetry, and in order for the four-dimensional theory to have supersymmetry the internal manifold must support supersymmetry spinors. Supersymmetry is naturally incorporated in string theory, and how much supersymmetry that is preserved in the effective action, i.e. the number of $\mathcal{N}$ supersymmetry spinors admitted on $\mathcal{M}_4$, is important for phenomenological reasons. Namely, it is desirable to preserve a minimal amount of supersymmetry, i.e. $\mathcal{N} = 1$, as that is compatible with minimal extensions of the Standard Model. Central to this thesis will be how the conditions of supersymmetry translate into topological and differential conditions on the internal manifold, which will strongly constrain its geometry. A powerful tool to use is an extension of differential geometry called generalised geometry which will be introduced in the context of flux compactifications.

1.1 Outline

This thesis is devoted to the study of supersymmetric compactifications with non-trivial fluxes. The thesis is organised as follows. In chapter 2 we introduce the basic notions of supersymmetry and supersymmetric Lagrangians as well as Kaluza-Klein compactification. In chapter 3 the type II superstring theories are constructed. Chapter 4 discusses type II backgrounds with $\mathcal{N} = 1$ with and without fluxes as well as introduces the formalism of complex generalised geometry used in describing the internal geometries of $\mathcal{N} = 1$ vacua. In particular we show how the G-structures are conveniently used to describe fluxless compactifications, but how they fail to capture all the information when fluxes are present. In chapter 5 we study the four-dimensional effective theories obtained from Calabi-Yau and Calabi-Yau orientifold projections of both type II theories. Flux-generated scalar potentials and their corresponding superpotentials are discussed and in the final section we also discuss mirror symmetry. In chapter 6 we derive the classical type II no-go theorems for compactifications with fluxes and discuss how they can be circumvented. In chapter 7 we study moduli stabilisation by fluxes on four different orientifold backgrounds. Chapter 8 is reserved for introducing corrections to the low-energy effective action which may be used to stabilise moduli as well as obtain classical toy models of de Sitter vacua. The structure of chapters 4–8 is heavily inspired by the fine review of [1] but aims to be more explicit. In chapter 9 we broaden the concept of the internal space by allowing non-geometric fluxes, i.e. fluxes that make the metric of the internal space $\mathcal{M}_6$ globally ill-defined. We show how these fluxes are naturally incorporated in double field theory, which is an $O(d, d)$ covariant extension of supergravity. In the final section we discuss consistent truncations of double field theory. In chapter 10 the construction of exceptional field theory is reviewed, a theory that covariantises U-duality by extending spacetime such that one completes the fundamental representation of $E_{d(d)}$ with extra coordinates. The supergravity theories can then be recovered by solving the so-called section constraint in two different ways. In chapter 11 we introduce exceptional generalised geometry, which is an extension of complex generalised geometry which ”geometrices” all the type II supergravity fluxes. The G-structure will be generalised to elegantly encode flux compactifications. In chapter 12 we use the notion of generalised G-structures to describe half-maximal supergravity obtained from exceptional field theory, and how consistent truncations can be formulated. Conclusions and discussion on future work are found in the last chapter 13. In Appendix A some of the conventions used throughout the thesis are stated. In Appendix B some basic concepts of differential geometry and topology are introduced. Appendix C complements some calculations in chapter 4, and Appendix D complements chapter 5 with an explicit computation of type IIB supergravity theory compactified on a Calabi-Yau manifold with and without fluxes. Finally, Appendix E contains some explicit calculations from chapter 9 relating to double field theory.

2
Basics of Supersymmetric Lagrangians and Kaluza-Klein Theory

In this chapter we review some important aspects of supersymmetry and how one constructs supersymmetric Lagrangians. In section 2.3 we review a classic method of dimensional compactification which was formulated in the early 1900's by Kaluza and Klein.

A supersymmetry transformation transforms a bosonic state into a fermionic one and vice versa;

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle \ , \quad Q\,|\text{boson}\rangle = |\text{fermion}\rangle \ , \tag{2.1}$$

where Q is a spinor, and together with its Hermitian conjugate $Q^\dagger$ it makes up the generators of the supersymmetry algebra. The supersymmetry generators are special in that they do not form a Lie algebra with ordinary commutation relations, but with anticommutation relations. They therefore fulfil a *graded* Lie algebra, the supersymmetry algebra;

$$\{Q_a, Q^\dagger_{\dot b}\} = 2\sigma^\mu_{a\dot b} P_\mu \ , \tag{2.2}$$

$$\{Q_a, Q_b\} = \{Q^\dagger_{\dot a}, Q^\dagger_{\dot b}\} = 0 \ , \tag{2.3}$$

$$[Q_a, P^\mu] = [Q^\dagger_{\dot a}, P^\mu] = 0 \ , \tag{2.4}$$

$$[Q_a, M^{\mu\nu}] = (\sigma^{\mu\nu})_a{}^b Q_b \ , \tag{2.5}$$

$$[Q_a, \text{gauge symmetry}] = 0 \ , \tag{2.6}$$

where $Q_a/Q^\dagger_{\dot a}$ is a right/left-handed spinor, P^μ are the translation generators, $M^{\mu\nu}$ are the generators of Lorentz transformations, and $\sigma_{\mu\nu} = \frac{i}{4}(\sigma_\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$, $\sigma^\mu = (1, \sigma^i)$ where $\sigma^{i=1,2,3}$ are the Pauli matrices. Particles in a supersymmetric theory fall into irreducible representations of the supersymmetry algebra called supermultiplets. These contain both fermions and bosons which are said to be superpartners to each other. There are traditionally two types of supermultiplets; the chiral supermultiplet and the vector supermultiplet. The chiral one consists of a Weyl fermion and a complex scalar field. The vector multiplet consists of a Weyl fermion and a massless vector boson.

2.1 The Wess-Zumino model

The simplest supersymmetry model contain a single, non-interacting chiral multiplet, consisting of a complex scalar field ϕ and a left-handed 2-component Weyl fermion ψ. The spinors are assumed to be 4D. The simplest action contain only the kinetic terms of the fermion and scalar;

$$S = \int \mathrm{d}^4x\,(\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}}) \ , \quad \text{where} \quad \begin{cases} \mathcal{L}_{\text{scalar}} & = -\partial^\mu\phi^*\partial_\mu\phi \\ \mathcal{L}_{\text{fermion}} & = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi \end{cases} . \tag{2.7}$$

This is called the massless, non-interacting *Wess-Zumino model*. We wish to study the supersymmetry transformations of this theory. The action of eq. (2.7) above should be invariant under a supersymmetry transformation;

$$S[\phi,\psi] = S[\phi+\delta_\epsilon\phi, \psi+\delta_\epsilon\psi] \quad \Leftrightarrow \quad \delta_\epsilon S = S[\delta_\epsilon\phi, \delta_\epsilon\psi] = 0 \ , \tag{2.8}$$

where ϵ is the supersymmetry parameter. The transformation should also turn the scalar boson field into something involving the fermion field ψ_a and vice versa. The simplest possibility for the transformation of the scalar field is

$$\delta_\epsilon\phi = \epsilon\psi \ , \qquad \delta_\epsilon\phi^* = \epsilon^\dagger\psi^\dagger \ , \tag{2.9}$$

where our supersymmetry parameter ϵ^a is an infinitesimal, anticommuting 2-component Weyl object. For a global symmetry, ϵ^a is constant, and so $\partial_\mu\epsilon^a = 0$. The relation between this parameter and the supersymmetry generators is $\delta_\epsilon X = (\epsilon Q + \epsilon^\dagger Q^\dagger)X$, with algebra $[\epsilon Q, \epsilon^\dagger Q^\dagger] = 2\epsilon\sigma^\mu\epsilon^\dagger P^\mu$ since ϵ anticommutes. Transforming the scalar part of the Lagrangian, we get

$$\begin{aligned}\delta_\epsilon\mathcal{L}_{\text{scalar}} &= \delta_\epsilon(-\partial^\mu\phi^*\partial_\mu\phi)\\ &= -\partial^\mu\delta_\epsilon\phi^*\partial_\mu\phi - \partial^\mu\phi^*\partial_\mu\delta_\epsilon\phi\\ &= -\epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi - \epsilon\partial^\mu\phi^*\partial_\mu\psi \ .\end{aligned} \tag{2.10}$$

Our goal is that this is cancelled fully, or up to a total derivative, by $\delta_\epsilon\mathcal{L}_{\text{fermion}}$. For this to happen $\delta_\epsilon\psi$ must be linear in ϵ and ϕ, as well as include a spacetime derivative;

$$\delta_\epsilon\psi_a = -i(\sigma^\mu\epsilon^\dagger)_a\partial_\mu\phi \ , \qquad \delta_\epsilon\psi^\dagger_b = i(\epsilon\sigma^\mu)_b\partial_\mu\phi^* \ . \tag{2.11}$$

With this guess the transformation of the fermionic part of the Lagrangian becomes

$$\begin{aligned}\delta_\epsilon\mathcal{L}_{\text{fermion}} &= \delta_\epsilon(i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi)\\ &= i\delta_\epsilon\psi^\dagger\bar\sigma^\mu\partial_\mu\psi + i\psi^\dagger\bar\sigma^\mu\partial_\mu\delta_\epsilon\psi\\ &= -\epsilon\sigma^\mu\partial_\mu\phi^*\bar\sigma^\nu\partial_\nu\psi + \psi^\dagger\bar\sigma^\nu\partial_\nu\sigma^\mu\epsilon^\dagger\partial_\mu\phi\\ &= -\epsilon\sigma^\mu\bar\sigma^\nu\partial_\mu\phi^*\partial_\nu\psi + \epsilon^\dagger\psi^\dagger\bar\sigma^\nu\sigma^\mu\partial_\mu\partial_\nu\phi \ .\end{aligned}$$

The Pauli matrices satisfy the anticommutation relations

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = -2\eta^{\mu\nu} \ , \tag{2.12}$$

which may allow some simplification. The product $\bar\sigma^\nu\sigma^\mu\partial_\mu\partial_\nu\phi$ of the second term above can be rewritten using the fact that the double derivative on ϕ is symmetric; $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$. We may divide $\bar\sigma^\nu\sigma^\mu$ into a symmetric and antisymmetric part respectively as $\bar\sigma^\nu\sigma^\mu = \frac{1}{2}(\bar\sigma^\nu\sigma^\mu - \sigma^\mu\bar\sigma^\nu) + \frac{1}{2}(\bar\sigma^\nu\sigma^\mu - \sigma^\mu\bar\sigma^\nu)$. A purely antisymmetric tensor $X^{\mu\nu}$ contracted with a symmetric tensor $Z^{\mu\nu}$ will vanish as $X^{\mu\nu}Z_{\mu\nu} = -X^{\nu\mu}Z_{\mu\nu} = -X^{\nu\mu}Z_{\nu\mu} = -X^{\mu\nu}Z_{\mu\nu}$, where in the last step we simply renamed the indices $\eta \to n \to \mu$, and $\mu \to m \to \nu$. We end up with $X^{\mu\nu}Z_{\mu\nu} = -X^{\mu\nu}Z_{\mu\nu}$ meaning that $X^{\mu\nu}Z_{\mu\nu} = 0$. Thus only the symmetric part of $\bar\sigma^\nu\sigma^\mu$ contracted with $\partial_\mu\partial_\nu$ will survive, a part that also may be simplified with the Dirac anticommutation relations:

$$\bar\sigma^\nu\sigma^\mu \to \tfrac{1}{2}(\bar\sigma^\nu\sigma^\mu + \sigma^\mu\bar\sigma^\nu) = \tfrac{1}{2}(-2\eta^{\mu\nu}) = -\eta^{\mu\nu} \ .$$

Thus our expression becomes

$$\delta_\epsilon\mathcal{L}_{\text{fermion}} = -\epsilon\sigma^\mu\bar\sigma^\nu\partial_\mu\phi^*\partial_\nu\psi - \epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi \ , \tag{2.13}$$

where we have contracted an index using the metric $\eta^{\mu\nu}$. Again, we wish to write this on a form which cancels the scalar part of the Lagrangian up to a total derivative. This is done by

$$\delta_\epsilon\mathcal{L}_{\text{fermion}} = \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi + \epsilon\partial^\mu\phi^*\partial_\mu\psi - \partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^* + \epsilon\psi\partial^\mu\psi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi) \ , \tag{2.14}$$

which we see matches our earlier expression:

$$
\begin{aligned}
&\epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi + \epsilon\partial^\mu\phi^*\partial_\mu\psi \\
&\quad -\partial_\mu\left(\epsilon\sigma^\nu\bar{\sigma}^\mu\psi\partial_\nu\phi^* + \epsilon\psi\partial^\mu\phi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi\right) = \cancel{\epsilon\partial^\mu\psi\partial_\mu\phi^*} + \bcancel{\epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\psi\partial_\nu\phi^* - \xcancel{\epsilon\sigma^\nu\bar{\sigma}^\mu\psi\partial_\mu\partial_\nu\phi^*} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \cancel{\epsilon\partial_\mu\psi\partial^\mu\phi^*} - \xcancel{\epsilon\psi\partial_\mu\partial^\mu\phi^*} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \bcancel{\epsilon^\dagger\partial_\mu\psi^\dagger\partial^\mu\phi} - \epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\psi\partial_\nu\phi^* - \epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\epsilon\sigma^\mu\bar{\sigma}^\nu\partial_\mu\phi^*\partial_\nu\psi - \epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi \ .
\end{aligned}
$$

Here terms crossed out in the same way cancel. The trick for $\sigma^\nu\bar{\sigma}^\mu\partial_\mu\partial_\nu$ has been used in the double line or "X" cancellation. We have seen that

$$
\delta_\epsilon\mathcal{L}_{\text{scalar}} + \delta_\epsilon\mathcal{L}_{\text{fermion}} = \partial_\mu\left(\epsilon\sigma^\nu\bar{\sigma}^\mu\psi\partial_\nu\phi^* + \epsilon\psi\partial^\mu\psi^* + \epsilon^\dagger\psi^\dagger\partial^\mu\phi\right) \ , \tag{2.15}
$$

and so we arrive at

$$
\delta_\epsilon S = \int \mathrm{d}^4x(\delta_\epsilon\mathcal{L}_{\text{scalar}} + \delta_\epsilon\mathcal{L}_{\text{fermion}}) = 0 \ . \tag{2.16}
$$

In order to complete our proof that the theory of eq. (2.7) is supersymmetric, our last thing to check is that the supersymmetry algebra closes. That is, to check that the commutator of two supersymmetry transformations with different supersymmetry parameters form another symmetry of the theory. The commutator acting on the scalar field results in that

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi &= \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) - \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) \\
&= \delta_{\epsilon_1}(\epsilon_2\psi) - \delta_{\epsilon_2}(\epsilon_1\psi) \\
&= -i\epsilon_2\sigma^\mu\epsilon_1^\dagger\partial_\mu\phi + i\epsilon_1\sigma^\mu\epsilon_2^\dagger\partial_\mu\phi \\
&= i\left(-\epsilon_2\sigma^\mu\epsilon_1^\dagger + \epsilon_1\sigma^\mu\epsilon_2^\dagger\right)\partial_\mu\phi \ ,
\end{aligned}
$$

where we have used the transformation relation of eq. (2.9) on the second line and the transformations of eq. (2.11) on the third line. Now turning to the fermion field ψ, we have that

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_a &= \delta_{\epsilon_1}\left(-i(\sigma^\mu\epsilon_2^\dagger)_a\partial_\mu\phi\right) - \delta_{\epsilon_2}\left(-i(\sigma^\mu\epsilon_1^\dagger)_a\partial_\mu\phi\right) \\
&= i\left(-(\sigma^\mu\epsilon_2^\dagger)_a\epsilon_1 + (\sigma^\mu\epsilon_1^\dagger)_a\epsilon_2\right)\partial_\mu\psi \\
&= i\left(-\epsilon_2\sigma^\mu\epsilon_1^\dagger + \epsilon_1\sigma^\mu\epsilon_2^\dagger\right)\partial_\mu\psi_a + i\left((\epsilon_2)_a\epsilon_1^\dagger\bar{\sigma}^\mu - (\epsilon_1)_a\epsilon_2^\dagger\bar{\sigma}^\mu\right)\partial_\mu\psi \ ,
\end{aligned}
$$

where on the last line we have used the Fierz identity $(A)_aBC = -(B)_aAC - BA(C)_a$. The last two terms vanish on-shell, i.e. when the equation of motion $\bar{\sigma}^\mu\partial_\mu\psi = 0$ is satisfied. Then the remaining terms is exactly the translation as for the scalar field.

The supersymmetry algebra only seem to close on-shell, but one usually would like for it to hold quantum mechanically as well. Because of this, one usually introduces a new complex scalar field F, which has no kinetic term. This field is referred to as an *auxiliary* field, whose only purpose is to allow the algebra to close off-shell. The Lagrangian density is given by

$$
\mathcal{L}_{\text{auxiliary}} = F^*F \ , \tag{2.17}
$$

with equations of motion being $F = F^* = 0$. This takes us to the free Wess-Zumino model;

$$
\mathcal{L} = -\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + |F|^2 \ , \tag{2.18}
$$

which is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta_\epsilon \phi &= 0 \ , \\
\delta_\epsilon \psi_a &= -i(\sigma^\mu \epsilon)_a \partial_\mu \phi + (\epsilon)_a F \ , \\
\delta_\epsilon F &= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \ .
\end{aligned} \tag{2.19}$$

Hence, whenever we are on-shell the equations of motion $\bar{\sigma}^\mu \partial_\mu \psi = 0$ and $F = 0$ are fulfilled, so F only comes into the picture when off-shell to close the algebra. The supersymmetry algebra closes for the auxiliary field as well;

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]F &= \delta_{\epsilon_1}(-i\epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi) - \delta_{\epsilon_2}(-i\epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi) \\
&= -i\epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu(-i\sigma^\nu \epsilon_1^\dagger \partial_\nu \psi + \epsilon_1 F) + i\epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu(-i\sigma^\nu \epsilon_2^\dagger \partial_\nu \phi + \epsilon_2 F) \\
&= -i(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger)\partial_\mu F - \epsilon_2^\dagger \bar{\sigma}^\mu \sigma^\nu \epsilon_1^\dagger \partial_\mu \partial_\nu \phi + \epsilon_1^\dagger \bar{\sigma}^\mu \sigma^\nu \epsilon_2^\dagger \partial_\mu \partial_\nu \phi \\
&= -i(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger)\partial_\mu F \ .
\end{aligned}$$

So for the fields $X = \{\phi, \phi^\dagger, \psi, \psi^\dagger, F, F^*\}$ we have that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]X = -i\left(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger\right)\partial_\mu X \ , \tag{2.20}$$

and we have shown that the theory is supersymmetric.

2.2 Superfield formalism

One may construct more complicated supersymmetric Lagrangians with the same method as in the previous section, but it would quickly become very complicated. There are also interactions that can not appear in a supersymmetric Lagrangian. To resolve these issues, a new formalism can be introduced where superpartners are treated as a single field; the *superfield*. The scalar and fermion related via supersymmetry are then considered as different components of this superfield, much like the spin up and down states of a single fermion.

2.2.1 Superspace and supersymmetry transformations

To develop the formalism of the superfield, one introduces the concept of the superspace, where an equal number of commuting and anticommuting coordinates are added to the ordinary space-time coordinates. In 4D the coordinates are $\{x^\mu, \theta^{\dot{\alpha}}, \bar{\theta}_{\dot{\alpha}}\}$, where $\bar{\theta}_{\dot{\alpha}} = (\theta_\alpha)^*$ and fulfil the anti-commutation relations

$$\{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = \{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0 \ . \tag{2.21}$$

The integrals over the superspace are

$$\begin{gathered}
\int \mathrm{d}\theta = \int \mathrm{d}\bar{\theta} = \int \mathrm{d}\bar{\theta}\bar{\theta} = \bar{\theta}\theta = 0 \ , \qquad \int \mathrm{d}\theta^\alpha \theta_\beta = \delta^\alpha_\beta \ , \\
\int \mathrm{d}^2\theta\theta^2 = \int \mathrm{d}^2\bar{\theta}\bar{\theta}^2 \ , \qquad \int \mathrm{d}^4\theta\theta^2\bar{\theta}^2 = 1 \ , \qquad \int \mathrm{d}\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} \ ,
\end{gathered} \tag{2.22}$$

defined as in [2], where the double differentials are given by

$$\mathrm{d}^2\theta \equiv -\tfrac{1}{4}\epsilon_{\alpha\beta}\,\mathrm{d}\theta^\alpha\,\mathrm{d}\theta^\beta \ , \qquad \mathrm{d}^2\bar{\theta} \equiv -\tfrac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\,\mathrm{d}\theta_{\dot{\alpha}}\,\mathrm{d}\theta_{\dot{\beta}} \ , \qquad \mathrm{d}^4\theta \equiv \mathrm{d}^2\bar{\theta}\,\mathrm{d}^2\theta \ . \tag{2.23}$$

Any supermultiplet can be expressed as a single superfield which depends on the superspace coordinates. A superfield with lowest component being a scalar field, i.e. a scalar superfield, can be Taylor expanded to take the general form of

$$\begin{aligned} S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) \\ + \bar{\theta}\bar{\sigma}^\mu\theta V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}D(x) \ . \end{aligned} \tag{2.24}$$

Note that the expansion in power series terminates at order $\theta^2\bar{\theta}^2$, which is common to all functions of superspace coordinates. To see that there are no other contributions, one can note the identities

$$\theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta \ , \qquad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \tfrac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \ , \qquad \theta_\alpha\bar{\theta}_{\dot{\beta}} = \tfrac{1}{2}\sigma^\mu_{\alpha\dot{\beta}}(\bar{\theta}\bar{\sigma}_\mu\theta) \ , \tag{2.25}$$

which can be used to rewrite any term into the forms given in eq. (2.24). In order to formulate the supersymmetry transformations in superspace, one defines the differential operators

$$\begin{aligned} \mathcal{Q}_\alpha &= i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\bar{\theta})_\alpha\partial_\alpha \ , & \overline{\mathcal{Q}}_{\dot{\alpha}} &= -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \ , \\ \mathcal{Q}^\alpha &= -i\frac{\partial}{\partial\theta^\alpha} + (\bar{\theta}\bar{\sigma}^\mu)^\alpha\partial_\alpha \ , & \overline{\mathcal{Q}}^{\dot{\alpha}} &= i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - (\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu \ . \end{aligned} \tag{2.26}$$

With these definitions, the anticommutation relations are given by

$$\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu = -2\sigma^\mu_{\alpha\dot{\beta}}\mathcal{P}_\mu \ , \quad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\overline{\mathcal{Q}}_{\dot{\alpha}}, \mathcal{Q}_{\dot{\beta}}\} = 0 \ , \tag{2.27}$$

where $\mathcal{P}_\mu \equiv -i\partial_\mu$ is the spacetime translation generator. The differential operators of eq. (2.26) obey the graded Leibniz/product rule, which acts as the normal product rule for derivatives but adds a minus sign for anticommutation through a Grassmann-odd object. That is, for two superfields S and T we have that $\mathcal{Q}_\alpha(ST) = (\mathcal{Q}_\alpha S)T + (-1)^S S(\mathcal{Q}_\alpha T)$ with $(-1)^S = \pm 1$ if S is Grassmann even or odd respectively. The supersymmetry transformations parameterised by the infinitesimal ϵ for any superfield S are then given by

$$\begin{aligned} \sqrt{2}\delta_\epsilon S &= -i(\epsilon\mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})S \\ &= \left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + \bar{\epsilon}_{\dot{\alpha}} + i\left(\epsilon\sigma^\mu\bar{\theta} + \bar{\epsilon}\bar{\sigma}^\mu\theta\right)\partial_\mu\right)S \\ &= S(x^\mu + i\epsilon\sigma^\mu\bar{\theta} + i\bar{\theta}\bar{\sigma}^\mu\theta, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) - S(x^\mu, \theta, \bar{\theta}) \ , \end{aligned} \tag{2.28}$$

where the factor $\sqrt{2}$ is chosen by convention and the last line is obtained by Taylor expansion to first order in ϵ and $\bar{\epsilon}$. The last line of eq. (2.28) tells us that a supersymmetry transformation can be viewed as a translation in superspace according to

$$\begin{aligned} x^\mu &\to x^\mu + i\epsilon\sigma^\mu\bar{\theta} + i\bar{\epsilon}\bar{\sigma}^\mu\theta \ , \\ \theta^\alpha &\to \theta^\alpha + \epsilon^\alpha \ , \\ \bar{\theta}_{\dot{\alpha}} &\to \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} \ . \end{aligned} \tag{2.29}$$

The supersymmetry transformations of all of the components in the scalar superfield of eq. (2.24) become [3]

$$
\begin{aligned}
\delta_\epsilon\varphi &= \tfrac{1}{\sqrt{2}}\left(\epsilon\psi + \bar{\epsilon}\bar{\chi}\right) \ , \\
\delta_\epsilon\psi_\alpha &= \tfrac{1}{\sqrt{2}}\left(2\epsilon_\alpha M - (\sigma^\mu\bar{\epsilon})_\alpha(V_\mu + i\partial_\mu\varphi)\right) \ , \\
\delta_\epsilon\bar{\chi}^{\dot{\alpha}} &= \tfrac{1}{\sqrt{2}}\left(2\bar{\epsilon}^{\dot{\alpha}}N + (\bar{\sigma}^\mu\epsilon)^{\dot{\alpha}}(V_\mu - i\partial_\mu\varphi)\right) \ , \\
\delta_\epsilon M &= \tfrac{1}{\sqrt{2}}\left(\bar{\epsilon}\bar{\lambda} - \tfrac{i}{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi\right) \ , \\
\delta_\epsilon N &= \tfrac{1}{\sqrt{2}}\left(\epsilon\rho - \tfrac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi}\right) \ , \\
\delta_\epsilon V^\mu &= \tfrac{1}{\sqrt{2}}\left(\epsilon\sigma^\mu\bar{\lambda} - \bar{\epsilon}\bar{\sigma}^\mu\rho - \tfrac{i}{2}\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu\psi + \tfrac{i}{2}\bar{\epsilon}\bar{\sigma}^\nu\sigma^\mu\partial_\nu\bar{\chi}\right) \ , \\
\delta_\epsilon\rho_\alpha &= \tfrac{1}{\sqrt{2}}\left(2\epsilon_\alpha D - i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu N - \tfrac{i}{2}(\sigma^\nu\bar{\sigma}^\mu\epsilon)_\alpha\partial_\mu V_\nu\right) \ , \\
\delta_\epsilon\bar{\lambda}^{\dot{\alpha}} &= \tfrac{1}{\sqrt{2}}\left(2\bar{\epsilon}^{\dot{\alpha}}D - i(\bar{\sigma}\epsilon)^{\dot{\alpha}}\partial_\mu M + \tfrac{i}{2}(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon})^{\dot{\alpha}}\partial_\mu V_\nu\right) \ , \\
\delta_\epsilon D &= \tfrac{1}{\sqrt{2}}\left(-\tfrac{i}{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\rho - \tfrac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\lambda}\right) \ .
\end{aligned}
\tag{2.30}
$$

In order to construct Lagrangians in superspace with superfields, one will need derivatives with respect to the superspace coordinates. The derivative $\partial/\partial\theta^\alpha$ is however not appropriate since it is not supersymmetrically covariant, i.e.

$$
\delta_\epsilon\left(\frac{\partial S}{\partial\theta^\alpha}\right) \neq \frac{\partial(\delta_\epsilon S)}{\partial\theta^\alpha} \ , \tag{2.31}
$$

and the equivalent applies for $\partial/\partial\bar{\theta}_{\dot{\alpha}}$. As a consequence, the derivative of superfields are not superfields themselves since they do not have the right transformation properties. To resolve this issue one defines the supersymmetric chiral covariant derivative

$$
D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \quad \bar{D}_\alpha \equiv -\frac{\partial}{\partial\theta_\alpha} + i(\bar{\theta}\bar{\sigma}^\mu)^\alpha\partial_\mu \ , \tag{2.32}
$$

which then do fulfil the transformation propertiy $\delta_\epsilon(D_\alpha S) = D_\alpha(\delta_\epsilon S)$. The anti-chiral covariant derivative can for Grassmann-even superfields be defined to obey $\bar{D}_{\dot{\alpha}}S^* \equiv (D_\alpha S)^*$, so that

$$
\bar{D}^{\dot{\alpha}} \equiv \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i(\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu \ , \quad \bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu \ , \tag{2.33}
$$

and hence $\delta_\epsilon(\bar{D}_{\dot{\alpha}}S) = \bar{D}_{\dot{\alpha}}(\delta_\epsilon S)$. All supersymmetric covariant derivatives anticommute with the $\mathcal{Q}$s;

$$
\{\mathcal{Q}_\alpha, D_\beta\} = \{\overline{\mathcal{Q}}_{\dot{\alpha}}, D_\beta\} = \{\mathcal{Q}_\alpha, \bar{D}_{\dot{\beta}}\} = \{\overline{\mathcal{Q}}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \ , \tag{2.34}
$$

whereas the chiral and anti-chiral covariant derivatives fulfil

$$
\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \ , \quad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \ . \tag{2.35}
$$

The superfield S is not an irreducible representation of the supersymmetry algebra, and so some components may be eliminated with S still being a superfield. Smaller superfields that are irreducible and fulfil different properties are for example the chiral Φ superfield where $\bar{D}_{\dot{\alpha}}\Phi = 0$, the anti-chiral $\bar{\Phi}$ where $D_\alpha\bar{\Phi} \equiv 0$, the vector (or real) superfield $V = V^*$, and the linear superfield L such that $DDL = 0$ and $L = L^\dagger$.

2.2.2 The chiral superfield

To describe the Wess-Zumino model we may use the chiral and anti-chiral superfields Φ and $\bar{\Phi}$. In order to solve the chiral superfield constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$ it is convenient to define

$$y^\mu \equiv x^\mu + i\bar{\theta}\bar{\sigma}^\mu\theta \ , \tag{2.36}$$

changing the superspace coordinates to the set $\{y^\mu, \theta^\alpha, \bar{\sigma}_{\dot{\alpha}}\}$. In terms of these variables, the chiral covariant derivatives become

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar{\theta})_\alpha\frac{\partial}{\partial y^\mu} \ , \qquad D^\alpha = -\frac{\partial}{\partial\theta_\alpha} + 2i(\bar{\theta}\bar{\sigma}^\mu)^\alpha\frac{\partial}{\partial y^\mu} \ , \tag{2.37}$$

$$\bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \ , \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \ . \tag{2.38}$$

By eq. (2.38) it is clear that the chiral superfield constraint is solved by any function of y^μ and θ but not by $\bar{\theta}$. Hence, the chiral superfield can be written on the form

$$\Phi = \phi(y) + \sqrt{2}\theta\xi(y) + \theta\theta F(y) \ , \tag{2.39}$$

and similarly for $\bar{\Phi}$. Here ϕ is a complex scalar field, ξ a 2-component fermion and F an auxiliary field. Expanding this superfield in terms of the anticommuting coordinates, and rewriting it in terms of the original coordinates $\{x^\mu, \theta^\alpha, \bar{\sigma}_{\dot{\alpha}}\}$, we arrive at

$$\Phi = \phi(x) + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi(x) + \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\phi(x) + \sqrt{2}\theta\xi(x) - \tfrac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\xi(x) + \theta\theta F(x) \ . \tag{2.40}$$

Comparing the terms with the general superfield of eq. (2.24), we have that

$$\begin{aligned} &\varphi = \phi \ , \quad \psi_\alpha = \xi_\alpha \ , \quad M = F \ , \quad \bar{\chi}^{\dot{\alpha}} = 0 \ , \quad N = 0 \ , \\ &V_\mu = i\partial_\mu\phi \ , \quad \rho_\alpha = 0 \ , \quad \bar{\lambda}^{\dot{\alpha}} = -\tfrac{i}{\sqrt{2}}(\bar{\sigma}^\mu\partial_\mu\xi)^{\dot{\alpha}} \ , \quad D = \tfrac{1}{4}\partial_\mu\partial^\mu\phi \ , \end{aligned} \tag{2.41}$$

and the supersymmetry transformations can then be shown in the same manner as before to give us

$$\begin{aligned} \delta_\epsilon\phi &= \epsilon\xi \ , \\ \delta_\epsilon\xi_\alpha &= -i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi + \epsilon_\alpha F \ , \\ \delta_\epsilon F &= -i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\xi \ . \end{aligned} \tag{2.42}$$

These are in agreement with the variations of eq. (2.19).

Our next interest is to write supersymmetric Lagrangians. The supersymmetry transformation of the auxiliary field F in eq. (2.42) is a total derivative. Hence, the 4D spacetime integral of the F-term, i.e. the one involving the θ^2 component, of any chiral field is invariant under supersymmetry. The product of chiral superfields of the same chirality gives another chiral superfield, so the 4D spacetime integral of the F-term of an arbitrary polynomial of chiral superfields is also invariant under supersymmetry. The most general renormalisable supersymmetric couplings involving chiral superfields Φ_i have the form

$$\mathcal{L}_W = \int \mathrm{d}^2\theta W(\Phi_i) + \text{h.c.} \ , \tag{2.43}$$

where the $W(\Phi_i)$ is an analytic function of chiral superfield, known as the *superpotential*, and the integration over $\mathrm{d}^2\theta$ selects its F-term. The superpotential allows us to introduce a broad set of supersymmetric interactions. Since the θ^2 component of W does not involve any spacetime derivatives, it will not lead to any kinetic terms of the theory. By the total derivative variation of the $\delta_\epsilon D$ term in eq. (2.30), one can conclude that the $\theta^2\bar{\theta}^2$ component of a real function of chiral superfields is also a supersymmetric invariant upon integration over 4D spacetime. This term, called the D-term, gives rise to canonical kinetic terms and is a component of the real function called the *Kähler potential* denoted $\mathcal{K}(\Phi_i, \bar{\Phi}_i)$. Its simplest variant it is the product

$$\mathcal{K} = \sum_i \Phi_i \bar{\Phi}_i \ , \tag{2.44}$$

of chiral and anti-chiral superfields. The corresponding term in the Lagrangian shows the kinetic terms

$$\mathcal{L}_\mathcal{K} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\mathcal{K} = \sum_i \left(|\partial_\mu \phi_i|^2 + i\bar{\xi}_i \partial_\mu \bar{\sigma}^\mu \xi_i + |F_i|^2 \right) \ . \tag{2.45}$$

A general Kähler potential $\mathcal{K}(\Phi_i, \bar{\Phi}_i)$ leads to more complicated terms as

$$\mathcal{L}_\mathcal{K} \supset K^{ij} \left(\partial_\mu \bar{\phi}_i \partial^\mu \phi_j + i\bar{\xi}_i \bar{\sigma}^\mu \partial_\mu \xi_j + \bar{F}_i F_j \right) \ , \tag{2.46}$$

where $K^{ij} \equiv \partial^2\mathcal{K}/\partial\bar{\Phi}_i\partial\Phi_j|_{\Phi=\phi}$ is a Kähler metric which depends both on the fields and parameters of the theory. Note that $\mathcal{K}$ is not uniquely defined, but only up to *Kähler transformations*

$$\mathcal{K}(\bar{\Phi}, \Phi) \to \mathcal{K}(\bar{\Phi}, \Phi) + f(\Phi) + \bar{f}(\bar{\Phi}) \ , \tag{2.47}$$

where the fs are arbitrary analytic functions. Since $\int \mathrm{d}^4\theta f(\Phi) = \int \mathrm{d}^4\theta \bar{f}(\bar{\Phi}) = 0$, the quantity $\int \mathrm{d}^4\theta\mathcal{K}(\bar{\Phi}, \Phi)$ is conserved.

Now, having discussed the invariant terms upon integration over 4D spacetime, the general form of the Lagrangian in an interacting theory of chiral superfields can be written

$$\begin{aligned} \mathcal{L} &= \int \mathrm{d}^4\theta\mathcal{K}(\bar{\Phi}_i, \Phi_i) + \int \mathrm{d}^2\theta W(\Phi_i) + \int \mathrm{d}^2\bar{\theta} W(\bar{\Phi}_i) \\ &= K^{ij} \left(\partial_\mu \bar{\phi}_i \partial^\mu \phi_j + i\bar{\xi}_i \bar{\sigma}^\mu \partial_\mu \xi_j + \bar{F}_i F_j \right) - \left(\frac{1}{2} \frac{\partial^2 W}{\partial\Phi_i \partial\Phi_j} \xi_i \xi_j - \frac{\partial W}{\partial \Phi_i} F_i + \text{h.c.} \right) + \dots \ , \end{aligned} \tag{2.48}$$

with the ellipsis denoting higher order terms. By solving the F-term equations of motion

$$\bar{F}_i = -\frac{\partial W}{\partial \Phi_i} \ , \tag{2.49}$$

the F-term contribution to the scalar potential of the theory is given by

$$V_F = \sum_{i,j} \frac{\partial \overline{W}}{\partial \bar{\Phi}_i} K_{ij} \frac{\partial W}{\partial \Phi_j} = \sum_i \left| \frac{\partial W}{\partial \Phi_i} \right|^2 \ . \tag{2.50}$$

In general the Kähler potential is assumed non-singular, and so the extrema of the superpotential correspond to the supersymmetric ground states of the theory.

The Wess-Zumino model is constructed with the simplest choice of Kähler potential and superpotential. It consists of the Kähler potential in eq. (2.44) and a superpotential containing only a single chiral field with general expression

$$W = \alpha + \beta\Phi + \tfrac{m}{2}\Phi^2 + \tfrac{g}{3}\Phi^3 \ . \tag{2.51}$$

The reason for these choices being the simplest is a consequence of dimensional analysis. For the theory to be renormalisable we know that $[\mathcal{L}] = 4$ and $[\Phi] = [\phi] = 1$ as well as $[\xi] = 3/2$ for the fermions. Hence the expansion of eq. (2.40) implies that $[\theta] = -1/2$ and $[F] = 2$. The Lagrangian depends linearly on the D-term Kähler potential and the F-term superpotential, so we must have the conditions $\mathcal{K}_D \in \mathcal{K} \leq 4$ and $W_F \in W \leq 4$. Then as $[\theta^2\bar{\theta}^2] = -2$, the D-term of the Kähler potential contributes with dimension -2 to the total Kähler potential, so that $[\mathcal{K}] \leq 2$. For the superpotential, $[\theta^2] = -1$ so that $[W] \leq 3$, and the polynomial of Φ in eq. (2.51) follows as the superpotential must also be holomorphic. The Kähler potential should depend on both chiral and anti-chiral fields and the first combination with dimension 2 that comes to mind is $\mathcal{K} = \bar{\Phi}\Phi$.

With $\alpha = \beta = 0$ in the general superpotential in eq. (2.51) above, the interacting Wess-Zumino model is realised by the Lagrangian

$$\mathcal{L}_{\mathrm{WZ}} = |\partial_\mu\phi|^2 + i\bar{\xi}\partial_\mu\bar{\sigma}^\mu\xi + |F|^2 + \left(mF\phi + gF\phi^2 - \tfrac{m}{2}\xi^2 - g\phi\xi^2\right) . \tag{2.52}$$

In many cases, Lagrangians obtained from superpotentials are invariant under $U(1)$ global supersymmetries, which act differently on the fermionic and scalar components of chiral multiplets. This can be encoded as an $U(1)$ charge assignment to the superspace coordinates θ via

$$\theta \to e^{-i\gamma}\theta \ , \quad \Phi_i \to e^{a_i\gamma}\Phi_i \ , \quad \bar{\Phi}_i \to e^{-a_i\gamma}\bar{\Phi}_i \ , \tag{2.53}$$

with γ and a being real numbers. These continuous $U(1)$ symmetries are called *R-symmetries*. The R-charge R is the sum $R = \sum_i a_i$. The superspace coordinates have $R(\theta) = 1$ which implies that $R(\mathrm{d}^2\theta) = -2$ and $R(\mathrm{d}^4\theta) = 0$ which makes the Kähler potential invariant under R-symmetry transformations, however the superpotential is restricted to interactions that fulfil $R(W) = 2$. The R-symmetry can be equivalently expressed regarding the supercharges, i.e. the superalgebra has an $U(1)$ automorphism fulfilling $\mathcal{Q} \to e^{-i\alpha}\mathcal{Q}$ and $\overline{\mathcal{Q}} \to e^{i\alpha}\overline{\mathcal{Q}}$ with $\alpha \in \mathbb{R}$, which in turn reproduces the results for the superspace and superfields.

2.2.3 The vector superfield

As mentioned earlier, a vector (or real) superfield V is obtained by imposing the constraint $V = V^*$. This constraint forces the terms of the general superfield in eq. (2.24) to satisfy

$$\psi = \varphi^* \ , \quad \bar{\chi} = \bar{\psi} \ , \quad N = M^* \ , \quad V_\mu = V_\mu^* \ , \quad \bar{\lambda} = \bar{\rho} \ , \quad D = D^* \ . \tag{2.54}$$

It will later prove convenient to define

$$\rho_\alpha \equiv \lambda_\alpha - \tfrac{i}{2}(\sigma^\mu\partial_\mu\bar{\psi})_\alpha \ , \quad V_\mu \equiv A_\mu \ , \quad D \equiv \tfrac{1}{2}\tilde{D} + \tfrac{1}{4}\partial_\mu\partial^\mu\varphi \ . \tag{2.55}$$

With these definitions, the component expansion of the vector superfield takes the form

$$\begin{aligned} V(x^\mu, \theta, \bar{\theta}) = \varphi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\theta M + \bar{\theta}\bar{\theta}M^* + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu + \bar{\theta}\bar{\theta}\theta(\lambda - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar{\psi}) \\ + \theta\theta\bar{\theta}(\bar{\lambda} - \tfrac{i}{2}\bar{\sigma}^\mu\partial_\mu\psi) + \theta\theta\bar{\theta}\bar{\theta}(\tfrac{1}{2}\tilde{D} - \tfrac{1}{4}\partial_\mu\partial^\mu\varphi) \ . \end{aligned} \tag{2.56}$$

Again, the supersymmetry transformation is obtained from $\sqrt{2}\delta_\epsilon V = -i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})V$ or by using the transformation results in eq. (2.30) from the general superfield but for the fields in eqs. (2.54), (2.55). Either way, the results are

$$\begin{aligned} \delta_\epsilon\varphi &= \tfrac{1}{\sqrt{2}}(\epsilon\psi - \bar{\epsilon}\bar{\psi}) \ , \\ \delta_\epsilon\psi_\alpha &= \tfrac{1}{\sqrt{2}}\left(2\epsilon_\alpha M - (\sigma^\mu\bar{\epsilon})_\alpha(A_\mu + i\partial_\mu\varphi)\right) \ , \\ \delta_\epsilon M &= \tfrac{1}{\sqrt{2}}(\bar{\epsilon}\bar{\lambda} - i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi) \ , \\ \delta_\epsilon A^\mu &= \tfrac{1}{\sqrt{2}}(i\epsilon\partial^\mu\psi - i\bar{\epsilon}\partial^\mu\bar{\psi} + \epsilon\sigma^\mu\bar{\lambda} - \bar{\epsilon}\bar{\sigma}^\mu\lambda) \ , \\ \delta_\epsilon\lambda_\alpha &= \tfrac{1}{\sqrt{2}}(\epsilon_\alpha\tilde{D} + \tfrac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha(\partial_\mu A_\nu - \partial_\nu A_\mu)) \ , \\ \delta_\epsilon\tilde{D} &= \tfrac{1}{\sqrt{2}}(-i\epsilon\sigma^\mu\partial_\mu\bar{\lambda} - i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\lambda) \ . \end{aligned} \tag{2.57}$$

Note that by comparing the vector constraints in eq. (2.54) and the chiral ones of eq. (2.41), one can conclude that a superfield can not be chiral and real at the same time unless it is a constant, i.e. independent of $x^\mu, \theta, \bar{\theta}$. The vector superfield is used to represent a gauge supermultiplet which contains a gauge boson A_μ, a gaugino λ and a gauge auxiliary field D as components. As seen in eq. (2.56) there are additional auxiliary fields; the real scalar φ, the 2-component fermion ψ_α and complex scalar M. These can however be removed by a gauge transformation, using the fact that the massless vector field has an $U(1)$ gauge invariance. Consider the "supergauge" transformation

$$V \to V + i(\Lambda^* - \Lambda) \ , \tag{2.58}$$

where the gauge transformation parameter Λ is a chiral superfield with components as in eq. (2.40). For the components of the vector field, this transformation corresponds to

$$\begin{aligned}
\varphi &\to \varphi + i(\phi^* - \phi) \ , \\
\psi_\alpha &\to \psi_\alpha - i\sqrt{2}\xi_\alpha \ , \\
M &\to M - iF \ , \\
A_\mu &\to A_\mu + \partial_\mu(\phi + \phi^*) \ , \\
\lambda_\alpha &\to \lambda_\alpha \ , \\
\tilde{D} &\to \tilde{D} \ .
\end{aligned} \tag{2.59}$$

The supergauge transformation for a vector superfield in eq. (2.58) makes the superspace Lagrangian invariant in the Abelian case. Imposing this gauge choice and eliminating the extra auxiliary fields φ, ψ_α, M, the vector superfield takes the form

$$V_{\text{WZ-gauge}} = \bar{\theta}\bar{\sigma}^\mu\theta A_\mu + \bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\lambda} + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\tilde{D} \ , \tag{2.60}$$

which is said to be in *Wess-Zumino gauge*. When imposing Wess-Zumino gauge on the vector superfield, one must remember that it is no longer consistent with the general supersymmetry transformations of eq. (2.57). Namely, the supersymmetry transformations are proportional to

$$\sqrt{2}\delta_\epsilon V_{\text{WZ-gauge}} \sim \bar{\theta}\bar{\sigma}^\mu\epsilon A_\mu - \theta\sigma^\mu\bar{\epsilon}A_\mu + \theta\theta\bar{\epsilon}\bar{\lambda} + \bar{\theta}\bar{\theta}\epsilon\lambda \ , \tag{2.61}$$

and so the supersymmetry transformations is not in Wess-Zumino gauge themselves. However, a supergauge transformation can always restore $\delta_\epsilon V_{\text{WZ-gauge}}$ to Wess-Zumino gauge.

Continuing with the Abelian case, the gauge invariant field strength is defined as

$$\mathcal{W}_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}D_\alpha V \ , \qquad \overline{W}_{\dot{\alpha}} = -\tfrac{1}{4}DD\bar{D}_{\dot{\alpha}}V \ , \tag{2.62}$$

with covariant derivatives defined as earlier and the vector superfield as in eq. (2.56). The field strengths are chiral and anti-chiral by construction, since any chiral and anti-chiral field can be written $\Phi = \bar{D}\bar{D}S$ and $\bar{\Phi} = DD\bar{S}$ for any superfield S. In the non-Abelian case they read

$$\mathcal{W}_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V \ , \qquad \bar{\mathcal{W}}_\alpha = -\tfrac{1}{4}DDe^{-V}\bar{D}_\alpha e^V \ , \tag{2.63}$$

where now the vector superfield is $V = T_a V^a$ with T_a being the gauge generators. The field strengths arise in the kinetic terms for gauge bosons and gauginos as well as their interactions, which comes from

$$\mathcal{L} = \tfrac{1}{4}\text{tr}\int \mathrm{d}^2\theta W^\alpha W_\alpha + \text{h.c.} = \text{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda} + \tfrac{1}{2}D^2\right) \ . \tag{2.64}$$

This term is invariant under the gauge transformations

$$\Phi(x,\theta) \to e^{-i\Lambda}\Phi(x,\theta) \ , \qquad e^V \to e^{-i\bar{\Lambda}}e^V e^{i\Lambda} \ , \tag{2.65}$$

where Φ transforming in some representation of the gauge group.

2.2.4 The F-term $\mathcal{N} = 1$ scalar potential in supergravity

In the last part of this section we will comment on some results from the generalisation to supergravity. We have seen that a superfield transforms like

$$\delta_\epsilon \Phi = i(\epsilon \mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})\Phi \ , \tag{2.66}$$

and when going to supergravity one turns the constant supersymmetry parameter ϵ into a function of the local spacetime coordinates $\epsilon(x^\mu)$, thus extending to a local symmetry. We are interested in the scalar potential of supergravity, which will be central in later chapters. Hence, we turn our focus to the chiral scalar part of the supergravity Lagrangian in superspace, which can be written

$$\mathcal{L} = -\frac{3}{\kappa^2}\int \mathrm{d}^4\theta \mathbb{E} e^{-\frac{\kappa^2}{3}\mathcal{K}} + \int \mathrm{d}^4 \mathcal{E} W + \text{h.c.} \ , \tag{2.67}$$

where $\kappa \equiv 8\pi G = 1/M_{\mathrm{Pl}}$, and $\mathbb{E}$ is the determinant of the supervielbein, i.e. the vierbein $\sqrt{-g} = e = e^\mu_a$ superspace generalisation, which corresponds to a superspace density. $\mathcal{E}$ is defined by $2\mathcal{R}\mathcal{E} = \mathbb{E}$ with $\mathcal{R}$ being the curvature superfield. Note that one can make the κ-expansion

$$e^{-\frac{\kappa^2}{3}\mathcal{K}} = 1 - \frac{\kappa^2}{3}\mathcal{K} + \mathcal{O}(\kappa^4) \ , \tag{2.68}$$

where the flat space limit corresponds to $\kappa \to 0$, $\int \mathrm{d}^2\bar{\theta}\mathcal{E} \to 1$ and $\mathbb{E} \to 1$ which reproduces the supersymmetric action in terms of $\mathcal{K}$ and superpotential W. For any finite value of κ the fact that $\mathcal{K}$ appears explicitly in the pure supergravity part of the Lagrangian implies that the coefficient of the Einstein term, which is the effective Planck mass, depends on the chiral matter fields. In order to go to the Einstein frame, which has constant M_{Pl}, a rescaling of the metric needs to be done. However, in turn one needs to rescale the fermionic fields of the theory which will complicate the derivation of the action of each component. To avoid this one can introduce compensator superfields ζ, known as *Weyl compensators*, which indeed are unphysical and hence required not to propagate. They are introduced such that the action is invariant under scale and conformal transformations [4]. After having calculated the component actions, the Weyl compensator field is fixed to a value that makes the Einstein term, i.e. the first one in eq. (2.67), canonical. This breaks the imposed scale invariance and reproduces the searched-for component actions. The Lagrangian above is then modified to

$$\mathcal{L} = -3\int \mathrm{d}^4\theta \mathbb{E}\zeta\bar{\zeta}e^{-\mathcal{K}/3} + \int \mathrm{d}^4\theta \mathcal{E}\zeta^3 W + \text{h.c.} \ , \tag{2.69}$$

which is invariant under the metric rescalings $\mathbb{E} \to e^{2(\Phi+\bar{\Phi})}\mathbb{E}$ and $\mathcal{E} \to e^{6\Phi}\mathcal{E}$ for some chiral superfield Φ when $\zeta \to e^{-2\Phi}\zeta$. It is clear that we must fix the Weyl compensator so that $\zeta\bar{\zeta}e^{-\mathcal{K}/3} = 1/M_{\mathrm{Pl}}$. Now, to obtain the scalar potential it is sufficient to consider only flat spacetime, so that we may take the limit $\mathbb{E} = 1$ and $\int \mathrm{d}^2\theta\mathcal{E} = 1$, so that the Lagrangian in eq. (2.69) takes the form

$$\mathcal{L} = -3\int \mathrm{d}^4\theta\zeta\bar{\zeta}e^{-\mathcal{K}/3} + \int \mathrm{d}^2\theta\zeta^3 W + \text{h.c.} \ , \tag{2.70}$$

where the covariant derivative reduces to global covariant derivatives. If we ignore the fermionic components we can integrate over half the superspace, so that the Lagrangian of eq. (2.70) becomes

$$\begin{aligned}\mathcal{L} &= -3\int \mathrm{d}^2\bar{\theta}\left(\bar{\zeta}e^{-\mathcal{K}/3}F^\zeta - \tfrac{1}{3}\bar{\zeta}\zeta e^{-\mathcal{K}/3}K_i F^i\right) + 3\zeta^2 F^\zeta W + \zeta^3 F^i W_i + \int \mathrm{d}^2\bar{\theta}\bar{\zeta}^3\overline{W} \ , \\ &= -e^{-\mathcal{K}/3}\left(3F^{\bar{\zeta}\zeta} - \bar{\zeta}K_{\bar{\jmath}}F^{\bar{\jmath}}F^\zeta - \zeta K_i F^i F^{\zeta} - \zeta\bar{\zeta}K_{i\bar{\jmath}}F^i F^{\jmath} + \tfrac{1}{3}\bar{\zeta}\zeta K_i F^i K_{\bar{\jmath}}F^{\bar{\jmath}}\right) \\ &\quad + 3\zeta^2 F^\zeta W + \zeta^3 F^i W_i + 3\bar{\zeta}^2 F^{\bar{\zeta}}\overline{W} + \bar{\zeta}^3 F^{\bar{\imath}}\overline{W}_{\bar{\imath}} \ .\end{aligned} \tag{2.71}$$

From this we get the equations of motion for the auxiliary F-fields;

$$\begin{aligned} -3e^{-\mathcal{K}/3}\left(F^{\zeta} - \tfrac{1}{3}\zeta K_i F^i\right) + 3\bar{\zeta}^2\overline{W} = 0 \ , \\ \zeta^3 W_i - 3e^{-\mathcal{K}/3}\left(-\tfrac{1}{3}\zeta K_i F^{\bar{\zeta}} - \tfrac{1}{3}\zeta\bar{\zeta}K_{i\bar{\jmath}}F^{\bar{\jmath}} + \tfrac{1}{9}\bar{\zeta}\zeta K_i K_{\bar{\jmath}} F^{\bar{\jmath}}\right) = 0 \\ \zeta^3 W_i + \zeta^3 W K_i + e^{-\mathcal{K}/3}\zeta\bar{\zeta}K_{i\bar{\jmath}}F^{\bar{\jmath}} = 0 \\ \zeta^3 D_i W + e^{-\mathcal{K}/3}\zeta\bar{\zeta}K_{i\bar{\jmath}}F^{\bar{\jmath}} = 0 \ . \end{aligned} \tag{2.72}$$

Solving this for F we get that

$$F^i = e^{\mathcal{K}/2M_{\text{Pl}}} K^{i\bar{\jmath}} D_{\bar{\jmath}}\overline{W} \ , \tag{2.73}$$

and then plugging it into the Lagrangian again, the result will be of the form

$$\mathcal{L} = \ldots + \zeta^2\bar{\zeta}^2 e^{\mathcal{K}/3}(K^{i\bar{\jmath}} D_i W D_{\bar{\jmath}}\overline{W} - 3|W|^2) \ . \tag{2.74}$$

In order to determine the value of the Weyl compensator field we demand that the Einstein term has constant Planck mass, i.e. cancels the varying effect of the Kähler potential in the exponent. Therefore we may take $\zeta = \bar{\zeta} = e^{\mathcal{K}/6}$, so the potential term in eq. (2.74) becomes

$$V_F = e^{\mathcal{K}/M_{\text{Pl}}}\left(K^{i\bar{\jmath}} D_i W D_{\bar{\jmath}}\overline{W} - 3\frac{|W|^2}{M_{\text{Pl}}^2}\right) \ , \tag{2.75}$$

with covariant derivatives given by

$$D_i W \equiv \partial_i + \frac{1}{M_{\text{Pl}}^2}(\partial_i \mathcal{K})W \ . \tag{2.76}$$

The F-type parameters for supersymmetry breaking now involve covariant derivatives $F_i = D_i W$. With F-terms being auxiliary fields, their equations of motion should vanish, and so in supersymmetric vacua one must have $D_i W = 0$. Supersymmetry breaking occurs when this identity does not apply, i.e. when eq. (2.73) does not equal zero. As seen by eq. (2.75), the vacuum expectation value of the scalar potential, i.e. the cosmological constant, can (only including the F-term) be either zero or negative depending on the vacuum expectation value of W.

Note that in the limit $M_{\text{Pl}} \to \infty$ gravity is decoupled from the theory and we are left with $V_F = K^{i\bar{\jmath}} D_i W D_{\bar{\jmath}}\overline{W}$ corresponding to the F-term scalar potential in eq. (2.50) with global supersymmetry.

2.3 Kaluza-Klein compactifications

In 1921 Kaluza in [5] showed that general relativity in five dimensions contained both a four dimensional theory of gravity and electromagnetism. Kaluza imposed a restriction, known as the cylinder condition, as to effectively make any effects from the fifth dimension to not appear in our common 4D physical laws. Namely, without anything implying a fifth dimension, he suggested that derivatives with respect to the fifth dimension vanish, thus effectively avoiding the question of the extra dimension. In 1926 Klein suggested [6] compactification of the fifth dimension as a more natural restriction for it not appearing in our 4D laws of physics. He justified Kaluza's condition with two main conditions on the fifth coordinate. First, if the fifth coordinate had a circular topology. Fields could depend on it periodically, allowing for them to be Fourier expanded. Second if the scale of the circular topology was small enough as to make the energies of the Fourier modes above the ground state be so large they be unobservable, the theory would become effectively 4D.

2.3.1 General features of the Kaluza-Klein mechanism

Starting with a theory in dimension $D = d + n$, the first assumption made in KK theory is that the n extra dimensions which are to be compactified have positive signature. This is done in order to avoid ghosts (fields with negative kinetic energy density) and tachyons (potential fields without minimum) in the theory. The next step is to consider ground state solutions of the field equations for the metric $\langle g_{MN} \rangle$ and collective matter fields $\langle \Phi \rangle$ which exhibit *spontaneous compactification*. In this case the metric takes the form of a product space $\mathcal{M}_D = \mathcal{M}_d \oplus \mathcal{M}_n$, whose ground state can be written

$$g^{(0)}_{MN}(x^d, y^n) = \begin{pmatrix} g^{(0)}_{\mu\nu}(x^d) & 0 \\ 0 & g^{(0)}_{mn}(y^n) \end{pmatrix} , \tag{2.77}$$

where the spacetime $\mathcal{M}_d$ are spanned by coordinates x^d and $\mathcal{M}_n$ is a compact space with coordinates y^n and Euclidean signature. In order to obtain the spectrum of the d-dimensional theory, we consider fluctuations of the D-dimensional fields around their ground state values to linear order, i.e.

$$\begin{aligned} g_{MN}(x,y) &= g^{(0)}_{MN}(x,y) + \delta g_{MN}(x,y) , \\ \Phi(x,y) &= \Phi^{(0)}(x,y) + \delta\Phi(x,y) , \end{aligned} \tag{2.78}$$

which we may use in the equations of motion. With the internal manifold being compact, one can expand the fields in terms of eigenfunctions of the corresponding mass operator $M^2 Y^{(i)} = m_i^2 Y^{(i)}$ such that

$$\begin{aligned} \delta\Phi(x,y) &= \sum_i^\infty \phi^{(i)}(x) E^{(i)}(y) , \\ \delta g_{\mu\nu}(x,y) &= \sum_i^\infty h^{(i)}_{\mu\nu}(x) Y^{(i)}(y) , \\ \delta g_{\mu m} &= \sum_i^\infty A^{(i)}_\mu(x) Y^{(i)}_m(y) , \\ \delta g_{mn} &= \sum_i^\infty X^{(i)}(x) Y^{(i)}_{mn}(y) . \end{aligned} \tag{2.79}$$

This results in a theory which has an infinite tower of massive modes, known as the *Kaluza-Klein tower*. The masses are inversely proportional to the scale of compactification $m_i \sim R^{-1}$, where R can be seen as the size of the compact manifold $\mathcal{M}_n$ [7]. For small R, the n extra dimensions need not conflict with every-day life (given $d = 4$) and its inverse square gravitational law. The masses of the extra dimensional fields would become so large as to put them beyond the reach of experiment, so only the finite zero-modes independent of extra dimensions would be observable. The massless mode eigenfunctions, being harmonic forms, will play an important role as internal metric deformations of a Calabi-Yau 3-fold which admits a finite number of them. After finding a spontaneous compactification one should also control that the vacuum is stable, i.e. that the massive tower states have positive energy. For Minkowski vacua this simply corresponds to $m_i^2 \geq 0$, but the conditions for other type of vacua are more subtle.

Choosing $\mathcal{M}_d$ to be Minkowski and $\mathcal{M}_n$ to be a Ricci-flat manifold, the mass operator for the bosonic eigenfunctions $E^{(i)}(y), Y^{(i)}(y)$ and $Y^{(i)}_m(y)$ will be a Laplace operator. If instead $E^{(i)}(y)$ is fermionic then the mass operator is a Dirac operator. The mass operator of $Y^{(i)}_{mn}(y)$ is a so-called Lichnerowicz operator and gives the Lichnerowicz equation for a field, which will return in chapter 5 when performing metric deformations.

2.3.2 Some examples in 5D

In this section we review some of the simpler examples of Kaluza-Klein compactifications in order to develop some intuition for the mechanism.

The massless scalar field. We start with a simple example in $D = 5$ where a massless scalar field has action

$$S_0 = -\tfrac{1}{2} \int \mathrm{d}^5 x \partial_M \phi \partial^N \phi \ , \tag{2.80}$$

where $M, N = 0, \ldots, 4$, $\partial^M = \eta^{MN}\partial_N$, and $\eta_{MN} = \eta^{MN} = \mathrm{diag}(-1,1,1,1,1)$ is a flat space. The equation of motion, found by the principle of stationary action, is

$$\partial_M \partial^M \phi = \Box \phi = 0 \ , \tag{2.81}$$

i.e. the massless Klein-Gordon equation. Setting the 5D space $\mathcal{M}_5$ to be a product space $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$ is still consistent with a flat metric. $\mathcal{M}_4$ is a 4D Minkowski space and S^1 is a circle of radius R. The coordinates of this space can be written as $x_M = (x^\mu, y)$, where $\mu = 0, \ldots, 3$ and the circle coordinate is $y \in [0, 2\pi R]$. The equation of motion may now be written as

$$\partial_M \partial^M \phi = \partial_\mu \partial^\mu \phi + \partial_y^2 \phi = 0 \ . \tag{2.82}$$

Since y is a periodic coordinate $\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi R)$, we may Fourier expand the field such that

$$\phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_{-\infty}^{\infty} \phi^{(n)}(x^\mu) e^{iny/R} \ . \tag{2.83}$$

Using this expression of ϕ in eq. (2.82), the equation of motion now takes the form

$$\partial_\mu \partial^\mu \phi^{(n)}(x^\mu) - \frac{n^2}{R^2}\phi(x^\mu) = 0 \ . \tag{2.84}$$

This is the Klein-Gordon equation for scalar fields $\phi_n(x^\mu)$ with masses n/R, which are 4D as $\mu = 0, \ldots, 3$. With usage of eq. (2.83) in the action of eq. (2.80) and integrating over y using the orthonormality of the eigenfunctions $\frac{1}{\sqrt{2\pi R}} e^{iny/R}$ of ∂_y^2 [8], the action can be written

$$S_0 = -\tfrac{1}{2} \sum_{-\infty}^{\infty} \int \mathrm{d}^4 x \Big(\partial_\mu \phi^{(n)} \partial^\mu \phi^{(n)*} - \frac{n^2}{R^2} \phi^{(n)*} \phi^{(n)} \Big) \ . \tag{2.85}$$

It is clear that there is one massless scalar $\phi^{(0)}$ and an infinite amount of massive scalars $\phi^{(n \neq 0)}$ with masses $2n/R$, i.e. the tower of massive Kaluza-Klein modes. We have now obtained the effective 4D action from the original 5D action. At energies lower than the $1/R$ scale, only the zero mode is kept as the massive modes become infinite in mass as $R \to 0$. The massive modes are truncated, and the physics become four dimensional. This is known as *dimensional reduction*, or Kaluza-Klein reduction. Would we keep the massive modes, we say that we have *compactified* the 5D theory, as the extra dimension is compact and its existence is taken into account as long as the Fourier modes are included.

The Maxwell vector field. Our next example is the 5D Maxwell action, whose fifth dimensional term we write explicitly for later convenience, i.e.

$$
\begin{aligned}
S &= \int \mathrm{d}x^5 \Big[-\tfrac{1}{4} F_{MN} F^{MN} \Big] \\
&= \int \mathrm{d}x^5 \Big[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} F_{\mu 4} F^{\mu 4} \Big] \\
&= \int \mathrm{d}x^5 \Big[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} (\partial_\mu A_4 - \partial_4 A_\mu)(\partial^\mu A^4 - \partial^4 A^\mu) \Big] \\
&= \int \mathrm{d}x^5 \Big[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} (\partial_\mu A_4 \partial^\mu A^4 - \partial_\mu A_4 \partial^4 A^\mu - \partial_4 A_\mu \partial^\mu A^4 + \partial_4 A_\mu \partial^4 A^\mu) \Big] \\
&= \int \mathrm{d}x^5 \Big[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} ((\partial_\mu A_4)^2 + (\partial_4 A_\mu)^2 - \partial_\mu A_4 \partial^4 A^\mu - \partial_4 A_\mu \partial^\mu A^4) \Big] \ .
\end{aligned} \tag{2.86}
$$

Once again compactifying the fifth dimension on a circle, we have that $A(x^M) = A(x^\mu, y) = A(x^\mu, y + 2\pi R)$ which allows for the Fourier decomposition

$$
\begin{aligned}
A(x^\mu, y) &= \frac{1}{\sqrt{2\pi R}} \sum_{-\infty}^{\infty} A^{(n)}(x^\mu) e^{iny/R} \\
&= \frac{1}{\sqrt{2\pi R}} \Big[A^{(0)}(x^\mu) + \sum_{n=1}^{\infty} (A^{(n)}(x^\mu) e^{iny/R} + \text{c.c.}) \Big] \ .
\end{aligned} \tag{2.87}
$$

Our dimensional reduction would be rather simple could we choose an axial gauge $A_4 = 0$, however it is not possible to remove the zero mode $A_4^{(0)}$ of the expansion above. The closest we get to axial gauge is therefore $A_4(x^\mu, y) = A_4^{(0)}(x^\mu)$. Making use of our expansion and choice of gauge, we insert it in the action of eq. (2.86), which becomes

$$
\begin{aligned}
S &= \frac{1}{2\pi R} \int \mathrm{d}x^4 \int_0^{2\pi R} \mathrm{d}y \Big[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} (\partial_\mu A_4^{(0)})^2 - \tfrac{1}{2} (\partial_4 A_\mu)^2 \Big] \qquad (2.88) \\
&= \int \mathrm{d}x^4 \Big[-\tfrac{1}{4} F_{\mu\nu}^{(0)} F^{\mu\nu(0)} - \tfrac{1}{2} (\partial_\mu A_4^{(0)})^2 + \sum_{n=1}^{\infty} \big(-\tfrac{1}{2} |\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}|^2 + \frac{n^2}{R^2} |A_\mu^{(n)}|^2 \big) \Big] \ .
\end{aligned}
$$

On the first line the cross terms from the fifth dimension have been integrated over $e^{iny/R}$ to zero. After integration the term $(\partial_\mu A_4^{(0)})^2$ remains unchanged, and $(\partial_4 A_\mu)^2$ becomes a mass term $\frac{n^2}{R^2} |A_\mu^{(n)}|^2$ as the derivative can be replaced by $\partial \to in/R$ under Fourier decomposition. The zero mode part vanishes since $\partial_4 A_\mu^{(0)} = \partial_y A_\mu^{(0)}(x^\mu) = 0$.

Thus the 5D Maxwell theory has been compactified to a 4D Maxwell theory, a scalar field $A_4^{(0)}$, and an infinite set of 4D gauge fields $A_\mu^{(n)}$ and $A_\mu^{(n)*}$ with masses $2n/R$.

Pure gravity. Another example is that of 5D pure Einstein gravity with one spatial dimension compactified on a circle of radius r with coordinate $y \in [0, 2\pi r]$. The corresponding action, the Einstein-Hilbert action, is given by

$$
S = \tfrac{1}{2\pi} \int \mathrm{d}^4 x \, \mathrm{d}y \sqrt{-g_5} R_5 \ , \tag{2.89}
$$

where $g_5 = \det g_{MN}$ with $M, N = 0, \ldots, 4$ is the 5D metric and R_5 is the 5D Ricci scalar. With no matter terms, the corresponding equations of motion are just

$$
R_{MN} = 0 \ , \tag{2.90}
$$

which tell us that the metric is flat. We may compactify the five dimensions into a 4D Minkowski and circle according to $\mathcal{M}_5 = \mathcal{M}_4 \times S^1$, as in the previous example. In this case the metric is of the type

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{d}y^2\ , \tag{2.91}$$

so that the metric g_{MN} is given by

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}\ , \tag{2.92}$$

where $\mu, \nu = 0, \ldots, 3$. Note that a compactification $\mathcal{M}_5 = \mathcal{M}_3 \times S^1 \times S^1$ is equally valid. To study the dynamics and spectrum of this theory we start by considering linear fluctuations around the ground state in eq. (2.92). The metric can then be written as

$$g_{MN} = \phi^{-1/3}\begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}\ , \tag{2.93}$$

which is known as the Kaluza-Klein ansatz. Here $\phi(x^\mu, y)$ is a scalar field and $A_\mu(x^\mu, y)$ a vector field. Any 5D metric can be written in the form of eq. (2.93). As the metric and added fields depend on the periodic coordinate y we may Fourier expand them, so that

$$g_{\mu\nu}(x^\mu, y) = \sum_{-\infty}^{\infty} g_{\mu\nu}^{(n)}(x^\mu)e^{iny/r}\ , \tag{2.94}$$

$$\phi(x^\mu, y) = \sum_{-\infty}^{\infty} \phi^{(n)}(x^\mu)e^{iny/r}\ , \tag{2.95}$$

$$A_\mu(x^\mu, y) = \sum_{-\infty}^{\infty} A_\mu^{(n)}(x^\mu)e^{iny/r}\ , \tag{2.96}$$

with ground state values according to the original metric;

$$g_{\mu\nu}^{(0)} = \eta_{\mu\nu}\ , \qquad \phi^{(0)} = 1\ , \qquad A_\mu^{(0)} = 0\ . \tag{2.97}$$

The complete dimensional reduction is achieved by keeping only the zero mode of g_{MN}, which gives us the action

$$S = -\int \mathrm{d}^4x\sqrt{-g_4}\left(R_4^{(0)} + \tfrac{1}{4}F_{\mu\nu}^{(0)}F^{\mu\nu(0)} - \tfrac{1}{6}\frac{\partial_\mu\phi^{(0)}\partial^\mu\phi^{(0)}}{\phi^{(0)2}}\right)\ . \tag{2.98}$$

Except for the 4D pure gravity action, we have also obtained the Maxwell action of electromagnetism as well as a massless scalar field ϕ.

3

Superstrings and Supergravity

Bosonic string theories are unsatisfactory in two aspects. First, the spectra for both closed and open strings, contain tachyons. The second feature is of course that it does not contain any fermions. It turns out that the incorporation of fermions in string theories requires supersymmetry and the resulting theories are called superstring theories.

3.1 The low-energy effective action of the bosonic string

In this section we describe some important properties of the bosonic action which will extend to the type II superstrings central to this thesis.

3.1.1 The massless bosonic spectrum and symmetries

Strings sweep the 2D worldsheet spanned by the coordinates σ along the string and τ along its perpendicular direction. The evolution of the worldsheet is given by the Polyakov action in D flat dimensions as

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma\, \mathrm{d}\tau \sqrt{-\gamma}\gamma^{\alpha\beta}\eta_{MN}\partial_\alpha X^M \partial_\beta X^N \ , \tag{3.1}$$

where $\gamma^{\alpha\beta}$ with $\alpha, \beta = \{\sigma, \tau\}$ is the worldsheet metric, η_{MN} is the Minkowski metric, and X^M with $M = 0 \ldots, D-1$ are the functions defining the embedding of the worldsheet in spacetime. The string tension T is related to α' via

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_\mathrm{s}^2} = \frac{M_\mathrm{s}^2}{2\pi} \ , \tag{3.2}$$

where l_s is the string scale whose inverse gives the string mass M_s. Before quantisation of the theory it is important to be aware of some global and local symmetries held by the Polyakov action, namely

- D-dimensional Poincaré invariance; a global symmetry acting on the worldsheet theory which leaves the coordinates τ, σ invariant while acting on the X^M fields according to

$$X^M(\tau,\sigma) \to \Lambda^M_N X^M(\tau,\sigma) + a^M \ , \qquad \gamma^{\alpha\beta}(\tau,\sigma) \to \gamma^{\alpha\beta}(\tau,\sigma) \ . \tag{3.3}$$

- Invariance under local worldsheet reparametrisations, for which the $X^M(\tau,\sigma)$ behave as scalars and $\gamma^{\alpha\beta}$ as a 2-index tensor. Given $\sigma^\alpha \equiv (\tau,\sigma)$ we have

$$\begin{aligned} \sigma^\alpha \to \sigma'^\alpha(\sigma^\alpha) \ , \qquad X^M \to X'^M(\sigma'^\alpha) = X^M(\sigma^\alpha) \ , \\ \gamma_{\alpha\beta}(\sigma^\alpha) \to \gamma'_{\alpha\beta}(\sigma'^\alpha) = \frac{\partial\sigma^\lambda}{\partial\sigma'^\alpha}\frac{\partial\sigma^\rho}{\partial\sigma'^\beta}\gamma_{\lambda\rho} \ , \end{aligned} \tag{3.4}$$

- Weyl invariance, i.e. invariance under local rescalings of the worldsheet metric;

$$X^M(\tau,\sigma) \to X^M(\tau,\sigma) \ , \qquad \gamma^{\alpha\beta}(\tau,\sigma) \to \Omega(\tau,\sigma)\gamma^{\alpha\beta}(\tau,\sigma) \ . \tag{3.5}$$

In order to quantise the system one needs to gauge fix the local invariances. The local symmetries of τ and σ can be used to remove degrees of freedom from $\gamma^{\alpha\beta}$ by imposing what is known as conformal gauge

$$\gamma^{\alpha\beta} = \eta^{\alpha\beta} \ . \tag{3.6}$$

This is always possible to do locally on the worldsheet and even globally for an infinite cylinder topology. In this gauge the Polyakov action may be written

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \, \mathrm{d}\sigma \eta^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \eta_{MN} \ . \tag{3.7}$$

The equations of motion obtained from varying this action is the 2D wave equation

$$(\partial_\tau^2 - \partial_\sigma^2) X^M(\tau, \sigma) = 0 \ . \tag{3.8}$$

Closed strings satisfy the boundary conditions $X^M(\tau, 0) = X^M(\tau, 2\pi)$ and $\partial_\sigma X^M(\tau, 0) = \partial_\sigma X^M(\tau, 2\pi)$. Imposing these conditions enables the mode expansion

$$X^M(\tau, \sigma) = X_R^M(\sigma^-) + X_L^M(\sigma^+) \ , \tag{3.9}$$

where

$$\begin{aligned} X_R^M(\sigma^-) &\equiv \tfrac{1}{2} x^M + \alpha' p^M \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^M}{n} e^{-2in\sigma^-} \ , \\ X_L^M(\sigma^+) &\equiv \tfrac{1}{2} x^M + \alpha' p^M \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^M}{n} e^{-2in\sigma^+} \ , \end{aligned} \tag{3.10}$$

and $\sigma^\pm \equiv \tau \pm \sigma$ are left-moving (σ^+) and right-moving (σ^-) worldsheet coordinates, the x^M and p^M are the centre of mass position and momentum respectively. For the solution to be real one may further impose the conditions $\alpha_{-n}^M = (\alpha_n^M)^*$ and $\tilde{\alpha}_{-n}^M = (\tilde{\alpha}_{-n}^M)^*$ on the mode expansion coefficients. Another constraint comes from the vanishing of the energy momentum tensor, obtained by varying the worldsheet metric, resulting in that

$$T^{\alpha\beta} \equiv \frac{2\pi}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma_{\alpha\beta}} = -\frac{1}{\alpha'} \left(\partial^\alpha X^M \partial^\beta X_M - \tfrac{1}{2} \gamma^{\alpha\beta} \gamma_{\lambda\rho} \partial^\lambda X^M \partial^\rho X_M \right) = 0 \ , \tag{3.11}$$

which in turn results in the constraints

$$\eta_{MN} \partial_\sigma + X_L^M \partial_\sigma + X_L^N = \eta_{MN} \partial_\sigma - X_R^M \partial_\sigma - X_R^N = 0 \ . \tag{3.12}$$

Note that with left- and right-moving coordinates $\sigma^\pm$, the conformal metric gauge in eq. (3.6) can be written $\mathrm{d}s^2 = -\,\mathrm{d}\sigma^+ \, \mathrm{d}\sigma^-$, which is unchanged by the change of coordinates

$$\sigma^+ \to f^+(\sigma^+) \ , \qquad \sigma^- \to f^-(\sigma^-) \ , \tag{3.13}$$

in combination with a suitable Weyl rescaling. Reparametrisations of the type in eq. (3.13) preserve the conformal structure, are known as conformal transformations. In the case of the 2D worldsheet the conformal structure corresponds to its angles. Conformally invariant theories like the Polyakov action are known as conformal field theories.

Quantising the action in eq. (3.7) results in the commutators

$$[\alpha_n^M, \alpha_m^N] = [\tilde{\alpha}_n^M, \tilde{\alpha}_m^N] = n \delta_{n+m,0} \eta^{MN} \ , \qquad [x^M, p^N] = i \eta^{MN} \ , \tag{3.14}$$

so that $\alpha_m, \tilde{\alpha}_m$ can be interpreted as annihilation operators and $\alpha_{-n}, \tilde{\alpha}_{-n}$ with $n > 0$ as creation operators. The modes annihilate or create a left- or right-moving excitation at the level n, which carries an energy proportional to the level. The mass of each state is given by

$$M^2 = \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty}(\alpha_n \cdot \alpha_{-n} + \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}) - 2\right) \equiv \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) , \tag{3.15}$$

with -2 being a normal ordering contribution to the zero-point energy. The left- and right-moving sectors are related by the level-matching condition

$$(L_0 - \tilde{L}_0)\,|\phi\rangle = 0 , \tag{3.16}$$

where

$$L_0 \equiv \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_{m-n} \cdot \alpha_n , \qquad \tilde{L}_0 \equiv \tfrac{1}{2}\tilde{\alpha}_0 + \sum_{n=1}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n , \tag{3.17}$$

for some physical state $|\phi\rangle$. The states are invariant under translations in σ when requiring that an equal amount of oscillators are excited on the left and right, i.e. $N = \tilde{N}$. The massless states $|\xi_{MN}\rangle$ for instance will then have one left-moving and one right-moving excitation

$$|\xi_{MN}\rangle \equiv \xi_M \tilde{\xi}_N \alpha_1^M \tilde{\alpha}_1^N\,|0\rangle , \tag{3.18}$$

as well as a centre of mass momentum k^M with $k \cdot k = 0$. It is clear that for the norm of the state to be positive, ξ and $\tilde{\xi}$ must be space-like vectors. The condition of eq. (3.12) implies that $\xi \cdot k = \tilde{\xi} \cdot k = 0$, meaning that the polarisation vectors must be orthogonal to the momentum. By choosing the frame in which $k^M = (k, k, 0 \ldots, 0)$ then the $\xi^M, \tilde{\xi}^M$ must belong to the space with $M = 2, \ldots, D-1$ and so these states are classified by their $SO(D-2)$ representation. The tensor ξ_{MN} can be combined with the metric to create a scalar, a symmetric and antisymmetric tensor by

$$\xi_t \equiv \tfrac{1}{D}\eta^{MN}\xi_{MN} , \qquad \xi^s_{MN} \equiv \tfrac{1}{2}(\xi_{MN} + \xi_{NM} - 2\xi_t\eta_{MN}) , \qquad \xi^a_{MN} \equiv \tfrac{1}{2}(\xi_{MN} - \xi_{NM}) , \tag{3.19}$$

so that the ξ_{MN} can be written as

$$\xi_{MN} = \xi^s_{MN} + \xi_t\eta_{MN} + \xi^a_{MN} . \tag{3.20}$$

The state corresponding to the scalar ξ_t is known as the *dilaton*, the one created by the symmetric tensor ξ^s_{MN} is the *graviton* with spin 2, and the one based on the antisymmetric tensor ξ^a_{MN} is the *B-field*. In the next section we will consider the effects of having a curved background instead of the flat η_{MN}.

3.1.2 Conformal invariance and the Einstein equations

There is an obvious generalisation of the Polyakov action to describe a string moving in curved spacetime, namely

$$S_\sigma = \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{-\gamma}\gamma^{\alpha\beta}g_{MN}\partial_\alpha X^M \partial_\beta X^N , \tag{3.21}$$

with g_{MN} being a general metric. This is also known as the sigma model action. While a flat metric had one graviton in its spectrum, a curved background can be viewed as a coherent state of gravitons, which can be seen by first considering a small deviation from the flat space; $g_{MN} = \eta_{MN} + h_{MN}$. Using this perturbation in the path integral, the corresponding partition function reads

$$Z = \int \mathcal{D}X\mathcal{D}\gamma e^{-S} = \int \mathcal{D}X\mathcal{D}\gamma e^{-(S_0+V)} = \int \mathcal{D}X\mathcal{D}\gamma e^{-S_0}\left(1 - V + \tfrac{1}{2}V^2 + \ldots\right) , \tag{3.22}$$

where the vertex operator associated to the graviton state is given by

$$V = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma\, \mathrm{d}\tau \sqrt{-\gamma}\gamma^{\alpha\beta} h_{MN} \partial_\alpha X^M \partial_\beta X^N \ . \tag{3.23}$$

A single copy of V in the path integral corresponds to the introduction of a single graviton state, and the inclusion of e^{-V} results in a coherent state of gravitons.

In conformal gauge, the Polyakov action in flat space reduces to a free theory, but in a curved background this is no longer true. Imposing conformal gauge of the σ-model in eq. (3.21), the worldsheet theory is described by an interacting 2D field theory, namely

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma\, \mathrm{d}\tau g_{MN} \partial_\alpha X^M \partial^\alpha X^N \ , \tag{3.24}$$

having raised the second derivative index. The interactions can be more easily considered by expanding around a classical solution in which the string sits at a point x^M, so that

$$X^M(\tau, \sigma) = x^M + \sqrt{\alpha'} Y^M(\tau, \sigma) \ , \tag{3.25}$$

where $Y^M(\tau, \sigma)$ are dynamical fluctuations around x^M that are assumed to be small [9]. The Lagrangian will then take the form

$$\begin{aligned} g_{MN}(X)\partial_\alpha X^M \partial^\alpha X^N = \alpha' \Big[& g_{MN}(x) + \sqrt{\alpha'} \partial_P g_{MN}(x) Y^P \\ & + \tfrac{\alpha'}{2} \partial_P \partial_Q g_{MN}(x) Y^P Y^Q + \ldots \Big] \partial_\alpha Y^M \partial^\alpha Y^N \ . \end{aligned} \tag{3.26}$$

Here each of the coefficient terms $\partial_P g_{MN}$, $\partial_P \partial_Q g_{MN}, \ldots$ are coupling constants for the interactions of the Ys, and it is clear that there is an infinite amount of them.

Classically the action in eq. (3.24) is conformally invariant, although not necessarily when quantising the theory. A UV cut-off is usually introduced to regulate divergences, and after renormalisation the physical quantities generally depend on the scale of a given process μ, thus breaking conformal invariance. The Yang-Mills theory is a typical example of such a theory in which conformal invariance is broken quantum mechanically. The β-function is an object which describes how couplings depend on a scale μ. In our case with the metric g_{MN}, the β-function is rather a functional of the form

$$\beta_{MN}(g) \sim \mu \frac{\partial g_{MN}(X, \mu)}{\partial \mu} \ . \tag{3.27}$$

For the theory to be conformally invariant the β-functional is required to vanish. Hence it is in order to study what the condition $\beta_{MN}(g) = 0$ imposes on the coupling. The usual strategy is to add a counterterm to the UV divergence in the action in eq. (3.24), and the β-functional will then vanish when the counterterm does. Around any point x one can always choose Riemann normal coordinates such that the expansion in eq. (3.25) becomes

$$g_{MN}(X) = \delta_{MN} - \tfrac{\alpha'}{3} R_{MPNQ}(x) Y^P Y^Q + \mathcal{O}(Y^3) \ . \tag{3.28}$$

Using this expression in the action, it will, up to quartic order in the fluctuations, take the form

$$S = \frac{1}{4\pi} \int \mathrm{d}\tau\, \mathrm{d}\sigma \left(\partial Y^M \partial Y^N \delta_{MN} - \tfrac{\alpha'}{3} R_{MPNQ} Y^P Y^Q \partial Y^M \partial Y^N \right) \ . \tag{3.29}$$

Treating this as an interacting 2D quantum field theory, the quartic interaction gives a vertex with Feynman rule giving a contribution $\sim R_{MPNQ}(p^M \cdot p^N)$, with p^M_α being the 2D momentum for the scalar field Y^M. The divergence in the theory comes from the 1-loop diagram, which can seen by considering the scalar propagator in position space

$$\langle Y^P(\sigma^\alpha) Y^Q(\tau', \sigma') \rangle = -\tfrac{1}{2}\delta^{PQ} \ln |\sigma^\alpha - \sigma'^\alpha|^2 \ , \tag{3.30}$$

which diverges as $\sigma^\alpha \to \sigma'^\alpha$. A scalar running in the loop have its start and end coincide. To isolate this divergence we use dimensional regularisation, where we have that

$$\langle Y^P(\sigma^\alpha) Y^Q(\sigma'^\alpha) \rangle = 2\pi\delta^{PQ} \int \frac{\mathrm{d}^{2+\epsilon}p}{(2\pi)^{2+\epsilon}} \frac{e^{ip(\sigma^\alpha - \sigma'^\alpha)}}{p^2} \quad \xrightarrow[\sigma^\alpha \to \sigma'^\alpha]{} \quad \frac{\delta^{PQ}}{\epsilon} \ . \tag{3.31}$$

In the action we may replace $Y^P Y^Q$ with $\langle Y^P Y^Q \rangle$, and so having isolated the divergence we may add a counterterm such that

$$R_{MPNQ} Y^P Y^Q \partial Y^M Y^N \to R_{MPNQ} Y^P Y^Q \partial Y^M Y^N - \frac{1}{\epsilon} R_{MN} \partial Y^M \partial Y^N \ . \tag{3.32}$$

This change can actually be absorbed in a renormalisation of the fluctuations $Y^M \to Y^M - \frac{\alpha'}{6\epsilon} R^M{}_N Y^N$ in combination with a renormalisation of the coupling constants

$$g_{MN} \to g_{MN} + \frac{\alpha'}{\epsilon} R^{MN} \ . \tag{3.33}$$

With this expression of the metric, the condition of vanishing β-functional results in that

$$\beta_{MN}(g) \sim \alpha' R_{MN} = 0 \ . \tag{3.34}$$

Hence, the requirement that the σ-model is conformally invariant implies that the background spacetime must be Ricci flat.

3.1.3 The non-linear sigma model

In the previous sections we have seen how strings couple to the background metric g_{MN} and also that the bosonic string has massless states associated with an antisymmetric tensor and scalar, known as the Kalb-Ramond field B_{MN} and dilaton ϕ respectively. In this section we will study how the inclusion of these fields in spacetime affects the theory.

Starting with the B-field, one needs to construct an action to describe the interaction or coupling. The vertex operator V_B for the field is of the form

$$V_B \sim \int \mathrm{d}\tau\, \mathrm{d}\sigma : e^{ik\cdot x} \partial X^M \bar{\partial} X^N : \zeta^{\mathrm{a}}_{MN} \ , \tag{3.35}$$

with ζ^{a}_{MN} being the antisymmetric part of a constant tensor ζ_{MN}. In the same manner the vertex operator for the gravitational field looks the same as the one in eq. (3.35) but with ζ^{a}_{MN} replaced by the traceless symmetric part ζ^{s}_{MN} of the same constant tensor. For the dilaton field one would have that

$$V_\phi \sim \int \mathrm{d}\tau\, \mathrm{d}\sigma : e^{ik\cdot x} \partial X^M \bar{\partial} X_M : \zeta \ , \tag{3.36}$$

with $\zeta \equiv \zeta^N{}_N$. Exponentiating the B-field vertex operator of eq. (3.35) results in an expression which describes how strings propagate in a B-field background, whose action is then given by

$$S_B = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\gamma} \left(iB_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \epsilon^{\alpha\beta} \right) \ , \tag{3.37}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita 2-tensor normalised such that $\sqrt{-\gamma}\epsilon^{12} = 1$. Adding this contribution to the action of eq. (3.24), we have

$$S_{g+B} = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\gamma} \left(\gamma^{\alpha\beta} g_{MN}(X) \partial_\alpha X^M \partial_\beta X^N - i B_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \epsilon^{\alpha\beta} \right) \ , \quad (3.38)$$

where it can be shown that this new addition still retains invariance under worldsheet reparametrisations and Weyl rescalings.

Proceeding to the dilaton field, the same incorporation method as for the B-field can not actually be used. Since the vertex operator is not a primary, it is not possible to obtain the coupling by simply exponentiating the vertex operator. Instead it turns out that the correct expression for the action describing the coupling is given by

$$S_\phi = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\gamma} \left(\alpha' \phi(X) R^{\mathrm{ws}} \right) \ , \quad (3.39)$$

with R^{ws} being the Ricci scalar on the worldsheet [10]. The dilaton coupling will therefore vanish on a Ricci flat worldsheet. Remarkably, the dilaton coupling is not generally Weyl invariant. An exception being if the dilaton is constant, i.e. if $\phi(X) = \phi_0 \in \mathbb{C}$. In this case the dilaton coupling action becomes

$$S_{\phi_0} = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\gamma} \left(\alpha' \phi_0 R^{\mathrm{ws}} \right) = \phi_0 \chi \ , \quad (3.40)$$

with $\chi = 2 - 2g$ being the Euler characteristic and is related to the genus g of the worldsheet. The coupling implies that the constant mode of the dilaton $\langle \phi \rangle$ determines the coupling constant g_{s}, i.e.

$$g_{\mathrm{s}} = e^{\langle \phi \rangle} \ . \quad (3.41)$$

Hence the string coupling is the expectation value of a field rather than a parameter. The full action including the graviton, B-field and dilaton then becomes

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\gamma} \left(\gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N g_{MN} + i B_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \epsilon^{\alpha\beta} + \alpha' \phi(X) R^{\mathrm{ws}} \right) \ . \quad (3.42)$$

Again, the presence of α' allows for a loop expansion in the non-linear σ-model of eq. (3.42). The broken Weyl invariance in the dilaton coupling may be compensated for by a 1-loop contribution arising from the couplings to g_{MN} and B_{MN}. The β-functions for the theory in eq. (3.42) would show this explicitly. The three β-functions from the three different fields add three contributions to the stress-energy tensor such that

$$\langle T^\alpha{}_\alpha \rangle = -\frac{1}{2\alpha'} \beta_{MN}(g) \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N - \tfrac{i}{2\alpha'} \beta_{MN}(B) \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N - \tfrac{1}{2} \beta(\phi) R^{\mathrm{ws}} \ , \quad (3.43)$$

where each β-function is given by

$$\begin{aligned} \beta_{MN}(g) &= \alpha' R_{MN} + 2\alpha' \nabla_M \nabla_N \phi - \frac{\alpha'}{4} H_{MPQ} H_N{}^{PQ} \ , \\ \beta_{MN}(B) &= -\frac{\alpha'}{2} \nabla^P H_{PMN} + \alpha' \nabla^P \phi H_{PMN} \ , \\ \beta(\phi) &= -\frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_M \phi \nabla^M \phi - \frac{\alpha'}{24} H_{MNP} H^{MNP} \ . \end{aligned} \quad (3.44)$$

These were derived in [11] where again $H_3 = dB_2$ is the field strength of the B-field. In order to preserve Weyl invariance we must then have that

$$\beta_{MN}(g) = \beta_{MN}(B) = \beta(\phi) = 0 \ . \quad (3.45)$$

These equations can be viewed as equations of motion for the background in which the string propagates. A $D = 26$ dimensional background which reproduces these equations of motion for the β-functions is the low-energy effective action of the bosonic string;

$$S = \frac{1}{2\kappa_0^2} \int \mathrm{d}^{26}X \sqrt{-g} e^{-2\phi} \left(R - \tfrac{1}{12} H_{MNP} H^{MNP} + 4 \partial_M \phi \partial^M \phi \right) + \mathcal{O}(\alpha') \ , \tag{3.46}$$

where $\kappa_0^2 = 8\pi G_{26}$. Varying this action with respect to the gravitational, B- and dilaton fields will indeed reproduce the results of eq. (3.44). To obtain the the conventional Einstein-Hilbert kinetic term, i.e. the one without the dilaton factor, the action of eq. (3.46) is transformed from the string frame to the Einstein frame via a Weyl rescaling $g_\mathrm{s} \to e^{2\phi} g_\mathrm{E}$. This results in

$$S = \frac{1}{2\kappa_0^2} \int \mathrm{d}^{26}X \sqrt{-g} \left(R - \tfrac{1}{12} e^{-\phi} H_{MNP} H^{MNP} + \tfrac{1}{2} \partial_M \phi \partial^M \phi \right) + \mathcal{O}(\alpha') \ , \tag{3.47}$$

where now the new $g = g_\mathrm{E}$ denotes the Einstein frame metric. This is called a low-energy effective action as one takes the energy to be much smaller than α', which is equivalent to fixing the energy and letting $\alpha' \to 0$. In this limit massive modes decouple, leaving the only importance to the massless ones and so the higher order terms of α' may be neglected.

3.1.4 Open strings and D-branes

So far we have discussed closed strings, and in this section we will make a brief comment on open strings. With boundary conditions $\sigma \in [0, \pi]$ and $\partial_\sigma X^M(\tau, \sigma) = \partial_\sigma X^M(\tau, \pi) = 0$ imposed on the equations of motion in eq. (3.8), the coordinates X^M can be mode expanded according to

$$X^M(\tau, \sigma) = x^M + 2\alpha' p^M \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^M}{n} e^{-in\tau} \cos(n\sigma) \ . \tag{3.48}$$

The corresponding mass operator is given by

$$M^2 = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} \alpha_n \cdot \alpha_{-n} - 1 \right) \ , \tag{3.49}$$

where we see that the vacuum state has negative mass; $\alpha' M^2 = -1$, which is a state known as a tachyon. The massless states are given by

$$|\xi_M\rangle \equiv \xi_M \alpha_1^M |0\rangle \ , \tag{3.50}$$

where then ξ_M is a 1-form gauge field. Denoting this field by A_M, the σ-model action for this field correspond to an interaction with the boundary

$$S_A = \int_{\partial M} \mathrm{d}\tau A_M \partial_\tau X^M \ . \tag{3.51}$$

Its corresponding equations of motion can be derived from the spacetime low-energy effective action for open bosonic strings, namely

$$S = \frac{1}{2\kappa_0} \int \mathrm{d}^{26}X \sqrt{-g} \left(-\tfrac{1}{4} e^{-\phi} F_{MN} F^{MN} \right) \ , \tag{3.52}$$

in string frame and where $F = \mathrm{d}A$. This action is the Yang-Mills action with field-dependent coupling constant which is the square root that of closed strings.

Open strings have degrees of freedom at their endpoints which are encoded in *Chan-Paton factors*. The endpoint dynamics can be seen in a dual picture of hyperplanes on which the open string endpoints can end, i.e. D-branes. The Chan–Paton indices are labels for the different D-branes on which the open strings can end. With N D-branes stacked on top of each other, the gauge group represented by the open strings is $U(N)$, making the action in eq. (3.52) non-Abelian. The low-energy action in eq. (3.52) is the low-energy limit $\alpha' \to 0$ of the *Dirac-Born-Infeld action*, which is given by

$$S_{\mathrm{BI}} = -T_p \int \mathrm{d}^{p+1}\xi e^{-\phi}\sqrt{\det(g_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \equiv -T_p \int \mathrm{d}^{p+1}\xi e^{-\phi}\sqrt{\det(g_{ab} + 2\pi\alpha'\mathscr{F})} \ . \tag{3.53}$$

This action describes open string dynamics to all orders in α'. The quantity $\mathscr{F}$ is gauge invariant in the worldvolume. The coefficient T_p is the Dp-brane tension given by $T_p = (2\pi)^{-p}(\alpha')^{-(p+1)/2}$, and the fields are pulled back on the D-brane metric, i.e. $g_{ab} = \frac{\partial X^M}{\partial \xi^a}\frac{\partial X^N}{\partial \xi^b} g_{MN}$.

D-branes in bosonic string theory are not stable as they decay by tachyon condensation, but they are so in superstring theories which are introduced in the next section. The D-branes couple to gauge potential fields in the superstring massless spectrum. The topological coupling to these fields adds a *Chern-Simons* term to the action, which reads

$$S_{\mathrm{CS}} = iT_p \int_{\Sigma_{p+1}} \mathrm{tr}(e^{2\pi\alpha' F + B} C) \ , \tag{3.54}$$

where the $(p+1)$-form gauge potential C_{p+1} from the superstring spectra couples to Dp-branes and Σ_{p+1} the Dp-brane world volume.

3.2 The type II superstrings

One of the basic approaches to incorporate supersymmetry into string theory is known as the Ramond-Neveu-Schwarz (RNS) formalism, which is supersymmetric on the string worldsheet. In the RNS formalism bosonic fields $X^\mu(\sigma,\tau)$ are paired up with fermionic partners $\psi^\mu(\sigma,\tau)$, where the fermionic fields are two-component spinors on the worldsheet. The action is given by

$$S = -\tfrac{1}{2\pi}\int \mathrm{d}^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\mu) \ , \tag{3.55}$$

where ρ^α with $\alpha = 1, 2$ are 2D Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{3.56}$$

satisfying the Clifford algebra $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$. The two-component spinor ψ^μ_A with $A = \pm$ can be written

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \ , \qquad \psi_\pm = \tfrac{1}{\sqrt{2}}(\psi^0 \pm \psi^1) \ , \tag{3.57}$$

where we define the Dirac conjugate of the spinor as

$$\bar{\psi} = i\psi^\dagger \rho^0 \ . \tag{3.58}$$

Both X^μ and ψ^μ are spacetime vectors and transform under the vector representation of the $SO(D-1,1)$ Lorentz group. While X^μ is simply a scalar on the worldsheet, the ψ^μ is a Majorana two-component spinor, i.e. a *real* two-component spinor which fulfils $\psi^*_\pm = \psi_\pm$. The action in eq. (3.55) is invariant under the supersymmetry transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu \ , \tag{3.59}$$

$$\delta_\epsilon \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \ , \tag{3.60}$$

where ϵ is a real spinor and $\bar{\epsilon} = i\epsilon^\dagger \rho^0$.

Introducing light-cone coordinates $\sigma^\pm$, the differential operators become $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ and $\bar{\psi} = i\psi^\dagger \rho^0 = i(\psi_+, -\psi_-)$, so the action for the fermionic fields in light-cone coordinates reads

$$S_f = \int \mathrm{d}^2\sigma (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \ , \tag{3.61}$$

where the Lorentz index μ has been suppressed. The equation of motion is the Dirac equation, which for the two spinor components takes the form

$$\partial_+ \psi_- = 0 \ , \qquad \partial_- \psi_+ = 0 \ , \tag{3.62}$$

as there is no mass term in the action. The equations describe left- and right-moving waves, and are in two dimensions better known as the Weyl conditions. The spinors $\psi_\pm$ are therefore sometimes called Majorana-Weyl spinors. The different sectors of the RNS formalism arise when considering the boundary terms in the variation of the fermionic action in eq. (3.62). They read

$$\delta S \sim \int \mathrm{d}\tau (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|_{\sigma=\pi} - (\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\Big|_{\sigma=0} \ , \tag{3.63}$$

and must vanish. There are several ways for them to do so, depending on whether the string is open or closed. For open strings the two terms in eq. (3.63) represents the two ends of the string, so they must vanish separately. This is achieved if both string ends fulfil

$$\psi_+^\mu = \pm \psi_-^\mu \ , \tag{3.64}$$

where the sign is a matter of convention. Say we choose $\psi_+^\mu|_{\sigma=0} = \psi_-^\mu|_{\sigma=0}$ at one end, then we still have two choices for the other end. Whether the two ends have the same choice of boundary condition, or with a relative sign, will become meaningful. The cases, or sectors, correspond to

- Ramond boundary condition: $\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi}$. The mode expansion in the R sector is

$$\psi_-^\mu(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \ , \tag{3.65}$$

$$\psi_+^\mu(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \ . \tag{3.66}$$

- Neveu-Schwarz boundary condition: $\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi}$. The mode expansion in the NS-sector takes the form

$$\psi_-^\mu(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)} \ , \tag{3.67}$$

$$\psi_+^\mu(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau+\sigma)} \ . \tag{3.68}$$

The Fourier modes d_n^μ, b_r^μ are Grassman numbers which anticommute. For the closed string, on the other hand, there are two possible periodicity conditions;

$$\psi_\pm^\mu(\sigma) = \pm \psi_\pm^\mu(\sigma + \pi) \ , \tag{3.69}$$

each of which makes the boundary term disappear. A positive sign describes a periodic boundary condition and a negative sign an antiperiodic boundary condition. The fermionic fields ψ^μ are either right- or left-moving, and the type of state they will represent depends on which type of periodicity that has been imposed. That is, for the right-movers, one can choose

$$\psi_-^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad \text{or} \quad \psi_-^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-2ir(\tau-\sigma)} \ , \tag{3.70}$$

while for the left-movers the choice is

$$\psi_+^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad \text{or} \quad \psi_+^\mu(\sigma,\tau) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)} \ . \tag{3.71}$$

Now, depending on which combination of right-and left-movers one chooses the states will correspond to spacetime fermions or bosons. States in the NSNS and RR sectors are bosons, and NSR and RNS are fermions, summing up to four distinct closed-string sectors.

As for the bosonic part of the action, $X^\mu(\sigma,\tau)$ is strictly periodic on a closed string; $X^\mu(0,\tau) = X^\mu(\pi,\tau)$, so it may also be Fourier expanded according to

$$X^\mu(\sigma,\tau)_R = \sum_{n\in\mathbb{Z}} \alpha_n e^{-2in(\tau-\sigma)} \ , \qquad X^\mu(\sigma,\tau)_L = \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n e^{-2in(\tau+\sigma)} \ . \tag{3.72}$$

The Forier modes of the bosonic and fermionic parts fulfil the commutation and anticommutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \ , \qquad \{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{n+m,0} \ , \qquad \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \ . \tag{3.73}$$

The fact that the spacetime vectors X^μ and ψ^μ are real implies that the modes satisfy

$$(\alpha_n)^\dagger = \alpha_{-n} \ , \qquad (d_n)^\dagger = d_{-n} \ , \qquad (b_r)^\dagger = b_{-r} \ . \tag{3.74}$$

Now, promoting the Fourier modes to operators that fulfil the conditions in eq. (3.73) and eq. (3.74), they may now be used to construct the states of the theory. For all parts, operators with positive-valued subscripts are annihilation operators and operators with negative-valued subscripts are creation operators. The ground state $|0\rangle$ in each sector is defined as the state that is annihilated when an annihilation operator acts on it, i.e.

$$\alpha_n |0\rangle = d_n |0\rangle = b_r |0\rangle = 0 \ , \qquad \forall n, r > 0. \tag{3.75}$$

The quantised open string states have masses

$$(\text{NS}) \qquad \alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{m=1}^{\infty} m d_{-m}^i d_m^i + a \ , \tag{3.76}$$

$$(\text{R}) \qquad \alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i + a \ , \tag{3.77}$$

where α' is the string constant and a is a normal ordering constant which is

$$a = \begin{cases} -\frac{1}{2} \ , & \text{for NS modes,} \\ 0 \ , & \text{for R modes.} \end{cases} \tag{3.78}$$

The closed string states includes left- and right-movers, where the mass operators are given by

$$(\text{NS}) \qquad \alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{m=1}^{\infty} m d_{-m}^i d_m^i + \sum_{l=1}^{\infty} \tilde{\alpha}_{-l}^i \tilde{\alpha}_l^i + \sum_{k=1}^{\infty} k \tilde{d}_{-k}^i \tilde{d}_k^i + a + \tilde{a} \ , \tag{3.79}$$

$$(\text{R}) \qquad \alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i + \sum_{s=1/2}^{\infty} r \tilde{b}_{-s}^i \tilde{b}_s^i \ . \tag{3.80}$$

The normal ordering constant for the NS sector can be calculated via

$$a_{\mathrm{NS}} = \tilde{a}_{\mathrm{NS}} = \frac{1}{2}(d-2)\Big(\sum_{n=0}^{\infty} n - \sum_{r=1/2}^{\infty} r\Big) = \frac{1}{2}(d-2)\Big(-\frac{1}{12} - \frac{1}{24}\Big) = -\frac{1}{16}(d-2) \ , \tag{3.81}$$

and we see that for $a_{\mathrm{NS}} = -1/2$ the critical dimension is $d = 10$. For the R sector on the other hand, all modes are integer modes, so the bosonic and fermionic parts cancel and we are left with $a_{\mathrm{R}} = \tilde{a}_{\mathrm{R}} = 0$.

Considering the mass term of the NS sector in eq. (3.76) it is clear that if there are no excitations then the ground state mass is given by $\alpha' M_0^2 = -\frac{1}{2}$. That is, the ground state is a tachyon. Since there is no anomaly in eq. (3.77) the ground state of the R sector is massless. The massless state of the R sector is also degenerate, as the ground state $|0\rangle$ has the same mass as $d_0 |0\rangle$. This is not the case in the NS sector as there are no fermionic oscillators with zero-valued subscripts ($r \in \mathbb{Z} + 1/2$). The modes d_0 also satisfies the anticommutation relations in eq. (3.73), which is a Clifford algebra up to a factor, and so the ground state modes d_0 may be identified with gamma matrices. The gamma matrices in turn are a set of tensor operators and transform according to the d-dimensional representation of $SO(d-1,1)$. Therefore the degenerate R sector ground state transforms as a spinor of $SO(d-1,1)$. Since we have $D = 10$, the R sector ground state is a 10D spinor which has $2^{D/2} = 2^5 = 32$ components.

In order to make RNS string theory a consistent theory that eliminates the tachyon, the spectrum can be truncated, or projected, in a specific way by a mechanism known as the *GSO projection*. To do so, we start by introducing an operator G called *G-parity*. In the NS sector, it is given by

$$G_{\mathrm{NS}} = (-1)^{F+1} \ , \quad \text{where} \quad F = \sum_{r=1/2}^{\infty} b^i_{-r} b^i_r \ . \tag{3.82}$$

F is the number of b-oscillator excitations, i.e. the worldsheet fermion number. The G-parity operator thus specifies if a state has an even or odd number of worldsheet fermion excitations. In the R sector we have that

$$G_{\mathrm{R}} = \Gamma_{11}(-1)^E \ , \quad \text{with} \quad E = \sum_{n=1}^{\infty} d^i_{-n} d^i_n \ , \tag{3.83}$$

and $\Gamma_{11} = \Gamma_0 \dots \Gamma_9$ is the 10D analogue of γ_5 in four dimensions. Similar to γ_5 it also satisfies $\{\Gamma_{11}, \Gamma^\mu\} = 0$ and $(\Gamma_{11})^2 = 1$. In 10D the chiral projection operators are defined as $P_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$. Note that a spinor fulfilling $\Gamma_{11}\psi = \pm\psi$ are said to have positive/negative chirality, and a spinor with a definite chirality is known as a Weyl spinor.

As for the GSO projection, it consists of only keeping the states with positive G-parity in the NS sector. That is, we keep only the states satisfying

$$(-1)^{F+1} = +1 \quad \Leftrightarrow \quad (-1)^F = -1 \ , \tag{3.84}$$

and eliminates those with negative G-parity. This mean that we require an odd number of b-oscillator excitations. For the R sector we may choose to project onto state of either positive or negative G-parity, depending on the chirality of the ground state spinor. The choice is a matter of convention.

Since the open string tachyon has negative G-parity, i.e. $G\,|0\rangle_{\mathrm{NS}} = -\,|0\rangle_{\mathrm{NS}}$, it will be eliminated from the spectrum. The first excited state, the massless vector boson $b^i_{-1/2}\,|0\rangle_{\mathrm{NS}}$, has positive parity and survives the projection, hence it becomes the ground state of the NS sector.

In order to analyse the closed string spectrum we need to consider left- and right-movers again. Thus there are four different sectors; NS-NS, NS-R, R-NS, and R-R to consider. We have concluded that a GSO projection onto states with positive G-parity eliminates the tachyon. For the R sector we choose a projection onto either states with negative or positive G-parity depending on the spinor ground state. The choice of whether the G-parity of the left- and right-moving R sectors is the same or opposite gives rise to two different theories.

In type IIB theory the left-and right-moving R sector ground states have the same chirality, chosen to be positive. Thus the two R sectors, each denoted $|+\rangle_R$, have the same G-parity. The opposite applies for type IIA, where the left- and right-moving ground states in the R sector have the opposite chirality. The massless states in the spectrum of each theory is summarised in table 3.1.

IIA	IIB
$\|-\rangle_{\rm R} \otimes \|+\rangle_{\rm R}$	$\|+\rangle_{\rm R} \otimes \|+\rangle_{\rm R}$
$\tilde{b}^i_{-1/2}\,\|0\rangle_{\rm NS} \otimes b^j_{-1/2}\,\|0\rangle_{\rm NS}$	$\tilde{b}^i_{-1/2}\,\|0\rangle_{\rm NS} \otimes b^j_{-1/2}\,\|0\rangle_{\rm NS}$
$\tilde{b}^i_{-1/2}\,\|0\rangle_{\rm NS} \otimes \|+\rangle_R$	$\tilde{b}^i_{-1/2}\,\|0\rangle_{\rm NS} \otimes \|+\rangle_R$
$\|-\rangle_R \otimes b^i_{-1/2}\,\|0\rangle_{\rm NS}$	$\|+\rangle_R \otimes b^i_{-1/2}\,\|0\rangle_{\rm NS}$

Table 3.1: Massless states of the type II theories. The sign $\pm$ denotes the chirality of each state. It can be seen that the RR and RNS sectors have different chiralities in type IIA and IIB, while the states in the NSNS and NSR sectors are the same for both theories.

The massless states of the two theories are very similar, only difference being that in type IIA the fermionic states come with two different chiralities. There are 64 states in each of the four massless sectors, adding up to a total of 256 states in each theory.

The massless string states transform under the little group SO(8), which is a subgroup of the 10D Lorentz group SO(9,1). The 8D representations of SO(8) are given by a vector representation $\mathbf{8_v}$ and two spinor representations $\mathbf{8_s}$ and $\mathbf{8_c}$. These are related to each other by the triality automorphism group. From them the whole closed massless type II superstring spectrum can be built. The R sector is built out of spinorial representations and the NS sector from the vector representation. The RR/RNS/NSR/NSNS sectors are each built by tensoring the different representations, and a following decomposition of the tensor product into irreducible SO(8) representations gives us the constituents of each sector. The NSNS sector is the same for type IIA and IIB, and consists of

$$\mathbf{8_v} \otimes \mathbf{8_v} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \equiv \phi \oplus B_{MN} \oplus g_{MN} \ , \tag{3.85}$$

where the scalar ϕ is called the dilaton, the 2-form field B_{MN} is a gauge field, and g_{MN} is a symmetric field known as the graviton. The irreps are obtained by decomposing a second rank tensor $(\mathbf{8_v} \otimes \mathbf{8_v})$ into a trace, an antisymmetric part and a symmetric traceless part respectively.

We have seen that the type IIA theory is a chiral theory on the worldsheet and non-chiral in spacetime, while the opposite applies for the type IIB theory. This means that they will each be built by spinors of opposite and equal chirality respectively, so that for the RR sectors they are given by

$$\text{IIA}: \qquad \mathbf{8_s} \otimes \mathbf{8_c} = \mathbf{8_v} \oplus \mathbf{56_v} \equiv C_M \oplus C_{MNP} \ , \tag{3.86}$$

$$\text{IIB}: \qquad \mathbf{8_s} \otimes \mathbf{8_s} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_c} \equiv C_0 \oplus C_{MN} \oplus C^+_{MNPQ} \ , \tag{3.87}$$

where the Cs are gauge fields of different degrees $(0, \ldots, 4)$, referred to as the RR gauge fields. The scalar C_0 is called the axion. The plus superscript on C_4 denotes that it is Hodge dual to

	IIA		IIB	
Sector	Quantity	Fields	Quantity	Fields
NSNS	1	dilaton ϕ	1	dilaton ϕ
	28	2-form gauge field B_2	28	2-form gauge field B_2
	35	graviton g_{MN}	35	graviton g_{MN}
RNS and RNS	112	gravitinos ψ_M^A	112	gravitinos ψ_M^A
	16	dilatinos λ^A	16	dilatinos λ^A
RR	8	1-form gauge field C_1	1	axion scalar C_0
	56	3-form gauge field C_3	28	2-form gauge field C_2
			35	4-form gauge field C_4

Table 3.2: The massless closed string spectra of the type IIA and type IIB theories.

itself. As for the mixed sectors NSR and RNS, the decomposition is as follows:

$$\text{IIA}: \quad \begin{aligned} \mathbf{8_v} \oplus \mathbf{8_c} = \mathbf{8_c} \oplus \mathbf{56_c} \equiv \lambda^1 \oplus \psi_M^1 \ , \quad &(\text{NSR}) \\ \mathbf{8_s} \oplus \mathbf{8_v} = \mathbf{8_s} \oplus \mathbf{56_s} \equiv \lambda^2 \oplus \psi_M^2 \ , \quad &(\text{RNS}) \end{aligned} \tag{3.88}$$

$$\text{IIB}: \quad \begin{aligned} \mathbf{8_v} \oplus \mathbf{8_s} = \mathbf{8_s} \oplus \mathbf{56_s} \equiv \lambda^1 \oplus \psi_M^1 \ , \quad &(\text{NSR}) \\ \mathbf{8_s} \oplus \mathbf{8_v} = \mathbf{8_s} \oplus \mathbf{56_s} \equiv \lambda^2 \oplus \psi_M^2 \ , \quad &(\text{RNS}) \end{aligned} \tag{3.89}$$

where the fermions in the **8** representation, λ^A, $A = 1, 2$ are called dilatinos, and the fermions ψ_M^A in the **56** representation are known as gravitinos. The complete massless spectrum of each of the theories are summarised in table 3.2. The NSNS sector is the same for the two theories. In the NSR and RNS sectors the gravitinos have the same chirality in the IIB case, and the opposite chirality in the type IIA case. By tensoring a pair of Majorana-Weyl spinor the RR sector states obtained are bosonic.

As a final remark we reconnect with section 3.1.4, and note that Dp-branes of odd p couple to the even potentials of type IIB superstring theory, while those of even p couple to the odd potentials of type IIA. These stable D-brane configurations are BPS states, which conserve half of the supersymmetry.

3.3 Type II low-energy effective actions and supergravity

The low-energy effective actions of the superstring can be obtained in a similar manner as for the bosonic string, but is a lot harder to do. Nevertheless, it turns out that the resulting low-energy effective actions are equivalent to the 10D supergravity theories, which are supersymmetric extensions of Einstein gravity. Hence the supergravity theories are low-energy limits of string theories, which we will briefly review in this next section.

3.3.1 The type II supergravity theories

The 11D supergravity theory was first formulated in 1978 by Cremmer, Julia and Scherk [12]. The 10D type II supergravity theories can be derived from 11D supergravity by a Kaluza-Klein compactification of the 11$^{\text{th}}$ dimension on a circle, which was first done in [13] in 1985. The 11D supergravity theory has two bosonic fields; the metric $G_{\tilde{M}\tilde{N}}$, with $\tilde{M}, \tilde{N} = 0, \ldots, 10$, and a 3-form potential $A_{\tilde{M}\tilde{N}\tilde{P}} \equiv A_3$ with field strength $F_4 = \mathrm{d}A_3$. The bosonic part of the action reads [14]

$$S_{11} = \frac{1}{2\kappa_{11}} \int \mathrm{d}^{11}x\sqrt{-G}\left(R - \tfrac{1}{2}|F_4|^2\right) - \frac{1}{6}\int \mathrm{d}^{11}x A_3 \wedge F_4 \wedge F_4 \ , \tag{3.90}$$

where κ_{11} is the 11D gravitational coupling constant. To dimensionally reduce this the starting point is writing a general metric that is invariant under translations in the 11^{th} direction, namely

$$\begin{aligned} \mathrm{d}s^2 &= G^{(11)}_{\tilde{M}\tilde{N}}(x^M)\,\mathrm{d}x^{\tilde{M}}\,\mathrm{d}x^{\tilde{N}} \\ &= G^{10}_{MN}(x^M)\,\mathrm{d}x^M\,\mathrm{d}x^N + e^{2\sigma(x^M)}\left(\mathrm{d}x^{10} + A_N(x^M)\,\mathrm{d}x^N\right)^2 \ , \end{aligned} \tag{3.91}$$

where $\sigma \equiv 2\phi/3$. The 11D metric thus reduces to a scalar σ and a 1-form gauge field A_1. The potential A_3 will reduce to potentials A_3 and A_2 where the A_2 comes from components with one index along the compact 11^{th} dimension. In summary the three terms in eq. (3.90) become respectively

$$S_1 = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\left(e^{\sigma}R - \tfrac{1}{2}e^{3\sigma}|F_2|^2\right) \ , \tag{3.92}$$

$$S_2 = -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\left(e^{-\sigma}|F_3|^2 + e^{\sigma}|\tilde{F}_4|^2\right) \ , \tag{3.93}$$

$$S_3 = -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x A_2 \wedge F_4 \wedge F_4 = -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x A_3 \wedge F_3 \wedge F_4 \ , \tag{3.94}$$

where we have compactified the theory on a circle of coordinate period $2\pi R$ so that $\kappa_{10}^2 = \kappa_{11}^2/2\pi R$. Note that the normalisation of the kinetic terms is canonical for $2\kappa_{10}^2 = 1$. The p-form field strengths are generally defined as the exterior derivative of a $(p-1)$-form gauge potential, according to $F_p = \mathrm{d}A_{p-1}$. In eq. (3.93) we have defined

$$\tilde{F}_4 \equiv \mathrm{d}A_3 - A_1 \wedge F_3 \ . \tag{3.95}$$

Terms like the one in eq. (3.94), i.e. with appearing p-form potentials rather than their exterior derivatives, are known as *Chern-Simons* terms. To make contact with string theory, we redefine the metric as

$$G_{MN} = e^{-\phi}G_{MN}(\text{new}) \ , \tag{3.96}$$

and reintroduce the dilaton ϕ in standard form into the theory. Some of the fields will be redefined as to match the ones appearing in the string worldsheet σ-model action. The fields of the reduced theory will then be the same as those of 10D type IIA string theory. The action of the type IIA supergravity theory is given by

$$S_{\text{IIA}} = S_{\text{NSNS}} + S_{\text{RR}} + S_{\text{CS}} \ , \tag{3.97}$$

where the action is grouped according to whether the fields belong to the NSNS sector or the RR sector. The Chern-Simons action contain fields from both these sectors. The constituent actions of eq. (3.97) are

$$S_{\text{NSNS}} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}e^{-2\phi}\left(R + 4\partial_M\phi\partial^M\phi - \tfrac{1}{2}|H_3|^2\right) \ , \tag{3.98}$$

$$S_{\text{RR}} = -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\left(|F_2|^2 + |\tilde{F}_4|^2\right) \ , \tag{3.99}$$

$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x B_2 \wedge F_4 \wedge F_4 \ , \tag{3.100}$$

where $F_p = \mathrm{d}C_{p-1}$ with C being the familiar RR gauge potential from the previous section, $H_3 = \mathrm{d}B_2$ for the NSNS sector, and $\tilde{F}_4 \equiv \mathrm{d}C_3 - C_1 \wedge F_3$.

The 11D supergravity action is dimensionally reduced to the type IIA supergravity action, but the type IIB supergravity cannot be obtained in the same way. Type IIA and IIB *string theory* are related by T-duality, so in principle the type IIB *supergravity* action could be obtained by T-dualising the type IIA string theory and thereafter taking the low-energy limit. It is not however an easy task because of the self-dual 5-form. Due to the presence of the self-dual 5-form field strength, there is no standard manifestly covariant action for this theory, but the following action comes close:

$$S_{\text{IIB}} = S_{\text{NSNS}} + S_{\text{RR}} + S_{\text{CS}} \ , \tag{3.101}$$

$$S_{\text{NSNS}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x\sqrt{-G}e^{-2\phi}\left(R + 4\partial_M\phi\partial^M\phi - \tfrac{1}{2}|H_3|^2\right) \ , \tag{3.102}$$

$$S_{\text{RR}} = -\frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \tfrac{1}{2}|\tilde{F}_5|^2\right) \ , \tag{3.103}$$

$$S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x C_4 \wedge H_3 \wedge F_3 \ , \tag{3.104}$$

where

$$\tilde{F}_3 \equiv F_3 - C_0 \wedge H_3 \ , \tag{3.105}$$

$$\tilde{F}_5 \equiv F_5 - \tfrac{1}{2}C_2 \wedge H_3 + \tfrac{1}{2}B_2 \wedge F_3 \ . \tag{3.106}$$

Note that the NSNS sector actions are the same for both theories, as should be expected from the field content in table 3.2. The self-duality of the 5-form field-strength, i.e. $\tilde{F}_5 = \star\tilde{F}_5$, must be added as a constraint to the equations of motion of the action in eq. (3.101). Imposing the constraint directly onto the action would result in the wrong equations of motion.

3.3.2 Type IIB $SL(2,\mathbb{R})$-invariance and S-duality

The type IIB action can be rewritten in a form which clearly shows that it is invariant under an $SL(2,\mathbb{R})$ symmetry transformation. Rewritten in Einstein frame, it is invariant under the transformations

$$\phi \to -\phi \ , \qquad B_2 \to -C_2 \ , \qquad C_2 \to B_2 \ , \tag{3.107}$$

and the axion C_0, the 4-form C_4 and the Einstein frame metric left as they are. Such a symmetry maps the string coupling according to $g_{\text{s}} \to 1/g_{\text{s}}$, which in the limit $g_{\text{s}} \to \infty$ relates the strong and weak coupling regimes. The RR and NSNS 2-forms in eq. (3.107) have a sign difference as to make the 10D Chern-Simons coupling invariant. This symmetry is called S-symmetry, a name motivated by its formal analogy with the perturbative T-duality $R \to 1/R$ and by the fact that it acts on the dilaton ϕ (which is sometimes denoted S). Since it relates the strong and weak coupling regimes it is also sometimes called the strong-weak duality.

The type IIB action has a larger $SL(2,\mathbb{R})$ symmetry. By combining the axion and the dilaton to the complex axion-dilaton

$$\tau \equiv C_0 + ie^{-\phi} \ , \tag{3.108}$$

this symmetry acts on the axion-dilaton and the 2-form potentials according to

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \qquad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \ , \tag{3.109}$$

where $ad - bc = 1$. The S-duality is then given as a specific $SL(2,\mathbb{R})$ transformation with $a = d = 0$ and $b = -c = 1$, reproducing eq. (3.107). Quantum mechanics requires the charge, with respect to the NSNS 2-form of the basic object of string theory – the fundamental string – to be quantised. This breaks the continuous group $SL(2,\mathbb{R})$ symmetry to the discrete subgroup $SL(2,\mathbb{Z})$, which is conjectured to be a symmetry of the full type IIB string theory. It is generated by the actions $\tau \to -\tau$ and $\tau \to \tau + 1$. In later chapters we will see that this is a version of a so-called axionic shift-symmetry, which in general is a continuous symmetry that is broken to a discrete subgroup by non-perturbative effects. As a consequence, $SL(2,\mathbb{Z})$ has also been conjectured to be the symmetry group of non-perturbative IIB superstring theory, and is also formally identical to the 2-torus modular group.

As a final note we note another common rewriting of the IIB action. With $G_3 \equiv F_3 - \tau H_3$ the type IIB action in eq. (3.101) can, after a Weyl rescaling to Einstein frame, be rewritten as

$$S_{\text{IIB}} = \frac{1}{\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g}\left(R - \frac{|\partial\tau|^2}{2(\mathrm{Im}\tau)^2} - \frac{|G_3|^2}{2(\mathrm{Im}\tau)^2} - \frac{|\tilde{F}_5|^2}{4} - \frac{i}{4\kappa_{10}^2}\int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\mathrm{Im}\tau}\right) , \quad (3.110)$$

where the field strengths are defined as earlier and with $\tilde{F}_5$ as in eq. (3.106). The $SL(2,\mathbb{R})$ transformation is then given with τ as in eq. (3.109) and $G_3 \to G_3/(c\tau + d)$.

