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Two-spinors and Symmetry Operators

An Investigation into the Existence of Symmetry Operators for the Massive Dirac Equation using Spinor Techniques and Computer Algebra Tools

Master's thesis in Physics

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DEPARTMENT OF MATHEMATICAL SCIENCES

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Abstract

This thesis employs spinor techniques to find what conditions a curved spacetime must satisfy for there to exist a second order symmetry operator for the massive Dirac equation. Conditions are of the form of the existence of a set of Killing spinors satisfying some set of covariant differential equations. These conditions then describe the most general form of such an operator. Partial results for the zeroth and first order are presented and interpreted as well. Computer algebra tools from the Mathematica package suite *xAct* are used to perform decompositions and simplifications of expressions involving spinors.

Keywords: symmetry operator, dirac equation, spin 1/2, classical quantum gravity, Killing spinor, two-spinor, spinor algebra, computer algebra, Mathematica, xAct.

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Simon Stefanus Jacobsson, Gothenburg, 2021

Contents

1
Introduction

General relativity is a geometrically and analytically rich field of physics. Explicit solutions to Einstein's field equations and their stability are actively researched and each of these solutions is a different spacetime manifold with its own physical phenomena to be explored. There are also more general questions being researched, such as the *cosmic censorship hypothesis*, relating to if general relativity is a deterministic theory, and the *Penrose inequality*, a theorem expressing a lower bound for the total mass of a spacetime in terms of the total area of its black holes [14].

While numerical computations have a long history of being applied in general relativity and in physics generally [11], modern developments of computer algebra systems (CAS) tackle more abstract problems by automating not only things like addition and multiplication, but manipulation of symbolic expressions [8]. This lets computers be applied to solving physics problems more theoretical in nature [17, 16].

With this thesis, I aim to explore the possibilities of applying CAS to a complex problem in general relativity.

This thesis is a continuation of the work done by the thesis supervisor Thomas Bäckdahl and co-authors in the article Second Order Symmetry Operators [3], where they examine what conditions must be satisfied by a four-dimensional Lorentzian spacetime for there to exist second order symmetry operators on the solution space of the Dirac–Weyl equation. The conditions found are expressed as the existence of Killing tensors or Killing spinors satisfying some equation relating the spacetime geometry to the solution space geometry. Here, I present similar conditions for the existence of up to second order symmetry operators on the solution space of the massive Dirac equation.

Killing tensors and *Killing spinors* are generalizations of Killing vectors, which are vector fields that preserve the metric, and so they are generators of the Lie algebra of isometries on a manifold. Hence, they are closely related to the symmetries of the space they live on.

A *symmetry operator* is a linear differential operator mapping solutions to solutions of a differential equation. Such operators can be very useful for detailed studies of the solutions of the differential equation. However, the existence of such operators is not trivial and is linked to the existence of different kinds of symmetries of the curved spacetime geometry the differential equation is defined on. Among other things, they can relate to conserved quantities through Noether's theorem. They can also relate to when separation of variables is possible for a differential equation.

To the best of my knowledge, the conditions I derive here for the existence of a second order symmetry operator are novel, while the conditions I derive for the existence of a first order symmetry operator have not been presented in frame-independent formalism before.

Spinor techniques are used abundantly throughout this thesis. The spacetimes under study are $(3 + 1)$-dimensional Lorentzian and they allow a spin structure. This spin structure may be used to decompose tensors into irreducible components, which encode information like symmetry and trace. These decompositions then help examine what conditions must be satisfied by the spacetime for there to exist symmetry operators.

It is in general a time-consuming and nontrivial task to find these irreducible decompositions. Thus, for this task, CAS such as Mathematica package *SymManipulator* [6] have been developed. While there is considerable of power in basic Mathematica, SymManipulator lets the user explicitly indicate the symmetries present in the tensor indices and will be more mindful of what symmetries composed tensors have. Also, in general, when using CAS to perform calculations, users can work much faster and at higher abstraction levels.

The calculations done in this thesis fall under the label "classical quantum gravity" in the sense that these are quantum mechanical equations considered in a curved spacetime. However, the spacetime is not coupled to the fields, as a consistent theory of quantum gravity is famously nowhere near being solved. But by considering these fields on an uncoupled spacetime, we will still get accurate results, since the gravitational forces of a single particle are often negligible.

2

Mathematical preliminaries

In this chapter, the mathematics necessary for understanding the methods and results are presented. Its prerequisites are mainly abstract linear algebra, a bit of differential geometry, and some familiarity with index notation.

Sections 2.2 and 2.3 give a brief explanation of Lorentzian and spinorial tensors. Tensors are an essential tool for dealing with general relativity and it is to great advantage that we can work with them abstractly, without reference to a basis.

Section 2.4 shows some of the common geometric tools used in general relativity. Most importantly, the covariant (spinor) derivative is defined.

Section 2.5 presents some basic representation theory tools. It also presents a result, theorem 15, which is used thoroughly in the rest of the thesis, putting tight constraints on what decompositions of spinorial tensors may look like. This theorem is then applied to decompose the spinorial Riemann tensor.

Lastly, in section 2.6, fundamental derivatives are presented. They are the irreducible parts of the covariant spinor derivative. They are also the first subject presented that is not standard textbook material

2.1 Notation

2.1.1 Abstract indices

The index notation used in relativistic physics builds on Einstein's summation convention: that repeated indices are summed over. So if $v \in V$ is a vector with components v_α in an orthonormal basis B and $L\colon V \to V$ is a linear map with components ${L^\alpha}_\beta$ in B, then $L(v)$ has components

$$\sum_\alpha {L^\alpha}_\beta v_\alpha,$$

or, for short,

$${L^\alpha}_\beta v_\alpha.$$

The drawbacks of this notation is that it is basis dependent. We will later, in sections 2.2.1 and 2.3.1, define tensors and spinors[1] more generally in terms of abstract linear spaces, so we do not want our calculations to require a basis. In this thesis, I will hence use *abstract indices*, where indices no longer denote components,

[1]The following terms will be defined properly there: tensor, spinor, spinorial tensor, valence, conjugate space.

but rather valence. If $v_a \in V$ is a vector and $L^a{}_b \colon V \to V$ is a linear map, then the action of $L^a{}_b$ on v_a is still written

$$L^a{}_b v_a,$$

but the contraction does not mean summation anymore. It just means *action*.

The advantage of abstract indices over not having any indices at all is both that the valence of tensors is always visible and, more importantly, that products between higher valence tensors can be performed in several ways, indistinguishable without indices. For example, try to concisely convey the difference between $T_{ab}{}^p U_{cdp}$ and $T_a{}^{pq} U_{bpq}$ without using indices!

Latin lowercase letters are used for tensor indices. Latin uppercase letters are used for spinor indices in the base space and primed Latin uppercase letters for the conjugate space. Example:

$$\psi^{AB}{}_{CD'}$$

is a valence $(2, 1; 0, 1)$ spinorial tensor.

The font `mathfrak` is used to denote compound indices. So if $\mathfrak{A} = (A_1, \ldots, A_k)$, $\mathfrak{B} = (B_1, \ldots, B_l)$, and $\mathfrak{C}' = (C'_1, \ldots, C'_{m'})$, then

$$\psi^{\mathfrak{A}}{}_{\mathfrak{B}}{}^{\mathfrak{C}'}$$

is a valence $(k, l; m', 0)$ spinorial tensor. The space of such spinorial tensors is profitably written

$$\mathbb{C}^{\mathfrak{A}}{}_{\mathfrak{B}}{}^{\mathfrak{C}'}.$$

2.1.2 Sources and further reading

In [27, ch. 2], Penrose and Rindler axiomatically explain abstract index notation.

2.2 Tensors

If you ask a physics professor what a tensor is, they will tell you “it is something that transforms like a tensor”. In this section I will attempt to give a more rigorous and, importantly, basis-independent definition of a tensor.

2.2.1 What is a tensor?

A very general class of quantities in physics is vectors and a very general class of maps in physics is *multilinear maps*, mapping ordered sets of vectors to other ordered sets of vectors linearly in each argument. Tensor (or outer) products naturally identify multilinear maps defined on Cartesian product spaces with linear maps defined on tensor product spaces. The following definition captures this in a universal property.

Definition 1. Let U, V, and T be vector spaces over a field[2] R and let $\pi\colon U \times V \to T$ be R-bilinear maps. Then π has (or T has) the *universal property for tensor products* if, for any R-vector space W and R-bilinear map $f\colon U \times V \to W$, there exists a unique R-linear map $g\colon T \to W$ such that $f = g \circ \pi$.

If two R-vector spaces both have the universal property for tensor products for U and V, then it follows that they are isomorphic.

It can also be shown that $U \otimes V$, defined as $F(U \times V)/\!\sim$ with $F(U \times V)$ being the free module generated by $U \times V$ and $\sim$ being the equivalence relation $(u, v) + (u', v) \sim (u + u', v)$, $r(u, v) \sim (ru, v) \sim (u, rv)$, indeed has the universal property for tensor products. The tensor product is also associative, and so may be chained together.

Furthermore, we may speak about symmetric and antisymmetric bilinear maps. If the word *bilinear* in definition 1 is replaced by *symmetric bilinear* we get the symmetric tensor product $U \odot U$, and if it is replaced by *antisymmetric bilinear* we get the antisymmetric tensor product $U \wedge U$.

A valence (k, l) *tensor* is multilinear map

$$f\colon \underbrace{V \times \cdots \times V}_{\times l} \to \underbrace{V \times \cdots \times V}_{\times k}$$

or, equivalently, a multilinear map[3]

$$f\colon \underbrace{V \times \cdots \times V}_{\times l} \times \underbrace{V^* \times \cdots \times V^*}_{\times k} \to R.$$

Here V^* is the dual space of V. Hence tensors are naturally identified with elements of tensor product spaces:

$$\begin{aligned}
&\{\, f\colon \underbrace{V \times \cdots \times V}_{\times l} \times \underbrace{V^* \times \cdots \times V^*}_{\times k} \to R \quad \text{s.th.} \quad f \text{ is multilinear} \,\} \\
\sim\ &\{\, f\colon \underbrace{V \otimes \cdots \otimes V}_{\times l} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\times k} \to R \quad \text{s.th.} \quad f \text{ is linear} \,\} \\
=\ &\Big(\underbrace{V \otimes \cdots \otimes V}_{\times k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\times l}\Big)^* \\
\sim\ &\underbrace{V^* \otimes \cdots \otimes V^*}_{\times k} \otimes \underbrace{V \otimes \cdots \otimes V}_{\times l}.
\end{aligned} \tag{2.1}$$

[2]The word field is being used here in the algebraic sense of “a commutative division ring” rather than in the physics sense of “a map from spacetime”.

[3]To see the isomorphism between these two classes of multilinear maps, one can use the argument of partial application discussed shortly, along with the identity $(V^*)^* = V$. For the special case of a map $f\colon V \to V$, the argument goes like this. Map $f\colon V \to V$ to the multilinear $f'\colon V \times V^* \to R$ by $f'(v, u^*) = u^*(f(v))$. Then map $f'\colon V \times V^* \to R$ to $f''\colon V \to (V^*)^*$ by $f''(v) = (f'(v, \cdot))$. But since $(V^*)^* \sim V$, $f''\colon V \to V$. The kernels of each of the two maps $f \mapsto f'$ and $f' \mapsto f''$ are clearly 0 and they are both clearly linear, so each of them must be an isomorphism.

Here $\sim$ is an isomorphism of vector spaces. The last step is not hard to believe, and it is not difficult to prove[4]. Hence the class of tensors is a realization of a very general class of maps of physical quantities as linear maps.

A way to view contractions of higher valence tensors is that they are "partial application" of a tensor viewed as a multilinear function. As an example, consider, in abstract index notation, a valence $(1, 2)$ tensor $X^a{}_{bc}\colon V \times V \to V$ and a valence $(2, 0)$ tensor $Y^{ab}\colon 0 \to V \times V$. Acting with $X^a{}_{bc}$ on Y^{ab}, written

$$X^a{}_{bc} Y^{bc},$$

yields a valence $(1, 0)$ tensor (vector in V). But by only providing the first argument, $X^a{}_{bc}$ can be seen as a map $: V \to \mathrm{Hom}(V, V)$. Applying $X^a{}_{bc}\colon V \to \mathrm{Hom}(V, V)$ to the first part of Y^{ab}, written

$$X^a{}_{bc} Y^{cd},$$

yields a valence $(2, 1)$ tensor.

Tensor products provide a bridge between these two views by telling us that the maps $X^a{}_{bc}\colon V \times V \to V$ and $X^a{}_{bc}\colon V \to \mathrm{Hom}(V, V)$ are the same element in $V^* \otimes V^* \otimes V$. Partial application used in this way is the definition of contraction when we are working with abstract indices.

A *metric* is an invertible symmetric linear map g_{ab} from a vector space to its dual space. Since the metric is symmetric, it is diagonalizable. *Minkowsky space* is a real vector space equipped with a metric that either has one positive and three negative or three positive and one negative eigenvalues, depending on convention. It is often written $\mathbb{R}^{1+3}$ or $\mathbb{R}^{3+1}$. It is the tangent space to spacetime. Hence V is in general relativity often Minkowsky space. The group of isometries of Minkowsky space is called the *Lorentz group* and tensors on it are called *Lorentz tensors*. We say that Lorentz tensors *transform under* the Lorentz group. This will later be contrasted with *spinorial tensors*, that act on a different space and transform under a different isometry group.

All but a handful of equations in this thesis are relations between Lorentzian or spinorial tensors.

2.2.2 Sources and further reading

A longer discussion of this definition of tensor products in terms of a universal property and how it relates to other definitions can be found in Hjalmar Rosengren's lecture notes [28, ch. 4] from his course in representation theory at Gothenburg University. The statements in the paragraph following definition 1 are discussed and proved there also.

[4]The idea is that $\phi\colon V^* \otimes W^* \to (V \otimes W)^*$ defined by $\phi(f \otimes g)(v \otimes w) = f(v)g(w)$ is an isomorphism (for finite-dimensional spaces) [1].

2.3 Spinors

Spinors are a fundamental concept in quantum field theory. The Dirac equation for example, can't be formulated covariantly using Lorentzian tensors. It is only possible with spinorial tensors.

2.3.1 What is a spinor?

Let W be a two-dimensional[5] vector space over $\mathbb{C}$. Then, just as we may define the *dual space* W^* as the space of linear maps from W to $\mathbb{C}$, we may define the *conjugate dual space* $\overline{W}^*$ as the space of antilinear maps from W to $\mathbb{C}$, and the *conjugate space* $\overline{W}$ as $(\overline{W}^*)^*$. A *spinor* is an element of such a space W equipped with an antisymmetric nonzero bilinear map $\epsilon_{AB} \colon W \times W \to \mathbb{C}$ called *spin metric.*

A *valence* $(k, l; k', l')$ *spinorial tensor* is an element of

$$\underbrace{W \otimes \cdots \otimes W}_{\times k} \otimes \underbrace{W^* \otimes \cdots \otimes W^*}_{\times l} \otimes \underbrace{\overline{W} \otimes \cdots \otimes \overline{W}}_{\times k'} \otimes \underbrace{\overline{W}^* \otimes \cdots \otimes \overline{W}^*}_{\times l'}.$$

These objects will later on, when it becomes tedious to say "spinorial tensor", also be called spinors.

Since the space of antisymmetric bilinear maps from $W \times W$ to $\mathbb{C}$ is one-dimensional, requiring that the spin metric ϵ_{AB} is nonzero is the same as requiring it to be invertible. And so we may define ϵ^{AB} to be minus the inverse of ϵ_{AB} seen as a map $W \to W^*$,

$$\epsilon^{AB}\epsilon_{BC} = -\delta^A{}_C. \tag{2.2}$$

ϵ^{AB} is then a valence $(2, 0; 0, 0)$ spinorial tensor. Since ϵ_{AB} is totally antisymmentric, we also may define the determinant

$$\det\{L\} = \frac{1}{2}\epsilon_{AB}\epsilon^{CD}L^A{}_C L^B{}_D. \tag{2.3}$$

With this definition, we now have a concept of $\mathrm{SL}(2, \mathbb{C})$.

For the purpose of establishing the relation between W and the Lorentz group $\mathrm{O}(3, 1, \mathbb{R})$, let o^A and ι^B be a basis for W satisfying

$$o_A \iota^A = 1 \tag{2.4}$$

and define

$$t^{AA'} = \frac{1}{\sqrt{2}}\left(o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}\right) \tag{2.5a}$$

$$x^{AA'} = \frac{1}{\sqrt{2}}\left(o^A\bar{\iota}^{A'} + \iota^A\bar{o}^{A'}\right) \tag{2.5b}$$

$$y^{AA'} = \frac{i}{\sqrt{2}}\left(o^A\bar{\iota}^{A'} - \iota^A\bar{o}^{A'}\right) \tag{2.5c}$$

$$z^{AA'} = \frac{i}{\sqrt{2}}\left(o^A\bar{o}^{A'} - \iota^A\bar{\iota}^{A'}\right). \tag{2.5d}$$

[5] In this section, we are defining *two-spinors.* There exist spinors of other dimensions, but they are not relevant for this thesis.

These comprise a basis of the four-dimensional vector space of valence $(1,0;1,0)$ spinorial tensors.

Each basis vector in (2.5a) to (2.5d) is real in the sense that it is mapped to itself under conjugation. They hence also comprise a basis for a four-dimensional real vector space. Call this space V. The tensor

$$g_{AA'BB'} \coloneqq \epsilon_{AB}\bar{\epsilon}_{A'B'} \tag{2.6}$$

is then a real tensor in $V^* \otimes V^*$. It has signature $+---$ and so V equipped with the metric $g_{AA'BB'}$ is $\mathrm{GL}(1,3,\mathbb{R})$. But if now $L^A{}_B \in \mathrm{SL}(2,\mathbb{C})$ then, by definition, it preserves ϵ_{AB}, and so

$$\lambda^{AA'}{}_{BB'} \coloneqq L^A{}_B \overline{L}^{A'}{}_{B'} \tag{2.7}$$

preserves $g_{AA'BB'}$. But the subset of $\mathrm{GL}(1,3,\mathbb{R})$ preserving the metric is precisely the proper Lorentz group $\mathrm{SO}^+(1,3,\mathbb{R})$. Since $L^A{}_B$ and $-L^A{}_B$ both give the same $\lambda^{AA'}{}_{BB'}$, and since it can be shown that any $\lambda^{AA'}{}_{BB'}$ can be factored into two such $L^A{}_B \in \mathrm{SL}(2,\mathbb{C})$, $\mathrm{SL}(2,\mathbb{C})$ is a double cover of $\mathrm{SO}^+(1,3,\mathbb{R})$.

We may define a tensor-spinor hybrid

$$\sigma^a{}_{AA'} = t^a t_{AA'} + x^a x_{AA'} + y^a y_{AA'} + z^a z_{AA'} \tag{2.8}$$

called the *soldering form*. It is the explicit map between tensors and spinors.

It turns out that the appropriate framework in which to define spinors on curved spacetime is a fiber bundle. A fiber bundle is a topological space that locally looks like a Cartesian product space. The Möbius band is the classic example, looking like $\mathbb{R}^2$ locally but not globally. The following definition makes this precise.

Definition 2. A *fiber bundle* is a collection of three topological spaces, the *total space* E, the *base space* B, and the *fiber* F, along with a continuous *projection* map $\pi\colon E \to B$ such that for each $b \in B$ there exists an open neighbourhood U of b and a homeomorphism $h\colon U \times F \to \pi^{-1}(U)$ on which $\pi \circ h$ is a projection to U.

The fiber bundle is *smooth* if E, B, and F are smooth manifolds, π is a smooth map, and h can be chosen as a diffeomorphism.

The fiber bundle is a *vector bundle* if the fiber F and each $\pi^{-1}(x)$ are vector spaces and if h can be chosen so that each $h(x,\cdot)\colon F \to \pi^{-1}(x)$ is a linear bijection.

Spinor fields are defined formally as sections of fiber bundles that are associated to *principal bundles* defined with $\mathrm{SL}(2,\mathbb{C})$ as their fibers. appendix A presents these concepts in a bit more rigour. But they are not required to understand the rest of the thesis.

2.3.2 Sources and further reading

The information presented here can be found largely in Wald's General Relativity [29, ch. 13]. The definition of fiber bundle comes from Madsen and Tornehave's From Calculus to Cohomology [21, ch. 15].

In appendix B, a more geometric relation between the proper Lorentz group and two-spinors is set up and explored using complex analysis.

2.4 The Riemann curvature tensor

The Riemann curvature tensor relates the commutator of covariant derivatives to the curvature of spacetime. It shows up in Einstein's field equations and later in section 2.6 when commuting fundamental derivatives. In this section, we also define Killing tensors and Killing spinors. They will appear as parts of the conditions for the existence of second order symmetry operators. There, they will bridge the conceptual gap between tensor algebra and spacetime geometry.

2.4.1 The tensor derivative

Let M be a manifold over $R = \mathbb{R}$ and let $\mathfrak{X}(M)$ be the set of vector fields on M.

Definition 3. A *covariant derivative* is an operator

$$\nabla_c \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$$

satisfying

1. linearity, $\nabla_c(aA^{\mathfrak{a}}{}_{\mathfrak{b}} + bB^{\mathfrak{a}}{}_{\mathfrak{b}}) = a\nabla_c A^{\mathfrak{a}}{}_{\mathfrak{b}} + b\nabla_c B^{\mathfrak{a}}{}_{\mathfrak{b}}$
2. Leibniz rule, $\nabla_c(A^{\mathfrak{a}}{}_{\mathfrak{b}} B^{\mathfrak{a}}{}_{\mathfrak{b}}) = \nabla_c(A^{\mathfrak{a}}{}_{\mathfrak{b}})B^{\mathfrak{a}}{}_{\mathfrak{b}} + A^{\mathfrak{a}}{}_{\mathfrak{b}}\nabla_c(B^{\mathfrak{a}}{}_{\mathfrak{b}})$
3. commutes with contraction, $\nabla_c(A^{\mathfrak{a}d}{}_{\mathfrak{b}d}) = \nabla_c A^{\mathfrak{a}d}{}_{\mathfrak{b}d}$
4. consistent with taking the directional derivative on scalars, $\frac{\partial}{\partial s}A(C(s))|_{s=0} = t^c\nabla_c A$, where t^c is a tangent vector of the curve C at $s = 0$
5. no torsion, $\nabla_a\nabla_b A = \nabla_b\nabla_a A$.

Proposition 4. *If ∇_a and ∇'_a are two such covariant derivatives, then their difference*

$$\nabla_a A_b - \nabla'_a A_b$$

at a point $p \in M$ depends only on the value of A_b at p. Specifically, since $A_b \mapsto \nabla_a A_b - \nabla'_a A_b$ is linear, we have that

$$\nabla_a A_b - \nabla'_a A_b = C^c{}_{ab} A_c \tag{2.9}$$

for some tensor $C^c{}_{ab}$.

Proposition 5. *There exists a unique covariant derivative such that*

$$\nabla_a g_{bc} = 0.$$

From now on, we will assume our covariant derivative satisfies this.

Definition 6. A tensor $T^{\mathfrak{a}}{}_{\mathfrak{b}}$ is *parallelly transported* along a curve C with tangent t^a if

$$t^a\nabla_a T^{\mathfrak{a}}{}_{\mathfrak{b}} = 0$$

at each point $p \in C$.

Using item 5 and the same argument as above, we see that $(\nabla_a\nabla_b - \nabla_a\nabla_b)A_c$ too depends only on the value of A_c. We may thus define a tensor ${R_{abc}}^d$ by

$$\nabla_{[b}\nabla_{a]}A_c = {R_{abc}}^d A_d \tag{2.10}$$

called the *Riemann curvature tensor*[6].

The geometric interpretation of the Riemann tensor is that it measures how much a given tensor changes when parallelly transported around an infinitesimal closed curve. Wald [29, sec. 3.2] has a discussion and proof of this.

2.4.2 The spinor derivative

Let M be a manifold over $\mathbb{R}$ with a spin structure and let $\mathfrak{S}(S)$ be the set of spinor fields.

Definition 7. A *covariant spinor derivative* is an operator

$$\nabla_{CC'} \colon \mathfrak{S}(M) \to \mathfrak{S}(M)$$

satisfying items 1 to 5 of definition 3 and

6. reality, $\overline{\nabla_{CC'}{\psi^{\mathfrak{A}}}_{\mathfrak{B}}} = \nabla_{CC'}\overline{{\psi^{\mathfrak{A}}}_{\mathfrak{B}}}$
7. it vanishes when acting on the spin metric, $\nabla_{CC'}\epsilon_{AB} = 0$.

Similarly to proposition 4, we have that there exists a spinor ${\chi_{AA'BB'C}}^D$ such that

$$(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})\psi_C = {\chi_{AA'BB'C}}^D\psi_D \tag{2.11}$$

for all dual spinors ψ_C. By item 6 and the notational fact that the relative order between primed and unprimed indices does not matter,

$$\begin{aligned}(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})\overline{\psi}_{C'} &= \overline{(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})\psi_C} \\ &= {\overline{\chi}_{AA'BB'C'}}^{D'}\overline{\psi}_{D'}.\end{aligned} \tag{2.12}$$

Acting with $(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})$ on a vector $\psi_C\overline{\psi}_{C'}$, we may identify the *spinorial Riemann curvature tensor*:

$$\begin{aligned}(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})\psi_C\overline{\psi}_{C'} &= {\chi_{AA'BB'C}}^D\psi_D\overline{\psi}_{C'} + {\overline{\chi}_{AA'BB'C'}}^{D'}\psi_C\overline{\psi}_{D'} \\ &= \left({\chi_{AA'BB'C}}^D{\overline{\epsilon}_{C'}}^{D'} + {\overline{\chi}_{AA'BB'C'}}^{D'}{\epsilon_C}^D\right)\psi_D\overline{\psi}_{D'}\,, \\ {R_{AA'BB'CC'}}^{DD'} &:= {\chi_{AA'BB'C}}^D{\overline{\epsilon}_{C'}}^{D'} + {\overline{\chi}_{AA'BB'C'}}^{D'}{\epsilon_C}^D.\end{aligned} \tag{2.13}$$

The Riemann tensor and, analogously, the spinorial Riemann tensor have a well-known set of symmetries,

[6] Please do not confuse the Riemann tensor with a tensor product space over a ring R. Even though their notation collide.

- $R_{abcd} = R_{cdab}$
- $R_{abcd} = R_{bacd}$
- $R_{abcd} = R_{abdc}$
- $R_{abcd} + R_{acdb} + R_{adbc} = 0$

that will be derived and used in section 2.5.3.

2.4.3 Einstein's field equations

The geometry of spacetime is described by the metric g_{ab}. *Einstein's field equations*,

$$R_{ab} - \frac{1}{2} g_{ab} R = -8\pi G T_{ab} \,, \tag{2.14}$$

describe the metric given the *stress-energy tensor* T_{ab}. In (2.14), G is the gravitational constant and

$$R \coloneqq R_a{}^a$$

is the *Ricci scalar*, and

$$R_{ab} \coloneqq R^a{}_{bcd}$$

is the *Ricci tensor*, defined here in terms of the Riemann curvature tensor.

The stress-energy tensor T_{ab} is the gravitational source term of (2.14) and hence its interpretation is that it's the matter distribution of the universe.

2.4.4 Killing tensors

If M is a manifold and $\xi^a \colon M \to TM$ is a vector field on M, then the *Lie derivative* at some point $p \in M$ with respect to a vector ξ^a measures the amount that some tensor quantity changes in a given direction compared to how it would change in the direction of ξ^a. More formally,

Definition 8. Let M be a Lie group. Then its tangent space T_pM at a point $p \in M$ is a Lie algebra with a bilinear antisymmetric operation $[\cdot,\cdot] \colon T_pM \times T_pM \to T_pM$ called *Lie bracket*. Let ξ^a, $\eta^b \in T_pM$. The *Lie derivative* of η^a with respect ξ^b is

$$\mathcal{L}_{\xi^a} \eta^b = [\xi^a, \eta^b] \,.$$

We get the Lie derivative of a general tensor over T_pM by demanding that it satisfies the Leibniz rule

$$\mathcal{L}_{\xi^a}(T \bullet U) = (\mathcal{L}_{\xi^a} T) \bullet U + T \bullet (\mathcal{L}_{\xi^a} U)$$

where $\bullet$ is some inner or outer product.

If ∇_a is a connection on M we may define a Lie bracket $[\xi^a, \eta^b] = \xi^a\nabla_a\eta^b - \eta^a\nabla_a\xi^b$. With this Lie bracket, we have

$$\begin{aligned}\mathcal{L}_{\xi^a}\eta^b &= \xi^a\nabla_a\eta^b - \eta^a\nabla_a\xi^b \\ \mathcal{L}_{\xi^a}\eta_b &= \xi^a\nabla_a\eta_b + \eta_a\nabla_b\xi^a.\end{aligned}$$

Definition 9. A *Killing vector* is a vector field K^c such that taking the derivative of the metric with respect to it yields zero,

$$\mathcal{L}_{K^c}g_{ab} = 0. \tag{2.15}$$

What definition 9 says geometrically is that translation in the direction of K^c is an infinitesimal isometry. Writing out (2.15) explicitly in the case where we have a connection, we get

$$\begin{aligned}K^c\nabla_c g_{ab} + g_{ac}\nabla_b K^c + g_{cb}\nabla_a K^c &= 0 \\ \nabla_b K_a + \nabla_a K_b &= 0 \\ \nabla_{(a}K_{b)} &= 0.\end{aligned}$$

And so the following two definitions are natural generalizations,

Definition 10. A totally symmetric tensor $K_{\mathfrak{b}}$ is a *Killing tensor* if

$$\nabla_{(a}K_{\mathfrak{b})} = 0. \tag{2.16}$$

Definition 11. A totally symmetric spinor $S_{\mathfrak{B}}{}^{\mathfrak{B}'}$ is a *Killing spinor* if

$$\nabla_{(A}{}^{(A'}S_{\mathfrak{B})}{}^{\mathfrak{B}')} = 0. \tag{2.17}$$

Another type of geometrical quantitity of interest is Killing–Yano tensors. They can be used to construct Killing tensors of valence $(2,0)$ and sometimes they are easier to find than the Killing tensors they correspond to.

Definition 12. A totally antisymmetric tensor $f_{\mathfrak{b}}$ is a *Killing–Yano tensor* if

$$\nabla_{(a}f_{b_0)b_1\ldots b_n} = 0. \tag{2.18}$$

Killing–Yano tensors can be thought of as square roots of valence $(2,0)$ Killing tensors in the following sense: $K_{bc} := f_{bb_1\ldots b_n}f_c{}^{b_1\ldots b_n}$ is a Killing tensor. For, consider

$$\begin{aligned}\nabla_a K_{bc} &= \left(\nabla_a f_{bb_1\ldots b_n}\right) f_c{}^{b_1\ldots b_n} + f_{bb_1\ldots b_n}\left(\nabla_a f_c{}^{b_1\ldots b_n}\right) \\ &\overset{(2.18)}{=} \left(\nabla_{[a}f_{b]b_1\ldots b_n}\right) f_c{}^{b_1\ldots b_n} + f_b{}^{b_1\ldots b_n}\left(\nabla_{[a}f_{c]b_1\ldots b_n}\right).\end{aligned}$$

Symmetrizing both sides of this yields (2.16).

2.4.5 Sources and further reading

Many of the arguments presented in sections 2.4.1 and 2.4.2 can be found in Wald's book on general relativity [29, ch. 3 & 13], including proofs of propositions 4 and 5. Some further information about Einstein's field equations may be found in Wald [29, ch. 4] and in Weinberg [30, ch. 7].

Definition 8 is taken from Penrose [26, sec. 14.6]. In Nakahara [24, sec. 5.3], a more explicit definition of the Lie derivative in terms of flows generated by vector fields may be found.

2.5 Representations

Definition 13. A *representation* of a group G is a homomorphism π from G to the general linear group, $\mathrm{GL}(V)$, of some vector space V. V is then a *representation space* of G.

The image $\pi(G)$ is then a subgroup of $\mathrm{GL}(V)$ and we have the usual homomorphism results:

$$\pi(ab) = \pi(a)\pi(b), \quad \pi(g^{-1}) = \pi(g)^{-1}, \quad \pi(1_G) = 1_{\mathrm{GL}(V)} = \mathrm{id}_V \, .$$

It is common to also call V representation. Representations may be defined analogously for associative algebras and Lie algebras.

2.5.1 Reducibility and decomposability

Definition 14. Let π be a representation of a group G and let V be the representation space. A linear subspace $W \subset V$ is *G-invariant* if $\pi(G)W \subset W$. The restriction of the action of G to $\mathrm{GL}(W)$ is then a *subrepresentation* of π. π is *reducible* if it has a nontrivial subrepresentation. Otherwise, *irreducible.*

Explicitly, π is reducible if the matrices $\pi(g)$ can be simultaneously block-upper triangulized,

$$\pi(g) = \begin{bmatrix} \pi_{11}(g) & \pi_{12}(g) \\ 0 & \pi_{22}(g) \end{bmatrix} .$$

π is *decomposable* if $W^\perp$ is a subrepresentation as well. Then the matrices $\pi(g)$ are simultaneously block-diagonalizable.

For finite groups over $\mathbb{C}$ and for semi-simple Lie groups over characteristic 0-fields, the notions of irreducibility and decomposability are equivalent. Some books use the word reducible to mean what I have defined as decomposable and do not care about the difference.

2.5.2 Symmetrization

The different irreducible parts of a representation encode such things as symmetries and trace. Which is why it is an interesting question to ask: what are the irreducible parts of a given representation? Something that will greatly simplify our discussion of this is that, when working over $\mathrm{SL}(2, \mathbb{C})$, the only spinorial tensor that is antisymmetric in more than two indices is 0 and the only spinorial tensor antisymmetric in two indices is ϵ_{AB} and its multiples.

We can *symmetrize* tensors and spinors, written as

$$A_{(ab\ldots z)} = \frac{1}{n!} \sum_{\sigma \in S_n} A_{\sigma(a)\sigma(b)\ldots\sigma(z)} .$$

Such a resulting tensor or spinor is *symmetric*. We may also *antisymmetrize* tensors and spinors, written as

$$A_{[ab\dots z]} = \frac{1}{n!}\sum_{\sigma\in S_n} \operatorname{sgn}\sigma A_{\sigma(a)\sigma(b)\dots\sigma(z)}.$$

Such a resulting tensor or spinor is *antisymmetric*. If we don't want to symmetrize over all the indices, we may exclude some with vertical bars

$$A_{(a|b|c\dots z)} = \frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} A_{\sigma(a)b\sigma(c)\dots\sigma(z)}.$$

Symmetrization and antisymmetrization are idempotents. Symmetrization and antisymmetrization do not necessarily commute, and so expressions like $A_{(a[b)c]}$ do not make sense without context. A tensor or spinor with more than one type of index is *(anti-) symmetric* if it is (anti-) symmetric in each of its index sets.

It is convenient to denote the linear space of valence $(p,q;p',q')$ spinors over a ring R as

$$R^{A_1\dots A_p}{}_{B_1\dots B_q}{}^{A'_1\dots A'_{p'}}{}_{B'_1\dots B'_{q'}},$$

and the linear space of valence $(p,q;p',q')$ spinors over a ring R that are (anti-) symmetric over indices $A_1, \dots, A_l$, indices $B_1, \dots, B_m$, indices $A'_1, \dots, A'_{l'}$, and indices $B'_1, \dots, B'_{m'}$ as

$$R^{(A_1\dots A_l)A_{l+1}\dots A_p}{}_{(B_1\dots B_m)B_{m+1}\dots B_q}{}^{(A'_1\dots A'_{l'})A_{l'+1}\dots A'_{p'}}{}_{(B'_1\dots B'_{m'})B'_{m'+1}\dots B'_{q'}}, \tag{2.19}$$

$$R^{[A_1\dots A_l]A_{l+1}\dots A_p}{}_{[B_1\dots B_m)]B_{m+1}\dots B_q}{}^{[A'_1\dots A'_{l'}]A_{l'+1}\dots A'_{p'}}{}_{[B'_1\dots B'_{m'}]B'_{m'+1}\dots B'_{q'}}. \tag{2.20}$$

We may make this notation more concise by composite indices $\mathfrak{A}_1 = (A_1, \dots, A_l)$, $\mathfrak{A}_2 = (A_{l+1}, \dots, A_p)$, and so on. (2.19) and (2.20) may then be written as

$$R^{(\mathfrak{A}_1)\mathfrak{A}_2}{}_{(\mathfrak{B}_1)\mathfrak{B}_2}{}^{(\mathfrak{A}'_1)\mathfrak{A}'_2}{}_{(\mathfrak{B}'_1)\mathfrak{B}'_2},$$
$$R^{[\mathfrak{A}_1]\mathfrak{A}_2}{}_{[\mathfrak{B}_1]\mathfrak{B}_2}{}^{[\mathfrak{A}'_1]\mathfrak{A}'_2}{}_{[\mathfrak{B}'_1]\mathfrak{B}'_2}.$$

If each linear space R^A has dimension d, then $R^{(A_1\dots A_l)}$ has dimension $\binom{d+l-1}{l}$ and $R^{[A_1\dots A_l]}$ has dimension $\binom{d}{l}$. Especially, if $d=2$, we see why no nonzero spinor can be antisymmetric in more than two indices and the space of two-index antisymmetric spinors is one-dimensional. The spinor ϵ_{AB} is in this context called the *Levi-Civita symbol*. From this follows a very useful theorem.