3.3.3 The democratic formulation

In this section we will describe a uniform formulation of the two type II supergravities in 10D in a way which is symmetric between IIA and IIB and utilises all RR forms as well as their duals, which was first introduced in [15]. It also applies to the massive type IIA theory by including the 9-form field C_9 as well as its dual field strength $F_0 = m$ equal to the mass. The extended field content is then

$$\begin{aligned} \text{IIA}: &\quad \{g_{MN}, B_{MN}, \phi, C_1, C_3, C_5, C_7, C_9, \psi_M, \lambda\} , \\ \text{IIB}: &\quad \{g_{MN}, B_{MN}, \phi, C_0, C_2, C_4, C_6, C_8, \psi_M, \lambda\} , \end{aligned} \quad (3.111)$$

where IIA has fermions of both chiralities while in IIB we have that $\Gamma^{11}\psi_M = \psi_M$ and $\Gamma^{11}\lambda = -\lambda$. The uniform bosonic action may be written

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^{10}x\sqrt{-g}\left(e^{-2\phi}\left[R - 4(\partial\phi)^2 + \tfrac{1}{2}|H|^2\right] + \tfrac{1}{4}\sum_{n=0,\frac{1}{2}}^{5,\frac{9}{2}} F_{2n}\cdot F_{2n}\right) , \quad (3.112)$$

where n is summed over integers $0,\ldots,5$ in IIA and half-integers $\frac{1}{2},\ldots,\frac{9}{2}$ in IIB. All of the potentials and field strengths may be grouped together as

$$C = \sum_{n=0,\frac{1}{2}}^{5,\frac{9}{2}} C_{2n-1} , \qquad F = \sum_{n=0,\frac{1}{2}}^{5,\frac{9}{2}} F_{2n} , \quad (3.113)$$

where the NSNS and RR field strength are given as

$$H = \mathrm{d}B , \qquad F = \mathrm{d}C - H \wedge C + me^B . \quad (3.114)$$

Again the last term in F above exists only in massive type IIA theory. Self-duality conditions are imposed to reduce the degrees of freedom to the physical ones, and are in this formulation given by

$$F_{2n} = (-1)^{\lfloor n \rfloor} \star F_{10-2n} , \quad (3.115)$$

with $\lfloor n \rfloor$ being the integer part of n and $\star$ the 10D Hodge star. The Bianchi identities for the NSNS flux and the RR fluxes are

$$\mathrm{d}H = 0 \ , \qquad \mathrm{d}F - H \wedge F = 0 \ , \tag{3.116}$$

respectively. The democratic formulation will for instance be used in the next chapter expressing the supersymmetry transformations of the type II fermionic fields.

4

Type II Supersymmetric Backgrounds with Maximal Spacetime Symmetry

The cosmological principle states that on large enough scales, i.e. about 500 Mpc, the universe is homogeneous and isotropic. By homogeneity we mean that the properties of the universe are the same at every point in space, i.e. it is translation invariant. Isotropy is being in a given point, in every direction we look at, the properties of the universe look the same. This means that it is invariant under rotations. A space that is both homogeneous and isotropic is maximally symmetric. In this chapter we will discuss compactifications from 10D to 4D in which the vacuum external geometry is maximally symmetric. There exist three different possible spaces that are so; the Minkowski, anti-de Sitter (AdS) and de Sitter (dS) spaces. These spaces have Poincaré-, $SO(1,4)$- and $SO(2,3)$-invariance respectively. The most general 10D metric including a 4D maximally symmetric space is

$$\mathrm{d}s^2 = e^{2A(y)}\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n \ , \quad m,n=1,\ldots,6 \ , \tag{4.1}$$

where A is a function of the internal, or compactified, coordinates called a *warp factor*, $\hat{g}_{\mu\nu}$ is our 4D Minkowski, AdS or dS metric and g_{mn} is an arbitrary 6D metric.

The first requirement of maximal spacetime symmetry is usually said to be that the vacuum expectation value (VEV) of the fermionic fields is zero. This is because of the supersymmetry requirement $\delta_\epsilon\psi \sim \partial_M\phi$, i.e. that a fermionic supersymmetry transformation is proportional to the derivative of a scalar field, which, in a maximally symmetric spacetime, has to be constant. Hence only a vacuum configuration $(\phi,\psi)=(\phi_0,0)$ with a constant scalar field ϕ_0 is allowed in a maximally symmetric space to conserve supersymmetry. The background must therefore be purely bosonic.

In 10D the type II supersymmetry variations for the fermions, i.e. the two gravitinos ψ^A_M with $A=1,2$ and the two dilatinos λ^A in the democratic formulation of the RR fields, are

$$\delta_\epsilon\psi_M = \nabla_M\epsilon + \tfrac{1}{4}\not{H}_M\mathcal{P}\epsilon + \tfrac{1}{16}e^\phi\sum_n \not{F}_{2n}\Gamma_M\mathcal{P}_n\epsilon \ , \tag{4.2}$$

$$\delta_\epsilon\lambda = \left(\not{\partial}\phi + \tfrac{1}{2}\not{H}\mathcal{P}\right)\epsilon + \tfrac{1}{8}e^\phi\sum_n(-1)^{2n}(5-2n)\not{F}_{2n}\mathcal{P}_n\epsilon \ . \tag{4.3}$$

Here the RR field strengths F are given in eq. (3.114), and the slash is defined according to $\not{F}_{2n} = \frac{1}{(2n)!}F_{P_1\ldots P_N}\Gamma^{P_1\ldots P_N}$. The 3-form field H_3 is contracted according to $\not{H}_M \equiv \frac{1}{2}H_{MNP}\Gamma^{NP}$ and the $\mathcal{P}$'s are given for type IIA and IIB respectively as

$$\text{IIA:} \qquad \mathcal{P} = \Gamma_{11} \ , \qquad \mathcal{P}_n = (\Gamma_{11})^n\sigma^1 \ , \tag{4.4}$$

$$\text{IIB:} \qquad \mathcal{P} = -\sigma^3 \ , \qquad \mathcal{P}_n = \begin{cases}\sigma^1 & n+\frac{1}{2}\ \text{odd}\\ i\sigma^2 & n+\frac{1}{2}\ \text{even}\end{cases} \ , \tag{4.5}$$

where the σ^is are the Pauli matrices. The vanishing of the fermionic supersymmetry variations will guarantee the background to be maximally symmetric and supersymmetric. Given a metric of the form in eq. (4.1), one can show that the supersymmetry variations in combination with the Bianchi identities of the NSNS and RR fields will imply all other equations of motion. Hence, the supersymmetry variations will be central in our study of supersymmetric backgrounds.

4.1 Supersymmetric backgrounds without flux

We begin our analysis with the simplest case in which we take all fluxes to vanish. With no fluxes present, demanding a vanishing VEV for the gravitino variation in eq. (4.2) reduces the equation to

$$\nabla_M \epsilon = 0 \ , \tag{4.6}$$

i.e. the requirement that there is a covariantly constant spinor ϵ on the manifold. The covariant derivative on a spinor acts like

$$\nabla_M \to \partial_M + \tfrac{1}{4}\omega_M{}^{\bar{M}\bar{N}}\Gamma_{\bar{M}}\Gamma_{\bar{N}} = \partial_M + \omega_M \ , \tag{4.7}$$

where the barred indices $\bar{M}, \bar{N}$ denote locally flat coordinates and $\omega_M{}^{\bar{M}\bar{N}}$ is the spin connection given in terms of vielbeins $e_M{}^{\bar{M}}$ as

$$\omega_M{}^{\bar{M}\bar{N}} = \tfrac{1}{2}e^{N\bar{M}}(\partial_M e_N{}^{\bar{N}} - \partial_N e_M{}^{\bar{N}}) - \tfrac{1}{2}e^{N\bar{N}}(\partial_M e_N{}^{\bar{M}} - \partial_N e_M{}^{\bar{M}}) \tag{4.8}$$

$$- \tfrac{1}{2}e^{P\bar{M}}e^{Q\bar{N}}(\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}})e_M{}^{\bar{R}} \ , \tag{4.9}$$

and the vielbeins satisfy

$$g_{MN} = e_M{}^{\bar{M}}e_N{}^{\bar{N}}\eta_{\bar{M}\bar{N}} \ , \quad e_M{}^{\bar{M}}e^M{}_{\bar{N}} = \delta^{\bar{M}}_{\bar{N}} \ , \quad \Gamma^M = e^M{}_{\bar{M}}\Gamma^{\bar{M}} \ . \tag{4.10}$$

The 11D gamma matrices Γ^M are given by

$$\Gamma^M = \begin{pmatrix}\Gamma^\mu \\ \Gamma^m\end{pmatrix} \ , \quad \text{where} \quad \begin{cases}\mu = 0,\dots,3, \\ m = 4,\dots,10.\end{cases} \tag{4.11}$$

With the metric of eq. (4.1), the vierbeins and gamma matrices are defined by

$$e_M{}^{\bar{M}} = \begin{pmatrix} e^A\hat{e}_\mu{}^{\bar{\nu}} & 0 \\ 0 & e^A\hat{e}_m{}^{\bar{\nu}}\end{pmatrix} \ , \tag{4.12}$$

$$\Gamma^M = e^M{}_{\bar{M}}\Gamma^{\bar{M}} = \begin{pmatrix} e^{-A}\hat{e}^\mu{}_{\bar{\nu}} & 0 \\ 0 & e^{-A}\hat{e}^m{}_{\bar{\xi}}\end{pmatrix}\begin{pmatrix}\Gamma^{\bar{\nu}} \\ \Gamma^{\bar{\xi}}\end{pmatrix} = e^{-A}\begin{pmatrix}\Gamma^\mu \\ \Gamma^m\end{pmatrix} \ . \tag{4.13}$$

We are now ready to calculate the spin connection ω_M. Starting by noticing that

$$\begin{aligned}\partial_M e_N{}^{\bar{N}} &= \begin{pmatrix}\partial_M(e^{A(y)}\hat{e}_\mu{}^{\bar{\nu}}) & 0 \\ 0 & \partial_M(e^{A(y)}\hat{e}_m{}^{\bar{\nu}})\end{pmatrix} \\ &= \begin{pmatrix}(\partial_m A)e^A\hat{e}_\mu{}^{\bar{\nu}} & 0 \\ 0 & (\partial_m A)e^A\hat{e}_m{}^{\bar{\nu}}\end{pmatrix} + \begin{pmatrix}e^A\partial_M\hat{e}_\mu{}^{\bar{\nu}} & 0 \\ 0 & e^A\partial_M\hat{e}_m{}^{\bar{\nu}}\end{pmatrix}\end{aligned}$$

the calculation of ω_M can be split according to $\omega_M = \hat{\omega}_M + \tilde{\omega}_M$, with $\hat{\omega}_M$ being the spin connection of the 4D metric $\hat{g}_{\mu\nu}$, not including the warp factor $A(y^m)$, and the $\tilde{\omega}_M$ of the internal manifold g_{mn}. Hence directing our attention to the $\tilde{\omega}_M$, which has only derivatives of m, the spin connection becomes

$$\begin{aligned}
\tilde{\omega}_M^{\bar{M}\bar{N}} &= \tfrac{1}{2}e^{N\bar{M}}(\partial_M e_N{}^{\bar{N}} - \partial_N e_M{}^{\bar{N}}) - \tfrac{1}{2}e^{N\bar{N}}(\partial_M e_N{}^{\bar{M}} - \partial_N e_M{}^{\bar{M}}) \\
&\quad - \tfrac{1}{2}e^{P\bar{M}}e^{Q\bar{N}}(\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}})e_M{}^{\bar{R}} \\
&= \tfrac{1}{2}(e^{N\bar{M}}\partial_M e_N{}^{\bar{N}} - e^{N\bar{N}}\partial_M e_N{}^{\bar{M}}) + \tfrac{1}{2}(-e^{N\bar{M}}\partial_N e_M{}^{\bar{N}} + e^{N\bar{N}}\partial_N e_M{}^{\bar{M}}) \\
&\quad - \tfrac{1}{2}e^{P\bar{M}}e^{Q\bar{N}}(\partial_P e_{Q\bar{R}} - \partial_Q e_{P\bar{R}})e_M{}^{\bar{R}} \\
\partial_K \to \partial_m A \quad &= \tfrac{1}{2}(\partial_m A)\Big[(e^{N\bar{M}}e_N{}^{\bar{N}} - e^{N\bar{N}}e_N{}^{\bar{M}}) + (-e^{m\bar{M}}e_M{}^{\bar{N}} + e^{m\bar{N}}e_M{}^{\bar{M}}) \\
&\quad - (e^{m\bar{M}}e^{Q\bar{N}}e_{Q\bar{R}}e_M{}^{\bar{R}} - e^{P\bar{M}}e^{m\bar{N}}e_{P\bar{R}}e_M{}^{\bar{R}})\Big] \\
&= \tfrac{1}{2}(\partial_m A)\Big[(-e^{m\bar{M}}e_M{}^{\bar{N}} + e^{m\bar{N}}e_M{}^{\bar{M}}) - (e^{m\bar{M}}e^{Q\bar{N}}e_{Q\bar{R}}e_M{}^{\bar{R}} - e^{P\bar{M}}e^{m\bar{N}}e_{P\bar{R}}e_M{}^{\bar{R}})\Big] \\
&= \tfrac{1}{2}(\partial_m A)\Big[(-e^{m\bar{M}}e_M{}^{\bar{N}} + e^{m\bar{N}}e_M{}^{\bar{M}}) - (e^{m\bar{M}}\delta^{\bar{N}}_{\bar{R}}e_M{}^{\bar{R}} - e^{m\bar{N}}\delta^{\bar{M}}_{\bar{R}}e_M{}^{\bar{R}})\Big] \\
&= \tfrac{1}{2}(\partial_m A)\Big[(-e^{m\bar{M}}e_M{}^{\bar{N}} + e^{m\bar{N}}e_M{}^{\bar{M}}) - (e^{m\bar{M}}e_M{}^{\bar{N}} - e^{m\bar{N}}e_M{}^{\bar{M}})\Big] \\
&= (\partial_m A)(-e^{m\bar{M}}e_M{}^{\bar{N}} + e^{m\bar{N}}e_M{}^{\bar{M}}) \ ,
\end{aligned}$$

where the first of the three terms inside the big parenthesis cancel as $e^{N\bar{M}}e_N{}^{\bar{N}} - e^{N\bar{N}}e_N{}^{\bar{M}} = \eta^{\bar{M}\bar{N}} - \eta^{\bar{N}\bar{M}} = 0$ since η^{MN} is symmetric. Adding the gamma matrices to this, some simplifications can be made:

$$\begin{aligned}
\tilde{\omega}_M^{\bar{M}\bar{N}}\Gamma_{\bar{M}}\Gamma_{\bar{N}} &= (\partial_m A)(-e^{m\bar{M}}e_M{}^{\bar{N}} + e^{m\bar{N}}e_M{}^{\bar{M}})\Gamma_{\bar{M}}\Gamma_{\bar{N}} \\
&= (\partial_m A)(-\Gamma^m\Gamma_M + \Gamma_M\Gamma^m) \\
&= -2(\partial_m A)\Gamma^m\Gamma_M \ .
\end{aligned}$$

Thus we have shown that $\tilde{\omega}_M = \frac{1}{4}\tilde{\omega}_M{}^{\bar{M}\bar{N}}\Gamma_{\bar{M}}\Gamma_{\bar{N}} = -\frac{1}{2}(\partial_m A)\Gamma^m\Gamma_M$. Notice that $\Gamma^m\Gamma_M = e^{-A}\Gamma^{\bar{m}}e^A\Gamma_{\bar{M}} = \Gamma^{\bar{m}}\Gamma_{\bar{M}}$, but from here on we will drop the bar on the indices. Using a decomposition of the gamma matrix $\Gamma^m = \gamma_5 \otimes \gamma^m$ we may write

$$\tilde{\omega}_M = -\tfrac{1}{2}\gamma_5 \otimes \gamma^m(\partial_m A)\Gamma_M = -\tfrac{1}{2}\gamma_5 \otimes (\partial\!\!\!/ A)\Gamma_M \ , \tag{4.14}$$

so that the covariant derivative on the spinor ϵ can be written

$$\begin{aligned}
\nabla_M \epsilon &= (\partial_M + \omega_M)\epsilon \\
&= \partial_M \epsilon - \tfrac{1}{2}(\gamma_5 \otimes \partial\!\!\!/ A\Gamma_M)\epsilon + \hat{\omega}_M \epsilon \\
&\equiv \hat{\nabla}_M \epsilon - \tfrac{1}{2}(\gamma_5 \otimes \partial\!\!\!/ A\Gamma_M)\epsilon \ .
\end{aligned} \tag{4.15}$$

The covariant derivative with the hat is the one related to the 4D metric. Considering only at the exterior 4D maximally symmetric space $\Gamma_M \to \Gamma_\mu$, we may use another gamma matrix decomposition $\Gamma_\mu = \gamma_\mu \otimes \mathbb{1}$, so that

$$\hat{\nabla}_\mu \epsilon + \tfrac{1}{2}(\gamma_\mu\gamma_5 \otimes \partial\!\!\!/ A)\epsilon = 0 \ , \tag{4.16}$$

where we used that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$. With this result we may express the commutator $[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon$ according to

$$\begin{aligned}[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon &= \tfrac{1}{4}(\gamma_\mu\gamma_5\gamma_\nu\gamma_5 \otimes \not{\nabla}A\not{\nabla}A - \gamma_\nu\gamma_5\gamma_\mu\gamma_5 \otimes \not{\nabla}A\not{\nabla}A)\epsilon \\ &= \tfrac{1}{4}(-\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) \otimes \not{\nabla}A\not{\nabla}A\epsilon \\ &= -\tfrac{1}{2}\gamma_{\mu\nu}\not{\nabla}A\not{\nabla}A\epsilon \\ &= -\tfrac{1}{2}\gamma_{\mu\nu}(\nabla_m A)(\nabla^m A)\epsilon \ , \end{aligned} \tag{4.17}$$

where we have used that $\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and $\not{P}\not{P} = P^2$. The condition in eq. (4.17) is known as an *integrability condition.* On the other hand, we know from general relativity the definition of the Riemann tensor

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]\epsilon = \tfrac{1}{4}\hat{R}_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}\epsilon \tag{4.18}$$

$$= \tfrac{k}{2}\gamma_{\mu\nu}\epsilon \ , \tag{4.19}$$

where we have used that the Riemann tensor of a maximally symmetric space is just

$$R_{\mu\nu\lambda\rho} = k(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \ . \tag{4.20}$$

As familiar k is negative for AdS, zero for Minkowski and positive for dS. Combining these facts, our integrability condition can now be expressed as

$$k + \nabla_m A\nabla^m A = 0 \ , \tag{4.21}$$

since $\gamma_{\mu\nu}$ is non-zero. The only possible constant value of $(\nabla A)^2$ on a compact manifold is zero, because the warp function A will have a minimum on a compact manifold where ∇A will vanish. As a result, the warp factor has to be constant and the external 4D space can only be Minkowski.

To study the internal component of the same supersymmetry variation, we split the supersymmetry spinors into 4D and 6D spinors. Without fluxes the gravitino variation is

$$\delta_\epsilon\psi_M^A = \nabla_M\epsilon^A = 0 \ . \tag{4.22}$$

In type IIA the gravitinos have opposite chiralities, and in type IIB the chiralities are the same. The type IIB spinor can therefore be decomposed as

$$\epsilon_{\text{IIB}}^A = \xi_+^A \otimes \eta_+ + \xi_-^A \otimes \eta_- \ , \tag{4.23}$$

where ξ^A is the 4D spinor and η the one in 6D. For type IIA the chirality makes the spinors decompose according to

$$\epsilon_{\text{IIA}}^1 = \xi_+^1 \otimes \eta_+ + \xi_-^1 \otimes \eta_- \ , \tag{4.24}$$

$$\epsilon_{\text{IIA}}^2 = \xi_+^2 \otimes \eta_- + \xi_-^2 \otimes \eta_+ \ , \tag{4.25}$$

where the spinors fulfil $\xi_-^A = (\xi_+^A)^*$ and $\eta_- = (\eta_+)^*$. Inserting these definitions into our gravitino variation in eq. (4.22), the condition for the internal component reads

$$\nabla_m\eta_\pm = 0 \ . \tag{4.26}$$

This means that the internal manifold has a covariantly constant spinor. This requirement forces the manifold to have a reduced geometry. Generally 6D manifolds have holonomy group $SU(3)$

or a subgroup of this, and are known as a Calabi-Yau manifolds, which admits one covariantly constant spinor.

In our decomposition of the supersymmetry spinors ϵ we have only used a single spinor for the internal manifold, but two for the external 4D space. The reason for this is that one internal spinor is the minimum requirement for decomposition, and gives the minimum amount of supersymmetries preserved. Having more than one covariant constant spinor the holonomy group of the manifold should be smaller than $SU(3)$, which results in a larger number of supersymmetries preserved. Thus assuming more internal spinors mean assuming a higher degree of supersymmetry. In the 4D external space we have two supersymmetry parameters ξ^1 and ξ^2, corresponding to eight conserved supercharges and $\mathcal{N} = 2$ supersymmetry.

In conclusion we have seen that the only solution of supersymmetric compactifications without fluxes, while requiring the external manifold to be maximally symmetric, is an external Minkowski space with a Calabi-Yau manifold as internal space. These compactifications preserve $\mathcal{N} = 2$ in 4D. The internal manifold should admit at least one supersymmetry parameter. In later sections we will see that fluxes can break the $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$, or break the supersymmetry completely, in a stable way.

4.2 Conditions on the internal manifold with fluxes present

In the absence of fluxes we have seen that supersymmetry requires a spinor to be covariantly constant on the internal manifold. This condition can be seen to be two-fold; first the very existence of a non-vanishing globally defined spinor, and second; that it is covariantly constant. The first condition can be viewed as a topological requirement on the internal manifold, while the second on the metric, or rather its connection, is a differential condition. We will analyse these conditions separately, starting with the first one.

4.2.1 Implications of the existence of a globally well-defined spinor

A spinor that is globally well-defined and non-vanishing exists only on a manifolds that have a *reduced structure*, i.e. a reduced structure group. Riemannian manifolds have structure group $SO(n)$, where n is the dimension of the manifold. All vectors, tensors and spinors on such a manifold can therefore be decomposed in representations of $SO(n)$.

Let us turn back to our 6D internal manifold. In 6D the structure group G is that of $SU(3)$. The spinor representation $\mathbf{4}$ of $SO(6)$ can be decomposed in representations of $SU(3)$ according to $\mathbf{4} \to \mathbf{3} + \mathbf{1}$. The fact that there is a $SU(3)$ singlet in the decomposition means that there is a spinor which depends trivially on the tangent bundle of the manifold, and is therefore well-defined and non-vanishing. Other $SO(6)$ representations such as the vector $\mathbf{6}$, the 2-form $\mathbf{15}$ and the 3-form $\mathbf{20}$ can be decomposed under $SU(3)$ according to

$$\mathbf{6} \to \mathbf{3} + \bar{\mathbf{3}} \ , \tag{4.27}$$

$$\mathbf{15} \to \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} \ , \tag{4.28}$$

$$\mathbf{20} \to \mathbf{6} + \bar{\mathbf{6}} + +\mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1} \ . \tag{4.29}$$

There are singlets in the decomposed 2-form and 3-form as well, so there is a globally well-defined real 2-form J and complex 3-form Ω. There are no singlets in the vector decomposition, so there are no invariant vectors. There is also no 5-form, so we know that the wedge product of J and Ω is zero. The 3-form Ω is also related to the volume form of J. Namely, the J and Ω fulfil the *compatibility constraints*

$$J \wedge \Omega = 0 \ , \qquad J \wedge J \wedge J = \tfrac{3i}{4} \Omega \wedge \bar{\Omega} \ . \tag{4.30}$$

By raising one of the indices of J an *almost complex structure* is obtained, i.e. a (1,1)-form fulfilling $J_m{}^p J_p{}^n = -\delta_m{}^n$. The eigenvalues of a real-valued matrix that squares to minus identity are $\pm i$. An almost complex structure allows us to introduce local holomorphic and antiholomorphic vectors $\partial_{z^i}, \partial_{z^{\bar{\imath}}}$, with $i = 1, 2, 3$, which act as the local eigenvectors with eigenvalues $+i$ and $-i$ respectively.

The integrability condition of the almost complex structure can be restated as the vanishing of the *Nijenhuis tensor*, i.e.

$$N_{mn}{}^p \equiv 2(J_m{}^q \nabla_{[q} J_{n]}{}^p - J_n{}^q \nabla_{[q} J_{m]}{}^p) = 0 \ . \tag{4.31}$$

The antisymmetrisation in the above equation actually allows for the covariant derivative to be replaced by an ordinary derivative. The structure group $SU(3)$ of our 6D manifold is determined by our invariant spinor η, or by J and Ω. J and Ω can be defined in terms of η as

$$J_{mn} = \mp 2i \eta_\pm^\dagger \gamma_{mn} \eta_\pm \ , \qquad \Omega_{mnp} = -2i \eta_-^\dagger \gamma_{mnp} \eta_+ \ . \tag{4.32}$$

where Ω_{mnp} is a (3,0)-form and J_{mn} a (1,1)-form with respect to the complex structure $J_m{}^n$.

4.2.2 Implications of a covariantly constant spinor

We proceed to the second condition stating that our globally well-defined spinor is covariantly constant and consider the implications of this. A manifold with $SU(3)$-structure with an $SU(3)$-invariant spinor that is also covariantly constant is the Calabi-Yau 3-fold. By 3-fold, we mean that the Calabi-Yau manifold has three complex dimensions, corresponding to six real dimensions. The Levi-Civita connection of the metric is said to have $SU(3)$ holonomy. The holonomy group of a connection consists of all possible changes of direction a vector makes when being parallel transported on a closed loop on some manifold. It is a subgroup of $O(n)$. The Levi-Civita connection is an affine connection (∇) that preserves the metric; $\nabla_m g_{np} = 0$ and is torsion-free;

$$[\nabla_m, \nabla_n] V_p = -R_{mnp}{}^q V_q - 2T_{mn}{}^q \nabla_q V_p \ , \tag{4.33}$$

where V_p is some vector and T_{mn} the torsion tensor. A connection fulfilling the first condition of $\nabla_m g_{np} = 0$, but that is not necessarily torsion-free, is sometimes referred to as a *metric compatible connection*.

In general on some manifold with $SU(3)$-structure, there is always a metric compatible connection which has $SU(3)$ holonomy and covariantly constant spinor $\nabla_m \eta = 0$. In the case when the connection is also torsion-free (Levi-Civita), the manifold is a Calabi-Yau. Therefore, there are other possible manifolds with torsion to fulfil our demands of $SU(3)$-structure with $\nabla_m \eta = 0$, which we ought to investigate. The torsion tensor is a natural starting point. It is in the following space:

$$T_{mn}{}^p \in \Lambda^1 \otimes \left(su(3) \oplus su(3)^\perp\right) \ , \tag{4.34}$$

where the index p spans the space of 1-forms Λ^1 and the indices m, n span the space of 2-forms. The space of 2-forms is isomorphic to, i.e. have the same form as, the Lie algebra $so(6)$ of the group $SO(6)$. However here we have used the decomposition $so(6) = so(3) \oplus so(3)^\perp$. As the torsion tensor is acting on $SU(3)$-invariant forms, the $su(3)$ piece may be dropped. The torsion tensor can now be called the *intrinsic torsion*, containing the following five representations:

$$\begin{aligned} T_{mn}{}^p \in \Lambda^1 \otimes su(3)^\perp &= (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}) \\ &= (\mathbf{1} \oplus \mathbf{1}) \oplus (\mathbf{8} \oplus \mathbf{8}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \\ &\equiv W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \ . \end{aligned} \tag{4.35}$$

The $W_1, \ldots, W_5$ are five *intrinsic torsion classes*, and appear in the covariant derivative of the spinor η, 2-form J and 3-form Ω. The W_4 and W_5 look the same but come from different spaces. In terms of forms, the torsion class

W_1 is a complex scalar,
W_2 is a complex primitive (1,1)-form,
W_3 is a real primitive (2,1)+(1,2)-form,
W_4 is a real vector,
W_5 is a complex (1,0)-form.

If some class W_i is primitive it means that $W_i \wedge J = 0$. These classes can be used rewrite $\mathrm{d}J$ and $\mathrm{d}\Omega$. In order to do so we first need to have a look at how the exterior derivative acts on a (p, q)-form $A_{p,q} \in X_{p,q}(M)$. It will decompose in form-spaces on the manifold according to

$$\mathrm{d}A_{p,q} \in X_{p+2,q-1}(M) \cup X_{p+1,q}(M) \cup X_{p,q+1}(M) \cup X_{p-1,q+2}(M) \ , \tag{4.36}$$

where $X_{p,q}(M)$ is the space of (p, q)-forms on the manifold M. We know that the exterior derivative acting on some p-form results in another $(p + 1)$-form, and so this decomposition is a natural extension of that. The exterior derivative of our fundamental $(1, 1)$-form J will thus decompose according to

$$\mathrm{d}J_{1,1} \in X_{3,0}(M) \cup X_{2,1}(M) \cup X_{1,2}(M) \cup X_{0,3}(M) \ . \tag{4.37}$$

The (3,0)-form and (0,3)-form parts of $\mathrm{d}J$ are described by W_1 and transforms in the $\mathbf{1} \oplus \mathbf{1}$ of $SU(3)$. The (2,1)-form transforms under $SU(3)$ according to

$$\mathbf{3} \oplus \mathbf{3} = (\mathbf{3} \oplus \mathbf{3})_S \oplus (\mathbf{3} \oplus \mathbf{3})_A = \mathbf{6} \oplus \bar{\mathbf{3}} \ , \tag{4.38}$$

and the (1,2)-form results in $\bar{\mathbf{6}} \oplus \mathbf{3}$ in the same way. In total, the (2,1)-form and (1,2)-form parts transform as $(\mathbf{3} \oplus \bar{\mathbf{3}}) \oplus (\mathbf{6} \oplus \bar{\mathbf{6}})$. Out of this, the $(\mathbf{3} \oplus \bar{\mathbf{3}})$ part is described by W_4 and $(\mathbf{6} \oplus \bar{\mathbf{6}})$ by W_3. To remove the $(\mathbf{3} \oplus \bar{\mathbf{3}})$ part W_3 must satisfy the *primitivity condition*

$$W_3 \wedge J = 0 \ . \tag{4.39}$$

Moving on to the fundamental $(3, 0)$-form Ω, the same method as for $\mathrm{d}J$ results in that $\mathrm{d}\Omega$ consists of a (4,0)-form, a (3,1)-form and a (2,2)-form. The first one is described by W_1 in the same way as the (3,0)-part of $\mathrm{d}J$, but the $(3, 1)$-form transforms as $\bar{\mathbf{3}}$ and is described by W_5. The $(2, 2)$-form transforms according to

$$\bar{\mathbf{3}} \oplus \mathbf{3} = \mathbf{8} + \mathbf{1} \ , \tag{4.40}$$

where the $\mathbf{8}$ is described by W_2, which is primitive: $W_2 \wedge J = 0$, and again W_1 describes $\mathbf{1}$. This may be summarised so that

$$\mathrm{d}J \in W_1 \oplus W_3 \oplus W_4 \ , \tag{4.41}$$

$$\mathrm{d}\Omega \in W_1 \oplus W_2 \oplus W_5 \ . \tag{4.42}$$

Knowing this, one can make the definition [1]

$$\mathrm{d}J = \tfrac{3}{2}\mathrm{Im}(\bar{W}_1\Omega) + W_4 \wedge J + W_3 \ , \tag{4.43}$$

$$\mathrm{d}\Omega = W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega \ . \tag{4.44}$$

The pieces in $\mathrm{d}\Omega$ that contain W_1 and W_2 are $(2,2)$-forms, while Ω itself is a $(3,0)$-form. On a complex manifold, the exterior derivative acting on a (p,q)-form results in $(p+1,q)$-form and $(p,q+1)$-form pieces, which result in $\mathrm{d}\Omega$ having $(4,0)$-form and $(3,1)$-form constituent. Thus it becomes clear that for the manifold to be complex the $(2,2)$-forms must vanish, i.e. we must have $W_1 = W_2 = 0$. Note that this condition is equivalent to requiring that the Nijenhuis tensor in eq. (4.31) vanishes, since we then are dealing with a complex manifold.

In a *symplectic* manifold the fundamental 2-form J is closed, i.e. $\mathrm{d}J = 0$. This results in that $W_1 = W_3 = W_4 = 0$ in accordance with eq. (4.43).

The Kähler manifold is both complex and symplectic, so it must have $W_1 = W_2 = W_3 = W_4 = 0$ and only $W_5 \neq 0$. The Calabi-Yau is a type of Kähler manifold with, as stated earlier, has vanishing torsion; $W_1 = W_2 = W_3 = W_4 = W_5 = 0$. These and some other manifolds are collected in table 4.1 as stated in [1].

Manifold	**Vanishing torsion classes**
Complex	$W_1 = W_2 = 0$
Symplectic	$W_1 = W_3 = W_4 = 0$
Half-flat	$\mathrm{Im}(W_1) = \mathrm{Im}(W_2) = W_4 = W_5 = 0$
Special Hermitian	$W_1 = W_2 = W_4 = W_5 = 0$
Nearly Kähler	$W_2 = W_3 = W_4 = W_5 = 0$
Almost Kähler	$W_1 = W_3 = W_4 = W_5 = 0$
Kähler	$W_1 = W_2 = W_3 = W_4 = 0$
Nearly Calabi-Yau	$W_1 = \mathrm{Im}(W_2) = W_3 = W_4 = W_5 = 0$
Calabi-Yau	$W_1 = W_2 = W_3 = W_4 = W_5 = 0$
Conformal Calabi-Yau	$W_1 = W_2 = W_3 = 0,\ 3W_4 - 2W_5 = 0$

Table 4.1: Vanishing torsion classes for some manifolds with $SU(3)$-structure.

4.3 Generalised complex geometry descriptions of the internal manifold

To attain a geometric description of the internal manifold, we begin by introducing the basic concepts of generalised complex geometry, which will be used in the later sections of this chapter. It will also provide us with a foundation for geometric descriptions in the extended formalism of exceptional generalised geometry, to be introduced in chapter 11.

4.3.1 Basic formalism of almost complex structures

This section is devoted to introduce the basic formulations and ideas of generalised complex geometry that are useful in the context of flux compactifications. Generalised complex geometry was originally formulated by Hitchin [16] and developed by his student Gualtieri [17].

In ordinary complex geometry one usually deals with the tangent and cotangent bundle of a manifold separately. The bundle of interest in generalised complex is the sum of these two, i.e. $TM \oplus T^*M$, which is called the *generalised tangent bundle*. The sections (see Appendix B) of the tangent bundle are vectors X, and the cotangent sections are 1-forms ξ. The generalised tangent bundle then have sections $\mathbb{X}$ consisting of a vector plus a 1-form $\mathbb{X} = X + \xi$.

The generalised almost complex structure $\mathcal{J}$ is the generalised complex geometry equivalent to the ordinary almost complex structure ${I_m}^n$. It is defined as a linear map from the generalised tangent bundle to itself, i.e.

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M \ , \tag{4.45}$$

and fulfils $\mathcal{J}^2 = -\mathbb{1}_{2d}$, where d is the dimension of the manifold. Note that the ordinary almost complex structure fulfils $I^2 = -\mathbb{1}_d$. Moreover, $\mathcal{J}$ respects the bundle structure

$$\pi(\mathcal{J}\mathbb{X}) = \pi(\mathbb{X}) \ , \tag{4.46}$$

where π is the projection operator and preserves the natural inner product of two generalised tangent vectors $\mathbb{X}$ and $\mathbb{Y} \equiv Y + \zeta$ according to

$$\langle \mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}\rangle = \langle \mathbb{X}, \mathbb{Y}\rangle = \mathcal{I} \ , \tag{4.47}$$

i.e. the metric is Hermitian. The inner product can be seen as the natural metric on the generalised bundle, and is given by

$$\mathcal{I} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ . \tag{4.48}$$

The condition for $\mathcal{J}$ then translates into $\mathcal{J}\mathcal{I}\mathcal{J} = \mathcal{I}$. A generalised almost complex structure has the form [18]

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix} \ , \tag{4.49}$$

where the constituents maps the tangent and cotangent bundle according to

$$\begin{aligned} J &: TM \to TM \ , \\ P &: T^*M \to TM \ , \\ L &: TM \to T^*M \ , \\ K &: T^*M \to T^*M \ . \end{aligned} \tag{4.50}$$

The Hermiticity condition $\mathcal{J}\mathcal{I}\mathcal{J} = \mathcal{I}$ leads to the constraints $K = -J^T$, $P = -P^T$ and $L = -L^T$, so that $\mathcal{J}$ now reads

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & -J^T \end{pmatrix} \ , \tag{4.51}$$

where P and L are antisymmetric matrices. The condition $\mathcal{J}^2 = -\mathbb{1}_{2d}$ imposes $J^2 + LP = -\mathbb{1}_d$. The ordinary almost complex structures are naturally embedded in the generalised ones, so we may construct

$$\mathcal{J}_1 \equiv \begin{pmatrix} I & 0 \\ 0 & -I^T \end{pmatrix} \ , \tag{4.52}$$

which fulfils $\mathcal{J}_1^2 = -\mathbb{1}_{2d}$ and $\mathcal{J}_1\mathcal{I}\mathcal{J}_1 = \mathcal{I}$. Another example of a generalised almost complex structure built with a non-degenerate 2-form ω_{mn} is

$$\mathcal{J}_2 \equiv \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \ . \tag{4.53}$$

The projection operators on an ordinary almost complex structure I is given by $\pi_\pm \equiv \frac{1}{2}(\mathbb{1}_d \pm iI)$, where the plus sign represents a holomorphic projection and the minus sign an antiholomorphic projection. For a generalised almost complex structure the definition of the projection operators follows rather naturally as $\Pi_\pm = \frac{1}{2}(\mathbb{1}_{2d} \pm i\mathcal{J})$. The integrability condition for an ordinary almost complex structure is the vanishing of the Nijenhuis tensor. The same condition can also be written in terms of the projection operators according to

$$\pi_{\mp}[\pi_{\pm}X, \pi_{\pm}Y]_{\mathrm{L}} = 0 \ , \tag{4.54}$$

where $[\cdot,\cdot]_{\mathrm{L}}$ is the Lie bracket. This bracket acting on some function f is defined as

$$[X, Y]_{\mathrm{L}} f \equiv X(Y(f)) - Y(X(f)) \ . \tag{4.55}$$

When X and Y are smooth vectors they can be seen as differential operators, so for a smooth function $f : M \to \mathbb{R}$ then Xf and Yf are again smooth functions that maps as $M \to \mathbb{R}$. The commutator of any two derivations is again a derivation, which is the case of the Lie bracket. The bracket is skew-symmetric $[X, Y]_{\mathrm{L}} = -[Y, X]_{\mathrm{L}}$ and satisfies the Jacobi identity. The Lie bracket can also be viewed in terms of flow. Namely, $[X, Y]_{\mathrm{L}}$ can be seen as the derivative of Y along the flow generated by X. The *Lie derivative* $\mathcal{L}$ is a generalisation that allows differentiation of any tensor field along the flow generated by X. In terms of this derivative the Lie bracket can be defined as $[X, Y]_{\mathrm{L}} = \mathcal{L}_X Y$.

For the generalised complex structure the integrability condition is defined as

$$\Pi_{\mp}[\Pi_{\pm}\mathbb{X}, \Pi_{\pm}\mathbb{Y}]_{\mathrm{C}} = 0 \ , \tag{4.56}$$

where the Lie bracket has been replaced by the *Courant bracket* $[\cdot,\cdot]_{\mathrm{C}}$. It is defined as

$$[\mathbb{X}, \mathbb{Y}]_{\mathrm{C}} \equiv [X + \xi, Y + \zeta]_{\mathrm{C}} \equiv [X, Y]_{\mathrm{L}} + \mathcal{L}_X\zeta - \mathcal{L}_Y\xi - \tfrac{1}{2}\,\mathrm{d}(\iota_X\zeta - \iota_Y\xi) \ , \tag{4.57}$$

where ι_X is the interior product, or interior derivative, which is defined as the contraction of a differential form with a vector field X. With X being a vector field on the manifold then $\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$. From its antisymmetric properties it follows that $\iota_X\iota_Y A = -\iota_Y\iota_X A$. It is also nilpotent; $\iota_X^2 = 0$, just as the exterior derivative d. Introducing the notation $\lrcorner$, a p-form A can contract a $(p+n)$-form B into a n-form such that

$$(A \lrcorner B)_{i_1,\ldots,i_n} = \tfrac{1}{p!} A^{j_1,\ldots,j_p} B_{j_1,\ldots,j_p,i_1,\ldots,i_n} \ ,$$

and the interior product is then written $\iota_X\zeta = X \lrcorner \zeta$. The interior product is related to the exterior derivative and the Lie derivative via $\mathcal{L}_X A = \mathrm{d}(\iota_X A) + \iota_X\,\mathrm{d}A$.

Returning to the examples of $\mathcal{J}_1$ and $\mathcal{J}_2$, the integrability conditions on $\mathcal{J}_1$ and $\mathcal{J}_2$ impose the corresponding integrability conditions on its building blocks, i.e. the ordinary almost complex structures I and ω. That is, integrability of $\mathcal{J}_1$ enforces I to be an integrable almost complex structure on TM which is equivalent to I being a complex structure, and thus the manifold is complex. For $\mathcal{J}_2$, integrability implies that $\mathrm{d}\omega = 0$, making ω a closed almost complex structure. These are the conditions put on symplectic forms, so the 2-form ω and its corresponding manifold must be symplectic.

These are just examples. More general types of generalised almost complex structures are partially complex and partially symplectic. It is worth noting that the introduction of the generalised tangent bundle and the exchange of the Lie bracket to the Courant bracket are perhaps the two most fundamental points of generalised complex geometry.

4.3.2 Conditions on the defining $SU(3)$ forms as pure spinors

There is an algebraic one-to-one correspondence between generalised almost complex structures and Clifford(6,6) pure spinors. Spinors on the generalised tangent bundle transform under Clifford(6), which has algebra $\{\gamma^m, \gamma^n\} = 2g^{mn}$. There is also a representation of this algebra in terms of forms with $\gamma^m \equiv \mathrm{d}x^m \wedge + g^{mn}\iota_n$, which satisfies a Clifford(d) algebra. The algebra for Clifford(d,d) is in turn given by

$$\{\Gamma^m, \Gamma^n\} = 0 \ , \qquad \{\Gamma_m, \Gamma_n\} = 0 \ , \qquad \{\Gamma^m, \Gamma_n\} = \mathcal{I}^m{}_n \ , \tag{4.58}$$

where the Γ^m and Γ_m are completely independent of each other, i.e. the index cannot be raised or lowered with a metric. Therefore the number of gamma matrices is twice the dimension of the manifold, which in our case is twelve. $\mathcal{I}^m{}_n$ is the $(6+6)$-dimensional natural metric of the generalised bundle defined in eq. (4.48). The representation also exists in terms of forms, given by

$$\Gamma^m = \mathrm{d}x^m \wedge \ , \qquad \Gamma_m = \iota_m \ . \tag{4.59}$$

A *pure* spinor is a spinor that is annihilated by a space of half the dimension of the algebra it lives in. Specifically a pure Clifford(6,6) spinor is one who is annihilated by two 6D gamma matrices Γ^m, Γ_n (or linear combinations of them) out of a total of four 6D gamma matrices $\Gamma^m, \Gamma_m, \Gamma^n, \Gamma_n$ which build the Clifford(6,6) algebra. On a manifold with $SU(3)$-structure there are two natural pure spinors. The first on is the 3-form Ω, which is annihilated by Γ^i and $\Gamma_{\bar{\imath}}$ according to

$$\Gamma^i \Omega = \mathrm{d}z^i \wedge (\Omega_{jkl}\, \mathrm{d}z^j \wedge \mathrm{d}z^k \wedge \mathrm{d}z^l) = 0 \ ,$$

since the index i has to be either j, k or l and $\mathrm{d}z^i \wedge \mathrm{d}z^i = 0$ for any choice of i since the wedge product is antisymmetric. For the $\Gamma_{\bar{\imath}}$ we have

$$\Gamma_{\bar{\imath}} \Omega = \iota_{\bar{\imath}} (\Omega_{jkl}\, \mathrm{d}z^j \wedge \mathrm{d}z^k \wedge \mathrm{d}z^l) = 0 \ ,$$

where $\iota_j\, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} = p\delta_j^{[i_1}\, \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_p]}$. As for the other gamma matrices, we have

$$\Gamma_i \Omega = \iota_i (\Omega_{jkl}\, \mathrm{d}z^j \wedge \mathrm{d}z^k \wedge \mathrm{d}z^l) = 3\Omega_{ikl} \wedge \mathrm{d}z^k \wedge \mathrm{d}z^l \ , \tag{4.60}$$

$$\Gamma^{\bar{\imath}} \Omega = \mathrm{d}z^{\bar{\imath}} \wedge (\Omega_{jkl}\, \mathrm{d}z^j \wedge \mathrm{d}z^k \wedge \mathrm{d}z^l) = \Omega_{jkl}\, \mathrm{d}z^{\bar{\imath}} \wedge \mathrm{d}z^j \wedge \mathrm{d}z^k \wedge \mathrm{d}z^l \ , \tag{4.61}$$

where wee see that $\Gamma_i \Omega$ in eq. (4.60) have no restrictions on the holomorphic indices and could be any 2-form. The same applies for the $\Gamma^{\bar{\imath}}\Omega$ in eq. (4.61) which is some $(3,1)$-form. Thus we have seen that Ω is indeed annihilated by half of the gamma matrices and is thus a pure Clifford(6,6) spinor. Acting with different combinations of the creation operators Γ_i and $\Gamma^{\bar{\imath}}$ on Ω, it is clear that the result can be forms of all possible degrees. Therefore we may say that Clifford(6,6) spinors are equivalent to (p,q)-forms.

As for the second pure spinor on a $SU(3)$-structure manifold, one might guess it to be the other invariant 2-form J, however it is actually given by

$$e^{iJ} \equiv 1 + iJ - \tfrac{1}{2} J \wedge J - \tfrac{i}{6} J \wedge J \wedge J \ . \tag{4.62}$$

It is annihilated by the linear combination $\Gamma_m - iJ_{mn}\Gamma^n = \iota_m - iJ_{mn}\, \mathrm{d}x^n \wedge$, which can be seen, term by term, from

$$\begin{aligned}
(\iota_m - iJ_{mn}\, \mathrm{d}x^n \wedge) 1 &= -iJ_{mn}\, \mathrm{d}x^n \\
i(\iota_m - iJ_{mn}\, \mathrm{d}x^n \wedge) J &= iJ_{mn}\, \mathrm{d}x^n + J_{mn}\, \mathrm{d}x^n \wedge J \\
-\tfrac{1}{2}(\iota_m - iJ_{mn}\, \mathrm{d}x^n \wedge) J \wedge J &= -\tfrac{1}{2} 2J_{mn}\, \mathrm{d}x^n \wedge J + \tfrac{i}{2} J_{mn}\, \mathrm{d}x^n \wedge J \wedge J \\
-\tfrac{i}{6}(\iota_m - iJ_{mn}\, \mathrm{d}x^n \wedge) J \wedge J \wedge J &= -\tfrac{i}{6} 3J_{mn}\, \mathrm{d}x^n \wedge J \wedge J
\end{aligned}$$

where we see that the first term cancels against the first term on the second line, and so on.

The Clifford map can map a Clifford(6,6) spinor (form) to a bispinor, i.e. a spinor that consists of two other spinors, via

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \quad \longleftrightarrow \quad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k}_{\alpha\beta} \ . \tag{4.63}$$

As we have established in previous sections, a space with $SU(3)$-structure has a nowhere vanishing $SU(3)$-invariant spinor η. This spinor is also a Clifford(6) spinor as it is annihilated by γ^i and $\gamma_{\bar{i}}$ in a similar fashion to the Clifford(6,6) spinors. From η we may construct two $SU(3,3)$-invariant bispinors, by tensoring η with its dagger. That is, we may construct two bispinors

$$\Phi_\pm \equiv \eta_+ \otimes \eta_\pm^\dagger \ . \tag{4.64}$$

This tensor product can be written, using Fierz identities, as

$$\Phi_\pm \equiv \eta_+ \otimes \eta_\pm^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_\pm^\dagger \gamma_{i_1 \ldots i_k} \eta_+ \gamma^{i_k \ldots i_1} \ . \tag{4.65}$$

Now, using the Clifford map in eq. (4.63) backwards, the bispinors can be identified with regular forms. The subscripts plus/minus in $\Phi_\pm$ denote the $Spin(6,6)$ chirality; plus denote an even form, and minus an odd form. The Clifford(6,6) spinors in terms of the fundamental forms defining the $SU(3)$-structure can be obtained using the expression of J and Ω in terms of η according to eq. (4.32), and combining it with the Fierz arrangement in eq. (4.65). This results in

$$\Phi_+ = \eta_+ \otimes \eta_+^\dagger = \tfrac{1}{8} e^{-iJ} \ , \tag{4.66}$$

$$\Phi_- = \eta_+ \otimes \eta_-^\dagger = -\tfrac{i}{8} \Omega \ . \tag{4.67}$$

It it clear that these spinors are pure and annihilated by $\Gamma_m - iJ_{mn}\Gamma^n$ as there is only a sign difference in the exponent from the pure spinor e^{iJ} in eq. (4.62). The one-to-one correspondence between a pure spinor and a generalised almost complex structure stated earlier, is given as

$$\Phi_+ = \tfrac{1}{8} e^{-iJ} \quad \leftrightarrow \quad \mathcal{J}_2 \ , \tag{4.68}$$

$$\Phi_- = -\tfrac{i}{8} \Omega \quad \leftrightarrow \quad \mathcal{J}_1 \ , \tag{4.69}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ was defined in eq. (4.52) and eq. (4.53) respectively.

Imposing the integrability condition on the generalised complex structure $\mathcal{J}$ corresponds for the spinor Φ that there exists a vector v and 1-form ξ such that $\mathrm{d}\Phi = (v\llcorner + \xi\wedge)\Phi$. A manifold that fulfils this condition is the *generalised Calabi-Yau*, as it has a closed pure spinor, i.e. $\exists\Phi$ pure such that $\mathrm{d}\Phi = 0$. Unlike an ordinary Calabi-Yau the generalised Calabi-Yau need not have vanishing torsion. For example it may be a complex (or symplectic) manifold with a trivial W_5 torsion class, say $W_5 = \bar{\partial} f$, then $\Phi = e^{-f}\Omega$ and $\mathrm{d}\Phi = 0$.

A closed 3-form flux H may "twist" the generalised Calabi-Yau. Adding such a form modifies the Courant bracket by adding a term $\iota_X \iota_Y H$ to the normal Courant bracket, i.e. so that it is now defined as

$$[X + \xi, Y + \zeta]_H \equiv [X, Y]_{\mathrm{C}} + \iota_X \iota_Y H \ . \tag{4.70}$$

This in turn changes the integrability condition so that $\mathrm{d}\Phi \to (\mathrm{d} - H\wedge)\Phi$, i.e. the spinor integrability condition now reads

$$(\mathrm{d} - H\wedge)\Phi = (v\llcorner + \xi\wedge)\Phi \ . \tag{4.71}$$

A *twisted generalised Calabi-Yau* has a pure spinor such that it fulfils

$$(\mathrm{d} - H\wedge)\Phi = 0 \ , \tag{4.72}$$

and hence H can be seen as to have "twisted" the integrability condition. Would one decompose the pure spinor into forms according to $\Phi = \sum_k \phi_k$, then this twisted generalised Calabi-Yau condition would read $\mathrm{d}\phi_k - H_3 \wedge \phi_{k-2}$ for all k. Note that a twisted exterior derivative condition such as this one appeared in the definition of the democratic RR fields in eq. (3.114) and in their corresponding Bianchi identity in eq. (3.116).

4.4 Consequences of imposing $\mathcal{N} = 1$ on the background

In this section we will collapse the $\mathcal{N} = 2$ supersymmetry of our Minkowski background to $\mathcal{N} = 1$, and see what consequences that follow. Our two 4D spinors $\xi^{1,2}$ in the spinor decomposition of eqs. (4.23)–(4.25) are the ones preserving the $\mathcal{N} = 2$ supersymmetry in the 4D external space. In order to have $\mathcal{N} = 1$ we need a relation between ξ^1 and ξ^2, a relation which maximal symmetry demands to be trivial, i.e. to be proportionality. We may therefore decompose the spinors according to

$$\begin{aligned} \xi^1_+ &\to a\xi_+ \ , \qquad & \xi^2_+ &\to b\xi_+ \ , \\ \xi^1_- &\to \bar{a}\xi_- \ , \qquad & \xi^2_- &\to \bar{b}\xi_- \ , \end{aligned} \tag{4.73}$$

where a and b are some complex functions of the internal space. This means that instead of our supersymmetry spinors $\epsilon^{1,2}$ consisting of two different 4D spinors $\xi^{1,2}$ (with chiralities $\pm$), they now consist of a single spinor ξ and some complex functions a and b. Hence, for the different type II theories, the supersymmetry spinors $\epsilon^{1,2}$ now look like

$$\begin{aligned} \epsilon^1_{\mathrm{IIA}} &= \xi_+ \otimes (a\eta_+) + \xi_- \otimes (\bar{a}\eta_-) \ , \qquad & \epsilon^1_{\mathrm{IIB}} &= \xi_+ \otimes (a\eta_+) + \xi_- \otimes (\bar{a}\eta_-) \ , \\ \epsilon^2_{\mathrm{IIA}} &= \xi_+ \otimes (\bar{b}\eta_-) + \xi_- \otimes (b\eta_+) \ , \qquad & \epsilon^2_{\mathrm{IIB}} &= \xi_+ \otimes (b\eta_+) + \xi_- \otimes (\bar{b}\eta_-) \ . \end{aligned} \tag{4.74}$$

In this section we will use these $\mathcal{N} = 1$ spinor decompositions in the type II supersymmetry variations, and relate them to torsion via the NSNS flux, from which one obtains conditions on the internal manifold.

4.4.1 Supersymmetry equations in terms of pure spinors

Using the decompositions of the supersymmetry spinors in eq. (4.74), the gravitino and dilatino supersymmetry conditions of eq. (4.2) and eq. (4.3) may be expressed in terms of 4D and 6D spinors separately. In the next our interest lies in the $SU(3)$-structure internal manifold, which admits the spinor η, and so we may neglect the 4D ζ spinor part. The procedure of rewriting the supersymmetry conditions as differential conditions on pure spinors in the context of flux backgrounds was first done in [19], whose results are reproduced in this section 4.4. In section 4.4.1 the necessary constraints on the background are obtained and are then solved in the next section 4.4.2.

We begin by turning to the supersymmetry transformations, where the gravitino variation for type IIA reads

$$\delta_\epsilon \psi_M = \left(D_M + \tfrac{1}{4} \not{H}_M \Gamma_{11} + \tfrac{1}{16} e^{\phi} \sum_{n=0}^{5} \not{F}_{2n} (\Gamma_{11})^n \sigma^1 \Gamma_M \right) \epsilon = 0 \ , \tag{4.75}$$

and the two supersymmetry parameters have different chirality according to

$$\Gamma_{11}\epsilon_1 = \epsilon_1 \ , \qquad \Gamma_{11}\epsilon_2 = -\epsilon_2 \ . \tag{4.76}$$

The fluxes in eq. (4.75) can be chosen to be purely internal as $F_{2n}^{(10)} = \tilde{F}_{2n} + \mathrm{vol}_4 \hat{F}_{2n-4}$ by 4D Poincaré invariance and $\hat{F}_{2n-4} = (-1)^{\lfloor n \rfloor} \star_6 \tilde{F}_{10-2n}$ by self-duality. The tilde denoting the internal fluxes may then be dropped. The external type IIA gravitino variation is then reduced to

$$D_\mu \epsilon + \tfrac{1}{8} e^\phi \left(\not{F}_0 + \not{F}_2 \Gamma_{11} + \not{F}_4 \Gamma_{11}^2 \right) \sigma^1 \Gamma_\mu \epsilon = 0 \ , \tag{4.77}$$

where we take $H_\mu = 0$ and use the self-duality condition of the fluxes. The Pauli matrices effect on the supersymmetry parameter via the $\mathcal{P}$s in eqs. (4.2) and (4.3) are summarised here for clarity as

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \ , \qquad \sigma^1 \epsilon = \begin{pmatrix} \epsilon_2 \\ \epsilon_1 \end{pmatrix} \ , \qquad i\sigma^2 \epsilon = \begin{pmatrix} \epsilon_2 \\ -\epsilon_1 \end{pmatrix} \ , \qquad -\sigma^3 \epsilon = \begin{pmatrix} -\epsilon_1 \\ \epsilon_2 \end{pmatrix} \ . \tag{4.78}$$

where again σ^1 appear in type IIA and the other two in the IIB theory. Inserting these in the gravitino variation of eq. (4.2), the type IIA variations read

$$(D_M + \tfrac{1}{4} \not{H}_M) \epsilon_1 + \tfrac{1}{8} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4) \Gamma_M \epsilon_2 = 0 \ , \tag{4.79}$$

$$(D_M - \tfrac{1}{4} \not{H}_M) \epsilon_2 + \tfrac{1}{8} e^\phi (\not{F}_0 + \not{F}_2 + \not{F}_4) \Gamma_M \epsilon_1 = 0 \ . \tag{4.80}$$

Next we will consider the external and internal space variations separately. Starting with the external, relevant expressions are $D_\mu \epsilon = [\hat{\nabla}_\mu \otimes \mathbb{1} + \frac{1}{2}(\gamma_\mu \gamma_5 \otimes \not{\partial} A)]\epsilon$ as in eq. (4.16) and $\Gamma_\mu = \gamma_\mu \otimes \mathbb{1}$. Using the spinor decompositions in eq. (4.74) and considering the external space to be Minkowski, then $\hat{\nabla}_\mu \epsilon = 0$ and eq. (4.79) is divided into two equations according to

$$\begin{aligned} 0 &= \tfrac{1}{2} \gamma_\mu \gamma_5 \zeta_+ \otimes \not{\partial} A a \eta_+ + \tfrac{1}{2} \gamma_\mu \gamma_5 \zeta_- \otimes \not{\partial} A \bar{a} \eta_- + \tfrac{1}{8} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4) \gamma_\mu \zeta_+ \otimes \bar{b} \eta_- \\ &\quad + \tfrac{1}{8} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4) \gamma_\mu \zeta_- \otimes b \eta_+ \ , \\ 0 &= \gamma_\mu \zeta_+ \otimes [\tfrac{1}{2} \not{\partial} A a \eta_+ + \tfrac{1}{8} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4) \bar{b} \eta_-] \\ &\quad + \gamma_\mu \zeta_- \otimes [-\tfrac{1}{2} \not{\partial} A \bar{a} \eta_- + \tfrac{1}{8} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4)] b \eta_+ \ , \end{aligned} \tag{4.81}$$

using $\gamma_5 \zeta_+ = \zeta_+$ and $\gamma_5 \zeta_- = -\zeta_-$. We see that the above equation can be written as a sum of two terms that are in the product space of $\gamma_\mu \zeta_+$ respectively $\gamma_\mu \zeta_-$. These must vanish separately as the 4D spinor ζ is arbitrary, i.e. we must have that

$$\not{\partial} A a \eta_+ + \tfrac{1}{4} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4) \bar{b} \eta_- = 0 \ , \tag{4.82}$$

$$\not{\partial} A \bar{a} \eta_- - \tfrac{1}{4} e^\phi (\not{F}_0 - \not{F}_2 + \not{F}_4) b \eta_+ = 0 \ . \tag{4.83}$$

In the exact same manner the ϵ_2-variation of eq. (4.80) results in the two equations

$$\not{\partial} A \bar{b} \eta_- + \tfrac{1}{4} e^\phi (\not{F}_0 + \not{F}_2 + \not{F}_4) a \eta_+ = 0 \ , \tag{4.84}$$

$$\not{\partial} A b \eta_+ - \tfrac{1}{4} e^\phi (\not{F}_0 + \not{F}_2 + \not{F}_4) \bar{a} \eta_- = 0 \ . \tag{4.85}$$

By making a linear combination of eq. (4.82) and eq. (4.85) we have that

$$(4.82) + i \times (4.85) = \alpha \not{\partial} A \eta_+ + \tfrac{i}{4} e^\phi \not{F}_{\mathrm{A1}} \eta_- = 0 \ , \tag{4.86}$$

where $\alpha \equiv a + ib$, $\beta \equiv a - ib$, and the flux is given by

$$\begin{aligned} -F_{\mathrm{A1}} &\equiv (\bar{a} + i\bar{b}) F_0 + (\bar{a} - i\bar{b}) F_2 + (\bar{a} + i\bar{b}) F_4 \\ &\equiv \beta^* F_0 + \alpha^* F_2 + \beta^* F_4 \ . \end{aligned} \tag{4.87}$$

The linear combination (4.82) $- i \times$ (4.85) corresponds to making the exchange $\alpha \to \beta$ and $F_{\mathrm{A}1} \to F_{\mathrm{A}2}$ in eq. (4.86), where

$$F_{\mathrm{A}2} \equiv \alpha^* F_0 + \beta^* F_2 + \alpha^* F_4 \ . \tag{4.88}$$

Eq. (4.86) is the IIA external gravitino variation we will use in later analysis. Continuing to the supersymmetry variation of the internal space, we have that $\Gamma_m = \gamma_5 \otimes \gamma_m$ and eqs. (4.79), (4.80) become respectively

$$\begin{aligned}
\epsilon_1 :& \begin{cases} D_m(a\eta_+) + \frac{1}{4}\not{H}_m a\eta_+ + \frac{1}{8}e^{\phi}(\not{F}_0 - \not{F}_2 + \not{F}_4)\gamma_m \bar{b}\eta_- = 0 \ , \\ D_m(\bar{a}\eta_-) + \frac{1}{4}\not{H}_m \bar{a}\eta_- - \frac{1}{8}e^{\phi}(\not{F}_0 - \not{F}_2 + \not{F}_4)\gamma_m b\eta_+ = 0 \ , \end{cases} \\
\epsilon_2 :& \begin{cases} D_m(\bar{b}\eta_-) - \frac{1}{4}\not{H}_m \bar{b}\eta_- + \frac{1}{8}e^{\phi}(\not{F}_0 + \not{F}_2 + \not{F}_4)\gamma_m a\eta_+ = 0 \ , \\ D_m(b\eta_+) - \frac{1}{4}\not{H}_m \bar{a}\eta_- - \frac{1}{8}e^{\phi}(\not{F}_0 + \not{F}_2 + \not{F}_4)\gamma_m \bar{a}\eta_- = 0 \ . \end{cases}
\end{aligned} \tag{4.89}$$

Like the external gravitino variations, adding the first ϵ_1-equation in eq. (4.89) and the last ϵ_2-equation multiplied with i, we get for the internal supersymmetry variation that

$$(\alpha D_m + \partial\alpha - \tfrac{1}{4}\beta\not{H}_m)\eta_+ + \tfrac{i}{8}e^{\phi}\not{F}_{\mathrm{A}1}\gamma_m\eta_- = 0 \ . \tag{4.90}$$

This one has the equivalent symmetry $\alpha \leftrightarrow \beta$ and $F_{\mathrm{A}1} \leftrightarrow F_{\mathrm{A}2}$ as the external variation in eq. (4.86).

Further, it will be useful to combine the gravitino and dilatino variations eqs. (4.2), (4.3) so that the RR flux terms cancel. This is achieved by the combination

$$\Gamma_M \delta\psi_M - \delta\lambda = \left(\not{D} - \not{\partial}\phi + \tfrac{1}{4}\not{H}\mathcal{P}\right)\epsilon = 0 \ , \tag{4.91}$$

which we refer to as the modified dilatino equation. Again for IIA we have $\mathcal{P} = \Gamma_{11}$, so the two equations for each ϵ become

$$\begin{aligned}
\epsilon_1 :& \begin{cases} (\not{D} - \not{\partial}\phi + \frac{1}{4}\not{H})a\eta_+ = 0 \ , \\ (\not{D} - \not{\partial}\phi + \frac{1}{4}\not{H})\bar{a}\eta_- = 0 \ , \end{cases} \\
\epsilon_2 :& \begin{cases} (\not{D} - \not{\partial}\phi - \frac{1}{4}\not{H})\bar{b}\eta_- = 0 \ , \\ (\not{D} - \not{\partial}\phi - \frac{1}{4}\not{H})b\eta_+ = 0 \ . \end{cases}
\end{aligned} \tag{4.92}$$

The second ϵ_1-equation is the complex conjugate of the first one and the first ϵ_2 equation is the complex conjugate of the last one. Hence we need only consider the first and last equation in eq. (4.92). Combining them in the usual manner give us

$$(\alpha\not{D} + \alpha\partial(2A - \phi + \ln\alpha) + \tfrac{1}{4}\beta\not{H})\eta_+ = 0 \ , \tag{4.93}$$

where we have expanded the derivative and used that $\partial\alpha = \alpha \ln\alpha$.

We are now ready to proceed to the IIB case. The gravitino variations reads

$$\begin{aligned}
(D_M - \tfrac{1}{4}\not{H}_M)\epsilon_1 + \tfrac{1}{8}e^{\phi}(\not{F}_1 + \not{F}_3 + \not{F}_5)\Gamma_M\epsilon_2 = 0 \ , \\
(D_M + \tfrac{1}{4}\not{H}_M)\epsilon_2 + \tfrac{1}{8}e^{\phi}(-\not{F}_1 + \not{F}_3 - \not{F}_5)\Gamma_M\epsilon_1 = 0 \ ,
\end{aligned} \tag{4.94}$$

where the spinors are given in eq. (4.74). The analysis is equivalent to the IIA case so we only list the results as

$$\begin{aligned}
(\alpha\not{\partial}A + \tfrac{i}{4}e^{\phi}F_{\mathrm{B}1})\eta_+ = 0 \ , \\
(\alpha D_m + \partial\alpha - \tfrac{1}{4}\beta\not{H}_m)\eta_+ - \tfrac{i}{8}e^{\phi}F_{\mathrm{B}1}\gamma_m\eta_+ = 0 \ , \\
(\alpha\not{D} + \alpha\not{\partial}(2A - \phi + \ln\alpha) - \tfrac{1}{4}\beta\not{H})\eta_+ = 0 \ ,
\end{aligned} \tag{4.95}$$

the first equation being the one coming from the external variation, the second from the internal variation, and the third one is the modified dilatino equation. The combined fluxes are given by

$$F_{\mathrm{B1}} \equiv \alpha F_1 - \beta F_3 + \alpha F_5 \ . \tag{4.96}$$

As in the IIA case, another set of equations can be found by the exchange $\alpha \leftrightarrow \beta$ and $F_{\mathrm{B1}} \leftrightarrow F_{\mathrm{B2}}$ where $-F_{\mathrm{B2}} \equiv \beta F_1 - \alpha F_3 + \beta F_5$.

The equations can be written in a basis $\eta_\pm, \gamma\eta_\pm, \gamma^m\eta_\pm$ since anything else in the Clifford algebra acting on $\eta_\pm$, say $\gamma^{m_1 \dots m_n}$, can be re-expressed in this basis. However, since $\eta_\pm$ is a Clifford vacuum, we have that $\gamma^i\eta_+ = \gamma^{\bar{\imath}}\eta_- = 0$, so we are left with $\eta_\pm$ and $\gamma^m\eta_\pm$. Out of these the pair $\{\eta_+, \gamma^m\eta_-\}$ have positive chirality and $\{\gamma^m\eta_+, \eta_-\}$ have negative chirality. In these bases, the supersymmetry conditions will then take the form

$$\delta\Psi_\mu = S\eta_- + (S_m + A_m)\gamma^m\eta_+ = 0 \ , \tag{4.97}$$
$$\delta\Psi_m = i(Q_m + R_m)\eta_+ + i(Q_{mn} + R_{mn})\gamma^n\eta_- = 0 \ , \tag{4.98}$$
$$\Gamma^M\delta\psi_M - \delta\lambda = T\eta_- + T_m\gamma^m\eta_+ = 0 \ , \tag{4.99}$$

where the Ψ denotes the appropriate linear combination of supersymmetry variations of each theory, e.g. $\delta\Psi_\mu = (\delta_{\epsilon_1} + i\delta_{\epsilon_2^*})\psi_\mu$ in IIA, etc.

Next, we are to derive the expression of each term in eqs. (4.97)–(4.99) in terms of the two pure spinors e^{iJ} and Ω. To do so we start by using the Fierz rearrangement in eq. (4.65), restated again as

$$\eta_\pm \otimes \eta_+^\dagger = \frac{1}{4}\sum_{k=0}^{6} \frac{i}{k!}\eta_+^\dagger\gamma_{i_1\dots i_k}\eta_\pm\gamma^{i_k\dots i_1} \ . \tag{4.100}$$

Using this, pure spinors can be constructed from tensor products of the standard $SU(3)$ admitted spinor η, so that

$$\begin{aligned} \eta_\pm \otimes \eta_\pm^\dagger &= \frac{1}{8}\not{e}^{\mp i\not{J}} \ , \\ \eta_+ \otimes \eta_-^\dagger &= -\frac{i}{8}\not{\Omega} \ , \\ \eta_- \otimes \eta_+^\dagger &= -\frac{i}{8}\not{\bar{\Omega}} \ , \end{aligned} \tag{4.101}$$

where the slash denotes the spinor equivalent rather than form in the Clifford map of eq. (4.63). We begin our analysis with the type IIA case, where we wish to rewrite eqs. (4.86), (4.90) and (4.93) on their respective form in eqs. (4.97)–(4.99). Starting with the external gravitino variation in eq. (4.86), the goal is to find S_{A1}, $S_{\mathrm{A}m}$ and A_m such that

$$\alpha\not{\partial}A\eta_+ + \tfrac{i}{4}e^\phi\not{F}_{\mathrm{A1}}\eta_- = S_{\mathrm{A1}}\eta_- + (S_{\mathrm{A}m} + A_m)\gamma^m\eta_+ \ . \tag{4.102}$$

These can be obtained by multiplying the gravitino variation with $\eta_\pm^\dagger$ and $\eta_\pm^\dagger\gamma_n$ from the left. For instance, multiplying by $\eta_-^\dagger$ from the left on both sides of eq. (4.102), the first term on the left-hand side and the second one on the right-hand side both vanish. With normalisation $\eta_\pm^\dagger\eta_\pm = \frac{1}{2}$, we are left with $\frac{1}{2}S_{\mathrm{A1}}$ on the right-hand side. The left-hand side is evaluated to

$$\eta_-^\dagger\not{F}_{\mathrm{A1}}\eta_- = \mathrm{tr}(\eta_-^\dagger\not{F}_{\mathrm{A1}}\eta_-) = \tfrac{1}{8}\mathrm{tr}(\not{F}_{\mathrm{A1}}\not{e}^{i\not{J}}) = \tfrac{1}{2}(\not{F}_{\mathrm{A1}}\not{e}^{i\not{J}})_0 \ , \tag{4.103}$$

where in the first equality a trace was inserted since the left-hand side is a constant and in the second equality we have inserted the first identity from eq. (4.101). In the last equality we used that the trace of products consisting of antisymmetric gamma matrices vanish, and so all products of gamma matrices in $\not{F}_{\mathrm{A1}}\not{e}^{i\not{J}}$ vanish, leaving only a single product that is proportional to identity rather than a gamma matrix. Higher dimensional gamma matrices are $N\times N$ matrices where $N=2^{\lfloor 2/n\rfloor}=8$ in $n=6$ dimensions, but since the spinors η are chiral the associated gamma matrices reduce to 4×4 matrices. Hence we get a factor $\mathrm{tr}(\mathbb{1})=4$, and the subscript $(\ldots)_0$ denotes the term in the Clifford product that does not contain any gamma matrices. In conclusion we have determined

$$S_{\mathrm{A}1}=\tfrac{i}{4}e^{\phi}(\not{F}_{\mathrm{A}1}\not{e}^{i\not{J}})_0\ , \tag{4.104}$$

where $\not{F}_{\mathrm{A}1}$ and $\not{e}^{i\not{J}}$ are contracted by the 6D metric.

In order to determine $S_{\mathrm{A}1m}$ and A_m, we return to eq. (4.102) but this time we multiply with $\eta_+^\dagger\gamma_p$ from the left on both sides. This will make the term containing $S_{\mathrm{A}1}$ vanish, and the RR flux term can be evaluated as in the previous case, where now

$$\eta_+^\dagger\gamma_p=\mathrm{tr}(\eta_+^\dagger\gamma_p\not{F}_{\mathrm{A}1}\eta_-)=-\tfrac{i}{8}\mathrm{tr}(\bar{\not{\Omega}}\gamma_p\not{F}_{\mathrm{A}1})=-\tfrac{i}{2}(\not{F}_{\mathrm{A}1}\bar{\not{\Omega}})_p\ , \tag{4.105}$$

where we used eq. (4.101) and the $(\ldots)_p$ denotes the term that gets multiplied with γ^p in the gamma matrix products. The right-hand side with $S_{\mathrm{A}1m}$ should be thought of as $\bar{P}_p{}^mS_m$ which projects onto antiholomorphic coordinates, where the projector is defined as

$$\bar{P}_p{}^m\equiv\tfrac{1}{2}(\delta_p{}^m+iJ_p{}^m)\ . \tag{4.106}$$

Gathering all the expressions, the type IIA spacetime gravitino variation of eq. (4.86) now reads

$$\tfrac{i}{4}e^{\phi}(\not{F}_{\mathrm{A}1}\not{e}^{i\not{J}})_0\eta_-+\left(\tfrac{1}{8}e^{\phi}(\not{F}_{\mathrm{A}1}\bar{\not{\Omega}})_m+\alpha\partial_mA\right)\gamma^m\eta_+=0\ , \tag{4.107}$$

form which one may read off the expressions of $S_{\mathrm{A}1}, S_{\mathrm{A}1m}$ and A_m. There is also another equation like this one but with exchanges $\alpha\to\beta$ and $F_{\mathrm{A}1}\to F_{\mathrm{A}2}$.

To rewrite the internal gravitino variation in eq. (4.90) on the form of eq. (4.98), we multiply them with $\eta_-^\dagger$ and $\eta_-^\dagger\gamma_p$ from the left on both sides. This will give terms of the form $\frac{1}{2}R_m$ and $P_n{}^pR_{mp}$ in eq. (4.98), where $P_n{}^m=\frac{1}{2}(\delta_n{}^m-iJ_n{}^m)$ is the projector onto holomorphic indices. The procedure is the same as for the external variation, but in order to rewrite the RR flux term we make use of the identities

$$\begin{aligned}\eta_+^\dagger\not{F}_{\mathrm{A}1}\gamma_m\eta_-&=-\tfrac{i}{2}(\bar{\not{\Omega}}\not{F}_{\mathrm{A}1})_m\ ,\\ \eta_-^\dagger\gamma_p\not{F}_{\mathrm{A}1}\gamma_m\eta_-&=-(\not{F}_{\mathrm{A}1m}\not{e}^{i\not{J}})_p+\tfrac{1}{2}(\not{F}_{\mathrm{A}1}\not{e}^{i\not{J}})_0g_{mp}+(\not{F}_{\mathrm{A}1}\not{e}^{i\not{J}})_{mp}\ ,\end{aligned} \tag{4.108}$$

where g is the internal 6D metric. These will contribute to the R-terms in eq. (4.98), as they are related to the RR flux. The Qs can be related to the NSNS flux and torsion, as we will describe next.

As familiar, a manifold with $SU(3)$ holonomy has a covariantly constant spinor $D_m\eta=0$, which may be translated into the closure of the forms J and Ω, and where the failure of this closure is measured by torsion. The J and Ω are given in terms of η as in eq. (4.32). In spinorial basis we may express the covariant derivative as

$$D_m\eta=(\tilde{q}_m+iq_m\gamma+iq_{mn}\gamma^n)\eta\ , \tag{4.109}$$

where γ is the 6D chirality operator. By choosing the normalisation of η to have constant norm, one can set $\tilde{q}_m=0$ [19]. The qs here are in fact another definition of torsion. In particular, the q_m is a vector, i.e. part of the $(3\oplus\bar{3})$ representation, and so q_{mn} lies in

$$(3\oplus\bar{3})\otimes(3\oplus\bar{3})=(6\oplus\bar{3})\oplus(\bar{6}\oplus 3)\oplus(8\oplus 1)\oplus(8\oplus 1)\ . \tag{4.110}$$

Hence we see that all representations of the Ws are present. Comparing eq. (4.109) with the exterior derivatives of J and Ω in eqs. (4.43)–(4.44), it was first found in [19] that the coefficients of eq. (4.109) can be written in holomorphic/antiholomorphic indices as

$$q_i = \tfrac{i}{2}(W_5 - W_4) , \qquad q_{ij} = -\tfrac{i}{8}W_{3ikl}\Omega^{kl}{}_j - \tfrac{1}{8}\Omega_{ijk}\bar{W}_4^k , \qquad q_{i\bar{j}} = -\tfrac{i}{4}\bar{W}_{2i\bar{j}} + \tfrac{1}{4}\bar{W}_1 g_{i\bar{j}} . \quad (4.111)$$

In relation to J and $\Omega \equiv \Omega_{\mathrm{R}} - i\Omega_{\mathrm{I}}$ then $q_{mn} = \frac{i}{8}\nabla_m J_{pq}\Omega_{\mathrm{I}}^{pq}{}_n$. There is reason to involve the NSNS flux H_3 to these equations since, as will be discussed in later chapters, some torsion and NSNS flux are exchanged under mirror symmetry. It is therefore customary to add H_3 to the covariant derivative in eq. (4.109) to behave better under mirror symmetry. The addition changes eq. (4.109) to

$$D_m^H \eta = i(Q_m\gamma + Q_m\gamma^n)\eta , \qquad (4.112)$$

where the most suitable derivative turns out to be $D_m^H \equiv D_m + \frac{1}{8}H_{mnp}\gamma^{np}$ [20]. The new coefficients are then given in holomorphic/antiholomorphic indices according to

$$\begin{aligned} Q_i &= \tfrac{i}{2}(W_5 - W_4 - iH^{(3)}) , \\ Q_{ij} &= -\tfrac{i}{8}(W_3 + iH^{(6)})_{ij} - \tfrac{1}{8}\Omega_{ijk}(\bar{W}_4 + i\bar{H}^{(3)})^k , \\ Q_{i\bar{j}} &= -\tfrac{i}{4}\bar{W}_{2i\bar{j}} - \tfrac{1}{4}(\bar{W}_1 + 3i\bar{H}^{(1)})g_{i\bar{j}} . \end{aligned} \qquad (4.113)$$

The addition of H essentially complexifies the torsion like $W \to W + iH$. To have supersymmetry with only NSNS flux we have from eq. (4.109) that

$$D_i^H\eta_+ = iQ_i\eta_+ + iQ_{ij}\gamma^j\eta_- , \qquad D_{\bar{\imath}}^H\eta_+ = iQ_{\bar{\imath}}\eta_+ + iQ_{\bar{\imath}j}\gamma^j\eta_- , \qquad (4.114)$$

where again it is enough that one chirality is annihilated by D^H. The $Q_{i\bar{j}}$ and $Q_{\bar{\imath}\bar{j}}$ disappear from $D_i^H\eta_+$ as η_-, being a Clifford vacuum, is annihilated by $\gamma^{\bar{\imath}}$.

Using these identities, the IIA internal gravitino variation can be written

$$\begin{aligned} &\tfrac{i}{8}e^{\phi}\Big[\tfrac{1}{2}(\not{F}_{\mathrm{A}1}e^{i\not{J}})_0 g_{mp} + (\not{F}_{\mathrm{A}1}e^{i\not{J}})_{mp} - (\not{F}_{\mathrm{A}1m}e^{i\not{J}})_p\Big]\gamma^p\eta_- \\ &+ (\partial_m\alpha + \tfrac{i}{2}J_m{}^n\alpha(W_5 - W_4)_n + \tfrac{i}{2}\beta H_m^{(3)})\eta_+ \\ &+ i\mathrm{Re}\Big[\tfrac{1}{2}(\alpha W_1 + 3i\beta H^{(1)})\bar{P}_{mn} - \tfrac{1}{4}\Omega_{mnp}(\alpha W_4 + i\beta H^{(3)})^p \\ &\qquad - \tfrac{i}{8}(\alpha W_3 + i\beta H^{(6)})_{mn} + \tfrac{i}{2}\alpha\bar{P}_m{}^p W_{2pn}\Big]\gamma^p\eta_- \\ &+ \tfrac{1}{8}e^{\phi}(\not{\Omega}\not{F}_{A1})_m\eta_+ = 0 . \end{aligned} \qquad (4.115)$$

The same procedure can be done for the modified dilatino equation, which will gives

$$\begin{aligned} &\Big[i\alpha q_m + \tfrac{i}{2}\alpha q_{nr}\Omega^{nr}{}_m + \tfrac{1}{48}\beta(\not{H}e^{-i\not{J}})_m + \alpha\partial_m(2A - \phi + \ln\alpha)\Big]\gamma^m\eta_+ \\ &+ \Big[2i\alpha P^{mn}q_{mn} - \tfrac{i}{24}\beta(\not{H}\Omega)_0\Big]\eta_- = 0 . \end{aligned} \qquad (4.116)$$

Hence, the three type IIA supersymmetry variations of eqs. (4.86), (4.90) and (4.93) can be written on their respective form in eqs. (4.97)–(4.99) where the coefficients are given by

$$
\begin{aligned}
A_m &= \alpha\partial_m A \ , \\
S &= \tfrac{i}{4}e^{\phi}(\not{F}_{\mathrm{A}_1}\not{e}^{i\not{J}})_0 \ , \\
S_m &= \tfrac{1}{8}e^{\phi}\mathrm{Re}((\not{F}_{\mathrm{A}_1}\bar{\not{\Omega}})_m) \ , \\
Q_m &= -i\partial_m\alpha + \tfrac{1}{2}J_m{}^n(\alpha W_5 - \alpha W_4)_n + \tfrac{1}{2}\beta H^{(3)}_m \ , \\
Q_{mn} &= \mathrm{Re}(\tfrac{1}{2}(\alpha W_1 + 3i\beta H^{(1)})\bar{P}_{mn} - \tfrac{1}{4}\Omega_{mnp}(\alpha W_4 + i\beta H^{(3)})^p - \tfrac{i}{8}(\alpha W_3 + i\beta H^{(1)})_{mn} + \tfrac{i}{2}\bar{P}_m{}^p\alpha W_{2pn}) \ , \\
R_m &= -\tfrac{i}{8}e^{\phi}(\bar{\not{\Omega}}\not{F}_{\mathrm{A}_1})_m \ , \\
R_{mn} &= \tfrac{1}{4}e^{\phi}\mathrm{Re}(-(\not{F}_{\mathrm{A}_1 m}\not{e}^{i\not{J}})_n + \tfrac{1}{2}(\not{F}_{\mathrm{A}_1}\not{e}^{i\not{J}})_0 g_{mn} + (\not{F}_{\mathrm{A}_1}\not{e}^{i\not{J}})_{mn}) \ , \\
T &= \tfrac{3}{2}(i\alpha W_1 - \beta H^{(1)}) \ , \\
T_m &= \alpha\partial_m(2A - \phi + \ln\alpha) + \alpha(W_{4m} + \tfrac{i}{2}J_m{}^n(W_5 - W_4)_n) - \tfrac{1}{2}J_{mn}\beta H^{(3)}_n \ ,
\end{aligned}
\tag{4.117}
$$

The same analysis applies to the type IIB case. In this case one uses the following identities for the RR flux terms:

$$
\begin{aligned}
\eta_-^\dagger\not{F}_{\mathrm{B}}\eta_+ &= -\tfrac{i}{2}(\not{F}_{\mathrm{B}}\not{\Omega})_0 \ , \\
\eta_+^\dagger\gamma_m\not{F}_{\mathrm{B}}\eta_+ &= \tfrac{1}{2}(\not{F}_{\mathrm{B}}\not{e}^{-i\not{J}})_m \ , \\
\eta_+^\dagger\not{F}_{\mathrm{B}}\gamma_m\eta_+ &= \tfrac{1}{2}(\not{e}^{-i\not{J}}\not{F}_{\mathrm{B}})_m \ , \\
\eta_-^\dagger\gamma_p\not{F}_{\mathrm{B}}\gamma_m\eta_+ &= -i(\not{F}_{\mathrm{B}m}\not{\Omega})_p + \tfrac{i}{2}(\not{F}_{\mathrm{B}}\not{\Omega})_0 g_{mp} + i(\not{F}_{\mathrm{B}}\not{\Omega})_{mp} \ ,
\end{aligned}
\tag{4.118}
$$

with $F_{\mathrm{B}} = F_1 + F_3 + F_5$. The type IIB supersymmetry variations in eq. (4.95) can thus be written on the form of eqs. (4.97)–(4.99) where the coefficients are given by

$$
\begin{aligned}
A_m &= \alpha\partial_m A \ , \\
S &= \tfrac{1}{4}e^{\phi}(\not{F}_{\mathrm{B}_1}\not{\Omega})_0 \ , \\
S_m &= \tfrac{1}{4}e^{\phi}\mathrm{Re}((\not{F}_{\mathrm{B}_1}\not{e}^{-i\not{J}})_0) \ , \\
Q_m &= -i\partial_m\alpha + \tfrac{1}{2}J_m{}^n(\alpha W_5 - \alpha W_4) - \tfrac{1}{2}\beta H^{(3)}_m \ , \\
Q_{mn} &= \mathrm{Re}(\tfrac{1}{2}(\alpha W_1 - 3i\beta H^{(1)})\bar{P}_{mn} - \tfrac{1}{4}\Omega_{mnp}(\alpha W_4 - i\beta H^{(3)})^p - \tfrac{i}{8}(\alpha W_3 - i\beta H^{(6)})_{mn} + \tfrac{i}{2}\bar{P}_m{}^p\alpha W_{2pn}) \ , \\
R_m &= -\tfrac{1}{8}e^{\phi}(\not{e}^{-iJ}\not{F}_{\mathrm{B}_1})_m \ , \\
R_{mn} &= \tfrac{1}{4}e^{\phi}\mathrm{Re}(i(\not{F}_{\mathrm{B}_1 m}\not{\Omega})_n - i(\not{F}_{\mathrm{B}_1}\not{\Omega})_{mn} - \tfrac{i}{2}(\not{F}_{\mathrm{B}_1}\not{\Omega})_0 g_{mn}) \ , \\
T &= \tfrac{3}{2}(i\alpha W_1 + \beta H^{(1)}) \ , \\
T_m &= \alpha\partial_m(2A - \phi + \ln\alpha) + \alpha(W_{4m} + \tfrac{i}{2}J_m{}^n(W_5 - W_4)_n) + \tfrac{1}{2}J_{mn}\beta H^{(3)}_n \ .
\end{aligned}
\tag{4.119}
$$

One can write the explicit expressions of the Clifford products in eq. (4.108) and eq. (4.118), i.e. in terms of J and Ω as well as the $SU(3)$ representations of the fluxes, and use them to obtain the matrices S, R, Q in terms of $SU(3)$ representations. While this alternative form of the matrix expressions is informative, it is not necessary for our analysis in the next section but may be found in Appendix C.

4.4.2 $\mathcal{N}=1$ constraints on fluxes and intrinsic torsion in Minkowski vacua

In order for the supersymmetry conditions in eqs. (4.97)–(4.99) to be satisfied, the coefficients in each representation must vanish separately. These equations will then give us a relation between the intrinsic torsion, fluxes, and warp factor in each representation. The results, as first

IIA			IIB		
$Q_i \in \mathbf{3}$	$R_i \in \mathbf{3}$	$S_i, T_i, A_i \in \mathbf{3}$	$Q_i \in \mathbf{3}$	$R_i \in \mathbf{3}$	$S_i, T_i, A_i \in \mathbf{3}$
$Q_{ij} \in \mathbf{6} \oplus \bar{\mathbf{3}}$	–	–	$Q_{ij} \in \mathbf{6} \oplus \bar{\mathbf{3}}$	$R_{ij} \in \mathbf{6} \oplus \bar{\mathbf{3}}$	–
$Q_{\bar{\imath}j} \in \mathbf{1} \oplus \mathbf{8}$	$R_{\bar{\imath}j} \in \mathbf{1} \oplus \mathbf{8}$	$S, T \in \mathbf{1}$	$Q_{\bar{\imath}j} \in \mathbf{1} \oplus \mathbf{8}$	–	$S, T \in \mathbf{1}$

Table 4.2: Matrices of eqs. (4.97)–(4.99) in their corresponding decomposed $SU(3)$ representation. Columns with Qs represent the NSNS sector and are the same for each theory, and the RR sector equivalent are given by the other two columns.

obtained in [19], are summarised in table 4.3 and 4.4, which give all possible $\mathcal{N} = 1$ Minkowski vacua for the type IIA and IIB theories respectively. In the next we start by discussing to which representation the coefficients in eqs. (4.97)–(4.99) belong.

Again, the NSNS flux and torsion contribution lies in the Q matrices and the RR contribution in the R and S matrices. The type IIA RR sector consists of a 0-form, a 2-form, a 4-form and a 6-form. The 0-form and 6-form have one component each whereas the 2- and 4-forms have 15 components each, summing up to a total of 32 components. Under $SU(3) \subset SO(6)$ the 0- and 6-forms are singlets and the other two decompose according to $\mathbf{15} \to \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{8}$.

In type IIB we naturally also have 32 components but they are distributed among a 1-form, 3-form and 5-form. The 1- and 5-form decompose as $\mathbf{3} \oplus \bar{\mathbf{3}}$ and the 3-form as $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{6}$+conjugates.

By switching to holomorphic indices we need only analyse half of the components. While the NSNS Q matrices are the same for both theories, there are no general R matrices as the representations are different, i.e. there is no $\mathbf{6}$ in IIA and no $\mathbf{8}$ in IIB. The representations of each type of matrix is summarised in table 4.2.

Turning to the supersymmetry conditions of eqs. (4.97)–(4.99), we are to analyse solutions to these equations using eq. (4.117) and eq. (4.119) (or eq. (C.9) and eq. (C.10) in Appendix C with holomorphic indices). The solutions give *necessary* constraints on any type of $\mathcal{N} = 1$ Minkowski background, as Bianchi identities still need to be imposed. Before proceeding a quick comment on the phase freedom in the 3-form Ω is in order. The $SU(3)$-structure requirements $J \wedge \Omega$ and $i\Omega \wedge \bar{\Omega} = \frac{2}{3!}J^3$ are left invariant by the redefinition $\Omega \to e^{i\varphi}\Omega$. This definition shifts the torsion $W_5 \to W_5 + i\,\mathrm{d}\varphi$ and the spinors $\eta_+ \to e^{i\varphi}\eta_+$ or simply $\alpha \to \alpha e^{i\varphi}$ and $\beta \to \beta e^{i\varphi}$. To avoid false solutions with W_5, we fix it by setting $\arg(\alpha) + \arg(\beta) = 0$.

In the following we start analysing the supersymmetry conditions in each representation of the type IIA theory.

IIA scalars. By the vanishing of the supersymmetry variations, the scalars must vanish on their own according to

$$S = 0 \ , \qquad Q^{(1)}_{\bar{\imath}j} + R^{(1)}_{\bar{\imath}j} = 0 \ , \qquad T = 0 \ , \tag{4.120}$$

from each of the supersymmetry equations of eqs. (4.97)–(4.99). Remember that there is a second set of equations obtained by the exchange $\alpha \to \beta$, and so $T = 0$ results by eq. (4.117) in the equations $i\alpha W_1 - \beta H^{(1)} = 0$ and $i\beta W_1 - \alpha H^{(1)} = 0$. If $\alpha \neq \pm\beta$ then we must have $W_1 = H^{(1)} = 0$, which in turn renders $Q^{(1)}_{\bar{\imath}j} = 0$ by eq. (4.117). We are then left with the two other conditions $S = R^{(1)}_{\bar{\imath}} = 0$ on the RR fluxes. When $\alpha \neq \pm\beta$ the four resulting equations for the four RR fluxes $F^{(1)}_0, F^{(1)}_2, F^{(1)}_4$ and $F^{(1)}_6$ are independent of each other, and hence must vanish separately.

The $T = 0$ equations allows for non-zero $H^{(1)}$ flux and torsion W_1 when $\alpha = \pm\beta$, in which case $W_1 \mp iH^{(1)} = 0$. Combining the α and β variants of the equation $Q^{(1)}_{\bar{\imath}j} + R^{(1)}_{\bar{\imath}j} = 0$ such that the RR part vanishes, one gets $W_1 \pm 3iH^{(1)} = 0$, which can only be satisfied at the same time as the $T=0$ equation if $W_1 = H^{(1)} = 0$. Hence both cases of $\alpha = \pm\beta$ and $\alpha \neq \pm\beta$ leads to that $W_1 = H^{(1)} = 0$. However with $\alpha = \pm\beta$ the RR scalar equations are dependent of each other, leading to that they can be allowed if they are equal among themselves according to $F_0^{(1)} = \pm F_2^{(1)} = F_4^{(1)} = \pm F_6^{(1)}$.

IIA $\mathbf{8}\oplus\mathbf{8}$. According to table 4.2, the $Q^{(8)}_{\bar{\imath}j}$ and $R^{(8)}_{\bar{\imath}j}$ lie in the **8** representation. Hence the relevant supersymmetry equation for this representation is $Q^{(8)}_{\bar{\imath}j} + R^{(8)}_{\bar{\imath}j} = 0$, which by eq. (4.117) becomes

$$\begin{aligned} e^{\phi}(\alpha^* F^{(8)}_{2\bar{\imath}j} + i\beta F_{4\bar{\imath}j\bar{k}l}J^{\bar{k}l}) + i\alpha W_{2\bar{\imath}j} = 0 \ , \\ e^{\phi}(\beta^* F^{(8)}_{2\bar{\imath}j} + i\alpha F_{4\bar{\imath}j\bar{k}l}J^{\bar{k}l}) + i\beta W_{2\bar{\imath}j} = 0 \ , \end{aligned} \tag{4.121}$$

for each choice of α and β. Considering the real and imaginary parts of these equations, and using $W_2 \equiv W_2^+ - iW_2^-$, the result is a system with four equations with four real unknowns $F_2^{(8)}, F_4^{(8)}, W_2^+$ and W_2^-. The determinant of this equation system is proportional to $\mathrm{Re}(\alpha\bar{\beta})\mathrm{Re}(\alpha^2+\beta^2)$ and with the phase fixing mentioned before, the determinant can only vanish for $\alpha = ik\beta$ with k being some real constant. If so, there are solutions

$$W_2^+ = e^{\phi}\frac{\mathrm{Im}(\alpha^2)}{|\alpha|^2}F_2^{(8)} \ , \qquad W_2^- = e^{\phi}\frac{\mathrm{Re}(\alpha^2)}{|\alpha|^2}F_2^{(8)} \ . \tag{4.122}$$

For the special case of $k=1$ there is an independent solution $W_2^+ = F_4^{(8)}J$. For any other relations between α and β the solution is trivial, i.e. all of the fields in this representation vanish.

IIA $\mathbf{6}\oplus\bar{\mathbf{6}}$. By table 4.2 there is only Q_{ij} in the **6** representation in IIA, hence eq. (4.98) is reduced to $Q^{(6)}_{ij} = 0$ which corresponds to $(\alpha W_3 + \beta H^{(6)})_{ij} = 0$, as well as another equation with $\alpha \to \beta$. Hence we have non-trivial solutions $W_3 \pm iH^{(6)} = 0$ or $W_3 = \pm\star_6 H^{(6)}$ when $\alpha = \pm\beta$ and vanishing components with $\alpha \neq \pm\beta$.

IIA $\mathbf{3}\oplus\bar{\mathbf{3}}$. As in the previous case we only have a Q as matrix contribution, and so $Q^{(3)}_{ij} = 0$ results in that $\alpha W_4 + \beta H^{(3)} = 0$ with $W_4 = \pm iH^{(3)}$ for $\alpha = \pm\beta$ and $W_4 = H^{(3)} = 0$ for $\alpha \neq \pm\beta$. The vector contributions result in that we have equations $S_i + A_i = 0$, $Q_i + R_i = 0$ and $T_i = 0$. In the case $\alpha = \pm\beta$ we get solutions $2W_4 = 2 \pm 2iH^{(\bar{3})} = \bar{W}_5 = 2\bar{\partial}\phi$. When $\alpha \neq \pm\beta$ we have that $W_4 = H^{(3)} = 0$, $F_4^{(\bar{3})} = 0$, $F_2^{\bar{3}} = \frac{2i}{3}\bar{\partial}\phi$, $W_5 = \frac{1}{3}\bar{\partial}\phi$ and $\bar{\partial}A = -\frac{1}{3}\bar{\partial}\phi$.

It is clear that the solutions are highly dependent on the normalisation of α and β of the two spinors, since it is by this choice the set of equations coming from the exchange $\alpha \leftrightarrow \beta$ become either dependent or independent. The results obtained are summarised in table 4.3. Next, we proceed with the same analysis for the type IIB case.

IIB scalars. Requiring $S=0$ immediately sets $F_3^{(1)} = 0$. The remaining equations $T=0$ and $Q_{\bar{\imath}j} = 0$ result in the linear combinations $\alpha i W_1 + \beta H^{(1)} = 0$ and $\alpha W_1 + 3i\beta H^{(1)} = 0$ which are only fulfilled simultaneously when $W_1 = H^{(1)} = 0$.

IIB $\mathbf{8}\oplus\mathbf{8}$. The single equation $Q^{(8)}_{\bar{\imath}j} = 0$ results in $W_2 = 0$. This can be also be seen by the fact that there are no components of H or of the RR fluxes in the **8** representation.

IIA	(A): $a = 0$ or $b = 0$	(BC): $a = be^{i\beta}$
1	$W_1 = H_3^{(1)} = 0$ $F_0^{(1)} = \mp F_2^{(1)} = F_4^{(1)} = \mp F_6^{(1)}$	$W_1 = H_3^{(1)} = 0$ $F_{2n}^{(1)} = 0$
8	$W_2 = F_2^{(8)} = F_4^{(8)} = 0$	$W_2^+ = \begin{cases} e^{\phi} F_2^{(8)} \ , & \beta \neq 0 \\ e^{\phi} F_2^{(8)} + e^{\phi} F_4^{(8)} \ , & \beta = 0 \end{cases}$ $W_2^- = 0$
6	$W_3 = \mp \star_6 H_3^{(6)}$	$W_3 = H_3^{(6)} = 0$
3	$\bar{W}_5 = 2W_4 = \mp 2iH_3^{(3)} = \bar{\partial}\phi$ $\bar{\partial}A = \bar{\partial}a = 0$	$W_4 = 0$ $2i\bar{W}_5 = F_2^{(\bar{3})} = -2i\bar{\partial}A = \frac{2i}{3}\bar{\partial}\phi$

Table 4.3: Relations between intrinsic torsion W, fluxes F, H, warp factor e^{2A}, dilaton ϕ and complex functions a, b for type IIA $\mathcal{N} = 1$ vacua. Conditions A, B, C on the complex functions yield different results in each representation $\mathbf{1}, \mathbf{8}, \mathbf{6}, \mathbf{3}$ of the space containing the torsion.

IIB $\mathbf{6} \oplus \bar{\mathbf{6}}$. The relevant supersymmetry equation in this representation is $Q_{ij}^{(6)} + R_{ij}^{(6)} = 0$, which gives two (complex) equations from $\alpha \leftrightarrow \beta$ which depend on the three complex variables $W_3, F_3^{(6)}$ and $H_3^{(6)}$. These can be rewritten as three connected equations

$$\begin{aligned} (\alpha^2 - \beta^2)W_3 &= 2\alpha\beta e^{\phi} F_3^{(6)} \ , \\ (\alpha^2 + \beta^2)W_3 &= -2\alpha\beta \star_6 H^{(6)} \ , \\ (\alpha^2 - \beta^2)H^{(6)} &= (\alpha^2 + \beta^2)e^{\phi} \star_6 F_3^{(6)} \ , \end{aligned} \tag{4.123}$$

from which one sees that the variables vanish from the following choices of α, β: $W_1 = 0 \Leftrightarrow \alpha = 0$ or $\beta = 0$, $F_3^{(6)} = 0 \Leftrightarrow \alpha = \pm\beta$, and $H^{(6)} = 0 \Leftrightarrow \alpha = i\beta$.

IIB $\mathbf{3} \oplus \bar{\mathbf{3}}$. This representation contains a large amount of components, and unlike the previous cases there are no equations that directly imply a specific relation between α and β. However, using the previously common used relations, we may impose by hand three different cases $\alpha = \pm\beta$ (A), $\alpha = 0$ or $\beta = 0$ (B), or $\alpha = \pm i\beta$ (C) and see what the solutions look like. There is also the possibility to use the phase fixing $\arg(\alpha) + \arg(\beta) = 0$ mentioned earlier, in which case one obtains $F_1 = 0$ and the rest proportional to $\bar{\partial}\beta$ according to

$$\begin{aligned} W_4 &= \frac{4(\alpha^2 + \beta^2)^2 \bar{\partial}\beta}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \ , & e^{\phi} F_3^{(\bar{3})} &= -\frac{8\alpha\bar{\partial}\beta}{3\alpha^2 + \beta^2} \ , & \bar{\partial}A &= -\frac{2(\alpha^2 - \beta^2)\bar{\partial}\beta}{\beta(3\alpha^2 + \beta^2)} \ , \\ \bar{W}_5 &= \frac{2(3\alpha^2 + \beta^2)\bar{\partial}\beta}{\beta(\alpha^2 - \beta^2)} \ , & e^{\phi} F_5^{(\bar{3})} &= \frac{4i(\alpha^2 + \beta^2)\bar{\partial}\beta}{\beta(3\alpha^2 + \beta^2)} \ , & \bar{\partial}\phi &= \frac{16\alpha^2\beta\bar{\partial}\beta}{(3\alpha^2 + \beta^2)(\alpha^2 - \beta^2)} \ . \\ H^{(\bar{3})} &= -\frac{8i\alpha(\alpha^2 + \beta^2)\bar{\partial}\beta}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \ , \end{aligned} \tag{4.124}$$

The complex functions α and β are also related to the warp factor via $A = \ln(|\alpha|^2 + |\beta|^2)$, and can be seen as an interpolating solution between case A and B. We also note that the ratios between the fields above, say $W_4/H^{(\bar{3})}$ are singular in case A and B.

All type IIB solutions that are summarised in table 4.4 have been re-expressed in the original variables a and b for each relation A, B and C between them. Some of the solutions does not fit

IIB	(A): $a=0$ or $b=0$	(B): $a=\pm ib$	(C): $a=\pm b$	(ABC)
1	$W_1 = F_3^{(1)} = H_3^{(1)} = 0$			
8	$W_2 = 0$			
6	$W_3 = \pm\star H_3^{(6)}$ $F_3^{(6)} = 0$	$W_3 = 0$ $e^{\phi}F_3^{(6)} = \mp\star H_3^{(6)}$	$W_3 = \pm e^{\phi}\star F_3^{(6)}$ $H_3^{(6)} = 0$	(4.127)
3	$\bar{W}_5 = 2W_4 = \mp 2iH_3^{(\bar{3})} = 2\bar{\partial}\phi$ $\bar{\partial}A = \bar{\partial}a = 0$	(4.125)	(4.126)	(4.128)

Table 4.4: The type IIB analogue of table 4.3. The relations A,B and C between the complex functions are slightly different from type IIA, and the last column with ”ABC” conditions corresponds to *intermediate*, or mixed, solutions which are specified in eq. (4.127).

the table. For instance, eq. (4.125) in the type IIB table 4.4 is divided into two possible cases;

$$\begin{aligned}
&\text{Case 1:} \quad \begin{cases} \frac{2i}{3}\bar{W}_5 = iW_4 = e^{\phi}F_5^{(3)} = -2i\bar{\partial}A = -4i\bar{\partial}\ln a \ , \\ \bar{\partial}\phi = 0 \ , \end{cases} \\
&\text{Case 2:} \quad \begin{cases} i\bar{W}_5 = iW_4 = e^{\phi}F_1^{(\bar{3})} = 2e^{\phi}F_5^{(\bar{3})} = i\bar{\partial}\phi \ , \\ F_3^{(3)} = H^{(3)} = 0 \ . \end{cases}
\end{aligned} \tag{4.125}$$

Eq. (4.126) too is a bit long to fit into the table, but is given by

$$2i\bar{W}_5 = \pm e^{\phi}F_3^{(\bar{3})} = -2i\bar{\partial}A = -4i\bar{\partial}\ln a = -i\bar{\partial}\phi \ . \tag{4.126}$$

The ”intermediate” ABC solution of eq. (4.127) reads

$$\begin{aligned}
W_3 &= \frac{e^{\phi}}{2ab}(a^2+b^2)\star_6 F_3^{(6)} \ , \\
H_3^{(6)} &= -\frac{e^{\phi}}{2ab}(a^2-b^2)F_3^{(6)} \ ,
\end{aligned} \tag{4.127}$$

and is intermediate in the way that one of the three complex variables $W_3, F_3^{(6)}$ or $H_3^{(6)}$ vanishes for one of the conditions A, B or C. With condition A with $a=0$ or $b=0$, we see that $F_3^{(6)}=0$, for B when $a=\pm ib$ then $W_3=0$, and for condition C then $a=\pm b$ and so $H_3^{(6)}=0$. Now for the final cell in table 4.4, eq. (4.128) is given by eq. (4.124) which in the original variables read

$$\begin{array}{lll}
W_4 = \frac{2(a^2-b^2)^2}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a & e^{\phi}F_3^{(\bar{3})} = \frac{-4iab(a^2+b^2)}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a & \bar{\partial}A = \frac{-4(ab)^2}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a \\
\bar{W}_5 = \frac{2(a^4-4a^2b^2+b^4)}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a & e^{\phi}F_5^{(\bar{3})} = \frac{-4iab(a^2-b^2)}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a & \bar{\partial}\phi = \frac{2(a^2+b^2)^2}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a \\
 & H_3^{(\bar{3})} = \frac{-2i(a^2+b^2)(a^2-b^2)}{a^4-2ia^3b+2iab^3+b^4}\bar{\partial}a &
\end{array} \ . \tag{4.128}$$

Again, in this case the conditions A, B and C do not separate into giving three distinct equations. Namely, condition B cannot be separated from the other two. One might instead turn to the gauge fixing condition $\arg(\alpha)+\arg(\beta)=0$, with $\alpha=a+ib, \beta=a-ib$, which reproduces eq. (4.128).

Even though the condition BC in table 4.3 for type IIA implies interpolating parameters in the same sense as the ABC solutions of type IIB, the solutions do not depend directly on the ”interpolating” parameter β. This means that the solution is not intermediate in the same sense as those with ABC conditions in type IIB. The type A solutions of type IIA corresponds to NS flux only. Using table 4.1 of the previous section we can classify the manifolds for each type of theory. For condition A in IIA the vanishing torsion classes are $W_1 = W_2 = 0$, so the manifold is complex. For the type BC solutions of IIA the vanishing torsion classes are $W_1 = W_3 = W_4 = 0$, so the manifold is symplectic. Opposite to the case A solutions, the BC solutions has only RR fluxes.

Proceeding to the type IIB table 4.4, we see that $W_1 = W_2 = 0$ for all types of solutions. The first column with type A solutions is the same as the type A solutions of the type IIA theory. Type C has only RR flux and the same vanishing torsion classes as type A. The type B solutions have one RR 5-form flux, i.e. $F_5^{(3)}$ or $F_5^{(\bar{3})}$, one RR 3-form flux $F_3^{(6)}$ and one NSNS 3-form flux $H_3^{(6)}$. The 3-form RR and NSNS fluxes are related by a Hodge duality, and is usually expressed in terms of a complex 3-form flux

$$G_3 = F_3 - ie^{-\phi}H_3 \equiv \hat{F}_3 - \tau H_3 \ , \tag{4.129}$$

where $\tau = C_0 + ie^{-\phi}$ is a complex combination of an axion and a dilaton, usually referred to as the axion-dilaton. G_3 is *imaginary self-dual* meaning that $\star G_3 = iG_3$. It also does not have a singlet or vector $(0,3)$ representation, so it must therefore be a (2,1)-form and primitive with respect to Ω, i.e. $G\Omega = 0$. In fact there is always a complex flux combination that is a (2,1)-form and primitive for type A and C solutions. They are given by

$$\begin{aligned} \mathrm{A}: &\qquad \mathrm{d}J \pm iH_3 \ , \\ \mathrm{B}: &\qquad F_3 \mp ie^{-\phi}H_3 \ , \\ \mathrm{C}: &\qquad \mathrm{d}(e^{-\phi}J) \pm iF_3 \ , \end{aligned} \tag{4.130}$$

where the $\pm$ comes from the $\pm$ in the relations between a and b in each condition A, B and C.

4.5 $\mathcal{N} = 1$ flux vacua with generalised Calabi-Yau manifolds

In the previous section it stood clear that all the type IIB solutions described a complex internal manifold. For IIA the solution of type A also involved a complex internal manifold, while for the BC type it was symplectic. A single geometric description of the allowed internal manifolds that unifies the two would be practical, and is indeed possible, using generalised geometry with the pure Clifford(6,6) spinors $\Phi_\pm$ defined in eq. (4.66) and eq. (4.67). In this description both the complex and symplectic manifolds are special cases of generalised complex manifolds in a generalised complex geometry.

So far we have seen that demanding vanishing supersymmetry variations of the gravitino and dilatino fields, $\delta_\epsilon\psi = \delta_\epsilon\lambda = 0$, set differential conditions on the internal $SU(3)$ admitted spinor η. In generalised complex geometry these conditions may be translated into differential conditions on the pure spinors $\Phi_\pm$. We saw that this procedure explicitly illustrated how torsion, fluxes and scalars must balance against each other in a $\mathcal{N} = 1$ vacuum. For $\mathcal{N} = 1$ the differential conditions on $\Phi_\pm$ will translate into integrability conditions for an $SU(3)$-structure, as was first shown in [21] and which we will reconstruct in this section.

Previously we worked with the case of a 4D Minkowski vacuum mostly because it simplifies calculations while still illustrating the main principles. As we will see, the differential conditions on the pure Clifford(6,6) spinors $\Phi_\pm$ results in equivalent constraints but with less indigents, and so we may include the possibility of an AdS vacuum. In doing so, we may choose a particular basis of 4D spinors $\zeta_\pm$ which satisfies

$$\nabla_\mu \zeta_\mp = \tfrac{1}{2}\mu\gamma_\mu\zeta_\pm \ , \tag{4.131}$$

so that the cosmological constant becomes $\Lambda = -|\mu|^2$. Showing the IIB case (type IIA works analogously) we let our starting point be the gravitino variations in eq. (4.94). Using eq. (4.131), one of the external variations become

$$\tfrac{1}{2}\mu\eta^1_- + \tfrac{1}{2}e^A \slashed{\partial} A \eta^1_+ - \tfrac{1}{8}e^{A+\phi}\slashed{F}\eta^2_+ = 0 \ . \tag{4.132}$$

Here we have used that $D_\mu = \hat{\nabla}_\mu \otimes \mathbb{1} + \frac{1}{2}\gamma_\mu\gamma_5 \otimes \slashed{\partial} A = e^{-A}\nabla_\mu \otimes \mathbb{1} + \frac{1}{2}\gamma_\mu\gamma_5 \otimes \slashed{\partial} A$. The η_2-equation will not be needed in the analysis of this section. The internal variations can be written

$$\begin{aligned} (D_m - \tfrac{1}{4}H_m)\eta^1_+ + \tfrac{1}{8}e^\phi \slashed{F}\gamma_m\eta^2_+ &= 0 \ , \\ (D_m + \tfrac{1}{4}H_m)\eta^2_+ - \tfrac{1}{8}e^\phi \eta^1_+\gamma_m\slashed{F} &= 0 \ . \end{aligned} \tag{4.133}$$

The modified dilatino variation will also prove useful when evaluating the exterior derivatives of the pure spinors. Including the contribution from eq. (4.131), the η^1-equation reads

$$2\mu e^A \eta^1_- + \slashed{D}\eta^1_+ + (\slashed{\partial}(2A-\phi) - \tfrac{1}{4}\slashed{H})\eta^1_+ = 0 \ . \tag{4.134}$$

Starting with the Φ_+ spinor, we use the Fierz identity in eq. (4.65) so that its exterior derivative is conveniently written as an anticommutation relation according to

$$\begin{aligned} \slashed{\mathrm{d}\Phi_+} &= \{\gamma^m, D_m(\eta^1_+ \otimes \eta^2_+)\} \\ &= \slashed{D}\eta^1_+ \otimes \eta^{2\dagger}_+ + \gamma^m\eta^1_+ \otimes D_m\eta^{2\dagger}_+ + D_m\eta^1_+ \otimes \eta^{2\dagger}_+\gamma^m + \eta^1_+ \otimes \slashed{D}\eta^{2\dagger}_+ \\ &= \big(-2\mu e^{-A}\eta^1_- - \slashed{\partial}(2A-\phi)\eta^1_+ + \tfrac{1}{4}\slashed{H}\eta^1_+\big) \otimes \eta^{2\dagger}_+ + \gamma_m\eta^1_+ \otimes \big(\tfrac{1}{4}\eta^{2\dagger}_+ H_m + \tfrac{1}{8}e^\phi \eta^{1\dagger}_+\gamma^m\slashed{F}\big) \\ &\quad + \big(\tfrac{1}{4}H_m\eta^1_+ - \tfrac{1}{8}e^\phi\slashed{F}\gamma_m\eta^2_+\big) \otimes \eta^{2\dagger}_+\gamma^m + \eta^1_+ \otimes \big(-2\bar{\mu}e^{-A}\eta^{2\dagger}_- - \eta^{2\dagger}_+\slashed{\partial}(2A-\phi) + \tfrac{1}{4}\eta^{2\dagger}_+\slashed{H}\big) \\ &= -4\mathrm{Re}(\bar{\mu}e^{-A}\slashed{\Phi}_-) - \{\slashed{\partial}(2A-\phi), \slashed{\Phi}_+\} + \tfrac{1}{4}\big(\{\slashed{H}, \slashed{\Phi}_+\} + \gamma_m\slashed{\Phi}_+H^m + H_m\slashed{\Phi}_+\gamma^m\big) \\ &\quad + \tfrac{1}{8}e^\phi\gamma_m\eta^1_+\eta^{1\dagger}_+\gamma^m\slashed{F} - \tfrac{1}{8}e^\phi\slashed{F}\gamma_m\eta^2_+\eta^{2\dagger}_+\gamma^m \ , \end{aligned} \tag{4.135}$$

where $\eta^1_+ \otimes D_m\eta^{2\dagger}_+\gamma^m = \eta^1_+ \otimes \slashed{D}\eta^{2\dagger}_+$ and the expressions for $D_m\eta^{1,2\dagger}_+$ come from eq. (4.133) and $\slashed{D}\eta^{1,2\dagger}_+$ from eq. (4.134). It is possible to simplify the H and RR flux parts. In treating the H flux term, it will prove convenient to use some Clifford(d,d) techniques, in particular the formulas

$$\begin{aligned} \gamma^m\slashed{A}_p &= \slashed{(\mathrm{d}x^m \wedge + g^{mn}\iota_{\partial_n})A_p} \ , \\ \slashed{A}_p\gamma^m &= (-1)^p\slashed{(\mathrm{d}x^m \wedge - g^{mn}\iota_{\partial_n})A_p} \ , \end{aligned} \tag{4.136}$$

for some p-form A_p. The slash denotes the bispinor equivalent of a form according to the Clifford map. To shorten notation we may define $\lambda^m \equiv \mathrm{d}x^m\wedge$ and $\iota_m \equiv \iota_{\partial_m}$[1] so that $\gamma^m\slashed{A}_p = \slashed{(\lambda^m + \iota^m)A_p}$ and $\slashed{A}_p\gamma^m = (-1)^p\slashed{(\lambda^m - \iota^m)A_p}$. Hence, for any form of even degree, the H flux terms take the form

$$\begin{aligned} \{\slashed{H}, \slashed{A}_{\mathrm{ev}}\} + \gamma^m\slashed{A}_{\mathrm{ev}}H_m + H_m\slashed{A}_{\mathrm{ev}}\gamma^m &= H_{mnp}\Big(\tfrac{1}{6}[(\lambda+\iota)^3 + (\lambda-\iota)^3] \\ &\qquad + \tfrac{1}{2}[(\lambda+\iota)(\lambda-\iota)^2 + (\lambda+\iota)^2(\lambda-\iota)]\Big)^{mnp}\slashed{A}_{\mathrm{ev}} \\ &= H_{mnp}\big(\tfrac{1}{3}\lambda^3 + \lambda\iota^2 + \lambda^3 - \lambda\iota^2\big)\slashed{A}_{\mathrm{ev}} \\ &= \tfrac{4}{3}H_{mnp}\lambda^{mnp}\slashed{A}_{\mathrm{ev}} \\ &= 8\slashed{H \wedge A_{\mathrm{ev}}} \ , \end{aligned} \tag{4.137}$$

where we have used the identities in eq. (4.136) for all three gamma matrices contracted with H in the $\{\slashed{H}, \slashed{A}_{\mathrm{ev}}\}$ term. In eq. (4.135) the terms with H reduce to $8\slashed{H \wedge \Phi_+}$.

[1]The contraction operator on a p-form acts according to $\iota_{\partial_n}\,\mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p} = p\delta^{[m_1}_n\,\mathrm{d}x^{m_2} \wedge \ldots \wedge \mathrm{d}x^{m_p]}$.

Proceeding with the RR flux terms, it is useful to rewrite the product spaces with $\eta_+^{1,2}$ using the 6D chirality operator γ. Namely, the chirality *projector* $(1-\gamma)/2$ can be rewritten as

$$\frac{1-\gamma}{2} = \eta_- \otimes \eta_-^\dagger + \tfrac{1}{2}\gamma^m \eta_+ \otimes \eta_+^\dagger \gamma_m \ , \tag{4.138}$$

which can be seen by multiplying both sides with the complete basis of spinors $\{\eta_\pm, \gamma^m\eta_\pm\}$ [21]. With this form it is useful to temporarily write the spinors with norm one, i.e. by making the substitution $\eta_+^1 \to a\eta_+^1$ and $\eta_+^2 \to b\eta_+^2$. Using eq. (4.132) we can rewrite the $e^\phi \not{F}$-terms so that the RR flux part of eq. (4.135) becomes

$$\begin{aligned}
\tfrac{1}{8}e^\phi\Big[|a|^2((1-\gamma) - 2\eta_-^1\eta_-^{1\dagger})\not{F} - |b|^2\not{F}((1-\gamma) - 2\eta_-^2\eta_-^{2\dagger})\Big]& \\
= \tfrac{1}{8}\Big[|a|^2 e^\phi((1-\gamma)\not{F} - 2\cdot 4\cdot \bar{a}\eta_-^1(\bar{b}\mu e^{-A}\eta_-^{2\dagger} - b\eta_-^{2\dagger}\not{\partial}A)& \\
- |b|^2(1+\gamma)\not{F} + 2\cdot 4\cdot(\bar{a}\not{\partial}A\eta_-^1 - a\bar{\mu}e^{-A}\eta_+^1)b\eta_-^{2\dagger}\Big]& \\
= \tfrac{1}{8}e^\phi\Big[(|a|^2 - |b|^2)\not{F} - (|a|^2+|b|^2)\gamma\not{F}\Big] + \bar{a}b\{\not{\partial}A, \overline{\not{\Phi}_+}\}& \\
- 2\mathrm{Re}(\bar{\mu}e^{-A}ab\not{\Phi}_-) \ ,&
\end{aligned} \tag{4.139}$$

where we used that $\not{F}\gamma = -\gamma\not{F}$. Re-absorbing the factors of $\bar{a}b$ and ab into the pure spinors we regain the original unnormalised ones with $||\eta^1|| = a$ and $||\eta^2|| = b$. For the product $\gamma\not{F}$ one can use the formula $\not{A}\gamma = i\cancel{\lambda(\star A)}$ where $\cancel{\lambda(A_p)} = (-1)^{\lfloor p/2\rfloor}\not{A}_p$ denotes the self-duality relation and results in that $\gamma\not{F} = -i\cancel{\star\lambda(F)}$. When evaluating the complex conjugation on the pure spinor we recall that the gamma matrices are chosen to be purely imaginary, and so complex conjugation and slash commute on even forms and anticommute on odd ones. Hence we have that $\eta_-^1 \otimes \eta_-^{2\dagger} = \overline{\not{\Phi}_+} = \not{\bar{\Phi}}_+$ and $\eta_-^1 \otimes \eta_+^{2\dagger} = \overline{\not{\Phi}_-} = -\not{\bar{\Phi}}_-$. This also results in that $\mathrm{Re}(\mu\not{\Phi}_-) = i\cancel{\mathrm{Im}(\bar{\mu}\Phi_-)}$.

Inserting these alternative expressions of the H and RR flux terms into eq. (4.135), the equation for $\mathrm{d}\Phi_+$ now reads

$$\begin{aligned}
e^{-2A+\phi}(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_+) = -3ie^{-A}\mathrm{Im}(\bar{\mu}\Phi_-) + \mathrm{d}A\wedge\bar{\Phi}_+& \\
+ \tfrac{1}{16}e^\phi\Big[(|a|^2 - |b|^2)F + i(|a|^2+|b|^2)\star\lambda(F)\Big] \ .&
\end{aligned} \tag{4.140}$$

This equation can be rewritten slightly using the fact that eq. (4.132) and eq. (4.133) can be found to give the relations

$$\mathrm{d}|\eta^1|^2 = |\eta^2|^2\,\mathrm{d}A \ , \qquad \mathrm{d}|\eta^2|^2 = |\eta^1|^2\,\mathrm{d}A \ . \tag{4.141}$$

Since $|\eta^1|^2 = |a|^2$ and $|\eta^2|^2 = |b|^2$ this results in that

$$|a|^2 + |b|^2 = c_+e^A \ , \qquad |a|^2 - |b|^2 = c_-e^{-A} \ , \tag{4.142}$$

where the $c_+ > 0$ and $c_- \geq 0$ are two integration constants. Using them in eq. (4.140), we get

$$\begin{aligned}
e^{-2A+\phi}(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_+) = -3ie^{-A}\mathrm{Im}(\bar{\mu}\Phi_-) + \mathrm{d}A\wedge\bar{\Phi}_+& \\
+ \tfrac{1}{16}\Big[c_-e^{\phi-A}F + ic_+e^{\phi+A}\star\lambda(F)\Big] \ .&
\end{aligned} \tag{4.143}$$

The total norm of the pure spinor becomes $||\Phi||^2 = \frac{1}{8}||\eta^1||^2||\eta^2||^2 = \frac{1}{32}(c_+^2e^{2A} - c_-^2e^{-2A})$.