Theorem 15. *Any spinor $T^{\mathfrak{A}}{}_{\mathfrak{B}}{}^{\mathfrak{A}'}{}_{\mathfrak{B}'}$ is the sum of $T^{(\mathfrak{A})}{}_{(\mathfrak{B})}{}^{(\mathfrak{A}')}{}_{(\mathfrak{B}')}$ and a linear combinations of outer products of symmetric spinors of lower valence with Levi-Civita symbols.*

Proof. Consider a valence $(0,r;0,0)$ spinor, $T_{A\dots F}$. The proof goes analogously for other valences. We have

$$T_{(A\dots F)} = \frac{1}{r}\left(T_{A(B\dots F)} + \dots + T_{F(A\dots E)}\right).$$

The difference between any two such terms, e.g. $T_{A(BC...F)} - T_{B(AC...F)}$, is clearly antisymmetric in A and B, and so is proportional to ϵ_{AB}. Its proportionality constant is found by raising one index and taking the trace. We get

$$T_{A(BC...F)} - T_{B(AC...F)} = -\epsilon_{AB} T_X{}^X{}_{(C...F)}\,, \tag{2.21}$$

which is indeed an "outer product of a symmetric spinor of lower valence and a Levi-Civita symbol". So repeating this for all terms yields that

$$T_{(A...F)} = T_{A(B...F)} + \left\{\begin{matrix}\text{linear combinations of outer products of symmetric}\\ \text{spinors of lower valence and Levi-Civita symbols}\end{matrix}\right\}.$$

This shows the induction step in a proof by induction on r. □

For example, a valence $(0,3;0,0)$ spinor can be decomposed as

$$T_{ABC} = T_{(ABC)} - \frac{1}{3}\epsilon_{AB} T^D{}_{(DC)} - \frac{1}{3}\epsilon_{AC} T^D{}_{(DB)} - \frac{1}{2}\epsilon_{BC} T_A{}^D{}_D\,. \tag{2.22}$$

Each of these subspaces transform independently under spin transformations, symmetric in ABC, antisymmetric in AB, antisymmetric in AC, and antisymmetric in BC. It is also clear that they only have the zero element in common, since only the zero element is antisymmetric in three indices or symmetric and antisymmetric in two indices at once. Other than a subrepresentation being one-dimensional, it is usually difficult to tell whether it is irreducible or not. But it is possible to show that (2.22) is a decomposition of the representation of the spin group composed of valence $(0,3;0,0)$ spinors into irreducible components.

Another way of expressing irreducibility is to say that a spinor is are "saturated with symmetries". If any further symmetries are imposed, then either no information is lost, or the resulting spinor is zero.

2.5.3 Application to the spinorial Riemann tensor

In section 2.4.2, we derived the spinorial Riemann tensor $R_{AA'BB'CC'}{}^{DD'}$. It is clearly antisymmetric under $AA' \longleftrightarrow BB'$, so it may be written as

$$\begin{aligned}
&R_{AA'BB'CC'}{}^{DD'}\\
&= \frac{1}{2}\left(R_{AA'BB'CC'}{}^{DD'} - R_{BB'AA'CC'}{}^{DD'}\right)\\
&= \frac{1}{2}\left(R_{ABA'B'CC'}{}^{DD'} - R_{BAB'A'CC'}{}^{DD'}\right)\\
&= \frac{1}{4}\Big(R_{ABA'B'CC'}{}^{DD'} + R_{ABA'B'CC'}{}^{DD'} + R_{ABB'A'CC'}{}^{DD'} - R_{ABB'A'CC'}{}^{DD'}\\
&\qquad + R_{BAA'B'CC'}{}^{DD'} - R_{BAA'B'CC'}{}^{DD'} - R_{BAB'A'CC'}{}^{DD'} - R_{BAB'A'CC'}{}^{DD'}\Big)\\
&= R_{(AB)[A'B']CC'}{}^{DD'} - R_{[AB](B'A')CC'}{}^{DD'}.
\end{aligned} \tag{2.23}$$

Again, since each of these terms is antisymmetric in two indices we may express them as a trace times the Levi-Civita symbol:

$$R_{(AB)[A'B']CC'}{}^{DD'} = \frac{1}{2} R_{(AB)X'}{}^{X'}{}_{CC'}{}^{DD'} \bar{\epsilon}_{A'B'}$$
$$R_{[AB](B'A')CC'}{}^{DD'} = \frac{1}{2} R_{X}{}^{X}{}_{(B'A')CC'}{}^{DD'} \epsilon_{AB} \ .$$

The spinorial Riemann tensor is also antisymmetric under $CC' \longleftrightarrow DD'$. This is most easily seen by viewing $R_{AA'BB'CC'DD'}$ as an action on the spin metric:

$$\begin{aligned} & R_{AA'BB'CC'DD'} + R_{AA'BB'DD'CC'} \\ &= R_{AA'BB'CC'}{}^{XX'} \epsilon_{XD} \bar{\epsilon}_{X'D'} + R_{AA'BB'DD'}{}^{XX'} \epsilon_{CX} \bar{\epsilon}_{C'X'} \\ &= (\nabla_{AA'} \nabla_{BB'} - \nabla_{BB'} \nabla_{AA'}) \epsilon_{CD} \bar{\epsilon}_{C'D'} \\ &\overset{\substack{\text{item 7 of} \\ \text{definition 7}}}{=} 0. \end{aligned}$$

Hence we may further decompose (2.23) as

$$\begin{aligned} R_{AA'BB'CC'DD'} &= R_{(AB)[A'B'](CD)[C'D']} - R_{(AB)[A'B'][CD](C'D')} \\ &\quad - R_{[AB](A'B')(CD)[C'D']} + R_{[AB](A'B')[CD](C'D')} \\ &= \frac{1}{4} \Big[R_{(AB)X'}{}^{X'}{}_{(CD)Y'}{}^{Y'} \bar{\epsilon}_{A'B'} \bar{\epsilon}_{C'D'} - R_{(AB)X'}{}^{X'}{}_{X}{}^{X}{}_{(C'D')} \bar{\epsilon}_{A'B'} \epsilon_{CD} \\ &\quad - R_{X}{}^{X}{}_{A'B'(CD)X'}{}^{X'} \epsilon_{AB} \bar{\epsilon}_{C'D'} - R_{X}{}^{X}{}_{(A'B')Y}{}^{Y}{}_{(C'D')} \epsilon_{AB} \epsilon_{CD} \Big] \,. \end{aligned} \tag{2.24}$$

It is nice to see that (2.24) is of the form $R + \overline{R}$ since we defined the spinorial tensor to be real.

One last central property of the spinorial Riemann tensor that we want to use is that it is symmetric under $AA'BB' \longleftrightarrow CC'DD'$. To show this, we will use that

$$R_{[abc]d} = 0. \tag{2.25}$$

This follows from

$$\nabla_{[a} \nabla_{b} w_{c]} = 0$$

for any vector w_c, which is true by item 5 of definition 3. For, if we work in tensor indices, then (2.25) implies

$$\begin{aligned} R_{abcd} - R_{cdab} &= R_{abcd} + R_{acbd} - R_{acbd} + R_{dcba} - R_{dcba} - R_{cdab} \\ &= R_{abcd} + R_{acbd} + R_{acdb} + R_{dcba} + R_{dbca} + R_{dcab} \\ &= \frac{1}{2} [R_{abcd} + R_{acbd} + R_{acdb} - R_{abdc} - R_{acdb} - R_{acbd} \\ &\quad - R_{dbca} + R_{dcab} + R_{dcba} - R_{dcab} - R_{dbac} - R_{dcba}] \\ &= 0. \end{aligned}$$

In (2.24), we now have two kind of terms, $T_{(AB)(CD)}$ and $T_{(AB)(C'D')}$, the latter not being further reducible. But by the above mentioned symmetry, we may decompose the former some more. We do this by using the decomposition $T_{AB} = \frac{1}{2}T_X{}^X\epsilon_{AB} + T_{(AB)}$ of a 2-spinor into a symmetric and antisymmetric part. $\epsilon^{AC}R_{(AB)X'}{}^{X'}{}_{(CD)Y'}{}^{Y'}$ is antisymmetric in BD and so is proportional to ϵ_{BD}. $\epsilon^{BC}R_{(AB)X'}{}^{X'}{}_{(CD)Y'}{}^{Y'}$ is similarly proportional to ϵ_{AD}. The final term we want to consider in $R_{(AB)X'}{}^{X'}{}_{(CD)Y'}{}^{Y'}$ is then symmetric in AB, CD, and BD, and so it is symmetric in $ABCD$. In conclusion,

$$R_{(AB)X'}{}^{X'}{}_{(CD)Y'}{}^{Y'} = R_{(ABCD)X'}{}^{X'}{}_{Y'}{}^{Y'} + \frac{1}{6}R_X{}^Y{}_{X'}{}^{X'}{}_Y{}^X{}_{Y'}{}^{Y'}\left(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}\right)$$

and we may finally write down the full result

$$\begin{aligned} &R_{AA'BB'CC'DD'} \\ &= \frac{1}{4}\Bigg[R_{(ABCD)X'}{}^{X'}{}_{Y'}{}^{Y'}\epsilon_{A'B'}\epsilon_{C'D'} + \frac{1}{6}R_X{}^Y{}_{X'}{}^{X'}{}_Y{}^X{}_{Y'}{}^{Y'}\left(\epsilon_{AC}\epsilon_{BD} + \epsilon_{BC}\epsilon_{AD}\right)\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} \\ &\quad - R_{(AB)X'}{}^{X'}{}_X{}^X{}_{(C'D')}\bar{\epsilon}_{A'B'}\epsilon_{CD}\Bigg] + \text{complex conjugate.} \end{aligned} \tag{2.26}$$

This is in the form of theorem 15 that we wanted to achieve.

These different irreducible parts of the spinorial Riemann tensor all have direct geometrical interpretations themselves. The first part, $\Psi_{ABCD} = \frac{1}{4}R_{(ABCD)X'}{}^{X'}{}_{Y'}{}^{Y'}$, is called *Weyl spinor*. The second part, $\Lambda = \frac{1}{24}R_X{}^Y{}_{X'}{}^{X'}{}_Y{}^X{}_{Y'}{}^{Y'}$, is called *Ricci scalar*. The third part, $\Phi = \frac{1}{4}R_{(AB)X'}{}^{X'}{}_X{}^X{}_{(C'D')}$, is called *Ricci spinor*. They will show in something called *curvature terms*, introduced in section 2.6.2.

2.5.4 Sources and further reading

Fuchs and Schweigert [15, ch. 5] has a concise and clear description of the representation theory of Lie algebras while Penrose and Rindler [27, sec. 3.3] has a thorough description symmetrization. Section 2.5.3 is largely based on Wald [29, sec. 13.2 p. 370–371].

2.6 Fundamental derivatives

The fundamental derivatives are four covariant operators that map symmetric spinors to symmetric spinors. They are the irreducible parts of the covariant spinor derivative, which is why we, after this, will use them to express our covariant differential equations.

2.6.1 What is a fundamental derivative?

Definition 16. If $\psi_{A_1\ldots A_k}{}^{A'_1\ldots A'_l}$ is a totally symmetric valence $(0,k;l,0)$ spinor field, then we have four *fundamental derivatives*: the *divergence* $\mathscr{D}$ which maps symmetric valence $(0,k;l,0)$ spinor fields to symmetric valence $(0,k-1;l-1,0)$ spinor fields by

$$(\mathscr{D}\psi)_{A_1\ldots A_{k-1}}{}^{A'_1\ldots A'_{l-1}} = \nabla^B{}_{B'}\psi_{A_1\ldots A_{k-1}B}{}^{A'_1\ldots A'_{l-1}B'}, \tag{2.27a}$$

the *curl* $\mathscr{C}$ which maps symmetric valence $(0,k;l,0)$ spinor fields to symmetric valence $(0,k-1;l+1,0)$ spinor fields by

$$(\mathscr{C}\psi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}} = \nabla_{(A_1|B'|}\psi_{A_2\dots A_{k+1})}{}^{A'_1\dots A'_{l-1}B'}, \tag{2.27b}$$

the *curl-dagger* $\mathscr{C}^\dagger$ which maps symmetric valence $(0,k;l,0)$ spinor fields to symmetric valence $(0,k+1;l-1,0)$ spinor fields by

$$(\mathscr{C}^\dagger\psi)_{A_1\dots A_{k-1}}{}^{A'_1\dots A'_{l+1}} = \nabla^{B(A'_1}\psi_{A_1\dots A_{k-1}B}{}^{A'_2\dots A'_{l+1})}, \tag{2.27c}$$

and the *twist* $\mathscr{T}$ which maps symmetric valence $(0,k;l,0)$ spinor fields to symmetric valence $(0,k+1;l+1,0)$ spinor fields by

$$(\mathscr{T}\psi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l+1}} = \nabla_{(A_1}{}^{(A'_1}\psi_{A_2\dots A_{k+1})}{}^{A'_2\dots A'_{l+1})}. \tag{2.27d}$$

Notice that these operators map from a space of symmetric spinor fields to another space of symmetric spinor fields, and so they can be composed with each other. These operators are used to decompose the covariant derivative into irreducible parts by to the following proposition.

Proposition 17. *Let* $\psi_{A_1\dots A_k}{}^{A'_1\dots A'_l}$ *be totally symmetric. Then*

$$\begin{aligned}\nabla_{A_1}{}^{A'_1}\psi_{A_2\dots A_{k+1}}{}^{A'_2\dots A'_{l+1}} &= (\mathscr{T}\psi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l+1}} \\ &\quad - \frac{l}{l+1}\bar{\epsilon}^{A'_1(A'_2}(\mathscr{C}\psi)_{A_1\dots A_{k+1}}{}^{A'_3\dots A'_{l+1})} \\ &\quad - \frac{k}{k+1}\epsilon_{A_1(A_2}(\mathscr{C}^\dagger\psi)_{A_3\dots A_{k+1})}{}^{A'_1\dots A'_{k+1}} \\ &\quad + \frac{kl}{(k+1)(l+1)}\epsilon_{A_1(A_2}\bar{\epsilon}^{A'_1(A'_2}(\mathscr{D}\psi)_{A_3\dots A_{k+1})}{}^{A'_3\dots A'_{l+1})}.\end{aligned} \tag{2.28}$$

Proof. This is of the form of theorem 15, so all we need to do is to verify the coefficients. Since $\psi_{A_1\dots A_k}{}^{A'_1\dots A'_l}$ is totally symmetric, it is clear that no ϵ with both indices in $\{A_2,\dots,A_{k+1}\}$ or $\bar{\epsilon}$ with both indices in $\{A'_2,\dots,A'_{l+1}\}$ should appear. Now, as in theorem 15, consider

$$\begin{aligned}&\nabla_{(A_1}{}^{(A'_1}\psi_{A_2\dots A_{k+1})}{}^{A'_2\dots A'_{l+1})} \\ &= \frac{1}{k+1}\left(\nabla_{A_1}{}^{(A'_1}\psi_{(A_2\dots A_{k+1})}{}^{A'_2\dots A'_{l+1})} + \dots + \nabla_{A_{k+1}}{}^{(A'_1}\psi_{(A_2\dots A_k)}{}^{A'_2\dots A'_{l+1})}\right) \\ &\overset{(2.21)}{=} \nabla_{A_1}{}^{(A'_1}\psi_{(A_2\dots A_{k+1})}{}^{A'_2\dots A'_{l+1})} - \frac{k}{k+1}\epsilon_{A_1(A_2}\nabla^{B(A'_1}\psi_{|B|A_2\dots A_k)}{}^{A'_2\dots A'_{l+1})}.\end{aligned}$$

A similar expansion in the primed indices yields

$$\begin{aligned}
&\nabla_{A_1}{}^{A'_1}\psi_{A_2\ldots A_{k+1}}{}^{A'_2\ldots A'_{l+1}}\\
&= \nabla_{(A_1}{}^{(A'_1}\psi_{A_2\ldots A_{k+1})}{}^{A'_2\ldots A'_{l+1})}\\
&\quad - \frac{l}{l+1}\bar{\epsilon}^{A'_1(A'_2}\nabla_{(A_1|B'|}\psi_{A_2\ldots A_{k+1})}{}^{B'(A'_3\ldots A'_{l+1})}\\
&\quad - \frac{k}{k+1}\epsilon_{A_1(A_2}\nabla^{B(A'_1}\psi_{|B|A_3\ldots A_{k+1})}{}^{A'_2\ldots A'_{l+1})}\\
&\quad + \frac{kl}{(k+1)(l+1)}\epsilon_{A_1(A_2}\bar{\epsilon}^{A'_1(A'_2}\nabla^{|B}{}_{|B'}\psi_{B|A_3\ldots A_k)}{}^{B'|A'_3\ldots A'_{l+1})}.
\end{aligned} \tag{2.29}$$

If you squint for long enough at definition 16, you can see that (2.28) and (2.29) are the same statement. □

The *Bianchi identity*, $\nabla_{[a}R_{bc]de}$, is another symmetry of the Riemann tensor. It follows from antisymmetry of the first two indices and item 5 of definition 3. See [27, sec. 4.2] for details. The spinorial Bianchi identity may be formulated in terms of fundamental derivatives.

Lemma 18. *The Bianchi identity for the spinorial Riemann tensor can be written*

$$(\mathscr{D}\Phi)_A{}^{A'} = -3(\mathscr{T}\Lambda)_A{}^{A'}, \tag{2.30}$$

$$(\mathscr{C}^\dagger\Psi)_{ABC}{}^{A'} = (\mathscr{C}\Phi)_{ABC}{}^{A'}. \tag{2.31}$$

We will use this identity along with its complex conjugate to simplify and canonicalize expressions containing derivatives of the spinorial Riemann tensor.

Another observation that later will form the bridge between the Dirac equation solution space geometry and spacetime geometry is that definition 11 may be reformulated as

Proposition 19. *A totally symmetric valence $(0,k;l,0)$ spinor $S_{\mathfrak{B}}{}^{\mathfrak{B}'}$ is a Killing spinor if and only if*

$$(\mathscr{T}S)_{A\mathfrak{B}}{}^{A'\mathfrak{B}'} = 0. \tag{2.32}$$

2.6.2 Commutators

Definition 20. The *spinor box operators* are

$$\square_{AB} = \nabla_{(A|A'|}\nabla_{B)}{}^{A'}, \tag{2.33}$$

$$\square_{A'B'} = \nabla_{(A|A'|}\nabla^{A}{}_{B)}. \tag{2.34}$$

Note that both operators are contractions of the expression

$$\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'}.$$

For example,

$$\begin{aligned}\Box_{AB} &= \nabla_{(A|A'|}\nabla_{B)}{}^{A'} \\ &= \nabla_{AA'}\nabla_{B}{}^{A'} + \nabla_{BA'}\nabla_{A}{}^{A'} \\ &= \bar{\epsilon}^{B'A'}\nabla_{AA'}\nabla_{BB'} + \bar{\epsilon}^{A'B'}\nabla_{BB'}\nabla_{AA'} \\ &= \bar{\epsilon}^{B'A'}(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'}). \end{aligned} \tag{2.35}$$

Hence any box operator acting on a spinor may be re-expressed as some partial contraction between that spinor and the spinorial Riemann tensor.

The box operators appear when we form commutators of fundamental derivatives.

Lemma 21. *The fundamental derivatives satisfy the following relations*

$$\begin{aligned}(\mathscr{D}\mathscr{C}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_{l-2}} &= \frac{k}{k+1}(\mathscr{C}\mathscr{D}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_{l-2}} \\ &\quad - \Box_{B'C'}\psi_{A_1\dots A_k}{}^{A'_1\dots A'_{l-2}B'C'}, \end{aligned} \tag{2.36a}$$

$$\begin{aligned}(\mathscr{D}\mathscr{C}^{\dagger}\psi)_{A_1\dots A_{k-2}}{}^{A'_1\dots A'_l} &= \frac{k}{l+1}(\mathscr{C}^{\dagger}\mathscr{D}\psi)_{A_1\dots A_{k-2}}{}^{A'_1\dots A'_l} \\ &\quad - \Box^{BC}\psi_{A_1\dots A_{k-2}BC}{}^{A'_1\dots A'_l}, \end{aligned} \tag{2.36b}$$

$$\begin{aligned}(\mathscr{C}\mathscr{T}\psi)_{A_1\dots A_{k+2}}{}^{A'_1\dots A'_l} &= \frac{l}{l+1}(\mathscr{T}\mathscr{C}\psi)_{A_1\dots A_{k+2}}{}^{A'_1\dots A'_l} \\ &\quad - \Box_{(A_1A_2}\psi_{A_3\dots A_{k+2})}{}^{A'_1\dots A'_l}, \end{aligned} \tag{2.36c}$$

$$\begin{aligned}(\mathscr{C}^{\dagger}\mathscr{T}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_{l+2}} &= \frac{k}{k+1}(\mathscr{T}\mathscr{C}^{\dagger}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_{l+2}} \\ &\quad - \Box^{(A'_1A'_2}\psi_{A_1\dots A_k}{}^{A'_3\dots A'_{l+2})}, \end{aligned} \tag{2.36d}$$

$$\begin{aligned}(\mathscr{D}\mathscr{T}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} &= -\left(\frac{1}{k+1}+\frac{1}{l+1}\right)(\mathscr{C}\mathscr{C}^{\dagger}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} \\ &\quad + \frac{l(l+2)}{(l+1)^2}(\mathscr{T}\mathscr{D}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} \\ &\quad - \frac{l+2}{l+1}\Box^{B}{}_{(A_1}\psi_{A_2\dots A_k)B}{}^{A'_1\dots A'_l} \\ &\quad - \frac{l}{l+1}\Box^{B'(A'_1}\psi_{A_1\dots A_k}{}^{A'_2\dots A'_l)}{}_{B'}, \end{aligned} \tag{2.36e}$$

$$\begin{aligned}(\mathscr{D}\mathscr{T}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} &= -\left(\frac{1}{k+1}+\frac{1}{l+1}\right)(\mathscr{C}^{\dagger}\mathscr{C}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} \\ &\quad + \frac{k(k+2)}{(k+1)^2}(\mathscr{T}\mathscr{D}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} \\ &\quad - \frac{k+2}{k+1}\Box^{B'(A'_1}\psi_{A_1\dots A_k}{}^{A'_2\dots A'_l)}{}_{B'} \\ &\quad - \frac{k}{k+1}\Box^{B}{}_{(A_1}\psi_{A_2\dots A_k)B}{}^{A'_1\dots A'_l}, \end{aligned} \tag{2.36f}$$

$$\begin{aligned}(\mathscr{C}\mathscr{C}^\dagger\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} &= (\mathscr{C}^\dagger\mathscr{C}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} \\ &+ \left(\frac{1}{k+1} - \frac{1}{l+1}\right)(\mathscr{T}\mathscr{D}\psi)_{A_1\dots A_k}{}^{A'_1\dots A'_l} \\ &- \square_{(A_1}{}^{B}\psi_{A_2\dots A_k)B}{}^{A'_1\dots A'_l} \\ &+ \square^{B'(A'_1}\psi_{A_1\dots A_k}{}^{A'_1\dots A'_l)}{}_{B'}.\end{aligned} \tag{2.36g}$$

A proof of this may be found in [3, sec. 2.2]. As we have seen, the box operators may be expressed as partial contractions with the spinorial Riemann derivative. These terms are hence called *curvature terms*. Importantly, they are order zero in derivatives.

2.6.3 Decomposing equations

A set $\{ (\phi_i)_{\mathfrak{A}_i}{}^{\mathfrak{A}'_i}, i = 1, 2, \dots \}$ of field subject to a differential equation is an *exact* set if, at each spacetime point P,

1. the symmetrized derivatives $\nabla_{(A}{}^{(A'}(\phi_i)_{\mathfrak{A}_i)}{}^{\mathfrak{A}'_i)}$, $\nabla_{(B}{}^{(B'}\nabla_{A}{}^{A'}(\phi_i)_{\mathfrak{A}_i)}{}^{\mathfrak{A}'_i)}$, ... can take arbitrary values, and
2. the unsymmetrized derivatives are determined by the symmetrized derivatives.

For example, sets of free massless fields are exact in flat spacetime. A free massless field is a symmetric spinor field $\phi_{A_1A_2\dots A_k}$ subject to

$$\nabla^{A_1A'_1}\phi_{A_1A_2\dots A_k} = 0. \tag{2.37}$$

Some quick algebra shows that this is equivalent to

$$\nabla_{B}{}^{B'}\phi_{A_1\dots A_k} = \nabla_{(B}{}^{B'}\phi_{A_1\dots A_k)}. \tag{2.38}$$

This shows that item 2 holds for order one. But for a free massless spinor field in flat spacetime, the covariant derivatives commute (combine (2.11) and (2.37) to see this), so

$$\nabla_{B_l}{}^{B'_l}\dots\nabla_{B_2}{}^{B'_2}\nabla_{B_1}{}^{B'_1}\phi_{A_1\dots A_k} = \nabla_{(B_l}{}^{(B'_l}\dots\nabla_{B_2}{}^{B'_2}\nabla_{B_1}{}^{B'_1)}\phi_{A_1\dots A_k)}, \tag{2.39}$$

and item 2 holds for higher orders as well. Also, since (2.39) is completely saturated with symmetries, no other linear relations may exist. Hence item 1 holds as well.

The Dirac equations are an exact set. This is a consequence of (4.5) to (4.7) that we will show later.

For reasons explained in section 4.1 we will encounter equations of the types

$$S^{A_1\dots A_{k+1}}{}_{BA'_1\dots A'_k}(\underbrace{\mathscr{T}\mathscr{T}\dots\mathscr{T}}_{\times k}\phi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k} = 0, \tag{2.40a}$$

$$S^{A_1\dots A_{k+1}}{}_{A'_1\dots A'_k}{}^{B'}(\underbrace{\mathscr{T}\mathscr{T}\dots\mathscr{T}}_{\times k}\phi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k} = 0, \tag{2.40b}$$

$$S^{A_1\dots A_k}{}_{BA'_1\dots A'_{k+1}}(\underbrace{\mathscr{T}\mathscr{T}\dots\mathscr{T}}_{\times k}\chi)_{A_1\dots A_k}{}^{A'_1\dots A'_{k+1}} = 0, \tag{2.40c}$$

$$S^{A_1\dots A_k}{}_{A'_1\dots A'_{k+1}}{}^{B'}(\underbrace{\mathscr{T}\mathscr{T}\dots\mathscr{T}}_{\times k}\chi)_{A_1\dots A_k}{}^{A'_1\dots A'_{k+1}} = 0, \tag{2.40d}$$

where ϕ_A and $\chi_{A'}$ are the Dirac fields and S is a spinor field. S may without loss of generality be taken to be symmetric in the indices that are contracted since they are contracted with a symmetric spinor.

By item 1, the twists can take arbitrary values at P. Contracting, for example, (2.40a) with a test field T_B yields the scalar equation

$$S^{A_1\dots A_{k+1}B}{}_{A'_1\dots A'_k}(\underbrace{\mathscr{T}\mathscr{T}\dots\mathscr{T}}_{\times k}\phi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k}T_B = 0. \tag{2.41}$$

But since the test field also may take arbitrary values,

$$W_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k}{}_B := (\underbrace{\mathscr{T}\mathscr{T}\dots\mathscr{T}}_{\times k}\phi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k}T_B \tag{2.42}$$

span $\mathbb{C}_{(A_1\dots A_{k+1})B}{}^{(A'_1\dots A'_k)}$. By theorem 15, $W_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k}{}_B$ has two independent parts: $W_{(A_1\dots A_{k+1}}{}^{(A'_1\dots A'_k)}{}_{B)}$ and $W_{A_1\dots(A_k}{}^{CA'_1\dots A'_k}{}_{|C|}\,\epsilon_{A_{k+1})B}$. Hence (2.41) splits into

$$\begin{aligned} 0 &= S^{(A_1\dots A_{k+1}B)}{}_{(A'_1\dots A'_k)}W_{(A_1\dots A_{k+1}}{}^{(A'_1\dots A'_k)}{}_{B)} \\ &\quad -\frac{k}{k+1}S^{(A_1\dots A_k B)}{}_{B(A'_1\dots A'_k)}W_{A_1\dots A_k}{}^{CA'_1\dots A'_k}{}_C\,. \end{aligned} \tag{2.43}$$

The two independent parts of $W_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_k}{}_B$ may take arbitrary and independent values, and so

$$S^{(A_1\dots A_{k+1}B)}{}_{(A'_1\dots A'_k)} = 0, \tag{2.44a}$$

$$S^{(A_1\dots B)}{}_{B(A'_1\dots A'_k)} = 0. \tag{2.44b}$$

This technique will be used abundantly when analyzing the equations for the symmetry operators.

2.6.4 Index-free notation

Theorem 15 allows us to decompose spinors into linear combinations of outer products of symmetric spinors and Levi-Civita symbols, but if an expression is symmetric in all of its free indices, then, after applying theorem 15, every Levi-Civita symbol will have at least one index contracted. But then the expression may be written only in terms of partially contracted outer products of symmetric spinors. If two symmetric spinors are multiplied and partially contracted, it does not matter *which* indices are contracted, only how many.

This is of course applicable to the massive Dirac equation since it only has one free index and is hence trivially totally symmetric. Hence the following definition,

Definition 22. Let $T_{\mathfrak{A}}{}^{\mathfrak{A}'}$ and $S_{\mathfrak{B}}{}^{\mathfrak{B}'}$ be totally symmetric spinors of valence $(0,k;l,0)$ and $(0,m;n,0)$ respectively. Then the *symmetric pq-multiplication* of $T_{\mathfrak{A}}{}^{\mathfrak{A}'}$ with $S_{\mathfrak{B}}{}^{\mathfrak{B}'}$ is the totally symmetrized outer product of T with S where p unprimed indices are contracted and q primed indices are contracted:

$$(T \overset{p,q}{\odot} S)_{A_1 \dots A_{k+m-2p}}{}^{A'_1 \dots A'_{l+n-2q}}$$
$$= T_{(A_1 \dots A_{k-p}}{}^{B_1 \dots B_p (A'_1 \dots A'_{l-q}}{}_{|B'_1 \dots B'_q|} S_{A_{k-p+1} \dots A_{k+m-2p}) B_1 \dots B_p}{}^{A'_{l-q+1} \dots A'_{l+n-2q}) B'_1 \dots B'_q}.$$

Remark 23. *Symmetric pq-multiplication is not associative, and it is symmetric in $T \longleftrightarrow S$ only if there are an even number of indices being contracted. Otherwise it is antisymmetric. Hence using the symbol $\odot$ may be slightly misleading. In its defense, it does overlap with the symmetric tensor product $\odot$ from section 2.2 when zero indices are contracted.*

We call this the *index-free notation*. It is well-defined up to a sign, which we may remedy by using the convention that unprimed indices are contracted in a downwards slope and primed indices in an upwards slope.

For example, (2.36a) is written

$$\mathscr{D}\mathscr{C}\psi = \frac{k}{k+1}\mathscr{C}\mathscr{D}\psi - \square \overset{0,2}{\odot} \psi$$

in index-free notation[7].

2.6.5 Sources and further reading

The fundamental derivatives are defined by Bäckdahl and co-authors [3]. The symmetric pq-multiplication is introduced in another article by Bäckdahl and co-authors [2, def. 8]. Section 2.6.3 is based on Bäckdahl and co-authors [3, sec. 2.4] and on Penrose and Rindler [27, sec. 5.10].

[7] It is admittedly unclear, when removing the indices of a box operator, which one of the two is meant. However, this will not really be a problem since we will be expanding every box operator in terms of Weyl spinors, Ricci spinors, and Ricci scalars.

3

Background

In their article [3], Lars Andersson, Thomas Bäckdahl, and Pieter Blue explore, for three physical equations, which spacetimes allow for second order symmetry operators to exist. The three equations considered are the conformal wave equation, the Dirac–Weyl equation, and the Maxwell equation. This chapter starts by summarizing their results for the Dirac–Weyl equation. Then, some previous results for the massive Dirac equation are presented. Lastly, some context is given to symmetry operators in physics.

3.1 Previous results

Unless otherwise stated, we are working on a four-dimensional Lorentzian manifold.

3.1.1 The Dirac–Weyl equation

The following theorem, derived by Andersson, Bäckdahl, and Blue, constitutes a simplification and reformulation in terms of fundamental derivatives of a result by McLenaghan, Smith, and Walker [23].

Definition 24. A valence $(0,2;2,0)$ Killing spinor $L_{AB}{}^{A'B'}$ satisfies auxiliary condition

A0 if there exists a scalar field Q such that

$$\mathscr{T}Q = -\frac{1}{3}\Psi \overset{3,0}{\odot} \mathscr{C}L - \frac{1}{3}\overline{\Psi} \overset{0,3}{\odot} \mathscr{C}^{\dagger}L + L \overset{2,1}{\odot} \mathscr{C}\Phi + L \overset{1,2}{\odot} \mathscr{C}^{\dagger}\Phi$$

A1 if there exists valence $(0,1;1,0)$ spinor field $P_A{}^{A'}$ satisfying

$$\mathscr{T}P = L \overset{2,0}{\odot} \Phi - L \overset{0,2}{\odot} \overline{\Psi}.$$

A valence $(0,3;1,0)$ Killing spinor $L_{ABC}{}^{A'}$ satisfies auxiliary condition

B0 if

$$0 = \frac{3}{4}\Psi \overset{2,0}{\odot} \mathscr{D}L + \frac{5}{3}\Psi \overset{3,0}{\odot} \mathscr{C}L - \frac{6}{5}L \overset{2,1}{\odot} \mathscr{C}\Phi + \frac{4}{3}L \overset{3,1}{\odot} \mathscr{T}\Psi.$$

The *left and right Dirac–Weyl equations* are

$$(\mathscr{C}^{\dagger}\phi)_{A'} = 0 \quad \text{and} \tag{3.1a}$$

$$(\mathscr{C}\chi)_{A} = 0 \tag{3.1b}$$

respectively. These equations are complex conjugates of each other.

A *symmetry operator* is a linear differential operator mapping solutions to solutions of some differential equation. The *order* of a symmetry operator is the highest order of derivative that it contains. A symmetry operator of the Dirac–Weyl equation of *the first kind* is a symmetry operator mapping solutions of (3.1a) to solutions of (3.1a). A symmetry operator of *the second kind* is a symmetry operator mapping solutions of (3.1a) to solutions of (3.1b) and hence necessarily also maps solutions of (3.1b) to solutions of (3.1a).

Theorem 25.

1. *There exists a nontrivial second order symmetry operator of the Dirac–Weyl equation of the first kind if and only if there exists a nonzero valence* $(0, 2; 2, 0)$ *Killing spinor satisfying items A0 and A1 of definition 24.*

2. *There exists a nontrivial second order symmetry operator of the Dirac–Weyl equation of the second kind if and only if there exists a nonzero valence* $(0, 3; 1, 0)$ *Killing spinor satisfying item B0 of definition 24.*

3.1.2 The Dirac equation

Kamran and McLenaghan derived, in 1984, the form of the most general first order symmetry operator for the massive Dirac equation, [20, thm. II]. It was there done explicitly in the Dirac basis, where the decomposition of the state space into left- and right-handed spinors is manifest.

Benn and Kress showed in 2003 that McLenaghan's solution is the most general one of the first order in the sense that it extends to arbitrary spin manifolds, [7]. If an operator commutes with the Dirac operator, then it is a symmetry operator, and so the set of operators commutating with the Dirac equation is a subset of the set of symmetry operators. Cariglia et al. [9] explores what the most general such operator looks like.

Some work has also been done in a two-dimensional spacetime. This is relevant for for example graphene. Fatibene et al. [12] showed, in 2009, the form of the most general second order operator commuting with the massive Dirac operator on a two-dimensional spin manifold.

3.2 Why symmetry operators are interesting

Symmetries in physical equations are interesting for several reasons. Noether's theorem directly relates the symmetries of a physical system to its conserved quantities. One application of Noether's theorem using a second order symmetry is the *Carter* *constant*, discovered by Carter [10] in 1968, which is a non-obvious conserved quantity that exists for a rotating black hole (Kerr spacetime). It is of the form

$$C = K^{ab}u_a u_b$$

where K^{ab} is a Killing tensor and u_a is a tangent to a geodesic.

A more recent example is when Andersson and Blue [4] proved in 2015 the decay of solutions to the covariant wave equation for Kerr spacetime. They used the second order symmetry operator associated with the Carter constant to formulate the problem in a simpler and more physical way than preceding attempts, avoiding Fourier transforms.

Symmetry operators may also be used to interpret results obtained by such less direct methods. Many partial differential equations such as the Schrödinger and Helmholtz equations lend themselves naturally to separation of variables, but also the Dirac equation has been separated in some cases. In an article from 1992, for example, Kalnins et al. [19, sec. 3] explain the separation of the Dirac equation on Kerr spacetime in terms of the existence of nontrivial symmetry operators associated with a Killing tensor by identifying a set of separation constants as eigenvalues of said symmetry operators.

4

Method

This chapter starts by exemplifying the method of rewriting an equation in terms of linear combinations of twists that Bäckdahl and co-authors used to derive theorem 25. CAS SymManipulator is introduced and employed to perform this method algorithmically. There are lots of code snippets presented in this chapter as well. Please think of them more as pseudocode than something logically consistent.

4.1 Preparing the Dirac equation

The *left and right Dirac equations* are

$$(\mathscr{C}^\dagger \phi)_{A'} = m\chi_{A'} \quad \text{and} \tag{4.1a}$$

$$(\mathscr{C}\chi)_A = -m\phi_A. \tag{4.1b}$$

In contrast to the Dirac–Weyl equation, there is no concept of first or second kind of symmetry operator. We say that the fields ϕ_A and $\chi_{A'}$ are *coupled*. The condition that a differential operator $D\colon (\phi_A, \chi_{A'}) \mapsto (\lambda_A, \gamma_{A'})$ is a symmetry operator for the Dirac equation is that

$$(\mathscr{C}^\dagger \lambda)_{A'} = m\gamma_{A'}, \tag{4.2a}$$

$$(\mathscr{C}\gamma)_A = -m\lambda_A. \tag{4.2b}$$

for all $(\phi_A, \chi_{A'})$ satisfying (4.1a) and (4.1b).

Lemma 26. *Any symmetry operator D of the Dirac equation may be written only in terms of twists.*

Proof. We will show this by induction on the order of the differential operator.