We may now proceed to calculate the exterior derivative of the negative chirality pure spinor, which by the same Fierz arrangement in eq. (4.65) can be written as a commutator:

$$
\begin{aligned}
\slashed{\mathrm{d}\Phi_-} &= [\gamma^m, D_m(\eta^1_+ \otimes \eta^{2\dagger}_-)] \\
&= \slashed{D}\eta^1_+ \otimes \eta^{2\dagger}_- + \gamma^m \eta^1_+ \otimes \eta^{2\dagger}_-\gamma^m - D_m\eta^1_+ \otimes \eta^{2\dagger}_-\gamma^m - \eta^1_+ \otimes \slashed{D}\eta^{2\dagger}_- \\
&= \big(-2\mu e^{-A}\eta^1_- - \slashed{\partial}(2A-\phi)\eta^1_+ + \tfrac{1}{4}\slashed{H}\eta^1_+\big) \otimes \eta^{2\dagger}_- + \gamma^m\eta^1_+ \otimes \big(\tfrac{1}{4}\eta^{2\dagger}_- H_m + \tfrac{1}{8}\eta^{1\dagger}_-\gamma^m \slashed{F}\big) \\
&\quad\; - \big(\tfrac{1}{4}H_m\eta^1_+ - \tfrac{1}{8}\slashed{F}\gamma_m\eta^2_+\big) \otimes \eta^{2\dagger}_-\gamma^m - \eta^1_+ \otimes \big(2\mu e^{-A}\eta^{2\dagger}_+ - \eta^{2\dagger}_-\slashed{\partial}(2A-\phi) + \tfrac{1}{4}\eta^{2\dagger}_-\slashed{H}\big) \\
&= -4\mu e^{-A}\mathrm{Re}(\slashed{\Phi}_+) - [\slashed{\partial}(2A-\phi), \slashed{\Phi}_-] + \tfrac{1}{4}\big([\slashed{H}, \slashed{\Phi}_-] + \gamma_m\slashed{\Phi}_- H^m - H_m\slashed{\Phi}_-\gamma^m\big) \ .
\end{aligned}
\tag{4.144}
$$

The RR flux terms cancel since $\gamma_m\eta^i_+\eta^{i\dagger}_-\gamma^m = 0$ which follows from the fact that

$$
\gamma_m \slashed{A}_p \gamma^m = (-1)^p(6-2p)\slashed{A}_p \ , \tag{4.145}
$$

and Φ_- is a 3-form for an $SU(3)$-structure. Hence, we need only evaluate the H flux term to obtain an expression similar to the previous one. Again using eq. (4.136) one has for an odd form $\slashed{A}_{\mathrm{odd}}$ that

$$
\begin{aligned}
[\slashed{H}, \slashed{A}_{\mathrm{odd}}] + \gamma^m\slashed{A}_{\mathrm{odd}}H_m - H_m\slashed{A}_{\mathrm{odd}}\gamma^m &= H_{mnp}\Big(\tfrac{1}{6}[(\lambda+\iota)^3 + (\lambda-\iota)^3] \\
&\qquad + \tfrac{1}{2}[(\lambda+\iota)(\lambda-\iota)^2 + (\lambda+\iota)^2(\lambda-\iota)]^{mnp}\slashed{A}_{\mathrm{odd}} \\
&= H_{mnp}\Big(\tfrac{1}{3}\lambda^3 + \lambda\iota^2 + \lambda^3 - \lambda\iota^2\Big)\slashed{A}_{\mathrm{odd}} \\
&= \tfrac{4}{3}H_{mnp}\lambda^{mnp}\slashed{A}_{\mathrm{odd}} \\
&= 8\slashed{H\wedge A_{\mathrm{odd}}} \ ,
\end{aligned}
\tag{4.146}
$$

which is the same as for the previous case. This adds up so that eq. (4.144) becomes

$$
e^{-2A+\phi}(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_-) = -2\mu e^{-A}\mathrm{Re}(\Phi_+) \ . \tag{4.147}
$$

The next and final task is to prove how the obtained equations eq. (4.141), eq. (4.143) and eq. (4.147) imply the supersymmetry equations in eqs. (4.132)–(4.134). First, we know that the pair of pure spinors $\Phi_\pm$ defines the two Weyl spinors $\eta^{1,2}_+$ since the $\Phi_\pm$s define an $SU(3)\times SU(3)$-structure and the $\eta^{1,2}_+$ each define an $SU(3)$-structure. The next step is to expand the left-hand side of the supersymmetry variations in terms of a spinorial basis consisting of these $\eta^{1,2}_+$ spinors. Specifically, the basis can be written as a "pure spinor Hodge diamond", defined as

$$
\begin{array}{ccccccc}
 & & & \Phi_+ & & & \\
 & & \Phi_+\gamma^{i_2} & & \gamma^{\bar\imath_1}\Phi_+ & & \\
 & \Phi_-\gamma^{\bar\imath_2} & & \gamma^{\bar\imath_1}\Phi_+\gamma^{i_2} & & \gamma^{i_1}\bar\Phi_- & \\
\Phi_- & & \gamma^{\bar\imath_1}\Phi_-\gamma^{\bar\jmath_2} & & \gamma^{i_1}\bar\Phi_-\gamma^{j_2} & & \bar\Phi_- \ , \\
 & \gamma^{\bar\imath_1}\Phi_- & & \gamma^{i_1}\bar\Phi_+\gamma^{\bar\jmath_2} & & \bar\Phi_-\gamma^{i_2} & \\
 & & \gamma^{i_1}\bar\Phi_+ & & \Phi_+\gamma^{\bar\jmath_2} & & \\
 & & & \bar\Phi_+ & & &
\end{array}
$$

where we denote holomorphic and antiholomorphic indices $i_1, j_1, \dots$ and $\bar\imath_1, \bar\jmath_1, \dots$ with respect to the almost complex structure I_1 defined by η^1 and analogously $i_2, j_2, \dots$ and $\bar\imath_2, \bar\jmath_2, \dots$ with respect to the almost complex structure I_2 of η^2. In this basis the RR flux $F = F_1 + F_3 + F_5$ is written

$$\begin{aligned} \not{F} &= f^{10}_{i_2}\not{\Phi}_+\gamma^{i2} + f^{01}_{\bar{\imath}_1}\gamma^{\bar{\imath}_1}\not{\Phi}_+ \\ +f^{30}\not{\Phi}_- &+ f^{21}_{\bar{\imath}_1\bar{\jmath}_2}\gamma^{\bar{\imath}_1}\not{\Phi}_-\gamma^{\bar{\jmath}_2} + f^{12}_{i_1 j_2}\gamma^{i_1}\bar{\not{\Phi}}_-\gamma^{j_2} + f^{03}\bar{\not{\Phi}}_- \\ &+f^{32}_{i_1}\gamma^{i_1}\bar{\not{\Phi}}_+ + f^{23}_{\bar{\imath}_2}\bar{\not{\Phi}}_+\gamma^{\bar{\imath}_2} \ , \end{aligned} \tag{4.148}$$

where the fs are coefficients. The left-hand side of the supersymmetry variations can then be written

$$\begin{aligned} (\not{D} - \tfrac{1}{4}\not{H})\eta^1 &= (T^1_m\gamma^m + T^1_- + iT^1_-\gamma)\eta^1 \ , \\ (\not{D} + \tfrac{1}{4}\not{H})\eta^2 &= (T^2_m\gamma^m + T^2_- + iT^2_-\gamma)\eta^2 \ , \\ (D_m - \tfrac{1}{4}H_m)\eta^1 &= (iQ^1_{mn}\gamma^n + \partial_m \ln|a| + iQ^1_m\gamma)\eta^1 \ , \\ (D_m + \tfrac{1}{4}H_m)\eta^2 &= (iQ^2_{mn}\gamma^n + \partial_m \ln|b| + iQ^2_m\gamma)\eta^2 \ . \end{aligned} \tag{4.149}$$

Now, by expanding eq. (4.143) and eq. (4.147) in this basis, we get components

$$\begin{gathered} T^1_{\bar{\imath}_1} - iQ^2_{\bar{\imath}_1} = -\partial_{\bar{\imath}_1}(2A - \phi + \ln|b|) + \tfrac{1}{4}e^{\phi}|a|^2 f^{01}_{\bar{\imath}_1} \ , \qquad T^1 = -3\mu e^{-A} - \tfrac{1}{4}e^{\phi}|a|^2 R^{03} \ , \\ \tfrac{1}{4}e^{\phi}|b|^2 R^{01}_{\bar{\imath}_1} = \partial_{\bar{\imath}}A \ , \qquad iQ^1_{i_2 j_1} = \tfrac{1}{4}e^{\phi}|b|^2 f^{12}_{j_1 i_2} \end{gathered} \tag{4.150}$$

respectively

$$T^1 = -2\mu e^{-A} \ , \qquad T^1_{\bar{\imath}_1} + iQ^2_{\bar{\imath}_1} = -\partial_{\bar{\imath}_1}(2A - \phi + \ln|b|) \ , \qquad Q^1_{\bar{\imath}_2 j_1} = 0 \ . \tag{4.151}$$

There is also another set of these identities obtained by letting $1 \leftrightarrow 2$, $a \leftrightarrow b$ and $f^{01} \to -f^{23}$ in the above equations. The content in the supersymmetry equations expanded in this basis is recovered by making some appropriate linear combinations of eq. (4.150) and eq. (4.151) above as well as eq. (4.141), to obtain

$$\begin{gathered} Q^1_{\bar{\imath}_2 j_2} = 0 \ , \quad iQ^1_{i_2 j_1} = \tfrac{1}{4}e^{\phi}|b|^2 f^{12}_{j_1 j_2} \ , \quad iQ^1_{\bar{\imath}_2} + \partial_{i_2}\ln|a| = iQ^1_{i_2} + \partial_{i_2}\ln|a| + \tfrac{1}{4}e^{\phi}|b|^2 f^{10}_{i_2} = 0 \ , \\ \mu e^{-A} + \tfrac{1}{4}e^{\phi}|b|^2 f^{03} = 0 \ , \quad \tfrac{1}{4}e^{\phi}|b|^2 f^{01}_{\bar{\imath}_1} = \partial_{\bar{\imath}}A \ , \\ 2\mu e^{-A} + T^1 = 0 \ , \quad T^1_{\bar{\imath}_1} + \partial_{\bar{\imath}_1}(2A - \phi) = 0 \ , \end{gathered} \tag{4.152}$$

as argued in [21]. Again there is another set with $1 \leftrightarrow 2$, $a \leftrightarrow b$ and $f^{01} \to -f^{23}$. Since these equations have the same content as the supersymmetry variations of eqs. (4.132)–(4.134), the proof is finished.

In this chapter we have seen the fundamental geometric descriptions of supersymmetric backgrounds and how present fluxes must balance against the intrinsic torsion of the internal manifold, the warp factor and the assumed complex functions a and b. It is clear that the formalism of complex generalised geometry provides a simple but powerful description of the internal manifold.

5

4D Effective Theories

In this chapter we review the 4D effective theories arising from compactifications of 10D type II supergravity theories on Calabi-Yau 3-folds and orientifolds. In section 5.3 magnetic and electric fluxes are allowed to be present and we will discuss some of their effects on the fields in each theory. Section 5.4 discusses how the 4D theories can be written in terms of superpotentials, and section 5.5 makes some comments on the effects of fluxes on general $SU(3)$-structure manifolds.

5.1 Calabi-Yau compactifications of type II theories

In this section we discuss the compactification of the type II supergravity theories on a Calabi-Yau 3-fold and its orientifold variants. The 4D effective action is obtained by a Kaluza-Klein reduction, where we only keep a finite number of massless modes and discard all the massive ones. The massless modes for each field correspond to harmonic forms on the internal manifold.

5.1.1 The moduli space of Calabi-Yau manifolds

Being in the low-energy limit of type II string theory and considering internal manifolds of string scale sizes, it is natural to consider only the massless modes. The massless modes of the internal eigenfunctions correspond to a set of harmonic forms admitted by the manifold. The elements of the cohomology groups are forms, and they may be chosen to be harmonic because of the isomorphism between the space of harmonic p-forms and the p^{th} cohomology class.

The Hodge numbers of the Calabi-Yau manifold are rather simple. See Appendix B.4. The fact that $h^{(1,0)} = 0$ tells us that all closed 1-forms are also exact – they are "trivial in cohomology". The $h^{(2,0)}$ tells us the same thing about 2-forms. The Hodge number $h^{(3,0)} = 1$ has a representative Ω – the holomorphic 3-form known from earlier.

A Calabi-Yau manifold M is characterised by the Hodge numbers $h^{(1,1)}$ and $h^{(2,1)}$, which denote the dimension of the Dolbeaut cohomology group $H^{(1,1)}$ and $H^{(2,1)}$ respectively. The $H^{(1,1)}$ group is Hodge dual to $H^{(2,2)}$. The elements of $H^{(1,1)}$ are (1,1)-forms, and we may choose a basis of $H^{(1,1)}$ to consist of harmonic (1,1)-forms, denoted ω_a with $a = 1, \ldots, h^{(1,1)}$. The same can be done for $H^{(2,2)}$, whose basis then consists of harmonic (2,2)-forms $\tilde{\omega}^a$ which may be defined in relation to the ω_a to fulfil

$$\int_M \omega_a \wedge \tilde{\omega}^b = \delta_a^b \ , \qquad \int_M \omega_a \wedge \omega_b = \int_M \tilde{\omega}^a \wedge \tilde{\omega}^b = 0 \ . \tag{5.1}$$

The tilde of $\tilde{\omega}^a$ is there to make explicit the difference to ω_a, as no metric can raise or lower the index a into the other form.

For real 3-forms the elements are denoted α_C and β^C with $C = 0, \ldots, h^{(2,1)}$ and lie in the spaces of $H^{(2,1)} \oplus H^{(1,2)}$ and $H^{(3,0)} \oplus H^{(0,3)}$ respectively. They are chosen to satisfy

$$\int_M \alpha_C \wedge \beta^D = \delta_C^D \ , \qquad \int_M \alpha_C \wedge \alpha_D = \int_M \beta^C \wedge \beta^D = 0 \ . \tag{5.2}$$

Cohomology group	Basis	Index span
$H^{(0,0)}$	1	
$H^{(1,1)}$	ω_a	$a = 1, \ldots, h^{(1,1)}$
$H^{(2,1)}$	χ_c	$c = 1, \ldots, h^{(2,1)}$
$H^{(2,2)}$	$\tilde{\omega}^a$	$a = 1, \ldots, h^{(1,1)}$
$H^{(3,3)}$	$\mathrm{vol}(M)$	
$H^{(3)}$	(α_C, β^C)	$C = 0, \ldots, h^{(2,1)}$

Table 5.1: Cohomology groups of a Calabi-Yau 3-fold and a corresponding bases of harmonic forms.

One can also choose a basis of complex forms, which are only in $H^{(2,1)}$, and denote them by χ_c. Finally there is also a $(3,3)$-form, which is the volume of the Calabi-Yau space, denoted $\mathrm{vol}(M)$ or V. In table 5.1 we summarise all the non-trivial cohomology groups on M and their basis elements.

In the next section these harmonic forms will be used to expand the 10D supergravity fields in order to obtain 4D massless fields, which correspond to the zero-modes of these expansions. However, there are also additional massless modes arising from deformations of the metric of the manifold. The Calabi-Yau is Ricci flat; $R_{mn}(g) = 0$, and one might ask what deformations of the metric can be made that preserve Ricci flatness. That is, what δg fulfils

$$R_{mn}(g + \delta g) = 0 \ , \qquad R_{mn}(g) = 0 \ ? \tag{5.3}$$

If g is a Ricci-flat metric, then so is every metric related by diffeomorphism, i.e. a coordinate transformation. To search for deformations δg beyond those generated by a coordinate transformation we impose a *coordinate condition* in order to achieve a diffeomorphism invariance. This is analogous to fixing a gauge in electromagnetism. The appropriate choice is demanding

$$\nabla^m g_{mn} = 0 \ . \tag{5.4}$$

A δg that satisfies this will also satisfy

$$\int_M \mathrm{d}^d x \sqrt{g} \delta g^{mn} (\nabla_m \xi_n + \nabla_n \xi_m) = 0 \ , \tag{5.5}$$

for some vector field ξ_m [8]. This means that δg is orthogonal to any change of the metric induced by a diffeomorphism generated by the vector field ξ_m. Now expanding eq. (5.3) to first order in δg and using $R_{mn}(g) = 0$ as well as the coordinate condition of eq. (5.4), one obtains a differential equation

$$\nabla^\rho \nabla_\rho \delta g_{mn} - 2 R_m{}^\rho{}_n{}^\sigma \delta g_{\rho\sigma} = 0 \ , \tag{5.6}$$

called the *Lichnerowicz equation* for a metric deformation. Here we used that the manifold is compact, so a term $\nabla_m \nabla_n \mathrm{tr}(\delta g)$ vanishes.

Since the Calabi-Yau is a Hermitian manifold the metric components with two holomorphic or antiholomorphic indices vanishes; $g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0$, see Appendix B.4. Therefore the metric deformations will be on the form $g_{i\bar{\jmath}}$, so eq. (5.6) becomes

$$\nabla^\rho \nabla_\rho \delta g_{i\bar{\jmath}} - 2 R_i{}^k{}_{\bar{\jmath}}{}^{\bar{l}} \delta g_{k\bar{l}} = 0 \ , \tag{5.7}$$

where now $\rho = (k, \bar{k})$. With the definition of a Laplacian action acting on a form, see eq. (B.13), eq. (5.7) is equivalent to

$$(\Delta \delta g)_{i\bar{\jmath}} = 0 \ . \tag{5.8}$$

The metric variation $\delta g_{i\bar{\jmath}}$ may be seen as components of a (1,1)-form. This means that we may view harmonic (1,1)-forms as metric variations of the type $\delta g_{i\bar{\jmath}}$ and thus to cohomologically non-trivial changes to the Kähler form $J = ig_{i\bar{\jmath}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}$. Expanding $\delta g_{i\bar{\jmath}}$ in the basis ω_a of harmonic (1,1)-forms, the deformations to the Kähler structure on the metric has the form of

$$g + \delta g = -iJ = v^a \omega_a \ , \tag{5.9}$$

where $a = 1, \dots h^{(1,1)}$ and v^a are real scalars, known as *Kähler moduli*. In order for $g_{i\bar{\jmath}} + \delta g_{i\bar{\jmath}}$ to be a Kähler metric the moduli v^a has to be chosen such that the deformed metric is still positive definite. Positive definiteness of a metric with associated Kähler form J is equivalent to it fulfilling the conditions

$$\int_C J > 0 \ , \qquad \int_S J \wedge J > 0 \ , \qquad \int_M J \wedge J \wedge J > 0 \ , \tag{5.10}$$

for all curves C and surfaces S on the Calabi-Yau manifold M in question. The subset in $\mathbb{R}^{h^{(1,1)}}$ spanned by the parameters v^a such that eq. (5.10) is satisfied, is known as the *Kähler cone*.

The B field present in both type II theories can also be expanded in $h^{(1,1)}$ real scalars b^a. Using these it is customary to "complexify" the Kähler cone, by defining the complex scalar fields t^a such that

$$B_{i\bar{\jmath}} + iJ_{i\bar{\jmath}} = (b^a + iv^a)(\omega_a)_{i\bar{\jmath}} \equiv t^a (\omega_a)_{i\bar{\jmath}} \ . \tag{5.11}$$

One usually calls this the *complexified Kähler form* $\mathcal{J} \equiv B + iJ$ and the dimension of t^a is $h^{(1,1)}$.

In addition to finding metric deformations $\delta g_{i\bar{\jmath}}$ which fulfil eq. (5.3), there is actually another type of deformations that can be made. The purely holomorphic and antiholomorphic metric components g_{ij} and $g_{\bar{\imath}\bar{\jmath}}$ vanish on a Calabi-Yau, although we may consider deformations to non-zero values. This will change the complex structure of the manifold, thus bringing us to study variations of the complex structure. With each such variation one can associate the complex (2,1)-form

$$\Omega_{ijk} g^{k\bar{l}} \delta g_{\bar{l}\bar{m}}\, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{m}} \ , \tag{5.12}$$

which is harmonic if eq. (5.3) is fulfilled. Ω_{ijk} is the familiar unique holomorphic (3,0)-form. Turning to the space of complex structure deformations of the metric, we use the (2,1)-form of eq. (5.12) to introduce another set of (2,1)-forms that later will be associated with the metric deformations. These (2,1)-forms are

$$\chi_c = \tfrac{1}{2}(\chi_c)_{ij\bar{k}}\, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{k}} \ , \qquad \text{where} \qquad (\bar{\chi}_c)_{ij\bar{k}} = -\tfrac{1}{2}\Omega_{ij}{}^{\bar{l}} \frac{\partial g_{\bar{k}\bar{l}}}{\partial z^c} \ , \tag{5.13}$$

and the z^c, $c = 1, \dots, h^{2,1}$ are local coordinates for the complex structure moduli space, known as *complex structure moduli*. Indices are raised and lowered with the Hermitian metric, so that for example $g^{k\bar{l}}\Omega_{ijk} = \Omega_{ij}{}^{\bar{l}}$. Like the (2,1)-form in eq. (5.12) which the χ_c were constructed with, they are also harmonic. Inverting the relations of eq. (5.13), we obtain a formula for the metric deformations, namely

$$\delta g_{\bar{\imath}\bar{\jmath}} = -\frac{1}{||\Omega||^2} \bar{\Omega}_{\bar{\imath}}{}^{kl} (\chi_c)_{kl\bar{\jmath}} \delta z^c \ , \qquad \text{where} \qquad ||\Omega||^2 = \tfrac{1}{6}\Omega_{ijk}\bar{\Omega}^{ijk} \ . \tag{5.14}$$

The total moduli space of the metric deformations can be seen as a sum of the parameters t^a and z^c. It has a natural metric defined on it, which is given as a sum of two pieces corresponding to the deformations Kähler structure and complex structure, in turn generated by t^a and z^c. The metric can be written [22]

$$\mathrm{d}s^2 = \frac{1}{2V} \int \mathrm{d}x^6 \sqrt{g} g^{i\bar{\jmath}} g^{k\bar{l}} \left(\delta g_{ik} \delta g_{\bar{\jmath}\bar{l}} + (\delta g_{i\bar{l}} \delta g_{k\bar{\jmath}} - \delta B_{i\bar{l}} \delta B_{k\bar{\jmath}}) \right) \ , \tag{5.15}$$

where V denotes the volume of the Calabi-Yau. This diagonal form of the metric implies that we may write the moduli space on the product form

$$\mathcal{M} = \mathcal{M}^{\text{cs}} \oplus \mathcal{M}^{\text{Ks}} , \tag{5.16}$$

where $\mathcal{M}^{\text{cs}}$ is the complex structure moduli space and $\mathcal{M}^{\text{Ks}}$ the Kähler structure moduli space. These spaces will be described in a little more detail in the next two paragraphs.

The complex structure moduli space. The metric of the complex structure moduli space can be written in terms of the local coordinates z^c according to

$$\mathrm{d}s^2 = 2G_{c\bar{d}}\delta z^c \delta \bar{z}^{\bar{d}} . \tag{5.17}$$

With the aid of eq. (5.15) and eq. (5.14) this can be rewritten such that the metric is given by

$$G_{c\bar{d}}\delta z^c \delta \bar{z}^{\bar{d}} = -\left(\frac{i\int \chi_c \wedge \bar{\chi}_{\bar{d}}}{i\int \Omega \wedge \bar{\Omega}}\right)\delta z^c \delta \bar{z}^{\bar{d}} . \tag{5.18}$$

A change in the complex structure will make the (3,0)-form Ω transform into a linear combination of a (3,0)-form and a (2,1)-form since $\mathrm{d}z$ becomes a linear combination of $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. More specifically, we have a formula found by Kodaira stating that

$$\frac{\partial \Omega}{\partial z^c} = K_c \Omega + \chi_c , \tag{5.19}$$

where K_c is a coefficient which depends on the coordinates z^c, but not on the coordinates of the Calabi-Yau manifold. Recalling that $G_{c\bar{d}} = \partial_c \partial_{\bar{d}} \mathcal{K}$ for a Kähler space and combining eq. (5.18) and eq. (5.19), it becomes clear that the complex structure moduli space is a Kähler manifold with the Kähler potential

$$\mathcal{K}^{\text{cs}} = -\ln\left(i\int \Omega \wedge \bar{\Omega}\right) . \tag{5.20}$$

Combining this equation with eq. (5.19), K_c is determined to $K_c = -\partial\mathcal{K}/\partial z^c$.

It is useful to introduce another basis now consisting of 3-cycles A^C and B^C with $C = 0, \ldots, h^{(2,1)}$ in order to describe the complex structure moduli space in more detail. The 3-cycles are chosen so that their *intersection numbers* are

$$A^C \cap B_D = -B_D \cap A^C = \delta^C_D , \qquad \text{and} \qquad A^C \cap A^D = B_C \cap B_D = 0 . \tag{5.21}$$

They are related to the cohomology basis $\{\alpha_C, \beta^C\}$ introduced earlier via

$$\int_{A^C} \alpha_D = \int_M \alpha_D \wedge B^C = \delta^C_D , \qquad \text{and} \qquad \int_{B_C} \beta^D = \int_M \beta^D \wedge \alpha_C = -\delta^D_C . \tag{5.22}$$

With this notation we may define coordinates Z^C on the complex structure moduli space using the *A periods of* Ω such that

$$Z^C \equiv \int_{A^C} \Omega , \tag{5.23}$$

with $C = 0, \ldots, h^{(2,1)}$. Note that the number of coordinates defined like this is one more than the number of moduli fields. To compensate we define the quotient space

$$z^c = \frac{Z^C}{Z^0} , \tag{5.24}$$

where again $c = 1, \ldots, h^{(2,1)}$. For the B periods of Ω we have that

$$\mathcal{F}_C \equiv \int_{B^C} \Omega \ , \tag{5.25}$$

where $\mathcal{F}$ can be taken as a function of the Zs; $\mathcal{F}_C = \mathcal{F}_C(Z)$, since they span the same number of coordinates. With these coordinates Ω can be rewritten as

$$\Omega = Z^C \alpha_C - \mathcal{F}_C(Z)\beta^C \ . \tag{5.26}$$

Using eq. (5.19) it is straightforward to see that

$$\int \Omega \wedge \partial_C \Omega = 0 \ , \tag{5.27}$$

since there are only three holomorphic and antiholomorphic coordinates each. This may be used to show that

$$\mathcal{F}_C = Z^D \frac{\partial \mathcal{F}_D}{\partial Z^C} = \tfrac{1}{2}\partial_C (Z^D \mathcal{F}_D) \ , \tag{5.28}$$

which is equivalent to writing

$$\mathcal{F}_C = \partial_C \mathcal{F} \ , \qquad \text{with} \qquad \mathcal{F} = \tfrac{1}{2} Z^C \mathcal{F}_C \ . \tag{5.29}$$

This is useful because now all of the B periods can be expressed as derivatives of a single function $\mathcal{F}$, called the *prepotential*. Since $2\mathcal{F} = Z^C \partial_C \mathcal{F}$, $\mathcal{F}$ is homogeneous of degree two meaning that a coordinate rescaling $Z \to \lambda Z$ yields $\mathcal{F}(\lambda Z) = \lambda^2 \mathcal{F}(Z)$. The prepotential is only defined up to an overall scaling, so technically it is not a function but rather a *section of a line bundle* over the moduli space.

The Kähler potential can be rewritten in terms of this prepotential, so that the prepotential effectively determines the metric of the moduli space via $G_{c\bar{d}} = \partial_c \partial_{\bar{d}} \mathcal{K}$. This makes $\mathcal{M}^{\text{cs}}$ a *special Kähler* manifold. For closed 3-forms $\{\alpha, \beta\}$ there is a general rule stating that

$$\int_M \alpha \wedge \beta = -\int_{A^C} \alpha \int_{B_C} \beta - \int_{A^C} \beta \int_{B_C} \alpha \ , \tag{5.30}$$

where summation over C is implied, which enables us to rewrite the Kähler potential in eq. (5.20) according to

$$\mathcal{K}^{\text{cs}} = -\ln\left[i(\bar{Z}^C \mathcal{F}_C - Z^C \bar{\mathcal{F}}_C) \right] \ . \tag{5.31}$$

Note that Ω in eq. (5.26) is only defined up to complex rescalings by some holomorphic function $e^{-f(z)}$, which via eq. (5.20) also allows for changes to the Kähler potential according to

$$\Omega \to \Omega e^{-h(z)} \qquad \Rightarrow \qquad \mathcal{K}^{\text{cs}} \to \mathcal{K}^{\text{cs}} + f + \bar{f} \ . \tag{5.32}$$

This is known as a Kähler transformation. This is the reason we may choose the gauge $Z^0 = 1$, $Z^C = (1, z^c)$, since the rescaling symmetry of $\Omega(Z)$ renders one of the periods Z^C unphysical.

The Kähler structure moduli space. We start by looking at the inner product of the space of the (1,1) cohomology class $H^{(1,1)}$, which is given by

$$G(\rho, \sigma) = \frac{1}{2V} \int_M \mathrm{d}^6 x \sqrt{g} \rho_{i\bar{j}} \sigma_{k\bar{l}} g^{i\bar{l}} g^{k\bar{j}} = \frac{1}{2V} \int_M \rho \wedge \star\sigma \ , \tag{5.33}$$

where ρ and σ are real (1,1)-forms [23]. This metric can be rewritten entirely in terms of the cubic form

$$\kappa(\rho, \sigma, \tau) \equiv \int_M \rho \wedge \sigma \wedge \tau \ . \tag{5.34}$$

Recall that $V = \frac{1}{6}\int_M J \wedge J \wedge J$ is the volume of the Calabi-Yau manifold. There is also the identity

$$\begin{aligned} \star\sigma &= -J \wedge \sigma - \frac{3}{2}\frac{\kappa(\sigma, J, J)}{\kappa(J, J, J)} J \wedge J \\ &= -J \wedge \sigma + \frac{1}{4V}\kappa(\sigma, J, J) J \wedge J \ , \end{aligned} \tag{5.35}$$

which enables us to write the metric as

$$G(\rho, \sigma) = -\frac{1}{2V}\kappa(\rho, \sigma, J) + \frac{1}{8V^2}\kappa(\rho, J, J)\kappa(\sigma, J, J) \ . \tag{5.36}$$

Using the earlier defined cohomology basis ω_a of harmonic (1,1)-forms together with the complexified Kähler form $\mathcal{J} = B + iJ = t^a \omega_a$ where $a = 1, \ldots, h^{(1,1)}$, the metric on the moduli space is given by

$$G_{a\bar{b}} \equiv \tfrac{1}{2} G(\omega_a, \omega_{\bar{b}}) = \frac{\partial}{\partial t^a}\frac{\partial}{\partial \bar{t}^{\bar{b}}}\mathcal{K}^{\mathrm{Ks}} \ , \tag{5.37}$$

where the Kähler potential of the Kähler structure moduli space $\mathcal{K}^{\mathrm{Ks}}$ is

$$\mathcal{K}^{\mathrm{Ks}} = -\ln\left(\frac{4}{3}\int_M J \wedge J \wedge J\right) \ . \tag{5.38}$$

Thus we have just shown that the Kähler structure moduli space is a Kähler manifold. Similar to the previous case of the complex structure, we may introduce new intersection numbers for the Kähler structure moduli according to

$$\begin{aligned} &\kappa \equiv \kappa(J, J, J) \equiv \int_M J \wedge J \wedge J \ , && \kappa_a \equiv \kappa(\omega_a, J, J) \equiv \int_M \omega_a \wedge J \wedge J \ , \\ &\kappa_{ab} \equiv \kappa(\omega_a, \omega_b, J) \equiv \int_M \omega_a \wedge \omega_b \wedge J \ , && \kappa_{abc} \equiv \kappa(\omega_a, \omega_b, \omega_c) \equiv \int_M \omega_a \wedge \omega_b \wedge \omega_c \ , \end{aligned} \tag{5.39}$$

so that the Kähler potential in eq. (5.38) becomes

$$\mathcal{K}^{\mathrm{Ks}} = -\ln\left(\tfrac{i}{6}\kappa_{abc}(t^a - \bar{t}^a)(t^b - \bar{t}^b)(t^b - \bar{t}^b)\right) = -\ln(\tfrac{4}{3}\kappa) \ . \tag{5.40}$$

With this Kähler potential, the metric of eq. (5.37) can be written

$$G_{a\bar{b}} = \partial_{t^a}\partial_{\bar{t}^{\bar{b}}}\mathcal{K} = -\frac{3}{2}\left(\frac{\kappa_{ab}}{\kappa} - \frac{3}{2}\frac{\kappa_a\kappa_b}{\kappa^2}\right) = \frac{3}{2\kappa}\int \omega_a \wedge \star\omega_b \ . \tag{5.41}$$

Developing the analogy to the case of the complex structure moduli space, we define

$$\tilde{\mathcal{F}}(t) \equiv \frac{1}{6}\frac{\kappa_{abc}t^a t^b t^c}{t^0} = \frac{1}{6t^0}\int_M \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J} \ , \tag{5.42}$$

which is analogous to the prepotential $\mathcal{F}$ of the complex structure moduli space. The additional coordinate t^0 has been introduced in order to make $\tilde{\mathcal{F}}(t)$ a homogeneous function of degree two. Finally, we reach the Kähler potential expression

$$\mathcal{K}^{\mathrm{Ks}} = -\ln\left[i\left(t^A \frac{\partial\bar{\tilde{\mathcal{F}}}}{\partial \bar{t}^A} - \bar{t}^A \frac{\partial\tilde{\mathcal{F}}}{\partial t^A}\right)\right] \ , \tag{5.43}$$

where t^0 is included in the summation over $A = 0, \ldots, h^{(1,1)}$ but evaluated at $t^0 = 1$, as it is customary to take $t^A = (1, t^a)$. Note that this Kähler potential is the analogue of the complex structure Kähler potential in eq. (5.31), but with Kähler coordinates t^A instead of complex structure coordinates $Z^C = (1, z^c)$ and different prepotentials $\tilde{\mathcal{F}}$ versus $\mathcal{F}$. The manifold $\mathcal{M}^{\mathrm{Ks}}$ is also special Kähler, since the Kähler potential in eq. (5.40) can be derived from its prepotential in eq. (5.42), thus determining the metric.

5.1.2 Dimensional reduction to 4D

In this section we will apply the tools developed in the previous subsection in order to compactify the type II theories on Calabi-Yau manifolds. The type II supergravity theories are maximally symmetric in 10D and are naturally obtained as the low-energy limit of the type II superstring theories. The low-energy limit omits all massive modes so in the corresponding supergravity theories the spectrum only consists of massless string modes. The fermionic massless string modes are the gravitinos and dilatinos, which are in the NSR/RNS sectors. In type IIA the gravitinos have different chiralities and in type IIB they have the same chiralities. The bosonic massless fields common for both type II theories are the dilaton $\phi^{(10)}$, the graviton $g^{(10)}$ and the 2-form field $B_2^{(10)}$, which all come from the NSNS sector. Here the superscript (10) denotes 10D, and will be dropped when reaching the corresponding fields in 4D.

In type IIA 10D supergravity, the bosonic RR sector contains the form fields $C_1^{(10)}$ and $C_3^{(10)}$. Defining the field strengths of the different fields in type IIA according to

$$H_3^{(10)} \equiv \mathrm{d}B_2^{(10)} \ , \quad F_2^{(10)} \equiv \mathrm{d}C_1^{(10)} \ , \quad F_4^{(10)} \equiv \mathrm{d}C_3^{(10)} - C_1^{(10)} \wedge H_3^{(10)} \ , \tag{5.44}$$

the 10D action in the Einstein frame is given by [14]

$$\begin{aligned} S_{\mathrm{IIA}}^{(10)} = \int & -\tfrac{1}{2}R^{(10)} \star \mathbb{1} - \tfrac{1}{4}\,\mathrm{d}\phi^{(10)} \wedge \star\, \mathrm{d}\phi^{(10)} - \tfrac{1}{4}e^{-\phi^{(10)}} H_3^{(10)} \wedge \star H_3^{(10)} - \tfrac{1}{2}e^{3\phi^{(10)}/2} F_2^{(10)} \wedge \star F_2^{(10)} \\ & - \tfrac{1}{2}e^{\phi^{(10)}/2} F_4^{(10)} \wedge \star F_4^{(10)} - \tfrac{1}{2}B_2^{(10)} \wedge F_4^{(10)} \wedge F_4^{(10)} \ . \end{aligned} \tag{5.45}$$

The unique field strengths for type IIB, i.e. the ones constructed out of the axion $C_0^{(10)}$, 2-form $C_2^{(10)}$ and 4-form $C_4^{(10)}$, are defined as

$$F_1^{(10)} \equiv \mathrm{d}C_0^{(10)} \ , \qquad F_3^{(10)} \equiv \mathrm{d}C_2^{(10)} - C_0^{(10)} \wedge H_3^{(10)} \ , \qquad F_5^{(10)} \equiv \mathrm{d}C_4^{(10)} - C_2^{(10)} \wedge H_3^{(10)} \ . \tag{5.46}$$

The type IIB action in 10D is given in the Einstein frame by

$$\begin{aligned} S_{\mathrm{IIB}}^{(10)} = \int & -\tfrac{1}{2}R^{(10)} \star \mathbb{1} - \tfrac{1}{4}\,\mathrm{d}\phi^{(10)} \wedge \star\, \mathrm{d}\phi^{(10)} - \tfrac{1}{4}e^{-\phi^{(10)}} H_3^{(10)} \wedge \star H_3^{(10)} - \tfrac{1}{4}e^{2\phi^{(10)}} F_1^{(10)} \wedge \star F_1^{(10)} \\ & - \tfrac{1}{4}e^{\phi^{(10)}} F_3^{(10)} \wedge \star F_3^{(10)} - \tfrac{1}{8}F_5^{(10)} \wedge \star F_5^{(10)} - \tfrac{1}{4}C_4^{(10)} \wedge H_3^{(10)} \wedge F_3^{(10)} \ , \end{aligned} \tag{5.47}$$

where the self-duality condition $F_5^{(10)} = \star F_5^{(10)}$ is imposed at the level of the equations of motions.

The resulting 4D theory after compactification also has $\mathcal{N} = 2$. For the zero mode fields in the Kaluza-Klein compactification this means that they have to assemble into massless $\mathcal{N} = 2$ multiplets. The zero modes are in a one to one correspondence with harmonic forms on the Calabi-Yau manifold M, and thus their multiplicity is counted by the dimension of the non-trivial cohomologies on M.

For the Kaluza-Klein compactification one chooses a block diagonal 10D background metric according to

$$\mathrm{d}s^2 = \eta_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + g_{i\bar{\jmath}}(y)\,\mathrm{d}y^i\,\mathrm{d}y^{\bar{\jmath}} \ . \tag{5.48}$$

Some of the 4D fields arise as variations around this background metric. The fields in question correspond to the 4D graviton and the geometric deformations $v^a(x)$ and $z^c(x)$ defined in eq. (5.9) and eq. (5.13). Variations of off-diagonal entries in the metric in eq. (5.48) must vanish since a Calabi-Yau manifold does not admit harmonic 1-forms.

	IIA		IIB	
gravity multiplet	1	$\{g_{\mu\nu}, A^0\}$	1	$\{g_{\mu\nu}, V^0\}$
vector multiplets	$h^{(1,1)}$	$\{A^a, v^a, b^a\}$	$h^{(2,1)}$	$\{V^c, z^c\}$
hypermultiplets	$h^{(2,1)}$	$\{z^c, \xi^c, \tilde{\xi}_c\}$	$h^{(1,1)}$	$\{v^a, b^a, c^a, \rho_a\}$
tensor multiplet	1	$\{B_2, \phi, \xi^0, \tilde{\xi}_0\}$	1	$\{B_2, C_2, \phi, C_0\}$

Table 5.2: The 4D moduli fields of the compactified type IIA and IIB supergravity theories on a Calabi-Yau, arranged in $\mathcal{N} = 2$ multiplets. The middle column denote the dimension of each multiplet.

Now, the next step is to expand the 10D NSNS and RR fields in terms of harmonic cohomology basis defined in table 5.1. The NSNS sector is the same for type IIA and IIB, with expansions

$$\phi^{(10)}(x, y) = \phi(x) \ , \tag{5.49}$$

$$g^{(10)}_{i\bar{j}}(x, y) = iv^a(x)(\omega_a)_{i\bar{j}}(y) \ , \qquad g^{(10)}_{ij} = i\bar{z}^c(x)\left(\frac{(\chi_c)_{i\bar{k}\bar{l}}\Omega^{\bar{k}\bar{l}}{}_j}{|\Omega|^2}\right)(y) \ , \tag{5.50}$$

$$B^{(10)}_2(x, y) = B_2(x) + b^a(x)\omega_a(y) \ . \tag{5.51}$$

For type IIA, the RR gauge potentials $C^{(10)}_1$ and $C^{(10)}_3$ are expanded as

$$C^{(10)}_1 = A^0(x) \ , \tag{5.52}$$

$$C^{(10)}_3 = A^a(x) \wedge \omega_a + \xi^C(x)\alpha_C(y) - \tilde{\xi}_C(x)\beta^C(y) \ , \tag{5.53}$$

where again $a = 1, \dots, h^{(1,1)}$ and $C = 0, \dots, h^{(2,1)}$. Here b^a, ξ^C and $\tilde{\xi}_C$ are 4D scalars, A^0, A^a are 1-forms and B_2 is a 2-form. The A^0 in the expansion of $C^{(10)}_1$ is only 4D as a direct consequence of the Calabi-Yau not admitting harmonic 1-forms. As for the $\mathcal{N} = 2$ multiplets, they will be built from the geometric deformations v^a and z^c as well as the fields in the expansions of eqs. (5.51)–(5.53). First, the graviton $g_{\mu\nu}$ and 1-form A^0 form a *gravity multiplet* $\{g_{\mu\nu}, A^0\}$. Then there are $h^{(1,1)}$ *vector multiplets* $\{A^a, v^a, b^a\}$, a number of $h^{(2,1)}$ *hypermultiplets* $\{z^c, \xi^c, \tilde{\xi}_c\}$, and one *tensor multiplet* $\{B_2, \phi, \xi^0, \tilde{\xi}_0\}$ with only bosonic components. The $\mathcal{N} = 2$ multiplets for the type II theories compactified on a Calabi-Yau are summarised in table 5.2. The multiplets in table 5.2 consist of more than the coordinates t^a and z^c of the Calabi-Yau 3-fold's moduli spaces $\mathcal{M}^{\mathrm{Ks}}$ and $\mathcal{M}^{\mathrm{cs}}$ respectively. Therefore, the $\mathcal{N} = 2$ moduli space is said to be of the structure

$$\mathcal{M}^{\mathrm{SK}} \oplus \mathcal{M}^{\mathrm{Q}} \ , \tag{5.54}$$

where $\mathcal{M}^{\mathrm{SK}}$ is the moduli space of the vector multiplet, i.e. a special Kähler manifold with $\mathcal{M}^{\mathrm{Ks}} \in \mathcal{M}^{\mathrm{SK}}$. The $\mathcal{M}^{\mathrm{Q}}$ is a special *quaternionic* manifold spanned by the scalars in the hypermultiplet, where $\mathcal{M}^{\mathrm{cs}} \in \mathcal{M}^{\mathrm{Q}}$.

The low-energy effective action in 4D is usually displayed in a "standard $\mathcal{N} = 2$ form" and to obtain it we need to define the 4D dilaton D as [23]

$$e^D \equiv \sqrt{\frac{6}{\kappa}} e^{\phi} \ . \tag{5.55}$$

Here ϕ is the direct 4D equivalent of the $\phi^{(10)}$ in 10D, and κ is as defined in eq. (5.39) where we again have combined the real scalars b^a and v^a into the complex fields t^a introduced in eq. (5.11). The v^a is evaluated in the string frame, and because of this, so are $J = v^a\omega_a$ and $\kappa = \int_M J \wedge J \wedge J$. In the string frame the Einstein-Hilbert term takes the form $\frac{1}{2}e^{-2\phi^{(10)}}R \star \mathbb{1}$, so to obtain an action in the Einstein frame a Weyl rescaling $J_{\text{str}} = e^{\phi/2} J_E$ needs to be done. Inserting the expansions of eqs. (5.51)–(5.53) into the 10D action in eq. (5.45) and integrating over the Calabi-Yau manifold, then using the dilaton definition and Weyl rescaling, the 4D action becomes

$$\begin{aligned} S^{(4)}_{\text{IIA}} = \int_{\mathcal{M}_4} & -\tfrac{1}{2} R \star \mathbb{1} - \tfrac{1}{2} \text{Re}\mathcal{N}_{AB} F^A \wedge F^B + \tfrac{1}{2} \text{Im}\mathcal{N}_{AB} F^A \wedge \star F^B \\ & - G_{ab}\, \text{d}t^a \wedge \star \text{d}\bar{t}^b - h_{uv}\, \text{d}q^u \wedge \star \text{d}q^v \ , \end{aligned} \tag{5.56}$$

as first obtained in [24][25]. Here the indices span $A, B = 0, \ldots, h^{(1,1)}$ and $a, b = 1, \ldots, h^{(1,1)}$. The field strengths are $F^A = \text{d}A^A = (\text{d}A^0, \text{d}A^a)$. G_{ab} is the metric on the Kähler structure moduli space defined in eq. (5.37) which depends only on the coordinates t^a of $\mathcal{M}^{\text{Ks}}$. The metric G_{ab} together with $\mathcal{N}_{AB}$ encode the couplings of the vector multiplets in the action of eq. (5.56). The complex matrix $\mathcal{N}$ is known as a gauge coupling function, and is given in terms of the prepotential $\tilde{\mathcal{F}}$ according to

$$\mathcal{N}_{AB} = \overline{\tilde{\mathcal{F}}}_{AB} + 2i \frac{\text{Im}\tilde{\mathcal{F}}_{AC} \text{Im}\tilde{\mathcal{F}}_{BD} t^C t^D}{\text{Im}\tilde{\mathcal{F}}_{DC} t^C t^D} \ , \tag{5.57}$$

with $A, B, C, D = 0, \ldots, h^{(1,1)}$. Inserting the expression for the prepotential in eq. (5.42) into $\mathcal{N}_{AB}$, the real and imaginary parts of the matrix is determined to

$$\text{Re}\mathcal{N}_{AB} = \begin{pmatrix} -\frac{1}{3}\kappa_{abc} b^a b^b b^c & \frac{1}{2}\kappa_{abc} b^b b^c \\ \frac{1}{2}\kappa_{abc} b^b b^c & -\kappa_{abc} b^c \end{pmatrix} \ , \tag{5.58}$$

$$\text{Im}\mathcal{N}_{AB} = \begin{pmatrix} -\kappa + \left(\kappa_{ab} - \frac{1}{4}\frac{\kappa_a \kappa_b}{\kappa}\right) b^a b^b & \left(\kappa_{ab} - \frac{1}{4}\frac{\kappa_a \kappa_b}{\kappa}\right) b^b \\ \left(\kappa_{ab} - \frac{1}{4}\frac{\kappa_a \kappa_b}{\kappa}\right) b^b & \kappa_{ab} - \frac{1}{4}\frac{\kappa_a \kappa_b}{\kappa} \end{pmatrix} \ . \tag{5.59}$$

Here the (0,0), (0,1), (1,0), and (1,1) matrix elements correspond to the $(0,0)$, $(0,a)$, $(a,0)$ and (a,b) components of $\mathcal{N}$, so that for instance $\text{Re}\mathcal{N}_{00} = -\frac{1}{3}\kappa_{abc} b^a b^b b^c$, etc.

The couplings of the hypermultiplet are encoded in the quaternionic matrix h_{uv}, obtained from the Kaluza-Klein reduction as [25]

$$\begin{aligned} h_{uv}\, \text{d}q^u\, \text{d}q^v = (\text{d}D)^2 + G_{c\bar{d}}\, \text{d}z^c\, \text{d}\bar{z}^d + \tfrac{1}{4} e^D (\text{d}a - (\tilde{\xi}_C\, \text{d}\xi^C - \xi^C\, \text{d}\tilde{\xi}_C))^2 \\ - \tfrac{1}{2} e^{2D} (\text{Im}\mathcal{M})^{-1CD} (\text{d}\tilde{\xi}_C - \mathcal{M}_{CD}\, \text{d}\xi^D)(\text{d}\tilde{\xi}_E - \mathcal{M}_{EF}\, \text{d}\xi^F) \ , \end{aligned} \tag{5.60}$$

where $G_{c\bar{d}}$ is the metric of the moduli space of complex structure deformations, defined earlier in eq. (5.18). The complex coupling matrix $\mathcal{M}_{CD}$ is the complex structure equivalent of $\mathcal{N}_{AB}$, which can be determined by eq. (5.57) using the complex structure prepotential $\mathcal{F}$ and metric G_{cd} of $\mathcal{M}^{\text{cs}}$, i.e.

$$\mathcal{M}_{CD} = \overline{\mathcal{F}}_{CD} + 2i \frac{\text{Im}\mathcal{F}_{CE} \text{Im}\mathcal{F}_{DF} Z^E Z^F}{\text{Im}\mathcal{F}_{EF} Z^E Z^F} \ . \tag{5.61}$$

It can also defined in terms of the $H^{(3)}$ cohomology basis [26] according to

$$\begin{aligned} &\int_M \beta^C \wedge \star \beta^D = (\text{Im}\mathcal{M})^{-1CD} \ , \\ &\int_M \alpha^C \wedge \star \alpha^D = -\Big(\text{Im}\mathcal{M} + (\text{Re}\mathcal{M})(\text{Im}\mathcal{M})^{-1}(\text{Re}\mathcal{M})\Big)_{CD} \ , \\ &\int_M \alpha_C \wedge \star \beta^D = -\Big((\text{Re}\mathcal{M})(\text{Im}\mathcal{M})^{-1}\Big)_C^{\ \ D} \ . \end{aligned} \tag{5.62}$$

We proceed to the case of type IIB theory compactified on a Calabi-Yau. For clarity an explicit calculation of this case is found in Appendix D, but here we only present the main steps. The procedure for type IIA is analogous.

The NSNS field expansions for type IIB are the same as the ones for type IIA, i.e. eqs. (5.49)–(5.51) for the dilaton, metric and $B_2^{(10)}$ field. The RR fields however will be expanded according to

$$C_2^{(10)} = C_2(x) + c^a(x)\omega_a(y) \ , \tag{5.63}$$

$$C_4^{(10)} = D_2^a(x) \wedge \omega_a(y) + V^C(x) \wedge \alpha_C(y) - U_C(x) \wedge \beta^C(y) + \rho_a(x)\tilde{\omega}^a(y) \ . \tag{5.64}$$

In the field expansions for type IIB the 4D fields that appear are the scalars $b^a(x), c^a(x), \rho_a(x)$, the 1-forms $V^C(x), U_C(x)$ and the 2-forms $B_2(x), C_2(x), D_2^a(x)$. The self-duality condition $F_5^{(10)} = \star F_5^{(10)}$ eliminates half the degrees of freedom, which will allow us to drop D_2^a and U_c in favour of V^C and ρ_a in the expansion above.

Inserting the expansions of eqs. (5.63), (5.64) into the action in eq. (5.47) and integrating over the Calabi-Yau manifold, the resulting 4D action, see Appendix D or [27][28], is

$$\begin{aligned} S_{\rm IIB}^{(4)} = \int & -\tfrac{1}{2}R \star \mathbb{1} + \tfrac{1}{4}\mathrm{Re}\mathcal{M}_{CD}F^C \wedge F^D + \tfrac{1}{4}\mathrm{Im}\mathcal{M}_{CD}F^C \wedge \star F^D \\ & - G_{cd}\,\mathrm{d}z^c \wedge \star\,\mathrm{d}\bar{z}^d - G_{ab}\,\mathrm{d}t^a \wedge \star\,\mathrm{d}\bar{t}^b - \mathrm{d}D \wedge \star\,\mathrm{d}D - \tfrac{1}{24}e^{2D}\kappa\,\mathrm{d}C_0 \wedge \star\,\mathrm{d}C_0 \\ & - \tfrac{1}{6}e^{2D}\kappa G_{ab}(\mathrm{d}c^a - C_0\,\mathrm{d}b^a) \wedge \star(\mathrm{d}c^b - C_0\,\mathrm{d}b^b) \\ & - \tfrac{3}{8\kappa}e^{2D}G_{ad}(\mathrm{d}\rho_a - \kappa_{abc}c^b\,\mathrm{d}b^c) \wedge \star(\mathrm{d}\rho_d - \kappa_{def}c^e\,\mathrm{d}b^f) \\ & - \tfrac{1}{4}e^{-4D}\,\mathrm{d}B_2 \wedge \star\,\mathrm{d}B_2 - \tfrac{1}{24}e^{-2D}\kappa(\mathrm{d}C_2 - C_0\,\mathrm{d}B_2) \wedge \star(\mathrm{d}C_2 - C_0\,\mathrm{d}B_2) \\ & - \tfrac{1}{2}\,\mathrm{d}C_2 \wedge (\rho_a\,\mathrm{d}b^a - b^a\,\mathrm{d}\rho_a) + \tfrac{1}{2}\,\mathrm{d}B_2 \wedge c^a\,\mathrm{d}\rho_a - \tfrac{1}{4}\kappa_{abc}c^a c^d\,\mathrm{d}B_2 \wedge \mathrm{d}b^c \ . \end{aligned} \tag{5.65}$$

Here $F^C = \mathrm{d}V^C$, the G_{cd} is the metric of $\mathcal{M}^{\rm cs}$ and G_{ab} the metric of $\mathcal{M}^{\rm Ks}$. This action is usually written in a "standard $\mathcal{N}=2$ form" which does not include the 2-forms B_2 and C_2. In order to obtain it, these fields are *dualised* to scalar fields, which is allowed since they are massless and possess gauge symmetries $B_2 \to B_2 + \mathrm{d}\Lambda_2$ and $C_2 \to C_2 + \mathrm{d}\Lambda_1$ [29]. The C_2 field, with field strength $\mathrm{d}C_2$, is dualised to a scalar field c^0 by introducing a Lagrange multiplier $\frac{1}{2}c^0\,\mathrm{d}(\mathrm{d}C_2)$ and adding a term $\frac{1}{2}\,\mathrm{d}C_2 \wedge \mathrm{d}c^0$ to the Lagrangian. The terms in eq. (5.65) that contain C_2, including our newly added term, are

$$\mathcal{L}_{C_2} = -\tfrac{g}{4}(\mathrm{d}C_2 - C_0\,\mathrm{d}B_2) \wedge \star(\mathrm{d}C_2 - C_0\,\mathrm{d}B_2) - \tfrac{1}{4}\,\mathrm{d}C_2 \wedge J_1 + \tfrac{1}{2}\,\mathrm{d}C_2 \wedge \mathrm{d}c^0 \ , \tag{5.66}$$

where we have abbreviated $g = \frac{1}{6}e^{-2D}\kappa$ and $J_1 = \rho_a\,\mathrm{d}b^a - b^a\,\mathrm{d}\rho_a$. The equation of motion for c^0 implies $\mathrm{d}C_2 = \mathrm{d}C_2$ and the equation of motion for $\mathrm{d}C_2$ reads $\star\,\mathrm{d}C_2 = \frac{1}{g}(\mathrm{d}c^0 - \frac{1}{2}J_1)$. Inserting the expression for $\star\,\mathrm{d}C_2$ into the Lagrangian in eq. (5.66), it reads in terms of c^0

$$\mathcal{L}_{c^0} = -\tfrac{1}{4g}(\mathrm{d}c^0 - \tfrac{1}{2}J_1) \wedge \star(\mathrm{d}c^0 - \tfrac{1}{2}J_1) - \tfrac{1}{2}C_0\,\mathrm{d}B_2 \wedge (\mathrm{d}c^0 - \tfrac{1}{2}J_1) \ . \tag{5.67}$$

The same method is used to dualise B_2 to another scalar field b^0, and we refer to Appendix D for the details. The type IIB compactified action can then be written

$$\begin{aligned} S_{\rm IIB}^{(4)} = \int_{\mathcal{M}_4} & -\tfrac{1}{2}R \star \mathbb{1} + \tfrac{1}{4}\mathrm{Re}\mathcal{M}_{CD}F^C \wedge F^D + \tfrac{1}{4}\mathrm{Im}\mathcal{M}_{CD}F^C \wedge \star F^D \\ & - G_{cd}\,\mathrm{d}z^c \wedge \star\,\mathrm{d}\bar{z}^d - h_{pq}\,\mathrm{d}\tilde{q}^p \wedge \star\,\mathrm{d}\tilde{q}^q \ , \end{aligned} \tag{5.68}$$

where the $\tilde{q}^p$ denote the coordinates for all $h^{(1,1)}+1$ hypermultiplets, and the quarternionic metric is given by

$$\begin{aligned} h_{pq}\,\mathrm{d}\tilde{q}^p\,\mathrm{d}\tilde{q}^q &= (\mathrm{d}D)^2 + G_{ab}\,\mathrm{d}t^a\,\mathrm{d}\bar{t}^b + \tfrac{1}{24}e^{2D}\kappa(\mathrm{d}C_0)^2 \\ &+ \tfrac{1}{6}e^{2D}\kappa G_{ab}\left(\mathrm{d}c^a - C_0\,\mathrm{d}b^a\right)\left(\mathrm{d}c^b - C_0\,\mathrm{d}b^b\right) \\ &+ \tfrac{3}{8\kappa}e^{2D}G^{ad}\left(\mathrm{d}\rho_a - \kappa_{abc}c^b\,\mathrm{d}b^c\right)\left(\mathrm{d}\rho_d - \kappa_{def}c^e\,\mathrm{d}b^f\right) \\ &+ \tfrac{3}{2\kappa}e^{2D}\left(\mathrm{d}c^0 - \tfrac{1}{2}(\rho_a\,\mathrm{d}b^a - b^a\,\mathrm{d}\rho_a)\right)^2 \\ &+ \tfrac{1}{2}e^{4D}\left(\mathrm{d}b^0 + C_0\,\mathrm{d}c^0 + c^a\,\mathrm{d}\rho_a + \tfrac{1}{2}C_0(\rho_a\,\mathrm{d}b^a - b^a\,\mathrm{d}\rho_a) - \tfrac{1}{4}\kappa_{abc}c^a c^b\,\mathrm{d}b^c\right)\ . \end{aligned} \tag{5.69}$$

The to the 2-forms C_2 and B_2 dualised scalars c^0 and b^0 are functions of z^c. See also [30][31]. In type IIB the moduli space $\mathcal{M}^{\mathrm{SK}} \oplus \mathcal{M}^{\mathrm{Q}}$ is a little different from IIA. The $\tilde{q}^p$ span the quaternionic manifold $\mathcal{M}^{\mathrm{Q}}$ which now contains $\mathcal{M}^{\mathrm{Ks}}$. The prepotential $\tilde{\mathcal{F}}$ of $\mathcal{M}^{\mathrm{Ks}}$ can be chosen so that $\mathcal{M}^{\mathrm{Q}}$ is special quaternionic. The complex structure manifold is now in the special Kähler one, i.e. $\mathcal{M}^{\mathrm{cs}} \in \mathcal{M}^{\mathrm{SK}}$.

5.1.3 Mirror symmetry

In this section we are to briefly review mirror symmetry in Calabi-Yau compactifications. The mirror symmetry is a duality between the moduli space of different Calabi-Yau manifolds. It states that given a Calabi-Yau M, there exists a mirror Calabi-Yau $\tilde{M}$ such that

$$H^{(p,q)}(M) = H^{(3-p,q)}(\tilde{M})\ . \tag{5.70}$$

Note that there are however a few special cases where this fails; there is still a mirror manifold, but it is not a Calabi-Yau. In particular, eq. (5.70) implies that

$$h^{(1,1)}(M) = h^{(2,1)}(\tilde{M})\ , \quad \text{and} \quad h^{(1,1)}(\tilde{M}) = h^{(2,1)}(M)\ , \tag{5.71}$$

so for for example when M has $h^{(2,1)} = 0$, then $h^{(1,1)}$ of $\tilde{M}$ should vanish, but a Calabi-Yau always has $h^{(1,1)} \geq 0$, so $\tilde{M}$ can not be a Calabi-Yau. Eq. (5.71) corresponds to a reflection along the diagonal of the *Hodge diamond*, see Appendix B.4. Thus mirror symmetry relates the even and odd cohomologies of two topologically distinct Calabi-Yau manifolds via

$$H^{\mathrm{even}}(M) \simeq H^{\mathrm{odd}}(\tilde{M})\ , \qquad H^{\mathrm{even}}(\tilde{M}) \simeq H^{\mathrm{odd}}(M)\ , \tag{5.72}$$

where the even and odd cohomologies are defined by

$$H^{\mathrm{even}} = H^{(0,0)} \oplus H^{(1,1)} \oplus H^{(2,2)} \oplus H^{(3,3)}\ , \tag{5.73}$$

$$H^{\mathrm{odd}} = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}\ . \tag{5.74}$$

It also implies an identification of the moduli spaces of deformations of M and $\tilde{M}$. The dimensions of the complex structure and Kähler structure moduli spaces introduced in the previous section are exactly $h^{(2,1)}$ and $h^{(1,1)}$ respectively. Thus eq. (5.71) implies that

$$\mathcal{M}^{\mathrm{Ks}}(M) = \mathcal{M}^{\mathrm{cs}}(\tilde{M})\ , \qquad \mathcal{M}^{\mathrm{Ks}}(\tilde{M}) = \mathcal{M}^{\mathrm{cs}}(M)\ . \tag{5.75}$$

This statement is highly non-trivial since the Calabi-Yau manifolds M and $\tilde{M}$ in general are *very* topologically different. This can be seen from their Euler characteristics, which are related according to

$$\chi(M) = -\chi(\tilde{M})\ . \tag{5.76}$$

Recall that the geometry of $\mathcal{M}^{\rm cs}(M)$ is encoded in the variations of the holomorphic 3-form Ω of M. As seen earlier as in eq. (5.26) it can be expanded in a basis of $H^{(3)}(M)$ according to

$$\Omega = Z^C \alpha_C - \mathcal{F}_C(Z)\beta^C \ , \tag{5.77}$$

where $\mathcal{F}_C$ with $C = 0, \dots, h^{(2,1)}$ is related to the prepotential $\mathcal{F}$ via $\mathcal{F}_I = \partial_I \mathcal{F}$. The coordinates z^c are related to the Z^C via $z^c = Z^c/Z^0$ where $c = 1, \dots, h^{2,1}$. Under mirror symmetry, these coordinates are identified with the complex Kähler coordinates $\tilde{t}^a$ on the Kähler structure moduli space $\mathcal{M}^{\rm Ks}_{\tilde{M}}$ of the mirror manifold $\tilde{M}$. This also implies that the prepotential on these spaces should be equivalent, i.e. that $\mathcal{F}_M(z^c) = \tilde{\mathcal{F}}_{\tilde{M}}(\tilde{t}^a)$. However, $\mathcal{M}^{\rm Ks}$ and $\mathcal{M}^{\rm cs}$ should still obviously have different structures even if they are defined on different manifolds, so this can not be the full truth. In fact, one expects corrections to $\mathcal{F}_M(z^c)$ and $\tilde{\mathcal{F}}_{\tilde{M}}(\tilde{t}^a)$. When mirror symmetry is embedded into string theory, these corrections gain a physical interpretation as string wrapping 2-cycles in M, called worldsheet instantons. Schematically we have that

$$\mathcal{F}_M(z) = z^3 + \mathcal{O}(e^{-z}) = \tilde{\mathcal{F}}_{\tilde{M}}(\tilde{t}) \ , \qquad \tilde{\mathcal{F}}_M(t) = t^3 + \mathcal{O}(e^{-t}) = F_{\tilde{M}}(\tilde{z}) \ . \tag{5.78}$$

If one turns the argument around, you could instead use mirror symmetry to calculate the worldsheet instanton corrections $\mathcal{O}(e^{-z})$ and $\mathcal{O}(e^{-t})$. This has been done in [32]. Such a calculation is usually simpler than a direct calculation of the contributions from worldsheet instantons.

The multiplets of the compactified theories are mapped via mirror symmetry as well. In fact, the moduli in the effective actions experience the exchange

$$\{\xi^C, \tilde{\xi}_C\} \longleftrightarrow \{c^A, \rho_A\} = \{C_2, c^a, C_0, \rho_a\} \ . \tag{5.79}$$

The reasoning of mirror symmetry between moduli spaces can be expanded into perhaps the most prominent conjecture based on mirror symmetry. Namely, that the type IIA theory compactified on a manifold M is equivalent to type IIB theory compactified on a mirror manifold $\tilde{M}$, and vice versa. This can be seen explicitly by specific redefinitions of the type IIB hypermultiplet scalars in terms of the IIA vector multiplet scalars, namely

$$\begin{aligned} C_0 &\to \xi^0 \ , & C_0 b^a - c^a &\to \xi^a \ , \\ -c^0 + \tfrac{1}{2}\kappa_{abc} b^a b^b c^c - \tfrac{1}{6} C_0 \kappa_{abc} b^a b^b b^c &\to \tilde{\xi}_0 \ , & \rho_a + \tfrac{1}{2} C_0 \kappa_{abc} b^b b^c - \kappa_{abc} c^b b^c &\to \tilde{\xi}_a \ , \\ 2b^0 + C_0 c^0 + \rho(c^a - C_0 b^a) &\to a \ , & \mathrm{d}\rho_a - \kappa_{abc} c^b \, \mathrm{d}b^c &\to \mathrm{d}\tilde{\xi}_a + \mathrm{Re}\mathcal{N}_{aB} \, \mathrm{d}\xi^B \ . \end{aligned} \tag{5.80}$$

Using these redefinitions in the effective action of the type IIB theory, we can compare it with that of type IIA. It becomes clear that we can go from the type IIA to the type IIB compactification on a Calabi-Yau 3-fold by allowing the moduli space mapping

$$z^c \to t^a \ , \qquad G_{ab} \to G_{c\bar{d}} \ , \qquad \mathcal{N}_{AB} \to \mathcal{M}_{CD} \ , \tag{5.81}$$

where the quantities are defined as earlier.

T-duality is perhaps the most simple example of mirror symmetry. When the bosonic string is compactified on a circle of radius R, the perturbative string spectrum is given by

$$\alpha' M^2 = \alpha' \left[\left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2 \right] + 2N_L + 2N_R - 4 \ , \tag{5.82}$$

where N_L, N_R are the total number of oscillators of the left- and right movers, W is the winding number and K the number of momenta on the string. They fulfil the modified level-matching condition

$$N_R - N_L = WK \ , \tag{5.83}$$

and T-duality denotes the invariance of the above equations under the interchanges

$$W \leftrightarrow K \ , \qquad R \leftrightarrow \frac{\alpha'}{R} \ , \tag{5.84}$$

which must be imposed simultaneously.

5.2 Effective actions of type II Calabi-Yau orientifolds

The subject of this section is the 4D low-energy effective supergravity theory obtained by compactifying type IIA and type IIB string theory on Calabi-Yau orientifolds. Similar to D-branes, orientifold planes are hyperplanes in the 10D background.

5.2.1 The type II orientifold projections

Orientifolds are generalisations of orbifolds which in turn are generalisations of manifolds. In short, an orbifold ("orbit-manifold") is defined as the quotient space $\Gamma = M/G$ of a manifold M and a discrete group G which acts on M. Orientifold theories are *unoriented*, i.e. contain unoriented strings, strings that have no "arrow" and whose two opposite orientations are equivalent. So far we have only dealt with oriented strings, since a left to right direction of the string has been unambiguously defined by parameterising the spacelike coordinate with σ. The simplest example of an unoriented string theory is the type I theory, which can be obtained by orientifolding type IIB string theory.

A Calabi-Yau orientifold can be written as the quotient space

$$\frac{M}{S_1 \cup S_2 \Omega_p} \ , \tag{5.85}$$

where M is a Calabi-Yau manifold and S_1, S_2 are some discrete isometry (distance-preserving) groups. Ω_p is the *worldsheet parity operator* and acts on σ, which again signifies the position along the length of a string, such that $\Omega_p : \sigma \to \sigma_0 - \sigma$. For closed strings $\sigma_0 = 2\pi$ and for open strings $\sigma_0 = \pi$. This means that the parity operator exchanges left- and right-moving fields, which leaves the two σ-directions equivalent, thus resulting in unoriented strings. Note that if S_2 is empty, the quotient space is an orbifold.

The locus where the orientifold action reduces to the change of the string orientation is called the orientifold plane. In type II theories, Op-planes couple to the $(p+1)$-form RR gauge potentials C_{p+1}. Type IIA and IIB have different gauge potentials, so they will couple to different Op-planes accordingly; type IIA couples to O0-, O2-, O4-, O4- and O8-planes while type IIB couples to O(-1)-, O1-, O3-, O5-, O7-, and O9-planes. However, for the orientifolds to be consistent with supersymmetry in a Calabi-Yau, not all of the available O-planes will be viable. Namely, for type IIA only O6-planes and for type IIB only O3-, O5-, O7- and O9-planes, are allowed by supersymmetry [1].

It is important to note however that while the theory is unoriented on the O-plane, away from the fixed O-planes the local physics are those of the oriented string theory [33].

Calabi-Yau orientifolds are constructed by dividing, or modding, out a discrete symmetry $S_1 \cup S_2 \Omega_p$ that includes the worldsheet parity Ω_p and spacetime fermion number in the left-moving sector $(-1)^{\mathrm{FL}}$. This division of symmetries in the groups of $S_1 \cup S_2 \Omega_p$ are referred to as *orientifold projections* and, depending on their actions, they will project out parts of the spectra given in table 5.2. The S_2 group in eq. (5.85) for a Calabi-Yau is generated by an involution σ_{inv}, with the subscript "inv" to distinguish from the string length parameter σ. It acts only on the Calabi-Yau and thus leaves the external 4D space invariant. The involution is of course

	ϕ	g	B_2	C_0	C_1	C_2	C_3	C_4
$(-1)^{F_L}$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
Ω_p	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$

Table 5.3: Symmetry transformations of type II massless bosonic fields. The actions of $(-1)^{F_L}$ and Ω_p on the fields leave the type IIB supergravity action invariant but not the type IIA. The table should be read according to $(-1)^{F_L}B_2 = +B_2$, $\Omega_p B_2 = -B_2$, etc.

involutive; $\sigma^2 = \mathbb{1}$, but also isometric and holomorphic *or* antiholomorphic depending on the theory. It acts on the complex structure (1,1)-form J and the holomorphic 3-form Ω according to [34]

$$\text{IIA}: \qquad \sigma^*_{\text{inv}} J = -J \ , \qquad \sigma^*_{\text{inv}}\Omega = \bar{\Omega} \ , \tag{5.86}$$

$$\text{IIB}: \qquad \sigma^*_{\text{inv}} J = J \ , \qquad \sigma^*_{\text{inv}}\Omega = \pm\Omega \ . \tag{5.87}$$

From the involution's action on Ω, we see that the involution is antiholomorphic in type IIA and holomorphic in type IIB. The two possible actions on the Ω for type IIB splits the viable choices of O-planes. The plus sign leads to O5- or O9-planes and the minus sign leads to O3- or O7-planes. Allowed symmetry operations will be different for the two cases as well. With the O5/O9-planes the simplest symmetry transformation of S_2 is a target space symmetry $\mathcal{M}_{10} \to \mathcal{M}_{10}$, but for the O3/O7-planes there is an additional operation $(-1)^{F_L}$, where F_L is the spacetime fermion number in the left-moving sector. Under this action the RR and RNS states are odd and the NSNS and NSR states are even. Another symmetry of type IIB is Ω_p. Both symmetries are summarised in table 5.3. Depending on the transformation properties of Ω, there are two possible symmetry operations $\mathcal{O} = S_2\Omega_p$ that are possible [35][36]:

$$\mathcal{O}_1 = (-1)^{F_L}\Omega_p\sigma_{\text{inv}} \ , \qquad \text{and} \qquad \mathcal{O}_2 = \Omega_p\sigma_{\text{inv}} \ . \tag{5.88}$$

Modding out with $\mathcal{O}_1$, i.e. having $M/\mathcal{O}_1$, leads to the possibility of having O3/O7-planes, and modding out with $\mathcal{O}_2$ allows for O5/O9-planes. In summary we have that

$$\text{O3/O7}: \qquad \sigma^*_{\text{inv}}\Omega = -\Omega \qquad \to \qquad \mathcal{O}_1 = \mathcal{O}_{\text{O3/O7}} = (-1)^{F_L}\Omega_p\sigma_{\text{inv}} \ , \tag{5.89}$$

$$\text{O5/O9}: \qquad \sigma^*_{\text{inv}}\Omega = +\Omega \qquad \to \qquad \mathcal{O}_2 = \mathcal{O}_{\text{O5/O9}} = \Omega_p\sigma_{\text{inv}} \ , \tag{5.90}$$

i.e. the different actions of σ_{inv} on Ω lead to different possible symmetry operations $\mathcal{O} = S_2\Omega_p$. As for type IIA, the symmetry operation $\mathcal{O}_1$ also allows for O6-planes [29].

Wishing to compute the effective 4D action for type IIB supergravity on our orientifold planes, we proceed with the same methodology as for the compactification on Calabi-Yau manifolds. Having established some symmetry operations which are modded out from the Calabi-Yau manifold, the next step is to determine the massless spectrum after this projection has been taken into account. In the 4D compactified theory only states invariant under the projection are kept, so the massless states that survive the orientifold projection are the ones that are even under the actions of eq. (5.88). For example, B_2 is odd under the action of Ω_p and also under $(-1)^{F_L}\Omega$, so in the presence of any O-plane the only surviving components of B_2 are those that are odd under σ_{inv}. Generally the actions of σ_{inv} on the fields of table 5.3 will be different depending on which O-plane setup we use. For O3/O7-planes the invariant states are even under the transformation of $\mathcal{O}_{\text{O3/O7}}$, and invariant states on O5/O9-planes are even under the action of $\mathcal{O}_{\text{O5/O9}}$. Using table 5.3 and eqs. (5.89), (5.90), there are three fields that transform according to

$$\begin{aligned} \sigma^*\phi^{(10)} &= \phi^{(10)} \ , \\ \sigma^* g^{(10)} &= g^{(10)} \ , \\ \sigma^* B_2^{(10)} &= -B_2^{(10)} \ , \end{aligned} \tag{5.91}$$

O-plane setup	Cohomology group		Basis		Dimension	
O3/O7 and O5/O9	$H_+^{(1,1)}$	$H_-^{(1,1)}$	ω_α	ω_a	$h_+^{(1,1)}$	$h_-^{(1,1)}$
	$H_+^{(2,1)}$	$H_-^{(2,1)}$	χ_κ	χ_k	$h_+^{(2,1)}$	$h_-^{(2,1)}$
	$H_+^{(1,1)}$	$H_-^{(1,1)}$	ω^α	ω^a	$h_+^{(1,1)}$	$h_-^{(1,1)}$
O3/O7	$H_+^{(3)}$	$H_-^{(3)}$	$(\alpha_\kappa, \beta^\lambda)$	$(\alpha_{\hat{k}}, \beta^{\hat{l}})$	$2h_+^{(2,1)}$	$2h_-^{(2,1)}+2$
O5/O9	$H_+^{(3)}$	$H_-^{(3)}$	$(\alpha_{\hat{\kappa}}, \beta^{\hat{\lambda}})$	(α_k, β^l)	$2h_+^{(2,1)}+2$	$2h_-^{(2,1)}$

Table 5.4: Cohomology groups and their corresponding bases for orientifold setups of type IIB supergravity.

for both the O3/O7 and O5/O9 setups. For the other fields however, the actions of $\mathcal{O}_{\text{O3/O7}}$ and $\mathcal{O}_{\text{O5/O9}}$ will force σ_{inv} to act on the RR fields of type IIB (i.e. C_0, C_2, and C_4) according to

$$\begin{array}{c|c} \text{O3/O7} & \text{O5/O9} \\ \hline \sigma_{\text{inv}}^* C_0^{(10)} = C_0^{(10)} & \sigma_{\text{inv}}^* C_0^{(10)} = -C_0^{(10)} \\ \sigma_{\text{inv}}^* C_2^{(10)} = -C_2^{(10)} & \sigma_{\text{inv}}^* C_2^{(10)} = C_2^{(10)} \\ \sigma_{\text{inv}}^* C_4^{(10)} = C_4^{(10)} & \sigma_{\text{inv}}^* C_4^{(10)} = -C_4^{(10)} \end{array} . \tag{5.92}$$

Furthermore, we seek a to expand the fields and metric deformations in terms of harmonic forms, which turns our attention to the relevant cohomology groups. Since σ_{inv} is holomorphic for type IIB, the action of σ_{inv} on $H^{(p,q)}$ and with it the harmonic (p,q)-forms, will split into two eigenspaces;

$$\sigma_{\text{inv}}^* H^{(p,q)} = H_+^{(p,q)} \oplus H_-^{(p,q)} . \tag{5.93}$$

The cohomology eigenspaces have eigenvalues $+1$ and -1, as well as dimensions $h_+^{(p,q)}$ and $h_-^{(p,q)}$, respectively. The involution σ_{inv} respects the orientation and metric of the Calabi-Yau, which mean that the action σ_{inv} and the Hodge-$\star$ operator commute. This means that the Hodge numbers will obey $h_\pm^{(1,1)} = h_\pm^{(2,2)}$. Furthermore, the holomorphicity of σ_{inv} implies that $h_\pm^{(2,1)} = h_\pm^{(1,2)}$ and the fact that $\sigma_{\text{inv}}^*\Omega = \pm\Omega$ leads to that for the different O-planes we have

$$\begin{array}{c|c} \text{O3/O7} & \text{O5/O9} \\ \hline h_+^{(3,0)} = h_+^{(0,3)} = 0 & h_+^{(3,0)} = h_+^{(0,3)} = 1 \\ h_-^{(3,0)} = h_-^{(0,3)} = 1 & h_-^{(3,0)} = h_-^{(0,3)} = 0 \end{array} . \tag{5.94}$$

The volume form $\sim \Omega\wedge\bar{\Omega}$ is invariant under σ_{inv}^* and so $h_+^{(0,0)} = h_+^{(3,3)} = 1$ and $h_-^{(0,0)} = h_-^{(3,3)} = 1$. The non-trivial cohomology groups and basis elements can be constructed in the same way as for the Calabi-Yau 3-fold, and they are summarised in table 5.4 taken from [29], where $H_\pm^{(3)} = H_\pm^{(1,2)} \oplus H_\pm^{(2,1)}$.

Now, the 4D invariant type IIB spectrum is found by using the Kaluza-Klein expansion of the Kähler structure and complex structure metric deformations given earlier in eqs. (5.9), (5.14), as well as the field expansions in eqs. (5.63), (5.64), where only fields obeying eq. (5.92) are kept. However, some modifications of these expansions will be needed. For starters, σ_{inv} leaves the volume form invariant and, since $\Omega \wedge \bar{\Omega} \sim J \wedge J \wedge J$, J has to be invariant as well. As can be seen in the first equation in eq. (5.87), J transforms evenly under the action of σ_{inv}, therefore only the even $h^{(1,1)}$ Kähler deformations v^α can remain in the spectrum. Thus for the orientifolds we must expand the Kähler form as

$$J = v^\alpha \omega_\alpha , \tag{5.95}$$

with $\alpha = 1, \ldots, h_+^{(1,1)}$ where ω_α is the basis of $H_+^{(1,1)}$. The complex structure metric deformations depend on Ω, and so because of how the involution acts on it in eq. (5.87), deformations will depend on the orientifold setup. For O3/O7 the deformations kept in the spectrum are elements in $H_+^{(1,2)}$ and for O5/O9 they are elements in $H_-^{(1,2)}$. In summary, the complex structure metric deformations of eq. (5.14) will be replaced by

$$\text{O3/O7}: \qquad \delta g_{ij} = \frac{i}{||\Omega||^2} \bar{z}^k (\chi_k)_{i\bar{i}\bar{j}} \Omega^{\bar{i}\bar{j}}{}_j \ , \qquad k = 1, \ldots, h_-^{(1,2)} \ , \tag{5.96}$$

$$\text{O5/O9}: \qquad \delta g_{ij} = \frac{i}{||\Omega||^2} \bar{z}^\kappa (\chi_\kappa)_{i\bar{i}\bar{j}} \Omega^{\bar{i}\bar{j}}{}_j \ , \qquad k = 1, \ldots, h_+^{(1,2)} \ , \tag{5.97}$$

where χ_k and χ_κ denote the basis of $H_-^{(1,2)}$ and $H_+^{(1,2)}$ respectively. See table 5.4 for all bases. As for the $B_2^{(10)}$ expansion we know from eq. (5.91) that only odd elements can be kept. Therefore the original $B_2^{(10)}$ expansion for Calabi-Yau manifolds will for orientifolds be replaced by

$$B_2^{(10)} = b^a \omega_a \ , \qquad a = 1, \ldots, h_-^{(1,1)} \ . \tag{5.98}$$

Turning to the RR sector fields, it is clear from eq. (5.92) that they will be expanded differently as they transform with opposite signs under σ_{inv}. For O3/O7, the axion $C_0^{(10)}$ is even, $C_2^{(10)}$ is even and $C_4^{(10)}$ is odd. The opposite applies to the O5/O9-planes. This will be reflected in the harmonic basis. For the O3/O7 setup, the RR fields are expanded according to

$$\text{O3/O7}: \qquad \begin{aligned} C_0^{(10)} &= C_0 \ , \\ C_2^{(10)} &= c^a \omega_a \ , \\ C_4^{(10)} &= D_2^\alpha \wedge \omega_\alpha + V^\kappa \wedge \alpha_\kappa + U_\kappa \wedge \beta^\kappa + \rho_\alpha \tilde{\omega}^\alpha \ . \end{aligned} \tag{5.99}$$

Note that both the 4D 2-forms B_2 and C_2 have been projected out and only the scalars c^a and b^a remain. For the O5/O9 setup, the field expansions read

$$\text{O5/O9}: \qquad \begin{aligned} C_0^{(10)} &= 0 \ , \\ C_2^{(10)} &= C_2 + c^\alpha \omega_\alpha \ , \\ C_4^{(10)} &= D_{(2)}^a \wedge \omega_a + V^k \wedge \alpha_k + U_k \wedge \beta^k + \rho_a \tilde{\omega}^a \ . \end{aligned} \tag{5.100}$$

The axion $C_0^{(10)}$ is projected out, but is then replaced by the 4D tensor C_2 to form a *linear multiplet* $\{\phi, C_2\}$ [29]. Again the self-duality of $F_5^{(10)}$ eliminates half the number of degrees of freedom in the $C_4^{(10)}$ expansion, where one can choose to eliminate the pair $\{D_2^a, \rho_a\}$ or $\{V^k, U_k\}$.

It is clear that the orientifold projection splits the dimension of the moduli space, and with it the $\mathcal{N} = 2$ supersymmetry spontaneously breaks down to $\mathcal{N} = 1$. The IIB multiplets are summarised in table 5.5.

5.2.2 The 4D effective orientifold theories

Before performing dimensional reduction, we need to make an important note about the 10D metric. With localised sources, such as orientifold planes and D-branes, the metric structure of eq. (5.48) is no longer entirely valid. In fact, the supergravity theory with sources, as well as fluxes, does not have the metric of eq. (5.48) as a solution [37][38][39]. Instead a warp-factor e^{-2A} has to be included in the metric such that

$$\mathrm{d}s^2 = e^{2A(y)} \eta_{\mu\nu}(x)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + e^{-2A(y)} g_{i\bar{j}}(y)\, \mathrm{d}y^i\, \mathrm{d}y^{\bar{j}} \ . \tag{5.101}$$

IIB	O3/O7		O5/O9	
gravity multiplet	1	$g_{\mu\nu}$	1	$g_{\mu\nu}$
vector multiplets	$h_+^{(2,1)}$	V^κ	$h_-^{(2,1)}$	V^k
chiral multiplets	$h_-^{(2,1)}$	z^k	$h_+^{(2,1)}$	z^κ
	$h_+^{(1,1)}$	$\{v^\alpha, \rho^\alpha\}$	$h_+^{(1,1)}$	$\{v^\alpha, c^\alpha\}$
	$h_-^{(1,1)}$	$\{b^a, c^a\}$	$h_-^{(1,1)}$	$\{b^a, \rho_a\}$
	1	$\{\phi, C_0\}$	1	$\{\phi, C_2\}$

Table 5.5: The 4D moduli fields of compactified type IIB supergravity theory on Calabi-Yau orientifold setups O3/O7 and O5/O9 arranged in $\mathcal{N}=1$ multiplets.

However, one may work in the large radius limit where $A(y) \to 0$ so that the metric of eq. (5.101) coincides with the unwarped metric of eq. (5.48) [39]. In this case the metric of the moduli spaces agree and the effective actions will be the same.

The effective 4D actions for type IIB orientifold planes are found by using our harmonic metric and field expansions and inserting them into the type IIB supergravity action of eq. (5.47), integrate over the Calabi-Yau and evaluate the multiplet sectors separately. We refer to [29][40] for details and give the actions for the O3/O7 and O5/O9 orientifold setups as

$$
\begin{aligned}
S^{(4)}_{\mathrm{O3/O7}} = \int & -\tfrac{1}{2} R \star \mathbf{1} - G_{k\bar{l}}\,\mathrm{d}z^k \wedge \star\, \mathrm{d}\bar{z}^l - G_{\alpha\beta}\,\mathrm{d}v^\alpha \wedge \star\, \mathrm{d}v^\beta - G_{ab}\,\mathrm{d}b^a \wedge \star\, \mathrm{d}b^b \\
& - \mathrm{d}D \wedge \star\, \mathrm{d}D - \tfrac{1}{24} e^{2D} \kappa\, \mathrm{d}C_0 \wedge \star\, \mathrm{d}C_0 \\
& - \tfrac{1}{6} e^{2D} \kappa G_{ab} (\mathrm{d}c^a - C_0\,\mathrm{d}b^a) \wedge \star (\mathrm{d}c^b - C_0\,\mathrm{d}b^b) \\
& - \tfrac{3}{8\kappa} e^{2D} G^{\alpha\beta} (\mathrm{d}\rho_\alpha - \kappa_{\alpha ab} c^a\,\mathrm{d}b^b) \wedge \star (\mathrm{d}\rho_\beta - \kappa_{\beta cd} c^c\,\mathrm{d}b^d) \\
& + \tfrac{1}{4} \mathrm{Im}\mathcal{M}_{\kappa\lambda} F^\kappa \wedge \star F^\lambda + \tfrac{1}{4} \mathrm{Re}\mathcal{M}_{\kappa\lambda} F^\kappa \wedge F^\lambda \ ,
\end{aligned}
\tag{5.102}
$$

respectively

$$
\begin{aligned}
S^{(4)}_{\mathrm{O5/O9}} = \int & -\tfrac{1}{2} R \star \mathbf{1} - G_{\kappa\bar{\lambda}}\,\mathrm{d}z^\kappa \wedge \star\, \mathrm{d}\bar{z}^\lambda - G_{\alpha\beta}\,\mathrm{d}v^\alpha \wedge \star v^\beta \\
& - G_{ab}\,\mathrm{d}b^a \wedge \star\, \mathrm{d}b^b - \mathrm{d}D \wedge \star\, \mathrm{d}D - \tfrac{1}{6} e^{2D} \kappa G_{\alpha\beta}\,\mathrm{d}c^\alpha \wedge \star\, \mathrm{d}c^\beta \\
& - \tfrac{3}{2\kappa} e^{2D} \left(\mathrm{d}h + \tfrac{1}{2}(\mathrm{d}\rho_a b^a - \rho_a\,\mathrm{d}b^a)\right) \wedge \star \left(\mathrm{d}h + \tfrac{1}{2}(\mathrm{d}\rho_a b^a - \rho_a\,\mathrm{d}b^a)\right) \\
& - \tfrac{3}{8\kappa} e^{2D} G^{ab} (\mathrm{d}\rho_a - \kappa_{ac\alpha} c^\alpha\,\mathrm{d}b^c) \wedge \star (\mathrm{d}\rho_b - \kappa_{bd\beta} c^\beta\,\mathrm{d}b^d) \\
& + \tfrac{1}{4} \mathrm{Im}\mathcal{M}_{kl} F^k \wedge \star F^l + \tfrac{1}{4} \mathrm{Re}\mathcal{M}_{kl} F^k \wedge F^l \ .
\end{aligned}
\tag{5.103}
$$

Here $F^k = \mathrm{d}V^k$ and $F^\kappa = \mathrm{d}V^\kappa$ with the rest of the definitions the same as in the previous section, but expressed in the appropriate bases.

Turning to the type IIA case, the NSNS sector fields transform in the same way as for type IIB under the action of the involution σ_{inv}, given in eq. (5.91). As for the RR fields, they transform according to

$$\sigma^*_{\mathrm{inv}} C_1^{(10)} = -C_1^{(10)} \ , \tag{5.104}$$

$$\sigma^*_{\mathrm{inv}} C_3^{(10)} = C_3^{(10)} \ , \tag{5.105}$$

in accordance with eq. (5.86) and the action $\mathcal{O}_1$ of eq. (5.88). The Kähler form J is odd under the action of σ_{inv} according to eq. (5.86), so J must be expanded in a basis of odd harmonic (1,1)-forms:

$$J = v^a(x)\omega_a \ , \tag{5.106}$$

where $a = 1, \ldots, h_-^{(1,1)}$ again using the basis defined in table 5.4.

The number of complex structure deformations are reduced in a similar fashion. The projection in eq. (5.86) for the holomorphic 3-form Ω sets constraints on the deformations. The Ω can also be expanded in a basis of $H^{(3)} = H_+^{(3)} \oplus H_-^{(3)}$ with dimensions $h_+^{(3)} = h^{(3)} = h^{(2,1)} + 1$ and basis elements $a_C \in H_+^{(3)}$, $b^D \in H_-^{(3)}$, so that

$$\Omega(z) = Z^C(z)a_C - \mathcal{F}_D(z)b^D \ , \qquad C, D = 0, \ldots, h^{(2,1)} \ . \tag{5.107}$$

This is just like the Ω expansion in the previous section with $\mathcal{N} = 2$ Calabi-Yau manifold compactifications. Again the z^c with $c = 1, \ldots, h^{(2,1)}$ are coordinates on the complex structure moduli space. Generally eq. (5.86) for Ω reads $\sigma_{\text{inv}}^* \Omega = e^{2i\theta}\bar{\Omega}$ with θ being some arbitrary phase, though we have previously chosen $\theta = 0$. Including this phase invariance in the action of σ on Ω and using the expression in eq. (5.107), the involution's action on Ω implies

$$\text{Im}(e^{-i\theta} Z^C(z)) = 0 \ , \qquad \text{Re}(e^{-i\theta}\mathcal{F}_D) = 0 \ . \tag{5.108}$$

The first equation sets $h^{(2,1)}+1$ constraints on only $h^{(2,1)}$ scalars. The extra equation is redundant and due to a scale invariance of Ω. The first equation in eq. (5.108) implies however that $h^{(2,1)}$ scalars are projected out, which is half of the complex structure deformations. The involution's action on Ω in eq. (5.108) also reduces its complex rescaling freedom $\Omega \to e^{-h}\Omega$ to a real rescaling freedom; $\Omega \to e^{-\text{Re}(h)}\Omega$. This affects the Kähler potential so that $\mathcal{K}^{\text{cs}} \to \mathcal{K}^{\text{cs}} + 2\text{Re}(h)$ when in $\mathcal{M}_{\mathbb{R}}^{\text{cs}}$. This freedom can be used to set one component of $\text{Re}(e^{-i\theta} Z^C)$ equal to one, so that Ω depends only on $h^{(2,1)}$ real deformation parameters. It can however be more convenient to leave this gauge freedom intact and instead define a "compensator field" C which transforms like $C \to Ce^{\text{Re}(h)}$ and fulfils $C\Omega \to C\Omega$. Then $C\Omega$ depends on $h^{(2,1)} + 1$ real parameters and may be expanded according to

$$C\Omega = \text{Re}(CZ^C)a_C - i\text{Im}(C\mathcal{F}_C)b^C \ . \tag{5.109}$$

The expansion of the $\phi^{(10)}$, $g^{(10)}$ and $B_2^{(10)}$ fields are the same for type IIA as for IIB as they belong to the NSNS sector. Under the action of σ_{inv} they transform according to eq. (5.91). For the scalars the expansion is again simply $\phi^{(10)} = \phi$ and $g^{(10)} = g$. The $B_2^{(10)}$ field is odd under the action of σ_{inv} so the 4D field B_2 does not survive the projection, but we are left with the scalars b^a expanded in the odd basis ω_a;

$$B_2^{(10)} = b^a \omega_a \ , \qquad a = 1, \ldots, h_-^{(1,1)} \ . \tag{5.110}$$

Differences appear in the expansions of the RR sector fields. The $C_1^{(10)}$ field is odd under σ_{inv} according to eq. (5.104) and, combining the fact that a Calabi-Yau does not support any harmonic 1-forms and that σ_{inv} acts trivially on the flat 4D dimensions, the entire $C_1^{(10)}$ field is projected out. This leads to that the $\mathcal{N} = 2$ multiplet $\{g_{\mu\nu}, A^0\}$ in the Calabi-Yau 3-fold reduction is being reduced to an $\mathcal{N} = 1$ orientifold multiplet containing only $g_{\mu\nu}$. As for the $C_3^{(10)}$ field, which is even under σ_{inv} according to eq. (5.105), it is expanded as

$$C_3^{(10)} = C_3(x) + A^\alpha(x) \wedge \omega_\alpha + \xi^C a_C \ , \tag{5.111}$$

where $C_3(x)$ is the 3-form RR field in 4D, A^α with $\alpha = 1, \ldots, h_+^{(1,1)}$ are 1-forms and ξ^C are $C = 0, \ldots, h^{(2,1)}$ real scalars. The $\mathcal{N} = 1$ multiplets of IIA O6 compactifications are summarised in table 5.6, where $N^C = \frac{1}{2}(\xi^C + 2i\text{Re}Z^C)$; a field combination we will return to in section 5.4.

IIA	O6	
gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h_+^{(1,1)}$	A^α
chiral multiplets	$h_-^{(1,1)}$	t^a
	$h^{(2,1)}+1$	N^C

Table 5.6: The 4D moduli fields of compactified type IIA supergravity theory on an O6 Calabi-Yau orientifold arranged in $\mathcal{N}=1$ multiplets.

The effective 4D action for type IIA orientifolds, which only allow O6-planes by supersymmetry, is found by inserting the expansions into the type IIA supergravity action of eq. (5.45) and integrating over the Calabi-Yau. The resulting action reads

$$\begin{aligned} S_{\mathrm{O6}}^{(4)} = \int & -\tfrac{1}{2}R \star \mathbf{1} - G_{ab}\,\mathrm{d}t^a \wedge \star\,\mathrm{d}\bar{t}^b + \tfrac{1}{2}\mathrm{Im}\mathcal{N}_{\alpha\beta}F^\alpha \wedge \star F^\beta + \tfrac{1}{2}\mathrm{Re}\mathcal{N}_{\alpha\beta}F^\alpha \wedge \star F^\beta \\ & - \mathrm{d}D \wedge \star\,\mathrm{d}D - G_{cd}(q)\,\mathrm{d}q^c \wedge \star\,\mathrm{d}q^d + \tfrac{1}{2}e^{2D}\mathrm{Im}\mathcal{M}_{CD}\,\mathrm{d}\xi^C \wedge \star\xi^D \ , \end{aligned} \tag{5.112}$$

with $F^\alpha = \mathrm{d}A^\alpha$ and was first obtained in [41]. Here the first line, except for the Einstein-Hilbert term, arises from the projection of the $\mathcal{N}=2$ vector multiplets' action. The second line arises from the reduction of the $\mathcal{N}=2$ hypermultiplets' action which is determined by the quaternionic metric in eq. (5.60).

5.3 Implications of allowing background fluxes

In the type II theories it is possible to allow background fluxes on the Calabi-Yau manifold. When sources are present, there is no globally well-defined potential, which means that the integral of the field strength over a cycle is not necessarily zero. When this is the case one talks about a *non-zero flux*. A flux with a standard Bianchi identity[1] for either NSNS or RR fluxes should be a real number, i.e.

$$\frac{1}{(2\pi\sqrt{\alpha'})^{p-1}} \int_{\Sigma_p} F_p \ \in \mathbb{Z} \ , \tag{5.113}$$

for some p-cycle Σ_p and p-form field strength F_p. The fluxes for a Calabi-Yau manifold are constructed from the 3-cycles A and B. In order to keep track on from which cycle they come from, the fluxes from the A-cycles are called *magnetic* and the ones from B-cycles are called *electric*. For each field strength, they are defined as

$$\begin{aligned} &\frac{1}{(2\pi)^2\alpha'}\int_{A_C} H_3 \equiv m^C \ , && \frac{1}{(2\pi)^2\alpha'}\int_{B^C} H_3 \equiv e^C \ , && C = 0,\dots,h^{(2,1)} \ , \\ &\frac{1}{(2\pi)^2\alpha'}\int_{A_C} F_3^{(10)} \equiv m_{\mathrm{RR}}^C \ , && \frac{1}{(2\pi)^2\alpha'}\int_{B^C} F_3^{(10)} \equiv e_{\mathrm{RR}C} \ , && \\ &\frac{1}{2\pi\sqrt{\alpha'}}\int_{A_a} F_2^{(10)} \equiv m_{\mathrm{RR}}^a \ , && \frac{1}{(2\pi\sqrt{\alpha'})^3}\int_{B^a} F_4^{(10)} \equiv e_{\mathrm{RR}K} \ , && a = 1,\dots,h^{(1,1)} \ , \end{aligned} \tag{5.114}$$

written in terms of α' for clarity. In the basis (α_C, β^C) of $H^{(3)}$ introduced earlier the α_C and β^C are Poincaré duals to the cycles. The same goes for ω_a and $\tilde{\omega}^a$ with respect to $H^{(1,1)}$ and

[1] An example of a non-standard Bianchi identity is that for a field strength of a 3-form in 4D, where no 5-forms exist, since a Bianchi has normally the form $D(\mathrm{d}A_p) = 0$ with D being a covariant derivative.

$H^{(2,2)}$, so that again;

$$\begin{aligned} \int_{A_C} \alpha_D &= \int_M \alpha_D \wedge \beta^C = \delta^C_D \ , \qquad & \int_{B^C} \beta^D &= \int_M \beta^D \wedge \alpha_C = -\delta^D_C \ , \\ \int_{A_a} \omega_b &= \int_M \omega_a \wedge \tilde{\omega}^b = \delta^b_a \ , & \int_{B^a} \tilde{\omega}^b &= \int_M \tilde{\omega}^b \wedge \omega_a = -\delta^b_a \ , \end{aligned} \tag{5.115}$$

where M is the manifold in question. The field strengths can be expanded in terms of the magnetic and electric fluxes as well as the basis defined above according to

$$\begin{aligned} \frac{1}{(2\pi)^2\alpha'} H_3 &= m^C \alpha_C - e_C \beta^C \ , \qquad & \frac{1}{(2\pi)^2\alpha'} F_3^{(10)} &= m^C_{\mathrm{RR}} \alpha_C - e_{\mathrm{RR}C} \beta^C \ , \\ \frac{1}{2\pi\sqrt{\alpha'}} F_2^{(10)} &= m^a_{\mathrm{RR}} \omega_a \ , & \frac{1}{(2\pi\sqrt{\alpha'})^3} F_4^{(10)} &= -e_{\mathrm{RR}a} \tilde{\omega}^a \ , \\ 2\pi\sqrt{\alpha'} F_0^{(10)} &= m^0_{\mathrm{RR}} \ , & \frac{1}{(2\pi\sqrt{\alpha'})^5} F_6^{(10)} &= e_{\mathrm{RR}} \mathrm{vol}_6 \ . \end{aligned} \tag{5.116}$$

It is conventional to pick units in which $(2\pi)^2\alpha' = 1$.

We will start by considering the $\mathcal{N} = 2$ compactifications of Calabi-Yau manifolds. Turning on fluxes while still keeping the same massless spectrum of the Calabi-Yau compactifications will correspond to simply rewriting the fields in terms of a magnetic or an electric flux in the appropriate cohomology basis. In type IIA, turning on RR fluxes amounts to performing the replacements

$$\begin{aligned} \mathrm{d}C_1 &\to \mathrm{d}C_1 + m^a_{RR} \omega_a \ , \\ \mathrm{d}C_3 &\to \mathrm{d}C_3 + e_{RRa} \tilde{\omega}^a \ . \end{aligned} \tag{5.117}$$

With these extra terms, the 10D reduction to 4D will result in an additional term in the type IIA 4D effective action in eq. (5.112), namely [27]

$$S_{\mathrm{RR}} = \int -B_2 \wedge J_2 - \tfrac{1}{2} M^2 B_2 \wedge \star B_2 - \tfrac{1}{2} M_T^2 B_2 \wedge \star B_2 - V^{\mathrm{RR}}_{\mathrm{IIA}} \ , \tag{5.118}$$

written in the Einstein frame, where the constituents are given by

$$\begin{aligned} J_2 &= -e_{\mathrm{RR}A} F^A + m^A_{\mathrm{RR}} \left(\mathrm{Im}\mathcal{N}_{AB} F^B + \mathrm{Re}\mathcal{N}_{AB} F^B \right) \ , \\ M^2 &= -m^A_{\mathrm{RR}} \mathrm{Im}\mathcal{N}_{AB} m^B_{\mathrm{RR}} \ , \\ M_T^2 &= -m^A_{\mathrm{RR}} \mathrm{Re}\mathcal{N}_{AB} m^B_{\mathrm{RR}} + m^A_{\mathrm{RR}} e_{\mathrm{RR}A} \ , \\ V^{\mathrm{RR}}_{\mathrm{IIA}} &= -\tfrac{1}{2} e^{4\phi} \left(e_{\mathrm{RR}A} - \overline{\mathcal{N}}_{AC} m^C_{\mathrm{RR}} \right) (\mathrm{Im}\mathcal{N})^{AB} \left(e_{\mathrm{RR}B} - \overline{\mathcal{N}}_{BD} m^D_{\mathrm{RR}} \right) \ . \end{aligned} \tag{5.119}$$

Here we have again $F^A = \mathrm{d}A^A$ and $\mathcal{N}$ given as in eqs. (5.58), (5.59). The RR fluxes introduce terms $B \wedge J_2$ known as Green-Schwarz type couplings, which make B_2 massive! There are now regular as well as topological mass terms m and m_T for the B_2 field. The fluxes also produce a potential V which depends on the complexified Kähler deformations t^a, which belong to the vector multiplets.

In type IIB, the RR fluxes are introduced by the replacement

$$\mathrm{d}C_2 \to \mathrm{d}C_2 + m^C_{\mathrm{RR}} \alpha_C - e_{\mathrm{RR}C} \beta^C \ , \tag{5.120}$$

which will result in the addition of an extra action term with the same form as the one in eq. (5.118). The difference is the definitions in eq. (5.119) where $\mathcal{N}$ is replaced with $\mathcal{M}$ and the indices $A, B, \ldots$ with $C, D, \ldots$ which span $h^{(2,1)} + 1$ terms. The introduction of the RR fluxes of type IIB thus results the same new physics as RR fluxes in type IIA. Namely, the Green-Schwarz couplings, a massive B_2 field and a potential.

The NSNS fluxes are common to both type II theories and are introduced by modifying the B_2 field according to

$$\mathrm{d}B_2 \to \mathrm{d}B_2 + m^C \alpha_C - e_C \beta^C \ . \tag{5.121}$$

The fluxes will in general give gauge charges to scalars in the tensor and hypermultiplets for both theories. That is, the ordinary derivatives of the effective 4D actions in eq. (5.56) and eq. (5.68) will become covariant derivatives of the form

$$\partial_\mu q^u \to D_\mu q^u \equiv \partial_\mu q^u - k^u_A X^A_\mu \ . \tag{5.122}$$

Here the k^u_A are Killing vectors, or gauge charges, which are directly proportional to the fluxes, and the X^A_μ are the vectors in the vector multiplet that participate in the gauging. For type IIA, the scalars that get a charge are the ones dual to the B_2 field (belonging to the tensor multiplet) and the $\{\xi^C, \tilde{\xi}_C\}$ of the hypermultiplet. The gaugings are given by

$$k^b_0 = m^C \tilde{\xi}_C - e_C \xi^C \ , \qquad k^b_a = 2e_{\mathrm{RR}a} \ , \qquad k^{\xi^C}_0 = m^C \delta^{u\xi^C} \ , \qquad k^{\tilde{\xi}^C}_0 = e_C \ , \tag{5.123}$$

where b denotes the appropriate duals to B_2 and $\delta^{u\xi^C}$ is the delta function. The only vector field participating is the graviphoton A^0, which as a consequence acquires a mass. The fluxes also generate a potential for the scalars $\{\phi, z^c, \xi^C, \tilde{\xi}_C\}$ in the hyper and tensor multiplets, which is given by

$$\begin{aligned} V^{\mathrm{NSNS}}_{\mathrm{IIA}} &= -\frac{e^{2\phi}}{4\kappa} \left(e_K + \mathcal{M}_{KM} m^M \right) (\mathrm{Im}\mathcal{M})^{KL} \left(e_L + \overline{\mathcal{M}}_{LN} m^N \right) \\ &\quad + \frac{e^{4\phi}}{2\kappa} \left(m^K \tilde{\xi}_K - e_K \xi^K + e_0 \right)^2 \ , \end{aligned} \tag{5.124}$$

where $\mathcal{M}$ rather than $\mathcal{N}$ enters. Note that out of all combinations of the axions $\{\tilde{\xi}_C, \xi^C\}$ only the linear combination $m^C \tilde{\xi}_C - e_C \xi^C$ gets a potential. This will be important in the next chapter.

In type IIB, the electric NSNS fluxes gauge the scalars in the tensor multiplet, namely the dual b^0 of B_2 and c^0 of C_2. In contrast to the IIA case, the vectors present that will gauge these scalars are the ones in the vector multiplets, i.e. V^c and z^c, rather than the graviphoton V^0. One combination of these vectors will therefore acquire a mass. The magnetic fluxes give a mass to C_2. The electric fluxes also generate a potential for the scalars in the vector multiplet as well as the axion and dilaton, which is given by

$$V^{\mathrm{NSNS}}_{\mathrm{IIB}} = -\frac{e^{4\phi}}{2\kappa} \left(C_0^2 - \frac{e^{-2\phi}}{2\kappa} \right) e_C (\mathrm{Im}\mathcal{M})^{CD} e_D \ . \tag{5.125}$$

The Killing vectors for RR and NSNS fluxes are

$$k^{b^0}_C = 2e_{\mathrm{RR}C} + e_C \xi^0 \ , \qquad k^{\tilde{\xi}^0}_C = e_C \ . \tag{5.126}$$

The effect of introducing the different kinds of fluxes are summarised in table 5.7 as first obtained in [42]. In Appendix D.2 we perform a Kaluza-Klein compactification of type IIB supergravity on a Calabi-Yau with fluxes turned on, which for instance illustrates how the B_2 field becomes massive.

Orientifold compactifications in the presence of fluxes are studied in a similar manner. In the O-plane compactifications of type II, the 4D B_2 field is projected out of the spectrum, so no massive tensors will arise from the introduction of fluxes. However, the flux part of the B_2 field in eq. (5.121) will still be there, i.e. we have

$$\mathrm{d}B_2 \to m^\lambda \alpha_\lambda - e_k \beta^k \ , \tag{5.127}$$

Introduced flux	IIA	IIB
e_{RR}	Green-Schwarz coupling	Green-Schwarz coupling
m_{RR}	massive B_2	massive B_2
e_0	massive A^0	one massive V^c
m_0	massive A^0	massive C_2

Table 5.7: The effect of introducing electric and magnetic RR fluxes (e_{RR} resp. m_{RR}) and NSNS fluxes (e resp. m) on the type IIA and IIB theories.

with $h_-^{(3)} = h^{(2,1)} + 1$ real NSNS flux parameters $\{m^\lambda, e_k\}$. The RR fluxes may be turned on in the usual sense but with indices $a = 1, \ldots, h_-^{(1,1)}$, which results in potential terms for the scalars v^a and b^a with the form of the one in eq. (5.119). One can combine both the NSNS flux and the RR fluxes into a potential of the form [41]

$$\begin{aligned} V_{\mathrm{O6}} = &-\frac{9e^{2\phi}}{\kappa^2}\left[\mathrm{Im}\mathcal{M}_{\kappa\lambda}m^\kappa m^\lambda + (e_k - \mathrm{Re}\mathcal{M}_{k\lambda}m^\lambda)(\mathrm{Im}\mathcal{M})^{-1kl}(e_l - \mathrm{Re}\mathcal{M}_{l\lambda}m^\lambda)\right] \\ &+\frac{18e^{4\phi}}{\kappa^2}\left[(\tilde{e}_{\mathrm{RR}\hat{a}} - \mathrm{Re}\mathcal{N}_{\hat{a}\hat{c}}m_{\mathrm{RR}}^{\hat{c}})(\mathrm{Im}\mathcal{M})^{-1\hat{a}\hat{b}}(\tilde{e}_{\mathrm{RR}\hat{b}} - \mathrm{Re}\mathcal{N}_{\hat{b}\hat{c}}m_{\mathrm{RR}}^{\hat{c}})\right] \ , \end{aligned} \tag{5.128}$$

where the fluxes $\tilde{e}_{\mathrm{RR}\hat{a}} \equiv (e_{\mathrm{RR}0} + \xi_\lambda m^\lambda - \xi^{\hat{k}} e_{\hat{k}}, e_{\mathrm{RR}a})$ and $m_{\mathrm{RR}}^{\hat{a}} \equiv (m_{\mathrm{RR}}^0, m_{\mathrm{RR}}^a)$ are written in the bases ω_a of $H_+^{(1,1)}$ (with the dual $\tilde{\omega}^a$) and ω_α of $H_-^{(1,1)}$ (with the dual $\tilde{\omega}^\alpha$). The indices run as $a = 1, \ldots, h_+^{(1,1)}$ with $\alpha = 1, \ldots, h_-^{(1,1)}$ and $\hat{a} = 0, \ldots, h_+^{(1,1)}$ with $\hat{\alpha} = 0, \ldots, h_-^{(1,1)}$, as stated in table 5.4.

In type IIB, O3/O7 setups project out not only B_2 but C_2 and A^0 as well. With both NSNS fluxes and RR fluxes turned on, a combined potential can be obtained as [43][26]

$$V_{\mathrm{O3/O7}} = -\frac{9e^\phi}{\kappa^2}\left[m^{\hat{k}}(\mathrm{Im}\mathcal{M})_{\hat{k}\hat{l}}\bar{m}^{\hat{l}} + (e_{\hat{l}} - (m\mathrm{Re}\mathcal{M})_{\hat{k}})\,(\mathrm{Im}\mathcal{M})^{-1\hat{k}\hat{l}}\,(\bar{e}_{\hat{l}} - (\bar{m}\mathrm{Re}\mathcal{M})_{\hat{l}})\right] \ . \tag{5.129}$$

In O5/O9 setups the C_2 field is not projected out from the spectrum and will therefore acquire a mass in the presence of with NSNS magnetic fluxes. Like in the case of O3/O7, a truncated version of the potentials in eq. (5.119) and eq. (5.125) for the RR and NSNS fluxes respectively, can be found to be [40]

$$\begin{aligned} V_{\mathrm{O5/O9}} = &-\frac{9e^\phi}{\kappa^2}\left[m_F^{\hat{\kappa}}(\mathrm{Im}\mathcal{M})_{\hat{\kappa}\hat{\lambda}}m_F^{\hat{\lambda}} + \left(e_{\hat{\lambda}}^F - (m_F\mathrm{Re}\mathcal{M})_{\hat{\kappa}}\right)(\mathrm{Im}\mathcal{M})^{-1\hat{\kappa}\hat{\lambda}}\left(e_{\hat{\lambda}}^F - (m_F\mathrm{Re}\mathcal{M})_{\hat{\lambda}}\right)\right] \\ &-\frac{9e^\phi}{\kappa^2}\left[m_H^k(\mathrm{Im}\mathcal{M})_{kl}m_H^l + \left(e_k^H - (m_H\mathrm{Re}\mathcal{M})_k\right)(\mathrm{Im}\mathcal{M})^{-1kl}\left(e_l^H - (m_H\mathrm{Re}\mathcal{M})_l\right)\right] \ . \end{aligned} \tag{5.130}$$

Here the indices F and H of the fluxes denote the field strengths corresponding to the 3-form fluxes in $\mathrm{d}C_2$ and $\mathrm{d}B_2$ respectively, i.e.

$$\begin{aligned} F_3 = \mathrm{d}C_2 = m_F^{\hat{\kappa}}\alpha_{\hat{\kappa}} - e_{\hat{\kappa}}^F\beta^{\hat{\kappa}} \ , \\ H_3 = \mathrm{d}B_2 = m_H^k\alpha_k - e_k^H\beta^k \ , \end{aligned} \tag{5.131}$$

with $\hat{\kappa} = 0, \ldots, h_+^{(2,1)}$ and $k = 1, \ldots, h_-^{(2,1)}$. All scalar moduli that get a potential for both type II theories are summarised in table 5.8. It has been showed that the potentials in eqs. (5.128)–(5.130) can be derived from a superpotential, which will be the subject of the next section.

A final note is that in type IIB there are $4h^{(1,1)}$ moduli in the hypermultiplets, or $2h^{(1,1)}$ complexified Kähler moduli in the orientifold cases, that do not get a potential from fluxes.

	IIA		IIB		
SUSY	$\mathcal{N}=2$	$\mathcal{N}=1$ (O6)	$\mathcal{N}=2$	$\mathcal{N}=1$ (O3/O7)	$\mathcal{N}=1$ (O5/O9)
RR flux	$\{v^a, b^a\}$	$\{v^a, b^a\}$	z^k	z^k	z^k
NSNS flux	$\{z^k, \xi^0, \phi\}$	$\{N^k, \xi^0\}$	$\{z^k, \phi, C_0\}$	$\{z^c, \phi, C_0\}$	z^c

Table 5.8: Scalars that get a potential in the presence of background fluxes in Calabi-Yau compactifications.

In type IIA, $2h^{(2,1)}$ scalars in the hypermultiplets (or $h^{(2,1)}$ for orientifold compactifications) do not get a potential. However, on a rigid manifold we have $h^{(2,1)} = 0$, so there might be a possibility that fluxes may be the only ingredient needed to stabilise all moduli in type IIA compactifications. In type IIB on the other hand this is not the case, as $h^{(1,1)} \geq 1$ since there is always the volume modulus.

5.4 Flux-induced scalar potentials from superpotentials

In the previous section we saw that background fluxes induce a potential for certain moduli in type II compactifications on Calabi-Yau manifolds and Calabi-Yau orientifolds. In the orientifold compactifications yielding $\mathcal{N} = 1$ actions, the potential can be fully derived from a so-called superpotential. The effective action for both type II orientifold compactifications can be written on a common "standard $\mathcal{N} = 1$ supergravity form" which includes the superpotential and result in different potentials for type IIA and IIB.

The effective action in question is given by

$$S^{(4)}_{\mathcal{N}=1} = -\int \tfrac{1}{2}R\star\mathbf{1} + K_{I\bar{J}}DM^I\wedge\star D\bar{M}^{\bar{J}} + \tfrac{1}{2}\mathrm{Re}f_{\kappa\lambda}F^\kappa\wedge\star F^\lambda + \tfrac{1}{2}\mathrm{Im}f_{\kappa\lambda}F^\kappa\wedge F^\lambda + V\star\mathbf{1}\ , \quad (5.132)$$

where M^I collectively denote all complex scalars in the chiral multiplets and $K_{I\bar{J}}$ is a Kähler metric which satisfies $K_{I\bar{J}} = \partial_I\bar{\partial}_{\bar{J}}\mathcal{K}(M,\bar{M})$ with $\mathcal{K}$ being the Kähler potential. The gauge coupling matrix $f_{\kappa\lambda}$ is a truncated version of its $\mathcal{N} = 2$ counterparts $\mathcal{N}_{AB}$ and $\mathcal{M}_{CD}$. They give the couplings of the gauge fields A^α with $\alpha = 1,\ldots,h^{(1,1)}_+$ for type IIA. For type IIB the $f_{\kappa\lambda}$ encodes the couplings of the $h^{(2,1)}_+$ fields V^κ in O3/O7 projections and the $h^{(2,1)}_-$ fields V^k in O5/O9 projections. The V is a scalar potential given in terms of the superpotential W and "D-terms" D_κ by

$$V = e^K(K^{I\bar{J}}D_IWD_{\bar{J}}\overline{W} - 3|W|^2) + \tfrac{1}{2}(\mathrm{Re}f)^{-1\kappa\lambda}D_\kappa D_\lambda\ , \quad (5.133)$$

with Kähler covariant derivatives $D_IW \equiv \partial_IW + W\partial_IK$. The D-terms are defined by

$$K_{I\bar{J}}\bar{X}^{\bar{J}}_k = i\partial_ID_k\ , \quad (5.134)$$

where X^I is the Killing vector generating the $U(1)$ gauge transformations, where

$$\delta M^I = \Lambda^k_0X^J\partial_JM^I\ . \quad (5.135)$$

In order to write the Kähler potential $\mathcal{K}$ of all the scalars in the chiral multiplets, we must first identify their complex coordinates M^I. We have seen that the scalars which survive the orientifold projection are a subset of the scalars in the $\mathcal{N} = 2$ vector multiplets, namely $t^a = b^a + iv^a$ with $a = 1,\ldots,h^{(1,1)}_-$ in type IIA, z^k, $k = 1,\ldots,h^{2,1}_-$ for IIB O3/O7 and z^κ with $\kappa = 1,\ldots,h^{2,1}_+$ for IIB O5/O9. Their Kähler potentials have the same form as the ones in $\mathcal{N} = 2$ but with the new indices a and k,κ respectively. For chiral multiplets (not vector) this is more complicated. In type IIA, the $h^{(2,1)+1}$ real scalars ξ^C have to combine with the $h^{(2,1)}$ real complex structure deformations z^c and the dilaton ϕ to form chiral multiplets. It has been shown in [41] that appropriate complex fields – the new "Kähler coordinates" M^I of the chiral multiplets – are encoded in the expansion of a complex 3-form field

$$\Omega_C = \xi^C(x)a_C + 2i\mathrm{Re}(C\Omega) \ . \tag{5.136}$$

Changing the basis from $a_C \in H_+^{(3)}$ and $b^C \in H_-^{(3)}$ to $(\alpha_k, \beta^\lambda)$ and $(\alpha_\lambda, \beta^k)$ respectively, eq. (5.109) becomes $C\Omega = \mathrm{Re}(CZ^k)\alpha_k + i\mathrm{Im}(CZ^\lambda)\alpha_\lambda - \mathrm{Re}(C\mathcal{F}_\lambda)\beta^\lambda - i\mathrm{Im}(C\mathcal{F}_k)\beta^k$ and $\xi^C(x)a_C \to \xi^k\alpha_k - \tilde{\xi}_\lambda\beta^\lambda$. The 3-form Ω_c then becomes

$$\begin{aligned} \Omega_c &= (\xi^k + 2i\mathrm{Re}(CZ^k))\alpha_k + (\tilde{\xi}_\lambda + 2i\mathrm{Re}(C\mathcal{F}_\lambda))\beta^\lambda \\ &\equiv 2N^k\alpha_k + iT_\lambda\beta^\lambda \ , \end{aligned} \tag{5.137}$$

where $N_k = \frac{1}{2}\xi^k + i\mathrm{Re}(CZ^k)$ and $T_\lambda = i\tilde{\xi}_\lambda - 2\mathrm{Re}(C\mathcal{F}_\lambda)$ are the new Kähler coordinates. Here C is the field compensating the scale invariance of Ω as introduced in the previous section. The IIA Kähler potential is then given by

$$\mathcal{K}_{\mathrm{O6}} = -\ln\left[\tfrac{4}{3}\int J\wedge J\wedge J\right] - 2\ln\left[2\int \mathrm{Re}(C\Omega)\wedge\star\mathrm{Re}(C\Omega)\right] \ , \tag{5.138}$$

which is the sum of the Kähler potentials of $\mathcal{M}^{\mathrm{Ks}}$ and $\mathcal{M}^{\mathrm{cs}}$. The Kähler coordinates consist of the t^a as well as N_k and T_λ.

The type IIB Kähler coordinates depend on the orientifold setup. For O3/O7 projections they consist of the complex structure moduli z^k and three other, namely

$$\begin{aligned} \tau &= C_0 + ie^{-\phi} \ , \\ G^a &= c^a - \tau b^a \ , \\ T_\alpha &= \tfrac{1}{2}\kappa_\alpha + i\rho_\alpha - \tfrac{i}{2(\tau-\bar{\tau})}\kappa_{\alpha bc}G^b(G-\bar{G})^c \ , \end{aligned} \tag{5.139}$$

with the κs defined as in the $\mathcal{N}=2$ case but with an appropriate basis. The Kähler potential is given by

$$\mathcal{K}_{\mathrm{O3/O7}} = -\ln\left[-i\int\Omega(z)\wedge\bar{\Omega}(\bar{z})\right] - \ln[-i(\tau-\bar{\tau})] - 2\ln[\tfrac{1}{6}\kappa(\tau,G,T)] \ , \tag{5.140}$$

where $\kappa = \kappa_{\alpha\beta\gamma}v^\alpha v^\beta v^\gamma$ should be expressed in terms of τ, G and T_α. However, such an expression can only be done explicitly for a single v, i.e. for $\alpha = h^{(1,1)+} = 1$, in which case there is only one $T_\alpha = T$. If there in addition are no Gs, i.e. $a = h_-^{(1,1)} = 0$, κ has a particularly simple form as

$$-2\ln\kappa = -3\ln[T+\bar{T}] \ . \tag{5.141}$$

The two last terms in the Kähler potential of eq. (5.140) satisfy a *no-scale* type condition

$$\partial_I\mathcal{K}\partial_{\bar{J}}\mathcal{K}K^{I\bar{J}} = 4 \ , \tag{5.142}$$

with summation over $I = (\tau, G^a, T_\alpha)$. With a non-trivial superpotential this will cancel the term $-3|W|^2$ in eq. (5.133) by contributing a $+4|W|^2$ term, so that the resulting scalar potential V will be positive semi-definite. For the O5/O9 orientifold projections the Kähler coordinates consist of the complex structure moduli z^κ and

$$\begin{aligned} t^\alpha &= e^{-\phi}v^\alpha - ic^\alpha \ , \\ A_a &= \Theta_{ab}b^b + i\rho_a \ , \\ S &= \tfrac{1}{6}e^{-\phi}\kappa + ic^0 - \tfrac{1}{4}(\mathrm{Re}\Theta^{-1})^{ab}A_a(A+\bar{A})_b \ , \end{aligned} \tag{5.143}$$

with c^0 again being the dual scalar field to C_2 and we have defined $\Theta_{ab}(t) \equiv \kappa_{ab\alpha}t^\alpha$ and $\int C_6 \equiv c^0 + \frac{1}{2}\rho_a b^a$. The O5/O9 Kähler potential is given by

$$\mathcal{K}_{\mathrm{O5/O9}} = -\ln\left[-i\int \Omega\wedge\bar{\Omega}\right] - \ln\left[\tfrac{1}{6}\int e^{-3\phi}J\wedge J\wedge J\right] - \ln\left[\tfrac{1}{3}e^{-\phi}\kappa(t^\alpha, A_a, S)\right] , \qquad (5.144)$$

where the first term depends entirely on z^κ, the second one on t^α and the last one solves for κ in terms of $\{t^\alpha, A_a, S\}$ using eq. (5.143).

We will now turn to the flux-induced superpotentials for the different theories. In type IIA the scalar potential of eq. (5.128) can be shown to come from a superpotential of the form

$$\begin{aligned} W_{O6} &= \int H_3\wedge\Omega_c + F_A^{(10)}\wedge e^{B+iJ} \\ &= -2N^k e_k - iT_\lambda m^\lambda + e_{\mathrm{RR}0} + e_{\mathrm{RR}a}t^a + \tfrac{1}{2}\kappa_{abc}m^a_{\mathrm{RR}}t^b t^c + \tfrac{1}{6}m^0_{\mathrm{RR}}\kappa_{abc}t^a t^b t^c \end{aligned} \qquad (5.145)$$

where $F_A = F_0 + F_2 + F_4$ and $F_{p+1} = \mathrm{d}C_p$. See [41][44][45] for more details. It is clear that the superpotential depends on all O6 moduli, just as the corresponding scalar potential V_{O6}.

The type IIB superpotential for compactifications on both Calabi-Yau manifolds and Calabi-Yau O3/O7 orientifolds is given by

$$W_{\mathrm{O3/O7}} = \int G_3\wedge\Omega = (e_{\mathrm{RR}C} - i\tau e_C)Z^C - (m^C_{\mathrm{RR}} - i\tau m^C)\mathcal{F}_C \ . \qquad (5.146)$$

This superpotential depends on the z^c complex structure moduli through Ω as well as the axion and dilaton via the definition of G_3 as

$$G_3 \equiv F_3 - \tau H_3 \ , \qquad (5.147)$$

with $F_3 = \mathrm{d}C_2$ and $H_3 = \mathrm{d}B_2$. The Kähler moduli v^α, ρ_α and b^a, c^a coming from B_2 and C_2 however do not appear in the superpotential. The corresponding scalar potential $V_{\mathrm{O3/O7}}$ is obtained via eq. (5.133) where one needs to calculate the Kähler covariant derivatives. They depend on the Kähler coordinates M^I, i.e. the chiral multiplets in O3/O7 compactifications, and are given by

$$\begin{aligned} D_\tau W &= \tfrac{i}{2}e^\phi\int\bar{G}_3\wedge\Omega + iG_{ab}b^a b^b W \ , &\qquad D_{T_\alpha}W &= -\frac{2v^\alpha}{\kappa}W \ , \\ D_{G^a}W &= 2iG_{ab}b^b W \ , & D_{z^c}W &= \int G_3\wedge\chi_c \ , \end{aligned} \qquad (5.148)$$

where Kodaira's formula $\frac{\partial\Omega}{\partial z^k} = k_k\Omega + \chi_k$ has been used. Using this and the fact that the D-terms in this case are zero, the scalar potential obtained is

$$V_{\mathrm{O3/O7}} = \frac{18ie^\phi}{\kappa^2\int\Omega\wedge\bar{\Omega}}\left(\int\bar{G}_3\wedge\Omega\int G_3\wedge\bar{\Omega} + G^{kl}\int G_3\wedge\chi_k\int\bar{G}_3\wedge\bar{\chi}_l\right) \ . \qquad (5.149)$$

It can be rewritten to exactly match the scalar potential in eq. (5.129) given earlier by expanding G_3 in the basis of $H^{(3)}_-$ as $G_3 = m^{\hat{k}}\alpha_{\hat{k}} - e_{\hat{k}}\beta^{\hat{k}}$. Here $m^{\hat{k}} = m^{\hat{k}}_{F_2} - \tau m^{\hat{k}}_{H_3}$ and $m^{\hat{k}} = m^{\hat{k}}_{F_2} - \tau m^{\hat{k}}_{H_3}$ are linear combinations of the magnetic and electric fluxes belonging to the expansions of $F_3 = \mathrm{d}C_2$ and $H_3 = \mathrm{d}B_2$. It is also clear that the potential is positive semi-definite, as expected from the no-scale condition. The potential contains both RR and NSNS fluxes and depends on the axion, dilaton and complex structure moduli.

A superpotential can be validated using supersymmetry conditions. For example, a Minkowski vacuum should have vanishing superpotential and vanishing first derivative of the superpotential, i.e. $W = 0$ and $\partial_I W = 0$. From $W_{\mathrm{O3/O7}}$ in eq. (5.146) we see that the first requirement implies that

$$W_{\mathrm{O3/O7}} = 0 \qquad \Rightarrow \qquad \int G_3\wedge\Omega = 0 \ , \qquad (5.150)$$

and the demand on the derivative implies via eq. (5.148) that

$$
\begin{aligned}
D_\tau W &= 0 \qquad \Rightarrow \qquad \int \bar{G}_3 \wedge \Omega = 0 \ , \\
D_{z^k} W &= 0 \qquad \Rightarrow \qquad \int G_3 \wedge \chi_k = 0 \ .
\end{aligned}
\tag{5.151}
$$

These equations imply for G_3, in order, that it has no $(0,3)$-, $(3,0)$- or $(1,2)$-components. The only remaining option is that G_3 is a $(2,1)$-form, and when this is the case it is clear that the scalar potential in eq. (5.149) is zero. Again there are no non-trivial 1-forms on a Calabi-Yau, so G_3 is automatically primitive. Hence G_3 satisfies the supersymmetry condition of type B in section 4.4, table 4.4, and the supersymmetries preserved in a type B solution are the same as the ones of O3/O7 orientifolds.

At last the superpotential for the O5/O9 orientifold setup is given by

$$
W_{\mathrm{O5/O9}} = \int \Omega \wedge F_3 \ . \tag{5.152}
$$

Unlike the O3/O7 case, this superpotential does not generate the corresponding scalar potential on its own. With fluxes in the O5/O9 case, the dual scalar field c^0 to C_2 becomes charged under $U(1)$ gauge transformations and as a consequence there is a non-vanishing D-term in the scalar potential. The Kähler derivatives are given by [40]

$$
\begin{aligned}
D_{t^\alpha} W &= \frac{3}{2}\left(\mathcal{K}_\alpha + e^{-\phi/2}\frac{\kappa_{\alpha ab} b^a b^b}{\kappa}\right) W \ , \qquad & D_{A_a} W &= -\frac{3b^a}{2} e^{-\phi/2} W \ , \\
D_S W &= \frac{3}{\kappa} e^{-\phi/2} W \ , & D_{z^k} W &= \int F_3 \wedge \chi_\kappa \ ,
\end{aligned}
\tag{5.153}
$$

where $\mathcal{K}_\alpha$ is the t^α-dependent part of eq. (5.144). The resulting scalar potential is given by

$$
\begin{aligned}
V_{\mathrm{O5/O9}} = \; & \frac{18 i e^\phi}{\kappa^2 \int \Omega \wedge \bar{\Omega}} \left(\int \Omega \wedge F_3 \int \bar{\Omega} \wedge F_3 + G^{\kappa\lambda} \int \chi_\kappa \wedge F_3 \int \bar{\chi}_\lambda \wedge F_3 \right) \\
& - \frac{9 e^\phi}{\kappa^2} \left[m_H^k (\mathrm{Im}\mathcal{M})_{kl} m_H^l + \left(e_k^H - (m_H \mathrm{Re}\mathcal{M})_k \right) (\mathrm{Im}\mathcal{M})^{-1kl} \left(e_l^H - (m_H \mathrm{Re}\mathcal{M})_l \right) \right] \ ,
\end{aligned}
\tag{5.154}
$$

where the first term can be rewritten to exactly match the first term given in eq. (5.130). Imposing the same Minskowski supersymmetric vacuum conditions $W = 0$ and $\partial_I W = 0$ on this setup imply that F_3 should have no $(0,3)$ or $(1,2)$ parts. It can be shown that O5/O9-planes preserve type C supersymmetric solutions as discussed in chapter 4.4.

A final remark is that while the O6 superpotential and scalar potential depend on all moduli, the corresponding type IIB orientifold potentials do not. This will be central in the next chapter when discussing moduli stabilisation.

5.5 Note on fluxes effects on general $SU(3)$-structure manifolds

In this section we will briefly discuss the effect fluxes have on general manifolds with $SU(3)$-structure, their general superpotential, and the mirror symmetry on such manifolds.

In the case of manifolds with $SU(3)$-structure it is possible to perform a consistent truncation to a finite set of massless modes. These modes are obtained by expanding a set of p-forms of which some are not closed. The non-closure of these forms is proportional to the torsion which has been shown to play a very similar role as the fluxes. In fact, some of the torsion classes defined earlier in eq. (4.35) are mirror to NSNS flux. The torsion is encoded in the non-closure of the forms in the basis

$$\begin{aligned} \mathrm{d}\omega_a &= m_a^C \alpha_C - e_{aD}\beta^D \ , \\ \mathrm{d}\tilde{\omega}^a &= 0 \ , \\ \mathrm{d}\alpha_C &= -e_{aC}\tilde{\omega}^a \ , \\ \mathrm{d}\beta^C &= m_a^C \tilde{\omega}^a \ , \end{aligned} \tag{5.155}$$

where the indices span $a = 1, \ldots, b_J$ where b_J is the dimension of the finite set of 2-forms, and $C, D = 1, \ldots, b_\Omega$ with b_Ω is the dimension of the set of 3-forms. The spectrum has been shown to be analogous to the one of Calabi-Yau manifolds in table 5.2 but with different index spans. Namely in IIA the $h^{(1,1)}$ is replaced with b_J and the $h^{(2,1)}$ is replaced by b_Ω. In type IIB the two dimensions are exchanged. In [42] it is shown that in the type IIA $\mathcal{N} = 2$ action, the presence of both electric and magnetic torsion e_{aC} and m_a^C results in a massive A^a field. In type IIB fluxes make some moduli in the $C_4^{(10)}$ expansion of eq. (5.64) massive. Namely e_{aC} generates a massive V^k and m_a^C make the tensors D_2^a massive.

The $\mathcal{N} = 1$ superpotentials of the O6-, O3/O7- and O5/O9-planes given in eqs. (5.145), (5.146) and (5.152) respectively, can be obtained from the supersymmetry transformation of the gravitino in 4D. The generic form of it reads [46]

$$\delta_\xi \psi_\mu = \nabla_\mu \xi + i e^{\mathcal{K}/2} \gamma_\mu \xi^* \ , \tag{5.156}$$

with ξ being the 4D supersymmetry parameter, $\mathcal{K}$ the $\mathcal{N} = 1$ Kähler potential, and W the superpotential. Namely, starting from the 10D supersymmetry variation of the gravitino in eq. (4.2) and inserting the spinor decomposition in eq. (4.74) as well as using the Kähler potentials for the different O-plane theories in eqs. (5.138), (5.140), (5.144), their corresponding superpotentials can be obtained. In the spinor decomposition of eq. (4.74), one must relate the complex functions a and b such that $a = ib$ for the O3/O7 setup, $a = b$ for O5/O9 and $a = be^{i\beta}$ for O6-planes.

Hitchin showed in [16] that there is a special Kähler structure on the space of generalised almost complex structures. In the case of ordinary almost complex structures, this bundle is known as the *twistor bundle*. The space of generalised almost complex structures consists of stable[2], real, even or odd forms, i.e. $\{\mathrm{Re}\Phi_+\}$ and $\{\mathrm{Re}\Phi_-\}$. For $SU(3)$ this corresponds to the spaces of J and $\mathrm{Re}\Omega$. The spinor Φ_+ can be complexified by adding the B field, so that

$$\Phi_+ = \tfrac{i}{8} e^{-(B+iJ)} \ , \qquad \Phi_- = \tfrac{1}{8}\Omega \ . \tag{5.157}$$

The $\mathrm{Re}\Phi_+$ and $\mathrm{Re}\Phi_-$ are $Spin(6,6)$ representations where $\mathrm{Im}\Phi_\pm = \star \mathrm{Re}\Phi_\pm$. They are the constituents of our pure Clifford(6,6) spinors since $\Phi_\pm = \mathrm{Re}\Phi_\pm + i\mathrm{Im}\Phi_\pm$. For the space of generalised complex structures the Kähler metric is obtianed from the Kähler potential and given by

$$\mathcal{K}_\pm = -\ln \left[i \int \langle \Phi_\pm, \bar{\Phi}_\pm \rangle \right] \ , \tag{5.158}$$

where the bracket is a scalar product between spinors, called the *Mukai pairing*, defined as

$$\begin{aligned} \langle \Psi_+, \Phi_+ \rangle &\equiv \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6 \ , \\ \langle \Psi_-, \Phi_- \rangle &\equiv \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Psi_3 + \Psi_1 \wedge \Phi_5 \ , \end{aligned} \tag{5.159}$$

where the number subscript denotes the degrees of the component form. An interesting note is that the Kähler potential is very similar to the $\mathcal{N} = 2$ Calabi-Yau counterparts as seen in eq.

[2] A stable form is a real form Φ whose neighbouring elements are equivalent to Φ under $GL(6, \mathbb{R})$ transformations.

(5.20) and eq. (5.38). The $\Phi_\pm$ do however not have to be closed forms, which means that they need not correspond to an integrable structure, which the Calabi-Yau structures must do.

The spinor formalism above can be used to understand superpotentials. The general $\mathcal{N}=1$ superpotentials for unwarped compactifications of manifolds with $SU(3)$-structure in each theory, are given by

$$W_{\text{IIA}} = \int \bar{a}^2 e^{-\phi} \langle \Phi_+, \mathrm{d}\bar{\Phi}_- \rangle - \int \bar{b}^2 e^{-\phi} \langle \Phi_+, \mathrm{d}\bar{\Phi}_- \rangle + 2 \int \bar{a}\bar{b} \langle \Phi_+, F_{\text{IIA}} \rangle \ , \tag{5.160}$$

$$W_{\text{IIB}} = \int \bar{a}^2 e^{-\phi} \langle \Phi_-, \mathrm{d}\bar{\Phi}_+ \rangle + \int \bar{b}^2 e^{-\phi} \langle \Phi_-, \mathrm{d}\bar{\Phi}_+ \rangle - 2i \int \bar{a}\bar{b} \langle \Phi_-, F_{\text{IIB}} \rangle \ , \tag{5.161}$$

where again $F_{\text{IIA}} = F_0 + F_2 + F_4$ and $F_{\text{IIB}} = F_1 + F_3 + F_5$ are the sums of the RR fluxes in each respective theory. If we insert $a = ib$ into W_{IIB} in eq. (5.161), the result will match exactly with the O3/O7 superpotential in eq. (5.146). If one inserts $a = b$ then the obtained expression will consist only on the RR part of eq. (5.152) for O5/O9. To compensate for this the superpotential gets modified by adding a torsion piece

$$W^{\text{comp}}_{\text{O5/O9}} = \int (e^{-\phi}\,\mathrm{d}J + F_3) \wedge \Omega \ . \tag{5.162}$$

With $a = 0$ in either W_{IIA} or W_{IIB}, the superpotential for the *heterotic* string is obtained as

$$W_{\text{het}} = \int e^{-\phi}(\mathrm{d}J + iH) \wedge \Omega \ . \tag{5.163}$$

Last but not least, with $a = ib$ for W_{IIA}, the "torsional" O6 superpotential is obtained as

$$W^{\text{tors}}_{\text{O6}} = \int e^{-\phi}(\mathrm{d}J + iH) \wedge \mathrm{Re}\Omega + i \int F_{\text{IIA}} \wedge e^{B+iJ} \ , \tag{5.164}$$

where the NSNS part has been integrated by parts. The superpotentials have the right holomorphic dependence on the respective chiral multiplets.

We will now proceed to the topic of mirror symmetry again, but this time with fluxes present. Does mirror symmetry survive the presence of fluxes? An important aspect is that from eq. (5.70) one should expect that fluxes in even cohomologies are mapped to fluxes in odd cohomologies. Fluxes in the RR sector agree with this, since type IIA contains fluxes in even cohomologies while the ones in type IIB are odd. The NSNS fluxes however all belong to an odd cohomology, so its mirror should be an even NSNS "flux". It turns out that this mirror NSNS "flux" in an even cohomology corresponds to torsion [47]. As a consequence it is suitable to study mirror symmetry in the presence of fluxes in the setup of compactifications on manifolds with torsion, i.e. manifolds with $SU(3)$-structure. More precisely, it has been shown [47][48] that the mirror of the NSNS flux H_3 is the torsion of half-flat manifolds, i.e. $\mathrm{Re}W_1$ and $\mathrm{Re}W_2$.

As for the defining objects of the complex structure, i.e. J and Ω, it can be shown from the T-duality rules for the supersymmetry parameter that there is an exchange of the pure spinors η_+ and η_-. These are the same spinors that build the $\Phi_\pm$ ones via $\Phi_+ = \eta_+ \otimes \eta_+^\dagger$ and $\Phi_- = \eta_+ \otimes \eta_-^\dagger$. It is therefore natural to conjecture that mirror symmetry is an exchange of the pure spinors, i.e.

$$\begin{aligned} \Phi_+ &\longleftrightarrow \Phi_- \ , \\ e^{B+iJ} &\longleftrightarrow \Omega \ , \end{aligned} \tag{5.165}$$

where the last symmetry is specific to $SU(3)$-structure manifolds. Now, introducing fluxes on such manifolds, it is clear that the RR fluxes map amongst themselves, i.e the even ones of type IIA map to the odd ones of type IIB. The NSNS fluxes will be mixed with metric components via

T-duality. The explicit maps involving NSNS fluxes and torsion of $SU(3)$-structure manifolds are [20]

$$\begin{aligned} i(W_3+iH^{(6)})_{ij}+\Omega_{ijk}(\overline{W}_4+iH^{(\bar{3})})^k &\longleftrightarrow -2i\overline{W}^2_{i\bar{j}}-2g_{i\bar{j}}(\overline{W}_1+3iH^{(\bar{1})})\ , \\ (W_5-W_4-iH^{(3)})_i &\longleftrightarrow (W_5-W_4-iH^{(3)})_{\bar{\imath}}\ , \end{aligned} \tag{5.166}$$

which are often written in a more compact way as

$$(\nabla J+H)_{ijk}\longleftrightarrow(\nabla J+H)_{\bar{\imath}\bar{\jmath}\bar{k}}\ . \tag{5.167}$$

This short hand notation comes from the fact that the Qs in eq. (4.98) have mirror symmetries $Q_{ij}\leftrightarrow Q_{i\bar{j}}$ and $Q_i\leftrightarrow-\bar{Q}_{\bar{\imath}}$ as a consequence of the exchange of the η_+ and η_- under T-dualities. In fact, all the matrices in eq. (4.98) have the same mirror symmetries as Q if in addition the RR fields satisfy

$$F_{\mathrm{IIA}}\longleftrightarrow F_{\mathrm{IIB}}\ . \tag{5.168}$$

The mirror symmetry of eq. (5.166) and eq. (5.168) can be understood as the result of mirror symmetry which exchanges the $SU(3)$ representations according to $\mathbf{6}+\bar{\mathbf{3}}\leftrightarrow\mathbf{8}+\mathbf{1}$.

In a final remark we turn to mirror symmetry in the $\mathcal{N}=2$ effective actions from compactifications of $SU(3)$-structure manifolds with fluxes. The Kähler potential of the vector multiplet moduli space in $SU(3)$-structure compactifications are spanned by the pure spinors Φ_+ for type IIA and Φ_- for type IIB. From the Kähler potential defined in eq. (5.158) it is clear that these Kähler potentials will be mapped to each other. The $\mathcal{N}=2$ version of the superpotential has been shown to respect the mirror symmetry maps. For instance, in the presence of RR and only electric NSNS fluxes (which include torsion), the $\mathcal{N}=2$ superpotential is symmetric under the exchange of complex structure and Kähler coordinates; $Z^C=(Z^0,Z^0z^c)\leftrightarrow t^A=(1,t^a)$ and prepotentials; $\mathcal{F}_C\leftrightarrow\tilde{\mathcal{F}}^A$ given that the fluxes are mapped according to

$$\begin{aligned} \{e_{\mathrm{RR}C},m^C_{\mathrm{RR}}\} &\longleftrightarrow \{e_{\mathrm{RR}A},m^A_{\mathrm{RR}}\}\ , \\ e_{AC} &\longleftrightarrow e_{CA}\ . \end{aligned} \tag{5.169}$$

Here the NSNS electric flux combines flux and torsion as $e_{AC}\equiv(e_C,e_aC)$. This is true for both the IIA and IIB $\mathcal{N}=2$ superpotentials. These potentials are however not mirror symmetric in the presence of magnetic NSNS fluxes.

6

No-Go Theorems for Compact Manifolds with Fluxes

In this chapter we will review a set of general no-go theorems for type II compactifications to 4D. For instance we will see that integrability conditions on the type IIB flux equations of motion in combination with a warped metric rule out compactifications to Minkowski and de Sitter vacua. In section 6.1 we reconstruct a no-go theorem coming from the 10D Einstein equation and discuss the need to include localised sources. In the remaining sections 6.2 and 6.3 we discuss conditions on the solutions of the Bianchi identities.

6.1 A no-go theorem from the 4D Einstein equation

In this section we review the results of [49] which show that the contribution from fluxes to the stress-energy tensor is always positive, which in turn rules out compactifications to compact internal manifolds. The starting point is the trace reversed 10D Einstein equation

$$R_{MN} = T_{MN} - \tfrac{1}{8} g_{MN} T^L{}_L \ . \tag{6.1}$$

Given the metric ansatz of eq. (4.1) the 4D components $R_{\mu\nu}$ of the 10D Einstein equation read

$$\hat{R}_{\mu\nu} - \hat{g}_{\mu\nu}(\nabla^2 A + 2(\nabla A)^2) = T_{\mu\nu} - \tfrac{1}{8} e^{2A} \hat{g}_{\mu\nu} T^L{}_L \ . \tag{6.2}$$

Contracting with $\hat{g}^{\mu\nu}$ on both sides results in that

$$\hat{R} - 4(\nabla^2 A + 2(\nabla A)^2) = e^{2A} T^\mu{}_\mu - \tfrac{1}{2} e^{2A} T^L{}_L \ , \tag{6.3}$$

where $\hat{R}$ is the Ricci scalar of the 4D metric $\hat{g}_{\mu\nu}$. Here we have used that $T^\mu{}_\mu$ should be contracted with the raised 4D metric component, i.e $g^{\mu\nu} = e^{-2A}\hat{g}^{\mu\nu}$, so that $\hat{g}^{\mu\nu} T_{\mu\nu} = e^{2A} g^{\mu\nu} T_{\mu\nu} = e^{2A} T^\mu{}_\mu$. Rearranging eq. (6.3) to

$$\hat{R} + e^{2A}(-T^\mu{}_\mu + \tfrac{1}{2} T^L{}_L) = 4(\nabla^2 A + 2(\nabla A)^2) \ , \tag{6.4}$$

and defining

$$\check{T} \equiv -T^\mu{}_\mu + \tfrac{1}{2} T^L{}_L = \tfrac{1}{2}(-T^\mu{}_\mu + T^m{}_m) \ , \tag{6.5}$$

as well as rewriting the right-hand side of eq. (6.4) as $4(\nabla^2 A + 2(\nabla A)^2) = 2e^{-2A}\nabla^2 e^{2A}$, eq. (6.3) becomes

$$\hat{R} + e^{2A}\check{T} = 2e^{-2A}\nabla^2 e^{2A} \ . \tag{6.6}$$

The energy-momentum tensor coming from fluxes may be investigated by using the general expression for the energy-momentum tensor in terms of some n-form field, which is given by

$$T_{MN} = F_{MP_1\ldots P_{n-1}} F^{NP_1\ldots P_{n-1}} - \frac{1}{2n} g_{MN} F^2 \ . \tag{6.7}$$

This can be compared with the energy-momentum tensor in electromagnetism, where

$$T_{\mu\nu} = F_{\mu}{}^{i} F_{\nu i} - \tfrac{1}{4} g_{\mu\nu} F^{ij} F_{ij} \ , \tag{6.8}$$

and F is the 2-form electromagnetic field strength. Given this expression, eq. (6.5) takes the form

$$\begin{aligned}\check{T} &= -\frac{1}{2}\left(F_{\mu P_1 \dots P_{n-1}} F^{\mu P_1 \dots P_{n-1}} - \frac{1}{2n} g^{\mu}{}_{\mu} F^2\right) + \frac{1}{2}\left((F^2 - F_{\mu P_1 \dots P_{n-1}} F^{\mu P_1 \dots P_{n-1}}) - \frac{1}{2n} g^{m}{}_{m} F^2\right) \\ &= -F_{\mu P_1 \dots P_{n-1}} F^{\mu P_1 \dots P_{n-1}} + \frac{n-1}{2n} F^2 \ . \end{aligned} \tag{6.9}$$

When rewriting the $T^{m}{}_{m}$ part we used that $g^{\mu}{}_{\mu} = g^{\mu\nu} g_{\mu\nu} = 4$ in the external spacetime, $g^{m}{}_{m} = g^{mn} g_{mn} = 6$ in the internal and the fact that

$$F^2 \equiv F_{M P_1 \dots P_{n-1}} F^{M P_1 \dots P_{n-1}} = F_{\mu P_1 \dots P_{n-1}} F^{\mu P_1 \dots P_{n-1}} + F_{m P_1 \dots P_{n-1}} F^{m P_1 \dots P_{n-1}} \ . \tag{6.10}$$

In order to preserve maximal symmetry in the external space, there can either be no non-zero flux components (legs) in any of its four directions, or legs in all of the four directions. We will therefore consider these two cases separately, as they will give different contributions to the traced energy-momentum tensor $\check{T}$. In the first case, there can only be fluxes in the internal space $\mathcal{M}_6$. We do not demand that this internal manifold is maximally symmetric, so the fluxes can be in an arbitrary number of dimensions within the $\mathcal{M}_6$ manifold. In this case the first term of eq. (6.9) is zero, so that

$$\check{T}_{\text{int}} = \frac{n-1}{2n} F^2 \geq 0 \ , \tag{6.11}$$

since $F^2 \geq 0$ in the purely spatial 6D space with positive signature.

In the second case, the fluxes have to be in the all of the four dimensions of the external space $\mathcal{M}_4$, and may in addition have legs in some arbitrary number of the remaining six dimensions. The first term of eq. (6.9) can then be rewritten as

$$F_{\mu P_1 \dots P_{n-1}} F^{\mu P_1 \dots P_{n-1}} = \frac{4}{n} F^2 \ , \tag{6.12}$$

which we use to simplify eq. (6.9) to

$$\check{T}_{\text{ext}+} = \frac{n-9}{2n} F^2 \ . \tag{6.13}$$

With fluxes in the time direction, it is natural to assume the fluxes to be time-like, which implies that $F^2 \leq 0$ in this case. Inserting eq. (6.12) into eq. (6.9), $\check{T}_{\text{ext}+}$ will then be larger than zero, i.e.

$$\check{T}_{\text{ext}+} = -\frac{9-n}{2n} F^2 \geq 0 \ , \tag{6.14}$$

where $4 \leq n \leq 9$ and $F^2 \leq 0$. It is therefore clear that whether the fluxes are only internal, or propagate in at least the four external directions, their contribution to the traced energy-momentum tensor is positive semi-definite in both cases of eq. (6.11) and eq. (6.14). For example, the 9-form flux F_9 has vanishing contribution in eq. (6.14), and fortunately its purely internal dual flux F_1 vanishes in eq. (6.11).

If one multiplies eq. (6.6) with e^{2A} and integrates over the internal manifold $\mathcal{M}_6$, the right-hand side of eq. (6.6) will vanish. Since the internal manifold is assumed to be compact, it does not have a boundary, thus eq. (6.6) becomes

$$\int_{\mathcal{M}_6} e^{2A} \hat{R} + e^{4A} \check{T} = 0 \ . \tag{6.15}$$

The integration will not change any signs on the left-hand side, and we have just showed that $\check{T} \geq 0$ for all fluxes, so we see that for the equality to be true $\hat{R}$ can not be positive. This is referred to as a no-go theorem which was first found in [49]. It implies that without including localised sources or higher order derivative corrections to the equations of motion, a compactification to a de Sitter vacuum is not allowed. Anti-de Sitter spaces with $\hat{R} < 0$ may cancel the contribution from fluxes to the energy-momentum tensor, however Minkowski spaces with $\hat{R} = 0$ only allow 1-form fluxes.

As mentioned earlier an inclusion of a localised source may cancel positive terms on the left-hand side of eq. (6.15). The inclusion of such a localised source adds an extra term in the trace reversed Einstein equation of eq. (6.1), so that

$$R_{MN} = T_{MN} - \tfrac{1}{8} g_{MN} T^L{}_L + T^{\text{loc}}_{MN} - \tfrac{1}{8} g_{MN} T^{\text{loc}L}{}_L \ . \tag{6.16}$$

This adds another term to eq. (6.3), which then reads

$$\hat{R} + e^{2A} \left(\check{T}^{\text{flux}} + \check{T}^{\text{loc}} \right) = 2 e^{-2A} \nabla^2 e^{2A} \ , \tag{6.17}$$

with $\check{T}^{\text{flux}}$ defined as in eq. (6.9). The energy-momentum tensor of the sources is given by

$$T^{\text{loc}}_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{loc}}}{\delta g_{MN}} \ , \tag{6.18}$$

where S_{loc} is the action describing the sources and g_{MN} is the same 10D metric as before. For a Dp-brane that wraps a $(p-3)$-cycle Σ in $\mathcal{M}_6$ the localised source action is [22]

$$S_{\text{loc}} = -\int_{\mathcal{M}_4 \oplus \Sigma} \mathrm{d}^{p+1}\xi \sqrt{-g}\ T_p + \mu_p \int_{\mathcal{M}_4 \oplus \Sigma} C_{(p+1)} \ , \tag{6.19}$$

to leading order in α' and with fluxes vanishing on the brane. Generally there are other terms of higher order in α' that contribute to $g = \det g_{MN}$ under the square root. T_p is the brane tension, or energy density, given in Einstein frame by

$$T_p = \mu_p e^{(p-3)\phi/4} \ , \tag{6.20}$$

where $\mu_p = (2\pi\sqrt{\alpha'})^{-(p+1)}$ is the Dp-brane charge and C_{p+1} is an RR field which couples to the Dp-brane. The action of eq. (6.19) gives us the energy-momentum tensors [39]

$$T^{\text{loc}}_{\mu\nu} = -T_p e^{2A} \hat{g}_{\mu\nu} \delta(\Sigma) \ , \qquad T^{\text{loc}}_{mn} = -T_p \Pi^{\Sigma}_{mn} \delta(\Sigma) \ , \tag{6.21}$$

where $\delta(\Sigma)$ is the delta function of the $(p-3)$-cycle Σ which is also dual to the cycle and the Π is the projector onto the cycle Σ. Using this, it was found in [39] that

$$\check{T}^{\text{loc}} = \frac{7-p}{2} T_p \delta(\Sigma) \ . \tag{6.22}$$

This implies that Dp-branes with $p < 7$ also give a positive contribution to the Einstein equation in eq. (6.17) and a vanishing contribution for $p = 7$. To allow for compactifications we therefore need to include objects with negative tension. String theory does however have such objects, namely orientifolds, which will be discussed in later chapters.

6.2 Tadpole cancellation conditions from Bianchi identities and flux equations of motion

In this section we introduce the no-go theorem from the integrated Bianchi identity or equations of motion with fluxes and source terms present. The Bianchi identities for the NSNS flux and the democratic RR fluxes are given in eq. (3.116), repeated here for convenience as

$$\mathrm{d}H = 0 \ , \qquad \mathrm{d}F - H \wedge F = 0 \ . \tag{6.23}$$

The RR fluxes are constrained by the Hogde- or self-duality relation eq. (3.115), i.e.

$$F_n = (-1)^{\lfloor n/2 \rfloor} \star_{10} F_{10-n} \ . \tag{6.24}$$

As a consequence of this self-duality, the Bianchi identity also contains the equation of motion for the fluxes, which read

$$\mathrm{d}(\star_{10} F_n) \pm H \wedge F_{8-n} = 0 \ , \tag{6.25}$$

where we use the plus sign for type IIA and minus sign for type IIB. This equation can be written in a simpler form of [1]

$$\mathrm{d}(\star_{10} F_n) + H \wedge \star_{10} F_{n+2} = 0 \ , \tag{6.26}$$

for both type IIA and IIB. Throughout this thesis we wish to study flux backgrounds that preserve maximal 4D symmetry. This can be translated into a condition on the fluxes according to

$$F_n^{(10)} = F_n + \mathrm{vol}_4 \wedge \hat{F}_{n-4} \ , \tag{6.27}$$

where F_n denotes internal fluxes, i.e. fluxes inside the compact manifold $\mathcal{M}_6$, $\hat{F}_{4-n}$ denotes the external fluxes in $\mathcal{M}_4$ and vol_4 is the 4D volume. The duality relation in eq. (6.25) allows us to relate the internal and external components by

$$\hat{F}_{n-4} = (-1)^{\lfloor n/2 \rfloor} \star_6 F_{10-n} \ , \tag{6.28}$$

where $\star_6$ denotes the 6D Hodge star. Now, using the decomposition of eq. (6.27) and eq. (6.28) as well as the metric defined in eq. (4.1), the Bianchi identities and the equations of motion for the internal RR fluxes become

$$(\mathrm{d} - H\wedge)F = 0 \ , \qquad \text{and} \qquad (\mathrm{d} - H\wedge)(e^{4A} \star_6 F) = 0 \ , \tag{6.29}$$

respectively. In the presence of localised sources a source term will be added to the equations of eq. (6.29) so that the Bianchi identity takes the form

$$\mathrm{d}F_n = H_3 \wedge F_{n-2} + \rho_{8-n}^{\mathrm{loc}} \ , \tag{6.30}$$

where $\rho_{8-n}^{\mathrm{loc}}$ is the dimensionless charge density of the $(8-n)$-dimensional magnetic source for F_n, containing a delta function $\delta^{n+1}(\vec{x} - \vec{x}_i)$. Integrating this identity over cycles wrapped by branes results in a type of charge conservation equation or *tadpole cancellation condition.* In type IIA, the D-branes that can give tadpole cancellation conditions are the D4-, D6- and D8-branes that are extended along spacetime. Since Dp-branes wrap $(p-3)$-cycles in the internal manifold, the D4- and D8-branes will wrap 1- and 5-cycles respectively. However, taking the internal manifold to be a Calabi-Yau, these cycles are not allowed since a Calabi-Yau manifold does not admit 1- or 5-forms. Hence, for type IIA, the only relevant D-brane that can give tadpole cancellation conditions is the D6-brane. The D6-branes are electric sources for $F_8^{(10)}$ and magnetic sources for $F_2^{(10)}$, which is the dual field of $F_8^{(10)}$. With localised sources consisting of D6-branes and O6-planes that extend along spacetime, they will wrap 3-cycles Σ_3 in the internal manifold. If one now integrates the Bianchi identity of eq. (6.30) over the dual cycle $\tilde{\Sigma}_3$, the type IIA tadpole cancellation condition becomes

$$\mu_p^{\mathrm{D}} N_{\mathrm{D6}}(\Sigma_3) + \mu_p^{\mathrm{O}} N_{\mathrm{O6}}(\Sigma_3) + \mu_p^{\mathrm{D}} F_0 \int_{\tilde{\Sigma}_3} H_3 = 0 \ . \tag{6.31}$$

Here μ_p^{D} is the Dp-brane charge and the Op-plane charge density is

$$\mu_p^{\mathrm{O}} = -2^{p-5} \mu_p^{\mathrm{D}} \ . \tag{6.32}$$

The N_{D6} and N_{O6} are the number of D6-branes and O6-planes wrapped on the cycle Σ_3, which is dual to $\tilde{\Sigma}_3$. With $p = 6$ the O-plane charge is determined to $-2\mu_p^D$ so that the above equation takes the form

$$N_{\mathrm{D6}}(\Sigma_3) - 2N_{\mathrm{O6}}(\Sigma_3) + F_0 \int_{\tilde{\Sigma}_3} H_3 = 0 \ . \tag{6.33}$$

In type IIB, the tadpole conditions come from D3-, D5- and D7-branes. In the previous section we saw however that D7-branes do not contribute to the energy-momentum tensor, and neither does the $F_1^{(10)}$ flux for which they are magnetic sources. A wrapped D7-brane does on the other hand have induced D3-charge if one takes into account the first α'-correction to its action, so it will therefore contribute to the D3 tadpole.

The D5-branes extended along spacetime are wrapped around an internal 2-cycle Σ_2 and are electric sources of $F_7^{(10)}$ and magnetic sources of the $F_3^{(10)}$ fields. The 4-cycle $\tilde{\Sigma}_4$ is dual to the 2-cycle, so an integration over the Bianchi identity in eq. (6.29) over these cycles results in that

$$N_{\mathrm{D5}}(\Sigma_2) - N_{\mathrm{O5}}(\Sigma_2) + \int_{\tilde{\Sigma}_4} H_3 \wedge F_1 = 0 \ . \tag{6.34}$$

D3-branes are electric sources of $\tilde{F}_1$ and magnetic sources of the F_5 which are 6D Hodge dual to each other according to eq. (6.28). In the internal space the D3-branes are point-like objects and therefore the integration will be over the whole internal space, so that the tadpole cancellation condition becomes

$$N_{\mathrm{D3}} - \tfrac{1}{4} N_{\mathrm{O3}} + \int H_3 \wedge F_3 = 0 \ . \tag{6.35}$$

Here the integral fluxes can be written as

$$N_{\mathrm{flux}} \equiv \int H_3 \wedge F_3 = (e_C m^C_{\mathrm{RR}} - m^C e_{\mathrm{RR}C}) = N_{\mathrm{NSNS}} \eta N_{RR}^T \ , \tag{6.36}$$

with $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

As for the branes that are sources to NSNS fluxes, i.e. NS-branes, they are the same in type IIA and IIB. The NS5-branes are magnetic sources of the H_3 flux, whose Bianchi identity in the presence of fluxes is

$$\mathrm{d}H_3 = \rho_{\mathrm{NS5}} \ . \tag{6.37}$$

6.3 Special type IIB solutions to the Bianchi identities

In this section we will review some constraints on type IIB solutions of the Bianchi identities for the fluxes. We consider the type IIB action in eq. (3.110) with the addition S_{loc} from localised sources. Again, we consider the warped metric

$$\mathrm{d}s^2 = e^{2A(y)} \hat{g}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu + e^{-2A(y)} \tilde{g}_{mn} \, \mathrm{d}y^m \, \mathrm{d}y^n \ , \tag{6.38}$$

with warp factor $e^{2A(y)}$. Poincaré-invariance in 4D and the Bianchi identity restrict the allowed components of the flux. Namely, the self-dual 5-form flux should be of the form

$$F_5^{(10)} = (1 + \star_{10})[\mathrm{d}\alpha \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^3] \ , \tag{6.39}$$

or equivalently

$$F_5 = e^{-4A} \star_6 \mathrm{d}(e^{4A}) \ , \tag{6.40}$$

where $\alpha = \alpha(y)$ is some function of the internal coordinates. The 3-form flux G_3 is only allowed to have components along $\mathcal{M}_6$. With the reversed Einstein equation of eq. (6.1) and the stress tensor of the sources given in eq. (6.18), the non-compact 4D components of the Ricci tensor take the form

$$R_{\mu\nu} = -e^{2A} g_{\mu\nu} \left(\frac{G_{mnp}\bar{G}^{mnp}}{48 \mathrm{Im}\tau} + \frac{e^{-8A}}{4} \partial_m \alpha \partial^m \alpha \right) + \kappa_{10}^2 \left(T_{\mu\nu}^{\mathrm{loc}} - \tfrac{1}{8} g_{\mu\nu} T^{\mathrm{loc}} \right) \ . \tag{6.41}$$

With the metric in eq. (6.38) the external Ricci components are computed to

$$\begin{aligned} R_{\mu\nu} &= -g_{\mu\nu} e^{4A} \tilde{\nabla}^2 A \\ &= -\tfrac{1}{4} g_{\mu\nu} \left(\tilde{\nabla}^2 e^{4A} - e^{-4A} \partial_m e^{4A} \partial^m e^{4A} \right) \ , \end{aligned} \tag{6.42}$$

where the tilde denotes the use of the internal metric $\tilde{g}_{mn}$. Combining this equation with eq. (6.41), we arrive, after tracing both equations, at

$$\tilde{\nabla}^2 A = e^{-2A} \frac{G_{mnp}\bar{G}^{mnp}}{48 \mathrm{Im}\tau} + \frac{e^{-6A}}{4} \partial_m \alpha \partial^m \alpha + \frac{\kappa_{10}^2}{8} e^{-2A} (T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}} \ . \tag{6.43}$$

This is equivalently written using the second line of eq. (6.41) as

$$\begin{aligned} \tilde{\nabla}^2 e^{4A} &= e^{2A} \frac{G_{mnp}\bar{G}^{mnp}}{12 \mathrm{Im}\tau} + e^{-6A} [\partial_m \alpha \partial^m \alpha + \partial_m e^{4A} \partial^m e^{4A}] + \frac{\kappa_{10}^2}{2} e^{2A} (T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}} \\ &= e^{2A+\phi} \frac{|G_3|^2}{12} + e^{-6A} (|\partial\alpha|^2 + |\partial e^{4A}|^2) + \frac{\kappa_{10}^2}{2} e^{2A} (T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}} \ . \end{aligned} \tag{6.44}$$

Would one integrate eq. (6.43) and eq. (6.44) over the internal manifold $\mathcal{M}_6$, the left-hand side vanishes as long as the internal manifold is assumed compact. The flux and warp factor terms are positive semi-definite, and so these equations serve as stringent constraints on flux/brane configurations for warped type IIB compactifications. It is clear that without any localised sources, G_3 flux must vanish and the warp factor must be constant.

Giddings, Kachru and Polchinski (GKP) [39] was first to show that a condition on the source terms determines the form of the solution completely. See also [50]. The condition in question is that for all localised sources one assumes that

$$\tfrac{1}{4}(T^m{}_m - T^\mu{}_\mu) \geq T_3 \rho_3^{\mathrm{loc}} \ , \tag{6.45}$$

which resembles a BPS condition. With D3-branes and O3-planes present, which have integrated ρ_3 equal to $+1$ and $-1/4$ respectively, the stress tensor is

$$T^0{}_0 = T^1{}_1 = T^2{}_2 = T^3{}_3 = -T_3 \rho_3^{\mathrm{loc}} \ , \qquad T^m{}_m = 0 \ , \tag{6.46}$$

which saturates the inequality of eq. (6.45), i.e. makes it an equality.

Again, the Bianchi identity/equations of motion for the 5-form flux is

$$\mathrm{d}\hat{F}_5 = H_3 \wedge F_3 + 2\kappa_{10}^2 T_3 \rho_3^{\mathrm{loc}} \ , \tag{6.47}$$

where an integration over the internal manifold results in the tadpole condition

$$\frac{1}{2\kappa_{10}^2 T_3}\int H_3 \wedge F_3 + Q_3^{\mathrm{loc}} = 0 \ , \tag{6.48}$$

with Q_3^{loc} being the total charge coming from D3- or D7-branes and O3-planes. The charge density of the O-planes is again given by eq. (6.32), so for instance with only D3-branes and O3-planes it is given by $Q_3^{\mathrm{loc}} = \mu_3(N_{\mathrm{D3}} - \frac{1}{4}N_{\mathrm{O3}})$. Using the expression of the 5-form flux in eq. (6.39) and inserting it into the Bianchi identity in eq. (6.47), an expression for α can be obtained as

$$\tilde{\nabla}^2\alpha = ie^{2A}\frac{G_{mnp}\star_6\bar{G}^{mnp}}{12\mathrm{Im}\tau} + 2e^{-6A}\partial_m\alpha\partial^m e^{4A} + 2\kappa_{10}^2 e^{2A}T_3\rho_3^{\mathrm{loc}} \ . \tag{6.49}$$

This is very similar to the Einstein equation constraint of eq. (6.44) and, subtracting eq. (6.49) from it, one obtains

$$\begin{aligned}\tilde{\nabla}^2(e^{4A}-\alpha) = &\frac{e^{2A}}{6\mathrm{Im}\tau}|iG_3 - \star_6 G_3|^2 + e^{-6A}|\partial(e^{4A}-\alpha)|^2 \\ &+ 2\kappa_{10}^2 e^{2A}\left(\tfrac{1}{4}(T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}} - T_3\rho_3^{\mathrm{loc}}\right) \ .\end{aligned} \tag{6.50}$$

Integrating over the internal compact manifold, the left-hand side vanishes. With the assumption of eq. (6.45) regarding the sources, the right-hand side of eq. (6.50) is positive semi-definite. As a consequence the three terms must vanish on their own, namely we must have

- a self-dual 3-form field strength $\star_6 G_3 = iG_3$,
- a 4-form function related to the warp factor via $\alpha(y) = e^{4A(y)}$,
- saturated the inequality in eq. (6.45).

In light of these conditions we may review the field equations and Bianchi identities. Starting with the 5-form field strength, firstly it is self-dual by construction. Its Bianchi identity in eq. (6.49) is consistent and determines α and A provided that the total D3 charge vanishes in eq. (6.48). As for the 3-form field strengths, one must impose their Bianchi identities

$$\mathrm{d}F_3 = \mathrm{d}H_3 = 0 \ . \tag{6.51}$$

With these conditions, the equation of motion takes the form

$$\mathrm{d}\left(e^{4A}\star_6 G_3 - i\alpha G_3\right) + \frac{i}{\mathrm{Im}\tau}\mathrm{d}\tau\wedge\mathrm{Re}\left(e^{4A}\star_6 G_3 - i\alpha G_3\right) = 0 \ , \tag{6.52}$$

which is satisfied by $\star_6 G_3 = iG_3$ and $\alpha(y) = e^{4A(y)}$. The Ricci tensor equation for the internal coordinates and the axion-dilaton τ must satisfy the two conditions

$$\begin{aligned}\hat{R}_{mn} &= \kappa_{10}^2\frac{\partial_m\tau\partial_n\bar{\tau} + \partial_n\tau\partial_m\bar{\tau}}{4(\mathrm{Im}\tau)^2} + \kappa_{10}^2\left(\hat{T}_{mn}^{\mathrm{D7}} - \tfrac{1}{8}\hat{g}_{mn}\hat{T}^{\mathrm{D7}}\right) \ , \\ \tilde{\nabla}^2\tau &= \frac{\tilde{\nabla}\tau\cdot\tilde{\nabla}\tau}{i\mathrm{Im}\tau} - \frac{4\kappa_{10}^2(\mathrm{Im}\tau)^2}{\sqrt{-g}}\frac{\delta\tilde{S}_{\mathrm{D7}}}{\delta\bar{\tau}} \ ,\end{aligned} \tag{6.53}$$

which are written in Einstein frame. Summarising, given that all localised sources satisfy eq. (6.45), the necessary and sufficient conditions are that the internal metric fulfils eq. (6.53), that we have a 5-form flux given by eq. (6.39), an imaginary self-dual (ISD) complex 3-form flux and that the inequality of eq. (6.45) is saturated, i.e. has vanishing total D3 charge. Note that the first three of these requirements correspond to the type B solutions of chapter 4. The reasoning in this section does not however impose supersymmetry at all. The ISD condition on G_3 is less restrictive than what a supersymmetric type B solution requires, since it allows for a non-primitive (1,2) component as well as a singlet (0,3) term. The type B solutions demand that G_3 is purely a primitive (2,1)-form. However, one may consider our compact internal manifold to have $SU(3)$-structure where there can be no non-trivial closed 1-forms, thus forbidding the non-primitive (1,2)-form. If there further would not exist a (0,3) singlet piece then the above solution is exactly of the type B form. Without any D7-branes, the internal manifold is a conformal Calabi-Yau, i.e. with torsion classes $2W_5 = 3W_4$ and a constant dilaton. With D7-branes present, the internal space obeys the condition in eq. (6.53) and has $\nabla\tau \neq 0$. In this case the internal space is no longer conformal Calabi-Yau but has $W_4 = W_5 = \partial\phi$.

In type B solutions, F_5 is the relevant field for the Bianchi identity, and the above anlysis can be thought of as a type B solution. As a final remark we will briefly comment on the IIB type A and C solutions. In type A solutions, the relevant Bianchi identity is the one with NSNS flux. Choosing the positive sign relation in table 4.4, the NSNS field strength H_3 is related to the fundamental form by

$$H_3 = i(\partial - \bar{\partial})J_2 \ , \tag{6.54}$$

with ∂ being the holomorphic exterior derivative [51]. The Bianchi identity of eq. (6.37) will then result in that

$$\mathrm{d}H_3 = -2i\partial\bar{\partial}J_2 = \rho_{\mathrm{NS5}} \ . \tag{6.55}$$

As for the type C solutions, they are S-dual to the type A solutions, so the corresponding Bianchi identity becomes [1]

$$\mathrm{d}F_3 = 2i\partial\bar{\partial}(e^{-2A}J_2) = H_3 \wedge F_{(1)} + (2\pi\sqrt{\alpha'})^2\rho_5^{\mathrm{loc}} \ . \tag{6.56}$$

In the next chapter we will discuss moduli stabilisation on different Ricci flat geometries, but not always with $SU(3)$-structure as we will consider tori geometries.

7

Moduli Stabilisation by Fluxes

In order to stabilise moduli a common recipe is to consider a specific geometry and introduce fluxes on it in ways conveniently entwined with the geometry at hand. In this chapter we illustrate this by working through some of the most common examples of moduli stabilisation in the literature.

7.1 Moduli stabilisation in type IIB Calabi-Yau orientifolds on the deformed conifold

In this section we review the main mechanisms in the stabilisation of type IIB moduli on orientifolds using the setup of a *deformed conifold* to describe some singular points in the Calabi-Yau manifold which may be related to moduli.

A striking result is that the moduli spaces belonging to completely different Calabi-Yau manifolds touch at some points in their boundaries. It is possible to move between the moduli spaces which correspond to topologically distinct Calabi-Yau manifolds, and it has been shown that a large number (perhaps all) of the moduli spaces of simply connected Calabi-Yaus together form a connected web [52]. In the regions where the different moduli spaces meet, called phase- or geometric transitions, the respective metrics of the Calabi-Yau become singular. As a consequence Calabi-Yau manifolds are actually not entirely smooth, but contain conical singularities for special values of their moduli fields. The most generic manifold with singular points is a *conifold*. The local area around a singular point is generically described by a quadric in $\mathbb{C}^4$, which spans the space of a *cone*, such that

$$\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = 0 \ , \tag{7.1}$$

where the ωs denote the local complex coordinates. The singular point is at $(\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2) = 0$. This is a cone since for any ω fulfilling the above equation, so does $\lambda\omega$ with λ being some constant function, hence the space is made up of complex lines through the origin.

In order to determine the base of the cone it is useful to decompose the ωs into their real and imaginary parts. Viewing ω as a four-component vector; $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, the conifold singularity of eq. (7.1) is written $\omega^2 = 0$. In terms of real and imaginary parts; $\omega = \xi + i\eta$, eq. (7.1) can be written as the pair

$$\xi^2 - \eta^2 = 0 \ , \qquad \xi \cdot \eta = 0 \ . \tag{7.2}$$

The base is then the intersection between the cone and a sphere with its centre at the singularity. If the sphere has radius r so that $\xi^2 + \eta^2 = r^2$, the base is described by

$$\xi^2 = \tfrac{1}{2}r^2 \ , \qquad \eta^2 = \tfrac{1}{2}r^2 \ , \qquad \xi \cdot \eta = 0 \ . \tag{7.3}$$

It is clear that the space of ξs is an S^3 and taking this as a starting point it is also clear that for each point on it, e.g. $\xi = (r/\sqrt{2}, 0, 0, 0)$, then by the second equation in eq. (7.3) the

$\eta = (0, \eta_1, \eta_2, \eta_3)$ is effectively a three-vector. This makes $\eta^2 = r/2$ to be an S^2. The base of the cone is therefore a fibre bundle with base S^3 and fibre S^2. Since all S^2 bundles over S^3 are trivial, the base of the cone must be topologically equivalent to the product space $S^2 \oplus S^3$.

The conifold singularity can be repaired, or smoothed, in two different ways. The first one is by a *small resolution* in which eq. (7.1) is rewritten by a linear change of variables into

$$XY - UV = 0 \ , \tag{7.4}$$

and then making a small resolution in which eq. (7.4) is replaced by two equations

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \ , \tag{7.5}$$

where λ_1, λ_2 are not both zero. This equation is equivalent to eq. (7.4) away from $(X, Y, U, V) = 0$. The solutions of the λs are determined up to an overall multiplicative factor, i.e. $(\lambda_1, \lambda_2) \simeq \lambda(\lambda_1, \lambda_2)$ with $\lambda \in \mathbb{C}$. Thus all variables of eq. (7.5) belong to $\mathbb{C}^4 \oplus \mathbb{P}^1$ with $\mathbb{P}$ denoting the complex projective space. The matrix on the left of eq. (7.5) has either rank one or zero, although it is only zero at a singularity where all elements X, Y, U, V vanish. Because of this, eq. (7.5) determines a unique ratio λ_1/λ_2 and hence a unique point in $\mathbb{P}^1$. When $(X, Y, U, V) \neq 0$ in eq. (7.5) the space is the same as the singular conifold. However, since $(X, Y, U, V) = 0$ at the singularity, λ_1 and λ_2 are completely unconstrained so that the space they span, i.e. $\mathbb{P}^1$, is projected down to each node of M. Thus in passing from M to the resulting smooth manifold $M^\flat$ each conifold singularity is replaced with a copy of $\mathbb{P}^1 \equiv S^2$ generated by $\lambda_1/\lambda_2 = -U/X = -Y/V$. It is not obvious but nevertheless true that the resulting smooth manifold $M^\flat$ is a Calabi-Yau [53].

The other way of "desingularising" the conifold singularity is by *deformation*. Namely, eq. (7.3) is deformed, or perturbed, into

$$\xi^2 = \epsilon^2 \ , \qquad \eta = 0 \ , \tag{7.6}$$

where ϵ is a non-zero constant. This space describes an S^3, so the conifold singularity has been replaced by an S^3 and the resulting smooth space is a Calabi-Yau manifold denoted $M^\sharp$. Thus by smoothing the conifold singularity of M by either doing a small resolution or a deformation, it is possible to pass continuously from one Calabi-Yau to another via [54]

$$M^\flat \longleftrightarrow M \longleftrightarrow M^\sharp \ . \tag{7.7}$$

The smoothed Calabi-Yau manifolds $M^\flat$ and $M^\sharp$ are perhaps not too surprisingly topologically distinct since the singularities of the original conifold M have been replaced by the different spheres S^2 and S^3 respectively. However, it can nevertheless be seen from the Euler numbers of the two manifolds. The Euler number of an S^3 vanishes; $\chi(S^3) = 0$, for an S^2 it is $\chi(S^2) = 2$ and for a point it is $\chi(\text{point}) = 1$. For each singularity, or node N, in M there is an S^3 in $M^\sharp$ and an S^2 in $M^\flat$, so their Euler numbers are related via

$$\chi(M) - \chi(M^\flat) = \chi(\text{point})N - \chi(S^2)N = N - 2N = -N \ , \tag{7.8}$$

$$\chi(M) - \chi(M^\sharp) = \chi(\text{point})N - \chi(S^3)N = N - 0 = N \ , \tag{7.9}$$

or more compactly;

$$\chi(M^\flat) = \chi(M) + N = \chi(M^\sharp) + 2N \ . \tag{7.10}$$

The geometry of the deformed conifold can be used to stabilise moduli. Recasting eq. (7.6) into the original complex local coordinates ω, it takes the form

$$\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = z \ , \tag{7.11}$$

where the complex parameter z controlling the size of the S^3 can be taken to be our familiar complex structure modulus. While we only work with one complex structure moduli, it can be naturally generalised to include several. Recalling the 3-cycles A^C and their duals B^C defined in eq. (5.114), there are $2h^{(2,1)}+2$ such cycles on the manifold which intersects each other pair-wise once according to $A^I \cap B_I = \delta^I_J$. The complex structure moduli is again related to them via $z^c = Z^C/Z^0$ where $Z^C = \int_{A^I} \Omega$ and the dual cycle $F_J(Z) = \int_{B_J} \Omega$ is related to the prepotential via $F_J = \partial_J \mathcal{F}$. In the vicinity of the conifold singularity we may have two cycles present, namely one of the A cycles and one of the B cycles, say A^1 and B_1. If one for example takes the ωs and z to be real and positive, then the A cycle will intersect the dual B cycle exactly once if the B cycle is constructed with imaginary $\omega_{1,2,3}$ as well as a real and positive ω_4.

At a conifold singularity $Z^1 \to 0$ the A^1 cycle will shrink to zero volume. The A^1 cycle circles this point, but all we know of the B^1 cycle is that it intersects with the A^1 cycle once. Because of this one might say that the dual cycle will go as

$$B_1 \to B_1 + A^1 \ , \tag{7.12}$$

around $Z^1 = 0$. This is known as a monodromy transformation. For the corresponding period it is $F_1 \to F_1 + Z^1$ while $Z^1 \to Z^1$ which in our case is close to $Z^1 = 0$. Monodromies are however not always related to shrinking cycles, but may arise because the complex structure moduli space of a Calabi–Yau manifold is usually a quotient of a larger space, known as the Teichmüller space [55]. The monodromy of eq. (7.12) implies that the F_J can be written

$$F_1(Z) = \frac{Z^1}{2\pi i} \ln Z^1 + \text{const} \ , \tag{7.13}$$

near the singularity. The Kähler metric of the complex structure moduli space $\mathcal{M}^{\text{cs}}$ defined in eq. (5.31) is around this point equal to $\mathcal{K}^{\text{cs}} = -\ln[i(\bar{Z}^1 F_1 - Z^1 \bar{F}_1)]$, which using eq. (7.13) becomes proportional to

$$\mathcal{K}^{\text{cs}} \sim |Z^1|^2 \ln |Z^1| + \text{const} \ . \tag{7.14}$$

Since $G_{1\bar{1}} = \partial_1 \partial_{\bar{1}} \mathcal{K}$, it follows that the metric of $\mathcal{M}^{\text{cs}}$ is singular in $Z^1 = 0$. Since the curvature is also divergent at this point, there is a real singularity, i.e. not a mere coordinate singularity, in the complex structure moduli space.

As in the works of Klebanov and Strassler in [56], who first found solutions with fluxes which generate smooth supergravity solutions in the vicinity of conifold singularities, we define that M units of F_3 flux will go through the A cycle and $-K$ units of H_3 flux through the dual B cycle;

$$\int_{A^1} F_3 = M \ , \qquad \int_{B_1} H_3 = -K \ . \tag{7.15}$$

This allows us to rewrite the O3 superpotential of eq. (5.146) in the vicinity of the conifold singularity according to

$$W = \int G_3 \wedge \Omega = -M F_1(Z) + \tau K Z^1 \ , \tag{7.16}$$

where again $G_3 = F_3 - \tau H_3$. For supersymmetry to hold we require $D_z W = 0$, where the covariant derivative of the superpotential is defined as in eq. (5.148). By setting $C_0 = 0$, it follows that $\tau = i/g_{\text{s}}$ where g_{s} is the string coupling constant. The leading terms in the equation $D_z W = 0$ in the limit $K >> M g_{\text{s}}$ are

$$D_z W \sim -\frac{M}{2\pi i} \ln Z^1 + i \frac{K}{g_{\text{s}}} + \mathcal{O}(1) = 0 \ . \tag{7.17}$$

It follows that Z^1 and with it the complex structure moduli z^1 must stabilise at an exponentially small value of

$$z^1 \sim e^{-2\pi K/M g_{\text{s}}} \ . \tag{7.18}$$

The corresponding equation for τ and the dilaton reads [39]

$$D_\tau W \sim \frac{1}{\bar{\tau}-\tau}(-MF_1 + \bar{\tau}KZ^1) = 0 \ , \tag{7.19}$$

and can not be satisfied. The second term in the parenthesis containing Z^1 is exponentially small by eq. (7.18), however the first term with $F_1(Z)$ is not. The coordinate $F_1(Z)$ is generally non-vanishing, namely $F_1(0) \sim \mathcal{O}(1)$. This problem arises because at $z^1 = 0$, the superpotential of eq. (7.16) is independent of τ. The remedy for this is therefore to consider a configuration with additional τ-dependence, which may be added by turning on additional fluxes. Keeping for simplicity the case of a single complex structure modulus z^1, there will be $2h^{(2,1)} + 2 = 4$ 3-cycles. We have worked with two of these until now, i.e. the pair $\{A^1, B_1\}$, so we may add the last two and denote them $\{A^2, B_2\}$. Turning on an additional $-\tilde{K}$ units of H_3 flux through the B_2 cycle, the superpotential becomes

$$W = -MF_1(Z) + \tau(KZ^1 + \tilde{Z}\tilde{K}) \ , \tag{7.20}$$

where $\tilde{Z}(Z^1)$ is a generally non-vanishing function at the conifold singularity; $\tilde{Z}(0) \sim \mathcal{O}(1)$. Fixing $z^1 = 0$, eq. (7.16) becomes

$$D_\tau W \sim \frac{1}{\bar{\tau}-\tau}(-MF_1(0) + \bar{\tau}\tilde{K}\tilde{Z}) = 0 \ , \tag{7.21}$$

which then fixes the dilaton at

$$\bar{\tau} = \frac{MF_1(0)}{\tilde{K}\tilde{Z}} \ . \tag{7.22}$$

With this value the complex structure modulus z^1 will stabilise at

$$z^1 \sim e^{\frac{2\pi K}{\tilde{K}}\mathrm{Im}(F_1(0)/\tilde{Z}(0))} \ . \tag{7.23}$$

Hence we have fixed the dilaton and with it stabilised the complex structure modulus. However, the Kähler moduli is not stabilised by these fluxes. There are interesting phenomenological consequences for the complex structure moduli being stabilised at a small value. Close to the KS *throat*, i.e. the singularity, the warp-factor which solves eq. (6.44) will be very small. This is because the warp-factor scales like $e^{4A} \sim r^4$ for D3-branes, with r denoting the radial coordinate from the brane. The resolution of the conifold cuts this off at $r \propto \omega^{2/3} \propto z^{1/3}$, so that there is a minimum for the warp-factor at

$$e^{2A_{\min}} \sim z^{2/3} \sim e^{-4\pi K/3Mg_s} \ , \tag{7.24}$$

which generates a large hierarchy of scales.

7.2 Moduli stabilisation in type IIB on $T^6/\mathbb{Z}_2$ orientifolds

In this section we will review moduli stabilisation of type IIB on tori. One of the first examples of this was done by Kachru, Schulz and Trivedi [57] who compactified on a $T^6/\mathbb{Z}_2$ orientifold with NSNS flux H_3 and RR flux F_3. More precisely, the orientifold is given by the quotient space

$$\mathcal{M}_6 = \frac{T^6}{G_1 \cup G_2\Omega} = \frac{T^6}{\mathbb{Z}_2} \tag{7.25}$$

with $G_1 = \{\varnothing\}$ being empty and the cyclic group $\mathbb{Z}_2$ effectively being equal to $\mathbb{Z}_2 = \Omega(-1)^{F_L}R$, where R reflects the six internal coordinates, i.e. $y^m \to -y^m$.

In the case of Calabi-Yau O3 orientifold compactifications the moduli are given in table 5.5. The $T^6/\mathbb{Z}_2$ is however not a Calabi-Yau and a torus has trivial structure group so that besides $h_-^{(2,1)} = h_+^{(1,1)} = 9$ there are additional cohomologies $h_-^{(1,0)} = h_+^{(2,0)} = h_+^{(3,1)} = h_-^{(3,2)} = 3$ and their conjugates ($h_-^{(0,1)} = \ldots = 3$).

An essential difference between the T^6 and a Calabi-Yau 3-fold is that for the latter Yau's theorem implies that a complex structure or Kähler deformation corresponds to a non-trivial deformation of the Ricci-flat metric. This is not the case for the torus or $T^6/\mathbb{Z}_2$. Instead, as will be described next, only three out of nine complex structure parameters will correspond to metric deformations.

The complex structure of a T^6 is described by nine complex coordinates. Introducing six real coordinates x^i, y^i with $i = 1, 2, 3$ on the torus which are periodic; $x^i \equiv x^i + 1$, $y^i \equiv y^i + 1$, we define the holomorphic 1-forms to be

$$\mathrm{d}z^i \equiv \mathrm{d}x^i + \upsilon^{ij}\,\mathrm{d}y^j \ . \tag{7.26}$$

Here υ^{ij} is the period matrix specifying the complex structure of the torus. In these coordinates the holomorphic 3-form is

$$\Omega = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \ . \tag{7.27}$$

The coordinates are conveniently expressed in the $H^{(3)}$ cohomology basis (α_C, β^C), with $C = 0, \ldots, 8$, as

$$\begin{aligned} \alpha_0 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ , & \beta^0 &= \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 \ , \\ \alpha_{ij} &= \tfrac{1}{2}\epsilon_{ilm}\,\mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^j \ , & \beta^{ij} &= -\tfrac{1}{2}\epsilon_{jlm}\,\mathrm{d}y^l \wedge \mathrm{d}y^m \wedge \mathrm{d}x^i \ , \end{aligned} \tag{7.28}$$

and $i, j = 1, 2, 3$. In this basis the holomorphic 3-form can be written

$$\Omega = \alpha_0 + \alpha_{ij}\upsilon^{ij} - \beta^{ij}(\mathrm{cof}\upsilon)_{ij} + \beta^0 \det(\upsilon^{ij}) \ , \tag{7.29}$$

where

$$(\mathrm{cof}\upsilon)_{ij} \equiv \det(\upsilon^{ij})(\upsilon^{ij})^{-1T} = \tfrac{1}{2}\epsilon_{ikm}\epsilon_{jpq}\upsilon^{kp}\upsilon^{mq} \ , \tag{7.30}$$

and T denotes the transpose. We now turn on fluxes. Fluxes present after the orientifold projection must be even under the action of the symmetry group $\mathbb{Z}_2$. As can be seen in table 5.3, the B_2 and C_2 fields are both odd under the intrinsic parity action $\Omega(-1)^{F_L}$. The 3-form basis of eq. (7.28) does however transform oddly under the $\mathbb{Z}_2$ action; $(x^i, y^i) \to -(x^i, y^i)$. Since the Bianchi identities of their field strengths imply that they be closed; $\mathrm{d}F_3 = \mathrm{d}H_3 = 0$, it is possible to describe them as linear combinations in the basis of (7.28). This can be done according to

$$\begin{aligned} F_3 &= m^0_{\mathrm{RR}}\alpha_0 + m^{ij}_{\mathrm{RR}}\alpha_{ij} + e_{\mathrm{RR}ij}\beta^{ij} + e_{\mathrm{RR}0}\beta^0 \ , \\ H_3 &= m^0\alpha_0 + m^{ij}\alpha_{ij} + e_{ij}\beta^{ij} + e_0\beta^0 \ , \end{aligned} \tag{7.31}$$

where again $\{m_{\mathrm{RR}}, e_{\mathrm{RR}}, m, e\}$ are all integers. With these fluxes, the superpotential in eq. (5.146) becomes

$$W = (m^0_{\mathrm{RR}} - \tau m^0)\det(\upsilon^{ij}) - (m^{ij}_{\mathrm{RR}} - \tau m^{ij})(\mathrm{cof}\upsilon)_{ij} - (e_{\mathrm{RR}ij} - \tau e_{ij})\upsilon^{ij} - (e_{\mathrm{RR}0} - \tau e_0) \ . \tag{7.32}$$

The superpotential depends on the complex dilaton field τ as well as the nine components of υ^{ij}, adding up to a total of ten complex variables. In type B solutions G_3 is a $(2,1)$-form and when that happens one must have $W = 0$ and $D_I W = 0$ by eqs. (5.150), (5.151). Demanding supersymmetry along the Kähler moduli, i.e. $D_{T_\alpha}W = 0$, forces $W = 0$ according to eq.

(5.148). When $W=0$, the Kähler covariant derivatives can be replaced by ordinary derivatives; $D_I W \to \partial_I W$. Using

$$\partial_{\upsilon^{ij}}\Omega = k_{ij}\Omega + \chi_{ij} \ , \tag{7.33}$$

and the O3 superpotential $W = \int G_3 \wedge \Omega$, the Kähler derivatives for the complex structure and the τ are

$$\begin{aligned} \partial_{\upsilon^{ij}} W &= k_{ij} W + \int G_3 \wedge \chi_{ij} \ , \\ \partial_\tau W &= -H_3 \wedge \Omega = \frac{1}{\tau - \bar{\tau}} \int (G_3 - \bar{G}_{(3)}) \wedge \Omega \ . \end{aligned} \tag{7.34}$$

Now, requiring $W=0$ in addition to $\partial_{\upsilon^{ij}} W = 0$ and $\partial_\tau W = 0$ results in eleven complex-coupled non-linear equations for ten complex variables. Since all equations are independent, there are generally no solutions to them, and as a consequence the supersymmetry is broken. Some equations of motion arising from these conditions are

$$W - \partial_\tau W = 0 \quad \Rightarrow \quad m^0_{\mathrm{RR}}\det(\upsilon^{ij}) - m^{ij}_{\mathrm{RR}}(\mathrm{cof}\upsilon)_{ij} - e_{\mathrm{RR}ij}\upsilon^{ij} - e_{\mathrm{RR}0} = 0 \ , \tag{7.35}$$

$$\partial_{\upsilon^{ij}} W = 0 \quad \Rightarrow \quad m^0\det(\upsilon^{ij}) - m^{ij}(\mathrm{cof}\upsilon)_{ij} - e_{ij}\upsilon^{ij} - e_0 = 0 \ , \tag{7.36}$$

$$\begin{aligned} \partial_\tau W = 0 \quad \Rightarrow \quad & (m^0_{\mathrm{RR}} - \tau m^0)(\mathrm{cof}\upsilon)_{ij} - (m^{ij}_{\mathrm{RR}} - \tau m^{ij})\epsilon_{ikm}\epsilon_{jln}\upsilon^{mn} \\ & - (e_{\mathrm{RR}ij} - \tau e_{ij})\delta^i_k\delta^j_l = 0 \ , \end{aligned} \tag{7.37}$$

where in the last equation we have used the relations $\det(\upsilon^{ij}) = \frac{1}{3}\epsilon_{ikl}\epsilon_{jln}\upsilon^{ij}\upsilon^{kl}\upsilon^{mn}$ and $(\mathrm{cof}\upsilon)_{ij} = \frac{1}{2}\epsilon_{ikm}\epsilon_{jln}\upsilon^{kl}\upsilon^{mn}$.

In [57] there are several examples of supersymmetic solutions. Let us look at an example where the flux matrices are taken to be diagonal. In this case the fluxes are written

$$(m^{ij}_{\mathrm{RR}}, e_{\mathrm{RR}ij}, m^{ij}, e_{ij}) = (m_{\mathrm{RR}}, e_{\mathrm{RR}}, m, e)\delta_{ij} \ . \tag{7.38}$$

The period matrix υ^{ij} will then also be diagonal according to

$$\upsilon^{ij} = \upsilon\delta^{ij} \ . \tag{7.39}$$

With a diagonal period matrix the torus factorises as $T^6 = T^2 \oplus T^2 \oplus T^2$ with respect to the complex structure. With this "flux diagonalisation", the supersymmetry conditions of eq. (7.35)–(7.37) take the form

$$P_1(\upsilon) \equiv m^0_{\mathrm{RR}}\upsilon^3 - 3m_{\mathrm{RR}}\upsilon^2 - 3e_{\mathrm{RR}}\upsilon - e_{\mathrm{RR}0} = 0 \ , \tag{7.40}$$

$$P_2(\upsilon) \equiv m^0\upsilon^3 - 3m\upsilon^2 - 3e\upsilon - e_0 = 0 \ , \tag{7.41}$$

$$(m^0_{\mathrm{RR}} - \tau m^0)\upsilon^2 - 2(m_{\mathrm{RR}} - \tau m)\upsilon - (e_{\mathrm{RR}} - \tau e) = 0 \ . \tag{7.42}$$

Assuming υ to be complex, one can show that the cubic polynomials P_1 and P_2 can be written

$$P_1(\upsilon) = (A\upsilon + B)P(\upsilon) \ , \qquad P_2(\upsilon) = (C\upsilon + D)P(\upsilon) \ , \tag{7.43}$$

for some

$$P(\upsilon) = E\upsilon^2 + F\upsilon + G \ , \tag{7.44}$$

where $\{A, B, C, D, E, F, G\} \in \mathbb{Z}$. The coefficients are related to the fluxes via

$$\begin{aligned} AF + BE &= -3m_{\mathrm{RR}} \ , \qquad & CF + DE &= -3m \ , \\ AG + BF &= -3e_{\mathrm{RR}} \ , \qquad & CG + DF &= -3e \ , \end{aligned} \tag{7.45}$$

so that one has modulo 3 consistency conditions. As a concrete example, we may take

$$P_1(\upsilon) \equiv \upsilon^3 - 1 = 0 \ , \tag{7.46}$$
$$P_2(\upsilon) \equiv \upsilon^3 + 3\upsilon^2 + 3\upsilon + 2 = 0 \ , \tag{7.47}$$

where both polynomials share a common factor

$$P(\upsilon) = \upsilon^2 + \upsilon + 1 \ . \tag{7.48}$$

The cubical polynomials can then be expressed as

$$P_1(\upsilon) \equiv (\upsilon - 1)P(\upsilon) = 0 \ , \tag{7.49}$$
$$P_2(\upsilon) \equiv (\upsilon + 2)P(\upsilon) = 0 \ . \tag{7.50}$$

Now, solving $P(\upsilon) = 0$ with the condition of $\mathrm{Im}(\upsilon) > 0$ results in that

$$\upsilon = e^{2\pi i/3} \ . \tag{7.51}$$

With this value, eq. (7.42) solves for the axion-dilaton, which is

$$\tau = \upsilon = e^{2\pi i/3} \ . \tag{7.52}$$

Comparing the polynomials of eqs. (7.49), (7.50) to eqs. (7.43), (7.44), the coefficients read

$$(A, B, C, D, E, F, G) = (1, -1, 1, 2, 1, 1, 1) \ . \tag{7.53}$$

The fluxes can then be read off eq. (7.40) and eq. (7.41) by comparing with eq. (7.46) and eq. (7.47) respectively, and be determined to

$$(m^0_{\mathrm{RR}}, m_{\mathrm{RR}}, e_{\mathrm{RR}0}, e_{\mathrm{RR}}, m^0, m, e_0, e) = (1, 0, 1, 0, 1, -1, -2, -1) \ . \tag{7.54}$$

Most of the flux integers are odd, which can lead to complications. Namely, apart from the 3-cycles present on the T^6, there are additional 3-cycles on the orientifold $T^6/\mathbb{Z}_2$ which can be seen as "half cycles" on the T^6. If the flux integers are odd it is necessary to allow fluxes from the additional 3-cycles in order to meet the quantisation condition (i.e. the condition that the integer flux is indeed an integer). It is however not trivial to turn on these additional fluxes so that they fulfil the charge conservation (tadpole cancellation) condition. This can be avoided by simply assuming the integer fluxes on T^6 to be even. A discussion on this can be found in Appendix A of [57].

Hence, to simply avoid any complications we may instead simply multiply the flux integers by two, which will ensure all fluxes to be integers. Thus we have

$$\begin{aligned} (m^0_{\mathrm{RR}}, m_{\mathrm{RR}}, e_{\mathrm{RR}0}, e_{\mathrm{RR}}, m^0, m, e_0, e) &= (2, 0, 2, 0, 2, -2, -4, -2) \ , \\ (A, B, C, D, E, F, G) &= (2, -2, 2, 4, 1, 1, 1) \ , \end{aligned} \tag{7.55}$$

where the coefficients can be worked out by comparison.

For the fluxes in eq. (7.55), the complex structure modulus υ and the axion-dilaton τ are fixed according to eq. (7.51) and eq. (7.52). We will now turn to the Kähler moduli. In the Calabi-Yau orientifold case, there are no 1-forms allowed which means that for type B solutions, the (2,1)-form G_3 must be primitive. The manifold $T^6/\mathbb{Z}_2$ is not a Calabi-Yau and does in fact allow three 1-forms since $h^{(1,0)}_- = h^{(0,1)}_- = 3$ which means that G_3 need not be primitive. However by imposing G_3 to be primitive, i.e. by demanding

$$J_2 \wedge G_3 = 0 \ , \tag{7.56}$$

some of the Kähler moduli v^α in J_2 can be stabilised. Since G_3 is a (2,1)-form and J_2 a (1,1)-form, $J_2 \wedge G_3$ must be a (3,2)-form. On a $T^6/\mathbb{Z}_2$ there are $h_-^{(3,2)} = 3$ non-trivial such forms, so eq. (7.56) above will give three complex or six real equations. The number of Kähler moduli is $h_+^{(1,1)} = 9$, so generically six of these nine Kähler moduli are fixed by eq. (7.56), leaving three unfixed "flat" directions that are unchanged by the G_3 flux. With the diagonal flux of eq. (7.38) at hand, the explicit forms of F_3 and H_3 in torus coordinates are explicitly

$$\begin{aligned} F_3 = {} & m^0_{\mathrm{RR}}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + m_{\mathrm{RR}} \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + \text{cycl. perm. of } 123\right) \\ & - e_{\mathrm{RR}} \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + \text{cycl. perm. of } 123\right) + e_{\mathrm{RR}0}\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 \ , \end{aligned} \tag{7.57}$$

$$\begin{aligned} H_3 = {} & m^0\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + m \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + \text{cycl. perm. of } 123\right) \\ & - e \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 + \text{cycl. perm. of } 123\right) + e_0\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 \ . \end{aligned} \tag{7.58}$$

The three moduli in J_2 that are not fixed by eq. (7.56) are the diagonal elements, which are of the from

$$J_{\mathrm{diag}} = \sum_{i=1}^{3} r_i^2\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i \sim i \sum_{i=1}^{3} r_i^2\,\mathrm{d}x^i \wedge \mathrm{d}y^i \ , \tag{7.59}$$

where in the last expression one uses the fact that the complex structure $\upsilon = e^{2\pi i/3}$ of all the three T^2s are equal. It is clear that $J_{\mathrm{diag}} \wedge G_3 \equiv J_{\mathrm{diag}} \wedge (F_3 - \tau H_3) = 0$ is fulfilled without any constraints on the three moduli r_i, since the $\mathrm{d}x^i \wedge \mathrm{d}y^i$ will hit another $\mathrm{d}x^i$ or $\mathrm{d}y^i$ in each term of F_3 and H_3. Generically eq. (7.56) will set constraints on the other constituents of J apart from the diagonal parts. However, in this example there are an additional three moduli apart from the r_i that remain unfixed after imposing primitivity of G_3 in eq. (7.56). This is because the G_3 flux in this example is particularly simple and non-generic. The three extra moduli that remain unfixed can be seen in the form of the components $J_{1\bar{2}} + J_{\bar{1}2}$ and analogous terms with indices $\{1,3\}$ and $\{2,3\}$, since they have the form

$$J_{1\bar{2}} + J_{\bar{1}2} \sim \mathrm{d}z^1 \wedge \bar{z}^2 + \mathrm{d}z^2 \wedge \bar{z}^1 \sim i(\mathrm{d}x^1 \wedge \mathrm{d}y^2 + \mathrm{d}x^2 \wedge \mathrm{d}y^1) \ . \tag{7.60}$$

It is straight-forward to check that indeed $(J_{1\bar{2}} + J_{\bar{1}2}) \wedge G_3 = 0$, and so any coefficients in eq. (7.60), i.e. moduli, would not be constrained by this primitivity condition.

This example has $\mathcal{N} = 1$ supersymmetry. A higher degree of supersymmetry requires that additional choices of complex structure are possible, wherein G_3 is still (2,1) and primitive. For example $\mathcal{N} = 2$ and $\mathcal{N} = 3$ would require one respectively two additional choices of complex structure. The solution to the example above have $T^6 \equiv T^2 \oplus T^2 \oplus T^2$ and there is a complete permutation symmetry between the three 2-tori. This fact ensures that G_3 must have the form

$$G_3 \sim \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^3 + \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}\bar{z}^1 + \mathrm{d}z^3 \wedge \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^2 \ . \tag{7.61}$$

One could in fact make other choices of complex structure, e.g. by letting $z^i \to \bar{z}^i$ for some or all T^2s. However, this would not preserve the imposed nature of G_3 to be $(2,1)$ and primitive, which as a consequence would leave all Kähler moduli unfixed. Because of this our example has $\mathcal{N} = 1$ supersymmetry.

The number of units of flux is constrained by the tadpole cancellation condition. In a $T^6/\mathbb{Z}_2$ there are $N_{\mathrm{O3}} = 2^6$ O3-planes present which give a negative contribution of $-2^{-2}N_{\mathrm{O3}} = -16$ units of D-brane charge to the tadpole of eq. (6.33), leading to the condition

$$\tfrac{1}{2}\int_{T^6} H_3 \wedge F_3 + N_{\mathrm{D3}} = 16 \ . \tag{7.62}$$

Here the factor 1/2 in front of the flux contribution term comes from integration over T^6 rather than $T^6/\mathbb{Z}_2$, since T^6 has twice the volume. The flux contribution in this case is given by

$$\begin{aligned} N_{\text{flux}} = \int_{T^6} H_3 \wedge F_3 &= (e_{\text{RR}0} m^0 - m^0_{\text{RR}} e_0) + 3(e_{\text{RR}} m - m_{\text{RR}} e) \\ &= -\tfrac{1}{3}(AD - BC)(F^2 - 4EG) \\ &= 12 \ , \end{aligned} \tag{7.63}$$

using the numbers in eq. (7.55). With this result it follows from eq. (7.62) that the number of D-branes present must be $N_{\text{D3}} = 10$.

Conclusively, we have seen an example of moduli stabilisation in type IIB on $T^6/\mathbb{Z}_2$ with $\mathcal{N} = 1$ supersymmetry, where the complex structure moduli and axio-dilaton could be fixed. As for the Kähler moduli we have seen that six out of nine could be fixed, while the other three remain unconstrained. In the next section we consider moduli stabilisation in general type IIA Calabi-Yau (O6) orientifolds.

7.3 Moduli stabilisation in type IIA Calabi-Yau orientifolds

The O6 superpotential is given in eq. (5.145) where its proper Kähler coordinates are the Kähler moduli t^a as well as N^k and T_λ defined in eq. (5.137). Repeated here for convenience, the superpotential is given by

$$\begin{aligned} W_{\text{O6}} &= \int H_3 \wedge \Omega_c + \int \hat{F}_{(A)} \wedge e^{B+iJ} \\ &= -2N^k e_k - iT_\lambda m^\lambda + e_{\text{RR}0} + \int F_4 \wedge \mathcal{J} - \tfrac{1}{2} \int F_2 \wedge \mathcal{J} \wedge \mathcal{J} - \tfrac{1}{6} m^0_{\text{RR}} \int \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J} \\ &= -e_k \xi^k + m^\lambda \tilde{\xi}_\lambda + 2i(-e_k \text{Re}(CZ^k) + m^\lambda \text{Re}(C\mathcal{F}_\lambda)) \\ &\quad + e_{\text{RR}0} + e_{\text{RR}a} t^a + \tfrac{1}{2}\kappa_{abc} m^a_{\text{RR}} t^b t^c - \tfrac{1}{6} m^0_{\text{RR}} \kappa_{abc} t^a t^b t^c \ , \end{aligned} \tag{7.64}$$

with $N^k = \frac{1}{2}\xi^k + i\text{Re}(CZ^k)$, $T_\lambda = i\tilde{\xi}_\lambda - 2\text{Re}(C\mathcal{F}_\lambda)$ and $H_3 = m^\lambda \alpha_\lambda - e_k \beta^k$. As usual, supersymmetric vacua are characterised by the vanishing of the F-term conditions

$$D_{t^a} W = D_{T_\lambda} W = D_{N^k} W = 0 \ . \tag{7.65}$$

Two of these, namely $D_{N^k} W = 0$ and $D_{T_\lambda} W = 0$ become respectively [58]

$$e_k + 2ie^{2D} W \text{Im}(C\mathcal{F}_k) = 0 \ , \tag{7.66}$$

$$m^\lambda + 2ie^{2D} W \text{Im}(CZ^\lambda) = 0 \ , \tag{7.67}$$

where $e^{2D} = e^{2\phi}/\text{vol} = 6e^{2\phi}/\kappa = 6e^{2\phi}/(\int J \wedge J \wedge J)$ is a function of the dilaton and Kähler moduli t^a in J. The real and imaginary parts of the equations must vanish separately. Given that C and D are real, the vanishing of the imaginary parts of both eq. (7.66) and eq. (7.67) requires that the real part of the superpotential vanish, i.e. that

$$\text{Re}W = -e_k \xi^k + m^\lambda \tilde{\xi}_\lambda + \text{Re}\left(e_{\text{RR}0} + e_{\text{RR}a} t^a + \tfrac{1}{2}\kappa_{abc} m^a_{\text{RR}} t^b t^c - \tfrac{1}{6} m^0_{\text{RR}} \kappa_{abc} t^a t^b t^c\right) = 0 \ . \tag{7.68}$$

The coefficients e_k and m^λ are again real numbers. This equation is the only condition that involves the axions ξ^k and $\tilde{\xi}_\lambda$, or one specific combination of them. The axions constitute real parts of the superpotential. Hence, the vanishing of the real parts of eqs. (7.66), (7.67), leading to an equation involving ImW, does not involve ξ^k and $\tilde{\xi}_\lambda$. There are thus $h^{(2,1)}$ axions left that are not fixed by the fluxes, which have to be stabilised by other mechanisms. The real parts of eq. (7.66) and eq. (7.67) do however result in equations which constrain the complex structure moduli. Namely, with non-zero NSNS flux, the real parts of these equations must fulfil

$$e_k - 2e^{2D}\mathrm{Im}W\mathrm{Im}(C\mathcal{F}_k) = 0 \ , \tag{7.69}$$

$$m^\lambda - 2e^{2D}\mathrm{Im}W\mathrm{Im}(CZ^\lambda) = 0 \ , \tag{7.70}$$

where the imaginary part of eq. (7.64) is

$$\begin{aligned} \mathrm{Im}W = & -2e_k\mathrm{Re}(CZ^k) + 2m^\lambda\mathrm{Re}(C\mathcal{F}_\lambda) \\ & + \mathrm{Im}\left(e_{\mathrm{RR}0} + e_{\mathrm{RR}a}t^a + \tfrac{1}{2}\kappa_{abc}m^a_{\mathrm{RR}}t^bt^c - \tfrac{1}{6}m^0_{\mathrm{RR}}\kappa_{abc}t^at^bt^c\right) \ . \end{aligned} \tag{7.71}$$

From eqs. (7.69), (7.70), we may factorise $2e^{2D}\mathrm{Im}W$ from both equations so that the complex structure and NSNS fluxes are related via

$$\frac{e_k}{\mathrm{Im}(C\mathcal{F}_k)} = \frac{m^\lambda}{\mathrm{Im}(CZ^\lambda)} \ . \tag{7.72}$$

These are $h^{(2,1)}$ equations that, for certain values of flux, fix the complex structure moduli in Z^λ. Both of them are also equal to $2e^{2D}\mathrm{Im}W$; a term which includes the dilaton. Defining the compensator field C as $C \equiv e^{-D+\mathcal{K}^{\mathrm{cs}}/2}$ which fulfils the transformation properties of Ω discussed in section 5.2.2, and using for example the left-hand side of eq. (7.72), we have that

$$2e^{2D}\mathrm{Im}W = \frac{e_k}{\mathrm{Im}(e^{-D+\mathcal{K}^{\mathrm{cs}}/2}\mathcal{F}_k)} = \frac{e_k}{\mathrm{Im}\mathcal{F}_k}e^{D-\mathcal{K}^{\mathrm{cs}}/2} \ . \tag{7.73}$$

With $e^{2D} = 6e^{2\phi}/\kappa$ we obtain for the dilaton ϕ the expression

$$e^{-\phi} = \frac{2\sqrt{6}\mathrm{Im}W\mathrm{Im}\mathcal{F}_k}{e_k\sqrt{\kappa}}e^{\mathcal{K}^{\mathrm{cs}}/2} \ , \tag{7.74}$$

which is fixed when the complex structure as well as Kähler moduli are fixed. The complex structure moduli are fixed via eq. (7.72), and it remains to investigate the Kähler moduli. To do so, we consider the last supersymmetry condition $D_{t^a}W = 0$, which takes the form

$$\partial_{t^a}W + W\partial_{t^a}\mathcal{K} = 0 \ . \tag{7.75}$$

The first term in this equation affects only the Kähler part of the superpotential which depends on t^a. Because of this we may write $\partial_{t^a}W = \partial_{t^a}W^{\mathrm{Ks}}$ with the O6 Kähler superpotential

$$W^{\mathrm{Ks}} = e_{\mathrm{RR}0} + e_{\mathrm{RR}a}t^a + \tfrac{1}{2}\kappa_{abc}m^a_{\mathrm{RR}}t^bt^c - \tfrac{1}{6}m^0_{\mathrm{RR}}\kappa_{abc}t^at^bt^c \ , \tag{7.76}$$

being the last line of eq. (7.64). The total superpotential W in the second term of eq. (7.75) may actually be rewritten assuming that the complex structure supersymmetry conditions in eq. (7.66) and eq. (7.67) are fulfilled. Multiplying eq. (7.66) with $\mathrm{Re}(CZ^k)$ and eq. (7.67) with $\mathrm{Re}(C\mathcal{F}_\lambda)$ and then subtracting eq. (7.67) from eq. (7.66) results in that

$$e_k\mathrm{Re}(CZ^k) - m^\lambda\mathrm{Re}(C\mathcal{F}_\lambda) + 2ie^{2D}W\left[\mathrm{Im}(C\mathcal{F}_k)\mathrm{Re}(CZ^k) - \mathrm{Im}(CZ^\lambda)\mathrm{Re}(C\mathcal{F}_\lambda)\right] = 0 \ . \tag{7.77}$$

Again $C = e^{-D+\mathcal{K}^{\mathrm{cs}}/2}$ where the complex structure Kähler potential is given by

$$\mathcal{K}^{\mathrm{cs}} = -\ln\left(i\int\Omega\wedge\bar{\Omega}\right) = -\ln\left[2(\mathrm{Im}Z_\lambda\mathrm{Re}\mathcal{F}_\lambda - \mathrm{Re}Z_k\mathrm{Im}\mathcal{F}_k)\right] \ , \tag{7.78}$$

where in the last equality we have used the earlier stated relation

$$C\Omega = \mathrm{Re}(CZ^k)\alpha_k - i\mathrm{Im}(C\mathcal{F}_k)\beta^k + i\mathrm{Im}(CZ^\lambda)\alpha_\lambda - \mathrm{Re}(C\mathcal{F}_\lambda)\beta^\lambda \ . \tag{7.79}$$

The large parenthesis in eq. (7.77) is very similar to the last equality of eq. (7.78). Pulling out C^2 from this parenthesis and using the Kähler potential, we have that $C^2 = e^{-2D}[2(\mathrm{Im}Z_\lambda\mathrm{Re}\mathcal{F}_\lambda - \mathrm{Re}Z_k\mathrm{Im}\mathcal{F}_k)]^{-1}$ and so

$$\mathrm{Im}(C\mathcal{F}_k)\mathrm{Re}(CZ^k) - \mathrm{Im}(CZ^\lambda)\mathrm{Re}(C\mathcal{F}_\lambda) = e^{-2D}\frac{\mathrm{Im}\mathcal{F}_k\mathrm{Re}Z^k - \mathrm{Im}Z^\lambda\mathrm{Re}\mathcal{F}_\lambda}{2(\mathrm{Im}Z_\lambda\mathrm{Re}\mathcal{F}_\lambda - \mathrm{Re}Z_k\mathrm{Im}\mathcal{F}_k)} = -\tfrac{1}{2}e^{-2D} \ . \tag{7.80}$$

Reinserting this expression into eq. (7.77), it simplifies to

$$e_k\mathrm{Re}(CZ^k) - m^\lambda\mathrm{Re}(C\mathcal{F}_\lambda) = iW \ . \tag{7.81}$$

The left-hand side of eq. (7.81) is minus half the imaginary part of the complex structure part of the superpotential, i.e.

$$W^{\mathrm{cs}} = -e_k\xi^k + m^\lambda\tilde{\xi}_\lambda + 2i(-e_k\mathrm{Re}(CZ^k) + m^\lambda\mathrm{Re}(C\mathcal{F}_\lambda)) \ , \tag{7.82}$$

where $W = W^{\mathrm{cs}} + W^{\mathrm{Ks}}$. Eq. (7.81) is thus equivalent to

$$-\tfrac{1}{2}\mathrm{Im}W^{\mathrm{cs}} = iW \ . \tag{7.83}$$

With $W = W^{\mathrm{cs}} + W^{\mathrm{Ks}}$ we have that $\frac{1}{2}i\mathrm{Im}W^{\mathrm{cs}} = i\mathrm{Im}W^{\mathrm{cs}} + i\mathrm{Im}W^{\mathrm{Ks}}$, or $-\frac{1}{2}\mathrm{Im}W^{\mathrm{cs}} = \mathrm{Im}W^{\mathrm{Ks}}$, which by eq. (7.83) becomes equivalent to

$$W = -i\mathrm{Im}W^{\mathrm{Ks}} \ . \tag{7.84}$$

Using this expression for the superpotential in the second term in eq. (7.75), and where only W^{Ks} is left for the first term, eq. (7.75) takes the form

$$\partial_{t^a}W^{\mathrm{Ks}} - i\mathrm{Im}W^{\mathrm{Ks}}\partial_{t^a}\mathcal{K} = 0 \ . \tag{7.85}$$

The Kähler potential is $\mathcal{K} = -\ln(\frac{4}{3}\kappa_{abc}v^av^bv^c) - 2\ln[2\int\mathrm{Re}(C\Omega)\wedge\star\mathrm{Re}(C\Omega)]$ as given earlier in eq. (5.138). Taking the derivative ∂_{t^a} of this potential, the only t^a dependence is in $v^a = \mathrm{Im}(t^a)$, and since $\partial_{t^a} = \partial_{\mathrm{Re}(t^a)} - i\partial_{\mathrm{Im}(t^a)} = \partial_{b^a} - i\partial_{v^a}$, it is clear that the second part of eq. (7.85) is purely real. Thus eq. (7.85) is with advantage written as

$$\partial_{t^a}W^{\mathrm{Ks}} - \mathrm{Im}W^{\mathrm{Ks}}\partial_{v^a}\mathcal{K} = 0 \ . \tag{7.86}$$

As in the case of $D_{N^k}W = D_{T_\lambda}W = 0$, it is useful to consider the real and imaginary parts of the $D_{t^a}W = 0$ condition in eq. (7.85) separately. With the second term being real, the imaginary part of eq. (7.85) vanishes according to

$$\mathrm{Im}(\partial_{t^a}W^{\mathrm{Ks}}) = \kappa_{abc}v^b(m^c_{\mathrm{RR}} - m^0_{\mathrm{RR}}b^c) = 0 \ , \tag{7.87}$$

according to eq. (7.76). With $v^b \neq 0$, the real part of the Kähler moduli stabilises at

$$b^c = \frac{m^c_{\mathrm{RR}}}{m^0_{\mathrm{RR}}} \ . \tag{7.88}$$

Note that the case of $m^0_{\mathrm{RR}} = 0$ is not really interesting as it by eq. (7.87) would require either $v^b = 0$ or vanishing RR fluxes; $m^c_{\mathrm{RR}} = 0$. Considering the real part of eq. (7.85), we hope to get a condition for the other Kähler moduli v^a. The vanishing of the real part of eq. (7.85) reads

$$\mathrm{Re}(\partial_{t^a}W^{\mathrm{Ks}}) - \mathrm{Im}W^{\mathrm{Ks}}\partial_{v^a}\mathcal{K} = 0 \ , \tag{7.89}$$

which results in that

$$3(m^0_{\mathrm{RR}})^2\kappa_{abc}v^b v^c + 10 m^0_{\mathrm{RR}} e_{\mathrm{RR}a} + 5\kappa_{abc} m^b_{\mathrm{RR}} m^c_{\mathrm{RR}} = 0 \ . \tag{7.90}$$

Just like the b^a Kähler moduli in eq. (7.88), the v^a Kähler moduli in eq. (7.90) contain $a = 1, \ldots, h_-^{(1,1)}$ equations for $h_-^{(1,1)}$ moduli v^a, thus stabilising both Kähler moduli. One can show using eq. (7.90) that the superpotential becomes

$$W = -i\mathrm{Im}W^{\mathrm{Ks}} = \frac{2i}{15} m^0_{\mathrm{RR}} \kappa_{abc} v^a v^b v^c \ , \tag{7.91}$$

which is non-vanishing as long as the volume $\kappa_{abc}v^a v^b v^c \sim \int J \wedge J \wedge J$ does not vanish. As a consequence, this type of vacuum can not be Minkowski. If one does not avoid the no-go theorem of section 6.1 by taking perturbative effects into account, the vacuum has to be AdS.

With both the complex structure and Kähler moduli stabilised, by using eq. (7.74) and eq. (7.91) for the expression of ImW, the dilaton is fixed to

$$e^{-\phi} = \frac{4\sqrt{6}}{15} \frac{m^0_{\mathrm{RR}} \sqrt{\kappa} \mathrm{Im}\mathcal{F}_k}{e_k} e^{\mathcal{K}^{\mathrm{cs}}/2} \ , \tag{7.92}$$

with $\kappa = \kappa_{abc}v^a v^b v^c$. Again the complex structure moduli are stabilised by

$$e_k = 0 \ \Rightarrow \ \mathrm{Im}(\mathcal{F}_k) = 0 \ , \qquad m_\lambda = 0 \ \Rightarrow \ \mathrm{Im}(Z^\lambda) = 0 \ , \tag{7.93}$$

and eq. (7.72) for $e_k, m_\lambda \neq 0$. The axion combination in eq. (7.68) reduces to

$$e_k \xi^k - m^\lambda \tilde{\xi}_\lambda = \mathrm{Re}W^{\mathrm{Ks}} = e_{\mathrm{RR}0} + \frac{e_{\mathrm{RR}a} m^a_{\mathrm{RR}}}{m^0_{\mathrm{RR}}} + \frac{\kappa_{abc} m^a_{\mathrm{RR}} m^b_{\mathrm{RR}} m^c_{\mathrm{RR}}}{3(m^0_{\mathrm{RR}})^2} \ , \tag{7.94}$$

where the terms involving the v moduli have cancelled and the b^a are stabilised from eq. (7.88).

In conclusion we have seen that fluxes in type IIA O6 compactifications generally stabilise all the Kähler moduli $\{b^a, v^a\}$, all the complex structure moduli via $\mathrm{Re}(Z^k), \mathrm{Im}(\mathcal{F}_\lambda)$, the dilaton ϕ, but only one combination of axions $\{\xi^k, \tilde{\xi}_\lambda\}$ while $h^{(2,1)}$ axions remain unfixed. The moduli have been fixed under the assumptions that $v^a \neq 0$ and $\kappa_{abc}v^a v^b v^c \neq 0$. The resulting 4D vacuum must then be AdS. In general we need that $m^0_{\mathrm{RR}} \neq 0$ and at least one of e_k or m_λ to be non-vanishing for a stabilised vacuum. Otherwise, would one of these conditions fail, all fluxes must vanish and the moduli would go unstabilised. The minimum set of fluxes to stabilise the moduli always includes m^0_{RR}, one $e_{\mathrm{RR}a}$ or m^a_{RR} for each Kähler modulus, and one e_k or m_λ for the geometric moduli.

7.4 Moduli stabilisation in type IIA on $T^6/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ orientifolds

In this section we review moduli stabilisation in type IIA on O6-planes created from a $T^6/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ orbifold, which was first carried out in [58]. This orientifold is a particular case of the general class of $\mathcal{N} = 1$ supersymmetric orientifolds of type IIA Calabi-Yau compactifications. As a consequence the mechanisms of moduli stabilisation in the previous section will be used here as well.

Let us start by with a look at the orbifold. The torus may be parameterised by the three complex coordinates $\mathrm{d}z^i = \mathrm{d}x^i + i\,\mathrm{d}y^i$, which differ from the torus in the IIB example since the action of $\mathbb{Z}_3$ leaves no freedom of choice for the complex structure. The coordinates have periodicity conditions

$$z^i \sim z^i + 1 \sim z^i + \alpha \ , \tag{7.95}$$

where $\alpha = e^{i\pi/3}$. The $\mathbb{Z}_3$ have two symmetry actions T and Q, which act on the coordinates as

$$T : (z^1, z^2, z^3) \to \alpha^2 (z^1, z^2, z^3) \ , \tag{7.96}$$

which make T have $3^3 = 27$ fixed points, three for each T^2. There is also an additional $\mathbb{Z}_3$ symmetry Q which transforms the 2-torus coordinates according to

$$Q : (z^1, z^2, z^3) \to \left(\alpha^2 z^1 + \tfrac{1+\alpha}{3}, \alpha^4 z^2 + \tfrac{1+\alpha}{3}, z^3 + \tfrac{1+\alpha}{3} \right) \ , \tag{7.97}$$

and which has no fixed points. The additional action of Q on the coordinates reduces the number of fixed points to nine. These fixed points are singular points which have to be resolved in order for the orientifold to be smooth. This procedure can be done by a conifold deformation or resolution as outlined in section 7.1. Since a 6-torus has no complex structure moduli, i.e. $h^{(2,1)} = 0$, such a singularity resolution will not stabilise these moduli. Since we also have $h^{(1)} = 0$ there are no harmonic 1-forms, which implies that the moduli spaces and moduli fixing mechanisms work as in Calabi-Yau manifolds.

The orientifold is constructed by taking the orbifold $T^6/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$ and modding out by the orientifold action for O6-planes $\mathcal{O} = \Omega_p (-1)^{F_L} \sigma$, where the involution σ acts on the complex coordinates according to

$$\sigma : z_i \to -\bar{z}_i \ . \tag{7.98}$$

The O6-plane fills the non-compact 4D space and wraps 3-cycles on the T^6. Table 5.3 gives the action of the worldsheet parity Ω_p and the left-moving fermion number $(-1)^{F_L}$ on the fields present in the type II theories. The combined action of $\Omega(-1)^{F_L}$ makes the NSNS field B_2 odd, the IIA RR fields C_1 and C_5 are odd while C_3 is even.

As usual we are interested in the moduli that survive the orbifold and orientifold projection. The resulting metric of the orbifold should be invariant under the $\mathbb{Z}_3$ actions T and Q in eqs. (7.96), (7.97) above. Under the T transformation $z^i \to \tilde{z}^i = \alpha^2 z^i$ the metric transforms as $g_{ij} \to \frac{\partial \tilde{z}^k}{\partial z^i} \frac{\partial \tilde{z}^l}{\partial z^j} g_{kl} = \alpha^4 g_{ij} \neq g_{ij}$. With an antiholomorphic coordinate we have instead $\bar{z}^i \to \tilde{\bar{z}}^i = \alpha^{-2} \bar{z}^i$ so that $g_{i\bar{\jmath}} \to \frac{\partial \tilde{z}^k}{\partial z^i} \frac{\partial \tilde{\bar{z}}^l}{\partial \bar{z}^j} g_{kl} = \alpha^2 \alpha^{-2} g_{i\bar{\jmath}} = g_{i\bar{\jmath}}$. Thus a metric invariant under T must be off-diagonal. Turning to the Q action, which the resulting metric should be invariant under as well, we may use that $Q : \mathrm{d}z^i \to \alpha^{2i} \, \mathrm{d}z^i$. This means that a metric transformation will result in different factors of α depending on the value of i, namely with the same analysis as for the T action, the Q action would transform the diagonal components according to $g_{i\bar{\jmath}} \to \alpha^{i-j} g_{i\bar{\jmath}} = g_{i\bar{\jmath}} \Leftrightarrow i = j$. Thus the metric is not only diagonal but the metric of each T^2 is diagonal. We may therefore parameterise the metric of the compact space as

$$\mathrm{d}s^2 = \sum_{i=1}^{3} r_i \, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i = \sum_{i=1}^{3} r_i^2 [(\mathrm{d}x^i)^2 + (\mathrm{d}y^i)^2] \ . \tag{7.99}$$

There are three radial Kähler moduli r_i which determine the radius of each T^2. We have concluded that the only 2-forms invariant under both actions T and Q must be of the form $\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i$. These may be used to construct a basis ω_i for other 2-form fields, such that

$$\omega_i \equiv (\sqrt{3})^{1/3} i \, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i = 2(\sqrt{3})^{1/3} \, \mathrm{d}x^i \wedge \mathrm{d}y^i \ , \tag{7.100}$$

which we choose to satisfy the normalisation $\int_{T^6/\mathbb{Z}_3^2} \omega_1 \wedge \omega_2 \wedge \omega_3 = 1$. It will prove useful later to introduce the dual basis $\tilde{\omega}^i$ of even 4 cycles, where

$$\tilde{\omega}^i \equiv 3^{1/3} (i \, \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^j)(i \, \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^k) \ , \qquad \int_{T^6/\mathbb{Z}_3^2} \omega_i \wedge \tilde{\omega}^j = \delta_i^j \ , \tag{7.101}$$

with $i,j,k = 1,2,3$. The 2-form basis ω_i is odd under the action of σ in accordance with eq. (7.98), which means that the NSNS potential B_2, expressed as

$$B_2 = \sum_{i=1}^{3} b_i \omega_i \ , \tag{7.102}$$

is even under the orientifold action $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$ and is thus non-zero on $T^6/(\mathbb{Z}_3 \otimes \mathbb{Z}_3)$. The real b^i combine with $v^i \equiv r^i/[2(\sqrt{3})^{1/3}]$ to form the Kähler moduli t^i in its familiar form $t^i = b^i + iv^i$. The dual basis $\tilde{\omega}^i$ is even under σ.

The 3-forms invariant under T and Q are the holomorphic (3,0)-form

$$\Omega = 3^{1/4} i \, \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \ , \tag{7.103}$$

and its complex conjugate $\bar{\Omega}$. They are normalised as $i \int_{T^6/\mathbb{Z}_3^2} \Omega \wedge \bar{\Omega} = 1$. Decomposing Ω into real and imaginary components we have

$$\Omega = \tfrac{1}{\sqrt{2}} \alpha_0 + i \beta^0 \ , \tag{7.104}$$

where α_0 and its dual β^0 are given explicitly as

$$\alpha_0 = (12)^{1/4} \left(\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 - \tfrac{1}{2}\epsilon_{ijk} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}y^k \right) \ , \tag{7.105}$$

$$\beta^0 = (12)^{1/4} \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \tfrac{1}{2}\epsilon_{ijk} \, \mathrm{d}y^i \wedge \mathrm{d}y^j \wedge \mathrm{d}x^k \right) \ . \tag{7.106}$$

The involution acts on them like $\sigma : \alpha_0 \to +\alpha_0$ and $\sigma : \beta_0 \to -\beta_0$ so that $\sigma : \Omega \to \bar{\Omega}$. With only the 3-cycle α_0 being even under the involution one can conclude that the O6-plane is wrapped around the α_0 cycle.

Since $h^{(1)} = 0$ there are no moduli associated with the RR 1-form field C_1. There is the dilaton ϕ modulus and its axion partner ξ^0 which come from the C_3 field in one chiral multiplet, and another chiral multiplet $\{b^i, v^i\}$ with a total of six Kähler moduli. This adds up to eight real scalar moduli. See table 5.6 for comparison with the general case.

The next step is turning on fluxes on this orientifold. Since the NSNS field B_2 is odd under σ, its field strength flux H_3 will be so too. It is therefore convenient to take the flux to be along the β^0 cycle. The mass parameter of IIA, i.e. F_0, is even under the involution, and so is F_4. The other RR fluxes F_2 and F_6 are odd and so we may express their values in the appropriate basis as

$$H_3 = -e_0 \beta^0 \ , \qquad F_0 = m^0_{\mathrm{RR}} \ , \qquad F_2 = -m^i_{\mathrm{RR}} \omega_i \ , \qquad F_4 = e_{\mathrm{RR}i} \tilde{\omega}^i \ , \qquad F_6 = e_{\mathrm{RR}0} \ . \tag{7.107}$$

With D-branes present, the type IIA tadpole cancellation condition of eq. (6.33) will enforce that

$$m^0_{\mathrm{RR}} e_0 = -2 \ . \tag{7.108}$$

With this condition the NSNS flux and mass parameter are fixed to the four possible choices of $(m^0_{\mathrm{RR}}, e_0) = \pm(1,-2)$ or $(m^0_{\mathrm{RR}}, e_0) = \pm(2,-1)$, since the fluxes are integers. The tadpole condition does not however set any constraints on the other fluxes present.

From the $D_{t^a} W = 0$ supersymmetry condition we saw in the previous section that the Kähler moduli b^i are stabilised at

$$b^i = \frac{m^i_{\mathrm{RR}}}{m^0_{\mathrm{RR}}} \ . \tag{7.109}$$

As for the Kähler moduli v^i, determined by eq. (7.90), the indices $i = 1, 2, 3$ together with the fact that κ_{ijk} is symmetric in its indices results in three equations for each $e_{\mathrm{RR}i}$ flux;

$$\begin{aligned} 6(m^0_{\mathrm{RR}})^2\kappa_{123}v^2v^3 + 10m^0_{\mathrm{RR}}e_{\mathrm{RR}1} + 10\kappa_{123}m^2_{\mathrm{RR}}m^3_{\mathrm{RR}} = 0 \ , \\ 6(m^0_{\mathrm{RR}})^2\kappa_{123}v^1v^3 + 10m^0_{\mathrm{RR}}e_{\mathrm{RR}2} + 10\kappa_{123}m^1_{\mathrm{RR}}m^3_{\mathrm{RR}} = 0 \ , \\ 6(m^0_{\mathrm{RR}})^2\kappa_{123}v^1v^2 + 10m^0_{\mathrm{RR}}e_{\mathrm{RR}3} + 10\kappa_{123}m^1_{\mathrm{RR}}m^2_{\mathrm{RR}} = 0 \ . \end{aligned} \tag{7.110}$$

A single closed expression for the solution to v^i can be obtained as

$$v^i = \frac{1}{|\hat{e}_{\mathrm{RR}i}|}\sqrt{\frac{-5\hat{e}_{\mathrm{RR}1}\hat{e}_{\mathrm{RR}2}\hat{e}_{\mathrm{RR}3}}{3m^0_{\mathrm{RR}}\kappa_{123}}} \ , \tag{7.111}$$

where we have defined

$$\hat{e}_{\mathrm{RR}i} \equiv e_{\mathrm{RR}i} + \frac{\kappa_{123}m^j_{\mathrm{RR}}m^k_{\mathrm{RR}}}{m^0_{\mathrm{RR}}} \ , \tag{7.112}$$

with j, k being the other two values than i. In the hypermultiplet sector $h^{(2,1)} = 0$, so we only have one index $k = 0$ with no λ indices since there are no complex structure moduli. As such, there is only one unique electric NSNS flux e_0 and the moduli are simply the dilaton and its single axionic partner ξ^0. Eq. (7.94) then determine the axion moduli to

$$\xi^0 = \frac{1}{e_0}\mathrm{Re}W^{\mathrm{Ks}} = \frac{1}{e_0}\left(e_{\mathrm{RR}0} + \frac{e_{\mathrm{RR}i}m^i_{\mathrm{RR}}}{m^0_{\mathrm{RR}}} + \frac{6\kappa_{123}m^1_{\mathrm{RR}}m^2_{\mathrm{RR}}m^3_{\mathrm{RR}}}{3(m^0_{\mathrm{RR}})^2}\right) \ , \tag{7.113}$$

with the factor 6 coming from the 3! ways to arrange the numerator in the last term. The dilaton is stabilised according to eq. (7.92) which in this case has $\mathcal{K}^{\mathrm{cs}} = 0$, which in turn gives $\mathrm{Im}\mathcal{F}_0 = -1/\sqrt{2}$ so that

$$e^{-\phi} = -\frac{4\sqrt{3}}{15}\frac{m^0_{\mathrm{RR}}\sqrt{\kappa}}{e_0} \ , \tag{7.114}$$

where κ is calculated to

$$\kappa = \frac{10}{|m^0_{\mathrm{RR}}|}\sqrt{\frac{-5\hat{e}_{\mathrm{RR}1}\hat{e}_{\mathrm{RR}2}\hat{e}_{\mathrm{RR}3}}{3m^0_{\mathrm{RR}}\kappa_{123}}} \ . \tag{7.115}$$

The above analysis has considered only the cycles not in the vicinity of any singular point. When resolving or "blowing up" the singularity there will be blow-up modes which have associated Kähler moduli. Locally, each blow-up looks like a resolution of $\mathbb{C}/\mathbb{Z}_3$ and is parameterised by a scale modulus and a corresponding B field modulus. Cycles on the blow-ups are referred to as *twisted* cycles, and the cycles in eq. (7.107) with associated fluxes $e_{\mathrm{RR}i}, m^i_{\mathrm{RR}}$ are *untwisted.* The F_2 and F_4 fluxes associated with the twisted cycles are n_A and f^A, i.e.

$$F_2 = -n^A\omega'_A \ , \quad F_4 = f_A\tilde{\omega}'^A \ , \tag{7.116}$$

which should be compared with the untwisted fluxes m^i_{RR} and $e_{\mathrm{RR}i}$. The blow-up Kähler modes are denoted t_{B^A} with $A = 1, \ldots, 9$, and the volumes v_{B_A} satisfy

$$3(m^0_{\mathrm{RR}})^2\kappa_{AAA}v^2_{B_A} + 10m^0_{\mathrm{RR}}f^A + 5\kappa_{AAA}n^2_A = 0 \ , \tag{7.117}$$

for each blow-up mode A. The solutions for the complex blow-up Kähler moduli are then

$$t^{B_A} = \frac{n_A}{m^0_{\mathrm{RR}}} - i\sqrt{\frac{-10\hat{f}_A}{3\kappa_{AAA}m^0_{\mathrm{RR}}}} \ , \tag{7.118}$$

where

$$\hat{f}_A \equiv f_A + \frac{\kappa_{AAA}(n^A)^2}{2m^0_{\rm RR}} \; . \tag{7.119}$$

With this inclusion the the total volume becomes

$$\kappa = \frac{10}{|m^0_{\rm RR}|}\sqrt{\frac{-5\hat{e}_{\rm RR1}\hat{e}_{\rm RR2}\hat{e}_{\rm RR3}}{3m^0_{\rm RR}\kappa_{123}} + \kappa_{AAA}\sum_{A=1}^{9}\left(\frac{-10\hat{f}_A}{3\kappa_{AAA}m^0_{\rm RR}}\right)^{3/2}} \; , \tag{7.120}$$

and to the axion there will be additions so that

$$\xi_0 = \frac{1}{e_0}\left(e_{\rm RR0} + \frac{e_{{\rm RR}i}m^i_{\rm RR} + f_A n^A}{m^0_{\rm RR}} + \frac{6\kappa_{123}m^1_{\rm RR}m^2_{\rm RR}m^3_{\rm RR} + \kappa_{AAA}\sum_A (n^A)^3}{3(m^0_{\rm RR})^2}\right) \; . \tag{7.121}$$

The solution obtained is valid as long as the moduli v^i and v^{B^A} are large enough to be able to neglect α'-corrections, and as long as the string coupling is small enough so that quantum corrections can be neglected. With v^i in eq. (7.111) and t^{B^A} in eq. (7.118), these moduli are large when

$$|\hat{e}_{{\rm RR}i}|^2 \gg |m^0_{\rm RR}| \; , \qquad |\hat{f}_A| \gg |m^0_{\rm RR}| \; . \tag{7.122}$$

In order to have the Kähler moduli remain within the Kähler cone, the untwisted volumes must be larger than the blow-up volumes;

$$|\hat{e}_{{\rm RR}i}| \gg |\hat{f}_A| \gg |m^0_{\rm RR}| \; . \tag{7.123}$$

Unlike $m^0_{\rm RR}$ and e_0, the 4-form and 2-form fluxes are not constrained by the tadpole condition, we have freedom to scale them as large as we wish and may always choose fluxes such that eq. (7.123) is fulfilled. Assuming eq. (7.123) to be satisfied, the Kähler moduli will scale like $v^i \sim \sqrt{e_{{\rm RR}i}}$ by eq. (7.111). Taking the 4-form flux to be some large value $\hat{e}_{{\rm RR}i} \sim N$, then the T^2 radius scale like $R \sim \sqrt{v^i} \sim N^{1/4}$. In the same manner the volume and dilaton scale according to

$$\kappa \equiv {\rm vol} \sim N^{3/2} \; , \qquad e^{\phi} \sim N^{-3/4} \; , \qquad e^D \equiv \sqrt{8}e^{\phi+\mathcal{K}^K/2} \equiv \frac{e^{\phi}}{\sqrt{\kappa}} \sim N^{-3/2} \; , \tag{7.124}$$

so that for large N the 10D respectively 4D string couplings are indeed small.

Since $W \neq 0$ for these type of vacua they are also AdS. The 4D cosmological constant can be found by inserting the stabilised moduli in the superpotential, resulting in that

$$\Lambda = -3e^{\mathcal{K}^{\rm ks}+\mathcal{K}^{\rm cs}}|W|^2 \sim N^{-9/2} \; , \tag{7.125}$$

where $e^{\mathcal{K}^{\rm cs}} = -2\ln\left[\int {\rm Re}(C\Omega)\wedge\star{\rm Re}(C\Omega)\right] = 4D$ having used the definition of e^D in eq. (7.124) and that $\mathcal{K}^{\rm cs} = -\ln\left[i\int\Omega\wedge\bar{\Omega}\right] = -\ln\left[2({\rm Im}Z_\lambda{\rm Re}\mathcal{F}_\lambda - {\rm Re}Z_k{\rm Im}\mathcal{F}_k)\right]$ for the surviving fields.

It is clear that moduli stabilisation in type IIA works very differently from type IIB. In summary, the two main differences are that in IIB the fluxes which stabilise the moduli are constrained by the tadpole cancellation condition, stripping us of the freedom to make them as large we want. Secondly, the Kähler moduli are not stabilised in type IIB orientifold models at all. In the case of a torus some of them may however be stabilised by the primitivity condition $J\wedge G = 0$, but the volume modulus is always unfixed. In type IIA there is only one of the axionic partners that is fixed (apart from the other moduli that are fixed), although in manifolds with rigid complex structure there is only one axion, hence all moduli are fixed. In the examples covered in this chapter the 4D geometry of both the type IIB examples is Minkowski, while for the type IIA it is AdS. All examples have $\mathcal{N} = 1$ supersymmetry.

8

Moduli Stabilisation with Corrections and de Sitter Vacua

In the previous chapter it became clear that fluxes are generally not enough to stabilise all moduli. For instance, in the previous section we saw that in type IIB, NSNS and RR 3-form fluxes can in general stabilise the dilaton and complex structure moduli but leaves the Kähler moduli unfixed. However, more moduli can be stabilised by considering perturbative and non-perturbative corrections to the Kähler potential as well as the superpotential, which we are to discuss in this chapter. Considering these corrections, we review their effect on moduli stabilisation and how they may be used to construct de Sitter vacua in section 8.3. The focus will lie on type IIB Calabi-Yau O3/O7 orientifolds.

The perhaps most important effect in stabilising the Kähler moduli is the breaking of the Kähler potential's no-scale structure. As discussed in chapter 5, the O3/O7 Kähler potential is given by

$$\mathcal{K} = -\ln(\tau + \bar{\tau}) - 2\ln(\mathcal{V}) - \ln\left(-i\int_{\mathcal{M}_6} \Omega \wedge \bar{\Omega}\right) \ , \tag{8.1}$$

where we used the relation $-3\ln(T+\bar{T}) = -2\ln(\frac{1}{6}\kappa) \equiv -2\ln(\mathcal{V})$ in eq. (5.141). Again, this Kähler potential fulfils a no-scale structure condition (c.f. eq. (5.142))

$$\sum_{i,j=T_\alpha, G^a} \partial_i \mathcal{K} \partial_{\bar{j}} \mathcal{K} K^{i\bar{j}} = 3 \ , \tag{8.2}$$

with proper Kähler coordinates T_α, G^a defined as in eq. (5.139). In combination with the flux-induced GVW superpotential of eq. (5.146), i.e.

$$W_0 \equiv W_{\text{GVW}} = \int_{\mathcal{M}_6} G_3 \wedge \Omega \ , \tag{8.3}$$

the corresponding scalar potential is demonstrably positive semi-definite according to

$$V = e^{\mathcal{K}}\left(\sum_{a,b} K^{a\bar{b}} D_a W_0 \overline{D_b W_0} - 3|W_0|^2\right) \longrightarrow e^{\mathcal{K}}\left(\sum_{c,d} K^{c\bar{d}} D_c W_0 \overline{D_d W_0}\right) \ , \tag{8.4}$$

where $a, b = \tau, T^\alpha, G_\alpha$ contain all moduli and $c, d = \tau, G^a$. This cancellation can be seen as an effect of the Kähler moduli not being present in the GVW superpotential, which only depends on the dilaton and complex structure moduli. With 3-form fluxes present, the supersymmetry conditions of vanishing Kähler covariant derivative of the GVW superpotential fixes the moduli, given that G_3 is imaginary self-dual. In chapter 5 we saw that these conditions for the dilaton and complex structure moduli forces G_3 to have (2,1)- and/or (0,3)-form pieces, while the Kähler derivative with respect to T further forced G_3 to be (2,1). In this case the superpotential automatically vanishes and results in a $\mathcal{N}=1$ Minkowski vacuum. In this chapter we will see how the loss of no-scale structure opens up to alternative vacua.

8.1 Perturbative corrections to the low-energy action

Corrections to the low-energy effective supergravity action are governed by the Planck scale, which in string theory is given by

$$M_{\mathrm{P}}^{8}=\frac{1}{g_{\mathrm{s}}^{2}\alpha'^{4}}\ . \tag{8.5}$$

Perturbative corrections consist of the double series expansion of the α' and g_{s} parameters in the 10D effective theory. The α'-expansion of this action can be written on the form

$$S=S_{(0)}+\alpha'^{3}S_{(3)}+\ldots+\alpha'^{n}S_{(n)}+\ldots+S_{(0)}^{\mathrm{loc}}+\alpha'^{2}S_{(2)}^{\mathrm{loc}}\ , \tag{8.6}$$

where $S_{(0)}$ is given by eq. (5.47) including the Chern-Simons terms and the S^{loc} are localised p-brane actions. The subscript indicates the degree of α'-dependence.

After compactification, corrections of the 10D action leads to corrections of the 4D Kähler potentials. The perhaps most famous perturbative correction to the Kähler potential originates from an α'^{3} curvature correction, i.e. the *quartic invariant* $\mathcal{R}^{4}$, in the 10D Einstein-Hilbert action of type IIB. This action can be written [59]

$$S_{g}=\int \mathrm{d}^{10}x\sqrt{-g}\left(\frac{M_{\mathrm{P}}}{2}R+\frac{\xi(3)}{3\cdot 2^{5}}\frac{1}{M^{6}}\mathcal{R}_{4}+\ldots\right)\ , \tag{8.7}$$

where M_{P} is the Planck mass in 10D, the ζ is the Riemann-zeta function where $\zeta(3)=\sum_{k=1}^{\infty}1/k^{3}\approx 1.202$, and M denotes the mass of the first excited level of the type II superstring;

$$M^{2}=\frac{4}{\alpha'}\ . \tag{8.8}$$

The omitted terms are subleading in $1/M$ and/or the string coupling g_{s}. The quartic invariant $\mathcal{R}^{4}$ arises via a 4-loop correction to the β-function in the worldsheet σ-model [59]. In the next we will review how this correction comes about, following its discoverer in [60].

8.1.1 The α'^{3}-correction to the Kähler potential

The metric of Kähler deformations on the Calabi-Yau 3-fold receives perturbative α'^{3}-corrections from higher derivative terms appearing in the type II 10D effective action. At tree level these terms are the same in type IIA and IIB. The relevant terms for this correction of the Kähler moduli space in the metric are given by

$$S=-\frac{1}{2\kappa_{10}^{2}}\int \mathrm{d}^{10}x\sqrt{-g}e^{-2\phi}\left(R+4(\partial\phi)^{2}+\alpha'^{3}c_{1}J_{0}\right)\ , \tag{8.9}$$

where $c_{1}\equiv\frac{\zeta(3)}{3\cdot 2^{11}}$ [59] and the higher order term J_{0} is defined as [61]

$$J_{0}=3\cdot 2^{8}\left(R^{HMNK}R_{PMNQ}R_{H}{}^{RSP}R^{Q}{}_{RSK}+\tfrac{1}{2}R^{HKMN}R_{PQMN}R_{H}{}^{RSP}R^{Q}{}_{RSK}\right)\ . \tag{8.10}$$

The equation of motion for ϕ up to order $\mathcal{O}(\alpha'^{3})$ is then given by

$$R+4\nabla^{2}\phi-4(\nabla\phi)^{2}+\alpha'^{3}c_{1}J_{0}=0\ . \tag{8.11}$$

Note that J_{0} does not contribute to the above equation as it has the property of vanishing on Ricci flat Kähler metrics. If one introduces the complex coordinates

$$\lambda^{a}=\tfrac{1}{\sqrt{2}}(y^{2a-1}+iy^{2a})\ , \tag{8.12}$$

with $a = 1, 2, 3$ on the internal manifold, it has been shown [62] that the metric β-function for the $\mathcal{N} = 2$ non-linear σ-model in 2D up to four loops is given by

$$\beta_{a\bar{b}} = \frac{1}{2\pi} R_{a\bar{b}} + \frac{1}{8\pi}\zeta(3)\nabla_a\nabla_{\bar{b}} Q \; . \tag{8.13}$$

Here $2\pi\alpha' = 1$, and Q is defined as

$$Q \equiv \frac{1}{12(2\pi)^3} R_{IJ}{}^{KL} R_{KL}{}^{MN} R_{MN}{}^{IJ} - 2R_I{}^K{}_J{}^L R_K{}^M{}_L{}^N R_M{}^{IJ}{}_N \; , \tag{8.14}$$

which is a generalisation of a 6D Euler density; $\int_M \mathrm{d}^6x\sqrt{g}Q = \chi$. By demanding $\beta_{a\bar{b}} = 0$ and using eq. (8.13) in eq. (8.11), the dilaton equation of motion is satisfied up to order $\mathcal{O}(\alpha'^3)$ when

$$\phi = \phi_0 + \frac{\zeta(3)}{16} Q \; , \tag{8.15}$$

where ϕ_0 is a constant and the second term comes from perturbative quantum corrections on the worldsheet. Hence, a higher order term in eq. (8.9) can not result in a constant dilaton as a solution.

After compactification of the 10D theory to 4D on a Calabi-Yau 3-fold, the interactions of eq. (8.9) (and an additional term which will show up later) give the perturbative correction to the metric on the moduli space of the Kähler deformations. The prepotential $\tilde{\mathcal{F}}$ for these Kähler deformations, or hypermultiplets, receives both perturbative and non-perturbative corrections on the worldsheet. These corrections have been calculated in [32] using mirror symmetry, and the perturbative corrections has been identified with the α'-corrections determined in [59][63]. Since the Kähler deformations lie in $\mathcal{N} = 2$ hypermultiplets, a truncation needs to be done in order to obtain the $\mathcal{N} = 1$ theory, which is of primary interest. Including the α'^3-correction, the prepotential has been shown [32] to change so that it takes the form

$$\tilde{\mathcal{F}}(X) = \frac{i}{6}\kappa_{abc} X^a X^b X^c + (X^0)^2\xi \; , \tag{8.16}$$

where ξ is some constant and $t^a = X^t/X^0$ are the Kähler deformations with $a = 1, \ldots, h^{(1,1)}$, where we may take $X^0 = 1$. Since the Kähler potential is also given by

$$\mathcal{K} = -\ln[X^i \bar{F}_i(\bar{X}) + \bar{X}^i F(X)] \; , \tag{8.17}$$

inserting the prepotential in eq. (8.16) into the Kähler potential of eq. (8.17) gives the corrected Kähler potential for the Kähler moduli t^a as

$$\mathcal{K} = -\ln[-\tfrac{i}{6}\kappa_{abc}(t^a - \bar{t}^a)(t^b - \bar{t}^b)(t^b - \bar{t}^b) + 4\xi] \; . \tag{8.18}$$

In the next we turn to determine the constant ξ. This constant is independent of the Kähler moduli which allows us to choose a single Kähler modulus u to work with. We start by making the following ansatz for the metric:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + e^{2u}\tilde{g}_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n \; . \tag{8.19}$$

For simplicity we choose the volume of the Calabi-Yau $\tilde{g}$ by setting $(2\pi\alpha')^3 = 1$ so that $\kappa_4 = \kappa_{10}$, and then normalising so that $\mathcal{V} = \frac{1}{6}\kappa_{abc}v^a v^b v^c = e^{6u}$. The constant ξ may be determined by after dimensional reduction of the metric in eq. (8.19) to 4D.