For the base case, consider that by proposition 17,

$$\begin{aligned}\nabla_A{}^{A'}\phi_B &= (\mathscr{T}\phi)_{AB}{}^{A'} + \epsilon_{AB}(\mathscr{C}^\dagger\phi)^{A'} \\ &\overset{(4.1a)}{=} (\mathscr{T}\phi)_{AB}{}^{A'} + \epsilon_{AB} m \chi^{A'}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\nabla_A{}^{A'}\chi^{B'} &= (\mathscr{T}\chi)_A{}^{A'B'} + \epsilon^{A'B'}(\mathscr{C}\chi)_A \\ &\overset{(4.1b)}{=} (\mathscr{T}\chi)_A{}^{A'B'} - \epsilon^{A'B'} m\phi_A.\end{aligned} \tag{4.4}$$

For the induction step, we need only consider three cases. With some sloppy notation, where # stands for "an arbitrary coefficient", S for either ϕ or χ, and H for the induction hypothesis, and using that whenever the box operator appears we may write it as a partial contraction with the Riemann spinor,

$$\begin{aligned}\mathscr{C}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S &\overset{(2.36c)}{=} \#\mathscr{T}\mathscr{C}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \#\Box\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S \\ &= \#\mathscr{T}\mathscr{C}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \text{lower order terms} \\ &\overset{H}{=} \#\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S + \text{lower order terms,}\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathscr{C}^{\dagger}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S &\overset{(2.36d)}{=} \#\mathscr{T}\mathscr{C}^{\dagger}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \#\Box\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S \\ &= \#\mathscr{T}\mathscr{C}^{\dagger}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \text{lower order terms} \\ &\overset{H}{=} \#\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S + \text{lower order terms,}\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathscr{D}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S &\overset{(2.36e)}{=} \#\mathscr{C}\mathscr{C}^{\dagger}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \#\mathscr{T}\mathscr{D}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \#\Box\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S \\ &= \#\mathscr{C}\mathscr{C}^{\dagger}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \#\mathscr{T}\mathscr{D}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times(n-1)}S + \text{lower order terms} \\ &\overset{H}{=} \#\mathscr{C}\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S + \#\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S + \text{lower order terms} \\ &\overset{(4.5)}{=} \#\underbrace{\mathscr{T}\ldots\mathscr{T}}_{\times n}S + \text{lower order terms.}\end{aligned} \tag{4.7}$$

Note that the left-most sides of (4.5) to (4.7) all have one less order than the right-most sides. This is a consequence of using the Dirac equations in the base case, which relate two first order expressions to two zeroth order expressions. □

It is to great advantage that this proof is constructive. It lets us power through the first orders explicitly using CAS. The first order was shown as the base case. The second order comes out to

$$\mathscr{D}\mathscr{T}\phi = \frac{3}{2}m^2\phi - 6\Lambda\overset{0,0}{\odot}\phi, \tag{4.8a}$$

$$\mathscr{D}\mathscr{T}\chi = \frac{3}{2}m^2\chi - 6\Lambda\overset{0,0}{\odot}\chi, \tag{4.8b}$$

$$\mathscr{C}\mathscr{T}\phi = \Psi\overset{1,0}{\odot}\phi, \tag{4.8c}$$

$$\mathscr{C}\mathscr{T}\chi = -\frac{1}{2}m\mathscr{T}\phi + \Phi\overset{0,1}{\odot}\chi, \tag{4.8d}$$

$$\mathscr{C}^{\dagger}\mathscr{T}\phi = \frac{1}{2}m\mathscr{T}\chi + \Phi\overset{1,0}{\odot}\phi, \tag{4.8e}$$

$$\mathscr{C}^{\dagger}\mathscr{T}\chi = \overline{\Psi}\overset{0,1}{\odot}\chi, \tag{4.8f}$$

and third order comes out to

$$\begin{aligned}
\mathscr{D}\mathscr{T}\mathscr{T}\phi ={}& \frac{4}{3}m^2\mathscr{T}\phi + \frac{5}{6}\phi\overset{1,0}{\odot}\mathscr{C}\Phi + \frac{5}{18}\phi\overset{0,0}{\odot}\mathscr{D}\Phi + \frac{10}{3}\Phi\overset{1,1}{\odot}\mathscr{T}\phi - \frac{9}{2}\phi\overset{0,0}{\odot}\mathscr{T}\Lambda \\
&- 12\Lambda\overset{0,0}{\odot}\mathscr{T}\phi + \frac{3}{2}\Psi\overset{2,0}{\odot}\mathscr{T}\phi, && \text{(4.9a)}\\
\mathscr{D}\mathscr{T}\mathscr{T}\chi ={}& \frac{4}{3}m^2\mathscr{T}\chi + \frac{5}{6}\chi\overset{0,1}{\odot}\mathscr{C}^\dagger\Phi + \frac{5}{18}\chi\overset{0,0}{\odot}\mathscr{D}\Phi + \frac{10}{3}\Phi\overset{1,1}{\odot}\mathscr{T}\chi - \frac{9}{2}\chi\overset{0,0}{\odot}\mathscr{T}\Lambda \\
&- 12\Lambda\overset{0,0}{\odot}\mathscr{T}\chi + \frac{3}{2}\bar{\Psi}\overset{0,2}{\odot}\mathscr{T}\chi, && \text{(4.9b)}\\
\mathscr{C}\mathscr{T}\mathscr{T}\phi ={}& -\frac{1}{2}\phi\overset{1,0}{\odot}\mathscr{T}\Psi - \frac{1}{10}\phi\overset{0,0}{\odot}\mathscr{C}^\dagger\Psi + \frac{5}{2}\Psi\overset{1,0}{\odot}\mathscr{T}\phi + \frac{1}{4}m\Psi\overset{0,0}{\odot}\chi + \Phi\overset{0,1}{\odot}\mathscr{T}\phi, && \text{(4.9c)}\\
\mathscr{C}\mathscr{T}\mathscr{T}\chi ={}& -\frac{1}{3}m\mathscr{T}\mathscr{T}\phi - \frac{2}{3}\chi\overset{0,1}{\odot}\mathscr{T}\Phi - \frac{2}{9}\chi\overset{0,0}{\odot}\mathscr{C}\Phi + \frac{8}{3}\Phi\overset{0,1}{\odot}\mathscr{T}\chi - \frac{1}{3}m\Phi\overset{0,0}{\odot}\phi \\
&+ \Psi\overset{1,0}{\odot}\mathscr{T}\chi, && \text{(4.9d)}\\
\mathscr{C}^\dagger\mathscr{T}\mathscr{T}\phi ={}& \frac{1}{3}m\mathscr{T}\mathscr{T}\chi - \frac{2}{3}\phi\overset{1,0}{\odot}\mathscr{T}\Phi - \frac{2}{9}\phi\overset{0,0}{\odot}\mathscr{C}^\dagger\Phi + \frac{8}{3}\Phi\overset{1,0}{\odot}\mathscr{T}\phi + \frac{1}{3}m\Phi\overset{0,0}{\odot}\chi \\
&+ \bar{\Psi}\overset{0,1}{\odot}\mathscr{T}\phi, && \text{(4.9e)}\\
\mathscr{C}^\dagger\mathscr{T}\mathscr{T}\chi ={}& -\frac{1}{2}\chi\overset{0,1}{\odot}\mathscr{T}\bar{\Psi} - \frac{1}{10}\chi\overset{0,0}{\odot}\mathscr{C}\bar{\Psi} + \frac{5}{2}\bar{\Psi}\overset{0,1}{\odot}\mathscr{T}\chi - \frac{1}{4}m\bar{\Psi}\overset{0,0}{\odot}\phi + \Phi\overset{1,0}{\odot}\mathscr{T}\chi. && \text{(4.9f)}
\end{aligned}$$

These are shown in a Mathematica notebook [18] by repeated use of lemma 26 along with lemma 18 for simplifying some of the curvature terms.

From this, we see that we, for example, may write a general first order symmetry operator as

$$\begin{aligned}
\lambda_A ={}& -\chi^{A'}L0_{AA'} + K0_A{}^B\phi_B + L1_A{}^{BA'B'}(\mathscr{T}\chi)_{BA'B'} \\
&+ K1_A{}^{BCA'}(\mathscr{T}\phi)_{BCA'} && \text{(4.10a)}\\
\gamma_{A'} ={}& -\chi^{B'}N0_{A'B'} + M0^A{}_{A'}\phi_A + N1^A{}_{A'}{}^{B'C'}(\mathscr{T}\chi)_{AB'C'} \\
&+ M1^{AB}{}_{A'}{}^{B'}(\mathscr{T}\phi)_{ABB'}. && \text{(4.10b)}
\end{aligned}$$

I apologize for using several characters to denote a single variable, but the idea is that, for example, Kn is the coefficient of the n:th derivative of ϕ appearing in λ.

We now want to put (4.10a) and (4.10b) into (4.2a) and (4.2b) and use lemma 26 to rewrite everything in terms of twists on ϕ_A and $\chi_{A'}$. This will be done later in section 4.2.3 with the help of CAS. From there we will proceed to split each order into irreducible parts and use the arguments from section 2.6.3 to split (4.2a) and (4.2b) into collections of independent equations.

4.2 Computer algebra

A *Computer Algebra System* (CAS) is a piece of software that can manipulate mathematical expressions. In the same way that there are numerical algorithms such as Newton's method for finding roots to a given function, there are *symbolic* algorithms such as "completing the square" for finding roots to completely arbitrary functions within a given class. The advantage of symbolic algorithms are two-fold,

1. They are potentially more accurate than any human.
2. They are more scalable than human calculation.
3. They allow for greater abstraction and compartmentalization.

In the intersection between mathematics and physics, where this thesis aims to lie, it is important both to understand the general structures associated with the subject of study but also to be able to perform concrete calculations. CAS software profitable in this context, greatly facilitating calculations and hence allowing for more time and effort to be spent understanding general structures.

4.2.1 Mathematica

Reading this section and the ones after, you are assumed to be somewhat familiar with Mathematica syntax. If not, see section 4.2.4.

Mathematica is a CAS, running in the Wolfram Language, that aims to be a general-purpose computational tool for a vast amount of scientific branches from zoology to combinatorics. It is most commonly run in a notebook interface, with an interactive kernel. Many of Mathematica's features are designed to make this notebook interface look and feel like a calculation done with pen and paper. For example, shorthands such as `%` for "the output of the previous line" and `%%` for "the output of the line before the previous line" encourage a linear workflow where an expression is calculated, evaluated, and then the next line is inputed after considering what a reasonable next step is.

This, combined with the luxuries of a full programming language such as for loops and defining your own functions, makes Mathematica a great tool for performing physics calculations.

As an example, consider a quantum system with the wavefunction $\phi(x)$ where we want to calculate the probability of observing some eigenstate $n(x)$. We may solve this by defining a function `inner_product` and calculating the projection of ϕ on n by

```
inner_product[a_, b_] :=
  Integrate[a[x] b[x], {x, -Infinity, Infinity}]
inner_product[phi, n]
probability = Abs[%]^2
```

Not only have we vanished a possibly tedious computation, but when we later on need to compute the probability of observing spin up given a spin-state $|\phi\rangle$ or even if we just need to solve the previous problem numerically by projection down on some finite-dimensional subspace, then we only have to modify the first line:

```
inner_product[a_, b_] := Transpose[a].b
inner_product[phi, n]
probability = Abs[%]^2
```

This allows us to think stricly in terms of bras and kets while solving quantum mechanical problems.

4.2.2 SymManipulator

While there is considerable power in basic Mathematica, xAct package *SymManipulator* allows the user to explicitly indicate the symmetries present in the tensor indices and can be more mindful of what symmetries composed tensors have. We will use two other Mathematica packages from the xAct suite as well: *xTensor* which SymManipulator is dependent on, and *Spinors*.

The first step towards defining a spinor or tensor field is to define a manifold `M4` on which the tensors/spinors will live,

```
DefManifold[M4, 4, {a, b, c, d}]
```

`M` will by this definition be 4-dimensional and tensors on it will be able to use indices `a`, `b`, `c`, and `d`. Notice how bare-bones this definition is. We for example have to define our own metric

```
DefMetric[{1, 3, 0}, g[-a, -b], CD]
```

which by this definition will have one positive, three negative, and zero zero-valued eigenvalues. `CD` is the covariant derivative defined by its action on the metric being zero. Along with this covariant derivative, a collection of related quantities are defined, such as the Christoffel symbol and Riemann tensor. We may also assume that it is possible to define a spin structure on `M4`. The line

```
DefSpinStructure[g, SpinorBundle, {A, B, C, D}, epsilon, sigma, CDe]
```

defines a spin structure on `M4` consisting of a vector bundle `SpinorBundle` which contains the spinors, a set of spinor indices `{A, B, C, D}`, a spin metric `epsilon`, a soldering form `sigma`, and a covariant spinor derivative `CDe`. For the time being, `DefSpinStructure` has only been implemented for 4-dimensional Lorentzian manifolds. This is the case that we have considered above and where we have results such as theorem 15 and proposition 17.

With this spin structure, we may define both tensors and spinors using xTensor method `DefTensor` and Spinors method `DefSpinor` respectively. A minus sign is used for lowered indices, and a dagger for primed indices. So

```
DefTensor[T[a, b, -c, -d], M4]
DefSpinor[S[A, B, -C, A†, -B†, -C†, -D†], M4]
```

defines a valence $(2,2)$ tensor and a valence $(2,1;1,3)$ spinor on `M4`. If we want to specify some symmetry that our tensor/spinor has we may write e.g.

```
DefTensor[U[a, b, -c, -d], M4, Symmetric[{3, 4}]]
```

to make `U` symmetric in its last two indices.

4.2.3 Irreducible decompositions

In section 2.6.3, we discussed how irreducible decompositions may be used to split an equation into independent parts. In this section, we show how this is implemented in SymManipulator with a first order symmetry operator and (4.2a) as an example.

Theorem 15 has been implemented in SymManipulator method `IrrDecompose`. We may use it to decompose the coefficients in (4.10a) and (4.10b) by first defining them with `DefSpinor`. Each coefficient will have all but one index contracted with a completely symmetric spinor, and so we may without loss of generality assume that

$$K1_{ABC}{}^{A'} = K1_{A(BC)}{}^{A'}, \qquad L1_{AB}{}^{A'B'} = L1_{AB}{}^{(A'B')}, \tag{4.11}$$
$$M1_{AB}{}^{A'B'} = M1_{(AB)}{}^{A'B'}, \qquad N1_{A}{}^{A'B'C'} = N1_{A}{}^{A'(B'C')}. \tag{4.12}$$

We can thus write the definitions as

```
DefSpinor[K0Coeff[-A, -B], M4]
DefSpinor[L0Coeff[-A, A†], M4]
DefSpinor[M0Coeff[-A, A†], M4]
DefSpinor[N0Coeff[A†, B†], M4]
DefSpinor[K1Coeff[-A, -B, -C, -A†], M4, Symmetric[{2, 3}]]
DefSpinor[L1Coeff[-A, -B, -A†, -B†], M4, Symmetric[{3, 4}]]
DefSpinor[M1Coeff[-A, -B, -A†, -B†], M4, Symmetric[{1, 2}]]
DefSpinor[N1Coeff[-A, -A†, -B†, -C†], M4, Symmetric[{3, 4}]]
```

Now, applying `IrrDecompose` as

```
symOpCoeffIrrDecEqs = {
  IrrDecompose[K0Coeff[-A, -B]],
  IrrDecompose[L0Coeff[-A, A†]],
  ...
}
```

we get that

$$K0_{AB} = -\frac{1}{2}K0^{C}{}_{C}\epsilon_{AB} + K0_{(AB)}, \tag{4.13a}$$
$$L0_{A}{}^{A'} = L0_{A}{}^{A'}, \tag{4.13b}$$
$$M0_{A}{}^{A'} = M0_{A}{}^{A'}, \tag{4.13c}$$
$$N0^{A'B'} = -\frac{1}{2}N0^{C'}{}_{C'}\bar{\epsilon}^{A'B'} + N0^{(A'B')}, \tag{4.13d}$$
$$K1_{ABC}{}^{A'} = -\frac{1}{3}K1^{D}{}_{(CD)}{}^{A'}\epsilon_{AB} - \frac{1}{3}K1^{D}{}_{(BD)}{}^{A'}\epsilon_{AC} + K1_{(ABC)}{}^{A'}, \tag{4.13e}$$
$$L1_{AB}{}^{A'B'} = -\frac{1}{2}L1^{C}{}_{C}{}^{(A'B')}\epsilon_{AB} + L1_{(AB)}{}^{(A'B')}, \tag{4.13f}$$
$$M1_{AB}{}^{A'B'} = -\frac{1}{2}M1_{(AB)}{}^{C'}{}_{C'}\bar{\epsilon}^{A'B'} + M1_{(AB)}{}^{(A'B')}, \tag{4.13g}$$
$$N1_{AA'B'C'} = -\frac{1}{3}N1_{A}{}^{D'}{}_{(C'D')}\bar{\epsilon}_{A'B'} - \frac{1}{3}N1_{A}{}^{D'}{}_{(B'D')}\bar{\epsilon}_{A'C'} + N1_{A(A'B'C')}. \tag{4.13h}$$

We name these components

$$\begin{aligned}
\underset{0,0}{K0} &= K0^{A}{}_{A}, & \underset{2,0}{K0}_{AB} &= K0_{(AB)}, && \text{(4.14a)}\\
\underset{1,1}{K1}_{A}{}^{A'} &= K1^{B}{}_{AB}{}^{A'}, & \underset{3,1}{K1}_{ABC}{}^{A'} &= K1_{(ABC)}{}^{A'}, && \text{(4.14b)}\\
\underset{0,2}{L1}^{A'B'} &= L1^{A}{}_{A}{}^{A'B'}, & \underset{2,2}{L1}_{AB}{}^{A'B'} &= L1_{(AB)}{}^{(A'B')}, && \text{(4.14c)}\\
\underset{2,0}{M1}_{AB} &= M1_{AB}{}^{A'}{}_{A'}, & \underset{2,2}{M1}_{AB}{}^{A'B'} &= M1_{(AB)}{}^{(A'B')}, && \text{(4.14d)}\\
\underset{0,0}{N0} &= N0^{A'}{}_{A'}, & \underset{0,2}{N0}^{A'B'} &= N0^{(A'B')}, && \text{(4.14e)}\\
\underset{1,1}{N1}_{A}{}^{A'} &= N1_{A}{}^{B'A'}{}_{B'}, & \underset{1,3}{N1}_{A}{}^{A'B'C'} &= N1_{A}{}^{(A'B'C')}. && \text{(4.14f)}
\end{aligned}$$

Notice that our naming scheme implies that $\underset{k,l}{S}$ is a totally symmetric spinor of valence $(0,k;l,0)$.

There is also a SymManipulator method, `DefFundSpinOperators`, for defining the fundamental derivatives. The line

```
DefFundSpinOperators[CDe]
```

defines the fundamental derivatives as functions `DivCDe`, `CurlCDe`, `CurlDgCDe`, and `TwistCDe` using the covariant derivative `CDe` that we defined previously using `Def-SpinStructure`.