In the action of the hypermultiplets in type IIB compactifications the Kähler metric is the coefficient of the kinetic terms of the Kähler moduli t^a. Hence, with the Kähler potential of eq. (8.18) this Kähler metric $K_{a\bar{b}} = \partial^2\mathcal{K}/\partial t^a\partial\bar{t}^b$ gives us the following action for the kinetic term:

$$S = \frac{1}{\kappa_4^2}\int \mathrm{d}^4x\sqrt{-g}\left[-(3 - 6\xi e^{6u})\partial_\mu u\partial^\mu u\right] + \ldots \; . \tag{8.20}$$

This will be used for comparison with the kinetic term of the resulting 4D action in eq. (8.9) after dimensional reduction. In the process of doing so it is useful to re-express J_0 such that [64]

$$S = -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}e^{-2\phi}\alpha'^3 c_2\left(12Z - RS + 12R_{MN}S^{MN} + R^2\right) + \dots\ , \tag{8.21}$$

with the constant $c_2 \equiv \zeta(3)/(3\cdot 2^5)$, $Z = Z_{IJ}g^{IJ}$ and $S = S_{IJ}g^{IJ}$ where

$$Z_{IJ} \equiv R_{IKLR}R_{JMN}{}^{R}\left(R^{K}{}_{P}{}^{M}{}_{Q}R^{NPLQ} - \tfrac{1}{2}R^{KN}{}_{RQ}R^{MLPQ}\right)\ , \tag{8.22}$$

$$S_{IJ} \equiv -2R_{I}{}^{MKL}R_{J}{}^{P}{}_{K}{}^{Q}R_{LPMQ} + \tfrac{1}{2}R_{I}{}^{MKL}R_{JMPQ}R_{KL}{}^{PQ}R_{I}{}^{K}{}_{J}{}^{L}R_{KMNQ}R_{L}{}^{MNQ}\ , \tag{8.23}$$

$$S \equiv 12(2\pi)^3 Q + R^2\ . \tag{8.24}$$

The Q is defined as earlier in eq. (8.14). In order to evaluate eq. (8.21) for the metric in eq. (8.19), the non-vanishing parts of the Riemann tensor are needed. With the usual conventions

$$\begin{aligned} R^{M}{}_{NPQ} &\equiv \partial_P\Gamma^{M}_{QN} - \partial_Q\Gamma^{M}_{PN} + \Gamma^{R}_{QN}\Gamma^{M}_{PR} - \Gamma^{R}_{PN}\Gamma^{M}_{QR}\ , \\ \Gamma^{M}_{NP} &\equiv \tfrac{1}{2}g^{MQ}\left(\partial_N g_{PQ} + \partial_P g_{QN} - \partial_Q g_{NP}\right)\ , \end{aligned} \tag{8.25}$$

one can find that the non-vanishing components of the Riemann tensor are

$$\begin{aligned} R^{m}{}_{\mu n\nu} &= -\delta^m_n(\partial_\mu u\partial_\nu u + \partial_\mu\partial_\nu u)\ , \\ R^{\mu}{}_{m\nu n} &= -g_{mn}(\partial_\nu u\partial^\mu u + \partial_\nu\partial^\mu u)\ , \\ R^{k}{}_{mnp} &= \tilde{R}^{k}{}_{mnp} + \partial_\mu u\partial^\mu u\left(\delta^k_p g_{mn} - \delta^k_n g_{pm}\right)\ , \end{aligned} \tag{8.26}$$

and all other components, which are not related to any of the above ones by symmetry, vanish. The non-vanishing components of the Ricci tensor is in turn given by

$$\begin{aligned} R_{\mu\nu} &= -6(\partial_\mu u\partial_\nu u + \partial_\mu\partial_\nu u)\ , \\ R_{mn} &= -g_{mn}(6\partial_\mu u\partial^\mu u + \partial_\mu\partial^\mu u)\ , \end{aligned} \tag{8.27}$$

where the contribution of the background metric has been discarded as it is proportional to $\nabla_a\nabla_{\bar{b}}\tilde{Q} \sim \beta_{a\bar{b}}$ and thus vanishes as $\beta_{a\bar{b}} = 0$. The Ricci scalar is given by

$$R = -42\partial_\mu u\partial^\mu u - 12\partial_\mu\partial^\mu u\ . \tag{8.28}$$

Turning our attention to eq. (8.21) again, it is clear by the components of eq. (8.27) and eq. (8.28) that the RS and $R_{mn}S^{mn}$ terms contribute to the kinetic terms of u, while $R_{\mu\nu}S^{\mu\nu}$ and R^2 do not. As for the first term that is proportional to Z, it can be seen by the definition of Z in eq. (8.23) that only the internal components of the Riemann tensor contribute to a kinetic term of u. Using the results on the last line of eq. (8.26), the expression of Z that contributes to the kinetic terms of u can be written

$$Z = e^{-6u}\partial_\mu u\partial^\mu u\left[12(2\pi)^3\tilde{Q} + \tilde{R}^{i}{}_{qlr}(2\tilde{R}_{ipn}{}^{r}\tilde{R}^{nplq} - \tilde{R}_{inp}{}^{r}\tilde{R}^{nlpq} - \tilde{R}_{ipn}{}^{r}\tilde{R}^{qnlp})\right]\ . \tag{8.29}$$

This expression may be simplified further. Again using the complex coordinates in eq. (8.12) on the manifold, it is true that on a Kähler manifold, the only non-trivial independent Riemann tensor component $\tilde{R}^{a}{}_{bc\bar{d}}$ possesses the symmetry

$$\tilde{R}^{a}{}_{bc\bar{d}} = \tilde{R}^{a}{}_{cb\bar{d}}\ . \tag{8.30}$$

Hence we have that for the first term in eq. (8.29) that

$$\begin{aligned} 12(2\pi)^3\tilde{Q} &= 4\left(\tilde{R}_{a\bar{b}}{}^{c\bar{d}}\tilde{R}_{c\bar{d}}{}^{e\bar{f}}\tilde{R}_{e\bar{f}}{}^{a\bar{b}} - \tilde{R}_{a}{}^{c}{}_{b}{}^{d}\tilde{R}_{c}{}^{e}{}_{d}{}^{f}\tilde{R}_{e}{}^{a}{}_{f}{}^{b}\right) \\ &= \tilde{R}^{i}{}_{qlr}\left(2\tilde{R}_{ipn}{}^{r}\tilde{R}^{nplq} - \tilde{R}_{inp}{}^{r}\tilde{R}^{nlpq} - \tilde{R}_{ipn}{}^{r}\tilde{R}^{qnlp}\right) \ , \end{aligned} \tag{8.31}$$

which reduces the expression of Z to

$$Z = 24(2\pi)^3 e^{-6u}\partial_\mu u \partial^\mu u \tilde{Q} \ . \tag{8.32}$$

Now performing the dimensional reduction of the metric eq. (8.19), where we use the dilaton $\phi = \phi_0 + cQ$ with $c \equiv \zeta(3)/16$ as before. It has been argued that the Ricci tensor R_{MN} in eq. (8.21) actually appears in the form $R_{MN} + 2\nabla_M\nabla_N\phi$, which would result in the additional terms

$$S = -\frac{1}{2\kappa_{10}^2}\int \sqrt{-g^{(10)}}e^{-2\phi}\alpha'^3 c_2(-2(\nabla^2\phi)S + 24(\nabla_M\nabla_N\phi)S^{MN}) \ . \tag{8.33}$$

to the action. Here the second term does not contribute in the dimensional reduction, and the first extra term does not modify the equations of motion but it is necessary to get the right form of the resulting 4D action. Without it one would get unallowed cross terms involving $\phi^{(4)}$ and u. With this addition to the 10D action it becomes to order $\mathcal{O}(\alpha'^3)$ in perturbation theory

$$\begin{aligned} S = -\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g}e^{-2\phi}\Big[&(1-2cQ)\left(R^{(4)} - 42\partial_\mu u\partial^\mu u - 12\partial_\mu\partial^\mu u + 4\partial_\mu\phi_0\partial^\mu\phi_0\right) \\ &- 48cQ\phi_0\partial^\mu u - 48c_2 Q\partial_\mu\phi_0\partial^\mu\phi_0 + 12c_2Q(-R^{(4)} - 6\partial_\mu u\partial^\mu u)\Big] \ . \end{aligned} \tag{8.34}$$

Using the fact that $6c_2 = c$ and integrating over the internal coordinates, one gets that

$$\begin{aligned} S = -\frac{1}{2\kappa_4^2}\int \mathrm{d}^4x\sqrt{-g}e^{-2\phi_0}\Big[&(e^{6u} - 4c\chi)R^{(4)} + (e^{6u} - 2c\chi)(-42\partial_\mu u\partial^\mu u - 12\partial_\mu\partial^\mu u) \\ &+ (e^{6u} - 4c\chi)4\partial_\mu\phi_0\partial^\mu\phi_0 - 48c\chi\partial_\mu\phi_0\partial^\mu u - 12c\chi\partial_\mu u\partial^\mu u\Big] \ . \end{aligned} \tag{8.35}$$

The coefficient of $R^{(4)}$ tells us that the 4D dilaton is given by

$$e^{2\phi^{(4)}} = e^{-2\phi_0}(e^{6u} - 4c\chi) \ . \tag{8.36}$$

By performing a partial integration of the term $(-42\partial_\mu u\partial^\mu u - 12\partial_\mu\partial^\mu u)$ in eq. (8.35) we simplify the expression of the action to

$$S = \frac{1}{\kappa_4^2}\int \mathrm{d}^4x\sqrt{-g}e^{-2\phi^{(4)}}\left[\tfrac{1}{2}R^{(4)} + 2\partial_\mu\phi^{(4)}\partial^\mu\phi^{(4)} - (3 + 48c\chi e^{-6u})\partial_\mu u\partial^\mu u\right] \ . \tag{8.37}$$

Here a Weyl rescaling does not change the coefficient of the u kinetic term, so by comparison with eq. (8.20), the constant ξ is finally determined to

$$\xi = -8c\chi = -\frac{\zeta(3)}{2}\chi \ . \tag{8.38}$$

With a Weyl rescaling $g_\mathrm{s} = e^{1/2\phi_0}g_\mathrm{E}$, the 4D dilaton in eq. (8.36) can be rescaled to Einstein frame via

$$e^{2\phi^{(4)}} = e^{-2\phi_0}(\mathcal{V} + \tfrac{1}{2}\xi) \to e^{-1/2\phi_0}(\mathcal{V}' + \tfrac{1}{2}\xi') \tag{8.39}$$

with $\xi' = \xi e^{-3/2\phi_0}$ and $\mathcal{V}'$ formed by $v'^a = v^a e^{-\phi_0/2}$ so that $\mathcal{V}' = \frac{1}{6}\kappa_{abc}v^av^bv^ce^{-3/2\phi_0}$.

The $\mathcal{N}=2$ action for the hypermultiplets is given in eq. (5.68) and eq. (5.69). By performing an orientifold projection of the $\mathcal{N}=2$ action, in which the antisymmetric 2-form fields are projected out and using the v'^a and eq. (8.39), the $\mathcal{N}=2$ supersymmetry breaks down to $\mathcal{N}=1$. The quarternionic geometry of the hypermultiplet is then reduced to a Kähler geometry. With the Kähler coordinates defined in eq. (5.139) (where $G^a=0$ may be taken as there is only one radial Kähler modulus u), the corrected Kähler potential takes the form

$$\begin{aligned}\mathcal{K} &= -\ln[-i(\tau-\bar{\tau})] - 2\ln\left[-i(T^a-\bar{T}^a)v'_a + \xi\left(-i\frac{\tau-\bar{\tau}}{2}\right)^{3/2}\right] - \ln\left[-i\int_{\mathrm{CY}_3}\Omega\wedge\bar{\Omega}\right] \\ &= \phi_0 - 2\ln[\mathcal{V}' + \tfrac{1}{2}\xi e^{-3\phi_0/2}] - \ln[-i\int_M \Omega\wedge\bar{\Omega}] + \text{const.}\ ,\end{aligned} \tag{8.40}$$

where again $\xi \equiv -\frac{\zeta(3)}{2(2\pi)^3}\chi$ with normalisation $2\pi\alpha'=1$ such that $2\kappa_{10}^2=(2\pi)^7\alpha'^4=(2\pi)^3$.

8.1.2 String coupling corrections to the Kähler potential

There are also perturbative g_s-corrections arising from loop-effects in spacetime, i.e. from higher-genus string worldsheets, which generally break the Kähler potential's no-scale structure as well. Some explicit results of such loop corrections have been found for the $T^6/(\mathbb{Z}_2\otimes\mathbb{Z}_2)$ orientifold with $\mathcal{N}=2$ and $\mathcal{N}=1$ in [65][66]. The calculation of such corrections is out of scope for this thesis, although we may provide a qualitative picture of their effect. The loop corrections to the Kähler potential of eq. (8.1) can be divided into two different contributions;

$$\delta\mathcal{K}_{g_\mathrm{s}} = \delta\mathcal{K}_{g_\mathrm{s}}^{\mathrm{KK}} + \delta\mathcal{K}_{g_\mathrm{s}}^{W}\ , \tag{8.41}$$

where $\delta\mathcal{K}^{\mathrm{KK}}$ arises from the exchange of closed strings with Kaluza-Klein (KK) momentum between D3- and D7-branes (or O3- and O7-planes), while $\delta\mathcal{K}^{W}$ comes from the exchange of closed strings with non-vanishing winding (W). The $\delta\mathcal{K}_{g_\mathrm{s}}^{\mathrm{KK}}$ has the form [67]

$$\delta\mathcal{K}_{g_\mathrm{s}}^{\mathrm{KK}} = \sum_{i=1}^{3}\frac{\mathcal{E}_i^{\mathrm{KK}}(z,\bar{z})}{4g_\mathrm{s}^{-1}\sigma_i}\ , \tag{8.42}$$

where the Kähler moduli Re$T\equiv\sigma$ and σ_i is the Kähler modulus associated to the volume of the 4-cycle wrapped by the i^{th} D7-brane (or O7-plane). The $\mathcal{E}_i^{\mathrm{KK}}$ have a complicated dependence of the complex structure moduli z^k and the dilaton modulus is related via the string coupling. The winding contribution has the form

$$\delta\mathcal{K}_{g_\mathrm{s}}^{W} = \sum_{i\neq j\neq k}^{3}\frac{\mathcal{E}_i^{\mathrm{W}}(z,\bar{z})}{4\sigma_j\sigma_k}\ , \tag{8.43}$$

where both functions $\mathcal{E}_i^{\mathrm{KK}}(z,\bar{z})$ and $\mathcal{E}_i^{\mathrm{W}}(z,\bar{z})$ are found in [65][66]. Despite the complicated dependence of these functions on the complex structure moduli, the corrections have a simple scaling with the Kähler moduli σ_i which can be used tho stabilise the Kähler moduli.

8.1.3 The superpotential receives no perturbative corrections

While the $\mathcal{N}=1$ Kähler potentials receives perturbative corrections at every order in perturbation theory, the superpotential only receives non-perturbative corrections. This is largely due to a so-called Peccei-Quinn type shift-symmetry to be explained next. The classical 10D type IIB action is gauge invariant and there are pseudo-scalar fields of the gauge potentials that inherits a continuous shift-symmetry of an arbitrary constant according to

$$a \to a + \text{const} \ . \tag{8.44}$$

These pseudo-scalar fields are sometimes referred to as axions, which in IIB O3/O7 setups refers to not only C_0 but also to the b^a, c^a, ρ_α and the scalar duals b^0, c^0 of B_2, C_2 respectively. Again, the duals are $b^0 \equiv \int_{\Sigma_2} B_2$ and $c^0 \equiv \int_{\Sigma_2} C_2$ with Σ_2 being a 2-cycle of the internal space. In a background with vanishing fluxes, the type IIB action is independent of the potentials C_0, C_2, C_4, B_2 and only involves the associated field strengths of each field, thus preserving the shift-symmetry of these fields. This may be illustrated with an example using the B_2 field and its scalar dual b^0. The worldsheet coupling of B_2 in the non-linear σ-model action, i.e

$$S_\sigma \supset -\tfrac{1}{2} \int_{\Sigma_2} \mathrm{d}^2\sigma \epsilon^{ab} \partial_a X^M \partial_b X^N B_{MN}(X) \ , \tag{8.45}$$

can be written equivalently as

$$S_\sigma \supset - \int_{\Sigma_2} B_2 \equiv -b^0 \ , \tag{8.46}$$

where the integral runs over a worldsheet wrapping a 2-cycle Σ_2. Eq. (8.46) can be recognised as a topological coupling. Now, expanding the B_2 field around some point $X_0 = 0$ gives us

$$B_{MN}(X) = B_{MN}(X_0) + X^P \partial_P B_{MN}(X_0) + \dots \ , \tag{8.47}$$

where the first term is constant and give rise in eq. (8.45) to a worldsheet total derivative;

$$-\tfrac{1}{2} \int_{\Sigma_2} \mathrm{d}^2\sigma \partial_a \left(\epsilon^{ab} X^M \partial_b X^N B_{MN}(X_0) \right) \ . \tag{8.48}$$

It is clear that this term must vanish in order to preserve the axionic shift-symmetry . At any order of σ-model perturbation theory the 2-cycle is trivial, so that eq. (8.48) is integrated to zero. However, it is non-vanishing if either the worldsheet wraps a topologically *non-trivial* cycle,or has a boundary. The second term in eq. (8.47) involving the spacetime derivative of B_{MN} is in general non-vanishing, but will in this case be proportional to a spacetime derivative of an axion and will therefore preserve the axionic shift-symmetry.

The axion ρ_α in the Kähler coordinate T_α defined in eq. (5.139) is as familiar included in the Kähler potential of eq. (8.1). However, since $\mathcal{K} \sim \ln(T + \bar{T})$, the ρ being the imaginary part of T will cancel and thus preserve the shift-symmetry of the Kähler potential. The superpotential however is holomorphic and hence only dependent on T_α rather than $T_\alpha + \bar{T}_\alpha$, but there are no non-trivial polynomials in T_α that are invariant under the shift of the axion. Since the gauge invariant action is related to the superpotential via the scalar potential as in eq. (5.132), the superpotential should be invariant under axionic shift-symmetries as well. As a result, the superpotential can only depend on T_α non-perturbatively. The corrections in the α'-expansion must change in magnitude as the T_αs are varied since

$$-2 \ln \left(\mathcal{V} + \frac{\xi}{2 g_s^{2/3}} \right) \approx -2 \ln \mathcal{V} - \frac{\xi}{g_s^{2/3} \mathcal{V}} + \dots \ . \tag{8.49}$$

The shift-symmetry of the superpotential forces it to be independent of the T_αs to any order in perturbation theory, and as a consequence it will not receive any perturbative α'-corrections.

Turning to the string loop expansion, the argument can be based on R-symmetry and PQ-symmetry. Again, both R- and PQ-symmetry can be seen as subgroups of $SL(2, \mathbb{R})$. For R-symmetry one chooses the following 1-parameter family of $SL(2, \mathbb{R})$ transformations:

$$b = -|\tau| \ , \qquad c = \frac{1}{|\tau|} \ , \qquad d = 0 \ , \tag{8.50}$$

which satisfy $ad - bc = 1\ \forall \tau$. These choices also have the property that $c\tau + d = \tau/|\tau| = e^{-i\alpha}$ where $\alpha = -\mathrm{arg}\tau$. Under R-symmetry the G_3 field then transforms as $G_3 \to \frac{G_3}{c\tau+d} = e^{+i\alpha}G_3$. The GVW superpotential transforms in the same way as G_3; $W_{\text{GVW}} \to \frac{W_{\text{GVW}}}{c\tau+d}$. For the PQ-symmetry one has

$$a = d = 1\ , \qquad c = 0\ , \qquad \tau \to \tau + b\ . \tag{8.51}$$

The PQ-symmetry is preserved to all orders in perturbation theory and the R-symmetry to all orders in g_{s} to leading order in α'. Hence, in leading order of α', both PQ- and R-symmetry survives to all orders in the string coupling expansion. Note that only non-perturbative corrections break the $SL(2,\mathbb{R})$ symmetry to $SL(2,\mathbb{Z})$. The result obtained for the string loop corrections to leading order in α' is easily extended to all orders in α', since we earlier in this section argued that the superpotential can not receive any α'-corrections.

Using these symmetries one can derive a non-renormalisation theorem for type IIB. To start one considers how the field content of the 4D theory transforms under global symmetries of the 10D action. To do so two kinds of fields are good to keep track of. The first kind consists of fields that describe light degrees of freedom whose masses are smaller than the KK scale, which are then fully described by the 4D $\mathcal{N} = 1$ theory. These scalar fields, denoted φ^i, transform under supersymmetry as chiral matter. Our friends the Kähler moduli and complex structure moduli are included here.

The second kind of fields are called *spurions* and describe the transformation properties of the background flux VEVs under the symmetry transformation of interest. These fluxes lie in the background value of the G_3 field and may be regarded as the VEVs $\mathcal{G}^r$ of the collection of 4D scalar fields obtained when G_3 is dimensionally reduced. These VEVs transform under $SL(2,\mathbb{R})$ transformations (or R/PQ-transformations) like G_3.

The supersymmetry transformation parameter ϵ has R-charge $q_\epsilon = +\frac{1}{2}$, and so it follows from R-invariance of the IIB action that the superpotential must carry R-charge $q_W = +1$. The G_3 field also has R-charge $q_W = +1$, so we are free to take W to be proportional to one of the $\mathcal{G}^r$'s, say $\mathcal{G}^0$. Then we may write a general superpotential as

$$W(\varphi^i, \tau, \mathcal{G}_r) = \mathcal{G}^0 A\left(\varphi^i, \frac{\mathcal{G}^r}{\mathcal{G}^0}\right)\ , \tag{8.52}$$

where A is some function that can not depend explicitly on τ as it must be PQ-invariant and the φ^is do not shift under a PQ-transformation. The $\mathcal{G}^r$ on the other hand do depend on τ, but the overall τ-dependence of A is cancelled in the fraction $\mathcal{G}^r/\mathcal{G}^0$.

Earlier we have treated the string coupling constant $g_{\mathrm{s}} = e^\phi$ as the loop-counting parameter of the low-energy effective action. This is in fact technically not true in IIB, since its low-energy effective Lagrangian contains different powers of e^ϕ. It is true in the heterotic case however, where the dilaton appears in an overall factor $e^{-2\phi}$ in the string frame heterotic low-energy Lagrangian. As a consequence of the IIB case, it is convenient to organise the loop perturbation series by the rescalings

$$e^\phi \to \lambda e^\phi\ , \qquad C_p \to \frac{C_p}{\lambda}\ , \tag{8.53}$$

and F_p, G_3 (hence $\mathcal{G}^r$) transforming the same way as C_p. With this rescaling the total IIB action scales like $S \to S/\lambda^2$. After rescaling one expands the observables and low-energy effective 4D action in power series of λ, after which one takes the limit $\lambda \to 1$. The orders of λ is then understood as the string loop expansions, which however go hand in hand with e^ϕ.

Considering type IIB O3/O7 setups, the ordinary GVW flux superpotential is taken to be the lowest-order superpotential in the λ, or string loop, expansion. The lowest-order result corresponds to the function A in eq. (8.52) being linear in the arguments $\mathcal{G}^r/\mathcal{G}^0$. Hence it may be written

$$A\left(\varphi^i, \frac{\mathcal{G}^r}{\mathcal{G}^0}\right) = \sum_{r\geq 0} \frac{\mathcal{G}^r}{\mathcal{G}^0} A_r(\varphi^i) \ , \tag{8.54}$$

which inserted in eq. (8.52) makes the lowest-order superpotential take the form

$$W_{\text{GVW}} = \sum_{r\geq 0} \mathcal{G}^r A_r(\varphi^i) \ . \tag{8.55}$$

With this form, we see that would one expand the superpotential in powers of λ, the fraction $\mathcal{G}^r/\mathcal{G}^0 \to \mathcal{G}^r/\mathcal{G}^0$ is left invariant. Hence if W in eq. (8.52) is the general form of the superpotential for any power of λ, this fraction is the same as the lowest-order superpotential W_{GVW} and transforms like $W \to W/\lambda$. As such, we see that higher order loop corrections do not change the lowest-order approximation.

Conclusively, the non-renormalisation theorems for the flux superpotential leads to that it does not receive any perturbative corrections. A nice argument for this can also be found in [68] covering both the α'-expansion as well as the string loop corrections. Including both perturbative and non-perturbative effects, the Kähler potential and superpotential can therefore be written on the forms

$$\begin{aligned} \mathcal{K} &= \mathcal{K}_0 + \mathcal{K}_{\text{p}} + \mathcal{K}_{\text{np}} \ , \\ W &= W_0 + W_{\text{np}} \ . \end{aligned} \tag{8.56}$$

In the next section, we will review the non-perturbative effects on these potentials, which for $\mathcal{K}_{\text{np}}$ typically stems from string worldsheet instantons, and for W_{np} from D-brane instantons or gaugino condensation on D-branes.

8.2 Non-perturbative corrections to the superpotential

In this section we review the main contributors to non-perturbative corrections to the superpotential, namely instantons and gaugino condensation.

8.2.1 Worldsheet and D-brane instantons

Apart from string theory, instanton effects are already present in the setup of field theory. Instantons are semi-classical configurations providing saddle points in the Euclidean path integral of the spacetime fields of the theory. They are classical solutions to the Euclidean equations of motion in field theory, where the prototypical example is given by instantons in 4D gauge theories. In this case the gauge field configurations obey the self-duality condition in Euclidean space, i.e.

$$F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \quad \Leftrightarrow \quad F = \star_4 F \ . \tag{8.57}$$

For the $SU(2)$ gauge groups an explicit solution of the gauge potential A_μ is given, in the gauge $\partial_\mu A^a_\mu = 0$, by

$$A^a_\mu = 2\bar{\eta}^a_{\mu\nu} \frac{(x-x_0)_\nu}{(x-x_0)^2} \frac{\rho^2}{(x-x_0)^2+\rho^2} \ . \tag{8.58}$$

Here the x_0 and ρ are the position respectively the size of the instanton. The $\bar{\eta}^a_{\mu\nu}$ are *t'Hooft* symbols, which realise the a^{th} $SU(2)$ generator of the $SO(4) \simeq SU(2)^2$ rotation group. Generally an instanton configuration can be characterised by providing a map from S^3 at infinity in $\mathbb{R}^4$ to the gauge group G, and is therefore classified as the homotopy group $\Pi_3(G)$. If G is a simple group, then $\Pi_3(G) = \mathbb{Z}$, thus implying that the instanton configurations are labelled by an integer winding number, or topological charge, known as the *instanton number*

$$k = \frac{1}{8\pi^2}\int_{4D} \text{tr}(F \wedge F) \ . \tag{8.59}$$

It is said that anti-instantons realise negative values of k and satisfy an anti-self-duality relation. With the self-duality of eq. (8.57), the classical action for an instanton configuration is given by

$$S_{\text{cl}} = \frac{8\pi^2}{g_{\text{YM}}^2}|k| \ , \tag{8.60}$$

with g_{YM} being the 4D Yang-Mills coupling. Also familiar from quantum mechanics, the saddle point approximation to a tunneling process in the theory is described by a classical solution to the Euclidean equations of motion. The strength, given by $e^{-S_{\text{cl}}} \sim e^{-k/g_{\text{YM}}^2}$, is clearly a non-perturbative contribution to the theory's path integral. There is a discrete shift-symmetry is visible here as well, since an angle θ in the gauge-theory contributes to the instanton amplitude as

$$e^{-S_{\text{cl}}} = e^{-k\left(\frac{8\pi^2}{g_{\text{YM}}^2}+i\theta\right)} \ . \tag{8.61}$$

The 2π-periodicity then breaks a continuous shift-symmetry of the θ parameter to a discrete one; $\theta \to \theta + 2\pi$. The factor i in the exponent arises because of the theory being Euclidean [69].

Since the low-energy limit of string theory contains field theory (coupled to gravity), it is not surprising that there are also non-perturbative effects from Euclidean instanton configurations. As discussed before, a fundamental non-perturbative contribution in the σ-model is a worldsheet wrapping a non-trivial cycle Σ_2, which is also referred to as a *worldsheet instanton*. These instantons are BPS states when Σ_2 is a holomorphic 2-cycle and their topological charge is the 2-homology class $[\Sigma_2]$. The classical instanton action is given by the volume of the internal cycle. The cycles are usually complexified into chiral multiplets so that the instanton strength is given by

$$e^{S_{\text{inst}}} = e^{-\int_{\Sigma_2}(J+iB_2)} \sim e^{-ia} \ , \tag{8.62}$$

where J is the Calabi-Yau Kähler form. We see that the axionic shift-symmetry which was originally continuous is broken to form a discrete shift-symmetry $a \to a + 2\pi$, as in the case of the gauge theory parameter θ.

The strength of the worldsheet instantons depends on the 2D worldsheet area, and the preceding discussion needed no assumption about the genus of the worldsheet, so it must hold to any order in string loop expansion. There arises however a new possibility in that the closed string worldsheet may break open on a D-brane, corresponding to a g_{s} non-perturbative effect. Hence, while the worldsheet instantons are non-perturbative in α', they are perturbative (tree level in fact) in g_{s}. Instantons that are non-perturbative in α' and in g_{s} must therefore have tension of inverse powers of g_{s}. In string theory, branes have this property. Hence in 4D string compactifications there may be brane instanton effects from possible branes localised in the 4D spacetime that are wrapped around different cycles in the internal manifold.

Another type of instanton is Euclidean Dp-branes, i.e. D-branes that are not spacetime filling. The conditions for a wrapped Euclidean brane to define a BPS instanton are the same kind of supersymmetry conditions as for ordinary D-branes. In type IIA with D6-branes wrapped on 3-cycles, BPS instantons arise from special Lagrangian[1] 3-cycles with suitable phases. In type IIB orientifold setups with D3-branes at points and D7-branes wrapping 4-cycles, the BPS instantons are given by D(-1)-branes at points in the Calabi-Yau and Euclidean D3-branes wrapped on some 4-cycle. With D5/D9-branes the BPS instantons come from D1-branes on holomorphic 2-cycles and Euclidean D5-branes wrapped on the whole Calabi-Yau, which may have gauge backgrounds on its world-volume. It is interesting to note that the mirror symmetry between

[1] A special Lagrangian cycle Π is a volume-minimising cycle on which the 2-form Kähler form J and holomorphic 3-form Ω_3 satisfies $J|_\Pi = 0$ and $\text{Im}(e^{-i\varphi}\Omega_3)|_\Pi = 0$ for some fixed phase φ. The volume of the special Lagrangian cycle is given by $\text{vol}(\Pi) = \int_\Pi \text{Re}(e^{-i\varphi}\Omega_3)$.

IIA and IIB orientifold models also extends to the instantons involved. In particular, the fact that all BPS instantons of type IIA are Euclidean D2-branes may be used to simplify some type IIB analysis.

A Dp-brane wrapped on a $(p+1)$-cycle Σ_{p+1} has strength $e^{-S_{\mathrm{DBI}}}$. Similar to its strength, the contribution from a Euclidean D3-brane wrapping a 4-cycle to the superpotential is

$$W_{\mathrm{ED3}} = \mathcal{A} e^{-2\pi T} \ , \tag{8.63}$$

where the coefficient $\mathcal{A}$ can depend on complex structure moduli, the axion-dilaton and D-brane positions, but is independent of the Kähler moduli T. Some conditions necessary for non-vanishing ED3-brane superpotentials was found by [70]. Namely, in order for the Euclidean D3-brane contribution to the superpotential not to vanish, its associated 4-cycle Σ_4 should be a projection of some 6-cycle Σ_6 such that

$$\chi(\mathcal{O}_{\Sigma_6}) \equiv \sum_{i=0}^{3} (-1)^i h^{0,i}(\Sigma_4) \ , \tag{8.64}$$

where $\mathcal{O}_{\Sigma_6}$ is the trivial line bundle defined on the 6-cycle and $\chi(\mathcal{O}_{\Sigma_6})$ the holomorphic Euler characteristic of Σ_6. A sufficient condition for a non-vanishing ED3-brane superpotential is that

$$h^{(0,1)} = h^{(0,2)} = h^{(0,3)} = 0 \ , \tag{8.65}$$

i.e. that the 6-cycle Σ_6 is *rigid*, since the Hodge numbers count the independent deformations of Σ_6.

8.2.2 Gaugino condensation

Rather than Euclidean D3-branes wrapping the 4-cycle Σ_4, we may also consider the same cycle being wrapped by a stack of N D7-branes. The worldvolume theory of the D7-branes includes a Yang-Mills action for 4D gauge fields A_μ of the form

$$S = \frac{1}{2g_7^2} \int_{\Sigma_4} \mathrm{d}^4\sigma \sqrt{g_{\mathrm{ind}}} e^{-4A(y)} \cdot \int \mathrm{d}^4 x \sqrt{-g} \mathrm{tr}[F_{\mu\nu} F^{\mu\nu}] \ , \tag{8.66}$$

where g_{ind} is the induced metric on the D7-brane, $g_{\mu\nu}$ is the unwarped metric and g_7 is the gauge-coupling of the Yang-Mills theory in 7+1 dimensions;

$$g_7^2 = 2(2\pi)^5 (\alpha')^2 \ . \tag{8.67}$$

The gauge coupling of the Yang-Mills theory in 4D is given by

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{T_3 \mathcal{V}_4}{8\pi^2} \ ,$$

with $\mathcal{V}_4$ being the volume enclosed by the 4-cycle Σ_4, defined as

$$\mathcal{V}_4 \equiv \int_{\Sigma_4} \mathrm{d}^4\sigma \sqrt{g_{\mathrm{ind}}} e^{-4A(y)} \ . \tag{8.68}$$

As in the case of the Euclidean D3-branes, a topological condition is that Σ_4 should have no deformation that corresponds to charged matter fields, which implies that the arising 4D gauge theory is that of *pure glue* $\mathcal{N} = 1$ Yang-Mills. At low energies, this field theory generates a non-perturbative superpotential from gaugino condensation;

$$|W_{\lambda\lambda}| \sim e^{-\frac{8\pi^2}{N g_{\mathrm{YM}}^2}} \sim e^{-\frac{T_3 \mathcal{V}_4}{N}} \ . \tag{8.69}$$

Gaugino condensation [71] is a strong coupling effect where a product of gaugino fields acquire a vacuum expectation value. This vacuum expectation value $\langle\lambda\lambda\rangle$ adds to the superpotential, where $\langle\lambda\lambda\rangle = \Lambda^3$, $\Lambda \sim \mu e^{-1/g^2(\mu)}$ and μ is the scale at which one matches the couplings of high- and low-energy theories. In short, the gaugino condensate superpotential may be written

$$W_{\lambda\lambda} = \mathcal{A} e^{-2\pi T/N} \ , \tag{8.70}$$

much like the case of the ED3-brane with $\mathcal{A}$ independent of the Kähler moduli.

Conclusively, we have seen that non-perturbative effects like strong gauge dynamics on D7-branes, like gaugino condensation, and instanton contributions from Euclidean D3-branes generate superpotential terms including the Kähler moduli. Whichever source, the contribution to the superpotential leads in combination with the constant flux superpotential, to the general structure

$$W = W_0 + \sum_{i=1}^{h_+^{(1,1)}} \mathcal{A}_\alpha e^{-a_\alpha T_\alpha} + \dots \ , \tag{8.71}$$

assuming that there is a non-perturbative effect for all Kähler moduli involved. The ellipsis denote higher-order non-perturbative terms. With an arbitrary Kähler potential, this superpotential leads to the scalar potential

$$V_{\text{np}} = e^{\mathcal{K}} K^{i\bar{\jmath}} \Big[a_i \mathcal{A}_i a_{\bar{\jmath}} \bar{\mathcal{A}}_{\bar{\jmath}} e^{-(a_i T_i + a_{\bar{\jmath}} \bar{T}_{\bar{\jmath}})} - \Big(a_i \mathcal{A}_i e^{-a_i T_i} \bar{W} \partial_{\bar{\jmath}} \mathcal{K} + a_{\bar{\jmath}} \bar{\mathcal{A}}_{\bar{\jmath}} e^{-a_{\bar{\jmath}} \bar{T}_{\bar{\jmath}}} W \partial_i \mathcal{K} \Big) \Big] \ . \tag{8.72}$$

Having gathered perturbative and non-perturbative corrections to the Kähler and superpotential, we are interested in whether their effect can result in a stable, or at least metastable, vacuum. A very general problem including these corrections which may stabilise flat directions is known as the Dine-Seiberg problem [72]. The problem is sometimes summarised as "when corrections are important they are not computable and when they are computable they are not important". We elaborate this by first considering some modulus ρ, such as the volume $\mathcal{V}$ or the inverse string coupling, which both have the property that the limit $\rho \to \infty$ corresponds to the weakly coupled region where we may trust our tree level effective action. Including quantum corrections, they may induce a potential $V(\rho)$ in the 4D effective theory. This potential must however have the property $\lim_{\rho\to\infty} V(\rho) \to 0$, because of our assumption that at $\rho \to \infty$ we can trust the tree level effective theory which by definition has zero potential for ρ, as it is a modulus. As $\rho \to \infty$ there are then two possibilities, either the potential goes to zero from above, i.e. $V(\rho) > 0$, or we have that $V(\rho) < 0$, which makes the ρ go to stronger coupling *from* the $\rho \to \infty$ limit in order to minimise the scalar potential. A local minimum can only arise if higher order corrections are included; one needs two more corrections for the first case and one more for the second case. However, since we are including corrections important enough to cause a significant departure from the first order shape of $V(\rho)$, we can longer be in the weakly coupled region. Not being in this region implies that higher order corrections must be significant too, although without extended supersymmetry, i.e. $\mathcal{N} \geq 2$ in 4D, there are generally not enough tools available to compute more than a few orders in perturbation theory [73]. As a consequence one looses control, and so Dine and Seiberg concluded 1985 that our own string vacuum is probably strongly coupled.

In the next section we will review the two leading ideas for Kähler moduli stabilisation in the type IIB theory; the KKLT scenario and the large volume scenario, where the resulting vacua come from competition between known corrections.

8.3 The KKLT setting

By using the non-perturbative corrections to the superpotential discussed in the previous section, one can show that all moduli in type IIB Calabi-Yau orientifold compactifications can be stabilised. The construction was first carried out by Kachru, Kallosh, Linde and Trivedi (KKLT) [74], where they also showed it possible to obtain a metastable de Sitter vacuum by adding a small number of $\overline{D3}$-branes. In this section we will review both these topics, starting with the moduli stabilisation.

8.3.1 Kähler moduli stabilisation by non-perturbative effects

The KKLT procedure is based on type IIB flux compactifications on O3 Calabi-Yau orientifolds, where they assume an arbitrary number $h^{(2,1)}$ of complex structure moduli but only a single Kähler modulus T such that $h_+^{(1,1)} = 1$ and $h_-^{(1,1)} = 0$. The real part of the Kähler modulus is given by eq. (5.139) as $\mathrm{Re}T = \frac{1}{2}\kappa^{2/3}$, which makes it scale like $\mathrm{Re}T \sim R^4$ with R being the radius of compactification. Would G_3 be a (0,3)-form, supersymmetry would be broken as $D_T W_{\mathrm{GVW}} \neq 0$, and the superpotential would not vanish as $W_{\mathrm{GVW}} = W_0$.

As we move on to add a non-perturbative contribution to the superpotential, the no-scale structure will break and open up to new possible vacua. Only including a single Kähler modulus, the non-perturbative superpotential of eq. (8.71) reduces to

$$W = W_0 + ce^{-2\pi aT} , \tag{8.73}$$

where again $W_0 = W_{\mathrm{GVW}}$ is the superpotential contribution coming from the fluxes, a is a positive model-dependent quantity and c a holomorphic function depending on the complex structure fields. Since it is possible to stabilise the complex structure moduli and dilaton of type IIB O3/O7 setups by mechanisms reviewed in the previous chapter, we may assume them stabilised in this analysis and focus wholly on the Kähler moduli. Hence, the c can be regarded as a constant in the following analysis and we may consider only the Kähler moduli part of the Kähler potential

$$\mathcal{K}^K = -3\ln(T + \bar{T}) . \tag{8.74}$$

At the supersymmetric minimum, all Kähler covariant derivatives of the superpotential in eq. (8.73) vanishes, including $D_T W = 0$. Using the expression eq. (5.148) for the Kähler derivative, we have that

$$D_T W = -2\pi ace^{-2\pi aT} - \frac{3}{T+\bar{T}}\left(W_0 + ce^{-2\pi aT}\right) = 0 , \tag{8.75}$$

using only the T-dependent part of the Kähler potential in eq. (8.1) for simplicity. The other parts of the Kähler potential will contribute with a constant factor in eq. (8.75), since the complex structure and dilaton moduli are assumed to be fixed. If further one sets $\mathrm{Im}T = 0$ and $\mathrm{Re}T \equiv \sigma$, the supersymmetric minimum satisfying eq. (8.75) is found at

$$W_0 = -\left(\frac{4\pi a\sigma_0}{3} + 1\right) ce^{-2\pi a\sigma_0} , \tag{8.76}$$

with σ_0 being the value of σ which minimises W_0 above. With this value of the constant W_0, the superpotential of eq. (8.73) is $W_{\min} = -\frac{4\pi a\sigma}{3}ce^{-2\pi a\sigma}$ at the minimum. According to eq. (8.4), the resulting vacuum has minimum scalar potential, i.e. cosmological constant, at

$$V_{\min} = -3e^{\mathcal{K}}|W_{\min}|^2 = -\frac{2\pi^2 a^2 c^2}{3\sigma}e^{-4\pi a\sigma} , \tag{8.77}$$

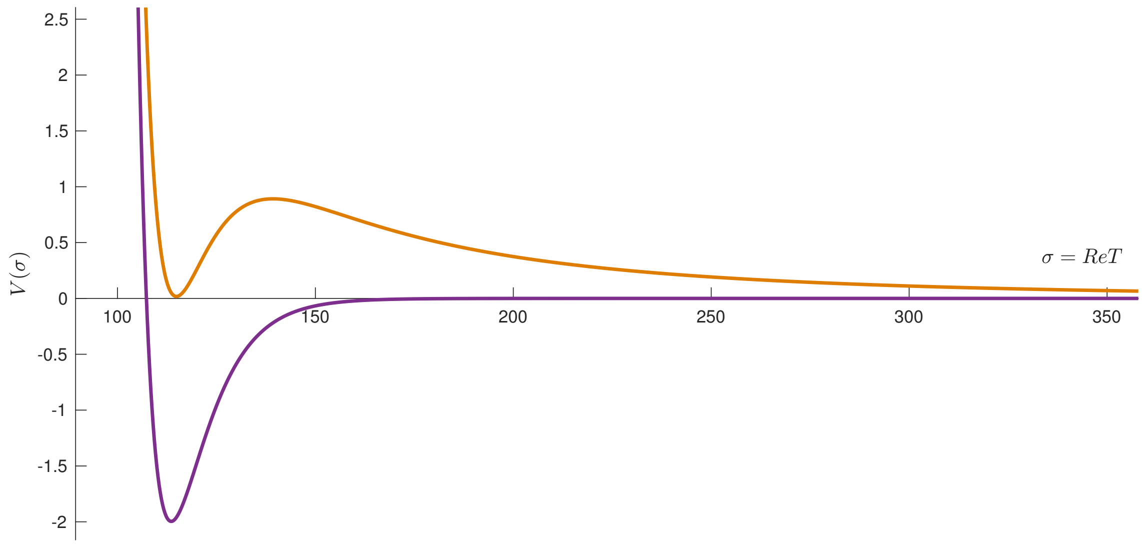

Figure 8.1: [Purple] AdS scalar potential of eq. (8.78), and [orange] supersymmetry-breaking dS scalar potential of eq. (8.80), as a function of Kähler moduli $\sigma \equiv \mathrm{Re}T$. Parameter choice in accordance with KKLT with $2\pi a = 0.1$, $c = 1$, $W_0 = -10^{-4}$ and $D = 3 \cdot 10^{-9}$. Both potentials are multiplied with a factor of 10^{15}.

which makes it a supersymmetric AdS_4. With the superpotential of eq. (8.73) and only including the Kähler moduli, the scalar potential of eq. (8.4) is given by

$$\begin{aligned} V = e^{\mathcal{K}^K}\left(K^{T\bar{T}}|D_T W|^2 - 3|W|^2\right) &= \frac{1}{(2\sigma)^3}\left(\frac{(2\sigma)^2}{3}\left(2\pi a c e^{-2\pi a\sigma} + \frac{3}{2\sigma}W\right)^2 - 3W^2\right) \\ &= \frac{\pi a c e^{-2\pi a\sigma}}{\sigma^2}\left(\frac{2\pi a c\sigma e^{-2\pi a\sigma}}{3} + W_0 + c e^{-2\pi a\sigma}\right) \, , \end{aligned} \tag{8.78}$$

with Kähler metric $K^{T\bar{T}} = \partial^T \partial^{\bar{T}} \mathcal{K}^K$. The behaviour of this scalar potential as a function of the volume modulus σ is illustrated in fig. 8.1. A final comment is that it is clear that the stabilisation only works for $W_0 \neq 0$, and as a consequence there must be a (0,3)-form piece of the G_3 flux which will result in a non-zero W_0. The above reasoning have showed that this will result in a 4D $\mathcal{N} = 1$ AdS vacuum.

8.3.2 Uplifting to de Sitter vacua

Not only do observations imply a universe with maximal spacetime symmetry, but data also seems to require a small positive cosmological constant [75] which suggests our universe to be (asymptotically) de Sitter. A dS vacuum can be constructed by a mechanism often referred to as *uplifting*, in which additional tension sources are added to the supersymmetric AdS vacuum constructed in section 8.3.1. In the previous AdS case, we assumed that the tadpole condition of eq. (6.35) with D3-branes, fluxes and O3-planes was saturated. If we choose to increase the amount of flux, additional $\overline{D}3$-brane sources may be added to keep the tadpole condition fulfilled. The addition of these branes adds some energy to the system, namely they add to the scalar potential the contribution

$$V_{\overline{D3}} = \frac{D'}{(T+\bar{T})^3} = \frac{D}{\sigma^3} \ , \tag{8.79}$$

where $D = D'/8$ is proportional to the number of added $\overline{D3}$-branes and warp factor e^{4A} at the position of the branes [74][76]. This addition however breaks the supersymmetry explicitly. By fine tuning the value of D, the potential

$$V = \frac{\pi a c e^{-2\pi a\sigma}}{\sigma^2}\left(\frac{2\pi a c\sigma e^{-2\pi a\sigma}}{3} + W_0 + c e^{-2\pi a\sigma}\right) + \frac{D}{\sigma^3} \ , \tag{8.80}$$

can become positive. It is clear from fig. 8.1 that the dS minimum value of σ_0 is only slightly shifted from the one of the AdS vacuum and the overall shapes of the potential minima are very similar. Note that the dS vacuum is only metastable, as there is a probability of tunnelling, i.e. a runaway behaviour at infinite volume. With the reasoning of the Dine-Seiberg problem discussed above, this may also be expected from any string theory. Nevertheless KKLT showed that the lifetime of the dS vacuum may be large in Planck times and can be longer than the cosmological time scale of $\sim 10^{10}$ years. Possible decay channels of the KKLT dS have been studied in [77], whose results are in agreement with the ones of KKLT.

8.4 The large volume scenario

A drawback of the KKLT setting is that it requires W_0 to be very small. Although achievable by suitably tuned fluxes, this raises the question if moduli stabilisation for generic values of W_0 is possible. The large volume scenario (LVS) [78] combines both the α'^3-correction to the Kähler potential in eq. (8.74) as well as the non-perturbative corrected superpotential of eq. (8.71). By combining W_0 with the α'^3-corrected Kähler potential the scalar potential receives the correction

$$\delta V_{\alpha'} = 3\hat{\xi} e^{\mathcal{K}K} \frac{\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})^2} W_0^2 \sim \frac{3\hat{\xi} W_0^2}{4\mathcal{V}^3} \ , \tag{8.81}$$

where $\hat{\xi} \equiv \xi/g_{\mathrm{s}}^{3/2}$ and ξ is defined as in eq. (8.38). The scalar potential is given by the sum of the non-perturbative contribution V_{np} of eq. (8.72) and the addition of eq. (8.81) above, i.e. by

$$V = e^{\mathcal{K}} K^{i\bar{\jmath}}\Big[a_i\mathcal{A}_i a_{\bar{\jmath}}\bar{\mathcal{A}}_{\bar{\jmath}} e^{-(a_iT_i + a_{\bar{\jmath}}\bar{T}_{\bar{\jmath}})} - \Big(a_i\mathcal{A}_i e^{-a_iT_i}\bar{W}\partial_{\bar{\jmath}}\mathcal{K} + a_{\bar{\jmath}}\bar{\mathcal{A}}_{\bar{\jmath}} e^{-a_{\bar{\jmath}}\bar{T}_{\bar{\jmath}}} W\partial_i\mathcal{K}\Big)\Big] + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3} \ . \tag{8.82}$$

In general the perturbative contribution of eq. (8.81) dominates over the non-perturbative terms, although competition between the two contributions can occur if one or more cycles are exponentially smaller than the largest cycles. In the LVS, the main mechanisms can be illustrated with two Kähler moduli T_b and T_s, where T_b controls the overall volume of the Calabi-Yau and the T_s controls the volume of a 4-cycle wrapped for example by ED3-brane instantons. These will be shown to stabilise at very large (big) and moderately large (small) values, as indicated by the subscripts.

LVS provide an explicit example with the Calabi-Yau $\mathbb{CP}^4_{(1,1,1,6,9)}$, which is a hypersurface of degree 18 with defining equation being a quintic polynomial set to zero, specifically

$$z_1^{18} + z_2^{18} + z_3^{18} + z_4^3 + z_5^2 - 18\psi z_1z_2z_3z_4z_5 - 3\phi z_1^6 z_2^6 z_3^6 = 0 \ . \tag{8.83}$$

Here the ψ and ϕ are two complex structure moduli which are invariant under the $\mathbb{Z}_6 \oplus \mathbb{Z}_{18}$ action. Flux compactifications on this manifold have been studied extensively in [79]. The volume given by this surface, in terms of 2-cycle volume moduli t_i with $i = 1, \dots, 5$, is given by [80]

$$\mathcal{V} = \frac{1}{6}\left(3t_1^2 t_5 + 18t_1 t_5^2 + 36t_5^3\right) \ . \tag{8.84}$$

By defining $\sigma_4 \equiv t_1^2$ and $\sigma_5 \equiv (t_1 + 6t_5)^2/2$, the volume may be written as

$$\mathcal{V} = \frac{1}{9\sqrt{2}}\left(\sigma_5^{3/2} - \sigma_4^{3/2}\right) \ . \tag{8.85}$$

σ_4 and σ_5 are 4-cycles, and the idea is to choose σ_4 to be small and σ_5 to be large. In the LVS this distinctive diagonal form of the volume in terms of a single "large" 4-cycle and a number of "small" 4-cycles is typical. Hence, by defining $\sigma_b \equiv \mathrm{Re}T_b$ and $\sigma_s \equiv \mathrm{Re}T_s$ the Kähler potential with volume of the form of eq. (8.85) and including the α'^3-correction, reads

$$\mathcal{K}^K = -2\ln\left(\frac{1}{9\sqrt{2}}(\sigma_b^{3/2} - \sigma_s^{3/2}) + \frac{\xi}{2g_s^{3/2}}\right) \ . \tag{8.86}$$

With this Kähler potential and $W = W_0 + \mathcal{A}_s e^{-a_s T_s}$, the scalar potential for the Calabi-Yau volume $\mathcal{V} \sim \sigma_b^{3/2}$ and with $\mathrm{Im}T_s = 0$ has the structure

$$V \sim A\frac{a_s^2|\mathcal{A}_s|^2\sqrt{\sigma_s}e^{-2a_s\sigma_s}}{\mathcal{V}} - B\frac{a_s W_0 \mathcal{A}_s \sigma_s e^{-a_s\sigma_s}}{\mathcal{V}^2} + C\frac{\xi|W_0|^2}{g_s^{3/2}\mathcal{V}^3} \ , \tag{8.87}$$

where A, B and C are $\mathcal{O}(1)$ constants. Note that the first two terms are analogous to the ones appearing in eq. (8.80). Now, taking the limit

$$\sigma_b \to \infty \ , \qquad \text{with} \qquad a_s\sigma_s = \ln\mathcal{V} \ , \tag{8.88}$$

the scalar potential in terms of $\mathcal{V}$ will go like $V \sim \frac{\sqrt{\ln\mathcal{V}}}{\mathcal{V}^3} - \frac{\ln\mathcal{V}}{\mathcal{V}^3} + \frac{1}{\mathcal{V}^3}$, so that the second term will dominate, making V approach zero from below when $\mathcal{V} \to \infty$. For smaller $\mathcal{V}$ however, the other terms dominate and are positive given that $\hat{\xi} > 0$. Hence, the scalar potential must have a local AdS minimum. This minimum must occur at intermediate values, specifically at

$$\mathcal{V} \propto e^{a_s\sigma_s} \gg 1 \ , \qquad \sigma_s \propto \frac{\xi^{2/3}}{g_s} \ . \tag{8.89}$$

The Kähler moduli are stabilised at the values that minimise this minimum. However, by the form of the scalar potential in eq. (8.87) it is also clear that at this minimum, the F-terms do not vanish, and consequently supersymmetry is spontaneously broken.

Conditions on the LVS minimum have been studied in [80]. A result for the small cycles is that at least one of them should be a *rigid exceptional divisor* arising from blowing up a singular point. Some background will be provided in the next. In a complex manifold M of complex dimension n, a point $p \in M$ can be replaced with a copy of $\mathbb{P}^{n-1}$ known as the *exceptional divisor*. If p is a singular point, the exceptional divisor can be more general. If the blow-up of M itself is a Calabi-Yau 3-fold, then the exceptional divisor is a 4-cycle of size parameterised by one of the Calabi-Yau's Kähler moduli [81]. Further, when the exceptional divisor satisfies the rigidity condition of eq. (8.65), the Euclidean D3-brane superpotential term is non-vanishing. The 4-cycle volume moduli are related to the 2-cycle volume moduli which encodes for the overall classic volume. Again, the classic volume is given by

$$\mathcal{V} = \tfrac{1}{6}\int_M J \wedge J \wedge J = \tfrac{1}{6}\kappa_{ijk}t^i t^j t^k \ , \tag{8.90}$$

where the Kähler form J can be written in the base $\{\hat{D}_i\} \in H^{(1,1)}(M,\mathbb{Z})$ with $\hat{D}_i$ being the 2-cycle divisor such that $J = \sum_i^{h^{(1,1)}} \hat{D}_i t^i$. $\hat{D}_i$ is Poincaré dual to the 4-cycle divisor $D_i \in H_4(M,\mathbb{Z})$, so the 4-cycles σ_i are related to the 2-cycles t^i via

$$\sigma_i = \partial_{t^i}\mathcal{V} = \frac{1}{2}\int_M \hat{D}_i \wedge J \wedge J = \frac{1}{2}\kappa_{ijk}t^i t^j \ . \tag{8.91}$$

Manifolds which are capable of supporting an appropriate structure of small and large cycles with volume

$$\mathcal{V} \sim \sigma_b^{3/2} - \sum_i \sigma_i^{3/2} \ , \tag{8.92}$$

are termed *Swiss cheese*, where the holes have the volumes $\sigma_i^{3/2}$. There are more possibilities of Swiss cheese type manifolds, and a manifold with volume modulus like eq. (8.92) are called *strong cheese* [82]. In the $\mathbb{CP}^4_{(1,1,1,6,9)}$ example above, the σ_4 and σ_5 are the volume moduli of the 4-cycles, i.e. divisors, $D_4, D_5 \in H_4(M,\mathbb{Z})$. It is clear that this is a single-hole strong Swiss cheese. The $\mathbb{CP}^4_{(1,1,1,6,9)}$ has Hodge numbers $h^{(1,1)} = 2$ and $h^{(2,1)} = 272$, which makes the Euler characteristic for a Calabi-Yau manifold $\chi(M) = 2(h^{(1,1)} - h^{(2,1)}) < 0$, so that indeed $\hat{\xi} > 0$ as required.

In final, we may summarise some differences between the LVS and KKLT senarios. For instance, in the LVS some cycles are exponentially larger than others while the KKLT scenario does not have any strict hierarchy between the sizes of the cycles. The classical flux superpotential W_0 has to be fine-tuned to be exponentially small in KKLT, while it is of $\mathcal{O}(1)$ in the LVS. The vacuum of KKLT is a supersymmetric AdS_4, while it is non-supersymmetric in the LVS, however both scenarios can be uplifted to a non-supersymmetric dS_4.

8.5 Critiques of de Sitter scenarios in string theory

A first remark is that the above models are obviously not constructed as general solutions describing a de Sitter space. The setting is minimal in the number of moduli present and does not include all quantum corrections. There is also difficulties in stabilising moduli in a controlled manner and a fine-tuning of D is required in order to make it sufficiently small in order to create a long-lived vacuum.

However, while there is a vast amount of de Sitter vacuum constructions, the KKLT and LVS setups are arguably the most well-understood. They are examples of a class of models that make up the vast majority of de Sitter vacuum constructions. In general these consist of classical flux compactifications with orientifold sources to which one adds quantum corrections. The corrections comprise virtually everything but the supergravity action and eventual localised sources at the two-derivative level, i.e. higher-derivative corrections, string loop corrections, non-perturbative corrections in the string coupling, and so on. In general these corrections are hard to calculate and, what is more troubling, non-perturbative corrections to the 4D scalar potential are not generally well-understood at the 10D level. Other more recent uplifting scenarios can be found in [83][84][85].

The uplifting mechanism of KKLT, i.e. the inclusion of $\overline{D3}$-branes, has been subject to a lot of study. In particular, there has been concerns regarding the anti-brane backreaction on the internal geometry. This critique mainly regards the consistency of the *probe approximation* which is most often used in describing the uplift by $\overline{D3}$-branes. When calculating the uplift energy and perturbative stability of the anti-branes the probe approximation corresponds to assuming that these calculations can be done in a background that is unaffected by the presence of the branes. Without this approximation one would need to consider the branes backreaction on the internal geometry, which is a hard task as there are no known Calabi-Yau metrics. To solve this it has been customary to assume a simpler background with known metric, in many cases the Klebaov-Strassler throat, to study these effects. In this discussion the seminal work was done in [86] and the discussion has been ongoing since.

Further, as first pointed out in [87], there have been concerns regarding the branes backreaction on the 4D moduli fields. That is, how much the moduli will shift after the uplift. The concern is essentially based on the possibility of an interplay between the gaugino condensation which stabilises the moduli and supersymmetry breaking. This has however been defended by proponents of a refined version of the KKLT in [88], and the debate is still ongoing. See also [89][90].

An alternative to the quantum correction approaches is what is called the *classical* de Sitter vacuum models. They are constructions at the two-derivative level of 10D supergravity and do not include any extra effects, the initial hope being to obtain a model that might be simple enough to allow fully explicit 10D solutions. However, trouble was experienced as new no-go theorems arose on top of the classical ones described in chapter 6, which excluded a large class of type II compactifications. Specifically the no-go theorems could only be avoided by adding orientifold planes, including RR fluxes and having a negatively curved internal space. So far de Sitter vacuum constructions have had tachyonic directions [91], but progress has been made.

Another class of de Sitter vacuum constructions are the ones that include *non-geometric* fluxes, i.e. fluxes whose presence make the internal metric globally ill-defined. These are the subject of the next section. Recent work on this theme include [92][93][94], however their uplift to string theory and consistent use in 4D supergravity has not been sufficiently understood so far. Out of the models which obtained a metastable de Sitter vacuum, none of them seemed to be locally geometric and hence lack a geometrical interpretation [95]. A more thorough review on the problems of de Sitter vacua constructions can be found in [95] and references therein.

The very existence of de Sitter vacua has also been questioned in [95], and recently it was suggested in [96] that all de Sitter spacetimes belong in the swampland. The swampland is the set of phenomenological models which cannot be derived as a low-energy effective theory of a quantum gravity, or as the set of apparently consistent effective field theories that cannot be completed into quantum gravity in the ultraviolet regime. One might think that the de Sitter conjecture stating that the de Sitter vacua are in the Swampland is ruled out by observation, since there is observational evidence of the universe experiencing acceleration most likely due to a positive cosmological constant. However, the inflation era of the early universe can be largely explained based on a scalar field in rolling down a potential, and so there may be a possibility of a similar mechanism describing the current phase of acceleration. The most well-known such model is known as *quintessence* describing a type of dynamical dark energy, see [97] for a review. The Swampland de Sitter conjecture has spurred new action in the field, where both refined conjectures of this kind have been proposed as well as counter examples have been researched. See [98] for a review of the Swampland.

9

Non-Geometric Fluxes and Double Field Theory

All fluxes previously considered have been defined on a manifold, but there is nothing that says that the target space in a compactification needs to have a conventional geometric description. In this sense non-geometric flux compactifications can mean all string compactifications except a set with a well-defined geometric background. In this chapter we will set the stage by considering how T-duality acts on a classically geometric background with H_3 flux.

9.1 Non-geometric fluxes on the twisted torus

In this section we introduce the notion of non-geometric fluxes using the canonical example of T-duality on a torus with NSNS flux. This knowledge can then be used in formulating a common superpotential of the effective field theory after compactification for both type II on a torus.

9.1.1 T-duality on a T^3 with H_3 flux

As is familiar, T-duality relates a string theory on a circle of radius R to string theory on a circle of radius α'/R. If we consider say a bosonic string on a circle, parametrised by θ, but with a more general metric, the action can be written

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z \partial\theta\bar{\partial}\theta g_{\theta\theta} \ . \tag{9.1}$$

If one introduce a Lagrange multiplier $\tilde{\theta}$ we may write the action as

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z [g_{\theta\theta}\partial\theta\bar{\partial}\theta + \tilde{\theta}(\partial(\bar{\partial}\theta) - \bar{\partial}(\partial\theta))] \ , \tag{9.2}$$

with $\mathrm{d}^2 z = \mathrm{d}\tau\,\mathrm{d}\sigma$ and where $\partial\mathcal{L}/\partial\tilde{\theta} = \partial(\bar{\partial}\theta) - \bar{\partial}(\partial\theta) = 0$ restores the original action in eq. (9.1). The constraint $\partial\mathcal{L}/\partial(\bar{\partial}\theta) = 0$ results in that $\partial\theta = \partial\tilde{\theta}/G_{\theta\theta}$, which gives the action

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z \partial\tilde{\theta}\bar{\partial}\tilde{\theta}\frac{1}{g_{\theta\theta}} \ , \tag{9.3}$$

and it becomes clear that two conformal field theories with target space metrics ($g_{\theta\theta}$ respectively $1/g_{\theta\theta}$) are classically equivalent. Here we used that the metric is independent of θ, i.e. that there is an isometry in this coordinate, since we assumed the string to be on a circle.

This procedure can be repeated when including a B field and dilaton, which was first done by Busher [99]. T-dualising in say the x-direction the resulting *Busher rules* transform the fields

$g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ into new ones according to

$$\begin{aligned} g_{xx} \to \frac{1}{g_{xx}} \,, \qquad & g_{x\mu} \to -\frac{B_{x\mu}}{g_{xx}} \,, \qquad B_{x\mu} \to -\frac{g_{x\mu}}{g_{xx}} \\ g_{\mu\nu} \to g_{\mu\nu} - \frac{g_{x\mu}g_{x\nu} - B_{x\mu}B_{x\nu}}{g_{xx}} \,, \qquad & B_{\mu\nu} \to B_{\mu\nu} - \frac{g_{x\mu}B_{x\nu} - B_{x\mu}g_{x\nu}}{g_{xx}} \,, \\ e^{\phi} \to \frac{e^{\phi}}{\sqrt{g_{xx}}} \,. & \end{aligned} \tag{9.4}$$

The T-duality rule for the dilaton can not be calculated by the above described procedure, but requires a 1-loop calculation. Note again that the procedure producing the Busher rules requires that there is an isometry in the x-direction; otherwise they are not valid. First consider a twisted simple torus in 3D with coordinates (x, y, z). Its metric is given by

$$\mathrm{d}s^2 = (\mathrm{d}x - f^x_{yz} z \,\mathrm{d}y)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{9.5}$$

This space can be compactified by making the identifications $x \sim x+1$ and $y \sim y+1$, which does not change eq. (9.5). The same can not be said about the z-coordinate; identifying $z \sim z+1$ makes the metric ill-defined. To compensate for this we can shift the x-coordinate by $f^x_{yz}\,\mathrm{d}y$ in combination with the shift in z. Hence, this space may be compactified with the identifications

$$(x, y, z) \sim (x+1, y, z) \sim (x, y+1, z) \sim (x + f^x_{yz} y, y, z+1) \,, \tag{9.6}$$

which keep the metric globally well-defined. The metric is topologically distinct from a T^3; it is a twisted torus. It can be viewed as a T^2 parametrised by (x, y) which is fibered over an S^1 base in the z-direction. Circling the S^1 the T^2 fiber shifts its complex structure as $\upsilon \to \upsilon + f^x_{yz}$. After one lap around S^1 the fiber should be the same, and hence the shift should be an $SL(2, \mathbb{Z})$ transformation so we must have $f^x_{yz} \in \mathbb{Z}$. A good way to think about the f^x_{yz} is by first defining globally invariant 1-forms such that

$$\eta_x \equiv \mathrm{d}x - f^x_{yz}\,\mathrm{d}y \,, \qquad \eta^y \equiv \mathrm{d}y \,, \qquad \eta^z \equiv \mathrm{d}z \,, \tag{9.7}$$

where obviously $\mathrm{d}\eta^y = \mathrm{d}\eta^z = 0$, but for the x-form we have that

$$\mathrm{d}\eta^x = f^x_{yz}\,\mathrm{d}y \wedge \mathrm{d}z = f^x_{yz}\eta^y \wedge \eta^z \,. \tag{9.8}$$

We see a clear similarity with Cartan's structure equations, to which the f^x_{yz} are components of the spin connection. If the manifold is a Lie group, then the f^x_{yz} are structure constants. This is easy to generalise to any manifold with globally defined 1-forms e^a, such that

$$\mathrm{d}e^a = f^a_{bc} e^b \wedge e^c \,, \tag{9.9}$$

where all f^a_{bc} being constants puts a non-trivial constraint on the manifold. Another constraint comes from $\mathrm{d}^2 = 0$, i.e.

$$\mathrm{d}^2 e^a = 2 f^a_{b[c} f^b_{dg]} e^d e^g e^c = 0 \,, \tag{9.10}$$

which is fulfilled when $f^a_{b[c} f^b_{dg]} = 0$. This is a Jacobi identity which indeed all structure constants of a Lie algebra should satisfy.

We are now ready for our example of the T^3 with H_3 flux. The Busher rules may be applied to this setup, which was first illustrated in [100]. To start, we declare the coordinates of the T^3 as $(x, y, z) \sim (x+1, y+1, z+1)$ and introduce N units of H_3-flux on the torus;

$$\int_{T^3} H_3 = N \,, \tag{9.11}$$

where $N \in \mathbb{Z}$. We may choose the gauge $B_{xy} = Nz$. The T^3 space can also be viewed as a T^2 in the (x, y) directions that is fibered over an S^1 in the z-direction. When $z \to z + 1$ the Kähler moduli T of the T^2 will then go as $T \to T + N$.

With our chosen gauge for the B field it is clear that nothing depends on the (x, y)-directions, hence we are free to T-dualise in those directions. The metric of the T^3 is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^3$. T-dualising the x-direction, we have that given $B_{xy} = Nz$, the Busher rules in eq. (9.4) gives $g_{xy} = -Nz$, $g_{yy} = (1 + (Nz)^2)\,\mathrm{d}y$ and all other diagonal elements unity. Hence the metric becomes

$$\mathrm{d}s^2 = (\mathrm{d}x - Nz\,\mathrm{d}y)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ , \tag{9.12}$$

and the B field vanishes. This metric is that of eq. (9.5) with $f^x_{yz} = N$, i.e. a twisted torus. The coordinate identification for the metric to be well-defined is $(x, y, z) \sim (x + Ny, y, z + 1)$. As with the twisted torus in IIA, encircling the S^1 base the T^2 fiber makes a shift in the complex structure $\tau \to \tau + N$, and thus the T-duality transformation has switched the Kähler modulus to a complex structure modulus. Conclusively we denote this T-duality transformation in the x-direction by the transformation

$$H_{xyz} \xrightarrow{T_x} f^x_{yz} \ , \tag{9.13}$$

and we may now proceed to T-dualise in the y-direction. The Busher rules on the metric of eq. (9.12) and $B = 0$ give us

$$\mathrm{d}s^2 = \frac{1}{1 + N^2 z^2}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2 \ , \tag{9.14}$$

and $B_{xy} = \frac{Nz}{1+(Nz)^2}$. There is no apparent coordinate identification that makes $z \sim z + 1$ well-defined on the metric. However, it is clear from the metric that the Kähler modulus has transformed to $T = \frac{1}{Nz-i}$, and so the identification $z \to z+1$ then takes it to $\frac{1}{T} \to \frac{1}{T} + N$. Since this is an $SL(2, \mathbb{Z})$ transformation it corresponds to the T^2 Kähler moduli changing as the z goes around the circle base space. Since the new Kähler modulus now includes flux, it is no longer a geometric quantity. We have seen that while the metric and flux are defined at every point, running around the z cycle mixes the metric and H flux through an $SL(2, \mathbb{Z})$ transformation. Extending the T-duality rule of eq. (9.13) to include the T-dualisation of the y-direction, we write

$$H_{xyz} \xrightarrow{T_x} f^x_{yz} \xrightarrow{T_y} Q^{xy}_z \ , \tag{9.15}$$

where Q^{xy}_z denotes the new non-geometric quantity. Since we have no isometry in the z-direction, we cannot perform another T-dualisation using the Busher rules. There is however reason to believe that this T-duality exists in some kind of sense, as we will motivate in the next section. Denoting this quantity by R^{xyz}, one completes the so-called *T-duality chain* for non-geometric fluxes;

$$H_{xyz} \xrightarrow{T_x} f^x_{yz} \xrightarrow{T_y} Q^{xy}_z \xrightarrow{T_z} R^{xyz} \ , \tag{9.16}$$

which is an important result. This will be used in the following two sections where it will be applied on two different type II tori.

9.1.2 Superpotentials of IIB on tori and IIA on twisted tori

To establish some notation we recall the moduli stabilisation of type IIB theory on the $T^6/\mathbb{Z}_2$ orientifold analysed in chapter 7. The superpotential as well as the included fluxes will be of primary interest. The IIB superpotential on this orientifold is given in eq. (7.32), and is for a diagonal torus $T^6 = T^2 \times T^2 \times T^2$ equal to

$$\begin{aligned} W_{\mathrm{IIB}} &= (m^0_{\mathrm{RR}}\upsilon^3 - 3m_{\mathrm{RR}}\upsilon^2 - 3e_{\mathrm{RR}}\upsilon - e_{\mathrm{RR}0}) + \tau(-m^0\upsilon^3 + 3m\upsilon^2 + 3e\upsilon + e_0) \\ &\equiv S_1(\upsilon) + \tau S_2(\upsilon) \ , \end{aligned} \tag{9.17}$$

flux integral	$\bar{F}_{\alpha\beta\gamma}$	$\bar{F}_{i\beta\gamma}$	$\bar{F}_{ij\gamma}$	$\bar{F}_{ijk}$	$\bar{H}_{\alpha\beta\gamma}$	$\bar{H}_{i\beta\gamma}$	$\bar{H}_{ij\gamma}$	$\bar{H}_{ijk}$
flux integer	m^0_{RR}	m_{RR}	e_{RR}	$e_{\mathrm{RR}0}$	m^0	m	e	e_0

Table 9.1: Fluxes in the IIB $T^6/\mathbb{Z}_2$ superpotential of eq. (9.17).

flux integral	F_0	$\bar{F}_{\alpha i}$	$\bar{F}_{\alpha i\beta j}$	$\bar{F}_{\alpha i\beta j\gamma k}$	f^{α}_{jk}	$\bar{H}_{ijk}$	$\bar{H}_{\alpha\beta k}$	$f^{k}_{j\alpha}$	$f^{i}_{\beta k}$	$f^{\alpha}_{\beta k}$
flux integer	m^0_{RR}	m_{RR}	e_{RR}	$e_{\mathrm{RR}0}$	e	e_0	e'_0	$\check{e}_1$	$\hat{e}_1$	$\tilde{e}_1$

Table 9.2: Type IIA fluxes in the twisted $T^6/\mathbb{Z}_2$ superpotential of eq. (9.20).

where τ is the axion-dilaton, υ is the complex structure modulus and the coordinates on each torus $i = 1, 2, 3$ have been defined as $z^i = x^i + \upsilon\delta_{ij}y^j$. The integer flux is as defined in eq. (5.114), the subscript RR corresponding to the flux of F_3 and the ones without subscript to H_3. In section 9.1.3 we will see that the superpotential constructed which is invariant under T-duality also incorporates the potential in eq. (9.17).

It will later prove useful to indicate which cycles on the T^6 give the different flux integers, and hence we may redefine the z^1, z^2, z^3 coordinates each parameterising a T^2 according to

$$z^1 \equiv x^i + \upsilon x^\alpha \ , \qquad z^2 \equiv x^j + \upsilon x^\beta \ , \qquad z^3 \equiv x^k + \upsilon x^\gamma \ , \tag{9.18}$$

with Latin and Greek indices, as in [101][102]. With this notation the integer fluxes in eq. (9.17) correspond to the 3-form fluxes trough certain 3-cycles spanned by the coordinates $x^{i,j,k}$ or $x^{\alpha,\beta,\gamma}$. For example the m_{RR} flux is

$$\bar{F}_{i\beta\gamma} \equiv \int_{\Sigma_{i\beta\gamma}} F_3 = m_{\mathrm{RR}} \ , \tag{9.19}$$

where $\Sigma_{i\beta\gamma}$ is a 3-cycle spanned by the coordinates x^i, x^β and x^γ in the six-torus. The integral fluxes with the corresponding integer fluxes in eq. (9.17) in this notation is gathered in table 9.1. As the T^2s are identical it is only necessary to write one combination of Latin and Greek indices, e.g. the integer flux through the $i\beta\gamma$ cycle is equivalent to the flux through the $j\alpha\gamma$ or $k\alpha\beta$ cycle. As usual the fluxes satisfy the tadpole condition of eq. (6.30).

Proceeding to the IIA case, we consider again the $T^6/\mathbb{Z}_2$, but twist it by adding some of the f^a_{bc} to the orientifold. This adds a rigid structure to the background metric and makes it different from the diagonal form $T^2 \times T^2 \times T^2$. The f^a_{bc}s that survive the orientifold projection are $f^\alpha_{jk}, f^i_{\beta k}, f^i_{j\gamma}$ and $f^\alpha_{\beta\gamma}$, i.e. the ones with an odd number of Greek indices. To find the superpotential for this setup, the same analysis as for the IIB case of chapter 7.2 has been carried out in [103][104], who obtain the superpotential for the type IIA twisted torus as

$$W_{\mathrm{IIA}} = (m^0_{\mathrm{RR}}\upsilon^3 - 3m_{\mathrm{RR}}\upsilon^2 - 3e_{\mathrm{RR}}\upsilon - e_{\mathrm{RR}0}) + \tau(3e\upsilon + e_0) + 3T(e'_0 + \upsilon(\check{e}_1 + \hat{e}_1 + \tilde{e}_1)) \ , \tag{9.20}$$

with T denoting the Kähler moduli. Using the notation of eq. (9.19), the corresponding flux cycle to the flux integers given in eq. (9.20) are given in table 9.2. These fluxes also need to satisfy some constraints. For instance, in the IIB case $\mathrm{d}H = 0$ was satisfied, however this will not be satisfied in this case as the fluxes are written in a basis of 1-forms such that $F_2 = \bar{F}_{ab}e^a \wedge e^b$, etc, where the 1-forms are not closed by eq. (9.9). Hence this condition becomes non-trivial and gives the constraint $\bar{H}_{a[bc}f^a_{dg]} = 0$ after integration over the (a, b, c, d, g)-cycle. In addition we have seen that the f^a_{bc}s must satisfy the Jacobi identity $f^a_{b[c}f^b_{dg]} = 0$. For the RR sector the Bianchi identities are of the form $(\mathrm{d} + H) \wedge F_{\mathrm{A}} = 0$, F_{A} being any of the IIA RR fluxes. The terms $\mathrm{d}F_2$ and $\mathrm{d}F_4$ results in non-trivial constraints, which after integration over the relevant cycles take the form

$$\begin{aligned} \bar{F}_{a[b}f^{a}_{cd]} + F_0\bar{H}_{bcd} &= 0 \ , \\ \bar{F}_{x[abc}f^{x}_{dg]} + \bar{F}_{ab}\bar{H}_{cdg]} &= 0 \ . \end{aligned} \tag{9.21}$$

Again supersymmetric vacua are found by imposing fluxes which satisfy the above constraints of the two NSNS and two RR fluxes, and then solving for $D_iW = 0$ for all moduli $i = \{\tau, \upsilon, T\}$.

9.1.3 A T-duality invariant superpotential

The relation between the IIA twisted tori and the IIB tori can be seen by T-dualising one leg in each of the T^2, in all of the Greek indices. This chain of three T-dualities will take the IIB O3-planes into IIA O6-planes. The T-duality rule in question for the RR fluxes is given by

$$\bar{F}_{x\alpha_1\ldots\alpha_p} \xleftrightarrow{T_x} \bar{F}_{\alpha_1\ldots\alpha_p} \ . \tag{9.22}$$

While there are generally other moduli-dependent terms in the T-duality rule [105], we choose to focus only on the topological part of the fluxes.

A superpotential which incorporates both superpotentials of eq. (9.17) and eq. (9.20) and is invariant under T-duality was first constructed in [102]. To do so we use these superpotentials and relate them via coordinate symmetry and T-duality. Following their reasoning, we start by taking the IIA theory on a twisted torus through the following steps. First, performing a T-dualisation on the three Greek indices, we arrive at IIB with O3-planes. Next, as the O3-planes in IIB are parallel to the compact space, i.e. they do not extend in any of the directions of the T^6, there is no geometrical distinction between the Greek and Latin indices. This may be interpreted as a rotational symmetry in these indices, and so we are allowed to rotate according to

$$x^\alpha + \upsilon x^i \longleftrightarrow x^i + \upsilon x^\alpha \ , \tag{9.23}$$

effectively exchanging the Latin and Greek indices. In the superpotential this amounts to the exchanges $1 \leftrightarrow \upsilon^3$ and $\upsilon \leftrightarrow \upsilon^2$. Finally, we may T-dualise the Greek directions again to get back to type IIA. From this one obtains the flux integer which the flux cycles of both type IIA and IIB correspond to, as is summarised in table (9.3). A superpotential for the symmetric torus in both the IIA and IIB theories is then given by

$$\begin{aligned} W = &(m^0_{\mathrm{RR}}\upsilon^3 - 3m_{\mathrm{RR}}\upsilon^2 - 3e_{\mathrm{RR}}\upsilon - e_{\mathrm{RR}0}) + \tau(-m^0\upsilon^3 + 3m\upsilon^2 + 3e\upsilon + e_0) \\ &+ 3T(\upsilon^3 e_3' + \upsilon^2(\check{e}_2 + \hat{e}_2 + \tilde{e}_2) - \upsilon(\check{e}_1 + \hat{e}_1 + \tilde{e}_1) - e_0') \ . \end{aligned} \tag{9.24}$$

As usual all fluxes in table 9.3 are antisymmetric in both upper and lower indices. The RR fluxes in type IIA which survive the orientifold projection have legs in both dimensions of each T^2, and hence a mixed pair of Greek and Latin indices. The fact that f and R fluxes do not appear in type IIB is due to that all the dimensions of the T^6 are odd under the orientifold projection, so the f and R would require an even number of odd indices to survive. While this is impossible on this geometry, it is still possible to have f and R fluxes in more general orientifolds. Conclusively, starting from the IIA superpotential on a twisted torus and using T-duality and rotational symmetry, the superpotential in eq. (9.24) could be obtained. As a final remark we briefly comment the constraints these fluxes must satisfy from the Bianchi identities. For the NSNS sector we have that $\mathrm{d}H = 0$ implies that in a geometric compactification we have

$$f^{x}_{[ab}\bar{H}_{cd]x} = 0 \ . \tag{9.25}$$

By simply T-dualising this equation, and T-dualising the obtained one and pursuing in this fashion, one obtains a set of Bianchi identities for each flux. They are given by

$$\begin{aligned} \bar{H}_{x[ab}f^c_{cd]} &= 0 \ , \\ f^a_{x[b}f^x_{cd]} + \bar{H}_{x[bc}Q^{ax}_{d]} &= 0 \ , \\ Q^{[ab]}_x f^x_{[cd} - 4f^{[a}_{x[c}Q^{b]x}_{d]} + \bar{H}_{x[cd]}R^{[ab]x} &= 0 \ , \\ Q^{[ab}_x Q^{c]x}_d + f^{[a}_{xd}R^{bc]x} &= 0 \ , \\ Q^{[ab}_x R^{cd]x} &= 0 \ , \end{aligned} \tag{9.26}$$

and since the f and Q fluxes should be T-dual to the H flux one further requires that

$$f^x_{xa} = Q^{ax}_x = 0 \ . \tag{9.27}$$

As for the RR fluxes, in the absence of localised sources the tadpole/Bianchi condition is given by $(\mathrm{d} + H)F = 0$, or

$$\mathrm{d}\tilde{F}_5 = \bar{F}_{[abc}\bar{H}_{def]} = 0 \ . \tag{9.28}$$

T-dualising in the same manner as before, various versions of this Bianchi identity containing each type of T-dual flux is obtained as

$$\begin{aligned} \bar{F}_{[abc}\bar{H}_{def]} &= 0 \ , \\ \bar{F}_{x[abc}f^x_{de]} - \bar{F}_{[ab}\bar{H}_{cde]} &= 0 \ , \\ \bar{F}_{xy[abc}Q^{xy}_{d]} - 3\bar{F}_{x[ab}f^x_{cd]} - 2\bar{F}_{[a}\bar{H}_{bcd]} &= 0 \ , \\ \bar{F}_{xyz[abc]}R^{xyz} - 9\bar{F}_{xy[ab}Q^{xy}_{c]} - 18\bar{F}_{x[a}f^x_{bc]} + 6F_0\bar{H}_{[abc]} &= 0 \ , \\ \bar{F}_{xyz[ab]}R^{xyz} + 6\bar{F}_{xy[a}Q^{xy}_{b]} - 6\bar{F}_x f^x_{[ab]} &= 0 \ , \\ \bar{F}_{xyza}R^{xyz} - 3\bar{F}_{xy}Q^{xy}_a &= 0 \ , \\ \bar{F}_{xyz}R^{xyz} &= 0 \ . \end{aligned} \tag{9.29}$$

In the presence of a localised source the right-hand side would be non-zero. In the next chapter we will see how the non-geometrical fluxes can be treated on an equal footing with the geometrical fluxes.

9.2 Double field theory

In order to naturally incorporate non-geometric fluxes, there was a development of the so-called double formalisms with the goal of finding a covariant description of geometric and non-geometric backgrounds. The two leading formalisms that aim to incorporate T-duality in this manner are generalised geometry and double field theory (DFT). In general T-duality relates winding modes and momentum of a closed string moving on some torus T^D via the group $O(D,D)$, which for this reason is referred to as the T-duality group. DFT assembles the NSNS sector of supergravity into a T-duality invariant formalism. In constructing the DFT it is central to rewrite the symmetries of the bosonic supergravity, i.e. diffeomorphisms and gauge transformations, in an $O(D,D)$-covariant way. Again, the diffeomorphisms, or change of coordinates, are parametrised by infinitesimal vectors λ^i with $i = 1, \ldots, D$, which transforms the NSNS field content according to

$$\begin{aligned} g_{ij} &\to g_{ij} + \mathcal{L}_\lambda g_{ij} \ , & \mathcal{L}_\lambda g_{ij} &\equiv \lambda^k\partial_k g_{ij} + g_{kj}\partial_i\lambda^k + g_{ik}\partial_j\lambda^k \ , \\ B_{ij} &\to B_{ij} + \mathcal{L}_\lambda B_{ij} \ , & \mathcal{L}_\lambda B_{ij} &\equiv \lambda^k\partial_k B_{ij} + B_{kj}\partial_i\lambda^k + B_{ik}\partial_j\lambda^k \ , \\ \phi &\to \phi + \mathcal{L}_\lambda\phi \ , & \mathcal{L}_\lambda\phi &\equiv \lambda^i\partial_i\phi \ . \end{aligned} \tag{9.30}$$

IIA flux integral	IIB flux integral	flux integer
F_0	$\bar{F}_{\alpha\beta\gamma}$	m^0_{RR}
$\bar{F}_{\alpha i}$	$\bar{F}_{i\beta\gamma}$	m_{RR}
$\bar{F}_{\alpha i\beta j}$	$\bar{F}_{ij\gamma}$	e_{RR}
$\bar{F}_{\alpha i\beta j\gamma k}$	$\bar{F}_{ijk}$	$e_{\mathrm{RR}0}$
$R^{\alpha\beta\gamma}$	$\bar{H}_{\alpha\beta\gamma}$	m^0
$Q^{\alpha\beta}_k$	$\bar{H}_{i\beta\gamma}$	m
f^{α}_{jk}	$\bar{H}_{\alpha jk}$	e
$\bar{H}_{ijk}$	$\bar{H}_{ijk}$	e_0
$\bar{H}_{\alpha\beta k}$	$Q^{\alpha\beta}_k$	e'_0
$f^j_{k\alpha}, f^i_{\beta k}, f^{\alpha}_{\beta\gamma}$	$Q^{\alpha j}_k, Q^{i\beta}_k, Q^{\beta\gamma}_{\alpha}$	$\check{e}_1, \hat{e}_1, \tilde{e}_1$
$Q^{\gamma i}_{\beta}, Q^{i\beta}_{\gamma}, Q^{ij}_k$	$Q^{i\beta}_{\gamma}, Q^{\gamma i}_{\beta}, Q^{ij}_k$	$\check{e}_2, \hat{e}_2, \tilde{e}_2$
$R^{ij\gamma}$	Q^{ij}_{γ}	e'_3

Table 9.3: Occuring fluxes in the T-duality invariant superpotential of eq. (9.24).

where the action of the Lie derivative amounts to diffeomorphism transformations, which leave the NSNS action in eq. (3.98) invariant. This means that the physics described remains unchanged under these coordinate changes. Further, the B field enjoys the gauge symmetry

$$B_{ij} \to B_{ij} + \partial_i \tilde{\lambda}_j - \partial_i \tilde{\lambda}_i \ , \tag{9.31}$$

parametrised by the infinitesimal 1-form $\tilde{\lambda}_i$. For a general vector we have

$$\mathcal{L}_\lambda V^\mu = \lambda^j \partial_j V^i - V^j \partial_j \lambda^i \ . \tag{9.32}$$

9.2.1 The double field theory action and its symmetries

T-duality is turned into a manifest symmetry by doubling the coordinates at the level of the the effective 4D action of a string theory. In DFT every conventional coordinate x^i associated to the momentum modes is paired with the T-dual coordinate $\tilde{x}_i$ associated to the winding modes, combining the coordinates to a $O(D,D)$ vector $X^M = (\tilde{x}_i, x^i)$ of dimension 2D. The index M is raised and lowered using the $O(D,D)$-invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix} \ , \tag{9.33}$$

and the partial derivative is given by $\partial^M = (\partial_i, \tilde{\partial}^i)$. An $O(D,D)$ coordinate transform is $X^M \to h^M{}_N X^N$ with $h^M{}_N \in O(D,D)$, hence mixing the coordinates x^i and $\tilde{x}_i$.

While there are several formulations of DFT, the earliest variant constructed in [106] used the combined field $\mathcal{E}_{ij} = g_{ij} + B_{ij}$ with background independent form found to be

$$\begin{aligned} S = \int \mathrm{d}x\, \mathrm{d}\tilde{x} e^{-2\phi'} \Big[& -\tfrac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl} \mathcal{D}_p \mathcal{E}_{ij} - \tfrac{1}{4} (\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki} \bar{\mathcal{D}}^i \mathcal{E}_{lj}) \\ & + \mathcal{D}^i \phi' \bar{\mathcal{D}}^j \mathcal{E}_{ij} - \bar{\mathcal{D}}^i \phi' \mathcal{D}^j \mathcal{E}_{ji} + 4 \mathcal{D}^i \phi' \mathcal{D}_i \phi' \Big] \ , \end{aligned} \tag{9.34}$$

with covariant derivatives $\mathcal{D}_i \equiv \frac{\partial}{\partial x^i} - \mathcal{E}_{ij} \frac{\partial}{\partial \tilde{x}_j}$ and $\bar{\mathcal{D}}_i \equiv \frac{\partial}{\partial x^i} + \mathcal{E}_{ij} \frac{\partial}{\partial \tilde{x}_j}$. The action is $O(D,D)$-invariant and the metric g_{ij} is responsible for raising and lowering indices. This action is gauge invariant under transformations

$$\delta_\zeta \mathcal{E}_{ij} = \mathcal{D}_i \tilde{\zeta}_j - \bar{\mathcal{D}}_j \tilde{\zeta}_j + \zeta^M \partial_M \mathcal{E}_{ij} + \mathcal{D}_i \zeta^k \mathcal{E}_{kj} + \bar{\mathcal{D}}_j \zeta^k \mathcal{E}_{ik} \ , \quad \delta_\zeta \phi' = -\tfrac{1}{2} \partial_M \zeta^M + \zeta^M \partial_M \phi' \ , \tag{9.35}$$

with gauge parameters $\zeta^M = (\zeta^i, \tilde{\zeta}_i)$. In order to make gauge invariance more apparent, another action was constructed in [107] consisting only of objects transforming in linear representations, as opposed to eq. (9.34). The fundamental objects consist of the generalised metric $\mathcal{H}_{MN}$ which combines the graviton and B field according to

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{ij} - B_{ik}g^{kl}B_{lj} & B_{ik}g^{kj} \\ -g^{ik}B_{kj} & g^{ij} \end{pmatrix} , \tag{9.36}$$

and the so-called *dilaton density*

$$e^{-2\phi'} \equiv \sqrt{-g}e^{-2\phi} , \tag{9.37}$$

which combines the metric and dilaton. Then the $O(D,D)$-invariant action is written

$$S = \int \mathrm{d}^{2D}X e^{-2\phi'}\mathcal{R} , \tag{9.38}$$

consisting only of covariant quantities, where the generalised Ricci scalar is given by

$$\begin{aligned} \mathcal{R} = 4\mathcal{H}^{MN}\partial_M\phi'\partial_N\phi' - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M\phi'\partial_N\phi' + 4\partial_M\mathcal{H}^{MN}\partial_N\phi' \\ +\tfrac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \tfrac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK} . \end{aligned} \tag{9.39}$$

Note that when the fields in eq. (9.34) or eq. (9.38) are independent of the dual coordinates $\tilde{x}^i$ the DFT action reduces to the NSNS supergravity action of eq. (3.98), which we will show in section 9.2.2. The generalised metric in combination with the $O(D,D)$-invariant metric fulfils the relations

$$\mathcal{H}^{MN}\eta_{ML}\mathcal{H}^{LK} = \eta^{NK} , \qquad \mathcal{H}^{MN} = \mathcal{H}^{NM} . \tag{9.40}$$

As for the $O(d,d)$ transformations, we have that $\mathcal{H}_{MN}(X^K) \to h_M{}^P h_N{}^Q \mathcal{H}_{PQ}(h^K{}_L X^L)$. For instance if h would correspond to a T-duality in some isometry direction, then the $O(D,D)$ transformation of $\mathcal{H}_{MN}(X^K)$ result in the Busher rules of eq. (9.4). Further, the level matching condition of eq. (5.83) implies that the metric of eq. (9.33) should satisfy the constraint

$$\eta^{MN}\partial_M\partial_N(A) = 0 , \tag{9.41}$$

where A is some arbitrary field. This is known as the *weak constraint* of DFT.

We now turn to the diffeomorphisms of DFT. From eq. (9.30) and eq. (9.31) we know that the diffeomorphisms of the field content are generated by a vector λ^i and the 2-form gauge transformations are generated by a 1-form $\tilde{\lambda}_i$. Having combined the metric and B field into the generalised metric $\mathcal{H}$, it is then natural to combine the gauge parameters into a generalised one; the $O(D,D)$ vector $\xi^M = (\tilde{\lambda}_i, \lambda^i)$ of generalised gauge parameters. Using this, one can construct the *generalised Lie derivative*

$$\hat{\mathcal{L}}_\xi A_M \equiv \xi^N\partial_N A_M + (\partial_M\xi^N - \partial^N\xi_M)A_N , \tag{9.42}$$

acting on arbitrary field A^M. It will then act on the generalised metric and dilaton according to

$$\begin{aligned} &\hat{\mathcal{L}}_\xi\mathcal{H}_{MN} = \xi^P\partial_P\mathcal{H}_{MN} + (\partial_M\xi^P - \partial^P\xi_M)\mathcal{H}_{PN} + (\partial_N\xi^P - \partial^P\xi_N)\mathcal{H}_{MP} , \\ &\hat{\mathcal{L}}_\xi(e^{-2\phi'}) = \partial_M(\xi^M e^{-2\phi'}) . \end{aligned} \tag{9.43}$$

From this it is clear that the generalised Lie derivative acting on the invariant metric vanishes, as

$$\hat{\mathcal{L}}\eta_{MN} = \xi^P\partial_P\eta_{MN} + (\partial_M\xi^P - \partial^P\xi_M)\eta_{PN} + (\partial_N\xi^P - \partial^P\xi_N)\eta_{MP} = 0 , \tag{9.44}$$

and hence we have that $\hat{\mathcal{L}}_\xi A^M = \eta^{MN}\hat{\mathcal{L}}_\xi A_N$. In order for the symmetry group to close, two consecutive transformations with parameters ξ_1 and ξ_2 should equal another transformation with parameter $\xi_3 \equiv \xi_3(\xi_1, \xi_2)$. To this end we may use the commutator of two gauge transformations and investigate whether

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}]A^M = \hat{\mathcal{L}}_{\xi_3}A^M \ . \tag{9.45}$$

By plugging in eq. (9.42), the commutator results in that

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}]A^M = \hat{\mathcal{L}}_{[\xi_1,\xi_2]_{\mathrm{C}}}A^M + F_M(\xi_1, \xi_2, A) \ , \tag{9.46}$$

where the transformation parameter is

$$\xi_3^M \equiv [\xi_1, \xi_2]_{\mathrm{C}}^M \equiv \xi_1^N \partial_N \xi_2^M - \tfrac{1}{2}\xi_{1N}\partial^M \xi_2^N - (\xi_2^N \partial_N \xi_1^M - \tfrac{1}{2}\xi_{2N}\partial^M \xi_1^N) \ . \tag{9.47}$$

The C-bracket $[\cdot,\cdot]_{\mathrm{C}}$ is the $O(D,D)$-covariant extension of the Courant bracket for doubled fields. The extra term F_M is given by

$$F_M = \tfrac{1}{2}\xi_1^N \partial^P \xi_{2N}\partial_P A^M - \tfrac{1}{2}\xi_2^N \partial^P \xi_{1N}\partial_P A^M - (\partial_N \xi_1^M \partial^N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P})A^P \ . \tag{9.48}$$

An explicit calculation of this can be found in Appendix E. For the algebra to close we must have $F_M = 0$, which implies what is known as the *strong constraint* of DFT;

$$\eta^{MN}\partial_M A \partial_N B = 0 \ , \tag{9.49}$$

for any fields A, B. This is a stronger form of the weak constraint in eq. (9.41).

The strong constraint is solved trivially by setting $\tilde{\partial}^i = 0$, which then recovers the supergravity frame. In this case the generalised Lie derivative of the *component* $\mathcal{H}^{ij}$ in eq. (9.43) reduces to

$$\hat{\mathcal{L}}_\xi \mathcal{H}^{ij} = \lambda^k \partial_k \mathcal{H}^{ij} - \partial_k \lambda^i \mathcal{H}^{kj} - \partial_k \lambda^j \mathcal{H}^{ik} = \mathcal{L}_\lambda g^{ij} \ , \tag{9.50}$$

i.e. the ordinary Lie derivative of g^{ij} along the vector field λ^i. In the same manner the generalised Lie derivative of the component $\mathcal{H}_{ij}$ becomes

$$\hat{\mathcal{L}}_\xi \mathcal{H}_{ij} = \mathcal{L}_\lambda B_{ij} + \partial_i \tilde{\lambda}_j - \partial_j \tilde{\lambda}_i \ , \tag{9.51}$$

when $\tilde{\partial}^i = 0$. Since the action of eq. (9.38) consists only of covariant quantities it posses a manifest global $O(D,D)$ symmetry. An interesting trait of the algebra in eq. (9.47) is that it does not in general satisfy the Jacobi identity. This can be seen by first noting that from the definition of the C-bracket, we have

$$\tfrac{1}{2}(\hat{\mathcal{L}}_{\xi_1}\xi_2^M - \hat{\mathcal{L}}_{\xi_2}\xi_1^M) = [\xi_1, \xi_2]_{\mathrm{C}}^M \ , \qquad \hat{\mathcal{L}}_{\xi_1}\xi_2^M + \hat{\mathcal{L}}_{\xi_2}\xi_1^M) = \partial^M(\xi_1^P \xi_{2P}) \ , \tag{9.52}$$

and so by eq. (9.45) we see that

$$\Big[[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}], \hat{\mathcal{L}}_{\xi_3}\Big] + \text{cycl. perm.} = \hat{\mathcal{L}}_{[\xi_1,\xi_2]_{\mathrm{C}} + \text{cycl. perm.}} \ . \tag{9.53}$$

This implies that the Jacobiator becomes

$$\begin{aligned} \mathrm{Jac}^M(\xi_1, \xi_2, \xi_3) &\equiv [[\xi_1, \xi_2]_{\mathrm{C}}, \xi_3]_{\mathrm{C}}^M + \text{cycl. perm.} \\ &= \tfrac{1}{6}\partial^M \Big([\xi_1, \xi_2]_{\mathrm{C}}^P \xi_3 + \text{cycl. perm.}\Big) \ . \end{aligned} \tag{9.54}$$

Peculiarly, this means that the generalised diffeomorphisms do not form a Lie group. However, since the Jacobiator is proportional to a total derivative, it will not generate a gauge transformation on fields fulfilling the strong constraint in eq. (9.49). Hence it is still consistent with the Jacobi identity for gauge transformations of physical fields, which always fulfil it.

Apart from generalised diffeomorphisms, the DFT action possesses another local symmetry, namely it is invariant under double Lorentz transformations. The invariance under a local double Lorentz transformation is conveniently written using veilbein formalism. The vielbein formalism was first developed in [108] and used in the context of DFT in [109]. The starting point is expressing the generalised metric in terms of the frame fields as

$$\mathcal{H}^{MN} = E^A{}_M N_{AB} E^B{}_N \ , \tag{9.55}$$

where N_{AB} is the flat generalised metric, given by

$$N_{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta_{ab} \end{pmatrix} \ , \tag{9.56}$$

where η_{ab} and inverse η^{ab} is the usual Minkowski metric in D dimensions. The indices $A, B, \ldots$ and $M, N, \ldots$ are flat respectively curved. In terms of the generalised vielbeins the $O(D, D)$-invariant metric of eq. (9.33) is given by

$$\eta_{MN} = E^A{}_M \eta_{AB} E^B{}_N \ , \qquad \text{where} \qquad \eta_{AB} = \begin{pmatrix} 0 & \delta_a^b \\ \delta_b^a & 0 \end{pmatrix} \ , \tag{9.57}$$

the flat metric η_{AB} being equivalent to the η_{MN} in curved indices. Having established veilbein notation, we consider the local double Lorentz transformation of the generalised veilbein, which may be written

$$E^A{}_M \to T^A{}_B E^B{}_M \ . \tag{9.58}$$

For this transformation to leave the generalised metric in eq. (9.55) invariant, the transformation has to satisfy

$$T^A{}_C N^{CD} T^B{}_D = N^{AB} \ . \tag{9.59}$$

Further it also has to fulfil the condition of eq. (9.57) for the $O(D, D)$-invariant metric, which requires that

$$T^A{}_C \eta^{CD} T^B{}_C = \eta^{AB} \ . \tag{9.60}$$

The next step is to find transformations $T^A{}_B$ of eq. (9.58) that fulfil the conditions of eq. (9.59) and eq. (9.60), which the DFT action would then be invariant under. To do so it will prove useful to rewrite the $O(D, D)$-invariant metric on a diagonal form, such that

$$S^{\bar{A}}{}_A \eta^{AB} S^{\bar{B}}{}_B = \eta^{\bar{A}\bar{B}} = \begin{pmatrix} -\eta_{\bar{a}\bar{b}} & 0 \\ 0 & \eta^{\bar{a}\bar{b}} \end{pmatrix} \ , \tag{9.61}$$

where the coordinate transformations S are

$$S^{\bar{A}}{}_B = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \delta_{\bar{a}}^b & -\eta_{\bar{a}b} \\ \eta^{\bar{a}b} & \delta_b^{\bar{a}} \end{pmatrix} \ , \qquad S_{\bar{A}}{}^B = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \delta_b^{\bar{a}} & -\eta^{\bar{a}b} \\ \eta_{\bar{a}b} & \delta_{\bar{a}}^b \end{pmatrix} \ . \tag{9.62}$$

The bared indices differentiate between the different representations of the $O(D, D)$-invariant metric η_{AB}. The bared variant of the flat generalised metric N^{AB} is given in the same manner by

$$S^{\bar{A}}{}_A N^{AB} S^{\bar{B}}{}_B = N^{\bar{A}\bar{B}} = \begin{pmatrix} \eta_{\bar{a}\bar{b}} & 0 \\ 0 & \eta^{\bar{a}\bar{b}} \end{pmatrix} \ . \tag{9.63}$$

With this diagonal form of the $O(D,D)$ metric in eq. (9.57), the constraints of eq. (9.59) and eq. (9.60) can be written simultaneously as

$$\begin{aligned}\begin{pmatrix}\pm\eta_{\bar a\bar b} & 0\\ 0 & \eta^{\bar a\bar b}\end{pmatrix} &= \begin{pmatrix}T_{\bar a}{}^{\bar c} & T_{\bar a\bar c}\\ T^{\bar a\bar c} & T^{\bar a}{}_{\bar c}\end{pmatrix}\begin{pmatrix}\pm\eta_{\bar c\bar d} & 0\\ 0 & \eta^{\bar c\bar d}\end{pmatrix}\begin{pmatrix}T_{\bar b}{}^{\bar d} & T^{\bar b\bar d}\\ T_{\bar b\bar d} & T^{\bar b}{}_{\bar d}\end{pmatrix}\\ &= \begin{pmatrix}\pm T_{\bar a}{}^{\bar c}\eta_{\bar c\bar d}T_{\bar b}{}^{\bar d} + T_{\bar a\bar c}\eta^{\bar c\bar d}T_{\bar b\bar d} & \pm T_{\bar a}{}^{\bar c}\eta_{\bar c\bar d}T^{\bar b\bar d} + T_{\bar a\bar c}\eta^{\bar c\bar d}T^{\bar b}{}_{\bar d}\\ \pm T^{\bar a\bar c}\eta_{\bar c\bar d}T_{\bar b}{}^{\bar d} + T^{\bar a}{}_{\bar c}\eta^{\bar c\bar d}T_{\bar b\bar d} & \pm T^{\bar a\bar c}\eta_{\bar c\bar d}T^{\bar b\bar d} + T^{\bar a}{}_{\bar c}\eta^{\bar c\bar d}T^{\bar b}{}_{\bar d}\end{pmatrix}\ ,\end{aligned} \tag{9.64}$$

the positive sign corresponding to eq. (9.59) and negative to eq. (9.60). This can be solved with

$$T_{\bar a\bar b} = T^{\bar a\bar b} = 0\ , \qquad \pm T_{\bar a}{}^{\bar c}\eta_{\bar c\bar d}T_{\bar b}{}^{\bar d} = \pm\eta_{\bar a\bar b}\ , \qquad T^{\bar a}{}_{\bar c}\eta^{\bar c\bar d}T^{\bar b}{}_{\bar d} = \eta^{\bar a\bar b}\ . \tag{9.65}$$

Note that since there is no metric to raise and lower the indices of the barred coordinate transformations, $T_{\bar a}{}^{\bar b}$ and $T^{\bar a}{}_{\bar b}$ are unrelated objects. Hence before writing the unbarred transformation $T^A{}_B$ we rename them according to $T_{\bar a}{}^{\bar b} = u_{\bar a}{}^{\bar b}$ and $T^{\bar a}{}_{\bar b} = v^{\bar a}{}_{\bar b}$, so that

$$T^A{}_B = S^A{}_{\bar A}T^{\bar A}{}_{\bar B}S^{\bar B}{}_B = \begin{pmatrix}\delta^{\bar a}_a & \eta_{a\bar a}\\ -\eta^{a\bar a} & \delta^a_{\bar a}\end{pmatrix}\begin{pmatrix}u_{\bar a}{}^{\bar b} & 0\\ 0 & v^{\bar a}{}_{\bar b}\end{pmatrix}\begin{pmatrix}\delta^b_{\bar a} & -\eta_{\bar b b}\\ \eta^{\bar b b} & \delta^b_{\bar b}\end{pmatrix} = \begin{pmatrix}u_a{}^b + v_a{}^b & -u_{ab} + v_{ab}\\ -u^{ab} + v^{ab} & u^a{}_b + v^a{}_b\end{pmatrix}\ . \tag{9.66}$$

Conclusively, we have identified a local double Lorentz transformation of the form of eq. (9.58) satisfying eqs. (9.59)–(9.60), which leaves the DFT action invariant.

The vielbein of the generalised metric naturally also combines the metric and B field. As an element of $O(D,D)$ it has $D(2D-1)$ independent degrees of freedom, however if we gauge fix the local double Lorentz transformations there will only remain D^2 independent degrees of freedom. From the antisymmetric B_{ij} field and metric vielbein $e^a{}_i$ where $e_a{}^i\eta_{ab}e^b{}_j = g_{ij}$, a possible parameterisation of the generalised vielbein is given by

$$E^A{}_M = \begin{pmatrix}e_a{}^i & e_a{}^jB_{ji}\\ 0 & e^a{}_i\end{pmatrix}\ . \tag{9.67}$$

If the $e^a{}_i$ has the form of an upper triangular matrix, then the double Lorentz symmetry is completely fixed by this form of $E_A{}^M$. If we do not gauge fix the double Lorentz symmetry, a general $O(D,D)$ vielbein can be written as

$$E_A{}^M = \begin{pmatrix}e_a{}^i & e_a{}^jB_{ji}\\ e^a{}_j\beta^{ji}e^a{}_i & e^a{}_j\beta^{jk}B_{ki}\end{pmatrix}\ , \tag{9.68}$$

where $e^a{}_i$ can take any form and β^{ij} is an antisymmetric bi-vector. Under generalised diffeomorphisms the generalised vielbein will transform according to

$$\mathcal{L}_\xi E^A{}_M = \xi^P\partial_P E^A{}_M + (\partial_M\xi^P - \partial^P\xi_M)E^A{}_P\ . \tag{9.69}$$

We will now proceed to the equations of motion for the DFT action. They are obtained by varying the DFT action in terms of the generalised metric. However, since the metric is constrained by eq. (9.40) we start by varying the generalised metric of this equation first, resulting in that

$$\delta\mathcal{H}^{LM}\mathcal{H}^K{}_M + \mathcal{H}^L{}_M\delta\mathcal{H}^{KM} = 0\ . \tag{9.70}$$

By using $\mathcal{H}^{ML}\mathcal{H}_{LN} = \delta^M_N$ we have that $\delta\mathcal{H}^{MN} = -\mathcal{H}^{MK}\delta\mathcal{H}_{KL}\mathcal{H}^{LN}$. The most general variation that satisfies eq. (9.70) is given by

$$\delta\mathcal{H}^{MN} = \bar P^{MK}\delta\mathcal{M}_{KL}P^{LN} + P^{MK}\delta\mathcal{M}_{KL}\bar P^{LN}\ , \tag{9.71}$$

where

$$\bar{P}^{MN} \equiv \tfrac{1}{2}(\eta^{MN} + \mathcal{H}^{MN}) \ , \qquad P^{MN} \equiv \tfrac{1}{2}(\eta^{MN} - \mathcal{H}^{MN}) \ , \tag{9.72}$$

and $\mathcal{M}_{MN}$ is some arbitrary symmetric variation not subject to any constraint. As such, the variation of the action can now be written as

$$\delta S_{\text{DFT}} = \int \mathrm{d}^{2D}X \mathcal{Q}^{MN} \delta\mathcal{H}_{MN} \equiv \int \mathrm{d}^{2D}X \mathcal{R}_{MN} \delta\mathcal{M}^{MN} \ , \tag{9.73}$$

where $\mathcal{R}_{MN}$ is the generalised Ricci tensor

$$\mathcal{R}_{MN} = P_{MK}\mathcal{Q}^{KL}\bar{P}_{LN} + \bar{P}_{MK}\mathcal{Q}^{KL}P_{LN} \ . \tag{9.74}$$

The equation of motion for the generalised metric is therefore given by $\mathcal{R}_{MN} = 0$. A common expression for $\mathcal{Q}^{MN}$ is obtained by varying the action with respect to the generalised vielbein, to give rise to

$$\begin{aligned}\mathcal{Q}_{MN} = &\tfrac{1}{8}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \tfrac{1}{4}(\partial_L - 2\partial_L\phi')(\mathcal{H}^{KL}\partial_K\mathcal{H}_{MN}) + 2\partial_M\partial_N\phi' \\ &- \tfrac{1}{2}\partial_{(M}\mathcal{H}^{KL}\partial_L\mathcal{H}_{N)K} + \tfrac{1}{2}(\partial_L - 2\partial_L\phi')(\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \mathcal{H}^{K}{}_{(M}\partial_K\mathcal{H}^{L}{}_{N)}) \ .\end{aligned} \tag{9.75}$$

Note that the indices in both $\mathcal{Q}_{MN}$ and $\mathcal{R}_{MN}$ is symmetric since the indices of $\mathcal{H}_{MN}$ are symmetric.

9.2.2 Reduction to supergravity

Letting the starting point be the old DFT action in eq. (9.34), we assume that no field depends on the dual winding $\tilde{x}$ coordinate, which effectively sets $\tilde{\partial}^i = 0$. This implies that $\mathcal{D}_i = \bar{\mathcal{D}}_i = \partial_i$ and as $\partial^i = g^{ij}\partial_j = \mathcal{D}^i = \bar{\mathcal{D}}^i$ the action takes the form

$$\begin{aligned} S\Big|_{\tilde{\partial}^i=0} &= \int \mathrm{d}x e^{-2\phi'}\big[- \tfrac{1}{4}g^{ik}g^{jl}g^{mn}(\partial_m\mathcal{E}_{kl}\partial_n\mathcal{E}_{ij} - \partial_i\mathcal{E}_{lm}\partial_j\mathcal{E}_{km} - \partial_i\mathcal{E}_{ml}\partial_j\mathcal{E}_{nk}) \\ &\qquad + 2\partial^i\phi'\partial^j g_{ij} + 4\partial^i\phi'\partial_i\phi'\big] \\ &= \int \mathrm{d}x e^{-2\phi'}\big[- \tfrac{1}{4}g^{ik}g^{jl}g^{mn}(\partial_m g_{kl}\partial_n g_{ij} - 2\partial_i g_{lm}\partial_j g_{kn} + \partial_m B_{kl}\partial_n B_{ij} - 2\partial_i B_{lm}\partial_j B_{kn}) \\ &\qquad + 2\partial^i\phi'\partial^j g_{ij} + 4(\partial\phi')^2\big] \ , \end{aligned} \tag{9.76}$$

where we have rewritten in terms of g_{ij} and B_{ij} on the last line. Next, we turn to rewrite the B_{ij} terms according to

$$\begin{aligned} -\tfrac{1}{4}g^{ik}g^{jl}g^{mn}(\partial_m B_{kl}\partial_n B_{ij} - 2\partial_i B_{lm}\partial_j B_{kn}) &= -\tfrac{1}{4}g^{ik}g^{jl}g^{mn}\partial_m B_{kl}(\partial_i B_{jn} + \partial_j B_{ni} + \partial_n B_{ij}) \\ &= -\tfrac{1}{12}g^{ik}g^{jl}g^{mn}H_{ijm}H_{kln} \\ &= -\tfrac{1}{12}H^2 \ , \end{aligned} \tag{9.77}$$

permuting indices and using $H_{ijk} = 3\partial_{[i}B_{jk]}$, which leaves us with

$$S\Big|_{\tilde{\partial}^i=0} = \int \mathrm{d}x e^{-2\phi'}\big[- \tfrac{1}{4}g^{ik}g^{jl}\partial^m g_{ij}\partial_m g_{kl} + \tfrac{1}{2}g^{mn}\partial^i g_{mj}\partial^j g_{ni} + 2\partial^i\phi'\partial^j g_{ij} + 4(\partial\phi')^2 - \tfrac{1}{12}H^3\big] \ . \tag{9.78}$$

Instead of further massaging this equation it is more convenient to rewrite the wanted NSNS action of eq. (3.98), i.e.

$$S_{\text{NSNS}} = \int \mathrm{d}x\sqrt{-g}e^{-2\phi}[R + 4(\partial\phi)^2 + \tfrac{1}{12}H^2] \ , \tag{9.79}$$

and show that eq. (9.79) is equivalent to eq. (9.78). To do so we use the dilaton redefinition of eq. (9.37) to write its derivative as

$$\partial_i\phi' = \partial_i\phi - \tfrac{1}{2}\Gamma_i \ , \qquad \text{with} \qquad \Gamma_i = \tfrac{1}{2}g^{kl}\partial_i g_{kl} \ , \tag{9.80}$$

and use in eq. (9.79) to obtain

$$S_{\text{NSNS}} = \int \mathrm{d}x e^{-2\phi'}\left[R + g^{ij}\Gamma_i\Gamma_j + 4\Gamma_i\partial^i\phi' + 4(\partial\phi')^2 - \tfrac{1}{12}H^3\right] \ . \tag{9.81}$$

The final step is simplifying the Einstein and Christoffel symbols. The Einstein term is

$$\begin{aligned} R &= g^{ij}(\partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^k_{ij}\Gamma_k - \Gamma^l_{ik}\Gamma^k_{jl}) \\ &= 2\partial^i\phi'(-\Gamma_i + g^{jk}\Gamma_{ijk}) + \partial_j g^{ij}\Gamma_i - \partial_k g^{ij}\Gamma^k_{ij} + g^{ij}(\Gamma^k_{ij}\Gamma_k - \Gamma^l_{ik}\Gamma^k_{jl}) \\ &= -4\partial^i\phi'\Gamma_i + 2\partial^i\phi'\partial^j g_{ij} + \partial_j g^{ij}\Gamma_i - \partial_k g^{ij}\Gamma^k_{ij} + g^{ij}(\Gamma^k_{ij}\Gamma_k - \Gamma^l_{ik}\Gamma^k_{jl}) \ , \end{aligned} \tag{9.82}$$

where $\Gamma_{ijk} = g_{il}\Gamma^l_{jk}$ and we have used the dilaton derivative on the last line. Inserting this in the action of eq. (9.81), the $4\Gamma_i\partial^i\phi'$ terms cancel and we get

$$\begin{aligned} S_{\text{NSNS}} = \int \mathrm{d}x e^{-2\phi'}\big[&\partial_j g^{ij}\Gamma_i - \partial_k g^{ij}\Gamma^k_{ij} + g^{ij}(\Gamma^k_{ij}\Gamma_k - \Gamma^l_{ik}\Gamma^k_{jl}) + g^{ij}\Gamma_i\Gamma_j \\ &+ 2\partial^i\phi'\partial^j g_{ij} + 4(\partial\phi')^2 - \tfrac{1}{12}H^3\big] \ . \end{aligned} \tag{9.83}$$

The last step is to evaluate the first line of eq. (9.83), which evaluates to

$$\partial_j g^{ij}\Gamma_j - \partial_k g^{ij}\Gamma^k_{ij} + g^{ij}(\Gamma^k_{ij}\Gamma_k - \Gamma^l_{ik}\Gamma^k_{jl}) + g^{ij}\Gamma_i\Gamma_j = -\tfrac{1}{4}g^{ik}g^{jl}\partial^m g_{ij}\partial_m g_{kl} + \tfrac{1}{2}g^{mn}\partial^i g_{mj}\partial^j g_{ni} \ , \tag{9.84}$$

as all terms of the form $g^{ij}\partial_k g_{ij}$ cancel against each other. Hence, we see that eq. (9.83) is identical to the DFT action in eq. (9.79) up to total derivatives.

9.2.3 Non-geometric fluxes in double field theory

It is possible to identify both geometric and non-geometric fluxes to DFT background fields and combine them into a single $O(D, D)$ tensor. Using the frame formalism, fluxes can be defined covariantly via the C-bracket and vielbein inner product according to

$$\begin{aligned} \mathcal{F}_{ABC} &= [E_A{}^L, E_B^L]_C E_{CL} \\ &= E_A{}^N\partial_N E_B{}^L E_{CL} - \tfrac{1}{2}E_{AN}\partial^L E_B{}^N E_{CL} - E_B{}^N\partial_N E_A{}^L E_{CL} + \tfrac{1}{2}E_{BN}\partial^L E_A{}^N E_{CL} \\ &\equiv \Omega_{ABC} + \tfrac{1}{2}\Omega_{CAB} - \Omega_{BAC} - \tfrac{1}{2}\Omega_{CBA} \ , \end{aligned} \tag{9.85}$$

using the definition of the C-bracket in eq. (9.47), the vielbein of eq. (9.67) and where we have introduced the generalised *Weitzenböck connection*

$$\Omega_{ABC} = E_A{}^M\partial_M E_B{}^N E_{CN} \ , \tag{9.86}$$

as is customary. The expression in eq. (9.85) can be further simplified as Ω_{ABC} is antisymmetric in its last two indices, which can be seen from

$$\Omega_{ABC} = E_A{}^M\partial_M E_B{}^N E_{CN} = -E_A{}^M\partial_M E_C{}^N E_{BN} = -\Omega_{ACB} \ , \tag{9.87}$$

since $\eta_{AB} = E_A{}^M E_{BM} = E_A{}^M\eta_{MN}E_B{}^N$ has $\partial_N\eta_{AB} = 0$. As such, eq. (9.85) reduces to a fully antisymmetric tensor

$$\mathcal{F}_{ABC} = \Omega_{ABC} + \Omega_{BCA} + \Omega_{CAB} = 3\Omega_{[ABC]} \ , \tag{9.88}$$

which, having three flat indices, are subject to double Lorentz transformations. The DFT action of eq. (9.38) can be rewritten in terms of two types of generalised fluxes, the first one being $\mathcal{F}_{ABC}$ in eq. (9.88), the second one being

$$\mathcal{F}_A = -e^{2\phi'}\hat{\mathcal{L}}_{E_A}e^{-2\phi'} = \Omega^B{}_{BA} + 2E_A{}^M\partial_M\phi' \ . \tag{9.89}$$

In terms of these the original DFT action of eq. (9.38) can be shown to be equivalent to

$$\begin{aligned} S = \int \mathrm{d}X e^{-2\phi'}\Big[&\mathcal{F}_{ABC}\mathcal{F}_{DEF}\left(\tfrac{1}{4}N^{AD}\eta^{BE}\eta^{CF} - \tfrac{1}{12}N^{AD}N^{BE}N^{CF} - \tfrac{1}{6}\eta^{AD}\eta^{BE}\eta^{CF}\right) \\ &+ \mathcal{F}_A\mathcal{F}_B(\eta^{AB} - N^{AB})\Big] \ . \end{aligned} \tag{9.90}$$

The $\mathcal{F}_{ABC}$ consist of eight $D \times D \times D$ blocks but, due to its antisymmetry, only four of these are independent from each other. The four independent components $\mathcal{F}_{abc}, \mathcal{F}^a{}_{bc}, \mathcal{F}^{ab}{}_c$ and $\mathcal{F}^{abc}$ correspond to the fluxes $H^{abc}, f^a{}_{bc}, Q^{ab}{}_c$ and R^{abc} respectively. In order to calculate them one may use the "overparameterised" generalised veilbein of eq. (9.68), which includes both B_{ij} and the bi-vector β^{ij}. It is overparameterised in the sense that physical fields are parametrised by *either* B_{ij} or β^{ij}. In light of this we rearrange the overparameterised veilbein of eq. (9.68) to be given by

$$E_A{}^M = \eta_{AB}E^B{}_N\eta^{NM} = \begin{pmatrix} e^a_i + e^a{}_j\beta^{jk}B_{ki} & e^a{}_j\beta^{ij} \\ e_a{}^jB_{ji} & e_a{}^i \end{pmatrix} \ . \tag{9.91}$$

Inserting this expression into the Ω_{abc} definition of eq. (9.87) enables us to evaluate $\mathcal{F}_{abc} = 3\Omega_{[abc]}$. The relevant component is calculated to

$$\begin{aligned} \Omega_{abc} &= E_a{}^M\partial_M E_b{}^N E_{cN} \\ &= E_a{}^m\partial_m E_b{}^n E_{cn} + E_{am}\tilde{\partial}^m E_b{}^n E_{cn} + E_a{}^m\partial_m\partial_m E_{bn}E_c{}^n + E_{am}\tilde{\partial}^m E_{bn}E_c{}^n \\ &= e_a{}^m\partial_m(e_b{}^n)e_c{}^jB_{jn} + e_a{}^k\tilde{\partial}^m(e_b{}^n)e_c{}^jB_{jn} + e_a{}^m\partial_m(e_b{}^jB_{jn})e_c{}^n + e_a{}^kB_{km}\tilde{\partial}^m(e_b{}^jB_{jn})e_c{}^n \\ &= e_a{}^m\partial_m(e_b{}^n)e_c{}^jB_{jn} + e_a{}^k\tilde{\partial}^m(e_b{}^n)e_c{}^jB_{jn} + e_a{}^m\partial_m(e_b{}^j)B_{jn}e_c{}^n + e_a{}^me_b{}^j\partial_m(B_{jn})e_c{}^n \\ &\quad + e_a{}^kB_{km}\tilde{\partial}^m(e_b{}^j)B_{jn}e_c{}^n + e_a{}^kB_{km}e_b{}^j\tilde{\partial}^m(B_{jn})e_c{}^n \\ &= e_a{}^ie_b{}^je_c{}^k[\partial_i(B_{jk}) + B_{im}\tilde{\partial}^m(B_{jk})] \ . \end{aligned} \tag{9.92}$$

Antisymmetrising this expression we obtain the H_{abc} flux in flat indices;

$$\mathcal{F}_{abc} = 3e_a{}^ie_b{}^je_c{}^k(\partial_{[i}B_{jk]} - B_{l[i}\tilde{\partial}^lB_{jk]}) = H_{abc} \ . \tag{9.93}$$

Next we move on to calculate $\mathcal{F}^a{}_{bc}$. The explicit calculations of the ingoing components $\Omega^a{}_{bc}, \Omega_a{}^b{}_c$ and $\Omega_{ab}{}^c$ are slightly messy, and they are therefore exiled to Appendix E, as well as the components of the other flux terms. Stating the results here, the for the $\mathcal{F}^a{}_{bc}$ relevant components of Ω are

$$\begin{aligned} &\Omega^a{}_{bc} = e^a{}_ie_b{}^je_c{}^k[\tilde{\partial}^iB_{jk} + \beta^{il}\Omega_{ljk}] \ , \\ &\Omega_a{}^b{}_c = e_a{}^i\partial_ie^b{}_je_c{}^j + e_a{}^iB_{ij}\tilde{\partial}^je^b{}_ke_c{}^k + e_a{}^ie^b{}_je_c{}^k\beta^{jl}\Omega_{ilk} \ , \\ &\Omega_{ab}{}^c = -\Omega_a{}^c{}_b \ , \end{aligned} \tag{9.94}$$

for which the covariant fluxes the $\mathcal{F}^a{}_{bc} = \Omega^a{}_{[bc]} + \Omega_{[c}{}^a{}_{b]} + \Omega_{[bc]}{}^a = \Omega^a{}_{[bc]} + 2\Omega_{[c}{}^a{}_{b]}$ become

$$\mathcal{F}^a{}_{bc} = e^a{}_ie_b{}^je_c{}^k(\tilde{\partial}^iB_{jk} + \beta^{il}H_{ljk}) + 2(e_{[b}{}^i\partial_ie^a{}_je_{c]}{}^j + e_a{}^iB_{ij}\tilde{\partial}^je^b{}_ke_c{}^k) = f^a{}_{bc} \ . \tag{9.95}$$

The equivalence between the geometric fluxes $f^a{}_{bc}$ becomes apparent when choosing a frame in which $\tilde{\partial}^i = 0$ and $\beta^{ij} = 0$, i.e. choosing a physical gauge for the vielbein, which reduces $\mathcal{F}^a{}_{bc}$ to

$$\mathcal{F}^a{}_{bc} = 2e_{[b}{}^i \partial_i e^a{}_j e_{c]}{}^j = f^a{}_{bc} \; . \tag{9.96}$$

This can be compared to the structure constants of eq. (9.8) in our previous torus example. Proceeding to the Q flux equivalent, we have that

$$\mathcal{F}^{ab}{}_c = \Omega^{[ab]}{}_c + \Omega_c{}^{[ab]} + \Omega^{[b}{}_c{}^{a]} = 2\Omega^{[ab]}{}_c + \Omega_c{}^{[ab]} \; , \tag{9.97}$$

where the components are given by

$$\begin{aligned} \Omega^{ab}{}_c &= e^a{}_i \tilde{\partial}^i e^b{}_j e_c{}^j + e^a{}_i e^b{}_j e_c{}^k \beta^{il} \Omega_l{}^j{}_k \; , \\ \Omega_a{}^{bc} &= e_a{}^i e_j{}^b e_k{}^c [\partial_i \beta^{jk} + B_{il} \tilde{\partial}^l \beta^{jk} + \beta^{jl} \beta^{km} \Omega_{ilm}] \; , \\ \Omega^a{}_b{}^c &= -\Omega^{ac}{}_b \; . \end{aligned} \tag{9.98}$$

With these components eq. (9.97) becomes

$$\mathcal{F}^{ab}{}_c = 2e^{[a}{}_i \tilde{\partial}^i e^{b]}{}_j e_c{}^j + e_i{}^{[a} e_j{}^{b]} e_c{}^k \big[\partial_k \beta^{ij} + B_{kl} \tilde{\partial}^l \beta^{ij} - \beta^{li}(2\Omega_l{}^j{}_k + \beta^{jn}\Omega_{kln})\big] = Q^{ab}{}_c \; , \tag{9.99}$$

which is equivalent to the non-geometric Q flux in flat indices. As with the previous f flux we may choose a physical gauge $\tilde{\partial}^i = B_{ij} = 0$ so that the Q flux takes the form

$$\mathcal{F}^{ab}{}_c = e_i{}^a e_j{}^b e_c{}^k [\partial_k \beta^{ij} - \beta^{l[i} f^{j]}{}_{kl}] = Q^{ab}{}_c \; . \tag{9.100}$$

Last but not least, the R flux equivalent only includes one component of Ω, namely

$$\Omega^{abc} = e^a{}_i e^b{}_j e^c{}_k [\tilde{\partial}^i \beta^{jk} + \beta^{il} \Omega_l{}^{jk}] \; , \tag{9.101}$$

which gives us the flux

$$\mathcal{F}^{abc} = 3e^a{}_i e^b{}_j e^c{}_k \big[\tilde{\partial}^{[i} \beta^{jk]} + \beta^{[il} \partial_l \beta^{jk]} + \beta^{il} B_{ln} \tilde{\partial}^n \beta^{jk} + \beta^{il} \beta^{jm} \beta^{kn} \mathcal{F}_{lmn}\big] \; . \tag{9.102}$$

With $\tilde{\partial}^i = B_{ij} = 0$ we get that this reduces to

$$\mathcal{F}^{abc} = 3e^a{}_i e^b{}_j e^c{}_k \beta^{[il} \partial_l \beta^{jk]} = R^{abc} \; . \tag{9.103}$$

9.2.4 The geometrical meaning of non-geometric fluxes

In this section we discuss the geometrical meaning of the non-geometric Q- and R fluxes, as they are not well-defined in terms of the metric and B field. It has been extensively proven in [110] that the DFT action is invariant under the field redefinition $(g, B, \phi') \to (\tilde{g}, \beta, \tilde{\phi})$ classically characterised in terms of the generalised metric as

$$\mathcal{H}^{MN} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lj} & B_{ik} g^{kj} \\ -g^{ik} B_{kj} & g^{ij} \end{pmatrix} = \begin{pmatrix} \tilde{g}^{ij} - \beta^{il} \tilde{g}_{lk} \beta^{kj} & -\beta^{il} \tilde{g}_{lj} \\ \tilde{g}_{ik} \beta^{kj} & \tilde{g}_{ij} \end{pmatrix} \; , \tag{9.104}$$

which was inspired from generalised geometry. This new metric also reproduces the NSNS supergravity action of eq. (9.79). This form of the generalised metric has been proven useful when investigating the geometrical meaning of the non-geometric Q and R fluxes, as they are not well-defined in terms of the ordinary field variables (g, B, ϕ').

In DFT gauge symmetries are spanned by $\xi^M = (\xi^i, \tilde{\xi}_i)$ which acts on the original field $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ according to

$$\delta \mathcal{E}_{ij} = \mathcal{L}_\xi \mathcal{E}_{ij} + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i + \mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} - \mathcal{E}_{ik}(\tilde{\partial}^k \xi^l - \tilde{\partial}^l \xi^k) \mathcal{E}_{lj} \; . \tag{9.105}$$

The ordinary Lie derivative with coordinate parameter ξ^i and dual winding coordinate $\tilde{\xi}_i$ are given by

$$\begin{aligned}\mathcal{L}_\xi \mathcal{E}_{ij} &= \xi^k \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \mathcal{E}_{kj} + \partial_j \xi^k \mathcal{E}_{ik} \ , \\ \mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} &= \tilde{\xi}_k \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \tilde{\xi}^i \mathcal{E}_{kj} - \tilde{\partial}^k \tilde{\xi}_j \mathcal{E}_{ik} \ ,\end{aligned} \tag{9.106}$$

respectively. The sign difference between these expressions reflects the fact that $\mathcal{E}_{ij}$ is a covaraint tensor with respect to the diffeomorphism group, but a contravariant tensor with respect to the dual diffeomorphisms with lower indices $\tilde{x}_i \to \tilde{x}_i - \tilde{\xi}_i$. While not apparent in this form, the gauge transformations in eq. (9.105) are $O(D,D)$-covariant. The transformation properties for $\mathcal{E}^{ij}$ are then obtained by the exchanges $\partial_i \to \tilde{\partial}^i$ and $\xi^i \to \tilde{\xi}^i$ so that

$$\delta \mathcal{E}^{ij} = \mathcal{L}_{\tilde{\xi}} \tilde{\mathcal{E}}^{ij} + \tilde{\partial}^i \xi^i - \tilde{\partial}^j \xi^i + \mathcal{L}_\xi \tilde{\mathcal{E}}^{ij} - \tilde{\mathcal{E}}^{ik} (\partial_k \tilde{\xi}_l - \partial_l \tilde{\xi}_k) \tilde{\mathcal{E}}^{lj} \ . \tag{9.107}$$

It is clear that the transformation is linear in ξ^i while it is non-linear in $\tilde{\xi}_i$. The gauge transformation is covariant if it is equal to the Lie derivative. I.e, considering only the ξ^i transformations, one has $\delta_\xi A = \mathcal{L}_\xi A$, $\forall A$. Expanding $\tilde{\mathcal{E}}^{ij} = \tilde{g}^{ij} + \beta^{ij}$ we have for the ξ^i transformation in eq. (9.107) that

$$\delta_\xi \tilde{g}^{ij} = \mathcal{L}_\xi \tilde{g}^{ij} \ , \qquad \delta_\xi \beta^{ij} = \tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i + \mathcal{L}_\xi \beta^{ij} \equiv \mathcal{L}_\xi \beta^{ij} + \Delta_\xi \beta^{ij} \ , \tag{9.108}$$

where $\Delta_\xi \beta^{ij} = \tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i$ denotes the non-covariant part. It is then natural to look for a connections $\tilde{\partial}^i \to \tilde{\nabla}^i$ that make the momentum diffeomorphisms covariant. To do so we also consider the gauge transformation of the dilaton scalar $\tilde{\phi}$, which transforms covariantly as

$$\delta_\xi \tilde{\phi} = \xi^i \partial_i \tilde{\phi} \ . \tag{9.109}$$

Its winding derivative $\tilde{\partial}^i \tilde{\phi}$ however does not, since

$$\delta_\xi (\tilde{\partial}^i \tilde{\phi}) = \tilde{\partial}^i (\delta_\xi \tilde{\phi}) = \tilde{\partial}^i (\xi^i \partial_\xi \tilde{\phi}) = \tilde{\phi}^i \xi^i \partial_j \tilde{\phi} + \xi^j \partial_j (\tilde{\partial}^i \tilde{\phi}) \ . \tag{9.110}$$

This transformation can be written in terms of the Lie derivative by adding the terms

$$-\partial_j \xi^i \tilde{\partial}^j \tilde{\phi} - \tilde{\partial}^j \xi^i \partial_j \tilde{\phi} = 0 \ , \tag{9.111}$$

which vanish by the strong constraint in eq. (9.49) and may so be added on the right-hand side of eq. (9.110). This results in that the transformation becomes

$$\begin{aligned}\delta_\xi (\tilde{\partial}^i \tilde{\phi}) &= \xi^i \partial_j (\tilde{\partial}^i \tilde{\phi}) - (\partial_j \xi^i) \tilde{\partial}^j \tilde{\phi} + (\tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i) \partial_j \tilde{\phi} \\ &\equiv \mathcal{L}_\xi (\tilde{\partial}^i \tilde{\phi}) + (\tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i) \partial_j \tilde{\phi} \ .\end{aligned} \tag{9.112}$$

The non-covariant part $\nabla_\xi (\tilde{\partial}^i \tilde{\phi}) = (\tilde{\partial}^i \xi^j - \tilde{\partial}^j \xi^i) \partial_j \tilde{\phi}$ is then seen to have the same form as $\nabla_\xi \beta^{ij}$ in eq. (9.108). Hence, these terms may be cancelled by introducing the covariant derivative

$$\tilde{D}^i \equiv \tilde{\partial}^i - \beta^{ij} \partial_j \ , \tag{9.113}$$

such that

$$\delta_\xi (\tilde{D}^i \tilde{\phi}) = \mathcal{L}_\xi (\tilde{D}^i \tilde{\phi}) \ . \tag{9.114}$$

The R flux derived in the previous section, given by eq. (9.103), can be written in curved indices via $R^{abc} = e^a{}_i e^b{}_j e^c{}_k R^{ijk}$ so that $R^{ijk} = 3\beta^{[il} \partial_l \beta^{jk]}$. The covariant derivative given by eq. (9.113) can be shown, using the strong constraint in eq. (9.49) and definitions of the R and Q fluxes in eq. (9.103) and eq. (9.100), to fulfil the commutation relation

$$[\tilde{D}^i, \tilde{D}^j] = -R^{ijk} \partial_k - Q_k{}^{ij} \tilde{D}^k \ , \tag{9.115}$$

where again $R^{ijk} \equiv 3\tilde{D}^{[i}\beta^{jk]} = 3(\tilde{\partial}^{[i}\beta^{jk]} + \beta^{p[i}\partial_p\beta^{jk]})$. The R flux can be shown to be covariant under eq. (9.108). To see this we determine the non-covariant terms of both terms in R, such that

$$\begin{aligned}\nabla_\xi(\tilde{\partial}^{[i}\beta^{jk]}) &= \tilde{\partial}^p\beta^{[ij}\partial_p\xi^{k]} + \partial_p\beta^{[ij}\tilde{\partial}^{k]}\xi^p + 2\tilde{\partial}^{[i}\partial_p\xi^j\beta^{k]p} \ , \\ \nabla_\xi(\beta^{p[i}\partial_p\beta^{jk]}) &= \partial_p\beta^{[ij|}\tilde{\partial}^p\xi^{|k]} - \partial_p\beta^{[ij}\tilde{\partial}^{k]}\xi^p - 2\tilde{\partial}^{[i}\partial_p\xi^j\beta^{k]p} \ ,\end{aligned} \tag{9.116}$$

which combine into

$$\nabla_\xi R^{ijk} = 3(\tilde{\partial}^p\beta^{[ij}\partial_p\xi^{k]} + \partial_p\beta^{[ij|}\tilde{\partial}^p\xi^{|k]}) = 0 \ . \tag{9.117}$$

Hence we see that the R^{ijk} flux is a covariant tensor and can viewed as the field strength of β^{ij}, since by comparison the NSNS H_3 flux is given by $H_{ijk} = 3\partial_{[i}B_{jk]}$ and $R^{ijk} = 3\tilde{\partial}^{[i}\beta^{jk]}$ when taking the alternative supergravity limit $\partial^i = 0$. To learn about the interpretation of the Q in this formulation, we proceed our consideration of covariant derivatives. For a vector V^i we may define in a common manner a covariant derivative that acts like

$$\tilde{\nabla}^i V^j = \tilde{D}^i V^j - \Gamma_k{}^{ij}V^k \ , \tag{9.118}$$

and since the $\tilde{D}$ derivative of V^i transforms non-covariantly as

$$\nabla_\xi(\tilde{D}^i V^j) = -\tilde{D}^i\partial_k\xi^j V^k \ , \tag{9.119}$$

we have that for eq. (9.118) to transform covariantly under gauge transformations, we must have that

$$\nabla_\xi\Gamma_k{}^{ij} = -\tilde{D}^i\partial_k\xi^j \ . \tag{9.120}$$

In order to determine the connection $\Gamma_k{}^{ij}$ in terms of the fields present it should satisfy some constraints. For instance the covariant derivative on the metric should vanish, i.e. metric compatibility implies that

$$\tilde{\nabla}^i\tilde{g}^{jk} = \tilde{D}^i\tilde{g}^{jk} - \Gamma_l{}^{ij}\tilde{g}^{lk} - \Gamma_l{}^{ik}\tilde{g}^{jl} = 0 \ . \tag{9.121}$$

With the metric being symmetric in its indices this condition will only be able to determine the symmetric part of the connection $\Gamma_k{}^{ij}$. The symmetric part can be written in terms of the antisymmetric part by writing eq. (9.121) as three equations by cycling each of its three indices, and then adding two of these equations and subtracting the last one. This gives us

$$2\tilde{g}^{jl}\Gamma_l{}^{ki} = \tilde{D}^i\tilde{g}^{jk} + \tilde{D}^k\tilde{g}^{ij} - \tilde{D}^j\tilde{g}^{ki} - 2(\Gamma_l{}^{[ij]}\tilde{g}^{lk} + \Gamma_l{}^{[kj]}\tilde{g}^{li}) \ , \tag{9.122}$$

so that

$$\begin{aligned}\Gamma_k{}^{(ij)} &= \tfrac{1}{2}\tilde{g}_{kl}(\tilde{D}^i\tilde{g}^{jl} + \tilde{D}^j\tilde{g}^{il} - \tilde{D}^l\tilde{g}^{ij}) - \tilde{g}_{kl}(\tilde{g}^{pi}\Gamma_p{}^{[jl]} + \tilde{g}^{pj}\Gamma_p{}^{[il]}) \\ &\equiv \check{\Gamma}_k{}^{ij} - \tilde{g}_{kl}(\tilde{g}^{pi}\Gamma_p{}^{[jl]} + \tilde{g}^{pj}\Gamma_p{}^{[il]}) \ .\end{aligned} \tag{9.123}$$

To determine the antisymmetric part we consider the commutator of the covariant derivative acting on $\tilde{\phi}$, where we get

$$\begin{aligned}[\tilde{\nabla}^i, \tilde{\nabla}^j]\tilde{\phi} &= \Big([\tilde{D}^i, \tilde{D}^j] - \Gamma_k{}^{ij}\tilde{D}^k + \Gamma_k{}^{ji}\tilde{D}^k\Big)\,\tilde{\phi} \\ &= -\Big(R^{ijk}\partial_k + (Q_k{}^{ij} + 2\Gamma_k{}^{[ij]})\tilde{D}^k\Big)\,\tilde{\phi} \ ,\end{aligned} \tag{9.124}$$

using eq. (9.115). We know that R^{ijk} transforms as a tensor, and so we may demand that this commutator transforms only in terms of R^{ijk}, so that a covariant condition is given by

$$[\tilde{\nabla}^i, \tilde{\nabla}^j]\tilde{\phi} = -R^{ijk}\partial_k \ , \tag{9.125}$$

which is then solved by

$$\Gamma_k{}^{[ij]} = -\tfrac{1}{2} Q_k{}^{ij} \ . \tag{9.126}$$

Hence the connection in eq. (9.118) is given by

$$\Gamma_k{}^{ij} = \check{\Gamma}_k{}^{ij} + \tilde{g}_{kl}\tilde{g}^{p(i}Q_p{}^{j)l} - \tfrac{1}{2} Q_k{}^{ij} \ . \tag{9.127}$$

As such it is clear that the Q flux can be regarded as a momentum covariant derivative. This precisely matches the dual situation for the geometric fluxes where H is a 3-form flux and f is related to the Levi-Civita spin connection. The covariant derivative and connection derived above can also be used to define invariant curvatures and Bianchi identities, as done in [110]. In turn, the curvatures can then be used in constructing a new action in terms of Ricci scalars and torsion which depends on the the non-geometric R and Q fluxes. This action is then proven in Appendix A of [110] to equal the DFT action in terms of $\tilde{g}$ and β.

Note that since in all above examples we have seen that the geometrical fluxes can be T-dualised into non-geometric fluxes, the converse also applies. The fluxes are said to be on the same $O(D,D)$ orbit. By turning on more fluxes, say one geometric and one non-geometric, these fluxes can not be transformed T-dualised into geometric fluxes at the same time, and so such a background is said to be purely non-geometric. See for instance [111] for an example of this.

9.2.5 Consistent truncations

The consistency of Kaluza-Klein compactifications, i.e. the matching of equations of motion of the original theory to the effective one, is in general very hard to establish. In this section we will briefly discuss the notion of Scherk-Schwarz compactifications [112][113] as well as generalised Scherk-Schwarz compactifications which are used in DFT, largely following [114].

In a Kaluza-Klein compactification on a T^d torus, the truncation of massive modes leaves an Abelian $U(1)^{2d}$ gauge symmetry of the resulting effective action. A more general reduction that does not have an Abelian gauge symmetry is provided by the Scherk-Schwarz compactification procedure. As usual, the first step is to split the external and internal coordinates as $x^i = (x^\mu, y^m)$ with spacetime indices $\mu = 1, \dots, d$ and internal indices $m = d+1, \dots, n$. The next step is to split the fields into external, internal and mixed components according to

$$\begin{aligned} g_{ij} &= \begin{pmatrix} g_{\mu\nu} + g_{pq}A^p{}_\mu A^q{}_\nu & A^p{}_\mu g_{pn} \\ g_{mp}A^p{}_\nu & g_{mn} \end{pmatrix} \ , \\ B_{ij} &= \begin{pmatrix} B_{\mu\nu} - \frac{1}{2}(A^p{}_\mu V_{p\nu} - A^p{}_\nu V_{p\mu}) + A^p{}_\mu A^q{}_\nu B_{pq} & V_{n\mu} - B_{np}A^p{}_\mu \\ -V_{m\nu} + B_{mp}A^p{}_\nu & B_{mn} \end{pmatrix} \ , \end{aligned} \tag{9.128}$$

with some vectors $A^p{}_\mu, V_{p\mu}$. The form of these blocks corresponds to the most general layouts needed to perserve the symmetric/antisymmetric properties of the fields. Now, for each component the SS reduction ansatzes specify the dependence of the external/internal coordinates and are given by

$$\begin{aligned} g_{\mu\nu}(x) &= \check{g}_{\mu\nu}(x) \ , & B_{\mu\nu}(x) &= \check{B}_{\mu\nu}(x) \ , \\ A^m{}_\mu &= u_a{}^m \check{A}^a{}_\mu(x) \ , & V_{m\mu} &= u^a{}_m(y)\check{V}_{a\mu}(x) \ , \\ g_{mn} &= u^a{}_m(y)u^b{}_n(y)\check{g}_{mn}(x) \ , & B_{mn} &= u^a{}_m(y)u^b{}_n(y)\check{B}_{mn}(x) + v_{mn}(y) \ , \end{aligned} \tag{9.129}$$

as well as $\phi = \check{\phi}(x)$ for the dilaton. Here the objects $u(y)$ and $v(y)$ are called twists, and after the truncation procedure is complete, there is no dependence on the internal coordinates but the twists will remain in the form of structure constants which parameterise the possible deformations of the effective action. The new effective fields consists of the d-dimensional metric $\check{g}_{\mu\nu}(x)$, 2-form $\check{B}_{\mu\nu}(x)$, $2n$ vector fields $\check{A}^a{}_\mu(x), \check{V}_{a\mu}(x)$ and n^2+1 scalars $\check{g}_{ab}, \check{B}_{ab}, \check{\phi}$.

As for the gauge transformations there must first be a split $\lambda^i = (\epsilon^\mu, \Lambda^m)$ and $\tilde{\lambda}_i = (\epsilon_\mu, \Lambda_m)$, which upon the SS ansatz takes the form

$$\lambda^i = (\hat{\epsilon}^\mu(x), u_a{}^m \hat{\Lambda}^a(x)) \ , \qquad \tilde{\lambda}_i = (\hat{\epsilon}^\mu(x), u_m{}^a \hat{\Lambda}_a(x)) \ . \tag{9.130}$$

The gauge transformations of the supergravity fields are given in eq. (9.30) and eq. (9.31) for the metric and 2-form field and eq. (9.32) for general vectors. Using the expansion of eq. (9.130) in these transformations, the gauge transformations of the effective theory can be obtained. For instance, for a vector field V^i splitted as $V^i = (\check{v}^\mu(x), u_a{}^m(y)\check{v}^a(x))$ the Lie derivative becomes

$$\mathcal{L}_\lambda V^\mu = \check{\epsilon}^\nu \partial_\nu \check{v}^\mu - \check{v}^\nu \partial_\nu \check{\epsilon}^\mu \equiv \check{\mathcal{L}}_{\check{\epsilon}} \check{v}^\mu \ , \tag{9.131}$$

for the external part of the vector, and corresponds to the d-dimensional Lie derivative of the effective theory. For the internal component we have

$$\mathcal{L}_\lambda V^m = u_a{}^m \mathcal{L}_{\check{\lambda}} \check{v}^a + F_{bc}{}^a \check{\Lambda}^b \check{v}^c \equiv u_a{}^m \check{L}_{\check{\lambda}} \check{v}^a \ , \tag{9.132}$$

where

$$F_{ab}{}^c = u_a{}^m \partial_m u_b{}^n u^c{}_n - u_b{}^m \partial_m u_a{}^n u^c{}_n \ , \tag{9.133}$$

are the structure-like constants mentioned earlier. They correspond to metric fluxes, as will become more apparent soon. Performing the same procedure one can obtain the new Lie derivatives of the metric and B field in the effective action. It is then convenient to combine them into parameters and fields of the effective theory. Specifically, for the gauge parameters we have

$$\check{\xi} = (\check{\epsilon}_\mu, \check{\epsilon}^\mu, \check{\Lambda}^A) \ , \quad \check{\Lambda}^A = (\check{\lambda}_a, \check{\lambda}^a) \ , \tag{9.134}$$

and for the vectors one sets

$$\check{A}^A{}_\mu = (\check{V}_{a\mu}, \check{A}^a{}_\mu) \ , \tag{9.135}$$

and finally for the scalars

$$\check{\mathcal{M}}_{AB} = \begin{pmatrix} \check{g}^{ab} & -\check{g}^{ac}\check{B}_{cd} \\ \check{B}_{ac}\check{g}^{cb} & \check{g}_{ab} - \check{B}_{ac}\check{g}^{cd}\check{B}_{db} \end{pmatrix} \ . \tag{9.136}$$

Notice that this is same the form as the generalised metric in eq. (9.36). Using these expressions, the gauge transformations of all the fields are given as

$$\begin{aligned}
\delta_{\check{\xi}} \check{g}_{\mu\nu} &= \mathcal{L}_{\check{\epsilon}} \check{g}_{\mu\nu} \ , \\
\delta_{\check{\xi}} \check{B}_{\mu\nu} &= \mathcal{L}_{\check{\epsilon}} \check{B}_{\mu\nu} + \partial_\mu \check{\epsilon}_\nu - \partial_\nu \check{\epsilon}_\mu \ , \\
\delta_{\check{\xi}} \check{A}^A{}_\mu &= \mathcal{L}_{\check{\epsilon}} \check{A}^A{}_\mu - \partial_\mu \check{\Lambda}^A + F_{BC}{}^A \check{\Lambda}^B \check{A}^C{}_\mu \ , \\
\delta_{\check{\xi}} \check{\mathcal{M}}_{AB} &= \mathcal{L}_{\check{\epsilon}} \check{\mathcal{M}}_{AB} + F_{AC}{}^D \check{\Lambda}^C \check{\mathcal{M}}_{DB} + F_{BC}{}^D \check{\Lambda}^C \check{\mathcal{M}}_{AD} \ .
\end{aligned} \tag{9.137}$$

From these transformations we can read off what type of transformation each parameter corresponds to in the effective theory. Namely, we see that the $\check{\epsilon}^\mu$ are diffeomorphism parameters, the $\check{\epsilon}_\mu$ are gauge parameters of the B field and the $\check{\Lambda}^A$ are the vectors' gauge transformation parameters. The $F_{AB}{}^C$ are referred to as "gaugings" or "fluxes" which consist of the components

$$\begin{aligned}
F_{abc} &= 3(\partial_{[a} v_{bc]} + F_{[ab}{}^d v_{c]d}) \ , \\
F_{ab}{}^c &= u_a{}^m \partial_m u_b{}^n u^c{}_n - u_b{}^m \partial_m u_a{}^n u^c{}_n \ , \\
F_a{}^{bc} &= 0 \ , \\
F^{abc} &= 0 \ .
\end{aligned} \tag{9.138}$$

As a final point the d-dimensional supergravity action is obtained by plugging in the SS reduction ansatz into the supergravity action, which results in

$$\begin{aligned} S = \int \mathrm{d}^d x \sqrt{-\check{g}} e^{-2\check{\phi}} \Big(& \mathbf{R} + 4\partial^\mu \check{\phi} \partial_\mu \check{\phi} - \tfrac{1}{4} \check{M}_{AB} \mathbb{F}^{A\mu\nu} \mathbb{F}^B{}_{\mu\nu} - \tfrac{1}{12} \mathbb{G}_{\mu\nu\rho} \mathbb{G}^{\mu\nu\rho} \\ & + \tfrac{1}{8} D_\mu \check{\mathcal{M}}_{AB} D^\mu \check{M}^{AB} + V \Big) \ , \end{aligned} \tag{9.139}$$

where $\mathbf{R}$ is the d-dimensional Ricci scalar and the fields are combined according to

$$\begin{aligned} \mathbb{F}^A{}_{\mu\nu} &= \partial_\mu \check{A}^A{}_\mu - \partial_\nu \check{A}^A{}_\mu - F_{BC} \check{A}^B{}_\mu \check{A}^C{}_\nu \ , \\ \mathbb{G}_{\mu\nu\rho} &= 3\partial_{[\mu} \check{B}_{\rho\lambda]} - F_{ABC} \check{A}^A{}_\mu \check{A}^B{}_\rho \check{A}^C{}_\lambda + 3\partial_{[\mu} \check{A}^A{}_\rho \check{A}_{\lambda]A} \ . \end{aligned} \tag{9.140}$$

The derivative D_μ acting on the scalar matrix is given by

$$D_\mu \check{\mathcal{M}}_{AB} = \partial_\mu \check{M}_{AB} - F_{AD}{}^C \check{A}^D{}_\mu \check{\mathcal{M}}_{CB} - F_{BD}{}^C \check{A}^D{}_\mu \check{\mathcal{M}}_{AC} \ , \tag{9.141}$$

and the potential is given as

$$V = -\tfrac{1}{4} F_{DA}{}^C F_{CB}{}^D \check{M}^{AB} - \tfrac{1}{12} F_{AC}{}^E F_{BD}{}^F \check{\mathcal{M}}^{AB} \check{\mathcal{M}}^{CD} \check{\mathcal{M}}_{EF} - \tfrac{1}{6} F_{ABC} F^{ABC} \ . \tag{9.142}$$

The consistency of this type of compactification is not obvious, and we refer the interested reader to [115] for a nice discussion on this and related topics. The action of eq. (9.139) obtained via SS reduction is an example of a *gauged* supergravity theory. By gauged one refers to the fact that the gauge transformations of the theory are not Abelian. For instance, if the global symmetry group is $O(6,6)$, an "ungauged" theory would have gauge transformations generating the Abelian group $U(1)^{12}$. If $(t_\alpha)_A{}^B$ with $\alpha = 1, \ldots, 66$ and $A = 1, \ldots, 12$ are $O(6,6)$ generators, then the gauge group generators are given by $\Theta_A{}^\alpha (t_\alpha)_B{}^C$ where $\Theta_A{}^\alpha$ is called the *embedding tensor* and encodes the possible gaugings of the theory.

Another important property is that SS reductions preserve all supersymmetries of the original theory. For instance, starting with a 10D theory with $\mathcal{N} = 1$ preserving 16 supercharges, in a reduction to $d = 4$ the effective theory still has 16 supercharges and is therefore $\mathcal{N} = 4$. This corresponds to half of the maximal amount of supersymmetry, and so the effective theory is referred to as a half-maximal gauged supergravity. See [116] for a thorough analysis on these.

A full review of gauged supergravities is out of the scope for this thesis, but for self-consistency it will be useful to point out some of the basics of its bosonic sector. In 4D, the bosonic field content of a half-maximal gauged supergravity is given by a metric $\check{g}_{\mu\nu}$, 12 vector fields $\check{A}^A{}_\mu$, and 38 scalars that are usually embedded in two objects. Two of the scalars, $\check{\phi}'$ and the scalar $\check{B}_0$ dual to the $\check{B}_{\mu\nu}$ in 4D, combines into $\check{\tau} = e^{-2\check{\phi}'} + i\check{B}_0$ and the other 36 scalars parameterise the coset space

$$\check{\mathcal{M}}_{AB} = \frac{O(6,6)}{O(6) \times O(6)} \ . \tag{9.143}$$

There is also a possibility to couple some N vector multiplets, however to do so the global symmetry group would has to be extended to $O(D, D+N)$, and it would also contain an $SL(2)$ factor related to S-duality. This factor is not captured by DFT, however S-duality can be incorporated, which results in what is called exceptional field theory, to be reviewed in the next chapter.

Next, we will discuss the interpretation of the SS ansatz in terms of a compactification and connect it with the previous example of the twisted torus in section 9.1.1. Indeed, we have already performed an SS compactification. By eq. (9.128) the internal metric is given by $g_{mn} = u^a{}_m(y) \check{g}_{ab}(x) u^b{}_n(y)$. The $\check{g}_{ab}$ can be then viewed as perturbations on the metric of the compact internal space which encoded by the twists $u^a{}_m$. When this ansatz is used in the supergravity action one will then obtain an effective theory of these perturbations which is deformed by objects depending on the internal metric. If the perturbations are fixed, say $\breve{g}_{ab} = \delta_{ab}$, then the metric of the background on which the compactification happens is given by

$$g_{mn} = u^a{}_m(y)\delta_{ab}(x)u^b{}_n(y) \ , \qquad \text{for} \qquad \breve{g}_{ab} = \delta_{ab} \ . \tag{9.144}$$

Similarly for the 2-form field, if we fix the perturbations at $B_{ab}(x) = 0$ then the B field in the internal space is given by

$$B_{mn} = v_{mn}(y) \ , \qquad \text{for} \qquad B_{ab}(x) = 0 \ . \tag{9.145}$$

From this we draw the conclusion that the $u^a{}_m(y)$ can be regarded as vielbeins of the interal metric, and the $v_{mn}(y)$ as the B field in the internal space.

Turning to the fluxes in eq. (9.138), using the SS ansatz and imposing the Busher rules in eq. (9.4), the metric and B field components are exchanged and the F_{abc} can be seen to transform into the $F_{ab}{}^c$. The non-vanishing flux components in eq. (9.138) are then identified as the geometrical fluxes discussed in section 9.1.2, i.e.

$$H_{abc} \equiv F_{abc} \ , \qquad f_{ab}{}^c \equiv F_{ab}{}^c \ . \tag{9.146}$$

Recalling the first step in the compactification on the T^3 torus example, the components in the SS reduction are given as

$$g_{mn} = \delta_{mn} \ , \qquad B_{23} = Ny^1 \ , \qquad \Leftrightarrow \qquad u_m{}^a = \delta^a{}_m \ , \qquad v_{23} = Ny^1 \ . \tag{9.147}$$

The corresponding flux components are then obtained by inserting these values into eq. (9.138), to obtain

$$H_{123} = N \ , \qquad f_{12}{}^3 = f_{23} = f_{31}{}^2 = 0 \ , \tag{9.148}$$

and so an H flux has been turned on in the effective action. Now, performing a T-dualisation in the y_2 direction by use of the Busher rules gives us the metric of eq. (9.12) as well as a vanishing B field, which corresponds to

$$\mathrm{d}s^2 = \mathrm{d}y_1^2 + \mathrm{d}y_2^2 + (\mathrm{d}y_3 + Ny_1\,\mathrm{d}y_2)^2 \ , \quad B_{mn} = 0 \ , \quad \Leftrightarrow \quad u^a{}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & Ny_1 & 0 \end{pmatrix} \ , \quad v_{mn} = 0 \ . \tag{9.149}$$

The flux components are then given by

$$H_{123} = f_{23}{}^1 = f_{31}{}^2 = 0 \ , \qquad f_{12}{}^3 = N \ . \tag{9.150}$$

which is consistent with the T-duality chain. From our previous example we know that an additional T-dualisation in one of the y_2 or y_3 directions leaves the metric ill-defined. However, we will now see how this is easily incorporated in DFT. For the twisted torus the generalised metric is given by

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{mn} & -g^{mp}B_{pn} \\ B_{mp}g^{pn} & g_{mn} - B_{mp}g^{pq}B_{qn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -Ny_1 & 0 & 0 & 0 \\ 0 & -Ny_1 & 1+(Ny_1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+(Ny_1)^2 & Ny_1 \\ 0 & 0 & 0 & 0 & Ny_1 & 1 \end{pmatrix} . \tag{9.151}$$

Acting with a T-duality, or $O(D,D)$, transformation on this metric we have that

$$\mathcal{H}_{MN} \to h^{(2)\,P}_{M} h^{(2)\,Q}_{N} \mathcal{H}_{PQ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+(Ny_1)^2 & 0 & 0 & 0 & Ny_1 \\ 0 & 0 & 1+(Ny_1)^2 & 0 & -Ny_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -Ny_1 & 0 & 1 & 0 \\ 0 & Ny_1 & 0 & 0 & 0 & 1 \end{pmatrix} , \qquad h^{(2)} = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \end{pmatrix} , \tag{9.152}$$

which give us the metric and B field

$$\mathrm{d}s^2 = g_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n = \mathrm{d}y_1^2 + \frac{1}{1+(Ny_1)^2}(\mathrm{d}y_2^2 + \mathrm{d}y_3^2) \ , \qquad B_{23} = -\frac{Ny_1}{1+(Ny_1)^2} \ , \tag{9.153}$$

which are associated to the Q flux.

In DFT the Scherk-Schwarz compactification is *generalised* to incorporate both geometric and non-geometric fluxes. In the double space we have coordinates $x^M = (x^i, \tilde{x}_i)$ and they are further split according to $x^i = (x^\mu, y^m)$ and $\tilde{x}_i = (\tilde{x}_\mu, \tilde{y}_m)$. From this the double external space can be denoted $\mathbb{X} = (x^i, \tilde{x}_i)$ and the double internal space $\mathbb{Y} = (y^m, \tilde{y}_m)$. The vielbein and dilaton are then expanded as

$$E^A{}_M(X) = E^A{}_I(\mathbb{X})U^I{}_M(\mathbb{Y}) \ , \qquad \phi'(X) = \check{\phi}'(\mathbb{X}) + \lambda(\mathbb{Y}) \ , \tag{9.154}$$

where $M, N = 1, \ldots, 2D$ are curved indices in the original theory and $I, J = 1, \ldots, 2D$ are curved indices in the effective theory. If this ansatz is plugged into the formula of generalised fluxes in eq. (9.88) and eq. (9.89) one finds

$$\begin{aligned} \mathcal{F}_{ABC} &= \check{F}_{ABC} + F_{IJK}\check{E}_A{}^I \check{E}_B{}^J \check{E}_C{}^K \ , \\ \mathcal{F}_A &= \check{F}_A + \check{F}_I E_A{}^I \ , \end{aligned} \tag{9.155}$$

where the $\mathbb{X}$ and $\mathbb{Y}$-dependent quantities are respectively given by

$$\begin{aligned} \check{F}_{ABC} &= 3\check{\Omega}_{[ABC]} \ , \qquad & F_A &= \check{F}_A + \check{F}_I E_A{}^I \ , \qquad & \check{\Omega}_{ABC} &= E_A{}^I \partial_I \check{E}_B{}^J \check{E}_{CJ} \ , \\ F_{IJK} &= 3\Omega_{[IJK]} \ , \qquad & F_I &= \Omega^J{}_{JI} + 2U_I{}^M \partial_M \lambda \ , \qquad & \Omega_{IJK} &= U_I{}^M \partial_M U_J{}^N U_{KN} \ . \end{aligned} \tag{9.156}$$

This coordinate splitting into $\mathbb{X}$- and $\mathbb{Y}$-dependent quantities is only possible if one requires for the twists that

$$U_I{}^M \partial_M \check{g} = \partial_I \check{g} \ , \qquad \partial_M U_I{}^N \partial_M \check{g} = 0 \ , \tag{9.157}$$

in order to preserve Lorentz invariance in the external dimensions. Further, in order to obtain a $\mathbb{Y}$-independent effective action the generalised fluxes $\mathcal{F}_{\bar{A}\bar{B}\bar{C}}$ and $\mathcal{F}_{\bar{A}}$ should also be $\mathbb{Y}$-independent. Their possible $\mathbb{Y}$-dependence can only come from the "gaugings" F_{IJK} and F_I, and so we must take them to be constant, i.e.

$$F_{IJK} = \text{const.} \ , \qquad F_I = \text{const.} \ . \tag{9.158}$$

This will in turn imply that the twists are globally defined, and the internal space is then called *paralellisable*. The condition in eq. (9.157) can be rewritten in terms of these gaugings as

$$F_{IJ}{}^K \partial_K \check{g} = 0 \ , \qquad F^I \partial_I \check{g} = 0 \ , \tag{9.159}$$

and by the second condition we notice that $\partial_I \check{g}$ can only be non-vanishing in the external directions, and hence the f^Is can not have any legs in these directions, otherwise Lorentz invariance would be broken. This again enforces the exclusive $\mathbb{Y}$-independence of the gaugings.

Now, using generalised fluxes of eq. (9.155) in the flux formulation of the DFT action eq. (9.90), and taking $F_I = 0$ for simplicity, one would eventually obtain

$$\begin{aligned} S_{\mathrm{gDFT}} = \int \mathrm{d}\mathbb{Y} e^{-2\lambda} \int \mathrm{d}\mathbb{X} e^{-2\check{\phi}'} \Big[& -\tfrac{1}{4}(\check{F}_{IK}{}^{L} + F_{IK}{}^{L})(\check{F}_{JL}{}^{K} + F_{JL}{}^{K})\check{\mathcal{H}}^{IJ} \\ & -\tfrac{1}{12}(\check{F}_{IJ}{}^{K} + F_{IJ}{}^{K})(\check{F}_{LH}{}^{G} + F_{LH}{}^{G})\check{H}^{IL}\check{\mathcal{H}}^{JH}\check{\mathcal{H}}_{KG} \\ & -\tfrac{1}{6}(\check{F}_{IJK} + F_{IJK})(\check{F}^{IJK} + F^{IJK}) + (\check{\mathcal{H}}^{IJ} - \eta^{IJ})\check{F}_I\check{F}_J \Big] \ . \end{aligned} \tag{9.160}$$

This corresponds to a gauged DFT. If we cancel the gaugings according to $F_{IJK} = 0$ then one would recover the usual DFT action in eq. (9.90).

As for the gauge parameters, the reduction ansatz reads

$$\xi^M(X) = \check{\xi}^I(\mathbb{X}) U_I{}^M(\mathbb{Y}) \ , \tag{9.161}$$

and the gauge transformations are generated by the generalised Lie derivative

$$\hat{\mathcal{L}}_\xi V^M = U_I{}^M \check{\mathcal{L}}_{\check{\xi}} \check{V}^I \ , \qquad \check{\mathcal{L}}_{\check{\xi}} \check{V}^I = \hat{\mathcal{L}}_{\check{\xi}} \check{V}^I - F^I{}_{JK}\check{\xi}^J \check{V}^K \ , \tag{9.162}$$

which generates a deformation due to the gaugings. The algebra closes when imposing two constraints, namely the strong constraint in the external space but also given a quadratic constraint on the gaugings;

$$\partial_I \check{V} \partial^I \check{W} = 0 \ , \qquad F_{K[IJ}F_{KL]}{}^H = 0 \ , \tag{9.163}$$

for arbitrary vectors $\check{V}, \check{W}$. With a specific parameterisation of the metric η_{IJ} and taking $F_{IJK} = F_{ABC}$, the generalised fluxes can be rewritten and inserted into the gauged DFT action of eq. (9.160) to eventually give the gauged supergravity action in eq. (9.139). From this one might think that the SS and generalised SS compactifications result in the same gauged supergravity action, but there are important differences. We have seen that the ordinary SS procedure only allows for geometric fluxes, i.e. the geometric gaugings are generated by $u^a{}_m$ and v_{mn} which can be combined into the twist matrix

$$U^A{}_M = \begin{pmatrix} u_a{}^m & u_a{}^n v_{nm} \\ 0 & u^a{}_m \end{pmatrix} \ . \tag{9.164}$$

The $u^a{}_m$ corresponds to the metric, the v_{mn} to the B field and $U^A{}_M$ to the metric vielbein as in eq. (9.67). A T-duality transformation would break the triangular form, and so backgrounds are then only allowed to transform as monodromies up to u/v-transformations, corresponding to the geometric subgroup of $O(n,n)$. The generalised SS compactification allows for generic $O(n,n)$ transformations, so the twist may be expanded as

$$U^A{}_M = \begin{pmatrix} u_a{}^m & u_a{}^n v_{nm} \\ u^a{}_n \beta^{nm} & u^a{}_m + u^a{}_n \beta^{np} v_{pm} \end{pmatrix} \ . \tag{9.165}$$

This corresponds to the generic DFT vielbein of eq. (9.68), which accounts for both geometric and non-geometric fluxes as seen in section 9.2.3. Conclusively, the generalised SS compactification allows for backgrounds that are globally ill-defined from the usual geometrical point of view.

10

Exceptional Field Theory

The double field theory generalisation of type II supergravities which incorporated T-duality can be generalised further to also incorporate S-duality. Exceptional field theory is an extension of 10 and 11-dimensional supergravity that makes $S+T=U$-duality manifest. In 1979 Cremmer and Julia [117] found that compactifications of 11D supergravity on tori T^n give rise to exceptional symmetries $E_{n(n)}$ in $D = 11 - n$ dimensions. In particular they found that the moduli of the effective theory parameterise the symmetric space $E_{n(n)}/K(E_{n(n)})$ where the $E_{n(n)}$ is the split real form of the complex Lie algebra e_7 and $K(E_{n(n)})$ its maximal compact subgroup. While the $E_{n(n)}$ are global symmetries, the maximal compact subgroups are generalisations of Lorentz symmetry. Table 10.1 summarises the global $E_{n(n)}$ and local $K(E_{n(n)})$ symmetries for $D = 11, \ldots, 3$.

The work of extending or embedding string/M-theory in a way to make these exceptional symmetries apparent eventually resulted in a formulation known as exceptional field theory, which was formulated rather recently in 2013 in [118]. As the resulting symmetries depend on the number of compactified dimensions according to table 10.1, the exceptional field theories are usually constructed separately for each exceptional group. There is however current research aiming for a unified formalism known as extended geometries [119]. In this chapter we describe the general ExFT construction for the group $E_{7(7)}$ largely following [120]. See [121] and [122] for the corresponding $E_{6(6)}$ and $E_{8(8)}$ theories respectively.

Dimension	U-duality group	Global symmetry	Local symmetry
11	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
10 (IIA)	$\mathbb{1}$	$SO(1,1,\mathbb{R})/\mathbb{Z}_2$	$\mathbb{1}$
10 (IIB)	$SL(2,\mathbb{Z})$	$SL(2,\mathbb{R})$	$SO(2)$
9	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$	$SL(2,\mathbb{R}) \times O(1,1,\mathbb{R})$	$SO(2)$
8	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$U(2)$
7	$SL(5,\mathbb{Z})$	$SL(5,\mathbb{R})$	$USp(4)$
6	$O(5,5,\mathbb{Z})$	$O(5,5,\mathbb{R})$	$USp(4) \times USp(4)$
5	$E_{6(6)}(\mathbb{Z})$	$E_{6(6)}(\mathbb{R})$	$USp(8)$
4	$E_{7(7)}(\mathbb{Z})$	$E_{7(7)}(\mathbb{R})$	$SU(8)$
3	$E_{8(8)}(\mathbb{Z})$	$E_{8(8)}(\mathbb{R})$	$Spin(16)$

Table 10.1: Symmetries present after compactification of M-theory on a T^n.

10.1 $E_{7(7)}$ generalised diffeomorphisms

The smallest fundamental representation of the $E_{7(7)}$ group is the **56**. The approach, being similar to DFT, is based on an extended spacetime consisting of $4+56$ dimensions. The external four dimensions are accompanied by 56 "internal" coordinates Y^M with $M = 1, \ldots, 56$ which transform in the fundamental representation. The adjoint representation of the $E_{7(7)}$ group is **133** and so its Lie algebra has dimension 133 and generators t_α with $\alpha = 1, \ldots, 133$. The exceptional group is embedded in the symplectic group $E_{7(7)} \subset Sp(56)$, which implies the existence of an invariant antisymmetric tensor ω^{MN}, with M, N being fundamental indices. This invariant tensor fulfils $\omega^{MK}\omega_{NK} = \delta_N{}^M$ and is used to raise and lower fundamental indices; $A^M = \omega^{MN}A_N$ and $A_M = A^N\omega_{MN}$. As for the adjoint indices, they are raised and lowered using the symmetric Cartan-Killing form $C_{\alpha\beta} \equiv (t_\alpha)_M{}^N(t_\alpha)_N{}^M$. The $(t_\alpha)_M{}^N$ are the gauge group generators in the fundamental representation. The invariance of ω^{MN} make the $(t_\alpha)_{MN}$ symmetric in their fundamental indices.

The ExFT variant of the strong/weak constraints of DFT are referred to as section constraints and are written in terms of the $E_{7(7)}$ generators $(t_\alpha)^{MN}$ as well as the invariant symplectic form ω^{MN} according to

$$\begin{aligned} &(t_\alpha)^{MN}\partial_M\partial_N A = 0 \ , \\ &(t_\alpha)^{MN}\partial_M A\partial_N B = 0 \ , \qquad \omega^{MN}\partial_M A\partial_N B = 0 \ , \end{aligned} \tag{10.1}$$

for any fields or gauge parameters A, B. Note that the first line corresponds to the ExFT equivalent of the weak constraint and the second line to strong constraints.

10.1.1 The generalised Lie derivative and E-bracket

An ExFT is uniquely determined by its bosonic gauge symmetries which consist of the generalised diffeomorphisms in its coordinates (x^μ, Y^M). We will start by considering the internal coordinates. As in the case of DFT, the diffeomorphisms take the form of a generalised Lie derivative $\mathbb{L}_\xi$ with parameter ξ. The action of a generalised Lie derivative with respect to a vector parameter ξ on a vector A of weight λ is written as

$$\begin{aligned} \delta_\xi A^M = \mathbb{L}_\xi A^M &\equiv \lambda^N\partial_N A^M - 12\mathbb{P}^M{}_N{}^P{}_Q\partial_P\xi^Q A^N + \lambda\partial_N\xi^N A^M \\ &= \xi^N\partial_N A^M + (\lambda - \tfrac{1}{2})\partial_N\xi^N A^M - \partial_N\xi^M A^N - 12(t^\alpha)^{MN}(t^\alpha)_{PQ}\partial_N\xi^P A^Q \\ &\quad - \tfrac{1}{2}\omega^{MN}\omega_{PQ}\partial_N\xi^P A^Q \ , \end{aligned} \tag{10.2}$$

where the first line is the general form of a general Lie derivative [123] with $\mathbb{P}^M{}_N{}^P{}_Q$ being a projector from fundamental onto adjoint indices. On the second line we have written out explicitly the projection operator for an $E_{7(7)}$ tensor, which is given by

$$\begin{aligned} \mathbb{P}^M{}_N{}^P{}_Q &\equiv (t_\alpha)_N{}^M(t^\alpha)_Q{}^P \\ &= \tfrac{1}{24}\delta_N^M\delta_Q^P + \tfrac{1}{12}\delta_N^P\delta_Q^M + (t_\alpha)_{NQ}(t^\alpha)^{MP} - \tfrac{1}{24}\omega_{NQ}\omega^{MP} \ , \end{aligned} \tag{10.3}$$

and satisfies $\mathbb{P}^M{}_N{}^N{}_M = 133$. The invariant tensor ω^{MN} is constant so that $\partial_P\omega^{MN} = 0$, and has weight $\lambda = 0$. With a gauge transformation of a tensor with n fundamental indices the Lie derivative of eq. (10.2) is generalised by having n projectors. Since ω^{MN} is antisymmetric in its indices and the projector is symmetric in its top and bottom indices, we have that

$$\mathbb{L}_\xi\omega^{MN} = 0 \ . \tag{10.4}$$

This also implies that ω can lower indices in the ordinary covariant manner as $A_M \equiv \omega_{NM}A^N$. Given the form of eq. (10.2), there are some choices of gauge parameters ξ^M which make the Lie derivative vanish as a consequence of the section constraints of eq. (10.1). Hence they do not generate a gauge transformation. These 'trivial' gauge transformations must have gauge parameters of the form

$$\xi^M = (t^\alpha)^{MN}\partial_N X_\alpha \ , \qquad \text{or} \qquad \xi^M = \omega^{MN}\partial_N X \ , \tag{10.5}$$

for arbitrary fields X_α, X. There is also another type of trivial parameter more general than eq. (10.5). This type is defined by

$$\xi^M = \omega^{MN} X_N \ , \tag{10.6}$$

with X_N being a field which satisfies the same section constraints as the ∂_M do in eq. (10.1), i.e. combinations like

$$\begin{aligned} (t_\alpha)^{MN} X_M A \partial_N B = 0 \ , \qquad (t_\alpha)^{MN} X_M A X_N B = 0 \ , \qquad (t_\alpha)^{MN} X_M X_N A = 0 \ , \ldots \\ \omega^{MN} X_M \partial_N A = 0 \ , \qquad \omega^{MN} X_M A \partial_N B = 0 \ , \qquad \omega^{MN} X_M A X_N B = 0 \ , \ldots \end{aligned} \tag{10.7}$$

The classical trivial parameter of eq. (10.5) is a special case of eq. (10.6) with $X_N = \partial_N X$. In DFT or $E_{6(6)}$ ExFT there is no real analogue of the more general trivial parameters of eq. (10.6). The reason for including them is that they will come in useful when checking the Jacobi identity later.

Moving on to the gauge algebra generated by eq. (10.2), the gauge transformations close according to

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1, \xi_2]_{\mathrm{E}}} \ , \tag{10.8}$$

where the so-called E-bracket $[\cdot,\cdot]_{\mathrm{E}}$ is given by

$$[\xi_1, \xi_2]^M_{\mathrm{E}} = 2\xi^N_{[2} \partial_N \xi^M_{1]} + 12 (t_\alpha)^{MN} (t^\alpha)_{PQ} \xi^P_{[2} \partial_P \xi^Q_{1]} - \tfrac{1}{4} \omega^{MP} \omega_{NQ} \partial_P (\xi^N_2 \xi^Q_1) \ . \tag{10.9}$$

The last term in this expression is of the trivial form of eq. (10.6), i.e. it can be written $\omega^{MP} \partial_P (\omega_{NQ} \xi^N_2 \xi^Q_1) - \omega^{MP} \partial_P (\omega_{NQ}) \xi^N_2 \xi^Q_1$ which indeed vanishes by eq. (10.7) according to the section constraint in eq. (10.1). There is however reason to keep this term, as will become apparent soon. Before checking that the E-bracket satisfies the Jacobi-identity, it is useful to introduce some notation, namely a Dorfman type product for vectors of weight 1/2;

$$\begin{aligned} (V \circ W)^M \equiv \mathbb{L}_V W^M &= V^N \partial_N W^M - W^N \partial_N V^M - 12 (t_\alpha)^{MN} (t^\alpha)_{PQ} \partial_N V^P W^Q \\ &\quad - \tfrac{1}{2} \omega^{MN} \omega_{PQ} \partial_N V^P W^Q \\ &= [V, W]^M_{\mathrm{E}} - 6 (t^\alpha)^{MN} \partial_N ((t_\alpha)_{PQ} W^P V^Q) \\ &\quad + \tfrac{1}{4} \omega^{MP} \omega_{NQ} (V^N \partial_P W^Q + W^N \partial_P V^Q) \ . \end{aligned} \tag{10.10}$$

The last term on the second line in eq. (10.10) can not be written as a total derivative as opposed to the DFT (and $E_{6(6)}$) case. It is however of a trivial form like in eq. (10.7) which vanishes due to section constraints. The same applies more clearly for the $-6(t^\alpha)^{MN}$ term in eq. (10.10) above. Hence these extra terms generate trivial gauge transformations and so

$$\mathbb{L}_{V \circ W} = \mathbb{L}_{[V,W]_{\mathrm{E}}} \ . \tag{10.11}$$

The Dorfman product of eq. (10.10) is also antisymmetric and can be shown to fulfil a Leibniz-like identity, so that for each property we have that

$$\tfrac{1}{2} (V \circ W - W \circ V)^M = [V, W]^M_{\mathrm{E}} \ , \tag{10.12}$$

$$U \circ (V \circ W) = (U \circ V) \circ W + V \circ (U \circ W) \ . \tag{10.13}$$

By the first identity $[V_1, V_2]_{\mathrm{E}} = [V_{[1}, V_{2]}]_{\mathrm{E}}$, and so using both these identities we have that the Jacobi identity is not closed;

$$\begin{aligned} \left[V_{[1}, [V_2, V_{3]}]_{\mathrm{E}}\right]_{\mathrm{E}} = \left[V_{[1}, V_2 \circ V_{3]}\right]_{\mathrm{E}} &= \tfrac{1}{2} \big(V_{[1} \circ (V_2 \circ V_{3]}) - (V_{[2} \circ V_3) \circ V_{1]} \big) \\ &= \tfrac{1}{2} \big((V_{[1} \circ V_2) \circ V_{3]} + V_{[2} \circ (V_1 \circ V_{3]}) - (V_{[2} \circ V_3) \circ V_{1]} \big) \\ &= -\tfrac{1}{2} V_{[1} \circ (V_2 \circ V_{3]}) \ , \end{aligned} \tag{10.14}$$

where we used the antisymmetric properties of the indices in the last step. From eq. (10.10) we also identify another expression as

$$\begin{aligned} \big[V_{[1},[V_2,V_{3]}]_\mathrm{E}\big]_\mathrm{E} = \big[V_{[1},V_2\circ V_{3]}\big]_\mathrm{E} = V_{[1}\circ(V_2\circ V_{3]}) + 6(t^\alpha)^{MN}\partial_N((t_\alpha)_{PQ}[V_2,V_3]^P_\mathrm{E}V_1^Q) \\ - \tfrac{1}{4}\omega^{MP}\omega_{NQ}(V_1^N\partial_P[V_2,V_3]^Q_\mathrm{E} + [V_2,V_3]^N_\mathrm{E}\partial_P V_1^Q)\ , \end{aligned} \tag{10.15}$$

and so comparing this expression with eq. (10.14) we have that

$$\begin{aligned} 3\big[V_{[1},[V_2,V_{3]}]_\mathrm{E}\big]_\mathrm{E} &= 6(t^\alpha)^{MN}\partial_N((t_\alpha)_{PQ}[V_2,V_3]^P_\mathrm{E}V_1^Q) \\ &\quad - \tfrac{1}{4}\omega^{MP}\omega_{NQ}(V_1^N\partial_P[V_2,V_3]^Q_\mathrm{E} + [V_2,V_3]^N_\mathrm{E}\partial_P V_1^Q)\ . \end{aligned} \tag{10.16}$$

Defining the Jacobiator, i.e. the failure of the E-bracket to fulfil the Jacobi-identity, as $J \equiv 3\big[[V_{[1},V_2]_\mathrm{E},V_{3]},\big]_\mathrm{E} = -3\big[V_{[1},[V_2,V_{3]}]_\mathrm{E}\big]_\mathrm{E}$, we have that

$$\begin{aligned} J &= -\tfrac{1}{2}(t^\alpha)^{MN}\partial_N((t_\alpha)_{PQ}[V_2,V_3]^P_\mathrm{E}V_1^Q + \text{ cycl.}) \\ &\quad + \tfrac{1}{12}\omega^{MP}\omega_{NQ}(V_1^N\partial_P[V_2,V_3]^Q_\mathrm{E} + [V_2,V_3]^N_\mathrm{E}\partial_P V_1^Q + \text{ cycl.})\ , \end{aligned} \tag{10.17}$$

writing out the total antisymmetrisation.

Now, having seen the action of the generalised Lie derivative on vectors with fundamental indices, we turn to the ones within the adjoint representation. With an adjoint vector V_α of weight λ, the projector in eq. (10.2) is replaced with structure constants $f_{\alpha\beta}{}^\gamma$ of $E_{7(7)}$ so that

$$\delta_\xi V_\alpha = \xi^N\partial_N V_\alpha + 12 f_{\alpha\beta}{}^\gamma (t^\beta)_Q{}^P\partial_P\xi^Q V_\gamma + \lambda\partial_P\xi^P V_\alpha\ . \tag{10.18}$$

10.1.2 Covariant derivatives

Since the gauge transformations introduced in the previous section depend on both external and internal indices, constructing covariant external derivatives will therefore require an associated gauge connection $A_\mu{}^M$. Covariantising in the usual manner we then have

$$\partial_\mu \to \mathcal{D}_\mu \equiv \partial_\mu - \mathbb{L}_{A_\mu}\ . \tag{10.19}$$

Acting on a vector B^M, we have by eq. (10.2) that

$$\begin{aligned} \mathcal{D}_\mu B^M &= \partial_\mu B^M - A^N_\mu\partial_N B^M + (\tfrac{1}{2}-\lambda)\partial_N A^N_\mu B^M + \partial_N A^M_\mu B^N \\ &\quad + 12(t^\alpha)^{MN}(t^\alpha)_{PQ}\partial_N A^P_\mu B^Q + \tfrac{1}{2}\omega^{MN}\omega_{PQ}\partial_N A^P_\mu B^Q \\ &\equiv D_\mu B^M - \lambda\partial_N A^N_\mu B^M\ , \end{aligned} \tag{10.20}$$

introducing D_μ for later use. Requiring that the covariant derivative transforms covariantly, i.e. $\mathbb{L}_\xi(\mathcal{D}_\mu B^M) = \delta\mathcal{D}_\mu\mathbb{L}_\xi B^M$, enables us to obtain the gauge variation of $A_\mu{}^M$ as

$$\begin{aligned} \delta_\xi A_\mu{}^M &= \partial_\mu\xi^M + \xi^N\partial_N A_\mu{}^M - \partial_N\xi^M A_\mu{}^N \\ &\quad + 12(t^\alpha)^{MN}(t^\alpha)_{PQ}\partial_N\xi^P A_\mu{}^Q + \tfrac{1}{2}\omega^{MN}\omega_{PQ}\partial_N\xi^P A_\mu{}^Q \\ &\equiv D_\mu\xi^M - \tfrac{1}{2}(\partial_N A_\mu{}^N)\xi^M \\ &\equiv \mathcal{D}_\mu\xi^M\ , \end{aligned} \tag{10.21}$$

which shows that the gauge parameter ξ^M is a vector of weight 1/2. As for the corresponding field strength

$$\begin{aligned} F_{\mu\nu}{}^M &= 2\partial_{[\mu}A_{\nu]}{}^M - [A_\mu,A_\nu]^M_\mathrm{E} \\ &= 2\partial_{[\mu}A_{\nu]}{}^M - 2A_{[\mu}{}^N\partial_N A_{\nu]}{}^M - \Big(12(t_\alpha)^{MP}(t^\alpha)_{NQ} - \tfrac{1}{2}\omega^{MP}\omega_{NQ}\Big)A_{[\mu}{}^N\partial_P A_{\nu]}{}^Q\ , \end{aligned} \tag{10.22}$$

it has gauge transformations of the general form

$$\begin{aligned}\delta_\xi F_{\mu\nu}{}^M &= 2D_{[\mu}\delta_\xi A_{\nu]}{}^M - \partial_N A_{[\mu}{}^N \delta_\xi A_{\nu]}{}^M - 12(t_\alpha)^{MP}(t^\alpha)_{NQ}\partial_P(A_{[\mu}{}^N\delta_\xi A_{\nu]}{}^Q) \\ &\quad - \tfrac{1}{2}\omega^{MP}\omega_{QN}\left(A_{[\mu}{}^N\partial_P\delta_\xi A_{\nu]}{}^Q - \partial_P A_{[\mu}{}^N\delta_\xi A_{\nu]}{}^Q\right) \ .\end{aligned} \tag{10.23}$$

The field strength is however not covariant under gauge transformations in the same manner as $A_\mu{}^M$ in eq. (10.21). The non-covariance of the field strength is a consequence of the failure of the E-bracket to fulfil the Jacobi-identity, i.e. $J \neq 0$ in eq. (10.17). In order to construct a covariant $F_{\mu\nu}{}^M$ it is customary to add a term to it, namely

$$\mathcal{F}_{\mu\nu}{}^M \equiv F_{\mu\nu}{}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} \ , \tag{10.24}$$

where $B_{\mu\nu\alpha}$ is a 2-form in the adjoint representation. The hope is that the transformation properties of this field may render $\mathcal{F}_{\mu\nu}{}^M$ covariant under gauge transformations. It turns out, however, that this addition will not be enough. While this works for the $E_{6(6)}$ case [121], the $\omega^{MP}\omega_{QN}$ terms in eq. (10.23) will not be absorbed by the transformation properties of $B_{\mu\nu\alpha}$, and so another term must be added to compensate for this. The result is obtained [120] as

$$\mathcal{F}_{\mu\nu}{}^M \equiv F_{\mu\nu}{}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \tfrac{1}{2}\omega^{MN}B_{\mu\nu N} \ , \tag{10.25}$$

where the $B_{\mu\nu N}$ is of the form eq. (10.6), and hence constrained in the same sense as X^N in eq. (10.7). With this new form the general gauge transformation of $\mathcal{F}_{\mu\nu}{}^M$ becomes

$$\begin{aligned}\delta_\xi \mathcal{F}_{\mu\nu}{}^M &= 2\mathcal{D}_{[\mu}\delta_\xi A_{\nu]}{}^M - 12(t^\alpha)^{MN}\partial_N\left((t_\alpha)_{PQ}A_{[\mu}{}^P\delta_\xi A_{\nu]}{}^Q + \delta_\xi B_{\mu\nu\alpha}\right) \\ &\quad - \tfrac{1}{2}\omega^{MP}\left(\omega_{NQ}(A_{[\mu}{}^N\partial_P\delta_\xi A_{\nu]}{}^Q - \partial_P A_{[\mu}{}^N\delta_\xi A_{\nu]}{}^Q) + \delta_\xi B_{\mu\nu P}\right) \ .\end{aligned} \tag{10.26}$$

Defining the $\delta_\xi B_{\mu\nu\alpha}$ and $\delta_\xi B_{\mu\nu P}$ such that

$$\begin{aligned}\Delta_\xi B_{\mu\nu\alpha} &\equiv (t_\alpha)_{PQ}A_{[\mu}{}^P\delta_\xi A_{\nu]}{}^Q + \delta_\xi B_{\mu\nu\alpha} = (t_\alpha)_{PQ}\xi^P\mathcal{F}_{\mu\nu}{}^Q \ , \\ \Delta_\xi B_{\mu\nu P} &\equiv \omega_{NQ}[A_{[\mu}{}^N\partial_P\delta_\xi A_{\nu]}{}^Q - \partial_P A_{[\mu}{}^N\delta_\xi A_{\nu]}{}^Q] + \delta_\xi B_{\mu\nu P} \\ &= -\omega_{PQ}[\mathcal{F}_{\mu\nu}{}^P\partial_M\xi^Q - \xi^Q\partial_M\mathcal{F}_{\mu\nu}{}^Q] \ ,\end{aligned} \tag{10.27}$$

we may use these expressions, eq. (10.21) and the fact that $\mathbb{L}_{F_{\mu\nu}}\xi^M = \mathbb{L}_{\mathcal{F}_{\mu\nu}}\xi^M$ to find the covariant transformation

$$\delta_\xi\mathcal{F}_{\mu\nu}{}^M = \partial_N\mathcal{F}_{\mu\nu}{}^M\xi^N - 12\mathbb{P}^M{}_N{}^P{}_Q\partial_P\xi^Q\mathcal{F}_{\mu\nu}{}^N + \tfrac{1}{2}\partial_N\xi^N\mathcal{F}_{\mu\nu}{}^M \ . \tag{10.28}$$

Hence the field strength also transforms as a vector of weight 1/2. The ΔBs will come in useful later. As in the case of the previous section we have that $F_{\mu\nu}{}^M$ and $\mathcal{F}_{\mu\nu}{}^M$ differ by terms that are trivial under gauge transformations, hence $\mathbb{L}_{F_{\mu\nu}} = \mathbb{L}_{\mathcal{F}_{\mu\nu}}$. The Bianchi identity is given by

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M = -12(t^\alpha)^{MN}\partial_N\mathcal{H}_{\mu\nu\rho\alpha} - \tfrac{1}{2}\omega^{MN}\mathcal{H}_{\mu\nu\rho N} \ , \tag{10.29}$$

where the 3-form field strengths $\mathcal{H}$ are defined by this equation up to terms that vanish under the action of $(t^\alpha)^{MN}\partial_N$.

10.2 Covariant $E_{7(7)}$ exceptional field theory

The action of the $E_{7(7)}$ ExFT is proposed to be

$$S = \int \mathrm{d}x^4\,\mathrm{d}Y^{56} e\Big(\hat{R} + \tfrac{1}{48}g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{D}_\nu\mathcal{M}_{MN} - \tfrac{1}{8}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N + e^{-1}\mathcal{L}_{\text{top}} - V(g_{\mu\nu},\mathcal{M}_{MN})\Big) \tag{10.30}$$

with e denoting the vielbein determinant. The form of this action is similar to the ones of gauged $\mathcal{N} = 8$ supergravities in 4D [124]. In this section we will discuss the ingoing terms of this action. The ExFT field content is

$$\{e_\mu{}^a \ , \ A_\mu{}^M \ , \ B_{\mu\nu\alpha}, \ B_{\mu\nu M} \ , \ \mathcal{M}_{MN}\} \ , \tag{10.31}$$

and the gauge fields fulfil the self-duality relation

$$\mathcal{F}_{\mu\nu}{}^M = -\tfrac{1}{2} e \epsilon_{\mu\nu\rho\sigma} \omega^{MN} \mathcal{M}_{NP} \mathcal{F}^{\rho\sigma P} \ , \tag{10.32}$$

which upon dropping the internal indices is known from $D = 11$ supergravity reduction to 4D. The ExFT construction will reproduce the 11D supergravity dynamics of these fields.

10.2.1 Kinetic and topological terms

The 4D vielbein $e_\mu{}^a$ is a scalar-density under gauge transformations with weight $\lambda = 1/2$, and hence the action of a covariant derivative on it leaves us with

$$\mathcal{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \partial_M e_\nu{}^a A_\mu{}^M - \tfrac{1}{2} \partial_M A_\mu{}^M e_\nu{}^a \ . \tag{10.33}$$

The Einstein-Hilbert term in eq. (10.30) is

$$\mathcal{L}_{\text{EH}} = e\hat{R} = e e_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab} \equiv e e_a{}^\mu e_b{}^\nu \left(R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^M e^{a\rho} \partial_M e_\rho{}^b \right) \ , \tag{10.34}$$

and has the property of being Lorentz invariant. $R_{\mu\nu}{}^{ab}$ is the curvature written in terms of the spin connection, where the ingoing derivative takes the form of eq. (10.33). The additional term $\mathcal{F}_{\mu\nu}{}^M e^{a\rho} \partial_M e_\rho{}^b$ in the above action is due to the non-commutivity of the covariant derivative $\mathcal{D}_\mu$. This results in that the corresponding covariant Riemann tensor does not transform as a tensor under local Lorentz transformations $\delta_\xi \omega_\mu{}^{ab} = \mathcal{D}_\mu \xi^{ab}$, and the addition above has been shown [125] to compensate for this. Under internal generalised diffeomorphisms it transforms as a density of weight 1 with the vielbein determinant contributing with 2 and the inverted vielbeins with $-1/2$ each.

Both the scalar and Yang-Mills kinetic terms include $\mathcal{M}_{MN}$, which is a symmetric 56×56 matrix describing the coset space $E_{7(7)}/SU(8)$ parametrised by the scalars of the theory. The inverse matrix is obtained as usual with the fundamental indices being raised with the $E_{7(7)}$ invariant form ω^{MN}, i.e. $\mathcal{M}^{MN} = \omega^{MP} \omega^{NQ} \mathcal{M}_{PQ}$. The covariant derivatives in the scalar kinetic part

$$\mathcal{L}_{\text{scalar}} = \tfrac{1}{48} e g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}_{MN} \mathcal{D}_\nu \mathcal{M}^{MN} \ , \tag{10.35}$$

is given by eq. (10.20) for which the matrix $\mathcal{M}_{MN}$ transforms as a tensor of weigth $\lambda = 0$. Further $\det \mathcal{M}_{MN} = 1$ and with e having weight 2 and $g^{\mu\nu}$ having -1 the total weight of the scalar action (10.35) is 1, as required by gauge invariance.

The covariant field strengths $\mathcal{F}^{\mu\nu M}$ transforms by eq. (10.28) as vectors of weight 1/2, and so the Yang-Mills term

$$\mathcal{L}_{\text{YM}} = -\tfrac{1}{8} e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu M} \mathcal{F}_{\mu\nu}{}^N \ , \tag{10.36}$$

also have the required weight of 1. The field equations from this term are of second order, for all 56 vector fields $A_\mu{}^M$. This is resolved with the first order self-duality relation in eq. (10.32) which leaves us with 28 independent propagating fields.

Moving on to the topological terms, they exist as a countermeasure to eq. (10.36), so that it does not yield inconsistent field equations. It is useful to build the term as a 5D boundary term of a manifestly gauge invariant form as

$$S_{\text{top}} = -\tfrac{1}{24} \int_{\Sigma_5} \mathrm{d}^5 x \int \mathrm{d}^{56} Y \epsilon^{\mu\nu\rho\sigma\tau} \mathcal{F}_{\mu\nu}{}^M \mathcal{D}_\rho \mathcal{F}_{\sigma\tau M} \equiv \int_{\partial\Sigma_5} \mathrm{d}^4 x \int \mathrm{d}^{56} Y \mathcal{L}_{\text{top}} \ . \tag{10.37}$$

The gauge transformation of $\mathcal{L}_{\rm top}$ is given by

$$\delta_\xi \mathcal{L}_{\rm top} = -\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}\Big[\delta_\xi A_\mu{}^M \mathcal{D}_\nu \mathcal{F}_{\rho\sigma M} + \mathcal{F}_{\mu\nu M}\big(6(t^\alpha)^{MN}\partial_N \Delta_\xi B_{\rho\sigma\alpha} - \tfrac{1}{4}\omega^{MN}\Delta_\xi B_{\rho\sigma N}\big)\Big] \ , \quad (10.38)$$

with the ΔBs defined as in eq. (10.27).

Combining the two Lagrangians $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_{\rm top}$ gives parts of the self-duality equation in eq. (10.32). Namely, varying with respect to $B_{\mu\nu\alpha}$ result in eq. (10.32) up to what vanishes under $(t^\alpha)^{MN}$, see eq. (10.29). Varying them with respect to $B_{\mu\nu M}$ results in eq. (10.32), but then one also must remember that $B_{\mu\nu M}$ is constrained by eq. (10.7). In the same sense as the democratic formulation of the type II supergravities, the actions $\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm top}$ needs to be complemented with the self-duality relation of eq. (10.32).

The second order field equations of the vector fields can be obtained by using the Bianchi identity in eq. (10.29) and taking the exterior derivative of the self-duality equation in eq. (10.32), which results in that

$$\mathcal{D}_\nu(e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}) = -2\epsilon^{\mu\nu\rho\sigma}(t^\alpha)_M{}^N\partial_N\mathcal{H}_{\mu\nu\rho\sigma\alpha} + \tfrac{1}{12}\epsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma M} \ . \quad (10.39)$$

This can be contrasted with the field equations obtained by varying the Yang-Mills and topological Lagrangian of eq. (10.36) and eq. (10.37), which turn out to be

$$\mathcal{D}_\nu(e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}) = 2e(J^\mu_{1\,M} + J^\mu_{2\,M}) - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{D}_\nu\mathcal{F}_{\rho\sigma M} \ , \quad (10.40)$$

where J_1 is the gravitational current and J_2 the matter current defined by

$$\delta_A\mathcal{L}_{\rm EH} \equiv J^\mu_{1\,M}\delta A_\mu{}^M \ , \qquad \delta\mathcal{L}_{\rm sc} \equiv J^\mu_{2\,M}\delta A_\mu{}^M \ , \quad (10.41)$$

e.g.

$$J^\mu_{2\,M} = e^{-1}\partial_N(e\mathcal{D}^\mu\mathcal{M}^{NP}\mathcal{M}_{MP}) - \tfrac{1}{24}\mathcal{D}^\mu\mathcal{M}^{KL}\partial_M\mathcal{M}_{KL} \ . \quad (10.42)$$

Now, combining eq. (10.39) and eq. (10.40) we will obtain a relation between the scalar and tensor fields as

$$e(J^\mu_{1\,M} + J^\mu_{2\,M}) = -2\epsilon^{\mu\nu\rho\sigma}(t^\alpha)_M{}^N\partial_N\mathcal{H}_{\nu\rho\sigma\alpha} + \tfrac{1}{12}\epsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma M} \ . \quad (10.43)$$

Using eq. (10.42) in this expression and multiplying with $(t^\alpha)^{MN}\partial_N$, it can be split into two equations

$$\begin{aligned} e(J^\mu_{1\,M} - \tfrac{1}{24}\mathcal{D}^\mu\mathcal{M}^{KL}\partial_M\mathcal{M}_{KL}) &= \tfrac{1}{12}\epsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma M} \ , \\ -\tfrac{1}{2}(t_\alpha)_K{}^L(e\mathcal{D}^\mu\mathcal{M}^{KP}\mathcal{M}_{LP}) &= \epsilon^{\mu\nu\sigma\rho}\mathcal{H}_{\nu\sigma\rho\alpha} \ . \end{aligned} \quad (10.44)$$

With this we finish our discussion on the kinetic and topological terms and we proceed to the potential term in the next section.

10.2.2 The potential and external diffeomorphisms

The potential in the ExFT action of eq. (10.30) consists of combinations with the internal derivative ∂_M on the external $g_{\mu\nu}$ and scalar metric $\mathcal{M}_{MN}$ such that

$$\begin{aligned} V(g_{\mu\nu},\mathcal{M}_{MN}) = &-\tfrac{1}{48}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_N\mathcal{M}_{PQ} + \tfrac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_Q\mathcal{M}_{NP} - \tfrac{1}{2}g^{\mu\nu}\partial_M g_{\mu\nu}\partial_N\mathcal{M}^{MN} \\ &- \tfrac{1}{4}\mathcal{M}^{MN}g^{\mu\nu}\partial_M g_{\mu\nu}g^{\mu\nu}\partial_N g_{\mu\nu} - \tfrac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu} \ , \end{aligned} \quad (10.45)$$

where each coefficient has been determined by ξ^M gauge invariance. This is shown more explicitly in [121] for the $E_{6(6)}$ case which works analogously. However in the next we will briefly describe how this comes about. An external derivative ∂_M which acts on a $E_{7(7)}$ scalar s adds a density weight $-1/2$. It will transform under gauge transformations as $\delta_\xi(\partial_M s) = \mathbb{L}_\xi(\partial_M s)$ with $\lambda = -1/2$, where the gauge variation of the single scalar is $\delta_\xi s = \xi^M \partial_M s$. In the same manner the $\mathcal{M}_{MN}$ have weight zero, and so $\partial_M \mathcal{M}_{PQ}$ has weight $-1/2$. In the action the potential term is multiplied with the vierbein determinant of weight 2, and so should be combined with other products of weight $-1/2$ to give the total weight of 1 in the action as required by gauge invariance. However, unlike the partial derivative of a scalar, $\partial\mathcal{M}$ can contain terms that are not covariant. Hence there is need for an explicit check of covariance for each of the terms. Starting with the first one, we have that since $\mathcal{M}^{-1}\partial\mathcal{M}$ have values in the Lie algebra, the projector will work as an identity operator: $\mathbb{P}^M{}_N{}^P{}_Q \mathcal{M}^{QR}\partial_S\mathcal{M}_{PR} = \mathcal{M}^{MR}\partial_S\mathcal{M}_{NR}$. Hence, calculating the variation of eq. (10.2), the result is

$$\delta_\xi(-\tfrac{1}{48}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_N\mathcal{M}_{PQ}) = e\partial_M\partial_N\xi^P\mathcal{M}^{MQ}\mathcal{M}^{RN}\partial_Q\mathcal{M}_{PR} \;, \tag{10.46}$$

up to boundary terms. The second term is calculated in the same manner, for which we have

$$\begin{aligned}
\delta_\xi(\tfrac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_Q\mathcal{M}_{NP}) &= -e\partial_M\partial_N\xi^P\mathcal{M}^{MQ}\mathcal{M}^{RN}\partial_Q\mathcal{M}_{PR} \\
&\quad + e\partial_M\partial_N\xi^P\partial_P\mathcal{M}^{MN} + e\partial_M\partial_N\xi^N\partial_P\mathcal{M}^{MP} \\
&\quad - 12e\partial_M\partial_N\xi^P(t^\alpha)^{QN}(t_\alpha)_{PR}\mathcal{M}^{RS}\mathcal{M}^{MT}\partial_S\mathcal{M}_{TQ} \\
&\quad - \tfrac{1}{2}e\partial_M\partial_N\xi^P\omega^{QN}\omega_{PR}\mathcal{M}^{RS}\mathcal{M}^{MT}\partial_S\mathcal{M}_{TQ} \\
&= -e\partial_M\partial_N\xi^P\mathcal{M}^{MQ}\mathcal{M}^{RN}\partial_Q\mathcal{M}_{PR} \\
&\quad + e\partial_M\partial_N\xi^P\partial_P\mathcal{M}^{MN} + e\partial_M\partial_N\xi^N\partial_P\mathcal{M}^{MP} \;.
\end{aligned} \tag{10.47}$$

The terms on the third and fourth line vanish because of the section constraint. Namely, the current term $(j_S)^M{}_Q \equiv \mathcal{M}^{MT}\partial_S\mathcal{M}_{TQ}$ on the third line is Lie algebra valued so we may expand it in terms of t_α and then contract the adjoint indices according to

$$2(j_S)^M{}_Q(t^\alpha)^{NQ} = 2(j_S)\beta(t_\beta)^M{}_Q(t^\alpha)^{NQ} = (j_S)^\beta f_{\beta\alpha}{}^\gamma(t_\gamma)^{MN} \;. \tag{10.48}$$

Then the term on the third line will be contracted with $\partial_M\partial_N\xi^P$ which by the first section constraint in eq. (10.1) vanishes. In a similar manner the current term on the fourth line in eq. (10.47) is contracted with $\partial_M\partial_N\xi^P$ after ω^{QN} raises one of the current indices, which then also vanishes by the section constraint. As for the surviving three terms, the first one clearly cancels with the one in eq. (10.46), and the two last ones can be shown straightforwardly to cancel against the resulting variations of the three last potential terms in eq. (10.45). This then proves the full gauge invariance of the potential.

We end this section by a final note on external diffeomorphisms. While the various terms in the ExFT action of eq. (10.30) are determined by the internal ξ^M generalised diffeomorphisms, its relative coefficients are determined by the external generalised diffeomorphisms with parameter $\zeta^\mu \equiv \zeta^\mu(x, Y)$. The theory is manifestly gauge invariant for $\zeta^\mu \equiv \zeta^\mu(x)$, i.e. Y-independent parameters, the gauge transformations of the Lie algebra valued vectors with general external parameters are given by

$$\begin{aligned}
\delta_\zeta e_\mu{}^a &= \zeta^\nu\mathcal{D}_\nu e_\mu{}^a + \mathcal{D}_\mu\zeta^\nu e_\nu{}^a \;, \\
\delta_\zeta\mathcal{M}_{MN} &= \zeta^\mu\mathcal{D}_\mu\mathcal{M}_{MN} \;, \\
\delta_\zeta A_\mu &= \zeta^\nu\mathcal{F}_{\nu\mu}{}^M + \mathcal{M}^{MN}g_{\mu\nu}\partial_N\zeta^\nu \;.
\end{aligned} \tag{10.49}$$

This is the general form of covariantised diffeomorphism apart from the $\mathcal{M}$ term addition in $\delta_\zeta A_\mu{}^M$. As for the 2-form tensors we have

$$\begin{aligned}
\Delta_\zeta B_{\mu\nu\alpha} &= \zeta^\rho\mathcal{H}_{\mu\nu\rho\alpha} \;, \\
\Delta_\zeta B_{\mu\nu M} &= \zeta^\rho\mathcal{H}_{\mu\nu\rho M} + 2e\epsilon_{\mu\nu\rho\sigma}g^{\sigma\tau}\mathcal{D}^\rho(g_{\tau\lambda}\partial_M\zeta^\lambda) \;,
\end{aligned} \tag{10.50}$$

The second term in $\Delta_\zeta B_{\mu\nu M}$ is non-covariant but required as to make the equations of motions of the theory gauge invariant. By the definition of the field strength in eq. (10.25), the variations of eq. (10.49) and eq. (10.50) results in that

$$\begin{aligned}\delta_\zeta \mathcal{F}_{\mu\nu} = \mathcal{L}_\zeta \mathcal{F}_{\mu\nu}{}^M + 2(\mathcal{D}_{[\mu}\mathcal{M}^{MN} g_{\nu]\rho} - 6(t^\alpha)^{MN}\mathcal{H}_{\mu\nu\rho\alpha})\partial_N \zeta^\rho \\ 2\mathcal{M}^{MN}\mathcal{D}_{[\mu}(g_{\nu]\rho}\partial_N\zeta^\rho) - e\epsilon_{\mu\nu\rho\sigma} g^{\sigma\tau}\omega^{MN}\mathcal{D}^\rho(g_{\tau\lambda}\partial_N\xi^\lambda)\ ,\end{aligned} \tag{10.51}$$

with $\mathcal{L}_\zeta = \zeta^\rho \mathcal{D}_\rho$ being the standard transformation under covariantised diffeomorphisms. This expression can be rewritten using the self-duality relation of eq. (10.44) and assuming an on-shell condition so that the current terms are positive, we have

$$\delta_\zeta \mathcal{F}_{\mu\nu}{}^M = \mathcal{L}_\zeta \mathcal{F}_{\mu\nu} + 2\mathcal{D}_{[\mu}(\mathcal{M}^{MN} g_{\nu]\rho}\partial_N\zeta^\rho) - e\epsilon_{\mu\nu\rho\sigma}\omega^{MN}\mathcal{M}_{NP}\mathcal{D}^{[\rho}(\mathcal{M}_{PR} g^{\sigma]\lambda}\partial^R\zeta_\lambda)\ . \tag{10.52}$$

From this we can see that when calculating the variation of the self-duality relation in eq. (10.32), the non-covariant terms of the variation of $\mathcal{F}_{\mu\nu}{}^M$ will vanish, which leaves us with

$$\delta_\zeta(\mathcal{F}_{\mu\nu}{}^M + \tfrac{1}{2}e\epsilon_{\mu\nu\rho\sigma}\omega^{MN}\mathcal{M}_{NP}\mathcal{F}^{\rho\sigma P}) = \mathcal{L}_\zeta(\mathcal{F}_{\mu\nu}{}^M + \tfrac{1}{2}e\epsilon_{\mu\nu\rho\sigma}\omega^{MN}\mathcal{M}_{NP}\mathcal{F}^{\rho\sigma P})\ . \tag{10.53}$$

Hence the self-duality relation in eq. (10.32) is also duality covariant. This also confirms that the extra terms in the second variation of eq. (10.50) are necessary for this covariance. The exact form of eq. (10.44) relating the scalars and tensors is needed to fix the coefficients in the ExFT action of eq. (10.30).

10.3 Embedding of M-theory and type IIB supergravity

Having discussed the covariant $E_{7(7)}$ ExFT theory, it remains to see how the $D = 11$ and type IIB supergravities fit into it. The embedding of M-theory and type IIB supergravity theories in exceptional field theory corresponds to two different solutions to the section condition. The fact that there are only two solutions can be seen from the E_7 dynkin diagram. The solution of the section constraint relates to different splitting of coordinates under two different maximal subgroups of $E_{7(7)}$.

Starting with $D = 11$ supergravity, the relevant maximal subgroup in question is $GL(7) \subset E_{7(7)}$, under which the $E_{7(7)}$ fundamental representation decomposes according to

$$\mathbf{56} \to 7_{+3} + \bar{21}_{+1} + 21_{-1} + \bar{7}_{-3}\ , \tag{10.54}$$

where the subscripts are the $GL(1)$ weights in the sense that $GL(7) = SL(7) \times GL(1)$. Barred numbers correspond to conjugate representations. The corresponding internal coordinate splitting is then defined as

$$Y^M \to y^m + y_{mn} + y^{mn} + y_m\ , \tag{10.55}$$

for each representation, where the $m, n = 1, \ldots, 7$ label the vector representations of $GL(7)$ and the double indices are antisymmetric: $y^{mn} \equiv y^{[mn]}$, $y_{mn} \equiv y_{[mn]}$. As for the adjoint representation, it decomposes according to

$$\mathbf{133} \to \bar{7}_{+4} + 35_{+2} + 1_0 + 48_0 + \bar{35}_{-2} + 7_{-4}\ . \tag{10.56}$$

Considering $(t_\alpha)^{mn}$, its weight should be $3 + 3$ according to the corresponding indices in eq. (10.55), but from eq. (10.56) we see that there is no representation with $GL(1)$ weight $+6$, and therefore we must have that

$$(t_\alpha)^{mn} = 0\ . \tag{10.57}$$

This also means that the section constraint in eq. (10.1) is trivially fulfilled for all derivatives ∂_m. Hence, we may truncate the coordinate dependence of Y^M to only include the coordinates of the 7_{+3}, i.e. we may choose

$$Y^M \to y^m \ , \qquad \partial^{mn} \to 0 \ , \qquad \partial_{mn} \to 0 \ , \qquad \partial^m \to 0 \ . \tag{10.58}$$

By doing so all fields will depend on the seven coordinates as $\phi(x^\mu, Y^M) \to \phi(x^\mu, y^m)$, for instance $B_{\mu\nu m}$ is kept but $B_{\mu\nu}{}^{mn}, B_{\mu\nu mn}$ and $B_{\mu\nu}{}^m$ are set to zero. The $A_\mu{}^M$ is split in the same manner as eq. (10.55) and $B_{\mu\nu\alpha}$ as eq. (10.56). In $D = 11$ supergravity the scalar matrix is parametrised with the group valued vielbein $\mathcal{V}$ as $\mathcal{M}_{MN} = (\mathcal{V}\mathcal{V}^T)_{MN}$, where

$$\mathcal{V} = e^{\phi t_{(0)}} \mathcal{V}_7 e^{c_{mnp} t_{(+2)}^{mnp}} e^{\epsilon^{mnpqrst} c_{mnpqrs} t_{(+4)t}} \ . \tag{10.59}$$

Here the $t_{(0)}$ is are the generators of the 1_0 and 48_0 representations in eq. (10.56), $\mathcal{V}_7$ is an element of the $SL(7)$ subgroup, the t_{+2} is a generator of the 35_{+2} and similarly the $t_{(+4)}$ term is associated to the $\bar{7}_{+4}$ representation. Keeping only the positively graded terms can be thought of as a gauge choice.

Choosing an explicit representation of the $(t_\alpha)_M{}^N$ in terms of $SL(7)$ invariant tensors, splitting the coordinates and fields in accordance with eq. (10.55) and eq. (10.56) and truncating the coordinates according to eq. (10.58), the field equations can be mapped into the ones of supergravity. To do this some redefinitions of the 3-form and 6-form fields of $D = 11$ supergravity are required, but we will not go into the specifics here, merely illustrate how the decomposition of the fundamental representation can give the coordinate reduction needed. In [126] this is done explicitly for the $E_{6(6)}$ case, resulting in the full action.

For the type IIB solution we have that the relevant maximal subgroup is $GL(6) \times SL(2) \subset E_{7(7)}$. The decomposition of the fundamental and adjoint representation under this subgroup is

$$\begin{aligned} \mathbf{56} &\to (6,1)_{+2} + (\bar{6},2)_{+1} + (20,1)_0 + (6,2)_{-1} + (\bar{6},1)_{-2} \ , \\ \mathbf{133} &\to (1,2)_{+3} + (\bar{15},1)_{+2} + (15,2)_{+1} + (35+1,1)_0 + (\bar{15},2)_{-1} + (15,1)_{-2} + (1,2)_{-3} \ , \end{aligned} \tag{10.60}$$

where the left-hand side in each parenthesis corresponds to the $GL(6)$ representation decomposition and the right-hand side to the one for $SL(2)$. As before the subscript is the $GL(1)$ weight, as it acts like a scaling factor to the maximal subgroup. As before the coordinate split is

$$Y^M \to y^m + y_{m\tilde{m}} + y_{mnp} + y^{m\tilde{m}} + y_m \ . \tag{10.61}$$

Here $m, n, p = 1, \ldots, 6$ mark the fundamental representations of $GL(6)$ and $\tilde{m} = 1, 2$ the fundamental representation of $SL(2)$. Coordinates and fields are again antisymmetric in their internal indices; $y_{mnp} \equiv y_{[mnp]}$. It is clear that as before there is no $(t^\alpha)^{mn}$ in the decomposed adjoint representation, and so the same argument as before allows us to keep ∂_m and let $\partial^{m\tilde{m}}, \partial^{mnp}, \partial_{m\tilde{m}}$ and ∂^m go to zero as well as the associated fields.

11

Calabi-Yau Structures in Exceptional Generalised Geometry

There is a natural extended geometry in which supersymmetry for a generic flux background corresponds to integrable and globally defined G-structures. In chapter 4 we studied flux backgrounds using complex generalised geometry, where the central object is the generalised tangent bundle $TM \oplus T^*M$ admitting an $O(6,6)$ metric. We saw that the symplectic form J and holomorphic form Ω could be generalised to a pair of $O(6,6)$ pure spinors $\Phi^{\pm}$ defining an $SO(3,3) \subset O(6,6)$ structure, although their compatibility conditions $J \wedge \Omega = 0$ and $J^3 = \frac{3i}{4}\Omega \wedge \bar{\Omega}$ implied the structure to be $SU(3) \times SU(3)$. With only NSNS fluxes, the supersymmetry variations result in the conditions $\mathrm{d}\Phi^{\pm} = 0$, called integrability conditions. Admitting RR fluxes was shown to partially break this integrability. This language is convenient in classifying backgrounds and has other applications related for instance to the AdS/CFT correspondence, marginal deformations of conformal field theories and holography. The NSNS fluxes were incorporated, or "geometrised", in the $O(6,6)$ pure spinor formalism. In this chapter we will consider $E_{d(d)} \times \mathbb{R}^+$ or exceptional generalised geometry which incorporates both NSNS and RR fluxes and provides an extension to M-theory. A lot of the analysis will be built upon the fine paper [127].

11.1 Generalising the Calabi-Yau structure

As we know from chapter 4, a Calabi-Yau manifold admits a single covariantly constant spinor under the action of the subgroup $SU(3)$ of $Spin(6) \simeq SU(4)$. The two corresponding invariants are the symplectic $(1,1)$-form J and holomorphic $(3,0)$-form Ω, which satisfy the compatibility conditions

$$J \wedge \Omega = 0 \ , \qquad J^3 = \tfrac{3i}{4}\Omega \wedge \bar{\Omega} \ . \tag{11.1}$$

If the internal manifold had an integrable torsion-free $SU(3)$-structure, the invariant forms satisfy $\mathrm{d}J = \mathrm{d}\Omega = 0$, which is also equal to the supersymmetry conditions without fluxes. Individually the 2-form J is in fact invariant under $Sp(6,\mathbb{R}) \subset GL(6,\mathbb{R})$ and the 3-form Ω under $SL(3,\mathbb{C}) \subset GL(6,\mathbb{R})$, so the structure groups embed according to

$$\begin{array}{ccc} GL(6,\mathbb{R}) & \supset & Sp(6,\mathbb{R}) \text{ for } J \\ \cup & & \cup \\ SL(3,\mathbb{C}) \text{ for } \Omega & \supset & SU(3) \text{ for } \{J,\Omega\} \end{array} \ . \tag{11.2}$$

When extending this result to include the B field and dilaton, we saw that complex generalised geometry effectively describes this using the bispinors $\Phi^{\pm}$. They can also be defined as $\Phi^+ = e^{-(B+\phi)e^{-iJ}}$ and $\Phi^- = ie^{-(B+\phi)}\Omega$, and fulfil the compatibility or consistency conditions

$$\langle \Phi^+, \bar{\Phi}^- \rangle = \langle \Phi^-, \bar{\Phi}^- \rangle \ , \qquad \langle \Phi^+, V \cdot \Phi^- \rangle = \langle \bar{\Phi}^+, V \cdot \Phi^- \rangle = 0 \ , \tag{11.3}$$

for all sections $V \equiv \xi + \lambda \in \Gamma(TM \oplus T^*M)$ of the complex generalised tangent bundle. The Clifford action is defined as $V \cdot \Phi^{\pm} = V^A \Gamma_A \Phi^{\pm} = \iota_\xi \Phi^{\pm} + \lambda \wedge \Phi^{\pm}$. The scalar product, or Mukai pairing, is defined as

$$\langle \Phi, \Psi \rangle \equiv \sum_p (-1)^{\lfloor (p+1)/2 \rfloor} \Phi_p \wedge \Psi_{6-p} \ , \tag{11.4}$$

with Φ_p labelling the p-form component of Φ. The integrability conditions were shown to read

$$\mathrm{d}\Phi^{\pm} = 0 \ , \tag{11.5}$$

which defined a generalised Calabi-Yau metric. Using this formalism and turning on RR fluxes we saw that the supersymmetry variations in terms of pure spinors then broke integrability. Central to complex generalised geometry is the generalised tangent bundle $TM \oplus T^*M$, which admits an $O(6,6)$ metric η. The pure spinors $\Phi^{\pm}$ are separately invariant under two different groups $SU(3,3)_{\pm}$ but together they define a $SU(3) \times SU(3)$-structure;

$$\begin{array}{ccc} O(6,6) \times \mathbb{R}^+ & \supset & SU(3,3)_+ \text{ for } \Phi^+ \\ \cup & & \cup \\ SU(3,3)_- \text{ for } \Phi^- & \supset & SU(3) \times SU(3) \text{ for } \{\Phi^+, \Phi^-\} \end{array} \ . \tag{11.6}$$

It is now natural to ask how complex generalised geometry is in turn generalised to also include RR fluxes, and in the next section we will describe the structures of exceptional generalised geometry that generalise the structures in eq. (11.2) and eq. (11.6).

11.2 $E_{7(7)} \times \mathbb{R}^+$-structures

In our analysis of the exceptional structures we will describe the structure group dynamics as well as the generalised tangent bundle which admits an action of the exceptional group.

11.2.1 Generalised G-structures

In ordinary differential geometry the typical structure group of the tangent bundle TM of a d-dimensional manifold M is $GL(d, \mathbb{R})$. If there exists a G-structure it means that the structure group reduces to a subgroup G of $GL(d, \mathbb{R})$. A G-structure on a manifold is also equivalent to it admitting globally defined tensors or spinors that are invariant under the group action of G. In generalised geometry one considers an extended tangent bundle that admits the action of a larger group than $GL(d, \mathbb{R})$. The factor $\mathbb{R}^+$ encodes what is called the trombone symmetry, which can be thought of as the supergravity equivalent of the conformal rescaling symmetry in relativity. In complex generalised geometry we have considered the extended tangent bundle $E = TM \oplus T^*M$ that admits the action of the group $O(d,d) \times \mathbb{R}^+$. In what is known as exceptional generalised geometry the tangent bundle is extended further, as will be specified in section 11.2.2, and the admitted group is $E_{7(7)} \times \mathbb{R}^+$. In 4D $\mathcal{N} = 2$ backgrounds there are two generalised G-structures known as the

$$\begin{array}{ll} \text{hypermultiplet structure } J_\alpha \ , & G = Spin^*(12) \ , \\ \text{vector multiplet structure } K \ , & G = E_{6(2)} \ , \end{array} \tag{11.7}$$

where $Spin^*(12)$ is the double cover of $SO^*(12)$ which in turn is a particular real form of the Lie algebra $so(12, \mathbb{C})$. The index $\alpha = 1, 2$ is a fundamental $SL(2, \mathbb{R})$ index and $E_{6(2)}$ is the quasi-split form of E_6. These will from now on be referred to as the H and V structure respectively. Together they can be defined as an exceptional Calabi-Yau structure given that their common stabiliser group[1] is $Spin^*(12) \cap E_{6(2)} = SU(6)$, i.e. $G = SU(6)$ for $\{J_\alpha, K\}$. We can now write

[1] Given an action of a group on some space, and say a point, then the stabiliser group of that point is the subgroup whose action leaves the point fixed, or invariant.

E	G_{frame}	G_{hyper}	Θ_{hyper}	G_{vector}	Θ_{vector}
TM	$GL(6)$	$Sp(6,\mathbb{R})$	J	$SL(3,\mathbb{C})$	Ω
$TM \oplus T^*M$	$O(6,6)\times\mathbb{R}^+$	$SU(3,3)_+$	Φ^+	$SU(3,3)_-$	Φ^-
$TM \oplus T^*M \oplus \Lambda^- T^*M \oplus \ldots$	$E_{7(7)}\times\mathbb{R}^+$	$Spin^*(12)$	J_α	$E_{6(2)}$	K

Table 11.1: Tangent bundles E in ordinary, complex generalised and exceptional generalised geometry for type IIB supergravity. The group G_{frame} acts on the (generalised) frame bundle, the objects Θ_{hyper} and Θ_{vector} are invariant under the action of the reduced structure groups G_{hyper} and G_{vector} of the corresponding hyper- and vector multiplet structures. Note that the choice of IIB (rather than IIA) is only relevant for exceptional generalised geometry.

the corresponding embeddings for the exceptional Calabi-Yau metric as

$$\begin{array}{ccc} E_{7(7)}\times\mathbb{R}^+ & \supset & Spin^*(12) \text{ for } J_\alpha \\ \cup & & \cup \\ E_{6(2)} \text{ for } K & \supset & SU(6) \text{ for } \{J_\alpha, K\} \end{array} . \tag{11.8}$$

This then generalises the embeddings of eq. (11.2) and eq. (11.6) as well as the symplectic and complex structures on Calabi-Yau manifolds. This is summarised in table 11.1 for the case of the IIB tangent bundle and hyper- and vector multiplet structures. For IIA the tangent bundle is slightly different and the generalising situation is opposite; the H structure generalises the symplectic structure J and pure spinor Φ^+, and the V structure generalises Ω and Φ^-.

11.2.2 $E_{7(7)} \times \mathbb{R}^+$ generalised geometry

The generalised geometry of $E_{d(d)} \times \mathbb{R}^+$ describes structures on its generalised tangent bundle E, which admits an action of the $E_{d(d)}$ group. The generalised tangent bundle is built from the decomposition of the coordinate representation, which corresponds to the fundamental representation, under different subgroups. For $D = 11$ supergravity one uses the decomposition of the $\mathbf{56}$ under $GL(7,\mathbb{R})$;

$$\mathbf{56} \to \mathbf{7} + \bar{\mathbf{7}} + \mathbf{21} + \bar{\mathbf{21}} , \tag{11.9}$$

where each term corresponds to a vector, 1-form, 2-form and 5-form respectively. Hence for $D = 11$ supergravity on a $(d \le 7)$-dimensional manifold the generalised tangent bundle is

$$E_{\text{M}} \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M) , \tag{11.10}$$

where Λ^k denotes the space of k-forms. In type II theory on a $(d-1)$-dimensional manifold M one decomposes the fundamental form under the subgroup $GL(d-1) \times SL(2)$, which makes the generalised tangent bundle take the form

$$E_{\text{II}} \simeq TM \oplus T^*M \oplus \Lambda^\pm T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) , \tag{11.11}$$

where $\Lambda^\pm$ labels the space of even/odd-degree forms for type IIA/IIB respectively. The type IIA is a direct dimensional reduction of the $D = 11$ case, see details in Appendix B of [128], however in this chapter we choose to focus mainly on the type IIB theory. Both generalised tangent bundles of eq. (11.10) and eq. (11.11) are $E_{d(d)} \times \mathbb{R}^+$ vector bundles which for $d = 7$ transform in the fundamental $\mathbf{56_1}$ representation, the subscript $\mathbf{1}$ denoting the $\mathbb{R}^+$ weight. For type IIB the generalised tangent bundle can be rewritten in a way which shows the $SL(2,\mathbb{R})$ symmetry as

$$\begin{aligned} E_{\text{IIB}} &\simeq TM \oplus T^*M \oplus (T^*M \oplus \Lambda^3 T^*M \oplus \Lambda^5 T^*M) \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \\ &\simeq TM \oplus (T^*M \otimes S) \oplus \Lambda^3 T^*M \oplus (\Lambda^5 T^*M \otimes S) \oplus (T^*M \otimes \Lambda^6 T^*M) , \end{aligned} \tag{11.12}$$

having expanded $\Lambda^{-}T^{*}M$ in odd forms and S is an $\mathbb{R}^2$ bundle which transforms as a doublet of $SL(2,\mathbb{R})$. In this formalism, generalised tensors are defined to be sections of the vector bundles associated with different $E_{d(d)} \times \mathbb{R}^{+}$ representations. The sections V of the IIB bundle in eq. (11.12) can be written as

$$V = v + \lambda^{i} + \rho + \sigma^{i} + \tau \ , \tag{11.13}$$

where each term is a section of a bundle component; $v \in \Gamma(TM)$, $\lambda^{i} \in \Gamma(T^{*}M \otimes S)$, $\rho \in \Lambda^{3}T^{*}M$, $\sigma \in \Gamma(\Lambda^{5}T^{*}M \otimes S)$ and $\tau \in \Gamma(T^{*}M \otimes \Lambda^{6}T^{*}M)$. Another bundle that will be used later is the generalised adjoint bundle created from the decomposition of the adjoint representation under the appropriate subgroup. The frame bundle $\tilde{F}$ is an $E_{d(d)} \times \mathbb{R}^{+}$ principal bundle constructed from frames of E. The adjoint bundle $\mathrm{ad}\tilde{F}$ is then a generalised frame bundle that is associated to the adjoint representation of $E_{d(d)} \times \mathbb{R}^{+}$. The adjoint representation decomposes under $GL(d,\mathbb{R})$ according to

$$\mathbf{133} \to \mathbf{1} \oplus \mathbf{48} \oplus \mathbf{7} \oplus \bar{\mathbf{7}} \oplus \mathbf{35} \oplus \bar{\mathbf{35}} \ , \tag{11.14}$$

to which the corresponding adjoint tangent bundle is given by

$$\mathrm{ad}\tilde{F} \simeq \mathbb{R} \oplus (T^{*}M \otimes TM) \oplus \Lambda^{3}T^{*}M \oplus \Lambda^{6}T^{*}M \oplus \Lambda^{3}TM \oplus \Lambda^{6}TM \ . \tag{11.15}$$

In type IIB we have that

$$\begin{aligned}\mathrm{ad}\tilde{F} &\simeq \mathbb{R} \oplus (\mathbb{R} \oplus \Lambda^{6}TM \oplus \Lambda^{6}T^{*}M) \oplus [(TM \otimes T^{*}M) \oplus \Lambda^{2}T^{*}M \oplus \Lambda^{2}TM] \oplus (\Lambda^{+}TM \oplus \Lambda^{+}T^{*}M)\\ &\simeq \mathbb{R} \oplus (TM \otimes T^{*}M) \oplus (S \otimes S^{*})_{0} \oplus (S \otimes \Lambda^{2}TM) \oplus (S \otimes \Lambda^{2}T^{*}M) \oplus \Lambda^{4}TM \oplus \Lambda^{4}T^{*}M\\ &\quad \oplus (S \otimes \Lambda^{6}TM) \oplus (S \otimes \Lambda^{6}T^{*}M) \ ,\end{aligned} \tag{11.16}$$

where $(S \otimes S^{*})_{0}$ denotes the traceless part. The type IIA adjoint bundle reads as the first line of eq. (11.16) but with $\Lambda^{-} \to \Lambda^{+}$. In $d = 7$ this bundle transforms in the $\mathbf{1_0} + \mathbf{133_0}$ representation of $E_{7(7)} \times \mathbb{R}^{+}$, where the singlet term is the one generating the $\mathbb{R}^{+}$ action. The sections of this adjoint bundle can be written

$$R = l + r + a + \beta^{i} + B^{i} + \gamma + C + \alpha^{i} + \tilde{a}^{i} \ , \tag{11.17}$$

where as usual $l \in \mathbb{R}^{+}$, $r \in \Gamma(TM \otimes T^{*}M)$, etc. The *adjoint action* of the adjoint section $R \in \Gamma(\mathrm{ad}\tilde{F})$ on the tangent section $V \in \Gamma(E)$ is defined as $V' = R \cdot V$, where each component of $V' = v' + \lambda'^{i} + \rho' + \sigma'^{i} + \tau'$ is given by

$$\begin{aligned} v' &= lv + r \cdot v + \gamma \lrcorner \rho + \epsilon_{ij}\beta^{i} \lrcorner \lambda^{j} + \epsilon_{ij}\alpha^{i} \lrcorner \sigma^{j} \ ,\\ \lambda'^{i} &= l\lambda^{i} + r \cdot \lambda^{i} + a^{i}{}_{j}\lambda^{j} - \gamma \lrcorner \sigma^{i} + v \lrcorner B^{i} + \beta^{i} \lrcorner \rho - \alpha^{i} \lrcorner \tau \ ,\\ \rho' &= l\rho + r \cdot \rho + v \lrcorner C + \epsilon_{ij}\beta^{i} \lrcorner \sigma^{j} + \epsilon_{ij}\lambda^{i} \wedge B^{j} + \gamma \lrcorner \tau \ ,\\ \sigma'^{i} &= l\sigma^{i} + r \cdot \sigma^{i} + a^{i}{}_{j}\sigma^{j} - C \wedge \lambda^{i} + \rho \wedge B^{i} + \beta^{i} \lrcorner \tau + v \lrcorner \tilde{a}^{i} \ ,\\ \tau' &= l\tau + r \cdot \tau + \epsilon_{ij} j\lambda^{i} \wedge \tilde{a}^{j} - j\rho \wedge C - \epsilon_{ij} j\sigma^{i} \wedge B^{j} \ . \end{aligned} \tag{11.18}$$

In turn, the adjoint action of R on another adjoint section R' is defined as the commutator $R'' = [R, R']$, whose components are

$$
\begin{aligned}
l'' &= \tfrac{1}{2}(\gamma \lrcorner C - \gamma' \lrcorner C) + \tfrac{1}{4}\epsilon_{ij}(\beta^i \lrcorner B'^j - \beta'^i \lrcorner B^j) + \tfrac{3}{4}\epsilon_{ij}(\alpha^i \lrcorner \tilde{a}'^j - \alpha'^i \lrcorner \tilde{a}^j) \ , \\
r'' &= (r \cdot r' - r' \cdot r) + \epsilon_{ij}(j\beta^i \lrcorner jB'^j - j\beta'^i \lrcorner jB^j) - \tfrac{1}{4}\mathbb{1}\epsilon_{ij}(\beta^k \lrcorner B'^j - \beta'^i \lrcorner B^j) \\
&\quad + (j\gamma \lrcorner jC' - j\gamma' \lrcorner jC) - \tfrac{1}{2}\mathbb{1}(\gamma \lrcorner C' - \gamma' \lrcorner C) \\
&\quad + \epsilon_{ij}(j\alpha^i \lrcorner j\tilde{a}'^j - j\alpha'^i \lrcorner j\tilde{a}^j) - \tfrac{3}{4}\epsilon_{ij}(\alpha^i \lrcorner \tilde{a}'^j - \alpha'^i \lrcorner \tilde{a}^j) \ , \\
a''^i{}_j &= (a \cdot a' - a' \cdot a)^i{}_j + \epsilon_{jk}(\beta^i \lrcorner B'^k - \beta'^i \lrcorner B^k) - \tfrac{1}{2}\delta^i{}_j\epsilon_{kl}(\beta^k \lrcorner B'^l - \beta'^k \lrcorner B^l) \\
&\quad + \epsilon_{jk}(\alpha^i \lrcorner \tilde{a}'^k - \alpha'^i \lrcorner \tilde{a}^k) - \tfrac{1}{2}\delta^i{}_j\epsilon_{kl}(\alpha^k \lrcorner \tilde{a}'^l - \alpha'^k \lrcorner \tilde{a}^l) \ , \\
\beta''^i &= (r \cdot B'^i - r' \cdot B^i) + (a \cdot \beta' - a' \cdot \beta)^i - (\gamma \lrcorner B'^i - \gamma' \lrcorner B^i) - (\alpha^i \lrcorner C' - \alpha'^i \lrcorner C) \ , \\
B''^i &= (r \cdot B'^i - r' \cdot B^i) + (a \cdot B' - a' \cdot B)^i + (\beta^i \lrcorner C' - \beta'^i \lrcorner C) - (\gamma \lrcorner \tilde{a}'^i - \gamma' \lrcorner \tilde{a}^i) \ , \\
\gamma'' &= (r \cdot \gamma' - r' \cdot \gamma) + \epsilon_{ji}\beta^i \wedge \beta'^j + \epsilon_{ij}(\alpha^i \lrcorner B'^j - \alpha'^i \lrcorner B^j) \ , \\
C'' &= (r \cdot C' - r' \cdot C) - \epsilon_{ij}B^i \wedge B'^j + \epsilon_{ij}(\beta^i \lrcorner \tilde{a}'^j - \beta'^i \lrcorner \tilde{a}^j) \ , \\
\alpha''^i &= (r \cdot \alpha'^i - r' \cdot \alpha^i) + (a \cdot \alpha' - a' \cdot \alpha)^i - (\beta^i \wedge \gamma' - \beta'^i \wedge \gamma) \ , \\
\tilde{a}''^i &= (r \cdot \tilde{a}'^i - r' \cdot \tilde{a}^i) + (a \cdot \tilde{a}' - a' \cdot \tilde{a})^i + (B^i \wedge C' - B'^i \wedge C) \ .
\end{aligned}
\tag{11.19}
$$

The generalised tangent bundle is defined together with some patching rules. For the overlap of two local covers $U_{(\alpha)} \cap U_{(\beta)}$ of M, a generalised vector or section $V \in \Gamma(E)$ is patched by

$$
V_{(\alpha)} = e^{\mathrm{d}\Lambda^i_{(\alpha\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} \cdot V_{(\beta)} \ , \tag{11.20}
$$

where $\Lambda^i_{(\alpha\beta)}$ and $\tilde{\Lambda}_{(\alpha\beta)}$ are a local 1-form and 3-form respectively and $\cdot$ denotes the adjoint action defined in eq. (11.18). The twisted and untwisted structures are then related according to

$$
V = e^{B^i + C}\hat{V} \ , \qquad R = e^{B^i + C}\hat{R}e^{-B^i - C} \ , \tag{11.21}
$$

where the corresponding 3-form and 5-form field strengths are defined as

$$
F^i = \mathrm{d}B^i \ , \qquad F = \mathrm{d}C - \tfrac{1}{2}\epsilon_{ij}F^i \wedge B^j \ ; \qquad B^i = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}^i \ , \tag{11.22}
$$

and the B field is written as an $SL(2,\mathbb{R})$ doublet so that $F^1 = \mathrm{d}B_2 \equiv H$ and $F^2 = F_3 = \mathrm{d}C_2$, i.e. the usual IIB supergravity fields. Comparing the action of eq. (11.20) with eq. (11.21), one has that

$$
\begin{aligned}
B^i_{(\alpha)} &= B^i_{(\beta)} + \mathrm{d}\Lambda^i_{(\alpha\beta)} \ , \\
C_{(\alpha)} &= C_{(\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} + \tfrac{1}{2}\epsilon_{ij}\, \mathrm{d}\Lambda^i_{(\alpha\beta)} \wedge B^j_{(\beta)} \ .
\end{aligned}
\tag{11.23}
$$

While the potentials are defined locally, their corresponding fields strengths are globally well-defined.

The generalised Lie derivative in exceptional generalised geometry takes the form

$$
\mathbb{L}_V V' = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N \ , \tag{11.24}
$$

similar to the $O(d,d)$ generalised case. The derivative operator $\partial_M = (\partial_m, 0, \dots, 0)$ is embedded in the 1-form component of the dual generalised tangent bundle E^* via the map $T^*M \to E^*$. The $\times_{\mathrm{ad}}$ denotes a projection to the adjoint according to

$$
\times_{\mathrm{ad}} : E^* \otimes E \to \mathrm{ad}\tilde{F} \ . \tag{11.25}
$$

The projection operator part in eq. (11.24) acts on a section according to

$$\partial \times_{\mathrm{ad}} V = \partial \otimes v + \mathrm{d}\lambda^i + \mathrm{d}\rho + \mathrm{d}\sigma^i \ , \tag{11.26}$$

and so $\partial \times_{\mathrm{ad}} V \in \Gamma(\mathrm{ad}\tilde{F})$. As such, the last term in eq. (11.24) can be expanded using the adjoint action in eq. (11.18). The generalised Lie derivative acting on a generalised vector V' is then

$$\begin{aligned}\mathbb{L}_V V' = {} & \mathcal{L}_v v' + (\mathcal{L}_v \lambda'^i - \iota_{v'}\mathrm{d}\lambda^i) + (\mathcal{L}_v\rho' - \iota_{v'}\mathrm{d}\rho + \epsilon_{ij}\,\mathrm{d}\lambda^i \wedge \lambda'^j) \\ & + (\mathcal{L}_v\sigma'^i - \iota_{v'}\mathrm{d}\sigma^i + \mathrm{d}\rho \wedge \lambda'^i - \mathrm{d}\lambda^i \wedge \rho') \\ & + (\mathcal{L}_v\tau' - \epsilon_{ij} j\lambda'^i \wedge \mathrm{d}\sigma^j + j\rho' \wedge \mathrm{d}\rho + \epsilon_{ij} j\sigma'^i \wedge \mathrm{d}\lambda^j) \ ,\end{aligned} \tag{11.27}$$

the $\mathcal{L}$ denoting the ordinary Lie derivative. Similarly, the Lie derivative acting on a section of the adjoint tangent bundle is given by

$$\begin{aligned}\mathbb{L}_V R = {} & (\mathcal{L}_v l + \tfrac{1}{2}\gamma\lrcorner\,\mathrm{d}\rho + \tfrac{1}{4}\epsilon_{ij}\beta^i\lrcorner\,\mathrm{d}\lambda^j + \tfrac{3}{4}\epsilon_{ij}\alpha^j\lrcorner\,\mathrm{d}\sigma^j) \\ & + (\mathcal{L}_v r + j\gamma\lrcorner j\,\mathrm{d}\rho - \tfrac{1}{2}\mathbb{1}\gamma\lrcorner\,\mathrm{d}\rho + \epsilon_{ij} j\beta^i\lrcorner j\,\mathrm{d}\lambda^j - \tfrac{1}{4}\mathbb{1}\epsilon_{ij}\beta^i\lrcorner\,\mathrm{d}\lambda^j + \epsilon_{ij} j\alpha^i\lrcorner j\,\mathrm{d}\sigma^j - \tfrac{3}{4}\mathbb{1}\epsilon_{ij}\alpha^i\lrcorner\,\mathrm{d}\sigma^j) \\ & + (\mathcal{L}_v a^i{}_j + \epsilon_{ij}\beta^i\lrcorner\,\mathrm{d}\lambda^j - \tfrac{1}{2}\delta^i{}_j\epsilon_{kl}\beta^k\lrcorner\,\mathrm{d}\lambda^l + \epsilon_{jk}\alpha^i\lrcorner\,\mathrm{d}\sigma^k - \tfrac{1}{2}\delta^i{}_j\epsilon_{kl}\alpha^k\lrcorner\,\mathrm{d}\sigma^l) \\ & + (\mathcal{L}_v\beta^i - \gamma\lrcorner\,\mathrm{d}\lambda^i - \alpha^i\lrcorner\,\mathrm{d}\rho) \\ & + (\mathcal{L}_v B^i + r\cdot\mathrm{d}\lambda^i + a^i{}_j\,\mathrm{d}\lambda^j + \beta^i\lrcorner\,\mathrm{d}\rho - \gamma\lrcorner\,\mathrm{d}\sigma^i) \\ & + (\mathcal{L}_v\gamma + \epsilon_{ij}\alpha^i\lrcorner\,\mathrm{d}\lambda^j) \\ & + (\mathcal{L}_v C + r\cdot\mathrm{d}\rho + \epsilon_{ij}\,\mathrm{d}\lambda^i \wedge B^j + \epsilon_{ij}\beta^i\lrcorner\,\mathrm{d}\sigma^j) \\ & + (\mathcal{L}_v\alpha^i) + (\mathcal{L}\tilde{a}^i + r\cdot\mathrm{d}\sigma^i + a^i{}_j\,\mathrm{d}\sigma^j - \mathrm{d}\lambda^i \wedge C + B^i \wedge \mathrm{d}\rho) \ .\end{aligned} \tag{11.28}$$

The hypermultiplet structure. In ordinary geometry the reduction of the structure group to a subgroup G, i.e. a G-structure, provides an elegant way to describe geometries that preserve some amount of supersymmetry. We saw this in chapter 4 with the reduction $SU(3) \simeq SO(4) \subset SO(6)$. In the exceptional generalised case we again search for invariants of reduced structures, but this time for subgroups of $E_{7(7)}$. As usual, the G-structure invariants correspond to singlets after having decomposed $E_{7(7)}$ representations under a subgroup G.

Starting by considering the adjoint representation **133** of $E_{7(7)}$, a subgroup is for instance $SU(2) \times Spin^*(12) \subset E_{7(7)}$ under which the adjoint representation decomposes as

$$\mathbf{133} \to (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}) + (\mathbf{3},\mathbf{1}) \ . \tag{11.29}$$

The hypermultiplet structure is defined as the three tensors invariant under $Spin^*(12)$ which have the highest weight subalgebra of $su(2)$, i.e. the last representation in eq. (11.29). The H structure thus corresponds to the triplet denoted J_α with $\alpha = 1,2,3$, and transforms in the $\mathbf{133_1}$ representation of $E_{7(7)} \times \mathbb{R}^+$, hence realising a $Spin^*(12) \subset E_{7(7)} \times \mathbb{R}^+$-structure. They are sections of the weighted adjoint bundle

$$J_\alpha \in \Gamma(\mathrm{ad}\tilde{F} \otimes (\mathrm{det}T^*M)^{1/2}) \ , \tag{11.30}$$

where $\alpha = 1,2,3$ label the $Spin^*(12)$ invariant triplet. Further, the hypermultiplet structure satisfies the $su(2)$ algebra

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma \ , \tag{11.31}$$

where the commutator is the commutator of the adjoint representation defined in eq. (11.19), and $\kappa \in \Gamma((\mathrm{det}T^*M)^{1/2})$. The norm of the J_αs are calculated using the Killing form, which for

$e_{d+1(d+1)}$ is given generally as

$$\begin{aligned}\mathrm{tr}(R,R') = \tfrac{1}{2}\Big[&\tfrac{1}{8-d}\mathrm{tr}(r)\mathrm{tr}(r') + \mathrm{tr}(rr') + \mathrm{tr}(aa') + \gamma\lrcorner C' + \gamma'\lrcorner C \\ &+ \epsilon_{ij}(\beta^i\lrcorner B'^j + \beta'^i\lrcorner B^j) + \epsilon_{ij}(\alpha^i\lrcorner\tilde{a}'^j + \alpha'^i\lrcorner\tilde{a}^j)\Big] \quad ,\end{aligned} \tag{11.32}$$

and which for the H structures reduces to

$$\mathrm{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta} \ . \tag{11.33}$$

The κ^2 is the $E_{d+1(d+1)}$ invariant volume form, which for compactifications of the form $g_{10} = e^{2A}g_{10-d} + g_d$ takes the form

$$\kappa = e^{-2\phi}e^{(8-d)A}\sqrt{g_d} \ , \tag{11.34}$$

including a dilaton dependence.

The space of all H structures is infinite, and has been proven to admit a hyper-Kähler metric. Since the J_α at all points $x \in M$ are $Spin^*(12)$ invariant, they can be seen to span a homogenous space $E_{7(7)} \times \mathbb{R}^+/Spin^*(12)$. Out of this space one can construct a fibre bundle $Z_H = \tilde{F}/G$ as the quotient of the generalised frame bundle $\tilde{F}$ and structure group $G = Spin^*(12)$. An H structure is then a section of this bundle, and $E_{7(7)} \times \mathbb{R}^+/Spin^*(12)$ its fiber. The infinite space of H structures A_H are then given by the space of smooth sections $A_H = \Gamma(Z_H)$.

The vector multiplet structure. There is also another type of G-structure admitted, the vector multiplet structure or V structure. In this case one decomposes the fundamental representation under $E_{6(2)}$ so that

$$\mathbf{56} \to \mathbf{27} + \bar{\mathbf{27}} + 2\cdot\mathbf{1} \ , \tag{11.35}$$

where we find two singlets which also transform in the $\mathbf{56_1}$ representation of $E_{7(7)} \times \mathbb{R}^+$. Naming these two singlets K and $\hat{K}$, they are almost generic tensors except that they are required to fulfil

$$q(K) > 0 \ , \tag{11.36}$$

where q is the *quartic invariant* of $E_{7(7)}$, which is a certain symmetric quadratic polynomial that is preserved. The $E_{7(7)}$ group can be defined as the group preserving a symplectic invariant s and a symmetric quartic invariant q whose explicit form will not be necessary here. The second singlet $\hat{K}$ can be defined in terms of the first one K according to

$$s(V,\hat{K}) = \frac{2}{\sqrt{q(K,K,K,K)}}q(V,K,K,K) \ , \tag{11.37}$$

for some $V \in \Gamma(E)$. The invariant tensors K and $\hat{K}$ are often combined to form the complex object

$$X \equiv K + i\hat{K} \ , \tag{11.38}$$

which will be used later. The vector multiplet is defined by a section of the generalised tangent bundle, i.e.

$$K \in \Gamma(E) \ . \tag{11.39}$$

Analogous to the H structure, the V structure $K|_x$ at point $x \in M$ stabilises a point in the homogeneous space $E_{7(7)} \times \mathbb{R}^+/E_{6(2)}$. A V structure is then a section of the fibre bundle $Z_V = \tilde{F}/G$ with $G = E_{6(2)}$, and the infinite-dimensional space of V structures is given by $A_V = \Gamma(Z_V)$ which it turns out admits a rigid special Kähler metric.

11.3 Exceptional Calabi-Yau structures

Having introduced generalised structures that give extensions to the symplectic and complex structures of Calabi-Yau manifolds for generic flux solutions, we now turn to the constraints they must fulfil in order to describe a supersymmetric background.

The compatibility conditions between the invariant tensors ensure that their structure groups have the overlap $SU(6) = Spin^*(12) \cap E_{6(2)}$. An exceptional Calabi-Yau structure is an $SU(6) \subset E_{7(7)} \times \mathbb{R}^+$ structure. As such, the H and V structures are required to fulfil

$$J_\alpha \cdot X = 0 \ , \qquad \mathrm{tr}(J_\alpha J_\beta) = -2\sqrt{q(K)}\delta_{\alpha\beta} \ , \tag{11.40}$$

where in the first condition one uses the adjoint $\cdot$ action on $\mathbf{56} \times \mathbf{133} \to \mathbf{56}$ as defined in eq. (11.19). They are equivalent to requiring

$$J_\pm \cdot X = 0 \ , \qquad \tfrac{i}{2} s(X, \bar{X}) = \kappa^2 \ , \tag{11.41}$$

where $s(X, \bar{X})$ denotes the symplectic invariant which can be written in terms of the generalised tangent bundle section of eq. (11.13) according to

$$s(V, V') = -\tfrac{1}{4}\Big[(\iota_v \tau' - \iota_{v'} \tau) + \epsilon_{ij}(\lambda^i \wedge \sigma'^j - \lambda'^i \wedge \sigma^j) - \rho \wedge \rho'\Big] \ . \tag{11.42}$$

Next, we will see how the familiar Calabi-Yau and generalised Calabi-Yau structures can be incorporated into this formalism.

11.3.1 Embedding of type IIB Calabi-Yau structures

As familiar the Calabi-Yau admits a covariantly constant spinor η defining an $SU(3) \subset Spin(6) \simeq SU(4)$-structure, or equivalently it admits the 2-form J and 3-form Ω which should be compatible. Choosing a frame $\{e^a\}$ on the metric of M we may set the invariant forms to be

$$J = e^{12} + e^{34} + e^{56} \ , \qquad \text{and} \qquad \Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \ , \tag{11.43}$$

in a short notation with $e^{mn} = e^m \wedge e^n$. It is clear that these forms fulfil the compatibility conditions of eq. (11.1) by construction. The almost complex structure of M is then as familiar obtained by raising one index of the symplectic 2-form;

$$I^m{}_n = -J^m{}_n = \tfrac{i}{8}(\bar{\Omega}^{mpq}\Omega_{npq} - \Omega^{mpq}\bar{\Omega}_{npq}) \ , \qquad \text{and} \qquad I^q{}_m \Omega_{qnp} = i\Omega_{mnp} \ . \tag{11.44}$$

In terms of G-structures each of the two invariant forms J and Ω define $Sp(6,\mathbb{R})$- and $SL(3,\mathbb{C})$-structures respectively. The compatibility conditions imply their common subgroup to be $Sp(6,\mathbb{R}) \cap SL(3,\mathbb{C}) = SU(3)$.

In terms of exceptional generalised geometry, the H structure is defined as the symplectic form J. The decomposition of the adjoint bundle is as in eq. (11.16) and the H structure triplet J_α is defined in [127] according to

$$\begin{aligned}
J_+ &= \tfrac{1}{2}\kappa n^i J - \tfrac{1}{2} i\kappa n^i J^\# + \tfrac{1}{12} i\kappa n^i J \wedge J \wedge J + \tfrac{1}{12}\kappa n^i J^\# \wedge J^\# \wedge J^\# \ , \\
J_- &= \tfrac{1}{2}\kappa \bar{n}^i J + \tfrac{1}{2} i\kappa \bar{n}^i J^\# - \tfrac{1}{12} i\kappa \bar{n}^i J \wedge J \wedge J + \tfrac{1}{12}\kappa \bar{n}^i J^\# \wedge J^\# \wedge J^\# \ , \\
J_3 &= \tfrac{1}{2}\kappa \tau^i{}_j - \tfrac{1}{4}\kappa J \wedge J + \tfrac{1}{4} J^\# \wedge J^\# \ ,
\end{aligned} \tag{11.45}$$

which we will prove to recover the familiar Calabi-Yau structures as well as their respective compatibility conditions. Here $J_\pm \equiv J_1 \pm iJ_2$, $n^i = (-i, 1)^i \in \Gamma(S)$ is an $SL(2,\mathbb{R})$ doublet,

$\tau = -i\sigma_2 \in \Gamma((S \otimes S^*)_0)$ with σ_2 being the second Pauli matrix and $\kappa^2 = \mathrm{vol}_6 = \frac{1}{6}J \wedge J \wedge J$ is the volume form of the Calabi-Yau.