We are now in a position to continue the decomposition of (4.2a) begun in section 4.1. We substitute (4.10a) and (4.10b) into (4.2a) by

```
lambda[-A] == K0Coeff[-A, B] phi[-B]
  + K1Coeff[-A, B, C, A†] (TwistCDe@phi)[-B, -C, -A†]
  + L0Coeff[-A, A†] chi[-A†]
  + L1Coeff[-A, B, A†, B†] (TwistCDe@chi)[-B, -A†, -B†]
lambdaRule = EqToRule@%
gamma[-A†] == M0Coeff[A, -A†] phi[-A]
  + M1Coeff[A, B, -A†, B†] (TwistCDe@phi)[-A, -B, -B†]
  + N0Coeff[-A†, B†] chi[-B†]
  + N1Coeff[A, -A†, B†, C†] (TwistCDe@chi)[-A, -B†, -C†]
gammaRule = EqToRule@%

leftDiracEq = 0 == (CurlDgCDe@lambda)[-A†] - mass gamma[-A†]
%/.{lambdaRule, gammaRule}
leftDiracEqExpanded = %
```

This gives, after also using (4.13a) to (4.13h) and then (4.14a) to (4.14f), that

$$\begin{aligned}
0 = {} & - L0 \overset{1,1}{\odot} \mathscr{T}\chi + \frac{1}{2} m L0 \overset{1,0}{\odot} \phi - \chi \overset{0,1}{\odot} \mathscr{C}^{\dagger} L0 - \frac{1}{2}\chi \overset{0,0}{\odot} \mathscr{D} L0 + \frac{1}{2}\phi \overset{1,0}{\odot} \mathscr{T} \underset{0,0}{K0}\\
& - \frac{1}{2} m \underset{0,0}{K0} \overset{0,0}{\odot} \chi - \underset{2,0}{K0} \overset{2,0}{\odot} \mathscr{T}\phi - \phi \overset{1,0}{\odot} \mathscr{C}^{\dagger} \underset{2,0}{K0} + \frac{1}{2}(\mathscr{T}\chi) \overset{1,2}{\odot} \mathscr{T} \underset{0,2}{L1} + \frac{1}{3}(\mathscr{T}\chi) \overset{1,1}{\odot} \mathscr{C} \underset{0,2}{L1}\\
& - \frac{1}{2} \underset{0,2}{L1} \overset{0,2}{\odot} \mathscr{C}^{\dagger} \mathscr{T}\chi - \frac{1}{3} \underset{0,2}{L1} \overset{0,1}{\odot} \mathscr{D} \mathscr{T}\chi - \underset{2,2}{L1} \overset{2,2}{\odot} \mathscr{T}\mathscr{T}\chi - \frac{2}{3} \underset{2,2}{L1} \overset{2,1}{\odot} \mathscr{C}\mathscr{T}\chi
\end{aligned}$$

$$
\begin{aligned}
&-(\mathscr{T}\chi)\overset{1,2}{\odot}\mathscr{C}^\dagger \underset{2,2}{L1} - \frac{2}{3}(\mathscr{T}\chi)\overset{1,1}{\odot}\mathscr{D}\underset{2,2}{L1} + \frac{2}{3}(\mathscr{T}\phi)\overset{2,1}{\odot}\mathscr{T}\underset{1,1}{K1} + \frac{1}{3}(\mathscr{T}\phi)\overset{2,0}{\odot}\mathscr{C}\underset{1,1}{K1} \\
&-\frac{2}{3}\underset{1,1}{K1}\overset{1,1}{\odot}\mathscr{C}^\dagger\mathscr{T}\phi - \frac{1}{3}\underset{1,1}{K1}\overset{1,0}{\odot}\mathscr{D}\mathscr{T}\phi - \underset{3,1}{K1}\overset{3,1}{\odot}\mathscr{T}\mathscr{T}\phi - \frac{1}{2}\underset{3,1}{K1}\overset{3,0}{\odot}\mathscr{C}\mathscr{T}\phi \\
&-(\mathscr{T}\phi)\overset{2,1}{\odot}\mathscr{C}^\dagger\underset{3,1}{K1} - \frac{1}{2}(\mathscr{T}\phi)\overset{2,0}{\odot}\mathscr{D}\underset{3,1}{K1} + \frac{1}{2}m\underset{0,0}{N0}\overset{0,0}{\odot}\chi + m\chi\overset{0,1}{\odot}\underset{0,2}{N0} + m\phi\overset{1,0}{\odot}M0 \\
&+\frac{2}{3}m\underset{1,1}{N1}\overset{1,1}{\odot}\mathscr{T}\chi + m(\mathscr{T}\chi)\overset{1,2}{\odot}\underset{1,3}{N1} + \frac{1}{2}m\underset{2,0}{M1}\overset{2,0}{\odot}\mathscr{T}\phi + m(\mathscr{T}\phi)\overset{2,1}{\odot}\underset{2,2}{M1}. \qquad (4.15)
\end{aligned}
$$

The commutators (2.36a) to (2.36g), have been implemented in SymManipulator method `CommuteFundSpinOp`, and so may be used to rewrite (4.2a) to the form of lemma 26,

```
diracEqs = {
  (CurlDgCDe@phi)[-A†] - mass chi[-A†] == 0,
  (CurlCDe@chi)[-A] + mass phi[-A] == 0
}
EqToRule/@%

toTwistRules = {
  DivCDe@TwistCDe@phi -> (DivCDe@TwistCDe@phi/.
    CommuteOp[DivCDe, TwistCDe, CurlCDe, CurlDgCDe]//.
    diracRules),
  DivCDe@TwistCDe@chi -> (DivCDe@TwistCDe@chi/.
    CommuteOp[DivCDe, TwistCDe, CurlDgCDe, CurlCDe]//.
    diracRules),
  CurlCDe@TwistCDe@phi -> (CurlCDe@TwistCDe@phi/.
    CommuteOp[CurlCDe, TwistCDe, TwistCDe, CurlCDe]//.
    diracRules),
  ...
}

leftDiracEqExpanded/.toTwistRules
```

The result is

$$
\begin{aligned}
0 = {}& \frac{1}{2}m\chi\overset{0,0}{\odot}\underset{0,0}{N0} + m\chi\overset{0,1}{\odot}\underset{0,2}{N0} - mM0\overset{1,0}{\odot}\phi + \frac{2}{3}m\underset{1,1}{N1}\overset{1,1}{\odot}\mathscr{T}\chi - m\underset{1,3}{N1}\overset{1,2}{\odot}\mathscr{T}\chi \\
&+\frac{1}{2}m\underset{2,0}{M1}\overset{2,0}{\odot}\mathscr{T}\phi - m\underset{2,2}{M1}\overset{2,1}{\odot}\mathscr{T}\phi - L0\overset{1,1}{\odot}\mathscr{T}\chi + \frac{1}{2}mL0\overset{1,0}{\odot}\phi - \chi\overset{0,1}{\odot}\mathscr{C}^\dagger L0 \\
&-\frac{1}{2}\chi\overset{0,0}{\odot}\mathscr{D}L0 + \frac{1}{2}\phi\overset{1,0}{\odot}\mathscr{T}\underset{0,0}{K0} - \frac{1}{2}m\chi\overset{0,0}{\odot}\underset{0,0}{K0} - \underset{2,0}{K0}\overset{2,0}{\odot}\mathscr{T}\phi - \phi\overset{1,0}{\odot}\mathscr{C}^\dagger\underset{2,0}{K0} \\
&+\frac{1}{2}(\mathscr{T}\chi)\overset{1,2}{\odot}\mathscr{T}\underset{0,2}{L1} + \frac{1}{3}(\mathscr{C}\underset{0,2}{L1})\overset{1,1}{\odot}\mathscr{T}\chi + \frac{1}{2}\underset{0,2}{L1}\overset{0,2}{\odot}\chi\overset{0,1}{\odot}\overline{\Psi} + \frac{1}{2}m^2\chi\overset{0,1}{\odot}\underset{0,2}{L1} \\
&+2\underset{0,2}{L1}\overset{0,1}{\odot}\chi\overset{0,0}{\odot}\Lambda - \underset{2,2}{L1}\overset{2,2}{\odot}\mathscr{T}\mathscr{T}\chi + \frac{1}{3}m\underset{2,2}{L1}\overset{2,1}{\odot}\mathscr{T}\phi + \frac{2}{3}\underset{2,2}{L1}\overset{2,1}{\odot}\chi\overset{0,1}{\odot}\Phi \\
&+(\mathscr{C}^\dagger\underset{2,2}{L1})\overset{1,2}{\odot}\mathscr{T}\chi - \frac{2}{3}(\mathscr{D}\underset{2,2}{L1})\overset{1,1}{\odot}\mathscr{T}\chi - \frac{2}{3}(\mathscr{T}\underset{1,1}{K1})\overset{2,1}{\odot}\mathscr{T}\phi + \frac{1}{3}(\mathscr{C}\underset{1,1}{K1})\overset{2,0}{\odot}\mathscr{T}\phi
\end{aligned}
$$

$$-\frac{1}{3}m\underset{1,1}{K1}\overset{1,1}{\odot}\mathscr{T}\chi+\frac{2}{3}\underset{1,1}{K1}\overset{1,1}{\odot}\phi\overset{1,0}{\odot}\Phi-\frac{1}{2}m^2\underset{1,1}{K1}\overset{1,0}{\odot}\phi+2\underset{1,1}{K1}\overset{1,0}{\odot}\Lambda\overset{0,0}{\odot}\phi$$
$$-\underset{3,1}{K1}\overset{3,1}{\odot}\mathscr{T}\mathscr{T}\phi+\frac{1}{2}\underset{3,1}{K1}\overset{3,0}{\odot}\phi\overset{1,0}{\odot}\Psi+(\mathscr{C}^\dagger\underset{3,1}{K1})\overset{2,1}{\odot}\mathscr{T}\phi-\frac{1}{2}(\mathscr{D}\underset{3,1}{K1})\overset{2,0}{\odot}\mathscr{T}\phi. \tag{4.16}$$

Now, collecting the terms with a second order derivative of ϕ, we get that

$$0=\underset{3,1}{K1}\overset{3,1}{\odot}\mathscr{T}\mathscr{T}\phi.$$

By section 2.6.3, we may contract this with a test spinor field $T^{B'}$, observe that $T_{B'}\,(\mathscr{T}\mathscr{T}\phi)_{ABC}{}^{A'B'}$ is irreducible by being totally symmetric, and deduce that

$$\underset{3,1}{K1}{}_{ABC}{}^{A'}=0. \tag{4.17}$$

Similarly, collecting the terms with a second order derivative of χ yields

$$\underset{2,2}{M1}{}_{AB}{}^{A'B'}=0. \tag{4.18}$$

Collecting the terms with a first order derivative of ϕ, we get that

$$0=-\underset{2,0}{K0}\overset{2,0}{\odot}\mathscr{T}\phi+\frac{1}{3}m\underset{2,2}{L1}\overset{2,1}{\odot}\mathscr{T}\phi+\frac{2}{3}(\mathscr{T}\phi)\overset{2,1}{\odot}\mathscr{T}\underset{1,1}{K1}+\frac{1}{3}(\mathscr{T}\phi)\overset{2,0}{\odot}\mathscr{C}\underset{1,1}{K1}$$
$$-\frac{1}{2}(\mathscr{T}\phi)\overset{2,0}{\odot}\mathscr{D}\underset{3,1}{K1}+\frac{1}{2}m\underset{2,0}{M1}\overset{2,0}{\odot}\mathscr{T}\phi+m(\mathscr{T}\phi)\overset{2,1}{\odot}\underset{2,2}{M1}. \tag{4.19}$$

We now want to start decomposing equations. To utilize the arguments from section 2.6.3, we must first decompose $T^{B'}(\mathscr{T}\phi)_{AB}{}^{A'}$. This may be done by

```
DefSpinor[TCoeff[B†], M4]
IrrDecompose[TCoeff[B†] (TwistCDe@phi)[-A, -B, A†]]
```

We get that

$$T^{B'}(\mathscr{T}\phi)_{AB}{}^{A'}=T^{(A'}(\mathscr{T}\phi)_{(AB)}{}^{B')}+\frac{1}{2}T^{C'}\bar{\epsilon}^{A'B'}(\mathscr{T}\phi)_{ABC'}.$$

Contracting (4.19) with $T^{B'}$, substituting in (4.20), and writing each irreducible part as a separate equation, we get that

$$\underset{2,0}{K0}-\frac{1}{2}m\underset{2,0}{M1}-\frac{1}{3}\mathscr{C}\underset{1,1}{K1}=0, \tag{4.20}$$
$$\frac{1}{3}m\underset{1,1}{K1}+L0-\frac{2}{3}m\underset{1,1}{N1}-\frac{1}{3}\mathscr{C}\underset{0,2}{L1}=0. \tag{4.21}$$

This is in general how it's possible to algorithmically go from (4.2a) to a set of linear differential conditions on the coefficients of the symmetry operator.

Following this algorithm through all three orders for both Dirac equations will yield

$$\underset{3,1}{K1}=0, \tag{4.22a}$$

$$\underset{2,2}{L1} = 0, \tag{4.22b}$$
$$\underset{2,2}{M1} = 0, \tag{4.22c}$$
$$\underset{1,3}{N1} = 0, \tag{4.22d}$$

$$0 = -\underset{2,0}{K0} + \frac{1}{2} m \underset{2,0}{M1} + \frac{1}{3} \mathscr{C} \underset{1,1}{K1}, \tag{4.23a}$$
$$0 = -\frac{1}{3} m \underset{1,1}{K1} - L0 + \frac{2}{3} m \underset{1,1}{N1} + \frac{1}{3} \mathscr{C} \underset{0,2}{L1}, \tag{4.23b}$$
$$0 = -\frac{2}{3} m \underset{1,1}{K1} - M0 + \frac{1}{3} m \underset{1,1}{N1} + \frac{1}{3} \mathscr{C}^\dagger \underset{2,0}{M1}, \tag{4.23c}$$
$$0 = -\frac{1}{2} m \underset{0,2}{L1} - \underset{0,2}{N0} + \frac{1}{3} \mathscr{C}^\dagger \underset{1,1}{N1}, \tag{4.23d}$$
$$0 = \mathscr{T} \underset{1,1}{K1}, \tag{4.23e}$$
$$0 = \mathscr{T} \underset{0,2}{L1}, \tag{4.23f}$$
$$0 = \mathscr{T} \underset{2,0}{M1}, \tag{4.23g}$$
$$0 = \mathscr{T} \underset{1,1}{N1}, \tag{4.23h}$$

$$0 = -m \underset{2,0}{K0} + \frac{1}{2} m^2 \underset{2,0}{M1} - 2\Lambda \overset{0,0}{\odot} \underset{2,0}{M1} + \frac{1}{2} \underset{2,0}{M1} \overset{2,0}{\odot} \Psi - \mathscr{C} M0, \tag{4.24a}$$
$$0 = \frac{1}{2} m^2 \underset{0,2}{L1} - 2\Lambda \overset{0,0}{\odot} \underset{0,2}{L1} + m \underset{0,2}{N0} + \frac{1}{2} \underset{0,2}{L1} \overset{0,2}{\odot} \overline{\Psi} - \mathscr{C}^\dagger L0, \tag{4.24b}$$
$$0 = \frac{1}{2} m \underset{0,0}{K0} - \frac{1}{2} m \underset{0,0}{N0} + \frac{1}{2} \mathscr{D} L0, \tag{4.24c}$$
$$0 = \frac{1}{2} m \underset{0,0}{K0} - \frac{1}{2} m \underset{0,0}{N0} + \frac{1}{2} \mathscr{D} M0, \tag{4.24d}$$
$$0 = \frac{1}{2} m^2 \underset{1,1}{K1} - \frac{1}{2} m L0 - 2\Lambda \overset{0,0}{\odot} \underset{1,1}{K1} + m M0 + \frac{2}{3} \underset{1,1}{K1} \overset{1,1}{\odot} \Phi - \mathscr{C}^\dagger \underset{2,0}{K0} + \frac{1}{2} \mathscr{T} \underset{0,0}{K0}, \tag{4.24e}$$
$$0 = -m L0 + \frac{1}{2} m M0 + \frac{1}{2} m^2 \underset{1,1}{N1} - 2\Lambda \overset{0,0}{\odot} \underset{1,1}{N1} + \frac{2}{3} \underset{1,1}{N1} \overset{1,1}{\odot} \Phi - \mathscr{C} \underset{0,2}{N0} + \frac{1}{2} \mathscr{T} \underset{0,0}{N0}. \tag{4.24f}$$

Please don't look too closely at this wall of equations. But just notice that (4.23e) to (4.23h) are the conditions that $\underset{1,1}{K1}$, $\underset{0,2}{L1}$, $\underset{2,0}{M1}$, and $\underset{1,1}{N1}$ are Killing spinors. The rest of the above equations are either

1. expressing some lower order coefficient in terms of $\underset{1,1}{K1}$, $\underset{0,2}{L1}$, $\underset{2,0}{M1}$, and $\underset{1,1}{N1}$, or
2. a differential condition on $\underset{1,1}{K1}$, $\underset{0,2}{L1}$, $\underset{2,0}{M1}$, and $\underset{1,1}{N1}$.

After some renaming and simplification, all of this reduces to theorem 29, presented in section 5.1.

4.2.4 Sources and further reading

A good resource for learning about Mathematica is its Documentation Center [31]. It contains a plethora of tutorials such as “Working with Operators” [32] and “Everything is an Expression” [33].

A good resource for learning about the xAct packages is the documentation notebooks provided with each package. They are available at [22].

For this thesis, I have used Mathematica version 12.0.0, xTensor version 1.1.4, Spinors version 1.0.6, SymManipulator version 0.9.4, and TexAct version 0.4.1. The notebooks used for creating all of the results presented in the next section are available in a GitHub repository [18]. The code snippets presented in this chapter are based on two sample notebooks `example_from_section_4-2-2.nb` and `example_from_section_4-2-3.nb`, also available at GitHub.

5
Results

In this chapter, we state the equations derived from the method described previously: substituting in the symmetry operator in the Dirac equation, rewriting the expressions to only contain twists, and then decomposing the resulting equations into irreducible parts. We then interpret these equations to see how they may put conditions on the spacetime they're defined on.

The main results are theorems 27, 29, and 31.

5.0 Zeroth order symmetry operator

Let $D\colon (\phi_A, \chi_{A'}) \mapsto (\lambda_A, \gamma_{A'})$ be of the form

$$\lambda_A = K_A{}^B \phi_B + L_A{}^{A'} \chi_{A'}, \tag{5.1a}$$

$$\gamma_{A'} = M^A{}_{A'} \phi_A + N_{A'}{}^{B'} \chi_{B'}. \tag{5.1b}$$

Substituting this into (4.2a) and (4.2b), collecting each order of derivative, and decomposing the resulting equations, we get that

Theorem 27. *The only zeroth order symmetry operators for the Dirac equation are proportional to the identity.*

5.1 First order symmetry operator

Let $D\colon (\phi_A, \chi_{A'}) \mapsto (\lambda_A, \gamma_{A'})$ be of the form

$$\lambda_A = K0_A{}^B \phi_B + L0_A{}^{A'} \chi_{A'} + K1_A{}^{BCA'} (\mathscr{T}\phi)_{BCA'} + L1_A{}^{BA'B'} (\mathscr{T}\chi)_{BA'B'}\,, \tag{5.2a}$$

$$\gamma_{A'} = M0^A{}_{A'} \phi_A + N0_{A'}{}^{B'} \chi_{B'} + M1^{AB}{}_{A'}{}^{B'} (\mathscr{T}\phi)_{ABB'} + N1^A{}_{A'}{}^{B'C'} (\mathscr{T}\chi)_{AB'C'}\,. \tag{5.2b}$$

As before, we substitute this into (4.2a) and (4.2b), collect each order of derivative, and decompose the resulting equations. There are then in total 18 equations and 12 variables. I will spare you from them, but after simplifying a bit, they may be expressed as theorem 29.

Definition 28. Let $S_A{}^{A'}$, $T^{A'B'}$, U_{AB}, and $R_A{}^{A'}$ be Killing spinors on a Lorentzian manifold M. They satisfy *auxiliary condition A* if

$$\mathscr{C}S = 0, \tag{5.3a}$$

$$\mathscr{C}^\dagger S = 0, \tag{5.3b}$$

$$\mathscr{C}T + \mathscr{C}^\dagger U = 0. \tag{5.3c}$$

The geometric interpretation of (5.3a) and (5.3b) is that $S_A{}^{A'}$ is closed vector field. The geometric interpretation of (5.3c) is that $f_A{}^{A'}{}_B{}^{B'} := U_{AB}\bar{\epsilon}^{A'B'} + \epsilon_{AB}T^{A'B'}$ is a Killing–Yano tensor. This is shown properly in a Mathematica notebook `small_killing-yano_calculation.nb` [18].

Theorem 29. *The massive Dirac equation has a first order symmetry operator if and only if there exist Killing spinors $S_A{}^{A'}$, $T^{A'B'}$, and U_{AB} satisfying auxiliary condition A and a Killing spinor $R_A{}^{A'}$ that is also a Killing vector (S, T, U, and R should not all be zero). The symmetry operator then takes the form*

$$\begin{aligned}\lambda = & -\frac{1}{3}m\chi\overset{0,1}{\odot}R + m\chi\overset{0,1}{\odot}S - \frac{1}{3}\chi\overset{0,1}{\odot}\mathscr{C}T + O\overset{0,0}{\odot}\phi - \frac{1}{4}\phi\overset{0,0}{\odot}\mathscr{D}S - \frac{1}{2}m\phi\overset{1,0}{\odot}U \\ & -\frac{1}{3}\phi\overset{1,0}{\odot}\mathscr{C}R - \frac{1}{2}T\overset{0,2}{\odot}\mathscr{T}\chi - \frac{2}{3}R\overset{1,1}{\odot}\mathscr{T}\phi - \frac{2}{3}S\overset{1,1}{\odot}\mathscr{T}\phi, \end{aligned} \tag{5.4a}$$

$$\begin{aligned}\gamma = & \chi\overset{0,0}{\odot}O + \frac{1}{4}\chi\overset{0,0}{\odot}\mathscr{D}S + \frac{1}{2}m\chi\overset{0,1}{\odot}T - \frac{1}{3}\chi\overset{0,1}{\odot}\mathscr{C}^\dagger R + \frac{1}{3}m\phi\overset{1,0}{\odot}R + m\phi\overset{1,0}{\odot}S \\ & -\frac{1}{3}\phi\overset{1,0}{\odot}\mathscr{C}^\dagger U - \frac{2}{3}R\overset{1,1}{\odot}\mathscr{T}\chi + \frac{2}{3}S\overset{1,1}{\odot}\mathscr{T}\chi - \frac{1}{2}U\overset{2,0}{\odot}\mathscr{T}\phi, \end{aligned} \tag{5.4b}$$

for some arbitrary constant O.

Note that the existence of R puts no further conditions on spacetime if U or T are nonzero. To convince ourselves of this, note that the condition that R is a Killing vector means that $\mathscr{T}R$ and $\mathscr{D}R = 0$. But $\mathscr{C}T$ (or, equivalently, $\mathscr{C}^\dagger U$) satisfies both of these equalities by (2.36a) and (2.36c) respectively.

I haven't talked about Petrov classification in this thesis. But it is worth mentioning anyways, that for vacuum spacetimes, the existence of a valence $(0,2;0,0)$ implies that the spacetime is of type D, N, or O [5, sec. 4.7].

A reflection we could make at this point is if theorem 29 is compatible with theorem 27. For, the set of symmetry operators is a linear space, so the equalities in theorem 29 must still hold after adding some multiple of the identity. But this is clearly the case since $O\phi_A$ appears as a term in (5.4a) and $O\chi_{A'}$ appears as a term in (5.4b).

Another reflection we could make is to compare this result with the previous result by Kamran and McLenaghan [20, thm. II] derived explicitly in the Dirac basis. A quick sanity check is that the number of derivatives in their conditions on the coefficients do not exceed one, which is true for my conditions as well.

We can also put down our fundamental derivatives, just for a moment, to state these results in terms of covariant spinor derivatives. We have that (5.3a) to (5.3c) become

$$\nabla_{(A}{}^{A'}S_{B)A'} = 0, \tag{5.5a}$$

$$\nabla^A{}_{(A'}S_{|A|B')} = 0, \tag{5.5b}$$

$$-\nabla_{AB'}T_{A'}{}^{B'} - \nabla_{BA'}U_A{}^B = 0. \tag{5.5c}$$

We also have that (5.4a) and (5.4b) become

$$\lambda_A = O\phi_A - \frac{1}{3}m\chi^{A'}R_{AA'} + m\chi^{A'}S_{AA'} - \frac{1}{2}m\phi^B U_{AB} + \frac{1}{3}\chi^{A'}\nabla_{AB'}T_{A'}{}^{B'}$$
$$- \frac{1}{4}\phi_A\nabla_{BA'}S^{BA'} - \frac{1}{2}T^{A'B'}\nabla_{A(A'}\chi_{B')} - \frac{2}{3}R^{BA'}\nabla_{(A|A'|}\phi_{B)}$$
$$- \frac{2}{3}S^{BA'}\nabla_{(A|A'|}\phi_{B)} - \frac{1}{3}\phi^B\nabla_{(A}{}^{A'}R_{B)A'}, \tag{5.6a}$$

$$\gamma_{A'} = \chi_{A'}O + \frac{1}{3}m\phi^A R_{AA'} + m\phi^A S_{AA'} + \frac{1}{2}m\chi^{B'}T_{A'B'} + \frac{1}{4}\chi_{A'}\nabla_{AB'}S^{AB'}$$
$$+ \frac{1}{3}\phi^A\nabla_{BA'}U_A{}^B - \frac{2}{3}R^{AB'}\nabla_{A(A'}\chi_{B')} + \frac{2}{3}S^{AB'}\nabla_{A(A'}\chi_{B')}$$
$$- \frac{1}{2}U^{AB}\nabla_{(A|A'|}\phi_{B)} - \frac{1}{3}\chi^{B'}\nabla^A{}_{(A'}R_{|A|B')}. \tag{5.6b}$$

5.2 Second order symmetry operator

Let $D\colon (\phi_A, \chi_{A'}) \mapsto (\lambda_A, \gamma_{A'})$ be of the form

$$\lambda_A = K0_A{}^B\phi_B + L0_A{}^{A'}\chi_{A'} \tag{5.7a}$$
$$+ K1_A{}^{BCA'}(\mathscr{T}\phi)_{BCA'} + L1_A{}^{BA'B'}(\mathscr{T}\chi)_{BA'B'} \tag{5.7b}$$
$$+ K2_A{}^{BCDA'B'}(\mathscr{T}\mathscr{T}\phi)_{BCDA'B'} + L2_A{}^{BCA'B'C'}(\mathscr{T}\mathscr{T}\chi)_{BCA'B'C'}, \tag{5.7c}$$
$$\gamma_{A'} = M0^A{}_{A'}\phi_A + \chi^{B'}N0_{A'B'} \tag{5.7d}$$
$$+ N1^A{}_{A'}{}^{B'C'}(\mathscr{T}\chi)_{AB'C'} + M1^{AB}{}_{A'}{}^{B'}(\mathscr{T}\phi)_{ABB'} \tag{5.7e}$$
$$+ N2^{AB}{}_{A'}{}^{B'C'D'}(\mathscr{T}\mathscr{T}\chi)_{ABB'C'D'} + M2^{ABC}{}_{A'}{}^{B'C'}(\mathscr{T}\mathscr{T}\phi)_{ABCB'C'}. \tag{5.7f}$$

Again, we substitute this into (4.2a) and (4.2b), collect each order of derivative, and decompose the resulting equations. There are then in total 26 equations and 20 variables. Simplifying those equations gives us theorem 31.

Definition 30. Let $V_{AB}{}^{A'B'}$, $W_{AB}{}^{A'B'}$, $X_{ABC}{}^{A'}$, and $Y_A{}^{A'B'C'}$ be Killing spinors on a Lorentzian manifold. They satisfy *auxiliary condition B* if there exist spinor fields $R_A{}^{A'}$, $T^{A'B'}$, U_{AB}, and $S_A{}^{A'}$ and a scalar field O such that

$$\mathscr{T}R = \frac{3}{8}\bar{\Psi}\overset{0,2}{\odot}W - \frac{3}{8}\Psi\overset{2,0}{\odot}W, \tag{5.8a}$$
$$\mathscr{D}R = 0, \tag{5.8b}$$
$$\mathscr{T}T = -\frac{1}{2}m\mathscr{C}^\dagger W, \tag{5.8c}$$
$$\mathscr{T}U = -\frac{1}{2}m\mathscr{C}W, \tag{5.8d}$$
$$\mathscr{C}T + \mathscr{C}^\dagger U = -m\mathscr{D}W - \frac{1}{5m}\Phi\overset{1,1}{\odot}\mathscr{D}W - \frac{9}{40m}\Psi\overset{3,0}{\odot}\mathscr{C}W + \frac{3}{8m}W\overset{2,1}{\odot}\mathscr{C}\Phi$$
$$- \frac{9}{20m}W\overset{2,2}{\odot}\mathscr{T}\Phi + \frac{3}{8m}W\overset{1,2}{\odot}\mathscr{C}^\dagger\Phi + \frac{3}{5m}W\overset{1,1}{\odot}\mathscr{T}\Lambda$$
$$- \frac{3}{10m}\Phi\overset{2,1}{\odot}\mathscr{C}W - \frac{3}{10m}\Phi\overset{1,2}{\odot}\mathscr{C}^\dagger W - \frac{9}{40m}\bar{\Psi}\overset{0,3}{\odot}\mathscr{C}^\dagger W$$
$$+ \frac{1}{20m}\mathscr{T}\mathscr{D}\mathscr{D}W, \tag{5.8e}$$

$$
\begin{aligned}
\mathscr{T}S ={}& \frac{3}{8}\overline{\Psi}\overset{0,2}{\odot}V - \frac{3}{8}\Psi\overset{2,0}{\odot}V + \frac{1}{2}m\mathscr{C}Y + \frac{1}{2}m\mathscr{C}^{\dagger}X, && (5.8\text{f})\\
\mathscr{C}S ={}& -m\mathscr{D}X - \frac{9}{20m}\Psi\overset{2,0}{\odot}\mathscr{D}X + \frac{3}{8}\mathscr{D}\mathscr{C}V - \frac{3}{2}\Phi\overset{1,2}{\odot}V - \frac{1}{m}\Psi\overset{3,0}{\odot}\mathscr{C}X\\
&+ \frac{18}{25m}X\overset{2,1}{\odot}\mathscr{C}\Phi - \frac{4}{5m}X\overset{3,1}{\odot}\mathscr{T}\Psi, && (5.8\text{g})\\
\mathscr{C}^{\dagger}S ={}& -m\mathscr{D}Y - \frac{9}{20m}\overline{\Psi}\overset{0,2}{\odot}\mathscr{D}Y - \frac{3}{8}\mathscr{D}\mathscr{C}^{\dagger}V + \frac{3}{2}\Phi\overset{2,1}{\odot}V - \frac{1}{m}\overline{\Psi}\overset{0,3}{\odot}\mathscr{C}^{\dagger}Y\\
&+ \frac{18}{25m}Y\overset{1,2}{\odot}\mathscr{C}^{\dagger}\Phi - \frac{4}{5m}Y\overset{1,3}{\odot}\mathscr{T}\overline{\Psi}, && (5.8\text{h})\\
\mathscr{T}O ={}& \frac{2}{3}m\Phi\overset{2,1}{\odot}X + \frac{2}{3}m\Psi\overset{3,0}{\odot}X - \frac{2}{3}m\overline{\Psi}\overset{0,3}{\odot}Y - \frac{2}{3}m\Phi\overset{1,2}{\odot}Y - \frac{1}{24}V\overset{2,1}{\odot}\mathscr{C}\Phi\\
&- \frac{4}{15}\Phi\overset{2,1}{\odot}\mathscr{C}V - \frac{3}{40}\Psi\overset{3,0}{\odot}\mathscr{C}V - \frac{1}{24}V\overset{1,2}{\odot}\mathscr{C}^{\dagger}\Phi - \frac{3}{40}\overline{\Psi}\overset{0,3}{\odot}\mathscr{C}^{\dagger}V\\
&- \frac{4}{15}\Phi\overset{1,2}{\odot}\mathscr{C}^{\dagger}V - \frac{1}{2}m^{2}\mathscr{D}V - \frac{8}{45}(\mathscr{D}V)\overset{1,1}{\odot}\Phi + \frac{1}{6}m\mathscr{D}\mathscr{C}Y\\
&- \frac{1}{6}m\mathscr{D}\mathscr{C}^{\dagger}X + \frac{8}{15}(\mathscr{T}\Lambda)\overset{1,1}{\odot}V - \frac{2}{5}V\overset{2,2}{\odot}\mathscr{T}\Phi. && (5.8\text{i})
\end{aligned}
$$

It is worth to note here that (5.8a) to (5.8e) and (5.8f) to (5.8i) are completely decoupled. They contain different variables from each other. Also, if V, W, X and Y are set to zero, we see that we get back the conditions for the first order operator, which is nice. Since, then, (5.8a), (5.8c), (5.8d), and (5.8f) would be the condition that S, R, U, and T are Killing spinors. (5.8b), (5.8e), (5.8g), and (5.8h) are then precisely auxiliary condition A, while (5.8i) is again just the existence of constant scalar field, so it adds no restrictions. It is hence comforting to see that the second order terms of (5.9a) and (5.9b) only have things involving V, W, X and Y as coefficients.

Theorem 31. *The massive Dirac equation has a second order symmetry operator if and only if there exist Killing spinors $V_{AB}{}^{A'B'}$, $W_{AB}{}^{A'B'}$, $X_{ABC}{}^{A'}$, and $Y_{A}{}^{A'B'C'}$ (not all zero) satisfying auxiliary condition B. The symmetry operator then takes the form*

$$
\begin{aligned}
\lambda ={}& -\frac{1}{6}m\chi\overset{0,1}{\odot}R + \frac{1}{2}m\chi\overset{0,1}{\odot}S + \frac{1}{3}\chi\overset{0,1}{\odot}\Phi\overset{1,2}{\odot}Y - \frac{1}{3}\chi\overset{0,1}{\odot}\mathscr{C}T - \frac{1}{6}\chi\overset{0,1}{\odot}\mathscr{C}\mathscr{D}Y\\
&+ \frac{1}{6}m\chi\overset{0,1}{\odot}\mathscr{D}W - \frac{1}{2}O\overset{0,0}{\odot}\phi - \frac{1}{20}(\mathscr{D}\mathscr{D}V)\overset{0,0}{\odot}\phi - \frac{1}{2}\phi\overset{0,0}{\odot}Q - \frac{1}{2}m\phi\overset{1,0}{\odot}U\\
&- \frac{1}{6}\phi\overset{1,0}{\odot}\mathscr{C}R - \frac{1}{6}\phi\overset{1,0}{\odot}\mathscr{C}S - \frac{1}{12}\phi\overset{1,0}{\odot}\mathscr{C}\mathscr{D}V - \frac{1}{12}\phi\overset{1,0}{\odot}\mathscr{C}\mathscr{D}W - \frac{1}{6}m\phi\overset{1,0}{\odot}\mathscr{D}X\\
&- \frac{1}{2}T\overset{0,2}{\odot}\mathscr{T}\chi - \frac{1}{2}(\mathscr{D}Y)\overset{0,2}{\odot}\mathscr{T}\chi + \frac{1}{4}mV\overset{1,2}{\odot}\mathscr{T}\chi - \frac{1}{2}mW\overset{1,2}{\odot}\mathscr{T}\chi\\
&+ \frac{1}{2}(\mathscr{C}Y)\overset{1,2}{\odot}\mathscr{T}\chi - \frac{1}{3}R\overset{1,1}{\odot}\mathscr{T}\phi - \frac{1}{3}S\overset{1,1}{\odot}\mathscr{T}\phi - \frac{1}{3}(\mathscr{D}V)\overset{1,1}{\odot}\mathscr{T}\phi\\
&- \frac{1}{3}(\mathscr{D}W)\overset{1,1}{\odot}\mathscr{T}\phi + \frac{2}{3}mX\overset{2,1}{\odot}\mathscr{T}\phi + \frac{1}{4}(\mathscr{C}V)\overset{2,1}{\odot}\mathscr{T}\phi + \frac{1}{4}(\mathscr{C}W)\overset{2,1}{\odot}\mathscr{T}\phi\\
&- \frac{2}{3}Y\overset{1,3}{\odot}\mathscr{T}\mathscr{T}\chi - \frac{3}{8}V\overset{2,2}{\odot}\mathscr{T}\mathscr{T}\phi - \frac{3}{8}W\overset{2,2}{\odot}\mathscr{T}\mathscr{T}\phi, && (5.9\text{a})
\end{aligned}
$$

$$\begin{aligned}\gamma = & -\frac{1}{2}\chi\overset{0,0}{\odot}O - \frac{1}{20}\chi\overset{0,0}{\odot}\mathscr{D}\mathscr{D}V + \frac{1}{2}\chi\overset{0,0}{\odot}Q + \frac{1}{2}m\chi\overset{0,1}{\odot}T - \frac{1}{6}\chi\overset{0,1}{\odot}\mathscr{C}^\dagger R + \frac{1}{6}\chi\overset{0,1}{\odot}\mathscr{C}^\dagger S \\ & - \frac{1}{12}\chi\overset{0,1}{\odot}\mathscr{C}^\dagger\mathscr{D}V + \frac{1}{12}\chi\overset{0,1}{\odot}\mathscr{C}^\dagger\mathscr{D}W + \frac{1}{6}m\chi\overset{0,1}{\odot}\mathscr{D}Y + \frac{1}{6}m\phi\overset{1,0}{\odot}R + \frac{1}{2}m\phi\overset{1,0}{\odot}S \\ & + \frac{1}{3}\phi\overset{1,0}{\odot}\Phi\overset{2,1}{\odot}X - \frac{1}{3}\phi\overset{1,0}{\odot}\mathscr{C}^\dagger U - \frac{1}{6}\phi\overset{1,0}{\odot}\mathscr{C}^\dagger\mathscr{D}X + \frac{1}{6}m\phi\overset{1,0}{\odot}\mathscr{D}W - \frac{1}{3}R\overset{1,1}{\odot}\mathscr{T}\chi \\ & + \frac{1}{3}S\overset{1,1}{\odot}\mathscr{T}\chi - \frac{1}{3}(\mathscr{D}V)\overset{1,1}{\odot}\mathscr{T}\chi + \frac{1}{3}(\mathscr{D}W)\overset{1,1}{\odot}\mathscr{T}\chi - \frac{2}{3}mY\overset{1,2}{\odot}\mathscr{T}\chi \\ & + \frac{1}{4}(\mathscr{C}^\dagger V)\overset{1,2}{\odot}\mathscr{T}\chi - \frac{1}{4}(\mathscr{C}^\dagger W)\overset{1,2}{\odot}\mathscr{T}\chi - \frac{1}{2}U\overset{2,0}{\odot}\mathscr{T}\phi - \frac{1}{2}(\mathscr{D}X)\overset{2,0}{\odot}\mathscr{T}\phi \\ & - \frac{1}{4}mV\overset{2,1}{\odot}\mathscr{T}\phi - \frac{1}{2}mW\overset{2,1}{\odot}\mathscr{T}\phi + \frac{1}{2}(\mathscr{C}^\dagger X)\overset{2,1}{\odot}\mathscr{T}\phi - \frac{3}{8}V\overset{2,2}{\odot}\mathscr{T}\mathscr{T}\chi \\ & + \frac{3}{8}W\overset{2,2}{\odot}\mathscr{T}\mathscr{T}\chi - \frac{2}{3}X\overset{3,1}{\odot}\mathscr{T}\mathscr{T}\phi. \end{aligned} \tag{5.9b}$$

These results are much more complicated than for the first order symmetry operator and stating them too in terms of covariant spinor derivatives would not be useful.