As for the V structure it is simply given as

$$X = \Omega \ , \tag{11.46}$$

and the decomposition in tensors of the tangent bundle is as in eq. (11.12). We may now check that these HV structures satisfy the constraints of eq. (11.31) and eq. (11.33). Starting with the normalisation condition of eq. (11.33), we identify the components of R in eq. (11.17) with the components of J_+ in eq. (11.45), where we have $\beta^i = -\frac{1}{2}i\kappa n^i J^\#$, $B^i = \frac{1}{2}\kappa n^i J$, $\alpha^i = \frac{1}{12}\kappa n^i J^\# \wedge J^\# \wedge J^\#$, $\tilde{a}^i = \frac{1}{12}i\kappa n^i J \wedge J \wedge J$ and $l = r = a = \gamma = C = 0$. Using the formula in eq. (11.32), we have that for J_+ and J_- the Killing form becomes

$$\begin{aligned}
\mathrm{tr}(J_+, J_-) &= \tfrac{1}{2}\Big[\epsilon_{ij}(\beta^i \lrcorner B'^j + \beta'^i \lrcorner B^j) + \epsilon_{ij}(\alpha^i \lrcorner \tilde{a}'^j + \alpha'^i \lrcorner \tilde{a}^j)\Big] \\
&= \tfrac{1}{2}\Big[\epsilon_{ij}(-\tfrac{1}{2}i\kappa n^i J^\# \lrcorner \tfrac{1}{2}\kappa \bar{n}^j J + \tfrac{1}{2}i\kappa \bar{n}^i J^\# \lrcorner \tfrac{1}{2}\kappa n^j J) \\
&\quad + \epsilon_{ij}(\tfrac{1}{12}\kappa n^i J^\# \wedge J^\# \wedge J^\# \lrcorner - \tfrac{1}{12}i\kappa \bar{n}^j J \wedge J \wedge J) \\
&\quad + \tfrac{1}{12}\kappa \bar{n}^i J^\# \wedge J^\# \wedge J^\# \lrcorner \tfrac{1}{12}i\kappa n^j J \wedge J \wedge J)\Big] \\
&= \tfrac{\kappa^2}{2}\Big[\tfrac{i}{4}\epsilon_{ij}(-n^i\bar{n}^j + \bar{n}^i n^j)J^\# \lrcorner J\Big] \\
&\quad + \tfrac{\kappa^2}{2}\Big[\tfrac{i}{(12)^2}\epsilon_{ij}(-n^i\bar{n}^j + \bar{n}^i n^j)J^\# \wedge J^\# \wedge J^\# \lrcorner J \wedge J \wedge J\Big] \\
&= \tfrac{\kappa^2}{2}\Big[\tfrac{i}{4}(2i + 2i)\cdot 3 + \tfrac{i}{(12)^2}(2i + 2i)\cdot 36\Big] = -2\kappa^2 \ ,
\end{aligned} \tag{11.47}$$

where we used that $\epsilon_{ij}n^i\bar{n}^j = -2i$, $J^\# \lrcorner J = 3$ and $J^\# \wedge J^\# \wedge J^\# \lrcorner J \wedge J \wedge J = 36$. This verifies the H structure normalisation condition in eq. (11.32).

Continuing with the $su(2)$ algebra condition in eq. (11.31), we begin with the commutator $J'' = [J_+, J_-]$. The components are identified in the same manner as before and the J_- have the same components except for the change in complex conjugation $i \to -i$, $n^i \to \bar{n}^i$. The resulting $J'' = [J_+, J_-]$ will then have components

$$\begin{aligned}
l'' &= \tfrac{1}{4}\epsilon_{ij}(\beta^i \lrcorner B'^j - \beta'^i \lrcorner B^j) + \tfrac{3}{4}\epsilon_{ij}(\alpha^i \lrcorner \tilde{a}'^j - \alpha'^i \lrcorner \tilde{a}^j) \sim \epsilon_{ij}(n^i\bar{n}^j + \bar{n}^i n^j) = 0 \ , \\
r'' &= \epsilon_{ij}(j\beta^i \lrcorner jB'^j - j\beta'^i \lrcorner jB^j) - \tfrac{1}{4}\mathbb{1}\epsilon_{ij}(\beta^i \lrcorner B'^j - \beta'^i \lrcorner B^j) \\
&\quad + \epsilon_{ij}(j\alpha^i \lrcorner j\tilde{a}'^j - j\alpha'^i \lrcorner j\tilde{a}^j) - \tfrac{3}{4}\epsilon_{ij}(\alpha^i \lrcorner \tilde{a}'^j - \alpha'^i \lrcorner \tilde{a}^j) \sim \epsilon_{ij}(n^i\bar{n}^j + \bar{n}^i n^j) = 0 \ , \\
a''^i{}_j &= \epsilon_{jk}(\beta^i \lrcorner B'^k - \beta'^i \lrcorner B^k) - \tfrac{1}{2}\delta^i{}_j\epsilon_{kl}(\beta^k \lrcorner B'^l - \beta'^k \lrcorner B^l) \\
&\quad + \epsilon_{jk}(\alpha^i \lrcorner \tilde{a}'^k - \alpha'^i \lrcorner \tilde{a}^k) - \tfrac{1}{2}\delta^i{}_j\epsilon_{kl}(\alpha^k \lrcorner \tilde{a}'^l - \alpha'^k \lrcorner \tilde{a}^l) \\
&= -i\kappa^2(\tfrac{1}{4}J^\# \lrcorner J + \tfrac{1}{(12)^2}J^\# \wedge J^\# \wedge J^\# \lrcorner J \wedge J \wedge J)\epsilon_{jk}(n^i\bar{n}^k + \bar{n}^i n^k) = -2i\kappa^2\tau^i{}_j \ , \\
\beta''^i &= 0 \ , \\
B''^i &= 0 \ , \\
\gamma'' &= \epsilon_{ij}\beta^i \wedge \beta'^j + \epsilon_{ij}(\alpha^i \lrcorner B'^j - \alpha'^i \lrcorner B^j) \\
&= \tfrac{1}{4}\kappa^2 J^\# \wedge J^\# \epsilon_{ij}n^i\bar{n}^j + \tfrac{1}{2\cdot 12}\kappa^2 J^\# \wedge J^\# \wedge J^\# \lrcorner J\epsilon_{ij}(-n^i\bar{n}^j + \bar{n}^i n^j) = -i\kappa^2 J^\# \wedge J^\# \ , \\
C'' &= \epsilon_{ij}B^i \wedge B'^j + \epsilon_{ij}(\beta^i \lrcorner \tilde{a}'^j - \beta'^i \lrcorner \tilde{a}^j) \\
&= -\tfrac{1}{4}\kappa^2 J \wedge J\epsilon_{ij}n^i\bar{n}^j + \tfrac{1}{2\cdot 12}\kappa^2 J^\# \lrcorner J \wedge J \wedge J\epsilon_{ij}(-n^i\bar{n}^j + \bar{n}^i n^j) = i\kappa^2 J \wedge J \ , \\
\alpha''^i &= 0 \ , \\
\tilde{a}''^i &= 0 \ .
\end{aligned} \tag{11.48}$$

Hence by comparing with J_3 in eq. (11.45), which has components $a^i{}_j = \frac{1}{2}\kappa\tau^i{}_j$, $\gamma = \frac{1}{4}\kappa J^\# \wedge J^\#$ and $C = -\frac{1}{4}\kappa J \wedge J$, it is clear that $J'' = -4i\kappa J_3$. This verifies the corresponding $su(2)$ algebra condition of eq. (11.31). The other combination that should be checked is $J'' = [J_+, J_3]$, and with the same identifications used before, the components of J'' become

$$
\begin{aligned}
l'' &= 0 \ , \\
r'' &= 0 \ , \\
a''^i{}_j &= 0 \ , \\
\beta''^i &= -(a' \cdot \beta)^i + \gamma' \lrcorner B^i - \alpha^i \lrcorner C' \\
&= \tfrac{1}{4}i\kappa^2\tau^i{}_j n^j J^\# + \tfrac{3}{8}\kappa^2 n^i J\# + \tfrac{1}{4\cdot 12}\kappa^2 n^i J^\# \wedge J^\# \wedge J = \kappa^2 n^i J^\# \ , \\
B''^j &= -(a' \cdot B)^i + \beta^i \lrcorner C' - \gamma' \lrcorner \tilde{a}^i \\
&= -\tfrac{1}{4}\kappa^2\tau^i{}_j n^j J + \tfrac{1}{8}i\kappa^2 J^\# \lrcorner J \wedge J - \tfrac{1}{4\cdot 12}i\kappa^2 J^\# \wedge J^\# \lrcorner J \wedge J \wedge J = i\kappa^2 n^i J \ , \\
\gamma'' &= 0 \ , \\
C'' &= 0 \ , \\
\alpha''^i &= -(a' \cdot \alpha)^i - \beta^i \wedge \gamma' \\
&= -\tfrac{1}{2\cdot 12}\kappa^2\tau^i{}_j n^j J^\# \wedge J^\# \wedge J^\# + \tfrac{1}{8}i\kappa^2 n^i J^\# \wedge J^\# \wedge J^\# = \tfrac{1}{6}i\kappa^2 n^i J^\# \wedge J^\# \wedge J^\# \ , \\
\tilde{a}''^i &= -(a' \cdot \tilde{a})^i + B^i \wedge C' \\
&= -\tfrac{1}{2\cdot 12}i\kappa^2\tau^i{}_j n^j J \wedge J \wedge J - \tfrac{1}{8}\kappa^2 n^i J \wedge J \wedge J = -\tfrac{1}{6}\kappa^2 n^i J \wedge J \wedge J \ ,
\end{aligned}
\tag{11.49}
$$

using $\tau^i{}_j n^j = -in^i$. Comparing with J_+ in eq. (11.45), we see that indeed $[J_+, J_3] = 2i\kappa J_+$. The equivalent applies for J_-.

Proceeding to the normalisation condition in eq. (11.41) for the V structure, the symplectic invariant in eq. (11.42) is straightforwardly calculated to

$$
s(\Omega, \bar{\Omega}) = \tfrac{1}{4}\Omega \wedge \bar{\Omega} \ , \tag{11.50}
$$

as the V structure only has one component which we identify as $\Omega = \rho \in \Gamma(\Lambda^3 T^*M)$ and $v = \lambda^i = \sigma^i = \tau = 0$ in the expression of eq. (11.13). Since $\mathrm{vol}_6 = \frac{1}{6}J^3 = \frac{i}{8}\Omega \wedge \bar{\Omega}$ it is clear that $\frac{i}{2}s(\Omega, \bar{\Omega}) = \kappa^2 = \mathrm{vol}_6$; thereby satisfying the normalisation condition.

Having checked the separate conditions on the H and V structure, the last thing to do is to check the compatibility conditions of eq. (11.41). In doing so we use the adjoint action on $X = \Omega$, with the same identifications of Ω and J_+ as before. The resulting components of the adjoint action $\Omega' = J_+ \cdot \Omega$ are then

$$
\begin{aligned}
v' &= 0 \ , \\
\lambda'^i &= -\tfrac{1}{2}i\kappa n^i J^\# \lrcorner \Omega \ , \\
\rho' &= 0 \ , \\
\sigma'^i &= \tfrac{1}{2}\kappa n^i \Omega \wedge J \ , \\
\tau' &= 0 \ .
\end{aligned}
\tag{11.51}
$$

It is clear that $J_+ \cdot \Omega = -\frac{1}{2}i\kappa n^i J^\# \lrcorner \Omega + \frac{1}{2}\kappa n^i \Omega \wedge J$ vanish only if

$$
J \wedge \Omega = J \wedge \bar{\Omega} = 0 \ , \tag{11.52}
$$

and as such the compatibility conditions of eq. (11.1) are recovered.

11.3.2 Embedding of type II generalised Calabi-Yau structures

In this section we will see how the structures of exceptional generalised geometry incorporate the $O(d,d)$ generalised geometry case, which was first done in [129]. The H structure is given by the pure spinors $\Phi^{\pm}$ each of which define $SU(3,3)_{\pm}$ structures, where the minus sign applies to the type IIA case and the plus sign to IIB. Recall again that the NSNS B field is included in the definition of the pure spinors.

The type II tangent bundle is given in eq. (11.11), to which we define sections

$$V_{\mathrm{II}} = v + \xi + \varsigma + \varrho + \tau \ , \tag{11.53}$$

where v is a vector $v \in \Gamma(TM)$, ξ a 1-form $\xi \in \Gamma(T^*M)$, the $\varsigma^{\pm} \in \Gamma(\Lambda^{\pm}T^*M)$ is a sum of even/odd forms, $\varrho \in \Gamma(\Lambda^5 T^*M)$ is a 5-form and $\tau \in \Gamma(T^*M \otimes \Lambda^6 T^*M)$ is a 1-form density.

Considering the decomposition of $E_{7(7)}$ under $SL(2,\mathbb{R}) \times O(6,6)$, we have that the adjoint representation becomes

$$\mathbf{133} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}^{\mp}) \ . \tag{11.54}$$

In the adjoint bundle, as given on the first line in eq. (11.16) and which includes both type II theories, i.e.

$$\mathrm{ad}\tilde{F} \simeq \mathbb{R} \oplus [\mathbb{R} \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M] \oplus [(TM \otimes T^*M) \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM] \oplus [\Lambda^{\mp} TM \oplus \Lambda^{\mp} T^*M] \ , \tag{11.55}$$

the three representations in eq. (11.54) correspond to the three terms in brackets. The first term $\mathbb{R}$ is the singlet $(\mathbf{1},\mathbf{1})$ which generates the $\mathbb{R}^+$ action. Since the general section R in eq. (11.17) applies specifically to the type IIB adjoint bundle, we need to introduce another section of the adjoint bundle in eq. (11.55). From the three decompositions of the adjoint representation in eq. (11.54) corresponding to the last three bundles in eq. (11.55) we define the section $\mu \in \Gamma(\mathrm{ad}\tilde{F})$ as

$$\mu = \mu^i{}_j + \mu^A{}_B + \mu^{i\pm} \ , \tag{11.56}$$

where $\mu^i{}_j \in \Gamma(\mathbb{R} \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M)$, $\mu^A{}_B \in \Gamma((TM \otimes T^*M) \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM)$ and $\mu^{i\mp} \in \Gamma(\Lambda^{\mp} TM \oplus \Lambda^{\mp} T^*M)$. The $SL(2,\mathbb{R})$ indices $i = 1,2$ label the specific bundle component according to $\mu^1{}_1 = -\mu^2{}_2 \in \mathbb{R}$, $\mu^1{}_2 \in \Lambda^6 T^*M$ and $\mu^2{}_1 \in \Lambda^6 TM$, as well as $\mu^{1\mp} = \Lambda^{\mp} T^*M$ and $\mu^{2\mp} = \Lambda^{\mp} TM$. The indices $A, B = 1, \ldots, 12$ are the fundamental $O(6,6)$ indices.

The pure spinors $\Phi^{\mp}$ lie in the $\mu^{i\mp}$, i.e. they are sections;

$$\Phi^{\mp} \in \Gamma(\Lambda^{\mp} T^*M) \ , \tag{11.57}$$

which may be seen as sections of $Spin(6,6) \times \mathbb{R}^+$ spinor bundles with positive and negaitve helicity. The $\mathbb{R}^+$ factor acts by rescaling. The spinors are pure in the sense that they are invariant under a common subgroup $SU(3,3) \subset Spin(6,6)$ and satisfy the compatibility conditions of eq. (11.3). The generalised almost complex structures $\mathcal{J}^{\mp} \in \Gamma(\mathrm{ad}\tilde{F})$ depend on the pure spinors according to

$$\mathcal{J}^{\mp A}{}_B = i \frac{\langle \Phi^{\mp}, \Gamma^A{}_B \bar{\Phi}^{\mp} \rangle}{\langle \Phi^{\mp}, \bar{\Phi}^{\mp} \rangle} \ , \tag{11.58}$$

where $\Gamma^A{}_B$ with $A, B = 1, \ldots, 12$ are the $O(6,6)$ gamma matrices. As such it belongs to the $\mu^A{}_B$ section above, i.e. $\mathcal{J}^{\pm}$ transforms in the $(\mathbf{1},\mathbf{66})$ representation of eq. (11.54).

By defining a determinant bundle section

$$u^i = \tfrac{1}{2} \begin{pmatrix} -i\kappa \\ \kappa^{-1} \end{pmatrix} \in \Gamma((\mathrm{det} T^*M)^{1/2} \otimes (\mathbb{R} \oplus \Lambda^6 T^*M)) \ , \tag{11.59}$$

where ϵ_{ij} raises and lowers indices according to $\epsilon_{ij}u^j = u_i$ and $u^i\bar{u}_i = -i/2$, the H structures can be defined as

$$\begin{aligned} J_+ &= u^i\Phi^\mp \ , \\ J_3 &= \kappa(u^i\bar{u}_j - \bar{u}^i u_j) - \tfrac{1}{2}\kappa\mathcal{J}^\mp \ . \end{aligned} \tag{11.60}$$

The almost complex structure $\mathcal{J}$ is given as in eq. (11.58), and the volume is

$$\kappa^2 = \tfrac{i}{8}\left\langle \Phi^\pm, \bar{\Phi}^\pm \right\rangle \ . \tag{11.61}$$

In order to check that these H structures work, it is necessary to check the $su(2)$ algebra of the H structure in eq. (11.31) and the normalisation condition in eq. (11.33). To do this we use the adjoint action and Killing form as first given in [129] for each respective check. The adjoint action is $\mu'' = [\mu, \mu']$ where the components are

$$\begin{aligned} \mu''^i{}_j &= \mu^i{}_k\mu'^k{}_j - \mu'^i_k\mu^k{}_j + \tfrac{1}{2}\epsilon_{jk}\left[\left\langle \mu^{i\mp}, \mu'^{k\mp}\right\rangle - \left\langle \mu'^{i\mp}, \mu^{k\mp}\right\rangle\right] \ , \\ \mu''^A{}_B &= \mu^A{}_C\mu'^C{}_B - \mu'^A{}_C\mu^C{}_B + \tfrac{1}{2}\epsilon_{ij}\left\langle \mu^{i\mp}, \Gamma^A{}_B\mu'^{j\mp}\right\rangle \ , \\ \mu''^{i\mp} &= \mu^i{}_j\mu'^{j\mp} - \mu'^i{}_j\mu^{j\mp} + \tfrac{1}{4}\mu_{AB}\Gamma^{AB}\mu'^{i\mp} - \tfrac{1}{4}\mu'_{AB}\Gamma^{AB}\mu^{i\mp} \ . \end{aligned} \tag{11.62}$$

The $su(2)$ algebra is satisfied when $[J_+, J_-] = -4i\kappa J_3$. The $J_\pm$ consist purely of the sections $\mu^{i\pm} = u^i\Phi^\mp$ and so the adjoint action $[J_+, J_-]$ must have components

$$\begin{aligned} \mu''^i{}_j &= \tfrac{1}{2}\epsilon_{jk}\left[\left\langle \mu^{i\pm}, \mu'^{k\pm}\right\rangle - \left\langle \mu'^{i\pm}, \mu^{k\pm}\right\rangle\right] = \tfrac{1}{2}\epsilon_{jk}(u^i\bar{u}^k - \bar{u}^i u^k)\left\langle \Phi^\mp, \bar{\Phi}^\mp\right\rangle = \tfrac{1}{2}(u^i\bar{u}_j - \bar{u}^i u_j)\left\langle \Phi^\mp, \bar{\Phi}^\mp\right\rangle \ , \\ \mu''^A{}_B &= \tfrac{1}{2}\epsilon_{ij}\left\langle \mu^{i\mp}, \Gamma^A{}_B\mu'^{j\mp}\right\rangle = \tfrac{1}{2}\epsilon_{ij}u^i\bar{u}^j\left\langle \Phi^\pm, \Gamma^A{}_B\bar{\Phi}^\pm\right\rangle = -\tfrac{i}{4}\left\langle \Phi^\pm, \Gamma^A{}_B\bar{\Phi}^\pm\right\rangle \ , \\ \mu''^{i\mp} &= 0 \ . \end{aligned} \tag{11.63}$$

From eq. (11.60) we know that for J_3 we identify $\mu^i{}_j = \kappa(u^i\bar{u}_j + \bar{u}^i u_j)$ and $\mu^A{}_B = -\frac{1}{2}\kappa\mathcal{J}^{\mp A}{}_B$ and so multiplying each component with $-4i\kappa$ and using the form of $\mathcal{J}^{\mp A}{}_B$ in eq. (11.58) we have that

$$-4i\kappa J_3 = -4i\kappa^2(u^i\bar{u}_j + \bar{u}^i u_j) - \tfrac{i}{4}\left\langle \Phi^\mp, \Gamma^A{}_B\bar{\Phi}^\mp\right\rangle \ . \tag{11.64}$$

It is clear that this exactly corresponds to the obtained expressions in eq. (11.63). For the second condition $[J_+, J_3] = 2i\kappa J_+$ one obtains $\mu''^i{}_j = 0$, $\mu''^A{}_B = 0$ and

$$\begin{aligned} \mu''^{i\pm} &= -\mu'^i{}_j\mu^{j\mp} - \tfrac{1}{4}\mu'_{AB}\Gamma^{AB}\mu^{i\mp} \\ &= -\kappa(u^i\bar{u}_j - \bar{u}^i u_j)u^j\Phi^\mp - \tfrac{1}{4}(-\tfrac{1}{2})\kappa\mathcal{J}^\mp_{AB}\Gamma^{AB}u^i\Phi^\mp \\ &= \tfrac{i}{2}\kappa u^i\Phi^\mp + \tfrac{3i}{2}\kappa u^i\Phi^\mp = 2i\kappa J_+ \ , \end{aligned} \tag{11.65}$$

where we have used that $u^i u_i = 0$ and the property

$$\tfrac{1}{4}\mathcal{J}^\mp_{AB}\Gamma^{AB}\Phi^\mp = 3i\Phi^\mp \ . \tag{11.66}$$

To check the normalisation in eq. (11.33), we use the Killing form

$$\mathrm{tr}(\mu, \mu) = \tfrac{1}{2}\mu^i{}_j\mu^j{}_i + \tfrac{1}{4}\mu^A{}_B\mu^B{}_A + \tfrac{1}{2}\epsilon_{ij}\left\langle \mu^{i\mp}, \mu^{j\mp}\right\rangle \ . \tag{11.67}$$

It is clear that $\mathrm{tr}(J_+, J_+) = \epsilon_{ij}u^i u^j\left\langle \Phi^\mp, \Phi^\mp\right\rangle = 0$ from $u^i u_i = 0$, and for $\mathrm{tr}(J_+, J_-) = -2\kappa^2$ we indeed have that

$$\mathrm{tr}(J_+, J_-) = \tfrac{1}{2}\epsilon_{ij}\left\langle u^i\Phi^\mp, \bar{u}^j\bar{\Phi}^\mp\right\rangle = -\tfrac{i}{4}\left\langle \Phi^\mp, \bar{\Phi}^\mp\right\rangle = -2\kappa^2 \ . \tag{11.68}$$

As for J_3, by eq. (11.33) it should fulfil $\mathrm{tr}(J_3, J_3) = -\kappa^2$, which is proven by

$$\mathrm{tr}(J_3, J_3) = \tfrac{1}{2}\kappa^2(u^i\bar{u}_j - \bar{u}^i u_j)(u^j\bar{u}_i - \bar{u}^j u_i) + \tfrac{1}{4}(-\tfrac{1}{2})\kappa^2\mathcal{J}^{\mp A}{}_B\mathcal{J}^{\mp B}{}_A = -\tfrac{1}{4}\kappa^2 - \tfrac{3}{4}\kappa^2 = -\kappa^2 \ . \tag{11.69}$$

We are now ready to proceed to the V structure, we first need to define sections of the fundamental representation decomposed under $SL(2,\mathbb{R}) \times O(6,6)$ according to

$$\mathbf{56} \to (\mathbf{2},\mathbf{12}) \oplus (\mathbf{1},\mathbf{32}) \ . \tag{11.70}$$

We may name the elements transforming in these representations

$$\lambda = \lambda^{iA} + \lambda^{\pm} \ , \tag{11.71}$$

where λ^{iA} transforms in the $(\mathbf{2},\mathbf{12})$ and $\lambda^{\pm}$ transforms in the $(\mathbf{1},\mathbf{32})$ as a 32-dimensional Weyl spinor with positive/negative chirality $\pm$. Again the chirality depends on each type II theory as follows: (λ^{+},μ^{i-}) for IIA and (λ^{-},μ^{i+}) for IIB. Now, the V structure can be defined as

$$X = \Phi^{\pm} \ , \tag{11.72}$$

and its corresponding normalisation condition in eq. (11.41) is checked with the symplectic invariant

$$s(\lambda,\lambda') = \tfrac{1}{4}\left(\epsilon_{ij}\eta_{AB}\lambda^{iA}\lambda'^{jB} + \langle\lambda^{\pm},\lambda'^{\pm}\rangle\right) \ , \tag{11.73}$$

with η_{AB} being the $O(6,6)$ metric. The V structure in eq. (11.72) is then defined by the sections $\lambda^{\pm} = \Phi^{\pm}$ and $\lambda^{iA} = 0$, so the normalisation condition in eq. (11.41) becomes

$$\tfrac{i}{2}s(\Phi^{\pm},\bar{\Phi}^{\pm}) = \tfrac{i}{8}\langle\Phi^{\pm},\bar{\Phi}^{\pm}\rangle = \kappa^2 \ , \tag{11.74}$$

which is expected.

Finally, the compatibility condition $J_{\pm}\cdot X$ in eq. (11.41) is checked using the adjoint action on a section of the fundamental representation $\lambda' = \mu\cdot\lambda$ as

$$\begin{aligned} \lambda'^{iA} &= \mu^{i}{}_{j}\lambda^{jA} + \mu^{A}{}_{B}\lambda^{iB} + \langle\mu^{i\mp},\Gamma^{A}\lambda^{\pm}\rangle \ , \\ \lambda'^{\pm} &= \tfrac{1}{4}\mu_{AB}\Gamma^{AB}\lambda^{\pm} + \epsilon_{ij}\lambda^{iA}\Gamma_{A}\mu^{j\mp} \ . \end{aligned} \tag{11.75}$$

With the same section identifications for $J_{\pm}$ as before, the only remaining component is

$$\lambda'^{iA} = \langle\mu^{i\mp},\Gamma^{A}\lambda^{\pm}\rangle \ , \tag{11.76}$$

and hence the compatibility condition results in that

$$J_{+}\cdot X = u^{i}\,\langle\Phi^{\mp},\Gamma^{A}\Phi^{\pm}\rangle \ , \qquad J_{-}\cdot X = \bar{u}^{i}\,\langle\bar{\Phi}^{\mp},\Gamma^{A}\Phi^{\pm}\rangle \ . \tag{11.77}$$

Hence, for the H and V structures to be compatible, we have

$$\langle\Phi^{\mp},\Gamma^{A}\Phi^{\pm}\rangle = 0 \ , \tag{11.78}$$

which indeed recovers the compatibility equations in eq. (11.3) above and so $\{\Phi^{+},\Phi^{-}\}$ define an $SU(3)\times SU(3)$-structure.

11.3.3 Type IIB on $M_{SU(2)} \times \mathbb{R}^2$ with RR flux

Having seen how our familiar type IIB Calabi-Yau structures and type II generalised Calabi-Yau structures are embedded in this formalism, we now turn to our final example. While the former example had no fluxes, and the latter NSNS flux, it is interesting to see how RR fluxes may be incorporated on the same footing. To do so we turn to a space that can be written $M = M_{SU(2)} \times \mathbb{R}^2$, where $M_{SU(2)}$ is a 4D *hyper-Kähler* space which has $SU(2)$-structure. Including a warp factor and RR 5-form flux F_5 we write the metric as

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\hat{s}^2(M_{SU(2)}) + e^{-2A}(\mathrm{d}x^2 + \mathrm{d}y^2) \\ &\equiv \mathrm{d}\hat{s}^2(M_{SU(2)}) + \zeta_1^2 + \zeta_2^2 \ , \end{aligned} \tag{11.79}$$

with $\mathrm{d}\hat{s}^2(M_{SU(2)})$ being the metric on the $SU(2)$-structure space. This hyper-Kähler space is determined by three 2-forms ω_α and the 1-forms ζ_1 and ζ_2. Given some frame $\{e^a\}$ on M we may choose them to be

$$\omega_1 = e^{14} + e^{23} \ , \qquad \omega_2 = e^{13} - e^{24} \ , \qquad \omega_3 = e^{12} + e^{34} \ , \qquad \zeta_1 = e^5 \ , \qquad \zeta_2 = e^6 \ , \tag{11.80}$$

with the same short notation $e^{mn} \equiv e^m \wedge e^n$ as before. The corresponding three complex structures are given by $(I_\alpha)^m{}_n = -(\omega_\alpha)^m{}_n$, and the volume form is given by

$$\tfrac{1}{2}\omega_\alpha \wedge \omega_\beta \wedge \zeta_1 \wedge \zeta_2 = \delta_{\alpha\beta}\mathrm{vol}_6 \ . \tag{11.81}$$

The integrability conditions for this structure are given in [127] as

$$\mathrm{d}(e^A\zeta_i) = 0 \ , \qquad \mathrm{d}(e^{2A}\omega_\alpha) = 0 \ , \qquad \mathrm{d}A = -\tfrac{1}{4} \star F_5 \ . \tag{11.82}$$

With this geometry we proceed to defining the exceptional structures. As we include the 5-form RR flux we work with twisted/untwisted structures. The untwisted H structure, which does not include any flux, is defined as

$$\hat{J}_\alpha = -\tfrac{1}{2}\kappa I_\alpha - \tfrac{1}{2}\kappa\omega_\alpha \wedge \zeta_1 \wedge \zeta_2 + \tfrac{1}{2}\kappa\omega_\alpha^\# \wedge \zeta_1^\# \wedge \zeta_2^\# \ , \tag{11.83}$$

where now $\kappa^2 = e^{2A}\mathrm{vol}_6$. The 5-form flux consists of the 4-form RR potential C_4 and 2-form potentials B^i like eq. (11.22) which "twists" the $\hat{J}_\alpha$ to

$$J_\alpha = e^{B^i + C_4}\hat{J}_\alpha e^{-B^i - C} \ . \tag{11.84}$$

As for the V structure, we define the untwisted V structure as

$$\hat{X} = \bar{n}^i e^A(\zeta_1 - i\zeta_2) + i\bar{n}^i e^A(\zeta_1 - i\zeta_2) \wedge \mathrm{vol}_4 \ , \tag{11.85}$$

where again $n^i = (-i, 1)^i$ and $\frac{1}{2}\omega_\alpha \wedge \omega_\beta = \zeta_{\alpha\beta}\mathrm{vol}_4$ is the volume of $M_{SU(2)}$. The present fluxes twist the V structure to

$$X = e^{B^i + C}\hat{X} \ . \tag{11.86}$$

From this we are ready to check the compatibility condition

$$\hat{J}_\alpha \cdot \hat{X} \sim -\bar{n}^i I_\alpha \cdot (\zeta_1 - i\zeta_2) - i\bar{n}(\omega_\alpha \wedge \zeta_1^\# \wedge \zeta_2^\#) \lrcorner ((\zeta_i - i\zeta_2) \wedge \mathrm{vol}_4) - i\bar{n}^i I_\alpha \cdot ((\zeta_1 - i\zeta_2) \wedge \mathrm{vol}_4) = 0 \ , \tag{11.87}$$

which is satisfied when $I_\alpha \cdot \mathrm{vol}_4 = 0$ and $\zeta_i^\# \lrcorner \omega_\alpha = 0$.

11.4 Integrability as vanishing of moment maps

In previous sections we have dealt with the algebraic conditions of HV structures, and we are now to turn to differential conditions on these invariant tensors. The conditions are necessary for the supersymmetry spinors to satisfy the supersymmetry variations, in turn preserving supersymmetry. We have seen before that in the absence of fluxes, the vanishing of intrinsic torsion preserved supersymmetry.

In order to define integrability conditions for the H structure one can introduce *moment maps*. A moment map is a map from the symplectic manifold M to the dual of the Lie algebra of the group acting on it, and can be viewed as a geometric generalisation of the classical linear and angular momentum. The moment maps for the action of the group of *generalised diffeomorphisms*, i.e. diffeomorphisms and form-field gauge transformations, on the infinite-dimensional space of H structures is reasoned in [127] to be defined as

$$\mu_\alpha(V) \equiv -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \mathrm{tr}(J_\beta(L_V J_\gamma)) \ . \tag{11.88}$$

The H structure J_α is then said to be integrable, or torsion-free, if it fulfils

$$\mu_\alpha(V) = 0 \ , \qquad \forall V \in \Gamma(E) \ , \tag{11.89}$$

with μ_α as in eq. (11.88). The integrability condition for the V structure is simpler since $K \in \Gamma(E)$, so we need only consider the generalised Lie derivative along K. As such, the V structure is integrable or torsion-free when fulfilling

$$L_K K = 0 \qquad \Rightarrow \qquad L_X \bar{X} = 0 \ , \tag{11.90}$$

where again $X = K + i\hat{K}$. This is equivalent to saying that the V structure should be invariant under generalised diffeomorphisms generated by itself. We are now ready to describe the integrability conditions for the combined HV structure, or exceptional Calabi-Yau structure, defined by the compatible pair $\{J_\alpha, K\}$. Firstly, both J_α and K should be integrable on their own, and secondly they should satisfy the combined action

$$L_X J_\alpha = 0 \ , \tag{11.91}$$

i.e. the J_α should also be invariant under generalised diffeomorphisms generated by $K, \hat{K}$.

In the next subsection we return to the examples of the previous section 11.3, and see what their integrability conditions look like. The integrability conditions of eqs. (11.89), (11.90), (11.91) are satisfied in each example. In order to show this, it is convenient to first start by rewriting the moment map of eq. (11.88) in terms of $\{J_\pm, J_3\}$ according to

$$\mu_+ \equiv -i\int_M \mathrm{tr}(J_3(L_V J_+)) = 0 \ , \qquad \mu_3 \equiv \tfrac{i}{2}\int_M \mathrm{tr}(J_-(L_V J_+)) = 0 \ . \tag{11.92}$$

Extending the generalised Lie derivative to act on generalised tensors rather than just vectors as in eq. (11.24), we have using the adjoint action in eq. (11.25) that it can be written

$$L_V A = \mathcal{L}_v A - P \cdot A \ , \qquad P \equiv \mathrm{d}\lambda^i + \mathrm{d}\rho + \mathrm{d}\sigma^i \ . \tag{11.93}$$

Here $P \in \Gamma(\mathrm{ad}\tilde{F})$, the $\cdot$ is the adjoint action, v is the vector component of $V \in \Gamma(E)$ and $\mathcal{L}_v$ is the ordinary Lie derivative. Using this form of the Lie derivative, the moment map in eq. (11.88) can be rewritten to

$$\begin{aligned}\mu_\alpha(V) &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \mathrm{tr}(J_\beta(\mathcal{L}_v J_\gamma - [P, J_\gamma])) \\ &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\int_M \mathrm{tr}(J_\beta \mathcal{L}_v J_\gamma) - 2\int_M \kappa\mathrm{tr}(P, J_\alpha) \ ,\end{aligned} \tag{11.94}$$

where on the second line we used the identity in eq. (11.31) and cyclicity of trace $\mathrm{tr}(A,[B,C]) = \mathrm{tr}(B[C,A])$. Hence the moment maps of eq. (11.92) become

$$\mu_+ = -i\int_M \mathrm{tr}(J_3, \mathcal{L}_v J_+) + 2\int_M \kappa\mathrm{tr}(P, J_+) \ , \tag{11.95}$$

$$\mu_3 = \tfrac{i}{2}\int_M \mathrm{tr}(J_-, \mathcal{L}_v J_+) - 2\int_M \kappa\mathrm{tr}(P, J_3) \ . \tag{11.96}$$

We are now ready to check the integrability conditions in each example. Later in section 11.5, we will show that these integrability conditions are equivalent to the existence of a generalised torsion-free connection that is compatible with the $SU(6)$-structure defined by J_α and K.

11.4.1 Integrability conditions on type IIB Calabi-Yau

With the general section $V \in \Gamma(E)$ as given in eq. (11.13), one can consider the moment maps of each of its components since the map is linear. Given the H structure in eq. (11.45), we identify the components of the J_α as well as P in eq. (11.93) above with the components of the general form of the adjoint section in eq. (11.17). Thereafter we calculate the components of the moment maps using the explicit form of the Killing form in eq. (11.32). Since J_3 and $J_\pm$ have components of different forms, all components of $\mathrm{tr}(J_3, \mathcal{L}_v J_+)$ vanish. Evaluating the second term in eq. (11.95) for the λ^i-component, we get the first integrability condition

$$\mu_+(\lambda^i) \sim \int_M \epsilon_{ij}\kappa^2 n^j J^\# \lrcorner\, \mathrm{d}\lambda^i \sim \int_M \epsilon_{ij} n^j J \wedge J \wedge \mathrm{d}\lambda^i \sim \int_M \epsilon_{ij} n^j\, \mathrm{d}J \wedge J \wedge \lambda^i = 0\ , \tag{11.97}$$

using $\kappa^2 \sim J^3$ so that $\kappa^2 J^\# \sim J \wedge J$. For the σ^i-component the same analysis result in that

$$\mu_+(\sigma^i) \sim \int_M \kappa^2 \epsilon_{ij} n^j J^\# \wedge J^\# \wedge J^\# \lrcorner\, \mathrm{d}\sigma^i \sim \int_M \epsilon_{ij} n^j \lrcorner\, \mathrm{d}\sigma^i = 0\ . \tag{11.98}$$

All other components vanish. As such we turn to the μ_3 map, where it is found for the ρ-component, the second term in eq. (11.96) becomes

$$\mu_3(\rho) \sim \int_M \kappa^2 J^\# \wedge J^\# \lrcorner\, \mathrm{d}\rho \sim \int_M J \wedge \mathrm{d}\rho \sim \int_M \mathrm{d}J \wedge \rho = 0\ . \tag{11.99}$$

The last non-vanishing component is the one for v, which gives a non-vanishing contribution from the first term in eq. (11.96), that is

$$\begin{aligned}\mu_3(v) = \int_M \kappa\Big(& J^\# \lrcorner \mathcal{L}_v(\kappa J) - \mathcal{L}_v(\kappa J^\#) \lrcorner \kappa J + \kappa J^\# \wedge J^\# \wedge J^\# \lrcorner \mathcal{L}_v(\kappa J \wedge J \wedge J) \\ & - \mathcal{L}_v(\kappa J^\# \wedge J^\# \wedge J^\#) \lrcorner \kappa J \wedge J \wedge J\Big) \\ \sim \tfrac{1}{2} \int_M & \Big(J \wedge J \wedge \mathcal{L}_v J + \mathcal{L}_v J \wedge J \wedge J\Big) = 0\ ,\end{aligned} \tag{11.100}$$

where the fourth term on the second line vanishes. With this last expression one may use the fact that $\mathcal{L}_v J = \iota_v\, \mathrm{d}\omega + \mathrm{d}\iota_v \omega$ to integrate by parts, so that this expression vanishes if $\mathrm{d}J = 0$. This is also required by the vanishing of most of the other moment maps above.

We now proceed to the integrability conditions of the V structure given in eq. (11.90). The generalised Lie derivative acting on a vector as given in eq. (11.27) becomes particularly simple with the Calabi-Yau V structure in eq. (11.46). The only non-vanishing component corresponds to the second last term in eq. (11.27), i.e.

$$L_X \bar{X} = j\bar{\Omega} \wedge \mathrm{d}\Omega = 0\ . \tag{11.101}$$

As for the final integrability condition of eq. (11.91), using the expression for the generalised Lie derivative acting on an adjoint section eq. (11.28), we find

$$L_X J_+ \sim i n^i J^\# \lrcorner\, \mathrm{d}\Omega - n^i J \wedge \mathrm{d}\Omega = 0\ , \tag{11.102}$$

$$L_X J_3 \sim -\tfrac{1}{2} J^\# \wedge J^\# \lrcorner\, \mathrm{d}\Omega - j J^\# \wedge J^\# \lrcorner j\, \mathrm{d}\Omega + \tfrac{1}{2}\mathbb{1} J^\# \wedge J^\# \lrcorner\, \mathrm{d}\Omega = 0\ . \tag{11.103}$$

As such we see that the familiar Calabi-Yau integrability conditions $\mathrm{d}J = 0$ and $\mathrm{d}\Omega = 0$ are recovered, as expected.

11.4.2 Integrability conditions of generalised type II Calabi-Yau

The generalised tangent bundle in eq. (11.11) essentially corresponds to a decomposition of the fundamental **56** under $GL(6,\mathbb{R}) \subset E_{7(7)}$. This embedding corresponds to the action of diffeomorphisms on the exceptional tangent space of eq. (11.11) in the exceptional generalised geometry [129]. The same applies to the adjoint bundle in eq. (11.55) and the decomposition of **133**. There is a subgroup of the adjoint bundle consisting of the p-form elements; $B \in \Lambda^2 T^*M$, $\tilde{B} \in \Lambda^6 T^*M$ and $C^\mp \in \Lambda^\mp T^*M$. Together they form an algebra $[B+\tilde{B}+C^\mp, B', \tilde{B}'+C'^\mp] = 2\langle C^\mp, C'^\mp\rangle + B\wedge C'^\mp - B'\wedge C^\mp$. These fields are in one-to-one correspondence with the form fields of type II supergravity. Specifically, the B is the internal NSNS B field, the $\tilde{B}$ is an internal 6-form field corresponding to the 10D dual of the external $B_{\mu\nu}$, and $C^\mp$ are the odd/even RR potentials of type IIA/IIB. The fields are also encoded in the $SL(2,\mathbb{R})\times O(6,6)$ decomposition of **133** in eq. (11.54). The embedding $GL(6,\mathbb{R}) \subset SL(2,\mathbb{R})\times O(6,6) \subset E_{7(7)}$ can be shown to break the $SL(2,\mathbb{R})$ symmetry, so that its elements are described by another $SL(2,\mathbb{R})$ vector $v^i = (1,0)$. The fields $B, \tilde{B}$ and $C^\mp$ can then be identified with the elements λ in eq. (11.56) as

$$\begin{aligned} \mu^i{}_j &= \tilde{B} v^i v_j \ , & B &\in \Lambda^2 T^*M \ , \\ \mu^A{}_B &= \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \ , & \tilde{B} &\in \Lambda^6 T^*M \ , \\ \mu^{i\mp} &= v^i C^\mp \ , & C^\mp &\in \Lambda^\mp T^*M \ . \end{aligned} \tag{11.104}$$

When there are non-trivial field strengths of these form fields the potentials $B, \tilde{B}$ and $C^\mp$ are defined locally. On each patch $U_{(\alpha)}$ on the manifold we have

$$\lambda_{(\alpha)} = e^{C^\mp_{(\alpha)} + \tilde{B}_{(\alpha)} - B_{(\alpha)}} \lambda \ , \tag{11.105}$$

so that on $U_{(\alpha)} \cap U_{(\beta)}$ the patching is given by gauge transformations

$$\lambda_{(\alpha)} = e^{\mathrm{d}\varsigma^\mp_{(\alpha\beta)} + \mathrm{d}\varrho_{(\alpha\beta)} - \mathrm{d}\xi_{(\alpha\beta)}} \lambda_{(\beta)} \ . \tag{11.106}$$

This implies for the potential fields on the patch that

$$\begin{aligned} B_{(\alpha)} &= B_{(\beta)} + \mathrm{d}\xi_{(\alpha\beta)} \ , \\ \tilde{B}_{(\alpha)} &= \tilde{B}_{(\beta)} + \mathrm{d}\varrho_{(\alpha\beta)} + \langle \mathrm{d}\varsigma^\pm_{(\alpha\beta)}, e^{-\,\mathrm{d}\xi_{(\alpha\beta)}} C^\mp_{(\beta)}\rangle \ , \\ C^\mp_{(\alpha)} &= C^\mp_{(\alpha)} + \mathrm{d}\varsigma^\pm_{(\alpha\beta)} + e^{-\,\mathrm{d}\xi_{(\alpha\beta)}} C^\mp_{(\beta)} \ , \end{aligned} \tag{11.107}$$

which makes the field strengths $H_3 = \mathrm{d}B_2$ and $F = e^B\,\mathrm{d}C^\mp$ gauge invariant. Given the section V_{II} of the type II generalised tangent bundle in eq. (11.53), the twisting $e^{-B+\tilde{B}+C^\mp}$ implies that P in eq. (11.94) will take the form

$$P = \mathrm{d}\xi + \mathrm{d}\varrho v^i v_j + v^i\,\mathrm{d}\varsigma^\pm \ . \tag{11.108}$$

The moment maps are again evaluated term-wise. Starting with eq. (11.95), the only non-vanishing component is the one for $\varsigma^\pm$ since $J_+ = u^i\Phi^\mp$. Using the Killing form in eq. (11.67), we have that

$$\mu_+(\varsigma^\pm) = -2\int_M \kappa \mathrm{tr}(PJ_+) = -\int_M \kappa\epsilon_{ij}\,\langle v^i\,\mathrm{d}\varsigma^\pm, u^j\Phi^\mp\rangle = -\int_M \langle \mathrm{d}\varsigma^\pm, \Phi^\mp\rangle = \int_M \langle \varsigma^\pm, \mathrm{d}\Phi^\mp\rangle = 0 \ , \tag{11.109}$$

using that $v^i u_i = \kappa^{-1}$ and partial integration in the second last step. We see that this is fulfilled when $\mathrm{d}\Phi^\mp = 0$. Next, the moment map for $J_3 =$ given by eq. (11.96) is evaluated term-wise.

Starting with the vector component ξ of the section P in eq. (11.108), only the second term of μ_3 in eq. (11.96) remains, and again using the Killing form in eq. (11.67) it takes the form

$$\mu_3(\xi) = -2\int_M \kappa \operatorname{tr}(P, J_3) = -2\int_M \kappa \tfrac{1}{4}\, \mathrm{d}\xi^A{}_B(-\tfrac{1}{2})\kappa \mathcal{J}^{\mp} = -\tfrac{1}{32}\int_M \mathrm{d}\xi^A{}_B\, \langle \Phi^{\mp}, \Gamma^B{}_A \bar{\Phi}^{\mp}\rangle = 0\ . \tag{11.110}$$

Here we have used the form of $\mathcal{J}^{\mp}$ as in eq. (11.58) and κ^2 as in eq. (11.61). Using the Clifford map this expression can be written

$$\mu_3(\xi) = -\tfrac{1}{32}\int_M \langle \Phi^{\mp}, \mathrm{d}\xi \wedge \bar{\Phi}^{\mp}\rangle = \tfrac{1}{32}\int_M \Big(\langle \mathrm{d}\Phi^{\mp}, \xi \wedge \bar{\Phi}^{\mp}\rangle + \langle \Phi^{\mp}, \xi \wedge \mathrm{d}\bar{\Phi}^{\mp}\rangle\Big) = 0\ , \tag{11.111}$$

where in the second step we have again used partial integration. This recovers the familiar integrability conditions $\mathrm{d}\Phi^{\mp} = 0$. For the 5-form component ϱ we have analogously that

$$\mu_3(\varrho) = -2\int_M \kappa \operatorname{tr}(\mathrm{d}\varrho v^i v_j, \kappa(u^j \bar{u}_i - \bar{u}^j u_i)) = -\int_M \kappa^2\, \mathrm{d}\varrho(\kappa^{-2} - \kappa^{-2}) = 0\ , \tag{11.112}$$

using that $v^i u_i = \kappa^{-1}$. Lastly we turn to the v-component. Only the first term in eq. (11.96) remains and becomes

$$\mu_3(v) = \tfrac{i}{2}\int_M \operatorname{tr}(J_-, \mathcal{L}_v J_+) = \tfrac{i}{2}\int_M \langle \bar{u}^i \bar{\Phi}^{\mp}, \mathcal{L}_v(u^j \Phi^{\mp})\rangle = \tfrac{1}{4}\int_M \langle \bar{\Phi}^{\mp}, \mathcal{L}_v \Phi^{\mp}\rangle = 0\ , \tag{11.113}$$

using $\bar{u}^i u_i = -i/2$ and partial integrating where $\mathcal{L}_v(u^i) = 0$. With $\mathcal{L}_v \Phi^{\mp} = \iota_v\, \mathrm{d}\Phi^{\mp} + \mathrm{d}\iota_v \Phi^{\mp}$ in the above expression and partial integrating the second term we find that eq. (11.113) becomes

$$\mu_3(v) = -\tfrac{1}{4}\int_M \langle \bar{\Phi}^{\mp}, \iota_v\, \mathrm{d}\Phi^{\mp}\rangle + \tfrac{1}{4}\int_M \langle \mathrm{d}\bar{\Phi}^{\mp}, \iota_v \Phi^{\mp}\rangle = 0\ . \tag{11.114}$$

From this we again recover the expected integrability conditions $\mathrm{d}\Phi^{\mp} = 0$. As for the integrability conditions on the V structure given in eq. (11.90) and compatibility condition in eq. (11.91), we use the generalised Lie derivative in eq. (11.93) to satisfy both conditions simultaneously. Since $X = \Phi^{\pm}$ does not have a v-component, we have that eq. (11.93) becomes

$$\mathbb{L}_X = -P \cdot A = -v^i\, \mathrm{d}\Phi^{\pm} \cdot A = 0\ , \tag{11.115}$$

having identified the section $\varsigma^{\pm} = \Phi^{\pm}$ in eq. (11.108). Since $\mathrm{d}\Phi^{\pm} = 0$ and A can be any generalised tensor, the conditions in eqs. (11.90), (11.91) are satisfied.

11.4.3 Integrability conditions on $M_{SU(2)} \times \mathbb{R}^2$ in type IIB with RR flux

With a non-vanishing RR flux F_5 we need to use the modified moment map and twisted generalised Lie derivative in eq. (11.93). The moment map is given by eq. (11.94) where we take P as given in eq. (11.93). Starting with the $\hat{\rho}$-component of P, the H structure is given in eq. (11.83) where we identify the 4-form $\frac{1}{2}\kappa\omega_\alpha^\# \wedge \zeta_1^\# \wedge \zeta_2^\#$ as γ in eq. (11.17), hence

$$\begin{aligned}\mu_\alpha(\hat{\rho}) = -2\int_M \kappa \operatorname{tr}(R, J_\alpha) &= -\int_M \kappa^2 \omega_\alpha^\# \wedge \zeta_1^\# \wedge \zeta_2^\# \lrcorner\, \mathrm{d}\hat{\rho} \\ &\sim \int_M e^{2A}\omega_\alpha \wedge \mathrm{d}\hat{\rho} \\ &\sim \int_M \mathrm{d}(e^{2A}\omega_\alpha) \wedge \hat{\rho} = 0\ ,\end{aligned} \tag{11.116}$$

again using the Killing form in eq. (11.32) and that $\kappa^2 = \frac{1}{2}e^{2A}\omega_\alpha \wedge \omega_\alpha \wedge \zeta_1 \wedge \zeta_2$. From this vanishing component of the moment map, we recover the integrability condition $\mathrm{d}(e^{2A}\omega_\alpha) = 0$

as given in eq. (11.82). As for the $\hat{v}$-component the calculation of the moment map is messier as it involves $\iota_{\hat{v}}F$ in P as well as the first term in the moment map of eq. (11.94). One eventually finds that

$$\mu_\alpha(\hat{v}) \sim \int_M \Big(e^{2A}\omega_\alpha \wedge \iota_{\hat{v}}F_5 + 2e^{2A}\epsilon_{\alpha\beta\gamma}\,\mathrm{d}A \wedge \omega_\alpha \wedge \iota_{\hat{v}}\omega_\gamma \wedge \zeta_1 \wedge \zeta_2\Big) = 0\ , \tag{11.117}$$

which is fulfilled given $\mathrm{d}A = -\frac{1}{4}F_5$. Continuing with the V structure, from eq. (11.93) we have that

$$\tilde{L}_{\hat{X}}A = -P \cdot A = 0\ . \tag{11.118}$$

The V structure is given in eq. (11.85), and its components are identified as $\hat{\lambda}^i = \bar{n}^i e^A(\zeta_1 - i\zeta_2)$ and $\hat{\sigma}^i = i\bar{n}^i e^A(\zeta_1 - i\zeta_2) \wedge \mathrm{vol}_4$ where $\hat{\lambda}, \hat{\sigma}$ are components of a generalised vector eq. (11.13). Hence, with the form of P in eq. (11.93), eq. (11.118) is fulfilled when

$$\hat{P} = \bar{n}^i\,\mathrm{d}\Big[e^A(\zeta_1 - i\zeta_2\Big] + i\bar{n}^i\,\mathrm{d}\Big[e^A(\zeta_1 - i\zeta_2) \wedge \mathrm{vol}_4\Big] + \bar{n}^i e^A(\zeta_1 - i\zeta_2) \wedge F_5 = 0\ . \tag{11.119}$$

The previously attained integrability condition can be rewritten as $\mathrm{d}(e^{4A}\mathrm{vol}_4) = 0$, which simplifies the second term in eq. (11.119) according to

$$\mathrm{d}\Big[e^{-4A}(e^A(\zeta_1 - i\zeta_2) \wedge e^{4A}\mathrm{vol}_4)\Big] = \mathrm{d}\Big[e^{-4A}(e^A(\zeta_1 - i\zeta_2)\Big] \wedge \mathrm{vol}_4\ . \tag{11.120}$$

From this the two other integrability conditions can be recovered. With $\mathrm{d}(e^A\zeta_i) = 0$ we see that the first term in eq. (11.119) vanishes and the second can be written

$$\begin{aligned}\mathrm{d}(e^A(\zeta_1 - i\zeta_2) \wedge \mathrm{vol}_4) &= \mathrm{d}(e^{-4A}(e^A(\zeta_1 - i\zeta_2) \wedge e^{4A}\mathrm{vol}_4)) \\ &= -4e^{-4A}\,\mathrm{d}A \wedge e^A(\zeta_1 - i\zeta_2) \wedge e^{4A}\mathrm{vol}_4\ ,\end{aligned} \tag{11.121}$$

so that eq. (11.119) becomes

$$-4i\,\mathrm{d}A \wedge (\zeta_1 - i\zeta_2) \wedge \mathrm{vol}_4 + (\zeta_1 - i\zeta_2) \wedge F_5 = 0\ . \tag{11.122}$$

This is solved with $\mathrm{d}A = -\frac{1}{4}\star F_5$ since generally $\lambda^{\#} \lrcorner \rho\mathrm{vol}_6 = \rho \wedge \star\lambda$, and so we recover all three integrability conditions of eq. (11.82).

11.5 Generalised intrinsic torsion

Like in the case of ordinary geometry, integrability is defined as the existence of a generalised torsion-free connection that is compatible with the G-structure, or equivalently as the vanishing of the generalised intrinsic torsion. A G-compatible connection is a connection that preserves tensors that are invariant under G, i.e. that preserve the ones that define the G-structure.

A general definition of generalised intrinsic torsion was constructed in [130], which we will formulate here. In order to define generalised intrinsic torsion, we start by defining a covariant derivative D acting on a vector $V \in \Gamma(E)$ according to

$$D_M \cdot V^N = \partial_N V^N + \Gamma_{MP}{}^N V^P\ , \tag{11.123}$$

with connection $\Gamma_{MP}{}^N \in \Gamma(E^* \otimes \mathrm{ad}\tilde{F})$ and $\partial_M = (\partial_m, 0, \ldots, 0)$. The generalised covariant derivative should preserve the $E_{7(7)} \times \mathbb{R}^+$-structure and also satisfy a Leibniz condition which states that given a function f and $V \in \Gamma(E)$, then

$$D(fV) = f(DV) + (\mathrm{d}f) \otimes V\ , \tag{11.124}$$

with $\mathrm{d}f$ being the 1-form in $T^*M \subset E^*$. This property implies that the action of the covariant derivative can be extended to sections of any $E_{d(d)} \times \mathbb{R}^+$ vector bundle, i.e. to any other generalised tensor field. The generalised torsion of the connection in the generalised derivative is then defined in the same manner as for ordinary geometry, i.e.

$$T(D) \cdot V' = L^D_V W - L_V V' \ , \tag{11.125}$$

where V' is some generalised tensor and L^D_V is the generalised Lie derivative with respect to the covariant derivative D rather than ∂. That is, eq. (11.24) is changed to

$$L^D_V V' = (V \cdot D)V' - (D \times_{\mathrm{ad}} V) \cdot V' \ . \tag{11.126}$$

Denoting $W \subset E^* \otimes \mathrm{ad}\tilde{F}$ as the space of generalised torsions, in [131] one classified which representations of $E_{d(d)} \times \mathbb{R}^+$ groups that appeared in W. For $E_{7(7)} \times \mathbb{R}^+$ the torsion lies in

$$W \subset \mathbf{56_{-1}} \oplus \mathbf{912_{-1}} \ , \tag{11.127}$$

again with subscript denoting the $\mathbb{R}^+$ weight. One can also define the generalised G-structure in terms of a principal bundle $\tilde{P}_G \subset \tilde{F}$ of the generalised frame bundle $\tilde{F}$. One talks of a connection such that G is compatible with $\tilde{P}_G$, which can always be found but will in general not be torsion-free. Given a G-compatible connection in D, any other G-compatible connection can be written as $D' = D + \Sigma$ where

$$\Sigma = D' - D \ , \qquad \Sigma \in \Gamma(E^* \oplus \mathrm{ad}\tilde{P}_G) \ . \tag{11.128}$$

To consider the corresponding torsion of these connections, we may define a map τ such that $\tau : E^* \oplus \mathrm{ad}\tilde{P}_G \to W$ so that

$$\tau(\Sigma) = T' - T \in \Gamma(W) \ , \tag{11.129}$$

as the difference in generalised torsions of the corresponding connections. If $\mathrm{im}\tau \subseteq W$ is the image of τ that corresponds to the space of torsions contained in τ, the *intrinsic torsion* is defined as the space of torsions not spanned by $\mathrm{im}\tau$;

$$W_{\mathrm{int}} \equiv \frac{W}{\mathrm{im}\tau} \ . \tag{11.130}$$

The map $\mathrm{im}\tau$ does in general not fill all of W. The generalised intrinsic torsion is hence defined as the part of a torsion that can not be removed by a change Σ of our G-compatible connection D. As such, the presence of intrinsic torsion is an obstruction to finding a G-compatible *and* torsion-free connection. In the next section this rather abstract concept will be applied to the H and V structures in our $E_{7(7)} \times \mathbb{R}^+$ generalised geometry, which will hopefully clarify these concepts.

11.5.1 Intrinsic torsion for hyper- and vector multiplet structures

Starting off by considering the H structure, we know that it is defined as a $Spin^*(12)$-structure. Decomposing the torsion representation under the $SU(2) \times Spin^*(12)$ subgroup gives

$$W = \mathbf{56} + \mathbf{912} \to 2(\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}) + (\mathbf{3}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}) + (\mathbf{2}, \mathbf{220}) \ . \tag{11.131}$$

The space of $Spin^*(12)$-compatible connections is constructed as the product space of the fundamental representation decomposed under $SU(2) \times Spin^*(12)$; $\mathbf{56} \to (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32})$, and the adjoint representation that is invariant under $SU(2)$ which by eq. (11.29) is $(\mathbf{1}, \mathbf{66})$. As such, we have the following representations

$$\begin{aligned} E^* \otimes \mathrm{ad}\tilde{P}_{Spin^*(12)} &= ((\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32})) \times (\mathbf{1}, \mathbf{66}) \\ &= (\mathbf{2}, \mathbf{12}) + (\mathbf{2}, \mathbf{220}) + (\mathbf{1}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}) \ . \end{aligned} \tag{11.132}$$

By comparing this equation with the allowed torsion representations in eq. (11.131), we see that all representations of $E^* \otimes \mathrm{ad}\tilde{P}_{Spin^*(12)}$ also appear in W, and so the $Spin^*(12)$-compatible torsions must be $W^{Spin^*(12)} \subseteq (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}) + (\mathbf{1}, \mathbf{352}) + (\mathbf{2}, \mathbf{220})$. This corresponds to $\mathrm{im}\tau(\Sigma)$ in eq. (11.129) above. Now taking the quotient of W and $W^{Spin^*(12)}$, the intrinsic torsion is determined to

$$W_{\mathrm{int}}^{Spin^*(12)} = (\mathbf{2}, \mathbf{12}) + (\mathbf{3}, \mathbf{32}) \ . \tag{11.133}$$

To prove the connection to the moment maps, we will now show that the H structure moment map will set constraints on the very same representations as the intrinsic torsion. With D being $Spin^*(12)$ compatible, then by definition we must have $DJ_\alpha = 0$. Given the moment map in eq. (11.94), the Lie derivative in eq. (11.93) can be re-expressed in terms of D using eq. (11.125) and eq (11.126). The moment map then takes the form

$$\begin{aligned} \mu_\alpha &= -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M \mathrm{tr}\Big[J_\beta\big((V \cdot D)J_\gamma - [(D \times_{\mathrm{ad}} V), J_\gamma] - [T(V), J_\gamma]\big)\Big] \\ &= -2 \int_M \kappa\Big[\mathrm{tr}(J_\alpha(D \times_{\mathrm{ad}} V)) + \mathrm{tr}(J_\alpha T(V))\Big] \\ &= -\int \kappa^2 T_{\mathrm{int}}^{Spin^*(12)}(J_\alpha \cdot V) - 2\int_M \kappa \mathrm{tr}(J_\alpha T_{\mathrm{int}}^{Spin^*(12)}(V)) \ , \end{aligned} \tag{11.134}$$

where the first term on the first line we partial integrate to get $DJ_\alpha = 0$ cancelling the term on the second line. On the third line we use the fact that the choice of compatible connection is arbitrary, so we may choose one that leaves only the intrinsic torsion. The first term on the third line is obtained by considering the torsion of D when doing an integration by parts. The H structure is defined as the $(\mathbf{3}, \mathbf{1})$ component under the decomposition of $SU(2) \times Spin^*(12)$, and so the moment map in eq. (11.134) can only vanish if the intrinsic torsion does not have a $(\mathbf{3}, \mathbf{1})$ component. This must hold for all values of V, which transforms in the $\mathbf{56} \to (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32})$ representation. By eq. (11.133) the intrinsic torsion also transforms in this representation, hence the $(\mathbf{3}, \mathbf{1})$ component of $T_{\mathrm{int}}^{Spin^*(12)}(V)$ vanishes if both $(\mathbf{2}, \mathbf{12})$ and $(\mathbf{1}, \mathbf{32})$ vanish. Then it is clear that for a $Spin^*(12)$-structure the moment map vanishes only if the intrinsic torsion vanish.

Proceeding to the V structure, the same analysis is made for the decomposition under $E_{6(2)}$. The torsion representations decompose as

$$W = \mathbf{56} + \mathbf{912} = \mathbf{1} + 2 \cdot \mathbf{27} + \mathbf{78} + \mathbf{351} + \text{c.c.} \ , \tag{11.135}$$

and the space of of $E_{6(2)}$-compatible connections is constructed like the product space of the fundamental representation decomposed under $E_{6(2)}$; $\mathbf{56} \to \mathbf{1} + \mathbf{27} + \text{c.c.}$ as in eq. (11.35) and the adjoint representation $\mathbf{133} \to \mathbf{78}$. Hence the space is

$$\begin{aligned} E^* \otimes \mathrm{ad}\tilde{P}_{E_{6(2)}} &= (\mathbf{1} + \mathbf{27} + \text{c.c.}) \times \mathbf{78} \\ &= \mathbf{27} + \mathbf{78} + \mathbf{351} + \mathbf{1728} + \text{c.c.} \ , \end{aligned} \tag{11.136}$$

which implies that $W^{E_{6(2)}} \subseteq \mathbf{27} + \mathbf{78} + \mathbf{351} + \text{c.c.}$. The intrinsic torsion is then what remains of W after quoting out $W^{E_{6(2)}}$, which is

$$W_{\mathrm{int}}^{E_{6(2)}} = \mathbf{1} + \mathbf{27} + \text{c.c.} \ . \tag{11.137}$$

As in the case for the H structure, we will now prove the equivalence between the V structure integrability condition of eq. (11.90) and the existence of a torsion-free $E_{6(2)}$-structure. Using eq. (11.125) and eq. (11.126), we have that

$$L_K K = L_K^D K - T(K) \cdot K = -T_{\text{int}}^{E_{6(2)}}(K) \cdot K \ , \tag{11.138}$$

where $DK = 0$. We know that K is a singlet under $E_{6(2)}$ and $L_K K$ is a generalised vector that transforms in the $\mathbf{56} \to \mathbf{1} + \mathbf{27} + \text{c.c.}$ representation. In order for the above equation to equal zero then the $\mathbf{1} + \mathbf{27} + \text{c.c.}$ part of $T_{\text{int}}^{E_{6(2)}}$ must vanish. Since these are the same representations as the intrinsic torsion in eq. (11.137), we conclude that the V structure integrability condition $L_K K = 0$ is equivalent to requiring a torsion-free $E_{6(2)}$-structure.

11.5.2 Intrinsic torsion for ECY structures and supersymmetry

In this section we will show the equivalence of integrability and $\mathcal{N} = 2$ supersymmetry with the usage of generalised intrinsic torsion.

In $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, a generalised metric is invariant under $H_d \subset E_{d(d)} \times \mathbb{R}^+$ transformations. In complex generalised geometry we have $H_d = O(d,d) \times \mathbb{R}^+$ which encompasses a generalised structure group $G \subset H_d$, which in this example is $G = SU(3) \times SU(3)$. A generalised metric can be viewed to define a H_d sub-bundle $\tilde{P} \subset \tilde{F}$ of the frame bundle $\tilde{F}$, where H_d is the maximal compact subgroup of G. This is a generalisation of ordinary geometry, where a metric can be defined as a $O(d)$ sub-bundle of the $GL(d, \mathbb{R})$ frame bundle for TM.

In $d = 7$ a generalised metric on a spin manifold, i.e. a manifold that admits spinors, is invariant under the double cover of the maximal compact subgroup $\tilde{H}_7 = SU(8)$. Its spinors, or fermionic degrees of freedom, will then be $SU(8)$ representations. In [130] it was shown that the fermion supersymmetry variations can be written in terms of an $SU(8)$-compatible connection D, so that

$$D\epsilon = 0 \ . \tag{11.139}$$

The supersymmetry parameter can be seen as a section of a spinor bundle S which transforms in the $\mathbf{8}$ representation of $SU(8)$. The internal gravitino variation can be seen as a section of a bundle J which transforms in the $\mathbf{56}$ of $E_{7(7)}$, hence the supersymmetry variations will transform in $S \oplus J$. For a single element in $\mathbf{8}$ the stabiliser group of $SO(8)$ is $SU(7)$, and for two elements the stabiliser group is $SU(6)$. This means that two supersymmetry spinors define a global $SU(6)$-structure. Generally in $d = 7$, $D = 4$ the structure group is $SO(7 - \mathcal{N})$ with $\mathcal{N}$ being the number of supersymmetry parameters preserved. Note that this only applies for $d = 7$, other values of d have different $\tilde{H}_d$ as well as structure groups G, as specified in [130]. Since the spinors define an $SO(6)$-structure, a $SO(6)$-compatible connection $\hat{D}$ can be found if $\hat{D}\epsilon = 0$, which is also equivalent to the intrinsic torsion of the connection vanishing. This is what we will show in this section and, as it turns out, the intrinsic torsion will be found to transform in the very same representations as $S \oplus J$, thus showing an equivalence between vanishing intrinsic torsion and preserved supersymmetry.

Starting with the latter, the representations of the supersymmetry variations $S \oplus J$ of decompose under $SU(2) \times SU(6)$ according to

$$S \oplus J = \mathbf{8} + \mathbf{56} = (\mathbf{2}, \mathbf{1}) + 2(\mathbf{1}, \mathbf{6}) + (\mathbf{2}, \mathbf{15}) + (\mathbf{1}, \mathbf{20}) \ . \tag{11.140}$$

These are thus the representations in which the supersymmetry variations transform. Turning to the torsion, a decomposition under $SU(2) \times SU(6)$ gives us

$$\begin{aligned} W = \mathbf{56} + \mathbf{912} = (\mathbf{1}, \mathbf{1}) + 2(\mathbf{1}, \mathbf{15}) + (\mathbf{1}, \mathbf{21}) + (\mathbf{1}, \mathbf{35}) + (\mathbf{1}, \mathbf{105}) + 3(\mathbf{2}, \mathbf{6}) \\ + (\mathbf{2}, \mathbf{20}) + (\mathbf{2}, \mathbf{84}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{15}) + \text{c.c.} \ . \end{aligned} \tag{11.141}$$

The space of $SU(6)$-compatible connections is formed as the product space of the fundamental form decomposed under $SU(2) \times SU(6)$; $\mathbf{56} \to (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + \text{c.c}$ and the adjoint representation that is invariant under $SU(2)$, i.e. $\mathbf{133} \to (\mathbf{1},\mathbf{35})$, so

$$\begin{aligned} E^* \otimes \mathrm{ad}\tilde{P}_{SU(6)} &= ((\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + \text{c.c}) \times (\mathbf{1},\mathbf{35}) \\ &= (\mathbf{1},\mathbf{15}) + (\mathbf{1},\mathbf{21}) + (\mathbf{1},\mathbf{35}) + (\mathbf{1},\mathbf{105}) + (\mathbf{1},\mathbf{384}) + (\mathbf{2},\mathbf{6}) \\ &\quad + (\mathbf{2},\mathbf{84}) + (\mathbf{2},\mathbf{120}) + \text{c.c.}\ , \end{aligned} \tag{11.142}$$

and hence $W^{SU(6)} = (\mathbf{1},\mathbf{15}) + (\mathbf{1},\mathbf{21}) + (\mathbf{1},\mathbf{35}) + (\mathbf{1},\mathbf{105}) + (\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{84})$. The intrinsic torsion is then given by

$$\begin{aligned} W_{\text{int}}^{SU(6)} &\subseteq (\mathbf{2},\mathbf{1}) \times (S + J) + \text{c.c.} \\ &= (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{1}) + 2(\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + (\mathbf{3},\mathbf{15}) + (\mathbf{2},\mathbf{20}) + \text{c.c.}\ . \end{aligned} \tag{11.143}$$

Note that the equality here holds given that there are no kernels in the map

$$\tau : K_{SU(6)} \equiv E^* \otimes \mathrm{ad}\tilde{P}_{SU(6)} \to W^{SU(6)}\ , \tag{11.144}$$

which is checked by constructing the explicit maps, as was done thoroughly in Appendix G of [127].

Now, decomposing the H and V structures under $SU(2) \times SU(6)$, the same analysis as above will result in the intrinsic torsions

$$\begin{aligned} W_{\text{int}}^{Spin^*(12)} &= (\mathbf{2},\mathbf{6}) + (\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{15}) + \text{c.c.}\ , \\ W_{\text{int}}^{E_{6(2)}} &= (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15}) + \text{c.c.}\ . \end{aligned} \tag{11.145}$$

It is clear that the sum of these torsion does not equal the one in eq. (11.143), as the $(\mathbf{2},\mathbf{20})$ component is missing. From this we draw the conclusion that the H and V structures being integrable separately is not enough to ensure that there is an integrable $SU(6)$-structure. This is solved by the compatibility condition $\mathbb{L}_X J_\alpha = 0$. From eq. (11.125) and eq. (11.126) and using $DK = DJ_\alpha = 0$ we get

$$\mathbb{L}_X J_\alpha = \mathbb{L}_X J_\alpha - [T(X), J_\alpha] = -[T_{\text{int}}^{SU(6)}(X), J_\alpha] = 0\ . \tag{11.146}$$

With X being a singlet and $\mathbb{L}_X J_\alpha$ transforming in $\mathbf{133}$ which under decomposition of $SU(6)$ includes the missing component $(\mathbf{2},\mathbf{20})$, which by this condition must vanish. This proves the one-to-one correspondence between $\mathcal{N} = 2$ supersymmetry and vanishing intrinsic torsion of the exceptional generalised G-structure.

As a final note we remark that while this analysis is made for a Minskowski vacuum, the same formalism and analysis has also been applied to the AdS case, see for instance [132][133]. This differs from our Minkowski case in that the momentum maps are non-vanishing. This formalism has also been used in the context of AdS/CFT duality in [134], where deformations of the CFT are compared with deformations of the H and V structures. In the next chapter we will see how to make use of the notion of intrinsic torsion of the generalised G-structure in the context of exceptional field theory, and see how it can be used in formulating a consistent truncation ansatz.

12

Half-Maximal Supersymmetry in 4D from Exceptional Field Theory

In this final chapter we will connect results from the previous chapter to $E_{7(7)}$ ExFT and see how the language of exceptional generalised G-structures can be used to construct vacua in exceptional field theory. In particular, in [135] this is done by interpreting the G-structure in terms of the ExFT analogue of differential forms, and describing the intrinsic torsion in terms of generalised tensors of the $E_{7(7)}$ representations. The vacua will then be defined in terms of compatibility and integrability conditions. These can then be used when constructing consistent truncations of exceptional field theory to 4D, or any other $D \geq 4$ background.

A lot like in the case of DFT, a consistent truncation in ExFT is also a type of generalised Scherk-Schwarz truncation made from twist matrices which requires globally well-defined generalised frame fields. This implies that all supersymmetries are preserved in the truncation. In attempting to construct more realistic models in the formalism of exceptional field theory it is of interest to construct models which do not preserve all supersymmetries. Half-maximal backgrounds were first constructed from $SL(5)$ ExFT in [136], and used to construct consistent truncations to half-maximal gauged supergravities in 7D. In [135] these results were extended to half-maximal supersymmetry in $D \geq 4$ dimensions. In this chapter we discuss results of the $D = 4$ case.

12.1 Half-maximal G-structures in terms of generalised differential forms

When compactifying on an internal manifold, a half-maximal theory will admit a half-maximal amount of spinors on it. In $d = 7$ with the group of the exceptional tangent bundle being $E_{7(7)}$, we recall that the spinors transform under $SU(8)$. The G-structure is the stabiliser group of $SU(8)$, which from the previous section 11.5.2 we recall as $SU(8 - \mathcal{N})$, with $\mathcal{N}$ being the amount of supersymmetry spinors preserved. In the half-maximal case $\mathcal{N} = 4$ we thus have that the corresponding G-structure is $G_{\text{half}} = SU(4)$. In turn, the half-maximal commutant of $SU(4) \subset SU(8)$ is $SU(4) \times U(1)$, which happen to correspond to the R-symmetry group for half-maximal supergravities in four dimensions.

A lot like in the previous chapter one will need constraints on the G_{half}-structures in order for the compactification on M to result in a 4D Minkowski or AdS vacuum. Again, there is no attempt to circumvent the no-go theorems required to construct a de Sitter vacuum.

As familiar by now, the G-structures can be defined in terms of a number of nowhere vanishing tensors which are stabilised by the group G. Such generalised tensors can be thought of as generalised differential forms which in turn can be connected to the sections of exceptional vector bundles appearing in the tensor hierarchy of exceptional field theory. This is what is used in [135] in order to describe the G-structures. The exceptional vector bundles $\mathcal{R}_i$ have fibres R_i

corresponding to the representations of $E_{7(7)}$, where

$$R_1 = \mathbf{56}\ , \qquad R_2 = \mathbf{133}\ , \qquad R_3 = \mathbf{912}\ , \tag{12.1}$$

are the representations that are used. With the three exceptional vector bundles $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ which have fibres R_1, R_2 and R_3 respectively, a type of wedge product $\tilde{\wedge}$ is defined according to

$$\begin{aligned} R_1 \tilde{\wedge} R_1 &\longrightarrow R_2\ , & \mathcal{R}_1 \tilde{\wedge} \mathcal{R}_1 &\longrightarrow \mathcal{R}_2\ , \\ R_1 \tilde{\wedge} R_2 &\longrightarrow R_3\ , & \mathcal{R}_1 \tilde{\wedge} \mathcal{R}_2 &\longrightarrow \mathcal{R}_3\ , \\ R_1 \tilde{\wedge}_s R_1 &\longrightarrow \mathbf{1}\ , & \mathcal{R}_1 \tilde{\wedge}_s \mathcal{R}_1 &\longrightarrow \mathbf{1}\ , \end{aligned} \tag{12.2}$$

where $\mathbf{1}$ is the singlet representation. For elements $A_1, A_2 \in R_1$ and $B \in R_2$, the wedge product in eq. (12.2) acts like

$$\begin{aligned} (A_1 \tilde{\wedge} A_2)^\alpha &= A_1{}^M A_2{}^N (t^\alpha)_{MN}\ , \\ (A_1 \tilde{\wedge}_s A_1) &= A_1{}^M A_1^N \omega_{MN}\ , \\ (A \tilde{\wedge} B)^{M\alpha} &= (\mathbb{P}_{\mathbf{912}})^{M\alpha}{}_{N\beta} A^N B^\beta\ , \end{aligned} \tag{12.3}$$

where again $M, N = 1, \ldots, 56$ are the fundamental indices of $E_{7(7)}$ and $\alpha = 1, \ldots, 133$ are indices of the adjoint representation. The t^α are the $E_{7(7)}$ generators and ω_{MN} the symplectic invariant. The projector $\mathbb{P}_{\mathbf{912}}$ onto the R_3 representation is given by

$$(\mathbb{P}_{\mathbf{912}})^{M\alpha}{}_{N\beta} = \tfrac{1}{7}\left(-12(t_\beta)^M{}_P (t^\alpha)_N{}^P + 4(t_\beta)_N{}^P (t^\alpha)_P{}^M + \delta_N^M \delta_\beta^\alpha\right)\ , \tag{12.4}$$

using the same conventions as in chapter 10. Having introduced some notation, we turn to the description of the G_{half}-structure. In $d = 7$ we have

$$G_{\text{half}} = SU(4) \simeq SO(6)\ . \tag{12.5}$$

This is embedded in the $SO(6,6)$ group which in turn lies in $E_{7(7)} \times \mathbb{R}^+$, i.e.

$$G_{\text{half}} = SO(6) \subset SO(6,6) \subset E_{7(7)} \times \mathbb{R}^+\ . \tag{12.6}$$

It will prove useful to start by describing the $SO(6,6)$-structures before reducing the structure group to $SO(6)$. A manifold with $SO(6,6)$-structure admits the following:

- a scalar field κ of weight $1/2$,
- a section $\mathcal{J}$ of the $\mathcal{R}_2$ bundle .

The maximal commutant of $SO(6,6) \subset E_{7(7)}$ is $SL(2)$, i.e. $SL(2)$ is the largest subgroup whose generators commute in $E_{7(7)}$ with generators of $SO(6,6)$; $[A_i, B_j] = 0$ if $A_i \in SO(6,6) \subset E_{7(7)}$ and $B_j \in SL(2) \subset E_{7(7)}$. In constructing the section $\mathcal{J}$ the $SL(2)$ commutant needs to be taken into account, namely since $SL(2)$ has three generators one will need to construct an $SL(2)$ triplet of sections on the $\mathcal{R}_2$ bundle. From eq. (12.1) we see that R_2 is the adjoint representation of $E_{7(7)}$, and so the $\mathcal{R}_2$ bundle corresponds to the adjoint bundle. The $SL(2)$ triple sections on the bundle will single out the $SO(6,6)$ dependence; specifically the sections are fields $\mathcal{J}_{ij}{}^\alpha \equiv \mathcal{J}_{(ij)}{}^\alpha$ where $i, j = 1, 2$ are the fundamental $SL(2)$ indices and α the adjoint indices of $E_{7(7)}$. This leaves the degrees of freedom in the $SO(6,6)$-structure. Decomposing the adjoint representation of $E_{7(7)}$ decompose under $E_{7(7)} \to SO(6,6) \times SL(2)$ according to

$$\mathbf{133} \longrightarrow (\mathbf{66}, \mathbf{1}) \oplus (\mathbf{32}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3})\ , \tag{12.7}$$

the sections $\mathcal{J}_{ij}$ correspond to the $(\mathbf{1},\mathbf{3})$ representation. These $SO(6,6)$-structure defining fields should satisfy the compatibility conditions

$$\begin{aligned}(\mathcal{J}_{ij}\otimes\mathcal{J}_{kl})_{\mathbf{1539}} &= 0\ ,\\ [\mathcal{J}_{ij},\mathcal{J}_{kl}] &= -2\kappa^2(\epsilon_{i(k}\mathcal{J}_{l)j}+\epsilon_{j(k}\mathcal{J}_{l)i})\ ,\\ \mathrm{tr}(\mathcal{J}_{ij}\mathcal{J}_{kl}) = \mathcal{J}_{ij}{}^{\alpha}\mathcal{J}_{kl\alpha} &= 12\kappa^4\epsilon_{i(k}\epsilon_{l)j}\ ,\end{aligned} \tag{12.8}$$

where $\epsilon_{ij}\equiv\epsilon_{[ij]}$ is the $SL(2)$-invariant tensor which is also used when raising and lowering $SL(2)$ indices and fulfils $\epsilon^{ik}\epsilon_{kj}=\delta^i_j$. As probably has been noted by now, these $\mathcal{J}_{ij}$ structures are very similar to the H structure in the previous chapter. They respectively define $SO(6,6)$- and $SO^*(12)$-structure groups, both of which are different real forms of $SO(12)$ and hence related by analytic continuation. The $\mathcal{J}_{ij}$ does not extend or generalise the H structure; would one extend $\mathcal{N}=2$ to $\mathcal{N}=4$ the $SO(6,6)$-structure does not embed the $SO^*(12)$-structure.

Proceeding to break the structure group down to $SO(6)\subset SO(6,6)\subset E_{7(7)}\times\mathbb{R}^+$, we first have that the fundamental representation $\mathbf{56}$ is decomposed under $SO(6,6)\times SL(2)$ according to

$$\mathbf{56}\to(\mathbf{12},\mathbf{2})+(\mathbf{32'},\mathbf{1})\ . \tag{12.9}$$

Decomposing the $SO(6,6)\to SO(6)\times SO(6)_R$ where $SO(6)$ is our wanted structure group and $SO(6)_R$ its R-symmetry group, the $(\mathbf{12},\mathbf{2})$ becomes a singlet under $SO(6)$. That is, decomposing under $SO(6)\times SO(6)_R\times SL(2)\subset SO(6,6)\times SL(2)\subset E_{7(7)}$ we have that

$$(\mathbf{12},\mathbf{2})\to(\mathbf{6},\mathbf{1},\mathbf{2})\oplus(\mathbf{1},\mathbf{6},\mathbf{2})\ . \tag{12.10}$$

The vector transforming in the singlet $(\mathbf{1},\mathbf{6},\mathbf{2})$ under $SO(6)$ is named $\mathcal{K}_{ui}$ where $u=1,\dots,6$ are $SO(6)_R$ fundamental indices and $i=1,2$ are the $SL(2)$ indices. Hence there are twelve such vectors, which can be seen to fulfil the compatibility conditions with the $\mathcal{J}_{ui}$s according to

$$\begin{aligned}J_{ui}\tilde{\wedge}K_{jk} &= 0\ ,\\ J_{ui}\tilde{\wedge}J_{vj} &= \delta_{uv}K_{ij}+\epsilon_{ij}J_{uv}\ .\end{aligned} \tag{12.11}$$

The first condition is a consequence of that under $SO(6,6)\times SL(2)$, the $\mathcal{J}_{ij}$ transform in the $(\mathbf{1},\mathbf{3})$ whereas the $\mathcal{K}_{ui}$ transform in the $(\mathbf{12},\mathbf{2})$. The second equations encodes for the decomposition in eq. (12.10). The $\mathcal{J}_{ij}$ are then seen to break the $E_{7(7)}$ to $SO(6,6)$, which in turn is broken to the $SO(6)$ by the $\mathcal{K}_{ui}$s. Combining the conditions in eq. (12.8) with those of eq. (12.11) we have that

$$\begin{aligned}\mathcal{J}_{ij} &= \tfrac{1}{6}\mathcal{K}^u{}_j\tilde{\wedge}J_{ui}\ ,\\ \mathcal{K}_{uv} &= \tfrac{1}{2}\mathcal{K}^u{}_j\tilde{\wedge}J_{ui}\ ,\\ \mathcal{K}_{ui}\tilde{\wedge}_S\mathcal{K}_{vj} &= -6\kappa^2\delta_{uv}\epsilon_{ij}\ ,\end{aligned} \tag{12.12}$$

i.e. so that the $\mathcal{J}_{ij}$ and $\mathcal{K}_{uv}$ are actually completely determined by the $\mathcal{K}_{ui}$s. Further the $\mathcal{J}_{ij}$ and $\mathcal{K}_{uv}$ act on the J_{ui} as $SL(2)$ and $SO(6)_R$ transformations respectively, so that

$$\begin{aligned}\mathcal{J}_{ij}\cdot\mathcal{K}_{uk} &= 2\kappa^2\epsilon_{k(i|}\mathcal{K}_{u|j)}\ ,\\ \mathcal{K}_{uv}\cdot\mathcal{K}_{wi} &= -\kappa^2\delta_{w[u}\mathcal{K}_{v]i}\ ,\end{aligned} \tag{12.13}$$

where the adjoint action is defined as

$$(\mathcal{J}_{ij}\cdot\mathcal{K}_{uk})^M = \mathcal{J}_{ij}{}^{\alpha}(t^{\alpha})^{MN}J_{ukN}\ . \tag{12.14}$$

12.2 Intrinsic torsion of the $SO(6)$-structure

Instrinsic torsion is the obstruction of having a G_{half}-compatible connection, which in turn implies a measure of supersymmetry breaking by the internal manifold. In our case, a G_{half}-compatible connection is defined as a connection of the covariant derivative D such that

$$D\mathcal{J} = D\hat{\mathcal{J}} = D\mathcal{K} = D\kappa = 0 \ . \tag{12.15}$$

The corresponding torsion of this connection is given by the ExFT analogue of the ordinary definition, namely

$$(\mathbb{L}^D_\xi - \mathbb{L}_\xi)V^M = T^M{}_{NP}\xi^N V^P \ , \tag{12.16}$$

with $\mathbb{L}_\xi$ being the generalised Lie derivative as defined in eq. (10.2) acting on the generalised vector field V^M, $\mathbb{L}^D_\xi$ is the Lie derivative with all ingoing derivatives replaced with covariant derivative D and T is the torsion. The space of torsions W lies in $W \subset R_1^* \otimes P \equiv \mathbf{56} \otimes \mathbf{133}$.

12.2.1 Intrinsic torsion of the $SO(6,6)$-structure

Turning to the intrinsic torsion of the $SO(6)$-structure, we are to first find the instrinsic torsion of the $SO(6,6)$-structure. With the same analysis as in the previous chapter, we decompose the torsion of $E_{7(7)}$ under $SO(6,6) \times SL(2)$ where it has been found that

$$W = \mathbf{56} \oplus \mathbf{912} = (\mathbf{32'}, \mathbf{1}) \oplus (\mathbf{352'}, \mathbf{1}) \oplus 2(\mathbf{12}, \mathbf{2}) \oplus (\mathbf{220}, \mathbf{2}) \oplus (\mathbf{32'}, \mathbf{3}) \ . \tag{12.17}$$

The space of $SO(6,6)$ connections is given by

$$\begin{aligned} K_{SO(6,6)} &= ((\mathbf{12}, \mathbf{2}) \oplus (\mathbf{32'}, \mathbf{1})) \oplus (\mathbf{66}, \mathbf{1}) \\ &= (\mathbf{32'}, \mathbf{1}) \oplus (\mathbf{352'}, \mathbf{1}) \oplus (\mathbf{1728'}, \mathbf{1}) \oplus (\mathbf{12}, \mathbf{2}) \oplus (\mathbf{220}, \mathbf{2}) \oplus (\mathbf{560}, \mathbf{2}) \ . \end{aligned} \tag{12.18}$$

Out of these, only the last four representations transform in the same representations as the torsion, as we see by comparison to eq. (12.17). Hence, the image of the torsion map $\mathrm{im}\tau_{SO(6,6)} : K_{SO(6,6)} \to W$ is

$$\mathrm{im}\tau_{SO(6,6)} = (\mathbf{32'}, \mathbf{1}) \oplus (\mathbf{352'}, \mathbf{1}) \oplus (\mathbf{12}, \mathbf{2}) \oplus (\mathbf{220}, \mathbf{2}) \ . \tag{12.19}$$

The subset of W that is independent of the choice of G_{half} is defined as the intrinsic torsion and hence given by

$$W^{\text{int}}_{SO(6,6)} = \frac{W}{\mathrm{im}\tau_{SO(6,6)}} = (\mathbf{12}, \mathbf{2}) \oplus (\mathbf{32'}, \mathbf{3}) \ . \tag{12.20}$$

Given two connections D created from two different G_{half}-structures, their difference $D - D'$ lies in the space of compatible connections $K_{G_{\text{half}}} \equiv R_1^* \otimes \mathrm{ad}(G_{\text{half}})$ and is tensor valued. Eq. (12.16) then defines the torsion of this tensor. The next task is to find an expression of the intrinsic torsion in terms of our different structures. From eq. (12.16) we see that the intrinsic torsion is a generalised tensor that has one derivative in it which by definition is independent of the G_{half}-structure.

To find an expression of the $SO(6,6)$ intrinsic torsion one considers derivatives of the $(\mathcal{J}_{ij}, \kappa)$ structure. It is natural to combine

$$(\mathrm{d}\mathcal{J}_{ij})^M = -12(t_\alpha)^{MN}\partial_N \mathcal{J}^\alpha_{ij} - \tfrac{1}{2}\omega^{MN}W_{ijN} \ , \tag{12.21}$$

where the W_{ijN}s are three compensator fields. These were introduced in chapter 10, where given any tensor that is a section of the $\mathcal{R}_2$ bundle, a covariant derivative can be constructed as in eq. (12.21) where the compensator field W must fulfil

$$(t_\alpha)^{MN}W_N\partial_N = \omega^{MN}W_M\partial_N = (t_\alpha)^{MN}W_M W_N = 0 \ . \tag{12.22}$$

It is generally not clear how to construct an explicit compensator field associated to a completely general tensor fulfilling eq. (12.21). However, since we have information about the triplet $\mathcal{J}_{ij}$ from its compatibility requirements in eq. (12.8), an appropriate compensator field is found in [135] to be

$$W_{ijM} = -\tfrac{1}{2\kappa^2}\mathcal{J}_{k(i}{}^{\alpha}\partial_M\mathcal{J}^k{}_{j)\alpha} \ , \tag{12.23}$$

which makes eq. (12.21) take the form

$$(\mathrm{d}\mathcal{J}_{ij})^M = -12(t_\alpha)^{MN}\partial_N\mathcal{J}^\alpha_{ij} + \tfrac{1}{4\kappa^2}\omega^{MN}\mathcal{J}_{k(i}{}^{\alpha}\partial_M\mathcal{J}^k{}_{j)\alpha} \ . \tag{12.24}$$

12.2.2 Reduction to the intrinsic torsion of the $SO(6)$-structure

Turning to the intrinsic torsion of the $SO(6)$-structure, decomposing $E_{7(7)} \to SO(6)\times SO(6)_R \times SL(2)$ will give the space of torsion

$$\begin{aligned} W &= \mathbf{56} \oplus \mathbf{912} \\ &= (\mathbf{15},\mathbf{6},\mathbf{2}) \oplus (\mathbf{6},\mathbf{15},\mathbf{2}) \oplus (\mathbf{10},\mathbf{1},\mathbf{2}) \oplus (\bar{\mathbf{10}},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{10},\mathbf{2}) \oplus (\mathbf{1},\bar{\mathbf{10}},\mathbf{2}) \\ &\quad \oplus 2(\mathbf{6},\mathbf{1},\mathbf{2}) \oplus 2(\mathbf{1},\mathbf{6},\mathbf{2}) \oplus \dots \ , \end{aligned} \tag{12.25}$$

where the dots are representations that only contain spinorial representations of $SO(6)$ and will not be needed here as they will vanish in a half-maximal background. The space of $SO(6)$ connections is given by

$$\begin{aligned} K_{SO(6)} &= ((\mathbf{6},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{6},\mathbf{2}) \oplus \dots) \otimes (\mathbf{15},\mathbf{1},\mathbf{1}) \\ &= (\mathbf{64},\mathbf{1},\mathbf{2}) \oplus (\mathbf{15},\mathbf{6},\mathbf{2}) \oplus (\mathbf{10},\mathbf{1},\mathbf{2}) \oplus (\bar{\mathbf{10}},\mathbf{1},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1},\mathbf{2}) \oplus \dots \ , \end{aligned} \tag{12.26}$$

so that the image of the torsion map becomes

$$\mathrm{im}\tau_{SO(6)} = (\mathbf{15},\mathbf{6},\mathbf{2}) \oplus (\mathbf{10},\mathbf{1},\mathbf{2}) \oplus (\bar{\mathbf{10}},\mathbf{1},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1},\mathbf{2}) \oplus \dots \ . \tag{12.27}$$

Hence, the intrinsic torsion is given by

$$W^{\mathrm{int}}_{SO(6)} = (\mathbf{6},\mathbf{15},\mathbf{2}) \oplus (\mathbf{10},\mathbf{1},\mathbf{2}) \oplus (\bar{\mathbf{10}},\mathbf{1},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{6},\mathbf{2}) \oplus \dots \ . \tag{12.28}$$

In constructing an explicit expression for the intrinsic torsion, one considers derivatives of the $SO(6)$-structure combined with different tensors. For instance we can use the expression for $\mathrm{d}\mathcal{J}_{ij}$ in eq. (12.24). Similarly as for $\mathrm{d}\mathcal{J}_{ij}$, one can define a derivative of the $\mathcal{K}_{uv}$ generators of $SO(6)_R$ such that

$$\mathrm{d}\mathcal{K}_{uv}{}^M = -12(t_\alpha)^{MN}\partial_N\mathcal{K}_{uv}{}^\alpha - \tfrac{1}{2}\omega^{MN}\omega_{MN}\mathcal{K}_{[u}{}^{iK}\partial_N\mathcal{K}_{v]i}{}^L \ , \tag{12.29}$$

which can be understood in the same way as for $\mathcal{K}_{uv}$ with the second term being a compensator field. Further, it is convenient to label the occurring tensors transforming in the $SO(6)\times SO(6)_R \times SL(2)$ representations according to

$$\begin{aligned} T_1^M &\in (\mathbf{6},\mathbf{1},\mathbf{2}) \ , & T_{2ui} &\in (\mathbf{1},\mathbf{6},\mathbf{2}) \ , \\ R_{1uv}{}^M &\in (\mathbf{6},\mathbf{15},\mathbf{2}) \ , & R_{2uvwi} &\in (\mathbf{1},\mathbf{10},\mathbf{2}) \oplus (\mathbf{1},\bar{\mathbf{10}},\mathbf{2}) \ , \end{aligned} \tag{12.30}$$

which implies that

$$T_1{}^M\mathcal{K}_{uM} = R_{1uv}{}^M\mathcal{K}_{uM} = 0 \ . \tag{12.31}$$

In order to find the possible tensorial combinations of $SO(6)$-structure derivatives, one begins by combining the compatibility conditions in eq. (12.8) and eq. (12.11) with the derivatives in eq. (12.24) and eq. (12.29) which yield

$$\mathbb{L}_{\mathcal{K}_{ui}}\mathcal{K}_{vj} + \mathbb{L}_{\mathcal{K}_{vj}}\mathcal{K}_{ui} = \delta_{uv}\,\mathrm{d}\mathcal{J}_{ij} + \epsilon_{ij}\,\mathrm{d}\mathcal{K}_{uv} \ . \tag{12.32}$$

We also know that the $\mathcal{J}_{ij}$ can be expressed in terms of $\mathcal{K}_{ui}$, and so $\mathbb{L}_{\mathcal{K}_{ui}}\mathcal{J}_{jk}$ would not be independent of the terms in eq. (12.32). Hence, these are then the only independent combinations of $SO(6)$ derivatives. In terms of the representations of eq. (12.30), they are given by

$$\begin{aligned} \mathrm{d}\mathcal{J}_{ij} &= -\tfrac{1}{2}\mathcal{J}_{ij}\cdot T_1 - \kappa\mathcal{K}_{u(i}T_2{}^u{}_{j)} + \dots\ , \\ \mathrm{d}\mathcal{K}_{uv} &= 2\kappa^2 R_{1uv} - \kappa R_{2uvw}\mathcal{K}^{wk} - \kappa T_{2[u}{}^k\mathcal{K}_{v]k} + \dots\ , \\ \mathbb{L}_{\mathcal{K}_{ui}}\mathcal{K}_{vj} - \mathbb{L}_{\mathcal{K}_{vj}}\mathcal{K}_{ui} &= -2\kappa R_{2uvw(i}\mathcal{K}^w{}_{j)} + \kappa\mathcal{J}_{ij}\cdot R_{1uv} + \kappa T_{2[u}{}^k\mathcal{K}_{v]k} - \kappa T_{2u(i}\mathcal{K}_{|v|j} \\ &\quad - 2\kappa\epsilon_{ij}T_{2(u}{}^k\mathcal{K}_{v)k} - \tfrac{1}{2}\kappa\delta_{uv}\epsilon_{ij}T_{2wk}J^{wk} + \tfrac{1}{2}\kappa\delta_{uv}\epsilon_{ij}\kappa^2 T_1 + \dots\ . \end{aligned} \tag{12.33}$$

However, since κ^2 also transforms in $(\mathbf{6},\mathbf{1},\mathbf{2})$, these extra representations can be set to vanish by assuming that $\mathbb{L}_{\mathcal{K}_{ui}}\kappa^2 = 0$. So together with the conditions in eq. (12.33), these equations define the intrinsic torsion.

12.2.3 Half-maximal flux vacua

Having identified the intrinsic torsion of the $SO(6)$-structure, we know from the previous chapter that the vanishing of the intrinsic torsion is required by a Minkowski vacuum preserving a half-maximal amount of supersymmetries, i.e.

$$\mathrm{d}\mathcal{J}_{ij} = \mathrm{d}\mathcal{K}_{uv} = \mathbb{L}_{\mathcal{K}_{ui}}\mathcal{K}_{ui} = \mathbb{L}_{\mathcal{K}_{ui}}\kappa^2 = 0\ . \tag{12.34}$$

The Minkowski$_4$ vacuum is then defined by an integrable $SO(6)$-structure. A half-maximal AdS$_4$ vacuum can be obtained in a similar manner by identifying in which representations the supersymmetry variations transform. It turns out that in an AdS$_4$ vacuum, only the $(\mathbf{1},\mathbf{10},\mathbf{2})$ component will vanish, so by eq. (12.33) we must have that the defining equations are given by

$$\mathrm{d}\mathcal{J}_{ij} = \mathbb{L}_{\mathcal{K}_{ui}}\kappa^2 = 0\ , \tag{12.35}$$

and

$$\mathrm{d}\mathcal{K}_{uv} = -\kappa R_{2uvwi}\mathcal{K}^{wi}\ , \qquad \mathbb{L}_{\mathcal{K}_{ui}}\mathcal{K}_{vj} = -\kappa R_{2uvwi}\mathcal{K}^w{}_j\ , \tag{12.36}$$

with $\kappa R_{2uvwi} = -\frac{1}{3!}\epsilon_{uvwxyz}\delta_{ij}\kappa R_2^{xyzj}$ being a constant. Since the intrinsic torsion does not vanish completely, the above structure can be called weakly integrable.

12.3 Application in consistent truncations

When studying half-maximal consistent truncations it has proven useful to reformulate the $E_{7(7)}$ ExFT in terms of the $SO(6)$-structure instead of the generalised metric. In doing so the ExFT will have manifest $\mathcal{N}=4$ before truncation. The scalar potential is of particular interest, and it may be found by comparing its general form with the established 4D half-maximal supergravity, to find that

$$\begin{aligned} V = -\tfrac{1}{4}\Big(&\tfrac{3}{4}T_2^{ui}T_{2u}{}^j\delta_{ij} - \tfrac{1}{16\kappa^2}T_1^M T_1^N\mathcal{J}_{ij}{}^\alpha(t_\alpha)_{MN}\delta^{ij} + \tfrac{1}{3}R_{2uvwi}R_2^{uvw}{}_j\delta^{ij} \\ &- \tfrac{1}{12\kappa^2}R_{1uv}{}^M R_1^{uvN}(t_\alpha)_{MN}\mathcal{J}_{ij}^\alpha\delta^{ij} - \tfrac{1}{9}R_{2uvwi}R_{2xyzj}\epsilon^{uvwxyz}\epsilon^{ij}\Big) + \dots\ . \end{aligned} \tag{12.37}$$

Again the ellipsis refers to the spinorial terms that vanish in the truncation.

The truncation ansatz is given by expanding all ExFT fields in terms of a background $SO(6)$-structure, which is defined by sections of three bundles;

$$n_{ij} \in \Gamma(\mathcal{R}_1^Y)\ , \qquad \Theta_{Aj} \in \Gamma(\mathcal{R}_2^Y)\ , \qquad \Upsilon \in \Gamma(\mathcal{S}^Y)\ , \tag{12.38}$$

where as usual i,j are $SL(2)$ indices and $A = 1,\dots,6-N$ are $SO(6-N)$ indices. The N will encode the number of vector multiplets in the theory. The $\mathcal{R}$ and $\mathcal{S}$ are bundles defined over the internal space on which the truncation happens, hence they depend only on the Y^M internal coordinates. The sections in eq. (12.38) have compatibility on their own, given by

$$\begin{aligned}
(n_{ij} \otimes n_{kl})|_{1539} &= 0 \ , \\
[n_{ij}, n_{kl}] &= -2\Upsilon^2(\epsilon_{i(k}n_{l)j} + \epsilon_{j(k}n_{l)i}) \ , \\
\mathrm{tr}(n_{ij}n_{kl}) &= 12\Upsilon^4\epsilon_{i(k}\epsilon_{l)j} \ , \\
\Theta_{Ai} \wedge n_{jk} &= 0 \ , \\
\Theta_{Ai} \wedge \Theta_{Bj} &= \eta_{AB}n_{ij} + \epsilon_{ij}\Theta_{AB} \ ,
\end{aligned} \tag{12.39}$$

which ensures that the background has $SO(6-N)$-structure. The truncation ansatz is then given as the expansion of the $SO(6)$-structure in terms of the background $SO(6-N)$ fields as follows:

$$\begin{aligned}
\mathcal{J}_{ij}(x,Y) &= a_i{}^k(x)a_j{}^l(x)n_{kl}(Y) \ , \\
\mathcal{K}_{ui}(x,Y) &= b_u{}^A(x)a_i{}^j(x)m_{Aj}(Y) \ , \\
\kappa(x,Y) &= \Upsilon(Y) \ , \\
g_{\mu\nu}(x,Y) &= \bar{g}_{\mu\nu}(x)\Upsilon(Y) \ .
\end{aligned} \tag{12.40}$$

As usual the coefficients $\bar{g}_{\mu\nu}(x)$, $a_i{}^k(x)$ and $b_u{}^A(x)$ will become scalar fields in the 4D effective theory. The expansion of $\kappa(x,Y)$ does not have a scalar coefficient as it would simply be a rescaling of $\Upsilon(Y)$ and be independent of Y^M. The compatibility conditions of $\mathcal{J}_{ij}$ in eq. (12.8) and $\mathcal{K}_{ui}$ in eq. (12.11) impose the conditions

$$a_i{}^k a_j{}^l \epsilon_{kl} = \epsilon_{ij} \ , \qquad b_u{}^A b_v{}^B \eta_{AB} = \delta_{uv} \ , \tag{12.41}$$

on the ingoing scalar fields. From these scalar fields one can construct combinations that are invariant under R-symmetry, namely

$$\mathcal{H}^{ij} = a_i{}^k a_j{}^l \delta_{kl} \ , \qquad P_-^{AB} = b_u{}^A b^{uB} = \tfrac{1}{2}(\eta^{AB} - \mathcal{H}^{AB}) \ , \tag{12.42}$$

which will be useful later. The $\mathcal{H}^{ij}$ and $\mathcal{H}^{AB}$ parametrise the coset spaces $SL(2)/U(1)$ and $SO(6,N)/(SO(6) \times SO(N))$ respectively, which makes the scalar manifold parametrise

$$\mathcal{M}_{\text{scalar}} = \frac{SO(6,N)}{SO(6) \times SO(N)} \times \frac{SL(2)}{U(1)} \ . \tag{12.43}$$

This is equivalent to the scalar manifold of half-maximal gauge supergravity with N vector multiplets. Further, in order to have a consistent truncation there are three conditions that need to be imposed on the intrinsic torsion of the $SO(6-N)$-structure. First, it should not include any spinor representation of $SO(6-N)$, and secondly we should be able to expand the intrinsic torsion in terms of a finite number of fields that define the $SO(6-N)$-structure background. The second condition implies that the vector representation of $SO(6-N)$ should vanish. With these conditions the $SO(6-N)$ intrinsic torsion can be written on the general form

$$\begin{aligned}
\mathrm{d}n_{ij} &= -\omega_{A(i}f^A{}_{j)} \ , \\
\mathrm{d}\Theta_{MN} &= -f_{ABCi}\Theta^{Ci} - f_{[A}{}^i\Theta_{b]i} \ , \\
\mathbb{L}_{\Theta_A}\Upsilon^2 &= 0 \ , \\
\mathbb{L}_{\Theta_{Ai}}\Theta_{Bj} &= -f_{ABC(i}\Theta^C{}_{j)} + \tfrac{1}{2}\left(f_{B(i|}\Theta_{A|j)} - f_{A(i|}\Theta_{B|j)} - f\eta_{AB}\Theta_{C(i}f^C{}_{j)}\right) \\
&\quad - \epsilon_{ij}\left(f_{A(}{}^i\Theta_{B)i} + \tfrac{1}{4}\eta_{AB}f_{Ci}\Theta^{Ci} - \tfrac{1}{2}f_{[A}{}^i\Theta_{B]i}\right) \ ,
\end{aligned} \tag{12.44}$$

for some fields f_{ABCi} and f_{Ai}. The third requirement is that these fields are constant. The f become the embedding tensors in the effective 4D gauged half-maximal supergravity, and have the most general form possible. As in the DFT case, the gaugings f also have to satisfy some quadratic constraints. In this case they will follow from the closure of the generalised Lie derivative, which in turn is fulfilled as long as the section constraint is fulfilled by our $SO(6-N)$-structures. This implies that the quadratic constraints are satisfied automatically.

From the conditions of eq. (12.44) one can calculate the components of the intrinsic torsion to find that

$$\begin{aligned} T_1 &= \Upsilon^{-2} P_+^{AB} f_A{}^i \Theta_{Bi} \ , \\ T_{2ui} &= \Upsilon^{-1} a_i{}^j b_u{}^A f_{Aj} \ , \\ R_{uv}{}^M &= \Upsilon^{-2} b_u{}^A b_v{}^B P_+{}^{CD} f_{ABD}{}^i \Theta_{Ci} \ , \\ R_{uvwi} &= \Upsilon^{-1} b_u{}^A b_v{}^B b_w{}^C a_i{}^j f_{ABCj} \ , \end{aligned} \tag{12.45}$$

where the other (spinorial) components vanish. Using this in the potential of eq. (12.37), as well as the formula

$$\left[P_-^{AD} P_-^{BE} (\tfrac{1}{3} P_-^{CF} + P_+^{CF}) - \tfrac{1}{12} \mathcal{H}^{AD} \mathcal{H}^{BE} \mathcal{H}^{CF} + \tfrac{1}{4} \mathcal{H}^{AD} \eta^{BE} \eta^{CF} - \tfrac{1}{6} \eta^{AD} \eta^{BE} \eta^{CF}\right] A_{ABC} A_{DEF} = 0 \ , \tag{12.46}$$

fulfilled by any antisymmetric tensor $A_{ABC} \equiv A_{[ABC]}$, the potential reduces to

$$\begin{aligned} V = -\tfrac{1}{4} \Upsilon^4 \Big[& f_{ABCi} f_{DEFj} \mathcal{H}^{ij} \left(\tfrac{1}{12} \mathcal{H}^{AD} \mathcal{H}^{BE} \mathcal{H}^{CF} - \tfrac{1}{4} \mathcal{H}^{AD} \eta^{BE} \eta^{CF} + \tfrac{1}{6} \eta^{AD} \eta^{BE} \eta^{CF} \right) \\ & - \tfrac{1}{9} f_{ABCi} f_{DEFj} \epsilon^{ij} \mathcal{H}^{ABCDEF} + \tfrac{3}{4} f_A{}^i f_B{}^j \mathcal{H}_{ij} \mathcal{H}^{AB} \Big] \ , \end{aligned} \tag{12.47}$$

where we have defined

$$\mathcal{H}^{ABCDEF} \equiv e^{uvwxyz} b_u{}^A b_v{}^B b_w{}^C b_x{}^D b_y{}^E b_z{}^F \ . \tag{12.48}$$

The scalar potential in eq. (12.37) agrees with the scalar potential in half-maximal gauged supergravity in 4D with N vector multiplets [116]. The only Y dependence is found in the conformal factors Υ, which guarantees a consistent truncation [135].

Connecting this with our previously found $SO(6)$-structures, we have found a consistent truncation to any half-maximal Minkowski$_4$ or AdS$_4$ vacuum of type II or $D = 11$ supergravity which only have the gravitional supermultiplet. The tensors $\mathcal{J}_{ij}$ and $\mathcal{K}_{ui}$ define an $SO(6)$-structure, which is integrable in the Minkowski case and weakly integrable in the AdS case, can be used in the truncation ansatz eq. (12.39) as

$$n_{ij}(Y) = \mathcal{J}_{ij}(Y) \ , \qquad \Theta_{ui} = \mathcal{K}_{ui}(Y) \ , \qquad \Upsilon(Y) = \kappa(Y) \ . \tag{12.49}$$

The components of the intrinsic torsion are then given by

$$\begin{aligned} \mathrm{d} n_{ij} &= \mathbb{L}_{\theta_{ui}} \Upsilon^2 = 0 \ , \\ \mathrm{d}\Theta_{uv} &= -\kappa R_{2uvwi} \Theta^{wi} \ , \\ \mathbb{L}_{\Theta_{ui}} \mathcal{K}_{wj} &= -\kappa R_{2uvwi} \Theta^w{}_j \ , \end{aligned} \tag{12.50}$$

in the AdS case, and with vanishing right-hand side for Minkowski. By comparing with eq. (12.44) we see that this indeed fulfils the conditions to be a consistent truncations. In this case the obtained 4D half-maximal supergravity obtained has embedding tensor $f_{uvwi} = \kappa R_{2uvwi}$. With this we end our discussion on consistent truncations in ExFT. This is an ongoing field of research. The tools regarding half-maximal G-structures has recently been used to construct supersymmetric warped AdS$_7$ vacua of massive IIA and AdS$_6$ vacua of IIB supergravity which include vector mutiplets [137][138]. Further, there have also been recent development in the discussion of orbifold and orientifold planes in the context of ExFT [139].

13
Conclusions and Outlook

The principal aim of this thesis was to study flux backgrounds. It is clear that the area of flux compactifications in string theory is a vast and varied field that has been very fruitful.

In this thesis we have covered a large scope, introducing basic notions of supersymmetry, Kaluza-Klein compactification and algebraic topology concepts, to explore various aspects of flux compactifications and duality-covariant extensions of supergravity. We have seen that compactifications of the two type II theories on Calabi-Yau manifolds result in effective actions with moduli fields, and how their solutions are related via mirror symmetry. Orientifold solutions project discrete symmetries, which mod out parts of the type II fields, and result in effective theories with a lower amount of preserved supersymmetry. Including a non-vanishing expectation values of the type II fields, i.e. flux, in the compactification, some moduli can be stabilised. In most cases the fluxes are not enough to stabilise all moduli, although there are examples of type IIA with all moduli fixed, for example on the orientifold $T^6/(\mathbb{Z}_2 \otimes \mathbb{Z}_2)$. Further, the inclusion of fluxes breaks supersymmetry partially or completely in a stable way which in turn generates warp factors that are used in finding large hierarchies of scales. Quantum corrections can be introduced to fix remaining moduli, however their incorporation is not always under theoretical control and is still a current area of research. Non-geometric compactifications are available but their geometrical interpretation is not always understood, and they are by this reason often neglected. Though vast progress has been made, it remains to be seen whether a de Sitter vacuum can be constructed using quantum corrections and if the underlying difficulties largely come from computational hardness or an underlying reason.

When considering moduli stabilisation by fluxes, their back-reaction on the geometry of the internal manifold is usually neglected, since all non-trivial flux backgrounds are manifolds with torsion. This is however neatly formulated in complex generalised geometry, which is a differential geometric description of the allowed internal manifolds where the back-reaction to the fluxes is taken to account by unifying complex and symplectic geometry. This provides a simple but powerful geometrical formulation of supersymmetric string backgrounds. In this formulation bosonic degrees of freedom are "geometrised" in the way that they are packaged into a generalised metric which is equivalent to a G-structure on the generalised tangent bundle. A geometrical interpretation of generic flux backgrounds in both type II and $D = 11$ supergravity with Minkowski spacetime was constructed using the language of $E_{7(7)} \times \mathbb{R}^+$ generalised geometry. Integrability of the G-structure is then defined as the existence of a generalised torsion-free connection that is compatible with the G-structure, which is equivalent to the vanishing of the generalised intrinsic torsion. The integrability was then shown to be in one-to-one correspondence with the Killing spinor equations, due to the fact that G is the stabiliser group of $\mathcal{N}$ Killing spinors. This defines what is called exceptional Calabi-Yau spaces, incorporating the usual compatibility conditions found in both ordinary and complex generalised geometry.

In the final chapter we studied half-maximal supersymmetric backgrounds in ExFT in 4D using the formalism of exceptional generalised G-structures. These backgrounds admitted globally defined nowhere vanishing tensors that are viewed as the ExFT analogue of differential forms, where the intrinsic torsion of G_{half}-structures could be written in terms of derivatives of these tensors. Using them, integrability or weak integrability conditions could be written down, implying a 4D half-maximal warped Minkowski or AdS vacuum.

The exceptional Calabi-Yau backgrounds, as well as their corresponding AdS solutions described by exceptional Sasaki-Einstein structures, both preserve $\mathcal{N} = 2$ supersymmetry. AdS backgrounds with $\mathcal{N} = 1$ have been described in exceptional generalised geometry [140]. Maximally symmetric backgrounds are described by parallelisations in both exceptional generalised geometry and ExFT. We have seen descriptions of half-maximal backgrounds in ExFT, though the corresponding picture in exceptional generalised geometry remains to be seen. A natural extension for future work is then to find a description of supergravity flux backgrounds in terms of both exceptional generalised geometry and ExFT preserving any amount of supersymmetry. The ultimate goal should be a classification of all supersymmetric backgrounds. Such a classification should also provide information on new examples of consistent truncations. Further, when structures preserving different amounts of supersymmetry are identified, the next task should be to classify them and find a coherent structure which describes them.

Further, these areas are also relevant in holography, field theory and pure mathematics as it covers various fields of algebraic geometry and topology as well as group theory. A fun thought regarding the recent developments in describing orientifold planes in ExFT, circumventing no-go theorems, is that it would be interesting to see if a de Sitter vacuum in ExFT could be constructed. It remains to be seen what the U-duality covariant approach of ExFT and exceptional generalised geometry might reveal, so that maybe one could better understand the geometrical nature of dualities in string theory. As such, generalised geometry and ExFT in this context are well-worth further studies.

A

Notation and Conventions

We gather some conventions used in this text even though they should be stated when used. Some clash of notation between chapters is unavoidable but their mening should be clear from statement and context.

- $c = \hbar = 1$ and when factors of α' are not explicitly stated we take $2\pi\alpha' = 1$.
- The following index notation is used:
  - $\mu, \nu, \ldots = 0, \ldots, 3$ are external spacetime indices,
  - $m, n \ldots = 4, \ldots, 9$ are real internal space indices,
  - $M, N, \ldots = 0, \ldots, 9$ label all spacetime coordinates,
  - $i, j, \ldots = 1, 2, 3,\ \bar{\imath}, \bar{\jmath}, \ldots = 1, 2, 3$ are holomorphic internal indices, though in chapter 9 i, j label all doubled coordinates, and in chapter 11 $i, j, \ldots = 1, 2$ are fundamental $SL(2, \mathbb{R})$ indices.
- The wedge products and contractions are as defined in Appendix B, i.e.
  - $(A \wedge A')^{p_1 \ldots p_{n+n'}} \equiv \frac{(n+n')!}{n!n'!} A^{[p_1 \ldots p_n} A'^{p_{n+1} \ldots p_{n'}]}$,
  - $(A \wedge A')_{p_1 \ldots p_{n+n'}} \equiv \frac{(n+n')!}{n!n'!} A_{[p_1 \ldots p_n} A'_{p_{n+1} \ldots p_{n'}]}$,
  - $(A \lrcorner A')^{p_1 \ldots p_{n-n'}} \equiv \frac{1}{n'!} A^{q_1 \ldots q_{n'} p_{n'+1} \ldots p_{n-n'}} A'_{q_1 \ldots q_{n'}}\ , \qquad n' < n$,
  - $(A \lrcorner A')_{p_1 \ldots p_{n'-n}} \equiv \frac{1}{n!} A^{q_1 \ldots q_n} A'_{q_1 \ldots q_n p_1 \ldots p_{n'-n}}\ , \qquad n < n'$,
  - $(jA \lrcorner jA')^i{}_j \equiv \frac{1}{(n-1)!} A^{ip_1 \ldots p_{n-1}} A'_{jp_1 \ldots p_{n-1}}$,
  - $(jA, A')_{p, p_1 \ldots p_d} \equiv \frac{d!}{(n-1)!(d+1-n)!} A_{p[p_1 \ldots p_{n-1}} A'_{p_n \ldots p_d]}$.
- A $\star$ denotes the Hodge operator, whose dimensionality is given by context if not explicitly named $\star_{10}$, $\star_6$, etc.
- The superscript (10) of the 10D flux field strength $F^{(10)}$ is not to be confused with the superscripts denoting specific $SU(3)$ representations, i.e. $F^{(2)}, F^{(4)}, F^{(6)}$ or $F^{(8)}$, as used in chapter 4 and Appendix C.
- In a decomposition of 10D gamma matrices $\Gamma^M = (\Gamma^\mu, \Gamma^m) = (\gamma^\mu \otimes 1, \gamma_5 \otimes \gamma^m)$ we have $\gamma_{11} = \gamma_5 \gamma_7$ where $\gamma_5 \equiv \frac{i}{4!} \epsilon_{\mu\nu\rho\lambda} \gamma^{\mu\nu\rho\lambda}$ and $\gamma_7 \equiv -\frac{i}{6!} \epsilon_{mnpqrs} \gamma^{mnpqrs}$.
- A slash is defined by $\not{F}_n \equiv \frac{1}{n!} F_{P_1 \ldots P_N} \Gamma^{P_1 \ldots P_N}$ where $\Gamma^{P_1 \ldots P_N} \equiv \Gamma^{[P_1} \ldots \Gamma^{P_N]}$.

B

Mathematical Preliminaries

In this appendix we collect some definitions and results in order to be self-contained. More rigorous definitions and explanations are for example found in [141][142].

B.1 Differential forms

In the generalisation from differentials to differential forms, as well as the associated vector calculus, three new operators will come to use: the wedge product, the exterior derivative, and the Hodge dual.

Definition: wedge product. The wedge product is an antisymmetrised tensor product. It is defined as to give differential elements the proper sign. For instance in 3D we have

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x \ .$$

This automatically gives the right orientation of a surface. A volume element becomes

$$V = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \ ,$$

which changes sign if any pair of the basis elements are switched. This is an example of a *differential form*, more precisely a volume form, which is a 3-form. There are other types of forms, such as a line integrand:

$$A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z \ ,$$

and a surface integrand:

$$A_z \mathrm{d}x \wedge \mathrm{d}y + A_y \mathrm{d}z \wedge \mathrm{d}x + A_x \mathrm{d}y \wedge \mathrm{d}z \ .$$

Line elements are 1-forms and surface elements are 2-forms and 0-forms are functions. In more than 3D there is a generalisation to p-forms, where p ranges from zero to the dimension D of the space. The wedge product is associative and obeys the usual distributive laws. The wedge product of a p-form with a q-form is a $(p+q)$-form.

Definition: exterior derivative. The exterior derivative d is an operator which when applied to a p-form gives an $(p+1)$-form. Consider a 1-form $A = A_i \mathrm{d}x^i$, we define

$$\begin{aligned} \mathrm{d}A &= \mathrm{d}A_i \wedge \mathrm{d}x^i \\ &= (\partial_j A_i \mathrm{d}x^j) \wedge \mathrm{d}x^i \\ &= \partial_j A_i \mathrm{d}x^j \wedge \mathrm{d}x^i \\ &= \tfrac{1}{2}(\partial_j A_i - \partial_i A_j) \mathrm{d}x^j \wedge \mathrm{d}x^i \ , \end{aligned} \tag{B.1}$$

since the product $\mathrm{d}x^j \wedge \mathrm{d}x^i$ is antisymmetric. Now suppose we are in 3D and we consider a 2-form expressed as

$$\mathrm{S} = A_z \mathrm{d}x \wedge \mathrm{d}y + A_y \mathrm{d}z \wedge \mathrm{d}x + A_x \mathrm{d}y \wedge \mathrm{d}z \ .$$

Applying the exterior derivative gives us

$$\begin{aligned}
\mathrm{d}S &= \mathrm{d}A_z \wedge \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}A_y \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}A_x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \partial_z A_z \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + \partial_y A_y \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + \partial_x A_x \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= (\partial_y A_y + \partial_y A_y + \partial_x A_x) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \ ,
\end{aligned} \tag{B.2}$$

so we see that the exterior derivative can produce the divergences for the corresponding form fields. For a general p-form field $A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$, the exterior derivative is defined as

$$\begin{aligned}
\mathrm{dA}_p = \mathrm{d} \wedge A_p &= \frac{1}{p!} \partial_\nu A_{\mu_1 \ldots \mu_p} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \\
&= \frac{1}{p!} \partial_{[\nu} A_{\mu_1 \ldots \mu_p]} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \ ,
\end{aligned}$$

resulting in a $(p+1)$-form. An important property that follows from this definition is that the exterior derivative is nilpotent, i.e. $d^2 = 0$. This can be shown by applying d^2 to a 0-form;

$$\begin{aligned}
\mathrm{dd}A_0 &= \mathrm{d} \left(\frac{\partial A_0}{\partial x^\mu} \mathrm{d}x^\mu \right) \\
&= \frac{\partial^2 A_0}{\partial x^\mu \partial x^\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ , \\
&= 0 \ ,
\end{aligned}$$

which vanishes as the wedge product is antisymmetric and the double derivative is symmetric in its indices. A p-form is called *closed* if it satisfies

$$\mathrm{d}A_p = 0 \ , \tag{B.3}$$

and *exact* if

$$A_p = \mathrm{d}A_{p-1} \ , \tag{B.4}$$

given that the $(p-1)$-form A_{p-1} is globally well-defined. A p-form may be written as eq. (B.4) *locally* but not globally, so a closed p-form need not be exact, but an exact p-form is always closed. A p-form A_p is said to be *co-exact* if $A_p = \mathrm{d}^\dagger A_{p+1}$.

Definition: Hodge dual. Taking the curl of the forms in eq. (B.1) and eq. (B.2) corresponds to turning the 2-form into a 1-form and the 3-form into a 0-form respectively. To accomplish this one introduces the Hodge dual, or star, operator; $\star$. In our examples with Cartesian coordinates we have that

$$\begin{aligned}
\star(\mathrm{d}x \wedge \mathrm{d}y) &= \mathrm{d}z \ , \\
\star(\mathrm{d}y \wedge \mathrm{d}z) &= \mathrm{d}x \ , \\
\star(\mathrm{d}z \wedge \mathrm{d}x) &= \mathrm{d}y \ , \\
\star(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) &= 1 \ .
\end{aligned}$$

In Cartesian coordinates the Hodge dual is its own inverse, so that $\star\star = 1$. The Hodge operator acting on a p-form is defined as

$$\star\left(\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}\right) = \frac{\epsilon^{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_n}}{(n-p)!\sqrt{g}} g_{\mu_{p+1}\nu_{p+1}} \cdots g_{\mu_n \nu_n} \, \mathrm{d}^{\nu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_n} \ , \tag{B.5}$$

where ϵ is the Levi-Civita symbol, transforming as a tensor density, and $\epsilon/\sqrt{g}$ is a tensor. Thus the Hodge operator acting on a p-form gives a $(n-p)$-form, with n being the dimension in question. For a general p-form, a double Hodge dual satisfies

$$\star\star A_p = (-1)^{p(n-p)+1} A_p \ , \tag{B.6}$$

for a Lorentzian signature and $\star\star A_p = (-1)^{p(n-p)} A_p$ for a Euclidean. The inner product for real p-forms is also defined using the Hodge-$\star$ operator, as

$$\int A_p \wedge \star A_p = \frac{1}{p!} \int A_{\mu_1 \ldots \mu_p} A^{\mu_1 \cdots \mu_p} \star \mathbf{1} \ , \tag{B.7}$$

where $\star\mathbf{1} = \mathrm{d}^n \sqrt{-G}$ is the n-dimensional measure.

B.2 Some homology and cohomology

Denoting the space of closed p-forms on a compact manifold M as $C^{(p)}(M)$, and the space of exact p-forms as $Z^{(p)}(M)$, then the p^{th} *de Rham cohomology group* $H^{(p)}(M)$ is defined as the *quotient space*

$$H^{(p)}(M) = C^{(p)}(M)/Z^{(p)}(M) \ . \tag{B.8}$$

A quotient space $Y = X/\sim$ is a set of equivalence classes of elements of X where $\sim$ is the equivalence relation on X. For example let $X = \mathbb{R}^1$ and $x \in X$ be a coordinate on the $\mathbb{R}^1$ real line. Let $x \sim x + 2\pi$ be the equivalence relation. Then the quotient space will be the sum of all possible lines of length 2π defined to start at some point $x \in \mathbb{R}^1$. In the same way, $H^{(p)}(M)$ is the space of closed forms in which two forms which differ by an exact form will be considered equivalent.

The dimension of our de Rham cohomology group $H^{(p)}(M)$ is called the *Betti number* b. They are topological invariants and characterise a manifold. Informally, the Betti number is the maximum number of cuts that can be made without dividing a surface into two separate pieces, and the k^{th} Betti number b_k refers to the number of k-dimensional holes on a manifold. For example, b_0 is the number of connected components, b_1 is the number of 1D holes, or "circular holes", b_2 is the number of 2D cavities, or voids. The sphere (S^2) is one connected component, so $b_0 = 1$, it does not have any "circular holes" so $b_1 = 0$, it has however one cavity enclosed within the surface, so $b_2 = 1$. For a T^2 torus it has one connected surface component, two circular holes; one external and one internal, and one cavity: the space embedded inside the torus. Thus the torus has Betti numbers $(b_0, b_1, b_2) = (1, 2, 1)$. Another topological invariant is the *Euler characteristic* χ, which can be expressed as an alternating sum of Betti numbers;

$$\chi(M) = \sum_{i=0}^{d} (-1)^i b_i(M) \ . \tag{B.9}$$

The Betti numbers of a manifold does not only give the dimension of the cohomology groups, but also the homology groups, which are defined in a similar way. Instead of using the exterior derivative d in its definition, the analogous boundary operator δ is introduced. The δ acts on submanifolds of M, e.g. if N is a submanifold of M then δN is its boundary. The δ is also nilpotent as the boundary of a boundary is an empty set, so $\delta^2 = 0$. Linear combinations of p-dimensional submanifolds are known as *p-chains*. A chain that has no boundary is called closed, and a closed chain z_p is known as a *cycle*, thus fulfilling

$$\delta z_p = 0 \ . \tag{B.10}$$

A chain that *is* a boundary is called exact. Thus the simplical homology group is defined in the analogous way as the quotient space $H_{(p)}(M)$ of equivalence classes of p-cycles;

$$H_{(p)}(M) = C_{(p)}(M)/Z_{(p)}(M) \ , \tag{B.11}$$

where $C_{(p)}(M)$ is the space of closed chains, and $Z_{(p)}(M)$ the space of exact chains. Thus two p-cycles are equivalent if their only difference is a boundary. As for the Betti numbers, the formal definition is that the k^{th} Betti number b_k is the rank of the k^{th} homology group of a manifold.

B.3 Harmonic forms and Hodge decomposition

The Laplace operator acting on p-forms in n-dimensional space is written

$$\Delta_p = \mathrm{d}^\dagger \, \mathrm{d} + \mathrm{d} \, \mathrm{d}^\dagger = (\mathrm{d} + \mathrm{d}^\dagger)^2 \ , \tag{B.12}$$

although a metric is needed to specify it. For example $\mathrm{d}^\dagger = (-1)^{np+n+1} \star \mathrm{d}\star$ for a metric with Euclidean signature, i.e. $(0, n)$ or as $\mathrm{d}^\dagger = (-1)^{np+n} \star \mathrm{d}\star$ for a Lorentzian signature $(1, n-1)$. Minkowski space for example is Lorentzian with the signature $(1, 3)$. Generally the action of Δ on some p-form A is given by

$$\Delta A_{\mu_1 \ldots \mu_p} = -\nabla^\nu \nabla_\nu A_{\mu_1 \ldots \mu_p} - p R_{\nu[\mu_1} A^\nu{}_{\mu_2 \ldots \mu_p]} - \tfrac{1}{2} p(p-1) R_{\nu\rho[\mu_1\mu_2} A^{\nu\rho}{}_{\mu_3 \ldots \mu_p]} \ , \tag{B.13}$$

where R is the Riemann tensor.

A p-form A is said to be *harmonic* if and only if

$$\Delta_p A = 0 \ . \tag{B.14}$$

Harmonic p-forms are in a one to one correspondence with the elements of the cohomology group $H^p(M)$, which can be seen by taking the scalar product

$$\langle A, \Delta A \rangle = \langle A | \Delta | A \rangle = \langle A | (\mathrm{d}^\dagger \, \mathrm{d} + \mathrm{d} \, \mathrm{d}^\dagger) | A \rangle = \langle \mathrm{d}A | \, \mathrm{d}A \rangle + \langle \mathrm{d}^\dagger A | \, \mathrm{d}^\dagger A \rangle \geq 0 \ , \tag{B.15}$$

since $\langle A | \, \mathrm{d}^\dagger \, \mathrm{d} | A \rangle = \langle \mathrm{d}A | \, \mathrm{d}A \rangle$. For $\langle A, \Delta A \rangle = 0$ then we must have $\mathrm{d}A = 0$ and $\mathrm{d}^\dagger A = 0$. A p-form that satisfies $\mathrm{d}^\dagger A = 0$ is said to be *co-closed*. In terms of the space of p-forms, Ω^p, the operators map according to

$$\begin{aligned} \mathrm{d} &: \Omega^p \to \Omega^{p+1} \ , \\ \mathrm{d}^\dagger &: \Omega^p \to \Omega^{p-1} \ , \\ \Delta &: \Omega^p \to \Omega^p \ . \end{aligned}$$

The *Hodge theorem* states that any p-form X_p can be decomposed uniquely into a harmonic, a (globally) exact, and a co-exact piece;

$$X_p = A_p + \mathrm{d}B_{p-1} + \mathrm{d}^\dagger C_{p+1} \ , \tag{B.16}$$

where $\Delta A_p = 0$. This is known as *Hodge decomposition.* When X_p is closed, then we must have $C_{p+1} = 0$, since

$$\mathrm{d}X_p = \mathrm{d}A_p + \mathrm{d}^2 B_{p-1} + \mathrm{d}\,\mathrm{d}^\dagger C_{p+1} = \mathrm{d}\,\mathrm{d}^\dagger C_{p+1} = 0 \ ,$$

where $\mathrm{d}A_p = 0$ since A_p is harmonic, $\mathrm{d}^2 B_{p-1} = 0$ since $\mathrm{d}^2 = 0$, thus forcing $C_{p+1} = 0$. If X_p is closed then it must also have the same cohomology class as A_p, having the same form. Each cohomology class therefore has precisely one harmonic form, which one can take to be a representative of the corresponding cohomology class. Thus the space of harmonic p-forms is isomorphic to the p^{th} cohomology;

$$\Omega^p_{\text{harm}} \simeq H^{(p)}(M) \ . \tag{B.17}$$

B.4 Manifolds

A manifold is a topological space which locally looks like $\mathbb{R}^m$, but not necessarily so globally. In this section we will list some of the appearing types of manifolds in this document.

Definition: complex manifold. The complex manifold of dimension n is a topological space M with a holomorphic map. A complex function $f = f_1 + i f_2$ is said to be holomorphic if it satisfies the *Cauchy-Riemann relations* for each complex coordinate $z^a \equiv x^a + i y^a$, $a = 1, \ldots, n$;

$$\frac{\partial f_1}{\partial x^a} = \frac{\partial f_2}{\partial y^a} \ , \quad \frac{\partial f_2}{\partial x^a} = -\frac{\partial f_1}{\partial y^a} \ . \tag{B.18}$$

We let z^a be local complex coordinates with complex-conjugates $\bar{z}^{\bar{a}}$. A complex manifold admits a tensor $J_a{}^b$ which has one covariant and one contravariant index (a respectively b). With our complex coordinates, the tensor components take the values

$$J_a{}^b = i\delta_a{}^b \ , \quad J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}} \ , \quad J_a{}^{\bar{b}} = J_{\bar{a}}{}^b = 0 \ . \tag{B.19}$$

These tensors are invariant under a holomorphic change of variables, so they describe a globally well-defined tensor J.

When given a real manifold with $D = 2n$ dimensions and wishing to determine whether the manifold is really complex, the first requirement is the existence of an *almost complex structure*, which is a tensor that satisfies

$$J_m{}^n J_n{}^p = -\delta_m{}^p \ , \tag{B.20}$$

which is preserved under a smooth change of coordinates. The second condition is of course that the almost complex structure is really a *complex structure.* This property is determined by the vanishing of the *Nijenhuis tensor*:

$$N^p{}_{mn} = J_m{}^q \partial_{[q} J_{n]}{}^p - J_n{}^q \partial_{[q} J_{m]}{}^p = 0 \ . \tag{B.21}$$

When eq. (B.21) is satisfied, it is possible to have complex coordinates on the manifold an so that J takes values as in eq. (B.19) with holomorphic mappings.

On a complex manifold a (p, q)-form can be defined with the complex coordinates, so that it has p holomorphic indices, and q antiholomorphic ones;

$$A_{p,q} = \frac{1}{p!q!} A_{a_1 \ldots a_p \bar{b}_1 \ldots \bar{b}_q} \, \mathrm{d}z^{a_1} \wedge \ldots \wedge \mathrm{d}z^{a_p} \wedge \mathrm{d}\bar{z}^{\bar{b}_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\bar{b}_q} \ . \tag{B.22}$$

In this formulation, the exterior derivative can be decomposed into an holomorphic and anti-holomorphic part, so that

$$\mathrm{d} = \partial + \bar{\partial} \equiv \mathrm{d}z^a \frac{\partial}{\partial z^a} + \mathrm{d}\bar{z}^{\bar{a}} \frac{\partial}{\partial \bar{z}^{\bar{a}}} \ . \tag{B.23}$$

The new holomorphic and antiholomorphic derivatives are called *Dolbeault operators*, and map (p,q)-forms to $(p+1,q)$-forms and $(p,q+1)$-forms respectively. They are nilpotent: $\partial^2 = \bar{\partial}^2 = 0$, and they anticommute: $\{\partial, \bar{\partial}\} = 0$.

Definition: Hermitian manifold. A Hermitian manifold is a special case of the complex Riemannian manifold. In terms of the complex coordinates introduced earlier on the complex manifold, the Riemannian metric can be written as

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}z^a\,\mathrm{d}z^b + g_{a\bar{b}}\,\mathrm{d}z^a\,\mathrm{d}z^{\bar{b}} + g_{\bar{a}b}\,\mathrm{d}z^{\bar{a}}\,\mathrm{d}z^b + g_{\bar{a}\bar{b}}\,\mathrm{d}z^{\bar{a}}\,\mathrm{d}z^{\bar{b}} \ . \tag{B.24}$$

Since $\mathrm{d}s^2$ is real, then $g_{\bar{a}\bar{b}}$ must be the complex conjugate of g_{ab}, and $g_{a\bar{b}}$ the complex conjugate of $g_{\bar{a}b}$. Now, a *Hermitian manifold* has metric condition

$$g_{ab} = g_{\bar{a}\bar{b}} = 0 \ . \tag{B.25}$$

These conditions are globally well-defined as they are invariant under holomorphic changes of variables.

The Dolbeault operators defined earlier form the Dolbeault cohomology group $H^{p,q}_{\bar{\partial}}(M)$ on a Hermitian manifold M, in the same way we have seen for the de Rahm cohomology group. It consists of equivalence classes of $\bar{\partial}$-closed (p,q)-forms, where two such forms are equivalent only if they differ with an exact $\bar{\partial}$-exact (p,q)-form. The dimension of $H^{(p,q)}_{\bar{\partial}}(M)$ is called the *Hodge number* $h^{(p,q)}$ (to be compared with the Betti number).

Definition: Kähler manifold. The Kähler manifold is defined as a Hermitian manifold on which the so-called *Kähler form* J;

$$J = ig_{a\bar{b}}\,\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}} \ , \tag{B.26}$$

is closed:

$$dJ = 0 \ . \tag{B.27}$$

The Kähler form actually comes from the complex structure we saw defined on the complex manifold, which on a Hermitian metric can be turned into a $(1,1)$-form, the Kähler form, defined as in eq. (B.26). The metric on Kähler manifolds satisfies $\partial_a g_{b\bar{c}} = \partial_b g_{a\bar{c}}$ and $\partial_a g_{\bar{b}c} = \partial_b g_{\bar{a}c}$, so locally it applies that

$$g_{a\bar{b}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^{\bar{b}}}\mathcal{K}(z,\bar{z}) \ , \tag{B.28}$$

where $\mathcal{K}$ known as the *Kähler potential* is a real-valued function. This allows the Kähler form of eq. (B.26) to be rewritten as

$$J = i\partial\bar{\partial}\mathcal{K} \ . \tag{B.29}$$

In the overlap of different coordinate charts, Kähler potentials may be related with additions of arbitrary holomorphic and antiholomorphic functions $f(z)$ and $\bar{f}(\bar{z})$;

$$\tilde{\mathcal{K}}(z,\bar{z}) = \mathcal{K}(z,\bar{z}) + f(z) + \bar{f}(\bar{z}) \ . \tag{B.30}$$

However they will still lead to the same metric. On Kähler manifolds the different cohomology groups, based on d, ∂ and $\bar{\partial}$, are identical:

$$H^{(p,q)}_{\bar{\partial}}(M) = H^{(p,q)}_{\partial}(M) = H^{(p,q)}(M) \ . \tag{B.31}$$

This can be seen as the various Laplacians become identical. As a consequence, the Hodge and Betti numbers are related via

$$b_k = \sum_{p=0}^{k} h^{(p,k-p)} \; . \tag{B.32}$$

If A is some (p,q)-form on a Kähler manifold of dimension n, then the complex conjugate form A^* is a (q,p)-form. Therefore we have that

$$h^{(p,q)} = h^{(q,p)} \; , \tag{B.33}$$

on a Kähler. In a similar sense, $\star A$ is a $(n-p, n-q)$-form, so we have that

$$h^{(n-p,n-q)} = h^{(p,q)} \; . \tag{B.34}$$

Definition: first Chern class. The Ricci tensor on a hermitan manifold has, in local complex coordinates, only mixed components that are non-vanishing. One can therefore define a $(1,1)$-form known as the *Ricci form*, as

$$\mathcal{R} = iR_{a\bar{b}}\,\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}} \; . \tag{B.35}$$

On a Hermitian manifold the exterior derivative of the Ricci form is proportional to the torsion. Since $\mathrm{d}J = 0$ on a Kähler manifold, the torsion vanishes, so the Ricci form must also be closed $\mathrm{d}\mathcal{R} = 0$. Because of this, the Ricci form is a representative belonging to the cohomology class $H^{1,1}(M)$. This class is known as the *first Chern class*

$$c_1 = \frac{1}{2\pi}[\mathcal{R}] \; . \tag{B.36}$$

Definition: symplectic manifold. A symplectic manifold (M,ω) is a manifold M equipped with a *non-degenerate* closed 2-form ω, known as a symplectic form. In local coordinates x^m on M we have that

$$\omega = \omega_{mn}(x)\,\mathrm{d}x^m \wedge \mathrm{d}x^n \; , \qquad \mathrm{d}\omega = 0 \; . \tag{B.37}$$

The condition of being non-degenerate means that the symplectic form is invertible, i.e. one can define an inverse ω^{mn} such that

$$\omega^{mn}\omega_{np} = \delta^m_p \; . \tag{B.38}$$

Since any invertible antisymmetric matrix has an even number of rows and columns, symplectic manifolds must therefore be of real and even dimension. This is equivalent to requiring that the n^{th} wedge product is nowhere vanishing;

$$\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega \neq 0 \; , \tag{B.39}$$

or that the determinant of the symplectic matrix is non-zero; $\det\omega_{mn} \neq 0$. An example of a symplectic manifold is $\mathbb{R}^{2n}$. Since we may write $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, with coordinates x^i and y_i on each product space respectively, the 2-form is given by $\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i$. This 2-form is clearly globally defined on $\mathbb{R}^{2n}$, closed, and non-degenerate. As a matrix it can be written

$$\omega \equiv \tfrac{1}{2}\omega_{mn}\,\mathrm{d}x^m \wedge \mathrm{d}x^n = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \; . \tag{B.40}$$

Definition: Calabi-Yau manifold. A Calabi-Yau n-fold is a compact Kähler manifold with vanishing first Chern class. A Calabi-Yau manifold admits a Kähler metric with $SU(n)$ holonomy. In turn, a manifold with $SU(n)$ holonomy admits a covariantly constant spinor field, which results in the manifold being Ricci flat. This is only valid for compact manifolds. For non-compact ones, additional boundary conditions at infinity need to be imposed. The converse is also true. A compact Ricci flat Kähler manifold of real dimension $2n$ has its holonomy group contained in $SU(n)$. This can be seen by considering the tangent vector $V = V^k \partial_k \in T_pM$, which we parallel transport along an infinitesimal parallelogram of area δa^{mn} with edges that are parallel to the vectors ∂_m and ∂_n. This transforms the tangent vector like

$$V^k \to V^k + \delta a^{mn} R_{mn}{}^k{}_l V^l \ . \tag{B.41}$$

The matrices $\delta^k_l + \delta a^{mn} R_{mn}{}^k{}_l$ are infinitesimally close to being identity and are elements of the holonomy group. For a Kähler metric the matrices $\delta a^{mn} R_{mn}{}^k{}_l$ are in the Lie algebra of $U(n)$. Close to the identity we have that $U(n) \simeq SU(n) \times U(1)$ with the $U(1)$ being generated by the trace

$$\delta a^{mn} R_{mn}{}^k{}_k = -4\delta^{\mu\bar{\nu}} R_{\mu\bar{\nu}} \ . \tag{B.42}$$

Hence we see that for a Ricci flat manifold the $U(1)$ part vanishes leaving us with $SU(n)$.

Hodge numbers of the Calabi-Yau n-fold. As we have seen earlier the Betti number b_p is the dimension of the p^{th} de Rahm cohomology $H^{(p)}(M)$ of the manifold M, and are topologically associated with the manifold. When the manifold has a metric the Betti numbers count the number of linearly independent harmonic p-forms on the manifold.

For Kähler manifolds the Betti numbers can be decomposed in terms of Hodge numbers as $b_k = \sum_{p=0}^{k} h^{(p,k-p)}$. The Hodge numbers in turn count the number of harmonic (p,q)-forms on the manifold. A Calabi-Yau is characterised by the values of its Hodge numbers. Note that they are not *defined* by it, as different (inequivalent) Calabi-Yaus can have the same Hodge numbers. The Hodge numbers of a Calabi-Yau n-fold satisfy

$$h^{(p,0)} = h^{(n-p,0)} \ , \tag{B.43}$$

which follows from the fact that the spaces $H^p(M)$ are isomorphic to $H^{n-p}(M)$. This can be proved by contracting a closed $(p,0)$-form with the complex conjugate of the holomorphic $(n,0)$-form and using the metric to make a closed $(0,n-p)$-form. As established earlier in eq. (B.33) and eq. (B.34), the Hodge numbers of a Kähler manifold also fulfil

$$h^{(p,q)} = h^{(q,p)} \ , \tag{B.44}$$

$$h^{(p,q)} = h^{(n-q,n-p)} \ . \tag{B.45}$$

A compact connected complex Kähler manifold has $h^{0,0} = 1$, which corresponds to the manifold having constant functions. A simply connected manifold is a connected manifold and every path between two points can be continuously transformed into any other such path while preserving the two endpoints in question. For example defining a loop on a space and then contracting into a point the point should still be on the space in question. A space is simply connected if it is connected and has vanishing *fundamental group*, i.e. the first *homotopy* group, which holds information about the basic shape, or holes, of a topological space. Because of this the manifold will also have vanishing first *homology* group, as a direct consequence of its definition, and thus resulting in

$$h^{(1,0)} = h^{(0,1)} = 0 \ . \tag{B.46}$$

Considering a Calabi-Yau 3-fold, which is an important case, the only Hodge numbers left to specify are $h^{(1,1)}$ and $h^{(2,1)}$. Because of the symmetry of eqs. (B.43)-(B.45), the Hodge numbers are often displayed in a *Hodge diamond*, which for the case of $n = 3$ is given by

$$\begin{array}{ccccccc}
 & & & h^{(3,3)} & & & \\
 & & h^{(3,2)} & & h^{(2,3)} & & \\
 & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\
h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\
 & h^{(2,0)} & & h^{(1,1)} & & h^{(1,3)} & \\
 & & h^{(1,0)} & & h^{(0,1)} & & \\
 & & & h^{(0,0)} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{(1,1)} & & 0 & \\
1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\
 & 0 & & h^{(1,1)} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

where $h^{(1,2)} = h^{(2,1)}$ according to eq. (B.44) and $h^{(2,2)} = h^{(1,1)}$ since $h^{(2,2)} = h^{(3-2,3-2)} = h^{(1,1)}$ according to eq. (B.45). The Euler characteristic for a Calabi-Yau 3-fold is thus given by

$$\chi = \sum_{p=0}^{6} (-1)^p b_p = 2(h^{(1,1)} - h^{(2,1)}) \ , \tag{B.47}$$

where the Betti numbers have been calculated using $b_k = \sum_{p=0}^{k} h^{(p,k-p)}$.

B.5 Fibre bundles

By introducing a chart on a manifold, a local Euclidean structure is created, enabling us to use conventional calculus of multiple variables. A fibre bundle looks a lot like a topological space consisting locally of a direct product of two topological spaces. For clarity we start off by having a look at the tangent bundle, which is a specific type of fibre bundle.

Definition: tangent bundle. A tangent bundle TM over an m-dimensional manifold M is a collection of all the tangent spaces of M;

$$TM \equiv \bigcup_{p \in M} T_p M \ . \tag{B.48}$$

The manifold M over which TM is defined is referred to as the *base space*. T_pM is the tangent space at a point p. If $\{U_i\}$ is an open covering of M, and $x^\mu = \varphi_i(p)$ is the coordinate on U_i, then an element of

$$TU_i \equiv \bigcup_{p \in U_i} T_p M$$

is specified by a point $p \in M$ and a vector $V = V^\mu(p) \frac{\partial}{\partial x^\mu}\big|_p \in T_pM$. U_i is homeomorphic to an open subset $\varphi_i(U_i)$ of $\mathbb{R}^m$ and each T_pM is homeomorphic to $\mathbb{R}^m$, so TU_i is identified with $\mathbb{R}^m \times \mathbb{R}^m$. By homeomorphic we mean possessing intrinsic topological equivalence. Two objects are homeomorphic if they can be deformed into each other by a continuous, invertible mapping. The TU_i itself is a smooth manifold whose dimension is $2m$. There is a projection $\pi : TU_i \to U_i$, so that for any point $u \in TU_i$, $\pi(u)$ is a a point $p \in U_i$ at which the vector V is defined. The projection π can also be defined globally since $\pi(u) = p$ does not depend on any special coordinate. Thus $\pi : TM \to M$ can be defined globally without any reference to local charts.

The *section* is an inverse map to the projection. The section s of TM is a smooth map that maps $s : M \to TM$ and fulfil $\pi s = \mathbb{1}_m$. It may also be defined locally on a chart U_i such that $s_i : U_i \to TU_i$. Given two charts U_i and U_j such that $U_i \cap U_j \neq \varnothing$, let $y^\mu = \psi(p)$ be the coordinate on U_j. Now a vector $V \in T_pM$, $p \in U_i \cap U_j$ has two coordinate presentations;

$$V = V^\mu \frac{\partial}{\partial x^\mu}\bigg|_p = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}\bigg|_p ,$$

so that $\tilde{V}^\nu = (\partial y^\nu/\partial x^\mu)_p V^\mu$. In order to have a functioning coordinate system with $\{x^\mu\}$ and $\{y^\nu\}$ the matrix $(G^\nu{}_\mu) \equiv (\partial y^\nu/\partial x^\mu)_p$ should be non-singular; $(G^\nu{}_\mu) \in GL(m, \mathbb{R})$. So that whenever we change fibre coordinates they are being rotated by an element of the group $GL(m, \mathbb{R})$, which is known as the *structure group* of TM.

Definition: cotangent bundle. The cotangent bundle

$$T^*M \equiv \bigcup_{p\in M} T_p^*M , \tag{B.49}$$

is defined very similarly to the tangent bundle. In this case the coordinates on a chart, or patch, U_i are given by x^μ. The basis of T_p^*M is then taken as $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^m\}$, which is dual $\{\partial/\partial x^\mu\}$. If y^μ are coordinates on U_j with $U_i \cap U_j \neq \varnothing$, then the coordinates are related via

$$\mathrm{d}y^\mu = \mathrm{d}x^\nu \left(\frac{\partial y^\mu}{\partial x^\nu}\right)_p . \tag{B.50}$$

A 1-form ω can then be written as

$$\omega = \omega_\mu = \tilde{\omega}_\mu \, \mathrm{d}y^\mu , \tag{B.51}$$

so that

$$\tilde{\omega}_\mu = G_\mu{}^\nu(p)\omega_\nu \equiv \left(\frac{\partial x^\nu}{\partial x^\mu}\right)_p \omega_\nu . \tag{B.52}$$

$G_\mu{}^\nu(p)$ then corresponds to the transition functions $t_{ij}(p)$. Just like the sections of tangent bundles are vector fields, the sections of the cotangent bundle are co-vector fields, i.e. 1-forms on M. The set of sections on T^*M therefore corresponds to the set of all 1-form fields on M, i.e. $\Gamma(T^*M) = X^{(1)}(X)$.

Definition: fibre bundle. A (differential) fibre bundle is defined by the elements (E, π, M, F, G), where

- E is a differentiable manifold known as the *total space*.
- M is a differentiable manifold known as the *base space*.
- F is a differentiable manifold known as the *fibre*.
- π is a surjection known as the *projection*, whose inverse image $\pi^{-1}(p) \equiv F_p$ is called the fibre at p.
- G is a Lie group known as the *structure group*, which acts on F from the left.

There is a set of open coverings $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\phi_i(p, f) = p$. Here the map ϕ_i is called the *local trivialism*. A diffeomorphism is a differentiable map between manifolds whose inverse is also differentiable. With $\phi_i(p, f) \equiv \phi_{i,p}(f)$ then the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. In $U_i \cap U_j \neq \varnothing$ one requires that $t_{ij}(p) \equiv \phi_{i,p}^{-1}\phi_{j,p} : F \to F$ is an element of G. Then the smooth map $t_{ij} : U_i \cap U_j \to G$ relates ϕ_i and ϕ_j by

$$\phi_i(p, t_{ij}(p)f) = \phi_j(p, f) \ .$$

The maps $\{t_{ij}\}$ are known as *transition functions.* It applies that $t_{ii}(p)$, $p \in U_i$ is the identity map, $t_{ij}(p) = t_{ji}(p)^{-1}$, $p \in U_i \cap U_j$, and $t_{ij}(p)t_{jk}(p) = t_{ik}(p)$, $p \in U_i \cap U_j \cap U_k$. A fibre bundle whose transition functions all can be taken to be identity maps is called a *trivial bundle.* A trivial bundle is just a direct product $M \times F$.

The section s is defined as a smooth map from the base space to the manifold $s : E \to M$ and satisfies $\pi s = \mathbb{1}_m$. It is clear that $s(p) = s|_p$ is an element of the inverse image $F_p = \pi^{-1}(p)$.

Definition: vector bundle. A vector bundle $E \xrightarrow{\pi} M$ is a fibre bundle whose fiber is a vector space. Given a m-dimensional manifold M and fibre $F = \mathbb{R}^k$ it is customary to call the fibre dimension dim$E = k$ even though the total space is $m + k$-dimensional. In this case the structure transition functions (similar to coordinate transformations between patches U_i) are elements of the structure group $G = GL(k, \mathbb{R})$ as it preserves the vector operations of addition and multiplication. Hence it maps a vector space isomorphically onto another vector space of the same dimension. With $F = \mathbb{R}^k$ being a real vector space the vector bundle is real, and if rather $F = \mathbb{C}^k$, the structure group is $G = GL(k, \mathbb{C})$ and one obtains complex vector bundles.

Definition: principal bundle. A principal bundle $P \xrightarrow{\pi} M$ is a fibre bundle whose fibre F is identical to its structure group G. It is therefore commonly called a G-bundle over M and denoted $P(M, G)$.

As usual the transition functions belongs to the structure group G, which then acts on the fibre $F = G$, i.e. on G itself, from the right. On a principal bundle there is also a left action of G on P, which is a map $P \times G \to P$ according to $(u, g) \to ug$ for some $u \in P$ and $g \in G$. It commutes with the projection according to $\pi(ug) = \pi(u)$. Given an open covering U_i of M, a point $p \in U_i$, and local trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ such that $\phi^{-1}(u) = (p, g_i)$ with $\pi(u) = p$ and $g_i \in G$. The right action of G on $\pi^{-1}(U_i)$ is then defined as

$$\phi^{-1}(ug) = (p, g_i g) \ , \qquad ug = \phi_i(p, g_i g) \ . \tag{B.53}$$

Hence, the right action of g translates a point on the fibre to a new point on the same fiber. In a patch overlap we have that

$$ug = \phi_j(p, g_j g) = \phi_j(p, t_{ji}(p)g_i g) = \phi_i(p, g_i g) \ , \tag{B.54}$$

and so the left action of G commutes with its right action. For two different elements $u_1, u_2 \in \pi^{-1}(p)$ there is a unique element $g \in G$ such that $u_2 = u_1 g$, which makes the right action *transitive*, and allows for a whole fibre to be constructed from an element of $\pi^{-1}(p)$ and the right action of G via $\pi^{-1}(p) = \{ug | g \in G\}$. Further, the right action on G is said to be free if it fulfils $ug = g$ for any $u = \phi_i(p, g_i)$ since g then has to be the identity element e of G.

Given a section $s_1(p)$ over U_i there is a preferred local trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ defined as follows. With $u \in \pi^{-1}(p)$ and $p \in U_i$ there exists a unique element $g_u \in G$ such that $u = s_i(p)g_u$. If one defines the local trivialisation ϕ_i as $\phi_i^{-1}(u) = (p, g_u)$, the section $s_i(p)$ is given by

$$s_i(p) = \phi_i(p, e) \ , \tag{B.55}$$

for which ϕ_i is called the canonical local trivialisation. If $p \in U_i \cap U_j$, the two corresponding sections $s_i(p)$ and $s_j(p)$ are related by the transition function $t_{ij}(p)$ according to

$$s_i(p) = \phi_i(p, e) = \phi_j(p, t_{ji}(p)e) = \phi_j(p, t_{ji}(p)) = \phi_j(p, e)t_{ji}(p) = s_j(p)t_{ji}(p) \ . \tag{B.56}$$

B.6 Holonomy groups

The holonomy group of a manifold of dimension n describes the way various objects, such as tensors or spinors, transform under parallel transport around a closed curve. The most general transformation of a vector for example, is rotation, which is an element of the group $SO(n)$. For spinors on the other hand, their corresponding transformation is an element of the group $Spin(n)$. A spinor ϵ being being parallel transported in a loop undergoes some sort of transformation

$$\epsilon \to U\epsilon \ ,$$

where U is an element of $Spin(n)$ in the spinor representation appropriate to ϵ. Now say that the spinor takes two consecutive closed but different paths and returns to the same point, then it will have transformed as

$$\epsilon \to U_1 U_2 \epsilon \ .$$

Thus the U matrices build a *holonomy group*, usually denoted $\mathcal{H}(M)$. As stated earlier, the generic holonomy group of a manifold that admits a spinor is $Spin(n)$. However, depending on the type of manifold, the holonomy group can be reduced to be only a subgroup of $Spin(n)$. Such manifolds is said to be of *special holonomy*. Some cases of interest in this thesis are

$$\begin{aligned}
\mathcal{H}(M) \subseteq U(n/2) &\Leftrightarrow M \text{ is Kähler} \ , \\
\mathcal{H}(M) \subseteq SU(n/2) &\Leftrightarrow M \text{ is Calabi-Yau} \ , \\
\mathcal{H}(M) \subseteq Sp(n/4) &\Leftrightarrow M \text{ is hyper-Kähler} \ , \\
\mathcal{H}(M) \subseteq Sp(n/4) \cdot Sp(1) &\Leftrightarrow M \text{ is quaternionic Kähler} \ .
\end{aligned}$$

where the dimension n has to be even.

B.7 G-structures on manifolds

In this section we will reuse much of the definitions of section B.5 but in a slightly different way in order to describe G-structures and provide an alternative view of previous definitions. Let M be a manifold of real dimension d and TM its tangent bundle. An arbitrary vector v at some point $p \in M$ can be written in a local basis as $v = v^a_{(\alpha)} e^{(\alpha)}_a$. This vector is defined on a chart U_α over M, and defining an other vector from the chart U_β, the coordinates of the two vectors are related by the local coordinate change

$$v^a_{(\alpha)} = M_{\alpha\beta}{}^a{}_b v^b_{(\beta)} \ , \tag{B.57}$$

where the transformation $M_{\alpha\beta} \in GL(d, \mathbb{R})$. Since this is the case for an arbitrary point $p \in M$, the transformation matrices $M_{\alpha\beta}$ can be seen as maps from the manifold to $GL(d, \mathbb{R})$;

$$p \to M_{\alpha\beta}(p) \ , \qquad M_{\alpha\beta} : M \to GL(d, \mathbb{R}) \ . \tag{B.58}$$

The transformation matrices $M_{\alpha\beta}$ are transition functions and contain information about non-trivial topologies in the tangent bundle. They are required to fulfil $M_{\alpha\beta}M_{\beta\gamma} = M_{\alpha\gamma}$ and $M_{\alpha\beta}M_{\beta\alpha} = 1$, which corresponds to the *closure* and *existence* of identity axioms of a group. As familiar the group of transition functions $GL(d, \mathbb{R})$ is the structure group of the tangent bundle.

A frame bundle on M is a principal bundle whose fibers at some point $p \in M$ are an ordered basis (frame) of the tangent space T_pM. That is $F = \cup_{p \in M} F_p$ with $F_p \equiv \{(p, \{e_a\}) | p \in M\}$. Identifying the fibre with the group $GL(d, \mathbb{R})$, it acts freely and transitively on each fibre on the right to result in another frame on the fibre. Hence, the right action of $GL(d, \mathbb{R})$ can be seen as a way of changing frames while keeping the point $p \in M$ fixed, limiting the transformation to the fibre only. If the frame $\{e_a^{(\alpha)}\}$ is globally defined over the entire manifold M, it is said to be parallelisable, which is a property of all Lie groups.

A manifold admits a G-structure if it is possible to reduce the structure group $GL(d, \mathbb{R})$ of TM to a subgroup $G \subset GL(d, \mathbb{R})$. In this case the transition functions takes values in the subgroup G and so the G-structure is a principal sub-bundle of the frame bundle $P \subset F$. Equally, a manifold M has a G-structure if and only if there exists globally defined tensors or spinors that are invariant under the group G. The equivalence between globally defined invariant tensors and G-structures also extends to other types of vector bundles. For instance, spin bundles and spin structure groups admits globally defined invariant spinors.

C

Supersymmetry Equations in Terms of $SU(3)$ Representations

In the calculations of the section 4.4.1 and in section 4.4.2 it can be convenient to use $SU(3)$ representations and holomorphic indices in the supersymmetry conditions of eq. (4.97)–(4.99). To start off, it is convenient to write out explicitly the expression for the Clifford products encountered so far;

$$
\begin{aligned}
(\not{F}_{\mathrm{A}}\not{e}^{iJ})_0 &= F_0 - \tfrac{i}{2}F^{ab}J_{ab} - \tfrac{1}{8}F^{abcd}J_{ab}J_{cd} + \tfrac{i}{48}F^{abcdef}J_{ab}J_{cd}J_{ef}\ ,\\
(\not{F}_{\mathrm{A}}\bar{\not{\Omega}})_m &= -\tfrac{1}{2}F^{ab}\bar{\Omega}_{abm} + \tfrac{1}{6}F^{abc}{}_m\bar{\Omega}_{abc}\ ,\\
(\bar{\not{\Omega}}\not{F}_{\mathrm{A}})_m &= -\tfrac{1}{2}F^{ab}\bar{\Omega}_{abm} - \tfrac{1}{6}F^{abc}{}_m\bar{\Omega}_{abc}\ ,\\
(\not{F}_{\mathrm{A}m}\not{e}^{iJ})_n &= 2F_{mp}P_n{}^p + iJ_{ab}F^{ab}{}_{mp}\bar{P}_n{}^p - \tfrac{1}{4}J_{ab}J_{cd}F^{abcd}{}_{mp}P_n{}^p\ ,\\
(\not{F}_{\mathrm{A}}e)_{mn} &= \tfrac{i}{2}F_0J_{mn} + \tfrac{1}{2}F_{mn} + iF^a{}_{[m}J_{n]a} + \tfrac{1}{4}F^{ab}J_{ab}J_{mn} - \tfrac{1}{2}F^{ab}J_{am}J_{bn}\\
&\quad - \tfrac{i}{4}F_{mn}{}^{ab}J_{ab} - \tfrac{1}{2}F^{abc}{}_{[m}J_{n]}J_{bc} - \tfrac{i}{16}F^{abcd}J_{ab}J_{cd}J_{mn} + \tfrac{i}{4}F^{abcd}J_{ab}J_{cm}J_{dn}\\
&\quad + \tfrac{1}{16}F^{abcd}{}_{mn}J_{ab}J_{cd}\ ,
\end{aligned}
\tag{C.1}
$$

where $F_{\mathrm{A}} = F_0 + F_2 + F_4 + F_6$. The Clifford products for the NSNS 3-form flux H_3 are given by

$$
\begin{aligned}
(\not{H}\not{\Omega})_0 &= -\tfrac{1}{6}H^{abc}\Omega_{abc}\ ,\\
(\not{H}\not{e}^{-iJ})_m &= iJ_{ab}H_n{}^{ab}\bar{P}_m{}^n\ ,
\end{aligned}
\tag{C.2}
$$

and is common for both type IIA and IIB. In terms of $SU(3)$ representations the NSNS flux is defined in the same way as the type IIB 3-form RR flux F_3, namely

$$
H_3 = -\tfrac{3}{2}\mathrm{Im}(H_3^{(1)}\bar{\Omega}) + H_3^{(3)}\wedge J + H_3^{(6)}\ . \tag{C.3}
$$

Here the components are explicitly given by

$$
\begin{aligned}
H_3^{(1)} &= -\tfrac{i}{36}H^{ijk}\Omega_{ijk}\ ,\\
H_{3i}^{(3)} &= \tfrac{1}{4}H_{imn}J^{mn}\ ,\\
H_{3ij}^{(6)} &= H^{kl}(i\Omega_{j)kl}\ .
\end{aligned}
\tag{C.4}
$$

The rest of the RR fluxes decomposes in $SU(3)$ representations according to

$$
\begin{aligned}
F_2 &= \tfrac{1}{3}F_2^{(1)} + \mathrm{Re}(F_2^{(3)}\llcorner\bar{\Omega}) + F_2^{(8)}\ ,\\
F_4 &= \tfrac{1}{6}F_4^{(1)}J\wedge J + \mathrm{Re}(F_4^{(3)}\wedge\bar{\Omega}) + F_4^{(8)}\ ,\\
F_6 &= \tfrac{1}{6}F_6^{(1)}J\wedge J\wedge J\ ,
\end{aligned}
\tag{C.5}
$$

where each component is given by

$$\begin{aligned} F_2^{(1)} &= \tfrac{1}{2}F_{mn}J^{mn} = F_{i\bar{j}}J^{i\bar{j}} \ , & F_4^{(1)} &= \tfrac{1}{8}F^{ij}\Omega_{ijk} \ , & & \\ F_{2k}^{(3)} &= \tfrac{1}{8}F^{ijk}\Omega_{ijk} \ , & F_{4k}^{(3)} &= \tfrac{1}{24}F_k{}^{ijl}\Omega_{ijl} \ , & F_6^{(1)} &= \tfrac{1}{48}F^{mnpqrs}J_{mn}J_{pq}J_{rs} \ . \end{aligned} \tag{C.6}$$

The occurring Clifford products for IIB are

$$\begin{aligned} (\not{F}_{\mathrm{B}}\not{\Omega})_0 &= -\tfrac{1}{6}F^{abc}\Omega_{abc} \ , \\ (\not{F}_{\mathrm{B}}\cancel{e^{-iJ}})_m &= 2F_n\bar{P}_m{}^n + iJ_{ab}F^{ab}{}_n\bar{P}_m{}^n - \tfrac{1}{4}J_{ab}J_{cd}F^{abcd}{}_n\bar{P}_m{}^n \ , \\ (\cancel{e^{-iJ}\not{F}_{\mathrm{B}}})_m &= 2F_nP_m{}^n + iJ_{ab}F^{ab}{}_nP_m{}^n - \tfrac{1}{4}J_{ab}J_{cd}F^{abcd}{}_nP_m{}^n \ , \\ (\not{F}_{\mathrm{B}m}\Omega)_n &= -\tfrac{1}{2}F^{ab}{}_m\Omega_{nab} - \tfrac{1}{6}F^{abc}{}_{mn}\Omega_{abc} \ , \\ (\not{F}_{\mathrm{B}}\Omega)_{mn} &= \tfrac{1}{2}F^a\Omega_{amn} - \tfrac{1}{2}F^{ab}{}_{[m}\Omega_{n]ab} - \tfrac{1}{12}F^{abc}{}_{mn}\Omega_{abc} \ , \end{aligned} \tag{C.7}$$

where the F_3 flux decomposes analogous to H_3 in eqs. (C.3), (C.4) and the 5-form flux is decomposed as

$$F_5 = F_5^{(3)} \wedge J \wedge J \ , \qquad F_{(5)i}^{(3)} = \tfrac{1}{16}F_i^{mnpq}J_{mn}J_{pq} \ . \tag{C.8}$$

Using the above identities, the matrices S, Q, R can be written in terms of $SU(3)$ representations. In the IIA case the coefficients of the supersymmetry conditions in eq. (4.97)–(4.99) obtained in [19] become

$$\begin{aligned} A_{\bar{\imath}} &= \alpha\partial_{\bar{\imath}}A \ , \\ S &= -\tfrac{i}{4}e^{\phi}\left(\beta^*F_0 - i\alpha^*F_2^{(1)} - \beta^*F_4^{(1)} + i\alpha^*F_6^{(1)}\right) \ , \\ S_{\bar{\imath}} &= \tfrac{1}{2}e^{\phi}\left(\alpha^*F_2^{(\bar{3})} + \beta^*F_4^{(\bar{3})}\right)_{\bar{\imath}} \ , \\ Q_{\bar{\imath}} &= -i\partial_{\bar{\imath}}\alpha - \tfrac{i}{2}\left(\alpha(W_5 - W_4) + i\beta H_3^{(3)}\right) \ , \\ Q_{ij} &= -\tfrac{1}{8}\left(\Omega_{ijk}(\alpha W_4 + i\beta H_3^{(3)})^k + \tfrac{i}{2}(\alpha W_3 + i\beta H_3^{(6)})_i{}^{kl}\Omega_{jkl}\right) \ , \\ Q_{\bar{\imath}j} &= \tfrac{1}{4}\left((\alpha W_1 - 3i\beta H^{(1)})g_{\bar{\imath}j} + i\alpha W_{2\bar{\imath}j}\right) \ , \\ R_i &= R_{ij} = 0 \ , \\ R_{\bar{\imath}} &= \tfrac{i}{2}e^{\phi}\left(\alpha^*F_2^{(\bar{3})} - \beta^*F_4^{(\bar{3})}\right)_{\bar{\imath}} \ , \\ R_{\bar{\imath}j} &= -\tfrac{1}{8}e^{\phi}\left(g_{\bar{\imath}j}\bar{S} - \tfrac{8}{3}g_{\bar{\imath}j}(\beta F_4^{(1)} - \alpha^*F_6^{(1)}) - 2\alpha^*F_{\bar{\imath}j}^{(8)} - 2i\beta F_{\bar{\imath}j\bar{k}l}^{(8)}J_{\bar{k}l}\right) \ , \\ T &= \tfrac{3}{4}\left(i\alpha W_1 - \beta H_{(3)}^{(1)}\right) \ , \\ T_{\bar{\imath}} &= \alpha\partial_{\bar{\imath}}(2A - \phi - \ln\alpha) + \tfrac{1}{2}\left(\alpha(W_4 + W_5) - i\beta H_3^{(3)}\right)_{\bar{\imath}} \ , \end{aligned} \tag{C.9}$$

where g is the 6D internal $SU(3)$-structure metric, and all fluxes come from the F_{A1} variant.

For IIB we get

$$
\begin{aligned}
A_{\bar{\imath}} &= \alpha \partial_{\bar{\imath}} A \ , \\
S &= \tfrac{3}{2} i \beta e^{\phi} F_3^{(1)} \ , \\
S_i &= \tfrac{1}{4} e^{\phi} \left(\alpha^* F_1^{(3)} + 2 i \beta^* F_3^{(3)} - 2 \alpha^* F_5^{(3)} \right) \ , \\
S_{\bar{\imath}} &= \tfrac{1}{4} e^{\phi} \left(\alpha F_1^{(\bar{3})} - 2 i \beta F_3^{(\bar{3})} - 2 \alpha F_5^{(\bar{3})} \right) \ , \\
Q_{\bar{\imath}} &= - i \partial_{\bar{\imath}} \alpha - \tfrac{i}{2} \left(\alpha (W_5 - W_4) - i \beta H_3^{(3)} \right) \ , \\
Q_{ij} &= - \tfrac{1}{8} \left(\Omega_{ijk} (\alpha W_4 - i \beta H_3^{(\bar{3})})^k + \tfrac{i}{2} (\alpha W_3 - i \beta H_3^{(6)})_i{}^{kl} \Omega_{jkl} \right) \ , \\
Q_{\bar{\imath} j} &= \tfrac{1}{4} \left((\alpha W_1 + 3 i \beta H^{(1)}) g_{\bar{\imath} j} + i \alpha W_{2 \bar{\imath} j} \right) \ , \\
R_i &= - \tfrac{i}{4} e^{\phi} \left(\alpha F_1^{(3)} - 2 i \beta F_3^{(3)} - 2 \alpha F_5^{(3)} \right)_i \ , \\
R_{\bar{\imath}} &= R_{\bar{\imath} j} = 0 \ , \\
R_{ij} &= - \tfrac{i}{16} e^{\phi} \left(\alpha F^{(\bar{3})k} \Omega_{ijk} - \beta F_{3ij}^{(6)} + 2 \alpha F_5^{(\bar{3})k} \Omega_{ijk} \right) \ , \\
T &= \tfrac{3}{4} \left(i \alpha W_1 + \beta H_3^{(1)} \right) \ , \\
T_{\bar{\imath}} &= \alpha \partial_{\bar{\imath}} (2A - \phi - \ln \alpha) + \tfrac{1}{2} \left(\alpha (W_4 + W_5) + i \beta H_3^{(3)} \right)_{\bar{\imath}} \ ,
\end{aligned}
\tag{C.10}
$$

where the fluxes are of the F_{B1} version.

D

Compactification of Type IIB Theory on a Calabi-Yau

In this chapter we make explicit the compactification of type IIB supegravity on a Calabi-Yau 3-fold. Section D.1 illustrates the classical Kaluza-Klein compactification as described in section 5.1.2. In section D.2 we allow non-trivial integer fluxes trough cycles on the Calabi-Yau, whose effects are discussed in section 5.3.

D.1 Compactification on the Calabi-Yau 3-fold

We start by considering the Einstein-Hilbert term, whose metric which upon imposing the Kaluza-Klein ansatz can be written

$$S_{\mathrm{EH}}^{(10)} = \int \mathrm{d}^{10}x\sqrt{-g^{(10)}}R^{(10)}\ , \qquad \int \mathrm{d}^{10}x\sqrt{-g^{(10)}} = \int \mathrm{d}^4x\sqrt{-g_4}\sqrt{g_6}\ . \tag{D.1}$$

The 10D Ricci scalar can be expanded according to

$$\begin{aligned} R^{(10)} = g^{MN}R_{MN} = g^{MN}{R_{MPN}}^P& \\ &= R + g^{\mu\nu}{R_{\mu i\nu}}^i + g^{i\bar{\jmath}}({R_{i\mu\bar{\jmath}}}^\mu + {R_{ik\bar{\jmath}}}^k + {R_{i\bar{k}\bar{\jmath}}}^{\bar{k}}) + g^{ij}{R_{i\mu j}}^\mu \\ &\quad + g^{ij}({R_{ikj}}^k + {R_{i\bar{k}j}}^{\bar{k}}) + \text{c.c.}\ , \end{aligned} \tag{D.2}$$

where here R can be shown to only be of 4D spacetime components, hence it is the true 4D Ricci scalar. Expanding the metric in terms of harmonic forms as given in eq. (5.50), i.e.

$$\begin{aligned} g_{ij} &= \bar{z}^c(\bar{b}_c)_{ij}\ , \\ g^{ij} &= -z^c(b_c)_{\bar{\imath}\bar{\jmath}}g^{i\bar{\imath}}g^{j\bar{\jmath}}\ , \end{aligned} \tag{D.3}$$

where $(\bar{b}_c)_{ij} = \frac{i}{||\Omega||^2}(\chi_c)_{i\bar{\jmath}\bar{k}}{\Omega^{\bar{\jmath}\bar{k}}}_j$ and the metric has been expanded as $g_{i\bar{\imath}} \to g_{i\bar{\imath}} - iv^a(\omega_a)_{\bar{\imath}\bar{\jmath}}$ and $g^{i\bar{\imath}} \to g^{i\bar{\imath}} - iv^a(\omega_a)_{j\bar{\jmath}}g^{i\bar{\jmath}}g^{j\bar{\imath}}$. Using these new metric deformations we are to evaluate the expression of the Ricci scalar in eq. (D.2). To do so we calculate the non-vanishing Christoffel symbols to second order in the moduli fields, with the results

$$\begin{aligned} \Gamma^j_{\mu i} &= -\tfrac{i}{2}(\omega_a)_i^j\partial_\mu v^a + \tfrac{1}{2}(\omega_a)^{\bar{\jmath}j}(\omega_a)_{i\bar{\jmath}}v^a\partial_\mu v^b - \tfrac{1}{2}(b_c)^{kj}(\bar{b}_c)_{ik}z^c\partial_\mu\bar{z}^d + \mathcal{O}(\text{moduli}^3)\ , \\ \Gamma^{\bar{\jmath}}_{\mu i} &= \tfrac{1}{2}(\bar{b}_c)_i^{\bar{\jmath}}\partial_\mu\bar{z}^c + \tfrac{i}{2}(\omega_a)^{j\bar{\jmath}}(\bar{b}_c)_{ij}v^a\partial_\mu\bar{z}^c + \tfrac{i}{2}(\omega_a)_{i\bar{k}}(\bar{b}_c)^{\bar{\jmath}\bar{k}}\bar{z}^c\partial_\mu v^a + \mathcal{O}(\text{moduli}^3)\ , \\ \Gamma^\mu_{ij} &= -\tfrac{1}{2}(\bar{b}_c)_{ij}\partial^\mu\bar{z}^c + \mathcal{O}(\text{moduli}^3)\ , \\ \Gamma^\mu_{i\bar{\jmath}} &= \tfrac{i}{2}(\omega_a)_{ij}\partial^\mu v^a + \mathcal{O}(\text{moduli}^3)\ , \end{aligned} \tag{D.4}$$

and the indices are raised and lowered with $g^{i\bar{\jmath}}$. Using these, the second term in eq. (D.2) is determined to

$$\begin{aligned} g^{\mu\nu}{R_{\mu i\nu}}^i = \tfrac{1}{4}(\omega_a)_i^j(\omega_b)_j^i\partial_\mu v^a\partial^\mu v^b - \tfrac{1}{4}(b_c)_i^j(\bar{b}_d)_j^i\partial_\mu z^c\partial^\mu\bar{z}^d + \tfrac{1}{2}(\omega_a)_i^j(\omega_b)_j^i v^a\nabla_\mu\partial^\mu v^b \\ - \tfrac{i}{2}(\omega_a)_{i\bar{\jmath}}g^{i\bar{\jmath}} - \tfrac{1}{2}(b_c)_i^j(\bar{b}_d)_j^i z^c\nabla_\mu\partial^\mu\bar{z}^b + \mathcal{O}(\text{moduli}^3)\ . \end{aligned} \tag{D.5}$$

This expression can be simplified by partial integrating the terms which includes the covariant derivative ∇. Introducing the short notation $(\omega_a\omega_b) \equiv (\omega_a)_i^{\ j}(\omega_b)_j^{\ i}$, $(b_c\bar{b}_d) \equiv (b_c)_i^{\ j}(\bar{b}_d)_j^{\ i}$ and $(\omega_a g) \equiv (\omega_a)_{i\bar{j}}g^{i\bar{j}}$, the third term in eq. (D.5) in the integral of eq. (D.1) becomes

$$\begin{aligned}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left(\tfrac{1}{2}(\omega_a\omega_b)v^a\nabla_\mu\partial^\mu v^b\right) &= -\tfrac{1}{2}\int \mathrm{d}^{10}x(\omega_a\omega_b)\nabla_\mu\left(\sqrt{-g^{(10)}}v^a\right)\partial^\mu v^b \\ &= -\tfrac{1}{2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}(\omega_a\omega_b)\partial_\mu v^a\partial^\mu v^b + \mathcal{O}(\text{moduli}^3)\ ,\end{aligned} \tag{D.6}$$

where in the last step we used that

$$\begin{aligned}\nabla_\mu\sqrt{-g^{(10)}} = \nabla_\mu(\sqrt{-g_4}\sqrt{g_6}) = \sqrt{-g_4}\partial_\mu\sqrt{g_6} &= \sqrt{-g_4}\left(\tfrac{1}{2}\sqrt{g_6}g^{mn}\partial_\mu g_{mn}\right) \\ &= \sqrt{-g^{(10)}}\left(g^{ij}\partial_\mu g_{ij} + g^{\bar{i}\bar{j}}\partial_\mu g_{\bar{i}\bar{j}} + 2g^{i\bar{j}}\partial_\mu g_{i\bar{j}}\right)\ .\end{aligned} \tag{D.7}$$

Proceeding to the fourth term in eq. (D.5), a partial integration gives us

$$\begin{aligned}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left(-\tfrac{i}{2}(\omega_a g)\nabla_\mu\partial^\mu v^a\right) &= \tfrac{i}{2}\int \mathrm{d}^{10}x(\omega_a g)\tfrac{1}{2}\sqrt{-g^{(10)}}\left(2g^{i\bar{\imath}}\partial_\mu g_{i\bar{j}}\right) + \mathcal{O}(\text{moduli}^3) \\ &= \tfrac{i}{2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}(\omega_a g)\left(g^{i\bar{\imath}} + iv^a(\omega_a)_{j\bar{j}}g^{i\bar{j}}g^{j\bar{\imath}}\right)\partial_\mu\left(g_{i\bar{\imath}} - iv^a(\omega_a)_{i\bar{\imath}}\right) \\ &= \tfrac{1}{2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}(\omega_a g)(\omega_b g)\partial_\mu v^a\partial^\nu v^b + \mathcal{O}(\text{moduli}^3)\ .\end{aligned} \tag{D.8}$$

The fifth term of eq. (D.5) is very similar to the third term, and so using the results in eq. (D.6) we have that

$$\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left(-\tfrac{1}{2}(b_c\bar{b}_d)z^c\nabla_\mu\partial^\mu\bar{z}^d\right) = \tfrac{1}{2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}(b_c\bar{b}_d)\partial_\mu z^c\partial^\mu\bar{z}^d + \mathcal{O}(\text{moduli}^3)\ . \tag{D.9}$$

Gathering these results, the second term in the Ricci scalar expression of eq. (D.5) takes the form

$$g^{\mu\nu}R_{\mu i\nu}{}^i = \tfrac{1}{2}\Big((\omega_a g)(\omega_b g) - \tfrac{1}{2}(\omega_a\omega_b)\Big)\partial_\mu v^a\partial^\mu v^b + \tfrac{1}{4}(b_c\bar{b}_d)\partial_\mu z^c\partial^\mu\bar{z}^d + \mathcal{O}(\text{moduli}^3)\ . \tag{D.10}$$

The remaining terms of eq. (D.2) are evaluated in the same manner, with the results being

$$\begin{aligned} g^{i\bar{j}}R_{i\mu\bar{j}}{}^\mu &= \tfrac{1}{2}\Big((\omega_a g)(\omega_b g) - \tfrac{1}{2}(\omega_a\omega_b)\Big)\partial_\mu v^a\partial^\mu v^b - \tfrac{1}{4}(b_c\bar{b}_d)\partial_\mu z^c\partial^\mu\bar{z}^d + \mathcal{O}(\text{moduli}^3)\ , \\ g^{i\bar{j}}R_{ik\bar{j}}{}^k &= -\tfrac{1}{4}\left((\omega_b g) - (\omega_a\omega_b)\right)\partial_\mu v^a\partial^\mu v^b + \mathcal{O}(\text{moduli}^3)\ , \\ g^{i\bar{j}}R_{i\bar{k}\bar{j}}{}^{\bar{k}} &= -\tfrac{1}{4}(\omega_a g\omega_b g)\partial_\mu v^a\partial^\mu v^b - \tfrac{1}{4}(b_c\bar{b}_d)\partial_\mu z^c\partial^\mu\bar{z}^d + \mathcal{O}(\text{moduli}^3)\ , \\ g^{ij}R_{i\mu j}{}^\mu &= \tfrac{1}{2}(b_c\bar{b}_d)\partial_\mu z^c\partial^\mu\bar{z}^d + \mathcal{O}(\text{moduli}^3)\ .\end{aligned} \tag{D.11}$$

The very last parenthesis in eq. (D.2) takes on a simple form since

$$g^{ij}(R_{ikj}{}^k + R_{i\bar{k}j}{}^{\bar{k}}) = g^{ij}R_{imj}{}^m = g^{ij}R_{ij}\ . \tag{D.12}$$

Since a Calabi-Yau is Ricci flat, i.e. $R_{ij} = 0$ in the internal indices, there will only be space-time dependent components. These components are found in the Γ^2 terms which are of order $\mathcal{O}(\text{moduli}^2)$, hence the combination with g^{ij} will result in third order moduli terms and will therefore not contribute. As such, the final expression for the Einstein-Hilbert term in eq. (D.1) can be written

$$S_{\text{EH}}^{(10)} = \int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\Big(R + ((\omega_a g)(\omega_b g) - \tfrac{1}{2}(\omega_a\omega_b))\partial_\mu v^a\partial^\mu v^b + \tfrac{1}{2}(b_c\bar{b}_d)\partial_\mu z^c\partial^\mu\bar{z}^d\Big)\ . \tag{D.13}$$

Further, if one defines

$$V_{ab} \equiv \int_{\mathrm{CY}_3} \sqrt{g_6}((\omega_a g)(\omega_b g) - \tfrac{1}{2}\omega_a\omega_b) \ , \qquad Z_{c\bar{d}} \equiv \tfrac{1}{2}\int_{\mathrm{CY}_3} \sqrt{g_6}(b_c\bar{b}_d) \ , \tag{D.14}$$

and uses that $\mathrm{vol}_6 = \int \mathrm{d}^6x\sqrt{g_6}$ the action of eq. (D.13) can be written

$$S_{\mathrm{EH}}^{(10)} = \int \mathrm{d}^4x\sqrt{-g_4}(\mathrm{vol}_6 R + V_{ij}\partial_\mu v^a\partial^\mu v^b + Z_{c\bar{d}}\partial_\mu z^c\partial_\mu\bar{z}^d) \ . \tag{D.15}$$

The triple intersection number of eq. (5.39) becomes

$$\begin{aligned} \kappa_{ab} &\equiv \int_{\mathrm{CY}_3} \omega_a\wedge\omega_b\wedge J \\ &\equiv \int_{\mathrm{CY}_3} \omega_a\wedge\omega_b\wedge v^c\omega_c = \kappa_{abc}v^c \\ &= \int_{\mathrm{CY}_3} \sqrt{g_6}(-(\omega_a g)(\omega_b g) + (\omega_a\omega_b)) \ . \end{aligned} \tag{D.16}$$

We now move on to consider the rest of the fields in the 10D type IIB supergravity action, where we expand the ingoing fields in terms of the harmonic forms admitted by a Calabi-Yau. The 10D action in string frame is written

$$\begin{aligned} S_{\mathrm{s}}^{(10)} = &\int e^{-2\phi^{(10)}}\left(-\tfrac{1}{2}R^{(10)}\star\mathbb{1} + 2\,\mathrm{d}\phi^{(10)}\wedge\star\,\mathrm{d}\phi^{(10)} - \tfrac{1}{4}H_3^{(10)}\wedge\star H_3^{(10)}\right) \\ &- \tfrac{1}{2}\int\left(\mathrm{d}C_0^{(10)}\wedge\star C_0^{(10)} + F_3^{(10)}\wedge\star F_3^{(10)} + \tfrac{1}{2}F_5^{(10)}\wedge\star F_5^{(10)} - \tfrac{1}{2}C_4^{(10)}\wedge H_3^{(10)}\wedge\mathrm{d}C_2^{(10)}\right) \ , \end{aligned} \tag{D.17}$$

where $F_3^{(10)} = \mathrm{d}C_2^{(10)} - C_0^{(10)}H_3^{(10)}$ and $F_5^{(10)} = \mathrm{d}C_4^{(10)} - H_3^{(10)}\wedge C_2^{(10)}$. As stated in eqs. (5.51), (5.63) and (5.64), the field expansions of type IIB are given by

$$\begin{aligned} B_2^{(10)} &= B_2 + b^a\wedge\omega_a \ , \\ C_2^{(10)} &= C_2 + c^a + \omega_a \ , \\ C_4^{(10)} &= D_2^a\wedge\omega_a + \rho_a\wedge\tilde{\omega}^a + V^C\wedge\alpha_C - U_C\wedge\beta^C \ , \end{aligned} \tag{D.18}$$

where again $\omega_a \in H^{(1,1)}(CY_3)$, $\tilde{\omega}_a \in H^{(2,2)}(CY_3)$, and $\alpha_C, \beta^C \in H^{(3)}(CY_3)$ where $C = 0, 1, \ldots, h^{(2,1)}$. The D_2^a are spacetime dependent 2-forms, the V^C, U_C are 1-forms, and b^a, c^a, ρ_a are scalars. As for the field strengths the expansions read

$$\begin{aligned} H_3^{(10)} &= H_3 + \mathrm{d}b^a\wedge\omega_a \ , \\ \mathrm{d}C_2^{(10)} &= \mathrm{d}C_2 + \mathrm{d}c^a\wedge\omega_a \ , \\ \mathrm{d}C_4^{(10)} &= \mathrm{d}D_2^a\wedge\omega_a + \mathrm{d}\rho_a\wedge\tilde{\omega}^a + F^C\wedge\alpha_C - G_c\wedge\beta^C \ , \end{aligned} \tag{D.19}$$

where we define

$$F^C \equiv \mathrm{d}V^C \ , \qquad G_C \equiv \mathrm{d}U_C \ . \tag{D.20}$$

Using these, the field strengths take the form

$$\begin{aligned} F_3^{(10)} &= \mathrm{d}C_2 + \mathrm{d}c^a\wedge\omega_a - C_0(H_3 + \mathrm{d}b^a\wedge\omega_a) \ , \\ F_5^{(10)} &= (D_2^a - \mathrm{d}b^a\wedge C_2 - c^a H_3)\wedge\omega_a + F^C\wedge\alpha_C \\ &\qquad - G_C\wedge\beta^C + \mathrm{d}\rho_a\wedge\tilde{\omega}^a - c^a\,\mathrm{d}b^b\wedge\omega_a\wedge\omega_b \ . \end{aligned} \tag{D.21}$$

As discussed in chapter 5.1.2, the self-duality condition $F_5^{(10)} = \star F_5^{(10)}$ of the 5-form flux implies for the 4-form potential $C_4^{(10)}$ in eq. (D.19) that only half the degrees of freedom are physical and that the occurring fields in $F_5^{(10)}$ consists of Poincaré dual pairs. Specifically D_2^a are 2-form duals to the ρ_a scalars, and V^C and U_C are magnetic/electric duals. Since we know that the self-duality condition can not be obtained from the action in eq. (D.17), it must be imposed in the expansion of eq. (D.21) for the equations of motion to be correct. Imposing the self-duality constraint using $F_5^{(10)}$ in eq. (D.21) results in the two conditions

$$\begin{aligned} \mathrm{d}\rho_a - \kappa_{abc}\,\mathrm{d}b^b c^c &= 4\mathrm{vol}_6 g^{ab} \star (D_2^a - \mathrm{d}b^a \wedge C_2 - c^a H_3) \ , \\ G_C &= \mathrm{Re}\mathcal{M}_{CD} F^D + \mathrm{Im}\mathcal{M}_{CD} \star F^D \ , \end{aligned} \tag{D.22}$$

each of which illustrates the duality of D_2^a/ρ_a and V^C/U_C respectively. Keeping all the fields for now, we turn to the integration over the CY_3 space, after which the conditions of eq. (D.22) will emerge again from adding Lagrange multipliers. Using the above field expansion we integrate the terms in eq. (D.17) over the Calabi-Yau one at a time. For the $\mathrm{d}\phi^{(10)}$-terms we get that

$$\int_{\mathrm{CY}_3} \mathrm{d}\phi^{(10)} \wedge \star\, \mathrm{d}\phi^{(10)} = \mathrm{vol}_6\, \mathrm{d}\phi \wedge \star\, \mathrm{d}\phi \ , \tag{D.23}$$

whereas for the NSNS field strength $H_3^{(10)}$ we have

$$\int_{\mathrm{CY}_3} H_3^{(10)} \wedge \star H_3^{(10)} = \mathrm{vol}_6 (H_3 \wedge \star H_3 + 4 g_{ab}\, \mathrm{d}b^a \wedge \star\, \mathrm{d}b^b) \ . \tag{D.24}$$

The integral of the axion-term becomes

$$\int_{\mathrm{CY}_3} \mathrm{d}C_0^{(10)} \wedge \star\, \mathrm{d}C_0^{(10)} = \mathrm{vol}_6\, \mathrm{d}C_0 \wedge \star\, \mathrm{d}C_0 \ , \tag{D.25}$$

and the terms including the 3-form and 5-form RR field strengths are

$$\int_{\mathrm{CY}_3} F_3^{(10)} \wedge \star F_3^{(10)} = \mathrm{vol}_6 \left[(\mathrm{d}C_2 - C_0 H_3) \wedge \star (\mathrm{d}C_2 - C_0 H_3) + 2 g_{ab} (\mathrm{d}c^a - C_0\, \mathrm{d}b^a) \wedge \star (\mathrm{d}c^b - C_0\, \mathrm{d}b^b) \right] \ , \tag{D.26}$$

respectively

$$\begin{aligned} \int_{\mathrm{CY}_3} F_5^{(10)} \wedge \star F_5^{(10)} = {} & 2\mathrm{vol}_6 g_{ab} (\mathrm{d}D_2^a - \mathrm{d}b^a \wedge C_2 - c^a\, \mathrm{d}B_2) \wedge \star (\mathrm{d}D_2^b - \mathrm{d}b^b \wedge C_2 - c^b\, \mathrm{d}B_2) \\ & + \frac{1}{8\mathrm{vol}_6} g^{ab} (\mathrm{d}\rho_a - \kappa_{acd} c^c\, \mathrm{d}b^d) \wedge \star (\mathrm{d}\rho_a - \kappa_{aef} c^e\, \mathrm{d}b^f) \\ & - \tfrac{1}{2} ((\mathrm{Im}\mathcal{M})^{-1})^{CD} (G_C - \mathcal{M}_{CE} F^E) \wedge \star (G_D - \mathcal{M}_{CF} F^F) \ . \end{aligned} \tag{D.27}$$

The integral of the final term in eq. (D.17) is evaluated to

$$\int_{\mathrm{CY}_3} C_4^{(10)} \wedge H_3^{(10)} \wedge \mathrm{d}C_2^{(10)} = \kappa_{abc} D_2^a \wedge \mathrm{d}b^b \wedge \mathrm{d}c^c + \rho_a (\mathrm{d}B_2 \wedge \mathrm{d}c^a + \mathrm{d}b^a \wedge \mathrm{d}C_2) \ . \tag{D.28}$$

Gathering everything, including the expression for the Einstein-Hilbert term in eq. (D.15), the resulting 4D type IIB action is given by

$$
\begin{aligned}
S_{\mathrm{s}}^{(4)} = &\int e^{-2\phi^{(10)}}\Big[-\tfrac{1}{2}\mathrm{vol}_6 R\star\mathbb{1} - \tfrac{1}{2}V_{ab}\,\mathrm{d}v^a\wedge\star\,\mathrm{d}v^b - \tfrac{1}{2}Z_{c\bar{d}}\,\mathrm{d}z^c\wedge\star\bar{z}^c + 2\,\mathrm{d}\phi\wedge\star\,\mathrm{d}\phi \\
&\qquad - \tfrac{1}{4}\mathrm{vol}_6 H_3\wedge\star H_3 - \mathrm{vol}_6 g_{ab}\,\mathrm{d}b^a\wedge\star\,\mathrm{d}b^b\Big] \\
&+\int\Big[-\tfrac{1}{2}\,\mathrm{d}C_0\wedge\star C_0 - \tfrac{1}{2}\mathrm{vol}_6(\mathrm{d}C_2 - C_0H_3)\wedge\star(\mathrm{d}C_2 - C_0H_3) \\
&\qquad - \mathrm{vol}_6 g_{ab}(c^a - C_0\,\mathrm{d}b^a)\wedge\star(c^b - C_0\,\mathrm{d}b^b) \\
&\qquad - \mathrm{vol}_6 g_{ab}(\mathrm{d}D_2^a - \mathrm{d}b^a\wedge C_2 - c^a\,\mathrm{d}B_2)\wedge\star(\mathrm{d}D_2^b - \mathrm{d}b^b\wedge C_2 - c^b\,\mathrm{d}B_2) \\
&\qquad - \tfrac{1}{16\mathrm{vol}_6}g^{ab}(\mathrm{d}\rho_a - \kappa_{acd}c^c\,\mathrm{d}b^d)\wedge\star(\mathrm{d}\rho_b - \kappa_{bef}c^e\,\mathrm{d}b^f) \\
&\qquad + \tfrac{1}{4}((\mathrm{Im}\mathcal{M})^{-1})^{CD}(G_C - \mathcal{M}_{CE}F^E)\wedge\star(G_D - \mathcal{M}_{DF}F^F)\Big] \\
&- \tfrac{1}{2}\int\Big[\kappa_{abc}D_2^a\wedge\mathrm{d}b^b\wedge\mathrm{d}c^c + \rho_a(\mathrm{d}B_2\wedge\mathrm{d}c^a + \mathrm{d}b^a\wedge\mathrm{d}C_2)\Big]\ .
\end{aligned}
\tag{D.29}
$$

Now, in order to obtain the traditionally written action, it is customary to add a total derivative term to the above action, specifically

$$
S_{\mathrm{s,\ addition}}^{(4)} = \tfrac{1}{2}(F^C\wedge G_C + \mathrm{d}D_2^a\wedge\rho_a)\ . \tag{D.30}
$$

The self-duality equations in eq. (D.22) are then recovered by varying $S_{\mathrm{s}}^{4\mathrm{D}} + S_{\mathrm{s,\ addition}}^{4\mathrm{D}}$ with respect to G_C and $\mathrm{d}D_2^a$. Using these and re-expressing G_C and $\mathrm{d}D_2^a$ in terms of V^C and ρ_a, then the action takes the form

$$
\begin{aligned}
S_{\mathrm{s}}^{(4)} = &\int e^{-2\phi^{(10)}}\Big[-\tfrac{1}{2}\mathrm{vol}_6 R\star\mathbb{1} - \tfrac{1}{2}V_{ab}\,\mathrm{d}v^a\wedge\star\,\mathrm{d}v^b - \tfrac{1}{2}Z_{c\bar{d}}\,\mathrm{d}z^c\wedge\star\bar{z}^c + 2\,\mathrm{d}\phi\wedge\star\,\mathrm{d}\phi \\
&\qquad - \tfrac{1}{4}\mathrm{vol}_6 H_3\wedge\star H_3 - \mathrm{vol}_6 g_{ab}\,\mathrm{d}b^a\wedge\star\,\mathrm{d}b^b\Big] \\
&+\int\Big[-\tfrac{1}{2}\,\mathrm{d}C_0\wedge\star C_0 - \tfrac{1}{2}\mathrm{vol}_6(\mathrm{d}C_2 - C_0H_3)\wedge\star(\mathrm{d}C_2 - C_0H_3) \\
&\qquad - \mathrm{vol}_6 g_{ab}(c^a - C_0\,\mathrm{d}b^a)\wedge\star(c^b - C_0\,\mathrm{d}b^b) \\
&\qquad - \tfrac{1}{2\mathrm{vol}_6}g^{ab}(\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{acd}c^c\,\mathrm{d}b^d)\wedge\star(\mathrm{d}\rho_b - \tfrac{1}{2}\kappa_{bef}c^e\,\mathrm{d}b^f) \\
&\qquad + \tfrac{1}{2}(\mathrm{Re}\mathcal{M}_{CD}F^C\wedge\star F^D + \mathrm{Im}\mathcal{M}_{CD}F^C\wedge\star F^D) \\
&\qquad - 2\,\mathrm{d}\rho_a\wedge(\mathrm{d}b^a\wedge C_2 + c^a\,\mathrm{d}B_2) - \tfrac{1}{2}\,\mathrm{d}B_2\wedge(\kappa_{abc}c^ac^b\,\mathrm{d}c^c)\Big]\ .
\end{aligned}
\tag{D.31}
$$

To obtain the corresponding action in Einstein frame, we perform a Weyl rescaling which effectively removes the dilaton factor in front of the Einstein-Hilbert term as well as the volume factor. Complexifying the Kähler moduli as $t^a \equiv b^a + iv^a$, the resulting action in the Einstein frame becomes

$$
\begin{aligned}
S_{\mathrm{E}}^{(4)} = \int\Big[&-\tfrac{1}{2}R\star\mathbb{1} - \tfrac{1}{2}g_{ab}\,\mathrm{d}t^a\wedge\star\,\mathrm{d}t^b - \tfrac{1}{2}g_{c\bar{d}}\,\mathrm{d}z^c\wedge\star\,\mathrm{d}\bar{z}^{\bar{d}} - \mathrm{d}\phi\wedge\star\,\mathrm{d}\phi - \tfrac{1}{4}e^{-4\phi}H_3\wedge\star H_3 \\
&- \tfrac{1}{2}\mathrm{vol}_6\left(e^{2\phi}\,\mathrm{d}C_0\wedge\star\,\mathrm{d}C_0 + e^{-2\phi}(\mathrm{d}C_2 - C_0H_3)\wedge\star(\mathrm{d}C_2 - C_0H_3)\right) \\
&- \tfrac{1}{8\mathrm{vol}_6}e^{2\phi}g^{ab}(\mathrm{d}\rho_a - \kappa_{acd}c^c\,\mathrm{d}b^d)\wedge\star(\mathrm{d}\rho_b - \kappa_{bef}c^e\,\mathrm{d}b^f) \\
&+ \tfrac{1}{2}(\mathrm{Re}\mathcal{M}_{CD}F^C\wedge\star F^D + \mathrm{Im}\mathcal{M}_{CD}F^C\wedge\star F^D) \\
&+ (\mathrm{d}\rho_a - \kappa_{abc}c^b\,\mathrm{d}c^c)\wedge(\mathrm{d}b^a\wedge C_2 + c^a\,\mathrm{d}B_2) + \tfrac{1}{2}\kappa_{abc}c^ac^b\,\mathrm{d}B_2\wedge\mathrm{d}b^c \\
&- 2\mathrm{vol}_6 e^{2\phi}g_{ab}(\mathrm{d}c^a - C_0\,\mathrm{d}b^a)\wedge\star(\mathrm{d}c^b - C_0\,\mathrm{d}b^b)\Big]\ .
\end{aligned}
\tag{D.32}
$$

A final thing to do in order to obtain the standard $\mathcal{N}=2$ action is to dualise the 2-form fields B_2 and C_2 into corresponding scalars b^0 respectively c^0. This dualisation is done by adding total derivative terms to the above action which contain these fields as well as their dual scalars. Starting with C_2, one adds the term

$$S_{C_2}^{\text{add}} = \int \mathrm{d}C_2 \wedge \mathrm{d}c^0 \ . \tag{D.33}$$

Gathering all C_2-dependent terms in the action of eq. (D.32) as well as the addition of eq. (D.33), we have that

$$S_{C_2} = \int \Big[-\tfrac{1}{2}\text{vol}_6 e^{-2\phi}(\mathrm{d}C_2 - C_0 H_3) \wedge \star(\mathrm{d}C_2 - C_0 H_3) - \mathrm{d}\rho_a \wedge \mathrm{d}C_2 b^a + \mathrm{d}c^0 \wedge \mathrm{d}C_2 \Big] \ . \tag{D.34}$$

The dualisation is made by varying the above action and thereafter re-expressing C_2 in terms of c^0, so that $S_{C_2} \to S_{c^0}$ where

$$S_{c^0} = \int \Big[-\tfrac{1}{2\text{vol}_6} e^{2\phi}(\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a) \wedge \star(\mathrm{d}c^0 - b^b\,\mathrm{d}\rho_b) + C_0\,\mathrm{d}B_2 \wedge (\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a) \Big] \ . \tag{D.35}$$

The same procedure should be made for B_2. The term to add is analogous to C_2, namely

$$S_{B_2}^{\text{add}} = \int \mathrm{d}B_2 \wedge \mathrm{d}b^0 \ , \tag{D.36}$$

so that all terms containing B_2 with the above action are

$$\begin{aligned} S_{B_2} = \int \Big[&-\tfrac{1}{4} e^{-4\phi} H_3 \wedge \star H_3 + (\mathrm{d}\rho_a - \kappa_{abc} c^b\,\mathrm{d}b^c) \wedge (c^a\,\mathrm{d}B_2) \\ &+ \tfrac{1}{2}\kappa_{abc} c^a c^b\,\mathrm{d}B_2 \wedge \mathrm{d}b^c + C_0\,\mathrm{d}B_2 \wedge (\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a) \Big] \ , \end{aligned} \tag{D.37}$$

the last term coming from the C_2 dualisation. Varying the above action and thereafter re-expressing $S_{B_2} \to S_{b^0}$ so that

$$\begin{aligned} S_{b^0} = \int &-e^{4\phi}(C_0\,\mathrm{d}c^0 + \mathrm{d}b^0 + (c^a - C_0 b^a)\,\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{abc} c^a c^b\,\mathrm{d}b^c) \\ &\wedge \star(C_0\,\mathrm{d}c^0 + \mathrm{d}b^0 + (c^a - C_0 b^a)\,\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{abc} c^a c^b\,\mathrm{d}b^c) \ . \end{aligned} \tag{D.38}$$

With these expressions, the 4D action for type IIB supergravity compactified on a Calabi-Yau 3-fold in Einstein frame, can finally be written down as

$$\begin{aligned} S_{\text{E}}^{(4)} = \int \Big[&-\tfrac{1}{2} R \star \mathbb{1} - \tfrac{1}{2} g_{c\bar{d}}\,\mathrm{d}z^c \wedge \star\,\mathrm{d}\bar{z}^{\bar{d}} - \tfrac{1}{2} g_{ab}\,\mathrm{d}t^a \wedge \star\,\mathrm{d}t^b - \mathrm{d}\phi \wedge \star\,\mathrm{d}\phi \\ &- \tfrac{1}{2}\text{vol}_6 e^{2\phi}\,\mathrm{d}C_0 \wedge \star\,\mathrm{d}C_0 - \tfrac{1}{8\text{vol}_6} e^{2\phi} g^{ab}(\mathrm{d}\rho_a - \kappa_{acd} c^c\,\mathrm{d}b^d) \wedge \star(\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{bef} c^e\,\mathrm{d}b^f) \\ &- 2\text{vol}_6 e^{2\phi} g_{ab}(\mathrm{d}c^a - C_0\,\mathrm{d}b^a) \wedge \star(\mathrm{d}c^b - C_0\,\mathrm{d}b^b) \\ &- \tfrac{1}{2\text{vol}_6} e^{2\phi}(\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a) \wedge \star(\mathrm{d}c^0 - b^b\,\mathrm{d}\rho_b) \\ &- e^{4\phi}(C_0\,\mathrm{d}c^0 + \mathrm{d}b^0 + (c^a - C_0 b^a)\,\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{abc} c^a c^b\,\mathrm{d}b^c) \\ &\wedge \star(C_0\,\mathrm{d}c^0 + \mathrm{d}b^0 + (c^a - C_0 b^a)\,\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{efg} c^e c^f\,\mathrm{d}b^g) \\ &+ \tfrac{1}{2}(\text{Re}\mathcal{M}_{CD} F^C \wedge \star F^D + \text{Im}\mathcal{M}_{CD} F^C \wedge \star F^D) \Big] \ . \end{aligned} \tag{D.39}$$

In summary, there are $4\times(h^{(1,1)}+1)$ ingoing scalars $\{\phi, C_0, v^a, b^0, b^a, c^0, c^a, \rho_a\}$ which are grouped up to form $h^{(1,1)}+1$ hypermultiplets. As for the fields, the graviton $g_{\mu\nu}$ and V^0 are grouped up as the gravity multiplet and the complex structure moduli z^c and vectors V^c are grouped together to form $h^{(2,1)}$ vector multiplets. The vector multiplets span a $2h^{(2,1)}$-dimensional special Kähler manifold and the hypermultiplets span a $4 \times (h^{(1,1)}+1)$-dimensional quaternionic manifold.

D.2 Turning on fluxes

We are now to turn on magnetic and electric flux as given in eq. (5.114) and perform the same analysis as in the previous section to illustrate the effect of the fluxes. Starting with purely NSNS fluxes on the Calabi-Yau, the H_3 takes the form

$$H_3^{(10)} = \mathrm{d}B_2 + \mathrm{d}b^a \wedge \omega_a + m^C \alpha_C - e_C \beta^C \ , \tag{D.40}$$

where m, e again corresponds to $h^{(2,1)} + 1$ units of quantised flux each. The original string frame action in eq. (D.17) will again serve as a starting point. This will also change the 3-form and 5-form field strengths in eq. (D.21) so that

$$\begin{aligned} F_3^{(10)} &= \mathrm{d}C_2 + \mathrm{d}c^a \wedge \omega_a - C_0(H_3 + \mathrm{d}b^a \wedge \omega_a + m^C \alpha_C - e_C \beta^C) \ , \\ F_5^{(10)} &= (D_2^a - \mathrm{d}b^a \wedge C_2 - c^a H_3) \wedge \omega_a + \tilde{F}^C \wedge \alpha_C - \tilde{G}_C \wedge \beta^C + \mathrm{d}\rho_a \wedge \tilde{\omega}^a - c^a\, \mathrm{d}b^b \wedge \omega_a \wedge \omega_b \ , \end{aligned} \tag{D.41}$$

where we have defined

$$\tilde{F}^C \equiv F^C - m^C C_2 \ , \qquad \tilde{G}^C \equiv G^C - e_C C_2 \ , \tag{D.42}$$

with F^C, G^C defined as in eq. (D.20). We are now in a position to investigate how this flux addition in $H_3^{(10)}$ affects the terms in the action of eq. (D.17). The kinetic term for $H_3^{(10)}$ will for instance become

$$\begin{aligned} \int_{\mathrm{CY}_3} H_3^{(10)} \wedge \star H_3^{(10)} &= \mathrm{vol}_6\, \mathrm{d}B_2 \wedge \star\, \mathrm{d}B_2 + 4\mathrm{vol}_4 g_{ab}\, \mathrm{d}b^a \wedge \star\, \mathrm{d}b^b \\ &\quad - (e_C + N^{\mathrm{cs}}_{CD} m^D)((\mathrm{Im}\mathcal{M})^{-1})^{CE}(e_E + \overline{\mathcal{M}_{EF}} m^F) \ . \end{aligned} \tag{D.43}$$

The kinetic terms for the 3-form and 5-form field strengths become

$$\begin{aligned} \int_{\mathrm{CY}_3} F_3^{(10)} \wedge \star F_3^{(10)} &= \mathrm{vol}_6(\mathrm{d}C_2 - C_0 H_3) \wedge \star(\mathrm{d}C_2 - C_0 H_3) \\ &\quad + 2\mathrm{vol}_6 g_{ab}(\mathrm{d}c^a - C_0\, \mathrm{d}b^a) \wedge \star(\mathrm{d}c^b - C_0\, \mathrm{d}b^b) \\ &\quad - C_0^2(e_C + N^{\mathrm{cs}}_{CD} m^D)((\mathrm{Im}\mathcal{M})^{-1})^{CE}(e_E + \overline{\mathcal{M}_{EF}} m^F) \ , \end{aligned} \tag{D.44}$$

respectively

$$\begin{aligned} \int_{\mathrm{CY}_3} F_5^{(10)} \wedge \star F_5^{(10)} &= \mathrm{vol}_6 g_{ab}(\mathrm{d}D_2^a - \mathrm{d}b^a \wedge C_2 - c^a\, \mathrm{d}B_2) \wedge \star(\mathrm{d}D_2^b - \mathrm{d}b^b \wedge C_2 - c^b\, \mathrm{d}B_2) \\ &\quad + \tfrac{1}{8\mathrm{vol}_6} g^{ab}(\mathrm{d}\rho_a - \kappa_{acd} c^c\, \mathrm{d}b^d) \wedge \star(\mathrm{d}\rho_b - \kappa_{bef} c^e\, \mathrm{d}b^f) \\ &\quad - \tfrac{1}{2}((\mathrm{Im}\mathcal{M})^{-1})^{CD}(\tilde{G}_C - \mathcal{M}_{CE}\tilde{F}^E) \wedge \star(\tilde{G}_D - \mathcal{M}_{DF}\tilde{F}^F) \ . \end{aligned} \tag{D.45}$$

The last term in eq. (D.17) also contain a $H_3^{(10)}$, which now becomes

$$\begin{aligned} \int_{\mathrm{CY}_3} C_4^{(10)} \wedge H_3^{(10)} \wedge \mathrm{d}C_2^{(10)} &= \kappa_{abc} D_2^a \wedge \mathrm{d}b^b \wedge \mathrm{d}c^c + \rho_a(\mathrm{d}B_2 \wedge \mathrm{d}c^a + \mathrm{d}b^a \wedge \mathrm{d}C_2) \\ &\quad + C_2 \wedge (F^C e_C - G_C m^C) \ . \end{aligned} \tag{D.46}$$

Conclusively these changes re-inserted in the action in eq. (D.17) amounts to that it effectively adds a potential term $S_{\mathrm{s}}^{(10)} \to S_{\mathrm{s}}^{(10)} + V^{\mathrm{NSNS}}$, which is given by

$$V^{\mathrm{NSNS}} = \tfrac{1}{2}(e^{4\phi} C_0^2 + \tfrac{1}{2\mathrm{vol}_6} e^{2\phi})(e_C + \mathcal{M}_{CD} m^D)((\mathrm{Im}\mathcal{M})^{-1})^{CE}(e_E + \overline{\mathcal{M}_{EF}} m^F) \ . \tag{D.47}$$

The next step is again to add a total derivative term like in eq. (D.30). Adding the first term in eq. (D.30) and varying with respect to G_A will result in the self-duality condition

$$\tilde{G}_C = \mathrm{Re}\mathcal{M}_{CD}\tilde{F}^D + \mathrm{Im}\mathcal{M}_{CD} \star \tilde{F}^D \ . \tag{D.48}$$

However, the second term in eq. (D.30) will not alter any condition. As such, it is possible to eliminate G_C in favor of F^C via

$$\tfrac{1}{2}F^C \wedge G_C - \tfrac{1}{2}C_2(F^C e_C - G_C m^C) \to \tfrac{1}{2}\mathrm{Re}\mathcal{M}_{CD}\tilde{F}^C \wedge \star\tilde{F}^D + \mathrm{Im}\mathcal{M}_{CD}\tilde{F}^C \wedge \star\tilde{F}^D - \tfrac{1}{2}(F^C e_C + \tilde{F}^C e_C) \wedge C_2 \ . \tag{D.49}$$

By the last term it is clear from eq. (D.42) defining $\tilde{F}_C$ that there is now a mass-term for C_2. The dualisation of C_2 is however generally done for a massless field, hence one may neglect the magnetic flux and set $m = 0$. As such, there will only be electric flux and $\tilde{F}_C = F_C$. The terms in the action containing C_2 terms are given by

$$S_{C_2} = \int -\tfrac{1}{2}\mathrm{vol}_6 e^{-2\phi}(\mathrm{d}C_2 - C_0 H_3) \wedge \star(\mathrm{d}C_2 - C_0 H_3) - \mathrm{d}\rho_a \wedge \mathrm{d}C_2 b^a + e_C F^C \wedge C_2 + \mathrm{d}c^0 \wedge \mathrm{d}C_2 \ , \tag{D.50}$$

and by comparison with eq. (D.34) we have that $\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a \to \mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a - e_C V^C$. The dualised action then takes the form

$$S_{c^0} = \int -\tfrac{1}{2\mathrm{vol}_6}e^{2\phi}(\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a) \wedge \star(\mathrm{d}c^0 - b^b\,\mathrm{d}\rho_b) - C_0\,\mathrm{d}B_2 \wedge (\mathrm{d}c^0 - b^a\,\mathrm{d}\rho_a - e_C V^C) \ . \tag{D.51}$$

The analogous procedure for S_{B_2} results in the dualised action

$$\begin{aligned} S_{b^0} = -\int e^{4\phi}&(C_0\,\mathrm{d}c^0 + \mathrm{d}b^0 + (c^a - C_0 b^a)\,\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{abc}c^a c^b\,\mathrm{d}b^c - C_0 e_C V^C) \\ &\wedge \star(C_0\,\mathrm{d}c^0 + \mathrm{d}b^0 + (c^b - C_0 b^b)\,\mathrm{d}\rho_a - \tfrac{1}{2}\kappa_{def}c^d c^e\,\mathrm{d}b^f - C_0 e_D V^D) \ . \end{aligned} \tag{D.52}$$

If we now consider turning on RR fluxes for the odd degree form field strengths of type IIB, we recall again that a Calabi-Yau does not admit harmonic 1-forms or 5-forms. Hence it is only possible to turn on fluxes for the 3-form RR field strength $\mathrm{d}C_2^{(10)}$. This will change $F_3^{(10)}$ so that now

$$F_3^{(10)} = \mathrm{d}C_2 + \mathrm{d}c^a \wedge \omega_a - C_0(H_3 + \mathrm{d}b^a \wedge \omega_a) + m^C\alpha_C - e_C\beta^C \ , \tag{D.53}$$

and the 5-form to

$$F_5^{(10)} = (D_2^a - \mathrm{d}b^a \wedge C_2 - c^a H_3) \wedge \omega_a + \tilde{F}^C \wedge \alpha_C - \tilde{G}_C \wedge \beta^C + \mathrm{d}\rho_a \wedge \tilde{\omega}^a - c^a\,\mathrm{d}b^b \wedge \omega_a \wedge \omega_b \ , \tag{D.54}$$

where now

$$\tilde{F}_C \equiv F^C + m^C C_2 \ , \qquad \tilde{G}_C \equiv G^C + e_C C_2 \ . \tag{D.55}$$

The affected kinetic terms in the action will then become

$$\begin{aligned} \int_{\mathrm{CY}_3} F_3^{(10)} \wedge \star F_3^{(10)} = \mathrm{vol}_6(&\mathrm{d}C_2 - C_0 H_3) \wedge \star(\mathrm{d}C_2 - C_0 H_3) \\ &+ 2\mathrm{vol}_6 g_{ab}(\mathrm{d}c^a - C_0\,\mathrm{d}b^a) \wedge \star(\mathrm{d}c^a - C_0\,\mathrm{d}b^a) \\ &- (e_C + \mathcal{M}_{CD}m^D)((\mathrm{Im}\mathcal{M})^{-1})^{CE}(e_E + \overline{\mathcal{M}_{EF}}m^F) \ , \end{aligned} \tag{D.56}$$

$$\begin{aligned} \int_{\mathrm{CY}_3} F_5^{(10)} \wedge \star F_5^{(10)} = 2\mathrm{vol}_6 g_{ab}(&\mathrm{d}D_2^a - \mathrm{d}b^a \wedge C_2 - c^a\,\mathrm{d}B_2) \wedge \star(\mathrm{d}D_2^b - \mathrm{d}b^b \wedge C_2 - c^b\,\mathrm{d}B_2) \\ &+ \tfrac{1}{8\mathrm{vol}_6}g^{ab}(\mathrm{d}\rho_a - \kappa_{acd}c^c\,\mathrm{d}b^d) \wedge \star(\mathrm{d}\rho_a - \kappa_{bef}c^e\,\mathrm{d}b^f) \\ &- \tfrac{1}{2}((\mathrm{Im}\mathcal{M})^{-1})^{CD}(\tilde{G}_C - \mathcal{M}_{CE}\tilde{F}^E) \wedge \star(\tilde{G}_D - \mathcal{M}_{DF}\tilde{F}^F) \ , \end{aligned} \tag{D.57}$$

$$\int_{\mathrm{CY}_3} C_4^{(10)} \wedge H_3^{(10)} \wedge \mathrm{d}C_2^{(10)} = \kappa_{abc} D_2^a \wedge \mathrm{d}b^b \wedge \mathrm{d}c^c + \rho_a (\mathrm{d}B_2 \wedge \mathrm{d}c^a + \mathrm{d}b^a \wedge C_2) \\ + (F^C e_C - G_C m^C) \wedge B_2 \ . \tag{D.58}$$

Effectively this will result in the addition of a potential in the action, namely

$$V^{\mathrm{RR}} = -\tfrac{1}{2} e^{4\phi} (e_C - \mathcal{M}_{CD} m^D)((\mathrm{Im}\mathcal{M})^{-1})^{CE} (e_E + \overline{\mathcal{M}_{EF}} m^F) \ . \tag{D.59}$$

E

Double Field Theory Calculations

Double field theory was introduced in chapter 9 and here we give details to some calculations discussed in the main text.

E.1 Closure of generalised diffeomorphisms

In this section we prove the form of the commutation relation of the generalised Lie derivative in eq. (9.42). Our starting point is eq. (9.45), repeated here for convenience as

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}]V^M = \hat{\mathcal{L}}_{\xi_3} V^M \ , \tag{E.1}$$

where the commutant acts on some generalised vector V^M and $\xi_3 \equiv \xi_3(\xi_1, \xi_2)$. The right-hand side of this equation is given by the definition in eq. (9.42) as

$$\hat{\mathcal{L}}_{\xi_3} V^M = \xi_3^N \partial_N V^M + (\partial^M \xi_{3N} - \partial_N \xi_3^M) V^N \ . \tag{E.2}$$

The left-hand side of eq. (E.1) is evaluated using the same formula, so that

$$\begin{aligned}
[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}]V^M &= \hat{\mathcal{L}}_{\xi_1}\hat{\mathcal{L}}_{\xi_2} V^M - \hat{\mathcal{L}}_{\xi_2}\hat{\mathcal{L}}_{\xi_1} V^M \\
&= \hat{\mathcal{L}}_{\xi_1}(\xi_2^N \partial_N V^M + (\partial^M \xi_{2N} - \partial_N \xi_2^M) V^M) - (1 \leftrightarrow 2) \\
&= (\xi_1^P \partial_P \xi_2^N + (\partial^N \xi_{1P} - \partial_P \xi_1^N)\xi_2^P)\partial_N V^M \\
&\quad + \xi_2^N \left(\xi_2^P \partial_P(\partial_N V^M)) + (\partial_N \xi_1^P - \partial^P \xi_{1N})\partial_P V^M + (\partial^M \xi_{1P} - \partial_P \xi_1^M)\partial_N V^P\right) \\
&\quad + V^P \left(\xi_1^N \partial_N(\partial^M \xi_{2P}) + (\partial^M \xi_{1N} - \partial_N \xi_1^M)(\partial^N \xi_{2P}) + (\partial_P \xi_1^N - \partial^N \xi_{1P})(\partial^M \xi_{2N})\right) \\
&\quad + (\partial^M \xi_{2P})(\xi_1^N \partial_N V^P + (\partial^P \xi_{1N} - \partial_N \xi_1^P) V^N) \\
&\quad - V^P \left(\xi_1^N \partial_N \partial_P \xi_2^M + (\partial_P \xi_1^N - \partial^N \xi_{1P})(\partial_N \xi_2^M) + (\partial^M \xi_{1N} - \partial_N \xi_1^M)(\partial_P \xi_2^M)\right) \\
&\quad - (\partial_P \xi_2^M)\left(\xi_1^N \partial_N V^P + (\partial^P \xi_{1N} - \partial_N \xi_1^P) V^N\right) - (1 \leftrightarrow 2) \\
&= (\xi_1^P \partial_P \xi_2^N - \xi_2^P \partial_P \xi_1^N)\partial_N V^M \\
&\quad + V^P \Big(\xi_1^N \partial_N \partial^M \xi_{2p} + \partial_P \xi_1^N \partial^M \xi_{2N} - \partial^N \xi_{1P} \partial^M \xi_{2N} - \xi_1^N \partial_N \partial_P \xi_2^M \\
&\qquad + \partial_N \xi_1^M \partial_P \xi_2^N . \xi_2^N \partial_N \partial^M \xi_{1P} - \partial_P \xi_2^N \partial^M \xi_{1N} + \partial^N \xi_{2P} \partial^M \xi_{1N} \\
&\qquad + \partial^N \xi_{2P} \partial^M \xi_{1N} + \xi_2^N \partial_N \partial_P \xi_1^M - \partial_N \xi_2^M \partial_P \xi_1^N \\
&\qquad - \partial_N \xi_1^M \partial^N \xi_{2P} + \partial_N \xi_2^M \partial^N \xi_{1P}\Big) \\
&= (\xi_1^P \partial_P \xi_2^N - \xi_2^P \partial_P \xi_1^N)\partial_N V^M \\
&\quad + \partial^M(\xi_{1N}\partial^N \xi_{2P} - \xi_{2N}\partial^N \xi_{1P} - \tfrac{1}{2}\xi_{1N}\partial_P \xi_2^N + \tfrac{1}{2}\xi_{2N}\partial_P \xi_{1N}) V^P \\
&\quad - \partial_P(\xi_{1N}\partial^N \xi_2^M - \xi_{2N}\partial^N \xi_1^M - \tfrac{1}{2}\xi_{1N}\partial^M \xi_2^N + \tfrac{1}{2}\xi_{2N}\partial^M \xi_{1N}) V^P \\
&\quad - (\partial_N \xi_1^M \partial^N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P}) V^P \ .
\end{aligned} \tag{E.3}$$

Now, adding and subtracting the terms $\frac{1}{2}(\xi_1^N \partial^P \xi_{2N} - \xi_2^N \partial^P \xi_{1N})\partial_P V^M$ to the above expression, we have that

$$\begin{aligned}
[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}]V^M = &\left(\xi_1^N \partial_N \xi_2^P - \xi_2^N \partial_N \xi_1^P - \tfrac{1}{2}\xi_1^N \partial^P \xi_{2N} + \tfrac{1}{2}\xi_2^N \partial^P \xi_{1N}\right)\partial_N V^M \\
&+ \partial^M(\xi_{1N}\partial^N \xi_{2P} - \xi_{2N}\partial^N \xi_{1P} - \tfrac{1}{2}\xi_{1N}\partial_P \xi_2^N + \tfrac{1}{2}\xi_{2N}\partial_P \xi_{1N})V^P \\
&- \partial_P(\xi_{1N}\partial^N \xi_2^M - \xi_{2N}\partial^N \xi_1^M - \tfrac{1}{2}\xi_{1N}\partial^M \xi_2^N + \tfrac{1}{2}\xi_{2N}\partial^M \xi_{1N})V^P \\
&- \left(\partial_N \xi_1^M \partial^N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P}\right)V^P + \tfrac{1}{2}\xi_1^N \partial^P \xi_{2N}\partial_P V^M - \tfrac{1}{2}\xi_2^N \partial^P \xi_{1N}\partial_P V^M \ .
\end{aligned} \tag{E.4}$$

Hence, by the definition of the C-bracket it is clear that the parameter ξ_3 should be

$$\xi_3^M = [\xi_1, \xi_2]_{\mathrm{C}}^M \ . \tag{E.5}$$

There is also terms in eq. (E.4) that does not fit the C-braket, namely the above calculation shows that

$$[\hat{\mathcal{L}}_{\xi_1}, \hat{\mathcal{L}}_{\xi_2}]V^M = \hat{\mathcal{L}}_{[]_{\mathrm{C}}} V^M + F^M \ , \tag{E.6}$$

where

$$F^M = \tfrac{1}{2}\xi_1^N \partial^P \xi_{2N}\partial_P V^M - \tfrac{1}{2}\xi_2^N \partial^P \xi_1 N \partial_P V^M - (\partial_N \xi_1^M \partial_N \xi_{2P} - \partial_N \xi_2^M \partial^N \xi_{1P})V^P \ . \tag{E.7}$$

In order for this algebra to close, i.e. in order fulfil eq. (E.1), we must have $F^M = 0$ which implies the strong constraint of double field theory.

E.2 Covariant fluxes

In this section we derive the expressions of the geometrical and non-geometrical flux components of $\mathcal{F}_{ABC}$ as they were stated in chapter 9.2.3. In DFT the covariant flux is defined as the scalar product

$$\mathcal{F}_{ABC} = [E_A{}^L, E_B^L]_C E_{CL} = 3\Omega_{ABC} \ , \tag{E.8}$$

with

$$\Omega_{ABC} \equiv E_A{}^M \partial_M E_B{}^N E_{CN} \ . \tag{E.9}$$

The flux calculation uses the non-gauge fixed vielbein of eq. (9.91), i.e.

$$E_A{}^M = \eta_{AB} E^B{}_N \eta^{NM} = \begin{pmatrix} e_i^a + e^a{}_j \beta^{jk} B_{ki} & e^a{}_j \beta^{ij} \\ e_a{}^j B_{ji} & e_a{}^i \end{pmatrix} \ , \tag{E.10}$$

The H-flux calculation is as in chapter 9.2.3, and using the above vielbein the single component Ω_{abc} becomes

$$\begin{aligned}
\Omega_{abc} &= E_a{}^M \partial_M E_b{}^N E_{cN} \\
&= E_a{}^m \partial_m E_b{}^n E_{cn} + E_{am} \tilde{\partial}^m E_b{}^n E_{cn} + E_a{}^m \partial_m \partial_m E_{bn} E_c{}^n + E_{am} \tilde{\partial}^m E_{bn} E_c{}^n \\
&= e_a{}^m \partial_m (e_b{}^n) e_c{}^j B_{jn} + e_a{}^k \tilde{\partial}^m (e_b{}^n) e_c{}^j B_{jn} + e_a{}^m \partial_m (e_b{}^j B_{jn}) e_c{}^n + e_a{}^k B_{km} \tilde{\partial}^m (e_b{}^j B_{jn}) e_c{}^n \\
&= e_a{}^m \partial_m (e_b{}^n) e_c{}^j B_{jn} + e_a{}^k \tilde{\partial}^m (e_b{}^n) e_c{}^j B_{jn} + e_a{}^m \partial_m (e_b{}^j) B_{jn} e_c{}^n + e_a{}^m e_b{}^j \partial_m (B_{jn}) e_c{}^n \\
&\quad + e_a{}^k B_{km} \tilde{\partial}^m (e_b{}^j) B_{jn} e_c{}^n + e_a{}^k B_{km} e_b{}^j \tilde{\partial}^m (B_{jn}) e_c{}^n \\
&= e_a{}^i e_b{}^j e_c{}^k [\partial_i (B_{jk}) + B_{im} \tilde{\partial}^m (B_{jk})] \ .
\end{aligned} \tag{E.11}$$

Antisymmetrising this expression we obtain the H_{abc}-flux in flat indices;

$$\mathcal{F}_{abc} = 3 e_a{}^i e_b{}^j e_c{}^k (\partial_{[i} B_{jk]} - B_{l[i} \tilde{\partial}^l B_{jk]}) = H_{abc} \ . \tag{E.12}$$

Continuing with the $f^a{}_{bc}$ flux, it is given by

$$\mathcal{F}^a{}_{bc} = \Omega^a{}_{bc} + \Omega_c{}^a{}_b + \Omega_{bc}{}^a = f^a{}_{bc} \ . \tag{E.13}$$

As Ω_{ABC} is antisymmetric we have that $\Omega_{bc}{}^a = -\Omega_b{}^a{}_c$, so we need only calculate $\Omega^a{}_{bc}$ and $\Omega_a{}^b{}_c$. Starting with the former, we have that

$$\begin{aligned}
\Omega^a{}_{bc} &= e^{aM}\partial_M E_b{}^N E_{cN} \\
&= E^{am}\partial_m E_b{}^n E_{cn} + E^a{}_m\tilde{\partial}^m E_b{}^n E_{cn} + E^{am}\partial_m E_{bn} E_c{}^n + E^a{}_m \tilde{\partial}^m E_{bn} E_c{}^n \\
&= e^a{}_i\beta^{im}(\partial_m e_b{}^j B_{jn})e_c{}^n + (e^a{}_m + e^a{}_i\beta^{ij}B_{jm})\tilde{\partial}^m(e_b{}^n)e_c{}^k B_{kn} \\
&\qquad + (e^a{}_i\beta^{im})\partial_m(e_b{}^j B_{jn})e_c{}^n + (e^a{}_m + e^a{}_i\beta^{ij}B_{jm})\tilde{\partial}^m(e_b{}^k B_{kn})e_c{}^n \\
&= e^a{}_i\beta^{im}\partial_m e_b{}^n e_c{}^n B_{kj} + e^a{}_m\tilde{\partial}^m e_b{}^n e_c{}^k B_{kn} + e^a{}_i\beta^{ij}B_{jm}\tilde{\partial}^m e_b{}^n e_c{}^k B_{kn} \\
&\quad\ e^a{}_i\beta^{im}\partial_m e_b{}^j B_{jn} e_c{}^n + e^a{}_i\beta^{im} e_b{}^j \partial_m B_{jn} e_c{}^n + e^a{}_m \tilde{\partial}^m e_b{}^k B_{kn} e_c{}^n \\
&\quad + e^a{}_m e_b{}^k \tilde{\partial}^m B_{kn} e_c{}^n + e^a{}_i\beta^{ij}B_{jm}\tilde{\partial}^m e_b{}^k B_{kn} e_c{}^n + e^a{}_i\beta^{ij}B_{jm} e_b{}^k \tilde{\partial}^l B_{kl} e_c{}^n \\
&= e^a{}_i e_b{}^j e_c{}^k \left(\tilde{\partial}^i B_{jk} + \beta^{im}(\partial_m B_{jk} - B_{lm}\tilde{\partial}^l B_{jk})\right) \\
&\equiv e^a{}_i e_b{}^j e_c{}^k \left(\tilde{\partial}^i B_{jk} + \beta^{im} D_m B_{jk}\right) \ .
\end{aligned} \tag{E.14}$$

For the other component $\Omega_a{}^b{}_c$ we get that

$$\begin{aligned}
\Omega_a{}^b{}_c &= E_a{}^M \partial_M E^{bN} E_{cN} \\
&= E_a{}^m\partial_m E^{bn}E_{cn} + E_{am}\tilde{\partial}^m E^{bn}E_{cn} + E_a{}^m\partial_m E^b{}_n E_c{}^n + E_{am}\tilde{\partial}^m E^b{}_n E_c{}^n \\
&= e_a{}^m\partial_m(e^b{}_j\beta^{jn})(e_c{}^k B_{kn}) + e_a{}^i B_{im}\tilde{\partial}(e^b{}_j\beta^{jn})e_c{}^k B_{kn} \\
&\quad + e_a{}^m\partial^m(e^b{}_n + e^b{}_j\beta^{jk}B_{kn})e_c{}^n + e_a{}^i B_{im}\tilde{\partial}^m(e^b{}_n + e^b{}_j\beta^{jk}B_{kn})e_c{}^n \\
&= e_a{}^m\partial_m e^b{}_j\beta^{jn}e_c{}^k B_{kn} + e_a{}^m e^b{}_j\partial_m\beta^{jn}e_c{}^k B_{kn} + e_a{}^i B_{im}\tilde{\partial}^m e^b{}_j\beta^{jn}e_c{}^k B_{kn} \\
&\quad + e_a{}^i B_{im} e^b{}_j\tilde{\partial}^m\beta^{jn}e_c{}^k B_{kn} + e_a{}^m\partial_m e^b{}_n e_c{}^n + e_a{}^m\partial_m e^b{}_j\beta^{jk}B_{kn}e_c{}^k \\
&\quad + e_a{}^m e^b{}_j\partial_m\beta^{jk}B_{kn}e_c{}^n + e_a{}^m e^b{}_j\beta^{jk}\partial_m B_{kn}e_c{}^n + e_a{}^i B_{im}\tilde{\partial}^m e^b{}_n e_c{}^n \\
&\quad + e_a{}^i B_{im}\tilde{\partial}^m e^b{}_j\beta^{jk}B_{kn}e_c{}^n + e_a{}^i B_{im} e_j\tilde{\partial}^m\beta^{jk}B_{kn}e_c{}^n + e_a{}^i B_{im} e^b{}_j\beta^{jk}\tilde{\partial}^m B_{kn}e_c{}^n \\
&= e_a{}^i\partial_i e^b{}_j e_c{}^j + e_a{}^i B_{im}\tilde{\partial}^m e^b{}_j e_c{}^j + e_a{}^i e^b{}_j\beta^{jk}(\partial_i B_{kn} + B_{im}\tilde{\partial}^m B_{kn})e_c{}^n \\
&\equiv e_a{}^i D_i e^b{}_j e_c{}^j + e_a{}^i e^b{}_j\beta^{jk} D_i B_{kn} e_c{}^n \ .
\end{aligned} \tag{E.15}$$

From this the $f^a{}_{bc}$ flux is then obtained as

$$\begin{aligned}
f^a{}_{bc} &= \Omega^a{}_{bc} + \Omega_c{}^a{}_b - \Omega_b{}^a{}_c \\
&= e^a{}_i e_b{}^j e_c{}^k \left(\tilde{\partial}^i B_{jk} + \beta^{im} D_m B_{jk}\right) + e_c{}^i D_i e^a{}_j e_b{}^j + e_c{}^i e^a{}_j \beta^{jk} D_i B_{kn} e_b{}^n \\
&\quad - e_b{}^i D_i e^a{}_j e_c{}^j - e_b{}^i e^a{}_j \beta^{jk} D_i B_{kn} e_c{}^n \\
&= 2e_{[c}{}^i D_i e^a{}_j e_{b]}{}^j + e^a{}_i e_b{}^j e_c{}^k \left(\tilde{\partial}^i B_{jk} + \beta^{im} H_{mjk}\right) \ .
\end{aligned} \tag{E.16}$$

Quickly proceeding to the Q-flux, again it is defined as

$$Q_a{}^{bc} = \Omega_a{}^{bc} + \Omega^{bc}{}_a + \Omega^c{}_a{}^b \ . \tag{E.17}$$

However, since Ω_{ABC} is antisymmetric in its indices, $\Omega^c{}_a{}^b = -\Omega^{bc}{}_a$, and we need only evaluate two components. Starting with $\Omega_a{}^{bc}$, we get

$$\begin{aligned}
\Omega_a{}^{bc} &= E_a{}^M \partial^M E^{bN} E^c{}_N \\
&= E_a{}^m \partial^m E^{bn} E^c{}_n + E_{am}\tilde{\partial}^m E^{bn} E^c{}_n + E_a{}^m \partial_m E^b{}_n E^{cn} + E_{am}\tilde{\partial}^m E^b{}_n E^{cn} \\
&= e_a{}^m \partial_m (e^b{}_i \beta^{in})(e^c{}_n + e^c{}_j \beta^{jk} B_{kn}) + e_a{}^i B_{im} \tilde{\partial}^m (e^b{}_j \beta^{jn})(e^c{}_n + e^c{}_k \beta^{kl} B_{ln}) \\
&\quad + e_a{}^m \partial_m (e^b{}_n + e^b{}_i \beta^{ij} B_{jn})(e^c{}_k \beta^{kn}) + e_a{}^i B_{im} \tilde{\partial}^m (e^b{}_n + e^b{}_j \beta^{jk} B_{kn}) e^c{}_l \beta^{ln} \\
&= e_a{}^m \partial_m e^b{}_i \beta^{in} e^c{}_n + e_a{}^m \partial_m e^b{}_i \beta^{in} e^c{}_j \beta^{jk} B_{kn} + e_a{}^m e^b{}_i \partial_m \beta^{in} e^c{}_n \\
&\quad + e_a{}^m e^b{}_i \partial_m \beta^{in} e^c{}_j \beta^{jk} B_{kn} + e_a{}^i B_{im} \tilde{\partial}^m e^b{}_j \beta^{jn} e^c{}_n + e_a{}^i B_{im} \tilde{\partial}^m e^b{}_j \beta^{jn} e^c{}_k \beta^{kl} B_{ln} \\
&\quad + e_a{}^i B_{im} e^b{}_j \tilde{\partial}^m \beta^{jn} e^c{}_n + e_a{}^i B_{im} e^b{}_j \tilde{\partial}^m \beta^{jn} e^c{}_k \beta^{kl} B_{ln} + e_a{}^m \partial_m e^b{}_n e^c{}_k \beta^{kn} \\
&\quad + e_a{}^m \partial_m e^b{}_i \beta^{ij} B_{jn} e^c{}_k \beta^{kn} + e_a{}^m e^b{}_i \partial_m \beta^{ij} B_{jn} e^c{}_k e^c{}_k \beta^{kn} + e_a{}^m e^b{}_i \beta^{ij} \partial_m B_{jn} e^c{}_k \beta^{kn} \\
&\quad + e_a{}^i B_{im} \tilde{\partial}^m e^b{}_j e^c{}_l \beta^{ln} + e_a{}^i B_{im} \tilde{\partial}^m e^b{}_j \beta^{jk} B_{kn} e^c{}_l \beta^{ln} + e_a{}^i B_{im} e^b{}_j \tilde{\partial}^m \beta^{jk} B_{kn} e^c{}_l \beta^{ln} \\
&\quad + e_a{}^i B_{im} e^b{}_j \beta^{jk} \tilde{\partial}^m B_{kn} e^c{}_l \beta^{ln} \\
&= e_a{}^i e^b{}_j e^c{}_k \left(\partial_i \beta^{jk} + B_{im} \tilde{\partial}^m \beta^{jk} + \beta^{jm} \partial_i B_{mn} \beta^{kn} + \beta^{jm} B_{il} \tilde{\partial}^l B_{mn} \beta^{kn} \right) \\
&\equiv e_a{}^i e^b{}_j e^c{}_k \left(D_i \beta^{jk} + \beta^{jm} D_i B_{mn} \beta^{kn} \right) \ .
\end{aligned} \tag{E.18}$$

The other component becomes

$$\begin{aligned}
\Omega^{ab}{}_c &= E^{aM} \partial_M E^{bN} E_{cN} \\
&= E^{am} \partial_m E^{bn} E_{cn} + E^a{}_m \tilde{\partial}^m E^{bn} E_{cn} + E^{am} \partial_m E^b{}_n E_c{}^n + E^a{}_m \tilde{\partial}^m E^b{}_n E_c{}^n \\
&= e^a{}_i \beta^{im} \partial_m (e^b{}_j \beta^{jn}) e_c{}^k B_{kn} + (e^a{}_m + e^a{}_i \beta^{ij} B_{jm}) \tilde{\partial}^m (e^b{}_k \beta^{kn}) e_c{}^l B_{ln} \\
&\quad + e^a{}_i \beta^{im} \partial_m (e^b{}_n e^b{}_j \beta^{jk} B_{kn}) e_c{}^n + (e^a{}_m + e^a{}_i \beta^{ij} B_{jm}) \tilde{\partial}^m (e^b{}_n e^b{}_k \beta^{kl}) e_c{}^n \\
&= e^a{}_i \beta^{im} \partial_m e^b{}_j \beta^{jn} e_c{}^k B_{kn} + e^a{}_i \beta^{im} e^b{}_j \partial_m \beta^{jn} e_c{}^k B_{kn} + e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kn} e_c{}^l B_{ln} \\
&\quad + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kn} e_c{}^l B_{ln} + e^a{}_i \beta^{ij} B_{jm} \tilde{\partial}^m e^b{}_k \beta^{kn} e_c{}^l B_{ln} + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \tilde{\partial}^{kn} e_a{}^l B_{ln} \\
&\quad + e^a{}_i \beta^{im} \partial_m e^b{}_n e_c{}^n + e^a{}_i \beta^{im} \partial_m e^b{}_j \beta^{jk} B_{kn} e_c{}^n + e^a{}_i \beta^{im} e^b{}_j \partial_m \beta^{jk} B_{kn} e_c{}^n \\
&\quad + e^a{}_i \beta^{im} e^b{}_j \beta^{jk} \partial_m B_{kn} e_c{}^n + e^a{}_m \tilde{\partial}^m e^b{}_n e_c{}^n + e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kl} B_{ln} e_c{}^n \\
&\quad + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kl} B_{ln} e_c{}^n + e^a{}_m e^b{}_k \beta^{kl} \tilde{\partial}^m B_{ln} e_c{}^n + e^a{}_i \beta^{ij} B_{jm} \tilde{\partial}^m e^b{}_n e_c{}^n \\
&\quad + e^a{}_i \beta^{ij} B_{jm} \tilde{\partial}^m e^b{}_k \beta^{kl} B_{ln} e_c{}^l + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \tilde{\partial}^m \beta^{kl} B_{ln} e_c{}^n \\
&\quad + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \beta^{kl} \tilde{\partial}^m B_{ln} e_c{}^n \\
&= e^a{}_i e_c{}^n \left(\tilde{\partial}^i e^b{}_n + \beta^{im} \partial_m e^b{}_n + \beta^{ij} B_{jm} \tilde{\partial}^m e^b{}_n + \beta^{ij} B_{jm} \tilde{\partial}^m e^b{}_k \beta^{kl} B_{ln} \right) \\
&\quad + e^a{}_i e^b{}_j e_c{}^k \left(\beta^{im} \beta^{jk} \partial_m B_{kn} + \beta^{jl} \tilde{\partial}^i B_{ln} \right) \\
&= e^a{}_i e^b{}_j e_c{}^k \beta^{jl} \tilde{D}^i B_{lk} + e^a{}_i e_c{}^n \left(\tilde{D}^i e^b{}_n + \beta^{ij} B_{jm} (\tilde{\partial}^m e^b{}_k \beta^{kl} B_{ln}) \right) \ ,
\end{aligned} \tag{E.19}$$

where $\tilde{D}^i \equiv \tilde{\partial}^i + \beta^{ij}\partial_j$. With this, the Q flux is given by

$$\begin{aligned}
Q_a{}^{bc} &= \Omega_a{}^{bc} + \Omega^c{}_a{}^b + \Omega^{bc}{}_a \\
&= \Omega_a{}^{bc} - \Omega^{bc}{}_a - \Omega^{cb}{}_a \\
&= e_a{}^i e^b{}_j e^c{}_k \left(D_i\beta^{jk} + \beta^{jm} D_i B_{mn}\beta^{kn} + \beta^{kl}\tilde{D}^j B_{li} - \beta^{jl}\tilde{D}^k B_{li} \right) \\
&\quad + e^b{}_j e_a{}^i \left(\tilde{D}^j e^c{}_i + \beta^{jp} B_{pm}(\tilde{\partial}^m e^c{}_i + \tilde{\partial}^m e^c{}_k \beta^{kl} B_{li}) \right) \\
&\quad - e^c{}_j e_a{}^i \left(\tilde{D}^j e^b{}_i + \beta^{jp} B_{pm}(\tilde{\partial}^m e^b{}_i + \tilde{\partial}^m e^c{}_k \beta^{kl} B_{li} \right) \\
&= e_a{}^i e^b{}_j e^c{}_k \left(D_i\beta^{jk} + \beta^{jm}\beta^{kn} D_i B_{mn} + 2\beta^{[k|l}\tilde{D}^{|j]} B_{li} \right) \\
&\quad + 2e_a{}^i \left(e^{[b}{}_j \tilde{D}^j e^{c]}{}_i + e^{[b}{}_j \beta^{jp} B_{pm}\tilde{\partial}^m e^{c]}{}_i + e^{[b}{}_j \beta^{jp} B_{pm}\tilde{\partial}^m e^{c]}{}_k \beta^{kl} B_{li} \right) \ .
\end{aligned} \tag{E.20}$$

Lastly, we turn to the R-fluxes defined by

$$R^{abc} = 3\Omega^{[abc]} \ . \tag{E.21}$$

The single component is calculated to

$$\begin{aligned}
\Omega^{abc} &= E^{aM}\partial_M E^{bN} E^c{}_n \\
&= E^{am}\partial_m E^{bn} E^c{}_n + E^a{}_m \tilde{\partial}^m E^{bn} E^c{}_n + E^{am}\partial_m E^b{}_n E^{cn} + E^a{}_m \tilde{\partial}^m E^b{}_n E^{cn} \\
&= e^a{}_i \beta^{im}\partial_m (e^b{}_j \beta^{jn})(e^c{}_n + e^c{}_l \beta^{lp} B_{pn} \\
&\quad + (e^a{}_m + e^a{}_i \beta^{ij} B_{jm})\tilde{\partial}^m (e^b{}_k \beta^{kn})(e^c{}_n + e^c{}_l \beta^{lp} B_{pn}) \\
&\quad + e^a{}_i \beta^{im}\partial_m (e^b{}_n + e^b{}_j \beta^{jk} B_{kn}) e^c{}_l \beta^{ln} \\
&\quad + (e^a{}_m + e^a{}_i \beta^{ij} B_{jm})\tilde{\partial}^m (e^b{}_n + e^b{}_k \beta^{kl} B_{ln}) e^c{}_p \beta^{pn} \\
&= e^a{}_i \beta^{im}\partial_m e^b{}_j \beta^{jn} e^c{}_n + e^a{}_i \beta^{im} e^b{}_j \partial_m \beta^{jn} e^c{}_n + e^a{}_i \beta^{im}\partial_m e^b{}_j \beta^{jn} e^c{}_k \beta^{kl} B_{ln} \\
&\quad + e^a{}_i \beta^{im} e^b{}_j \partial_m \beta^{jn} e^c{}_k \beta^{kl} B_{ln} + e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_n + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_n \\
&\quad + e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_l \beta^{lp} B_{pn} + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_l \beta^{lp} B_{pn} + e^a{}_i \beta^{ij} B_{jm}\tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_n \\
&\quad + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_n + e^a{}_i \beta^{ij} B_{jm}\tilde{\partial}^m e^b{}_k \beta^{kn} e^c{}_l \beta^{lp} B_{pn} \\
&\quad + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \tilde{\partial}^m \beta^{kn} e^c{}_l \beta^{lp} B_{pn} + e^a{}_i \beta^{im}\partial_m e^b{}_j e^c{}_l \beta^{ln} + e^a{}_i \beta^{im}\partial_m e^b{}_j \beta^{jk} B_{kn} e^c{}_l \beta^{ln} \\
&\quad + e^a{}_i \beta^{im} B_{jm}\partial_m \beta^{jk} B_{kn} e^c{}_l \beta^{ln} + e^a{}_i \beta^{im} e^b{}_j \beta^{jk}\partial_m B_{kn} e^c{}_l \beta^{ln} + e^a{}_m \tilde{\partial}^m e^b{}_n e^c{}_p \beta^{pn} \\
&\quad + e^a{}_m \tilde{\partial}^m e^b{}_k \beta^{kl} B_{ln} e^c{}_p \beta^{pn} + e^a{}_m e^b{}_k \tilde{\partial}^m \beta^{kl} B_{ln} e^c{}_p \beta^{pn} + e^a{}_m e^b{}_k \beta^{kl}\tilde{\partial}^m B_{ln} e^c{}_p \beta^{pn} \\
&\quad + e^a{}_i \beta^{ij} B_{jm}\tilde{\partial}^m e^b{}_n e^c{}_p \beta^{pn} + e^a{}_i \beta^{ij} B_{jm}\tilde{\partial}^m e^b{}_k \beta^{kl} B_{ln} e^c{}_p \beta^{pn} + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \tilde{\partial}^m \beta^{kl} B_{ln} e^c{}_p \beta^{pn} \\
&\quad + e^a{}_i \beta^{ij} B_{jm} e^b{}_k \beta^{kl}\tilde{\partial}^m B_{ln} e^c{}_p \beta^{pn} \\
&= e^a{}_i e^b{}_j e^c{}_k \Big(\tilde{\partial}^i \beta^{jk} + \beta^{im}\partial_m \beta^{jk} + \beta^{il} B_{lm}\tilde{\partial}^l \beta^{jk} + \beta^{im}\beta^{jl}\beta^{kn}\partial_m B_{ln} + \beta^{jl}\beta^{kn}\tilde{\partial}^i B_{ln} \\
&\quad + \beta^{im}\beta^{jl}\beta^{kn} B_{mp}\tilde{\partial}^b B_{ln} \Big) \\
&= e^a{}_i e^b{}_j e^c{}_k \left(\tilde{D}^i \beta^{jk} + \beta^{im} B_{ml}\tilde{\partial}^l \beta^{jk} + \beta^{im}\beta^{jl}\beta^{kn} D_m B_{ln} + \beta^{jl}\beta^{kn}\tilde{\partial}^i B_{ln} \right) \ .
\end{aligned} \tag{E.22}$$

Hence, we have

$$\begin{aligned}
R^{abc} &= 3e^a{}_i e^b{}_j e^c{}_k \left(\tilde{D}^{[i}\beta^{jk]} + \beta^{[i|m} B_{ml}\tilde{\partial}^l \beta^{|jk]} + \beta^{im}\beta^{jl}\beta^{kn} D_{[m} B_{ln]} + \beta^{[jl}\beta^{k]n}\tilde{\partial}^i B_{ln} \right) \\
&= 3e^a{}_i e^b{}_j e^c{}_k \left(\tilde{D}^{[i}\beta^{jk]} + \beta^{[i|m} B_{ml}\tilde{\partial}^l \beta^{|jk]} + \tfrac{1}{3}\beta^{im}\beta^{jl}\beta^{kn} H_{mln} + \beta^{[jl}\beta^{k]n}\tilde{\partial}^i B_{ln} \right) \ .
\end{aligned} \tag{E.23}$$

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