6
Conclusion

In conclusion, the massive Dirac equation is well-suited for applying CAS to finding symmetry operators. While the derivation of the unsimplified conditions such as (4.22a) to (4.24f) can be automated, the simplification and interpretation still needs to be performed by an interacting human. The latter two being time-consuming tasks in comparison.

While we have found that there are no nontrivial zeroth order symmetry operators, auxiliary condition A of definition 28 and auxiliary condition B of definition 30 are differential equations involving Killing spinors whose solvability are equivalent to the existence of first and second order symmetry operators respectively. We managed to interpret auxiliary condition A in fairly direct geometrical terms, but auxiliary condition B was too complex to be meaningfully interpreted.

There are several directions for further work visible from here. A naive direction would be to apply these same methods to find higher order symmetry operators. With the explosion in complexity that we saw when going from first to second order, this is not likely to yield informative results. However, if the simplification of such results is automated, the interpretation of them may become more feasible.

One other very conceivable idea for future work is to apply the same methods to other equations containing two-spinors. Since Bäckdahl and co-authors [3] already have explored up to spin-1, the *Rarita–Schwinger equation* comes to mind, as it describes spin-$\frac{3}{2}$ particles. Some work previous has been done on this subject as well [13].

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A

Spinors in curved spacetime

Definition 32. Let X be a path-connected topological space and $x \in X$. Then the *fundamental group* $\pi_1(X)$ of X is the group of equivalence classes under continuous deformation of closed loops in X through x. The group operation $*$ acts on two closed loops $\gamma_1,\ \gamma_2\colon [0,1] \to X$ by

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & , t \leq \frac{1}{2} \\ \gamma_2(2t) & , t > \frac{1}{2} \end{cases} .$$

The topological space X is simply connected if and only if its fundamental group is trivial for every x.

Definition 33. Let X be a topological space. A *Covering space* of X is a space $\widehat{X}$ together with a map $p\colon X \to \widehat{X}$ such that for each $x \in X$, there is a neighbourhood $U \subset X$ of x with each connected component of $p^{-1}(U)$ mapping homeomorphically onto U.

Definition 34. A covering space $(\widehat{X}, \pi)$ of X is a *universal covering space* if for any other covering space $(\widehat{X}, p)$ there exists a unique map $q\colon \widehat{X} \to \widehat{X}$ such that $q \circ \pi = p$.

Two universal covering spaces of the same space are isomorphic. It can be shown that, given a point $p \in X$, $\widehat{X}$ defined as $S/{\sim}$ with S being the set of curves with one end in p and $\sim$ being the equivalence relation of homotopy indeed is a universal covering space of X. The key property here being that $S/{\sim}$ is simply connected.

A covering space $(\widehat{X}, p)$ can be made to inherit various structures from X. If X is a manifold, $\widehat{X}$ becomes a manifold by requiring that p is a diffeomorphism on each connected component of $p^{-1}(U)$. Similarly, if X is a Lie group, $\widehat{X}$ can be made a Lie group by requiring p to be a group homomorphism. These are called the *universal covering manifold* and *universal covering group* of X respectively.

$\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of the proper Lorentz transformations. This is the reason we have defined spinors on $\mathrm{SL}(2,\mathbb{C})$ in the previous section. $\mathrm{SL}(2,\mathbb{C})$ is a double cover, which is why a rotation by 2π of a spinor will yield minus the original spinor.

Definition 35. A *section* in a fiber bundle (E, B, F, π) is a map $s\colon B \to E$ such that $\pi \circ s = \mathrm{id}$.

As an example, if $F = \mathbb{R}$ and $E = F \times B$, any section is just a graph of a real-valued function on B. More generally, if M is a manifold, T_pM denotes its tangent space at a point $p \in m$, and TM denotes the (disjoint) union of such tangent spaces, then the set of sections in the smooth vector bundle $(TM, M, T_pM, \text{“}T_pM \mapsto p\text{”})$ is the set of tangent vector fields on M.

Definition 36. If (E, B, F, π) is a vector bundle, then a *frame* is a set of sections s_i, $i = 1, 2, \ldots r$, such that for each $b \in B$ $s_i(b)$ is a basis for F.

If G is a Lie group and B is a manifold, then a smooth map $\phi\colon G \times B \to B$ is a *left action* if $g \mapsto \phi(g, \cdot)$ is a homomorphism and $b \mapsto \phi(g, b)$ is a diffeomorphism for each $g \in G$. A left action is *free* if only the action associated with the identity element has a fixed point.

Definition 37. A *principal fiber bundle* is a fiber bundle where the fiber is a Lie group G with a free left action ϕ that acts on the total space E so that the projection π is a bijection between the G-orbits of E and the base space B.

Given a principal smooth fiber bundle (E, B, G, π) and a free[1] left action χ of G on some manifold F we may build an associated smooth fiber bundle with F as its fiber. To this end, define the left action ψ on $E \times F$ by

$$\psi\colon (g, b, f) \mapsto (\phi(g, b), \chi(g, f)).$$

We then define a new total space E' as the orbits of $E \times F$ and $\pi'[(e, f)] = \pi(e)$. This is a well-defined map by definition 37. Then for each such $U \subset B$ from definition 2, $(\pi')^{-1}(U)$ is homeomorphic to $U \times F$, and so (E', B, F, π') is smooth a fiber bundle. This fiber bundle is called the *fiber bundle associated with* (E, G, B, ϕ).

Given a principal fiber bundle, we may pick $F = G$ and χ as just left action of G. The resulting E' is isomorphic to E and so every principal fiber bundle may also be viewed as a smooth fiber bundle. Some authors do not care about the difference and so include both F and G in their definition of fiber bundle.

We may now finally define spinors in curved spacetime. First, consider the principal fiber bundle (E, B, G, π) where G is the proper Lorentz group and B is the spacetime manifold. The universal covering group of the proper Lorentz group is $\mathrm{SL}(2, \mathbb{C})$. So consider the principal fiber bundle $(E, B, \mathrm{SL}(2, \mathbb{C}), \pi)$. Its associated fiber bundle is $(E', B, \mathbb{C}^2, \pi')$. *Spinor fields* are sections of this associated fiber bundle.

Visually, given a spacetime M, the idea is that we start with the principal fiber bundle of oriented time-oriented orthonormal bases of Minkowsky space. Each fiber is then diffeomorphic to the proper Lorentz group. Then we “unwrap” each fiber to produce a principal $\mathrm{SL}(2, \mathbb{C})$ bundle over M. The spinor bundle then is constructed as the fiber bundle associated to this principal bundle with fiber $F = \mathbb{C}^2$, where X is taken to be the natural action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$.

A.1 Sources and further reading

The information presented here can be found largely in Wald’s General Relativity [29, ch. 13] and in Madsen and Tornehave’s From Calculus to Cohomology [21,

[1] if χ is not free, just replace F with its orbits under G.

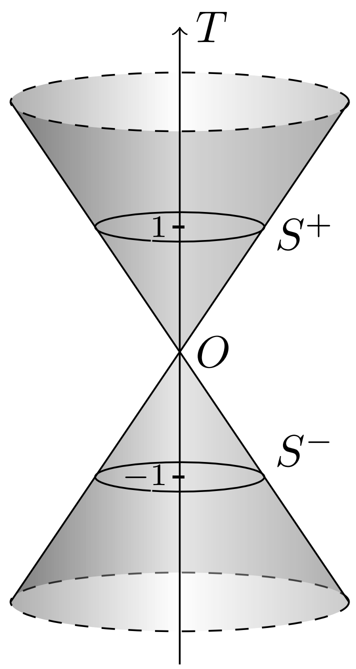
ch. 15]. In Nakahara's Geometry, Topology and Physics [24, ch. 4] there is an an encyclopedic description of homotopy groups in a physics context.

B

The celestial sphere

In this appendix, a detour is taken through geometric complex analysis in order to contextualize and interpret the spin transformation group as Möbius transformations.

Figure B.1: The null cone.



Any given observer O somewhere in spacetime defines a null cone with its vertex in O. Directions in the part of this cone with $T < 0$, the *past cone*, may naturally be identified with O's field of vision. We call the projective space of such directions *the celestial sphere*, S^-. The celestial sphere may be endowed with a set of (homogenous) coordinates by setting $T = -1$. Similarly, $T = 1$ defines a set, S^+, of future null directions. See figure B.1. Either of these may be regarded as a Riemann sphere Σ associated with a complex plane $\mathbb{C}$.

Definition 38. The *stereographic projection* $\mathcal{P}\colon \Sigma \to \mathbb{C}$ of a point P on the unit sphere Σ in $\mathbb{R}^3$ to a complex plane $\mathbb{C}$ that slices the sphere through its origion perpendicularly to its north pole N is the complex number ζ that lies on the intersection between $\mathbb{C}$ and line NP. See figure B.2a.

We may get explicit formulae for $\mathcal{P}$ by considering the plane with N, P, and Σ's origin O as in figure B.2b. Let $\zeta = x + iy$ and let $P = (X, Y, Z)$ in a coordinate

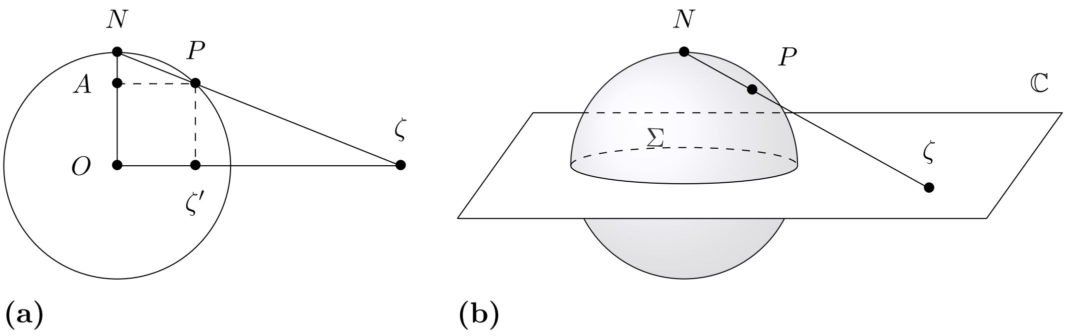


Figure B.2

system where the xy and XY coordinates overlap. Then $ON\zeta \sim ANP$, so by the power of similar triangles,

$$|\zeta| = \frac{|\zeta'|}{1 - Z}.$$

But $\zeta = x + iy$ and $\zeta' = X + iY$ have by construction the same argument. Thus

$$x + iy = \frac{X + iY}{1 - Z},$$

from which we can read off expressions for x and y in terms of X, Y, and Z. It is also possible to go in the other direction, and show that

$$X = \frac{\zeta + \bar{\zeta}}{\zeta\bar{\zeta} + 1}, \quad Y = \frac{\zeta - \bar{\zeta}}{\zeta\bar{\zeta} + 1}, \quad \text{and} \quad Z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}. \tag{B.1}$$

Definition 39. A *Möbius* transformation is a map on $\mathbb{C}$ of the form

$$z \mapsto \frac{a - bz}{c - dz}.$$

In order for the Möbius transformations to form a group, we demand that $ad - bc \neq 0$.

There is a remarkable connection between Lorentz transformations and Möbius transformations. We call a Lorentz transformation *restricted* if it preserves both space and time orientation. It is clear that such a transformation maps the celestial sphere to itself.

Theorem 40. *The complex mappings that correspond to the restricted Lorentz transformations on the celestial sphere are the Möbius transformations! More precisely, stereographic projection defines an isomorphism between the two Lie groups.*

Proof. We prove each direction separately. To show that each Möbius transformation maps homomorphically to a restricted Lorentz transformation, begin by considering a complex number ζ in homogenous coordinates $\zeta = \frac{\xi}{\eta}$. A Möbius transformation f acts on ζ by a complex matrix

$$f\colon \begin{bmatrix} \xi \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Since the four coeffients in a Möbius transformation are unique only up to scaling with a common complex factor, we may impose on this matrix that its determinant $ad - bc = 1$. The group, $\mathrm{SL}(2, \mathbb{C})$, of these matrices are the *spin transformations*. Since the two spin transformations A and $-A$ define the same Möbius transformation, the spin transformations are a double cover of the Möbius transformations. As we saw in appendix A, they are the universal covering group.

By (B.1), we may choose (real) coordinates

$$T = \xi\bar{\xi} + \eta\bar{\eta}, \quad X = \xi\bar{\eta} + \eta\bar{\xi}, \quad Y = \frac{1}{i}(\xi\bar{\eta} - \eta\bar{\xi}), \quad Z = \xi\bar{\xi} - \eta\bar{\eta}. \tag{B.2}$$

We now want to show that a Möbius transformation of ζ will yield a real linear transformation of (T, X, Y, Z). To this end, rewrite (B.2) as

$$\begin{bmatrix} T+Z & X+iY \\ X-iY & T-Z \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \begin{bmatrix} \bar{\xi} & \bar{\eta} \end{bmatrix}.$$

Then, if A is the matrix of f,

$$f\colon \begin{bmatrix} T+Z & X+iY \\ X-iY & T-Z \end{bmatrix} \mapsto A \begin{bmatrix} T+Z & X+iY \\ X-iY & T-Z \end{bmatrix} A^\dagger \tag{B.3}$$

This is a linear transformation, and it is real since it maps Hermitian matrices to Hermitian matrices. Also, taking the determinant of (B.3) shows that f preserves the form $T^2 - X^2 - Y^2 - Z^2$, even when it is nonzero, and so the action of f is a Lorentz transformation on Minkowsky space. It is a restricted Lorentz transformation since it is continuous with identity.

To show the converse, that each restricted Lorentz transformation maps homomorphically to a Möbius transformation, it is enough to show that it is the case for X-, Y-, Z-rotations and for Z-boosts. These transformations generate the restricted Lorentz group. A rotation around the z-axis is the same as rotating the $\mathbb{C}$ around 0, so this is clearly a Möbius transformation. Rotating around the X-axis maps

$$\begin{aligned} X &\mapsto X\cos\theta - Y\sin\theta \\ Y &\mapsto X\sin\theta + Y\cos\theta \end{aligned}$$

Looking at (B.3), this corresponds to

$$A = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}.$$

There is no conceptual difference for Y-rotation. We can rewrite a Z-boost as

$$\begin{aligned} T+Z &\mapsto w(T+Z) \\ T-Z &\mapsto \frac{1}{w}(T-Z) \end{aligned}$$

so that it's easy to identify

$$A = \begin{bmatrix} w^{\frac{1}{2}} & 0 \\ 0 & w^{-\frac{1}{2}} \end{bmatrix}$$

from (B.3). Both of these A yield Möbius transformations, and so we are done. □

This proof also shows that a restricted Lorentz transformation is uniquely determined by how it acts on the celestial sphere.

One quite visual consequence of theorem 40 is that the restricted Lorentz transformations are the most general conformal maps on the celestial sphere that preserve circles. It follows from the ensuing two lemmas, lemmas 42 and 43.

Remark 41. *Actually, conformality follows from the property of mapping circles to circles. This relation can be understood by convincing oneself that if a map* $f : \mathbb{C} \to \mathbb{C}$ *has continous partial derivatives, then it maps infinitesimal circles to infinitesimal ellipses, and that angles are preserved if and only if those infinitesimal ellipses are also circles.*

Lemma 42. *Stereographic projection bijectively maps circles between the Riemann sphere and* $\mathbb{C}$*.*

For a proof of this, see [25, p. 142].

Lemma 43. *The Möbius transformations are the most general invertible conformal maps on* $\mathbb{C}$ *that preserves circles*[1]*.*

Proof. Möbius transformations preserve circles since they can be thought of as compostions of combinations of the following four maps,

1. translation $z \mapsto z + a$
2. dilation and rotation $z \mapsto bz$
3. reflection in the real axis $z \mapsto \bar{z}$
4. reflection in the unit circle $z \mapsto \frac{1}{\bar{z}}$,

all of which preserve circles. It is trivial to see why this is true for items 1 to 3, but to see it for item 4, we will construct reflection in the unit circle from stereographic projections. For, consider the composition $\mathcal{PRP}^{-1} : \mathbb{C} \to \mathbb{C}$ of inverse stereographic projection, reflection of Σ in the XY-plane, and stereographic projection. $\mathcal{PRP}^{-1}$ is by lemma 42 anti-conformal and maps circles to circles. Especially, $\mathcal{PRP}^{-1}$ maps all points on the unit circle to themselves. But if two (anti-) conformal functions agree on a curve of of nonzero length then they agree on the whole of $\mathbb{C}$, and so $\mathcal{PRP}^{-1}(z) = \frac{1}{\bar{z}}$.

Now we want to show that Möbius transformations are the most general maps with this property. The Riemann mapping theorem states that between any two simple closed smooth curves in $\mathbb{C}$ there is a conformal bijection. Furthermore, this bijection is completely determined by the image of three distinct points on the first curve. But any three distinct points determine a Möbius transformation. □

what we have mentioned before that the spin transformations, $\mathrm{SL}(2, \mathbb{C})$, are the universal cover of the proper Lorentz group. We may summarize the connection between spin transformations and spinors by the following geometric statemental, “spinors are the things on Minkowsky space for which (ξ, η) are coordinates”.

[1] Lines are also circles in this context.

B.1 Sources and further reading

Many of these arguments are based on Penrose and Rindler [27, ch. 1], but also on Needham's treatment of Möbius transformations [25, ch. 3] in his excellent book on complex analysis through geometry.

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